THE UNIVERSITY OF CHICAGO

ON THE MOTIVIC FILTRATION OF TR

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
FAIDON ANDRIOPOULOS

CHICAGO, ILLINOIS
JUNE 2024

Στον πατέρα μου, Χρήστο,

εις μνήμην

Στη μητέρα μου, Χαρά, και στον αδελφό μου, Γιάννη,

για τα πάντα

Καθώς και στο δάσκαλό μου, Μιχάλη Πατσαλιά,

γιατί χωρίς την έμπνευση, την ενθάρρυνση και την υποστήριξή του,

αυτό το ταξίδι ενδέχεται να μην είχε ξεκινήσει ποτέ

Δοκίμαζε, συνέχιζε τά γυμνάσματά σου.
Κοίτα πού κ΄ ἡ θάλασσα ἀνακατεύει συνεχῶς
οὐρανό καί φύκια
πασχίζοντας νά βρεῖ το σωστό της χρῶμα.

Ἄρης Ἀλεξάνδρου, *Γύμνασμα*

CONTENTS

ACKNOWLEDGMENTS

Tired of lying in the sunshine/ Staying home to watch the rain/ You are young and life is long/ There is time to kill today/ And then one day you find/ Ten years have got behind you/ No one told you when to run/ You missed the starting gun. These lyrics of Pink Floyd, besides having a nontrivial connection with parts of this thesis, do bear some truth.

The end of my studies as a graduate student at the University of Chicago roughly coincides with the end of my first decade in adulthood, since the final year of high school and the first year of my undergraduate studies. It was by no means an easy period of time, but it was filled with the passion and anticipation of pursuing mathematics, dreaming of reaching the very core of the subject. It was mainly through these and because of these, that I had the extreme luck of meeting and interacting with several people, each of whom had a significant impact on my life, and in particular, on the production of this thesis.

First and foremost, I would like to express my deep gratitude towards my thesis advisor, Prof. Alexander Beilinson. Meeting Sasha and spending time with him was a revelation for me, a feeling that is difficult to express with words. His unique viewpoint, not only in mathematics, but life in general, had an enormous impact on me and our interaction was a formative experience. In short, I crossed the Atlantic with the hope of finding an academic advisor and I was extremely lucky to have found a Mentor, a Teacher, and a Friend.

I would like to thank Prof. Akhil Mathew for being in my Thesis Committee, for co-organizing a great conference in Oberwolfach back in October of 2019, which prompted me to start thinking seriously about THH and algebraic K-theory, as well as for suggesting that I look at Hesselholt's conjectures on the absolute de Rham–Witt complex; thinking about this helped me complete a picture. I would also like to thank Prof. Vladimir Drinfeld, for his beautiful work in the theory of prismatization. His lectures in the geometric Langlands seminar during the academic year 2018-2019 pushed me to think hard on the subject.

Special thanks to a small, yet extended, group of roommates, who were the closest thing to a family during my years in Chicago: Kostas Psaromiligkos, together with whom we first embarked on this journey, boarding that troublesome flight, back in September of 2017, Iason Kountouridis, who was an ideal companion on the difficult road to arithmetic geometry, back in the house, and in the office, Nikiforos Mimikos-Stamatopoulos, for the long walks, the fiery political conversations, and the deep, shared spirit of comradeship, the honorary Greek/honorary roommate Shago Chaves Aguilar, for his unique enthusiasm in math, football, politics, and life in general, as well as Chengyang Bao, who brought some cheerful vibes during the last year, in what otherwise was a dull group of boys.

Thanks to some great friends, from whom I learned a lot of math and with whom I share some amazing memories: my academic siblings Charly di Fiore, Olivier Martin, and Nacho Darago, as well as Gal Porat, Pallav Goyal, Billy Lee, Mathilde Gerbelli-Gauthier, Daniel Campos, Adan Medrano Martin del Campo, Mark Cerenzia, Antonis Zitridis, Oliver Wang, Peter Huxford, Henrik Matthiesen, Jamil Chaker, and the entire math community here.

An apology and a thanks to the local community, especially to the strangers of 53rd street: the black man who told me to keep my head high and the hispanic woman who urged me to smile. The solidarity they practice is rarely found in upper classes or academic circles [Gra19]. This thanks also extends to the friends from GSU.

Back in Greece, I would like to thank the teachers I've had in the public schools and universities. They know who they are and how great of an impact they've had on me. I would like to particularly mention my high school math teacher, Michalis Patsalias, who inspired me to start this journey many years ago. He faced several challenges in his own life, which deprived him from following a very promising academic path as a research mathematician. Yet, when I faced difficulties in my own life, he was there to encourage me and give me hope, strength, and support. I would also like to thank Prof. Apostolos Giannopoulos, whose tireless efforts inspired generations of young students in mathematics. His Real Analysis lectures, which I attended as a first-year undergraduate student, were a shining beacon during what otherwise was a dark time in my life.

Thanks to some unique friends with whom we've shared our common experiences, from our early undergraduate years to the realm of research mathematics: Marios Apetroaie and Angeliki Menegaki, the big group of Ilias Andreou, Dimitris Gazoulis, Miltos Karakikes, Theodoros Lagiotis, Spiros Lentas, and Stavros Papathanasiou, as well as Giorgos Politopoulos. Special thanks to Adrianos Athanasiadis-Fotopoulos for the many years of discussions and common explorations in math, physics, engineering, and the music of Bach, with the sole purpose of satisfying our curiosity.

Thanks to the many friends from back home who supported me and cheered me up, despite the years apart. I would like to particularly mention and thank Nikos Kantartzis, for his constant spamming as his way of showing affection, as well as Zoi Tzamtzi, who shared music and skyped from time to time, in order to check that I'm doing ok.

Thanks to Giorgos Sakellaris and Valia Gazaki for their kind help in preparing me for the many things I would encounter in Chicago. Sincere thanks to Takis Giannakopoulos, Malvina Papadaki, and the Karis and Andriotis families, for their generocity. In particular, Kostas Karis discussed with me and gave me my first serious science book many years ago, while Nikos Andriotis provided support of various kinds, during difficult times. Kind thanks to my uncle Yannis, for having me in his mind, during my stay at this side of the globe.

In conclusion, I would like to thank my family for inspiring and supporting me throughout the pursuit of my dreams, helping me overcome the obstacles of life. I would like to thank my mom, for her continuous love and care, and my bro, for being the best friend anyone could wish for. Finally, I would like to thank the person who taught me by example: to be kind, just, and generous, to work hard, to think, to question, to appreciate mathematical beauty, to love nature, and to enjoy life. Σ' ευχαριστώ μπαμπά. Το βάρος της απουσίας σου είναι αβάσταχτο. *Οἱ μὲν ἄλλοι πάντες ἄργυρον ἢ χρυσὸν ἤ τινα τῆς ἄλλης πολυτελείας ἔλαβον, Αἰνείας δὲ τὸν πατέρα γεγηρακότα τελέως ἀράμενος ἐπὶ τοὺς ὤμους ἐξήνεγκεν.* [Old39]

ABSTRACT

Given a p-complete, animated ring S, we define a filtration on the Nygaard-complete, absolute prismatic cohomology $\mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S$, which we call the *$r$-Nygaard filtration.* This is obtained by suitably gluing r-copies of the usual Nygaard filtration and, in the case that S is a perfectoid ring, it corresponds to the ξ_r-adic filtration on $\mathbb{A}_{\mathrm{inf}}$.

Using this, we study the motivic filtration of topological restriction homology $\mathrm{TR}^r(S;\mathbb{Z}_p)$ and of its S^1-homotopy fixed points. We also explore connections with topological cyclic homology. Finally, we apply our results to the case of $A\Omega$-cohomology, drawing comparisons to the first results of Bhatt–Morrow–Scholze, which also marked the beginning of the prismatic story.

CHAPTER 1

INTRODUCTION

1.1 Main results of the Thesis

Prismatic cohomology is a recently introduced p-adic cohomology theory due to Bhatt–Scholze, which is developed in their groundbreaking work [BS22] and relies on the notion of a *prism* (A, I). Given a p-complete ring S over $\overline{A} := A/I$, one is able to construct its *relative prismatic cohomology* $\Delta_{S/\overline{A}}$, via the prismatic site. These are the main objects of study of [BS22]. Prismatic theory has already proved invaluable in tackling several mathematical questions in various mathematical areas. This is a consequence of the fact that it is believed to be the *right* p-adic cohomology theory, as it specializes to and refines all other previously known ones: de Rham, Hodge–Tate, crystalline, étale, etc.

A first attempt to construct a good universal p-adic cohomology theory was made in the work of Bhatt–Morrow–Scholze [BMS18]. The authors mainly used techniques from perfectoid geometry to construct cohomological invariants over Fontaine's period ring $\mathbb{A}_{\mathrm{inf}}$. This was closely related to the theory of *Breuil–Kisin–Fargues modules* and, via Fargues' theorem [SW20], to *mixed characteristic shtukas* with one leg over spa $C^\flat$. Thus, this falls under the general scope of Scholze's insight, who suggested that p-adic cohomology theories are expected to have shtuka-like properties [Sch18].

Further algebro-geometric approaches are those of Drinfeld [Dri20] and Bhatt–Lurie [BL22a; BL22b; Bha22], which use the language of stacks in an essential way, in order to study prismatic invariants. Their viewpoint is based on a stacky reformulation of the prismatic site of Bhatt–Scholze, shedding new light on geometric phenomena of *absolute prismatic cohomology* and especially to structure associated with it, such as the *Hodge–Tate cohomology*, the *Nygaard filtration*, and the categories of *prismatic crystals/gauges*.

An approach to absolute prismatic cohomology, which is of different flavour, is the second work of Bhatt–Morrow–Scholze [BMS19], as it uses tools from homotopy theory. Interestingly enough, this story is opposite to the stacky one, as shown in the study of the *filtered prismatization stacks* of [Dri20] and [Bha22]. In [BMS19], the authors constructed and studied *motivic filtrations* of *topological Hochschild homology* THH and other associated invariants, in the case of p-complete rings. In particular, the associated graded pieces of these motivic filtrations can be expressed in terms of structure related to prismatic cohomology. The inspiration for this work came from the theory of motives, in which the algebraic K-theory of a well-behaved scheme carries a motivic filtration, whose graded pieces are identified with *motivic cohomology.* In fact one of the many applications of [BMS19], was the identification of syntomic cohomology with p-adic étale motivic cohomology, as a consequence of the approximation of algebraic K-theory using topological cyclic homology, via the trace map [CMM21].

The classical approach to topological cyclic homology TC was paved via the use of another invariant of THH, called *topological restriction homology* TR. This is a rich invariant which has a deep connection to algebraic K-theory, but is also closely related to the Witt vector functor. In particular, it is equipped with a Frobenius endomorphism $\mathrm{F} : \mathrm{TR} \to \mathrm{TR}$, which gives rise to TC, via the following homotopy fibre:

$$\mathrm{TC} = \mathrm{fib}\left(1 - \mathrm{F} : \mathrm{TR} \longrightarrow \mathrm{TR}\right)$$

In fact, there is more to say about the relation between TR and K-theory. On one hand, TR can be recovered via the so-called curves in K-theory [McC21], while on the other hand, it is related, via Goodwillie calculus, to another K-theoretic invariant, called cyclic K-theory $\mathrm{K}^{\mathrm{cyc}}$ [LM12; HS19; Nik20].

Attempting to see how these different approaches to prismatic cohomology and motivic phenomena in the p-adic world relate to each other, one is led to several natural questions:

Ques 1. What is the role of the Witt vector functor in prismatic cohomology, given their importance in the theory of δ-rings and the approach via prismatization?

Ques 2. What is the motivic filtration of TR^r and in terms of what structure, related to prismatic cohomology, can we interpret its graded pieces with?

Ques 3. How does the de Rham–Witt complex fit in the prismatic formalism?

Ques 4. If we wish to see the prismatic formalism as a variant of shtukas with one leg over Spa $C^\flat$, under their correspondence with Breuil–Kisin–Fargues modules, then is there any part of the prismatic formalism which captures the information of shtukas with more than one leg?

These questions, among others, motivated the author throughout his studies. In this work, we aim to tackle some of these. A general perspective, together with certain conjectures, may be found in the last chapter of the thesis. Expanded versions of some of the results we present here may be found in [And24a] and [And24b], which are currently in preparation.

In this thesis, using both algebraic and homotopy theoretic techniques, we study two problems, which are interlinked to each other. On the one hand, we study the motivic filtration of topological restriction homology $\mathrm{TR}^r(S;\mathbb{Z}_p)$ and of its S^1-homotopy fixed points $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$.

On the other hand, we introduce a filtration $\mathcal{N}_r^{\geq \bullet}\widehat{\mathbb{\Delta}}_S$, for every $1 \leq r \leq \infty$, by suitably gluing r copies of the Nygaard filtration $\mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_S$, and study some of its properties. It follows that we can express the graded pieces of the motivic filtrations of $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ and $\mathrm{TR}^r(S;\mathbb{Z}_p)$ through $\mathcal{N}_r^{\geq \bullet}\widehat{\mathbb{\Delta}}_S$, as well as those of some related homotopy-theoretic invariants. We also pursue connections with more classical cohomological invariants in the setting of positive/mixed characteristic.

Note that it is possible to define the r-Nygaard filtration on the non-completed version of prismatic cohomology $\mathcal{N}_r^{\geq \bullet}\mathbb{\Delta}_S$. We provide a brief explanation on how this is related to a relative de Rham–Witt comparison theorem, together with certain future possible directions of research, in the last chapter.

We break our main theorem in two parts: first the algebraic data and then their relation to the homotopy theoretic invariants.

Theorem 1.1.1 (The r-Nygaard filtration). *Let S be a p-complete, animated ring. For $1 \leq r \leq \infty$, via the following iterated product construction, we associate a certain filtration to the Nygaard-complete prismatic cohomology $\widehat{\Delta}_S$, which we call the* r-Nygaard filtration*:*

$$\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\} := \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \times_{\widehat{\Delta}_S\{i\}} \cdots \times_{\widehat{\Delta}_S\{i\}} \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \tag{1.1}$$

In this iterated product construction, the maps on the left are the canonical inclusion maps $\iota : \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \hookrightarrow \widehat{\Delta}_S\{i\}$, while the maps on the right correspond to the divided Frobenius $\varphi_i : \mathcal{N}^{\geq i}\widehat{\Delta}_S \to \widehat{\Delta}_S$.

The r-th iteration of Frobenius takes the i-th filtered piece of the r-Nygaard filtration $\mathcal{N}_r^{\geq i}\Delta_S$ to the i-th power $I_r^i\Delta_S$ of the generalized ideal $I_r := I \otimes \varphi^* I \otimes \cdots \otimes (\varphi^{r-1})^* I$, where the tensor product is taken over the Frobenius endomorphism φ of Δ_S. Therefore, we have a canonical inclusion map and a divided r-Frobenius *map:*

$$\iota : \mathcal{N}_r^{\geq i}\Delta_S\{i\} \hookrightarrow \Delta_S\{i\} \qquad \varphi_{r,i} : \mathcal{N}_r^{\geq i}\Delta_S\{i\} \longrightarrow \Delta_S\{i\}$$

For $1 \leq r < \infty$, we denote by $\Delta_S^{\mathrm{HT},r}$ the r-Hodge–Tate cohomology*, which lies in the following commutative square:*

$$\begin{tikzcd} \mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\} \arrow[r, "\varphi_{r,i}"] \arrow[d] & \widehat{\Delta}_S\{i\} \arrow[d] \\ \mathcal{N}_r^{i}\widehat{\Delta}_S\{i\} \arrow[r] & \Delta_S^{\mathrm{HT},r}\{i\} \end{tikzcd} \tag{1.2}$$

where $\mathcal{N}_r^i\Delta_S$ is the i-th graded piece of the r-Nygaard filtration and $\mathrm{gr}\,\varphi_{r,i}$ is the graded *version of the r-divided Frobenius, mapping to the r-Hodge–Tate cohomology.*

There exist natural Restriction, Frobenius, and Verschiebung maps for the filtered invariants and, therefore, for the graded ones as well:

$$\begin{tikzcd} \mathcal{N}^{\geq i}_{r+1}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{R},\ \mathrm{F}"] \arrow[d] & \mathcal{N}^{\geq i}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[d] \\ \mathcal{N}^{i}_{r+1}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{R},\ \mathrm{F}"] & \mathcal{N}^{i}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N}^{\geq i}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{V}"] \arrow[d] & \mathcal{N}^{\geq i}_{r+1}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[d] \\ \mathcal{N}^{i}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{V}"] & \mathcal{N}^{i}_{r+1}\widehat{\mathbb{\Delta}}_S\{i\} \end{tikzcd}$$

Taking the limit with respect to the Restriction maps $\mathcal{N}^{\geq i}_{\infty}\widehat{\mathbb{\Delta}}_S\{i\} \simeq \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}^{\geq i}_{r}\widehat{\mathbb{\Delta}}_S\{i\}$, *we obtain Frobenius endofunctors*

$$\begin{tikzcd} \mathcal{N}^{\geq i}_{\infty}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{F}"] \arrow[d] & \mathcal{N}^{\geq i}_{\infty}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[d] \\ \mathcal{N}^{i}_{\infty}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{F}"] & \mathcal{N}^{i}_{\infty}\widehat{\mathbb{\Delta}}_S\{i\} \end{tikzcd}$$

By passing to the 0*-th associated graded piece for the r-Nygaard filtration on* $\widehat{\mathbb{\Delta}}_S$, *one is able to obtain the r-truncated Witt vectors of S:*

$$\mathcal{N}^{0}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \simeq \mathrm{W}_r(S) \tag{1.3}$$

Notice that this equivalence respects the natural structure of Restriction, Frobenius, and Verschiebung, which we explained above.

Finally, as a simple example, notice that in the case S is a perfectoid ring, the r-Nygaard filtration on its prismatic cohomology is just the ξ_r-adic filtration on $\mathbb{A}_{\mathrm{inf}}(S)$. *The natural symmetries* R, F, V *interact well with the Fontaine-style maps* ϑ_r, $\widetilde{\vartheta}_r$, *as explained in [BMS18, Sec. 3].*

The structure of the r-Nygaard filtration arises in the theory of topological Hochschild homology, as expected from the work of Bhatt–Morrow–Scholze [BMS19]. In particular, one can read the data of the graded/filtered pieces of the r-Nygaard filtration from studying the motivic filtration of topological restriction homology and of its S^1-fixed points, respectively. The theorem reads as follows:

Theorem 1.1.2 (The motivic filtration of TR^r)**.** *Let S be a p-complete, quasisyntomic ring. The following hold:*

1) *For $1 \leq r \leq \infty$, the invariants $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(S;\mathbb{Z}_p)$ are equipped with complete, exhaustive, descending, multiplicative, $\mathbb{Z}$-indexed* motivic filtrations*:*

$$\mathrm{Fil}^{\bullet}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{Fil}^{\bullet}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)$$

which are the quasisyntomic sheafifications of their respective double speed Postnikov filtrations, from the quasiregular - semiperfectoid case. Passing to their associated graded pieces, these can be identified with:

$$\begin{cases} \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq R\Gamma_{syn}\Big(S, \tau_{[2i-1,2i]} \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1}\Big) \\ \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p) \simeq R\Gamma_{syn}\Big(S, \tau_{[2i-1,2i]} \mathrm{TR}^r(-;\mathbb{Z}_p)\Big) \end{cases}$$

by applying quasisyntomic descent from the quasiregular - semiperfectoid case, in which both are identified with two-term complexes.

