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EXOTIC SMOOTH EQUIVARIANT MANIFOLDS

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To my father

TABLE OF CONTENTS

LIST OF FIGURES

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ABSTRACT

In this thesis, we study the differences between smooth and topological manifolds in the equivariant setting. The central topic will be on smooth structures. Kirby–Siebenmann [KS77] showed (using work of Kervaire–Milnor [KM63]) that every closed high dimensional manifold has only finitely many smooth structures. In contrast, if G is a nontrivial finite group, we construct G-manifolds with infinitely many equivariant smooth structures. Our examples even include nonpositively curved manifolds on which G acts by isometries.

CHAPTER 1

INTRODUCTION

A classical theorem in geometric topology is Milnor's construction of exotic 7-spheres [Mil56]. Milnor showed that there are smooth manifolds Σ^7 homeomorphic to, but not diffeomorphic to, S^7. Kervaire–Milnor [KM63] later showed that spheres of dimension at least 5 have only finitely many smooth structures. Kirby–Siebenmann used this theorem to show that, if X is a closed manifold of dimension at least 5, then X has only finitely many smooth structures. In this thesis, we show that these results do not hold in the equivariant setting.

A *smooth structure* or a *smoothing* of a manifold X can be represented by a homeomorphism $f : Y \to X$ where Y is a smooth manifold. Smooth structures are typically considered up to isotopy or concordance (we will give a more thorough treatment of this in Section 2.5). Kirby–Siebenmann show that, in high dimensions, the two notions are equivalent. We denote isotopy classes of smooth structures on X by $TOP/O(X)$[1]. If $f_i : Y_i \to X$, $i = 0, 1$, determine isotopic smooth structures, then Y_0 and Y_1 are diffeomorphic. This follows from the fact (also proven by Kirby–Siebenmann) that smooth structures on $X \times I$ are product structures. The converse is not necessarily true; if Y_0 and Y_1 are diffeomorphic, the smooth structures determined by f_i need not be isotopic. Hence, the set of smooth manifolds homeomorphic to X, up to diffeomorphism, is a quotient of $TOP/O(X)$. We denote this by $\overline{TOP/O}(X)$.

If G is a finite group and X is G-manifold, then one may similarly define isotopy classes of G-smoothings $TOP/O_G(X)$. This is studied in [Las79] and [LR78]. One may also consider the set of smooth G-manifolds equivariantly homeomorphic to X. In the equivariant setting, it is no longer the case that G-smoothings of $X \times I$ are products (see [BH78, 262-267]). Hence, it is not clear whether this set is a quotient of $TOP/O_G(X)$. Nevertheless, we denote this set $\overline{TOP/O}_G(X)$.

1. Classically, this set is denoted $\mathcal{S}^{TOP/DIFF}(X)$.

This thesis addresses the following question.

Question 1. When is $\overline{TOP/O}_G(X)$ infinite?

Some examples of G-manifolds X with $\overline{TOP/O}_G(X)$ infinite have appeared in the literature. For infinitely many primes p, Schultz [Sch79] has shown that $\overline{TOP/O}_{\mathbb{Z}/p}(S^{2n})$ is infinite for certain $\mathbb{Z}/p$-actions on S^{2n} (here, n increases with p). In these examples, the fixed point set of the action is S^2 but the normal bundle of the fixed set varies as $\mathbb{Z}/p$-vector bundles. This construction relies on computations of Ewing [Ewi76] regarding arithmetic properties of the coefficients appearing in the Atiyah–Singer G-signature theorem.

In [BH78], it was shown that $\overline{TOP/O}_G(X \times I)$ could be infinite. The construction of these exotic G-manifolds do not change the normal bundle of the fixed set. The sphere bundle of the fixed set yields a lens space bundle and the construction of [BH78] involves using the Whitehead group of G to replace a neighborhood of the sphere bundle.

1.1 Exotic Normal Bundles

In Chapter 2, we generalize Schultz's work to the case where the fixed set is not necessarily S^2. In this chapter, we focus on the case $G = \mathbb{Z}/p$, where p is an odd prime. We prove the following theorem.

Theorem 1.1.1. *Suppose X is a closed, smooth $\mathbb{Z}/p$-manifold where 2 has odd order in $\mathbb{F}_p^\times$. Let M be a component of the fixed set such that the following hold.*

- *Each eigenbundle of the normal bundle of M is trivial as a complex vector bundle and has complex rank at least* $\dim M$*,*
- $H^2(M;\mathbb{Q}) \neq 0$.

Then, $\overline{TOP/O}_G(X)$ is infinite.

The condition on the prime p holds when $p \equiv 7$ modulo 8 but not when $p \equiv 3, 5$. The conditions on the normal bundle of M in Theorem 1.1.1 can be significantly improved and we discuss this further in Chapter 2. Even without these improvements, Theorem 1.1.1 provides many new examples of G-manifolds with infintely many G-smoothings.

Example 1. If M is a closed parallelizable manifold with $H^2(M;\mathbb{Q}) \neq 0$ (e.g. $M = T^2$), then $\overline{TOP/O}_G(M^{\times p})$ is infinite for p as in Theorem 1.1.1.

Example 2. Let M be a closed manifold with $H^2(M;\mathbb{Q}) \neq 0$. Let p be as in Theorem 1.1.1 and let $V = (\mathbb{R}[\mathbb{Z}/p]/\mathbb{R})^{\dim M}$ be $\dim M$ copies of the reduced regular representation. Let S^V denote the one point compactification of V. Then $\overline{TOP/O}_G(M \times S^V)$ is infinite.

1.2 Stably Trivial G-smoothings

The G-smoothings in Theorem 1.1.1 are detected by Chern classes of the normal bundle. In particular, they remain distinct under stabilization maps

$$\overline{TOP/O}_G(X) \to \overline{TOP/O}_G(X \times V)$$

where V is any representation. In Chapter 3, we construct infinitely many elements of $\overline{TOP/O}_G(X)$ which map to the same element under

$$\overline{TOP/O}_G(X) \to \overline{TOP/O}_G(X \times \mathbb{R}).$$

Moreover, we work in a more general setting than in Chapter 2; we consider semifree actions of odd order cyclic groups of X.

We construct elements of $\overline{TOP/O}_G(X)$ as follows. Let M be a component of the fixed set and let $S\nu$ denote the sphere bundle of the normal bundle. Then $S\nu/G$ is a lens space bundle. Let $(W; S\nu/G, S\nu/G)$ be an h-cobordism with both boundary components diffeomorphic to

$S\nu/G$ and let X_W denote the manifold obtained by removing an equivariant neighborhood of $S\nu$ and replacing it with the G-cover of W. Distinct h-cobordisms W give rise to distinct elements of $\overline{TOP/O}_G(X)$. These constructions do not change the normal bundle of the fixed set.

We use a swindling argument to define an equivariant homeomorphism when W is a controlled h-cobordism in the sense of Quinn [Qui82]. These are the h-cobordisms in the image of the assembly map

$$H_1(M; \mathbf{Wh}(G)) \to \mathrm{Wh}(G).$$

In order to detect elements W whose boundary components are both $S\nu/G$, we study an involution on these groups.

Our main theorem is the following.

Theorem 1.2.1. *Let G be an odd order cyclic group of order at least 5 and let X be a closed, smooth semifree G-manifold. Suppose M is a component of the fixed point set which is closed, aspherical and whose fundamental group satisfies the K-theoretic Farrell–Jones conjecture. Suppose either of the following hold.*

1. *M (and, hence X) is odd dimensional.*
2. *M is even dimensional, $H^2(M;\mathbb{Q}) \neq 0$ and there are distinct prime factors p_i, p_j of $|G|$ such that p_i has odd order in $(\mathbb{Z}/p_j)^\times$.*

Then there are infinitely many elements of $\overline{TOP/O}_G(X)$ which vanish under the stabilization map $\overline{TOP/O}_G(X) \to \overline{TOP/O}_G(X \times \mathbb{R})$.

Example 3. If G acts semifreely on X and X^G has a component which is a hyperbolic homology 3-sphere, then $\overline{TOP/O}_G(X)$ is infinite by Theorem 3.1.1.

1.3 Exotic Smooth G-homotopy equivalences

In Chapter 4, we investigate the difference between equivariant smooth homotopy equivalences and equivariant topological homotopy equivalences. The Borel conjecture states that, if X is aspherical, then the surgery theoretic topological structure set $\mathcal{S}^{TOP}(X)$ has only one element. Explicitly, this means that every homotopy equivalence $X' \to X$ from a topological manifold is homotopic to a homeomorphism. If the Borel conjecture holds for X, then there are only finitely many smooth homotopy equivalences $X' \to X$ from a smooth manifold, up to smooth homotopy [Wei90]. Our goal in Chapter 4 is to construct infinitely many smooth aspherical G-manifolds X_α and smooth G-homotopy equivalences $f_\alpha : X_\alpha \to X$ which are not smoothly G-homotopic to each other.

The main results of Chapter 4 can be summarized in the following statement.

Theorem 1.3.1. *There exist closed, equivariantly aspherical, smooth G-manifolds X which admit infinitely many smooth G-homotopy equivalences $\{f_\alpha : X_\alpha \to X\}_{\alpha \in A}$ such that the following hold.*

- *The maps f_α and f_β are not smoothly G-homotopic if $\alpha \neq \beta$,*
- *The maps f_α and f_β are G-homotopic.*

Unlike our constructions in Chapters 2 and 3, our construction of the manifolds X_α appearing in Theorem 1.3.1 use the odd dimensional cohomology of the fixed set M. We use elements of $H^{odd}(M; \mathbb{Q})$ to construct lens space PL-block bundles over $M \times I$ which are trivial over $M \times S^0$. Then we replace a neighborhood of $S\nu$ with the G-cover of these block bundles. We use the equivariant Novikov conjecture to distinguish these constructions from each other. Finally, we emphasize that Theorem 1.3.1 is a statement about the maps f_α rather than a statement about the underlying G-manifolds X_α; there may be an equivariant diffeomorphism $X_\alpha \cong X_\beta$.

CHAPTER 2

CHERN CLASSES AND EXOTIC NORMAL BUNDLES

2.1 Introduction

For a closed topological manifold X, a smoothing of X is defined to be a homeomorphism $Y \to X$ where Y is a smooth manifold. In [KS77], the set $TOP/O(X)$ of smoothings up to isotopy is studied and it is shown that there is a bijection $TOP/O(X) \cong [X, TOP/O]$ where TOP/O is an infinite loop space. In particular, this set is a cohomology group so some computational methods are available.

The group $\mathrm{Homeo}(X)$ acts on $TOP/O(X)$ and the quotient $\overline{TOP/O}(X)$ is the set of smooth manifolds homeomorphic to X up to diffeomorphism. This set is more difficult to compute. When $n \geq 5$, the group $[S^n, TOP/O]$ can be identified with the group of homotopy spheres of [KM63] and is therefore finite. For $n < 5$, Kirby–Siebenmann show that $[S^n, TOP/O]$ is finite by other means. Consequently $TOP/O(X)$ and $\overline{TOP/O}(X)$ are finite sets. We may define analogous sets in an equivariant setting.

Definition 2.1.1. Let G be a finite group and let X be a G-manifold. A *G-smoothing* of X is an equivariant homeomorphism $\alpha : Y \to X$. Two G-smoothings α_0 and α_1 are *isotopic* if α_0 is homotopic through G-homeomorphisms to α_0' where $\alpha_1^{-1} \circ \alpha_0'$ is a G-diffeomorphism. Define $TOP/O_G(X)$ to be the set of isotopy classes of G-smoothings of X. Define $\overline{TOP/O}_G(X)$ to be the equivariant diffeomorphism classes of smooth G-manifolds Y which are equivariantly homeomorphic to X.

Remark. If X is closed and $\overline{TOP/O}_G(X)$ is infinite, then Kirby–Siebenmann's result implies there is a smooth structure on X such that there are infinitely many periodic diffeomorphisms of X which are conjugate in $\mathrm{Homeo}(X)$ but not in $\mathrm{Diff}(X)$.

Schultz shows in [Sch79] that, for certain actions of $\mathbb{Z}/p\mathbb{Z}$ on S^{2n} with fixed point set S^2, the set $\overline{TOP/O}_G(S^{2n})$ is infinite contrary to the non-equivariant case. Our goal is to generalize Schultz's construction to smooth $\mathbb{Z}/p\mathbb{Z}$-actions on manifolds whose fixed point set is not necessarily S^2. Before explaining Schultz's work we introduce some notions fundamental to the study of smooth group actions.

For the remainder of the paper, let $G = \mathbb{Z}/p\mathbb{Z}$ where p is an odd prime and let g_0 be a fixed generator. If M is a connected component of X^G, then the normal bundle of M inherits the structure of a real G-vector bundle with fiber a real G-representation V. We call V the *normal representation of* M and we say that V is *free* if $V^G = \{0\}$. All representations obtained from normal bundles of fixed point sets are free. A bundle of G-representations is said to be free if its fibers are free.

The nontrivial real irreducible G-representations are isomorphic to $\mathbb{C}$ where g_0 acts via multiplication by a primitive p-th root of unity. Moreover, the representation determined by a primitive p-th root of unity ζ is isomorphic as a real representation to the one determined by the complex conjugate $\bar{\zeta}$. So there are $\frac{p-1}{2}$-many nontrivial irreducible free real G-representations. If ν is the normal bundle of M as above, then ν decomposes as a sum $\nu \cong \bigoplus_{k=1}^{\frac{p-1}{2}} \nu_k$ of eigenbundles for g_0. If $\varphi : Y \to X$ is an equivariant diffeomorphism, $\varphi^{-1}(M) \cong M$ and the normal bundle of $\varphi^{-1}(M)$ is $\varphi^* E$.

2.1.1 Actions on Spheres

In [AS68], Atiyah and Singer define the G-signature $\mathrm{sign}_G(X)$ of a smooth G-manifold X. This is an equivariant homotopy invariant of X valued in the real representation ring of G. For computational purposes, it is convenient to identify a representation with its character. The Atiyah–Singer G-signature theorem states

$$\mathrm{sign}_G(X)(g) = \langle A(g, V) L(X^g) \mathcal{M}(g, \nu), [X^g] \rangle$$

where $A(g,V) \in \mathbb{C}$, $L(X^g)$ is the L-genus and $\mathcal{M}(g,\nu) \in H^*(X^g;\mathbb{C})$ is a (non-homogeneous) characteristic class of the normal bundle ν of X^g. The class $\mathcal{M}(g,\nu)$ can be described in terms of the Chern classes of ν. As suggested by the notation, the number $A(g,V)$ depends only on the element $g \in G$ and the representation V. We call $A(g_0,V)\mathcal{M}(g_0,\nu)$ the Atiyah–Singer class. In Lemma 2.4.7 we show that the representation is determined by the Atiyah–Singer class.

Ewing [Ewi78] applies the Atiyah–Singer G-signature theorem to determine the possible Chern classes of the normal bundle of the fixed point set of a smooth, semifree action of a cyclic group. When X is a sphere, the G-signature always vanishes since spheres have no middle-dimensional cohomology. Moreover, the fixed point set is a rational homology $2n$-sphere by Smith theory so $\mathcal{L}(X^{g_0}) = 1$ and

$$\mathcal{M}(g_0,\nu) = 1 + \sum_{k=1}^{\frac{p-1}{2}} \Phi_{n,k} c_n(\nu_k)$$

where the $\Phi_{n,k}$ are elements of $\mathbb{Q}(\zeta)$. Ewing shows that, unless $n = 1$ and 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$, the elements $\Phi_{n,k}$ are $\mathbb{Q}$-linearly independent as k varies. Since the Atiyah–Singer class must vanish, this implies that, outside the special case, the Chern classes of the normal bundle must vanish.

In the case p has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$, Ewing shows that the set $\{\Phi_{1,k}\}_{k=1}^{\frac{p-1}{2}}$ is $\mathbb{Q}$-linearly dependent. If V is the normal representation of S^2 then this implies that, provided V contains enough nonzero eigenbundles, there are infinitely many G-vector bundles E over S^2 with fiber V whose Atiyah–Singer class vanishes. Schultz shows in [Sch79] that infinitely many of these G-vector bundles can be realized as normal bundles of $\mathbb{Z}/p\mathbb{Z}$-actions on homotopy $2n$-spheres.

The above results motivate the following questions.

Question 2. Suppose E is a free G-vector bundle over a CW-complex M. When does the

Atiyah–Singer class vanish?

Question 3. Given a free G-representation V, when are there infinitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class?

Question 4. Suppose G acts on X and let M be a component of X^G with trivial normal bundle $M \times V$. If there are infinitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class, are there infinitely many G-smoothings of X realizing these bundles as normal bundles?

2.1.2 Main Results

If E is a free G-vector bundle, we write $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ for the decomposition into eigenbundles where g_0 acts on the fiber of E_k via multiplication by ζ^k. In this case, we give a complete answer to Question 2.

Theorem 2.1.2. *Let $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ be a free G-vector bundle over a space M. There is an equality $A(g_0, V)\mathcal{M}(g_0, E) = 1$ if and only if both of the following hold.*

1. $\sum_{k=1}^{\frac{p-1}{2}} c_1(E_k)\Phi_{1,k} = 0 \in H^2(M; \mathbb{C})$;

2. *For each k and $n \geq 1$, $c_n(E_k) = \frac{1}{n!} c_1(E_k)^n$.*

When 2 has even order in $(\mathbb{Z}/p\mathbb{Z})^\times$, Ewing shows that the first condition above implies $c_1(E_k) = 0$ for all k. Otherwise, the $\mathbb{Q}$-span of $\{\Phi_{1,k}\}_{k=1}^{\frac{p-1}{2}}$ is a $\frac{(p-1)(t-1)}{2t}$-dimensional vector space where t is the order of 2 in $(\mathbb{Z}/p\mathbb{Z})^\times$. For simplicity, let $u := \frac{(p-1)(t-1)}{2t}$.

Definition 2.1.3. Let $G = \mathbb{Z}/p\mathbb{Z}$ where p is such that 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$. Suppose M is a space and V is a free G-representation. An element $\beta \in H^2(M; \mathbb{Z})$ is *sufficiently nilpotent with respect to V* if there is an $N > 0$ such that the following hold.

1. $\beta^{N+1} = 0$,

2. V contains $(u+1)$-many irreducible real representations with multiplicity at least N.

Example 4. If V is contains one copy of each nontrivial irreducible real representation, then every β satisfying $\beta^2 = 0$ is sufficiently nilpotent with respect to V.

We can now state a partial answer to Question 3.

Theorem 2.1.4. *Let $G = \mathbb{Z}/p\mathbb{Z}$ and let V be a free G-representation. Suppose M is homotopy equivalent to a finite CW-complex.*

1. *If 2 has even order in $(\mathbb{Z}/p\mathbb{Z})^\times$ or if $H^2(M;\mathbb{Q}) = 0$ then there are only finitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class.*

2. *If 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$ and there is a nonzero $\beta \in H^2(M;\mathbb{Q})$ sufficiently nilpotent with respect to V, then there are infinitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class.*

To more easily state our answer to Question 4 we introduce another auxiliary definition.

Definition 2.1.5. Suppose G acts smoothly on a manifold X and let M be a component of the fixed point set. A G-vector bundle E over M is an *exotic normal bundle of* (X, M) if G acts smoothly on a manifold Y and there is an equivariant homeomorphism $f : Y \to X$ such that $f^{-1}(M)$ has normal bundle E.

Theorem 2.1.6. *Suppose $G = \mathbb{Z}/p\mathbb{Z}$ acts smoothly on a manifold X. Let M be a component of X^G whose normal bundle is $M \times V$ with V a free G-representation. Suppose M is homotopy equivalent to a finite CW-complex and admits infinitely many G-vector bundles with fiber V and vanishing Atiyah–Singer class. Then,*

1. *Infinitely many of these vector bundles may be realized as exotic normal bundles of (X, M),*

2. The first Chern classes of these exotic normal bundles occupy infinitely many $\mathrm{GL}_{\dim_{\mathbb{Q}} H^2(M;\mathbb{Q})}(\mathbb{Z})$-orbits of $H^2(M;\mathbb{Q})$. In particular, $\overline{TOP/O}_G(X)$ is infinite.

Remark. There are infinitely many primes p where 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$. Indeed, this is true whenever $p \equiv 7$ modulo 8 and it occurs infinitely many times when $p \equiv 1$ modulo 8 (see [Ewi78]). Outside these cases, 2 always has even order in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Example 5. Suppose V is a $2n$-dimensional real representation of G which contains at least one copy of each nontrivial irreducible representation. Let DV denote the unit disk of V and let SV denote the unit sphere. Then, $S^{2n+2} = (S^2 \times DV) \cup_{S^2 \times SV} (D^2 \times S^V)$ has a G-action with fixed point set S^2. The normal representation is V and the generator of $H^2(S^2;\mathbb{Q})$ is sufficiently nilpotent with respect to V so $\overline{TOP/O}_G(S^{2n})$ is infinite. This is the example of Schultz.

Example 6. If M is smooth then the normal bundle of M diagonally embedded in $M^{\times p}$ is $\tau_M^{\oplus p-1}$ where τ_M is the tangent bundle. If $M^{\times p}$ is given a G-action via cyclically permuting coordinates, then the normal bundle of M is $\tau_M \otimes \mathbb{R}[G]/\mathbb{R}$. In particular, if M is also closed and parallelizable such that $H^2(M;\mathbb{Q})$ is nonzero and if p is a prime such that 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$ then Theorem 2.1.6 implies $\overline{TOP/O}_G(M^{\times p})$ is infinite.

Let V be the reduced regular representation and let S^V denote the one point compactification of V. If M is only stably parallelizable with $H^2(M;\mathbb{Q}) \neq 0$ and p is as above, then $\overline{TOP/O}_G(M^{\times p} \times S^V)$ is infinite.

Example 7. For a more complicated example, let M and V be as in Theorem 2.1.6. Suppose M is closed. Then $M \times SV$ is a closed manifold with a free G-action which nonequivariantly bounds. By equivariant cobordism theory [CF64] a disjoint union of $M \times SV$ bounds a smooth, compact manifold X' with free G-action. Define X to be the manifold obtained by gluing copies of $M \times V$ to each boundary component of X'. Then X will satisfy the hypotheses of Theorem 2.1.6.

Modifying this construction also shows that there are infinitely many G-manifolds X for which the hypotheses of 2.1.6 do not hold.

The hypotheses on the normal bundle of the fixed point set can be removed when we stabilize as in [Las79]. Lashof defines a stable G-smoothing of a G-manifold X is a G-smoothing of $X \times \rho$ for a finite dimensional G-representation ρ. Two stable G-smoothings $\alpha_i : Y_i \to X \times \rho_i$, $i = 0, 1$ are stably isotopic if there are representations σ_0 and σ_1 such that $\alpha_i \times \sigma_i : Y_i \times \sigma_i \to X \times \rho_i \times \sigma_i$ are isotopic. Let $TOP/O_G^{st}(X)$ denote the set of stable isotopy classes of stable G-smoothings of X.

Theorem 2.1.7. *Let $G = \mathbb{Z}/p\mathbb{Z}$ where p is such that 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$. Let X be a smooth G-manifold. If $H^2(X^G;\mathbb{Q})$ is nonzero for some component M of X^G homotopy equivalent to a finite CW-complex, then $TOP/O_G^{st}(X)$ is infinite. In particular, if X is closed and $H^2(X^G;\mathbb{Q}) \neq 0$ then $TOP/O_G^{st}(X)$ is infinite.*

2.1.3 Outline

The proof of Theorem 2.1.2 has a large computational component and will be the subject of Section 2.2. In [Sch79], Schultz exploits the fact that homotopy classes of maps from spheres into various classifying spaces have abelian group structures. We do not have this luxury at our level of generality. It turns out that the maps we are concerned with will factor through $\mathbb{C}\mathrm{P}^N$ for a sufficiently large N and self-maps of $\mathbb{C}\mathrm{P}^N$ serve as a replacement. We elaborate on this and prove Theorem 2.1.4 in Section 2.3.

