THE UNIVERSITY OF CHICAGO

SOME RESULTS IN MEAN CURVATURE FLOW AND MIN-MAX THEORY

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
CHUN PONG CHU

CHICAGO, ILLINOIS
JUNE 2024

Contents

List of Figures

List of Tables

Acknowledgement

First, I thank my advisor André Neves, an absolutely crucial figure for my academic success and independence as a researcher. Undoubtedly it was my fortune to be his student.

Second, I thank numerous seniors and colleagues, who nurtured, guided, and propelled me in the academia: John Boller, Eric Chen, Otis Chodosh, Hyun Chul Jang, Hojoo Lee, Martin Li, Fernando Marques, Antoine Song, Daniel Stern, Ao Sun, Zhizhang Xie, Xin Zhou, and many others.

Third, I thank my friends and fellow students. Gifted me five really memorable years, you all are awesome: Iqra Altaf, Jingwen Chen, Nikolay Grantcharov, Katie Gravel, Punghui How, Peter Huxford, DeVon Ingram, Sehyun Ji, Ruojing Jiang, Yulia Kotelnikova, Seraphina Lee, Yangyang Li, Ben Lowe, Jinyue Luo, Yao Luo, Xinchun Ma, Daniel Mitsutani, Hoan Nguyen, Chloe Postel-Vinay, Megan Roda, Linus Setiabrata, James Stevens, Jinwoo Sung, Chi Cheuk Tsang, Tin Yau Tsang, Zhihan Wang, Pranjal Warade, Antonios Zitridis, and many others.

Lastly, but deeply, I thank my parents for their unconditional love and support all these years.

Abstract

This thesis consists of four parts. In the first part, we introduce the results, and sketch the ideas behind the proofs. In the second part, we construct genus one singularities for mean curvature flow using an interpolation argument. In the third part, we bound some Almgren-Pitts min-max p-widths of the unit 3-sphere by proving a strong multiplicity one min-max theorem. In the fourth part, we present results concerning p-widths and minimal surfaces in the unit 3-ball.

1. Introduction

From a broad perspective, the topics of minimal surfaces, min-max theory, and mean curvature flow (MCF) arise from studying the space of all hypersurfaces in any given Riemannian manifold. Namely, on the space of all hypersurfaces, let us consider the area functional $\mathcal{A}$. Then a minimal surface can be viewed as a critical point of $\mathcal{A}$, while mean curvature flow can be viewed as the gradient flow of $\mathcal{A}$. As for min-max theory, it is a version of Morse theory for the infinite dimensional space of all hypersurfaces. By equipping the space of hypersurfaces with a suitable topology, min-max theory allows one to construct and study the critical points of $\mathcal{A}$.

1.1. Genus one singularities in MCF.

Results in this sections are joint works with Ao Sun.

Mean curvature flow (MCF) is the fastest way to decrease the area of a surface. It originated from applied science and has been attracting much attention recently because of its potential to study the geometry and topology of surfaces in three-manifolds. As a nonlinear geometric heat flow, MCF may have singularities, and the singularities may result in changes in the geometry and topology of the surfaces.

The blow-up method developed by Huisken [Hui90], Ilmanen [Ilm95], and White [Whi97] shows that the singularities are modeled by a special class of surfaces called *self-shrinkers*. They satisfy the equation $\vec{H} + \vec{x}^{\perp}/2 = 0$.

It is a difficult question to determine what singularity models can show up in an arbitrary MCF. With the convexity assumption, Huisken [Hui84] proved that the singularities must be modeled by spheres, and with the mean convexity assumption, White [Whi97, Whi00, Whi03] proved that the singularities must be modeled by spheres and cylinders.

In this section, we find a condition to ensure the appearance of a singularity modeled by a genus one self-shrinker. To our knowledge, this is the first result to ensure a self-shrinker with non-zero genus shows up as the singularity model.

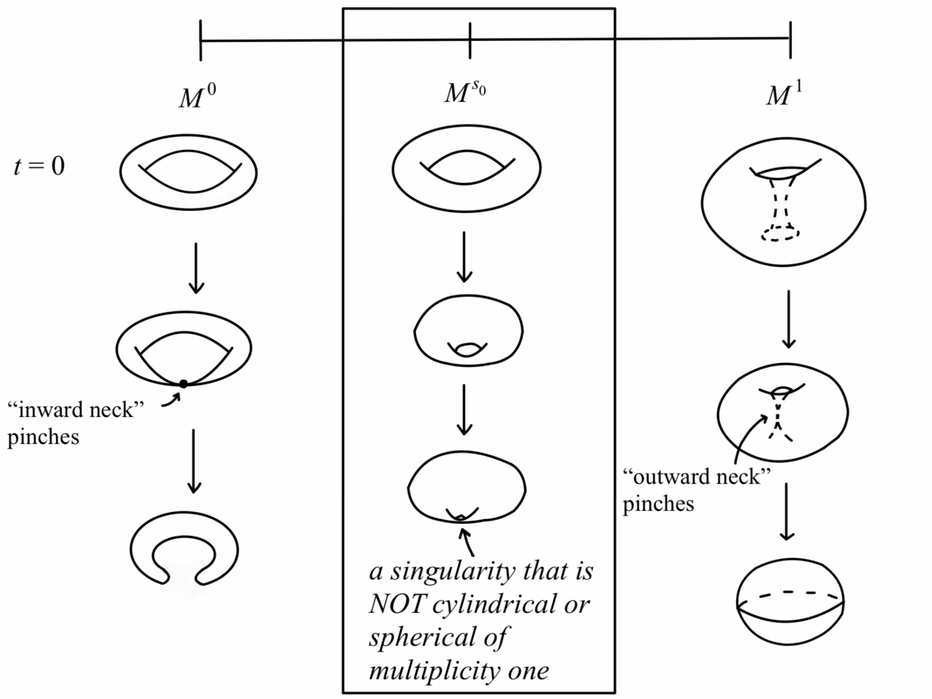


FIGURE 1. An interpolation argument to construct genus one singularities.

Let us first explain the heuristics, which involves an interpolation argument. In Figure 1, we have a one-parameter family $\{M^s\}_{s\in[0,1]}$ of tori at the top row. Suppose that the starting torus M^0 has a thin "inward neck", so that when we run MCF this neck will pinch. On the other hand, the ending torus M^1 has a thin "outward neck" at the middle, which will pinch under MCF. Then, there should be a critical $s_0 \in [0,1]$ such that for the torus M^{s_0}, both the inward neck and the outward neck will pinch under MCF, resulting in a genus one singularity (see Figure 1).

The following is our main theorem. We will define what "inward (or outward) torus neck will pinch" means precisely later (see Definition 1.1.8).

Theorem 1.1.1. *Let $\{M^s\}_{s\in[0,1]}$ be a smooth family of tori in $\mathbb{R}^3$ such that for the MCF starting from M^0 (resp. M^1), the inward (resp. outward) torus neck will pinch. Then there exists $s_0 \in [0,1]$ such that the MCF starting from M^{s_0} would develop a singularity that is not multiplicity one cylindrical or multiplicity one spherical.*

In particular, we can immediately rule out multiplicity if the *entropy* of each torus M^s is less than 2, i.e.

$$\mathrm{Ent}(M^s) := \sup_{x_0\in\mathbb{R}^3, t_0>0} (4\pi t_0)^{-1} \int_{M^s} e^{-\frac{|x-x_0|^2}{4t_0}} < 2.$$

Corollary 1.1.2. *In the setting of Theorem 1.1.1, if each initial torus M^s has entropy less than* 2*, then at the singularity concerned, every tangent flow is given by a multiplicity one, embedded, genus one self-shrinker.*

Recall that the tangent flow is a specific blow-up limit of a MCF at a singularity, see §2.1. Using Huisken's monotonicity formula [Hui90], Ilmanen [Ilm95] and White [Whi97] showed that the tangent flow must be a self-shrinker with multiplicity.

Let us explicitly provide such a family of tori as follow. Let $\mathbb{T}$ denote the rotationally symmetric, genus one self-shrinker in $\mathbb{R}^3$ constructed by Drugan-Nguyen [DN18]. Note that both $\mathbb{T}$ and the Angenent torus [Ang92] are called *shrinking doughnuts*, and it is possible that they are the same. By [DN18], $\mathbb{T}$ has entropy strictly less than 2. Berchenko-Kogan [BK21] showed that the Angenent torus has entropy approximated 1.85 using a numerical method.

Theorem 1.1.3. *Let $\{M^s\}_{s\in[0,1]}$ be a smooth family of tori in $\mathbb{R}^3$ that are sufficiently close in C^∞ to the shrinking doughnut $\mathbb{T}$, with M^0 strictly inside $\mathbb{T}$ while M^1 strictly outside. Then there exists $s_0 \in [0,1]$ such that the MCF starting from M^{s_0} would develop a singularity at which every tangent flow is given by a multiplicity one, embedded, genus one self-shrinker.*

The idea of Theorem 1.1.3 can be traced back to the work of Lin and the second author in [LS22]. In earlier work, Colding-Ilmanen-Minicozzi-White [CIMIW13] observed that one can perturb a closed embedded self-shrinker in $\mathbb{R}^3$ such that the MCF has only neck and spherical singularities. Lin and the second author observed a bifurcation: inward (resp. outward) perturbations make the MCF pinch from inside (resp. outside).

It is also interesting to compare our results with the recent development of generic MCF [CM12, CCMS20, CCMS21, SX21a, SX21b, CCS23, Sun23]: One can perturb a single MCF to avoid a singularity that is not spherical or cylindrical. In contrast, our results imply that for a certain one-parameter family of MCFs, a singularity that is modeled by a genus one shrinker is robust under perturbations.

It is natural to ask whether Theorem 1.1.1 would hold for surfaces with genus two or above. Actually, it would not: See a counterexample in Remark 2.5.2. Nevertheless, we believe a similar theory can be established for a multi-parameter family of higher genus surfaces, see Question 1.1.10

Let us now give several applications of the above theorems.

Theorem 1.1.4. *An embedded, genus one self-shrinker in $\mathbb{R}^3$ of the least entropy either is non-compact or has index 5.*

We remark that the existence of an entropy minimizer among all embedded, genus g self-shrinkers in $\mathbb{R}^3$, with a fixed g, is given by Sun-Wang [SW20].

Theorem 1.1.5. *There exists an ancient MCF through cylindrical and spherical singularities $\{M(t)\}_{t<0}$ in $\mathbb{R}^3$ such that:*

- *As $t \to -\infty$, $\frac{1}{\sqrt{-t}}M(t) \to \mathbb{T}$ smoothly.*
- *As $t \to 0$, $M(t)$ hits a singularity at which every tangent flow is given by a multiplicity one, embedded, genus one self-shrinker.*

In fact, Theorem 1.1.5 still holds with $\mathbb{T}$ replaced by any other closed, embedded, rotationally symmetric, genus one shrinker (if they actually exist), and the exact same proof will work.

Recalling that the rotationally symmetric shrinker $\mathbb{T}$ must have index at least 7 by Liu [Liu16], we see that both Theorem 1.1.4 and 1.1.5 implies the following.

Corollary 1.1.6. *There exists in $\mathbb{R}^3$ an embedded, genus one self-shrinker with entropy lower than $\mathbb{T}$.*

The three self-shrinkers with the lowest entropy are the plane, the sphere, and the cylinder ([CIMIW13, BW17]). All of them are rotationally symmetry. Kleene-Møller [KMl14] showed that all other rotationally symmetric smooth embedded self-shrinkers are closed with genus 1, for which we can apply Theorem 1.1.1.

It is known that the space of smooth embedded self-shrinkers in $\mathbb{R}^3$ with entropy less than some constant $\delta < 2$ is compact in C^∞_{loc} topology (see [Lee21]). Together with the rigidity of the cylinder as a self-shrinker by [CIM15], there exists a smooth embedded self-shrinker minimizing entropy among all the smooth embedded self-shrinkers with entropy larger than the cylinder.

Corollary 1.1.7. *A smooth embedded self-shrinker in $\mathbb{R}^3$ with the fourth lowest entropy is not rotationally symmetric.*

Main ideas: Change in homology under MCF. The major challenge of this work is to introduce some new concepts to rigorously state and prove the interpolation argument we outlined in page 1 and Figure 1. Particularly, we need to describe the topological change of the surfaces more precisely. Let $\mathcal{M} = \{M(t)\}_{t\geq 0}$ be a MCF in $\mathbb{R}^3$, where the initial condition $M(0)$ is a closed, smooth, embedded surface. Since we would allow $M(t)$ to have singularities and thus change its topology, $\mathcal{M}$ is, more precisely, a *level set flow. In this section, we often use the phrases MCF and level set flow interchangeably.*

It is known that the topology of $M(t)$ simplifies over time. In [Whi95], White focused on describing the complement $\mathbb{R}^3\backslash M(t)$ (instead of $M(t)$ itself), and how it changes over time. For example, he showed $\text{rank}(H_1(\mathbb{R}^3\backslash M(t)))$ is non-increasing in t, where H_1 denotes the first homology group in $\mathbb{Z}$-coefficients. Thus, heuristically, topology can only be destroyed but not created.

In this work, we will further describe this phenomenon by *keeping track of which elements of the initial homology group $H_1(\mathbb{R}^3\backslash M(0))$ are destroyed, and how they are destroyed.* To illustrate, let us use the flow in Figure 2 as an example.

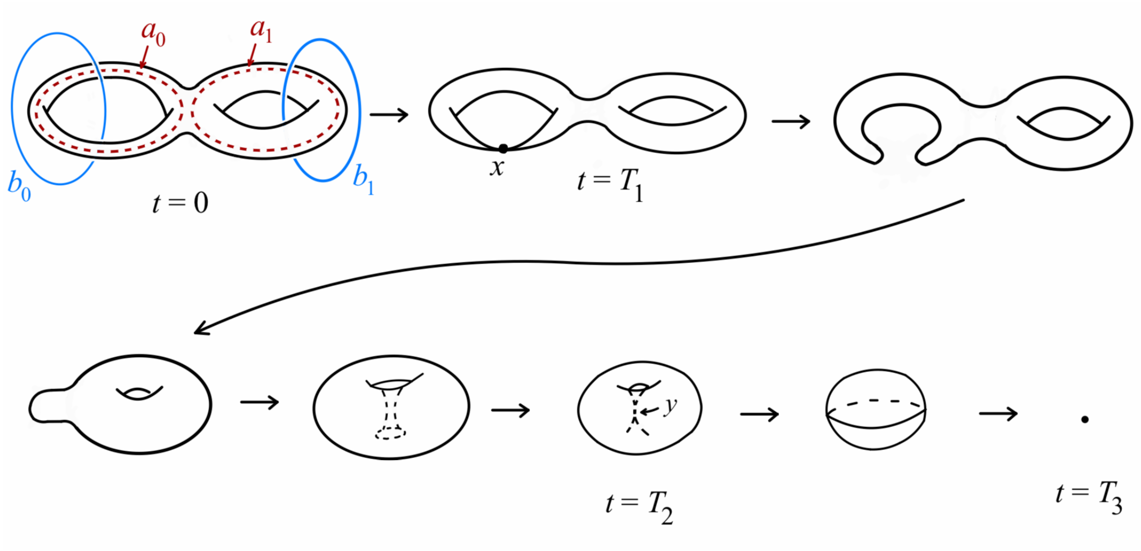


FIGURE 2. An example of MCF.

Heuristic observation. Let us first list some heuristic observations regarding Figure 2. We will describe them more precisely in a moment. We fix four elements of $H_1(\mathbb{R}^3 \backslash M(0))$ at time $t = 0$, as shown in the figure. Note that a_0 and a_1 belong to the bounded region *inside* the genus two surface $M(0)$, while b_0 and b_1 belong to the region *outside* $M(0)$.

(1) At time $t = T_1$, a_0 is "broken" by the cylindrical singularity x of the flow. As a result, for later time $t > T_1$, a_0 no longer exists. Apparently, it "terminates" at time T_1.

(2) On the other hand, a_1, b_0, and b_1 all can survive through time T_1. For example, for b_0, we can clearly have a *continuous* family of loops, $\{\beta_t\}_{t \geq 0}$, where $[\beta_0] = b_0$ and each β_t is a loop *outside* the surface $M(t)$. In this sense, b_0 will survive for all time, although it becomes *trivial* after time T_1.

(3) As for b_1, although it survives through $t = T_1$, it will terminate at $t = T_2$, when it is broken by the cylindrical singularity y.

Let us now make these observations precise.

Three new concepts. To our knowledge, these concepts are new, meanwhile, they seem natural in the context of geometric flows. We think these concepts may have independent interests.

To set up, for any two times $t_1 < t_2$, let us consider the *complement* of the spacetime track of the flow within the time interval $[t_1, t_2]$:

$$W[t_1, t_2] := \bigcup_{t \in [t_1, t_2]} (\mathbb{R}^3 \backslash M(t)) \times \{t\} \subset \mathbb{R}^3 \times [t_1, t_2].$$

To discuss the "termination" of an element $c_0 \in H_1(\mathbb{R}^3 \backslash M(0))$ under the flow, we first need to *relate elements of* $H_1(\mathbb{R}^3 \backslash M(0))$ *with elements of* $H_1(\mathbb{R}^3 \backslash M(t))$, at some later time $t > 0$.

> **Homology descent.** (Definition 2.3.1.) Given two elements $c_0 \in H_1(\mathbb{R}^3 \backslash M(t_0))$ and $c \in H_1(\mathbb{R}^3 \backslash M(t))$ with $t > 0$, we say that c *descends from* c_0, and denote
>
> $$c_0 \succ c,$$
>
> if the following holds: For every representative $\gamma_0 \in c_0$ and $\gamma \in c$, if we view them as subsets
>
> $$\gamma_0 \subset (\mathbb{R}^3 \backslash M(0)) \times \{0\}, \quad \gamma \subset (\mathbb{R}^3 \backslash M(t)) \times \{t\},$$
>
> then they bound some singular 2-chain $\Gamma \subset W[0, t]$, i.e. $\gamma_0 - \gamma = \partial\Gamma$. (See Figure 3.)

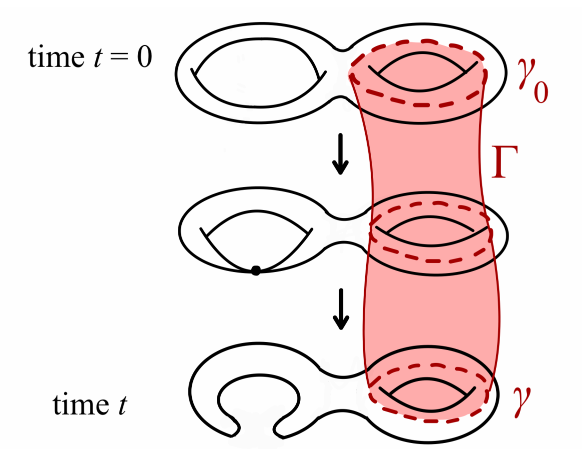


FIGURE 3. Homology descent.

As we will prove, the above notion satisfies some nice properties. For example, given a $c_0 \in H_1(\mathbb{R}^3 \backslash M(0))$, the element $c \in H_1(\mathbb{R}^3 \backslash M(t))$ described above, if exists, turns out to be *unique*: Thus we will denote this c by $c_0(t)$.

This enables us to further define:

Homology termination. (Definition 2.3.8.) Let $c_0 \in H_1(\mathbb{R}^3 \backslash M(0))$. If

$$\mathfrak{t}(c_0) := \sup\{t \geq 0 : c_0 \succ c \text{ for some } c \in H_1(\mathbb{R}^3 \backslash M(t))\}$$

is finite, then we say that c_0 *terminates at time* $\mathfrak{t}(c_0)$.

For example, in Figure 2, a_0 terminates at time T_1, and b_1 terminates at time T_2. On the other hand, b_0 *never* terminates even though $b_0(t)$ becomes trivial for $t > T_1$. a_1 also will never terminate, even though $a_1(t)$ becomes trivial for $t > T_2$.

Finally, we can describe what "a_0 breaks at a cylindrical singularity x" means.

Homology breakage. (Definition 2.3.12.) Let $c_0 \in H_1(\mathbb{R}^3 \backslash M(0))$, $T > 0$, and $x \in M(T)$. Suppose the following holds:

- For each $t \in [0, T)$, *the* element $c_0(t) \in H_1(\mathbb{R}^3 \backslash M(t))$ (such that $c_0 \succ c_0(t)$) exists.
- For every neighborhood $U \subset \mathbb{R}^3$ of x, for each $t < T$ sufficiently close to T, every element of $c_0(t)$ intersects U.

Then we say that c_0 *breaks at* (x, T). (See Figure 4.)

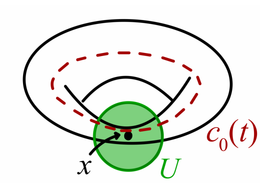


FIGURE 4. The picture at time t, for all $t < T$ sufficiently close to T.

For example, in Figure 2, a_0 breaks at (x, T_1), while b_1 breaks at (y, T_2).

As we will see, these three new concepts are quite useful. They satisfy some nice properties. To name a few:

- A homology class cannot break at a regular point, or a spherical singularity of the flow (Proposition 2.3.14 and 2.3.15).
- If the initial condition $M(0)$ is a closed surface, then some initial homology class must terminate at finite time (Remark 2.4.10).
- Suppose $\{M(t)\}_{t\geq 0}$ is a MCF with only spherical and cylindrical singularities. If a homology class terminates at some time T, then it must break at (x, T) for which some cylindrical singularity $x \in M(T)$ (Theorem 2.4.5).

These properties all are crucial in proving the main theorems.

Finally, let us now define precisely what "inward (or outward) torus neck will pinch" means in Theorem 1.1.1.

Definition 1.1.8. Given a torus M in $\mathbb{R}^3$, let a_0 (resp. b_0) be a generator of the first homology group of the interior (resp. exterior) region of M, which is isomorphic to $\mathbb{Z}$ (see Figure 5). We say that the *inward (resp. outward) torus neck of M will pinch* if a_0 (resp. b_0) will terminate under MCF.

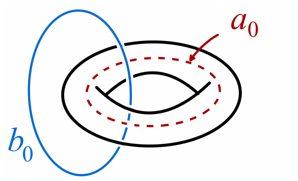


FIGURE 5. The loops a_0 and b_0.

Clearly, a_0 (and b_0) is unique up to a sign, and the above notion is independent of which sign we choose.

Structure of cylindrical singularities. Once we establish the topological concepts to keep track of the homology classes under the MCF, another challenge arises: we need to understand what happens to these homology classes as the MCF passes the cylindrical singularities.

Intuitively, a cylindrical singularity is just like a neck, and as we approach the singular time, the neck pinches as in Figure 1. However, the real situation can be much more complicated. For example, the MCF of the boundary of a tubular neighborhood of a rotationally symmetric S^1 in $\mathbb{R}^3$ will shrink to a singular set that is a rotationally symmetric S^1, where each singular point is cylindrical, but it does not look like a neck pinching.

There are two theories that we use to study the structure of cylindrical singularities. One is the partial regularity of the singular set of cylindrical singularities studied by White [Whi97] and Colding-Minicozzi [CM15, CM16]. This allows us to control the singular set. Most importantly, we obtain the compactness of the singular set of cylindrical singularities that are inward (or outward), and they can only show up for a zero-measured set of time.

Another important theory is the mean convex neighborhood theory of cylindrical singularities by Choi-Haslhofer-Heshkovitz [CHH22], and a generalized version by Choi-Haslhofer-Heshkovitz-White [CHHW22]. In [CHH22, CHHW22], they classified the possible limit flows at a cylindrical singularity. As a consequence, they obtain a canonical neighborhood theorem at a cylindrical singularity, to describe the local behavior of MCF.

We study the local behavior of MCF at cylindrical singularities based on these two theories. Nevertheless, the local behavior we need to understand does not come from [CHH22, CHHW22] directly. We record these results in §2.2.

Outline of proofs.

Theorem 1.1.1. We will prove by contradiction. For each $s \in [0,1]$, let $\mathcal{M}^s = \{M^s(t)\}_{t\geq 0}$ be the MCF (more precisely, level set flow) with $M^s(0) = M^s$ as its initial condition. Let a_0 (resp. b_0) be a generator of the first homology group of the inside (resp. outside) region of each torus M^s (recall Definition 1.1.8). Then, if Theorem 1.1.1 were false, $\mathcal{M}^s$ would be a MCF through cylindrical and spherical singularities for each s. This flow is unique and well-defined by Choi-Haslhofer-Hershkovits [CHH22]. Then, we show that for each s, either a_0 or b_0 will terminate, but *not both*. This claim relies on the fact, which we mentioned above, that if a homology class will terminate then it must break at a neck singularity. This fact uses crucially the mean convex neighborhood theorem and the canonical neighborhood theorem by Choi, Haslhofer, Hershkovits, and White [CHH22, CHHW22].

Thus, we can partition $[0, 1]$ into a disjoint union $A \sqcup B$, where A is the set of s for which a_0 will terminate, and B is the set of s for which b_0 will terminate. And we will show that A and B are both closed sets. Then, recall that we are given that $0 \in A$ and $1 \in B$. Since $[0, 1]$ is connected, a contradiction arises.

Theorem 1.1.3. We can apply Theorem 1.1.1 to prove Theorem 1.1.3, provided we know that the inward torus neck will pinch (i.e. a_0 will terminate, reusing the above notations) for the starting flow ($s = 0$), and the outward torus neck will pinch (i.e. b_0 will terminate) for the ending flow ($s = 1$). To prove, say, a_0 will terminate for the starting flow, we recall that $M^0(0)$ lies strictly inside the shrinker Σ. Then we will run MCF to these two, and *use the avoidance principle, which says the distance between the two surfaces will increase, to conclude that the* a_0 *must terminate.*

Theorem 1.1.4. Let Σ be an embedded, genus one shrinker with the least entropy. Suppose by contradiction that it is compact with index at least 6. Then disregarding the four (orthogonal) deformations induced by translation and scaling, there are still two others that decrease the entropy, one of which is the one-sided deformation given by the first eigenfunction of the Jacobi operator. Thus, we can construct a one-parameter family of tori with entropy less than Σ, such that the starting torus is inside Σ and the ending torus is outside Σ. Then, as in the proof of Theorem 1.1.3, we apply Theorem 1.1.1 to obtain *another* genus one shrinker with less entropy than Σ. This contradicts the definition of Σ.

Theorem 1.1.5. By Liu [Liu16], the shrinking doughnut $\mathbb{T}$ has index at least 7. Thus, by the result of Choi-Mantoulidis [CM22], there exists a one-parameter family of ancient *rescaled* MCF originating from $\mathbb{T}$ that decreases the entropy. As above, applying Theorem 1.1.1, we immediately obtain the desired genus one, self-shrinking tangent flow with lower entropy.

Open questions. We propose several open problems. The first one is motivated by generic MCF and min-max theory.

Conjecture 1.1.9. *There exists in* $\mathbb{R}^3$ *an embedded, genus one, index* 5 *self-shrinker that is the "second most generic".*

We say a self-shrinker Σ is the "second most generic", after the generic ones (the cylinder and the sphere), in the below sense: Suppose we have a one-parameter family of embedded surfaces $\{M^s\}_{s\in[0,1]}$ in $\mathbb{R}^3$. Then, we can perturb *this family* such that when we run MCF to every M^s, every singularity is either cylindrical, spherical, or modeled by Σ.

Note that Theorem 1.1.4 and its proof can be viewed as a verification of a very "local" version of this conjecture. Indeed, we can interpret Theorem 1.1.4 as the following: Any closed embedded genus one self-shrinker with index at least 6 is not the second most generic.

Now, we note that Theorem 1.1.1 does not hold for initial conditions with genus greater than one: See Remark 2.5.2.

Question 1.1.10. Can Theorem 1.1.1 be generalized to the higher genus case?

To do the higher genus case, one might need to consider higher parameter families of initial conditions.

Finally, we point out that many concepts that we introduce in this work highly rely on the extrinsic structure of mean curvature flow.

Question 1.1.11. Can the concepts of homology descent, homology termination, and homology breakage be adapted to the setting of Ricci flow?

Organizations. In §2.1, we will introduce the preliminary materials, which include a refined canonical neighborhood theorem. In §2.3, we will define the concepts of homology descent, homology termination, and homology breakage, and prove some relevant basic propositions. In §2.4, we focus on the case of MCF through cylindrical and spherical singularities, with torus as the initial condition. In §2.5, we prove the main theorems.

1.2. **A strong multiplicity one theorem in min-max theory.** Results in this section are joint works with Yangyang Li.

For every closed smooth Riemannian manifold (M, g) of dimension at least 2, a non-decreasing sequence of positive real numbers, $\{\omega_p(M, g)\}_{p=1,2,...}$, is associated with it. These numbers are referred to as the min-max *p-widths* of M and they can be heuristically defined as follows:

Using geometric measure theory, we consider the flat cycle space $\mathcal{Z}$, which consists of "all boundaryless geometric objects in M of codimension 1", with coefficients in $\mathbb{Z}_2$. F. Almgren [Alm62] showed that this space $\mathcal{Z}$ is weakly homotopy equivalent to $\mathbb{RP}^\infty$, and consequently, its cohomology ring $H^*(\mathcal{Z}; \mathbb{Z}_2)$ is $\mathbb{Z}_2[\bar{\lambda}]$, which is generated by an order-2 element $\bar{\lambda}$. Now, a continuous map $\Phi : X \to \mathcal{Z}$, where X is a finite simplicial complex, is called a *p-sweepout* if $\Phi^*(\bar{\lambda}^p) \neq 0$. The p-width of M is then defined as

$$\omega_p(M, g) := \inf_{\substack{p\text{-sweepout} \\ \Phi: X \to \mathcal{Z}}} \sup_{x \in X} \operatorname{area}(\Phi(x)) \,.$$

Note that sometimes the g in $\omega_p(M, g)$ is omitted. Remarkably, these p-widths also follow a Weyl law [LMN18], similar to the spectrum of the Laplace-Beltrami operator on M.

For the unit 3-sphere S^3, it is well-known that

$$\omega_1(S^3) = \omega_2(S^3) = \omega_3(S^3) = \omega_4(S^3) = 4\pi \,.$$

Building upon the resolution of the Willmore conjecture by Marques-Neves [MN14], C. Nurser [Nur16] showed that

$$\omega_5(S^3) = \omega_6(S^3) = \omega_7(S^3) = 2\pi^2, \;\; 2\pi^2 < \omega_9(S^3) < 8\pi, \;\; \omega_{13}(S^3) \leq 8\pi \,.$$

Recently, F. Marques [Mar23] proved that $\omega_8(S^3)$ is also equal to $2\pi^2$.

The following question was posed by Marques-Neves [MN17, §9].

Question. Which p-widths of S^3 lie strictly between $2\pi^2$ and 8π?

In this work, we prove that the 10th to the 13th widths of S^3 lie strictly between $2\pi^2$ and 8π, by establishing the following theorem:

Theorem 1.2.1. *$\omega_{13}(S^3) < 8\pi$ for the unit 3-sphere S^3.*

Note that it is still open whether the 14-width of S^3 is also strictly less than 8π.

Our pursuit of this improvement from $\omega_{13}(S^3) \leq 8\pi$ to Theorem 1.2.1 is also motivated by recent developments. In recent years, the p-widths have played a crucial role in constructing minimal hypersurfaces using the Almgren-Pitts min-max theory [MN17, IMN18, Zho20, MN21, Li23b]. In particular, they are crucial in Song's proof [Son23] of Yau's conjecture regarding the existence of infinitely many immersed closed minimal surfaces in 3-manifolds [Yau82, p.689]. In min-max theory, one subtle feature is that the minimal hypersurfaces obtained may have multiplicities. In a closed Riemannian manifold (M^{n+1}, g) with $3 \leq n+1 \leq 7$, the min-max theory yields a collection $\{\Sigma_1, \ldots, \Sigma_N\}$ of disjoint, closed, smooth, embedded, minimal hypersurfaces accompanied by a set $\{m_1, \ldots, m_N\}$ of positive integers. They constitute a varifold

$$m_1|\Sigma_1| + \cdots + m_N|\Sigma_N| \tag{1.1}$$

with a mass of $\omega_p(M)$. Note that any varifold of the form (1.1) is called an *embedded minimal cycle.*

Regarding the result $\omega_{13}(S^3) \leq 8\pi$, C. Nurser constructed an explicit 13-sweepout such that the supremum of the area is attained by multiplicity-two equatorial 2-spheres. In our proof of Theorem 1.2.1, we show that such a sweepout cannot be optimal, thereby leading to a strict inequality of the 13-width.

For other explicit computations of widths, it is worth noting that for the unit round 2-sphere,

- Aiex showed that the first three widths are 2π and the fourth to the eighth are 4π [Aie19];

- Chodosh-Mantoulidis showed that the p-width is given by $2\pi\lfloor\sqrt{p}\rfloor$ [CM23].

Readers interested can refer to [Lun19, BL22a, Chu22, Don22, BL23, Zhu23].

Furthermore, our techniques lead to a more general multiplicity one theorem, explained in the following. When the ambient manifold has a bumpy metric or positive Ricci curvature, X. Zhou [Zho20] proved the existence of a multiplicity-one, two-sided, minimal hypersurface with area given by the p-width. However, it is important to note that this does not hold for general metrics, as shown by Wang-Zhou [WZ22]. In our work, we strengthen Zhou's multiplicity one theorem as follows:

Theorem 1.2.2 (Strong multiplicity one theorem)**.** *Consider a closed Riemannian manifold (M^{n+1}, g) ($3 \leq n+1 \leq 7$) equipped with a bumpy metric or a metric of positive Ricci curvature. Let $p \in \mathbb{N}^+$. Then there exists a pulled-tight minimizing sequence $(\Phi_i)_i$ for the p-width $\omega_p(M, g)$ such that for every embedded minimal cycle V in the critical set $\mathbf{C}((\Phi_i)_i)$, there exists some current $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $V = |T|$.*

Remark 1.2.3. In particular, the varifold V in Theorem 1.2.2 is two-sided and multiplicity one.

Precise definitions of the terminologies will be provided in §3.1.

It is worth highlighting that very recently, Wang-Zhou [WZ23] established a multiplicity one theorem in the Simon-Smith min-max setting, which yielded four embedded, minimal 2-spheres in every S^3 with a bumpy metric or positive Ricci curvature. Their work builds on Sarnataro-Stryker's work [SS23] on the regularity for minimizers of the prescribed mean curvature functional.

Main ideas. In this work, we employ various variants of min-max theory. Instead of doing min-max over the class of maps homotopic to a given sweepout, we will, for example, consider the class of maps *homologous* to a given sweepout, or restrict the class of maps by imposing an upper bound on mass (a technique previously developed by the second author in [Li23b]). These novel min-max theorems are detailed in §3.3.

Let us outline their applications in the main theorems.

Theorem 1.2.1. Let us sketch the proof that $\omega_{13}(S^3) < 8\pi$. By C. Nurser [Nur16], there exists a 13-sweepout Φ_0 such that $\mathbf{M} \circ \Phi_0 \leq 8\pi$ and its critical set consists of multiplicity two equatorial 2-spheres. Suppose by contradiction that $\omega_{13}(S^3) = 8\pi$.

Let us take a sequence of bumpy metrics g_i that tend to the round metric $\bar{g}$. Take a sequence of positive numbers $\delta_i \to 0$. We will run for each i a *restrictive homological min-max with a mass upper bound* $8\pi + \delta_i$ under the metric g_i. More precisely, for each i, we will consider the class $\mathcal{H}_i$ of all maps that is homologous to Φ_0 through some "cobordism" (in the space of cycles) whose mass is bounded from above by $8\pi + \delta_i$, and then do min-max in this class. The min-max width L_i will tend to 8π.

By X. Zhou's multiplicity one theorem, for each g_i, the width L_i corresponds to some min-max minimal hypersurface Σ_i with multiplicity one. And by Marques-Neves [MN21], we can assume that Σ_i is the only minimal hypersurface in (M, g_i) that has area L_i, even if multiplicity is allowed. Since $\bar{g}$ has positive Ricci curvature, by Sharp's compactness theorem we can assume Σ_i tends to some minimal hypersurface Σ smoothly, with multiplicity one.

Now, let $\mathcal{C}^1$ (resp. $\mathcal{C}^{>1}$) be the set of $\bar{g}$-minimal hypersurfaces with multiplicity one (resp. greater than one) and area 8π. For some sufficiently large i, we choose an "optimal" 13-sweepout Φ_i in $\mathcal{H}_i$, and an "optimal" cobordism Ψ_i between Φ_0 and Φ_i, so that by [MN21, Theorem 4.7] we would have, heuristically:

- If $\Phi_i(x)$ has high area (i.e. has area greater than $L_i - \epsilon$ for some $\epsilon > 0$), then $\Phi_i(x)$ is close to Σ_i, and thus to $\Sigma \in \mathcal{C}^1$.
- If $\Psi_i(x)$ has high area, then $\Psi_i(x)$ is close to $\mathcal{C}^1 \cup \mathcal{C}^{>1}$.

We will now derive a contradiction by constructing some $\Xi \in \mathcal{H}_i$ such that $\mathbf{M}_{g_i} \circ \Xi < L_i$. For simplicity, *let us suppose for now there is a Morse index upper bound for every element* $\mathcal{C}^1$ *and* $\mathcal{C}^{>1}$. Then, since $\bar{g}$ has a positive Ricci curvature, the closure of $\mathcal{C}^1$ and $\mathcal{C}^{>1}$ are separated from each other by Sharp's compactness. Thus, viewing $\Phi_i, \Psi_i, \mathcal{C}^1$, and $\mathcal{C}^{>1}$ all as sets of currents by abuse of notation, we can choose a subset $A \subset \Psi_i$ that is away from $\mathcal{C}^{>1}$ and contains all elements of Ψ_i which are close to $\mathcal{C}^1$: See Figure 6. Now, we remove from Φ_i the part $\Phi_i \cap A$, and glue back in a cap $\partial A \backslash \Phi_i$. The new sweepout is our Ξ: See Figure 6. Now, Ξ is away from $\mathcal{C}^1$ and $\mathcal{C}^{>1}$. Thus, by the two bullet points in the last paragraph, $\mathbf{M}_{g_i} \circ \Xi < L_i$. Contradiction arises.

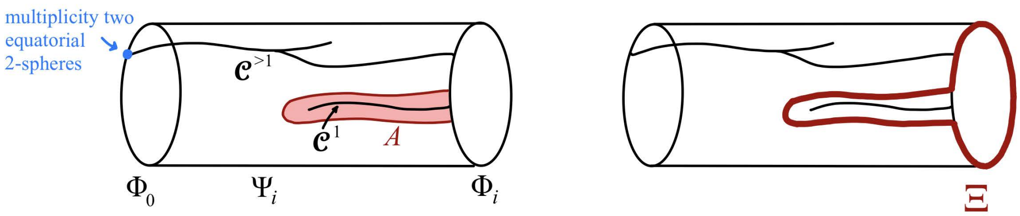


FIGURE 6. Constructing the family Ψ.

In reality, *we do not have an upper bound on Morse index* for elements of $\mathcal{C}^1$ and $\mathcal{C}^{>1}$. In fact, the sets $\mathcal{C}^1$ and $\mathcal{C}^{>1}$ will be defined in a different way, using Pitts' notion of almost-minimizing, and the fact that they are separated would be deduced by examining properties of annular replacements.

Theorem 1.2.2. Let $\mathcal{C}^1$ (resp. $\mathcal{C}^{>1}$) be the set of $\bar{g}$-minimal hypersurfaces with multiplicity one (resp. greater than one) and area $\omega_p(M)$. As above, for simplicity, we assume these two sets are separated. In the spirit of [MN21, Theorem 4.7], choose an "optimal" p-sweepout Φ for $\omega_p(M)$ such that if x is such that $\Phi(x)$ has high area, then $\Phi(x)$ is close to $\mathcal{C}^1 \cup \mathcal{C}^{>1}$. Let W be the set of x such that $\Phi(x)$ is in fact close to $\mathcal{C}^{>1}$. We will do a relative, homological min-max process, by considering *the set $\mathcal{H}$ of maps that is homological to $\Phi|_W$ relative to $\Phi|_{\partial W}$*.

To prove Theorem 1.2.2, it suffices to show that the min-max width for the relative, homological min-max class $\mathcal{H}$ is less than $\omega_p(M)$, because then we can "lower" $\Phi|_W$, and thus Φ, away from $\mathcal{C}^{>1}$. Suppose the otherwise, so that $\Phi|_W$ is an "optimal" sweepout for $\mathcal{H}$ such that elements of high area are close to $\mathcal{C}^{>1}$. Then we are in a situation analogous to the proof of Theorem 1.2.1, where the 13-sweepout Φ_0 is an optimal sweepout whose critical set lie in $\mathcal{C}^{>1}$. Thus, we can argue as in the proof of Theorem 1.2.1 to get a contradiction.

Organization. While Theorem 1.2.1 will be a direct consequence of Theorem 1.2.2, for the sake of clarity in presentation, we choose to first prove Theorem 1.2.1, which is simpler, and then prove Theorem 1.2.2.

In §3.1, we include some preliminary materials. In §3.2 we introduce the notion of (m, r)-almost minimizing varifold. In §3.3, we prove some restrictive min-max theorems, which include a homotopic and a homological version. In §3.4, we prove Theorem 1.2.1. In §3.5, we prove some technical propositions used in the previous section. In §3.6, we prove the strong multiplicity one theorem, Theorem 1.2.2.