2) *Let us denote by $\mathrm{gr}^{i,\mathrm{odd}}_{\mathcal{M}}$ the corresponding quasisyntomic sheafification of odd homotopy groups π_{2i-1} and by $\mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}}$ the corresponding quasisyntomic sheafification of even homotopy groups π_{2i}, of either $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ or $\mathrm{TR}^r(S;\mathbb{Z}_p)$. Then $\mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}}$ is the 0^{th} cohomology group of the two-term complex, while $\mathrm{gr}^{i,\mathrm{odd}}_{\mathcal{M}}$ is the 1^{st} cohomology group. For these, the following identifications hold:*

For $1 \leq r < \infty$, the even parts can be expressed in terms of the r-Nygaard filtration on

Nygaard-completed prismatic cohomology $\widehat{\mathbb{\Delta}}_S$*:*

$$\begin{cases} \mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}^{\geq i}_r \widehat{\mathbb{\Delta}}_S\{i\}[2i] \\ \mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \mathcal{N}^{i}_r \widehat{\mathbb{\Delta}}_S\{i\}[2i] \end{cases}$$

On the other hand, the odd parts $\mathrm{gr}^{i,\mathrm{odd}}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ *and* $\mathrm{gr}^{i,\mathrm{odd}}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)$*, which correspond to the odd homotopy groups, locally vanish for the quasisyntomic topology. Hence, these invariants are, locally with respect to the quasisyntomic topology, even.*

Taking the limit over Restriction maps, we pass to the case of $r=\infty$*, for which we have the following identifications:*

$$\begin{cases} \mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} \mathrm{TR}(S;\mathbb{Z}_p)^{hS^1} \simeq \lim_{\mathrm{R}} \mathcal{N}^{\geq i}_r \widehat{\mathbb{\Delta}}_S\{i\}[2i] \\ \mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} \mathrm{TR}(S;\mathbb{Z}_p) \simeq \lim_{\mathrm{R}} \mathcal{N}^{i}_r \widehat{\mathbb{\Delta}}_S\{i\}[2i] \end{cases}$$

Locally in the quasisyntomic topology, we have that

$$\begin{cases} \mathrm{gr}^{i}_{\mathcal{M}} \mathrm{TR}(-;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}^{\geq i}_\infty \widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] := \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}^{\geq i}_r \widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] \\ \mathrm{gr}^{i}_{\mathcal{M}} \mathrm{TR}(-;\mathbb{Z}_p) \simeq \mathcal{N}^{i}_\infty \widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] := \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}^{i}_r \widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] \end{cases}$$

Thus, locally for the quasisyntomic topology, the odd parts correspond to the $\lim^1_{\mathrm{R}}$ *terms.*

For $1 \leq r \leq \infty$*, the canonical map*

$$\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TR}^r(S;\mathbb{Z}_p) \tag{1.4}$$

gives rise to the map from the i-th filtered to the i-th graded piece for the r-Nygaard *filtration:*

$$\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\}[2i] \longrightarrow \mathcal{N}_r^{i}\widehat{\Delta}_S\{i\}[2i]$$

Restricting to the case $1 \leq r < \infty$, the commutative diagram involving higher Frobenii

$$\begin{tikzcd}
\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(S;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(S;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^r(S;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(S;\mathbb{Z}_p)^{hC_{p^{r-1}}} \arrow[r, "\varphi^{hC_{p^{r-1}}}"] & \mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd} \tag{1.5}$$

gives rise to the commutative diagram on r-Nygaard filtered prismatic cohomology groups, involving the r-divided Frobenius, together with its graded version:

$$\begin{tikzcd}
\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\}[2i] \arrow[r, "\varphi_{r,i}"] \arrow[d] & \widehat{\Delta}_S\{i\}[2i] \arrow[d] \\
\mathcal{N}_r^{i}\widehat{\Delta}_S\{i\}[2i] \arrow[r] & \Delta_S^{\mathrm{HT},r}\{i\}[2i]
\end{tikzcd}$$

Also, the symmetries of Restriction, Frobenius, and Verschiebung on TR^r and of its S^1-homotopy fixed points, give rise to corresonding symmetries on the graded and filtered pieces for the r-Nygaard filtered $\widehat{\Delta}_S$.

3) From the vanishing of odd homotopy groups, locally for the quasisyntomic topology, by applying quasisyntomic descent, we obtain multiplicative spectral sequences, for $1 \leq r \leq \infty$:

$$\begin{cases} E_2^{ij} = H^{i-j}\left(\mathcal{N}_r^{\geq -j}\widehat{\Delta}_{(-)}\right) \Rightarrow \pi_{-i-j}\,\mathrm{TR}^r((-);\mathbb{Z}_p)^{hS^1} \\ E_2^{ij} = H^{i-j}\left(\mathcal{N}_r^{-j}\widehat{\Delta}_{(-)}\right) \Rightarrow \pi_{-i-j}\,\mathrm{TR}^r(-;\mathbb{Z}_p) \end{cases}$$

Thus, locally in the quasisyntomic topology, we can identify the r-Nygaard filtration on $\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\}[2i]$ as the one coming from the degeneration of the S^1-homotopy fixed points spectral sequence.

4) By left Kan extending, all statements can be extended to the case S is a p-complete animated ring. One of the failures, as pointed out in [BL22a], is that the associated motivic filtrations are not exhaustive.

Topological restriction homology was the invariant used in the first attempt to understand topological cyclic homology, via the formula:

$$\mathrm{TC}(-;\mathbb{Z}_p) \simeq \mathrm{fib}\left(1-\mathrm{F} : \mathrm{TR}(-;\mathbb{Z}_p) \longrightarrow \mathrm{TR}(-;\mathbb{Z}_p)\right)$$
$$\simeq \lim_{\mathrm{R}} \mathrm{fib}\left(\mathrm{R}-\mathrm{F} : \mathrm{TR}^{r+1}(-;\mathbb{Z}_p) \longrightarrow \mathrm{TR}^{r}(-;\mathbb{Z}_p)\right) =: \lim_{\mathrm{R}} \mathrm{TC}_r(-;\mathbb{Z}_p)$$

However, TR is a cyclotomic spectrum (with Frobenius lifts) on its own, therefore it is natural to attempt to study $\mathrm{TC}\left(\mathrm{TR}\right)$, but also $\widetilde{\mathrm{TC}}\left(\mathrm{TR}\right)$, where $\widetilde{\mathrm{TC}}(-) := \mathrm{map}_{\mathrm{CycSp}^{\mathrm{Fr}}}(\mathbb{S},-)$ is a version of topological cyclic homology for cyclotomic spectra with Frobenius lifts. The following theorem is concerned with these questions:

Theorem 1.1.3 (Topological cyclic homology of TR)**.** *Applying the previous results to the setup of topological cyclic homology, the following are true:*

1) Topological cyclic homology for cyclotomic spectra with Frobenius lifts of $\mathrm{TR}(-;\mathbb{Z}_p)$ *is equivalent to:*

$$\widetilde{\mathrm{TC}}\left(\mathrm{TR}(-;\mathbb{Z}_p)\right) \simeq$$
$$\mathrm{fib}\left(1-\mathrm{F}^{hS^1} : \mathrm{TR}(-;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TR}(-;\mathbb{Z}_p)^{hS^1}\right) \simeq \lim_{\mathrm{R}} \widetilde{\mathrm{TC}}^{r}\left(\mathrm{TR}(-;\mathbb{Z}_p)\right)$$

where we define the family of spectra $\widetilde{\mathrm{TC}}^{r}\left(\mathrm{TR}(-;\mathbb{Z}_p)\right)$ *to be:*

$$\widetilde{\mathrm{TC}}^{r}\left(\mathrm{TR}(-;\mathbb{Z}_p)\right) := \mathrm{fib}\left(\mathrm{R}^{hS^1}-\mathrm{F}^{hS^1} : \mathrm{TR}^{r}(-;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TR}^{r-1}(-;\mathbb{Z}_p)^{hS^1}\right)$$

Mapping further to TP*, one obtains the following family of spectra, defined as:*

$$\mathrm{TC}^r\left(\mathrm{TR}(-;\mathbb{Z}_p)\right) := \mathrm{fib}\left(\mathrm{can} - \varphi^{hS^1} : \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(-;\mathbb{Z}_p)\right)$$

which "interpolates" between $\mathrm{TC}\left(\mathrm{TR}(-;\mathbb{Z}_p)\right)$*, for* $r=\infty$*, and* $\mathrm{TC}(-;\mathbb{Z}_p)$*, for* $r=1$*:*

$$\begin{gathered}\mathrm{TC}^r\left(\mathrm{TR}(-;\mathbb{Z}_p)\right) \simeq \\ \mathrm{fib}\left(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TR}^{r-1}(-;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TP}(-;\mathbb{Z}_p)\right) \simeq \\ \mathrm{fib}\left(\mathrm{can} - \varphi^{hS^1} : \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(-;\mathbb{Z}_p)\right)\end{gathered}$$

2) For $1 \leq r \leq \infty$ *and the input of a p-quasisyntomic ring* S*, topological cyclic homology* $\mathrm{TC}_r(-;\mathbb{Z}_p)$*, as well as the spectra of part (1) are equipped with exhaustive, decreasing, multiplicative,* $\mathbb{Z}$*-indexed motivic filtrations. Locally for the quasisyntomic topology, these can be expressed as follows:*

$$\begin{cases}\mathrm{gr}^i_{\mathcal{M}}\,\mathrm{TC}_r(-;\mathbb{Z}_p) \simeq \mathrm{fib}\left(\mathrm{R} - \mathrm{F} : \mathcal{N}^i_{r+1}\widehat{\Delta}_{(-)}\{i\}[2i] \longrightarrow \mathcal{N}^i_r\widehat{\Delta}_{(-)}\{i\}[2i]\right) \\ \mathrm{gr}^i_{\mathcal{M}}\,\widetilde{\mathrm{TC}}^r\left(\mathrm{TR}(-;\mathbb{Z}_p)\right) \simeq \mathrm{fib}\left(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathcal{N}^{\geq i}_r\widehat{\Delta}_{(-)}\{i\}[2i] \longrightarrow \mathcal{N}^{\geq i}_{r-1}\widehat{\Delta}_{(-)}\{i\}[2i]\right) \\ \mathrm{gr}^i_{\mathcal{M}}\,\mathrm{TC}^r\left(\mathrm{TR}(-;\mathbb{Z}_p)\right) \simeq \mathrm{fib}\left(\mathrm{can} - \varphi^{hS^1} : \mathcal{N}^{\geq i}_r\widehat{\Delta}_{(-)}\{i\}[2i] \longrightarrow \widehat{\Delta}_{(-)}\{i\}[2i]\right)\end{cases}$$

Locally for the quasisyntomic topology, these give rise to spectral sequences which degenerate. Of course, as in the main theorem, the statements above can also be suitably extended to the case we work with a p-complete animated ring.

Let us, finally, specialize to the cases of mixed and positive characteristic. In the former, we recover the relation to the objects $\widetilde{W_r\Omega}_S$ and $A\Omega \simeq \lim_r \widetilde{W_r\Omega}_S$, which were introduced and studied in [BMS18]. In particular, the authors explain that they were able to build these complexes by studying TR^r of perfectoid rings. This was indeed the precursor of the prismatic theory.

Theorem 1.1.4 (Mixed characteristic)**.** *Let S be a p-completely smooth ring over a perfectoid base R_0. Then the r-Nygaard filtration can be expressed in terms of the décalage functor:*

$$\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\} \simeq L\eta_{\xi_r}^{\geq i}\widehat{\Delta}_S\{i\}$$

In the setting of [BMS18], we recover a relation to the $A\Omega$-cohomology: If $R_0 = \mathcal{O}_C$, where C is a perfectoid field containing all p-roots of unity, then $\mathcal{N}_r^{\geq i}\widehat{\Delta}_S \simeq L\eta_{\xi_r}^{\geq i}A\Omega_S$. Then, the motivic filtration of TR^r can be expressed as:

$$\mathrm{gr}_{\mathcal{M}}^{i,\mathrm{even}}\,\mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \tau^{\leq i}\widetilde{W_r\Omega}_S$$

Taking the limit with respect to the Frobenius maps, we recover $A\Omega$-cohomology via the motivic filtration of topological Frobenius homology:

$$\mathrm{gr}_{\mathcal{M}}^{i,\mathrm{even}}\,\mathrm{TF}(S;\mathbb{Z}_p) \simeq \tau^{\leq i}A\Omega_S$$

Passing to the latter case of positive characteristic, we show that in the quasiregular - semiperfect setting, TR^r satisfies odd-vanishing. Therefore, things work as in the case of [BMS19] and, ultimately, we obtain a relation with the de Rham–Witt complex over $\mathbb{F}_p$.

Theorem 1.1.5 (Positive Characteristic case)**.** *Let S be a quasiregular - semiperfect $\mathbb{F}_p$-algebra. Then the odd homotopy groups of the following invariants vanish, for $1 \leq r \leq \infty$:*

$$\begin{cases} \pi_{\mathrm{odd}}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq 0 \\ \pi_{\mathrm{odd}}\,\mathrm{TR}^r(S;\mathbb{Z}_p) \simeq 0 \end{cases}$$

In particular, and in analogy with [BMS19], we can identify the r-Nygaard filtration on $\widehat{\Delta}_S \simeq \widehat{\mathbb{A}}_{\mathrm{crys}}(S)$ as the one coming from the S^1-homotopy fixed points spectral sequence, for $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$, which degenerates. This was also recently discussed in [DR23] and [DM23].

It follows that if we assume S to be a smooth algebra over $\mathbb{F}_p$, then we have the identification:

$$\mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \tau^{\leq i} W_r\Omega_S$$

1.2 Proof outline

The main motive of this work is, instead of directly dealing with $\mathrm{TR}^r(-;\mathbb{Z}_p)$, to first study its S^1-homotopy fixed points $\mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1}$, for $1 \leq r < \infty$. This is where the r-Nygaard filtration on completed prismatic cohomology arises from. Then, we use a trick of Nikolaus–Scholze [AN21], where by taking quotient with the class generating $\pi_{-2}\,\mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1}$, we are able to pass to $\mathrm{TR}^r(-;\mathbb{Z}_p)$. Thus, this enables us to identify the relevant structure of the latter with the graded pieces for the r-Nygaard filtration. This is a pattern systematically used in [BMS19], where $\mathrm{TC}^-(-;\mathbb{Z}_p)$ gives rise to the Nygaard filtration on completed prismatic cohomology, while passing to $\mathrm{THH}(-;\mathbb{Z}_p)$ gives rise to its associated graded pieces.

As in [BMS19], the first step is to try and make precise calculations in the perfectoid case. This is, indeed, possible as for a perfectoid ring R_0, the following identification holds:

$$\pi_*\,\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \mathbb{A}_{\mathrm{inf}}(R_0)[u_r,v_r]/(u_rv_r-\xi_r)$$
$$\deg(u_r)=2, \deg(v_r)=-2, \deg(\xi_r)=0$$

Remember that $\widetilde{\vartheta}_r : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathrm{W}_r(R_0)$ is the usual projection to the r-truncated Witt vectors, while $\vartheta_r := \widetilde{\vartheta}_r \circ \varphi^r : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathrm{W}_r(R_0)$ is twisted by the r-the iterated Frobenius. The kernel of the latter is generated by the element ξ_r, while of the former by $\widetilde{\xi}_r = \varphi^r(\xi_r)$.

Taking quotient with respect to $v_r \in \pi_{-2}\,\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$ takes us back to $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$, thus giving rise to the following equivalence:

$$\pi_* \operatorname{TR}^r(R_0; \mathbb{Z}_p) \simeq \mathrm{W}_r(R_0)[u_r]$$

Equivalently, we can reformulate these as:

$$\begin{cases} \pi_{2i} \operatorname{TR}^r(R_0; \mathbb{Z}_p)^{hS^1} \simeq \xi_r^i \, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \\ \pi_{2i} \operatorname{TR}^r(R_0; \mathbb{Z}_p) \simeq \xi_r^i / \xi_r^{i+1} \, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \end{cases}$$

By taking the limit over Restriction maps, we have the following identification for the two-term complexes:

$$\begin{cases} \tau_{[2i-1,2i]} \operatorname{TR}(R_0; \mathbb{Z}_p)^{hS^1} \simeq \operatorname{Rlim}_{\mathrm{R}} \xi_r^i \, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \\ \tau_{[2i-1,2i]} \operatorname{TR}(R_0; \mathbb{Z}_p) \simeq \operatorname{Rlim}_{\mathrm{R}} \xi_r^i / \xi_r^{i+1} \, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \end{cases}$$

For this, we use the following iterated product formula of the S^1-homotopy fixed points, for $1 \leq r \leq \infty$:

$$\begin{aligned} \operatorname{TR}^r(-; \mathbb{Z}_p)^{hS^1} &\simeq \operatorname{TC}^-(-; \mathbb{Z}_p) \times_{\operatorname{TP}(-;\mathbb{Z}_p)} \cdots \times_{\operatorname{TP}(-;\mathbb{Z}_p)} \operatorname{TC}^-(-; \mathbb{Z}_p) \\ &\simeq \operatorname{fib}\left(\prod_{1 \leq k \leq r} \operatorname{TC}^-(-; \mathbb{Z}_p) \to \prod_{1 \leq k \leq r-1} \operatorname{TP}(-; \mathbb{Z}_p) \right) \end{aligned}$$

Following [BMS19], we then pass to the study of quasiregular - semiperfectoid rings, which provide a basis for applying quasisyntomic descent. Suppose we have a quasirgular - semiperfectoid ring over a fixed perfectoid base $R_0 \to S$. Using the presentation for $\operatorname{TR}^r(S; \mathbb{Z}_p)^{hS^1}$, we have the following identification for the even homotopy groups, with the

r-Nygaard filtered complete prismatic cohomology:

$$\pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\}$$

The filtration $\mathcal{N}_r^{\geq \bullet}\widehat{\mathbb{\Delta}}_S$ is a decreasing, multiplicative, complete filtration on $\widehat{\mathbb{\Delta}}_S$, with the following iterated product description:

$$\mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\} := \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\} \times_{\widehat{\mathbb{\Delta}}_S\{i\}} \cdots \times_{\widehat{\mathbb{\Delta}}_S\{i\}} \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\}$$

Equivalently, we have the following more descriptive definition, in analogy with the classical Nygaard filtration:

$$\mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S = \left\{ x \in \widehat{\mathbb{\Delta}}_S \;\middle|\; \varphi^{ri}(x) \in \widetilde{\xi}_r^i \widehat{\mathbb{\Delta}}_S \right\}$$

Passing back to $\mathrm{TR}^r(S;\mathbb{Z}_p)$ via the Nikolaus-Scholze trick, we are able to identify its even homotopy groups with the associated graded pieces of the r-Nygaard filtration:

$$\pi_{2i}\,\mathrm{TR}(S;\mathbb{Z}_p) \simeq \mathcal{N}_r^{i}\widehat{\mathbb{\Delta}}_S\{i\}$$

Between them, via the map $\varphi^{hS^1} : \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \to \mathrm{TC}^-(S;\mathbb{Z}_p) \to \mathrm{TP}(S;\mathbb{Z}_p)$, we have a divided r-Frobenius:

$$\varphi_{r,i} : \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\} \to \widehat{\mathbb{\Delta}}_S\{i\}$$

which is related to the r-th iterated Frobenius, via the formula: $\varphi_{r,i} = \varphi^{ri}/\widetilde{\xi}_r^i$

In order to move forward, as we already mentioned, we need to use the quasisyntomic topology. Under this scope, using the vanishing result of [BS22], the odd homotopy groups

$$\pi_{2i-1}\,\mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1}, \quad \pi_{2i-1}\,\mathrm{TR}^r(-;\mathbb{Z}_p)$$

vanish locally in the quasisyntomic topology. Notice that for quasiregular - semiperfect rings, this vanishing holds without requiring to pass to a suitable quasisyntomic cover.