The idea of the proof of Theorem 2.1.6 is as follows: if $M \subseteq X^G$ has normal bundle $M \times V$, we remove $M \times V$ and glue in E where E is the total space of some free G-vector bundle on M. To do this, we need $M \times SV$ to be equivariantly diffeomorphic to the unit sphere bundle SE. We introduce block bundles in Section 2.4 and apply result of Cappell–Weinberger to show that, if E has vanishing Atiyah–Singer class, then SE/G is almost equivalent to $M \times SV/G$ as lens space block bundles over M. In Section 2.5, we show that this equivalence can be taken to be a diffeomorphism and we prove Theorem 2.1.6. In Section 2.6 we give some remarks on the necessity of the trivial normal bundle hypothesis in our theorems and we prove Theorem 2.6.2 which is a more general version of Theorem 2.1.6. This theorem is used to prove Theorem 2.1.7.

2.2 Exponential Vector Bundles

In this section, we determine necessary and sufficient conditions for the vanishing of the Atiyah–Singer class. The main result of this section is Theorem 2.1.2.

2.2.1 The Atiyah–Singer Classes

For the convenience of the reader and to establish notation, we review Hirzebruch's theory of multiplicative sequences and its application in the Atiyah–Singer G-signature formula. Details can be found in [Hir66, Chapter 1] and [AS68, Section 6] Recall that, if $E \to X$ is a complex rank n vector bundle, then the splitting principle asserts there is a space $P(E)$ with a map $f : P(E) \to X$ where f^*E splits into a sum of line bundles $L_1 \oplus \cdots \oplus L_n$ and $f^* : H^*(X) \to H^*(P(E))$ is injective. So in $H^*(P(E))$, the total Chern class of E factors as

$$c_*(E) = 1 + c_1(E) + \cdots + c_n(E) = \prod_{j=1}^{n} (1 + c_1(L_j)).$$

This motivates the use of formal factorizations used below.

Fix a commutative ring R and let $R[c_1, c_2, \cdots]$ be the graded commutative ring of polynomials in c_j where c_j has grading $2j$ (we deviate slightly from the notation of [Hir66] here). Similarly, we consider $R[c_1, \cdots, c_j]$ as a graded commutative ring and, for convenience, we set $c_0 = 1 \in R$. A *multiplicative sequence* $\{K_j\}$ is a sequence of polynomials where $K_j \in R[c_1, \cdots, c_j]$ is homogeneous of degree j, where $K_0 = 1$ and such that, if there is a formal factorization

$$1 + c_1 z + c_2 z^2 + \cdots = (1 + c_1' z + c_2' z^2 + \cdots)(1 + c_1'' z + c_2'' z^2 + \cdots),$$

then

$$\sum_{j=0}^{\infty} K_j(c_1, \cdots, c_j) z^j = \sum_{j=0}^{\infty} K_j(c_1', \cdots, c_j'') z^j \sum_{k=0}^{\infty} K_k(c_1'', \cdots, c_k'') z^k.$$

Suppose $Q(z) = \sum_{j=0}^{\infty} b_j z^j$ is a formal power series with coefficients in R. If $b_0 = 1$, then we can assign a multiplicative sequence $\{K_j\}$ as follows. To determine K_j, let $m \geq j$ and suppose there is a formal factorization

$$1 + b_1 z + \cdots + b_m z^m = \prod_{k=1}^{m} (1 + \beta_k z)$$

where each β_k is of degree 1. Suppose $j_1 \geq j_2 \geq j_3 \geq \cdots \geq j_r$ and that $j_1 + \cdots + j_r = j$. Then, the coefficient of $c_{j_1} c_{j_2} \cdots c_{j_r}$ in $K_j(c_1, \cdots, c_j)$ is the sum of *distinct* S_j-translates of $\beta_{j_1} \cdots \beta_{j_r}$. As an example, the coefficient of c_j in K_j is $\beta_1^j + \beta_2^j + \cdots + \beta_m^j$ and the coefficient of c_1^j is the j-th elementary symmetric polynomial on $\beta_1, \cdots, \beta_m$. So long as $m \geq j$, these coefficients are well-defined. We will let $\tau(j_1, \cdots, j_r)$ denote the coefficient of $c_{j_1} \cdots c_{j_r}$.

Consider a free $\mathbb{Z}/p\mathbb{Z}$-vector bundle E over a manifold M. This breaks into a sum of eigenbundles $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ where a given generator $g \in \mathbb{Z}/p\mathbb{Z}$ acts by a primitive p-th root of unity ζ^k on E_k and such that $\zeta^k \neq \zeta^{k'}, \bar{\zeta}^{k'}$ for $k \neq k'$.

Let $\{\mathcal{M}_r^{\zeta^k}(c_1, \cdots, c_r)\}$ be the multiplicative sequence determined by the power series associated to

$$\left(\frac{\zeta^k - 1}{\zeta^k + 1}\right)\left(\frac{\zeta^k e^z + 1}{\zeta^k e^z - 1}\right).$$

Define

$$\mathcal{M}^{\zeta^k}(E_k) := \sum_{r=0}^{\infty} \mathcal{M}_r^{\zeta^k}(c_1(E_k), \cdots, c_r(E_k))$$

where $c_1(E_k), \cdots, c_r(E_k) \in H^*(M;\mathbb{C})$ are Chern classes of the vector bundle E_k. The complex number showing up in the Atiyah–Singer index theorem is

$$A(g,V) = \prod_{k=1}^{\frac{p-1}{2}} \left(\frac{\zeta^k+1}{\zeta^k-1}\right)^{\mathrm{rank}_{\mathbb{C}}(E_k)}$$

and the class $\mathcal{M}(g,E)$ is

$$\mathcal{M}(g,E) := \prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k).$$

Choose an integer m such that $m > \mathrm{rank}_{\mathbb{C}}(E_k)$ for all k and consider a formal factorization $\prod_{j=1}^{m}(1+\beta_{j,k}z)$ of the first m terms of the power series $\left(\frac{\zeta^k-1}{\zeta^k+1}\right)\left(\frac{\zeta^k e^z+1}{\zeta^k e^z-1}\right)$. We may write $\mathcal{M}^{\zeta^k}(E_k)$ as

$$\mathcal{M}^{\zeta^k}(E_k) = \sum_{r=0}^{m} \sum_{\substack{j_1\geq\cdots\geq j_\ell>0\\ j_1+\cdots+j_\ell=r}} \tau(j_1,\cdots,j_\ell)(\zeta^k)c_{j_1}(E_k)\cdots c_{j_\ell}(E_k).$$

Remark. The $\Phi_{n,k}$ in the introduction and in Theorem 2.1.2 are the numbers $\tau(n)(\zeta^k)$.

We will rely on results from [Ewi78], summarized below, for our analysis of the Atiyah–Singer class.

Lemma 2.2.1. *If $r > 1$ then $\{\tau(r)(\zeta), \tau(r)(\zeta^2), \cdots, \tau(r)(\zeta^{\frac{p-1}{2}})\}$ is a $\mathbb{Q}$-linearly independent set. If $r = 1$, then this set is $\mathbb{Q}$-linearly independent if and only if 2 has even order in $(\mathbb{Z}/p\mathbb{Z})^\times$. Moreover, when 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$, the span of this set has dimension $\frac{(p-1)(t-1)}{2t}$ where t is the order of 2 in $(\mathbb{Z}/p\mathbb{Z})^\times$.*

The following observation will be important later.

Lemma 2.2.2. *The numbers $\tau(j_1,\cdots,j_\ell)$ are in $\mathbb{Q}(\zeta)$. Moreover, if $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ then $\sigma(\tau(j_1,\cdots,j_\ell)(\zeta^k)) = \tau(j_1,\cdots,j_\ell)(\sigma\cdot\zeta^k)$.*

Proof. Let σ be the field automorphism determined by $\zeta \mapsto \zeta^n$. Suppose

$$\left(\frac{\zeta^k - 1}{\zeta^k + 1}\right)\left(\frac{\zeta^k e^z + 1}{\zeta^k e^z - 1}\right) = 1 + b_{1,k}z + b_{2,k}z^2 + b_{3,k}z^3 + \cdots$$

is a power series expansion. Each $b_{j,k} \in \mathbb{Q}(\zeta)$ and $\sigma \cdot b_{j,k} = b_{j,nk}$, i.e. σ sends the power series for $\left(\frac{\zeta^k-1}{\zeta^k+1}\right)\left(\frac{\zeta^k e^z+1}{\zeta^k e^z-1}\right)$ to the power series for $\left(\frac{\zeta^{nk}-1}{\zeta^{nk}+1}\right)\left(\frac{\zeta^{nk} e^z+1}{\zeta^{nk} e^z-1}\right)$.

Let $e_{\ell,k}$ denote the ℓ-th elementary symmetric polynomial of the $\beta_{j,k}$ The factorization $\prod_{j=1}^{m}(1 + \beta_{j,k}z)$ of the first m terms implies that $e_{\ell,k} = b_{\ell,k}$ for $\ell \leq m$ so $\sigma \cdot e_{\ell,k} = e_{\ell,nk}$. Since $\tau(j_1, \cdots, j_\ell)(\zeta^k)$ is an algebraic combination of the $e_{\ell,k}$, we see that it is indeed in $\mathbb{Q}(\zeta)$ and that

$$\sigma(\tau(j_1, \cdots, j_\ell)(\zeta^k)) = \tau(j_1, \cdots, j_\ell)(\zeta^{nk})$$

as desired. □

Theorem 2.1.2. *Let $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ be a free G-vector bundle over a space M. There is an equality $A(g_0, V)\mathcal{M}(g_0, E) = 1$ if and only if both of the following hold.*

1. $\sum_{k=1}^{\frac{p-1}{2}} c_1(E_k)\Phi_{1,k} = 0 \in H^2(M; \mathbb{C})$;

2. *For each k and $n \geq 1$, $c_n(E_k) = \frac{1}{n!}c_1(E_k)^n$.*

Proof. Suppose $\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k) = 1$. It is clear that the first condition must hold since the sum is the part of $\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)$ in cohomological degree 2. To prove that $c_n(E_k) = \frac{1}{n!}c_1(E_k)^n$, we use induction. The case $n = 1$ is vacuous.

Suppose that $n \geq 2$ and that $c_j(E_k) = \frac{1}{j!}c_1(E_k)^j$ for all k and all $j \leq n - 1$. In degree $2n$, the product $\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)$ can be expressed as

$$\left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)\right)_{2n} = \sum_{\substack{\ell_1, \cdots, \ell_{\frac{p-1}{2}} \geq 0 \\ \ell_1 + \cdots + \ell_{\frac{p-1}{2}} = n}} \prod_{k=1}^{\frac{p-1}{2}} \sum_{\substack{j_1 \geq \cdots \geq j_r > 0 \\ j_1 + \cdots + j_r = \ell_k}} \tau(j_1, \cdots, j_r)(\zeta^k)c_{j_1}(E_k) \cdots c_{j_r}(E_k)$$

where the inner sum is taken to be 1 if $\ell_k = 0$ and the subscript on the left hand side indicates that we are restricting to the cohomological degree $2n$ part. It follows from the definition of $\tau(1)(\zeta^k)$ that

$$\tau(1)(\zeta^k)^{\ell_k} c_1(E_k)^{\ell_k} = \sum_{\substack{j_1,\cdots,j_n \geq 0 \\ j_1+\cdots+j_n=\ell_k}} \frac{\ell_k!}{j_1!\cdots j_n!} \beta_{1,k}^{j_1} \cdots \beta_{n,k}^{j_n} c_1(E_k)^{\ell_k}.$$

If $\ell_k < n$, then the inductive hypothesis and the definition of $\tau(j_1, \cdots, j_r)$ gives

$$\begin{aligned} \tau(1)(\zeta^k)^{\ell_k} c_1(E_k)^{\ell_k} &= \sum_{\substack{j_1,\cdots,j_n \geq 0 \\ j_1+\cdots+j_n=\ell_k}} \ell_k! \beta_{1,k}^{j_1} \cdots \beta_{n,k}^{j_n} c_{j_1}(E_k) \cdots c_{j_n}(E_k) \\ &= \sum_{\substack{j_1 \geq \cdots \geq j_r > 0 \\ j_1+\cdots+j_r=\ell_k}} \ell_k! \tau(j_1, \cdots, j_r) c_{j_1}(E_k) \cdots c_{j_r}(E_k). \end{aligned}$$

From this, we conclude

$$\sum_{\substack{j_1 \geq \cdots \geq j_r > 0 \\ j_1+\cdots+j_r=\ell_k}} \tau(j_1, \cdots, j_r) c_{j_1}(E_k) \cdots c_{j_r}(E_k) = \frac{1}{\ell_k!} \tau(1)(\zeta^k)^{\ell_k} c_1(E_k)^{\ell}_k. \tag{2.1}$$

Similarly, if $\ell_k = n$, we get

$$\begin{aligned} \tau(1)(\zeta^k)^n c_1(E_k)^n &= n!\tau(n)(\zeta^k) c_1(E_k)^n + \sum_{\substack{n > j_1,\cdots,j_n \geq 0 \\ j_1+\cdots+j_n=n}} n! \beta_{1,k}^{j_1} \cdots \beta_{n,k}^{j_n} c_{j_1}(E_k) \cdots c_{j_n}(E_k) \\ &= n!\tau(n)(\zeta^k) c_1(E_k)^n + \sum_{\substack{n > j_1 \geq \cdots \geq j_r > 0 \\ j_1+\cdots+j_r=n}} n! \tau(j_1, \cdots, j_r) c_{j_1}(E_k) \cdots c_{j_r}(E_k). \end{aligned}$$

This implies

$$\sum_{\substack{n\geq j_1\geq\cdots\geq j_r>0\\ j_1+\cdots+j_r=n}} \tau(j_1,\cdots,j_r)c_{j_1}(E_k)\cdots c_{j_r}(E_k) = \frac{1}{n!}\tau(1)(\zeta^k)^n c_1(E_k)^n$$

$$- \tau(n)(\zeta^k)c_1(E_k)^n + \tau(n)(\zeta^k)c_n(E_k). \tag{2.2}$$

Using Equations (2.1) and (2.2) above, we may rewrite $\left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)\right)_{2n}$ as follows.

$$\left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)\right)_{2n} = \sum_{k=1}^{\frac{p-1}{2}} \left(\tau(n)(\zeta^k)c_n(E_k) - \frac{1}{n!}\tau(n)(\zeta^k)c_1(E_k)^n\right) \tag{2.3}$$

$$+ \sum_{\substack{\ell_1,\cdots,\ell_n\geq 0\\ \ell_1+\cdots+\ell_{\frac{p-1}{2}}=n}} \prod_{k=1}^{\frac{p-1}{2}} \frac{1}{\ell_k!}\tau(1)(\zeta^k)^{\ell_k} c_1(E_k)^{\ell_k}$$

Since $\sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)c_1(E_k) = 0$,

$$\sum_{k_1,\cdots,k_n\in\left\{1,\cdots,\frac{p-1}{2}\right\}} \prod_{m=1}^{n} \tau(1)(\zeta^{k_m})c_1(E_m) = 0. \tag{2.4}$$

Suppose $(k_1,\cdots,k_n) \in \left\{1,\cdots,\frac{p-1}{2}\right\}^n$. Define a map $(k_1,\cdots,k_n) \mapsto (\ell_1,\cdots,\ell_{\frac{p-1}{2}})$ where ℓ_k is the amount of times k appears in $(k_1,\cdots,k_n)$. Clearly, $\ell_1+\cdots+\ell_{\frac{p-1}{2}} = n$ for $(\ell_1,\cdots,\ell_{\frac{p-1}{2}})$ in the image and this assignment is invariant under the S_n action on the domain. Using this to re-index the sum above, we see that

$$\sum_{k_1,\cdots,k_n\in\left\{1,\cdots,\frac{p-1}{2}\right\}}\prod_{m=1}^{n}\tau(1)(\zeta_{k_m})c_1(E_m)=$$

$$\sum_{\substack{\ell_1,\cdots,\ell_n\geq 0\\ \ell_1+\cdots+\ell_{\frac{p-1}{2}}=n}}\frac{n!}{\ell_1!\cdots\ell_{\frac{p-1}{2}}!}\prod_{k=1}^{\frac{p-1}{2}}\tau(1)(\zeta^k)^{\ell_k}c_1(E_k)^{\ell_k}.$$

Using Equations (2.3) and (2.4), we conclude

$$\left(\prod_{k=1}^{\frac{p-1}{2}}\mathcal{M}^{\zeta^k}(E_k)\right)_{2n}=\sum_{k=1}^{\frac{p-1}{2}}\left(\tau(n)(\zeta^k)c_n(E_k)-\frac{1}{n!}\tau(n)(\zeta^k)c_1(E_k)^n\right).$$

Setting this to 0 and using Lemma 2.2.1 shows that $c_n(E_k)=\frac{1}{n!}c_1(E_k)^n$.

For the converse, note that the computations above show the two conditions in the proposition imply $\prod_{k=1}^{\frac{p-1}{2}}\mathcal{M}^{\zeta^k}(E_k)=1$. □

The second condition of Theorem 2.1.2 can be written as $c(E_k)=e^{c_1(E_k)}$. This motivates the following definition.

Definition 2.2.3. A complex vector bundle E is *exponential* if its Chern classes satisfy $c_k(E)=\frac{1}{k!}c_1(E)^k$ or, equivalently, if the total Chern class is $e^{c_1(E)}$.

Example 8. Suppose $E=L_1\oplus\cdots\oplus L_d$ is a sum of line bundles such that $c_1(L_j)^2=0$ for each $j=1,\cdots,d$. It follows from the additivity of the total Chern class that $c_m(E)$ is the m-th elementary symmetric polynomial on $c_1(L_1),\cdots,c_1(L_d)$ when $m\leq d$. The hypothesis that $c_1(L_j)^2=0$ for each j implies that $c_1(E)^m=m!c_m(E)$. This example will be generalized in Proposition 2.3.8 below.

We record the following observations.

Proposition 2.2.4. *Exponential vector bundles satisfy the following properties.*

1. *If E_1 and E_2 are exponential vector bundles then so is $E_1 \oplus E_2$.*

2. *The pullback of an exponential vector bundle is an exponential vector bundle.*

3. *Exponential vector bundles have trivial Pontryagin classes.*

Proof. 1. Since $c_1(E_1 \oplus E_2) = c_1(E_1) \oplus c_1(E_2)$,

$$c(E_1 \oplus E_2) = c(E_1)c(E_2) = e^{c_1(E_1)}e^{c_1(E_2)} = e^{c_1(E_1)+c_1(E_2)} = e^{c_1(E_1 \oplus E_2)}.$$

2. This follows from the naturality of Chern classes.

3. The total Pontryagin class is given by the formula

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots = (1 + c_1(E) + c_2(E) + \cdots)(1 - c_1(E) + c_2(E) - \cdots).$$

It follows that

$$\begin{aligned} p_k(E) = \sum_{i+j=k} (-1)^j c_i(E)c_j(E) &= \sum_{i+j=k} (-1)^j \frac{1}{i!j!} c_1(E)^k \\ &= \frac{1}{k!} \sum_{i+j=k} (-1)^j \binom{k}{j} c_1(E)^k = 0. \end{aligned}$$

Alternatively, we may write $p(E) = e^{c_1(E)}e^{-c_1(E)} = e^0 = 1$.

□

We will only need the second and third property in Proposition 2.2.4.

Proposition 2.2.5. *Suppose E is a free G-vector bundle over M with vanishing Atiyah–Singer class. Then E has vanishing Euler class.*

Proof. Let d be the rank of E and let d_k be the rank of each eigenbundle. By Theorem 2.1.2 the Euler class is scalar multiple of $\prod_{k=1}^{\frac{p-1}{2}} c_1(E_k)^{d_k}$ and $c_1(E_k)^{d_k+1} = 0$. Also,

$$0 = \left(\sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k) c_1(E_k) \right)^d = \prod_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)^{d_k} c_1(E_k)^{d_k}$$

where the first equality follows from Theorem 2.1.2. Since $\tau(1)(\zeta^k) \neq 0$, this implies $\prod_{k=1}^{\frac{p-1}{2}} c_1(E_k)^{d_k} = 0$ as desired. □

We conclude this section with a homotopical characterization of exponential vector bundles. We will not use this result but it may be of independent interest.

Proposition 2.2.6. *A complex vector bundle E over M is exponential if and only if its Chern character is contained in $H^0(M;\mathbb{Q}) \oplus H^2(M;\mathbb{Q})$.*

Proof. Using the splitting principle, write $c_1(E) = x_1 + \cdots + x_d$ where d is the rank of E. Generally, $c_n(E)$ is the n-th elementary symmetric polynomial on $x_1, \cdots, x_d$ and the Chern character is

$$\mathrm{ch}(E) = \sum_{j=1}^{d} e^{x_j} = d + \sum_{j=1}^{d} x_j + \frac{1}{2} \sum_{j=1}^{d} x_j^2 + \cdots .$$

To alleviate notation, let e_m denote the m-th elementary symmetric polynomial on $x_1, \cdots, x_d$ and let $P_m := \sum_{j=1}^{d} x_j^m$. There are formal relations between elementary symmetric polynomials and sums of powers.

$$P_n = \sum_{m=1}^{n} (-1)^{m-1} e_m P_{n-m} \qquad e_n = \frac{1}{n} \sum_{m=1}^{n} (-1)^{m-1} e_{n-m} P_m$$

Suppose E is exponential. We must show that $P_n = 0$ for $n \geq 2$. By hypothesis, we have

$$c_n(E) = \frac{1}{n!} e_1^n = e_n.$$

We proceed by induction on n. First, note that when $n = 2$, the above equation becomes $\frac{1}{2}P_2 + e_2 = e_2$ so this case follows immediately. Assume by induction that $P_m = 0$ whenever $2 \leq m \leq n$. Then,

$$P_{n+1} = (-1)^n e_{n-1}P_1 + (-1)^{n+1}e_n P_0 = (-1)^n \frac{e_1^n}{(n-1)!} + (-1)^{n+1} n \frac{e_1^n}{n!} = 0$$

as desired.

For the converse, suppose $P_m = 0$ for $m \geq 2$. We must show $e_n = \frac{1}{n!}e_1^n$. We proceed by induction with the case $n = 1$ being vacuous. Then,

$$e_n = \frac{1}{n}e_{n-1}P_1 = \frac{1}{n}e_{n-1}e_1 = \frac{1}{n!}e_1^n$$

which completes the proof. □

2.3 Construction of Vector Bundles

Our goal in this section is to construct exponential vector bundles with a prescribed first Chern class β. If we require our exponential vector bundle to have rank N, then β must satisfy $\beta^{N+1} = 0$. We show in Proposition 2.3.8 that, up to multiplying β by a nonzero integer, this is the only requirement. Using these vector bundles and Theorem 2.1.2 we then prove Theorem 2.1.4.

Obstruction theory will play an important role in promoting rational nullhomotopies to integral nullhomotopies. We summarize the version of obstruction theory we need below. We refer to [Bau77] for a obstruction theory when the target space is not necessarily simply connected.

Theorem 2.3.1. *Suppose $E \to B$ is a fibration with connected fiber F. Suppose (X, A) is a relative CW-complex and $f : X \to B$ is a map. Let $g : X^{(n)} \cup A \to E$ be a lift* *of f on the relative n-skeleton of (X, A). If $n \geq 2$, there is an obstruction class $\mathcal{O}(g) \in H^{n+1}(X, A; \pi_n F_\rho)$ where $\pi_n F_\rho$ denotes a local coefficient system with stalk $\pi_n F$. This class vanishes if and only if g can be redefined over the relative n-skeleton leaving the relative $(n-1)$-skeleton fixed so that g extends as a lift to the relative $(n+1)$-skeleton.*

Moreover, if $h : (Y, C) \to (X, A)$ is a cellular map then $g \circ h : Y^{(n)} \cup C \to E$ is a lift of $f \circ h : Y \to B$ defined on the relative n-skeleton of (Y, C) and $\mathcal{O}(g \circ h) = h^*\mathcal{O}(g) \in H^{n+1}(Y, C; h^*\pi_n F_\rho)$.