1.3. **A free boundary minimal surface via a 6-sweepout.** Given a compact Riemannian 3-manifold M, one can *relate the topology of the space of all surfaces in M to minimal surfaces in M via Morse theory.* One may even obtain information about the genus, Morse index, and area of the minimal surfaces. In this work, we will illustrate this phenomenon by looking at surfaces with low genus and area in the compact Euclidean unit 3-ball $\mathbb{B}^3$, via the Almgren-Pitts and the Simon-Smith min-max theory.

Let $\mathcal{E}$ denote the set of all surfaces, possibly with boundary, in $\mathbb{B}^3$ that are smooth and properly embedded *except possibly at finitely many points* (see §4.1 for details). The reason for allowing singularities is that we want to study the space of all surfaces, regardless of their genus or number of connected components, as a whole. In fact, let us define on $\mathcal{E}$ the following topology inspired by the Simon-Smith min-max theory: For each finite set $P \subset \mathbb{B}^3$, we define on the subset

$$\{S \in \mathcal{E} : S \backslash P \text{ is smooth and properly embedded}\} \tag{1.2}$$

the topology induced by the graphical C^∞-convergence within open sets $U \subset\subset \mathbb{B}^3 \backslash P$ (meaning $\overline{U} \subset \mathbb{B}^3 \backslash P$). Now, we collect all open sets in (1.2) for all possible P to form a base, thereby defining a topology on $\mathcal{E}$. Note that under this topology, one has continuous paths in $\mathcal{E}$ of surfaces with different genus or number of connected components via neck-pinching. Then our first main result is the following.

Let $\mathcal{E}_g \subset \mathcal{E}$ be the subset of *smooth* surfaces with genus g, and $\mathcal{E}^a \subset \mathcal{E}$ the subset of surfaces with area less than a for $a \in (0, \infty]$. Note that $\mathcal{E}^\infty \neq \mathcal{E}$, since an element of $\mathcal{E}$ can have an infinite area, concentrated near a singularity. And as in the Simon-Smith min-max theory, the genus of a disconnected smooth surface is defined as the sum of the genus of each of its connected components.

Theorem 1.3.1. *The first to the sixth cohomology groups of*

$$\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$$

in $\mathbb{Z}_2$-coefficients are non-trivial: In fact, the cup-length of this space is at least 6. And the same is true for any subspace of $\mathcal{E}^\infty$ that contains $\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$.

Note that $\overline{\mathcal{E}_0 \cup \mathcal{E}_1}$ denotes the closure of $\mathcal{E}_0 \cup \mathcal{E}_1$ in $\mathcal{E}$, and the *cup-length* of a space X is defined as the maximum number of elements in the cohomology ring of X with degree at least 1 such that their cup product is non-trivial. We remark that 2π is twice the area of the equatorial disk in $\mathbb{B}^3$.

Let us mention the following results. In his celebrated work [Hat83], Hatcher proved the Smale conjecture, implying that the space of smoothly embedded 2-spheres in the (round) 3-sphere deformation retracts to the subspace of great 2-spheres, which is homeomorphic to $\mathbb{RP}^3$ and thus has cup-length 3. Moreover, based on Marques-Neves' ground-breaking resolution of the Willmore conjecture [MN14], Nurser showed that the space of *flat 2-cycles* in the unit round 3-sphere with area at most $2\pi^2$ (which is the area of the Clifford torus) has cup-length in $\mathbb{Z}_2$-coefficients at least 7 [Nur16].

Theorem 1.3.1 follows immediately from the result below, of which the terminologies will be defined precisely in §4.1.

Theorem 1.3.2. *There exists in the Euclidean unit* 3*-ball a family* Ψ *of surfaces such that:*

(A) Ψ *is a* 6*-sweepout in the sense of Almgren-Pitts min-max theory.*

(B) Ψ *is a smooth family of surfaces with genus at most* 1*, in the sense of Simon-Smith min-max theory.*

(C) The area of each element in Ψ *is less than* 2π*.*

Theorem 1.3.2 also gives the following result immediately.

Corollary 1.3.3. *The Almgren-Pitts* 6*-width of the Euclidean unit* 3*-ball is less than* 2π*.*

Currently, the Almgren-Pitts widths of $\mathbb{B}^3$ are not well-understood: While the first three widths are π and are detected by the equatorial disk (since the collection of flat disks in $\mathbb{B}^3$ is a 3-sweepout), the fourth already seems to be unknown. Regarding computations of Almgren-Pitts widths of other manifolds, see also [Aie19, BL22b, CM21, Don22, Zhu22].

Let us now turn to the other side of the story: Free boundary minimal surfaces in the $\mathbb{B}^3$. In recent years, besides the two most basic examples, the equatorial disk and the critical catenoid, an abundance of free boundary minimal surfaces in $\mathbb{B}^3$ were constructed. For example, by solving extremal eigenvalue problems, Fraser-Schoen constructed examples with genus 0 and arbitrary number of boundary components [FS16]. Using gluing techniques, Kapouleas-Li constructed embedded free boundary minimal surfaces of large genus that desingularize the union of the equatorial disk and the critical catenoid [KL17]. (See [CFS20, CSW22, FPZ17, KM20, KZ21, Ket16a, Ket16b, KW17] for more examples.)

We will use min-max theory to produce a free boundary minimal surface. The advantage of this approach is that one can upper bound the Morse index of the minimal surface because of the work of Marques-Neves [MN16]. In general, Morse index is difficult to compute. For example, to our best knowledge, in $\mathbb{B}^3$ the only embedded free boundary minimal surfaces whose index are known are the equatorial disk and the critical catenoid:

They have index 1 and 4 respectively [Dev19, SZ19, Tra20]. In addition, from the recent resolution of the multiplicity one conjecture in the free boundary setting by Sun-Wang-Zhou [SWZ20] based on the work of Zhou [Zho20], we know there exists a sequence $\{\Sigma_k\}$ of embedded free boundary minimal surfaces in $\mathbb{B}^3$ with area growth of order $k^{1/3}$ and index at most k. However, using the Almgren-Pitts min-max theory, one cannot control the genus of the surfaces. In this work, we apply the Simon-Smith min-max theory to the family Ψ in Theorem 1.3.2 to construct an example with index, genus, and area bound:

Theorem 1.3.4. *There exists in the Euclidean unit* 3*-ball an embedded free boundary minimal surface with genus* 0 *or* 1*, Morse index* 4 *or* 5*, and area in the range* $(\pi, 2\pi)$*, that is not the equatorial disk or the critical catenoid.*

In fact, using the results of Sargent [Sar17] and Ambrozio-Carlotto-Sharp [ACS18b] that lower bound the index of a free boundary minimal surface by its genus and number of boundary components, we know that the surface in Theorem 1.3.4 has at most 16 boundary components. But we believe this bound is far from optimal (see §1.3 below). We also note that, since we have to prove the index bound, we cannot use the equivariant min-max theory of Ketover [Ket16a].

Remark 1.3.5. We remark that Carlotto-Franz-Schulz [CFS20] showed, using equivariant min-max theory, there exists a free boundary minimal surface in $\mathbb{B}^3$ that has genus 1, area less than 3π, a connected boundary, and symmetry group D_2, where $D_2 \subset SO(3)$ denotes the dihedral group with four elements (see the Geometric Analysis Gallery by Schulz [Sch]). In fact, as we will see in §4.2, Theorem 1.3.2 can reproduce their result and slightly improve the area bound from 3π to 2π.

The family Ψ in Theorem 1.3.2 can be modified to become a desirable 6-sweepout in $\mathbb{R}^3$ equipped with the Gaussian metric $\frac{1}{4\pi}e^{-|\mathbf{x}|^2/4}g_0$, in which g_0 denotes the Euclidean metric, allowing one to construct a self-shrinker with genus, index and Gaussian area control. However, the Gaussian metric has a singularity at infinity, which poses some challenges in carrying out the min-max theory. We plan to address this in our upcoming work.

Open questions. Regarding Theorem 1.3.1, it would be interesting to change the genus 0 and 1 constraint, the area bound 2π, or the ambient space $\mathbb{B}^3$ (to the round 3-sphere for example), and investigate the topology of the corresponding space of surfaces. It will be nice to have more examples of k-sweepouts for $k > 6$ that are smooth families. And for each k, among k-sweepouts Φ that are smooth families, is there a non-trivial lower bound for the maximum of the genus of elements in Φ?

We conjecture that the free boundary minimal surface in Theorem 1.3.4, denoted Σ, has index 5. One can also ask if Σ has the third lowest area among all free boundary minimal surfaces in $\mathbb{B}^3$, after the equatorial plane and the critical catenoid. Moreover, we speculate that Σ is the same as the free boundary minimal surface constructed by Carlotto-Franz-Schulz [CFS20] mentioned in Remark 1.3.5.

Concerning the Almgren-Pitts min-max theory in $\mathbb{B}^3$, we conjecture the 4-width is detected by the critical catenoid $\mathbb{K}$. In particular, showing the 4-width is at least area($\mathbb{K}$) seems challenging, as it may depend on the conjecture that the second least area of an immersed free boundary minimal surface in $\mathbb{B}^3$ is realized by the critical catenoid [Li19, §7]. As for the 5-width and the 6-width, it will be interesting to know if they are detected by the free boundary minimal surface of Theorem 1.3.4.

Overview of proofs. Let us outline the construction of the smooth family Ψ in Theorem 1.3.2. We first consider the saddle surface $\{x^2 - y^2 + z = 0\}$ in $\mathbb{R}^3$, and then translate, rescale, and rotate it arbitrarily: We even allow the scaling factor to be 0 or $\pm\infty$. Then we collect all such surfaces, and it turns out this collection can be parametrized by a 7-dimensional quotient space of some D_2-action on $\mathbb{RP}^4 \times SO(3)$. This is actually due to the D_2-symmetry of the saddle. However, this collection contains intersecting planes like $\{x^2 - y^2 = 0\}$, the blow down of the saddle, which has a singular line and thus is not allowed in the Simon-Smith setting. To resolve this, we *desingularize the intersecting* *planes by adding a small z^3 term to their defining equations* (e.g. see Figure 16 and Table 1), so that only isolated singularities appear. Finally, we intersect all surfaces with $\mathbb{B}^3$ to define Ψ.

Theorem 1.3.1 is an immediate consequence of Theorem 1.3.2: See §4.2.

Finally, for Theorem 1.3.4, we will use the smooth family Ψ in Theorem 1.3.2 as follows. Let $\Psi^{(5)}$ denote the subfamily of Ψ parametrized by a 5-skeleton of the parameter space of Ψ. By applying the Simon-Smith min-max theorem to $\Psi^{(5)}$, we obtain a free boundary minimal surface Γ with genus at most 1, index at most 5, and area less than 2π. Note that, although it is not known if the multiplicity one conjecture holds in the Simon-Smith setting, we can guarantee that Γ has multiplicity one because $\text{area}(\Gamma) < 2\pi$ and the least possible area of a free boundary minimal surface in $\mathbb{B}^3$ is π. Then by the fact that Ψ is a 6-sweepout and topological arguments of Lusternik-Schnirelmann, we show that the method above, with some modifications, produces a free boundary minimal surface with the desired properties that is not the equatorial disk or the critical catenoid.

Organization. We introduce some preliminaries in §4.1, and in §4.2 prove the main results. The proofs of some propositions used in §4.2 will be postponed to §4.3.

2. Genus one singularities in MCF

This section is from a joint work with Ao Sun.

2.1. **Preliminaries.**

In §2.1 let us set up the language to define MCF through cylindrical and spherical singularities.

Weak solutions of MCF. Throughout this work, we will be focused on two different types of weak solutions of MCF. One is a set-theoretic weak solution defined by the **level set flow**, and another one is a geometric measure theoretic weak solution called **Brakke flow**. We refer the readers to [ES91, Ilm92] for detailed discussions of level set flows, and we refer the readers to [Bra78, Ilm94] for detailed discussions of Brakke flow.

The level set flow equation is a degenerate parabolic equation

$$\partial_t u = \Delta u - \left(\frac{D^2u(Du, Du)}{|Du|^2}\right). \tag{2.1}$$

Suppose $M(0)$ is a closed hypersurface in $\mathbb{R}^{n+1}$, then if $u(\cdot, t)$ solves (2.1) with $M(0) = \{x \in \mathbb{R}^{n+1} : u(\cdot, 0) = 0\}$, then $M(t) := \{x \in \mathbb{R}^{n+1} : u(\cdot, 0) = 0\}$ can be viewed as a weak solution to MCF. In particular, when $M(t)$ is smooth, this weak solution coincides with the classical solution of MCF.

The level set flow was introduced by Osher-Sethian in [OS88]. The solution to (2.1) may not be smooth, but it suffices to use Lipschitz solutions to define weak MCF. Chen-Giga-Goto [CGG91] and Evans-Spruck [ES91] introduced the viscosity solutions to (2.1), and these solutions are Lipschitz. Throughout this work, when we say u is a **level set function** or a solution to the level set flow equation, u is actually a viscosity solution to (2.1).

The set-theoretic solution of a MCF will be called the **level set flow** or **biggest flow**. These notions are used by Ilmanen [Ilm92] and White [Whi00, Whi03]. The term "biggest flow" is used to avoid the ambiguity of the weak solution for noncompact flow.

Brakke flow is defined using geometric measure theory. Let X be a complete manifold without boundary. The Brakke flow is a family of Radon measures $\{\mu_t\}_{t\geq 0}$, such that for any test function $\phi \in C_c^2(X)$ with $\phi \geq 0$,

$$\limsup_{s\to t} \frac{\mu_s(\phi) - \mu_t(\phi)}{s-t} \leq \int (-\phi H^2 + \nabla^\perp \cdot \vec{H}) d\mu_t,$$

where $\vec{H}$ is the mean curvature vector of μ_t whenever μ_t is rectifiable and has L^2-mean curvature in the varifold sense. Otherwise, the right-hand side is defined to be $-\infty$.

In general, the Brakke flow starting from a given initial data is not unique. We will be interested in unit regular cyclic integral Brakke flows. We refer the readers to [Whi09] for detailed discussions of these notions. The existence of such a flow starting from a smooth surface is guaranteed by Ilmanen's elliptic regularization, see [Ilm94]. Such flows have good compactness theory.

Setting and notations. Let $M(0)$ be a closed smooth n-dimensional hypersurface in $\mathbb{R}^{n+1}$ that bounds a compact set $K_{\text{in}}(0)$. Let $K_{\text{out}}(0) = \overline{\mathbb{R}^{n+1} \backslash K_{\text{in}}(0)}$. Now, denote by

$$\{M(t)\}_{t\geq 0}, \{K_{\text{in}}(t)\}_{t\geq 0}, \text{ and } \{K_{\text{out}}(t)\}_{t\geq 0}$$

respectively the level set flow (i.e. the biggest flow) with initial condition $M(0), K_{\text{in}}(0)$, and $K_{\text{out}}(0)$. Then, define their *spacetime tracks*

$$\mathcal{M} = \{(x,t) : x \in M(t), t \geq 0\},$$

$$\mathcal{K}_{\text{in}} = \{(x,t) : x \in K_{\text{in}}(t), t \geq 0\},$$

$$\mathcal{K}_{\text{out}} = \{(x,t) : x \in K_{\text{out}}(t), t \geq 0\}.$$

We then define the *inner flow* of $M(0)$,

$$M_{\text{in}}(t) = \{x : (x,t) \in \partial\mathcal{K}_{\text{in}}\}$$

and the *outer flow* of $M(0)$,

$$M_{\text{out}}(t) = \{x : (x,t) \in \partial\mathcal{K}_{\text{out}}\}.$$

Lemma 2.1.1. *Let $u : \mathbb{R}^{n+1} \times [0,\infty) \to \mathbb{R}$ be a level set function of $\mathcal{M}$, with $u(\cdot,0) \leq 0$ on $K_{\text{in}}(0)$. Then*

$$\mathbb{R}^{n+1} \backslash K_{\text{in}}(t) = \{x : u(x,t) > 0\}, \;\; \mathbb{R}^{n+1} \backslash K_{\text{out}}(t) = \{x : u(x,t) < 0\}.$$

Proof. For the first claim, we let $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(x) = x$ if $x > 0$ and $\Phi(x) = 0$ otherwise. By the relabelling lemma ([Ilm92, Lemma 3.2]), $v := \Phi \circ u$ also satisfies the level set equation. Noting $v(\cdot,0) = 0$ precisely on $K_{\text{in}}(0)$, which is compact, we know by the uniqueness of level set flow that v is a level set function of $\mathcal{K}_{\text{in}}$. Hence,

$$\mathbb{R}^{n+1} \backslash K_{\text{in}}(t) = \{x : u(x,t) > 0\}.$$

The second claim is similar. We let $\Psi : \mathbb{R} \to \mathbb{R}$ by $\Psi(x) = x$ if $x < 0$ and $\Psi(x) = 0$ otherwise. Then $v = \Psi \circ u$ satisfies the level set equation by the relabelling lemma, and $\{x : u(x,t) \geq 0\} = \{x : v(x,t) = 0\}$, which is non-compact. Nevertheless, by Ilmanen [Ilm92], because any level sets other than K_{out} are compact, $\{x : v(x,t) = 0\}$ is the biggest flow, which is unique. Then the second claim will follow. □

Finally, we denote

$$W_{\text{in}}(t) = \mathbb{R}^{n+1} \backslash K_{\text{out}}(t), \;\; W_{\text{out}}(t) = \mathbb{R}^{n+1} \backslash K_{\text{in}}(t), \;\; W(t) = W_{\text{in}}(t) \cup W_{\text{out}}(t).$$

In fact, we will furthur define the spacetime track

$$W_{\text{in}}[t_0,t_1] = \bigcup_{t \in [t_0,t_1]} W_{\text{in}}(t) \times \{t\},$$

and we can similarly define $W_{\text{out}}[t_0,t_1]$ and $W[t_0,t_1]$. The reason we care above these sets is that their topological changes are described by White [Whi95], which will be crucial for us later. We remark that, when we need to specify the flow $\mathcal{M}$, we will add a superscript $^{\mathcal{M}}$ to the symbols: e.g. we will write $W_{\text{in}}^{\mathcal{M}}(t)$ in place of $W_{\text{in}}(t)$.

Let (x, T) be a singularity of $\mathcal{M}$, and $\lambda_j \to \infty$. Then any subsequential limit, in the sense of Brakke flow (see [Ilm94, Section 7]), of the rescaled flows

$$\{\lambda_j(M(\lambda_j^{-2}t + T) - x)\}_{-\lambda_j^2 T < t < 0}$$

is called a *tangent flow* at (x, T). Presumably, the tangent flow need not be unique. Nevertheless, Colding-Minicozzi [CM15] proved that if one tangent flow is the cylinder, then the tangent flow is unique. And the convergence is in C^∞_{loc} by Brakke's regularity theorem (see [Whi05]).

Now, following [CHHW22], we call (x, T) an *inward neck singularity* of $\mathcal{M}$ if as $\lambda \to \infty$ the rescaled flows

$$\{\lambda(K_{\text{in}}(\lambda^{-2}t + T) - x)\}_{-\lambda^2 T < t < 0}$$

converge locally smoothly with multiplicity one to the solid shrinking cylinder

$$\{B^n(\sqrt{-2(n-1)t}) \times \mathbb{R}\}_{t<0}$$

up to rotation and translation. Similarly, we can define an *outward neck singularity.* If, instead, those rescaled flows converge with multiplicity one to the solid shrinking ball

$$\{B^{n+1}(\sqrt{-2nt})\}_{t<0}$$

up to translation, then we call (x, T) an *inward spherical singularity.* We can again similarly define an *outward spherical singularity.*

2.2. **MCF through cylindrical and spherical singularities.** If every singularity of $\mathcal{M}$ is a neck or a spherical singularity, then we call $\mathcal{M}$ a *MCF through cylindrical and spherical singularities.* In this case, building on Hershkovits-White [HW20], Choi-Haslhofer-Hershkovits-White showed $M(t), M_{\text{in}}(t)$, and $M_{\text{out}}(t)$ are all the same [CHHW22, Theorem 1.19], i.e. fattening does not occur.

Neck singularities are well-understood after the work of many [HS99a, HS99b, Whi00, Whi03, SW09, Wan11, And12, Bre15, CM15, HK17, ADS19, ADS20, CHH22, CHHW22], among others. We will first in Theorem 2.2.3 state the canonical neighborhood theorem of Choi-Haslhofer-Hershkovits-White [CHHW22]. Using that, we obtain a more detailed topological description of neck singularities in Theorem 2.2.4.

Definition 2.2.1. Let $X = (x, T)$ be a regular point in a level-set flow $\mathcal{M}$. Let $\lambda := |\mathbf{H}(x)|$. Suppose there exists an ancient MCF $\{\Sigma(t)\}$ that is, up to spacetime translation and parabolic rescaling, one of the following:

- the shrinking sphere,
- the shrinking cylinder with axis ℓ,
- the translating bowl with axis ℓ,
- the ancient oval with axis ℓ,

and furthermore satisfies that: For each $t \in (-1/\epsilon^2, 0]$ and inside $B_{1/\epsilon}(0) \subset \mathbb{R}^{n+1}$,

$$\lambda(M(\lambda^{-2}t + T) - x) \text{ and } \Sigma(t)$$

are ϵ-close in $C^{\lfloor 1/\epsilon \rfloor}$. Then, we call

$$\left(T - \frac{1}{\lambda^2 \epsilon^2}, T\right] \times B_{\frac{1}{\lambda\epsilon}}(x)$$

an *ϵ-canonical neighborhood of X with axis ℓ.*

We will also have a weaker definition, for situations when we focus on a time-slice:

Definition 2.2.2. Let x be a regular point in a subset M. Let $\lambda := |\mathbf{H}(x)|$. Suppose there exists a hypersurface Σ that is, up to translation and rescaling, a time-slice of one of the following:

- the shrinking sphere,
- the shrinking cylinder with axis ℓ,
- the translating bowl with axis ℓ,

- the ancient oval with axis ℓ,

and furthermore satisfies that: Inside $B_{1/\epsilon}(0) \subset \mathbb{R}^{n+1}$, $\lambda(M - x)$ and Σ are ϵ-close in $C^{\lfloor 1/\epsilon \rfloor}$. Then, we call $B_{\frac{1}{\lambda\epsilon}}(x)$ an *ϵ-canonical neighborhood of x with axis ℓ.*

One can compare the above with the notion of ϵ-canonical neighborhoods in 3-dimensional Ricci flow [MF10, Lecture 2].

Theorem 2.2.3 (Canonical neighborhood)**.** *Let (x, T) be a neck singularity of a MCF through cylindrical and spherical singularities $\mathcal{M}$, and ℓ be the axis of the cylindrical tangent flow at (x, T). Then for every $\epsilon > 0$, there exists $\delta, \bar{\delta} > 0$ such that every regular point of $\mathcal{M}$ in $B_{2\delta}(x) \times (T - \bar{\delta}, T + \bar{\delta})$ has an ϵ-canonical neighborhood with axis ℓ in the sense of Definition 2.2.1.*

We used balls of radius 2δ (instead of δ): This is solely for the sake of notational convenience, so that it can be directly quoted in Theorem 2.2.4.

Proof. This is from [CHHW22, Corollary 1.18]: Note that all limit flows at (x, T) have the same axis (see the end of §1 of their paper). □

In dimension $n = 2$ or 3. In the cases $n = 2$ or 3, at almost every time, the time-slice of a MCF through cylindrical and spherical singularities is smooth, by Colding-Minicozzi [CM16, Corollary 0.6]. Based on this, in items (3) - (6) of the following theorem, we will obtain a topologically more refined picture of neck-pinches. The shapes of the surfaces described in items (3) - (6) are illustrated in Figure 7.

Theorem 2.2.4. *There exists a universal constant $R_0 = R_0(n)$ with the following significance. Let (x, T) be an inward neck singularity of a MCF through cylindrical and spherical singularities $\mathcal{M}$ in $\mathbb{R}^{n+1}$, with $n = 2$ or 3, and ℓ be the axis of the cylindrical tangent flow at (x, T). For every $\delta_0 > 0$ and every $R > R_0$, there exists $\delta \in (0, \delta_0)$ and $\bar{\delta} > 0$ such that:*

(1) Let $B = B_\delta(x)$. Then the set $M(T - \bar{\delta}) \cap B$

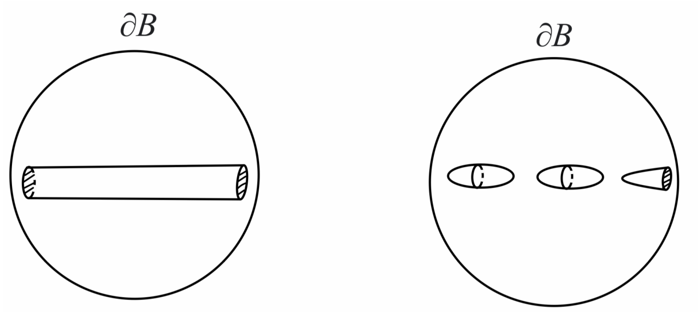


FIGURE 7. Possible topological pictures near a neck singularity.

- *is up to scaling and translation* $\frac{1}{R}$*-close in* C^∞ *to the cylinder (*$\cong S^{n-1}\times\mathbb{R}$*) in* $B_R(0)$ *with axis* ℓ *and radius* 1*,*
- *and as a topological cylinder has* $K_{\text{in}}(T-\bar\delta)\cap B$ *on its inside.*

(2) (Mean convex neighborhood) For every $T-\bar\delta<t_1<t_2<T+\bar\delta$*,*

$$K_{\text{in}}(t_2)\cap B\subset K_{\text{in}}(t_1)\backslash M(t_1).$$

Moreover, there exists some countable dense set $J\subset[T-\bar\delta,T+\bar\delta]$ *with* $T-\bar\delta\in J$ *such that we have for every* $t\in J$*:*

(3) $M(t)$ *is smooth, and intersects* ∂B *transversely.*

(4) Each connected component of $K_{\text{in}}(t)\cap\partial B$ *is a convex n-ball in* ∂B*.*

(5) Denote the two connected components of $K_{\text{in}}(T-\bar\delta)\cap\partial B$ *by* D_1 *and* D_2*. Then* $M(t)\cap D_i$ *has at most one connected component for* $i=1,2$*.*

(6) Let K *be a connected component of* $K_{\text{in}}(t)\cap B$*. Then* K *satisfies one of the following:*

- ∂K *is a connected component of* $M(t)\cap B$ *that is a sphere.*
- ∂K *consists of a connected component of* $M(t)\cap B$ *that is an n-ball and another ball on* ∂B*.*

- *∂K consists of a connected component of $M(t) \cap B$ that is a cylinder $\cong S^{n-1} \times (0,1)$ and two balls on ∂B.*

And the case for outward neck singularities is analogous.

Proof. We will just do the case of inward neck singularity.

To obtain (1) and (2). Let us first arbitrarily pick some $\epsilon, R > 0$, which we will further specify later. Let $\delta, \bar{\delta} > 0$ be obtained from applying the canonical neighborhood theorem (Theorem 2.2.3) to (x, T) and ϵ. We can decrease $\bar{\delta}$ such that it lie in the range $(0, \delta_0)$.

By possibly further decreasing $\delta, \bar{\delta}$, we can guarantee (2) by the mean convex neighborhood theorem of Choi-Haslhofer-Hershkovits-White [CHHW22, Theorem 1.17]. In fact, further decreasing $\delta, \bar{\delta}$, we can by the definition of neck singularity assume that $M(T - \bar{\delta}) \cap B_{2\delta}(x)$

- is, up to scaling and translation, $\frac{1}{R}$-close in C^∞ to the cylinder ($\cong S^{n-1} \times \mathbb{R}$) in $B_{2R}(0)$ with axis ℓ and radius 1,
- and as a topological cylinder has $K_{\text{in}}(T - \bar{\delta}) \cap B_{2\delta}(x)$ on its inside.

In particular, (1) is fulfilled.

To define J and obtain (3). Note that using [CM16, Corollary 0.6], for some set $I_1 \subset [T - \bar{\delta}, T + \bar{\delta}]$ of full measure, $M(t)$ is smooth for all $t \in I_1$. Then (3) just follows from a standard transversality argument. Namely, for each $t \in I_1$, via the transversality theorem, $B_r(x)$ intersects $M(t)$ transversely for a.e. $r \in (\delta/2, \delta)$. Hence, for some countable dense subset $J \subset I_1$ and some set $I_2 \subset (\delta/2, \delta)$ of full measure, for all $(t, r) \in J \times I_2$, $B_r(x)$ intersects $M(t)$ transversely. Hence, by slightly decreasing δ, (3) can be fulfilled.

To obtain (4). Let us first state a lemma, which gives us the constant R_0 we need.

Lemma 2.2.5. *There exist constants $R_0 > 2$, and $\epsilon_0, \epsilon_1 > 0$, all depending only on n, with the following significance.*

- *Consider some ball $B_{2R_0}(x)$, and fix a diameter line ℓ. Let $\mathcal{C} \subset B_{2R_0}(x)$ be the solid cylinder with radius 2 and axis ℓ.*

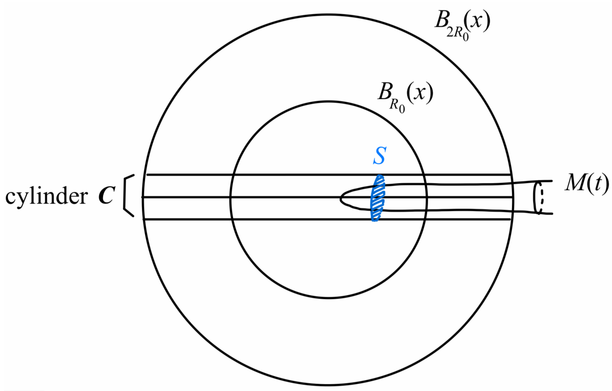


FIGURE 8. The cylinder $\mathcal{C}$.

- *Let x' be a regular point of some time-slice $M(t)$ of a level set flow in $\mathbb{R}^{n+1}$, and x' has an ϵ_0-canonical neighborhood with axis ℓ.*
- *Assume $x' \in B_{R_0}(x)$, $M(t) \cap B_{2R_0}(x) \subset \mathcal{C}$.*
- *Let S be a smooth n-disc properly embedded in $\mathcal{C}$, with ∂S lying on and transversely intersecting the cylindrical part of $\partial\mathcal{C}$, and $x' \in S$, such that:*
- *S is ϵ_1-close in C^∞ to some planar n-disc perpendicular to ℓ. (See Figure 8.)*

Then we have:

- *If $M(t)$ intersects S transversely at x', then the connected component D of $K_{\text{in}}(t) \cap S$ that contains x' is a convex n-disc in S, and $M(t) \cap D = \partial D$ with the intersection being transverse.*
- *If $M(t)$ does not intersect S transversely at x', then D is just the point x'.*

Proof. By an inspection of the geometry of the sphere, cylinder, bowl, and ancient oval, for all sufficiently large R_0 and small ϵ_0, if $M(t) \cap B_{2R_0}(x) \subset \mathcal{C}$ then

$$M(t) \cap B_{2R_0}(x) \cap (\epsilon_0\text{-canonical neighborhood of } x')$$

has curvature $|A| > 1/2$. Thus, if the smooth n-disc S is sufficiently planar, the desired claim follows easily. □

Now, we begin proving (4). Let us assume the R, ϵ we chose satisfy $R > R_0$ and $\epsilon < \epsilon_0$, with R_0, ϵ_0 from the above lemma. By how we chose R in the proof of (1) above, we can rescale $M(T - \bar{\delta})$ by some factor λ such that

$$\lambda(M(T - \bar{\delta}) - x) \cap B_{2R}(0)$$

lies in the solid cylinder $C \subset B_{2R}(0)$ with axis ℓ and radius 2. Thus, by the mean convex neighborhood property (2), for all $t \in (T - \bar{\delta}, T + \bar{\delta})$,

$$\lambda(M(t) - x) \cap B_{2R}(0) \subset C.$$

Now, remember that we should focus on those $t \in J \subset (T - \bar{\delta}, T + \bar{\delta})$. By Theorem 2.2.3 and $\epsilon < \epsilon_0$, $M(t)$ has an ϵ_0-canonical neighborhood with ℓ, and thus so does $\lambda(M(t) - x)$ since the property is independent of scaling and translation. Let S be a connected component of $\partial B_R(0) \cap C$. By increasing R, we can make S arbitrarily close to being planar. Hence, we can apply Lemma 2.2.5. Then (4) follows immediately.

To obtain (5). We will just do the case for D_1. Let

$$T_1 := \sup\{t \in J : M(t) \cap D_1 \text{ has only one connected component}\}.$$

Note that $T_1 > T - \bar{\delta}$ by (1) and $T - \bar{\delta} \in J$. To prove that $M(t) \cap D_1$ has at most one connected component for each $t \in J$, it suffices to prove that $T_1 = T + \bar{\delta}$. Suppose the otherwise, i.e. $T_1 < T + \bar{\delta}$ so that there exists a sequence in J, $t_1, t_2, ... \downarrow T_1$, such that $M(t_i) \cap D_1$ contains at least two components.

Now, let

$$K_1 = \bigcap_{T - \bar{\delta} < t < T_1} K_{\text{in}}(t) \cap D_1, \quad K_2 = K_{\text{in}}(T_1) \cap D_1, \quad K_3 = \bigcup_i K_{\text{in}}(t_i) \cap D_1.$$

Note that $K_1 \supset K_2 \supset K_3$ by the mean convex neighborhood property (2).

Proposition 2.2.6. *K_1 is a convex n-ball in ∂B, $K_1 = K_2$, and K_3 is dense in K_1.*

Proof. By the mean convex property,

$$K_1 = \bigcap_{t \in J, t < T_1} K_{\text{in}}(t) \cap D_1.$$

Then by (4), K_1 is a convex n-ball.

To prove $K_1 = K_2$, it suffices to prove $K_1 \subset K_2$. Note that by Lemma 2.1.1, for every $x \in K_1$ and $t \in (T - \bar{\delta}, T_1)$ we have $u(x,t) \leq 0$, where u is a level set function for $\mathcal{M}$. Since u is continuous, $u(x, T_1) \leq 0$, implying $x \in K_2$ by Lemma 2.1.1.

Finally, to prove K_3 is dense in K_1, it suffices to prove $K_1 \backslash K_3$ has empty interior (as a subset of ∂B) since K_1 is a convex n-ball. We claim that $K_2 \backslash K_3 \subset M_{\text{in}}(T_1)$. Indeed, if $x \in K_2 \backslash K_3$, then for every spacetime neighborhood U of (x, T_1) in $\mathbb{R}^{n+1} \times \mathbb{R}$, for each i, U contains the point

$$(x, t_i) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \backslash \mathcal{K}_{\text{in}}.$$

Thus, $(x, T_1) \in \partial \mathcal{K}_{\text{in}}$, and so $x \in M_{\text{in}}(T_1)$.

As a result,

$$K_1 \backslash K_3 = K_2 \backslash K_3 \subset M_{\text{in}}(T_1) \cap D_1 = M(T_1) \cap D_1,$$

where the last equality is by the non-fattening of $\mathcal{M}$ [CHHW22, Theorem 1.19]. We will prove that $M(T_1) \cap D_1$ consists entirely of singularities (of $\mathcal{M}$), and then immediately we would know $M(T_1) \cap D_1$ has empty interior using [CM16, Theorem 0.1], which says that the singular set of $\mathcal{M}$ is contained in finitely many compact embedded Lipschitz submanifolds each of dimension at most $n-1$ together with a set of dimension $n-2$.

Suppose by contradiction that $M(T_1) \cap D_1$ contains some regular point p. So around some neighborhood of p in $\mathbb{R}^{n+1}$, $M(T_1)$ is a smooth surface, with $K_{\text{in}}(T_1)$ on one side. Thus, we have $p \in \partial K_2$, with K_2 a convex n-ball. Then we repeat the argument in the above proof of (4) to apply Lemma 2.2.5 around the point p, and conclude:

- ∂K_2 is a smooth $(n-1)$-sphere and consists entirely of regular points.
- The interior of K_2 does not intersects $M(T_1)$.
- $M(T_1)$ intersects D_1 transversely along ∂K_2.

So, for some short amount of time after T_1, $M(T_1) \cap D_1$ would still have only one connected component by pseudolocality of (locally) smooth MCF (see [INS19, Theorem 1.5]). This contradicts the definition of T_1. □

Let us continue the proof of (5). Now, for each i, $K_{\text{in}}(t_i) \cap D_1$ has finitely many connected components by transversality (3). Let E_i be the one with the maximal diameter (measured inside ∂B), denoted d_i. Then by the canonical neighborhood property Theorem 2.2.3, assuming ϵ small, for some geodesic ball $\widetilde{E}_i \subset \partial B$ of diameter $3d_i$, $\widetilde{E}_i \cap K_{\text{in}}(t_i) = E_i$.

Now, note d_i is increasing in i by the mean convex neighborhood property (2). Let $d = \lim_i d_i$. There are two cases: (a) $d \geq \text{diam}(K_1)/2$, and (b) $d < \text{diam}(K_1)/2$. For case (a), by the definition of t_i, we know for sufficiently large i, the neighborhood $\widetilde{E}_i$ would then need to contain a connected component of $K_{\text{in}}(t_i) \cap D_1$ other than E_i, contradicting the definition of $\widetilde{E}_i$. So case (a) is impossible. Case (b) is also impossible since it, together with the existence of $\widetilde{E}_i$, violates Proposition 2.2.6 which says K_3 is dense in K_1. This finishes the proof of (5).

To obtain (6). Choose a connected component K of $K_{\text{in}}(t) \cap B_\delta(x)$. Let us foliate $B_{2\delta}(x)$ with planar n-discs that are perpendicular to the axis ℓ. Then as in the proof of (4), we apply Lemma 2.2.5 to characterize the intersection of K with every such planar n-discs. Namely, every such set of intersection consists of convex n-discs and isolated points. Viewing these sets of intersection as level sets of some function defined on K, Morse theory then immediately implies (6).

This finishes the proof of Theorem 2.2.4. □

Finally, we discuss some convergence theorems of MCF through cylindrical and spherical singularities.

Proposition 2.2.7. *Let $\mathcal{M}^i = \{M^i(t)\}_{t\geq 0}$, with $i = 1, 2, \ldots$, and $\mathcal{M} = \{M(t)\}_{t\geq 0}$ be MCF through neck and spherical singularities in $\mathbb{R}^{n+1}$ with $n = 2$ or 3. Assume that each $M^i(0)$ and $M(0)$ are smooth, closed hypersurfaces, with $M^i(0) \to M(0)$ in C^∞. Then*

(1) For a.e. t, $M^i(t) \to M(t)$ in C^∞.

(2) The spacetime tracks $\mathcal{M}^i \to \mathcal{M}$ in the Hausdorff sense.

Proof. By Ilmanen's elliptic regularization (see [Ilm94, Whi09]), for any closed smooth hypersurface $M^i(0)$, there exists a unit regular cyclic Brakke flow $\{\mu_t^i\}_{t\geq 0}$ such that $\mu_0^i = M^i(0)\lfloor\mathcal{H}^n$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. By the mean convex neighborhood theorem [CHH22] and the nonfattening of level set flow with singularities that have mean convex neighborhood [HW20], $\{\mu_t^i\}_{t\geq 0}$ is supported on $\mathcal{M}^i$. Then the compactness of Brakke flows ([Ilm94, Whi09]) implies that $\{\mu_t^i\}_{t\geq 0}$ subsequentially converges to a limit unit regular cyclic Brakke flow $\{\mu_t^\infty\}_{t\geq 0}$.

Because $M^i(0) \to M(0)$ smoothly, $\mu_0^\infty = \mu_0$, and by the uniqueness of unit regular cyclic Brakke flow, $\mu_t^\infty = \mu_t$ a.e. for all $t \geq 0$. In particular, the regular part of μ_t^∞ equals the regular part of μ_t. Then by Brakke's regularity theorem and a.e. time regularity of $\mathcal{M}^i$ with neck and spherical singularities, we have for a.e. t, $M^i(t) \to M(t)$.

The compactness of weak set flow shows that $\mathcal{M}^i$ subsequentially converges to a limit weak set flow $\mathcal{M}^\infty$ in Hausdorff distance. Because $\{\mu_t\}_{t\geq 0}$ is supported on $\mathcal{M}^\infty$, we have $\mathcal{M} \subset \mathcal{M}^\infty$. Meanwhile, $\mathcal{M}$ is the biggest flow, therefore $\mathcal{M}^\infty \subset \mathcal{M}$. Thus, $\mathcal{M}^\infty = \mathcal{M}$. This also shows the uniqueness of the limit. Therefore, $\mathcal{M}^i$ converges to $\mathcal{M}$ in Hausdorff distance.

□

2.3. **Homology descent, homology termination, and homology breakage.** In this section, we consider general level set flows $\mathcal{M} = \{M(t)\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$, in which $M(0)$ need not be a closed hypersurface. We will develop three new concepts in this section. For a heuristic description of them, see §1.1.

Let $H_k(\cdot)$ denotes the k-th homology group in $\mathbb{Z}$-coefficients.

Definition 2.3.1 (Homology descent)**.** We define a relation $\succ$ on the *disjoint* union

$$\bigsqcup_{t\geq 0} H_{n-1}(W(t))$$

as follows. Given two times $T_0 \leq T_1$, and two homology classes $c_0 \in H_{n-1}(W(T_0))$ and $c_1 \in H_{n-1}(W(T_1))$, we say that c_1 *descends from* c_0, and denote

$$c_0 \succ c_1,$$

if every representative $\gamma_0 \in c_0$ and $\gamma_1 \in c_1$ together bound some n-chain $\Gamma \subset W[T_0, T_1]$, i.e. $\gamma_0 - \gamma_1 = \partial\Gamma$. (Recall Figure 3.)