It follows that, locally for the quasisyntomic topology, we can identify the graded pieces for the motivic filtrations (which come from the double-speed Postnikov filtrations) as:

$$\begin{cases} \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] \\ \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(-;\mathbb{Z}_p) \simeq \mathcal{N}_r^{i}\widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] \\ \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}(-;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] = \mathcal{N}_\infty^{\geq i}\widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] \\ \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}(-;\mathbb{Z}_p) \simeq \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}_r^{i}\widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] = \mathcal{N}_\infty^{i}\widehat{\mathbb{\Delta}}_{(-)}\{i\}[2i] \end{cases}$$

The remaining results regarding TR and its related invariants follow from these identifications.

1.3 Overview of the Thesis

The format of this thesis follows the natural succession that we just described.

In particular, in chapter 2, we recall the main definitions and properties regarding topological Hochschild homology, topological restriction homology, and related invariants. Additionally, we provide a quick review of the main results of the second work of Bhatt–Morrow–Scholze [BMS19], together with the basic properties of the invariants associated to the plain Hochschild homology and the identification of their respective motivic filtrations in terms of the Hodge completed derived de Rham complex.

In chapter 3, we gather the calculations of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(R_0;\mathbb{Z}_p)$, in the case R_0 is a perfectoid ring. We provide explicit presentations for their motivic filtrations, which for finite $1 \leq r < \infty$ coincide with the double-speed Postnikov filtration, as well as polynomial presentations of their cohomology rings.

In chapter 4, which is the main one of this work, we first start with calculations in the case of quasi-regular semiperfectoid rings. As we already described, the even homotopy groups carry the structure relevant to the r-Nygaard filtration. Next, we use quasisyntomic descent to pass to the wider class of quasisyntomic rings (and left-Kan extending to go to the case of any animated ring). We apply these results to the study of topological cyclic homology of TR, providing an interpolating family of spectra between $\widetilde{\mathrm{TC}}(\mathrm{TR})$ and $\mathrm{TC}(\mathrm{TR})$.

In chapter 5, we explore applications towards mixed and positive characteristic. The main ingredient here is the identification of the r-Nygaard filtration with the filtration associated to the décalage functor $L\eta_{\xi_r}$, for the element $\xi_r \in \mathbb{A}_{\mathrm{inf}}$.

Finally, in chapter 6 we discuss ongoing work in relating the r-Nygaard filtration with the de Rham-Witt complex and pursuing a proof of Hesselholt's conjectures over $\mathcal{O}_C$. In addition, we explain some future research directions involving the theory of prismatization, in an attempt to also view our results under the heuristics of Scholze's ICM address [Sch18].

1.4 Conventions

Regarding the theory of ∞-categories and higher algebra, we follow standard conventions, as they are presented in Lurie's treatises [Lur09; Lur17]. Regarding the theory of cyclotomic spectra, topological Hochschild homology and topological restriction homology, we treat the p-typical case, following the works of Nikolaus–Scholze [NS18] and Antieau–Nikolaus [AN21]. Finally, regarding the theory of perfectoid rings, prismatic cohomology and how these relate to the theory of THH, we follow the recent works of Bhatt–Morrow–Scholze [BMS18; BMS19], Bhatt–Scholze [BS22], and Bhatt–Lurie [BL22a]. In the next chapter, we briefly recall some of that background.

CHAPTER 2

PRELIMINARIES

In this chapter, we briefly gather some background regarding topological Hochschild homology, topological restriction homology, and other related invariants. We mostly present the p-typical aspects of the story, providing references of the integral aspects, for the interested reader. We end with a brief discussion of the work of [BMS19] and other on the motivic filtrations of THH and HH related invariants.

2.1 Cyclotomic spectra and topological Hochschild homology

The main advantage of the theory of topological Hochschild homology, as opposed to the original story of Hochschild homology, is the existence of the *higher Frobenius* maps. In order to study them, Nikolaus-Scholze [NS18] constructed the ∞-category of *cyclotomic spectra*; here we remind the reader of the p-typical story, the version outlined by Antieau-Nikolaus [AN21].

Definition 2.1.1 (p-typical cyclotomic spectra)**.** We fix a prime number p. A *p-typical cyclotomic spectrum X* is a spectrum equipped with an S^1-action and an S^1-equivariant *Frobenius* map to its C_p-Tate fixed points $\varphi_p : X \to X^{tC_p}$. These assemble into the presentable, stable ∞-category of *p-typical cylotomic spectra* CycSp_p, which is defined to be the lax equalizer of the higher Frobenius and identity maps in $\mathrm{Sp}^{\mathrm{B}S^1}$.

Remark 2.1.2 (Comparison to the genuine theory)**.** As shown in [NS18], this approach to the theory of cyclotomic spectra is particularly well-behaved in the bounded-below case, where it actually identifies with the more classical theory of genuine cyclotomic spectra.

Example 2.1.3 (Topological Hochschild homology)**.** The main example of a p-typical cyclotomic spectrum, comes from the theory of *topological Hochschild homology*. Since we are interested in THH of classical/animated rings, we can restrict ourselves to the following definition. Given a connective, $\mathbb{E}_\infty$-ring spectrum $A \in \mathrm{CAlg}^{\geq 0}$, topological Hochschild homology of A is defined to be:

$$\mathrm{THH}(A) := A \otimes_{A \otimes A^{\mathrm{op}}} A$$

The S^1-action on $\mathrm{THH}(A)$ can be seen via the presentation of the simplicial circle as a push-out:

$$S^1 \simeq * \amalg_{* \amalg *} *$$

A geometric approach to topological Hochschild homology can be achieved through the theory of factorization homology.

We can associate more invariants to $\mathrm{THH}(A)$, by playing with the S^1-action. Its S^1-homotopy fixed points constitute the *negative topological cyclic homology* $\mathrm{TC}^-(A)$ and its S^1-Tate fixed points constitute the *periodic topological cyclic homology* $\mathrm{TP}(A)$. Between those two, there exists the canonical map $\mathrm{can} : \mathrm{TC}^-(A) \to \mathrm{TP}(A)$ and the associated Frobenius $\varphi^{hS^1} : \mathrm{TC}^-(A) \to \mathrm{TP}(A)$.

Construction 2.1.4 (Topological cyclic homology)**.** Given a p-typical cyclotomic spectrum $X \in \mathrm{CycSp}_p$, mapping out of the sphere spectrum $\mathbb{S}$, which is also an element of CycSp_p with trivial S^1-action and Frobenius, produces *topological cyclic homology*:

$$\mathrm{TC}(X) := \mathrm{map}_{\mathrm{CycSp}_p}(\mathbb{S}, X)$$

Using the lax equalizer presentation of CycSp_p, Nikolaus-Scholze produce a formula for computing topological cyclic homology:

$$\mathrm{TC}(X) \simeq \mathrm{eq}\left(\mathrm{can}, \varphi^{hS^1} : X^{hS^1} \rightrightarrows (X^{tC_p})^{hS^1}\right)$$

In particular, for $\mathbb{Z}_p$-coefficients due to the Tate orbit lemma, we are able to compute the topological cyclic homology of A, using $\mathrm{TC}^-(A)$ and $\mathrm{TP}(A)$:

$$\mathrm{TC}(A;\mathbb{Z}_p) := \mathrm{TC}(\mathrm{THH}(A;\mathbb{Z}_p)) \simeq \left(\mathrm{can}, \varphi^{hS^1} : \mathrm{TC}^-(A;\mathbb{Z}_p) \rightrightarrows \mathrm{TP}(A;\mathbb{Z}_p)\right)$$

2.2 Topological restriction homology

The classical theory of cyclotomic spectra paves an approach to topological cyclic homology through genuine fixed points, with respect to finite subgroups of S^1. Since we restrict to the bounded-below case, genuine fixed points identify with topological restriction homology, as shown in [NS18].

Definition 2.2.1 (Topological restriction homology)**.** Given a bounded-below $X \in \mathrm{CycSp}_p$, we can construct its *r-truncated topological restriction homology*, which identifies with the $C_{p^{r-1}}$-genuine fixed points of X and is given by the following iterated product formula, for $r \geq 1$:

$$\mathrm{TR}^r(X) := X^{hC_{p^r}} \times_{(X^{tC_p})^{hC_{p^{r-1}}}} X^{hC_{p^{r-1}}} \times_{(X^{tC_p})^{hC_{p^{r-2}}}} \cdots \times_{X^{tC_p}} X \simeq X^{C_{p^{r-1}}}$$

where the maps on the right are $\varphi^{hC_{p^k}}$, for $0 \leq k \leq r-1$, and the maps on the left are the canonical ones. In the case $X = \mathrm{THH}(A)$, for some $A \in \mathrm{CAlg}^{\geq 0}$, we simply write $\mathrm{TR}^r(A) := \mathrm{TR}^r(\mathrm{THH}(A))$.

There is a deep analogy between topological restriction homology and the theory of Witt vectors. In particular, there exist *Restriction* maps $\mathrm{R} : \mathrm{TR}^{r+1}(X) \to \mathrm{TR}^r(X)$, for $r \geq 1$ by forgetting the leftmost factor in the iterated product presentation, *Frobenius* maps $\mathrm{F} : \mathrm{TR}^{r+1}(X) \to \mathrm{TR}^r(X)$, for $r \geq 1$ by forgetting the rightmost factor together with part of the homotopy fixed points information, and *Verschiebung* maps $\mathrm{V} : \mathrm{TR}^r(X) \to \mathrm{TR}^{r+1}(X)$, for $r \geq 1$ by moving everything one place to the left, applying $(\cdot)^{hC_p}$, and introducing a copy of X in the rightmost place.

Taking the limit with respect to the Restriction maps yields the *topological restriction homology* of X:

$$\mathrm{TR}(X) := \lim_{\mathrm{R}} \mathrm{TR}^r(X) \simeq \cdots \times_{(X^{tC_p})^{hC_{p^r}}} X^{hC_{p^r}} \times_{(X^{tC_p})^{hC_{p^{r-1}}}} \cdots \times_{X^{tC_p}} X$$

In this case, Frobenius and Verschiebung become endofunctors of TR.

The relation between topological restriction homology and the Witt vectors becomes explicit via the following theorem of Hesselholt-Madsen:

Theorem 2.2.1 (Hesselholt-Madsen [HM97])**.** *Let $A \in \mathrm{CAlg}^{\geq 0}$ be a connective $\mathbb{E}_\infty$-ring spectrum. Then $\mathrm{TR}^r(A)$ is also a connective $\mathbb{E}_\infty$-ring spectrum. Applying the 0-th homotopy group functor produces the following equivalence, which maps the structure associated to topological restriction homology to the usual structure associated to the Witt vectors of $\pi_0 A$:*

$$\pi_0 \,\mathrm{TR}^r(A) \simeq \mathrm{W}_r(\pi_0 A)$$

Remark 2.2.2. An equivalent way to present topological restriction homology is either as an equalizer, for $1 \leq r \leq \infty$:

$$\mathrm{TR}^r(X) \simeq \mathrm{eq}\left(\prod_{0 \leq k \leq r-1} X^{hC_{p^{i-1}}} \rightrightarrows \prod_{1 \leq k \leq r-1} (X^{tC_p})^{hC_{p^{i-1}}} \right)$$

or, inductively, via the pullback square, for $r \geq 1$:

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(X) \arrow[r, "\mathrm{R}"] \arrow[dd] & \mathrm{TR}^r(X) \arrow[dd] \arrow[dr, dotted] & \\
& & X^{hC_{p^{r-1}}} \arrow[dl, dotted, "\varphi^{hC_{p^{r-1}}}"] \\
X^{hC_{p^r}} \arrow[r, "\mathrm{can}"'] & (X^{tC_p})^{hC_{p^r}} &
\end{tikzcd}$$

For $\mathbb{Z}_p$-coefficients, observe that due to Tate orbit lemma [NS18], the C_{p^k}-Tate fixed points of $\mathrm{TR}^r(A;\mathbb{Z}_p)$ reduce to the following, via the natural projection map to $\mathrm{THH}(A;\mathbb{Z}_p)$, for $1 \leq k \leq \infty$:

$$\mathrm{TR}^r(A;\mathbb{Z}_p)^{tC_{p^k}} \simeq \mathrm{THH}(A;\mathbb{Z}_p)^{tC_{p^k}}$$

In fact, the iterated pullback formula simplifies to the following, for $1 \leq r \leq \infty$:

$$\mathrm{TR}^r(A;\mathbb{Z}_p) \simeq \mathrm{THH}(A;\mathbb{Z}_p)^{hC_{p^r}} \times_{\mathrm{THH}(A;\mathbb{Z}_p)^{tC_{p^{r-1}}}} \cdots \times_{\mathrm{THH}(A;\mathbb{Z}_p)^{tC_p}} \mathrm{THH}(A;\mathbb{Z}_p)$$

There are similar simplifications in ways of presenting $\mathrm{TR}^r(A;\mathbb{Z}_p)$.

Finally, applying S^1-homotopy fixed points to the last formula, we have the following very important identification for $\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1}$, for $1 \leq r \leq \infty$:

$$\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{TC}^-(A;\mathbb{Z}_p) \times_{\mathrm{TP}(A;\mathbb{Z}_p)} \cdots \times_{\mathrm{TP}(A;\mathbb{Z}_p)} \mathrm{TC}^-(A;\mathbb{Z}_p)$$

where the maps on the left are the canonical ones $\mathrm{can} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p)$, while the maps on the right $\varphi^{hS^1} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p)$ are the S^1-homotopy fixed points of Frobenius.

In particular, we highlight two maps who have $\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1}$ as their source. In particular, the first one corresponds to the C_{p^r}-homotopy fixed points of the higher Frobenius. It is obtained by mapping to $\mathrm{TC}^-(A;\mathbb{Z}_p)$ on the left and, afterwards, to $\mathrm{TP}(A;\mathbb{Z}_p)$ via the S^1-homotopy fixed points of higher Frobenius:

$$\varphi^{hS^1} : \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^-(A;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)$$

And, thus, it fits in the following commutative diagram:

$$\begin{tikzcd} \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(A;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(A;\mathbb{Z}_p) \arrow[d] \\ \mathrm{TR}^r(A;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(A;\mathbb{Z}_p)^{hC_{p^{r-1}}} \arrow[r, "\varphi^{hC_{p^{r-1}}}"] & \mathrm{THH}(A;\mathbb{Z}_p)^{tC_{p^r}} \end{tikzcd}$$

In addition to this, we also have a counterpart of the inclusion map, by mapping first to $\mathrm{TC}^-(A;\mathbb{Z}_p)$ on the right and then to $\mathrm{TP}(A;\mathbb{Z}_p)$ via the inclusion map:

$$\mathrm{can}: \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^-(A;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)$$

Note that the aforementioned description for the S^1-homotopy fixed points of TR can be also given in the form of an equalizer:

$$\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{eq}\left(\prod_{1\le k\le r-1} \mathrm{TC}^-(A;\mathbb{Z}_p) \rightrightarrows \prod_{2\le k\le r-1} \mathrm{TP}(A;\mathbb{Z}_p)\right)$$

Of course, there exist such presentations in general, without the need to restrict to $\mathbb{Z}_p$ coefficients. However, in our study, we mostly restrict to the case of working over $\mathbb{Z}_p$.

2.3 Cyclotomic spectra with Frobenius lifts

If cyclotomic spectra provide a framework for studying the properties of higher Frobenius, then there should be a good notion of Frobenius lift. Such a notion does, in fact, exist and the prime example is given by $\mathrm{TR}(X)$, for $X \in \mathrm{CycSp}_p$; it happens that Frobenius factors through the C_p-homotopy fixed points of $\mathrm{TR}(X)$.

Definition 2.3.1 (p-typical cyclotomic spectra with Frobenius lifts)**.** A *p-typical cyclotomic spectrum with a Frobenius lift* is a spectrum Y with an S^1-action and a *Frobenius lift* map to its C_p-homotopy fixed points $F_p : Y \to Y^{hC_p}$. The assemble into the presentable, stable

∞-category of *p-typical cyclotomic spectra with Frobenius lifts* $\mathrm{CycSp}_p^{\mathrm{Fr}}$, which is defined to be the lax equalizer of the Frobenius lift and identity maps in $\mathrm{Sp}^{\mathrm{B}\,S^1}$.

Remark 2.3.2. Any $Y \in \mathrm{CycSp}_p^{\mathrm{Fr}}$ is also in CycSp_p, by considering the Frobenius map to be the Frobenius lift, postcomposed with the canonical map to the C_p-Tate fixed points:

$$\varphi_p = \mathrm{can} \circ \mathrm{F}_p : Y \xrightarrow{\mathrm{F}_p} Y^{hC_p} \xrightarrow{\mathrm{can}} Y^{tC_p}$$

Theorem 2.3.1 (TR is a p-typical cyclotomic spectrum with Frobenius lifts, [KN18])**.** *The forgetful functor provided by the previous remark admits a right adjoint, given by* TR*. Therefore, the ∞-categories* $\mathrm{CycSp}_p^{\mathrm{Fr}}$ *and* CycSp_p *fit into the following adjunction:*

$$\begin{tikzcd} \mathrm{CycSp}_p^{\mathrm{Fr}} \arrow[rr, bend left, "\mathrm{forg.}"] \arrow[rr, phantom, "\perp"] & & \mathrm{CycSp}_p \arrow[ll, bend left, "\mathrm{TR}"] \end{tikzcd}$$

In fact, there is more to the story, as also the Verschiebung map of $\mathrm{TR}(X)$, for $X \in \mathrm{CycSp}_p$, seems to factor through the C_p-homotopy orbits.

Definition 2.3.3 (p-typical topological Cartier modules, [AN21])**.** There exists a presentable, stable ∞-category TCart_p of *p-typical topological Cartier modules*, an element M of which, is defined to be a spectrum equipped with and S^1-action, whose C_p-norm has a factorization, as follows:

$$\begin{tikzcd} M_{hC_p} \arrow[r, "\mathrm{V}"] \arrow[rr, bend right, "\mathrm{Nm}"'] & M \arrow[r, "\mathrm{F}"] & M^{hC^p} \end{tikzcd}$$

Remark 2.3.4. There exists an obvious chain of forgetful functors:

$$\mathrm{TCart}_p \xrightarrow{\mathrm{forg.}} \mathrm{CycSp}_p^{\mathrm{Fr}} \xrightarrow{\mathrm{forg.}} \mathrm{CycSp}_p$$

Remark 2.3.5. Let us denote with $(\cdot)/\,\mathrm{V}$ the cofiber of the Verschiebung map in TCart_p.

Given an $M \in \mathrm{TCart}_p$, consider the spectrum M/V. Then M/V is a cyclotomic spectrum.

Theorem 2.3.2 (Antieau-Nikolaus adjunction, [AN21])**.** *The functor* $(\cdot)/\mathrm{V}$ *provided by the previous remark admits a right adjoint, given by* TR. *Therefore, the* ∞*-categories* TCart_p *and* CycSp_p *fit into the following adjunction:*

$$\begin{tikzcd} \mathrm{TCart}_p \arrow[rr, bend left, "(\cdot)/\mathrm{V}"] \arrow[rr, phantom, "\perp"] & & \mathrm{CycSp}_p \arrow[ll, bend left, "\mathrm{TR}"] \end{tikzcd}$$

In fact, there exists a so-called cyclotomic t-structure *on* CycSp_p*, for which the functor* TR *is* t*-exact. In addition, in the bounded-below case, it is also fully faithful, thus identifying* CycSp_p *with the* V-complete p-typical topological Cartier modules $\widehat{\mathrm{TCart}}_p \subset \mathrm{TCart}_p$.

2.4 Topological cyclic homology

As we already noted, topological restriction homology served as the initial tool for approaching topological cyclic homology.