2.3.1 Obstruction Theory for $\mathbb{C}P^N$

Generally, it can be difficult to show that obstructions vanish when the relevant cohomology group is nonzero. When a space has sufficiently nice self-maps, however, pulling back cocycles allows us to find maps with vanishing obstruction cocycles. We record some useful observations when the space X is $\mathbb{C}\mathrm{P}^N$.

An element $t \in H^2(\mathbb{C}\mathrm{P}^N; \mathbb{Z})$ determines a map $\mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^\infty$ where the induced map on cohomology sends a generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ to the element t. After pushing the map into a $2N$-skeleton, we obtain a map $\lambda : \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$. On $H^{2k}(\mathbb{C}\mathrm{P}^N; \mathbb{Z})$, λ induces multiplication by t^k. We say λ is a *scaling* map of $\mathbb{C}\mathrm{P}^N$ if the corresponding integer t is nonzero.

Lemma 2.3.2. *Suppose $E \to B$ is a fibration with connected fiber F. Let $f : \mathbb{C}P^N \to B$ be a map whose restriction to the 2-skeleton lifts to E. If, for $n \geq 2$, each $\pi_n F$ consists only of torsion elements then there is a scaling map $\lambda : \mathbb{C}P^N \to \mathbb{C}P^N$ such that $f \circ \lambda$ lifts to E.*

Proof. Suppose there is a lift $g : (\mathbb{C}\mathrm{P}^N)^{(n)} \to E$ of $f|_{(\mathbb{C}\mathrm{P}^N)^{(n)}}$. The obstruction to extending this to a lift over $(\mathbb{C}\mathrm{P}^N)^{(n+1)}$ is an element of $H^{n+1}(\mathbb{C}\mathrm{P}^N; \pi_n F)$ (we use that $\mathbb{C}\mathrm{P}^N$ is simply connected to justify constant coefficients and we use that $\mathbb{C}\mathrm{P}^N$ has only even dimensional cells to ignore the subtlety of having to redefine g over the n-cells).

For a suitable scaling map λ_n of $\mathbb{C}\mathrm{P}^N$, $\lambda_n^*\mathcal{O}(g) = 0$ so there is a lift of $f \circ \lambda_n$ over the $(n+1)$-skeleton of $\mathbb{C}\mathrm{P}^N$. Continuing this way shows that there is a lift of $f \circ \lambda$ for a suitable scaling map λ. $\square$

Lemma 2.3.3. *Suppose $E \to B$ is a fibration with connected fiber F. Let $f : \mathbb{C}P^N \to E$ be a map such that the composite $\mathbb{C}P^N \to E \to B$ is nullhomotopic. If, for $n \geq 2$, each $\pi_n F$ consists only of torsion elements, then there is a scaling map $\lambda : \mathbb{C}P^N \to \mathbb{C}P^N$ such that $f \circ \lambda$ is nullhomotopic.*

Proof. Let $C(\mathbb{C}\mathrm{P}^N)$ denote the cone on $\mathbb{C}\mathrm{P}^N$. The nullhomotopy in the hypothesis gives the following diagram.

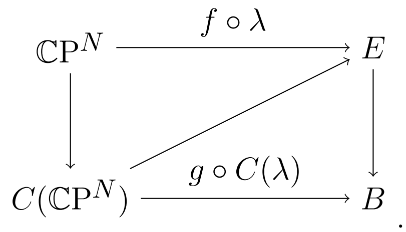
$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^N & \xrightarrow{\quad f \quad} & E \\ \downarrow & \nearrow & \downarrow \\ C(\mathbb{C}\mathrm{P}^N) & \xrightarrow{\quad g \quad} & B \end{array}$$

We would like to find a map $C(\mathbb{C}\mathrm{P}^N) \to E$ making the diagram commute.

Let X_n denote the relative n-skeleton of the pair $(C(\mathbb{C}\mathrm{P}^N), \mathbb{C}\mathrm{P}^N)$. Note that X_2 consists of only $\mathbb{C}\mathrm{P}^N$, the cone point, and an edge connecting the cone point to a $\mathbb{C}\mathrm{P}^N$. By the assumption that F is connected, the path in B determined by the edge lifts to a path in E. Hence there is a map $g_2 : X_2 \to E$ lifting the map $C(\mathbb{C}\mathrm{P}^N) \to B$. Let Σ denote the suspension. Theorem 2.3.1 states that there is an obstruction

$$\mathcal{O}(g_2) \in H^3(C(\mathbb{C}\mathrm{P}^N), \mathbb{C}\mathrm{P}^N; \pi_2 F) \cong H^3(\Sigma(\mathbb{C}\mathrm{P}^N); \pi_2 F) \cong H^2(\mathbb{C}\mathrm{P}^N; \pi_2 F)$$

which vanishes if and only if g_2 can be redefined over the relative 1-skeleton and extended to the relative 3-skeleton. As in the proof of Lemma 2.3.2, we can consider a scaling map $\lambda_2 : \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ such that the induced map on $H^2(\mathbb{C}\mathrm{P}^N; \pi_2 F)$ eliminates the obstruction. Con-ing λ_2 gives a map of relative CW-complexes $C(\lambda_2) : (C(\mathbb{C}\mathrm{P}^N), \mathbb{C}\mathrm{P}^N) \to (C(\mathbb{C}\mathrm{P}^N), \mathbb{C}\mathrm{P}^N)$. Since $\lambda_2(g_2)$ vanishes, we may redefine $g_2 \circ C(\lambda_2) : X_2 \to E$ over the relative 1-skeleton so that there is a map $g_3 : X_3 \to E$ lifting $g \circ C(\lambda_2) : C(\mathbb{C}\mathrm{P}^N) \to B$. Continuing this way shows that, after a suitable self-map λ of $\mathbb{C}\mathrm{P}^N$, there is a lift in the diagram



which proves the Lemma. □

2.3.2 Characteristic Classes

Recall the isomorphism $H^*(BU(N);\mathbb{Z}) \cong \mathbb{Z}[c_1, \cdots, c_N]$ where each c_m has degree $2m$. Since $c_m \in H^{2m}(BU(N);\mathbb{Z})$, Brown representability identifies c_m with a map $c_m : BU(N) \to K(\mathbb{Z}, 2m)$ so the elements $c_1, \cdots, c_N$ together determine a map

$$c_* : BU(N) \to \prod_{m=1}^{N} K(\mathbb{Z}, 2m).$$

By abuse of notation, we will use c_m to denote both the element in $H^{2m}(BU(N);\mathbb{Z})$ and the map above. Let $x_m \in H^{2m}(K(\mathbb{Z}, 2m);\mathbb{Z})$ be a generator of the cohomology group. Then, $c_m^* x_N = c_m$.

Rationally, there are isomorphisms

$$H^*(K(\mathbb{Z}, 2m);\mathbb{Q}) \cong \mathbb{Q}[x_m]$$

and

$$H^*(\prod_{m=1}^{N} K(\mathbb{Z}, 2m);\mathbb{Q}) \cong \bigotimes_{m=1}^{N} H^*(K(\mathbb{Z}, 2m);\mathbb{Q}) \cong \mathbb{Q}[x_1, \cdots x_N].$$

The map $(c_*)^*$ sends x_m to c_m and so induces an isomorphism on rational cohomology groups. Therefore, it induces an isomorphism on rational homotopy groups. In particular, we have

Lemma 2.3.4. *The fiber of c_* is connected with torsion homotopy groups.*

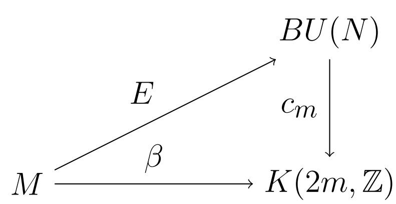
One can perform a similar analysis with $BSO(2N)$. Rationally, the cohomology ring is $H^*(BSO(2N);\mathbb{Q}) \cong \mathbb{Q}[p_1, \cdots, p_{N-1}, e]$ where $p_m \in H^{4m}(BSO(2N);\mathbb{Q})$ are the Pontryagin classes and $e \in H^{2N}(BSO(2N);\mathbb{Q})$ is the Euler class. These classes exist integrally and so determine maps $p_* : BSO(2N) \to \prod_{m=1}^{N-1} K(\mathbb{Z}, 4m)$ and $e : BSO(2N) \to K(\mathbb{Z}, 2N)$. As in the case of $BU(N)$, the map

$$e \times p_* : BSO(2N) \to K(\mathbb{Z}, 2N) \times \prod_{m=1}^{N-1} K(\mathbb{Z}, 4m)$$

induces an isomorphism of rational cohomology rings and, therefore, an isomorphism of rational homotopy groups. This shows

Lemma 2.3.5. *The fiber of $e \times p_*$ is connected with torsion homotopy groups.*

Suppose E is a complex vector bundle over M. By abuse of notation, identify E with a map $E : M \to BU(N)$. If $\beta \in H^{2m}(M;\mathbb{Z})$ is an element, then to say that $c_m(E) = \beta$ is to say that the diagram



commutes. There is a similar interpretation of the Euler and Pontryagin classes.

2.3.3 Construction of Exponential Vector Bundles

We first study the special case of exponential vector bundles on $\mathbb{C}\mathrm{P}^N$.

Proposition 2.3.6. *Let $N' \geq N$ be an integer and let $\alpha \in H^2(\mathbb{C}P^N;\mathbb{Z})$ be a generator. There is an integer $t > 0$ and a rank N' exponential vector bundle E on $\mathbb{C}P^N$ such that $c_1(E) = t\alpha$.*

Proof. First, define $\beta := N!\alpha$, so that the classes $\frac{\beta^m}{m!}$ exist integrally. These classes define a map

$$\beta_* : \mathbb{C}\mathrm{P}^N \to \prod_{m=1}^{N'} K(\mathbb{Z}, 2m).$$

We would like to find a lift in diagram

$$\begin{array}{ccc} & & BU(N') \\ & \nearrow & \downarrow c_* \\ \mathbb{C}\mathrm{P}^N & \xrightarrow{\beta_*} & \prod_{m=1}^{N'} K(\mathbb{Z}, 2m) \end{array}$$

where c_* denotes the map determined by the Chern classes. Such a lift need not exist but, by Lemma 2.3.2, a lift of $\beta_* \circ \lambda$ exists where λ is a scaling map of $\mathbb{C}\mathrm{P}^N$. Let E denote the vector bundle defined by this lift. Then, $c_1(E) = t\beta$ for some nonzero integer t and $c_m(E) = \frac{1}{m!}c_1(E)^m$ for $m \leq N'$. When $m > N'$ then, by our assumption that $N' \geq N$, $c_m(E) = 0 = \frac{1}{m!}c_1(E)^m$. $\square$

Proposition 2.2.4 states that pullbacks of exponential vector bundles are exponential. We obtain the following from taking further pullbacks along self-maps of $\mathbb{C}\mathrm{P}^N$.

Proposition 2.3.7. *Let $N' \geq N$ be an integer and let $t\alpha \in H^2(\mathbb{C}P^N;\mathbb{Z})$ denote the class in Proposition 2.3.6. Then, any integer multiple of $t\alpha$ can be realized as $c_1(E)$ where E is a rank N' exponential vector bundle.*

The pullback property also allows us to construct exponential vector bundles over more general spaces.

Proposition 2.3.8. *Let M be homotopy equivalent to a finite complex and let $N' \geq N$. Then for every $\beta \in H^2(M;\mathbb{Z})$ satisfying $\beta^{N+1} = 0$, there is an integer $t > 0$ such that, for all integers u, there is a rank N' exponential vector bundle E on M with $c_1(E) = ut\beta$. Moreover, the classifying maps $M \to BU(N')$ associated to these bundles factor through $\mathbb{C}P^N$.*

Proof. By Proposition 2.3.7 and the pullback property, it suffices to show that there is a map $f : M \to \mathbb{C}\mathrm{P}^N$ such that $f^*\alpha$ is some nonzero integer multiple of β.

We first reduce to the case that M is simply connected. Attach 2-cells to M in order to obtain a simply connected finite complex M' such that $M/M' \simeq \bigvee S^2$. Note that $H^2(M';\mathbb{Z})$ surjects onto $H^2(M;\mathbb{Z})$ and $H^j(M';\mathbb{Z}) \cong H^j(M;\mathbb{Z})$ for all $j > 2$. So, there is an element $\beta' \in H^2(M';\mathbb{Z})$ mapping to β and such that $(\beta')^{N+1} = 0$. If the result holds for M', then pulling the exponential vector bundle back along the inclusion $M \subseteq M'$ shows the result also holds for M.

Assuming M is simply connected, there is an element in the Sullivan algebra (Λ_M, d_M) of M representing β. We will also use β to denote this element. The nilpotence hypothesis on β implies there is a degree $2N+1$ element $\gamma \in \Lambda_M$ such that $d_M\gamma = (\beta)^{N+1}$. These elements determine a map of differential graded algebras $(\Lambda_{\mathbb{C}\mathrm{P}^N}, d_{\mathbb{C}\mathrm{P}^N}) \to (\Lambda_M, d_M)$ which yields a map of rationalizations

$$M_{(0)} \to \mathbb{C}\mathrm{P}^N_{(0)}.$$

We may assume M is a finite complex so that the map $M \to M_{(0)} \to \mathbb{C}\mathrm{P}^N_{(0)}$ has image in a finite subcomplex of $\mathbb{C}\mathrm{P}^N_{(0)}$. The rationalization $\mathbb{C}\mathrm{P}^N_{(0)}$ can be constructed as a homotopy colimit of the diagram

$$\mathbb{C}\mathrm{P}^N \xrightarrow{\lambda^2} \mathbb{C}\mathrm{P}^N \xrightarrow{\lambda^3} \mathbb{C}\mathrm{P}^N \xrightarrow{\lambda^4} \mathbb{C}\mathrm{P}^N \to \cdots$$

where λ^t is the scaling map corresponding to the integer t. In particular, it is an infinite mapping telescope so the map $M \to \mathbb{C}\mathrm{P}^N_{(0)}$ factors through some finite mapping telescope

$$T_q = \varinjlim(\mathbb{C}\mathrm{P}^N \xrightarrow{\lambda^2} \mathbb{C}\mathrm{P}^N \xrightarrow{\lambda^3} \cdots \xrightarrow{\lambda^q} \mathbb{C}\mathrm{P}^N).$$

There is a homotopy equivalence $T_q \to \mathbb{C}\mathrm{P}^N$ so we have constructed a map $M \to \mathbb{C}\mathrm{P}^N$.

We now check that the pullback of α under this map is of the form $t\beta$. Since M is simply connected, $H^2(M;\mathbb{Q})$ can be identified with $\mathrm{Hom}(\pi_2(M);\mathbb{Q})$. Let β^* denote the element in $\pi_2(M)$ dual to β under this identification. Similarly, let $\alpha^* \in \pi_2(\mathbb{C}\mathrm{P}^N)$ denote the dual to α and let α_0^* denote the image of α^* in $\pi_2(\mathbb{C}\mathrm{P}^N_{(0)})$. Under the identification $\mathbb{C}\mathrm{P}^N \simeq T_q$, the map $\mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N_{(0)}$ sends α^* to $\frac{1}{q!}\alpha_0^*$. But the map $M \to \mathbb{C}\mathrm{P}^N_{(0)}$ sends β^* to α_0^*. It follows that the map $M \to \mathbb{C}\mathrm{P}^N$ sends β^* to $\frac{1}{q!}\alpha^*$. Hence, α pulls back to $\frac{1}{q!}\beta$. □

Theorem 2.1.4. *Let $G = \mathbb{Z}/p\mathbb{Z}$ and let V be a free G-representation. Suppose M is homotopy equivalent to a finite CW-complex.*

1. *If 2 has even order in $(\mathbb{Z}/p\mathbb{Z})^\times$ or if $H^2(M;\mathbb{Q}) = 0$ then there are only finitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class.*

2. *If 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$ and there is a nonzero $\beta \in H^2(M;\mathbb{Q})$ sufficiently nilpotent with respect to V, then there are infinitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class.*

Proof of Theorem 2.1.4. Under the hypotheses of the first part, the Chern classes of each eigenbundle vanish. Since the Chern classes determine a $BU(N) \to \prod_{m=1}^N K(\mathbb{Z}, 2m)$ whose fiber has finite homotopy groups and M is homotopy equivalent to a finite complex, there are only finitely many complex vector bundles of a fixed rank with prescribed Chern classes.

For the second part, suppose 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$. If $\beta \in H^2(M;\mathbb{Z})$ is sufficiently nilpotent with respect to V, we may apply Proposition 2.3.8 to take exponential vector bundles E_k such that $c_1(E_k) = u_k t\beta$ where the u_k realize the linear relation $\sum_{k=1}^{\frac{p-1}{2}} u_k \tau(1)(\zeta^k) = 0$. This proves the second part of Theorem 2.1.4. □

In order to address Question 4, we would like a converse to Theorem 2.1.4. The difficulty in obtaining a converse is number theoretic; we know that any subset of $\{\tau(1)(\zeta^k)\}$ of size $u+1$ has a $\mathbb{Q}$-linear relation but it is not clear whether there exist smaller subsets which are $\mathbb{Q}$-linearly dependent. However, we can say the following.

Proposition 2.3.9. *Suppose M is homotopy equivalent to a finite complex and that there are infinitely many G-vector bundles with vanishing Atiyah–Singer class. Then, infinitely many of these G-vector bundles are pulled back from G-vector bundles over $\mathbb{C}P^N$.*

Proof. We may assume that there is a vector bundle E with vanishing Atiyah–Singer class and such that some of the $c_1(E_k)$ are nonzero rationally.

Let $\beta \in H^2(M;\mathbb{Q})$ be one of the nonzero $c_1(E_k)$ where the N such that $c_1(E_k)^{N+1} = 0$ is minimal. By projecting the relation $\sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)c_1(E_k) = 0$ to the $\mathbb{Q}(\zeta)$-subspace of $H^2(M;\mathbb{Q}(\zeta))$ spanned by β, we see that there is a linear relation $\sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)\beta_k$ where β_k is a rational multiple of β and $\beta_k = 0$ if $c_1(E_k) = 0$. Moreover, our choice of β ensures that the dimension of the eigenspace V_k is at least N when $\beta_k \neq 0$. By scaling the β_k simultaneously, we may use Proposition 2.3.8 to realize β_k as the first Chern class of an exponential bundle factoring through $\mathbb{C}\mathrm{P}^N$. Adding these together gives a bundle over M with vanishing Atiyah–Singer class which factors through $\mathbb{C}\mathrm{P}^N$. Composing with self-maps of $\mathbb{C}\mathrm{P}^N$ gives infinitely many such vector bundles. $\square$

2.4 Block Bundles

We recall some definitions and facts about block bundles. We refer to [Cas96] and [RS71] for a more detailed treatment.

Definition 2.4.1. Let K be a finite simplicial complex and let Y be a polyhedron. Let $\pi : E \to |K|$ be a continuous map. A *block chart* for a simplex $\sigma \subseteq K$ is a PL-homeomorphism

$$h_\sigma : \pi^{-1}(\sigma) \to \sigma \times Y$$

such that, for each face $\tau \leq \sigma$, the restriction $h_\sigma|_{\pi^{-1}(\tau)}$ is a PL-homeomorphism $\pi^{-1}(\tau) \to \tau \times Y$. We say that $\pi : E \to |K|$ is a *block bundle with fiber* Y if there is a block chart for every simplex $\sigma \subseteq K$.

It is not true that, for a PL-block bundle, $\pi^{-1}(x) \cong F$ for an arbitrary point $x \in |K|$; this distinguishes block bundles from fiber bundles. One can define block bundles with other structure groups. We will only be concerned with PL-block bundles. Our block bundles will typically be over smooth manifolds in which case we give the manifold a PL-structure compatible with the smoothing.

Definition 2.4.2. Let $\pi_i : E_i \to |K|$ be PL-block bundles for $i = 0, 1$. An *isomorphism* of PL-block bundles is a PL-homeomorphism $H : E_0 \to E_1$ such that $H(\pi_0^{-1}(\sigma)) = \pi_1^{-1}(\sigma)$ for all simplices $\sigma \subseteq K$.

The block bundles π_0 and π_1 are *equivalent* if there is a subdivision K' of K such that π_0 and π_1 determine isomorphic block bundles over K'.

Casson shows in [Cas96] that equivalence of PL-block bundles is an equivalence relation.

Definition 2.4.3. Let Y be a polyhedron. Define $\widetilde{PL}(Y)$ to be the simplicial group whose d-simplices are the PL-homeomorphisms $f : \Delta^d \times Y \to \Delta^d \times Y$ such that, for each face $\sigma \subseteq \Delta^d$,

$$f(\pi_{\Delta^d}^{-1}(\sigma)) \subseteq \pi_{\Delta^d}^{-1}(\sigma)$$

where $\pi_{\Delta^d} : \Delta^d \times Y \to \Delta^d$ is the projection. If Y is an orientable PL-manifold, define $\widetilde{SPL}(Y)$ to be the simplicial group whose d-simplices are the orientation preserving PL-homeomorphisms $f : \Delta^d \times Y \to \Delta^d \times Y$ satisfying the above property.

One can construct classifying spaces $B\widetilde{PL}(Y)$ for PL-block bundles with fiber Y. The following is [Cas96, Theorem 2].

Theorem 2.4.4. *There is a bijection between equivalence classes of PL-block bundles over K with fiber Y and homotopy classes of maps $[K, B\widetilde{PL}(Y)]$.*

Finally, we record a consequence of the fact that block bundles are controlled over the base space. Let $SO^G(V)$ denote the group of G-equivariant orientation preserving linear transformations of V. Suppose E_0 and E_1 are two G-vector bundles such that the composites $M \to BSO^G(V) \to B\widetilde{SPL}(SV/G)$ are homotopic. Let D_0 and D_1 be the respective unit disk bundles and let SE_0 and SE_1 denote the respective unit sphere bundles. The homotopy $M \times I \to B\widetilde{SPL}(SV/G)$ gives a concordance W of PL-block bundles SE_1/G and SE_2/G. Let $\widetilde{W}$ denote the G-cover. We may form the G-manifold $E' := \widetilde{W} \cup_{SE_1} D_1$ which has boundary SE_0.

Proposition 2.4.5. *In the situation above, suppose $f : SE_1/G \to SE_0/G$ is an equivalence of PL-block bundles over M. Let $\tilde{f}$ denote the map on covers. Then there is an equivariant homeomorphism $E' \to D_0$ restricting to $\tilde{f}$ on the boundary.*

Proof. First, note that $D_0 \setminus M$ is equivariantly homeomorphic to $SE_0 \times [0, \infty)$. We construct an equivariant homeomorphism $W \cup_{SE_1} D_1 \setminus M \to SE_0 \times [0, \infty)$ such that the restriction to the boundary is $\tilde{f}$ and we show that this homeomorphism extends to M.

By hypothesis, W, SE_0/G and SE_1/G have the same classifying map. So after taking a subdivision of $M \times I$, there is an isomorphism of PL-block bundles $F : W/G \to SE_0/G \times [0, 1]$ which restricts to f on $SE_0/G \times \{0\}$. Let M_0 denote the triangulation of $M \times \{0\}$ and let M_1 denote the triangulation of $M \times \{1\}$. Let f_1 denote the isomorphism $F|_{SE_1/G} : SE_1/G \to SE_0/G$.

Write W_j for the trivial PL-block bundle over $M \times [j, j+1]$ where we equip M with a triangulation subordinate to the barycentric subdivision of M_j. Subdivide W_j so that there is an isomorphism of PL-block bundles $F_j : W_j \to M \times [j, j+1] \times SE_0/G$ restricting to f_j on the part over $M \times \{j\}$. Define M_{j+1} to be the triangulation on $M \times \{j+1\}$ and define f_{j+1} to be the restriction of F_j to the part over $M \times \{j+1\}$.