Clearly, in the above definition, we can equivalently replace "every representative" with "some representative". Note that, we use singular homology, so that γ_0, γ_1, and Γ are just singular chains.

Remark 2.3.2. The relation $\succ$ is a partial order. Indeed, let $c_i \in H_{n-1}(W(T_i))$ for $i = 0, 1, 2$. Clearly $c_0 \succ c_0$. If $c_0 \succ c_1$ and $c_1 \succ c_0$, then $T_0 = T_1$ and thus $c_0 = c_1$. If $c_0 \succ c_1$ and $c_1 \succ c_2$, then $T_0 \leq T_2$ and it follows easily $c_0 \succ c_2$.

It turns out this relation has some nice properties.

Proposition 2.3.3. *Let $c_0 \in H_{n-1}(W(T_0))$ and $T_0 \leq T_1$. Then there exists at most one $c_1 \in H_{n-1}(W(T_1))$ such that $c_0 \succ c_1$.*

Proof. Suppose $c_1, c_2 \in H_{n-1}(W(T_1))$ are such that $c_0 \succ c_1$ and $c_0 \succ c_2$. We wish to show $c_1 = c_2$. Choose $\gamma_i \in c_i$ for $i = 0, 1, 2$. Then by definition, $\gamma_0 - \gamma_1 = \partial A$ for some $A \subset W[T_0, T_1]$, and $\gamma_0 - \gamma_2 = \partial B$ for some $B \subset W[T_0, T_1]$. Thus, γ_1 and γ_2 bounds $A - B \subset W[T_0, T_1]$. Since the map

$$H_{n-1}(W(T_1)) \to H_{n-1}(W[T_0, T_1])$$

induced by the inclusion $W(T_1) \to W[T_0, T_1]$ is injective by White [Whi95, Theorem 1 (iii)], we know γ_1 and γ_2 are homologous within $W(T_1)$. Hence, $c_1 = c_2$. □

Remark 2.3.4. Note that in the above it is possible that no $c_1 \in H_{n-1}(W(T_1))$ is such that $c_0 \succ c_1$: In Figure 9, after time T, no homology class c_1 satisfies $a_0 \succ c_1$.

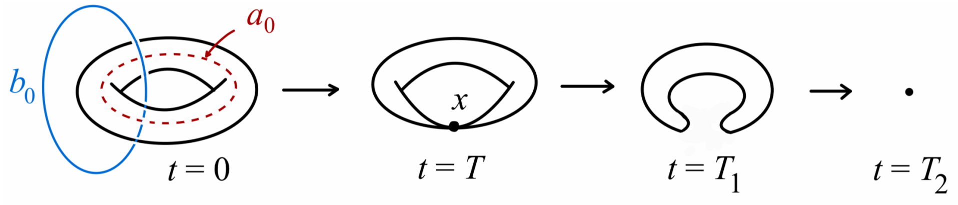


FIGURE 9. An inward neck pinch.

Remark 2.3.5. On the other hand, it is possible that there are multiple homology classes $c_0 \in H_1(W(T_0))$ such that $c_0 \succ c_1$. For the flow in Figure 9, both $b_0 \in H_1(W_{\text{out}}(0))$ and the trivial element of $H_1(W_{\text{out}}(0))$ descend to the trivial element of $H_1(W_{\text{out}}(T_1))$.

In fact, precisely because of Proposition 2.3.3 and Remark 2.3.5, we chose the symbol $\succ$ (instead of $\prec$) to pictographically reflect that more than one homology classes may descend into one, but not the other way around.

Proposition 2.3.6. *We focus on the case $n = 2$: Let $c_1 \in H_1(W(T_1))$ and $T_0 \leq T_1$. Then there exists at least one $c_0 \in H_1(W(T_0))$ such that $c_0 \succ c_1$.*

Proof. Choose some $\gamma \in c_1$. By White [Whi95, Theorem 1 (ii)], γ can be homotoped through $W[T_0, T_1]$ to some loop γ' in $W(T_0)$. So $c_0 := [\gamma'] \succ c_1$. □

The following proposition says that a homology class cannot disappear and then reappear later.

Proposition 2.3.7. *Let $T_0 < T_1$, $c_0 \in H_{n-1}(W(T_0))$, and $c_1 \in H_{n-1}(W(T_1))$ with $c_0 \succ c_1$. Then for every $t \in [T_0, T_1]$ there exists a unique $c \in H_1(W(t))$ such that $c_0 \succ c \succ c_1$.*

Proof. We just need to prove existence, as then uniqueness would follow from Proposition 2.3.3.

By our assumption, there exist $\gamma_0 \in c_0$ and $\gamma_1 \in c_1$ that together in $W[T_0, T_1]$ bound some n-chain C. Without loss of generality we can assume that $\beta_t := \{x : (x, t) \in C\}$ is an $(n-1)$-chain without boundary for each $t \in [T_0, T_1]$ (?): This is because we can tilt the faces to make them not lie in any slice $\mathbb{R}^3 \times \{t\}$. Then $[\beta_t] \in H_{n-1}(W(t))$ and $c_0 \succ [\beta_t] \succ c_1$. □

Based on Proposition 2.3.7, the following definition is well-defined.

Definition 2.3.8 (Homology termination)**.** Let $c_0 \in H_{n-1}(W(T_0))$.

- If

$$\mathfrak{t}(c_0) := \sup\{t \geq T_0 : c_0 \succ c \text{ for some } c \in H_{n-1}(W(t))\}$$

  is finite, then we say that c_0 *terminates at time* $\mathfrak{t}(c_0)$, otherwise we say c_0 *never terminates.*

- And for each $t \geq T_0$, the unique $c \in H_{n-1}(W(t))$ such that $c_0 \succ c$, if exists, is denoted $c_0(t)$.

If needed, we use $\mathfrak{t}^{\mathcal{M}}$ in place of $\mathfrak{t}$ to specify the flow.

Note that since W is open, if c_0 terminates at time $\mathfrak{t}(c_0)$ then there is no $c \in H_{n-1}(W(\mathfrak{t}(c_0)))$ such that $c_0 \succ c$, i.e. $c_0(\mathfrak{t}(c_0))$ is never well-defined. In particular, any $c_0 \in H_{n-1}(W(T_0))$ cannot terminate at time T_0. Thus, one can think of the time interval $[T_0, \mathfrak{t}(c_0))$ as the "maximal interval of existence" of c_0.

Remark 2.3.9 (Trivial homology classes)**.** Let us also elaborate on trivial homology classes. For each time t, $H_{n-1}(W(t))$ has a *unique* trivial homology class 0_t. This is true even for situations like Figure 9 when the surfaces have inside and outside regions: The trivial elements of $H_1(W_{\text{in}}(t))$ and $H_1(W_{\text{out}}(t))$ are viewed as the same.

However, 0_t are viewed as different for different t, because we used disjoint union in Definition 2.3.1. Nonetheless, for any $t_1 < t_2$, it is vacuously true that $0_{t_1} \succ 0_{t_2}$. Thus, we can as well write each 0_t as $0(t)$, following the notation in Definition 2.3.8. In addition, clearly, *the trivial homology class never terminates.*

Example 2.3.10. Let us revisit Figure 9. It can be easily seen that a_0 terminates at time T, while b_0 does not. In fact, b_0 will never terminate: $b_0(t)$ would just become trivial for each $t > T$.

Example 2.3.11. Let us now instead consider the flow in Figure 10. At time T, b_0 terminates while a_0 does not. In fact, $a_0(t)$ becomes trivial after time T, and thus will never terminate.

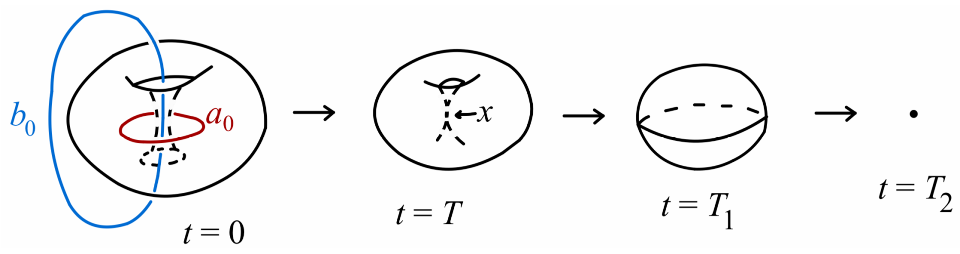


FIGURE 10. An outward neck pinch.

Now we define another concept. In Figure 9, a_0 terminates at time T because, heuristically, it "breaks" at the cylindrical singularity x. And in Figure 10, b_0 terminates at time T because it "breaks" at the outward cylindrical singularity. The following definition makes this phenomenon of breakage precise.

Definition 2.3.12 (Homology breakage)**.** Let $c_0 \in H_{n-1}(W(T_0))$, $T_1 > T_0$, and $K \subset M(T_1)$ be a compact set. Suppose the following holds:

- For each $T_0 \leq t < T_1$, there exists $c_0(t) \in H_{n-1}(W(t))$ such that $c_0 \succ c_0(t)$.
- For every neighborhood $U \subset \mathbb{R}^{n+1}$ of K, for each $t < T_1$ sufficiently close to T_1, every element of $c_0(t)$ intersects U. (Recall Figure 4.)

Then we say that c_0 *breaks in* (K, T_1). We will often concern the case when K is just a point $x \in M(T)$, for which we say that c_0 *breaks at* (x, T_1).

One might wonder why Definition 2.3.12 does not require c_0 to terminate at time T_1. This is because it is unnecessary:

Proposition 2.3.13. *If a homology class $c_0 \in H_{n-1}(W(T_0))$ breaks in some (K, T_1), then c_0 terminates at time T_1.*

Proof. Suppose the otherwise: There exists $T_2 > T_1$ and $c_2 \in H_{n-1}(W(T_2))$ such that $c_0 \succ c_2$. Then there exists $\gamma_0 \in c_0$ and $\gamma_2 \in c_2$ that together in $W[T_0, T_2]$ bound some

n-chain C. Without loss of generality we can assume that $\beta_t := \{x : (x,t) \in C\}$ is an $(n-1)$-chain without boundary for each $t \in [T_0, T_2]$ (?). Then $c_0(t) = [\beta_t] \in H_{n-1}(W[t])$ satisfies $c_0 \succ c_0(t)$.

By assumption c_0 breaks in some (K, T_1) with $K \subset M(T_1)$. So $K \cap C = \emptyset$. Then since K is compact and C is closed, there exists some neighborhood of K in $\mathbb{R}^{n+1} \times \mathbb{R}$ of the form $B_r(K) \times [T_1 - \delta, T_1 + \delta]$ that does not intersect C. So for all $t \in [T_1 - \delta, T_1 + \delta]$, β_t avoids $B_r(K)$. This contradicts the assumption that c_0 breaks at (K, T_1). $\square$

Note that, vacuously, the trivial homology class does not break in any (K, T). Moreover, if a homology class breaks in (K_1, T) and $K_1 \subset K_2 \subset M(T)$, then it also breaks in (K_2, T).

On might wonder whether the converse of the above proposition is true. Actually, it is true that for 2-dimensional MCF through cyindrical and spherical singularities, if a homology class terminates at some time T, then it actually breaks at some cylindrical singularity (x, T). This is the statement of Theorem 2.4.5, which is one of the main result in §2.4. However, we are unsure whether the converse is true in general.

Proposition 2.3.14. *No homology class breaks at a regular point.*

Proof. Suppose (x, T) is a regular point. Then there exists a small ball B around x such that for all t close to T, $M_t \cap B$ is a smooth n-disk. Then it is clear every n-chain can be homotoped to avoid B. So no homology class breaks at (x, T). $\square$

Proposition 2.3.15. *No homology class breaks at a spherical singularity.*

Proof. Suppose otherwise. Without loss of generality, suppose some $c_0 \in H_{n-1}(W(T_0))$ breaks at some spherical singularity (x, T). Then there exists a small ball B around x such that for all $t < T$ close to T, $M(t) \cap B$ is a smooth sphere. For each such t, pick some $\gamma \in c_0(t)$. By removing the components of γ inside the sphere $M(t) \cap B$, we can assume that γ lies outside the sphere. Thus clearly γ can be homotoped within $W(t)$ to avoid B. This again contradicts that c_0 breaks at (x, T). $\square$

Lastly, we conclude this section with the following proposition, which provides us a scenario where we know the inside homology classes must terminate. Namely, *if we take a compact shrinker and push it inward, then all non-trivial inside homology classes will terminate, while the outward ones will not.* This proposition will be crucial for us when we use Theorem 1.1.1 to prove other main theorems.

Proposition 2.3.16. *The setting is as follows.*

- *Let Σ be a smooth, embedded, compact shrinker in $\mathbb{R}^3$.*
- *Let $S^0(-1)$ be a surface, lying strictly inside Σ, given by deforming Σ within the inside region of Σ.*
- *Let $S^1(-1)$ be a surface, lying strictly outside Σ, given by deforming Σ within the outside region of Σ.*
- *Note that the first homology groups of*

$$\mathbb{R}^3 \backslash \Sigma, \quad \mathbb{R}^3 \backslash S^0(-1), \quad \textit{and } \mathbb{R}^3 \backslash S^1(-1)$$

  can be canonically identified.
- *Let*

$$\mathcal{S} = \{\sqrt{-t}\Sigma\}_{-1 \le t \le 0}, \quad \mathcal{S}^0 = \{S^0(t)\}_{t \ge -1}, \quad \textit{and } \mathcal{S}^1 = \{S^1(t)\}_{t \ge -1}$$

  be the associated level set flows.

Then there exist times $T, \widetilde{T} \in (-1, 0)$ such that

(1) For each non-trivial element $a_0 \in H_1(W_{\text{in}}^{\mathcal{S}^0}(-1))$, $\mathfrak{t}(a_0) \le \widetilde{T}$.

(2) For each element $b_0 \in H_1(W_{\text{out}}^{\mathcal{S}^0}(-1))$, $b_0(\widetilde{T})$ exists and is trivial.

(3) For each element $a_1 \in H_1(W_{\text{in}}^{\mathcal{S}^1}(-1))$, $a_1(T)$ exists and is trivial.

(4) For each non-trivial element $b_1 \in H_1(W_{\text{out}}^{\mathcal{S}^1}(-1))$, $\mathfrak{t}(b_1) \le T$.

Proof. For the first claim, note that:

- $S^0(-1)$ is inside Σ.

- $\operatorname{dist}(\sqrt{-t}\Sigma, S^0(t))$ is non-decreasing in t by [ES91, Theorem 7.3].
- Σ shrinks self-similarly under the flow.

Thus, we know that there exists $\widetilde{T} < 0$ such that for every $t \geq \widetilde{T}$, $S^0(t)$ is empty. Thus, for any non-trivial element $a_0 \in H_1(W_{\text{in}}^{\mathcal{S}^0}(-1))$, either $\mathfrak{t}(a_0) \leq \widetilde{T}$, or $a_0(\widetilde{T})$ still exists but is trivial. Suppose by contradiction that the latter holds. Then we can pick some $\alpha_0 \in a_0$ such that $\alpha_0 = \partial A$ for some

$$A \subset W_{\text{in}}^{\mathcal{S}^0}([-1, \widetilde{T}]) \subset W_{\text{in}}^{\mathcal{S}}([-1, \widetilde{T}]).$$

Thus, rescaling each time slice of A, we can have α_0 bounding some

$$\widetilde{A} \subset (\text{interior region of } \Sigma) \times [-1, \widetilde{T}].$$

Projecting $\widetilde{A}$ into the interior region of Σ, we have that α_0 is homologically trivial, contradicting the definition of α_0. This finishes the proof of the first claim.

For the second claim, since Σ just shrinks self-similarly under the flow, we know b_0 has not terminated yet by time $\widetilde{T}(< 0)$ for the flow $\{\sqrt{-t}\Sigma\}$. Then by the fact that $S^0(t)$ lies inside $\sqrt{-t}\Sigma$ for each $t \in [-1, \widetilde{T}]$, which comes from the avoidance principle, we know $b_0(\widetilde{T})$ also still exists for the flow $\mathcal{S}^0$. But since $S^0(\widetilde{T})$ is empty, $b_0(\widetilde{T})$ must be trivial.

Let us prove the fourth claim before the third. We first let

$$\epsilon = \operatorname{dist}(\Sigma, S^1(-1))\}.$$

Pick some loop $\beta_1 \in b_1$. Let $B_\epsilon(\sqrt{-t}\Sigma)$ be the ϵ-neighborhood of $\sqrt{-t}\Sigma$, and denote

$$Y(t) := \mathbb{R}^3 \backslash B_\epsilon(\sqrt{-t}\Sigma)$$
$$Y[t_1, t_2] := \bigcup_{t \in [t_1, t_2]} (\mathbb{R}^3 \backslash B_\epsilon(\sqrt{-t}\Sigma)) \times \{t\}.$$

Now, to prove the fourth claim, it suffices to show for some $-1 < T < 0$, there does not exist a 2-chain $C \subset W^{\mathcal{S}^1}_{\text{out}}[0, T]$ such that $\partial C = \beta_1 - \beta_2$ for some closed 1-chain β_2 outside $S^1(T)$. Noting $S^1(-1)$ is outside Σ, by the avoidance principle it suffices to show that:

Lemma 2.3.17. *For some $-1 < T < 0$, there does not exists a 2-chain $C \subset Y[-1, T]$ such that $\partial C = \beta_1 - \beta_2$ for some closed 1-chain $\beta_2 \subset Y(T)$.*

Proof. Fix a T sufficiently close to 0 such that

$$\operatorname{diam}(\sqrt{-T}\Sigma) < \epsilon,$$

then $B_\epsilon(\sqrt{-T}\Sigma)$ is star-shaped (with respect to any point on $\sqrt{-T}\Sigma$). Hence, $\partial B_\epsilon(\sqrt{-T}\Sigma)$ has genus 0.

Suppose by contradiction that there exists a 2-chain $C \subset Y[-1, T]$ such that $\partial C = \beta_1 - \beta_2$ for some closed 1-chain $\beta_2 \subset Y(T)$. By rescaling C at each time slice t, we can construct another 2-chain $\widetilde{C}$ outside Σ such that $\partial\widetilde{C} = \beta_1 - \sqrt{-T}\beta_2$.

Since β_1, which lies outside Σ, is homologically non-trivial, we can pick a non-trivial loop α inside Σ such that

$$[\beta_1] \in H_1(\mathbb{R}^3 \backslash \alpha)$$

is non-trivial. Then by the existence of $\widetilde{C}$, we have $[\beta_2] \neq 0$ in $H_1(\mathbb{R}^3 \backslash \alpha)$ too. However, this is impossible because $\sqrt{-T}\beta_2$ is outside $B_{\epsilon/\sqrt{-T}}(\Sigma)$ while α is inside, and $\partial B_{\epsilon/\sqrt{-T}}(\Sigma)$ has genus 0 by the first paragraph of this proof. ☐

This finishes proving the fourth claim of Proposition 2.3.16.

Finally, for the third claim, since $a_1(T)$ exists for the flow $\{\sqrt{-t}\Sigma\}_{t\leq 0}$, by the avoidance principle we know $a_1(T)$ exists for $\mathcal{S}^1$. Moreover, since the inside of $S^1(T)$ contains $B_\epsilon(\sqrt{-T}\Sigma)$, which is star-shaped, we know $a_1(T) = 0$ in $H_1(W^{\mathcal{S}^1}(T))$. ☐

2.4. **Homology breakage of MCF through cylindrical and spherical singularities.**

MCF through cylindrical and spherical singularities. In this section, we focus on 2-dimensional MCF $\mathcal{M} = \{M(t)\}_{t\geq 0}$ through cylindrical and spherical singularities in $\mathbb{R}^3$, with $M(0)$ a smooth closed surface.

Proposition 2.4.1. *For any $T_0 \geq 0$, no element of $H_1(W_{\text{out}}(T_0))$ can break at an inward neck singularity, and no element of $H_1(W_{\text{in}}(T_0))$ can break at an outward neck singularity.*

Proof. Let us just prove the first claim. Suppose by contradiction some $c_0 \in H_1(W_{\text{out}}(T_0))$ breaks at an inward neck singularity (x, T), with $T > T_0$. Applying Theorem 2.2.4 to (x, T) with $\delta_0 = 1$ and any $R > R_0$, we obtain constants $\delta, \bar{\delta} > 0$ and a dense subset $J \subset [T - \bar{\delta}, T + \bar{\delta}]$ satisfying the properties in Theorem 2.2.4. Let $B = B_\delta(x)$.

Pick a time $t \in J \cap [T - \bar{\delta}, T)$. Since c_0 breaks at T, $c_0(t)$ still exists. Pick a loop $\gamma \in c_0(t)$. By Theorem 2.2.4 (6) (and recall Figure 7), we can homotope γ within $W_{\text{out}}(t)$ to avoid B. This can be done for all t in $J \cap [T - \bar{\delta}, T)$, which is dense in $[T - \bar{\delta}, T)$. So we obtain a contradiction to the fact that c_0 breaks at (x, T). □

Let us now in the following proposition describe more precisely the shape around a neck pinch at which some homology class *breaks*. Namely, in this case, before the singular time, only the last bullet point of Theorem 2.2.4 (6), i.e. $M(t) \cap B$ being a cylinder, can occur.

Proposition 2.4.2. *There exists a universal constant $R_0 > 0$ with the following significance. Suppose $c_0 \in H_1(W_{\text{in}}(T_0))$ breaks at some inward neck singularity (x, T). Let $\delta_0 > 0$. Then for each $R > R_0$, there exists some constants $\delta \in (0, \delta_0)$, $\bar{\delta} > 0$, and some dense subset $J \subset (T - \bar{\delta}, T + \bar{\delta})$ with $T - \bar{\delta} \in J$, such that:*

1. *The first five items of Theorem 2.2.4 holds.*
2. *For each $t \in J \cap [T - \bar{\delta}, T)$, $K_{\text{in}}(t) \cap B_\delta(x)$ is a solid cylinder such that its boundary consists of a connected component of $M(t) \cap B_\delta(x)$ that is a cylinder and two disks D_1, D_2 on $\partial B_\delta(x)$.*

(3) And for such t, every element $\gamma \in c_0(t)$ has a non-zero intersection number (in $\mathbb{Z}$-coefficient) with each D_i.

And the outward case is analogous.

Proof. We will just prove the inward case. Let us apply Theorem 2.2.4 to (x, T) to obtain the constants $\delta, \bar{\delta}$ and the subset $J \subset [T-\bar{\delta}, T+\bar{\delta}]$. Let $B = B_\delta(x)$. And then the first five items of Theorem 2.2.4 will hold.

Then, we need to show that for each $t \in J \cap (T_0, T)$ sufficiently close to T, $K_{\text{in}}(t) \cap B_\delta(x)$ satisfies the description in (2): After that we could just shrink $\bar{\delta}$ and the set J to guarantee (2). Suppose by contradiction that there exists a sequence in J, $t_1, t_2, ... \uparrow T$, such that $K_{\text{in}}(t_i) \cap B_\delta(x)$ violates the description in (2). Fix any t_i. Note that Theorem 2.2.4 (5) and (6) together imply that $K_{\text{in}}(t_i) \cap B$ can have *at most one* cylindrical component. Thus, in our case, $K_{\text{in}}(t_i) \cap B$ actually has no cylindrical component. Hence, any connected component K of $K_{\text{in}}(t_i) \cap B$ satisfies either one of the following by Theorem 2.2.4 (6):

- ∂K is a connected component of $M(t) \cap B$ that is a sphere.
- ∂K consists of a connected component of $M(t) \cap B$ that is an disc and another disc on ∂B.

In any case, it is clear that one perturb any element of $c_0(t_i)$ to avoid B. Arguing for each t_i, this contradicts that c_0 breaks at (x, T).

Finally, to prove (3), it suffices to show that for each $t \in J \cap (T_0, T)$ sufficiently close to T, $c_0(t)$ satisfies the description of (3): Then we could just shrink J, and we would be done. Suppose the otherwise, so that there exists a sequence in J, $t_1, t_2, ... \uparrow T$, such that $c_0(t_i)$ violates the description of (3). Then for each t_i, we can find a loop $\gamma \in c_0(t_i)$ with intersection number zero with some connected component of $K_{\text{in}}(t_i) \cap \partial B$. In fact, since $K_{\text{in}}(t_i) \cap B$ is a cylinder by (2), γ has intersection number zero with *both* connected components D_1, D_2 of $K_{\text{in}}(t_i) \cap \partial B$ (which are discs). To contradict the fact that c_0 breaks at (x, T), it suffices to find another element of $c_0(t_i)$ that avoids B.

Indeed, this can be done: We can assume γ intersects ∂B transversely. Since γ has intersection number zero with D_1, we can pair up each positive intersection point of $\gamma \cap D_1$ with a negative one. Now fix a pair, and draw a line segment L on D_1 to connect the pair of points. Adding L and $-L$ to γ, and slightly pushing the resulting curve away from D_1 around $L, -L$, we can obtain another representative of $c_0(t_i)$ that avoids this pair of intersection points. And we do this for each pair. Then at the end we get a curve belonging to $c_0(t_i)$ that avoids D_1 completely. Then, we repeat this process with D_2, to get a the curve that avoids D_2 too. Lastly we discard all connected components of the curve that are in K, which are all trivial as K is a solid cylinder, to obtain an element of $c_0(t_i)$ that avoids B, as desired. □

Denote by $\mathcal{S}^{\text{in}}_{\text{sphere}}$ the set of inward spherical singularities of $\mathcal{M}$, and by $\mathcal{S}^{\text{in}}_{\text{neck}}$ the set of inward neck singularities of $\mathcal{M}$. Similarly, we define $\mathcal{S}^{\text{out}}_{\text{sphere}}$ and $\mathcal{S}^{\text{out}}_{\text{neck}}$. Then, we denote by $S^{\text{in}}_{\text{sphere}}(t) \subset \mathbb{R}^3$ the slice of $\mathcal{S}^{\text{in}}_{\text{sphere}}$ at time t, and proceed similarly for the other three sets.

Lemma 2.4.3. *$S^{\text{in}}_{\text{neck}}(T)$ and $S^{\text{out}}_{\text{neck}}(T)$ are compact sets.*

Proof. We only show $S^{\text{in}}_{\text{neck}}(T)$ is compact and the proof for $S^{\text{out}}_{\text{neck}}(T)$ is the same. It suffices to show $\overline{S^{\text{in}}_{\text{neck}}(T)} = S^{\text{in}}_{\text{neck}}(T)$. By the semi-continuity of the Gaussian density, a limit point p of $S^{\text{in}}_{\text{neck}}(T)$ must be a neck singularity. Hence it suffices to show $p \in S^{\text{in}}_{\text{neck}}(T)$. We prove by contradiction: suppose not, then $p \in S^{\text{out}}_{\text{neck}}(T)$, and by mean convex neighbourhood theorem, there is a neighborhood U of p and $\delta > 0$ such that the MCF $\{M_t\}_{t \in [T-\delta, T+\delta]}$ in U moves outward. This contradicts the assumption that p is a limit point of $S^{\text{in}}_{\text{neck}}(T)$. □

Proposition 2.4.4. *Suppose $c_0 \in H_1(W_{\text{in}}(T_0))$ terminates at some time $T > T_0$. Then c_0 breaks in $(S^{\text{in}}_{\text{neck}}(T), T)$.*

And the outward case is analogous.

Proof. We will just prove the inward case. Suppose the otherwise: There exist a neighborhood U of $S^{\text{in}}_{\text{neck}}(T)$ in $\mathbb{R}^3$, an increasing sequence of times $t_1, t_2, ... \uparrow T$, and elements $\gamma_i \in c_0(t_i)$ such that each γ_i is disjoint from U.

Now, by the mean convex neighborhood theorem and the compactness of $S^{\text{in}}_{\text{neck}}(T)$ and $S^{\text{out}}_{\text{neck}}(T)$ from Lemma 2.4.3, we can further pick open neighborhoods $U_{\text{in}}, \widetilde{U}_{\text{in}}$ with

$$S^{\text{in}}_{\text{neck}}(T) \subset U_{\text{in}} \subset\subset \widetilde{U}_{\text{in}} \subset\subset U,$$

an open neighborhood U_{out} of $S^{\text{out}}_{\text{neck}}(T)$, and two times $T_1 < T < T_2$ such that:

- $\widetilde{U}_{\text{in}}$ and U_{out} are disjoint.
- In the time interval (T_1, T_2), $M(t) \cap \widetilde{U}_{\text{in}}$ *evolves inward* (i.e.

$$K_{\text{in}}(t_2) \cap \widetilde{U}_{\text{in}} \subset K_{\text{in}}(t_1) \backslash M(t)$$

  for every $T_1 < t_1 < t_2 < T_2$) while $M(t) \cap U_{\text{out}}$ *evolves outward.*

By Huisken's analysis of spherical singularities (see also the special case of [CM16, Theorem 4.6]), each spherical singularity is isolated in spacetime. Therefore, the limit points of spherical singularities can only be cylindrical singularities.

We claim that after appropriately shrinking the time interval $[T_1, T_2]$,

$$(\mathbb{R}^3 \backslash (U_{\text{in}} \cup U_{\text{out}})) \times [T_1, T_2]$$

has only finitely many singular points, and we can thus assume such singular points all are spherical singularities at time T. In fact, suppose not, there exists a sequence of distinct singular points $\{p_i\}_{i=1}^{\infty}$ outside $U_{\text{in}} \cup U_{\text{out}}$, with singular time $t_i \to T$. Then by compactness of the singular set of $\mathcal{M}$ and the previous paragraph, there is a subsequence converging to a cylindrical singularity in $(S^{\text{in}}_{\text{neck}}(T) \cup S^{\text{out}}_{\text{neck}}(T)) \times \{T\}$. This contradicts our choice of the p_i's.

As a consequence of the claim, by shrinking $[T_1, T_2]$ and the neighborhoods $\widetilde{U}_{\text{in}}$ and U_{out}, we can assume

$$\overline{\widetilde{U}_{\text{in}} \backslash U_{\text{in}}} \times [T_1, T_2]$$

consists only of smooth points. Furthermore, we can choose a neighborhood V_{in} of $S^{\text{in}}_{\text{sphere}}(T)$ such that $M(t) \cap V_{\text{in}}$ is a finite union of convex smooth spheres for each $t \in [T_1, T_2]$. Similarly, we can find a neighborhood V_{out} for $S^{\text{out}}_{\text{sphere}}(T)$ with analogous properties. We can assume the closures of $\widetilde{U}_{\text{in}}, U_{\text{out}}, V_{\text{in}}, V_{\text{out}}$ are all disjoint. Moreover, $M(t)\backslash(U_{\text{in}} \cup U_{\text{out}} \cup V_{\text{in}} \cup V_{\text{out}})$ evolves smoothly for $t \in [T_1, T_2]$.

To derive a contradiction to $\mathfrak{t}(c_0) = T$, we are going to prove that for some t_i, there exists a smooth deformation of γ_i, $\{\gamma^t \subset W_{\text{in}}(t)\}_{t\in[t_i,T]}$, with $\gamma^{t_i} = \gamma_i$, thereby letting γ_i "survive" up to time T. Note that:

- By the smoothness of $M(t)$ in $\overline{\widetilde{U}_{\text{in}} \backslash U_{\text{in}}}$ for $t \in [T_1, T_2]$,

  $$C := \sup_{t\in[T_1,T_2],\, x\in M(t)\cap\overline{\widetilde{U}_{\text{in}}\backslash U_{\text{in}}}} |A| < \infty.$$

  Thus, the velocity of the flow in this spacetime region is bounded by C. Thus, since γ_i avoids $\widetilde{U}_{\text{in}}$, we can take a $t_i \in (T_1, T)$ sufficiently close to T such that there is not enough time for any point of $M(t_i)\backslash\widetilde{U}_{\text{in}}$ to be pushed into U_{in} by time T.
- Note that $M(t)$ evolves outward in $\widetilde{U}_{\text{out}}$ for $t \in [T_1, T_2]$.
- Since V_{in} and V_{out} consists of spheres, we can remove the components of γ_i inside the spheres, so we may assume γ_i avoids V_{in} and V_{out}.

Combining the above observations, we can construct a smooth deformation of γ_i, $\{\gamma^t \subset W_{\text{in}}(t)\}_{t\in[t_i,T]}$, using the evolution of MCF, with $\gamma^{t_i} = \gamma_i$. This contradicts that $\mathfrak{t}(c_0) = T$. □

Here comes a key theorem, which supports that our definition of homology termination and breakage actually describes the heuristic phenomenon in Figure 9.

Theorem 2.4.5. *Suppose $c_0 \in H_1(W_{\text{in}}(T_0))$ terminates at some time $T > T_0$. Then c_0 breaks at some inward neck singularity (x, T).*

And the outward case is analogous.

Note that such x may be non-unique: Consider a flow that is a thin torus collapsing into a closed curve consisting entirely of neck singularities.

Proof. Let us just do the inward case. We will prove by contradiction. Suppose the theorem is false. Namely:

Assumption (*): For every inward neck singularity (x, T), there is a neighborhood U_x of x such that it is not true that "for every time $t < T$ close enough to T, every element of $c_0(t)$ intersects U_x".

Applying Theorem 2.2.4 to each inward neck singularity (x, T), with a constant $\delta_0(x) > 0$ such that $B_{\delta_0(x)}(x) \subset U_x$ and an $R > \max\{R_0, 100\}$, we obtain constants $\delta(x), \bar{\delta}(x) > 0$ and a set of full measure $J(x) \subset [T - \bar{\delta}(x), T + \bar{\delta}(x)]$ satisfying the properties of Theorem 2.2.4.

Since $S^{\text{in}}_{\text{neck}}(T)$ is compact by Lemma 2.4.3, there exist $x_1, ..., x_n \in S^{\text{in}}_{\text{neck}}(T)$ such that

$$B_{\delta(x_1)/2}(x_1), ..., B_{\delta(x_n)/2}(x_n)$$

cover $S^{\text{in}}_{\text{neck}}(T)$. For simplicity, we denote those balls by $\frac{1}{2}B_1, ..., \frac{1}{2}B_n$, while

$$B_1 := B_{\delta(x_1)}(x_1), ..., B_n := B_{\delta(x_n)}(x_n)$$

Since c_0 terminates at time T, we know c_0 breaks in $(S^{\text{in}}_{\text{neck}}(T), T)$ by Proposition 2.4.4. Thus, by definition, there exists a time T_1 with $\max_i T - \bar{\delta}(x_i) < T_1 < T$ such that for each $t \in [T_1, T)$, every element of $c_0(t)$ intersects $\cup_i \frac{1}{2} B_i$. We can assume $T_1 \in \cap_i J(x_i)$ so that $M(T_1)$ is smooth and intersects each ∂B_i transversely by Theorem 2.2.4 (3).

Lemma 2.4.6. *Let D be a connected component of $K_{\text{in}}(T_1) \cap \partial B_i$ (which there are at most two by Theorem 2.2.4 (5)), and $\gamma \in c_0(T_1)$. Then the linking number* $\text{link}(\gamma, \partial D) = 0$

Proof. Suppose the otherwise, that there exists some D as above and $\gamma \in c_0(t_0)$ such that $\text{link}(\gamma, \partial D) \neq 0$. Now, pick any $t_1 \in [T_1, T)$ and $\gamma_1 \in c_0(t_1)$. By definition, γ_1 is homologous to γ within $W_{\text{in}}[T_1, t_1]$. Thus, γ_1 is homologous to γ within $\mathbb{R}^3 \backslash \partial D$, since by the mean convex neighborhood property Theorem 2.2.4 (2) we know $\partial D \subset \mathbb{R}^3 \backslash W_{\text{in}}(t)$ for all $t \in [T_1, t_1]$. Therefore, $\text{link}(\gamma_1, \partial D) \neq 0$, meaning γ_1 must intersect D. But $D \subset \bar{B}_i \subset U_{x_i}$, so we have shown that for all $t_1 \in [T_1, T)$, any element of $c_0(t_1)$ must intersect U_{x_i}. This contradicts assumption (*). □

Let $\epsilon_1 := \min_i \delta(x_i)/2$. Let $\gamma \in c_0(T_1)$ be such that

$$\text{length}(\gamma) < \inf_{\gamma' \in c_0(T_1)} \text{length}(\gamma') + \epsilon_1/100 \tag{2.2}$$

Without loss of generality, we can assume γ intersects all ∂B_i transversely. To finish the proof, it suffices to show that γ avoids $\cup_i \frac{1}{2} B_i$: This would contradict the definition of T_1.

Lemma 2.4.7. *γ does not intersect $\cup_i \frac{1}{2} B_i$.*

Proof. We prove by contradiction. Suppose that γ intersects some $\frac{1}{2} B_i$. We will produce an element of $c_0(T_1)$ whose length is too small.

Without loss of generality, we can assume that no connected component of $\gamma \cap B_i$ is a closed loop. This is because we could just remove all such loops from γ, and the resulting curve is still in $c_0(T_1)$ by Theorem 2.2.4 (6). Hence, letting β be a connected component of $\gamma \cap B_i$, we can assume that β is a line segment.

Now, by Theorem 2.2.4 (5) and our choice that $T_1 \in \cap_i J(x_i)$, $W_{\text{in}}(T_1) \cap \partial B_i$ consists of at most two disks. There are two cases: either (1) β starts and ends on the same disk, say D_1, or (2) β starts and ends on different disks, D_1 and D_2. We will show that both are impossible.

For case (1), since β intersects $\frac{1}{2} B_i$, whose distance to ∂B_i is $\delta(x_i)/2$, we know that $\text{length}(\beta)$ is at least $\delta(x_i)$. On the other hand, note that by Theorem 2.2.4 (1), (2), and (4), D_1 is a convex disc on ∂B_i with diameter less than $\delta(x_i)/50$ (recall $R > 100$). Thus, we can join the end points of β, from $\beta(1)$ to $\beta(0)$, by a segment β_1 on D_1 of length less than $\delta(x_i)/50$: See Figure 11. Then, we consider the new loop $\gamma - \beta - \beta'$, which replaces $\beta \subset \gamma$ with β'. This loop lies in $c_0(T_1)$, because $\beta + \beta'$ bounds a disc in $W_{\text{in}}(T_1) \cap \bar{B}_i$ by Theorem 2.2.4 (6).

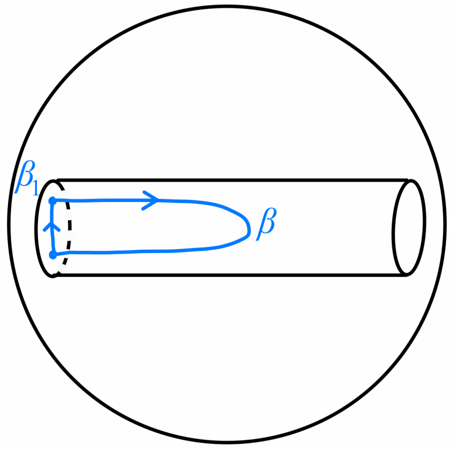


FIGURE 11. Shortening γ in case (1).

Moreover, this new loop is impossibly short:

$$\begin{aligned}
\text{length}(\gamma - \beta - \beta') &\leq \text{length}(\gamma) - \delta(x_i) + \delta(x_i)/50 \\
&< \text{length}(\gamma) - \delta(x_i)/2 \\
&\leq \text{length}(\gamma) - \epsilon_1 \\
&< \inf_{\gamma' \in c_0(T_1)} \text{length}(\gamma'),
\end{aligned}$$

in which the last inequality is from the definition of γ. Thus, a contradiction arises, and case (1) is impossible.

For case (2), suppose the starting point $\beta(0)$ is in D_1 and the ending point $\beta(1)$ is in D_2. We claim that there is another connected component $\hat{\beta}$ of $\gamma \cap B_i$ such that starting point $\hat{\beta}(0)$ is in D_2 and ending point $\hat{\beta}(1)$ is in $\in D_1$. This claim follows immediately from:

- By Theorem 2.2.4 (6), $M(T_1) \cap \partial B_i$ is a cylinder.
- By Lemma 2.4.6, $\text{link}(\gamma, \partial D_1) = \text{link}(\gamma, \partial D_2) = 0$.
- Case (1) was proven impossible.

Finally, let β_1 be a segment on D_1 connecting $\hat{\beta}(1)$ to $\beta(0)$, and β_2 be a segment on D_2 connecting $\hat{\beta}(0)$ to $\beta(1)$ (see Figure 12). As in case (1), we can guarantee $\text{length}(\beta_1), \text{length}(\beta_2) < \delta(x_i)/50$. Hence, we consider the new loop $\gamma - \beta - \hat{\beta} - \beta_1 - \beta_2$, which replaces $\beta + \hat{\beta} \subset \gamma$ with $-\beta_1 - \beta_2$. This new loop lies in $c_0(T_1)$, because $\beta + \hat{\beta} + \beta_1 + \beta_2$ bounds a disc in $W_{\text{in}}(T_1) \cap \bar{B}_i$ by Theorem 2.2.4 (6). Moreover, as in case (1), we can show that

$$\text{length}(\gamma - \beta - \hat{\beta} - \beta_1 - \beta_2) < \inf_{\gamma' \in c_0(T_1)} \text{length}(\gamma'),$$

which is a contradiction. So case (2) is impossible either.