Construction 2.4.1 (TC via TR, [NS18])**.** Let X be a bounded below p-typical cyclotomic spectrum. Then, we have the following equivalence, for $r \geq 1$:

$$\mathrm{TC}_r(X) := \mathrm{fib}\left(\mathrm{R}-\mathrm{F} : \mathrm{TR}^{r+1}(X) \to \mathrm{TR}^r(X)\right) \simeq$$
$$\mathrm{fib}\left(\mathrm{can} - \varphi^{hC_{p^{r+1}}} : X^{hC_{p^{r+1}}} \longrightarrow (X^{tC_p})^{hC_{p^r}}\right)$$

It follows that, over $\mathbb{Z}_p$-coefficients, the following holds:

$$\mathrm{TC}(X) \simeq \lim_{\mathrm{R}} \mathrm{TC}_r(X) \simeq \mathrm{fib}\left(1 - \mathrm{F} : \mathrm{TR}(X) \longrightarrow \mathrm{TR}(X)\right)$$

Therefore, understanding TR could possibly lead to a better understanding of TC. In addition, TR has the structure of a p-typical cyclotomic spectrum with a Frobenius lift, and as a result via the forgetful functor, the structure of a p-typical cyclotomic spectrum. Thus, trying to understand TC(TR) would also be an interesting goal, in its own sake.

Definition 2.4.2 (Topological cyclic homology $\widetilde{\mathrm{TC}}$ for $\mathrm{CycSp}_p^{\mathrm{Fr}}$)**.** Given a p-typical cyclotomic spectrum with a Frobenius lift $Y \in \mathrm{CycSp}_p^{\mathrm{Fr}}$, mapping out of the sphere spectrum $\mathbb{S}$, which is also an element of $\mathrm{CycSp}_p^{\mathrm{Fr}}$ with trivial S^1-action and Frobenius lift, produces *topological cyclic homology for Frobenius lifts*:

$$\widetilde{\mathrm{TC}}(Y) := \mathrm{map}_{\mathrm{CycSp}_p^{\mathrm{Fr}}}(\mathbb{S}, Y)$$

Construction 2.4.3 (Computing $\widetilde{\mathrm{TC}}$ for $\mathrm{CycSp}_p^{\mathrm{Fr}}$)**.** In complete analogy with the standard TC, we can use the lax equalizer approach to defining $\mathrm{CycSp}_p^{\mathrm{Fr}}$, in order to have a formula for computing topological cyclic homology for Frobenius lifts:

$$\widetilde{\mathrm{TC}}(Y) \simeq \mathrm{eq}\left(\mathrm{id}, \mathrm{F}^{hS^1} : Y^{hS^1} \rightrightarrows Y^{hS^1}\right)$$

Proof. The process is similar to the one in [NS18, Prop. II.1.9], for TC. □

The natural question to ask is what happens in the case of the standard TC of p-typical cyclotomic spectra with Frobenius lifts.

Construction 2.4.4 (Computing TC of $\mathrm{TR}(A; \mathbb{Z}_p)$)**.** Let A be a connective $\mathbb{E}_\infty$-ring spectrum and consider $\mathrm{TR}(A; \mathbb{Z}_p)$. Then, the following equivalences hold, regarding topological cyclic homology:

$$\widetilde{\mathrm{TC}}\Big(\mathrm{TR}(A; \mathbb{Z}_p)\Big) \simeq \lim_{\mathrm{R}} \widetilde{\mathrm{TC}}^r(A; \mathbb{Z}_p)$$

for the spectra

$$\widetilde{\mathrm{TC}}^r(A;\mathbb{Z}_p) := \mathrm{fib}\left(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^{r-1}(A;\mathbb{Z}_p)^{hS^1}\right)$$

Mapping further to the Tate constructions, we get that:

$$\mathrm{TC}\left(\mathrm{TR}(A;\mathbb{Z}_p)\right) \simeq \left(1 - \mathrm{F}^{hS^1} : \mathrm{TR}(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)\right) \simeq \lim_{\mathrm{R}} \mathrm{TC}^r(A;\mathbb{Z}_p)$$

for the spectra:

$$\begin{aligned}\mathrm{TC}^r(A;\mathbb{Z}_p) &:= \mathrm{fib}\left(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TR}^{r-1}(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)\right)\\ &\simeq \mathrm{fib}\left(\mathrm{can} - \varphi^{hS^1} : \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^-(A;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)\right)\end{aligned}$$

Note that $\mathrm{TC}^r(A;\mathbb{Z}_p)$ interpolates between $\mathrm{TC}(A;\mathbb{Z}_p)$, for $r = 1$, and $\mathrm{TC}\left(\mathrm{TR}(A;\mathbb{Z}_p)\right)$, for $r = \infty$.

Proof. These essentially follow from the commutativity of the diagrams, below. The diagrams on the left correspond to the action of the Restriction operators above/ Frobenius operators below, for $\mathrm{TR}^r(A;\mathbb{Z}_p)$. Applying S^1-homotopy fixed points, we obtained the diagrams on the right.

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(A;\mathbb{Z}_p) \arrow[r, "\mathrm{R}"] \arrow[d] & \mathrm{TR}^{r}(A;\mathbb{Z}_p) \arrow[d] \\
\mathrm{THH}(A;\mathbb{Z}_p)^{hC_{p^r}} \arrow[r, "\mathrm{can}"] & \mathrm{THH}(A;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{TR}^{r+1}(A;\mathbb{Z}_p)^{hS^1} \arrow[r, "\mathrm{R}^{hS^1}"] \arrow[d] & \mathrm{TR}^{r}(A;\mathbb{Z}_p)^{hS^1} \arrow[d] \\
\mathrm{TC}^{-}(A;\mathbb{Z}_p) \arrow[r, "\mathrm{can}"] & \mathrm{TP}(A;\mathbb{Z}_p)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(A;\mathbb{Z}_p) \arrow[r, "\mathrm{F}"] \arrow[d] & \mathrm{TR}^{r}(A;\mathbb{Z}_p) \arrow[d] \\
\mathrm{THH}(A;\mathbb{Z}_p)^{hC_{p^r}} \arrow[r, dotted] \arrow[d, "\varphi^{hC_{p^r}}"'] & \mathrm{THH}(A;\mathbb{Z}_p)^{tC_{p^r}} \\
\mathrm{THH}(A;\mathbb{Z}_p)^{tC_{p^{r+1}}} \arrow[ur, "\mathrm{proj.}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{TR}^{r+1}(A;\mathbb{Z}_p)^{hS^1} \arrow[r, "\mathrm{F}^{hS^1}"] \arrow[d] & \mathrm{TR}^{r}(A;\mathbb{Z}_p)^{hS^1} \arrow[d] \\
\mathrm{TC}^{-}(A;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] & \mathrm{TP}(A;\mathbb{Z}_p)
\end{tikzcd}$$

Taking homotopy fibres, it follows that:

$$\begin{aligned}
\mathrm{TC}^r(A;\mathbb{Z}_p) &:= \mathrm{fib}\Big(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TR}^{r-1}(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)\Big) \\
&\simeq \mathrm{fib}\Big(\mathrm{can} - \varphi^{hS^1} : \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^{-}(A;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(A;\mathbb{Z}_p)\Big)
\end{aligned}$$

□

We have only presented the p-typical story, which is going to suffice for the majority of this work. For the reader who wishes to view the integral statements, we direct them to works such as [NS18], [KN18], [KMN23], [McC21].

Let us finally note that one of the reasons for the importance of the theory of topological cyclic homology is its proximity to algebraic K-theory, as a result of the theory of traces and the story of curves in K-theory.

2.5 Motivic filtrations following [BMS19]

For the full approach to prismatic cohomology, via the theory of THH, as well as for the basics of quasiregular - semiperfectoid rings and p-quasisyntomic descent, we direct the reader to [BMS19]. A brief recount can also be found in [Ant+22]. Here, we provide a reminder on the basics of quasisyntomic descent and focus on the main commutative square involving invariants of THH, which gathers the substance of [BMS19] ideas.

Definition 2.5.1 (Quasisyntomic rings and quasisyntomic topology, [BMS19, Sect. 4])**.** 1) We call a p-complete ring, S *quasisyntomic*, if it has bounded p^{∞} torsion and its cotangent complex $\mathbb{L}_{S/\mathbb{Z}_p}$ has p-complete Tor amplitude concentrated in $[-1,0]$. The category of p-quasisyntomic rings is denoted by QSyn.
2) A map of p-complete rings is called a *quasisyntomic map/quasisyntomic cover* if B is p-completely flat/faithfully flat over A and the cotangent complex $\mathbb{L}_{B/A}$ has p-complete Tor amplitude concentrated in $[-1,0]$.
3) The category $\mathrm{QSyn}^{\mathrm{op}}$ obtains the structure of a site, when equipped with the quasisyntomic covers. We call this the *quasisyntomic site*.

Let us recall some important, special cases of quasisyntomic rings, the first of which is the class of perfectoid rings.

Definition 2.5.2 (Perfectoid rings [BMS18, Sec. 3])**.** A ring R_0 is called *perfectoid* if and only if it is π-adically complete for some element $\pi \in R_0$, for which π^p divides p, the Frobenius map $\varphi : R_0/p \to R_0/p$ is surjective, and the kernel of Fontaine's map $\vartheta : \mathbb{A}_{\mathrm{inf}}(R_0) \to R_0$ is principal, where $\mathbb{A}_{\mathrm{inf}}(R_0) := \mathrm{W}(R_0^{\flat})$.

Certain quotients of perfectoid rings are also included in the category of quasisyntomic rings:

Definition 2.5.3 (Quasiregular - semiperfectoid rings, following [BMS19])**.** 1) We call a ring R quasiregular - semiperfectoid if it is quasisyntomic, has a perfectoid base $R_0 \to R$ and its reduction S/p is semiperfect.
2) We denote by QRSPerfd the category of quasiregular - semiperfectoid rings. In particular, $\mathrm{QRSPerfd}^{\mathrm{op}}$ becomes a site, when equipped with quasisyntomic covers.

The significance of quasiregular - semiperfectoid rings stems from the fact that on the one hand are quite computable, while on the other, they provide a basis for the quasisyntomic topology:

Proposition 2.5.1 (Quasisyntomic descent, following [BMS19])**.** *The natural map:*

$$u : \mathrm{QRSPerfd}^{\mathrm{op}} \longrightarrow \mathrm{QSyn}^{\mathrm{op}}$$

identifies the class of quasiregular - semiperfectoid rings as a basis for the quasisyntomic topology.

For further details, the interested reader is directed to [BMS19, Sect. 4].

Let S be a p-quasisyntomic ring, for which we consider the following commutative square of [BMS19]:

$$\begin{tikzcd}
\mathrm{TC}^{-}(S;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(S;\mathbb{Z}_p) \arrow[d] \\
\mathrm{THH}(S;\mathbb{Z}_p) \arrow[r, "\varphi"] & \mathrm{THH}(S;\mathbb{Z}_p)^{tC_p}
\end{tikzcd}$$

The invariants of this square are equipped with the complete, exhaustive, decreasing, multiplicative, $\mathbb{Z}$-indexed *motivic filtrations*, which induce associated *motivic spectral sequences.* In the case S is a quasiregular - semiperfectoid ring, the filtrations are nothing but the double-speed Postnikov filtrations; in fact, the general case follows from this, via quasisyntomic descent. Passing to i-th graded pieces for the motivic filtrations, we obtain the following commutative square:

$$\begin{tikzcd}
\mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\}[2i] \arrow[r, "\varphi_i"', "\text{divided Frobenius}"] \arrow[d] & \widehat{\Delta}_S\{i\}[2i] \arrow[d] \\
\mathcal{N}^{i}\widehat{\Delta}_S[2i] \arrow[r] & \overline{\Delta}_S\{i\}[2i]
\end{tikzcd}$$

In this THH-approach to prismatic cohomology, all invariants happen to be complete, with respect to the *Nygaard filtration*. This is a filtration detected by $\mathrm{TC}^{-}(S;\mathbb{Z}_p)$, as a result of the S^1-homotopy fixed points spectral sequence: On the upper row, we have the divided Frobenius φ_i which takes the i-th *Nygaard filtered* piece $\mathcal{N}^{\geq i}\widehat{\Delta}_S$ to $\widehat{\Delta}_A$. For example, if we work over a perfectoid base $R_0 \to A$, then this is related to the prismatic Frobenius, which maps $\xi^i\widehat{\Delta}_S$ to $\widetilde{\xi}^i\widehat{\Delta}_S$, via the formula:

$$\varphi = \widetilde{\xi}^i\varphi_i$$

On the lower row, we have the graded counterpart of the aforementioned situation, where the n-th graded piece of the Nygaard filtration $\mathcal{N}^i\widehat{\Delta}_S$ maps to the *Hodge-Tate cohomology* $\overline{\Delta}_S$.

Again, working over a fixed perfectoid base $R_0 \to S$, we can equip the Hodge-Tate cohomology with the *conjugate filtration*. This is an increasing filtration on $\overline{\Delta}_S$, whose i-th filtered piece can be identified with:

$$\mathrm{Fil}_i^{\mathrm{conj}}\,\overline{\Delta}_S\{i\} \simeq \mathcal{N}^i\widehat{\Delta}_S\{i\}$$

Taking homotopy fibres, one can have an explicit formula calculating the associated graded pieces for the motivic filtration of topological cyclic homology:

$$\mathrm{gr}^i_{\mathcal{M}}\,\mathrm{TC}(A;\mathbb{Z}_p) \simeq \mathrm{fib}\left(\mathrm{can} - \varphi_i : \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \to \widehat{\Delta}_S\{i\}\right)$$

The starting point for the calculations of [BMS19] is the case when $R = R_0$ a perfectoid ring. Let us recall this, as this is essentially the basis for our discussions in the next chapter. In particular, passing to even homotopy groups, the commutative square gives rise to:

$$\begin{tikzcd}
\mathbb{A}_{\mathrm{inf}}(R_0)[u,v]/(uv-\xi) \arrow[r, "u\mapsto\sigma,\ v\mapsto\widetilde{\xi}\sigma^{-1}", "\varphi\text{-linear (divided Frob.)}"'] \arrow[d, "\vartheta\text{-linear}"', "u\mapsto u,\ v\mapsto 0"] & \mathbb{A}_{\mathrm{inf}}(R_0)[\sigma,\sigma^{-1}] \arrow[d, "\widetilde{\vartheta}\text{-linear}"', "\sigma\mapsto\sigma"] \\
R_0[u] \arrow[r, "u\mapsto\sigma", "R_0\text{-linear}"'] & R_0[\sigma,\sigma^{-1}]
\end{tikzcd}$$

2.6 Hochschild homology and the HKR-filtration

As shown in the works of Antieau [Ant19], Bhatt-Morrow-Scholze [BMS19], Moulinos–Robalo–Toën [MRT22], and Raksit [Rak20], the invariants that are associated to Hochschild homology are equipped with integral motivic filtrations, whose graded pieces can be computed in terms of the Hodge-completed derived de Rham complex.

Theorem 2.6.1 (HKR filtration)**.** *The following invariants associated to Hochschild homology are equipped with integral, complete, exhaustive, decreasing, multiplicative, $\mathbb{Z}$-indexed motivic filtrations, for an animated ring A. Passing to their associated graded pieces, these can be expressed in terms of the Hodge-completed derived de Rham complex:*

$$\mathrm{gr}^n_{\mathcal{M}} \mathrm{HH}(A) \simeq \wedge^n \mathbb{L}_A[2n]$$

$$\mathrm{gr}^n_{\mathcal{M}} \mathrm{HC}^-(A) \simeq \widehat{\mathrm{dR}}^{\geq n}_A[2n]$$

$$\mathrm{gr}^n_{\mathcal{M}} \mathrm{HP}(A) \simeq \widehat{\mathrm{dR}}_A[2n]$$

$$\mathrm{gr}^n_{\mathcal{M}} \mathrm{HC}(A)[1] \simeq \widehat{\mathrm{dR}}_A/\widehat{\mathrm{dR}}^{\geq n}_A[2n-1]$$

CHAPTER 3

THE PERFECTOID CASE

In this chapter we begin by recalling some basic constructions regarding perfectoid rings, mainly following [BMS18, Sec. 3]. We apply these in the study of the invariants TR^r and $(\mathrm{TR}^r)^{hS^1}$, in the case of perfectoid rings, building on ideas of [BMS19, Sec. 6] and [Mat21, Sec. 7].

3.1 Perfectoid rings

Here we gather some background regarding perfectoid rings. In what follows, R_0 always denotes a perfectoid ring.

Construction 3.1.1 (Fontaine-style maps $\vartheta_r, \widetilde{\vartheta}_r$ [BMS18, Lem. 3.2])**.** Let R_0 be a perfectoid ring. Then the following equivalence holds:

$$\mathbb{A}_{\mathrm{inf}}(R_0) \simeq \lim_{\mathrm{F}} \mathrm{W}_r(R_0)$$

As a result of this, the Frobenius automorphism φ on $\mathbb{A}_{\mathrm{inf}}(R_0)$ is identified with the Witt vector Frobenius $\mathrm{F} : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathbb{A}_{\mathrm{inf}}(R_0)$, while its inverse φ^{-1} is identified with the restriction map $\mathrm{R} : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathbb{A}_{\mathrm{inf}}(R_0)$.

Under this identification, we can consider the projection maps to the r-truncated Witt vectors of R_0:

$$\widetilde{\vartheta}_r : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathrm{W}_r(R_0)$$

as well as their twists by the r-th iteration of φ^{-1}:

$$\vartheta_r := \widetilde{\vartheta}_r \circ \varphi^r : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathrm{W}_r(R_0)$$

In particular, for $r = 1$, we obtain the aforementioned Fontaine map $\vartheta_1 = \vartheta$, whose kernel is generated by the degree 1 distinguished element $\xi \in \ker \vartheta$.

From this, one is able to also deduce that the kernel of the map ϑ_r is generated by the element:

$$\xi_r := \xi\varphi^{-1}(\xi)\dots\varphi^{-(r-1)}(\xi)$$

and the kernel of $\widetilde{\vartheta}_r$ is generated by

$$\widetilde{\xi}_r := \varphi^r(\xi_r) = \varphi(\xi)\varphi^2(\xi)\dots\varphi^r(\xi)$$

Finally, taking the derived limit over the Restriction mas, we write $\xi_\infty := \mathrm{R}\lim_{\mathrm{R}} \xi_r$. It follows that:

$$\mathbb{A}_{\inf}(R_0)/\xi_\infty \simeq \mathrm{R}\lim_{\mathrm{R}} \mathbb{A}_{\inf}(R_0)/\xi_r \simeq \mathrm{W}(R_0)$$

In particular, if $R_0 = \mathcal{O}_K$, for a spherically complete perfectoid ring K, we get that $\xi_\infty = \lim_{\mathrm{R}} \xi_r = \mu$ and $\mathbb{A}_{\inf}(R_0)/\mu \simeq \mathrm{W}(R_0)$.