Continuing this way, we obtain a homeomorphism

$$W/G \cup_{SE_1/G} (SE_1/G \times [1,\infty)) \to SE_0/G \times [0,\infty).$$

Lifting to the G-cover gives an equivariant homeomorphism

$$W \cup_{SE_1} (SE_1 \times [1,\infty)) \to SE_0 \times [0,\infty).$$

This extends continuously to M; any sequence in $W \cup_{SE_1} (SE_1 \times [1,\infty))$ approaching a point $m \in M$ will get sent to a sequence on the right hand side approaching the same point. $\square$

2.4.1 The Rational Homotopy Type of $B\widetilde{SPL}(SV/G)$

In [CW91], Cappell–Weinberger describe $B\widetilde{SPL}(SV/G)$ rationally. Let $\tilde{L}^s_k(G)$ denote the reduced simple L-space of G; this is a space satisfying $\pi_n \tilde{L}^s_k(G) = \tilde{L}^s_{n+k}(G)$ where the right hand side denotes the reduced simple L-groups.

Theorem 2.4.6 (Cappell–Weinberger)**.** *There is a map*

$$B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV) \times \tilde{L}^s_{\dim_{\mathbb{R}} V}(G)_{(0)}.$$

whose fiber is connected with torsion homotopy groups.

Remark. Cappell–Weinberger state that the map in Theorem 2.4.6 is a $\frac{1}{2|G|}$-equivalence. They do not show that $B\widetilde{SPL}(SV/G)$ is simply connected. Their proof shows that the fundamental group is a finite solvable group with a composition series having $|G|$-torsion abelian subquotients. They also show that the map on higher homotopy groups is an equivalence after inverting $2\,|G|$.

In Theorem 2.4.6, the map $B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV)$ is given by pulling back a homeomorphism of $\Delta^d \times SV/G$ to $\Delta^d \times SV$. In particular, if an equivariant vector bundle is non-equivariantly trivial, then the composite

$$M \to BSO^G(V) \to B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV)$$

is nullhomotopic.

The other component of the map involves the Atiyah–Singer class and an argument with the Conner-Floyd isomorphism (a more detailed treatment of an analogous argument may be found in [MM79, Chapter 4]). First recall the rational equivalence $\tilde{L}^s_{\dim_{\mathbb{R}} V}(G)_{(0)} \simeq BO(\widetilde{RO}(G))_{(0)} \times \Omega^2 BO(\widetilde{RO}(G))_{(0)}$ where $BO(\widetilde{RO}(G))$ denotes Ω^∞ of KO smashed with Moore spectrum. To define a map $B\widetilde{SPL}(SV/G) \to BO(\widetilde{RO}(G))_{(0)}$ it suffices to define an element of $KO^0(B\widetilde{SPL}(SV/G); \widetilde{RO}(G)_{(0)})$.

The universal coefficients theorem for KO gives an isomorphism

$$KO^0(X; \widetilde{RO}(G)_{(0)}) \to \mathrm{Hom}(KO_0(X), \widetilde{RO}(G)_{(0)})$$

for finite complexes X. An inverse limit argument shows that, for infinite X, there is a surjection

$$KO^0(X; \widetilde{RO}(G)_{(0)}) \to \varprojlim \mathrm{Hom}(KO_0(X^{(i)}), \widetilde{RO}(G)_{(0)})$$

where the limit on the right is taken over skeleta.

The Conner-Floyd isomorphism states that $\Omega^{SO}_{4*+i}(X) \otimes_{\Omega^{SO}_*(*)} \mathbb{Z}[\frac{1}{2}] \cong KO_i(X; \mathbb{Z}[\frac{1}{2}])$. So given an element of

$$\mathrm{Hom}(\Omega^{SO}_{4*}(X) \otimes_{\Omega^{SO}_*(*)} \mathbb{Z}[\frac{1}{2}], \widetilde{RO}(G)_{(0)})$$

we obtain an element of $KO^0(X; \widetilde{RO}(G)_{(0)})$ and hence a map

$$X \to BO(\widetilde{RO}(G))_{(0)}.$$

This map is unique up to homotopy if X is a finite complex.

We now define the homomorphism $AS \in \mathrm{Hom}(\Omega_{4*}^{SO}(X) \otimes_{\Omega_*^{SO}(*)} \mathbb{Z}[\frac{1}{2}], \widetilde{RO}(G)_{(0)})$ giving rise to the map $B\widetilde{SPL}(SV/G) \to BO(\widetilde{RO}(G))_{(0)}$. Suppose $f : M \to B\widetilde{SPL}(SV/G)$ represents an element of $\Omega^{SO}(B\widetilde{SPL}(SV/G))$. Let $SE/G \to M$ be the corresponding block bundle. Then SE/G has a G-cover SE which is a block bundle over M with fiber SV. Since G acts freely on SE and because SE bounds non-equivariantly, there is an integer $r > 0$ such that r-many copies of SE bounds a manifold X on which G acts freely. Define

$$AS([f]) := \frac{1}{r}\,\mathrm{sign}_G(X) - \mathrm{sign}(E) \cdot \mathrm{triv}$$

where sign_G denotes the $\widetilde{RO}(G)$-valued multisignature, $\mathrm{sign}(E)$ denotes the (non-equivariant) signature of the block bundle obtained by coning the sphere bundle and triv denotes the trivial representation.

So far, only "half" of the map $B\widetilde{SPL}(SV/G) \to L^s_{\dim_{\mathbb{R}} V}(G)_{(0)}$ has been defined; we still need to define a map

$$B\widetilde{SPL}(SV/G) \to \Omega^2 BO(\widetilde{RO}(G))_{(0)}.$$

This is equivalent to a map $\Sigma^2 B\widetilde{SPL}(SV/G) \to BO(\widetilde{RO}(G))_{(0)}$. As above, we obtain such a map from a group homomorphism

$$\Omega_{4*}(\Sigma^2 B\widetilde{SPL}(SV/G)) \otimes_{\Omega_*^{SO}(*)} \mathbb{Z}[\frac{1}{2}] \cong \Omega_{4*+2}(B\widetilde{SPL}(SV/G)) \otimes_{\Omega_*^{SO}(*)} \mathbb{Z}[\frac{1}{2}] \xrightarrow{AS} \widetilde{RO}(G)_{(0)}.$$

This homomorphism is defined using the Atiyah–Singer invariant in an identical manner.

Suppose the $f : M \to B\widetilde{SPL}(SV/G)$ factors through $BSO^G(V)$. Then we may regard SE above as the sphere bundle of the corresponding G-vector bundle and, by [AS68, Section 7], $AS([f])$ is the element of $\widetilde{RO}(G)$ with character

$$AS([f])(g) = \langle A(g,V)L(M)\mathcal{M}(g,E), [M]\rangle\,.$$

The following lemma asserts that the character is determined by its value on a generator of G.

Lemma 2.4.7. *Suppose* $\xi = \bigoplus_{k=1}^{\frac{p-1}{2}} \xi_k$ *and* $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ *are two* G*-vector bundles with fiber* V *and with eigenbundle decompositions associated to a generator* g_0 *of* G*. If*

$$\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(\xi_k) = \prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)$$

then $AS(M,\xi) = AS(M,E)$*.*

Proof. For an arbitrary $g_0^n \in G$, we have

$$AS(M,E)(g_0^n) = \left\langle A(g_0^n, V)\mathcal{L}(M) \prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^{nk}}(E_k), [M] \right\rangle.$$

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ be the automorphism defined by $\zeta \mapsto \zeta^n$. By Lemma 2.2.2,

$$\begin{aligned} AS(M,E)(g_0^n) &= \left\langle A(g_0^n, V)\mathcal{L}(M)\sigma\left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)\right), [M] \right\rangle \\ &= \left\langle A(g_0^n, V)\mathcal{L}(M)\sigma\left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(\xi_k)\right), [M] \right\rangle \\ &= AS(M,\xi)(g_0^n). \end{aligned}$$

□

Proposition 2.4.8. *Suppose* M *is homotopy equivalent to a finite complex and* $\nu_0, \nu_1 : M \to BSO^G(V)$ *determine* G*-vector bundles with equal Atiyah–Singer classes. Then the compositions*

$$M \to BSO^G(V) \to B\widetilde{SPL}(SV/G) \to \tilde{L}^s_{\dim_{\mathbb{R}} V}(G)_{(0)}$$

is are homotopic.

Proof. Since M is homotopy equivalent to a finite complex, if the two induced maps

$$\mathrm{Hom}(\Omega^{SO}_{4*}(B\widetilde{SPL}(SV/G)) \otimes_{\Omega^{SO}_*(*)} \mathbb{Z}[\frac{1}{2}], \widetilde{RO}(G)) \to \mathrm{Hom}(\Omega^{SO}_{4*}(M) \otimes_{\Omega^{SO}_*(*)} \mathbb{Z}[\frac{1}{2}], \widetilde{RO}(G))$$

send the map AS to the same element, they determine the same map $M \to BO(\widetilde{RO}(G))_{(0)}$. For $i = 0, 1$, let $\tilde{\nu}_i$ denote the composition

$$M \xrightarrow{\nu_i} BSO^G(V) \to B\widetilde{SPL}(SV/G).$$

Suppose $h : M' \to M$ represents an element of $\Omega^{SO}_{4*}(M)$. Then $\nu_0 \circ h$ and $\nu_1 \circ h$ classify G-vector bundles over M' with equal Atiyah–Singer classes. So the Atiyah–Singer invariants of the two SV/G-block bundles over M' are equal. It follows that $\tilde{\nu}_i \circ h$ represent elements of $\Omega^{SO}_{4*}(B\widetilde{SPL}(SV/G))$ such that $AS([\tilde{\nu}_0 \circ h]) = AS([\tilde{\nu}_1 \circ h])$.

A similar argument shows that ν_0 and ν_1 determine the same map

$$M \to \Omega^2 BO(\widetilde{RO}(G))_{(0)}.$$

□

Using Proposition 2.4.8 we can show that vector bundles with vanishing Atiyah–Singer class can be taken to have lens space bundles which are trivial as PL-block bundles.

Proposition 2.4.9. *Suppose M is homotopy equivalent to a finite complex. Suppose there are infinitely many G-vector bundles over M with fiber V and vanishing Atiyah–Singer class. Then, infinitely many of these G-vector bundles E satisfy the following properties.*

1. *The classifying map of E factors through $\mathbb{C}P^N$,*

2. *The corresponding lens space bundle SE/G is isomorphic as a PL-block bundle to the trivial bundle $M \times SV/G$.*

Proof. As before, we begin with the case $M = \mathbb{C}\mathrm{P}^N$. By hypothesis and Proposition 2.4.8, the composite

$$\mathbb{C}\mathrm{P}^N \xrightarrow{E} BSO^G(V) \to B\widetilde{SPL}(SV/G) \to \tilde{L}^s_{\dim_{\mathbb{R}} V}(G)_{(0)}$$

is nullhomotopic.

We show that the composite

$$\mathbb{C}\mathrm{P}^N \xrightarrow{E} BSO^G(V) \to B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV)$$

also vanishes. There is a commuting diagram

$$\begin{array}{ccccc} \mathbb{C}\mathrm{P}^N & \xrightarrow{E} & BSO^G(V) & \longrightarrow & B\widetilde{SPL}(SV/G) \\ & & \downarrow & & \downarrow \\ & & BSO(V) & \longrightarrow & B\widetilde{SPL}(SV) \end{array}$$

where the left vertical map is obtained by forgetting the action. So it suffices to show that E is trivial as a (non-equivariant) real vector bundle. The Pontryagin classes of E vanish by Proposition 2.2.4 and the Euler class vanishes by Proposition 2.2.5. We have shown that the composition

$$\mathbb{C}\mathrm{P}^N \xrightarrow{E} BSO^G(V) \to BSO(V) \xrightarrow{e \times p_*} K(\mathbb{Z}, \dim_{\mathbb{R}} V) \times \prod_{m=1}^{\dim_{\mathbb{R}} V/2-1} K(\mathbb{Z}, 4m)$$

is nullhomotopic. Applying Lemma 2.3.2 to the fibration $BSO(2d) \xrightarrow{e \times p_*} K(\mathbb{Z}, 2d) \times$

$\prod_{m=1}^{d-1} K(\mathbb{Z}, 4m)$ shows that, after composing with a scaling map λ of $\mathbb{C}\mathrm{P}^N$, the composite $E \circ \lambda : \mathbb{C}\mathrm{P}^N \to BSO^G(V) \to BSO(2d)$ is nullhomotopic.

Replace E with the $\lambda^* E$ so that it is trivial as a (non-equivariant) real vector bundle. So far, we have constructed an $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ such that some $c_1(E_k)$ is a nonzero and such that the composite

$$\mathbb{C}\mathrm{P}^N \xrightarrow{E} BSO^G(V) \to B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV) \times \tilde{L}^s_{2d}(G)_{(0)}$$

is nullhomotopic. It remains to show that we can modify E so that the map is nullhomotopic integrally. We apply Lemma 2.3.3 to the fibration $B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV) \times \tilde{L}^s_{\dim_{\mathbb{R}} V}(G)_{(0)}$ to see that, for a scaling map $\lambda : \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$,

$$\mathbb{C}\mathrm{P}^N \xrightarrow{\lambda} \mathbb{C}\mathrm{P}^N \xrightarrow{E} BSO^G(V) \to B\widetilde{SPL}(SV/G)$$

is nullhomotopic.

For the general case, apply Proposition 2.3.9 to the result on $\mathbb{C}\mathrm{P}^N$. □

2.5 Smoothing PL-Concordances

In this section, we would like to take advantage of the fact that a closed manifold of dimension at least 5 has only finitely many smooth structures in order to show that the vector bundles constructed above, whose lens space bundles have homeomorphic total spaces, can be made to have diffeomorphic total spaces.

Let us first recall some classical facts about smoothing. We refer the reader to [HM74] for details.

Definition 2.5.1. Given a closed PL-manifold M, a *smoothing* of M (or a *smooth structure* on M) is a smooth manifold M_0 and a PL-homeomorphism $f_0 : M_0 \to M$. Two smooth structures $f_i : M_i \to M$, $i = 0, 1$, are *concordant* if there is a smooth structure on the PL-manifold $M \times I$ and a PL-homeomorphism $F : M \times I \to M \times I$ such that the following hold.

- F restricts to f_0 on $M \times \{0\}$;
- F restricts to $f_1 \circ \phi$ on $M \times \{1\}$ for some diffeomorphism ϕ of M_1.

Let $PL/O(M)$ denote the concordance classes of smoothings of M. When $\dim M \geq 5$, this set can be identified with concordance classes of linear structures on the stable tangent PL-microbundle. Specifying an initial smooth structure η on M gives a bijection $PL/O(M) \cong [M, PL/O]$ and PL/O has an H-space structure induced by Whitney sum. Let $PL/O(M, \eta)$ denote the set of concordance classes of smoothings with a specified smooth structure η. Thus $PL/O(M, \eta)$ is a homotopy functor from the category of smooth manifolds and continuous maps to abelian groups.

2.5.1 Differentiable Vector Bundles

Suppose $p : E \to M$ is a vector bundle where M is smoothable and let $\mathcal{U}$ be an atlas on M such that p is locally trivial over each $U \in \mathcal{U}$. Given a smooth structure η on M, we may assume that the transition maps $U_\alpha \cap U_\beta \to GL_k(\mathbb{R})$ are smooth for U_α and U_β in some subatlas $\mathcal{U}'$. The vector bundle equipped with a maximal subatlas satisfying this property is called a *differentiable vector bundle*. By [HM74, p.89, Theorem 1.9], every vector bundle over a smooth manifold admits the structure of a differentiable vector bundle and this structure is unique.

The total space of a differentiable vector bundle has a unique smooth structure such that the local trivializations $p^{-1}U \to U \times \mathbb{R}^n$ are smooth. It turns out that this assignment yields a well-defined bijection $p^! : PL/O(M) \to PL/O(E)$ [HM74, p. 93, Theorem 2.6]. If we wish to consider these as pointed sets, then there is a well-defined bijection $p^! : PL/O(M, \eta) \to$

$PL/O(E, p^! \eta)$. It is important to note that there are generally multiple ways of making the total space E a vector bundle over M and different ways of doing so result in different bijections.

Using the structure group $SO^G(V)$, can define differentiable G-vector bundles over a smooth manifold (with trivial G-action) similarly and the proof of [HM74, p.89, Theorem 1.9] shows that every G-vector bundle admits a unique differentiable G-vector bundle structure. When V is a free representation, this gives the corresponding lens space bundle a smooth structure.

2.5.2 Functoriality

Suppose $f : (M, \eta) \to (N, \omega)$ is a continuous map between smooth manifolds. Hirsch-Mazur [HM74, p. 111] give the following description of the induced map $PL/O(f) : PL/O(N, \omega) \to PL/O(M, \eta)$. Let $\varphi : (N, \beta) \to (N, \omega)$ represent a smooth structure in $PL/O(N, \omega)$ which we will denote $[\beta]$. Let ρ denote the standard smooth structure on $\mathbb{R}$. For some sufficiently large integer d, there is a smooth embedding

$$\psi : (M, \eta) \to (N \times \mathbb{R}^d, \omega \times \rho^d)$$

such that $\pi_N \circ \psi$ is homotopic to f. Then the normal bundle ν of $\psi(M) \subseteq (N \times \mathbb{R}^d, \omega \times \rho^d)$ determines a vector bundle on the PL-submanifold $\psi(M) \subseteq (N \times \mathbb{R}^d, \beta \times \rho^d)$. The total space $E(\nu)$ has a smooth structure $\nu^! \eta$ coming from the smooth structure η on M and the vector bundle structure. This space can also be identified with an open subset of $(N \times \mathbb{R}^d, \beta \times \rho^d)$ hence it inherits a smooth structure $[\beta \times \rho^d] \in \mathcal{S}^{PL/O}(E(\nu), \nu^! \eta)$. Since $\nu^! : PL/O(M, \eta) \to PL/O(E(\nu), \nu^! \eta)$ is a bijection, there is a unique smooth structure $[\alpha] \in PL/O(M, \eta)$ such that $\nu^!([\alpha]) = [\beta \times \rho^d]$. The smoothing α represents $PL/O(f)([\beta])$.

The next result essentially states that given a smooth map between manifolds, the smooth structure on a pullback PL-block bundle is the pullback smooth structure induced by maps of total spaces.

Proposition 2.5.2. *Suppose F, M and N are smooth manifolds. Let E_0 and E_1 be F-bundles over N such that E_0 and E_1 are smooth and the projections to N are smooth. Let $\varphi : E_1 \to E_0$ be an isomorphism of PL-block bundles and let $f : M \to N$ be smooth. Let η denote the given smooth structure on E_0 and let $f^*\eta$ denote the smooth structure on f^*E_0 making it a smooth submanifold of $M \times E_0$ and let $f^* : PL/O(E_0, \eta) \to PL/O(f^*E_0, f^*\eta)$ be the induced map on smooth structures. Then, $f^*[\varphi]$ is represented by the induced map on pullbacks $f^*E_1 \to f^*E_0$.*

Proof. Let $\psi : M \to \mathbb{R}^d$ be a smooth embedding. This determines a smooth embedding $f^*E_0 \to E_0 \times \mathbb{R}^d$ which sends $x \in f^*E_0$ to $(f(x), \psi \circ \pi_{f^*E_0} x)$ where $\pi_{f^*E_0} : f^*E_0 \to M$ is the bundle projection. Let ν_0 denote the normal bundle of this embedding. The total space $E(\nu_0)$ inherits a smooth structure as an open subset of $E_0 \times \mathbb{R}^d$. Let us call this structure γ. Also, there is the smooth structure $\nu_0^! f^*\eta$ coming from the vector bundle structure. By Hirsch-Mazur's description of the induced map,

$$\nu_0^!(PL/O(f)([SE/G])) = [\gamma]$$

in the set $PL/O(E(\nu_0), \nu_0^! f^*\eta)$.

Let W denote a concordance of PL-block bundles between E_0 and E_1. Then, f^*W is a concordance of PL-block bundles between f^*E_0 and f^*E_1. Moreover, there is an isomorphism $F : W \to E_1 \times I$ of PL-block bundles over $N \times I$. Let π_M denote the composite $f^*W \to M \times I \to M$ and consider the PL-embedding

$$(F \times \mathrm{id}_{\mathbb{R}^d}) \circ (f \times (\psi \circ \pi_M)) : f^*W \to W \times \mathbb{R}^d \to E_1 \times I \times \mathbb{R}^d.$$

Over $0 \in I$, this restricts to the embedding $f^*E_0 \to E_0 \times \mathbb{R}^d$ above and over $1 \in I$, this restricts to a smooth embedding $f^*E_1 \to E_1 \times \mathbb{R}^d$. Let ν_1 denote the normal bundle of the second embedding. By taking an open neighborhood of the image of $F(f \times (\psi \circ \pi_M))$, we see that the smooth structure $[\gamma]$ above is the same as $\nu_1^![f^*\varphi]$. So it suffices to show that ν_0 and ν_1 are isomorphic as vector bundles.

Let ν denote the normal bundle of the smooth embedding $f \times \psi : M \to N \times \mathbb{R}^d$. The pullback of ν to W restricts to ν_0 over $0 \in I$ and ν_1 over $1 \in I$ which shows that ν_0 and ν_1 are isomorphic vector bundles. $\square$

2.5.3 Smooth Trivialization of SE/G

We now show that the isomorphism $SE/G \to M \times SV/G$ of lens space block bundles over M can be made into a diffeomorphism of total spaces. As before, the main tool will be the use of scaling maps of $\mathbb{C}\mathrm{P}^N$.

Proposition 2.5.3. *Suppose E is a G-vector bundle over $\mathbb{C}P^N$ and let $f : SE/G \to \mathbb{C}P^N \times SV/G$ be an equivalence of PL-block bundles. Then, there is a scaling map $\lambda : \mathbb{C}P^N \to \mathbb{C}P^N$ such that $\lambda^* f : \lambda^* SE/G \to \lambda^*(\mathbb{C}P^N \times SV/G)$ is PL-isotopic to a diffeomorphism.*

Proof. Identify $[\mathbb{C}\mathrm{P}^N \times SV/G, PL/O]$ with the smoothings of $\mathbb{C}\mathrm{P}^N \times SV/G$ by specifying the product smooth structure. The isomorphism of PL-block bundles $f : SE/G \to \mathbb{C}\mathrm{P}^N \times SV/G$ determines an element $[f] \in [\mathbb{C}\mathrm{P}^N \times SV/G, PL/O]$. Since PL/O is an infinite loop space, there is a generalized cohomology theory $\mathbf{E}^*$ such that $\mathbf{E}^0(X) = [X, PL/O]$. Explicitly, if $PL/O = \Omega^n \mathbf{E}_n$, then for $n \geq 0$, $\mathbf{E}^n(X) = [X, \mathbf{E}_n]$ and $\mathbf{E}^{-n}(X) = [X, \Omega^n PL/O]$. In particular, there is an Atiyah-Hirzebruch-Serre spectral sequence

$$H^i(\mathbb{C}\mathrm{P}^N; \mathbf{E}^j(SV/G)) \Rightarrow \mathbf{E}^{i+j}(\mathbb{C}\mathrm{P}^N \times SV/G).$$

Let X_n denote $\pi^{-1}_{\mathbb{C}\mathrm{P}^N}((\mathbb{C}\mathrm{P}^N)^{(n)})$, the preimage of the n-skeleton of $\mathbb{C}\mathrm{P}^N$ under the projection. Convergence means that there is a filtration

$$\cdots F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_0 = \mathbf{E}^{i+j}(\mathbb{C}\mathrm{P}^N \times SV/G)$$

where, for $n > 0$, F_n is the kernel of the restriction

$$\mathbf{E}^{i+j}(\mathbb{C}\mathrm{P}^N \times SV/G) \to \mathbf{E}^{i+j}(X_{n-1})$$

such that the E_∞-terms of the spectral sequence are subquotients of the filtration. We will only be interested in the case where $i+j=0$ in which case $E_\infty^{n,-n} = F_n/F_{n+1}$. We may assume that over a vertex x_0 of $\mathbb{C}\mathrm{P}^N$, f restricts to a diffeomorphism so $[f]$ vanishes under the restriction

$$[\mathbb{C}\mathrm{P}^N \times SV/G, PL/O] \xrightarrow{x_0^*} [SV/G, PL/O].$$

In particular, $[f] \in F_1$.