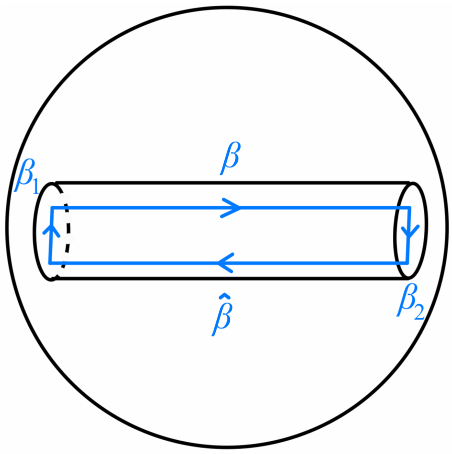


FIGURE 12. Shortening γ in case (2).

Therefore, we have reached a contradiction. □

This finishes the proof of Theorem 2.4.5. □

MCF through cylindrical and spherical singularities from torus. In §2.4 , we will focus on 2-dimensional MCF $\mathcal{M} = \{M(t)\}_{t \geq 0}$ through cylindrical and spherical singularities in $\mathbb{R}^3$, where $M(0)$ is a smooth *torus*. The main goal of §2.4 is to prove the following.

Theorem 2.4.8. *The setting is as follows.*

- *Let $\{M(t)\}_{t\geq 0}$ be a MCF through cylindrical and spherical singularities with $M(0)$ a smooth torus in $\mathbb{R}^3$.*
- *Let a_0 be a generator of $H_1(W_{\text{in}}(0)) \cong \mathbb{Z}$, and b_0 be a generator of $H_1(W_{\text{out}}(0)) \cong \mathbb{Z}$.*
- *Let $T = \min\{\mathfrak{t}(a_0), \mathfrak{t}(b_0)\}$.*

Then $T < \infty$, and $\text{genus}(M(t)) = 1$ *for a.e. $t < T$, while* $\text{genus}(M(t)) = 0$ *for a.e. $t > T$.*

Throughout §2.4, we will retain the notations in this theorem.

Let us first sketch the proof. By [CM16], $M(t)$ is smooth for a.e. time. And by [Whi95], $\text{genus}(M(t))$, when well-defined, is non-increasing in t. Thus, there exists some time T_g such that $\text{genus}(M(t)) = 1$ for a.e. $t < T_g$, while $\text{genus}(M(t)) = 0$ or $M(t)$ is empty for a.e. $t > T_g$. Our goal is to show $T = T_g$.

The proof consists of proving the following six claims one-by-one:

- $T < \infty$.
- Let $t \geq 0$. If $M(t)$ is a smooth torus and $a_0(t)$ exists, then $a_0(t)$ generates $H_1(W_{\text{in}}(t))$. And the case for b_0 is analogous.
- $T_g \geq T$.
- $\mathfrak{t}(a_0) \neq \mathfrak{t}(b_0)$.
- If $\mathfrak{t}(a_0) < \mathfrak{t}(b_0)$, then $b_0(t)$ is trivial for each $t > \mathfrak{t}(a_0)$. And if $\mathfrak{t}(b_0) < \mathfrak{t}(a_0)$, then $a_0(t)$ is trivial for each $t > \mathfrak{t}(b_0)$.
- $T_g \leq T$.

We now begin the proof.

Proposition 2.4.9. *$T < \infty$.*

Proof. Suppose otherwise, i.e. a_0 and b_0 both never terminate. Since $M(0)$ is compact, eventually $K_{\text{out}}(t) = \mathbb{R}^3$. So $a_0(T)$ and $b_0(T)$ both become trivial for some large $T > 0$. As a result, if we pick some loops $\alpha_0 \in a_0$ and $\beta_0 \in b_0$, then there exist 2-chain $A \subset W_{\text{in}}[0, T]$ and $B \subset W_{\text{out}}[0, T]$ such that $\partial A = \alpha_0$ and $\partial B = \beta_0$.

Now, denote by $\hat{B} \subset \mathbb{R}^3 \times [-T, 0]$ the reflection of B across $\mathbb{R}^3 \times \{0\}$. Let $\widetilde{B} = B \cup \hat{B}$, which can be viewed as a *closed* 2-chain in $\mathbb{R}^4$. Then we view $A \subset \mathbb{R}^4 \backslash \widetilde{B}$. Thus, to derive a contradiction, it suffices to show that α_0 is homologically non-trivial in $\mathbb{R}^4 \backslash \widetilde{B}$.

Without loss of generality, we can assume $\widetilde{B}$ is connected by discarding all those connected components that do not contain β_0. By Alexander duality,

$$H_1(\mathbb{R}^4 \backslash \widetilde{B}) \cong H^2(\widetilde{B}) \cong \mathbb{Z}.$$

One can check that $\alpha_0 \subset \mathbb{R}^4 \backslash \widetilde{B}$ actually generates $\mathbb{Z}$ as the linking number $\text{link}(a_0, b_0) = 1$ (?). This shows α_0 is homologically non-trivial in $\mathbb{R}^4 \backslash \widetilde{B}$, contradicting the existence of A. □

Remark 2.4.10. Note that the above proof works also in the case when $M(0)$ is a closed surface of any genus with $a_0 \in H_1(W_{\text{in}}(0))$ and $b_0 \in H_1(W_{\text{out}}(0))$ linked, and the flow $\{M(t)\}_{t \geq 0}$ is a general level set flow (whose singularities are not necessarily cylindrical or spherical).

Proposition 2.4.11. *Let $t \geq 0$. If $M(t)$ is a smooth torus and $a_0(t)$ exists, then $a_0(t)$ generates $H_1(W_{\text{in}}(t))$. And the case for b_0 is analogous.*

Proof. We will just prove the case for a_0. Let $\bar{a}$ be a generator of $H_1(W_{\text{in}}(t)) \cong \mathbb{Z}$. It suffices to show $\bar{a} = a_0(t)$ up to a sign.

By definition, there exists $\alpha_0 \in a_0$, $\alpha_1 \in a_0(t)$ such that $\alpha_0 - \alpha_1 = \partial A$ for some $A \subset W[0, t]$. On the other hand, pick a loop $\bar{\alpha}_1 \in \bar{a}$, then by [Whi95, Theorem 1 (ii)], there exists a homotopy H in $W[0, T]$ joining $\bar{\alpha}_1$ back to some loop $\bar{\alpha}_0 \subset W(0)$ (which means $\partial H = \bar{\alpha}_1 - \bar{\alpha}_0$). So $[\bar{\alpha}_0] = ka_0$ for some integer k, and so $\bar{\alpha}_0 - k\alpha_0 = \partial A_0$ for some $A_0 \subset W(0)$. If we manage to show $a_0 = [\bar{\alpha}_0]$ or $-[\bar{\alpha}_0]$, then by the fact that a_0 can only descend into one class at time t (Proposition 2.3.3), we would know $a_0(t) = \bar{a}$ or $-\bar{a}$, as desired. Hence, it suffices to show that $k = \pm 1$.

Let us glue H, A_0, and kA together, so that we have

$$\bar{\alpha}_1 - k\alpha_1 = \partial(H + A_0 + kA).$$

Thus, since the inclusion $H_1(W_{\text{in}}(t)) \to H_1(W_{\text{in}}[0,t])$ is injective by [Whi95, Theorem 1 (iii)], $\bar{a} = k\alpha_0(t)$ in $H_1(W_{\text{in}}(t))$. Since $\bar{a}$ is a generator by definition, $k = \pm 1$, as desired. □

Proposition 2.4.12. *$T_g \geq T$.*

Proof. Let us assume $T = \mathfrak{t}(a_0)$, as the other case $T = \mathfrak{t}(b_0)$ is analogous. Recall that we have shown $T < \infty$. Since genus$(M(t))$, if well-defined, is non-increasing in t, it suffices to prove that there exists $T_1 < T$ such that for a dense set of $t \in (T_1, T)$, genus$(M(t)) = 1$.

By Theorem 2.4.5, $T = \mathfrak{t}(a_0)$ implies a_0 breaks at some inward neck singularity (x, T). Then, applying Proposition 2.4.2 to (x, T) with $\delta_0 = 1$ and an $R > R_0$, we obtain constants $\delta, \bar{\delta}$ and a dense set $J \subset [T - \bar{\delta}, T + \bar{\delta}]$ with $T - \bar{\delta} \in J$. We let $T_1 = T - \bar{\delta}$, and $B = B_\delta(x)$.

Now, fix any $t \in (T_1, T)$, and D let be one of the two connected component of $K_{\text{in}}(t) \cap \partial B$: Recall that $K_{\text{in}}(t) \cap B$ is a solid cylinder by Proposition 2.4.2. By Proposition 2.4.2, some element $\alpha \in a_0(t)$ has a non-zero intersection number with D. Now, we push ∂D slightly into $K_{\text{out}}(t) \cap B$ and call that loop β. Then the linking number link(β, α) is non-zero, with α inside $M(t)$ and β outside $M(t)$. Hence, genus$(M(t))$ is non-zero, and thus has to be one, as desired. □

Proposition 2.4.13. *$\mathfrak{t}(a_0) \neq \mathfrak{t}(b_0)$.*

Proof. If $\mathfrak{t}(b_0) < \mathfrak{t}(a_0)$, we are done. So let us assume $\mathfrak{t}(a_0) \leq \mathfrak{t}(b_0)$ and aim to show $\mathfrak{t}(b_0) > \mathfrak{t}(a_0)$.

Let us focus at the time $t = T_1$, with $T_1 := T - \bar{\delta}$, as defined in the proof of Proposition 2.4.12. We know genus$(M(T_1)) = 1$ from before. Now, consider the loops $\alpha \in a_0(T_1)$ and $\beta \subset W_{\text{out}}(T_1) \cap B$ defined in the previous proof. Then by Proposition 2.4.11, α is a generator of $H_1(W_{\text{in}}(T_1))$, and from the construction of β it is clear $\text{link}(\beta, \alpha) = \pm 1$. So β actually generates $H_1(W_{\text{out}}(T_1))$. Then by Proposition 2.4.11 again and the assumption $\mathfrak{t}(b_0) \geq \mathfrak{t}(a_0)$, we have $[\beta] = b_0(T_1)$, possibly after changing the orientation of β.

Finally, by the mean convex neighborhood property, $\beta \subset W_{\text{out}}(T_1) \cap B$ will survive after time T. So $\mathfrak{t}(b_0) > \mathfrak{t}(a_0)$. □

Proposition 2.4.14. *If* $\mathfrak{t}(a_0) < \mathfrak{t}(b_0)$*, then* $b_0(t)$ *exists and is trivial for each* $t > \mathfrak{t}(a_0)$*. And if* $\mathfrak{t}(b_0) < \mathfrak{t}(a_0)$*, then* $a_0(t)$ *exists and is trivial for each* $t > \mathfrak{t}(b_0)$*.*

Proof. Let us just prove the first statement.

We will retain the notation from the previous proof. By Proposition 2.4.2, $M(T_1) \cap B$ (recall $T_1 = T - \bar{\delta}$) is close to a round cylinder. Now, enclose this cylinder by an Angenent torus, and run the MCF. Note that:

- Since the time interval around T given by the mean convex neighborhood property is independent of R (in Proposition 2.4.2), we can, by making R very large and thus the Angenent torus very small, assume that the mean convex neighborhood property still holds at the moment the Angenent torus vanishes.
- By the avoidance principle, the distance between the Angenent torus and $M(t)$ is non-decreasing.

Thus, at the moment the Angenent torus vanishes, the neck $M(t) \cap B$ has been "cut into disconnected pieces" already. Therefore, the loop β, which stays disjoint from the surface, would have become trivial at the moment the Angenent torus vanishes.

Finally, note that as $R \to \infty$, by the definition of cylindrical singularity we know $T_1 = T - \bar{\delta} \to T$ and $M(T - \bar{\delta}) \cap B$ tends to an actual round cylinder. This shows the moment the Angenent torus vanishes will tend to T (?). Thus, $b_0(t)$ is trivial for each $t > T$. □

Finally, since we have already proven $T_g \geq T$, to finish the proof of Theorem 2.4.8, it suffices to prove:

Proposition 2.4.15. $T_g \leq T$.

Proof. Suppose by contradiction $T_g > T$. Again, we assume the case $\mathfrak{t}(a_0) < \mathfrak{t}(b_0)$.

By our last proposition, we can pick a time $T_2 \in (T, T_g)$ when $M(T_2)$ is a smooth torus and $b_0(T_2)$ exists and is trivial. This contradicts Proposition 2.4.11, which says $b_0(T_2)$ generates $H_1(W_{\text{out}}(T_2))$. □

This finishes the proof of Theorem 2.4.8.

Termination time of limit of MCF. Finally, in §2.4, let us mention a proposition that describes a relationship between termination time and a convergent sequence of initial conditions.

Proposition 2.4.16. *The setting is as follows.*

- *Let* $\mathcal{M}^i = \{M^i(t)\}_{t\geq 0}$, $i = 1, 2, \ldots$, *and* $\mathcal{M} = \{M(t)\}_{t\geq 0}$ *all be MCF through cylindrical and spherical singularities, such that each* $M^i(0)$ *and* $M(0)$ *are smooth, close hypersurfaces.*
- *For each* i, *assume* $M^i(0)$ *is sufficiently close in* C^∞ *to* $M(0)$ *such that each* $H_1(W^{\mathcal{M}^i}(0))$ *can be canonically identified with* $H_1(W^{\mathcal{M}}(0))$. *Moreover,* $M^i(0) \to M(0)$ *in* C^∞.
- *Let* $c_0 \in H_1(W^{\mathcal{M}}(0))$. *Note that* c_0 *can be viewed as an element of* $H_1(W^{\mathcal{M}^i}(0))$ *for each* i *too.*

Then

$$\liminf_i \mathfrak{t}^{\mathcal{M}^i}(c_0) \geq \mathfrak{t}^{\mathcal{M}}(c_0).$$

Proof. Let $T = \mathfrak{t}^{\mathcal{M}}(c_0)$, which may be infinite. Suppose by contradiction that there exists a subsequence $\{i_k\}_k$ and some $T_1 < T$ such that $\mathfrak{t}^{\mathcal{M}^{i_k}}(c_0) \leq T_1$ for each k. Pick some element $\gamma_0 \subset W^{\mathcal{M}}(0)$ with $[\gamma_0] = c_0$, and $\gamma_1 \subset W^{\mathcal{M}}(\frac{T_1+T}{2})$ with $[\gamma_1] = c_0(\frac{T_1+T}{2})$. By definition, γ_0 and γ_1 together bound some $\Gamma \subset W^{\mathcal{M}}[0, \frac{T_1+T}{2}]$.

Now, recall that $\mathcal{M}^i \to \mathcal{M}$ in the Hausdorff sense by Proposition 2.2.7. Thus, since Γ is compact, for all sufficiently large i, $\Gamma \subset W^{\mathcal{M}^i}[0, \frac{T_1+T}{2}]$. Moreover, γ_0 represents $c_0 \in H_1(W^{\mathcal{M}^i}(0))$ for such large i. This contradicts that $\mathfrak{t}^{\mathcal{M}^{i_k}}(c_0) \leq T_1$ for each k. □

2.5. **Proof of main theorems.**

Proof of Theorem 1.1.1. Suppose by contradiction that for each $s \in [0,1]$, $\{M^s(t)\}_{t\geq 0}$ is a MCF through cylindrical and spherical singularities. For each $s \in [0,1]$, let

$$T^s = \min\{\mathfrak{t}^{\mathcal{M}^s}(a_0), \mathfrak{t}^{\mathcal{M}^s}(b_0)\}.$$

Note that $T^s < \infty$ by Proposition 2.4.9. In fact, by Proposition 2.4.13 and 2.4.14, either a_0 or b_0 will terminate, but not both. Thus, we can write $[0,1]$ as a disjoint union $A \sqcup B$, where A is the set of s for which $T^s = \mathfrak{t}^{\mathcal{M}^s}(a_0)$ while B is the set of s for which $T^s = \mathfrak{t}^{\mathcal{M}^s}(b_0)$. Note that by assumption, $0 \in A$ and $1 \in B$. Hence, the following lemma immediately gives a contradiction.

Lemma 2.5.1. *The sets A and B are both closed.*

Proof. We will just prove that A is closed. Let $s \in [0,1]$ be an accumulation point of A, and pick a sequence s_i in A with $s_i \to s$. Note that:

- For each i, by Theorem 2.4.8, $\text{genus}(M^{s_i}(t)) = 1$ for a.e. $t < T^{s_i}$ and $\text{genus}(M^{s_i}(t)) = 0$ for a.e. $t > T^{s_i}$.
- Similarly, $\text{genus}(M^s(t)) = 1$ for a.e. $t < T^s$ and $\text{genus}(M^s(t)) = 0$ for a.e. $t > T^s$.

Thus, together with Proposition 2.2.7, which says $M_i^s(t) \to M^s(t)$ in C^∞ for a.e. $t \geq 0$, we know $T^{s_i} \to T^s$. Hence,

$$T^s = \liminf_i T^{s_i} = \liminf_i \mathfrak{t}^{\mathcal{M}^{s_i}}(a_0) \geq \mathfrak{t}^{\mathcal{M}^s}(a_0).$$

Note that the second equality holds because $s_i \in A$, and the inequality holds by Proposition 2.4.16. Thus, we know $T^s = \mathfrak{t}^{\mathcal{M}^s}(a_0)$, which means for the flow $\mathcal{M}^s$, a_0 will terminate but b_0 will not. So $s \in A$. This shows A is closed. □

This finishes the proof of Theorem 1.1.1.

Remark 2.5.2. Let us explain why Theorem 1.1.1 would not hold if the initial conditions have genus greater than one. For example, consider the genus two surface in Figure 13, with a_0 and b_0 as shown, which are linked. Then, the MCF actually could develop an inward cylindrical singularity and an outward cylindrical singularity at the same time, with a_0 breaking at the inward one and b_0 breaking at the outward one. This phenomenon may prevent a genus one singularity to appear in any intermediate flow between $\{M^0(t)\}_{t\geq 0}$ and $\{M^1(t)\}_{t\geq 0}$, in the setting of Theorem 1.1.1.

One might think if we choose a_0 and b_0 better, like in Figure 14, then the conclusion of Theorem 1.1.1 may hold. However, Figure 13 and 14 are actually homotopic to each other. In conclusion, in a genus two surface, we cannot force a genus one singularity to appear just by topology: The geometry of the initial conditions must play a role.

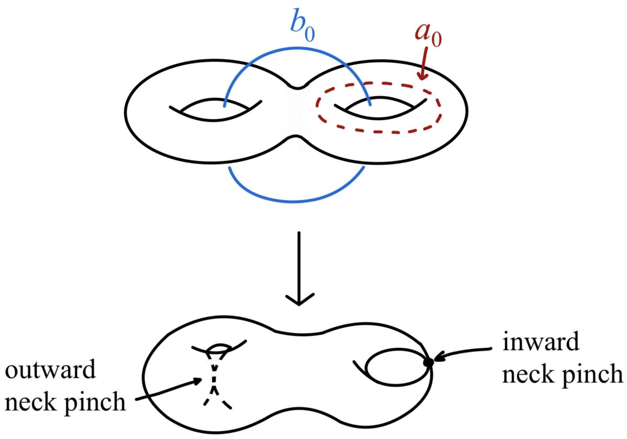


FIGURE 13. Simultaneous inward and outward neck pinches.

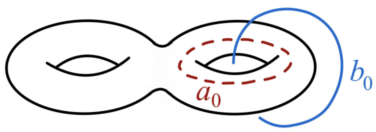


FIGURE 14. A topological equivalent picture.

Proof of Corollary 1.1.2. Let $\mathcal{M}^s := \{M^s(t)\}_{t\geq 0}$ be the level set flow starting from $M^s(0) := M^s$. We can apply Theorem 1.1.1 to the flows $\mathcal{M}^s$, $s \in [0,1]$, which shows there exists $s_0 \in [0,1]$ such that $\mathcal{M}^{s_0}$ has a singularity (x,T) that is not (multiplicity one) cylindrical or spherical. In other words, every tangent flow $\mathcal{M}'$ at (x,T) is not the shrinking cylinder or sphere of multiplicity one. Recall that by [Ilm95], $\mathcal{M}'$ is some smooth, embedded, self-shrinking flow $\{\sqrt{-t}m\Sigma'\}_{t<0}$ with genus at most one and possibly have multiplicity m. But the multiplicity can only be 1 by the entropy bound $\mathrm{Ent}(M^{s_0}) < 2$ and the monotonicity formula. Thus, Σ' has genus 1.

Proof of Theorem 1.1.3. Note that we have $\text{Ent}(M^s) < 2$ for each s as M^s is close to $\mathbb{T}$, which has entropy less than 2. To apply Corollary 1.1.2, it suffices to show that for the MCF starting from M^0 (resp. M^1), the inward (resp. outward) torus neck will pinch. But this is given by Proposition 2.3.16.

Proof of Theorem 1.1.4. Let Σ_1 be a genus one embedded shrinker in $\mathbb{R}^3$ with the least entropy. Recall that by [CM12] $\text{index}(\Sigma_1) \geq 5$. Hence, to prove Theorem 1.1.4, let us suppose by contradiction that Σ_1 is compact with index at least 6.

We first need a family of initial conditions to run MCF. That will be provided by the following lemma.

Lemma 2.5.3. *Let Σ^n by any smooth, embedded, compact, n-dimensional shrinker in $\mathbb{R}^{n+1}$ with index at least 6. Let $\epsilon > 0$ be sufficiently small. Then there exists a one-parameter family of smooth, compact, embedded surfaces $\{M^s(0)\}_{s\in[0,1]}$ such that:*

(1) The family varies continuous in the C^∞-topology, and each $M^s(0)$ is ϵ-close to C^∞ to Σ.

(2) Each $M^s(0)$ has entropy less than that of Σ.

(3) $M^0(0), M^1(0)$, and Σ are all disjoint, with $M^0(0)$ inside Σ and $M^1(0)$ outside.

Proof. Fix an outward unit normal vector field $\mathbf{n}$ to Σ. Since $\text{index}(\Sigma) \geq 6$, the eigenfunctions of its Jacobi operator, with respect to the Gaussian metric, that have negative eigenvalues include:

- three induced by translation in $\mathbb{R}^3$,
- one by scaling,
- the unique one-sided one which has the lowest eigenvalue, denoted ϕ_0,
- and at least one more, denoted ϕ_1,

all of which are orthonormal under the L^2-inner product. We will choose $\phi_0 > 0$.

Let $\epsilon > 0$, and define $M^s(0)$ to be the following perturbation of Σ:

$$M^s(0) := \Sigma + \epsilon(-\cos(s\pi)\phi_0 + \sin(s\pi)\phi_1)\mathbf{n}.$$

Thus, if $\epsilon > 0$ is sufficiently small, clearly the family $\{M^s(0)\}_{s\in[0,1]}$ is smooth. Item (3) holds because $\phi_0 > 0$. Finally, (2) holds because ϕ_0, ϕ_1 are not induced by translation or scaling (see Theorem 0.15 in [CM12]). $\square$

Applying the above lemma to Σ_1, we obtain a one-parameter family $\{M^s(0)\}_{s\in[0,1]}$ of tori. Then

$$\mathrm{Ent}(M^s(0)) < \mathrm{Ent}(\Sigma_1) \leq \mathrm{Ent}(\mathbb{T}) < 2.$$

Thus, applying Corollary 1.1.2, and by the monotonicity formula, we obtain another embedded genus one shrinker with entropy less than Σ_1, which contradicts the definition of Σ_1.

Proof of Theorem 1.1.5. Since $\mathbb{T}$ is rotationally symmetric, by [Liu16], it has index at least 7. Again, we need a family of MCF. We will apply [CM22, Theorem 1.6] of Choi-Mantoulidis. Namely, since $\mathbb{T}$ is a minimal surface with index at least 6 under the Gaussian metric, it has, as we saw in the proof of Lemma 2.5.3, two orthonormal eigenfunctions ϕ_0, ϕ_1 to the Jacobi operator that

- have negative eigenvalues,
- and are both orthogonal to the other 4 eigenfunctions induced by translation and scaling.

Now, pick an $\epsilon > 0$. Applying [CM22, Theorem 1.6] to the 2-dimensional function space spanned by ϕ_0 and ϕ_1, we obtain a one-parameter family of smooth ancient *rescaled MCF* (i.e. MCF under the Gaussian metric) $\widetilde{\mathcal{M}}^s = \{\widetilde{M}^s(\tau)\}_{\tau\leq 0}$, $s \in [0,1]$, such that:

- For each s, $\widetilde{M}^s(t) \to \mathbb{T}$ in C^∞ as $t \to -\infty$.
- $\widetilde{M}^0(0)$ lies inside $\mathbb{T}$ while $\widetilde{M}^1(0)$ lies outside.
- $\{\widetilde{M}^s(0)\}_{s\in[0,1]}$ is a smooth family of tori, each being ϵ-close to $\mathbb{T}$ in C^∞ (see [CM22, Corollary 3.4]).

If ϵ is small enough, we can apply Theorem 1.1.3 to the family $\{\widetilde{M}^s(0)\}_{s\in[0,1]}$ to obtain an $s_0 \in [0,1]$ such that the level set flow $\{M(t)\}_{t\geq 0}$ with initial condition $M(0) = \widetilde{M}^s(0)$

would develop a singularity at which every tangent flow is given by a multiplicity one, embedded, genus one self-shrinker.

Finally, we define an ancient smooth MCF $\{M(t)\}_{t\leq -1}$ by rescaling the rescaled MCF $\{\widetilde{M}^{s_0}(\tau)\}_{\tau\leq 0}$:

$$M(t) = \sqrt{-t}\widetilde{M}(-\log(-t)), \ \ t \leq -1.$$

Note that $M(-1) = \widetilde{M}(0) = M(0)$. Hence, combining the two flows $\{M(t)\}_{t\leq -1}$ and $\{M(t)\}_{t\geq 0}$, we obtain an ancient MCF satisfying Theorem 1.1.5.

Proof of Corollary 1.1.7. Let Σ be an embedded shrinker with the fourth least entropy in $\mathbb{R}^3$. Suppose by contradiction that Σ is rotationally symmetric. Then by Kleene-Møller [KMl14], Σ is closed with genus one. Moreover, Σ has entropy less than 2 since the shrinking doughnut $\mathbb{T}$ in [DN18] does, and by [Liu16], Σ has index at least 7. Thus, Theorem 1.1.5 still holds with $\mathbb{T}$ replaced by Σ: The exact same proof will work. Thus, we obtain a genus one shrinker with entropy strictly lower than Σ. However, the self-shrinkers with the three lowest entropy are the plane, the sphere, and the cylinder ([CIMIW13, BW17]). Contradiction arises.

3. A strong multiplicity one theory in min-max theory

This section is from a joint work with Yangyang Li.

3.1. **Preliminaries.** Throughout this section, unless specified otherwise, the ambient Riemannian manifolds (M^{n+1}, g) we consider will always be smooth and closed, with $3 \leq n+1 \leq 7$.

Notations.

- $\mathbf{I}_k(M;\mathbb{Z}_2)$: the set of integral k-dimensional currents in M with $\mathbb{Z}_2$-coefficients.
- $\mathcal{Z}_k(M;\mathbb{Z}_2) \subset \mathbf{I}_k(M;\mathbb{Z}_2)$: the subset that consists of elements T such that $T = \partial Q$ for some $Q \in \mathbf{I}_{k+1}(M;\mathbb{Z}_2)$ (such T are also called *flat k-cycles*).
- $\mathcal{Z}_k(M;\nu;\mathbb{Z}_2)$ with $\nu = \mathcal{F}, \mathbf{F}, \mathbf{M}$: the set $\mathcal{Z}_k(M;\mathbb{Z}_2)$ equipped with the three common topologies given respectively by the *flat* metric $\mathcal{F}$, the $\mathbf{F}$-*metric*, and the *mass* $\mathbf{M}$ (see, for example, the survey [MN20]).
- $\mathcal{V}_n(M)$ or $\mathcal{V}(M)$: the closure, in the varifold weak topology, of the space of n-dimensional rectifiable varifolds in M.
- $\|V\|$: the Radon measure induced on M by $V \in \mathcal{V}_n(M)$.
- For any a, the varifold topology on $\{V \in \mathcal{V}_n(M) : \|V\|(M) \leq a\}$ can be induced by an $\mathbf{F}$-*metric* defined by Pitts in [Pit81, p.66].
- $|T| \in \mathcal{V}_n(M)$: the varifold induced by a current $T \in \mathcal{Z}_k(M;\mathbb{Z}_2)$, or a submanifold T.
- In the same spirit, given a map Φ into $\mathcal{Z}_n(M;\mathbb{Z}_2)$, the associated map into $\mathcal{V}_n(M)$ is denoted $|\Phi|$.
- $\mathrm{spt}(\cdot)$: the support of a current or a varifold.
- $\mathbf{B}^{\nu}_{\epsilon}(\cdot)$: the open ϵ-neighborhood of an element or a subset in $\mathcal{Z}_n(M;\nu;\mathbb{Z}_2)$.
- $\mathbf{B}^{\mathbf{F}}_{\epsilon}(\cdot)$: the open ϵ-neighborhood of an element or a subset of $\mathcal{V}_n(M)$ under the $\mathbf{F}$-metric.

- $I(1,j)$: the cubical complex on $I := [0,1]$ whose 1-cells and 0-cells are respectively

$$[0,1/3^j],[1/3^j,2/3^j],\dots,[1-1/3^j,1] \quad \text{and} \quad [0],[1/3^j],[2/3^j],\dots,[1].$$

- $I(m,j)$: the cubical complex structure

$$I(m,j) = I(1,j) \otimes \cdots \otimes I(1,j) \ \ (m \text{ times})$$

  on I^m.
- X_q: the set of q-cells of X.
- $X(q)$ for a cubical subcomplex of $I(m,j)$: the subcomplex of $I(m,j+q)$ with support X.
- $\mathbf{f}(\Phi)$: the *fineness* of a map $\Phi : X_0 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$,

$$\sup\{\mathbf{M}(\Phi(x)-\Phi(y)) : x, y \text{ belong to some common cell}\}.$$

- $\mathbf{n}(i,j) : I(m,i)_0 \to I(m,j)_0$ for $j \le i$: the map such that $\mathbf{n}(i,j)(x)$ is the closest vertex in $I(m,j)_0$ to x.
- $\Gamma^\infty(M)$: the set of smooth Riemannian metrics on M.
- $B_g(p,r)$: the open r-neighborhood of a point p in metric g.

In the subsequent discussion, for the sake of simplicity, we consistently consider a complex and its underlying space as identical.

p-width. By the Almgren isomorphism theorem [Alm62] (see also [LMN18, §2.5]), when equipped with the flat topology, $\mathcal{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopic equivalent to $\mathbb{RP}^\infty$. Thus we can denote its cohomology ring by $\mathbb{Z}_2[\bar{\lambda}]$.

Definition 3.1.1. Let $\mathcal{P}_p$ be the set of all $\mathbf{F}$-continuous maps $\Phi : X \to \mathcal{Z}_2(M;\mathbb{Z}_2)$, where X is a finite simplicial complex, such that $\Phi^*(\bar{\lambda}^p) \neq 0$. Elements of $\mathcal{P}_p$ are called *p-sweepouts.*

Remark 3.1.2. Note that every finite cubical complex is homeomorphic to a finite simplicial complex and vice versa (see [BP02, §4]). So when X is a finite cubical complex in above, the notion of p-sweepout still makes sense.

Definition 3.1.3. Denoting by $\mathrm{dmn}(\Phi)$ the domain of Φ, the *p-width* of (M, g) is defined by

$$\omega_p(M, g) := \inf_{\Phi \in \mathcal{P}_p} \sup_{x \in \mathrm{dmn}(\Phi)} \mathbf{M}(\Phi(x)).$$

We may write $\omega_p(M)$ for $\omega_p(M, g)$ if no confusion is caused.

Definition 3.1.4. A sequence $(\Phi_i)_i$ in $\mathcal{P}_p$ is called a *minimizing sequence* for $\mathcal{P}_p$, or the p-width $\omega_p(M)$, if

$$\limsup_{i \to \infty} \max_{x \in \mathrm{dmn}(\Phi_i)} \mathbf{M}(\Phi_i(x)) = \omega_p(M).$$

For a minimizing sequence $(\Phi_i)_i$, we define its *critical set* by

$$\mathbf{C}_g((\Phi_i)_i) := \left\{V = \lim_j |\Phi_{i_j}(x_j)| : \{i_j\}_j \subset \mathbb{N}, x_j \in \mathrm{dmn}(\Phi_{i_j}), \|V\|_g(M) = \omega_p(M)\right\}.$$

We will often omit the subscript $_g$ if no confusion is caused. This will also be the case for other variants of min-max theory in §3.3. Now, a sequence is called *pulled-tight* if every varifold in $\mathbf{C}((\Phi_i)_i)$ is stationary.

Remark 3.1.5. There is an equivalent definition of p-widths from [MN21, Remark 5.7]: First, an $\mathcal{F}$-continuous map $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is said to have *no concentration of mass* if

$$\lim_{r \to 0} \sup_{x \in X, p \in M} \|\Phi(x)\|(B_r(p)) = 0.$$

Then, when defining the p-width, instead of using the collection $\mathcal{P}_p$, we use the collection of all $\mathcal{F}$-continuous maps Φ with no concentration of mass such that $\Phi^*(\bar{\lambda}^p) \neq 0$.

Interpolations. In this subsection, we collect some interpolation results in the literature. Let (M, g) be a closed manifold and m be a positive integer.

Proposition 3.1.6 ([MN17, Theorem 3.7]). *There exist positive constants* $C_{3.1.6} = C_{3.1.6}(M, g, m)$ *and* $\delta_{3.1.6} = \delta_{3.1.6}(M, g, m)$ *with the following property:*

If X *is a cubical subcomplex of* $I(m, l)$ *for some* $l \in \mathbb{N}^+$ *and*

$$\phi : X_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2)$$

has $\mathbf{f}(\phi) < \delta_{3.1.6}$, *then there exists a map (called the Almgren extension)*

$$\Phi : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

continuous in the mass norm and satisfying

(1) $\Phi(x) = \phi(x)$ *for all* $x \in X_0$;

(2) if α *is some* j*-cell in* X_j, *then* Φ *restricted to* α *depends only on the values of* ϕ *assumed on the vertices of* α;

(3) $\sup\{\mathbf{M}(\Phi(x) - \Phi(y)) : x, y$ *lie in a common cell of* $X\} \leq C_{3.1.6}\mathbf{f}(\phi)$.

Proposition 3.1.7 ([MN21, Proposition 3.2]). *There exist positive constants* $C_{3.1.7} = C_{3.1.7}(M, g, m)$ *and* $\delta_{3.1.7} = \delta_{3.1.7}(M, g, m)$ *with the following property:*

If X *is a cubical subcomplex of* $I(m, l)$ *for some* $l \in \mathbb{N}^+$ *and two continuous maps*

$$\Phi_0, \Phi_1 : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

satisfy

$$\sup_{x \in X} \mathbf{M}_g(\Phi_0(x) - \Phi_1(x)) < \delta_{3.1.7}\,,$$

then there exists a homotopy

$$H : [0, 1] \times X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

with $H(0, \cdot) = \Phi_0$ *and* $H(1, \cdot) = \Phi_1$ *and such that*

$$\sup_{(t,x) \in [0,1] \times X} \mathbf{M}_g(H(t, x) - \Phi_0(x)) \leq C_{3.1.7} \sup_{x \in X} \mathbf{M}_g(\Phi_0(x) - \Phi_1(x))\,.$$

In particular, for all $(t,x) \in [0,1] \times X$,

$$\mathbf{M}_g(H(t,x)) \leq \mathbf{M}_g(\Phi_0(x)) + C_{3.1.7} \sup_{x \in X} \mathbf{M}_g(\Phi_0(x) - \Phi_1(x))\,.$$

Proposition 3.1.8 (Improved version of [MN17, Proposition 3.8])**.** *There exist positive constants* $\eta_{3.1.8} = \eta_{3.1.8}(M,g,m)$ *and* $C_{3.1.8} = C_{3.1.8}(M,g,m)$ *with the following property:*

Suppose that X *is a cubical subcomplex in* $I(m,q)$*, and* $\phi_0 : X(l_0)_0 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ *is* $(X, \mathbf{M}_g)$*-homotopic to* $\phi_1 : X(l_1)_0 \to \mathcal{Z}_n(M;\mathbb{Z}_1)$ *through a discrete homotopy map*

$$h : I(1, q+l)_0 \times X(l)_0 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$$

with fineness $\mathbf{f}(h) < \eta_{3.1.8}$ *and such that if* $i = 0, 1$ *and* $x \in X(l)_0$*, then*

$$h([i], x) = \phi_i(\mathbf{n}(q+l, q+l_i)(x))\,.$$

Then the Almgren extensions

$$\Phi_0, \Phi_1 : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

of ϕ_0*,* ϕ_1*, respectively, are homotopic through a* $\mathbf{M}_g$*-continuous homotopy map*

$$H : [0,1] \times X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

with $H(0,\cdot) = \Phi_0$ *and* $H(1,\cdot) = \Phi_1$*. Furthermore, for all* $(t,x) \in [0,1] \times X$*, there exists* $(t_0, x_0) \in I(1, q+l)_0 \times X(l)_0$ *such that* x *and* x_0 *are in the same cell of* $X(l)$*, and*

$$\mathbf{M}_g(H(t,x)) \leq h(t_0, x_0) + C_{3.1.8}\mathbf{f}(h)\,.$$

Proof. Set $\eta_{3.1.8} = \min\left(\frac{\delta_{3.1.7}}{2C_{3.1.6}}, \delta_{3.1.6}\right)$ and $C_{3.1.8} = C_{3.1.6} + C_{3.1.7} \cdot 2C_{3.1.6}$.

For $i = 0, 1$, let $\phi_i' : X(l)_0 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ be given by $\phi_i'(x) = h([i], x)$. Since

$$\mathbf{f}(\phi_i), \mathbf{f}(\phi_i') \leq \mathbf{f}(h) < \eta_{3.1.8}\,,$$

by Proposition 3.1.6, for $i = 0, 1$, let $\Phi_i, \Phi'_i : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$ be the Almgren extensions of ϕ_i, ϕ'_i, respectly, and it follows that

$$\mathbf{M}_g(\Phi_i(x) - \Phi'_i(x)) \leq 2C_{3.1.6}\mathbf{f}(h) < 2C_{3.1.6}\eta_{3.1.8} \leq \delta_{3.1.7}\,.$$

Hence, for $i = 0, 1$, by Proposition 3.1.7, there exists a $\mathbf{M}_g$-continuous

$$H'_i : [0,1] \times X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)\,,$$

with $H'_i(0, \cdot) = \Phi_i$ and $H'_i(1, \cdot) = \Phi'_i$, and such that for all $(t, x) \in (0, 1) \times X$,

$$\begin{aligned}\mathbf{M}_g(H'_i(t,x)) &\leq \mathbf{M}_g(\Phi'_i(x)) + C_{3.1.7} \cdot 2C_{3.1.6}\mathbf{f}(h)\\ &\leq \mathbf{M}_g(\phi'_i(x_0)) + C_{3.1.6}\mathbf{f}(\phi'_i) + C_{3.1.7} \cdot 2C_{3.1.6}\mathbf{f}(h)\\ &\leq h([i], x_0) + (C_{3.1.6} + C_{3.1.7} \cdot 2C_{3.1.6})\mathbf{f}(h)\\ &\leq h([i], x_0) + C_{3.1.8}\mathbf{f}(h)\,.\end{aligned}$$

as long as $x_0 \in X(l)_0$ and x are in the same cell of $X(l)$.

By Proposition 3.1.6 again, the Almgren extension

$$H' : [0,1] \times X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

of h is a homotopy between Φ'_0 and Φ'_1 such that for all $(t, x) \in (0, 1) \times X$,

$$\mathbf{M}_g(H'(t,x)) \leq \mathbf{M}_g(h(t_0, x_0)) + C_{3.1.6}\mathbf{f}(h) \leq \mathbf{M}_g(h(t_0, x_0)) + C_{3.1.8}\mathbf{f}(h)\,,$$

as long as $(t_0, x_0) \in I(1, q+l)_0 \times X(l)_0$ and (t, x) are in the same cell of $I(1, q+l) \times X(l)$.