The interactions between the maps ϑ_r, $\widetilde{\vartheta}_r$ and the Restriction, Frobenius maps on the Witt vectors are documented in the following lemma:

Lemma 3.1.1 (The action of R, F following [BMS18, Lem. 3.4])**.** *Consider the Witt vector Restriction and Frobenius maps. Under their action, the following diagrams are commutative for ϑ_r:*

$$\begin{tikzcd}
\mathbb{A}_{\inf}(R_0) \arrow[r, "\mathrm{id}"] \arrow[d, "\vartheta_{r+1}"'] & \mathbb{A}_{\inf}(R_0) \arrow[d, "\vartheta_r"] \\
\mathrm{W}_{r+1}(R_0) \arrow[r, "\mathrm{R}"] & \mathrm{W}_r(R_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}_{\inf}(R_0) \arrow[r, "\varphi"] \arrow[d, "\vartheta_{r+1}"'] & \mathbb{A}_{\inf}(R_0) \arrow[d, "\vartheta_r"] \\
\mathrm{W}_{r+1}(R_0) \arrow[r, "\mathrm{F}"] & \mathrm{W}_r(R_0)
\end{tikzcd}$$

Also, the following diagrams are commutative for $\widetilde{\vartheta}_r$:

$$\begin{tikzcd}
\mathbb{A}_{\mathrm{inf}}(R_0) \arrow[r, "\varphi^{-1}"] \arrow[d, "\widetilde{\vartheta}_{r+1}"'] & \mathbb{A}_{\mathrm{inf}}(R_0) \arrow[d, "\widetilde{\vartheta}_r"] \\
\mathrm{W}_{r+1}(R_0) \arrow[r, "\mathrm{R}"] & \mathrm{W}_r(R_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}_{\mathrm{inf}}(R_0) \arrow[r, "\mathrm{id}"] \arrow[d, "\widetilde{\vartheta}_{r+1}"'] & \mathbb{A}_{\mathrm{inf}}(R_0) \arrow[d, "\widetilde{\vartheta}_r"] \\
\mathrm{W}_{r+1}(R_0) \arrow[r, "\mathrm{F}"] & W_r(R_0)
\end{tikzcd}$$

3.2 TR of perfectoid rings

Now we move on to study the invariants of TR^r of perfectoid rings. Given a perfectoid ring R_0, an important property is that most invariants of THH are concentrated on even degrees. This is also true for $\mathrm{TR}^r(R_0; \mathbb{Z}_p)$, whose properties are documented in the following proposition.

Proposition 3.2.1 (The properties of $\mathrm{TR}^r(R_0; \mathbb{Z}_p)$, following [BMS19, Sec. 6], [Mat21, Sec. 7])**.** *Consider the following maps, defined for each $1 \le r < \infty$:*

$$\mathrm{TR}^r(R_0; \mathbb{Z}_p) \longrightarrow \mathrm{THH}(R_0; \mathbb{Z}_p)^{hC_{p^{r-1}}} \xrightarrow{\varphi^{hC_{p^{r-1}}}} \mathrm{THH}(R_0; \mathbb{Z}_p)^{tC_{p^r}}$$

Then, these are equivalences on connective covers. In particular, the following hold for $\mathrm{TR}^r(R_0; \mathbb{Z}_p)$*:*

1) The spectrum $\mathrm{TR}^r(R_0; \mathbb{Z}_p)$ is concentrated in even degrees, for $1 \le r < \infty$.

2) The object $\pi_2 \mathrm{TR}^r(R_0; \mathbb{Z}_p)$ is an invertible module over $\pi_0 \mathrm{TR}^r(R_0; \mathbb{Z}_p) \simeq \mathrm{W}_r(R_0)$, as a result of which, the following multiplication map is an isomorphism for $i \ge 0$:

$$\mathrm{Syn}^i \pi_2 \mathrm{TR}^r(R_0; \mathbb{Z}_p) \longrightarrow \pi_{2i} \mathrm{TR}^r(R_0; \mathbb{Z}_p)$$

As we noted in the introduction, our investigations follow a somewhat indirect course. In particular, the plan is to first look at the S^1-homotopy fixed points $\mathrm{TR}^r(R_0; \mathbb{Z}_p)^{hS^1}$ and then pass to $\mathrm{TR}^r(R_0; \mathbb{Z}_p)$ itself, for $1 \le r < \infty$. The following two propositions are devoted to these. In particular, we first identify the structure of the homotopy groups of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$ and then explain how to pass from the S^1-homotopy fixed points to the original spectrum $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$.

We are now able to identify the structure of the spectra $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(R_0;\mathbb{Z}_p)$, the bulk of which is concentrated in the following theorem:

Proposition 3.2.2 (Understanding $\pi_* \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$). *The spectrum* $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$, $1 \leq r < \infty$ *is concentrated in even degrees. In particular, the S^1-homotopy fixed points spectral sequence degenerates, yielding the following identification for the even homotopy groups:*

$$\pi_{2i} \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \begin{cases} \xi_r^i \, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}, & i \geq 0 \\ \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}, & i < 0 \end{cases}$$

Proof. Since the spectrum $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$ is concentrated in even degrees, the same happens for the spectra $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{tS^1}$, from say [BL22a, Rem. 6.1.7]. However, from the Tate orbit lemma of Nikolaus–Scholze [AN21, Lem. I.2.1] we know that $\mathrm{TR}^r(-;\mathbb{Z}_p)^{tC_p} \simeq \mathrm{THH}(-;\mathbb{Z}_p)^{tC_p}$, and therefore, $\mathrm{TR}^r(-;\mathbb{Z}_p)^{tS^1} \simeq \mathrm{TP}(-;\mathbb{Z}_p)$. From [BMS19, Sec. 6], it follows that $\mathrm{TP}(R_0;\mathbb{Z}_p)$ has the following identification for its even homotopy groups:

$$\pi_{2i} \mathrm{TP}(R_0;\mathbb{Z}_p) \simeq \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

Hence, forgetting the multiplicative structure of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$, the same is *non-canonically* true for its even homotopy groups, as well:

$$\pi_{2i} \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

Since the spectrum $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$ is concentrated even degrees, the associated S^1-homotopy fixed points spectral sequence degenerates, thus endowing $\pi_{2i} \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$ with a filtration. Applying π_0 to the map $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(R_0;\mathbb{Z}_p)$, this is identified with the map:

$$\vartheta_r : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathrm{W}_r(R_0)$$

whose kernel is generated by the element ξ_r. This follows by the commutative square:

$$\begin{tikzcd}
\mathbb{A}_{\mathrm{inf}}(R_0) \arrow[r, "\varphi^r"] \arrow[d, "\vartheta_r"] & \mathbb{A}_{\mathrm{inf}}(R_0) \arrow[d, "\widetilde{\vartheta}_r"] \\
\mathrm{W}_r(R_0) \arrow[r] & \mathrm{W}_r(R_0)
\end{tikzcd}$$

which is obtained by applying π_0 to the commutative square:

$$\begin{tikzcd}
\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(R_0;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(R_0;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^r(R_0;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{hC_{p^{r-1}}} \arrow[r, "\varphi^{hC_{p^{r-1}}}"] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd}$$

This is true for $r = 1$, by [BMS19]. For general $r \geq 1$, the claim follows by induction. One needs to use the fact that the rightmost map $\mathrm{TP}(R_0;\mathbb{Z}_p) \to \mathrm{THH}(R_0;\mathbb{Z}_p)^{tC_{p^r}}$ always gives rise to $\widetilde{\vartheta}_r : \mathbb{A}_{\mathrm{inf}}(R_0) \to \mathrm{W}_r(R_0)$, upon applying π_0 [BMS19, Sec. 6].

It follows that the filtration on $\pi_0 \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$ coming from the S^1-homotopy fixed points spectral sequence is the ξ_r-adic filtration on $\mathbb{A}_{\mathrm{inf}}(R_0)$. It propagates to all even homotopy groups of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$, via its multiplicative structure. The canonical identification follows:

$$\pi_* \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \begin{cases} \xi_r^i \,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\}, & * = 2i \geq 0 \\ \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}, & * = 2i < 0 \\ 0, & \text{otherwise} \end{cases}$$

This identification can also be shown via the iterated product description:

$$\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{TC}^-(R_0;\mathbb{Z}_p) \times_{\mathrm{TP}(R_0;\mathbb{Z}_p)} \cdots \times_{\mathrm{TP}(R_0;\mathbb{Z}_p)} \mathrm{TC}^-(R_0;\mathbb{Z}_p)$$

which gives rise to the short exact sequences:

$$0 \longrightarrow \pi_{2i}\,\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \longrightarrow \prod_{1\le k\le r} \pi_{2i}\,\mathrm{TC}^-(R_0;\mathbb{Z}_p)$$

$$\longrightarrow \prod_{1\le k\le r-1} \pi_{2i}\,\mathrm{TP}(R_0;\mathbb{Z}_p) \longrightarrow 0$$

$\square$

A direct reformulation of this proposition, using the graded ring descriptions from [BMS19, Sec. 6]:

$$\begin{cases} \pi_*\,\mathrm{TC}^-(R_0;\mathbb{Z}_p) \simeq \mathbb{A}_{\mathrm{inf}}(R_0)[u,v]/(uv-\xi), & \deg(u)=2,\ \deg(v)=-2,\ \deg(\xi)=0 \\ \pi_*\,\mathrm{TP}(R_0;\mathbb{Z}_p) \simeq \mathbb{A}_{\mathrm{inf}}(R_0)[\sigma,\sigma^{-1}], & \deg(\sigma)=2 \end{cases}$$

is the following corollary, regarding the multiplicative structure of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$:

Corollary 3.2.3. *The graded ring associated to* $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$ *has the following description, for generating elements with* $\deg(u_r)=2$, $\deg(v_r)=-2$, *and* $\deg(\xi_r)=0$*:*

$$\pi_*\,\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \mathbb{A}_{\mathrm{inf}}(R_0)[u_r,v_r]/(u_rv_r-\xi_r)$$

The following proposition is the main trick that allows us to go from the S^1-homotopy fixed points $\mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1}$, back to the original spectrum $\mathrm{TR}^r(-;\mathbb{Z}_p)$. It will be used repeatedly throughout the work, as this is paves a way to go from fairly more accessible calculations via $\mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1}$, back to identifying the structure of $\mathrm{TR}^r(-;\mathbb{Z}_p)$.

Proposition 3.2.4 (Back to $\mathrm{TR}^r(-;\mathbb{Z}_p)$)**.** *Let A be a connective $\mathbb{E}_\infty$-R_0-algebra. Then the following natural map is an equivalence of $\mathbb{E}_\infty$-ring spectra, for any $1 \leq r < \infty$:*

$$\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1}/v_r \simeq \mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \otimes_{\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}} \mathrm{TR}^r(R_0;\mathbb{Z}_p) \xrightarrow{\simeq} \mathrm{TR}^r(A;\mathbb{Z}_p)$$

Proof. The proof is a direct analogue of the one provided in [BMS19, Prop 6.4], which is an application of [AN21, Lem IV.4.12]. $\square$

Now, we are able to state our main theorem in the case of perfectoid rings.

Theorem 3.2.5 (Main result in the perfectoid case)**.** *Given a fixed perfectoid ring R_0, the following are true for $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$ and $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$:*

1) Consider the following commutative diagrams for the Restriction and Frobenius maps of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$ and its S^1-homotopy fixed points $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$, for $1 \leq r < \infty$.

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p)^{hS^1} \arrow[r, "\mathrm{R}^{hS^1}"] \arrow[d] & \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \arrow[d] \\
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p) \arrow[r, "\mathrm{R}"] & \mathrm{TR}^r(R_0;\mathbb{Z}_p)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p)^{hS^1} \arrow[r, dashed, "\mathrm{F}^{hS^1}"] \arrow[d] & \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \arrow[d] \\
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p) \arrow[r, dashed, "\mathrm{F}"] & \mathrm{TR}^r(R_0;\mathbb{Z}_p)
\end{tikzcd}$$

Passing to homotopy groups, they give rise to the following:

$$\begin{tikzcd}
\dfrac{\mathbb{A}_{\mathrm{inf}}(R_0)[u_{r+1},v_{r+1}]}{(u_{r+1}v_{r+1}-\xi_{r+1})} \arrow[r, "u_{r+1}\mapsto\varphi^{-r}(\xi)u_r,\ v_{r+1}\mapsto v_r", "\mathbb{A}_{\mathrm{inf}}(R_0)\text{-}\mathit{linear}"'] \arrow[d, "\vartheta_{r+1}\text{-}\mathit{linear}"', "u_{r+1}\mapsto u_{r+1},\ v_{r+1}\mapsto 0"] & \dfrac{\mathbb{A}_{\mathrm{inf}}(R_0)[u_r,v_r]}{(u_rv_r-\xi_r)} \arrow[d, "\vartheta_r-\mathit{linear}"', "u_r\mapsto u_r,\ v_r\mapsto 0"] \\
\mathrm{W}_{r+1}(R_0)[u_{r+1}] \arrow[r, "u_{r+1}\mapsto\varphi^{-r}(\xi)u_r"] & \mathrm{W}_r(R_0)[u_r]
\end{tikzcd}$$

$$\begin{tikzcd}
\dfrac{\mathbb{A}_{\mathrm{inf}}(R_0)[u_{r+1},v_{r+1}]}{(u_{r+1}v_{r+1}-\xi_{r+1})} \arrow[r, dashed, "u_{r+1}\mapsto u_r,\ v_{r+1}\mapsto\varphi(\xi)v_r", "\varphi\text{-}\mathit{linear}"'] \arrow[d, "\vartheta_{r+1}\text{-}\mathit{linear}"', "u_{r+1}\mapsto u_{r+1},\ v_{r+1}\mapsto 0"] & \dfrac{\mathbb{A}_{\mathrm{inf}}(R_0)[u_r,v_r]}{(u_rv_r-\xi_r)} \arrow[d, "\vartheta_r\text{-}\mathit{linear}"', "u_r\mapsto u_r,\ v_r\mapsto 0"] \\
\mathrm{W}_{r+1}(R_0)[u_{r+1}] \arrow[r, dashed, "u_{r+1}\mapsto u_r"] & \mathrm{W}_r(R_0)[u_r]
\end{tikzcd}$$

2) Mapping further to the Tate fixed points, we obtain the following commutative diagrams:

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(R_0;\mathbb{Z}_p) \arrow[r, "\mathrm{can}"] \arrow[d] & \mathrm{TP}(R_0;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{hC_{p^r}} \arrow[r, "\mathrm{can}"] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(R_0;\mathbb{Z}_p) \arrow[r, dashed, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(R_0;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^{r+1}(R_0;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{hC_{p^r}} \arrow[r, dashed, "\varphi^{hC_{p^r}}"] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{tC_{p^{r+1}}}
\end{tikzcd}$$

Passing to homotopy groups, they give rise to the following:

$$\begin{tikzcd}
\frac{\mathbb{A}_{\mathrm{inf}}(R_0)[u_{r+1},v_{r+1}]}{(u_{r+1}v_{r+1}-\xi_{r+1})} \arrow[rr, "u_{r+1}\mapsto\xi_{r+1}\sigma,\ v\mapsto\sigma^{-1}"] \arrow[d, "u_{r+1}\mapsto u_{r+1},\ v_{r+1}\mapsto 0", "\vartheta_{r+1}\text{-linear}"'] & & \mathbb{A}_{\mathrm{inf}}(R_0)[\sigma,\sigma^{-1}] \arrow[d, "\sigma\mapsto\sigma", "\widetilde{\vartheta}_{r+1}\text{-linear}"'] \\
\mathrm{W}_{r+1}(R_0)[u_{r+1}] \arrow[rr, "u_{r+1}\mapsto\xi_{r+1}\sigma"] & & \mathrm{W}_{r+1}(R_0)[\sigma,\sigma^{-1}]
\end{tikzcd}$$

$$\begin{tikzcd}
\frac{\mathbb{A}_{\mathrm{inf}}(R_0)[u_{r+1},v_{r+1}]}{(u_{r+1}v_{r+1}-\xi_{r+1})} \arrow[rr, dashed, "u_{r+1}\mapsto\sigma,\ v_{r+1}\mapsto\widetilde{\xi}_r\sigma^{-1}"] \arrow[d, "u_{r+1}\mapsto u_{r+1},\ v_{r+1}\mapsto 0", "\vartheta_{r+1}\text{-linear}"'] & & \mathbb{A}_{\mathrm{inf}}(R_0)[\sigma,\sigma^{-1}] \arrow[d, "\sigma\mapsto\sigma", "\widetilde{\vartheta}_{r+1}\text{-linear}"'] \\
\mathrm{W}_{r+1}(R_0)[u_{r+1}] \arrow[rr, dashed, "u_{r+1}\mapsto\sigma"] & & \mathrm{W}_{r+1}(R_0)[\sigma,\sigma^{-1}]
\end{tikzcd}$$

Proof. This essentially follows from iteration of the arguments presented in the proof of Proposition 3.2.2, together with the results of Proposition 3.2.4. First of all, via applying the latter, we pass from $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$ to $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$, by mapping $v_r \mapsto 0$. It follows that the graded ring of $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$ is equivalent to:

$$\pi_* \mathrm{TR}^r(R_0;\mathbb{Z}_p) \simeq \mathrm{W}_r(R_0)[u_r]$$

The results follow after passage to homotopy groups and applying [BMS19, Sec. 6], Proposition 3.2.1, as well as from the effect of the diagrams in 3.1.1, for the Fontaine style maps ϑ_r, $\widetilde{\vartheta}_r$.

In particular, on π_0, the diagrams in (1) correspond to the following diagrams for ϑ_r and its relation to the Restriction and Frobenius maps:

$$\begin{tikzcd}
\mathbb{A}_{\mathrm{inf}}(R_0) \arrow[r, "\mathrm{id}\,,\,\varphi"] \arrow[d, "\vartheta_r"'] & \mathbb{A}_{\mathrm{inf}}(R_0) \arrow[d, "\vartheta_r"] \\
\mathrm{W}_r(R_0) \arrow[r, "\mathrm{R}\,,\ \mathrm{F}"'] & \mathrm{W}_r(R_0)
\end{tikzcd}$$

On the other hand, passing to the Tate constructions, the diagrams we obtain on π_0 correspond to passing from ϑ_r to $\widetilde{\vartheta}_r$:

$$\begin{tikzcd}
\mathbb{A}_{\mathrm{inf}}(R_0) \arrow[r, "\varphi^r"] \arrow[d, "\vartheta_r"] & \mathbb{A}_{\mathrm{inf}}(R_0) \arrow[d, "\widetilde{\vartheta}_r"] \\
\mathrm{W}_r(R_0) \arrow[r] & \mathrm{W}_r(R_0)
\end{tikzcd}$$

$\square$

The following is a reformulation of our results, in the style of motivic filtrations:

Proposition 3.2.6 (Motivic filtrations in the perfectoid case)**.** *For a perfectoid ring R_0, the spectra* $\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(R_0;\mathbb{Z}_p)$ *are equipped with* motivic filtrations*, which in this case are the double-speed Postnikov filtrations.*

1) For $1 \le r < \infty$, we define a multiplicative, decreasing, complete filtration on $\mathbb{A}_{\mathrm{inf}}(R_0)$, which we call the r-Nygaard filtration*. This is defined as:*

$$\mathcal{N}_r^{\ge i}\,\mathbb{A}_{\mathrm{inf}}(R_0) := \begin{cases} \xi_r^i\,\mathbb{A}_{\mathrm{inf}}(R_0), & i \ge 0 \\ \\ \mathbb{A}_{\mathrm{inf}}(R_0), & i < 0 \end{cases}$$

The even homotopy groups (graded pieces for the motivic filtrations) of topological restriction homology and of its S^1-homotopy fixed points can be interpreted as follows, using the

r-Nygaard filtration on $\mathbb{A}_{\inf}(R_0)$:

$$\pi_* \operatorname{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\} = \bigoplus_{i\in\mathbb{Z}} \xi_r^i\,\mathbb{A}_{\inf}(R_0)\{i\}$$

$$\pi_* \operatorname{TR}^r(R_0;\mathbb{Z}_p) = \bigoplus_{i\geq 0} \mathcal{N}_r^{i}\,\mathbb{A}_{\inf}(R_0)\{i\} = \bigoplus_{i\geq 0} \xi_r^i/\xi_r^{i+1}\,\mathbb{A}_{\inf}(R_0)\{i\}$$

The restriction and Frobenius maps

$$\mathrm{R},\mathrm{F} : \operatorname{TR}^{r+1}(R_0;\mathbb{Z}_p) \longrightarrow \operatorname{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$$

induce natural maps on π_{2i}:

$$\mathrm{R},\mathrm{F} : \mathcal{N}_{r+1}^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\} \longrightarrow \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\}$$