Now, if $\lambda : \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ is a scaling map, it induces a map of fiber bundles

$$\lambda : \lambda^*(\mathbb{C}\mathrm{P}^N \times SV/G) \cong \mathbb{C}\mathrm{P}^N \times SV/G \to \mathbb{C}\mathrm{P}^N \times SV/G$$

and hence a morphism of spectral sequences. For $n > 1$, this induces multiplication by t^n on $H^n(\mathbb{C}\mathrm{P}^N; \mathbf{E}^{-n}(SV/G))$ for some integer t. Since the homotopy groups of PL/O are finite, so are the groups $[SV/G, \Omega^n PL/O] = \mathbf{E}^{-n}(SV/G)$. It follows that, by choosing an appropriate λ, F_1 is in the kernel of the induced map

$$\lambda^* : \mathbf{E}^0(\mathbb{C}\mathrm{P}^N \times SV/G) \to \mathbf{E}^0(\mathbb{C}\mathrm{P}^N \times SV/G).$$

In particular, $[f] \in [\mathbb{C}\mathrm{P}^N \times SV/G, PL/O]$ vanishes after pulling back along λ. The result now follows from Proposition 2.5.2. □

Using the case for $\mathbb{C}\mathrm{P}^N$, we can give an analogous statement for bundles over M.

Proposition 2.5.4. *Suppose a G-vector bundle over a smooth manifold M given by Proposition 2.4.9 is classified by the composite*

$$M \xrightarrow{\beta} \mathbb{C}P^N \xrightarrow{E'} BSO^G(V).$$

Then, there is a scaling map λ of $\mathbb{C}P^N$ such that the G-vector bundle E classified by

$$M \xrightarrow{\beta} \mathbb{C}P^N \xrightarrow{\lambda} \mathbb{C}P^N \xrightarrow{E'} BSO^G(V)$$

gives a lens space bundle SE/G which is equivalent as a PL-block bundle to $M \times SV/G$. Moreover this equivalence is PL-isotopic to a diffeomorphism.

Proof. Proposition 2.5.3 shows that there is a positive degree self-map λ of $\mathbb{C}\mathrm{P}^N$ such that $\lambda^* E'/G$ is equivalent as a PL-block bundle to $\mathbb{C}\mathrm{P}^N \times SV/G$ and that this equivalence is PL-isotopic to a diffeomorphism. Applying Proposition 2.5.2 to $\beta^* : [\mathbb{C}\mathrm{P}^N \times SV/G, PL/O] \to [M \times SV/G, PL/O]$ gives the desired result. □

2.5.4 Proof of Theorem 2.1.6

We can now prove

Theorem 2.1.6. *Suppose $G = \mathbb{Z}/p\mathbb{Z}$ acts smoothly on a manifold X. Let M be a component of X^G whose normal bundle is $M \times V$ with V a free G-representation. Suppose M is homotopy equivalent to a finite CW-complex and admits infinitely many G-vector bundles with fiber V and vanishing Atiyah–Singer class. Then,*

1. *Infinitely many of these vector bundles may be realized as exotic normal bundles of (X, M),*

2. *The first Chern classes of these exotic normal bundles occupy infinitely many* $\mathrm{GL}_{\dim_{\mathbb{Q}} H^2(M;\mathbb{Q})}(\mathbb{Z})$*-orbits of* $H^2(M;\mathbb{Q})$*. In particular,* $\overline{TOP/O}_G(X)$ *is infinite.*

Proof. Let X and $M \subseteq X^G$ be as in the theorem. Define $\bar{X}$ to be the complement of an equivariant tubular neighborhood of M in X. Then $\bar{X}$ has a G action and $\partial\bar{X}$ is equivariantly diffeomorphic to $M \times SV$.

By Proposition 2.4.9 and Proposition 2.5.4, for infinitely many of these vector bundles E, there are isomorphisms of PL-block bundles $SE/G \to M \times SV/G$ which are PL-concordant to diffeomorphisms. Let $f : SE/G \to M \times SV/G$ denote the diffeomorphism and let $\tilde{f}$ denote its lift on G-covers. Define the smooth G-manifold $Y := \bar{X} \cup_{\tilde{f}} E$.

It remains to construct an equivariant homeomorphism $g : Y \to X$. On $\bar{X}$, we take g to be the identity so we just need to construct an equivariant homeomorphism $g : DE \to M \times DV$ where D denotes the unit disk bundle and such that g restricts to $\tilde{f}$ on the boundary. Since f is PL-concordant to an equivalence of PL-block bundles, there is a PL-isomorphism $F : SE/G\times I \to M\times SV/G\times I$ such that $F|_{SE/G\times\{0\}} = f$ and $F|_{SE/G\times\{1\}}$ is an equivalence of PL-block bundles over M. Writing $DE = SE \times I \cup DE$ and defining g to be the lift of F on $SE \times I$, we may assume instead that f is an equivalence of PL-block bundles over M. Proposition 2.4.5 shows that $\tilde{f}$ may be extended to an equivariant homeomorphism. This proves the first part.

For the second part, just note that by taking scaling maps of $\mathbb{CP}^N$, the first Chern classes are being multiplied by constants t with $|t| > 1$. □

2.6 Nontrivial Normal Bundles

So far, we have concentrated on the case where the normal bundle of M is trivial as a G-vector bundle. If this assumption is removed, the characteristic class computations become much more difficult and there is not much we are able to say. Suppose $\xi = \bigoplus_{k=1}^{\frac{p-1}{2}} \xi_k$ and $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ are G-vector bundles over M. In order for the Atiyah–Singer classes to be equal, one sees that, in cohomological degree 2,

$$\sum \tau(1)(\zeta^k)(c_1(\xi_k) - c_1(E_k)) = 0$$

so the classes $c_1(\xi_k) - c_1(E_k)$ must realize the linear relation between the $\tau(1)(\zeta^k)$. One can also derive a condition for c_2.

Proposition 2.6.1. *If ξ and E have the same Atiyah–Singer class, then*

$$c_2(\xi_k) - c_2(E_k) = \frac{1}{2}c_1(\xi_k)^2 - \frac{1}{2}c_1(E_k)^2.$$

Proof. In cohomological degree 4, we have

$$\begin{aligned} 0 &= \left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(\xi_k)\right)_4 - \left(\prod_{k=1}^{\frac{p-1}{2}} \mathcal{M}^{\zeta^k}(E_k)\right)_4 \\ &= \sum_{k=1}^{\frac{p-1}{2}} \tau(2)(\zeta^k)(c_2(\xi_k) - c_2(E_k)) + \sum_{k=1}^{\frac{p-1}{2}} \tau(1,1)(\zeta^k)(c_1(\xi_k)^2 - c_1(E_k)^2) \\ &\quad + \sum_{k_1 \neq k_2} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})(c_1(\xi_{k_1})c_1(\xi_{k_2}) - c_1(E_{k_1})c_1(E_{k_2})). \end{aligned}$$

We use the conditions on the first Chern class to simplify the expression.

$$\begin{aligned} 0 &= \left(\sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)(c_1(\xi_k) - c_1(E_k))\right)^2 \\ &= \sum_{k_1,k_2=1}^{\frac{p-1}{2}} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})(c_1(\xi_{k_1})c_1(\xi_{k_2}) - c_1(E_{k_1})c_1(\xi_{k_2}) \\ &\quad - c_1(\xi_{k_1})c_1(E_{k_2}) + c_1(E_{k_1})c_1(E_{k_2})) \end{aligned}$$

We can square

$$2\sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)c_1(\xi_k) = \sum_{k=1}^{\frac{p-1}{2}} \tau(1)(\zeta^k)(c_1(\xi_k) + c_1(E_k))$$

to obtain

$$\begin{aligned}
&4\sum_{k_1,k_2=1}^{\frac{p-1}{2}} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})c_1(\xi_{k_1})c_1(\xi_{k_2}) \\
&= \sum_{k_1,k_2=1}^{\frac{p-1}{2}} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})(c_1(\xi_{k_1})c_1(\xi_{k_2}) + c_1(\xi_{k_1})c_1(E_{k_2}) \\
&+ c_1(E_{k_1})c_1(\xi_{k_2}) + c_1(E_{k_1})c_1(E_{k_2})).
\end{aligned}$$

Now, subtracting $2\sum_{k_1,k_2=1}^{\frac{p-1}{2}} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})(c_1(\xi_{k_1})c_1(\xi_{k_2}) + c_1(E_{k_1})c_1(E_{k_2}))$ from both sides gives

$$\begin{aligned}
&2\sum_{k_1,k_2=1}^{\frac{p-1}{2}} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})(c_1(\xi_{k_1})c_1(\xi_{k_2}) - c_1(E_{k_1})c_1(E_{k_2})) \\
&= \sum_{k_1,k_2}^{\frac{p-1}{2}} \tau(1)(\zeta_{k_1})\tau(1)(\zeta_{k_2})(-c_1(\xi_{k_1})c_1(\xi_{k_2}) + c_1(\xi_{k_1})c_1(E_{k_2}) \\
&+ c_1(E_{k_1})c_1(\xi_{k_2}) - c_1(E_{k_1})c_1(E_{k_2})).
\end{aligned}$$

Our previous computation shows that this is 0.

We use this to cancel out many of the classes showing up in the Atiyah–Singer formula.

We obtain

$$0 = \sum_{k=1}^{\frac{p-1}{2}} \tau(2)(\zeta^k)(c_2(\xi_k) - c_2(E_2)) - \tau(1)(\zeta^k)^2(c_1(\xi_k)^2 - c_1(E_k)^2)$$
$$+ \tau(1,1)(\zeta^k)(c_1(\xi_k)^2 - c_1(E_k)^2)$$
$$= \sum_{k=1}^{\frac{p-1}{2}} \tau(2)(\zeta^k)(c_2(\xi_k) - c_2(E_k)) - \frac{1}{2}(c_1(\xi_k)^2 - c_1(E_k)^2).$$

By linear independence of $\{\tau(2)(\zeta^k)\}$, we obtain $c_2(\xi_k) - c_2(E_k) = \frac{1}{2}(c_1(\xi_k)^2 - c_1(E_k)^2)$. $\square$

This condition on c_2 shows that our analysis of the trivial bundle case does not naïvely extend to the nontrivial case.

Example 9. Let $M = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$. The cohomology ring of M is

$$H^*(M;\mathbb{Z}) \cong \mathbb{Z}[a,b]/(a^3 = b^3 = 0, ab = 0, a^2 = -b^2).$$

Now, let $\xi = \bigoplus_{k=1}^{\frac{p-1}{2}} \xi_k$ where each eigenbundle ξ_k is a line bundle with $c_1(\xi_k) = a$. Suppose $E = \bigoplus_{k=1}^{\frac{p-1}{2}} E_k$ has the same Atiyah–Singer class. Then each E_k must be a line bundle so Proposition 2.6.1 gives

$$c_1(E_k)^2 = c_1(\xi_k)^2 = a^2.$$

Writing $c_1(E_k) = xa + yb$, this equation becomes

$$(x^2 - y^2)a^2 = a^2.$$

Since $x^2 - y^2 = 1$ has only finitely many integer solutions, we conclude that there only finitely many cohomology classes appear as $c_1(E_k)$. Finally, note that $(a+b)^2 = 0$ so there is a nonzero element sufficiently nilpotent with respect to the representation.

Proposition 2.6.1 suggests a condition such as

$$c_m(\xi_k) - c_m(E_k) = \frac{1}{m!}c_1(\xi_k)^m - \frac{1}{m!}c_1(E_k)^m$$

is the correct way of generalizing the exponential condition. However, we have not been able to modify the proof of 2.1.2 to this more general case. In order to show that sums of exponential vector bundles give rise to trivial lens space block bundles, we also had to show that the Pontryagin classes and the Euler class vanish. It follows from Proposition 2.6.1 that $p_1(\xi_k) = p_1(E_k)$ when ξ and E have the same Atiyah–Singer class. In general, there is no reason to expect the Pontryagin classes or the Euler class of ξ and E to be equal when ξ and E have the same Atiyah–Singer class.

Our methods do give smoothings when the normal bundle has a large trivial summand. By factoring a normal bundle $M \to BSO^G(V)$ through $Y \times \mathbb{C}\mathrm{P}^N$ where Y is a product of Grassmannians, we can prove the following.

Theorem 2.6.2. *Let $G = \mathbb{Z}/p\mathbb{Z}$ where p is such that 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$. Let X be a smooth G-manifold and let M be a component of X^G.*

Suppose the normal bundle ν of M is of the form $\nu_0 \oplus \varepsilon_{V_1}$ where ε_{V_1} denotes the trivial G-vector bundle for some representation V_1. If there is a nonzero element $\beta \in H^2(M;\mathbb{Q})$ sufficiently nilpotent with respect to V_1, then $\overline{TOP/O}_G(X)$ is infinite.

Proof. Let E be a G-vector bundle over M with fiber V whose lens space bundle is trivial as a PL-block bundle constructed as in Theorem 2.1.4. Since $SO^G(V_0)$ is a product of unitary groups, there is a product of Grassmannian manifolds Y such that the classifying map for

$\nu \oplus E$ factors as follows.

$$\begin{array}{ccccc} M \longrightarrow Y \times \mathbb{C}\mathrm{P}^N & \xrightarrow{\gamma \times E'} & BSO^G(V_0) \times BSO^G(V_1) & \longrightarrow & BSO^G(V_0 \oplus V_1) \\ & & \downarrow & & \downarrow \\ & & B\widetilde{SPL}(SV_0/G) \times B\widetilde{SPL}(SV_1/G) & \longrightarrow & B\widetilde{SPL}(S(V_0 \oplus V_1)/G) \end{array}$$

The map $M \to \mathbb{C}\mathrm{P}^N$ is determined by a nonzero multiple of the cohomology class β. By abuse of notation, we will denote this map by β. The map $E' : \mathbb{C}\mathrm{P}^N \to BSO^G(V_1)$ defines a G-vector bundle whose lens space bundle is trivial as a PL-block bundle. We may assume that the ζ^k-eigenbundle E'_k of E' (corresponding to the eigenbundle decomposition with respect to some fixed generator $g_0 \in \mathbb{Z}/p\mathbb{Z}$) has nonzero first Chern class.

We first consider the bundle $\gamma \times E'$ over $M = Y \times \mathbb{C}\mathrm{P}^N$. Since the lens space bundle of E' gives a trivial PL-block bundle, we see that the lens space bundle of the G-vector bundle $\gamma \times E'$ is isomorphic as a PL-block bundle to $\gamma \times \varepsilon_{V_1}$. This gives us an element of $PL/O(S(\gamma \times \varepsilon_{V_1})/G, \eta)$ where η is the smooth structure on the lens space bundle given by considering $\gamma \times \varepsilon_{V_1}$ as a differentiable G-vector bundle. Note that $S(\gamma \times \varepsilon_{V_1})/G$ is a bundle over $\mathbb{C}\mathrm{P}^N$ with fiber the total space of $S(\gamma \times \varepsilon_{V_1}|_{Y \times \{*\}})/G$. In particular, we may apply the Atiyah-Hirzebruch-Serre spectral sequence

$$H^i(\mathbb{C}\mathrm{P}^N; \mathbf{E}^{-i}(S(\gamma \times \varepsilon_{V_1}|_{Y \times \{*\}})/G)) \Rightarrow [S(\gamma \times \varepsilon_{V_1})/G, PL/O]$$

and argue as in Proposition 2.5.3 to see that, for some scaling map λ of $\mathbb{C}\mathrm{P}^N$, there is a PL-block bundle isomorphism $\lambda^* S(\gamma \times E')/G \to S(\gamma \times \varepsilon_{V_1})/G$ which is PL-isotopic to a diffeomorphism. Moreover, $\lambda^*(\gamma \times E') = \gamma \times \lambda^* E'$ so $c_1(\lambda^*(\gamma \times E')) = c_1(\gamma) + t c_1(E')$ for some nonzero t. This equation also holds on eigenbundles so $c_1(\lambda^*(\gamma \times E')) \neq c_1((\gamma \times \varepsilon_{V_1})_k)$.

As a consequence, $\nu_0 \oplus \beta^* \lambda^* E'$ is a G-vector bundle over M such that $c_1((\nu_0 \oplus \beta^* \lambda^* E')_k) -$

$c_1((\nu_0 \oplus \varepsilon_{V_1})_k)$ is a nonzero integer multiple of β. Also, there is an isomorphism of PL-block bundles $S(\nu_0 \oplus \beta^* \lambda^* E')/G \to S(\nu_0 \oplus \varepsilon_{V_1})/G$ which is PL-isotopic to a PL-diffeomorphism. Proceeding as in the proof of Theorem 2.1.6, we see that $\nu_0 \oplus \beta^* \lambda^* E'$ is an exotic normal bundle of (X, M). Finally, taking further compositions with scaling maps $Y \times \mathbb{C}\mathrm{P}^N \to Y \times \mathbb{C}\mathrm{P}^N$ shows that there are infinitely many such bundles whose ζ^k-eigenbundles have distinct first Chern classes. □

2.6.1 Stable Smoothings

Definition 2.6.3. Let X be a G-manifold. A *stable G-smoothing* of X is a G-smoothing $\alpha : Y \to X \times \rho$ where ρ is some G-representation. Two stable G-smoothings $\alpha_i : Y_i \to X \times \rho_i$, $i = 0, 1$ are *stably isotopic* if there are representations σ_i such that $\rho_0 \oplus \sigma_0 \cong \rho_1 \oplus \sigma_1$ and the smooth G-structures $\alpha_i \times \mathrm{id}_{\sigma_i} : Y_i \times \sigma_i \to X \times \rho_i \times \sigma_i$ are G-isotopic. Let $TOP/O_G^{st}(X)$ denote the stable isotopy classes of stable G-smoothings.

Theorem 2.1.7. *Let $G = \mathbb{Z}/p\mathbb{Z}$ where p is such that 2 has odd order in $(\mathbb{Z}/p\mathbb{Z})^\times$. Let X be a smooth G-manifold. If $H^2(X^G; \mathbb{Q})$ is nonzero for some component M of X^G homotopy equivalent to a finite CW-complex, then $TOP/O_G^{st}(X)$ is infinite. In particular, if X is closed and $H^2(X^G; \mathbb{Q}) \neq 0$ then $TOP/O_G^{st}(X)$ is infinite.*

Proof. By Theorem 2.6.2, after taking the product with a sufficiently large representation ρ, the set $TOP/O_G(X \times \rho)$ is infinite. For a smoothing $\alpha : Y \to X \times \rho$, let $(\nu_{\alpha^{-1}} M)_k$ denote the ζ^k-eigenbundle of the normal bundle of $\alpha^{-1}(M)$. The construction of these smoothings yields an infinite set $\{\alpha_j : Y_j \to X \times \rho\}$ such that $c_1((\nu_{\alpha_i^{-1} M})_k) \neq c_1((\nu_{\alpha_j^{-1} M})_k)$ for some k and whenever $i \neq j$. If σ is a G-representation,

$$c_1((\nu_{\alpha_i^{-1} M})_k) = c_1((\nu_{(\alpha_i \times \mathrm{id}_\sigma)^{-1} M})_k)$$

so we see that

$$c_1((\nu_{(\alpha_i \times \mathrm{id}_\sigma)M})_k) \neq c_1((\nu_{(\alpha_j \times \mathrm{id}_\sigma)M})_k)$$

whenever $i \neq j$. Therefore, the smoothings constructed in Theorem 2.6.2 are not stably isotopic. $\square$

CHAPTER 3

THE WHITEHEAD GROUP AND STABLY TRIVIAL G-SMOOTHINGS

3.1 Introduction

A smooth structure, or a smoothing, of a manifold X can be represented by a homeomorphism $f : Y \to X$ where Y is smooth. Two such homeomorphisms f_0, f_1 determine *isotopic* smooth structures if there is a smooth structure on $Y \times I$ and a homeomorphism $F : Y \times I \to X \times I$ satisfying the following.

- F restricts to f_0 over $Y \times \{0\}$,
- F restricts to $f_1 \circ \phi$ over $Y \times \{1\}$ where $\phi : Y_1 \to Y_1$ is a diffeomorphism,
- F restricts to a homeomorphism $Y \times \{t\} \to X \times \{t\}$.

When we discuss smooth structures, we consider homeomorphisms as above under this equivalence relation. If G is a finite group (or, more generally, a compact Lie group), one defines isotopy classes of equivariant smooth structures for G-manifolds analogously.

Isotopy classes of equivariant smooth structures differ from the non-equivariant counterpart in important ways. First, if $f_i : Y_i \to X$ are isotopic smooth structures of high dimensional manifolds, then Y_0 and Y_1 are diffeomorphic. This is because every smooth structure on the product $Y_0 \times I$ is diffeomorphic to a product smooth structure. Equivariantly, this is not true [BH78].

Another important difference is the number of smooth structures. Kirby–Siebenmann showed in [KS77] that a closed manifold of dimension at least 5 has only finitely many smooth structures up to isotopy. If X is a closed G-manifold, there need not be finitely many equivariant smoothings. In [Sch79], Schultz gives S^{2n} the structure of a $\mathbb{Z}/p$-manifold and shows that, when 2 has odd order in $(\mathbb{Z}/p)^\times$ and n is sufficiently large, the resulting $\mathbb{Z}/p$-manifold has infinitely many $\mathbb{Z}/p$-smoothings. Schultz's example is generalized in Chapter 2 where infinitely many exotic $\mathbb{Z}/p$-smoothings are constructed using the second cohomology of the fixed point set. The constructions in [Sch79] and Chapter 2 are done by changing the normal bundle of the fixed point set in a way that does not change the underlying equivariant topological structure. One differentiates between infinitely many $\mathbb{Z}/p$-smoothings by examining the Chern classes of these bundles. If $Y_1 \to X$ and $Y_2 \to X$ are distinct $\mathbb{Z}/p$-smoothings given by this construction, then $Y_1 \times \rho \to X \times \rho$ and $Y_2 \times \rho \to X \times \rho$ are distinct for any $\mathbb{Z}/p$-representation ρ.

A key theorem in smoothing theory, proven by Kirby–Siebenmann, is the product structure theorem. This states that smooth structures of X are in bijection with smooth structures of $X \times \mathbb{R}$. It is shown in Chapter 2 that an equivariant version of the stabilization map in the product structure theorem is not generally surjective. Indeed, if M is a $\mathbb{Z}/p$-manifold with a trivial action, then it has only finitely many $\mathbb{Z}/p$-smoothings. But, if $H^2(M;\mathbb{Q}) \neq 0$ and 2 has odd order in $(\mathbb{Z}/p)^\times$, then $M \times \rho$ has infinitely many $\mathbb{Z}/p$-smoothings for a sufficiently large representation ρ (one may take ρ to be $(\mathbb{R}[\mathbb{Z}/p]/\mathbb{R})^{\dim M}$).

In the present chapter, we construct infinitely many equivariant smoothings $X_W \to X$ of $\mathbb{Z}/m$-manifolds X where W varies over certain elements of a Whitehead group. The X_W constructed will not be equivariantly diffeomorphic to each other but they will all be stably trivial in the sense that $X_W \times \mathbb{R}$ is equivariantly diffeomorphic to $X \times \mathbb{R}$. The constructions come from h-cobordisms of the sphere bundle of the normal bundle of a component M of the fixed set and will be done away from the normal bundle. Our main theorem is the following.

Theorem 3.1.1. *Let G be an odd order cyclic group of order at least 5 and let X be a closed, smooth semifree G-manifold. Suppose M is a component of the fixed point set which is closed, aspherical and whose fundamental group satisfies the K-theoretic Farrell–Jones conjecture. Suppose either of the following hold.*

1. *M (and, hence X) is odd dimensional.*

2. *M is even dimensional,* $H^2(M;\mathbb{Q}) \neq 0$ *and there are distinct prime factors* p_i, p_j *of* $|G|$ *such that* p_i *has odd order in* $(\mathbb{Z}/p_j)^\times$.