Finally, concatenating H'_0, H' and $H'_1(1-t, x)$, we obtain a homotopy

$$H : [0,1] \times X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

between Φ_0 and Φ_1. By the previous estimates, for all $(t,x) \in [0,1] \times X$,

$$\mathbf{M}_g(H(t,x)) \leq \mathbf{M}_g h(t_0, x_0) + C_{3.1.8}\mathbf{f}(h)\,.$$

for some $(t_0, x_0) \in I(1, q+l)_0 \times X(l)_0$ where x and x_0 are in the same cell of $X(l)$. □

Proposition 3.1.9 (Simplicial variant of [MN21, Proposition 3.7])**.** *Let X be a finite simplicial complex, and $\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$ be a continuous map. Then for every $\varepsilon > 0$ there exists an $\mathbf{M}_g$-continuous map*

$$\Phi' : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

and an $\mathbf{F}_g$-continuous homotopy $H : [0,1] \times X \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$ with $H(0,\cdot) = \Phi$ and $H(1,\cdot) = \Phi'$, and such that

$$\sup_{(t,x)\in(0,1)\times X} \mathbf{F}_g(H(t,x), \Phi(x)) < \varepsilon\,,$$

$$\sup_{(t,x)\in(0,1)\times X} \mathbf{M}_g(H(t,x)) < \sup_{x\in X} \mathbf{M}_g(\Phi(x)) + \varepsilon\,.$$

Proof. By [BP02, Chapter 4], the finite simplicial complex X is homeomorphic to a cubical subcomplex of some I^N. It follows from that [MN21, Proposition 3.7] for every $\varepsilon > 0$, we have a desired homotopy map H with

$$\sup_{(t,x)\in(0,1)\times X} \mathbf{F}_g(H(t,x), \Phi(x)) < \varepsilon\,.$$

Furthermore, by the definition of $\mathbf{F}_g$, for every $\varepsilon' > 0$, we can select even smaller ε ensuring that the aforementioned inequality leads to

$$\sup_{(t,x)\in(0,1)\times X} |\mathbf{M}_g(H(t,x)) - \mathbf{M}_g(\Phi(x))| < \varepsilon'\,.$$

This concludes the proof. □

3.2. $(m,r)_g$**-almost minimizing varifolds.** Let M^{n+1} ($3 \le n+1 \le 7$) be a closed smooth manifold, and $\Gamma^\infty(M)$ be the set of all the Riemannian metrics on M. Let us first recall the definitions of almost minimizing varifolds in [Pit81, MN21].

Definition 3.2.1. For each pair of positive numbers ε, δ, an open subset $U \subset (M^{n+1}, g)$, and $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$, an (ε,δ)-*deformation* of T in U is a finite sequence $(T_i)_{i=0}^q$ in $\mathcal{Z}_n(M;\mathbb{Z}_2)$ with

(1) $T_0 = T$ and $\operatorname{spt}(T - T_i) \subset U$ for all $i = 1, \cdots, q$;

(2) $\mathbf{M}_g(T_i - T_{i-1}) \le \delta$ for all $i = 1, \cdots, q$;

(3) $\mathbf{M}_g(T_i) \le \mathbf{M}_g(T) + \delta$ for all $i = 1, \cdots, q$;

(4) $\mathbf{M}_g(T_q) < \mathbf{M}_g(T) - \varepsilon$.

We define $\mathfrak{a}_g(U;\varepsilon,\delta)$ to be the set of all flat cycles $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ that do not admit (ε,δ)-deformations in U.

Definition 3.2.2. For an open set $U \subset (M^{n+1}, g)$, a varifold $V \in \mathcal{V}_n(M)$ is *almost minimizing* if for every $\varepsilon > 0$, we can find $\delta > 0$ and

$$T \in \mathfrak{a}_g(U;\varepsilon,\delta)$$

with $\mathbf{F}_g(V, |T|) < \varepsilon$.

In the following, for $m \in \mathbb{N}^+$, we set $I_m := 3^{m3^m}$. We now define a quantitative almost minimizing condition inspired by Pitts' combinatorial arguments.

Definition 3.2.3. Let $m \in \mathbb{N}^+$ and $r \in \mathbb{R}^+$. A varifold V in (M,g) is $(m,r)_g$-*almost minimizing* if the following holds. For any point $p \in M$ and any I_m concentric annuli $\{\mathrm{An}_g(p, r_i - s_i, r_i + s_i)\}_{i=1}^{I_m}$, where $\{r_i\}$ and $\{s_i\}$ satisfy

$$r_i - 2s_i > 2(r_{i+1} + 2s_{i+1}),\ i = 1, \dots, I_m - 1,$$

$$r_{I_m} - 2s_{I_m} > 0,$$

$$r_1 + s_1 < r,$$

V is almost minimizing [Pit81, Definition 3.1] in at least one of the annuli.

Remark 3.2.4. The definition presented here closely resembles the Property (m) in [Li23a] with the added requirement of a radius bound assumption.

Theorem 3.2.5 ([SS81, Theorem 4])**.** *In $(M^{n+1}, g)(3 \leq n+1 \leq 7)$, if V is $(m, r)_g$-almost minimizing and stationary, then V is a stationary integral varifold whose support* $\operatorname{spt}(V)$ *is a smooth, embedded, closed, minimal hypersurface.*

When the metric g is obvious from the context, we might omit the subscript $_g$ for simplicity.

In the subsequent subsections, we establish a set of technical lemmas concerning the two essential concepts of the previous definitions: annular replacements and the almost minimizing property.

Annular replacement.

Lemma 3.2.6. *Let $D, L \in \mathbb{R}^+$, $m \in \mathbb{N}^+$, and $K \subset \Gamma^\infty(M)$ be a compact set of C^∞ Riemannian metrics on M. There exists $\eta_{3.2.6} = \eta_{3.2.6}(M, K, D, L, m) > 0$ for which the following hold.*

If $g, \bar{g} \in K$ and $V_0, V_1, V_2 \in \mathcal{V}_n(M)$ satisfy

- $\|g - \bar{g}\|_{C^\infty, \bar{g}} \leq \eta_{3.2.6}$,
- $\|V_0\|_g(M), \|V_1\|_g(M), \|V_2\|_g(M) \leq 2L$,
- $\mathbf{F}_g(V_0, V_1) \leq \eta_{3.2.6}$,
- *V_0 stationary in $(M, \bar{g})$,*
- *$V_1 = V_2$ on $M \backslash (\overline{B}_g(p_1, 2\eta_{3.2.6}) \cup \cdots \cup \overline{B}_g(p_t, 2\eta_{3.2.6}))$ for some collection $\{p_1, \cdots, p_t\} \subset M, t \leq 3^{2m}$,*
- $\|V_1\|_g(M) - \eta_{3.2.6} \leq \|V_2\|_g(M) \leq \|V_1\|_g(M) + \eta_{3.2.6}$,

then $\mathbf{F}_g(V_1, V_2) < D/2$.

Proof. The proof is essentially the same as that of [MN21, Lemma 4.5].

If this is false for any $\eta_i = \frac{1}{i}$ $(i = 1, 2, 3, \cdots)$, we obtain the existence of $g_i, \bar{g}_i \in K$, $V_0^i, V_1^i, V_2^i \in \mathcal{V}_n(M)$ satisfying all the conditions in the lemma with $(\eta_i, g_i, \bar{g}_i, V_0^i, V_1^i, V_2^i)$ in place of $(\eta, g, \bar{g}, V_0, V_1, V_2)$ but

$$\mathbf{F}_g(V_1^i, V_2^i) \geq D/2 \,.$$

Up to a subsequence, by taking a limit, we obtain $g', \bar{g}' \in K$, $V_0', V_1', V_2' \in \mathcal{V}_n(M)$ and $\{p_1, \cdots, p_{t'}\} \subset M$, $t' \leq 3^{2m}$, such that

- i $g' = \bar{g}'$;
- ii $V_1' = V_0'$ is stationary;
- iii $\|V_0'\|(M) = \|V_2'\|(M)$ and $V_0' = V_2'$ on $M \setminus (\overline{B}_{g'}(p_1, r) \cup \cdots \cup \overline{B}_{g'}(p_{t'}, r))$ for all $r > 0$;
- iv $\mathbf{F}_g(V_0', V_2') \geq D/2$.

It follows from the monotonicity formula for stationary varifolds that there exists $C > 0$ such that

$$\|V_0'\|_{g'}(B_{g'}(q, r)) \leq Cr^n$$

for all r sufficiently small. Hence, (ii) implies that $V_0' = V_2'$, which contradicts (iii). $\square$

Lemma 3.2.7. *For any $m \in \mathbb{N}^+$ and $g \in \Gamma^\infty(M)$, there exists a positive constant $\eta_{3.2.7} = \eta_{3.2.7}(M, g, m)$ with the following property:*

If $V \in \mathcal{V}_n(M)$ is a stationary varifold on (M, g), $\{p_i\}_{i=1}^t \subset M$ with $t \leq 4 \cdot 3^{2m}$ is a collection of points, and $\{r_i\}_{i=1}^t \subset (0, 4\eta_{3.2.7})$, then

$$\operatorname{spt}(V) \setminus \bigcup_{i=1}^t \overline{B}_g(p_i, r_i) \neq \emptyset \,.$$

Proof. Using the monotonicity formula, we can find positive constants $R > s > 0$ which only depend on M, g and m, such that for any $p \in M$ and for any stationary varifold V,

$$\|V\|_g(B_g(p, R)) \geq 8 \cdot 3^{2m} \|V\|_g(B_g(p, s)) \,.$$

We can set $\eta_{3.2.7} := s/4$.

Now, assume for the sake of contradiction that there exists a stationary varifold V such that

$$\operatorname{spt}(V) \subset \bigcup_{i=1}^{t} \overline{B}_g(p_i, r_i)\,.$$

Then, there must exist a ball $\overline{B}_g(p_{i_0}, r_{i_0})$ satisfying

$$\|V\|_g(\overline{B}_g(p_{i_0}, r_{i_0})) > \frac{\|V\|_g(M)}{4 \cdot 3^{2m}}\,.$$

Since $r_{i_0} < 4\eta_{3.2.7} \leq s$, it follows that

$$\|V\|_g(\overline{B}_g(p_{i_0}, R)) \geq (8 \cdot 3^{2m})\|V\|_g(\overline{B}_g(p_{i_0}, r_{i_0})) > \|V\|_g(M)\,.$$

This leads to a contradiction. □

Lemma 3.2.8. *For any $m \in \mathbb{N}^+$ and $g \in \Gamma^\infty(M)$, there exists a positive constant $\eta_{3.2.8} = \eta_{3.2.8}(M, g, m)$ with the following property:*

For some $k \in \{0, 1, \cdots, n\}$, let $T, S \in \mathbf{I}_k(M; \mathbb{Z}_2)$ be two flat chains with $\partial T = \partial S = 0$, $\{p_i\}_{i=1}^t \subset M$ with $t \leq 4 \cdot 3^{2m}$ be a collection of points, and $\{r_i\}_{i=1}^t \subset (0, 4\eta_{3.2.8})$. If $T = S$ on $M \setminus \bigcup_{i=1}^t \overline{B}_g(p_i, r_i)$, then

$$T \in \mathcal{Z}_k(M; \mathbb{Z}_2) \iff S \in \mathcal{Z}_k(M; \mathbb{Z}_2)\,.$$

Proof. Since (M, g) is closed, we can find a positive constant $\eta_1 = \eta_1(M, g)$ such that for every $p \in M$ and $s \in (0, 4\eta_1)$, $\overline{B}_g(p, s)$ is diffeomorphic to a ball.

Claim 3.2.9. *If every $r_i < 4\eta_1$ and all the balls $\{\overline{B}_g(p_i, r_i)\}_i$ are disjoint from each other, then*

$$T \in \mathcal{Z}_n(M; \mathbb{Z}_2) \iff S \in \mathcal{Z}_n(M; \mathbb{Z}_2)\,.$$

Proof. If $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$, then there exists $R \in \mathbf{I}_{k+1}(M; \mathbb{Z}_2)$ such that $T = \partial R$.

Since $T = S$ on $M \setminus \bigcup_{i=1}^t \overline{B}_g(p_i, r_i)$, then

$$\operatorname{spt}(T - S) \subset \bigcup_{i=1}^t \overline{B}_g(p_i, r_i)\,.$$

In particular, since $\partial(T - S) = 0$ and the closed balls are disjoint from each other, for each i, $P_i = (T - S)\llcorner\overline{B}_g(p_i, r_i)$ also satisfies

$$\partial P_i = 0\,.$$

As $\overline{B}_g(p_i, r_i)$ is contractible, we can find $R_i \in \mathbf{I}_k(M; \mathbb{Z}_2)$ with $\partial R_i = P_i$.

Therefore, for $R' \coloneqq R + \sum_i R_i$, we have

$$\partial R' = \partial R + \sum_i \partial R_i = T + (T - S) = S\,,$$

i.e., $S \in \mathcal{Z}_k(M; \mathbb{Z}_2)$.

By symmetry, if $S \in \mathcal{Z}_k(M; \mathbb{Z}_2)$, then $T \in \mathcal{Z}_k(M; \mathbb{Z}_2)$ as well. This completes the argument. □

In general, we have the following covering results.

Claim 3.2.10. *There exists $\eta_2 = \eta_2(M, g, m) > 0$ such that if $r_i \in (0, 4\eta_2)$, then any collection of balls $\{\overline{B}_g(p_i, r_i)\}_{i=1}^t$ with $t \leq 4 \cdot 3^{2m}$, can be covered by $4 \cdot 3^{2m}$ many pairwise disjoint closed balls of radius at most η_1, $\{\overline{B}_g(p'_i, r'_i)\}_{i=1}^{t'}$.*

Proof. If false, we obtain a sequence of balls $\{\overline{B}_g(p_i^j, r_i^j)\}_{i=1}^{t_j}$ such that

(i) $t_j \leq 4 \cdot 3^{2m}$;

(ii) $\lim_j \sup_i r_i^j = 0$;

(iii) $\{\overline{B}_g(p_i^j, r_i^j)\}_{i=1}^{t_j}$ cannot be covered by $4 \cdot 3^{2m}$ disjoint closed balls of radius at most $r/2$.

Up to a subsequence, $\{\overline{B}_g(p_i^j, r_i^j)\}_{i=1}^{t_j}$ converges to a set of points $\{p_i'\}_{i=1}^{t'}$ in the Hausdorff sense and $t' \leq 4 \cdot 3^{2m}$. Obviously, we can choose $4 \cdot 3^{2m}$ pairwise disjoint closed balls of radius at most η_1 whose interiors cover $\{p_i'\}_{i=1}^{t'}$. Hence, these balls also cover $\{\overline{B}_g(p_i^j, r_i^j)\}_{i=1}^{t_j}$ for sufficiently large j. This contradicts our assumption (iii). □

Now, let $\eta_{3.2.8} = \min(\eta_1, \eta_2)$ defined above, and then $\{\overline{B}_g(p_i, r_i)\}_{i=1}^{t}$ can be covered by pairwise disjoint balls of radii no greater than η_1,

$$\{\overline{B}_g(p_i', r_i')\}_{i=1}^{t'} .$$

Since $T = S$ on $M \setminus \bigcup_{i=1}^{t'} \overline{B}_g(p_i', r_i')$, it follows from Claim 3.2.9 that

$$T \in \mathcal{Z}_k(M; \mathbb{Z}_2) \iff S \in \mathcal{Z}_k(M; \mathbb{Z}_2) .$$

□

We denote by $\mathcal{M}_g^L \subset \mathcal{V}_n(M)$ the set of all the embedded minimal cycles with total measure L, i.e. all the stationary integral varifolds with total measure L, and supported on a smooth, closed, embedded minimal hypersurface in (M, g).

Lemma 3.2.11. *Let $m \in \mathbb{N}^+$, $L \in \mathbb{R}^+$ and $g \in \Gamma^\infty(M)$ be a C^∞ Riemannian metrics on M. Let $\mathcal{G} \subset \mathcal{M}_g^L$ be a compact subset such that for every $W \in \mathcal{G}$, there exists $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $W = |T|$. Let $r := \min(\eta_{3.2.7}(M, g, m), \eta_{3.2.8}(M, g, m))$ from Lemmas 3.2.7 and 3.2.8. Then there exists $\eta_{3.2.11} = \eta_{3.2.11}(M, g, m, \mathcal{G}) > 0$ with the following property.*

If $V_1, V_2 \subset \mathcal{V}_n(M)$ satisfy

- $V_1 \in \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.2.11}}(\mathcal{G})$;
- $V_2 \in \mathcal{M}_g^L$;
- $V_2 = V_1$ *on* $M \setminus (\overline{B}_g(p_1, 2r) \cup \cdots \cup \overline{B}_g(p_t, 2r))$ *for some collection* $\{p_1, \cdots, p_t\} \subset M, t \leq 3^{2m}$;

Then there exists $T_2 \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $V_2 = |T_2|$.

Proof. The lemma follows from the following claim.

Claim 3.2.12. *For every $W \in \mathcal{G}$, there exists $\eta_1 = \eta_1(M, g, r, W) > 0$ such that for any $V \in \mathcal{M}_g^L$, any $s \in [2r, 5r/2]$, and any collection $\{p_1, \cdots, p_t\} \subset M, t \leq 3^{2m}$, if*

$$\mathbf{F}_{g, M\setminus(\overline{B}_g(p_1,s)\cup\cdots\cup\overline{B}_g(p_t,s))}(W, V) < \eta_{W,r}\,, \tag{3.1}$$

then there exists $T' \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $V = |T'|$. In particular, since r depends on M, g, and m, so $\eta_1 = \eta_1(M, g, m, W)$.

Proof. Let $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $W = |T|$. By Allard's ε-regularity theorem [All72], there exists an $\eta_1(M, g, r, W) > 0$ independent of the choice of balls, such that for any V satsfying (3.1), we have

$$\operatorname{spt}(V) \cap M \setminus (\overline{B}_g(p_1, 3r) \cup \cdots \cup \overline{B}_g(p_t, 3r))$$

is a subset of a (multiplicity-one) minimal graph over W.

Hence, as $r \leq \eta_{3.2.7}(M, g, m)$, by Lemma 3.2.7, V has multiplicity one and thus,

$$\mathcal{M}_g^L \ni V = |T'|\,,$$

for some current $T' \in \mathbf{I}_n(M; \mathbb{Z}_2)$ with $\partial T' = 0$.

Furthermore, the graphical property implies the existence of $R \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ such that

$$\operatorname{spt}(\partial R - T' - T) \subset (\overline{B}_g(p_1, 7r/2) \cup \cdots \cup \overline{B}_g(p_t, 7r/2))\,.$$

i.e., $\partial R + T = T'$ in $M \setminus (\overline{B}_g(p_1, 7r/2) \cup \cdots \cup \overline{B}_g(p_t, 7r/2))$. It follows from Lemma 3.2.8 and $r \leq \eta_{3.2.8}(M, g, m)$ that

$$T' \in \mathcal{Z}_n(M; \mathbb{Z}_2)\,.$$

□

Indeed, there exists $s \in [2r, 5r/2]$ such that $\operatorname{spt}(W)$ intersects every $\partial B_g(p_i, s)$ transversally. Hence, there exists $\eta'_W > 0$ such that for every $V' \in \mathcal{V}_n(M)$,

$$\mathbf{F}_g(W, V') < \eta'_W \implies \mathbf{F}_{g,M_s}(W, V') < \eta_W\,. \tag{3.2}$$

Since $\mathcal{G}$ is compact, there exists a finite subset $\{W_i\}_{i=1}^N$ such that

$$\mathcal{G} \subset \bigcup_{i=1}^{N} \mathbf{B}^{\mathbf{F}_g}_{\eta'_{W_i}}(W_i)\,,$$

and furthermore, we can choose $\eta_{3.2.11}$ such that

$$\mathbf{B}^{\mathbf{F}_g}_{\eta_{3.2.11}}(\mathcal{G}) \subset \bigcup_{i=1}^{N} \mathbf{B}^{\mathbf{F}_g}_{\eta'_{W_i}}(W_i)\,. \tag{3.3}$$

Now if V_1 and V_2 satisfies all the conditions in the lemma, then

$$\begin{aligned} V_1 \in \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.2.11}}(\mathcal{G}) &\implies \exists\, W_i \in \mathcal{G},\ V_1 \in \mathbf{B}^{\mathbf{F}_g}_{\eta'_{W_i}}(W_i) \quad \text{(by (3.3))} \\ &\implies \exists\, s \in [2r, 5r/2],\ \mathbf{F}_{g,M_s}(W_i, V_1) < \eta_{W_i} \quad \text{(by (3.2))} \\ &\implies \mathbf{F}_{g,M_s}(W_i, V_2) < \eta_{W_i} \quad \text{(by the third bullet point)} \\ &\implies \exists\, T_2 \in \mathcal{Z}_n(M; \mathbb{Z}_2),\ V_2 = |T_2| \quad \text{(by Claim 3.2.12)}\,. \end{aligned}$$

□

Lemma 3.2.13. *Let $m \in \mathbb{N}^+$, $L \in \mathbb{R}^+$, $g \in \Gamma^\infty(M)$ be a C^∞ Riemannian metrics on M, $r := \min(\eta_{3.2.7}(M, g, m), \eta_{3.2.8}(M, g, m))$ and $\mathcal{G}, \mathcal{B} \subset \mathcal{M}^L_g$ be compact subsets such that*

- *For every $V \in \mathcal{G}$, there exists $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $V = |T|$;*
- *For every $V \in \mathcal{B}$, no $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $V = |T|$.*

Then there exists $\eta_{3.2.13} = \eta_{3.2.13}(M, g, m, L, \mathcal{G}, \mathcal{B}) > 0$ with the following property:

If $g', g'' \in \Gamma^\infty(M)$, and $\{V_i\}_{i=1}^6 \cup \{W_i\}_{i=1}^6 \subset \mathcal{V}_n(M)$ satisfy

- *$\|g' - g\|_{C^\infty, g} < \eta_{3.2.13}$;*
- *$\|V_i\|_g(M), \|W_i\|_g(M) < 2L$ for any i;*

- $V_1 \in \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.2.13}}(\mathcal{G}), W_1 \in \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.2.13}}(\mathcal{B})$;
- $V_2 = V_1$ *on* $M \setminus (\overline{B}_{g'}(p_1, 2r) \cup \cdots \cup \overline{B}_{g'}(p_t, 2r))$ *for some collection* $\{p_1, \cdots, p_t\} \subset M, t \leq 3^{2m}$;
- $W_2 = W_1$ *on* $M \setminus (\overline{B}_{g'}(\bar{p}_1, 2r) \cup \cdots \cup \overline{B}_{g'}(\bar{p}_{\bar{t}}, 2r))$ *for some collection* $\{\bar{p}_1, \cdots, \bar{p}_{\bar{t}}\} \subset M, \bar{t} \leq 3^{2m}$;
- $\mathbf{F}_{g'}(V_3, V_2) < \eta_{3.2.13}$;
- $\mathbf{F}_{g'}(W_3, W_2) < \eta_{3.2.13}$;
- $\mathbf{F}_{g'}(V_4, V_3) < \eta_{3.2.13}$;
- $\mathbf{F}_{g'}(W_4, W_3) < \eta_{3.2.13}$;
- $V_5 = V_4$ *on* $M \setminus (\overline{B}_{g''}(p'_1, 2r) \cup \cdots \cup \overline{B}_{g''}(p'_{t'}, 2r))$ *for some collection* $\{p'_1, \cdots, p'_{t'}\} \subset M, t' \leq 3^{2m}$;
- $W_5 = W_4$ *on* $M \setminus (\overline{B}_{g''}(\bar{p}'_1, 2r) \cup \cdots \cup \overline{B}_{g''}(\bar{p}'_{\bar{t}'}, 2r))$ *for some collection* $\{\bar{p}'_1, \cdots, \bar{p}'_{\bar{t}'}\} \subset M, \bar{t}' \leq 3^{2m}$;
- $\mathbf{F}_{g''}(V_6, V_5) < \eta_{3.2.13}$;
- $\mathbf{F}_{g''}(W_6, W_5) < \eta_{3.2.13}$;

Then $\mathbf{F}_g(V_6, W_6) \geq \eta_{3.2.13}$.

In particular, we have: Let $\mathcal{G}'_{3.2.13} = \mathcal{G}'_{3.2.13}(M, g, m, L, \mathcal{G}, \mathcal{B})$ *be the set of all* V_3, $\mathcal{G}''_{3.2.13} = \mathcal{G}''_{3.2.13}(M, g, m, L, \mathcal{G}, \mathcal{B})$ *be the set of all* V_6, $\mathcal{B}'_{3.2.13} = \mathcal{B}'_{3.2.13}(M, g, m, L, \mathcal{G}, \mathcal{B})$ *be the set of all* W_3, *and* $\mathcal{B}''_{3.2.13} = \mathcal{B}''_{3.2.13}(M, g, m, L, \mathcal{G}, \mathcal{B})$ *be the set of all* W_6, *which satisfy the conditions above. Then,*

(1) $\mathcal{G}'_{3.2.13}, \mathcal{G}''_{3.2.13}, \mathcal{B}'_{3.2.13}$ *and* $\mathcal{B}''_{3.2.13}$ *are open;*

(2) $\mathcal{G} \subset \mathcal{G}'_{3.2.13} \subset \mathcal{G}''_{3.2.13}$ *and* $\mathcal{B} \subset \mathcal{B}'_{3.2.13} \subset \mathcal{B}''_{3.2.13}$;

(3) $\mathbf{F}_g(\mathcal{G}''_{3.2.13}, \mathcal{B}''_{3.2.13}) \geq \mathbf{F}_g(\mathcal{G}'_{3.2.13}, \mathcal{B}'_{3.2.13}) \geq \eta_{3.2.13}$.

Remark 3.2.14. (1) This lemma essentially shows that through a finite number of annular replacements and approximations, it is not possible for two elements from sets $\mathcal{G}$ and $\mathcal{B}$, respectively, to approach each other closely.

(2) For every $V \in \mathcal{B}$, either one connected component of $\mathrm{spt}(V)$ has multiplicity, or there exists $T' \in \mathbf{I}_n(M;\mathbb{Z}_2) \setminus \mathcal{Z}_n(M;\mathbb{Z}_2)$ such that $V = |T'|$ and $\partial T' = 0$.

Proof. If this is false, by compactness, there exists $V_1 \in \mathcal{G}$ and $W_1 \in \mathcal{B}$ such that

$$V_1 = W_1$$

on $M \setminus (\overline{B}_g(p_1, 3r) \cup \cdots \cup \overline{B}_g(p_t, 3r))$ for some collection $\{p_1, \cdots, p_t\} \subset M, t \leq 4 \cdot 3^{2m}$;

Suppose for the sake of contradiction that one connected component $\widetilde{\Sigma}$ of $\mathrm{spt}(V_2)$ has multiplicity at least two, then by Lemma 3.2.7 and $r \leq \eta_{3.2.7}(M, g, m)$,

$$\widetilde{\Sigma}' := \widetilde{\Sigma} \setminus \bigcup_{i=1}^{t} \bar{B}_g(p_i, 3r) \neq \emptyset\,.$$

Therefore, $V_1 \llcorner \widetilde{\Sigma}' = W_1 \llcorner \widetilde{\Sigma}'$, which contradicts the multiplicity one property of V_1.

Hence, both V_1 and W_1 have multiplicity one. In particular, there exists $T \in \mathbf{I}_n(M;\mathbb{Z}_2)$ with $\partial T = 0$ and $S \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ such that

$$V_1 = |T|, \quad W_1 = |S|\,.$$

Since $T = S$ on $M \setminus (\overline{B}_g(p_1, 3r) \cup \cdots \cup \overline{B}_g(p_t, 3r))$, and $r \leq \eta_{3.2.8}(M, g, m)$, $S \in \mathcal{Z}_n(M;\mathbb{Z}_2)$, which contradicts the definition of $\mathcal{B}$. □

Almost minimizing property.

Lemma 3.2.15. *Let $U \subset (M, g)$ be an open subset and $L, \varepsilon > 0$ be positive numbers. There exists $\eta_{3.2.15} = \eta_{3.2.15}(M, g, L, U, \varepsilon) > 0$ for which the following hold.*

If $g' \in \Gamma^\infty(M)$ and $V, V' \in \mathcal{V}_n(M)$ satisfy

- $\|g' - g\|_{C^\infty, g} \leq \eta_{3.2.15}$,
- $\|V\|_g(M) \leq 2L$,
- $T \notin \mathfrak{a}_g(U; 2\varepsilon, \delta)$ *for any* $\delta > 0$ *and* $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ *with* $\mathbf{F}_g(V, |T|) < 2\varepsilon$,
- $\mathbf{F}_g(V, V') \leq \eta_{3.2.15}$,

then for any $\delta > 0$ *and any* $T' \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ *with* $\mathbf{F}_{g'}(V', |T'|) < \varepsilon$*,*

$$T' \notin \mathfrak{a}_{g'}(U;\varepsilon,\delta)$$

Proof. If this is false for any $\eta_i = \frac{1}{i}(i = 1, 2, 3, \cdots)$, we obtain the existence of g_i', V_i, V_i' satisfying all the conditions in the lemma with $(\eta_i, g_i', V_i, V_i')$ in place of (η, g', V, V') but

$$T_i' \in \mathfrak{a}_{g_i'}(U;\varepsilon,\delta_i)$$

for some $\delta_i > 0$ and $T_i' \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ with $\mathbf{F}_{g_i'}(V_i', |T_i'|) < \varepsilon$.

For sufficiently large i, $\mathbf{F}_{g_i}(V_i, |T_i'|) < 2\varepsilon$, so $T_i' \notin \mathfrak{a}_{g_i}(U; 2\varepsilon, \delta_i/2)$, i.e., there exists a finite sequence $(T_j)_{j=0}^q$ in $\mathcal{Z}_n(M;\mathbb{Z}_2)$ with

(i) $T_0 = T_i'$ and $\text{spt}(T - T_j) \subset U$ for all $j = 1, \cdots, q$;

(ii) $\mathbf{M}_g(T_j - T_{j-1}) \le \delta_i/2$ for all $j = 1, \cdots, q$;

(iii) $\mathbf{M}_g(T_j) \le \mathbf{M}_g(T) + \delta_i/2$ for all $j = 1, \cdots, q$;

(iv) $\mathbf{M}_g(T_q) < \mathbf{M}_g(T) - 2\varepsilon$.

Hence, if i is sufficiently large, the above conditions induce the similar ones with respect to g_i',

(i) $T_0 = T_i'$ and $\text{spt}(T - T_j) \subset U$ for all $j = 1, \cdots, q$;

(ii) $\mathbf{M}_{g_i'}(T_j - T_{j-1}) \le \delta_i$ for all $j = 1, \cdots, q$;

(iii) $\mathbf{M}_{g_i'}(T_j) \le \mathbf{M}_{g_i'}(T) + \delta_i$ for all $j = 1, \cdots, q$;

(iv) $\mathbf{M}_{g_i'}(T_q) < \mathbf{M}_{g_i'}(T) - \varepsilon$;

which contradicts $T_i' \in \mathfrak{a}_{g_i'}(U;\varepsilon,\delta_i)$. □

Lemma 3.2.16. *Let* $m \in \mathbb{N}^+$*,* $r, d, L \in \mathbb{R}^+$*,* $\mathcal{SV}_g^L$ *be the space of all stationary varifolds in* (M^{n+1}, g) $(3 \le n+1 \le 7)$ *with volume* L*, and* $\mathcal{W}_g^L \subset \mathcal{SV}_g^L$ *be the subset of all the* $(m,r)_g$*-almost minimizing, stationary integral varifolds whose support is a smooth, closed minimal hypersurface. There exist positive constants* $\bar{\varepsilon}_{3.2.16} = \bar{\varepsilon}_{3.2.16}(M, g, m, r, d, L)$*,* $\bar{s}_{3.2.16} = \bar{s}_{3.2.16}(M, g, m, r, d, L)$ *and* $\eta_{3.2.16} = \eta_{3.2.16}(M, g, m, r, d, L)$ *for which the following hold.*

If $g' \in \Gamma^\infty(M)$ *and* $V \in \mathcal{V}_n(M)$ *satisfy*

- $\|g - g'\|_{C^\infty,g} < \eta$,
- $V \in \mathbf{B}^{\mathbf{F}_g}_\eta(\mathcal{SV}^L_g)$, *and*
- $V \notin \mathbf{B}^{\mathbf{F}_g}_d(\mathcal{W}^L_g)$,

then there exists $p \in M$ *and* I_m *concentric annuli* $\{\mathrm{An}_{g',i}(V)\}_{i=1}^{I_m} \equiv \{\mathrm{An}_{g'}(p, r_i - s_i, r_i + s_i)\}_{i=1}^{I_m}$ *such that*

(1) $\{r_i\}$ *and* $\{s_i\}$ *satisfy*

$$
\begin{aligned}
r_i - 2s_i &> 2(r_{i+1} + 2s_{i+1}),\ i = 1, \dots, I_m - 1,\\
r_{I_m} - 2s_{I_m} &> 0,\\
r_1 + s_1 &< r,\\
\min_i\{s_i\} &> \bar{s}_{3.2.16};
\end{aligned}
$$

(2) For any $i \in \{1, \cdots, I_m\}$, $\delta > 0$ *and* $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$, *if* $\mathbf{F}_{g'}(V, |T|) < \bar{\varepsilon}_{3.2.16}$, *then* $T \notin \mathfrak{a}_{g'}(\mathrm{An}_{g',i}(V), \delta, \bar{\varepsilon})$.

Proof. Let $\mathcal{K} := V \in \mathcal{SV}^L_g \setminus \mathbf{B}^{\mathbf{F}_g}_d(\mathcal{W}^L_g)$.

For every $V \in \mathcal{K}$, there exist $\varepsilon_V > 0$, $p_V \in M$, and I_m concentric annuli $\{\widetilde{\mathrm{An}}_{g,i}(V)\}_{i=1}^{I_m} \equiv \{\mathrm{An}_g(p_V, r_{V,i} - s_{V,i}, r_{V,i} + s_{V,i})\}_{i=1}^{I_m}$ such that

(i) $\{r_{V,i}\}$ and $\{s_{V,i}\}$ satisfy

$$
\begin{aligned}
r_{V,i} - 2s_{V,i} &> 2(r_{V,i+1} + 2s_{V,i+1}),\ i = 1, \dots, I_m - 1\,,\\
r_{V,I_m} - 2s_{V,I_m} &> 0\,,\\
r_{V,1} + s_{V,1} &< r\,;
\end{aligned}
$$

(ii) For any $i \in \{1, \cdots, I_m\}$, $\delta > 0$ and $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$, if $\mathbf{F}_g(V, |T|) < \varepsilon_V$, then

$$T \notin \mathfrak{a}_g(\widetilde{\mathrm{An}}_{g,i}(V), \delta, \varepsilon_V)\,.$$

Otherwise, V is both $(m,r)_g$-almost minimizing and stationary, so by Theorem 3.2.5, $V \in \mathcal{W}_g^L$, which contradicts $V \notin \mathbf{B}_d^{\mathbf{F}_g}(\mathcal{W}_g^L)$.

Since $\mathcal{K}$ is compact, there exists a finite subset $\{V_j\}_{j=1}^N \subset \mathcal{K}$ such that

$$\mathcal{K} \subset \bigcup_{j=1}^N \mathbf{B}^{\mathbf{F}_g}_{\varepsilon_{V_j}/2}(V_j)\,.$$

In other words, for every $V \in \mathcal{K}$, there exists some j such that $V \in \mathbf{B}^{\mathbf{F}_g}_{\varepsilon_{V_j}/2}(V_j)$, so for any $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ with $\mathbf{F}_g(V,|T|) < \varepsilon_{V_j}/2$, any $i \in \{1,\cdots,I_m\}$ and any $\delta > 0$,

$$T \notin \mathfrak{a}_g(\widetilde{\mathrm{An}}_{g,i}(V_j), \delta, \varepsilon_{V_j}/2)\,.$$

Moreover, there exists a positive constant η' such that for any $g' \in \Gamma^\infty(M)$ with $\|g'-g\|_{C^\infty,g} < \eta'$, and any $j \in \{1,\cdots,N\}$, there exist concentric annuli $\{\widetilde{\mathrm{An}}_{g',i}(V_j)\}_{i=1}^{I_m} \equiv \{\mathrm{An}_{g'}(p_{V_j}, r_{g',V_j,i} - s_{g',V_j,i}, r_{g',V_j,i} + s_{g',V_j,i})\}_{i=1}^{I_m}$ in (M,g') such that

(i) The radii $\{r_{g',V_j,i}\}_{i=1}^{I_m}$ and $\{s_{g',V_j,i}\}_{i=1}^{I_m}$ satisfy

$$r_{g',V_j,i} - 2s_{g',V_j,i} > 2(r_{g',V_j,i+1} + 2s_{g',V_j,i+1}),\ i = 1,\ldots,I_m - 1\,,$$
$$r_{g',V_j,I_m} - 2s_{g',V_j,I_m} > 0\,,$$
$$r_{g',V_j,1} + s_{g',V_j,1} < r\,;$$

(ii) $\min_i(s_{g',V_j,i}) > \min_j \min_i \frac{s_{V_j,i}}{2}$;

(iii) $\widetilde{\mathrm{An}}_{g',i}(V_j) \supset \widetilde{\mathrm{An}}_{g,i}(V_j)$;

Now, we set

$$\bar\varepsilon_{3.2.16} := \min_j \frac{\varepsilon_{V_j}}{10}\,,$$
$$\bar s_{3.2.16} := \min_j \min_i \frac{s_{V_j,i}}{2}\,,$$
$$\eta_{3.2.16} := \min\{d, \min_j \min_i \eta_{3.2.15}(M,g,L,\widetilde{\mathrm{An}}_{g,i}(V_j), \bar\varepsilon_{3.2.16})\}\,.$$

To see that these constants fulfill our requirements, let $g' \in \Gamma^\infty(M)$ and $W \in \mathcal{V}_n(M)$ with $\|g - g'\|_{C^\infty, g} < \eta$, $W \in \mathbf{B}^{\mathbf{F}_g}_\eta(\mathcal{SV}^L_g)$, and $W \notin \mathbf{B}^{\mathbf{F}_g}_d(\mathcal{W}^L_g)$. Therefore, there exists $V \in \mathcal{K}$ and V_j such that

$$\mathbf{F}_g(W, V) < \eta_{3.2.16}\,,$$

$$V \in \mathbf{B}^{\mathbf{F}_g}_{\varepsilon_{V_j}/2}(V_j)\,.$$

Since V satisfies that for any $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $\mathbf{F}_g(V, |T|) < 2\bar{\varepsilon}_{3.2.16}$, any $i \in \{1, \cdots, I_m\}$ and any $\delta > 0$,

$$T \notin \mathfrak{a}_g(\widetilde{\mathrm{An}}_{g,i}(V_j), \delta, 2\bar{\varepsilon}_{3.2.16})\,,$$

it follows from Lemma 3.2.15 that for any $T' \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $\mathbf{F}_{g'}(W, |T'|) < \bar{\varepsilon}_{3.2.16}$, any $i \in \{1, \cdots, I_m\}$ and any $\delta > 0$ that

$$T' \notin \mathfrak{a}_{g'}(\widetilde{\mathrm{An}}_{g,i}(V_j), \delta, \bar{\varepsilon}_{3.2.16})\,,$$

and thus,

$$T' \notin \mathfrak{a}_{g'}(\widetilde{\mathrm{An}}_{g',i}(V_j), \delta, \bar{\varepsilon}_{3.2.16})\,,$$

as $\widetilde{\mathrm{An}}_{g',i}(V_j) \supset \widetilde{\mathrm{An}}_{g,i}(V_j)$. Letting

$$\mathrm{An}_{g',i}(W) \coloneqq \widetilde{\mathrm{An}}_{g',i}(V_j)\,,$$

for each i, the conclusions (1) and (2) follow immediately. □

Lemma 3.2.17. *Let $m \in \mathbb{N}^+$, $r \in \mathbb{R}^+$, $(M^{n+1}, g)(3 \leq n+1 \leq 7)$ be a closed Riemannian manifold, and $(g_i)_{i=1}^\infty$ be a sequence of metrics with $g_i \to g$ in C^∞. For each $i \in N^+$, let Σ_i be an $(m, r)_{g_i}$-almost minimizing minimal hypersurface in (M, g_i). Then Σ_i subsequentially converges graphically in C^∞ to some $(m, r)_g$-almost minimizing minimal hypersurface Σ in (M, g).*

Furthermore, suppose that g is a metric with positive Ricci curvature or a bumpy metric. If $(\Sigma_i)_{i=1}^\infty$ is multiplicity-one, then Σ is also multiplicity-one; If $(\Sigma_i)_{i=1}^\infty$ is two-sided, then Σ is also two-sided.

Proof. By Allard's compactness theorem [All72], Σ_i subsequentially converges, in the varifold sense, to a stationary integral varifold V in (M,g). Without loss of generality, by relabelling, we may assume that Σ_i converges to V in the varifold sense.

To see that V is $(m,r)_g$ almost-minimizing, we take any $p \in M$ and I_m concentric annuli $\{\mathrm{An}_g(p, r_j - s_j, r_j + s_j)\}$ where $\{r_j\}$ and $\{s_j\}$ satisfy

$$r_j - 2s_j > 2(r_{j+1} + 2s_{j+1}),\ j = 1, \dots, I_m - 1,$$

$$r_{I_m} - 2s_{I_m} > 0,$$

$$r_1 + s_1 < r\,.$$

Since each Σ_i is almost minimizing in at least one of these finitely many annuli, there exist a $j_0 \in \{1, \cdots, I_m\}$ and a subsequence $(\Sigma_{i_k})_{k=1}^\infty$ such that every Σ_{i_k} is almost minimizing in $\mathrm{An}_g(p, r_{j_0} - s_{j_0}, r_{j_0} + s_{j_0})$. Consequentially, their limit V is also almost minimizing in $\mathrm{An}_g(p, r_{j_0} - s_{j_0}, r_{j_0} + s_{j_0})$ by Lemma 3.2.15, so V is $(m,r)_g$-almost minimizing and its support is a smooth embedded minimal hypersurface, denoted by Σ, i.e., $V = m|\Sigma|$ for some $m \in \mathbb{N}^+$.

If $m = 1$, then by Allard's regularity theorem, Σ_i subsequentially converges in C^∞. Therefore, it suffices to show that $m \geq 2$ is impossible provided that g is a metric with positive Ricci curvature or a bumpy metric.

Note that the almost minimizing property implies stability. By Schoen-Simon's regularity theory, the convergence above in $\mathrm{An}_g(p, r_{j_0} - s_{j_0}, r_{j_0} + s_{j_0})$ is locally smooth and graphical.