In particular, Restriction induces the natural embedding:

$$\mathrm{R} : \mathcal{N}_{r+1}^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\} \hookrightarrow \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\}$$

while Frobenius corresponds to the following map:

$$\mathrm{F} : \mathcal{N}_{r+1}^{\geq n}\,\mathbb{A}_{\inf}(R_0) \longmapsto \big(\varphi(\xi_{r+1})\big)^i\,\mathbb{A}_{\inf}(R_0)\{i\} =$$
$$\big(\varphi(\xi)\xi_r\big)^i\,\mathbb{A}_{\inf}(R_0)\{i\} \hookrightarrow \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\}$$

Passing to graded pieces, we also obtain graded versions of these:

$$\mathrm{R},\mathrm{F} : \mathcal{N}_{r+1}^{i}\,\mathbb{A}_{\inf}(R_0)\{i\} \longrightarrow \mathcal{N}_r^{i}\,\mathbb{A}_{\inf}(R_0)\{i\}$$

2) Further mapping to $\mathrm{TP}(R_0;\mathbb{Z}_p)$*, the canonical and higher Frobenius maps*

$$\mathrm{can}, \varphi^{hS^1} : \mathrm{TR}^r(R_0;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(R_0;\mathbb{Z}_p) \longrightarrow \mathrm{TP}(R_0;\mathbb{Z}_p)$$

induce natural maps on π_{2i}*:*

$$\mathrm{can}, \varphi_{r,i} : \mathcal{N}_r^{\geq i}\, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \longrightarrow \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

The canonical map induces the natural embedding:

$$\mathrm{can} : \mathcal{N}_r^{\geq i}\, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \hookrightarrow \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

On the other hand, the higher Frobenius φ^{hS^1} *induces a* divided r-Frobenius*:*

$$\varphi_{r,i} : \mathcal{N}_r^{\geq i}\, \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \longrightarrow \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

This is defined away from ξ_r

$$\varphi_{r,i} : \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}\left[\frac{1}{\xi_r}\right] \longrightarrow \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}\left[\frac{1}{\widetilde{\xi}_r}\right]$$

and is related to the r*-th iterated Frobenius via the formula:*

$$\varphi^r = \xi_r^i \varphi_{r,i}$$

The divided r*-Frobenius naturally comes from the following commutative diagram, where* *the top row gives rise to filtered invariants, while the lower one gives rise to graded ones:*

$$\begin{tikzcd}
\mathrm{TR}^r(R_0; Zp)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(R_0;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(R_0;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^r(R_0;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{hC_{p^{r-1}}} \arrow[r, "\varphi^{hC_{p^{r-1}}}"'] & \mathrm{THH}(R_0;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd}$$

3) Taking the limit over restriction maps yields equivalences:

$$\begin{cases} \tau_{[2i-1,2i]}\,\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \simeq \operatorname{Rlim}_{\mathrm{R}}\ \xi_r^i\,\mathbb{A}_{\inf}(R_0)\{i\} =: \mathcal{N}_\infty^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\} \\ \tau_{[2n-1,2n]}\,\mathrm{TR}(R_0;\mathbb{Z}_p) \simeq \operatorname{Rlim}_{\mathrm{R}}\ \xi_r^i/\xi_r^{i+1}\,\mathbb{A}_{\inf}(R_0)\{i\} =: \mathcal{N}_\infty^{i}\,\mathbb{A}_{\inf}(R_0)\{i\} \end{cases}$$

In particular, we can identify:

$$\begin{cases} \pi_{2i}\,\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \simeq \lim_{\mathrm{R}} \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\inf}(R_0)\{i\} \simeq \lim_{\mathrm{R}} \xi_r^i\,\mathbb{A}_{\inf}(R_0)\{i\} \\ \pi_{2i-1}\,\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \simeq \lim_{\mathrm{R}}{}^1\,\mathcal{N}_r^{i}\,\mathbb{A}_{\inf}(R_0)\{i\} \simeq \lim_{\mathrm{R}}{}^1\,\xi_r^n\,\mathbb{A}_{\inf}(R_0)\{i\} \\ \pi_{2i}\,\mathrm{TR}(R_0;\mathbb{Z}_p) \simeq \lim_{\mathrm{R}} \mathcal{N}_r^{i}\,\mathbb{A}_{\inf}(R_0)\{i\} \\ \pi_{2i-1}\,\mathrm{TR}(R_0;\mathbb{Z}_p) \simeq \lim_{\mathrm{R}}{}^1\,\mathcal{N}_r^{i}\,\mathbb{A}_{\inf}(R_0)\{i\} \end{cases}$$

Notice that the Rlim^1 *terms, and therefore the odd homotopy groups, vanish in the case* $R_0 = \mathcal{O}_K$ *is the ring of integers of a spherically complete perfectoid field* K*, thus identifying* $\mu = \lim_{\mathrm{R}} \xi_r$*.*
The Frobenius maps

$$\begin{tikzcd}
\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \arrow[r, "\mathrm{F}^{hS^1}"] \arrow[d] & \mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \arrow[d] \\
\mathrm{TR}(R_0;\mathbb{Z}_p) \arrow[r, "\mathrm{F}"] & \mathrm{TR}(R_0;\mathbb{Z}_p)
\end{tikzcd}$$

induce natural Frobenius endofunctors:

$$\begin{tikzcd}
\mathcal{N}^{\geq i}_{\infty} \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \arrow[r, "\mathrm{F}"] \arrow[d] & \mathcal{N}^{\geq i}_{\infty} \mathbb{A}_{\mathrm{inf}}(R_0) \arrow[d] \\
\mathcal{N}^{i}_{\infty} \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \arrow[r, "\mathrm{F}"] & \mathbb{A}_{\mathrm{inf}}(R_0)\{i\}
\end{tikzcd}$$

Proof. The statements for finite $1 \leq r < \infty$ are a direct reformulation of Theorem 3.2.5. For the case $r = \infty$, taking the limit over restriction maps, we get that the graded pieces for the double-speed Postnikov filtrations give rise to the following two-term complexes for TR and its S^1-homotopy fixed points:

$$\tau_{[2i-1,2i]} \mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1} \longrightarrow \tau_{[2i-1,2i]} \mathrm{TR}(R_0; \mathbb{Z}_p)$$

Remember that the iterated pullback description of $\mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1}$ is:

$$\mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1} \simeq \cdots \times_{\mathrm{TP}(R_0;\mathbb{Z}_p)} \mathrm{TC}^-(R_0; \mathbb{Z}_p) \times_{\mathrm{TP}(R_0;\mathbb{Z}_p)} \cdots \times_{\mathrm{TP}(R_0;\mathbb{Z}_p)} \mathrm{TC}^-(R_0; \mathbb{Z}_p)$$

Passing to homotopy groups, we obtain the following exact sequences:

$$0 \longrightarrow \pi_{2i} \mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1} \longrightarrow \prod \pi_{2i} \mathrm{TC}^-(R_0; \mathbb{Z}_p) \longrightarrow$$

$$\longrightarrow \prod \pi_{2i} \mathrm{TP}(R_0; \mathbb{Z}_p) \longrightarrow \pi_{2i-1} \mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1} \longrightarrow 0$$

This is equivalent to:

$$0 \longrightarrow \pi_{2i} \mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1} \longrightarrow \prod \mathcal{N}^{\geq i} \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \longrightarrow$$

$$\longrightarrow \prod \pi_{2i} \mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \longrightarrow \pi_{2i-1} \mathrm{TR}(R_0; \mathbb{Z}_p)^{hS^1} \longrightarrow 0$$

which we identify with a $\lim^1$ sequence. In particular, we have the following equivalences:

$$\tau_{[2i-1,2i]}\,\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\} =: \mathcal{N}_\infty^{\geq i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

$$\begin{cases} \pi_{2i}\,\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \simeq \lim_{\mathrm{R}} \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \\ \pi_{2i-1}\,\mathrm{TR}(R_0;\mathbb{Z}_p)^{hS^1} \simeq \lim_{\mathrm{R}}^1 \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \end{cases}$$

Passing to $\mathrm{TR}^r(R_0;\mathbb{Z}_p)$ by taking quotient with v_r and taking the limit with respect to the Restriction maps, we also have that:

$$\tau_{[2i-1,2i]}\,\mathrm{TR}(R_0;\mathbb{Z}_p) \simeq \mathrm{R}\lim_{\mathrm{R}} \mathcal{N}_r^{i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\} =: \mathcal{N}_\infty^{i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

$$\begin{cases} \pi_{2i}\,\mathrm{TR}(R_0;\mathbb{Z}_p) \simeq \lim_{\mathrm{R}} \mathcal{N}_r^{i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \\ \pi_{2i-1}\,\mathrm{TR}(R_0;\mathbb{Z}_p) \simeq \lim_{\mathrm{R}}^1 \mathcal{N}_r^{i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\} \end{cases}$$

The remaining statements are a direct consequence of these identifications. $\square$

CHAPTER 4

THE GENERAL CASE

In this chapter we focus on explaining the main theorem of this work. Based on the calculations for perfectoid rings and some general properties of TR, we understand the motivic filtrations in the case of quasiregular - semperfectoid rings, which we later extend via quasisyntomic descent.

4.1 Calculations for quasiregular - semiperfectoid rings

In what follows, we study the motivic filtrations of invariants associated to TR of quasiregular - semiperfectoid rings. Let us fix a quasiregular - semiperfectoid ring S. Suppose we also make a choice of a perfectoid base $R_0 \to S$, mostly for simplicity.

For what follows, given any R_0-algebra A, we view:

$$\begin{cases} \pi_* \operatorname{TR}^r(A;\mathbb{Z}_p)^{hS^1} & \text{as a graded algebra over } \pi_* \operatorname{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \dfrac{\mathbb{A}_{\inf}(R_0)[u_r, v_r]}{(u_r v_r - \xi_r)} \\ \pi_* \operatorname{TR}^r(A;\mathbb{Z}_p) & \text{as a graded algebra over } \pi_* \operatorname{TR}^r(R_0;\mathbb{Z}_p) \simeq \mathrm{W}_r(R_0)[u_r] \end{cases}$$

Theorem 4.1.1 (Motivic filtration in the QRSPerfd case)**.** *Let S be a quasiregular - semiperfectoid ring over a fixed perfectoid base $R_0 \to S$. The following hold:*

1) For $1 \leq r < \infty$, the spectra $\operatorname{TR}^r(S;\mathbb{Z}_p)^{hS^1} \to \operatorname{TR}^r(S;\mathbb{Z}_p)$ admit functorial, complete and exhaustive, descending, multiplicative $\mathbb{Z}$-indexed (resp. $\mathbb{N}$-indexed) motivic filtrations*, with:*

$$\begin{cases} \operatorname{gr}^i_{\mathcal{M}} \operatorname{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \tau_{[2i-1,2i]} \operatorname{TR}^r(S;\mathbb{Z}_p)^{hS^1} \\ \operatorname{gr}^i_{\mathcal{M}} \operatorname{TR}^r(S;\mathbb{Z}_p) \simeq \tau_{[2i-1,2i]} \operatorname{TR}^r(S;\mathbb{Z}_p) \end{cases}$$

In particular, the associated spectral sequences calculating $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ *and* $\mathrm{TR}^r(S;\mathbb{Z}_p)^{tS^1} \simeq \mathrm{TP}(S;\mathbb{Z}_p)$ *equip:*

$$\widehat{\mathbb{\Delta}}_S \simeq \pi_0 \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \pi_0 \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{tS^1} \simeq \pi_0 \, \mathrm{TP}(S;\mathbb{Z}_p)$$

with the same complete, descending $\mathbb{N}$*-indexed filtration* $\mathcal{N}_r^{\geq \bullet}\widehat{\mathbb{\Delta}}_S$*, which we call the* r*-Nygaard filtration.*

2) Via the multiplicative structure of $\pi_* \, \mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1}$*, one can identify* $\mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S \subset \widehat{\mathbb{\Delta}}_S = \pi_0 \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$*, with* $\pi_{2i} \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$*, via multiplication with* $v_r^i \in \pi_{-2i} \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$*. In particular, we have the following descriptions of the even homotopy groups, setting* $\mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S = \widehat{\mathbb{\Delta}}_S$*, for* $i \leq 0$*:*

$$\pi_{2i} \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\}$$

Taking quotient with $v_r \in \pi_{-2} \, \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$*, we pass to* $\mathrm{TR}^r(S;\mathbb{Z}_p)$*, thus obtaining the following identification for its even homotopy groups:*

$$\pi_{2i} \, \mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \mathcal{N}_r^{i}\widehat{\mathbb{\Delta}}_S\{i\}$$

where, in particular, we have that:

$$\mathcal{N}_r^{0}\widehat{\mathbb{\Delta}}_S \simeq \mathrm{W}_r(S)$$

3) The Restriction and Frobenius maps

$$\begin{tikzcd} \mathrm{TR}^{r+1}(S;\mathbb{Z}_p)^{hS^1} \arrow[r, "\mathrm{R}^{hS^1}\,,\ \mathrm{F}^{hS^1}"] \arrow[d] & \mathrm{TR}^{r}(S;\mathbb{Z}_p)^{hS^1} \arrow[d] \\ \mathrm{TR}^{r+1}(S;\mathbb{Z}_p) \arrow[r, "\mathrm{R}\,,\ \mathrm{F}"] & \mathrm{TR}^{r}(S;\mathbb{Z}_p) \end{tikzcd}$$

induce natural maps on the filtered and graded pieces of the r-Nygaard filtered $\widehat{\mathbb{\Delta}}_S$*:*

$$\begin{tikzcd} \mathcal{N}^{\geq i}_{r+1}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{R}\,,\ \mathrm{F}"] \arrow[d] & \mathcal{N}^{\geq i}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[d] \\ \mathcal{N}^{i}_{r+1}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{R}\,,\ \mathrm{F}"] & \mathcal{N}^{i}_{r}\widehat{\mathbb{\Delta}}_S\{i\} \end{tikzcd}$$

Mapping further to $\mathrm{TP}(S;\mathbb{Z}_p)$*:*

$$\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow \mathrm{TC}^-(S;\mathbb{Z}_p) \xrightarrow{\mathrm{can}\,,\ \varphi^{hS^1}} \mathrm{TP}(S;\mathbb{Z}_p)$$

we obtain maps on even homotopy groups $\mathrm{can}, \varphi_{r,i} : \pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \to \mathrm{TP}(S;\mathbb{Z}_p)$*, which are equivalent to the canonical injection*

$$\mathrm{can} : \mathcal{N}^{\geq i}_r\widehat{\mathbb{\Delta}}_S\{i\} \hookrightarrow \widehat{\mathbb{\Delta}}_S\{i\}$$

and to the divided r-Frobenius*:*

$$\varphi_{r,i} : \mathcal{N}^{\geq i}_r\widehat{\mathbb{\Delta}}_S\{i\} \longrightarrow \widehat{\mathbb{\Delta}}_S\{i\}$$

which relates to the r-th iteration of Frobenius, which maps the r-Nygaard filtration to *the $\widetilde{\xi}_r$-adic filtration, via the formula:*

$$\varphi^r = \widetilde{\xi}_r^i \varphi_{r,i}$$

Using the commutative diagram

$$\begin{tikzcd}
\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(S;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(S;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^r(S;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(S;\mathbb{Z}_p)^{hC_{p^{r-1}}} \arrow[r, "\varphi^{hC_{p^{r-1}}}"] & \mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd}$$

*one can identify the effect of $\varphi^{hC_{p^{r-1}}}$, as the graded version of φ^{hS^1}. In particular, passing to even homotopy groups, we obtain a graded Frobenius map from the associated graded pieces of the r-Nygaard filtration to the r-*Hodge–Tate cohomology $\Delta_S^{\mathrm{HT},r} \simeq \widehat{\Delta}_S/\widetilde{\xi}_r$, *obtained from* $\mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^r}}$ *:*

$$\mathrm{gr}\,\varphi_{r,i} : \mathcal{N}_r^i\widehat{\Delta}_S\{i\} \longrightarrow \Delta_S^{\mathrm{HT},r}\{i\} = \widehat{\Delta}_S/\widetilde{\xi}_r\{i\}$$

4) Taking the limit over the Restriction maps yields the following identifications:

$$\begin{cases} \pi_{2i}\,\mathrm{TR}(S;\mathbb{Z}_p)^{hS^1} \simeq \lim\limits_{\mathrm{R}} \mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\} \\ \pi_{2i}\,\mathrm{TR}(S;\mathbb{Z}_p) \simeq \lim\limits_{\mathrm{R}} \mathcal{N}_r^{i}\widehat{\Delta}_S\{i\} \end{cases}$$

We have natural Frobenius maps:

$$\begin{tikzcd} \mathrm{TR}(S;\mathbb{Z}_p)^{hS^1} \arrow[r, "\mathrm{F}^{hS^1}"] \arrow[d] & \mathrm{TR}(S;\mathbb{Z}_p)^{hS^1} \arrow[d] \\ \mathrm{TR}(S;\mathbb{Z}_p) \arrow[r, "\mathrm{F}"] & \mathrm{TR}(S;\mathbb{Z}_p) \end{tikzcd}$$

Passing to even homotopy groups, induces Frobenius endofunctors on filtered and graded pieces:

$$\begin{tikzcd} \lim\limits_{\mathrm{R}} \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{F}^{hS^1}"] \arrow[d] & \lim\limits_{\mathrm{R}} \mathcal{N}_r^{\geq i}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[d] \\ \lim\limits_{\mathrm{R}} \mathcal{N}_r^{i}\widehat{\mathbb{\Delta}}_S\{i\} \arrow[r, "\mathrm{F}"] & \lim\limits_{\mathrm{R}} \mathcal{N}_r^{i}\widehat{\mathbb{\Delta}}_S\{i\} \end{tikzcd}$$

Proof. The basics of all calculations follow from the iterated pullback descriptions:

$$\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{TC}^-(S;\mathbb{Z}_p) \times_{\mathrm{TP}(S;\mathbb{Z}_p)} \cdots \times_{\mathrm{TP}(S;\mathbb{Z}_p)} \mathrm{TC}^-(S;\mathbb{Z}_p)$$

and

$$\mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \mathrm{THH}(S;\mathbb{Z}_p)^{hC_{p^{r-1}}} \times_{\mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^{r-1}}}} \cdots \times_{\mathrm{THH}(S;\mathbb{Z}_p)^{tC_p}} \mathrm{THH}(S;\mathbb{Z}_p)$$

These can be re-written as the fibers:

$$\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow \prod_{1\leq k\leq r} \mathrm{TC}^-(S;\mathbb{Z}_p) \longrightarrow \prod_{1\leq k\leq r-1} \mathrm{TP}(S;\mathbb{Z}_p)$$

and

$$\mathrm{TR}^r(S;\mathbb{Z}_p) \longrightarrow \prod_{1\leq k\leq r} \mathrm{THH}(S;\mathbb{Z}_p)^{hC_{p^k}} \longrightarrow \prod_{1\leq k\leq r-1} \mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^k}}$$

In particular, the motivic filtrations come from the double-speed Postnikov filtrations, whose graded pieces are the two term complexes:

$$\tau_{[2i-1,2i]}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}, \qquad \tau_{[2i-1,2i]}\,\mathrm{TR}(S;\mathbb{Z}_p)$$

More specifically, the homotopy groups fit in exact sequences:

$$0 \longrightarrow \pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow \prod_{1\le k\le r} \pi_{2i}\,\mathrm{TC}^-(S;\mathbb{Z}_p) \longrightarrow$$
$$\longrightarrow \prod_{1\le k\le r-1} \pi_{2i}\,\mathrm{TP}(S;\mathbb{Z}_p) \longrightarrow \pi_{2i-1}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow 0$$

or, equivalently

$$0 \longrightarrow \pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow \prod_{1\le k\le r} \mathcal{N}^{\ge i}\widehat{\mathbb{\Delta}}_S\{i\} \longrightarrow$$
$$\longrightarrow \prod_{1\le k\le r-1} \widehat{\mathbb{\Delta}}_S\{i\} \longrightarrow \pi_{2i-1}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow 0$$

and

$$0 \longrightarrow \pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p) \longrightarrow \prod_{1\le k\le r} \pi_{2i}\,\mathrm{THH}(S;\mathbb{Z}_p)^{hC_{p^k}} \longrightarrow$$
$$\longrightarrow \prod_{1\le k\le r-1} \pi_{2i}\,\mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^k}} \longrightarrow \pi_{2i-1}\,\mathrm{TR}^r(S;\mathbb{Z}_p) \longrightarrow 0$$

From the first of these exact sequences it follows that $\pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ is identified as the i-th layer of the r-Nygaard filtration on $\widehat{\mathbb{\Delta}}_S\{i\}$, which is defined via the following iterated pullback diagram:

$$\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\} \simeq \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \times_{\widehat{\Delta}_S\{i\}} \cdots \times_{\widehat{\Delta}_S\{i\}} \mathcal{N}^{\geq i}\widehat{\Delta}_S$$

If, for simplicity, we assume that to be working over a fixed perfectoid $R_0 \to S$, it follows that we have the equivalent description:

$$\mathcal{N}_r^{\geq i}\widehat{\Delta}_S \simeq \left\{ x \in \widehat{\Delta}_S \;\middle|\; \varphi^{ri}(x) \in \widetilde{\xi}_r^i \widehat{\Delta}_S \right\}$$

To go back to $\mathrm{TR}^r(S;\mathbb{Z}_p)$, we let $v_r \mapsto 0$. Through this, we obtain the identification:

$$\pi_{2i}\,\mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \mathcal{N}_r^i\widehat{\Delta}_S\{i\}$$

Finally, the identification regarding $\widehat{\Delta}_S/\xi_r$ is a direct corollary of Proposition **??**, in analogy with [BMS19].