Then there are infinitely many elements of $\overline{TOP/O}_G(X)$ *which vanish under the stabilization map* $\overline{TOP/O}_G(X) \to \overline{TOP/O}_G(X \times \mathbb{R})$.

The K-theoretic Farrell–Jones conjecture for M allows us to understand parts of the $\mathrm{Wh}(\pi_1 M \times G)$ by considering the homology of M with coefficients in the lower K-theory of $\mathbb{Z}[G]$. The G-smoothings in the first case of Theorem 3.1.1 come from $H_0(M; \mathrm{Wh}(G))$ whereas the G-smoothings in the second case come from $H_2(M; K_{-1}(\mathbb{Z}[G]))$.

Remark. In Theorem 3.1.1, we require X be closed so that there are only finitely many components of the fixed set homeomorphic to M.

Remark. Both the smoothings constructed in Theorem 3.1.1 and those constructed in [Sch79] and [Wan23] involve both the second cohomology of the fixed point set and the order of elements in $(\mathbb{Z}/p)^\times$. We believe this is coincidental though it would be very interesting if there were some deeper number theoretic or homotopy theoretic reason.

3.1.1 Outline

In Section 3.2, we describe the construction giving rise to the G-smoothings in Theorem 3.1.1. In Section 3.3, we reduce the proof of Theorem 3.1.1 to an analysis of the involution on $K_{-1}(\mathbb{Z}[G])$. In the appendix, we elaborate on Madsen–Rothenberg's analysis of the involution on $K_{-1}(\mathbb{Z}[G])$. The results of this section are qualitative in the sense that we do not compute the dimension of the eigenspaces of the involution though we expect a computation to be feasible.

3.2 The Construction of Smoothings

3.2.1 Whitehead Torsion

Recall that, for a ring R, $K_1(R) := \mathrm{GL}(R)_{ab}$ and that, for a group G, the Whitehead group is $\mathrm{Wh}(G) := K_1(\mathbb{Z}[G])/\langle \pm g\rangle$. There is an involution τ_1 on $K_1(R[G])$ defined by sending a matrix M to the inverse of its conjugate transpose. This induces an involution on $\mathrm{Wh}(G)$ which we also denote by τ_1.

Remark. The involution τ_1 is the negative of the involution considered in [Mil66]. We will let τ_1 be our "standard" involution as it behaves better with the involution on $K_0(R[G])$ defined by dualization.

Let M_0 be a closed, connected n-dimensional CAT-manifold where CAT is the category TOP, PL or $DIFF$. A cobordism over M_0 consists of a tuple $(W; M_0, f_0, M_1, f_1)$ where W is an $(n+1)$-manifold with $\partial W = \partial_0 W \coprod \partial_1 W$ and the maps $f_0 : M_0 \to \partial_0 W$ and $f_1 : -M_1 \to \partial_1 W$ are CAT-isomorphisms. Here, $-M_1$ denotes M_1 with a reversed orientation. An h-cobordism is a cobordism such that f_0 and f_1 are homotopy equivalences. We will usually write $(W; M_0, M_1)$ instead and suppress the f_i from the notation. Two h-cobordisms $(W; M_0, M_1)$ and $(W'; M_0, M_2)$ over M_0 are isomorphic if there is a CAT isomorphism $F : W_0 \to W_1$ of manifolds with boundary such that $F \circ f_0 = f_1$. When $n \geq 5$, there is a bijection between isomorphism classes of h-cobordisms over M_0 and the Whitehead group given by Whitehead torsion $(W; M_0, M_1) \mapsto \tau(W, M_0)$.

The following formula can be found in [Mil66, Section 10].

$$\tau(W, M_0) = (-1)^{n+1}\tau_1 \cdot \tau(W, M_1)$$

We will be interested in h-cobordisms where $M_0 \cong M_1$, which are called *inertial*. A slightly more convenient class of h-cobordisms are the *strongly inertial* h-cobordisms. These are the inertial h-cobordisms such that the map $M_0 \to M_1$ is homotopic to a homeomorphism. The set of strongly inertial cobordisms forms a subgroup and it is a homotopy invariant of M. Neither of these are necessarily true for inertial cobordisms. It turns out that strongly inertial cobordisms are a finite index subgroup of the invariant subgroup $\mathrm{Wh}(\pi_1 M)^{(-1)^{n+1}}$. This holds for any choice of CAT [JK18, Proposition 5.2]. We refer to [JK18] for more details on inertial and strongly inertial h-cobordisms.

3.2.2 Controlled h-Cobordisms

We will be interested in h-cobordisms over lens space bundles over a manifold M. Hence our notation here will differ from our notation above.

Definition 3.2.1. Let (M, d) be a metric space and let $\varepsilon > 0$. Suppose $p : E \to M$ and $p' : E' \to M$ are functions.

1. A function $f : E \to E'$ is *ε-controlled* if, for all $x \in E$, $d(p(x), p' \circ f(x)) < \varepsilon$.

2. A homotopy $H : E \times I \to E'$ is *ε-controlled* if, for all $x \in E$, the set $p' \circ H(x, I)$ has diameter less than ε.

Definition 3.2.2. Let $p : E \to M$ be a (not necessarily continuous) function where M is a compact metric ANR. Let $(W; E, E')$ be an h-cobordism and let $\varphi : W \to M$ be a map. We say that $\varphi : (W; E, E') \to M$ is a *controlled h-cobordism with respect to p* if, for all $\varepsilon > 0$, there is a deformation retraction of W to E which is ε-controlled.

Two controlled h-cobordisms $\varphi_i : (W_i; E_i, E_i') \to M$, $i = 0, 1$, are *controlled isomorphic* if, for all $\varepsilon > 0$, there is an isomorphism of h-cobordisms $\Phi : W_0 \to W_1$ which is ε-controlled over M.

Proposition 3.2.3. *Suppose $\xi \to M$ is a G-vector bundle whose fibers are G-representations whose only fixed point is the origin. Let $\tilde{E}$ denote the sphere bundle of ξ and let $p : E \to$*

M denote the lens space bundle obtained by quotienting. Let $(W; E, E)$ be a controlled h-cobordism with respect to p and let $\tilde{W}$ denote the G-cover. Then there is an equivariant homeomorphism $\tilde{W} \cup_{\tilde{E}} D\xi \to D\xi$ where $D\xi$ denotes the disk bundle. If $f : \tilde{E} \to \tilde{E}$ is a controlled equivariant homeomorphism, then the homeomorphism can be assume to restrict to f on the boundary.

Proof. Let ε_n be a sequence such that $\sum \varepsilon_n < \infty$. Write $(W_0; E_0, E_1) := (W; E, E)$ and let $(W_1; E_1, E_2)$ denote a controlled h-cobordism such that $(W_0 \cup W_1; E_0, E_2)$ is controlled isomorphic to $(E \times I; E, E)$. Let $F_1 : W_0 \cup W_1 \to E \times I$ be an ε_1-controlled isomorphism and let f_1 denote the restriction of F_1 on E_2. Inductively, define

- $(W_n; E_n, E_{n+1})$ to be a controlled h-cobordism such that

  $$(W_{n-1} \cup_{f_{n-1}} W_n; E_{n-1}, E_{n+1}) \cong (E \times I; E, E)$$

  as controlled h-cobordisms,

- $F_n : (W_{n-1} \cup_{f_{n-1}} W_n; E_{n-1}, E_{n+1}) \to (E \times I; E, E)$ to be a an ε_n-controlled isomorphism and

- f_n to be the restriction of F_n on E_{n+1}.

All E_n are of course diffeomorphic to E.

Define

$$Y := W_0 \cup W_1 \cup_{f_1} W_2 \cup_{f_2} W_3 \cup \cdots .$$

Clearly, Y is homotopy equivalent to E so we may take a G-cover $\tilde{Y}$. Define $p_Y : Y \to M$ as follows. For $x \in W_n \setminus E_{n+1}$, let $p_Y(x)$ be the composition $W_n \to E_n \xrightarrow{p} M$ where the first map comes from an ε_n-deformation retraction. Note that p_Y is not, in general, continuous.

Topologize $\tilde{Y} \cup M$ by declaring that a sequence of points $x_n \in W_n$ converges to $m \in M$ if $p_Y(x_n)$ converges to m. Let $F : Y \to E \times [0, \infty)$ be defined to be F_{2n+1} on $W_{2n} \cup_{f_{2n}} W_{2n+1}$

and let $G : Y \to W \cup_E E \times [0,\infty)$ be defined to be the identity $W_0 \to W$ and F_{2n} on $W_{2n-1} \cup_{f_{2n-1}} W_{2n}$. Then $\tilde{F}$ and $\tilde{G}$ are equivariant homeomorphisms

$$\tilde{W} \cup_{\tilde{E}} \tilde{E} \times [0,\infty) \xleftarrow{\tilde{G}} \tilde{Y} \xrightarrow{\tilde{F}} \tilde{E} \times [0,\infty)$$

which extends to equivariant homeomorphisms

$$\tilde{W} \cup_{\tilde{E}} D\xi \leftarrow \tilde{Y} \cup M \to D\xi.$$

□

3.2.3 The Construction of X_W

Suppose X is a smooth, semifree G-manifold and let M be a component of X^G. Let ν denote the normal bundle of M, let $S\nu$ denote the sphere bundle, let $D\nu$ denote the disk bundle and let $\mathring{D}\nu$ denote the interior of $D\nu$. Then $S\nu$ has a free G-action and $E := S\nu/G$ is a lens space bundle over M. Define $X' := X \setminus \mathring{D}\nu$.

Let $(W; E, E)$ be a smooth inertial h-cobordism controlled over M and let $\tilde{W}$ be the G-cover. Define

$$X_W := X' \cup \tilde{W} \cup D\nu.$$

By Proposition 3.2.3, there is an equivariant homeomorphism $X_W \to X$. Moreover, if $(W'; E, E)$ is another h-cobordism as above such that $\varphi\tau(W', E) \neq \tau(W, E)$ for any automorphism φ of $\mathrm{Wh}(\pi_1 E)$, then there is no equivariant diffeomorphism $X_W \cong X_{W'}$ sending M to M. Indeed, if there were an equivariant diffeomorphism, then there would be a diffeomorphism $W \cong W'$ which is a contradiction.

Remark. An important detail used here is that equivariant diffeomorphisms respect the normal bundle of M, unlike equivariant homeomorphisms.

As a consequence of the above discussion, we have the following.

Proposition 3.2.4. *Let X be a closed smooth G-manifold with E and M as above. Suppose* $\mathrm{Wh}(\pi_1 E)$ *is finitely generated and that some nonzero free abelian subgroup can be represented by inertial h-cobordisms over E controlled over M. Then the constructions X_W above give infinitely many smooth G-manifolds which are equivariantly homeomorphic but not equivariantly diffeomorphic.*

We also record the following.

Proposition 3.2.5. *There is are equivariant diffeomorphisms $X_W \times S^1 \to X \times S^1$ and $X_W \times \mathbb{R} \to X \times \mathbb{R}$ where S^1 and $\mathbb{R}$ are given trivial actions.*

Proof. Let $(W; E_0, E_1)$ be an h-cobordism. Since the Euler characteristic of S^1 vanishes, there is an isomorphism

$$F : W \times S^1 \xrightarrow{\cong} E_0 \times I \times S^1.$$

Taking the $\mathbb{Z}$-cover shows that $W \times \mathbb{R} \cong E_0 \times I \times \mathbb{R}$ as h-cobordisms over E_0. The proposition follows from the construction of X_W. □

3.3 Control and Assembly

In this section, we use the assembly map and a result of Quinn to find inertial h-cobordisms over E controlled over M.

3.3.1 Controlled h-Cobordisms and Homology

Let $p : E \to M$ be a bundle with connected fiber F and suppose M is connected. Denote $\pi := \pi_1 M$. Following [FLS18], define a functor $\underline{E} : Or(\pi) \to Top$ by sending each orbit π/H to the pullback bundle over the cover of M corresponding to H. Let $\mathbf{E} : Top \to Sp$ be a functor from spaces to spectra. Define $\mathbf{E}(p)$ to be the composite $\mathbf{E}\circ\underline{E}$. For a π-CW-complex X, we may define the Davis–Lück equivariant homology groups $H_*^\pi(X;\mathbf{E}(p))$.

We are primarily interested in the case $\mathbf{E}$ is the Whitehead spectrum $\mathbf{Wh}$. Recall that this is defined as follows. For a space X, let $\mathbf{A}(X)$ denote the nonconnective A-theory spectrum of X. Then $\mathbf{Wh}(X)$ is defined to be the cofiber of the assembly $X_+ \wedge \mathbf{A}(*) \to \mathbf{A}(X)$.

One may alternatively define a Whitehead spectrum using algebraic K-theory instead. Let $\mathbf{Wh}_{\mathbf{K}}(X)$ be the cofiber of the assembly $B\pi_1X_+ \wedge \mathbf{K}(\mathbb{Z}) \to \mathbf{K}(\mathbb{Z}[\pi_1 X])$ where $\mathbf{K}$ denotes the nonconnective algebraic K-theory spectrum. The linearization map $\mathbf{A}(X) \to \mathbf{K}(\pi_1 X)$ is a map of spectra with involution [Vog85, Proposition 2.11] and it induces isomorphisms of groups with involution

$$\pi_n\mathbf{Wh}(X) \to \pi_n\mathbf{Wh}_{\mathbf{K}}(X)$$

for $n \leq 1$.

In [Qui82], Quinn defines homology with coefficients in a spectrum valued functor $\mathbf{E} : Top \to Sp$. Let $\mathbb{H}(M;\mathbf{E})$ denote this homology spectrum and let $\mathbb{H}_k(M;\mathbf{E})$ denote the homotopy groups. He shows that a particular homology group $\mathbb{H}_1(M;\mathcal{S}(p))$ is in bijection with h-cobordisms $(W;E,E')$ controlled over M where $p : E \to M$. Farrell–Lück–Steimle compare Quinn's homology group with the Davis–Lück equivariant homology theory.

Proposition 3.3.1. *Suppose M is an aspherical manifold and E is a closed manifold. Let $\tilde{M}$ be the universal cover of M and let $\pi = \pi_1 M$. Let $p : E \to M$ be a bundle with connected fiber F and let $\varphi : (W;E,E') \to M$ be a controlled h-cobordism. There is an invariant $q(\varphi,p) \in H_1^\pi(\tilde{M};\mathbf{Wh}(p))$ such that the following hold.*

1. *Two controlled h-cobordisms are controlled isomorphic if and only if their invariants are equal.*

2. *When $\dim E \geq 5$, all invariants in this group can be realized.*

Proof. This follows from [Qui82, 1.2] and the identification of Quinn's homology group with

$H_1^\pi(\tilde{M}; \mathbf{Wh}(p))$ in [FLS18, Lemma 4.9]. $\square$

3.3.2 Assembly

Quinn also defined an assembly map $\mathbb{H}_1(M; \mathcal{S}(p)) \to \mathrm{Wh}(\pi_1 E)$ which can be compared to the Farrell–Jones assembly in the Davis–Lück formulation. Geometrically, Quinn's assembly sends a controlled h-cobordism $(W; E, E')$ to the torsion $\tau(W, E)$ where we consider $(W; E, E')$ as an "uncontrolled" h-cobordism. Farrell–Lück–Steimle show that, when M is aspherical, the Quinn assembly map has the same image as the Davis–Lück assembly map [FLS18, Lemma 4.9.iii]. Finally, they show that the Davis–Lück assembly map

$$H_1^\pi(\tilde{M}; \mathbf{Wh}(p)) \to H_1^\pi(pt; \mathbf{Wh}(p)) = \pi_1(\mathbf{Wh}(E))$$

is split injective provided M is aspherical, $p : E \to M$ is π_1-surjective and π satisfies the K-theoretic Farrell–Jones conjecture.

3.3.3 Some Additional Simplifications

Returning to our geometric situation, we have a closed aspherical n-manifold M whose fundamental group π satisfies the K-theoretic Farrell–Jones conjecture. Moreover, the map $p : E \to M$ is a lens space bundle with fiber F. The only orbits involved in the construction of the Davis–L uck homology spectrum is the orbit G/pt. Since $\mathbf{Wh}(p)(G/pt) = \mathbf{Wh}(F)$, there is an isomorphism $H_1^\pi(\tilde{M}; \mathbf{Wh}(p)) \cong H_1(M; \mathbf{Wh}(F))$ where the right hand side should be thought of as a generalized twisted homology group.

We may simplify this further. Let G denote $\pi_1 F$. Since M is aspherical, there is an extension

$$G \to \pi_1 E \to \pi.$$

Let $\tilde{E}$ denote the G-cover of E. The covering map $\tilde{E} \to E$ exhibits G as a quotient $\pi_1 E/\pi$.

Therefore, $\pi_1 E \cong G \times \pi$. In particular, the action of π on the fundamental group $\pi_1 F$ is trivial. Linearization gives an isomorphism

$$H_1(M; \mathbf{Wh}(F)) \to H_1(M; \mathbf{Wh}_{\mathbf{K}}(F))$$

of twisted generalized homology groups. But since the action of π on $\mathbf{Wh}_{\mathbf{K}}(F)$ is determined entirely by its action on $\pi_1 F$, the homology group on the right hand side is untwisted.

The following proposition follows from Proposition 3.2.4, Proposition 3.3.1 and the above discussion.

Proposition 3.3.2. *If $H_1(M; \mathbf{Wh}_{\mathbf{K}}(F))^{(-1)^{n+1}\tau_1}$ has nonzero rank then there are infinitely many distinct G-smoothings of X. Here, the homology group is untwisted.*

3.3.4 Involutions on $H_1(M; \mathbf{Wh}_{\mathbf{K}}(F))$

We now reduce the study of the involution τ_1 on $H_1(M; \mathbf{Wh}_{\mathbf{K}}(F))$ to the study of the involution on $K_{-1}(\mathbb{Z}[G])$.

Proposition 3.3.3. *Suppose X is a CW complex. Then*

$$H_1(X; \mathbf{Wh}_{\mathbf{K}}(F))_{(0)} \cong H_0(X; \mathrm{Wh}(G))_{(0)} \oplus H_2(X; K_{-1}(\mathbb{Z}[G]))_{(0)}.$$

Proof. Since we are only interested in the first homology group, the Atiyah-Hirzebruch spectral sequence is easy to analyze. Its E^2-page is

$$\begin{array}{ccc} H_0(X; \mathrm{Wh}(G)) & H_1(X; \mathrm{Wh}(G)) & H_2(X; \mathrm{Wh}(G)) \\ \\ H_0(X; \tilde{K}_0(\mathbb{Z}[G])) & H_1(X; \tilde{K}_0(\mathbb{Z}[G])) & H_2(X; \tilde{K}_0(\mathbb{Z}[G])) \\ \\ H_0(X; K_{-1}(\mathbb{Z}[G])) & H_1(X; K_{-1}(\mathbb{Z}[G])) & H_2(X; K_{-1}(\mathbb{Z}[G])) \end{array}$$

but the left column splits off, $\tilde{K}_0(\mathbb{Z}[G])$ is finite and Carter's vanishing theorem implies that there are no lower rows. Therefore, $E^{\infty}_{0,1} = E^2_{0,1} \cong \mathrm{Wh}(G)$, $E^{\infty}_{1,0}$ is a finite group and $E^{\infty}_{2,-1} = E^2_{2,-1} \cong H_2(X; K_{-1}(\mathbb{Z}[G]))$. $\square$

We would like to endow the right hand side of the expression in Proposition 3.3.3 with an involution such that the decomposition of $H_1(X; \mathbf{Wh_K}(F))_{(0)}$ above respects the involution. On $H_0(X; \mathrm{Wh}(G))$, the involution is just given by τ_1 on $\mathrm{Wh}(G)$. The map $H_0(X; \mathrm{Wh}(G)) \to H_1(X; \mathbf{Wh_K}(F))$ respects the involution since it is induced by the inclusion of a point.

We show there is an involution on $H_2(X; K_{-1}(\mathbb{Z}[G]))$ and a quotient map

$$H_1(X; \mathbf{Wh_K}(F)) \to H_2(X; K_{-1}(\mathbb{Z}[G]))$$

respecting the involution. We do this by considering the filtration of the left hand side. Recall that Atiyah–Hirzebruch spectral sequence is given by a filtration arising from skeleta of X. If $X^{(i)}$ denotes the i-skeleton, then the filtration on $H_1(X; \mathbf{Wh_K}(F))$ is given by

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq H_1(M; \mathbf{Wh_K}(F))$$

where $F_i = \mathrm{im}(H_1(X^{(i)}; \mathbf{Wh_K}(F)) \to H_1(X; \mathbf{Wh_K}(F)))$ and $E^{\infty}_{i,1-i} = F_i/F_{i-1}$. In particular, $F_i/F_{i-1} = 0$ for $i \geq 3$. This implies $F_2 = F_3 = \cdots = H_1(X; \mathbf{Wh_K}(F))$. So

$$H_2(X; K_{-1}(\mathbb{Z}[G])) \cong H_1(X; \mathbf{Wh_K}(F))/H_1(X^{(1)}; \mathbf{Wh_K}(F)). \tag{3.1}$$

The following proposition becomes immediate.

Proposition 3.3.4. *If $X \to Y$ is a map of CW complexes then there is a commuting diagram* *of abelian groups with involution*

$$\begin{array}{ccccc} H_0(X;\mathrm{Wh}(G)) & \longrightarrow & H_1(X;\mathbf{Wh_K}(F)) & \longrightarrow & H_2(X;K_{-1}(\mathbb{Z}[G])) \\ \downarrow & & \downarrow & & \downarrow \\ H_0(Y;\mathrm{Wh}(G)) & \longrightarrow & H_1(Y;\mathbf{Wh_K}(F)) & \longrightarrow & H_2(Y;K_{-1}(\mathbb{Z}[G])) \end{array}$$

where the left horizontal maps are injective, the right horizontal maps are surjective, the horizontal composites are trivial and the rows are exact after rationalizing.

Note that the involution on $H_0(X;\mathrm{Wh}(G))$ is given by its identification with the homology group $H_1(\pi_0 X;\mathbf{Wh_K}(F))$. So, understanding the involution on this homology group amounts to understanding the involution on the spectrum $\mathbf{Wh_K}(F)$. The involution on the group $H_2(X;K_{-1}(\mathbb{Z}[G]))$ is defined by the identification (3.1) above. To compute the involution, we reduce to the case where X is a surface by noting that every element of $H_2(X;\mathbb{Z})$ is of the form $f_*[\Sigma_g]$ where $f:\Sigma_g \to M$ is a map from a closed oriented surface. Moreover, every closed oriented surface admits a map to T^2 which is an isomorphism on H_2. By considering these maps, Proposition 3.3.4 gives the following result.

Proposition 3.3.5. *Suppose $H_2(X;\mathbb{Z})$ is a finitely generated group of rank r. There is a map of abelian groups with involution*

$$H_2(T^2;K_{-1}(\mathbb{Z}[G]))^r \to H_2(X;K_{-1}(\mathbb{Z}[G]))$$

which is an isomorphism when restricted to the torsion free part.

Remark. In the statement of Proposition 3.3.5, we are implicitly using that $K_{-1}(\mathbb{Z}[G])$ is finitely generated for a finite group G [Car80b].

We have now reduced the computation of the involution on $H_2(M;K_{-1}(\mathbb{Z}[G]))$ to the computation of the involution on $H_2(T^2;K_{-1}(\mathbb{Z}[G]))$ but this is just the involution on

$K_{-1}(\mathbb{Z}[G])$.

We may now prove the following.

Theorem 3.3.6. *Suppose G is a finite cyclic group of order at least 5. The involution on $H_1(X; \mathbf{Wh_K}(F))_{(0)}$ has a -1-eigenspace. It has a 1-eigenspace if and only if $H_2(X; \mathbb{Q}) \neq 0$ and there are distinct prime factors p_i and p_j of $|G|$ such that p_i has odd order in $(\mathbb{Z}/p_j)^\times$.*

Proof. By our assumption on the order of G, the Whitehead group is infinite. By [Bak77], the involution on $\mathrm{Wh}(G)$ is multiplication by -1. So $H_0(X; \mathrm{Wh}(G))_{(0)}$ is nontrivial and the involution is multiplication by -1.