Let $\mathcal{S}$ be the set of open subsets of M such that for each $A \in \mathcal{S}$, there exists a subsequence $(\Sigma_{i_k})_{k=1}^\infty$ converges locally smoothly and graphically in A. Clearly, every non-empty totally ordered subset $\mathcal{T}$ of $\mathcal{S}$ has an upper bound (simply by taking the union of all the sets in $\mathcal{T}$), so by Zorn's lemma, $\mathcal{S}$ has at least one maximal element. Let us denote one of these maximal elements by R.

Now, we shall show that $M \setminus R$ has at most finitely many points. Suppose not and there exists a sequence of distinct points $(p_l)_{l=1}^{\infty}$ in $M \setminus R$ which converges to some $p \in M$. Let $(\Sigma_{i_k})_{k=1}^{\infty}$ be the subsequence which converges locally smoothly and graphically in R. We can choose I_m concentric annuli $\{\mathrm{An}_g(p, r_j - s_j, r_j + s_j)\}$ satisfying the relations as above and further such that each annuli contains at least one p_l. By the $(m, r)_g$-almost minimizing property at p, $(\Sigma_{i_k})_{k=1}^{\infty}$ has a subsequence $(\Sigma_{i'_k})_{k=1}^{\infty}$, each of which is also almost minimizing in a concentric annulus, say, $\mathrm{An}_g(p, r_{j_0} - s_{j_0}, r_{j_0} + s_{j_0})$ containing p_{l_0}. Therefore, $(\Sigma_{i'_k})_{k=1}^{\infty}$ converges locally smoothly and graphically in $R \cup \mathrm{An}_g(p, r_{j_0} - s_{j_0}, r_{j_0} + s_{j_0}) \supsetneq R$, contradicting the maximality of R.

It follows from [ACS18a, Theorem 5] that when $m \geq 2$, Σ (or its double cover $\widetilde{\Sigma}$ if one-sided) is stable and has nullity 1, which is impossible since g is either bumpy or of positive Ricci curvature. □

In [Li23a], it was shown that every p-width $\omega(M, g)$ can be realized by a varifold with Property $(2p + 1)_g$. An important ingredient of its proof is the following lemma, which was established in Proposition 3.2 therein. For the sake of completeness, we present the proof here.

Lemma 3.2.18. *If X is a k-dimensional finite simplicial complex, there exists a cubical subcomplex Y of $I(2k + 1, l)$ for some $l \in \mathbb{N}^+$ for which the following hold.*

(1) If we regard Y as a simplicial complex, X can be viewed as a subcomplex of some refinement of Y. In particular, there exists an embedding

$$\iota : X \to Y;$$

(2) There exists a retraction map

$$r : Y \to X\,.$$

Proof. Note that the underlying set of a finite simplicial complex is also a compact polyhedron (See [RS82, 1.8]). Applying the general position theorem for maps [RS82, Theorem 5.4] with $M = I^{2k+1}$ (endowed with the Euclidean metric), $P = X$, $P_0 = \emptyset$, $\varepsilon = 1/10$ and the closed map $f : X \to I^{2k+1}$ defined by

$$f(p) \equiv c(I^{2k+1})$$

where $c(I^{2k+1})$ is the center point of I^{2k+1}, we obtain a piecewise-linear embedding

$$f' : X \to I^{2k+1}\,.$$

f' is an embedding since it is nondegenrate and $\dim(S(f')) \leq 2k - (2k+1) < 0$.

By [RS82, p.33], there exists a regular neighborhood U, and by [RS82, Corollary 3.30], $f'(X)$ is a deformation retract of Z, i.e., there exists a retraction map

$$\widetilde{r} : U \to f'(X)\,.$$

Since $d = \mathrm{dist}(f'(X), \partial U) > 0$, we can find a large integer $l = l(k, d) \in \mathbb{N}^+$ such that for every (closed) $(2k+1)$-cell α of $I(2k+1, l)$, if $|\alpha| \cap f'(X) \neq \emptyset$, then $|\alpha| \subset U$, and we set Y to be the union of all such $|\alpha|$. It is obvious that

$$f'(X) \subset Y \subset U\,.$$

Therefore, f' and $\widetilde{r}$ induce the embedding $\iota : X \to Y$ and the retraction $r := f'^{-1} \circ \widetilde{r}|_Y : Y \to X$. □

3.3. **Restrictive min-max theory.** The concept of restrictive min-max theory was originally introduced by the second author in [Li23b, Section 2.3] with the purpose of generating CMC hypersurfaces. In this work, we extend and apply this theory to our specific setting.

In the following, let X be a k-dimensional, finite, simplicial complex and Z be a subcomplex of a refinement of X. Note that Z can be the empty set.

Deformations. In this subsection, we adapt some crucial technical deformation constructions from Pitts [Pit81] and Marques-Neves [MN21] to our setting. Later, we will apply these constructions to improve sweepouts.

Lemma 3.3.1 (Pull-tight)**.** *Given $c > 0$, we define*

$$\mathcal{V}^{\leq c} := \{V \in \mathcal{V}_n(M) : \|V\|_g(M) \leq c\},$$

$$\mathcal{SV}^{\leq c} := \{V \in \mathcal{V}^{\leq c} : V \text{ is stationary in } (M, g)\},$$

$$\mathcal{Z}^{\leq c} := \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2) \cap \{T : |T| \in \mathcal{V}^{\leq c}\}.$$

Then there exist continuous maps,

$$\bar{F}^{\mathrm{PT}} : [0,1] \times \mathcal{V}^{\leq c} \to \mathcal{V}^{\leq c},$$

$$F^{\mathrm{PT}} : [0,1] \times \mathcal{Z}^{\leq c} \to \mathcal{Z}^{\leq c},$$

such that

(1) For all $V \in \mathcal{V}^{\leq c}$, $\bar{F}^{\mathrm{PT}}(0, V) = V$;

(2) For all $t \in [0,1]$, $\bar{F}^{\mathrm{PT}}(t, V) = V$ if $V \in \mathcal{SV}^{\leq c}$;

(3) For all $t \in (0,1]$, $\|\bar{F}^{\mathrm{PT}}(t, V)\|_g(M) < \|V\|_g(M)$ if $V \notin \mathcal{SV}^{\leq c}$;

(4) Furthermore, for each $t \in [0,1]$ and each $S \in \mathcal{Z}^{\leq c}$,

$$|F^{\mathrm{PT}}(t, S)| = \bar{F}^{\mathrm{PT}}(t, |S|).$$

Proof. The proof is essentially the same as that of [Pit81, Theorem 4.3] (see also Sect. 15 of [MN14]). □

Corollary 3.3.2 (Pulled-tight sequence)**.** *Given a constant $c > 0$ and a sequence of finite simplicial complices $(X_i)_{i=1}^{\infty}$, let $(\Phi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_{i=1}^{\infty}$ be a sequence of $\mathbf{F}_g$-continuous maps such that*

$$L = \limsup_i \sup_{x \in X_i} \mathbf{M}_g \circ \Phi_i(x) > 0,$$

$$\sup_i \sup_{x \in X_i} \mathbf{M}_g \circ \Phi_i(x) \leq c.$$

Then there exists a sequence $(\Phi'_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_{i=1}^{\infty}$ *such that*

(1) *For each* $i \in \mathbb{N}^+$, *there exists a homotopy map*

$$H_i^{\mathrm{PT}} : [0,1] \times X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$$

with $H_i^{\mathrm{PT}}(0, \cdot) = \Phi_i(\cdot), H_i^{\mathrm{PT}}(1, \cdot) = \Phi'_i(\cdot)$ *satisfying that for each* $(t, x) \in [0,1] \times X_i$,

$$\mathbf{M}_g(H_i^{\mathrm{PT}}(t, x)) \leq \mathbf{M}_g(\Phi_i(x))\,;$$

(2) *The set of varifolds*

$$\mathcal{V}^L((\Phi'_i)_i) := \{V = \lim_j |\Phi'_{i_j}(x_j)| : \mathbb{N}^+ \ni i_j \nearrow \infty, x_j \in X_{i_j}, \|V\|_g(M) = L\}$$

is a subset of

$$\mathcal{V}^L((\Phi_i)_i) := \{V = \lim_j |\Phi_{i_j}(x_j)| : \mathbb{N}^+ \ni i_j \nearrow \infty, x_j \in X_{i_j}, \|V\|_g(M) = L\};$$

(3) $\mathcal{V}^L((\Phi'_i)_i) \subset \mathcal{SV}^{\leq c}$, *i.e.,* $\mathcal{V}^L((\Phi'_i)_i)$ *only contains stationary varifolds.*

Proof. By Lemma 3.3.1, we can define H_i^{PT} by

$$H_i^{\mathrm{PT}}(t, x) := F^{\mathrm{PT}}(t, \Phi_i(x))\,.$$

The properties of F^{PT} and $\bar{F}^{\mathrm{PT}}$ immediately yield all the stated conclusions. □

The following is a continuous version of [MN21, Theorem 4.6].

Lemma 3.3.3 ((ε, δ)-deformation)**.** *Let* $R, \bar{\varepsilon}, \eta, s > 0$ *be constants such that* $\bar{\varepsilon} < 2R$, *and* $\mathcal{W} \subset \mathcal{V}_n(M)$. *Let* $\Phi : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$ *be a continuous map and* $L = \sup_{x \in X} \mathbf{M}_g(\Phi(x))$ *such that if* $x \in X$ *satisfies*

$$\mathbf{M}_g(\Phi(x)) \geq L - \bar{\varepsilon}, \quad \mathbf{F}_g(|\Phi(x)|, \mathcal{W}) \geq R\,,$$

then $\Phi(x)$ *satisfies* annular $(\bar{\varepsilon}, \delta)$-deformation conditions, *i.e., there exist* $p(x) \in M$ *and* I_{2k+1} *positive numbers*

$$r_1(x), \cdots, r_{I_{2k+1}}(x), s_1(x), \cdots, s_{I_{2k+1}}(x)$$

satisfying

$$s_i(x) \geq s, \quad i = 1, \cdots, I_{2k+1} - 1$$

$$r_i(x) - 2s_i(x) > 2(r_{i+1}(x) + 2s_{i+1}(x)), \quad i = 1, \cdots, I_{2k+1} - 1$$

$$r_1(x) + 2s_1(x) < \eta,$$

$$r_{I_{2k+1}}(x) - 2s_{I_{2k+1}}(x) > 0,$$

such that $\Phi(x)$ *admits an* $(\bar{\varepsilon}, \delta)$*-deformation in each annulus*

$$\mathrm{An}_g(p(x), r_i(x) - s_i(x), r_i(x) + s_i(x)) \cap M,$$

$i = 1, \cdots, I_{2k+1}$, *for every* $\delta > 0$.

Then for any $\bar{\delta} > 0$, *there exists a continuous map*

$$\Phi^* : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

for which the following hold.

(1) There exists a homotopy map

$$H^{\mathrm{DEF}} : [0,1] \times X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

with $H^{\mathrm{DEF}}(0, \cdot) = \Phi$ *and* $H^{\mathrm{DEF}}(1, \cdot) = \Phi^*$ *satisfying that for each* $(t, x) \in [0,1] \times X$,

$$\mathbf{M}_g(H^{\mathrm{DEF}}(t, x)) < \mathbf{M}_g(\Phi(x)) + \bar{\delta};$$

(2) For any $x \in X$ and $t \in [0,1]$, there exist $\hat{x} = \hat{x}(x) \in X$ and $T_{t,x} \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ such that

$$\mathbf{M}_g(H^{\mathrm{DEF}}(t,x), T_{t,x}) < \bar{\delta}\,,$$

$$\mathbf{M}_g(\Phi(\hat{x}), \Phi(x)) < \bar{\delta}\,,$$

$$\mathbf{M}_g(\Phi(\hat{x})) > \mathbf{M}_g(H^{\mathrm{DEF}}(t,x)) - \bar{\delta}\,,$$

and

$$T_{t,x} \llcorner (M \setminus (\overline{B}_g(p_1,\eta) \cup \cdots \cup (\overline{B}_g(p_m,\eta))) = \Phi(\hat{x}) \llcorner (M \setminus (\overline{B}_g(p_1,\eta) \cup \cdots \cup (\overline{B}_g(p_m,\eta)))$$

for some collection $\{p_1, \cdots, p_m\} \subset M$, $m \leq 3^{2k+1}$;

(3) If $\mathbf{M}_g(\Phi^*(x)) \geq L - \bar{\varepsilon}/10$, then

$$\mathbf{F}_g(|\Phi(\hat{x})|, \mathcal{W}) \leq 2R\,,$$

where $\hat{x}$ is the same as that in (2).

Proof. Fix $\bar{\delta} > 0$.

Since X is a k-dimensional finite simplicial complex, by 3.2.18, there exists a cubical subcomplex Y of $I(2k+1, l)$ for some $l \in \mathbb{N}^+$ and a retraction map

$$r : Y \to X\,.$$

Define $\Psi : Y \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$ by $\Psi = \Phi \circ r$. It is easy to verify that Ψ also satisfies all the assumptions for Φ in the lemma.

For each $q \in \mathbb{N}^+$, we define $\psi_q : Y(q)_0 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$, by

$$\psi_q = \Psi|_{Y(q)_0}\,.$$

Since Ψ is continuous in the $\mathbf{M}_g$-topology, the fineness $\mathbf{f}(\psi_q) \to 0$ as $q \to \infty$. In the following, we shall subsequently choose q larger and larger so as to apply interpolation propositions from the previous section.

First, we choose $N_1 \in \mathbb{N}^+$, such that for all $q \geq N_1$,

$$\mathbf{f}(\psi_q) < \min(\delta_{3.1.6}, \min(\delta_{3.1.7}, \bar{\delta}/(5C_{3.1.7}))/(2C_{3.1.6}))$$

and for all $x, y \in Y$, if x and y lie in a common cell of $Y(q)$,

$$\mathbf{M}_g(\Psi(x), \Psi(y)) < \min(\delta_{3.1.7}, \bar{\delta}/(5C_{3.1.7}))/2\,.$$

Then by Proposition 3.1.6, ψ_q has the Almgren extension Ψ_q and

$$\sup_{y \in Y} \mathbf{M}_g(\Psi(y) - \Psi_q(y)) < \min(\delta_{3.1.7}, \bar{\delta}/(5C_{3.1.7})\,.$$

It follows from Proposition 3.1.7, there exists a homotopy map

$$H_q^{(1)} : [0,1] \times Y \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

with $H_q^{(1)}(0, \cdot) = \Psi$ and $H_q^{(1)}(1, \cdot) = \Psi_q$, and for all $t \in [0,1]$ and $y \in Y$,

$$\begin{aligned}\mathbf{M}_g(H_q^{(1)}(t,y)) &\leq \mathbf{M}_g(\Psi(y)) + C_{3.1.7} \sup_{y \in Y} \mathbf{M}_g(\Psi(y) - \Psi_q(y)) \\ &< \mathbf{M}_g(\Psi(y)) + \bar{\delta}/5\,.\end{aligned}$$

Secondly, we choose $N_2 \in \mathbb{N}^+$ with $N_2 > N_1$, such that for all $q \geq N_2$, the following condition from [MN21, Theorem 4.6] holds,

$$(2k+1)\mathbf{f}(\psi_q)(1 + 4(3^{2k+1} - 1)s^{-1}) < \min\{\frac{\bar{\varepsilon}}{3^{2(2k+1)}8}, \gamma_{\text{iso}}\}\,,$$

Consequently, [MN21, Theorem 4.6] implies that there exists $C = C(k,s) > 0$, an integer $q' > q$, and a map

$$\psi_q^* : Y(q')_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2)$$

such that

(i) ψ_q^* is $(Y, \mathbf{M}_g)$-homotopic to ψ_q, through a discrete homotopy

$$h_q : I(1, l+q')_0 \times Y(q')_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2)$$

with fineness $\mathbf{f}(h_q) \leq C\mathbf{f}(\psi_q)$;

and for all $(t, y) \in I(1, l+q')_0 \times Y(q')_0$, if $\hat{y} = \mathbf{n}(l+q', l+q)(y)$, then

(ii) $h_q(t,y)(M \setminus (\overline{B}_\eta(p_1) \cup \cdots \cup (\overline{B}_\eta(p_m))) = \psi_q(\hat{y}) \llcorner (M \setminus (\overline{B}_\eta(p_1) \cup \cdots \cup (\overline{B}_\eta(p_m)))$ for some collection $\{p_1, \cdots, p_m\} \subset M$, $m \leq 3^{2k+1}$;

(iii) $\mathbf{M}_g(h_q(t,y)) \leq \mathbf{M}_g(\psi(\hat{y})) + 2 \cdot 3^{2(2k+1)}(2k+1)(1 + 4(3^{2k+1}-1)s^{-1})\mathbf{f}(\psi_q)$;

(iv) if $\mathbf{M}_g(\psi_q^*(y)) \geq L - \bar{\varepsilon}/5$, then $\mathbf{F}_g(|\psi_q(\hat{y})|, \mathcal{W}) \leq 2R$.

Thirdly, we choose $N_3 \in \mathbb{N}^+$ with $N_3 > N_2$, such that for all $q \geq N_3$,

$$\mathbf{f}(h_q) \leq C\mathbf{f}(\psi_q) < \min(\eta_{3.1.8}, \bar{\delta}/(5C_{3.1.8}))\,,$$

$$2 \cdot 3^{2(2k+1)}(2k+1)(1 + 4(3^{2k+1}-1)s^{-1})\mathbf{f}(\psi_q) < \bar{\delta}/5\,,$$

and for all x and y which lie in the same cell of $Y(q)$,

$$\mathbf{M}_g(\Psi(x) - \Psi(y)) < \bar{\delta}/5\,.$$

Applying Proposition 3.1.8 to (i) above, we obtain a $\mathbf{M}_g$-continuous homotopy map

$$H_q^{(2)} : [0,1] \times Y \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

with $H_q^{(2)}(0, \cdot) = \Psi_q$ the Almgren extension of ψ_q, and $H_q^{(2)}(1, \cdot) = \Psi_q^*$ the Almgren extension of ψ_q^*. Furthermore, for all $t \in [0,1]$, $y \in Y$, there exists $(t_0, y_0) \in I(1, l+q')_0 \times$

$Y(q')_0$ such that y and y_0 are in the same cell of $Y(q')$, and

$$\begin{aligned}\mathbf{M}_g(H_q^{(2)}(t,y)) &\le \mathbf{M}_g(h_q(t_0,y_0)) + C_{3.1.8}\mathbf{f}(h_q)\\ &\le \mathbf{M}_g(\psi_q(\hat{y}_0)) + 2\cdot 3^{2(2k+1)}(2k+1)(1+4(3^{2k+1}-1)s^{-1})\mathbf{f}(\psi_q) + C_{3.1.8}\mathbf{f}(h_q)\\ &< \mathbf{M}_g(\psi_q(\hat{y}_0)) + 2\bar{\delta}/5\\ &< \mathbf{M}_g(\Psi(y)) + 3\bar{\delta}/5\end{aligned}$$

where we use (iii) in the second line and the fact that y and $\hat{y}_0$ are in the same cell of $Y(q)$ in the last line.

Now, concatenating $H_q^{(1)}$ and $H_q^{(2)}$, we obtain a homotopy

$$H_q : [0,1]\times Y \to \mathcal{Z}_n(M;\mathbf{M}_g;\mathbb{Z}_2)$$

between Ψ and Ψ_q^*, and for all $(t,y)\in[0,1]\times Y$,

$$\mathbf{M}_g(H_q(t,y)) < \mathbf{M}(\Psi(y)) + \bar{\delta}\,. \tag{3.4}$$

Finally, we choose $N_4 \in \mathbb{N}^+$ with $N_4 > N_3$ such that for all $q \ge N_4$,

$$\mathbf{f}(h_q) \le C\mathbf{f}(\psi_q) < \min(\bar{\varepsilon},\bar{\delta})/(10C_{3.1.6})\,,$$

and for all x and y which lie in the same cell of $Y(q)$,

$$\mathbf{F}_g(|\Psi(x)|,|\Psi(y)|) < R\,.$$

Hence, we can fix a $q \ge N_4$, and set

$$H^{\mathrm{DEF}} := H_q|_{[0,1]\times X}\,,\quad \Phi^* := \Psi_q^*|_X\,.$$

The statement (1) follows immediately from (3.4).

To see (2), for any $x\in X$ and $t\in[0,1]$, choose $y_0 \in Y(q')_0$ such that x and y_0 lie in the same cell of $Y(q')$. If $H^{\mathrm{DEF}}(x,t) = H_q^{(1)}(x,t_1)$, then we set $t_0 := 0$; Otherwise

$H^{\mathrm{DEF}}(x,t) = H_q^{(2)}(x,t_2)$, and we choose $t_0 \in [0,1](l+q')_0$ such that t_0 and t_2 lie in the same cell of $[0,1](1+q')$. We let $T_{t,x} \coloneqq h_q(t_0, y_0)$. By Proposition 3.1.6 and Proposition 3.1.7,

$$\begin{aligned}\mathbf{M}_g(H^{\mathrm{DEF}}(t,x), T_{t,x}) &= \mathbf{M}_g(H_q(t,x), h_q(t_0,y_0)) \\ &< \max(C_{3.1.7} \sup_{y\in Y} \mathbf{M}_g(\Psi(y) - \Psi_q(y)), C_{3.1.6}\mathbf{f}(h_q)) \\ &< \bar{\delta}/5\,.\end{aligned}$$

It follows from (ii) above that

$$h_q(t_0,y_0)(M \setminus (\overline{B}_\eta(p_1) \cup \cdots \cup (\overline{B}_\eta(p_m))) = \psi_q(\hat{y}_0) \llcorner (M \setminus (\overline{B}_\eta(p_1) \cup \cdots \cup (\overline{B}_\eta(p_m)))$$

for some collection $\{p_1, \cdots, p_m\} \subset M$, $m \leq 3^{2k+1}$. Using the retraction map r, we let $\hat{x} = r(\hat{y}_0)$, then we obtain

$$T_{t,x} \llcorner (M \setminus (\overline{B}_\eta(p_1) \cup \cdots \cup (\overline{B}_\eta(p_m))) = \Phi(\hat{x}) \llcorner (M \setminus (\overline{B}_\eta(p_1) \cup \cdots \cup (\overline{B}_\eta(p_m)))\,.$$

In addition, by (iii),

$$\begin{aligned}\mathbf{M}_g(\Phi(\hat{x})) &= \mathbf{M}_g(\Psi(\hat{y}_0)) \\ &= \mathbf{M}_g(\psi_q(\hat{y}_0)) \\ &\geq \mathbf{M}_g(h_q(y_0,t_0)) - \bar{\delta}/5 \\ &> \mathbf{M}_g(H^{\mathrm{DEF}}(x,t)) - \bar{\delta}\,.\end{aligned}$$

Since x and y_0 lie in the same cell of $Y(q')$

$$\mathbf{M}_g(\Phi(\hat{x}), \Phi(x)) = \mathbf{M}_g(\Psi(\hat{y}_0), \Psi(x)) < bar\delta$$

As for (3), if $\mathbf{M}_g(\Phi^*(x)) \geq L - \bar{\varepsilon}/10$, we choose $y_0 \in Y(q')_0$ again such that x and y_0 lie in the same cell of $Y(q')$. Since

$$\begin{aligned}\mathbf{M}_g(\psi_q^*(y_0)) &\geq \mathbf{M}_g(\Psi(x)) - C_{3.1.6}\mathbf{f}(h_q)\\ &\geq \mathbf{M}_g(\Phi(x)) - \frac{\bar{\varepsilon}}{10}\\ &\geq L - \frac{\bar{\varepsilon}}{5},\end{aligned}$$

by (iv) above, there exist $\hat{y}_0 \in Y(q)_0$, such that

$$\mathbf{F}_g(|\Phi(\hat{x})|, \mathcal{W}) = \mathbf{F}_g(|\Psi(\hat{y}_0)|, \mathcal{W}) \leq 2R\,.$$

□

Corollary 3.3.4 ((ε, δ)-deformed sequence)**.** *For any $c, D > 0$, there exists a positive constant $\eta_{3.3.4} = \eta_{3.3.4}(M, g, D, c) \in (0, D)$ with the following property.*

Let $\mathcal{W} \subset \mathcal{V}_n(M)$, $(X_i)_{i=1}^\infty$ a sequence of k-dimensional finite simplicial complices and $(\Phi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_{i=1}^\infty$ be a sequence of $\mathbf{F}_g$-continuous maps such that

- $L = \limsup_i \sup_{x\in X_i} \mathbf{M}_g \circ \Phi_i(x) \in (0, c)$;
- $\mathcal{V}^L((\Phi_i)_i) \subset \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.3.4}}(\mathcal{S}\mathcal{V}^L)$;
- *No varifold in* $\mathcal{V}^L((\Phi_i)_i) \setminus \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.3.4}}(\mathcal{W})$ *is* $(2k+1, \eta_{3.3.4})_g$*-almost minimizing.*

For any sequence $(\delta_i)_{i=1}^\infty > 0$, there exists a sequence $(\Phi_i^* : X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_{i=1}^\infty$ such that

(1) For each $i \in \mathbb{N}^+$, there exists a homotopy map

$$H_i^{\mathrm{DEF}} : [0,1] \times X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$$

with $H_i^{\mathrm{DEF}}(0,\cdot) = \Phi_i$, $H_i^{\mathrm{DEF}}(1,\cdot) = \Phi_i^*$ satisfying that for each $(t,x) \in [0,1] \times X_i$,

$$\mathbf{M}_g(H_i^{\mathrm{DEF}}(t,x)) < \mathbf{M}_g(\Phi_i(x)) + \delta_i\,;$$

(2) $\mathcal{V}^L((\Phi_i^*)_i) \subset \mathbf{B}^{\mathbf{F}_g}_D(\mathcal{W})$.

Proof. By Proposition 3.1.9, for each $i \in \mathbb{N}^+$, there exists a $\mathbf{M}_g$-continuous sequence $(\Psi_i : X \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2))_{i=1}^\infty$ such that Φ_i and Ψ_i are homotopic through a $\mathbf{F}_g$-continuous homotopy $\widetilde{H}_i$ which satisfies

$$\mathbf{M}_g(\widetilde{H}_i(x,t)) < \mathbf{M}_g(\Phi_i(x)) + \min(\delta_i, 1/i)$$

for all x and t. Clearly,

$$L = \limsup_i \sup_{x \in X_i} \mathbf{M}_g \circ \Psi_i(x) \equiv \limsup_i L_i \,,$$

and

$$\mathcal{V}^L((\Phi_i)_i) = \mathcal{V}^L((\Psi_i)_i) \,.$$

We define

$$R \equiv \eta_{3.3.4} := \min(D/6, \eta_{3.2.6}(M, \{g\}, D, c, 2k+1)) \,,$$

$$\bar{\varepsilon}_1 := \bar{\varepsilon}_{3.2.16}(M, g, 2k+1, r, r, c) \,,$$

$$\eta := \eta_{3.2.16}(M, g, 2k+1, r, r, c) \,,$$

$$s := \bar{s}_{3.2.16} \,,$$

from Lemma 3.2.6 and Lemma 3.2.16.

Since $\mathcal{V}^L((\Psi_i)_i)$ is compact and $\mathbf{B}^{\mathbf{F}_g}_{\eta_{3.3.4}}(\mathcal{S}\mathcal{V}^L_n)$ is open, for sufficiently large i, there exists $\bar{\varepsilon} \in (0, \bar{\varepsilon}_1)$, such that if $\mathbf{M}_g(\Psi_i(x)) \geq L_i - \bar{\varepsilon}$, then

$$\Psi_i(x) \in \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.3.4}}(\mathcal{S}\mathcal{V}^L_n) \,.$$

By Lemma 3.2.16, Ψ_i satisfies the annular $(\bar{\varepsilon}, \delta)$-deformation assumptions of Lemma 3.3.3 with $R, \bar{\varepsilon}, \eta, s$ and $\mathcal{W}$ defined above.

Therefore, for each i, we can choose $\bar{\delta} \in (0, (\delta_i, \eta_{3.3.4}))$, we obtain a homotopy map

$$\widetilde{H}_i^{\mathrm{DEF}} : [0,1] \times X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$$

with $\widetilde{H}_i^{\mathrm{DEF}}(0,\cdot) = \Psi_i$, $\widetilde{H}_i^{\mathrm{DEF}}(1,\cdot) = \Psi_i^*$ satisfying that for each $(t,x) \in [0,1] \times X_i$,

$$\mathbf{M}_g(\widetilde{H}_i^{\mathrm{DEF}}(t,x)) < \mathbf{M}_g(\Psi(x)) + \bar{\delta} < \mathbf{M}_g(\Psi(x)) + \delta_i\,.$$

Hence, concatenating $\widetilde{H}_i$ and $\widetilde{H}_i^{\mathrm{DEF}}$ implies the conclusion (1).

Moreover, if $\Psi_i^*(x) \geq L_i - \bar{\varepsilon}/10$, by Lemma 3.3.3, there exist $\hat{x} \in X_i$ and $T_{1,x} \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ with the following properties:

(i) $\mathbf{M}_g(\Psi_i^*(x), T_{1,x}) < \bar{\delta} < \eta_{3.3.4}$ and $\mathbf{F}_g(\Psi_i^*(x), T_{1,x}) < \eta_{3.3.4}$;

(ii) $T_{1,x} = \Psi_i^*(x)$ on $M \setminus \bigcup_{i=1}^m \overline{B}_g(p_1,\eta)$, $m \leq 2k+1$;

(iii) $\mathbf{M}_g(\Psi_i(\hat{x})) > \mathbf{M}_g(\Psi_i^*(x)) - d \geq L_i - \bar{\varepsilon}$;

(iv) $\mathbf{F}_g(|\Psi_i(\hat{x})|, \mathcal{W}) \leq 2R < D/3$.

By (iii), $\Psi_i(\hat{x}) \in \mathbf{B}^{\mathbf{F}_g}_{\eta_{3.3.4}}(\mathcal{SV}_n^L)$. By (ii) and Lemma 3.2.6,

$$\mathbf{F}_g(|\Psi_i(\hat{x})|, |T_{1,x}|) < D/2\,.$$

By definition of $\eta_{3.3.4}$, we conclude that

$$|\Psi_i^*(x)| \subset \mathbf{B}^{\mathbf{F}_g}_D(\mathcal{W})\,.$$

which is the conclusion (2). □

Restrictive homotopic min-max theory.

Definition 3.3.5. In a closed Riemannian manifold (M,g), given $\delta > 0$ and an $\mathbf{F}_g$-continuous map $\Phi_0 : X \to \mathcal{Z}_n(M;\mathbf{F}_g;\mathbb{Z}_2)$, we define the *restrictive* (X,Z)*-homotopy class of* Φ_0 *with an upper bound* δ, denoted by $\Pi_g^\delta(\Phi_0)$, to be the set of $\mathbf{F}_g$-continuous maps $\Psi : X \to \mathcal{Z}_n(M;\mathbf{F}_g;\mathbb{Z}_2)$ satisfying the following conditions:

(1) Each Ψ is homotopic to Φ_0 in the $\mathbf{F}_g$-topology, and

(2) The homotopy map $H : [0,1] \times X \to \mathcal{Z}_n(M;\mathbf{F}_g;\mathbb{Z}_2)$ satisfies

$$\sup_{t\in[0,1],z\in Z} \mathbf{M}_g(H(t,z)) < \sup_{z\in Z} \mathbf{M}_g(\Phi_0(z)) + \delta\,, \tag{3.5}$$

and

$$\sup_{t\in[0,1],x\in X}\mathbf{M}_g(H(t,x))<\sup_{x\in X}\mathbf{M}_g(\Phi_0(x))+\delta\,. \tag{3.6}$$

Definition 3.3.6. The *restrictive min-max width of* $\Pi_g^\delta(\Phi_0)$ is defined as

$$\mathbf{L}(\Pi_g^\delta(\Phi_0)):=\inf_{\Psi\in\Pi_g^\delta(\Phi_0)}\sup_{x\in X}\mathbf{M}_g\circ\Psi(x)\,.$$

Definition 3.3.7. A sequence of maps $(\Phi_i)_{i=1}^\infty$ in $\Pi_g^\delta(\Phi_0)$ is called a *minimizing sequence* for $\Pi_g^\delta(\Phi_0)$ if

$$\mathbf{L}(\Pi_g^\delta(\Phi_0))=\limsup_{i\to\infty}\sup_{x\in X}\mathbf{M}_g\circ\Phi_i(x)\,.$$

For a minimizing sequence $(\Phi_i)_{i=1}^\infty$ for $\Pi_g^\delta(\Phi_0)$, we define its *critical set* by

$$\mathbf{C}((\Phi_i)_i):=\{V=\lim_j|\Phi_{i_j}(x_j)|:\mathbb{N}^+\ni i_j\nearrow\infty,x_j\in X,\|V\|_g(M)=\mathbf{L}(\Pi_g^\delta(\Phi_0))\}\,.$$

Furthermore, the sequence is called *pulled-tight* if every varifold in $\mathbf{C}((\Phi_i)_i)$ is stationary.

Theorem 3.3.8 (Restrictive homotopic min-max Theorem for $\Pi_g^\delta(\Phi_0)$)**.** *Given a closed Riemannian manifold* (M^{n+1},g) $(2\le n\le 6)$, $\delta>0$, $D>0$, *and* $\Phi_0:X\to\mathcal{Z}_n(M;\mathbf{F}_g;\mathbb{Z}_2)$, *if*

$$\mathbf{L}(\Pi_g^\delta(\Phi_0))>\max(0,\sup_{x\in Z}\mathbf{M}_g(\Phi_0(x))+\delta)\,,$$

then there exists some sequence $(\Phi_i)_{i=1}^\infty$ *in* $\Pi_g^\delta(\Phi_0)$ *such that:*

(1) $(\Phi_i)_{i=1}^\infty$ *is a pulled-tight minimizing sequence for* $\Pi_g^\delta(\Phi_0)$.

(2) The critical set $\mathbf{C}((\Phi_i)_i)$ *contains a* $(2k+1,r)_g$*-almost minimizing varifold* V *with* $\|V\|_g(M)=\mathbf{L}(\Pi_g^\delta(\Phi_0))$ *for some* $r>0$*;*

(3) $\mathrm{spt}(V)$ *is a smooth, embedded, minimal hypersurface.*

(4) If $\mathcal{W}_L$ *is the set of all the embedded minimal cycles of area* L*, then*

$$\mathbf{C}((\Phi_i)_i)\subset\mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}_L)\,.$$

Proof of Theorem 3.3.8. First, we pick a minimizing sequence $(\Phi_i' : X \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_{i=1}^\infty$ for $\Pi_g^\delta(\Phi_0)$. since

$$\mathbf{L}(\Pi_g^\delta(\Phi_0)) > 0\,,$$

we can apply Corollary 3.3.2 to $(\Phi_i')_i$ with $c = \sup_x \mathbf{M}_g(\Phi_0(x)) + \delta$, and then we obtain $(\Phi_i : X \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_i \subset \Pi_g^\delta(\Phi_0)$. Note that in this case, $L = \limsup_i \sup_{x\in X} \mathbf{M}_g(\Phi'(x)) = \mathbf{L}(\Pi_g^\delta(\Phi_0))$, and

$$\limsup_i \sup_{x\in X} \mathbf{M}_g(\Phi(x)) \le \limsup_i \sup_{x\in X} \mathbf{M}_g(\Phi'(x)) = \mathbf{L}(\Pi_g^\delta(\Phi_0))\,.$$

The definition of restrictive min-max width implies $L = \limsup_i \sup_{x\in X} \mathbf{M}_g(\Phi(x))$. Therefore, $\mathcal{V}_n^L((\Phi_i)_i) = \mathbf{C}((\Phi_i)_i)$ and $\mathcal{V}_n^L((\Phi_i')_i) = \mathbf{C}((\Phi_i')_i) \subset \mathcal{SV}_n^{\le c}$ which implies that $(\Phi_i : X \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))$ is a pulled-tight minimizing sequence.

We choose a sequence $(\delta_i)_{i=1}^\infty$ such that for each $i \in \mathbb{N}^+$,

$$\delta_i < \delta - \max\left(\sup_{z\in Z} \mathbf{M}_g(\Phi_i(z)) - \sup_{z\in Z} \mathbf{M}_g(\Phi_0(z)), \sup_{x\in X} \mathbf{M}_g(\Phi_i(x)) - \sup_{x\in X} \mathbf{M}_g(\Phi_0(x))\right)$$

and $\lim_{i\to\infty} \delta_i = 0$. In addition, let $c = 2L$, $\eta = \eta_{3.3.4}(M, g, D, 2L)$ from Corollary 3.3.4, $\mathcal{W}'$ be the set of all the $(m, \eta)_g$ almost-minimizing varifolds and

$$\mathcal{W} = \mathcal{W}' \cap \mathbf{C}((\Phi_i')_i)\,.$$

Hence, we can apply Corollary 3.3.4 to obtain a new sequence of sweepout $(\Psi_i)_{i=1}^\infty$ for $\Pi_g^\delta(\Phi_0)$, such that

$$\mathbf{C}((\Psi_i)_i) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}_L)\,.$$

In particular, this also implies that $\mathcal{W}' \cap \mathbf{C}((\Phi_i')_i) \ne \emptyset$, and analogously, by applying Corollary 3.3.4 to $(\Psi_i)_i$, we also have

$$\mathcal{W}' \cap \mathbf{C}((\Phi_i')_i) \ne \emptyset\,,$$

which concludes (2) and thus, by Theorem 3.2.5, (3).

Finally, we apply Corollary 3.3.2 again to $(\Psi_i)_i$ to obtain a pulled-tight sequence $(\Psi'_i)_i$ which satisfies (2), (3) and (4) as well. □

Restrictive homological min-max theory. Recall that in (M^{n+1}, g), for each $p \in \mathbb{N}^+$, the min-max p-width is defined by

$$\omega_p(M, g) = \inf_{\Phi \in \mathcal{P}_p} \sup_{x \in \mathrm{dmn}(\Phi)} \mathbf{M}_g \circ \Phi(x)\,,$$

where $\mathcal{P}_p = \{\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathcal{Z}_2) | X$ is a finite simplicial complex and $\Phi^*(\bar{\lambda}^p) \neq 0\}$.

The effectiveness of the homotopic min-max theory in producing minimal hypersurfaces of p-width can be attributed to the following rationale. Given two continuous maps $\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ and $\Psi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$, by the homotopy theory, if there exists a homotopy map $H : [0, 1] \times X \to \mathcal{Z}_n(M; \mathbf{F}; \mathcal{Z}_2)$ such that $H(0, \cdot) = \Phi(\cdot)$ and $H(1, \cdot) = \Psi(\cdot)$, then

$$\Phi \in \mathcal{P}_p \iff \Psi \in \mathcal{P}_p\,. \tag{3.7}$$

However, from the definition of the admissible set $\mathcal{P}_p$, the condition $\Phi^*(\bar{\lambda}^p) \neq 0$ suggests that we should appeal to a homology/cohomology theory. In particular, we have the following two observations.

Lemma 3.3.9. *Given $p \in \mathbb{N}^+$, let X be a finite simplicial $(p+1)$-chain such that $\partial X = X^\alpha + X^\omega$ (with $\mathbb{Z}_2$ coefficients) where X^α and X^ω are both simplicial p-cycles. Given a continuous map $\Psi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$, we set $\Psi^\alpha := \Psi|_{X^\alpha}$ and $\Psi^\omega := \Psi|_{X^\omega}$. Then we have*

$$\Psi^\alpha \in \mathcal{P}_p \iff \Psi^\omega \in \mathcal{P}_p\,.$$

Proof. Let $A := \Psi(X^\alpha)$, $B := \Psi(X^\omega)$ and $C := \Psi(X)$ be the corresponding singular chains in $C_*(\mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2); \mathbb{Z}_2)$. Since both X^α and X^ω are cycles, $\partial A = \partial B = 0$ and we obtain

$$a := [A], b := [B] \in H_p(\mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2); \mathbb{Z}_2)\,.$$

Moreover, $\partial X = X^\alpha + X^\omega$ implies that

$$\partial C = A + B\,,$$

and thus,

$$a = b\,.$$

Hence, we have

$$\Psi^\alpha \in \mathcal{P}_p \iff \langle \bar{\lambda}^p, a\rangle \neq 0 \iff \langle \bar{\lambda}^p, a\rangle \neq 0 \iff \Psi^\omega \in \mathcal{P}_p\,.$$

□

Lemma 3.3.10. *Given $k \in \mathbb{N}^+$, let W be a finite simplicial k-chain with boundary $Z = \partial W$ (possibly empty). Then $X = [0,1] \times [0,1]$ is a finite simplicial $(k+1)$-chain such that $\partial X = X^\alpha + X^\omega$ (with $\mathbb{Z}_2$ coefficients) where*

$$X^\alpha := \{0\} \times W$$

and

$$X^\omega := \{1\} \times W + [0,1] \times Z$$

are both finite simplicial k-chain with boundary $\{0\} \times Z$.