The rest of the claims are a direct application of induction on $1 \leq r < \infty$, using the iterated pullback descriptions for $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ and $\mathrm{TR}^r(S;\mathbb{Z}_p)$, together with the base cases on perfectoids:

$$\mathrm{gr}^i_{\mathcal{M}}\,\mathrm{TR}^r(R_0;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}_r^{\geq i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\}, \qquad \mathrm{gr}^i_{\mathcal{M}}\,\mathrm{TR}^r(R_0;\mathbb{Z}_p) \simeq \mathcal{N}_r^{i}\,\mathbb{A}_{\mathrm{inf}}(R_0)\{i\}$$

□

4.2 Applying quasisyntomic descent

In the previous section we managed to provide an overview of the nature of the invariants $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^r(S;\mathbb{Z}_p)$, $1 \leq r \leq \infty$, for a given quasiregular - semiperfectoid ring S. Following the road paved by [BMS19], we can now extend to the quasisyntomic (and even animated) case. The main steps are provided in what follows.

Proof of Theorem 1.1.1. For simplicity, we work over a fixed perfectoid ring $R_0 \to S$. As the category of quasiregular - semiperfectoid rings provide a basis for the quasisyntomic topology, most of the claims follow in a completely analogous way to the main theorem of [BMS19]. In particular, we apply quasisyntomic descent to the two term complexes:

$$\tau_{[2i-1,2i]} \operatorname{TR}^r(S;\mathbb{Z}_p)^{hS^1}, \qquad \tau_{[2i-1,2i]} \operatorname{TR}^r(S;\mathbb{Z}_p)$$

for S quasiregular - semiperfectoid and the associated results of the previous section.

Regarding the odd homotopy groups of our invariants, remember that for S quasiregular - semiperfectoid, these fit in the exact sequence:

$$0 \longrightarrow \mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\} \longrightarrow \prod_{1\leq k\leq r} \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \xrightarrow{\alpha}$$
$$\xrightarrow{\alpha} \prod_{1\leq k\leq r-1} \widehat{\Delta}_S\{i\} \longrightarrow \pi_{2i-1} \operatorname{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow 0$$

Pre-composing the map α, with the diagonal map:

$$\operatorname{diag} : \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\} \longrightarrow \prod_{1\leq k\leq r} \mathcal{N}^{\geq i}\widehat{\Delta}_S\{i\}$$

we are able to use the vanishing theorem of Bhatt-Scholze [BS22, Sec. 14]. In particular, by directly applying that result, there exists a suitable quasisyntomic cover $S' \to S$, for which $\alpha \circ \operatorname{diag}$ is surjective. Hence, the same is true for α, as well, from which the local vanishing of the cokernel follows.

Therefore, locally for the quasisyntomic topology, we have that:

$$\operatorname{gr}^i_{\mathcal{M}} \operatorname{TR}^r(-;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}_r^{\geq i}\widehat{\Delta}_{(-)}\{i\}[2i]$$

Hence, by letting $v_r \mapsto 0$, we also have the analogous result for TR^r, locally in the quasisyntomic topology:

$$\mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(-;\mathbb{Z}_p) \simeq \mathcal{N}^i_r \widehat{\Delta}_{(-)}\{i\}[2i]$$

Taking the limit over restriction maps, in analogy with the perfectoid case, we also obtain:

$$\begin{cases} \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}(-;\mathbb{Z}_p)^{hS^1} \simeq \operatorname*{Rlim}_{\mathrm{R}} \mathcal{N}^{\geq i}_r \widehat{\Delta}_{(-)}\{i\}[2i] =: \mathcal{N}^{\geq i}_{\infty} \widehat{\Delta}_{(-)}\{i\}[2i] \\ \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}(-;\mathbb{Z}_p) \simeq \operatorname*{Rlim}_{\mathrm{R}} \mathcal{N}^i_r \widehat{\Delta}_{(-)}\{i\}[2i] =: \mathcal{N}^i_{\infty} \widehat{\Delta}_{(-)}\{i\}[2i] \end{cases}$$

As in [BL22a, Sec. 6], it is possible to left Kan extend the motivic filtrations, and thus all associated prismatic invariants, from the quasisyntomic to the animated case. However, the motivic filtrations need not be exhaustive in this case. $\square$

Following this, the proof of Theorem B is a direct consequence of the above:

Proof of Theorem 1.1.2. We consider the invariant

$$\widetilde{\mathrm{TC}}^r(-;\mathbb{Z}_p) := \mathrm{fib}\left(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1} \to \mathrm{TR}^{r-1}(-;\mathbb{Z}_p)^{hS^1}\right)$$

Since, locally for the quasisyntomic topology we have that:

$$\mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(-;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}^{\geq i}_r \widehat{\Delta}_{(-)}\{i\}[2i]$$

it follows that by taking fibers, $\widetilde{\mathrm{TC}}^r$ is equipped with a motivic filtration, whose description, locally for the quasisyntomic topology, is the following:

$$\mathrm{gr}^i_{\mathcal{M}} \widetilde{\mathrm{TC}}^r(-;\mathbb{Z}_p) \simeq \left(\mathrm{R}^{hS^1} - \mathrm{F}^{hS^1} : \mathcal{N}^{\geq i}_r \widehat{\Delta}_{(-)}\{i\}[2i] \to \mathcal{N}_{r-1} \widehat{\Delta}_{(-)}\{i\}[2i]\right)$$

We proceed similarly for TC^r. $\square$

CHAPTER 5

THE CASES OF MIXED/POSITIVE CHARACTERISTIC

We know restrict to the case of mixed and positive characteristic.

5.1 The mixed characteristic case

Proposition 5.1.1. *Let S be a p-completely smooth ring over a fixed perfectoid base R_0. Then, the r-Nygaard filtration is identified with the filtration related to the décalage functor $L\eta_{\xi_r}\widehat{\mathbb{\Delta}}_{S/R_0}$, for the element $\xi \in \mathbb{A}_{\text{inf}}(R_0)$.*

Proof. This is a direct application of the description of the Nygaard filtration over a perfectoid base. Gluing r-copies of the Nygaard filtered prismatic cohomology gives the result. A related discussion is in [BMS19, Cor. 7.10]. □

The proof of 1.1.4 follows, as a direct application of the identification $\widehat{\mathbb{\Delta}}_{S/R_0} \simeq A\Omega_S$, for a p-completely smooth algebra over $\mathcal{O}_C$. In particular we have the following corollary, which follows from the properties of the décalage functor/the Beilinson t-structure.

Corollary 5.1.2. *Let S be a p-completely smooth algebra over $\mathcal{O}_C$. Then the commutative diagram:*

$$\begin{tikzcd}
\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \arrow[r] \arrow[d] & \mathrm{TC}^-(S;\mathbb{Z}_p) \arrow[r, "\varphi^{hS^1}"] \arrow[d] & \mathrm{TP}(S;\mathbb{Z}_p) \arrow[d] \\
\mathrm{TR}^r(S;\mathbb{Z}_p) \arrow[r] & \mathrm{THH}(S;\mathbb{Z}_p)^{hC_{p^{r-1}}} \arrow[r, "\varphi^{hC_{p^{r-1}}}"] & \mathrm{THH}(S;\mathbb{Z}_p)^{tC_{p^r}}
\end{tikzcd}$$

gives rise to the following commutative square, by passing to the even parts of the motivic *filtrations of the associated invariants:*

$$\begin{tikzcd} L\eta_{\xi_r}^{\geq i} A\Omega_S\{i\} \arrow[r, "\varphi_{r,i}"] \arrow[d] & A\Omega_S\{i\} \arrow[d] \\ \widetilde{W_r\Omega}_S^{\leq i}\{i\} \arrow[r, hook] & \widetilde{W_r\Omega}_S\{i\} \end{tikzcd}$$

where, in the lower row, we have the canonical injection of the i-th filtered piece of the conjugate filtration of the complex $\widetilde{W_r\Omega}_S$.

5.2 The positive characteristic case

Let us, finally, treat the case of $\mathbb{F}_p$-algebras. Given a quasisyntomic $\mathbb{F}_p$-algebra S, we know from [BMS19] that its prismatic cohomology is identified with the Nygaard-completed derived de Rham complex:

$$\widehat{\mathbb{\Delta}}_S \simeq \widehat{\mathrm{LW\Omega}}_S$$

In particular, if S is a quasiregular - semiperfect $\mathbb{F}_p$-algebra, we have that $\widehat{\mathbb{\Delta}}_S \simeq \widehat{\mathbb{A}}_{\mathrm{crys}}(S)$, with the Nygaard filtration identified as:

$$\mathcal{N}^{\geq i}\,\widehat{\mathbb{A}}_{\mathrm{crys}}(S) = \left\{ x \in \widehat{\mathbb{A}}_{\mathrm{crys}}(S) \;\middle|\; \varphi^i(x) \in p^i\,\widehat{\mathbb{A}}_{\mathrm{crys}}(S) \right\}$$

Using these, we can show the odd vanishing for the invariants of TR:

Proof of Theorem 1.1.5. Consider the exact sequence coming from the iterated pullback description of $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ for a quasiregular - semiperfect $\mathbb{F}_p$-algebra S:

$$0 \longrightarrow \mathcal{N}_r^{\geq i}\,\widehat{\mathbb{A}}_{\mathrm{crys}}(S)\{i\} \longrightarrow \prod_{1\leq k\leq r} \mathcal{N}^{\geq i}\,\widehat{\mathbb{A}}_{\mathrm{crys}}(S)\{i\} \xrightarrow{\alpha}$$

$$\xrightarrow{\alpha} \prod_{1\leq k\leq r-1} \widehat{\mathbb{A}}_{\mathrm{crys}}(S)\{i\} \longrightarrow \pi_{2i-1}\,\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \longrightarrow 0$$

Following [BMS19, Sec. 8], we know that the map

$$\alpha \circ \mathrm{diag} : \mathcal{N}^{\geq i} \widehat{\mathbb{A}}_{\mathrm{crys}}(S)\{i\} \to \prod_{1 \leq k \leq r} \widehat{\mathbb{A}}_{\mathrm{crys}}(S)\{i\}$$

is surjective. Therefore the same is also true for α itself. The vanishing of $\pi_{2i-1} \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ follows. Letting $v_r \mapsto 0$, we can also see that $\pi_{2i-1} \mathrm{TR}^r(S;\mathbb{Z}_p)$ is also even.

Taking the limit over restriction maps, since we are in the characteristic $p > 0$ case, notice that the $\lim_{\mathrm{R}}^1$ term vanishes, therefore $\mathrm{TR}(S;\mathbb{Z}_p)^{hS^1}$ and $\mathrm{TR}(S;\mathbb{Z}_p)$ are also concentrated on even degrees.

Hence, the discussion we had in the perfectoid case, also applies here. In particular, because of the vanishing of odd homotopy groups, it follows that the S^1-homotopy fixed points spectral sequence degenerates and the r-Nygaard filtration on $\widehat{\mathbb{A}}_{\mathrm{crys}}(S)$ is indeed the filtration coming from the spectral sequence.

The relation to the conjugate-filtered de Rham–Witt complex follows from Theorem 1.1.4. In particular, this is also related, and essentially explains, the presence of the Hodge filtration on TP of $\mathbb{F}_p$-algebras, as discussed by Antieau–Nikolaus [AN21, Sec. 6.3]. $\square$

Applying quasisyntomic descent, the following is a direct corollary of what we just discussed:

Corollary 5.2.1. *Let S be a quasisyntomic algebra over $\mathbb{F}_p$. Then the spectra $\mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1}$ and $\mathrm{TR}^r(S;\mathbb{Z}_p)$ are equipped with motivic filtrations, whose graded pieces can be identified with:*

$$\begin{cases} \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p)^{hS^1} \simeq \mathcal{N}_r^{\geq i} \widehat{\mathbb{\Delta}}_S\{i\}[2i] \\ \mathrm{gr}^i_{\mathcal{M}} \mathrm{TR}^r(S;\mathbb{Z}_p) \simeq \mathcal{N}_r^{i} \widehat{\mathbb{\Delta}}_S\{i\}[2i] \end{cases}$$

In an analogous manner, the graded pieces for the motivic filtrations of $\widetilde{\mathrm{TC}}^r\big(\mathrm{TR}(S;\mathbb{Z}_p)\big)$

and $\mathrm{TC}^r\left(\mathrm{TR}(S;\mathbb{Z}_p)\right)$ *can be shown to be equivalent to:*

$$\begin{aligned}\mathrm{gr}^i_{\mathcal{M}}\,\widetilde{\mathrm{TC}}^r\Big(\mathrm{TR}(S;\mathbb{Z}_p)\Big) &\simeq \mathrm{fib}\Big(\mathrm{R}^{hS^1}-\mathrm{F}^{hS^1}:\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\}[2i]\to\mathcal{N}_{r-1}^{\geq i}\widehat{\Delta}_S\{i\}[2i]\Big)\\ \mathrm{gr}^i_{\mathcal{M}}\,\mathrm{TC}^r\Big(\mathrm{TR}(S;\mathbb{Z}_p)\Big) &\simeq \mathrm{fib}\Big(\mathrm{can}-\varphi^{hS^1}:\mathcal{N}_r^{\geq i}\widehat{\Delta}_S\{i\}[2i]\to\widehat{\Delta}_S\{i\}[2i]\Big)\end{aligned}$$

CHAPTER 6

ONGOING WORK AND FUTURE RESEARCH DIRECTIONS

Associated to the Nygaard filtration on (absolute) prismatic cohomology, one can consider a number of related structure, such as the prismatization stacks Σ', Σ'', the Hodge–Tate locus Σ^{HT}, the diffracted complex and the absolute/relative de Rham–Witt comparisons, etc. In analogy with this, we expect some relevant structure to arise in relation to the r-Nygaard filtration on prismatic cohomology. In what follows, we describe some phenomena which shall be explained in forthcoming works, which are essentially extended versions of this thesis [And24a], [And24b], as well as work in progress.

6.1 A relative de Rham–Witt comparison

Let S be a p-complete animated ring. Following [BL22a], one has the divided Frobenius on the Nygaard filtered absolute prismatic cohomology, which fits in the following square:

$$\begin{tikzcd} \mathcal{N}^{\geq i}\Delta_S\{i\} \arrow[r, "\varphi_i"] \arrow[d] & \Delta_S\{i\} \arrow[d] \\ \mathcal{N}^i\Delta_S \arrow[r, "\mathrm{gr}\,\varphi_i"] & \overline{\Delta}_S\{i\} \end{tikzcd}$$

If we pass to the relative situation, where S lives over $\overline{A}$ for a bounded prism (A, I), then we can say more, as a relative Hodge–Tate comparison is easy to formulate:

$$\begin{tikzcd}
\mathcal{N}^{\geq i}\varphi^*{\mathbb{\Delta}}_{S/A}\{i\} \arrow[rr] \arrow[d] & & {\mathbb{\Delta}}_{S/A}\{i\} \arrow[d] \\
\mathcal{N}^{i}\varphi^*{\mathbb{\Delta}}_{S/A} \arrow[rr] \arrow[dr, bend right, "\simeq"'] & & \overline{\mathbb{\Delta}}_{S/A}\{i\} \\
& \mathrm{Fil}_i^{\mathrm{conj}}\,\overline{\mathbb{\Delta}}_{S/A}\{i\} \arrow[ur, hook, bend right] &
\end{tikzcd}$$

as Hodge–Tate cohomology is equipped with the increasing conjugate filtration, whose i-th filtered piece is identified with the i-th associated graded piece of the Nygaard filtration, as shown in the factorization of the diagram. The relative Hodge–Tate comparison is formulated as:

$$\mathrm{gr}_i^{\mathrm{conj}}\,\overline{\mathbb{\Delta}}_{S/A}\{i\} \simeq \mathrm{gr}^i_{\mathrm{Hod}}\,\mathrm{dR}_{S/\overline{A}}$$

where on the right hand side we have the i-th graded piece for the p-complete, Hodge-filtered, relative derived de Rham complex. In fact, we can reformulate this as a de Rham comparison:

$$\overline{A} \otimes_{\varphi} {\mathbb{\Delta}}_{S/A} \simeq \mathrm{dR}_{S/\overline{A}}$$

in which case the Nygaard filtration on the left maps to the Hodge filtration on the right.