The statement on 1-eigenspaces follows from Proposition 3.3.5 and Corollary 3.4.11. □

3.4 The Involution on $K_{-1}(\mathbb{Z}[G])$

3.4.1 Involutions on Spectra

It is well-known that there are involutions on the K-theory spectra of group rings (and more generally of rings with involution). Let $K(R[G])$ denote the connective K-theory spectrum of the group ring $R[G]$. By regarding this as a space via Quillen's +-construction, an involution is defined the involution $\mathrm{GL}(R[G]) \to \mathrm{GL}(R[G])$ defined by sending a matrix to the inverse of its conjugate transpose. Alternatively, one can also consider $K(R[G])$ as the K-theory of the symmetric monoidal category of finitely generated free R-modules. Then, an involution is induced by the contravariant functor sending a module to its dual.

Remark. These define the same involution on connective K-theory but, on $K_1(R[G])$, it is the negative of the involution considered in [Mil66].

These involutions extend to involutions on non-connective K-theory spectra in the following sense. Let $\mathbf{K}(R[G])$ denote the non-connective K-theory spectrum. Then there is an involution on $\mathbf{K}(R[G])$ such that $K(R[G]) \to \mathbf{K}(R[G])$ is a map of spectra with involution.

To be more explicit, one may consider, for instance, the Pedersen–Weibel model for $\mathbf{K}(R[G])$ [PW85]. They consider additive categories $\mathcal{C}_{\mathbb{R}^n}(R[G])$ of finitely generated free $R[G]$-modules locally finitely indexed by points in $\mathbb{R}^n$. Then, $\mathbf{K}(R[G])$ is defined to be an Ω-spectrum with n-th space $K(\mathcal{C}_{\mathbb{R}^n}(R[G]))$. One can define a contravariant functor on $\mathcal{C}_{\mathbb{R}^n}(R[G])$ which dualizes each module and preserves the coordinate in $\mathbb{R}^n$. This makes $\mathbf{K}(R[G])$ into a spectrum with involution in the sense that it is an Ω-spectrum whose spaces have involution and whose structure maps respect the involution.

3.4.2 Dual Representations, K_0 and K_1

If $x = \sum a_i g_i \in R[G]$, let $\overline{x} := \sum a_i g_i^{-1}$.

Definition 3.4.1. Let P be a finitely generated projective $R[G]$-module. Define the dual to be $P^* := \mathrm{Hom}_{R[G]}(P, R[G])$ where, for $g \in G$, $x \in P$ and $f \in P^*$,

$$(g \cdot f)(x) = f(x) \cdot g^{-1}.$$

Define $\tau_0 : K_0(R[G]) \to K_0(R[G])$ by $[P] \mapsto [P^*]$.

Let $A = (a_{ij})$ be a matrix with coefficients in $R[G]$. Define $A^* := (\overline{a_{ji}})$ and $\tau_1 : K_1(R[G]) \to K_1(R[G])$ by $[A] \mapsto -[A^*]$.

We note that P^* is isomorphic as an $R[G]$-module to $\mathrm{Hom}_R(P, R)$ with the action defined by $(g \cdot \varphi)(x) = \varphi(g^{-1} \cdot x)$ for $\varphi \in \mathrm{Hom}_R(P, R)$. Indeed, if $f(x) = \sum_{g \in G} a_{g,x} g$, the map $\psi : P^* \to \mathrm{Hom}_R(P, R)$ sending f to $\psi(f)(x) = a_{1,x}$ defines an isomorphism.

Proposition 3.4.2. *Let $\Phi : K_0(R[G]) \to K_1(R[G \times \mathbb{Z}])$ be the homomorphism sending $[P]$ to $[te + (1 - e)]$ where t is a generator of $\mathbb{Z}$ and $e : R[G]^n \to R[G]^n$ is an idempotent matrix* *corresponding to the projective module P. The following diagram is commutative.*

$$\begin{array}{ccc} K_0(R[G]) & \xrightarrow{\Phi} & K_1(R[G \times \mathbb{Z}]) \\ \tau_0 \downarrow & & \tau_1 \downarrow \\ K_0(R[G]) & \xrightarrow{\Phi} & K_1(R[G \times \mathbb{Z}]) \end{array}$$

Proof. The idempotent corresponding to P^* is e^* so

$$\Phi \circ \tau_0([P]) = \Phi([P^*]) = [te^* + (1 - e^*)].$$

On the other hand,

$$\tau_1 \circ \Phi([P]) = -[t^{-1}e^* + (1 - e^*)]$$

so $\Phi \circ \tau_0([P]) = \tau_1 \circ \Phi([P])$. □

3.4.3 K_{-1} and Localization Sequences

In order to compute negative K-groups of group rings, localization sequences are very useful. These sequences are obtained from a homotopy cartesian diagram of nonconnective K-theory spectra (see, for instance, [Wei13, V.7]). In our case, the maps of spectra are induced by maps of coefficient rings of group rings. So, the maps in the sequences below will respect the involution.

Carter's Sequence

Definition 3.4.3. Let S be a central multiplicative subset of a ring A. Define the category $\mathbf{H}_S(A)$ to be the S-torsion A modules M which have a finite length resolution of finitely generated projective A-modules.

Let $S \subseteq \mathbb{Z}$ be a multiplicative subset generated by a set of primes and let $\langle p \rangle$ denote the multiplicative subset generated by p. There is an equivalence of categories

$$\mathbf{H}_S(\mathbb{Z}[G]) \simeq \prod_{p \in S} \mathbf{H}_{\langle p \rangle}(\mathbb{Z}_p[G])$$

when G is noetherian group. This equivalence is given by sending an S-torsion $\mathbb{Z}[G]$-module to its p-primary parts.

Recall that, for a ring A, $K_{-1}(A)$ is defined to be the cokernel of $K_0(A[t]) \oplus K_0(A[t^{-1}]) \to K_0(A[t, t^{-1}])$. Moreover, the map $K_0(A[t, t^{-1}]) \to K_{-1}(A)$ naturally splits so we may regard $K_{-1}(A)$ as a subgroup of $K_0(A[t, t^{-1}])$. Carter [Car80a] provides a resolution of free abelian groups computing $K_{-1}(\mathbb{Z}[G])$ when G is finite of order n.

$$0 \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q}[G]) \oplus \bigoplus_{p|n} K_0(\mathbb{Z}_p[G]) \to \bigoplus_{p|n} K_0(\mathbb{Q}_p[G]) \xrightarrow{\partial} K_{-1}(\mathbb{Z}[G]) \to 0$$

The map $K_0(\mathbb{Q}_p[G]) \to K_{-1}(\mathbb{Z}[G])$ is defined using a connecting homomorphism

$$\partial : K_1(\mathbb{Q}_p[G \times \mathbb{Z}]) \to K_0(\mathbb{Z}[G \times \mathbb{Z}]).$$

This connecting homomorphism ∂ is defined to be a composite

$$K_1(\mathbb{Q}_p[G \times \mathbb{Z}]) \to K_0\mathbf{H}_{\langle p \rangle}(\mathbb{Z}_p[G \times \mathbb{Z}]) \to K_0\mathbf{H}_{\langle p \rangle}(\mathbb{Z}[G \times \mathbb{Z}]) \to K_0(\mathbb{Z}[G]).$$

Suppose $A \in \mathrm{GL}_n(\mathbb{Q}_p[G \times \mathbb{Z}])$ is a matrix representing an element of $K_1(\mathbb{Q}_p[G \times \mathbb{Z}])$. There is an $r \geq 0$ such that $p^r A$ has coefficients in $\mathbb{Z}_p[G \times \mathbb{Z}]$. The first map sends A to $[\mathrm{coker}(p^r A)] - [\mathrm{coker}(p^r I_n)]$. The second map sends a p-primary group regarded as a module over $\mathbb{Z}_p[G \times \mathbb{Z}]$ to the same group regarded as a module over $\mathbb{Z}[G \times \mathbb{Z}]$. The third map sends an S-torsion module with a finite length resolution to the Euler characteristic of the resolution.

Note that

$$\mathbb{Z}_p[G \times \mathbb{Z}]^n \xrightarrow{p^r A} \mathbb{Z}_p[G \times \mathbb{Z}]^n \to \operatorname{coker}(p^r A)$$

is a projective resolution of $\mathbb{Z}_p[G \times \mathbb{Z}]$-modules. The argument in the proof of [Car80a, Lemma 2.3] shows there is a projective resolution of $\mathbb{Z}[G \times \mathbb{Z}]$-modules

$$F \to \mathbb{Z}[G \times \mathbb{Z}]^m \to \operatorname{coker}(p^r A).$$

One can similarly describe the $\operatorname{coker}(p^r I_n)$ term and conclude that

$$\partial[A] = [\mathbb{Z}[G \times \mathbb{Z}]^m] - [F].$$

One can give $K_{-1}(\mathbb{Z}[G])$ and involution by restricting the involution on $K_0(\mathbb{Z}[G \times \mathbb{Z}])$. The following result shows that the Carter sequence respects this involution.

Proposition 3.4.4. *The following diagrams commute.*

$$\begin{array}{ccc} K_1(\mathbb{Q}_p[G \times \mathbb{Z}]) & \xrightarrow{\partial} & K_0(\mathbb{Z}[G \times \mathbb{Z}]) \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle \tau_0} \\ K_1(\mathbb{Q}_p[G \times \mathbb{Z}]) & \xrightarrow{\partial} & K_0(\mathbb{Z}[G \times \mathbb{Z}]) \end{array} \qquad \begin{array}{ccc} K_0(\mathbb{Q}_p[G]) & \xrightarrow{\partial} & K_{-1}(\mathbb{Z}[G]) \\ {\scriptstyle \tau_0}\downarrow & & \downarrow{\scriptstyle \tau_{-1}} \\ K_0(\mathbb{Q}_p[G]) & \xrightarrow{\partial} & K_{-1}(\mathbb{Z}[G]) \end{array}$$

Proof. The second diagram follows from the first and Proposition 3.4.2.

We show that the first diagram commutes. Let $[A] \in K_1(\mathbb{Q}_p[G \times \mathbb{Z}])$ and define $M := \operatorname{coker}(p^r A)$ Let

$$0 \to F \to \mathbb{Z}[G \times \mathbb{Z}]^m \to M \to 0 \tag{3.2}$$

be as above. It follows immediately that

$$\tau_0 \circ \partial[A] = [\mathbb{Z}[G \times \mathbb{Z}]^m] - [F^*].$$

Instead of evaluating $\partial \circ \tau_1[A]$, it will be slightly easier to evaluate $\partial \circ (-\tau_1)[A]$. There is an exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \to \mathbb{Z}_p[G \times \mathbb{Z}]^n \xrightarrow{A^*} \mathbb{Z}_p[G \times \mathbb{Z}]^n \to \mathrm{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \to 0.$$

The term $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ vanishes since M is torsion. So to compute $\partial \circ (-\tau_1)[A]$ we need a projective $\mathbb{Z}[G \times \mathbb{Z}]$-resolution of $\mathrm{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$.

Dualizing (3.2) above gives a projective $\mathbb{Z}[G \times \mathbb{Z}]$-resolution

$$0 \to \mathbb{Z}[G \times \mathbb{Z}]^m \to F^* \to \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \to 0$$

Since $\mathrm{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}_p)$ it suffices to show that $\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}_p) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z})$. This isomorphism follows by considering the injective resolutions

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$
$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to 0$$

and recalling that M is p-primary. □

The Madsen-Rothenberg Sequence

In [MR88], Madsen and Rothenberg regard the functor $K(R[-])$ as a Mackey functor. It follows that $K_n(R[G])$ has an action of the Burnside ring $A(G)$. Let $q(G, 0) \subseteq A(G)$ denote the ideal generated by the virtual finite G-sets whose G-fixed point set has order 0. If $\mathcal{M}$ is a Mackey functor, then localization at this ideal can be described as follows.

$$\mathcal{M}(G/G)_{q(G,0)} = \ker(\mathcal{M}(G/G)_{(0)} \to \bigoplus_{(H)} \mathcal{M}(G/H)_{(0)}) \tag{3.3}$$

Here, the H on the right hand side varies over conjugacy classes of proper subgroups of G. Heuristically, this localization is isolating the part of $\mathcal{M}(G/G)_{(0)}$ which does not come from a proper subgroup.

Let $G = \mathbb{Z}/m\mathbb{Z}$ be finite cyclic. For a subgroup H, the composite

$$\mathcal{M}(G/H)_{(0)} \to \mathcal{M}(G/G)_{(0)} \to \mathcal{M}(G/H)_{(0)}$$

is multiplication by the index so it is a vector space isomorphism.

Madsen–Rothenberg claim that localizing the Carter sequence at $q(G, 0)$ gives the following short exact sequence.

$$0 \to K_0(\mathbb{Q}(\zeta_m))_{(0)} \to \bigoplus_{p|m} K_0(\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_m))_{(0)} \to K_{-1}(\mathbb{Z}[G])_{q(0,2)} \to 0$$

Indeed, writing $\mathbb{Q}[G]$ as a product of cyclotomic fields, we see that only the summand $K_0(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_m))_{(0)}$ is in the kernel above. Additionally, if we write $m = p^r m_p$ where p does not divide m_p then

$$\begin{aligned} K_0(\mathbb{Z}_p[G]) &\cong K_0(\mathbb{Z}_p[\mathbb{Z}/p^r\mathbb{Z}][\mathbb{Z}/m_p\mathbb{Z}]) \cong K_0(\mathbb{F}_p[\mathbb{Z}/p^r\mathbb{Z}][\mathbb{Z}/m_p\mathbb{Z}]) \\ &\cong K_0(\mathbb{F}_p[x][\mathbb{Z}/m_p\mathbb{Z}]/(x^{p^r} - 1)) \cong K_0(\mathbb{F}_p[\mathbb{Z}/m_p\mathbb{Z}]) \cong K_0(\mathbb{Z}_p[\mathbb{Z}/m_p\mathbb{Z}]). \end{aligned}$$

The second and last isomorphisms follow from the fact that (p) is a complete ideal in $\mathbb{Z}_p$. The fourth isomorphism follows from the fact that the ideal $(x - 1)$ is nilpotent. Therefore, $K_0(\mathbb{Z}_p[G])_{q(G,0)} = 0$.

The action on the middle term is more complicated. We will need the following lemma.

Lemma 3.4.5. *Suppose $K/\mathbb{Q}$ is a finite Galois extension. Then $\mathbb{Q}_p \otimes_{\mathbb{Q}} K$ is a product of isomorphic fields.*

Proof. We may write $K = \mathbb{Q}[x]/f(x)$ and $\mathbb{Q}_p \otimes_{\mathbb{Q}} K = \mathbb{Q}_p[x]/f(x) = \mathbb{Q}_p[x]/f_1(x) \cdots f_s(x)$

where $f(x) = f_1(x) \cdots f_s(x)$ is a factorization into irreducible polynomials in $\mathbb{Q}_p$. So

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} K \cong \prod_{i=1}^{s} \mathbb{Q}_p[x]/f_i(x)$$

where each $\mathbb{Q}_p[x]/f_i(x)$ is a field. The Galois group of $K/\mathbb{Q}$ acts transitively on the roots of f so there is an automorphism σ sending a root of $f_a(x)$ to a root of $f_b(x)$. This induces a ring automomorphism of $\mathbb{Q}_p \otimes_{\mathbb{Q}} K$.

Consider the composite

$$\mathbb{Q}_p[x]/f_a(x) \to \prod_{i=1}^{s} \mathbb{Q}_p[x]/f_i(x) \xrightarrow{\sigma} \prod_{i=1}^{s} \mathbb{Q}_p[x]/f_i(x) \to \mathbb{Q}_p[x]/f_b(x).$$

The first map sends an element $g(x)$ to the element which is $g(x)$ in the coordinate indexed my a and 0 elsewhere. This is a non-unital ring homomorphism. The composite is a nonzero field homomorphism so it is injective. Similarly, σ^{-1} gives a nonzero field homomorphism going the other way. Since these are finite dimensional $\mathbb{Q}_p$-vector spaces, we see that $\mathbb{Q}_p[x]/f_a(x) \cong \mathbb{Q}_p[x]/f_b(x)$. □

In our case, we are interested in $K = \mathbb{Q}(\zeta)$.

Proposition 3.4.6. *Let ζ be an m-th root of unity and let p be a prime divisor of m. Write $m = p^r m_p$ where p does not divide m_p. There is an isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta) \cong \prod_{i=1}^{s} \mathbb{Q}_p(\zeta)$ where s is the index of p in $(\mathbb{Z}/m_p)^\times$.*

Proof. Let t denote the order of p in $(\mathbb{Z}/m_p)^\times$. The degree of the extension $\mathbb{Q}_p(\zeta)/\mathbb{Q}_p$ is $t(p-1)p^{r-1}$ (see [Ser79, IV.4]) and the degree of the extension $\mathbb{Q}(\zeta)$ is $\left|(\mathbb{Z}/m_p)^\times\right|(p-1)p^{r-1}$. The result follows from Lemma 3.4.5. □

Involutions on $K_0(\mathbb{Q}_p[G])$

An analysis of the involution on $K_0(\mathbb{Q}_p[G])$ follows easily from [Ser77, 12.4]. Let K be a field of characteristic 0 and G a finite group with order m. Define $L := K(\zeta_m)$ where ζ_m is a primitive m-th root of unity then $\mathrm{Gal}(L/K) \subseteq (\mathbb{Z}/m\mathbb{Z})^\times$. Let Γ_K denote the image of the Galois group in $(\mathbb{Z}/m\mathbb{Z})^\times$. Two elements s and s' of G are Γ_K conjugate if there is a $t \in \Gamma_k$ such that s^t and s' are conjugate in G. The following is [Ser77, 12.4 Corollary 1].

Corollary 3.4.7. *A class function $f : G \to K$ belongs to $K \otimes_{\mathbb{Z}} R_K(G)$ if and only if it is constant on Γ_K-classes of G.*

Lemma 3.4.8. *Let G be an odd order abelian group. Then $\mathbb{Z}[\mathbb{Z}/2]$-module $R_K(G)/\langle \mathrm{triv} \rangle$ is either free or a free abelian group with a trivial involution. In the first case, the set of nontrivial irreducible G-representations over K form a free $\mathbb{Z}/2$-set.*

Proof. If $-1 \in \Gamma_K$ then all characters χ satisfy $\chi(g) = \chi(g^{-1})$. Suppose $1 \notin \Gamma_K$. Since we have assumed $|G|$ is odd, there is no nontrivial $g \in G$ such that $g = g^{-1}$ so $K \otimes_{\mathbb{Z}} R_K(G)/\langle \mathrm{triv} \rangle$ is a free $K[\mathbb{Z}/2]$-module. Also, $R_K(G)$ is a finitely generated $\mathbb{Z}[\mathbb{Z}/2]$-module which is obtained by linearizing the $\mathbb{Z}/2$-set of irreducible G-representations over K. It follows that the set of nontrivial irreducible representations must be a free $\mathbb{Z}/2$-set. □

Let $G = \mathbb{Z}/m$ where m is odd and let ζ be a primitive m-th root of unity as before. In this case, $\Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p) \leq (\mathbb{Z}/m)^\times$. The following lemma records our knowledge of the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)$.

Lemma 3.4.9. *Suppose p divides m. The Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p) \leq (\mathbb{Z}/m)^\times$ contains -1 if and only if, for each prime factor p_j of m not equal to p, the group $\langle p \rangle \leq (\mathbb{Z}/p_j)^\times$ contains -1.*

Proof. Factor $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$. There is an injection of Galois groups

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p_1^{r_1}})/\mathbb{Q}_p) \times \cdots \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p_k^{r_k}})/\mathbb{Q}_p)$$

such that composition with each projection on the right hand side is a surjection. Under the isomorphism

$$(\mathbb{Z}/m)^\times \cong (\mathbb{Z}/p_1^{r_1})^\times \times \cdots \times (\mathbb{Z}/p_k^{r_k})^\times$$

-1 is mapped to $(-1, -1, \cdots, -1)$. For $p_j = p$, $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^r})/\mathbb{Q}_p) \cong (\mathbb{Z}/p^r)^\times$ so -1 is always in the image of this component.

Assume $p_j \neq p$. To prove the lemma, it suffices to show that -1 is in $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p_j^{r_j}})/\mathbb{Q}_p) \leq (\mathbb{Z}/p_j^{r_j})^\times$ if and only if $\langle p \rangle \leq (\mathbb{Z}/p_j)^\times$ contains -1. This group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p_j^{r_j}})/\mathbb{Q}_p)$ is cyclic with order equal to the order of p in $(\mathbb{Z}/p_j^{r_j})^\times$ [Ser79, IV.4]. It is straightforward to check that p has even order in $(\mathbb{Z}/p_j^{r_j})^\times$ if and only if it has even order in $(\mathbb{Z}/p_j)^\times$. $\square$

The abelian group $K_0(\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta))$ inherits an involution from the involution $[P] \mapsto [P^*]$ on $K_0(\mathbb{Q}_p[G])$.

Corollary 3.4.10. *The $\mathbb{Z}[\mathbb{Z}/2]$-module $K_0(\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta))$ is free if and only if, for each prime factor p_j of m, $p \neq p_j$, the order of p in $(\mathbb{Z}/p_j)^\times$ is odd. Otherwise the involution is trivial.*

Corollary 3.4.11. *The involution on $K_{-1}(\mathbb{Z}[G])_{(0)}$ has a -1-eigenspace if and only if there are distinct prime factors p_i, p_j of $|G|$ such that the order of p_i in $(\mathbb{Z}/p_j)^\times$ is odd. Otherwise the involution is trivial.*

CHAPTER 4

EXOTIC SMOOTH G-HOMOTOPY EQUIVALENCES

4.1 Introduction

The Borel conjecture states that the surgery theoretic topological structure set is trivial for a closed aspherical manifold. For an aspherical manifold satisfying the Borel conjecture, the smooth structure set will be finite [Wei90]. In this chapter, we investigate the following question.

Question 5. Suppose X is a closed, smooth, equivariantly aspherical G-manifold. Are there infinitely many smooth G-homotopy equivalences $X' \to X$ up to smooth G-homotopy?

We use the equivariant Novikov conjecture of [RW90] to construct examples of equivariantly aspherical manifolds for which infinitely many such G-homotopy equivalences exist. Before stating the main results, we establish some definitions.

Definition 4.1.1. Suppose X is a smooth G-manifold and suppose $f_i : X_i \to X$ are smooth G-maps where $i = 0, 1$. We say f_0 and f_1 are *smoothly G-homotopic* if there is a G-smooth structure on $X_0 \times I$, a smooth map $F : X_0 \times I \to X$ and a G-diffeomorphism $\phi : X_1 \to X_0 \times \{1\}$ such that $F|_{X_0 \times \{0\}} = f_0$ and $F_{X_0 \times \{1\}} \circ \phi = f_1$.

Definition 4.1.2. For a smooth G-manifold X, let $\mathsf{S}_G^{DIFF}(X)$ denote the set of smooth G-homotopy classes of G-homotopy equivalences $X' \to X$.

Remark. When G is the trivial group and X has dimension at least 5, $\mathsf{S}_G^{DIFF}(X)$ is just the smooth structure set $\mathcal{S}^{DIFF}(X)$ of X in the sense of surgery theory (see [Wal99]). Our notation is chosen to suggest that the set $\mathsf{S}_G^{DIFF}(X)$ is an equivariant analogue of $\mathcal{S}^{DIFF}(X)$. However, we have no reason to believe that this set belongs in a surgery exact sequence.

The topological analogue of $\mathsf{S}_G^{DIFF}(X)$ has been studied in [CDK14] and [CDK15]. They show that the topological equivariant structure set of manifolds with discrete fixed point sets can be infinite. Specifically, they identify this set with a sum of UNil terms. This motivates the following, more refined, question.