Proof. This follows immediately from the definition. □

Definition 3.3.11. Given $\delta > 0$ and an $\mathbf{F}$-continuous map $\Phi_0 : W \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$, where X is a finite simplicial k-chain with boundary $Z = \partial W$ (possibly empty), we define $\widetilde{\mathcal{H}}^\delta_g(\Phi_0)$ to be the set of all $\mathbf{F}_g$-continuous maps $\Psi : X \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$ such that:

- X is a finite simplicial $(k+1)$-chain such that $\partial X = X^\alpha + X^\omega$ (with $\mathbb{Z}_2$ coefficients) where $X^\alpha = W$ and X^ω is another finite, simplicial k-chain with $\partial X^\omega = Z$;
- $\Psi^\alpha := \Psi|_{X^\alpha} = \Phi_0$, and $\Psi^\omega := \Psi|_{X^\omega}$;

- $$\sup_{x\in X} \mathbf{M}_g \circ \Psi(x) < \sup_{w\in W} \mathbf{M}_g \circ \Phi_0(w) + \delta.$$

We define

$$\mathcal{H}^\delta_g(\Phi_0) := \{\Psi^\omega : \Psi \in \widetilde{\mathcal{H}}^\delta_g(\Phi_0)\}.$$

Remark 3.3.12. We should think of Ψ as a cobordism between "the beginning" Φ_0 and "the loose end" $\Psi|_{X^\omega}$. The set $\mathcal{H}^\delta_g(\Phi_0)$ may be viewed as the homology class represented by Φ_0.

Definition 3.3.13. The *restrictive min-max width of* $\mathcal{H}^\delta(\Phi_0)$ is defined as

$$\mathbf{L}(\mathcal{H}^\delta_g(\Phi_0)) := \inf_{\Phi\in\mathcal{H}^\delta_g(\Phi_0)} \sup \mathbf{M}\circ\Phi \quad \left(= \inf_{\Psi\in\widetilde{\mathcal{H}}^\delta_g(\Phi_0)} \sup \mathbf{M}\circ\Psi^\omega \right).$$

The previous lemma implies that this width is nontrivial provided that Φ_0 is some p-admissible sweepout.

Corollary 3.3.14. *If $\Phi_0 : W \to \mathcal{Z}_n(M;\mathbf{F}_g;\mathbb{Z}_2) \in \mathcal{P}_p$ with W a finite simplicial p-cycle, every $\Phi \in \mathcal{H}^\delta_g(\Phi_0)$ is also inside $\mathcal{P}_p$. In particular,*

$$\mathbf{L}(\mathcal{H}^\delta_g(\Phi_0)) \geq \omega_p(M,g)\,.$$

Definition 3.3.15. A sequence of maps $(\Phi_i)_{i=1}^\infty$ in $\mathcal{H}^\delta_g(\Phi_0)$ is called a *minimizing sequence* for $\mathcal{H}^\delta_g(\Phi_0)$ if

$$\mathbf{L}(\mathcal{H}^\delta_g(\Phi_0)) = \limsup_{i\to\infty} \sup_{x\in X} \mathbf{M}_g \circ \Phi_i(x)\,.$$

For a minimizing sequence $(\Phi_i)_{i=1}^\infty$ for $\mathcal{H}^\delta_g(\Phi_0)$, we define its *critical set* by

$$\mathbf{C}_g((\Phi_i)_i) := \{V = \lim_j |\Phi_{i_j}(x_j)| : \{i_j\}_j \subset \mathbb{N}, x_j \in \mathrm{dmn}(\Phi_{i_j}), \|V\|_g(M) = \mathbf{L}(\mathcal{H}^\delta_g(\Phi_0))\}\,.$$

Furthermore, the sequence is called *pulled-tight* if every varifold in $\mathbf{C}((\Phi_i)_i)$ is stationary.

Theorem 3.3.16 (Restrictive homological min-max theorem)**.** *Given $\delta > 0$, $D > 0$ and an $\mathbf{F}_g$-continuous map $\Phi_0 : W \to \mathcal{Z}_n(M;\mathbf{F}_g;\mathbb{Z}_2)$, where W is a finite simplicial k-chain* *with boundary $Z = \partial W$, suppose that*

$$L := \mathbf{L}(\mathcal{H}_g^\delta(\Phi_0)) > \max(\sup_{x\in Z} \mathbf{M}_g(\Phi_0(x)) + \delta, 0)\,.$$

Then there exists a minimizing sequence

$$(\Psi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2))_{i=1}^\infty$$

in $\widetilde{\mathcal{H}}_g^\delta(\Phi_0)$ such that:

(1) $(\Psi_i^\omega)_{i=1}^\infty$ is a pulled-tight minimizing sequence for $\mathcal{H}_g^\delta(\Phi_0)$.

(2) The critical set $\mathbf{C}((\Psi_i^\omega)_i)$ contains a $(2k+1, r)_g$-almost minimizing varifold V with $\|V\|_g(M) = \mathbf{L}(\Pi_g^\delta(\Phi_0))$ for some $r > 0$;

(3) $\mathrm{spt}(V)$ is a smooth, embedded, minimal hypersurface.

(4) If $\mathcal{W}_L$ is the set of all the embedded minimal cycles of area L, then

$$\mathbf{C}((\Psi_i^\omega)_i) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}_L)\,.$$

Proof. First, we pick a sequence $(\Phi_i : Y_i \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2))_{i=1}^\infty$ in $\widetilde{\mathcal{H}}_g^\delta(\Phi_0)$ such that $(\Phi_i^\omega)_i$ is a minimizing sequence for $\mathcal{H}_g^\delta(\Phi_0)$. Since

$$L := \mathbf{L}(\mathcal{H}_g^\delta(\Phi_0)) > 0\,,$$

we can apply Corollary 3.3.2 to each Φ_i^ω with $c = \sup_{x\in X} \mathbf{M}_g \circ \Phi_0(x) + \delta$, and we obtain

$$H_i^{\mathrm{PT}} : [0,1] \times Y_i^\omega \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$$

such that for all $(t, y) \times [0,1] \times Y_i^\omega$,

$$\mathbf{M}_g(H_i^{\mathrm{PT}}(t, y)) \le \mathbf{M}_g(\Phi_i^\omega(y))\,.$$

We define a space

$$X_i := Y_i \cup [0,1] \times Y_i^\omega,$$

by identifying Y_i^ω and $\{0\} \times Y_i^\omega$, and a $\mathbf{F}_g$-continuous map in $\widetilde{\mathcal{H}}_g^\delta(\Phi_0)$,

$$\Psi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$$

by concatenating Φ_i and H_i^{PT}. By Lemma 3.3.10, we have $X_i^\alpha = W$ and $X_i^\omega = [0,1] \times Z \cup \{1\} \times Y_i^\omega$. Again, (Ψ_i^ω) is pulled-tight minimizing sequence, since on X_i^ω,

$$\begin{aligned}\sup_{x \in [0,1]\times Z} \mathbf{M}_g(\Psi_i^\omega(x)) &= \sup_{x \in [0,1]\times Z} \mathbf{M}_g(H_i^{\mathrm{PT}}(x)) \\ &\le \sup_{z \in Z} \mathbf{M}_g(\Phi_0(z)) \\ &< L - \delta\,,\end{aligned}$$

For each i, we choose

$$\delta_i \le \min(\max(\sup_{x \in X_i^\omega} \mathbf{M}_g \circ \Psi_i^\omega(x) - \sup_{w \in W} \mathbf{M}_g \circ \Phi_0(w), 0), \delta/2)\,,$$

such that $\lim_i \delta_i = 0$. As in the proof of Theorem 3.3, let $c = 2L$, $\eta = \eta_{3.3.4}(M, g, D, 2L)$ from Corollary 3.3.4, $\mathcal{W}'$ be the set of all the $(m, \eta)_g$-almost minimizing varifolds and

$$\mathcal{W} = \mathcal{W}' \cap \mathbf{C}((\Psi_i^\omega)_i)\,.$$

Hence, we can apply Corollary 3.3.4 to obtain a new sequence of sweepout $(\Psi_i^*)_{i=1}^\infty$ and homotopy maps H_i^{DEF} such that

$$\mathcal{V}^L((\Psi_i^*)_i) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}_L)$$

and

$$\mathbf{M}_g(H_i^{\mathrm{DEF}}(t,x)) < \mathbf{M}_g(\Psi_i^\omega(x)) + \delta_i\,; \tag{3.8}$$

We define a space

$$\widetilde{X}_i := X_i \cup [0,1] \times X_i^\omega$$

by identifying X_i^ω and $\{0\} \times X_i^\omega$ and a $\mathbf{F}_g$-continuous map in $\widetilde{\mathcal{H}}_g^\delta(\Phi_0)$,

$$\widetilde{\Psi}_i : \widetilde{X}_i \to \mathcal{Z}_n(M; \mathbf{F}_g; \mathbb{Z}_2)$$

by concatenating Ψ_i and H_i^{DEF}. By Lemma 3.3.10, we have $\widetilde{X}_i^\alpha = W$ and $\widetilde{X}_i^\omega = [0,1] \times Z \cup \{1\} \times Y_i^\omega$.

It follow from (3.8) that $(\widetilde{\Psi}_i) \in \widetilde{\mathcal{H}}_g^\delta(\Phi_0)$ is a minimizing sequence. Moreover,

$$\mathbf{C}((\widetilde{\Psi}_i^\omega)_i) = \mathcal{V}^L((\widetilde{\Psi}_i^\omega)_i) = \mathcal{V}^L((\Psi_i^*)_i) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}) \subset \mathbf{B}_D^{\mathbf{F}_g}(\mathcal{W}_L)$$

since

$$\begin{aligned} \sup_{x \in [0,1] \times Z} \mathbf{M}_g(\Psi_i^\omega(x)) &= \sup_{x \in [0,1] \times Z} \mathbf{M}_g(H_i^{\mathrm{DEF}}(x)) \\ &\leq \sup_{z \in Z} \mathbf{M}_g(\Phi_0(z)) + \delta_i \\ &< L - \delta/2\,. \end{aligned}$$

In particular, this also implies that $\mathcal{W}' \cap \mathbf{C}((\Psi_i^\omega)_i) \neq \emptyset$, and analogously, by applying Corollary 3.3.4 to $(\widetilde{\Psi}_i^\omega)_i$, we also have

$$\mathcal{W}' \cap \mathbf{C}((\widetilde{\Psi}_i^\omega)_i) \neq \emptyset\,,$$

which concludes (2) and thus, by Theorem 3.2.5, (3).

Finally, we apply Corollary 3.3.2 again to $(\widetilde{\Psi}_i)_i$ as in the beginning to obtain a pulled-tight sequence $(\widetilde{\widetilde{\Psi}}_i)_i$ which satisfies (2), (3) and (4) as well. □

3.4. **Proof of Theorem 1.2.1.** Let $(S^3, \bar{g})$ be the unit 3-sphere.

Recall that C. Nurser [Nur16] showed the 13-width $\omega_{13}(S^3, \bar{g})$ is at most 8π. To prove Theorem 1.2.1, we assume, for the sake of contradiction, that the 13-width is precisely 8π on $(S^3, \bar{g})$. In fact, C. Nurser has constructed an $\mathcal{F}$-continuous map $\Phi_0 : \mathbb{RP}^{13} \to \mathcal{Z}_2(S^3; \mathbb{Z}_2)$ with no concentration of mass, such that

$$\sup_{x \in \mathbb{RP}^{13}} \mathbf{M}_{\bar{g}}(\Phi_0(x)) = 8\pi\,.$$

We define

$$r_0 := \min(\eta_{3.2.7}(S^3, \bar{g}, 29), \eta_{3.2.8}(S^3, \bar{g}, 29))$$

from Lemmas 3.2.7 and 3.2.8, and define the following sets of varifolds on $(S^3, \bar{g})$.

- $\mathcal{SV}_{\bar{g}}^{8\pi} \subset \mathcal{V}_2(S^3)$ is the set of all stationary n-varifolds on $(S^3, \bar{g})$ with total measure 8π;
- $\mathcal{M}_{\bar{g}}^{8\pi} \subset \mathcal{SV}_{\bar{g}}^{8\pi}$ is the subset consisting of all stationary integral varifolds whose support is a smooth, embedded, closed minimal surface;
- $\mathcal{G}_{\bar{g}} \subset \mathcal{M}_{\bar{g}}^{8\pi}$ is the (good) subset comprising all $(29, r_0)_{\bar{g}}$-almost minimizing varifolds V for which there is $T \in \mathcal{Z}_2(S^3; \mathbb{Z}_2)$ with $V = |T|$;
- $\mathcal{B}_{\bar{g}} \subset \mathcal{M}_{\bar{g}}^{8\pi}$ is the (bad) subset comprising all $(29, r_0)_{\bar{g}}$-almost minimizing varifolds V for which no $T \in \mathcal{Z}_2(S^3; \mathbb{Z}_2)$ with $V = |T|$.

With these definitions, it follows that $\mathcal{G}_{\bar{g}} \cup \mathcal{B}_{\bar{g}} \subset \mathcal{M}_{\bar{g}}^{8\pi}$ is the subset consisting of all $(29, r_0)_{\bar{g}}$-almost minimizing varifolds.

Additionally, in the case of $(S^3, \bar{g})$ where every closed surface has separation property and equators are the only smooth minimal surfaces with area 4π, it follows from Remark 3.2.14 that $\mathcal{B}_{\bar{g}}$ is the set of all the multiplicity-two equators. Furthermore, for the 13-sweepout Φ_0, the set

$$\{V = \lim_j |\Phi_0(x_j)| : x_j \in \mathbb{RP}^{13}, \|V\|_g(S^3) = 8\pi\}$$

is exactly $\mathcal{B}_{\bar{g}}$. Then by Marques-Neves [MN17, Corollary 3.9], there exists a pulled-tight sequence of $\mathbf{F}_{\bar{g}}$-continuous maps

$$(\Phi_{0,i} : \mathbb{RP}^{13} \to \mathcal{Z}_2(S^3; \mathbf{F}_{\bar{g}}; \mathbb{Z}_2))_{i=1}^{\infty}$$

such that the critical set

$$\mathbf{C}_{\bar{g}}((\Phi_{0,i})_{i=1}^{\infty}) \subset \mathcal{B}_{\bar{g}}\,. \tag{3.9}$$

Since $(S^3, \bar{g})$ has positive Ricci curvature, by Lemma 3.2.17, both $\mathcal{G}_{\bar{g}}$ and $\mathcal{B}_{\bar{g}}$ are compact in the varifold topology. Therefore, we can define

$$\begin{aligned}
d_0 &\coloneqq \eta_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})/10\,, \\
\varepsilon_0 &\coloneqq \bar{\varepsilon}_{3.2.16}(M, g, 29, r_0, d_0, 8\pi)\,, \\
s_0 &\coloneqq \bar{s}_{3.2.16}(M, g, 29, r_0, d_0, 8\pi)\,, \\
\eta_0 &\coloneqq \min(d_0, \varepsilon_0, \eta_{3.2.16}(M, g, 29, r_0, d_0, 8\pi))/10\,, \\
\widetilde{\mathcal{G}}_{\bar{g}} &\coloneqq \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_0}(\mathcal{G}_{\bar{g}})\,, \\
\widetilde{\mathcal{B}}_{\bar{g}} &\coloneqq \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_0}(\mathcal{B}_{\bar{g}})\,, \\
\hat{\mathcal{G}}_{\bar{g}} &\coloneqq \mathcal{G}'_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,, \\
\hat{\mathcal{B}}_{\bar{g}} &\coloneqq \mathcal{B}'_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,,
\end{aligned}$$

from Lemma 3.2.13 and Lemma 3.2.16. By (3.9), there exists $\bar{\varepsilon}_0 > 0$ such that for sufficiently large i,

$$\mathbf{M}_{\bar{g}} \circ \Phi_{0,i}(x) \geq 8\pi - \bar{\varepsilon}_0 \implies |\Phi_{0,i}(x)| \in \widetilde{\mathcal{B}}_{\bar{g}}\,. \tag{3.10}$$

Metric perturbations. We begin with a proposition.

Proposition 3.4.1. *Let M^{n+1} be a smooth closed manifold with $3 \leq n+1 \leq 7$. Then there exists a Baire residual set $\Gamma^{\infty}_{\text{uniq}}$ of C^∞ bumpy Riemannian metrics on M such that for any $g \in \Gamma^{\infty}_{\text{uniq}}$ and any $L \in \mathbb{R}^+$, there exists at most one combination of minimal hypersurfaces (with multiplicities) whose total areas sum up to L.*

Proof. For each metric g, let $\mathfrak{M}_g$ be the set of closed, embedded, smooth, minimal hypersurfaces in (M, g). For each $\alpha > 0$ and integer $p > 0$, let us denote by $\mathcal{U}_{p,\alpha}$ the set of smooth metrics on M such that:

(1) Every element of $\mathfrak{M}_g$ with index at most p and area at most α is nondegenerate.

(2) For any $p_1, \dots, p_N \in \mathbb{Z}$ and $\Sigma_1, \dots, \Sigma_N \in \mathfrak{M}_g$, where $|m_k| \leq p$, $\operatorname{index}(\Sigma_k) \leq p$, and $\operatorname{area}(\Sigma_k) \leq \alpha$ for each k, if

$$p_1 \operatorname{area}_g(\Sigma_1) + \cdots + p_N \operatorname{area}_g(\Sigma_N) = 0,$$

then $p_1 = \cdots = p_N = 0$.

By [MN21, Claim 8.6], for each p and α, the set $\mathcal{U}_{p,\alpha}$ is open and dense in the space of all smooth metrics in M. Thus, we can let $\Gamma^\infty_{\text{uniq}} := \cap_{n \in \mathbb{N}^+} \mathcal{U}_{n,n}$. □

Then, combined with X. Zhou's multiplicity one theorem [Zho20], this proposition implies that for each $g \in \Gamma^\infty_{\text{uniq}}$, every (restrictive) min-max width can be realized by a unique combination of multiplicity-one two-sided minimal hypersurfaces, which is also the boundary of a Caccioppoli set.

We choose a sequence $(g_i)_{i=1}^\infty$ in $\Gamma^\infty_{\text{uniq}}$ such that $\|g_i - \bar{g}\|_{C^\infty, \bar{g}} < \eta_0$, and

$$\lim_{i \to \infty} g_i = \bar{g}$$

in the C^∞ topology. We define for each $i \in \mathbb{N}^+$,

$$S_i := \sup_x \mathbf{M}_{g_i} \circ \Phi_{0,i}(x)\,.$$

Let $(\delta_i)_{i=1}^\infty$ be a decreasing sequence in $\mathbb{R}^+$ such that

$$\lim_{i \to \infty} \delta_i = 0\,.$$

Restrictive homological min-max. We define

$$\eta_2 := \eta_0\,.$$

For each $i \in \mathbb{N}^+$, we consider the restrictive homology class $\mathcal{H}^{\delta_i}_{g_i}(\Phi_{0,i})$ of $\Phi_{0,i}$. For simplicity, we denote

$$\widetilde{\mathcal{H}}_i = \widetilde{\mathcal{H}}^{\delta_i}_{g_i}(\Phi_{0,i}) \text{ and } \mathcal{H}_i = \mathcal{H}^{\delta_i}_{g_i}(\Phi_{0,i})\,.$$

Note that, since $\Phi_{0,i}$ is a 13-sweepout, by Coroally 3.3.14,

$$\lim_{i\to\infty} \mathbf{L}(\mathcal{H}_i) = \lim_{i\to\infty} S_i + \delta_i = 8\pi\,.$$

Thus, applying the restrictive min-max theorem, Theorem 3.3.16, to each $\mathcal{H}_i$, we obtain for each i a pulled-tight minimizing sequence

$$(\Psi_i^j : X_i^j \to \mathcal{Z}_2(S^3; \mathbf{F}_{g_i}; \mathbb{Z}_2))_{j=1}^\infty$$

in $\widetilde{\mathcal{H}}_i$ such that:

(i) $\lim_{j\to\infty} \sup_x \mathbf{M}_{g_i} \circ (\Psi_i^j)^\omega(x) = \mathbf{L}(\mathcal{H}_i)\,.$

(ii) There exists a constant $\varepsilon_{2,i} > 0$ such that for all j large enough,

$$\mathbf{M}_{g_i}((\Psi_i^j)^\alpha(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{2,i} \implies \mathbf{M}_{\bar{g}}((\Psi_i^j)^\alpha(x)) \geq 8\pi - \bar{\varepsilon}_0\,, \tag{3.11}$$

and

$$\mathbf{M}_{g_i}((\Psi_i^j)^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{2,i} \implies |(\Psi_i^j)^\omega(x)| \in \mathbf{B}^{\mathbf{F}_{g_i}}_{1/i}(\Sigma_i), \tag{3.12}$$

where Σ_i is the unique multiplicity one two-sided minimal surface with area $\mathbf{L}(\mathcal{H}_i)$, and it is the boundary of a Caccioppoli set, as $g_i \in \Gamma^\infty_{\mathrm{uniq}}$.

(iii) $|\Sigma_i|$ is $(27, r_0)_{g_i}$-almost minimizing and thus, $(29, r_0)_{g_i}$-almost minimizing.

Then by Proposition 3.2.17, after relabelling the i's, Σ_i converges in C^∞ (with multiplicity one) to some $\Sigma \in \mathcal{G}_{\bar{g}}$. Hence, we can assume for any $i \in \mathbb{N}^+$,

$$\Sigma_i \in \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_2/2}(\mathcal{G}_{\bar{g}})\,. \tag{3.13}$$

Furthermore, for each i we can take an $j(i) \in \mathbb{N}^+$ such that, by discarding finitely many g_i:

(i) $(\Psi_i^{j(i)})_{i=1}^\infty$ is a minimizing sequence for $\omega_{13}(S^3, \bar{g}) = 8\pi$.

(ii) For all $j \geq j(i)$,

$$\mathbf{M}_{g_i}((\Psi_i^j)^\alpha(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{2,i} \implies |(\Psi_i^j)^\alpha(x)| \in \widetilde{\mathcal{B}}_{\bar{g}}\,. \tag{3.14}$$

and

$$\mathbf{M}_{g_i}((\Psi_i^j)^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{2,i} \implies |(\Psi_i^j)^\omega(x)| \in \widetilde{\mathcal{G}}_{\bar{g}}\,. \tag{3.15}$$

Note that the first item follows from $\delta_i \to 0$, while the second from (3.10), (3.11), (3.12) and (3.13). For simplicity, we will denote

$$\Psi_i := \Psi_i^{j(i)}, \quad X_i := X_i^{j(i)}\,.$$

Pull-tight. In this part, our goal is to construct for each i some $\hat{\Psi}_i \in \widetilde{\mathcal{H}}_i$ such that $(\hat{\Psi}_i)_i$ is a minimizing sequence for the 13-width, and for some $\eta_3 \in (0, \eta_0)$ the critical set

$$\mathbf{C}_{\bar{g}}((\hat{\Psi}_i)_i) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}\mathcal{V}^{8\pi}_{\bar{g}})\,,$$

while preserving conditions similar to (3.14) and (3.15).

We define

$$\eta_3 := 3\eta_2\,.$$

Additionally, for any positive constant A, we introduce the set

$$\mathcal{V}^{\leq A} := \{V \in \mathcal{V}_2(S^3) \mid \|V\|_{\bar{g}}(S^3) \leq A\}\,.$$

We denote a subset of $\mathcal{V}^{\leq A}\bar{g}$ consisting of stationary varifolds with respect to $\bar{g}$ as $\mathcal{S}^{\leq A}\bar{g}$. Further define

$$\mathcal{U}^{\leq A} := \mathcal{V}^{\leq A} \setminus \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}^{\leq A})\,, \tag{3.16}$$

which is a compact set. Similarly, for cycles, we define

$$\mathcal{Z}_2^{\leq A} := \{T \in \mathcal{Z}_2(S^3; \mathbf{F}_{\bar{g}}; \mathbb{Z}_2) : \mathbf{M}_{\bar{g}}(T) \leq A\}\,.$$

Following Pitt's pull-tight construction [Pit81], we have the following proposition, with its proof postponed to §3.5.

Proposition 3.4.2. *After possibly discarding finitely many elements in the sequence $(g_i)_i$ of metrics, there exists an continuous deformation map*

$$H : [0,1] \times \mathcal{Z}_2^{\leq A} \to \mathcal{Z}_2^{\leq A}$$

satisfying the following properties:

(1) $H(0, \cdot) = \mathrm{id}$.

(2) If $|T| \in \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3/2}(\mathcal{S}^{\leq A})$ *then* $H(t,T) = T$ *for each* t.

(3) For each i and (t,T), $\mathbf{M}_{\bar{g}}(H(t,T)) \leq \mathbf{M}_{\bar{g}}(T)$ *and* $\mathbf{M}_{g_i}(H(t,T)) \leq \mathbf{M}_{g_i}(T)$. *And any of the equalities holds only if* $H(t,T) = T$.

(4) There exists $\varepsilon_3 > 0$, such that for each i and T, if $|H(1,T)| \notin \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}^{\leq A})$, *then*

$$\mathbf{M}_{\bar{g}}(H(1,T)) \leq \mathbf{M}_{\bar{g}}(T) - \varepsilon_3 \textit{ and } \mathbf{M}_{g_i}(H(1,T)) \leq \mathbf{M}_{g_i}(T) - \varepsilon_3.$$

We choose $A := 8\pi + 1$ and, in accordance with the proposition above, discard finitely many values of i. Without loss of generality, we can assume for each i,

$$\sup_x \mathbf{M}_{g_i} \circ \Phi_{0,i}(x) + \delta_i < A.$$

Next, we define for each i the space

$$\hat{X}_i := ([0,1] \times \mathbb{RP}^{13}) \cup X_i$$

where $\{1\} \times \mathbb{RP}^{13} \subset [0,1] \times \mathbb{RP}^{13}$ and $\mathbb{RP}^{13} \subset X_i$ are identified. Note that

$$\partial \hat{X}_i = (\{0\} \times \mathbb{RP}^{13}) \cup X_i^{\omega}.$$

We can apply Proposition 3.4.2 to obtain a deformation map H, and define

$$\hat{\Psi}_i : \hat{X}_i \to \mathcal{Z}_2(S^3; \mathbf{F}_{g_i}; \mathbb{Z}_2)$$

by

$$\hat{\Psi}_i|_{[0,1]\times\mathbb{RP}^{13}}(t,x) := H(t, \Phi_{0,i}(x))$$
$$\hat{\Psi}_i|_{X_i}(x) := H(1, \Psi_i(x)).$$

Now, note that:

(i) Since (S^3, g_i) and $(S^3, \bar{g})$ are diffeomorphic for each i, $\mathbf{F}_{g_i}$ and $\mathbf{F}_{\bar{g}}$ induces homeomorphic topologies. Thus, $\hat{\Psi}_i$ is still $\mathbf{F}_{g_i}$-continuous.

(ii) By definition, we have $\hat{\Psi}_i^\alpha = \Phi_{0,i}$.

(iii) By Proposition 3.4.2 (3),

$$\sup_{x\in\hat{X}_i} \mathbf{M}_{g_i} \circ \hat{\Psi}_i(x) \leq \sup_{x\in X_i} \mathbf{M}_{g_i} \circ \Psi_i(x) < \sup_{x\in\mathbb{RP}^{13}} \mathbf{M}_{g_i} \circ \Phi_{0,i}(x) + \delta_i.$$

Consequently, we can conclude that $\hat{\Psi}_i \in \widetilde{\mathcal{H}}_i$. Moreover, by (ii) and (iii), since $\delta_i \to 0$, $(\hat{\Psi}_i)_i$ is a minimizing sequence for the 13-width.

Additionally, we have the following lemma with proof postponed to §3.5.

Lemma 3.4.3. $\mathbf{C}_{\bar{g}}((\hat{\Psi}_i)_i) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{SV}^{8\pi}_{\bar{g}})$.

Hence, there exists an $\bar{\varepsilon}_3 > 0$ such that for every sufficiently large i and each $x \in \hat{X}_i$, if $|\hat{\Psi}_i(x)| \notin \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{SV}^{8\pi}_{\bar{g}})$, then

$$\mathbf{M}_{\bar{g}}(\hat{\Psi}_i(x)) < 8\pi - \bar{\varepsilon}_3\,, \quad \mathbf{M}_{g_i}(\hat{\Psi}_i(x)) < \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}_3\,. \tag{3.17}$$

Additionally, let $\varepsilon_{3,i} := \varepsilon_{2,i}$ and we observe that

$$\mathbf{M}_{g_i}(\hat{\Psi}_i^\alpha(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{3,i} \implies |\hat{\Psi}_i^\alpha(x)| \in \widetilde{\mathcal{B}}_{\bar{g}}\,, \tag{3.18}$$

and

$$\mathbf{M}_{g_i}(\hat{\Psi}_i^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{3,i} \implies |\hat{\Psi}_i^\omega(x)| \in \widetilde{\mathcal{G}}_{\bar{g}}\,. \tag{3.19}$$

Indeed, (3.18) trivially follows from (3.14) and the fact that $\hat{\Psi}_i^\alpha = \Phi_{0,i}$. Regarding (3.19), we can justify it step by step:

$$\begin{aligned}
\mathbf{M}_{g_i}(\hat{\Psi}_i^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{3,i} &\implies \mathbf{M}_{g_i}(\Psi_i^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{3,i} \quad \text{(by Proposition 3.4.2(3))}\\
&\implies |\Psi_i^\omega(x)| \in \widetilde{\mathcal{G}}_{\bar{g}}) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3/2}(\mathcal{S}\mathcal{V}^{8\pi}_{\bar{g}}) \quad \text{(be (3.15))}\\
&\implies \Psi_i^\omega(x) = \hat{\Psi}_i^\omega(x) \quad \text{(by Proposition 3.4.2 ((2)))}\\
&\implies |\hat{\Psi}_i^\omega(x)| \in \widetilde{\mathcal{G}}_{\bar{g}})\,.
\end{aligned}$$

For simplicity, we discard finitely many i such that (3.17), (3.18) and (3.19) hold for every i.

(ε, δ)-deformation. In this part, we aim to construct a sweepout $\hat{\hat{\Psi}} : \hat{\hat{X}} \to \mathcal{Z}_2(S^3; \mathbf{F}_{g_i}; \mathbb{Z}_2)$ in $\widetilde{\mathcal{H}}_i$ such that for each $x \in \hat{\hat{X}}$,

$$\mathbf{M}_{g_i}(\hat{\hat{\Psi}}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 \implies |\hat{\hat{\Psi}}(x)| \in \hat{\mathcal{G}}_{\bar{g}} \sqcup \hat{\mathcal{B}}_{\bar{g}}\,, \tag{3.20}$$

where i is some positive integer, and $\varepsilon_4 > 0$. Furthermore, similar to (3.18) and (3.19), it also satisfies

$$\mathbf{M}_{g_i}(\hat{\hat{\Psi}}^\alpha(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 \implies |\hat{\hat{\Psi}}^\alpha(x)| \in \hat{\mathcal{B}}_{\bar{g}}\,, \tag{3.21}$$

and

$$\mathbf{M}_{g_i}(\hat{\hat{\Psi}}^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 \implies |\hat{\hat{\Psi}}^\omega(x)| \in \hat{\mathcal{G}}_{\bar{g}}\,. \tag{3.22}$$

For each $i \in \mathbb{N}^+$, we choose an arbitrary $\varepsilon > 0$, which we will specify later, and apply Proposition 3.1.9 with this ε to $\hat{\Psi}_i(x)$ to obtain a $\mathbf{M}_{g_i}$-continuous map

$$\hat{\Psi}'_i(x) : \hat{X}_i \to \mathcal{Z}_2(S^3; \mathbf{M}_{g_i}; \mathbb{Z}_2)\,.$$

$\hat{\Psi}'_i(x)$ satisfies the conditions:

(i) There is an $\mathbf{F}_{g_i}$-continuous homotopy

$$\hat{H}_i : [0,1] \times \hat{X}_i \to \mathcal{Z}_2(S^3; \mathbf{F}_{g_i}; \mathbb{Z}_2)$$

with $\hat{H}_i(0,\cdot) = \hat{\Psi}_i$ and $\hat{H}_i(1,\cdot) = \hat{\Psi}'_i$;

(ii) $\sup_{(t,x)} \mathbf{F}_{g_i}(\hat{H}_i(t,x), \hat{\Psi}_i(x)) < \varepsilon$;

(iii) $\sup_{(t,x)} \mathbf{M}_{g_i}(\hat{H}_i(t,x)) < \sup_x \mathbf{M}_{g_i}(\hat{\Psi}_i(x)) + \varepsilon$.

We can take ε very small such that

$$\sup_{(t,x)} \mathbf{M}_{g_i}(\hat{H}_i(t,x)) < S_i + \delta_i\,, \tag{3.23}$$

and for $\eta_4 \coloneqq 2 \cdot \eta_3 \in (0, \min(\eta_0, \varepsilon_0))$, $\bar{\varepsilon}_4 \coloneqq \bar{\varepsilon}_3/2$, and $\varepsilon_{4,i} \coloneqq \varepsilon_{3,i}/2$,

$$\forall x \in \hat{X}_i,\ \mathbf{M}_{g_i}(\hat{H}_i(t,x)) \geq \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}_4 \implies |\hat{H}_i(t,x)| \in \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_4}(\mathcal{SV}^{8\pi}_{\bar{g}})\,, \tag{3.24}$$

$$\forall x \in \hat{X}^\alpha_i,\ \mathbf{M}_{g_i}(\hat{H}_i(t,x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{4,i} \implies |\hat{H}_i(t,x)| \in \widetilde{\mathcal{B}}_{\bar{g}}\,, \tag{3.25}$$

$$\forall x \in \hat{X}^\omega_i,\ \mathbf{M}_{g_i}(\hat{H}_i(t,x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{4,i} \implies |\hat{H}_i(t,x)| \in \widetilde{\mathcal{G}}_{\bar{g}}\,, \tag{3.26}$$

for any $t \in [0,1]$. These conditions are a direct result of (ii) and the previously established (3.17), (3.18), and (3.19).

Given that $\lim_i (S_i + \delta_i) = \lim_i \mathbf{L}(\mathcal{H}_i)$, for sufficiently large i, we have

$$\mathbf{L}(\mathcal{H}_i) - \min(\varepsilon_0, d_0, \bar{\varepsilon}_4)/100 > S_i + \delta_i - \min(\varepsilon_0, d_0, \bar{\varepsilon}_4)/10\,.$$

We can fix such an i for our subsequent construction.

Now, in (M, g_i), we apply Lemma 3.3.3 to $\hat{\Psi}'_i$ with

$$R = d_0\,,$$
$$\bar{\varepsilon} = \min(\varepsilon_0, d_0, \bar{\varepsilon}_4)\,,$$
$$\eta = r_0\,,$$
$$s = s_0\,,$$
$$\mathcal{W} = \mathcal{G}_{\bar{g}} \cup \mathcal{B}_{\bar{g}}\,.$$

To verify that $\hat{\Psi}'_i$ satisfies the assumptions of Lemma 3.3.3, let $x \in \hat{X}_i$ satisfy

$$\mathbf{M}_{g_i}(\hat{\Psi}'_i(x)) \geq L - \bar{\varepsilon}\,, \quad \mathbf{F}_{g_i}(|\hat{\Psi}'_i(x)|, \mathcal{W}) \geq R\,.$$

Since $L = \sup_{x\in\hat{X}_i} \mathbf{M}_g(\hat{\Psi}'_i(x)) \geq \mathbf{L}(\mathcal{H}_i)$ and $\bar{\varepsilon} \geq \bar{\varepsilon}_4$, we have

$$\mathbf{M}_{g_i}(\hat{\Psi}'_i(x)) \geq \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}_4\,,$$

and thus, by (3.24),

$$|\hat{\Psi}'_i(x)| \in \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_4}(\mathcal{SV}^{8\pi}_{\bar{g}}) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_0}(\mathcal{SV}^{8\pi}_{\bar{g}})\,.$$

By Lemma 3.2.16, given that

$$R = d_0 < \eta_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})$$

and

$$\|g_i - \bar{g}\|_{C^\infty, \bar{g}} < \eta_0 \leq d_0 < \eta_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,,$$

$|\hat{\Psi}'_i(x)|$ satisfies the annular $(\bar{\varepsilon}, \delta)$-deformation conditions as required by Lemma 3.3.3.

Consequently, for an arbitrary $\bar{\delta} > 0$, which will be specified later, we obtain a continuous map

$$\hat{\Psi}^*_i : \hat{X}_i \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)\,,$$

and a homotopy map

$$\hat{H}_i^{\mathrm{DEF}} : [0,1] \times \hat{X}_i \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2)$$

with $\hat{H}_i^{\mathrm{DEF}}(0,\cdot) = \hat{\Psi}_i'$ and $\hat{H}_i^{\mathrm{DEF}}(1,\cdot) = \hat{\Psi}_i^*$.

We can indeed choose $\bar{\delta}$ to satisfy the following conditions:

(i) $\bar{\delta} < \min(d_0, \varepsilon_{4,i}/10)$;

(ii)

$$\sup_{t,x} \mathbf{M}_{g_i} \circ \hat{H}_i^{\mathrm{DEF}}(t,x) < S_i + \delta_i\,, \tag{3.27}$$

(iii) For any $T, S \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $\mathbf{M}_{g_i}(T), \mathbf{M}_{g_i}(S) \le S_i + \delta_i$,

$$\mathbf{M}_{g_i}(T,S) < \bar{\delta} \implies \mathbf{F}_{g_i}(T,S) < d_0\,.$$

Claim 3.4.4. *For any $x \in \hat{X}_i$, if $\mathbf{M}_{g_i}(\hat{\Psi}_i^*(x)) \ge \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}/100$, then $|\hat{\Psi}_i^*(x)| \in \hat{\mathcal{G}}_{\bar{g}} \sqcup \hat{\mathcal{B}}_{\bar{g}}$*

Proof. $\mathbf{M}_{g_i}(\hat{\Psi}_i^*(x)) \ge \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}/100$ implies $\mathbf{M}_{g_i}(\hat{\Psi}_i^*(x)) \ge L - \bar{\varepsilon}/10$.

By (2) and (3) of Lemma 3.3.3, there exists $T_{1,x} \in \mathcal{Z}_2(S^3; \mathbb{Z}_2)$ and $\hat{x} \in \hat{X}_i$ satisfying

(i) $\mathbf{M}_{g_i}(\hat{\Psi}_i^*(x), T_{1,x}) < \bar{\delta}$, and thus, by Property (iii) of $\bar{\delta}$, $\mathbf{F}_{g_i}(\hat{\Psi}_i^*(x), T_{1,x}) < d_0$;

(ii)

$$T_{1,x} \llcorner (M \setminus (\overline{B}_{g_i}(p_1, \eta) \cup \cdots \cup (\overline{B}_{g_i}(p_m, r_0))) = \hat{\Psi}_i'(\hat{x}) \llcorner (M \setminus (\overline{B}_{g_i}(p_1, r_0) \cup \cdots \cup (\overline{B}_{g_i}(p_m, r_0)))$$

for some collection $\{p_1, \cdots, p_m\} \subset M$, $m \le 3^{29}$,

(iii) $\mathbf{F}_{g_i}(|\hat{\Psi}_i'(\hat{x})|, \mathcal{B}_{\bar{g}} \cup \mathcal{G}_{\bar{g}}) \le 2d_0$.

Since $d_0 = \eta_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})/10$, by Lemma 3.2.13 with $V_j = \hat{\Psi}_i'(\hat{x})$, $V_j' = T_{1,x}$ and $V_j'' = \hat{\Psi}_i^*(x)$ for $j = 1$ or 2, we have

$$|\hat{\Psi}_i^*(x)| \in \hat{\mathcal{G}}_{\bar{g}} \sqcup \hat{\mathcal{B}}_{\bar{g}}\,.$$

□

Claim 3.4.5. *For any $x \in \hat{X}_i^\alpha$ and any $t \in [0,1]$, if $\mathbf{M}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{4,i}/4$, then $|\hat{H}_i^{\mathrm{DEF}}(t,x)| \in \hat{\mathcal{B}}_{\bar{g}}$.*

Proof. By (2) of Lemma 3.3.3, there exists $T_{t,x} \in \mathcal{Z}_2(S^3;\mathbb{Z}_2)$ such that

(i) $\mathbf{M}_{g_i}(\hat{\Psi}_i'(x), \hat{\Psi}_i'(\hat{x})) < \bar{\delta}$

(ii) $\mathbf{M}_{g_i}(\hat{\Psi}_i'(\hat{x})) > \mathbf{M}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x)) - \bar{\delta}$;

(iii) $\mathbf{M}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x), T_{t,x}) < \bar{\delta}$;

(iv)

$$T_{t,x}\llcorner(M \setminus (\overline{B}_{g_i}(p_1,\eta) \cup \cdots \cup (\overline{B}_{g_i}(p_m,r_0))) = \hat{\Psi}_i'(\hat{x})\llcorner(M \setminus (\overline{B}_{g_i}(p_1,r_0) \cup \cdots \cup (\overline{B}_{g_i}(p_m,r_0)))$$

for some collection $\{p_1, \cdots, p_m\} \subset M$, $m \leq 3^{29}$.