It appears that a similar picture is possible for the r-Nygaard filtration. Using the iterated product description:

$$\mathcal{N}_r^{\geq i}{\mathbb{\Delta}}_S\{i\} \simeq \mathcal{N}^{\geq i}{\mathbb{\Delta}}_S\{i\} \times_{{\mathbb{\Delta}}_S\{i\}} \cdots \times_{{\mathbb{\Delta}}_S\{i\}} \mathcal{N}^{\geq i}{\mathbb{\Delta}}_S\{i\}$$

it is possible to have a divided r-Frobenius by mapping to ${\mathbb{\Delta}}_S\{i\}$ from the left, via the divided

Frobenius map:

$$\varphi_{r,i} : \mathcal{N}_r^{\geq i} {\mathbbl{\Delta}}_S\{i\} \longrightarrow \mathcal{N}^{\geq i} {\mathbbl{\Delta}}_S\{i\} \xrightarrow{\varphi_i} {\mathbbl{\Delta}}_S\{i\}$$

This happens to fit in the following commutative diagram, together with the r-Hodge–Tate cohomology:

$$\begin{tikzcd}
\mathcal{N}_r^{\geq i} {\mathbbl{\Delta}}_S\{i\} \arrow[r, "\varphi_{r,i}"] \arrow[d] & {\mathbbl{\Delta}}_S\{i\} \arrow[d] \\
\mathcal{N}_r^{i} {\mathbbl{\Delta}}_S\{i\} \arrow[r, "\mathrm{gr}\,\varphi_{r,i}"] & {\mathbbl{\Delta}}_S^{\mathrm{HT},r}\{i\}
\end{tikzcd}$$

Passing to the relative situation, where we suppose that S lives over $\overline{A}$, for a bounded prism (A, I), we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{N}_r^{\geq i} (\varphi^r)^* {\mathbbl{\Delta}}_{S/A}\{i\} \arrow[rr] \arrow[d] & & {\mathbbl{\Delta}}_{S/A}\{i\} \arrow[d] \\
\mathcal{N}_r^{i} (\varphi^r)^* {\mathbbl{\Delta}}_{S/A}\{i\} \arrow[rr] \arrow[dr, bend right, "\simeq"'] & & {\mathbbl{\Delta}}_{S/A}^{\mathrm{HT},r}\{i\} \\
& \mathrm{Fil}_i^{\mathrm{conj}} {\mathbbl{\Delta}}_{S/A}^{\mathrm{HT},r}\{i\} \arrow[ur, hook, bend right] &
\end{tikzcd}$$

where the lower row factors through the i-th filtered piece for the conjugate filtration on the r-Hodge–Tate cohomology. Passing to the associated graded pieces, we obtain the i-th associated graded piece for the p-complete, Hodge-filtered, animated relative de Rham–Witt complex of Langer–Zink:

$$\mathrm{gr}_i^{\mathrm{conj}} {\mathbbl{\Delta}}_{S/A}^{\mathrm{HT},r} \simeq \mathrm{gr}_{\mathrm{Hod}}^i LW_r\Omega_{S/\overline{A}}$$

In fact, it is possible to reformulate this as a relative de Rham–Witt comparison:

$$A/I_r \otimes_{\varphi^r} \Delta_{S/A} \simeq LW_r\Omega_{S/\overline{A}}$$

in which case, the r-Nygaard filtration on the left maps to the Hodge filtration on the right.

6.2 The absolute de Rham–Witt complex and higher Hodge–Tate loci

The absolute counterpart, for the relative de Rham–Witt complex of Langer–Zink, is given by the absolute de Rham–Witt complex of Hesselholt–Madsen [HM04; Hes05; HM03; Hes15]. The problem, when comparing the two, stands in the following asymmetry. Let S be a smooth p-complete ring. We consider p-complete versions of the absolute/relative de Rham–Witt complexes. Then $W_r\Omega_{S/\mathbb{Z}_p}$ is not isomorphic to $W_r\Omega_S$. We believe this asymmetry is analogous to the incompatibility between the conjugate filtrations on absolute Hodge–Tate cohomology versus relative Hodge–Tate cohomology and that therefore, one should have a connection between r-Hodge–Tate cohomology and the absolute de Rham–Witt complex.

In order to explain how this should arise, we need to consider higher versions of the Hodge–Tate locus on Σ. Consider the ideal sheaf $\mathcal{I}_r$ on Σ. In order to define this, it suffices to do the construction in a compatible way on transversal prisms $(A, I) \mapsto I_r$. We denote by $\Sigma^{\mathrm{HT},r}$ the locus on which $\mathcal{I}_r$ vanishes. Then we have the following squares, the left of which is Cartesian:

$$\begin{tikzcd} \operatorname{spf} A/I_r \arrow[r, "\rho_A^{\mathrm{HT},r}"] \arrow[d, hook] & \Sigma^{\mathrm{HT},r} \arrow[d, hook] \\ \operatorname{spf} A \arrow[r, "\rho_A"] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \operatorname{spf} A/I_r \arrow[r] \arrow[d, hook] & \Sigma^{\mathrm{HT},r} \arrow[d, hook] \\ \operatorname{spf} A/I_{r+k} \arrow[r] & \Sigma^{\mathrm{HT},r+k} \end{tikzcd}$$

In particular, for the prism $(\mathbb{Z}_p[[\widetilde{p}]], (\widetilde{p}))$, we obtain a morphism:

$$\eta_r : \mathbb{Z}_p[[\widetilde{p}]]/(\widetilde{p})_r \longrightarrow \Sigma^{\mathrm{HT},r}$$

which gives rise to a diffracted p-complete Hodge–Witt complex $W_r\Omega_S^{\not{D}}$. We expect these constructions to give rise to a Sen operator Θ, which is something natural to expect, since we are dealing with graded stacks for the I_r-adic filtration on Δ_S. Given this, one should have an absolute r-Hodge–Tate comparison:

$$\mathcal{N}_r^i \Delta_S\{n\} \longrightarrow \mathrm{Fil}_i^{\mathrm{conj}} W_r\Omega_S^{\not{D}} \xrightarrow{\Theta+m} \mathrm{Fil}_{m-1}^{\mathrm{conj}} W_r\Omega_S^{\not{D}}$$

As in [BL22a], we expect these invariants to give rise to an absolute de Rham–Witt comparison, in which the r-Nygaard filtered prismatic cohomology $\mathcal{N}_r^{\geq i}\Delta_S\{i\}$ maps to the Hodge filtered, p-complete, animated, absolute de Rham–Witt complex $\mathrm{Fil}^i_{\mathrm{Hod}} LW_r\Omega_S$, and thus to fit in the following diagram:

$$\begin{tikzcd}
\mathrm{Fil}^i_{\mathrm{Hod}} LW_r\Omega_S \arrow[d] & \mathcal{N}_r^{\geq i}\Delta_S\{i\} \arrow[l] \arrow[r, "\varphi_{r,i}"] \arrow[d] & \Delta_S\{i\} \arrow[d] \\
\mathrm{gr}^i_{\mathrm{Hod}} LW_r\Omega_S \arrow[d] & \mathcal{N}_r^i\Delta_S\{i\} \arrow[l] \arrow[r] \arrow[d] & \Delta_S^{\mathrm{HT},r}\{i\} \arrow[d] \\
LW_r\Omega^i_S[-i] & \mathrm{Fil}_i^{\mathrm{conj}} W_r\Omega_S^{\not{D}} \arrow[l] \arrow[r] & W_r\Omega_S^{\not{D}}
\end{tikzcd}$$

6.3 The big de Rham–Witt complex

A natural question to ask is whether it is possible to extend our results from the p-complete to the integral case. Remember that with integral coefficients, TR^r is related to the big de

Rham–Witt complex $\mathbb{W}_r\Omega$, as presented in [Hes15].

The idea is to build on the theory of global motivic complexes of Bhatt–Lurie [BL22a]. Let S be an animated ring. Then, we can associate to it its global prismatic complex $\mathbb{\Delta}_S^{\mathrm{gl}}$, which can be obtained via a gluing square over the different primes p and $\mathbb{Q}$. This is equipped with an integral version of a Nygaard filtration $\mathcal{N}^{\geq \bullet}\widehat{\mathbb{\Delta}}_S^{\mathrm{gl}}$, whose graded pieces sit in a fibre sequence which involves the integral diffracted Hodge complex, together with the action of the Sen operator:

$$\mathcal{N}^i\widehat{\mathbb{\Delta}}_S^{\mathrm{gl}}\{n\} \longrightarrow \mathrm{Fil}_i^{\mathrm{conj}}\,\Omega_S^{\not{D}} \xrightarrow{\Theta+i} \mathrm{Fil}_{i-1}^{\mathrm{conj}}\,\Omega_S^{\not{D}}$$

In analogy, we glue the r-Nygaard filtration over different primes. Its associated graded pieces should sit in a fibre sequence, with diffracted versions of the big de Rham–Witt complex:

$$\mathcal{N}_r^i\widehat{\mathbb{\Delta}}_S^{\mathrm{gl}}\{n\} \longrightarrow \mathrm{Fil}_i^{\mathrm{conj}}\,\mathbb{W}_r\Omega_S^{\not{D}} \xrightarrow{\Theta+i} \mathrm{Fil}_{i-1}^{\mathrm{conj}}\,\mathbb{W}_r\Omega_S^{\not{D}}$$

Going back to the homotopy theoretic picture, it is possible by gluing to obtain a relation between the graded pieces for the global Nygaard filtration $\mathcal{N}_r^i\widehat{\mathbb{\Delta}}_S^{\mathrm{gl}}$, as obtained from the motivic filtration of $\mathrm{TR}^r(S)$, and the big de Rham–Witt complex $\mathbb{W}_r\Omega_S$.

6.4 Hesselholt's conjectures on the absolute de Rham–Witt complex

In [Hes05], Hesselholt introduces the absolute de Rham–Witt complex and states a number of conjectures on how its structure relates to étale Tate twists. In particular given a smooth scheme X over $\mathcal{O}_K$, for K a p-adic field, one can associate to it the usual diagram describing the special/general fibres:

$$\begin{tikzcd}
Y \arrow[r, hook, "f"] \arrow[d] & X \arrow[d] & U \arrow[l, hook', "g"'] \arrow[d] \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} \mathcal{O}_K & \operatorname{Spec} K \arrow[l, hook']
\end{tikzcd}$$

Starting from this diagram, one is able to look at topological restriction homology with logarithmic poles $\mathrm{TR}^r(X|U;\mathbb{Z}_p)$. For a reminder of aspects of this theory, the interested reader could look at [HS19], [Bin+23]. Hesselholt's question is the following: when we view $\mathrm{TR}^r(X|U;\mathbb{Z}_p)$ as an étale sheaf, we would like to endow it with a motivic filtration $\mathrm{Fil}^{\bullet}_{\mathcal{M}} \mathrm{TR}^r(X|U;\mathbb{Z}_p)$, whose graded pieces $\mathrm{gr}^i_{\mathcal{M}}$ are complexes concentrated in degrees $[0,i]$, with the i-th part calculated by the log absolute de Rham–Witt complex $W_r\Omega_{(X,M_X)}$ and which should sit in a fiber sequence, as follows:

$$\tau_{\leq i} f^* Rg_* \mu_{p^v}^{\otimes i} \longrightarrow f^* \mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} /p^v \xrightarrow{1-\mathrm{F}} f^* \mathrm{gr}^{i,\mathrm{even}}_{\mathcal{M}} /p^v$$

This was the motivation in searching for the relationship between topological restriction homology and the absolute de Rham–Witt complex. We briefly comment on how this is resolved, as details may be found in [And24b]. We work over the perfectoid ring $\mathcal{O}_C$, since in this case it is less technical to tackle the amplitude questions, regarding the complexes coming from the motivic filtration of TR.

The idea is to construct motivic filtrations on TR with logarithmic poles and its S^1-homotopy fixed points, as in [Bin+23]. Then, we use the fibre sequences for TR, TC, and algebraic K-theory, which are constructed in [HM03], in order to have an understanding of the theory with logarithmic poles in terms of the non-logarithmic theory. This, together with the fibre sequence, through which one constructs TC via TR, yield the desired results.

6.5 On the prismatization stacks Σ'_r

As we already noted in the introduction, one of the main slogans of Scholze's ICM address [Sch18] was that p-adic cohomology theories should have certain similarities to shtukas. This heuristic is, in general, backed by one of the main slogans of the Langlands program, whose aim is to understand motivic phenomena through the lens of automorphic representations.

Evidence for such a claim lies in Fargues' result on the equivalence between Breuil–Kisin–Fargues modules and certain shtukas with one leg in p-adic geometry [SW20]. One way of viewing prismatic theory is as an attempt to generalize and geometrize BKF modules. Hence, in that regard we posed the question at the introduction on whether it is possible to also capture the information of shtukas with any number of legs.

We believe that this information is captured by the r-Nygaard filtration in prismatic cohomology. In particular, there exist prismatization stacks Σ'_r, which capture such information. Notice that quasicoherent sheaves give rise to correspondences, which behave like Hecke operators in the prismatic setting:

$$\begin{tikzcd} & \mathrm{QCoh}(\Sigma'_r) \arrow[dl] \arrow[dr] & \\ \mathrm{QCoh}(\Sigma) & & \mathrm{QCoh}(\Sigma) \end{tikzcd}$$

where the map on the left corresponds to the r-divided Frobenius $\varphi_{r,\bullet}$, while the one on the right corresponds to the canonical map. Taking the equalizer for the maps of ∞-categories of quasi-coherent sheaves (coequalizer on the level of stacks) should produce prismatic counterparts of shtukas with r-number of legs.

We hope to study the properties of these stacky constructions. Locally, they should give rise to prismatic counterparts of the Witt vector affine Grassmannians [Zhu17; BS17; SW20]. We believe that trying to formulate the playground for a geometric Satake, via suitable categories of perverse sheaves, coming from a suitable perverse t-structure. We hope this provides an approach to studying motives from the realization perspective.

From the homotopy theoretic viewpoint, remember that the trace map comes from analyzing cyclic K-theory, whose Goodwillie tower is formed by TR^r. The trace factors as:

$$\mathrm{K} \longrightarrow \widetilde{\mathrm{TC}}(\mathrm{K}^{\mathrm{cyc}}) \longrightarrow \widetilde{\mathrm{TC}}(\mathrm{TR}) \longrightarrow \mathrm{TC}$$

Notice that the THH-related invariants satisfy étale descent, so in this picture it does not hurt that much if we try to replace algebraic K-theory by its étale sheafification (or by Selmer K-theory). In particular, in lieu of the iterated product identification for the S^1-homotopy fixed points of TR^r:

$$\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1} \simeq \mathrm{TC}^-(A;\mathbb{Z}_p) \times_{\mathrm{TP}(A;\mathbb{Z}_p)} \cdots \times_{\mathrm{TP}(A;\mathbb{Z}_p)} \mathrm{TC}^-(A;\mathbb{Z}_p)$$

which reminds us of the Hecke correspondence that we observe in the stacky counterpart (especially when we pass to modules over $\mathrm{TR}^r(A;\mathbb{Z}_p)^{hS^1}$) and thus, taking equalizers in order to form versions of TC, produces shtuka like objects. To conclude, we would like to think of the trace map as a Galois-to-automorphic (or motivic-to-automorphic) map, under the auspices of the Langlands program. An interesting viewpoint on the Arin reciprocity maps, which makes use of these gadgets, has been discussed in [Cla17].

REFERENCES

[And24a] Faidon Andriopoulos. “TR and the r-Nygaard filtered prismatic cohomology”. In: *Preprint* (2024).

[And24b] Faidon Andriopoulos. “TR with logarithmic poles and the de Rham–Witt complex”. In: *Preprint* (2024).

[Ant19] Benjamin Antieau. “Periodic cyclic homology and derived de Rham cohomology”. In: *Annals of K-theory* 4.3 (2019), pp. 505–519.

[AN21] Benjamin Antieau and Thomas Nikolaus. “Cartier modules and cyclotomic spectra”. In: *Journal of the American Mathematical Society* 34.1 (2021), pp. 1–78.

[Ant+22] Benjamin Antieau et al. “On the Beilinson fiber square”. In: *Duke Mathematical Journal* 171.18 (2022), pp. 3707–3806.

[Bha22] Bhargav Bhatt. “Prismatic F-gauges”. In: *Lecture notes available at https://www. math. ias. edu/˜ bhatt/teaching/mat549f22/lectures. pdf* (2022).

[BL22a] Bhargav Bhatt and Jacob Lurie. “Absolute prismatic cohomology”. In: *arXiv preprint arXiv:2201.06120* (2022).

[BL22b] Bhargav Bhatt and Jacob Lurie. “The prismatization of p-adic formal schemes”. In: *arXiv preprint arXiv:2201.06124* (2022).

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. “Integral p-adic Hodge theory”. In: *Publications mathématiques de l’IHÉS* 128.1 (2018), pp. 219–397.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. “Topological Hochschild homology and integral p-adic Hodge theory”. In: *Publications mathématiques de l’IHÉS* 129.1 (2019), pp. 199–310.

[BS17] Bhargav Bhatt and Peter Scholze. “Projectivity of the Witt vector affine Grassmannian”. In: *Inventiones mathematicae* 209 (2017), pp. 329–423.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". In: *Annals of Mathematics* 196.3 (2022), pp. 1135–1275.

[Bin+23] Federico Binda et al. "Logarithmic prismatic cohomology via logarithmic THH". In: *International Mathematics Research Notices* 2023.22 (2023), pp. 19641–19696.

[Cla17] Dustin Clausen. "A K-theoretic approach to Artin maps". In: *arXiv: K-Theory and Homology* (2017).

[CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. "K-theory and topological cyclic homology of henselian pairs". In: *Journal of the American Mathematical Society* 34.2 (2021), pp. 411–473.

[DR23] Micah Darrell and Noah Riggenbach. "TR of quasiregular semiperfect rings is even". In: *arXiv preprint arXiv:2308.13008* (2023).

[DM23] Sanath K Devalapurkar and Shubhodip Mondal. "p-typical curves on p-adic Tate twists and de Rham-Witt forms". In: *arXiv preprint arXiv:2309.16623* (2023).

[Dri20] Vladimir Drinfeld. "Prismatization". In: *arXiv preprint arXiv:2005.04746* (2020).

[Gra19] David Graeber. "It Wasn't a Tenure Case–A Personal Testimony, with Reflections". In: *Public Anthropologist* 1.1 (2019), pp. 96–104.

[Hes05] Lars Hesselholt. *The absolute de Rham-Witt complex*. 2005.

[Hes15] Lars Hesselholt. "The big de Rham–Witt complex". In: (2015).

[HM97] Lars Hesselholt and Ib Madsen. "On the K-theory of finite algebras over Witt vectors of perfect fields". In: *Topology* 36.1 (1997), pp. 29–101.

[HM03] Lars Hesselholt and Ib Madsen. "On the K-theory of local fields". In: *Annals of mathematics* 158.1 (2003), pp. 1–113.

[HM04] Lars Hesselholt and Ib Madsen. "On the de Rham-Witt complex in mixed characteristic". In: *Annales scientifiques de l'Ecole normale supérieure*. Vol. 37. 1. 2004, pp. 1–43.

[HS19] Lars Hesselholt and Peter Scholze. "Arbeitsgemeinschaft: Topological cyclic homology". In: *Oberwolfach Reports* 15.2 (2019), pp. 805–940.

[KMN23] Achim Krause, Jonas McCandless, and Thomas Nikolaus. "Polygonic spectra and TR with coefficients". In: *arXiv preprint arXiv:2302.07686* (2023).

[KN18] Achim Krause and Thomas Nikolaus. "Lectures on topological Hochschild homology and cyclotomic spectra". In: *preprint* (2018).

[LM12] Ayelet Lindenstrauss and Randy McCarthy. "On the Taylor tower of relative K–theory". In: *Geometry & Topology* 16.2 (2012), pp. 685–750.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur17] Jacob Lurie. *Higher algebra*. 2017.

[Mat21] Akhil Mathew. "On K(1)-local TR". In: *Compositio Mathematica* 157.5 (2021).

[McC21] Jonas McCandless. "On curves in K-theory and TR". In: *arXiv preprint arXiv: 2102.08281* (2021).

[MRT22] Tasos Moulinos, Marco Robalo, and Bertrand Toën. "A universal Hochschild–Kostant–Rosenberg theorem". In: *Geometry & Topology* 26.2 (2022), pp. 777–874.

[Nik20] Thomas Nikolaus. "Characteristic polynomials and TR with coefficients, video from a talk". In: *BIRS Workshop on Equivariant Stable Homotopy Theory and p-adic Hodge Theory* (2020).

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: *Acta Mathematica* 221.2 (2018), pp. 203–409.

[Old39] CH Oldfather. "Diodorus Siculus, Library of History. Volume III: Books 4.59-8". In: (1939).

[Rak20] Arpon Raksit. "Hochschild homology and the derived de Rham complex revisited". In: *arXiv preprint arXiv:2007.02576* (2020).

[Sch18] Peter Scholze. "p-adic Geometry". In: *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*. World Scientific. 2018, pp. 899–933.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley Lectures on p-adic Geometry:(AMS-207)*. Princeton University Press, 2020.

[Zhu17] Xinwen Zhu. "Affine Grassmannians and the geometric Satake in mixed characteristic". In: *Annals of Mathematics* 185.2 (2017), pp. 403–492.