Question 6. Suppose X is a closed, smooth, equivariantly aspherical G-manifold. Are there infinitely many elements of $\mathsf{S}_G^{DIFF}(X)$ which are G-homotopic?

We now give a rigorous definition of equivariantly aspherical spaces. Given a G-space X, one may construct a G-space $B\pi(X)$ such that the following hold.

- For any subgroup H, the fixed set $B\pi(X)^H$ is aspherical,
- There is a natural G-map $f : X \to B\pi(X)$ such that the maps $f^H : X^H \to B\pi(X)^H$ induce bijections on π_0 and isomorphisms on π_1 of components.

For an explicit construction in the case G is a compact Lie group, see the appendix of [RW90]. We will only consider the case where G is finite.

Definition 4.1.3. A G-space X is *equivariantly aspherical* if the natural map $X \to B\pi(X)$ is a G-homotopy equivalence. An equivariantly aspherical space X is *good* if, for all fiber bundles $X \to X' \to S^1$, the total space X' satisfies the equivariant Novikov conjecture.

We postpone a discussion of the equivariant Novikov conjecture to Section 4.2.

Example 10. If X is a complete, nonpositively curved manifold with G acting by isometries, then Rosenberg–Weinberger [RW90] show that X is equivariantly aspherical and satisfies the equivariant Novikov conjecture. They do not show that such X are good in the sense of Definition 4.1.3 but we expect this additional condition to hold.

In the results below, we assume that X is a good, closed, equivariantly aspherical, smooth G-manifold.

Theorem 4.1.4. *Suppose $G = \mathbb{Z}/p$ and suppose 2 has odd order in $\mathbb{F}_p^\times$. Suppose there is a component M of X^G such that the following hold.*

- *The eigenbundles of the normal bundle of M have rank at least 1,*
- *$H^1(M;\mathbb{Q}) \neq 0$.*

Then there are infinitely many elements of $\mathsf{S}_G^{DIFF}(X)$ which are G-homotopic.

Theorem 4.1.5. *Suppose G is an odd order cyclic group and that G acts semifreely on X. Suppose there is a component M of X^G with normal bundle ν such that the following hold.*

- *There exists an eigenbundle ν_k of complex rank d_k,*
- *$H^{2m-1}(M;\mathbb{Q}) \neq 0$ for some $m > d_k$.*

Then there are infinitely many elements of $\mathsf{S}_G^{DIFF}(X)$ which are G-homotopic.

As a special case of Theorem 4.1.5, we have the following result.

Corollary 4.1.6. *Suppose G is an odd order cyclic group acting semifreely on X and suppose M is a unitypical component of X^G (i.e. the normal bundle of M has only one nontrivial eigenbundle). If M has odd dimensional rational cohomology then there are infintely many elements of $\mathsf{S}_G^{DIFF}(X)$ which are G-homotopic.*

4.2 Higher G-signatures

The signature of a closed $4m$-dimensional manifold X can be computed as $\langle \mathcal{L}(X)_{4m}, [X]\rangle$ where $\mathcal{L}(X)$ is the Hirzebruch L-class. This class is not homogeneous and $\mathcal{L}(X)_{4m}$ denotes the degree $4m$-part. One may instead consider the inhomogeneous homology class $\langle \mathcal{L}(X), [X]\rangle$. The parts in positive degree are the *higher signatures* of X.

In the equivariant setting, Atiyah–Singer show that there is the following formula for G-signatures in terms of characteristic classes. The G-signature is a virtual representation whose character is given, in our setting, by the formula

$$(\operatorname{sign}_G X)(g) = \langle A(g,\nu)(\mathcal{L}(M)\mathcal{M}(g,\nu))_{\dim M}, [X]\rangle.$$

Here, $A(g,\nu)$ is a complex number, $\mathcal{L}(M)$ is the Hirzebruch L-class and $\mathcal{M}(g,\nu) \in H^*(M;\mathbb{C})$ is some inhomogeneous class. We may similarly consider the evaluation of the entire inhomogeneous class of $A(g,\nu)\mathcal{L}(M)\mathcal{M}(g,\nu)$ against $[M]$. As a result, we obtain an element of $H_*(M;\mathbb{C})$ which we call the *higher G-signature*.

Remark. The higher G-signature may also be defined as the index of an equivariant elliptic operator (see [RW88] and [RW90]). With this definition, it is an element of the equivariant K-homology group $K_*^G(X)$. This is not equivalent to our definition of the higher G-signature as an element of the homology of the fixed point set. Our homological definition is obtained from the K-homology definition by localization and taking a Chern character.

To distinguish between these two notions, call the index of the G-signature operator the *K-theoretic higher G-signature*. We will denote this by $[D_X] \in K_*^G(X)$.

4.2.1 The Equivariant Novikov Conjecture

In analogy with the non-equivariant case, the G-signature is a G-homotopy invariant but the higher G-signature is not. We will need an equivariant analogue of the Novikov conjecture studied in [RW90].

Definition 4.2.1. A G-map $f : X \to X'$ is a *G-pseudoequivalence* if it is a non-equivariant homotopy equivalence.

The equivariant Novikov conjecture as stated by [RW90] is as follows.

Conjecture 4.2.2 (Equivariant Novikov conjecture). *Let $h : X \to X'$ is an orientation preserving G-pseudoequivalence of connected, closed, oriented G-manifolds. Consider the following commutative diagram.*

$$\begin{array}{ccc} K_*^G(X) & \xrightarrow{(f_X)_*} & K_*^G(B\pi(X)) \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ K_*^G(X') & \xrightarrow{(f_{X'})_*} & K_*^G(B\pi(X')) \end{array}$$

If $K_*^G(B\pi(X'))$ is finitely generated over $R(G)$ then the K-theoretic higher G-signatures agree. In other words,

$$h_* \circ (f_X)_*([D_X]) = (f_{X'})_*([D_{X'}]).$$

Rosenberg and Weinberger prove this conjecture in the case X is a complete, nonpositively curved manifold with G acting by isometries. When X is equivariantly aspherical and satisfies the equivariant Novikov conjecture, we see that $h_*[D_X] = [D_{X'}]$. The following summarizes this discussion.

Proposition 4.2.3. *Suppose X' is an equivariantly aspherical smooth G-manifold such that the equivariant Novikov conjecture holds. Let X be a smooth G-manifold and let $f : X \to X'$ be a G-pseudoequivalence. Then,*

$$A(g,\nu)\mathcal{L}(M)\mathcal{M}(g,\nu) = f^*(A(g,\nu')\mathcal{L}(M')\mathcal{M}(g,\nu')) \in H^*(M;\mathbb{C})$$

where $M = X^G$, ν is the normal bundle of M and M', ν' are similarly defined.

Remark. So far, our discussion of higher signatures has taken place in the smooth category where these signatures may be described in terms of characteristic classes of normal bundles. If G acts cellularly on a manifold X, then the symmetric signature of X gives an element of

$L^{\dim X}(\mathbb{C}[\Gamma])$ where Γ is the orbifold fundamental group. This maps to $K_0^G(C_r^*(\pi(X)))$ and so we obtain an element in the K-theory homology of X. By the equivariant Kaminker–Miller theorem [RW90, Theorem 3.6], this is the K-theoretic G-signature of X when X is smooth. Therefore, the higher G-signature of X may be defined when X is a G-PL-manifold. As with the smooth case, we will work with this higher G-signature as an element of the singular cohomology of the fixed set.

4.3 Blocked Surgery

In this section, we summarize some results of what we call blocked surgery.

If Y is a connected n-manifold, the topological structure set may be defined as homotopy groups of a semisimplicial set, which we also call $\mathcal{S}^{TOP}(Y)$. The m-simplicies are simple homotopy equivalences $Y' \to \Delta^m \times Y$ which restricts to a homeomorphism $\partial Y' = \partial(\Delta^m \times Y)$. These homeomorphisms must respect a decomposition of the boundary according to the combinatorics of Δ^m. This can be thoroughly written out in the language of n-ads (see [Wal99]). If K is a simplicial complex then a map $K \to \mathcal{S}^{TOP}(Y)$ determines a TOP-block bundle W over K with fiber Y and an equivalence of W with the $K \times Y$ as a block fibration.

There is a map of spaces $L^s_{n+1}(\pi_1 Y) \to \mathcal{S}^{TOP}(Y)$ and we will be interested in block bundles in the image of the induced map

$$[K, L^s_{n+1}(\pi_1 Y)] \to [K, \mathcal{S}^{TOP}(Y)].$$

Instead of working explicitly with these spaces as semisimplicial sets, we give an interpretation of this map in terms of the action of surgery groups on structure sets.

The map (of groups) $L^s_{n+1}(\pi_1 Y) \to \mathcal{S}^{TOP}(Y)$ can be described as follows. An element of $L^s_{n+1}(\pi_1 Y)$ may be represented by a map $\Phi : (W; \partial_0 W, \partial_1 W) \to (Y \times I; Y \times \{0\}, Y \times \{1\})$ where W is a manifold with boundary. The map F restricts to a homeomorphism Φ_0 :

$\partial_0 W \to Y \times \{0\}$ and a simple homotopy equivalence $\Phi_1 : \partial_1 W \to Y \times \{1\}$. The restriction Φ_1 determines an element of $\mathcal{S}^{TOP}(Y)$. The manifold W is obtained by taking $M \times I$ and attaching a sequence of handles to $M \times \{1\}$. When Y has boundary, a similar description can be given.

Consider the projection $\pi : K \times Y \to K$ where K and Y are smooth manifolds. We may consider surgeries done as follows. Over each vertex Δ^0 of K, we do surgery to obtain a map $W_{\Delta^0} \to \Delta^0 \times Y \times I$. Inductively, over each d-simplex Δ^d, we do surgery to obtain a map $W_{\Delta^d} \to \Delta^d \times Y \times I$ in a way that agrees with the surgeries done over $\partial\Delta^d$. Doing this over each simplex yields determines a map $K \to L^s_{n+1}(\pi_1 Y)$.

Proposition 4.3.1. *A map $K \to L^s_{n+1}(\pi_1 Y)$ determines a TOP-block bundle $W \to K$ with fiber $Y \times I$ such that the following hold.*

- *W is a manifold with boundary $\partial W = W_0 \cup W_1$,*
- *There is an equivalence $F : W \to K \times Y \times I$ of TOP-block bundles,*
- *$F|_{W_0}$ is a homeomorphism,*
- *$F|_{W_1} : W_1 \to K \times Y$ is a simple homotopy equivalence and a map of TOP-block bundles.*

4.3.1 The Case of Lens Spaces

We are particularly interested in the case where Y is a lens space SV/G which Cappell–Weinberger [CW91] studied in the PL-setting. As in Chapter 2, the fiber Y will be the lens space arising from the normal representation V at a component M of the fixed set. We will henceforth write SV/G for this lens space and we let n denote its dimension. There is an identification of $\mathcal{S}^{PL}(SV/G)$ with the fiber of

$$B\widetilde{SPL}(SV/G) \to BSF(SV/G)$$

where $F(SV/G)$ is the topological monoid of homotopy equivalences and the S denotes the restriction to orientation preserving maps.

Cappell–Weinberger show that the following maps of spaces becomes a split fiber sequence after rationalizing.

$$\tilde{L}^s_{n+1}(G) \to B\widetilde{SPL}(SV/G) \to B\widetilde{SPL}(SV)$$

Using that the first map factors through the PL-structure space $\mathcal{S}^{PL}(SV/G)$ and the fact that $\tilde{L}^s_{n+1}(G)_{(0)} \cong \prod_{j\geq 1} K(\widetilde{RO}(G), 2j)_{(0)}$, we obtain the following.

Proposition 4.3.2. *For each element in $\alpha \in \bigoplus_{j\geq 1} H^{2j}(M; \widetilde{RO}(G))/Torsion$, there is a PL-block bundle E_α over M such that the following hold.*

- *There is an equivalence of block fibrations $E_\alpha \to M \times SV/G$,*
- *If $\alpha \neq \beta$ then E_α and E_β are not equivalent as PL-block bundles.*

We will be interested in lens space block bundles over $M \times I$ which are trivial over $M \times S^0$ and which do not arise from G-vector bundles. Namely, we are interested in elements in the image of

$$\rho_M : [\Sigma M, \tilde{L}^s_{n+1}(G)] \to [\Sigma M, B\widetilde{SPL}(SV/G)]$$

up to an ambiguity detected by

$$\sigma_M : [\Sigma M, BSO^G(V)] \to [\Sigma M, B\widetilde{SPL}(SV/G)].$$

Since ΣM is a wedge of spheres, these sets are groups. Define

$$U(M) := \mathrm{im}(\rho_M)/(\mathrm{im}(\sigma_M) \cap \mathrm{im}(\rho_M)).$$

The maps above were studied in Chapter 2. If $V = \bigoplus_{k=1}^{\frac{p-1}{2}} V_k$ is an eigenbundle decomposition and $\dim_{\mathbb{C}} V_k = d_k$, then $BSO^G(V) \cong \prod_{k=1}^{\frac{p-1}{2}} BU(d_k)$. It follows form [CW91] that the composite map

$$\Psi_j : \pi_{2j}(BSO^G(V))_{(0)} \to \pi_{2j}(B\widetilde{SPL}(SV/G))_{(0)} \to \pi_{2j}(\tilde{L}^s(G))_{(0)}$$

can be identified with the map

$$(c_j(E_1), \cdots, c_j(E_k)) \mapsto \sum_{k=1}^{\frac{p-1}{2}} \Phi_{j,k} c_j(E_k)$$

where the $\Phi_{j,k}$ are the complex numbers of [Ewi76] and $\pi_{2j}(\tilde{L}^s(G))_{(0)}$ is identified with either the purely real or purely imaginary part of $\mathbb{Q}(\zeta)$. Note that these are maps between isomorphic vector spaces. In particular, when Ψ_j is not an isomorphism and $H^{2j-1}(M;\mathbb{Q}) \neq 0$, there are infinitely many elements in $U(M)$. By the results of Ewing, we can conclude the following.

Proposition 4.3.3. *1. Suppose $j > 1$. Then Ψ_j is an isomorphism if and only if $d_k \geq j$ for all k.*

2. Suppose $d_k \geq 1$ for all k. Then Ψ_1 is an isomorphism if and only if 2 has even order in $\mathbb{F}_p^\times$.

3. If $d_k < 1$ for some k then Ψ_1 is not an isomorphism.

Proposition 4.3.3 implies that there are two reasons an element may be in $U(\Sigma M)$. The dimensions of the eigenspaces of V may not be large enough to get the required Atiyah–Singer classes or the linear relations in the $\Phi_{1,k}$ do not allow certain cohomology classes to be Atiyah–Singer classes of G-vector bundles. We summarize this in the following proposition.

Proposition 4.3.4. *Then there are infinitely many elements of $U(M)$ in the following cases.*

1. There is a $j \geq 1$ such that $H^{2j-1}(M;\mathbb{Q}) \neq 0$ and $d_k < j$ for some $k = 1, \cdots, \frac{p-1}{2}$, *2. 2 has odd order in* $\mathbb{F}_p^\times$ *and* $H^1(M;\mathbb{Q}) \neq 0$.

Remark. Although our discussion of blocked surgery has taken place in the topological and PL categories, the manifolds obtained by this procedure can be assumed to be smooth provided M is smooth. Explicitly, if M is smooth, the lens space PL-block bundle corresponding to an element of $[\Sigma M, \tilde{L}^s_{n+1}(G)]$ may be represented by a smooth manifold E and the equivalence of block fibrations $E \to M \times I \times SV/G$ can be assumed to be smooth.

4.3.2 Construction of Exotic G-Homotopy eEquivalences

We now use the machinery developed to describe G-manifolds X_α and smooth equivariant homotopy equivalences $X_\alpha \to X$.

Suppose X is a smooth G-manifold and that a component M of the fixed set has a normal bundle with each eigenbundle a trivial complex vector bundle. Let W_α denote a lens space block bundle over $M \times I$ obtained from an element $\alpha \in [\Sigma M, \tilde{L}^s_{n+1}(G)]$. In particular, W_α is trivial over $M \times S^0$ and there is a smooth map $W_\alpha \to M \times I \times SV/G$ which is an equivalence of block fibrations. Let $\tilde{W}_\alpha$ denote the G-cover of W_α and define

$$X_\alpha := (X \setminus (S\nu \times I)) \cup_{S\nu \times S^0} \tilde{W}_\alpha.$$

Since W_α is obtained by blocked surgery, there is a smooth equivariant homotopy equivalence $f_\alpha : X_\alpha \to X$.

Remark. This construction is similar to the construction in Chapter 3 in that we are replacing the neighborhood of a sphere bundle. The difference is that we obtain a *smooth* equivariant homotopy equivalence here.

4.4 Proof of Theorems

The proof of Theorem 4.1.4 and Theorem 4.1.5 will follow from Proposition 4.3.4 and the following proposition.

Proposition 4.4.1. *Suppose X is a good, closed, equivariantly aspherical, smooth, semifree G-manifold. If $U(M)$ is infinite for a component M of the fixed set, then there are infinitely many elements of $\mathsf{S}_G^{DIFF}(X)$ which are G-homotopic.*

Proof. Let W_α and W_β denote two lens space block bundles over $M \times I$ obtained through blocked surgery from elements $\alpha, \beta \in [\Sigma M, \tilde{L}^s_{n+1}(G)]$ and suppose these elements map to different elements in $U(M)$. Suppose there exists a G-smoothing of $X_\alpha \times I$ and a smooth G-map $F : X_\alpha \times I \to X$ restricting to f_α over $X_\alpha \times \{0\}$ and to $f_\beta \circ \phi$ over $X_\alpha \times \{1\}$, where $\phi : X_\alpha \to X_\beta$ is some G-diffeomorphism.

Let X_0 denote $X \setminus \nu$. We will construct a smooth G-manifold with boundary Y_α and a smooth G-homotopy equivalence $Y_\alpha \to X_0 \times I$ by performing blocked surgery on the boundary of $X_0 \times I$. The boundary is

$$\partial(X_0 \times I) = (X_0 \times \{0\}) \cup (S\nu \times I) \cup (X_0 \times \{1\}).$$

Note that ∂X_0 has a collared neighborhood of the form $S\nu \times [0, \infty)$. Perform blocked surgery on $S\nu/G \times [1, 2] \subseteq X_0/G \times \{0\} \subseteq \partial(X_0/G \times I)$ using α to obtain a smooth G-manifold Y'_α which admits a smooth G-map to $X_0 \times I$. The boundary of this manifold is

$$\partial Y'_\alpha = (X_\alpha \setminus \nu) \cup (S\nu \times I) \cup (X_0 \times \{1\}).$$

Now, perform blocked surgery along $S\nu/G \times I$ using α to obtain Y_α. By construction, there is a smooth G-homotopy equivalence $F_\alpha : Y_\alpha \to X_0 \times I$ and

$$\partial Y_\alpha = (X_\alpha \setminus \nu) \cup \tilde{W}_\alpha \cup X_0.$$

Additionally,

$$F_\alpha|_{X_\alpha \setminus \nu} = f_\alpha \qquad F_\alpha|_{X_0} = \mathrm{id}$$

and $F_\alpha|_{\tilde{W}}$ becomes an equivalences of lens space block bundles after taking the quotient by G. Similarly, construct a smooth G-manifold Y_β with boundary

$$\partial Y_\beta = (X_\beta \setminus \nu) \cup \tilde{W}_\beta \cup X_0$$

and with a smooth G-homotopy equivalence $F_\beta : Y_\beta \to X_0 \times I$.

Our assumption on the G-smoothing of $X_\alpha \times I$ implies that there is a normal G-vector bundle ξ on $M \times I$ restricting to ν over $X_\alpha \times \{0\}$ and $(\phi^{-1})^*\nu$ over $X_\alpha \times \{1\}$.

Define

$$X' := ((X_\alpha \times I) \setminus \xi) \cup_{X_\alpha \setminus \nu} Y_\alpha \cup_{X_0} Y_\beta.$$

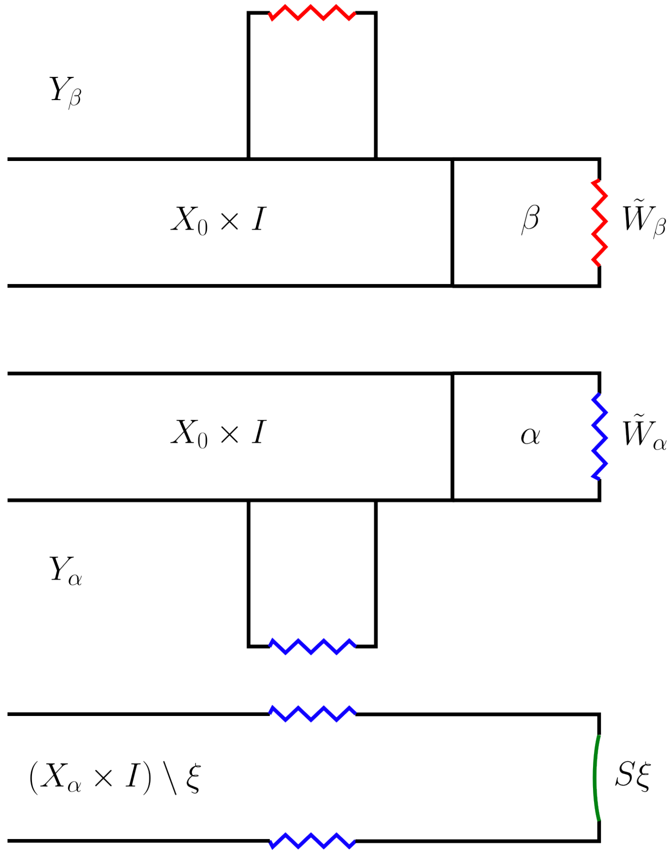
This has boundary $\partial X' = (X_\alpha \setminus \nu) \cup (X_\beta \setminus \nu)$. Define a smooth G-manifold X'' by identifying the boundary components by the G-diffeomorphism ϕ There is a smooth G-homotopy equivalence $X' \to X_0 \times I$ which induces a smooth G-homotopy equivalence

$$X'' \to X(\phi) \setminus \nu_\phi$$

where $X(\phi)$ denotes the mapping torus of ϕ and ν_ϕ is the neighborhood of the mapping torus $M(\phi)$ of $\phi|_M$.

The boundary of X'' is the G-cover of a lens space PL-block bundle over $M(\phi)$.

Figure 4.1: Construction of X''



Consider the diagram

$$\begin{array}{ccccc}
[\Sigma M, BSO^G(V)] & \xrightarrow{\sigma_M} & [\Sigma M, B\widetilde{SPL}(SV/G)] & \longrightarrow & [\Sigma M, L^s_{n+1}(G)_{(0)}] \\
\downarrow & & \downarrow \gamma & & \downarrow \\
[M(\phi), BSO^G(V)] & \xrightarrow{\sigma_\phi} & [M(\phi), B\widetilde{SPL}(SV/G)] & \xrightarrow{\delta} & [M(\phi), L^s_{n+1}(G)_{(0)}]
\end{array}.$$

The right vertical map splits by the Serre spectral sequence. The left vertical map splits after rationalizing by a similar argument. By construction, the lens space PL-block bundle $\partial X''/G$ can be identified with $\gamma(W + W')$ where W' is in the image of σ_M (since we are considering ΣM, there is a group structure so addition makes sense). It follows from these observations and some diagram chasing that $\delta(\partial X''/G)$ is not in the image of $\delta \circ \sigma_\phi$.

Now, let $\overline{X}$ be the G-manifold obtained by attaching an equivariant block neighborhood to X'' along $\partial X''$. The higher G-signature of $\overline{X}$ restricted to the component $M(\phi)$ is $\delta(\partial X'')$. The equivariant Novikov conjecture (along with our assumption that X is good) implies that this is $\delta(\nu_\phi)$. By the previous paragraph, we have a contradiction.

To show that $X_\alpha \to X$ is homotopic to a homeomorphism, use that W_α is trivial as an PL-block bundle over $M \times I$ (where we do not require ends to be fixed) and apply a swindling argument as in Chapters 2 and 3. $\square$

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