By (i) and (ii), we have

$$\mathbf{M}_{g_i}(\hat{\Psi}_i'(x)) > \mathbf{M}_{g_i}(\hat{\Psi}_i'(\hat{x})) - \bar{\delta} > \mathbf{M}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x)) - 2\bar{\delta} > \mathbf{L}(\mathcal{H}_i) - \varepsilon_{4,i}\,,$$

and it follows from (3.25) that

$$|\hat{\Psi}_i'(x)| \in \widetilde{\mathcal{B}}_{\bar{g}}\,.$$

By Property (iii) of $\bar{\delta}$, (i) and (iii) above imply $\mathbf{F}_{g_i}(\hat{\Psi}_i'(x), \hat{\Psi}_i'(\hat{x})) < d_0$ and $\mathbf{F}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x), T_{t,x}) < d_0$. Consequently,

$$|\hat{\Psi}_i'(\hat{x})| \in \mathbf{B}_{d_0}^{\mathbf{F}_{\bar{g}_i}}(\widetilde{\mathcal{B}}_{\bar{g}})\,.$$

Since $d_0 = \eta_{3.2.13}(M, \bar{g}, 29, 8\pi, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})/10$, by Lemma 3.2.13 with $V_2 = \hat{\Psi}_i'(\hat{x})$, $V_2' = T_{t,x}$ and $V_2'' = \hat{H}_i^{\mathrm{DEF}}(t,x)$, we conclude that

$$|\hat{H}_i^{\mathrm{DEF}}(t,x)| \in \hat{\mathcal{B}}_{\bar{g}}\,.$$

□

Claim 3.4.6. *For any $x \in \hat{X}_i^\omega$ and any $t \in [0,1]$, if $\mathbf{M}_{g_i}(\hat{\Psi}_i^*(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{4,i}/4$, then $|\hat{\Psi}_i^*(x))| \in \hat{\mathcal{G}}_{\bar{g}}$.*

Proof. By (2) of Lemma 3.3.3, there exists $T_{1,x} \in \mathcal{Z}_2(S^3;\mathbb{Z}_2)$ such that

(i) $\mathbf{M}_{g_i}(\hat{\Psi}'_i(x), \hat{\Psi}'_i(\hat{x})) < \bar{\delta}$

(ii) $\mathbf{M}_{g_i}(\hat{\Psi}'_i(\hat{x})) > \mathbf{M}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x)) - \bar{\delta}$;

(iii) $\mathbf{M}_{g_i}(\hat{\Psi}^*(x), T_{1,x}) < \bar{\delta}$;

(iv)

$$T_{1,x}\llcorner(M\setminus(\overline{B}_{g_i}(p_1,\eta)\cup\cdots\cup(\overline{B}_{g_i}(p_m,r_0))) = \hat{\Psi}'_i(\hat{x})\llcorner(M\setminus(\overline{B}_{g_i}(p_1,r_0)\cup\cdots\cup(\overline{B}_{g_i}(p_m,r_0)))$$

for some collection $\{p_1,\cdots,p_m\}\subset M$, $m\le 3^{29}$.

By (i) and (ii), we have

$$\mathbf{M}_{g_i}(\hat{\Psi}'_i(x)) > \mathbf{M}_{g_i}(\hat{\Psi}'_i(\hat{x})) - \bar{\delta} > \mathbf{M}_{g_i}(\hat{H}_i^{\mathrm{DEF}}(t,x)) - 2\bar{\delta} > \mathbf{L}(\mathcal{H}_i) - \varepsilon_{4,i}\,,$$

and it follows from (3.26) that

$$|\hat{\Psi}'_i(x)| \in \widetilde{\mathcal{G}}_{\bar{g}}\,.$$

By Property (iii) of $\bar{\delta}$, (i) and (iii) above imply $\mathbf{F}_{g_i}(\hat{\Psi}'_i(x), \hat{\Psi}'_i(\hat{x})) < d_0$ and $\mathbf{F}_{g_i}(\hat{\Psi}^*(x), T_{1,x}) < d_0$. Consequently,

$$|\hat{\Psi}'_i(\hat{x})| \in \mathbf{B}^{\mathbf{F}_{\bar{g}_i}}_{d_0}(\widetilde{\mathcal{G}}_{\bar{g}})\,.$$

Since $d_0 = \eta_{3.2.13}(M,\bar{g},29,8\pi,\mathcal{G}_{\bar{g}},\mathcal{B}_{\bar{g}})/10$, by Lemma 3.2.13 with $V_1 = \hat{\Psi}'_i(\hat{x})$, $V'_1 = T_{t,x}$ and $V''_1 = \hat{\Psi}^*(x)$, we conclude that

$$|\hat{\Psi}^*(x)| \in \hat{\mathcal{G}}_{\bar{g}}\,.$$

□

We define the space

$$\hat{\hat{X}} := ([0,2]\times\mathbb{RP}^{13})\cup\hat{X}_i$$

where $\{2\}\times\mathbb{RP}^{13}\subset[0,2]\times\mathbb{RP}^{13}$ and $\hat{X}_i^{\alpha}\subset\hat{X}_i$ are identified. On $\hat{\hat{X}}$, we define the map

$$\hat{\hat{\Psi}} : \hat{\hat{X}} \to \mathcal{Z}_2(S^3;\mathbf{F}_{g_i};\mathbb{Z}_2)$$

as follows

$$\hat{\hat{\Psi}}|_{[0,1]\times\mathbb{RP}^{13}}(t,x) := \hat{H}_i|_{[0,1]\times\hat{X}_i^\alpha}(t,x)$$

$$\hat{\hat{\Psi}}|_{[1,2]\times\mathbb{RP}^{13}}(t,x) := \hat{H}_i^{\mathrm{DEF}}|_{[0,1]\times\hat{X}_i^\alpha}(t-1,x)$$

$$\hat{\hat{\Psi}}|_{\hat{X}_i} := \Psi_i^* .$$

Since $\hat{\hat{\Psi}}^\alpha = \hat{H}_i|_{\{0\}\times\hat{X}_i^\alpha} = \hat{\Psi}_i^\alpha = \Phi_{0,i}$ and $\sup_x \mathbf{M}_{g_i}\circ\hat{\hat{\Psi}}(x) < S_i + \delta_i$ by (3.23) and (3.27), we can conclude that $\hat{\hat{\Psi}} \in \widetilde{H}^i$.

Moreover, for $\varepsilon_4 := \min(\varepsilon_{4,i}/4, \bar{\varepsilon}/100)$, the previous three claims imply (3.20), (3.21) and (3.22).

Constructing a map Ξ^ω homologous to $(\hat{\hat{\Psi}})^\omega$. Finally, for the i we fixed in the last part, we will now construct some map $\Xi \in \widetilde{\mathcal{H}}_i$ such that

$$\mathbf{M}_{g_i}\circ\Xi^\omega < \mathbf{L}(\mathcal{H}_i),$$

thereby arriving at a contradiction.

First, we define two subsets

$$\hat{\hat{\mathcal{G}}}_{\bar{g}} := \{V \in \hat{\mathcal{G}}_{\bar{g}} : \|V\|_{g_i}(M) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4\}, \quad \hat{\hat{\mathcal{B}}}_{\bar{g}} := \{V \in \hat{\mathcal{B}}_{\bar{g}} : \|V\|_{g_i}(M) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4\}.$$

Due to the fact that $\mathbf{F}_{\bar{g}}(\hat{\mathcal{G}}_{\bar{g}}, \hat{\mathcal{B}}_{\bar{g}}) > \eta_0$, (3.20), (3.21), and (3.22), we can proceed by subdividing the simplicial complex structure of $\hat{\hat{X}}$ to ensure that the subcomplex

$$A := \bigcup\{14\text{-cells } \alpha \subset \hat{\hat{X}} : |\hat{\hat{\Psi}}|_\alpha| \text{ intersects } \hat{\hat{\mathcal{G}}}_{\bar{g}}\}$$

of $\hat{\hat{X}}$ satisfies (see Figure 15):

(i) A is disjoint from $\hat{\hat{X}}^\alpha$.

(ii) $|\hat{\hat{\Psi}}|_A|$ is disjoint from $\hat{\hat{\mathcal{B}}}_{\bar{g}}$.

(iii) For every $x \in \partial A \backslash \hat{\hat{X}}^\omega$,

$$\mathbf{M}_{g_i}\circ\hat{\hat{\Psi}}(x) < \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 , \tag{3.28}$$

and thus,

$$|\hat{\hat{\Psi}}|_{\partial A\setminus\hat{\hat{X}}^\omega}| \text{ is disjoint from } \hat{\hat{\mathcal{G}}}_{\bar{g}} \cup \hat{\hat{\mathcal{B}}}_{\bar{g}}.$$

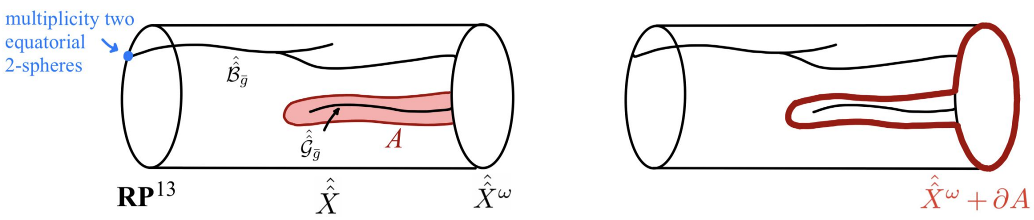


FIGURE 15. $\hat{\hat{X}}^\omega + \partial A$

Next, we define a space

$$X_A := \overline{\hat{\hat{X}}\setminus A}\,,$$

and a map

$$\Xi := \hat{\hat{\Psi}}|_{X_A}\ .$$

We will now show that Ξ leads to a contradiction. As A is disjoint from $\hat{\hat{X}}^\alpha \subset \hat{\hat{X}}$, we know that ∂X_A is a *disjoint* union of $\hat{\hat{X}}^\alpha$ and $\hat{\hat{X}}^\omega + \partial A$: Note that we used "+" in the sense of adding simplicial subcomplexes with $\mathbb{Z}_2$-coefficients, such that the part $\hat{\hat{X}}^\omega \cap \partial A$ cancels out. Consequently, we can conclude that $\Xi \in \widetilde{\mathcal{H}}_i$.

Now, let us consider $\Xi^\omega \in \mathcal{H}_i$. Its domain $\hat{\hat{X}}^\omega + \partial A$ is a union of $\partial A\setminus\hat{\hat{X}}^\omega$ and $\hat{\hat{X}}^\omega\setminus A$ (see Figure (15)). By (3.28),

$$\sup_{\partial A\setminus\hat{\hat{X}}^\omega} \mathbf{M}_{g_i} \circ \Xi < \mathbf{L}(\mathcal{H}_i) - \varepsilon_4\,,$$

and by (3.22),

$$\sup_{\hat{\hat{X}}^\omega\setminus\partial A} \mathbf{M}_{g_i} \circ \Xi < \mathbf{L}(\mathcal{H}_i) - \varepsilon_4\,.$$

In summary, we arrive at the conclusion

$$\mathbf{M}_{g_i} \circ \Xi^\omega < \mathbf{L}(\mathcal{H}_i) - \varepsilon_4,$$

leading to a contradiction. This completes the proof of Theorem 1.2.1.

3.5. **Technical ingredients.**

Proof of Proposition 3.4.2. Since $\mathcal{U}^{\leq A} = \mathcal{V}^{\leq A} \setminus \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}^{\leq A})$ (defined in (3.16)) is compact, there exists $\bar{\varepsilon}_{3,1} > 0$, such that

$$\inf_{V \in \mathcal{U}^{\leq A}} \|\delta_{\bar{g}} V\|_{\bar{g}} > 4\bar{\varepsilon}_{3,1}\,,$$

where

$$\|\delta_{\bar{g}} V\|_{\bar{g}} \coloneqq \sup\{\delta_{\bar{g}} V(Y) : Y \in \Gamma TS^3, \|Y\|_{C^1,\bar{g}} \leq 1\}\,,$$

with ΓTS^3 denoting the set of smooth vector fields on S^3.

Furthermore, we can choose

- a positive integer q,
- varifolds $\{V_j\}_{j=1}^q \subset \mathcal{U}^{\leq A}$,
- radii $\{r_j\}_{j=1}^q \subset (0, \eta_3)$,
- smooth vector fields $\{X_j\}_{j=1}^q \subset \Gamma TS^3$ with $\|X_j\|_{C^1,\bar{g}} \leq 1$,
- open balls $\{\mathcal{B}_j \coloneqq \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{r_j}(V_j)\}_{j=1}^q$ with $r_j < \eta_3/2$,

such that

$$\mathcal{U}^{\leq A} \subset \widetilde{\mathcal{U}}^{\leq A} \coloneqq \bigcup_{j=1}^q \mathcal{B}_j\,,$$

$$\emptyset = \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3/2}(\mathcal{S}^{\leq A}) \cap \widetilde{\mathcal{U}}^{\leq A}\,,$$

$$\delta_{\bar{g}} V(X_j) \leq -\frac{1}{2}\|\delta_{\bar{g}} V_j\| < -2\bar{\varepsilon}_{3,1} < 0,\ \forall V \in \mathcal{B}_j,\ \text{and}\ j = 1, 2, \cdots, q\,. \tag{3.29}$$

Hence, without loss of generality, by possibly discarding finitely many i's, for each $i \in \mathbb{N}^+$ and each $j = 1, 2, \cdots, q$, we have

$$\delta_{g_i} V(X_j) < -\bar{\varepsilon}_{3,1} < 0\,, \tag{3.30}$$

holds for all $V \in \mathcal{B}_j$.

For each $j = 1, 2, \cdots, q$, we define a continuous function

$$\psi_j : \widetilde{\mathcal{U}}^{\leq A} \to [0, \infty)\,,$$

by defining $\psi_j(V) := \mathbf{F}_{\bar{g}}(V, \mathcal{V}^{\leq A} \setminus \mathcal{B}_j)$, the $\mathbf{F}_{\bar{g}}$-distance of V from $\mathcal{V}^{\leq A} \setminus \mathcal{B}_j$.

Now, we can define vector fields associated with each varifold in $\widetilde{\mathcal{U}}^{\leq A}$, i.e.

$$X : \widetilde{\mathcal{U}}^{\leq A} \to \Gamma T S^3\,,$$
$$V \mapsto \sum_{j=1}^{q} \psi_j(V) X_j\,.$$

Note that X is continuous, and by compactness, there exists $\bar{\varepsilon}_{3,2} > 0$ such that for all $V \in \mathcal{U}^{\leq A}$,

$$\delta_{\bar{g}} V(X(V)) < -2\bar{\varepsilon}_{3,2} < 0\,. \tag{3.31}$$

Then, by discarding finitely many g_i, for each $i \in \mathbb{N}^+$,

$$\delta_{g_i} V(X(V)) < -\bar{\varepsilon}_{3,2} < 0\,, \tag{3.32}$$

We can extend X such that X becomes a continuous map $\mathcal{V}^{\leq A} \to \Gamma T S^3$ by putting $X(V)$ to be the zero vector field for each V outside $\widetilde{\mathcal{U}}^{\leq A}$.

Now, we are ready to define the desired map

$$H : [0,1] \times \{T \in \mathcal{Z}_2(S^3; \mathbf{F}_{\bar{g}}; \mathbb{Z}_2) : \mathbf{M}_{\bar{g}}(T) \leq A\} \to \{T \in \mathcal{Z}_2(S^3; \mathbf{F}_{\bar{g}}; \mathbb{Z}_2) : \mathbf{M}_{\bar{g}}(T) \leq A\}.$$

First we define a map

$$f : [0, \infty) \times \mathcal{V}^{\leq A} \to \mathrm{Diff}(S^3)$$

by letting $\{f(t, V)\}_t$ be the one-parameter family of diffeomorphisms on S^3 generated by the vector field $X(V)$. By (3.31) and (3.32), there exists a continuous function $h : \mathcal{V}^{\leq A} \to [0, 1]$ such that:

- $h > 0$ on $\widetilde{\mathcal{U}}^{\leq A}$ and $h = 0$ elsewhere.

- If $0 \leq t < s \leq h(V)$,

$$\|f(s,V)_{\#}(V)\|_{\bar{g}}(S^3) < \|f(t,V)_{\#}(V)\|_{\bar{g}}(S^3). \tag{3.33}$$

- For each i, if $0 \leq t < s \leq h(V)$,

$$\|f(s,V)_{\#}(V)\|_{g_i}(S^3) < \|f(t,V)_{\#}(V)\|_{g_i}(S^3). \tag{3.34}$$

Now, define the desired map H by

$$H(t,T) = \begin{cases} f(t,|T|)_{\#}(T) & \text{if } 0 \leq t \leq h(|T|), \\ f(h(|T|),|T|)_{\#}(T) & \text{if } h(|T|) \leq t \leq 1. \end{cases}$$

Clearly, $H(0,\cdot) = \mathrm{id}$, so H satisfies Proposition 3.4.2 (1). And because $h = 0$ on $\mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3/2}(\mathcal{S}^{\leq A})$, $H(t,\cdot)$ fixes T if $|T| \in \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3/2}(\mathcal{S}^{\leq A})$, so Proposition 3.4.2 (2) holds. Moreover, by the three bullet points in the definition of h above, we know for each (t,T),

$$\mathbf{M}_{\bar{g}}(H(t,T)) \leq \mathbf{M}_{\bar{g}}(T), \;\; \mathbf{M}_{g_i}(H(t,T)) \leq \mathbf{M}_{g_i}(T).$$

And any of the equalities hold only if $t = 0$ or $h(|T|) = 0$. So H satisfies Proposition 3.4.2 (3) too. Furthermore, we claim that there exists $\varepsilon_3 > 0$, such that for each T, if $|H(1,T)| \in \mathcal{U}^{\leq A}$, then

$$\mathbf{M}_{\bar{g}}(H(1,T)) \leq \mathbf{M}_{\bar{g}}(T) - 2\varepsilon_3.$$

Indeed, if not, then there exists a sequence $(T_j)_j$ such that $|H(1,T_j)| \in \mathcal{U}^{\leq A}$ and

$$\mathbf{M}_{\bar{g}}(H(1,T)) \geq \mathbf{M}_{\bar{g}}(T) - 1/j. \tag{3.35}$$

Then, by compactness and relabeling the j's, $|T_j|$ converges to some $V' \in \mathcal{V}^{\leq A}$. However, note that:

- Putting $T = T_j$ into (3.35) and taking $j \to \infty$, we have

$$\|f(h(V'), V')(V')\|(S^3) \geq \|V'\|(S^3),$$

which by (3.33) implies $h(V') = 0$. Thus, by the definition of h, $V' \notin \widetilde{\mathcal{U}}^{\leq A}$, so $V' \notin \mathcal{U}^{\leq A}$.

- On the other hand, taking $j \to \infty$ to $|H(1, T_j)| \in \mathcal{U}^{\leq A}$, we know

$$f(h(V'), V')_{\#}(V') \in \mathcal{U}^{\leq A}.$$

But we have shown $h(V') = 0$, so $V' \in \mathcal{U}^{\leq A}$.

Hence, contradiction arises. This proves our claim that if $|H(1, T)| \in \mathcal{U}^{\leq A}$, then

$$\mathbf{M}_{\bar{g}}(H(1, T)) \leq \mathbf{M}_{\bar{g}}(T) - 2\varepsilon_3.$$

Then, we also have

$$\mathbf{M}_{g_i}(H(1, T)) \leq \mathbf{M}_{g_i}(T) - \varepsilon_3$$

by discarding finitely many g_i. Therefore, H satisfies Proposition 3.4.2 (4).

This finishes the proof of Proposition 3.4.2.

Proof of Lemma 3.4.3. Note that:

- $\lim_i \sup \mathbf{M}_{\bar{g}} \circ \Phi_{0,i} = 8\pi$, which by Proposition 3.4.2 (3) implies

$$\lim_i \sup \mathbf{M}_{\bar{g}} \circ \hat{\Psi}_i|_{[0,1] \times \mathbb{RP}^{13}} = 8\pi.$$

- By $\mathbf{L}(\Pi_i) \to 8\pi$, $\lim_i \sup \mathbf{M}_{\bar{g}} \circ \Phi_{0,i} + \delta_i = 8\pi$, and Proposition 3.4.2 (3), we know

$$\lim_i \sup \mathbf{M}_{\bar{g}} \circ \hat{\Psi}_i|_{X_i} = 8\pi.$$

Hence,

$$\mathbf{C}_{\bar{g}}((\hat{\Psi}_i)_i) = \mathbf{C}_{\bar{g}}((\hat{\Psi}_i|_{[0,1] \times \mathbb{RP}^{13}})_i) \cup \mathbf{C}_{\bar{g}}((\hat{\Psi}_i|_{X_i})_i).$$

(Note that $(\hat{\Psi}_i|_{[0,1]\times\mathbb{RP}^{13}})_i$ and $(\hat{\Psi}_i|_{X_i})_i$ can be viewed as minimizing sequences for the 13-width, so the notion of critical set does make sense.)

To see that $\mathbf{C}_{\bar{g}}((\hat{\Psi}_i|_{[0,1]\times\mathbb{RP}^{13}})_i) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}_0)$, take $(t_i, x_i) \in [0,1] \times \mathbb{RP}^{13}$ such that, after passing to a subsequence,

$$|\hat{\Psi}_i|_{[0,1]\times\mathbb{RP}^{13}}(t_i, x_i)|$$

tends to some varifold V with mass 8π. Then by Proposition 3.4.2 (3) we know the mass of

$$\hat{\Psi}_i|_{[0,1]\times\mathbb{RP}^{13}}(0, x_i) = \Phi_{0,i}(x_i)$$

tends to 8π. Since $(\Phi_{0,i})_i$ is a pulled-tight minimizing sequence, we know $\Phi_{0,i}(x_i)$ subsequentially converge to some stationary varifold of mass 8π. By Proposition 3.4.2 (2), this stationary varifold is V. So $\mathbf{C}_{\bar{g}}((\hat{\Psi}_i|_{[0,1]\times\mathbb{RP}^{13}})_i) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}_0)$.

To see that $\mathbf{C}((\hat{\Psi}_i|_{X_i})_i) \subset \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}_0)$, suppose by contradiction that, after passing to a subsequence, $|\hat{\Psi}_i|_{X_i}(x_i)|$ converges to some V with mass 8π but $V \notin \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_3}(\mathcal{S}_0)$. Then for every large i, by Proposition 3.4.2 (4), $\mathbf{M}_{\bar{g}}(\hat{\Psi}_i(x_i)) \le \mathbf{M}_{\bar{g}}(\Psi(x_i)) - \varepsilon_3$. However, noting

$$\limsup_i \mathbf{M}_{\bar{g}} \circ \Psi_i = 8\pi,$$

contradiction arises. This finishes the proof.

3.6. **Proof of Theorem 1.2.2.** Let $(M^{n+1}, \bar{g})$ with $3 \le n+1 \le 7$ be a closed Riemannian manifold of positive Ricci curvature or bumpy metrics. Let $p \in N^+$ and $L_0 \coloneqq \omega_p(M, \bar{g})$ be the min-max p-width.

Analogously, we can define $r_0, \mathcal{SV}^{L_0}_{\bar{g}}$, $\mathcal{M}^{L_0}_{\bar{g}}$, $\mathcal{G}_{\bar{g}}$ and $\mathcal{B}_{\bar{g}}$ as follows:

- $r_0 \coloneqq \min(\eta_{3.2.7}(M, \bar{g}, 2p+3), \eta_{3.2.8}(M, \bar{g}, 2p+3))$ from Lemmas 3.2.7 and 3.2.8;
- $\mathcal{SV}^{L_0}_{\bar{g}} \subset \mathcal{V}_n(M)$ is the set of all stationary n-varifolds on $(M, \bar{g})$ with total measure L_0;
- $\mathcal{M}^{L_0}_{\bar{g}} \subset \mathcal{SV}^{L_0}_{\bar{g}}$ is the subset consisting of all stationary integral varifolds whose support is a smooth, embedded, closed minimal surface;

- $\mathcal{G}_{\bar{g}} \subset \mathcal{M}_{\bar{g}}^{L_0}$ is the (good) subset comprising all $(2p+3, r_0)_{\bar{g}}$-almost minimizing varifolds V for which there is $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $V = |T|$;
- $\mathcal{B}_{\bar{g}} \subset \mathcal{M}_{\bar{g}}^{L_0}$ is the (bad) subset comprising all $(2p+3, r_0)_{\bar{g}}$-almost minimizing varifolds V for which no $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $V = |T|$.

With these definitions, it follows that $\mathcal{G}_{\bar{g}} \cup \mathcal{B}_{\bar{g}} \subset \mathcal{M}_{\bar{g}}^{L_0}$ is the subset consisting of all $(2p+3, r_0)_{\bar{g}}$-almost minimizing varifolds.

Since $(M, \bar{g})$ has positive Ricci curvature or a bumpy metric, by Lemma 3.2.17, both $\mathcal{G}_{\bar{g}}$ and $\mathcal{B}_{\bar{g}}$ are compact in the varifold topology. Similarly, we define

$$
\begin{aligned}
d_0 &\coloneqq \min(\eta_{3.2.11}(M, \bar{g}, 2p+3, \mathcal{G}_{\bar{g}}), \eta_{3.2.13}(M, \bar{g}, 2p+3, L_0, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}}))/10\,, \\
\varepsilon_0 &\coloneqq \bar{\varepsilon}_{3.2.16}(M, g, 2p+3, r_0, d_0, L_0)\,, \\
s_0 &\coloneqq \bar{s}_{3.2.16}(M, g, 2p+3, r_0, d_0, L_0)\,, \\
\eta_0 &\coloneqq \min(d_0, \varepsilon_0, \eta_{3.2.16}(M, g, 2p+3, r_0, d_0, L_0))/10\,, \\
\widetilde{\mathcal{G}}_{\bar{g}} &\coloneqq \mathcal{G}'_{3.2.13}(M, \bar{g}, 2p+3, L_0, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,, \\
\widetilde{\mathcal{B}}_{\bar{g}} &\coloneqq \mathcal{B}'_{3.2.13}(M, \bar{g}, 2p+3, L_0, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,, \\
\hat{\mathcal{G}}_{\bar{g}} &\coloneqq \mathcal{G}''_{3.2.13}(M, \bar{g}, 2p+3, L_0, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,, \\
\hat{\mathcal{B}}_{\bar{g}} &\coloneqq \mathcal{B}''_{3.2.13}(M, \bar{g}, 2p+3, L_0, \mathcal{G}_{\bar{g}}, \mathcal{B}_{\bar{g}})\,,
\end{aligned}
$$

from Lemma 3.2.11, Lemma 3.2.13 and Lemma 3.2.16.

By Marques-Neves [MN21], there exists a pulled-tight minimizing sequence of $\mathbf{F}_{\bar{g}}$-continuous p-admissible maps $(\Phi_i : \bar{W}_i \to \mathcal{Z}_n(M; \mathbf{F}_{\bar{g}}; \mathbb{Z}_2))_{i=1}^\infty$ such that

$$\mathbf{C}_{\bar{g}}((\Phi_i)_{i=1}^\infty) \subset \mathcal{SV}_{\bar{g}}^{L_0}\,.$$

Therefore, by discarding finitely many i's and after relabelling, there exists $\varepsilon_0' \in (0, \varepsilon_0/2)$ such that for any $i \in \mathbb{N}^+$ and any $x \in \bar{W}_i$,

$$\mathbf{M}_{\bar{g}} \circ \Phi_i(x) \geq L_0 - \varepsilon_0' \implies |\Phi_i(x)| \in \mathbf{B}_{\eta_0}^{\mathbf{F}_{\bar{g}}}(\mathcal{SV}_{\bar{g}}^{L_0})\,.$$

In addition, by definition of $\mathcal{P}_p$, for every $i \in \mathbb{N}^+$, there exists a finite simplicial p-chain C_i of $\bar{W}_i$ such that

$$\langle \Phi_i^*(\bar{\lambda}^p), [C_i] \rangle = 1 \mod 2 .$$

Therefore, in the following, we may assume that the domain of each Φ_i, $\bar{W}_i$ is a finite simplicial p-chain.

Proposition 3.6.1. *There exists a pulled-tight minimizing sequence of $\mathbf{F}_{\bar{g}}$-continuous p-admissible maps $(\Phi_i : \bar{W}_i \to \mathcal{Z}_n(M; \mathbf{F}_{\bar{g}}; \mathbb{Z}_2))_{i=1}^{\infty}$, such that*

$$\mathbf{C}_{\bar{g}}((\Phi_i)_{i=1}^{\infty}) \subset (\widetilde{\mathcal{G}}_{\bar{g}} \cup \widetilde{\mathcal{B}}_{\bar{g}}) \cap \mathcal{SV}_{\bar{g}}^{L_0} .$$

In particular, there exists an positive constant $\bar{\varepsilon}_0 > 0$, such that

$$\mathbf{M}_{\bar{g}} \circ \Phi_i(x) \geq L_0 - \bar{\varepsilon}_0 \implies |\Phi_i(x)| \in (\widetilde{\mathcal{G}}_{\bar{g}} \cup \widetilde{\mathcal{B}}_{\bar{g}}) \cap \mathbf{B}_{\eta_0}^{\mathbf{F}_{\bar{g}}}(\mathcal{SV}_{\bar{g}}^{L_0}) .$$

Proof. By discarding finitely many i's, we may assume for every i,

$$L_i \coloneqq \sup_{x \in \bar{W}_i} \mathbf{M}_{\bar{g}} \circ \Phi_i(x) < L_0 + \varepsilon_0'/100 .$$

To see that we can apply Lemma 3.3.3 to each Φ_i with

$$R = d_0 ,$$

$$\bar{\varepsilon} = \varepsilon_0' ,$$

$$\eta = r_0 ,$$

$$s = s_0 ,$$

$$\mathcal{W} = \mathcal{G}_{\bar{g}} \cup \mathcal{B}_{\bar{g}} ,$$

it suffices to verify that Φ_i satisfies the assumption of Lemma 3.3.3, let $x \in \bar{W}_i$ satisfy

$$\mathbf{M}_{\bar{g}}(\Phi_i(x)) \geq L_i - \bar{\varepsilon} , \quad \mathbf{F}_{\bar{g}}(|\Phi_i(x)|, \mathcal{W}) \geq R .$$

. Then we have

$$\Phi_i(x) \in \mathbf{B}^{\mathbf{F}_{\bar g}}_{\eta_0}(\mathcal{SV}^{L_0}_{\bar g}) \setminus \mathbf{B}^{\mathbf{F}_{\bar g}}_{d_0}(\mathcal{W})\,.$$

By Lemma 3.2.16, $|\Phi_i(x)|$ satisfies the annular $(\bar\varepsilon, \delta)$-deformation conditions as required by Lemma 3.3.3.

Consequently, for any $\bar\delta_i > 0$, we obtain a new sweepout Φ_i^* homotopic to Φ_i in the $\mathbf{F}_{\bar g}$ topology such that

(i) $\mathbf{M}_{\bar g}(\Phi_i^*(x)) < \mathbf{M}_{\bar g}(\Phi_i(x)) + \bar\delta_i$;

(ii) There exists $T_{1,x} \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ and $\hat x \in \bar W_i$ such that

$$\mathbf{M}_{\bar g}(\Phi_i^*(x), T_{1,x}) < \bar\delta_i\,,$$

and $T_{1,x} = \Phi_i(\hat x)$ on $M \setminus \bigcup_{i=1}^m \overline{B}_{\bar g}(p_i, \eta)$ for some collection $\{p_i\}_{i=1}^m \subset M$, $m \le 3^{2p+1}$;

(iii) If $\mathbf{M}_{\bar g} \circ \Phi_i^*(x) \ge L_0 - \varepsilon_0'/100 > L_i - \varepsilon_0'/10$,

$$\mathbf{F}_{\bar g}(|\Phi_i(\hat x)|, \mathcal{W}) \le 2R\,;$$

Then, let $\bar\delta_i < \min(\eta_0, \frac{1}{i})$ and $\bar\varepsilon_0 = \varepsilon_0'/100$. Then if $\mathbf{M}_{\bar g} \circ \Phi_i^*(x) \ge L_0 - \bar\varepsilon_0$, by (ii), (iii) and Lemma 3.2.13, we have

$$\Phi_i^*(x) \in \widetilde{\mathcal{G}}_{\bar g} \cup \widetilde{\mathcal{B}}_{\bar g}\,.$$

In addition, $(\Phi_i^*)_{i=1}^\infty$ is a minimizing sequence.

Finally, applying Corollary 3.3.2, we obtain a pulled-tight minimzing sequence $(\Psi_i)_{i=1}^\infty$ such that

$$\mathbf{C}_{\bar g}((\Psi_i)_{i=1}^\infty) \subset (\widetilde{\mathcal{G}}_{\bar g} \cup \widetilde{\mathcal{B}}_{\bar g}) \cap \mathcal{SV}^{L_0}_{\bar g}\,.$$

The second part follows from the compactness of the critical set and the openness of $\widetilde{\mathcal{G}}_{\bar g} \cup \widetilde{\mathcal{B}}_{\bar g}$. □

Let (Φ_i) be a pulled-tight minimizing sequence from the previous Proposition. Since $\mathbf{F}_{\bar g}(\widetilde{\mathcal{G}}_{\bar g}, \widetilde{\mathcal{B}}_{\bar g}) > 0$, for each $\bar W_i$, we can refine $\bar W_i$ to obtain two p-chains W_i and W_i' with the following properties.

(i) $\bar{W}_i = W_i + W'_i$;

(ii) On W_i, $\mathbf{M}_{\bar{g}} \circ \Phi_i(x) \geq L_0 - \bar{\varepsilon}_0 \implies |\Phi_i(x)| \in \widetilde{\mathcal{B}}_{\bar{g}} \cap \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_0}(\mathcal{SV}^{L_0}_{\bar{g}})$;

(iii) On W'_i, $\mathbf{M}_{\bar{g}} \circ \Phi_i(x) \geq L_0 - \bar{\varepsilon}_0 \implies |\Phi_i(x)| \in \widetilde{\mathcal{G}}_{\bar{g}} \cap \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_0}(\mathcal{SV}^{L_0}_{\bar{g}})$;

(iv) For all $x \in \partial W_i$,

$$\mathbf{M}_{\bar{g}} \circ \Phi_i(x) \leq L_0 - \bar{\varepsilon}_0 \,.$$

Define $\Phi_{0,i} \coloneqq \Phi_i|_{W_i}$ and $\delta = \bar{\varepsilon}_0/10$.

If there exists $\delta_0 > 0$ such that

$$\limsup_i \mathbf{L}(\mathcal{H}^{\delta}_{\bar{g}}(\Phi_{0,i})) < L_0 - \delta_0 \,,$$

then for sufficiently large i, we obtain a $\Psi_i \in \widetilde{\mathcal{H}}^{\delta}_{\bar{g}}(\Phi_{0,i})$ with domain X_i such that

$$\sup_{x \in X^{\omega}_i} \mathbf{M}_g \circ \Psi^{\omega}_i(x) < L_0 - \delta_0 \,. \tag{3.36}$$

Therefore, we can define $\bar{W}'_i \coloneqq \bar{W}_i + \partial X_i$ and

$$\Phi'_i(x) \coloneqq \begin{cases} \Phi_i(x) & x \in W'_i \\ \Psi^{\omega}_i(x) & x \in X^{\omega} \,. \end{cases}$$

By Lemma 3.3.9, $\Phi'_i \in \mathcal{P}_p$. Moreover, by (3.36) and Property (iii) or Φ_i, for every $x \in \bar{W}'_i$,

$$\mathbf{M}_{\bar{g}} \circ \Phi'_i(x) \geq L_0 - \bar{\varepsilon}_0 \implies |\Phi'_i(x)| \in \widetilde{\mathcal{G}}_{\bar{g}} \,,$$

and thus, $\mathbf{C}(\Phi'_i) \subset \widetilde{\mathcal{G}}_{\bar{g}}$. By Lemma 3.2.11, it is a pulled-tight minimizing sequence such that every embedded cycle in its critical set is associated with a flat cycle. This concludes Theorem 1.2.2.

Otherwise, we have

$$\limsup_i \mathbf{L}(\mathcal{H}^{\delta}_{\bar{g}}(\Phi_{0,i})) \equiv L_0 \,, \tag{3.37}$$

and we shall deduce a contradiction from this by proceeding with the proof of Theorem 1.2.1. It suffices to consider the boundary part

$$Z_i \coloneqq \partial W_i$$

when attaching a homotopy map. Fortunately, due to the small mass of cycles on the boundary, they have no significant impact on the critical set. This will be elaborated upon in the following.

Metric perturbations. We make use of the same Proposition 3.4.1 to choose a sequence $(g_i)_{i=1}^\infty$ in $\Gamma^\infty_{\text{uniq}}$ such that $\|g_i - \bar{g}\|_{C^\infty,\bar{g}} < \eta_0$, and

$$\lim_{i\to\infty} g_i = \bar{g}$$

in the C^∞ topology. We define for each $i \in \mathbb{N}^+$,

$$S_i \coloneqq \sup_{x\in W_i} \mathbf{M}_{g_i} \circ \Phi_{0,i}(x)\,.$$

Let $(\delta_i)_{i=1}^\infty$ be a decreasing sequence in $(0, \bar{\varepsilon}_0/2)$ such that

$$\lim_{i\to\infty} \delta_i = 0\,.$$

Restrictive homological min-max. We define

$$\eta_2 \coloneqq \min(\eta_0, \varepsilon_0)/10\,.$$

For each $i \in \mathbb{N}^+$, we consider the restrictive homology class

$$\widetilde{\mathcal{H}}_i = \widetilde{\mathcal{H}}^{\delta_i}_{g_i}(\Phi_{0,i}) \text{ and } \mathcal{H}_i = \mathcal{H}^{\delta_i}_{g_i}(\Phi_{0,i})\,.$$

By (3.37),

$$\lim_{i\to\infty} \mathbf{L}(\mathcal{H}_i) = \lim_{i\to\infty} S_i + \delta_i = L_0\,.$$

Similarly, applying the restrictive min-max theorem, Theorem 3.3.16, to each $\mathcal{H}_i$, for each i, we obtain a sweepout $\Psi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}_{g_i}; \mathbb{Z}_2)$ in $\widetilde{\mathcal{H}}_i$ and $\varepsilon_{2,i} > 0$, such that

$$\mathbf{M}_{g_i}((\Psi_i)^\alpha(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{2,i} \implies |(\Psi_i)^\alpha(x)| \in \widetilde{\mathcal{B}}_{\bar{g}} \cap \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_2}(\mathcal{S}\mathcal{V}^{L_0}_{\bar{g}}),$$

and

$$\mathbf{M}_{g_i}((\Psi_i)^\omega(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{2,i} \implies |(\Psi^j_i)^\omega(x)| \in \widetilde{\mathcal{G}}_{\bar{g}} \cap \mathbf{B}^{\mathbf{F}_{\bar{g}}}_{\eta_2}(\mathcal{S}\mathcal{V}^{L_0}_{\bar{g}}).$$

Pull-tight. We define

$$\eta_3 \coloneqq 3\eta_2 .$$

For each i, after possibly discarding finitely many i, we define the space

$$\hat{X}_i \coloneqq ([0,1] \times W_i) \cup X_i ,$$

where $\{1\} \times W_i \subset [0,1] \times W_i$ and $X_i^\alpha \subset X_i$ are identified. Note that in this case,

$$(\hat{X}_i)^\alpha = \{0\} \times W_i \cong W_i, \quad (\hat{X}_i)^\omega = [0,1] \times Z_i \cup X_i^\omega .$$

We can apply the same Proposition 3.4.2 with $A = L_0 + 1$ to obtain a deformation map H, and define

$$\hat{\Psi}_i : \hat{X}_i \to \mathcal{Z}_2(S^3; \mathbf{F}_{g_i}; \mathbb{Z}_2)$$

by

$$\hat{\Psi}_i|_{[0,1]\times W_i}(t,x) \coloneqq H(t, \Phi_{0,i}(x))$$

$$\hat{\Psi}_i|_{X_i}(x) \coloneqq H(1, \Psi_i(x)).$$

Then, there exists an $\bar{\varepsilon}_3 \in \mathbb{R}^+$ and $(\varepsilon_{3,i})_{i=1}^{\infty} \subset \mathbb{R}^+$ such that

$$\mathbf{M}_{g_i}(\hat{\Psi}_i^{\alpha}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}_3 \implies |\hat{\Psi}_i^{\alpha}(x)| \in \mathbf{B}_{\eta_3}^{\mathbf{F}_{\bar{g}}}(\mathcal{S}\mathcal{V}_{\bar{g}}^{L_0}),$$

$$\mathbf{M}_{g_i}(\hat{\Psi}_i^{\alpha}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{3,i} \implies |\hat{\Psi}_i^{\alpha}(x)| \in \widetilde{\mathcal{B}}_{\bar{g}} \cap \mathbf{B}_{\eta_2}^{\mathbf{F}_{\bar{g}}}(\mathcal{S}\mathcal{V}_{\bar{g}}^{L_0}),$$

$$\mathbf{M}_{g_i}(\hat{\Psi}_i^{\omega}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_{3,i} \implies |\hat{\Psi}_i^{\omega}(x)| \in \widetilde{\mathcal{G}}_{\bar{g}} \cap \mathbf{B}_{\eta_2}^{\mathbf{F}_{\bar{g}}}(\mathcal{S}\mathcal{V}_{\bar{g}}^{L_0}).$$

Here, we use the inequality on $\hat{X}_i^{\omega}$,

$$\sup_{[0,1]\times Z_i} \mathbf{M}_{g_i} \circ \hat{\Psi}_i^{\omega} \leq \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}_0/2.$$

(ε, δ)-deformation. Fix such a sufficiently large i. In (M, g_i), we apply Lemma 3.1.9 to $\hat{\Psi}_i : \hat{X}_i \to \mathcal{Z}_n(M; \mathbf{F}_{g_i}; \mathbb{Z}_2)$ and obtain a $\mathbf{M}_{g_i}$-continuous map $\hat{\Psi}_i'$ and a $\mathbf{F}_{g_i}$-continuous homotopy map $\hat{H}_i$.

Similar arguments imply that we can apply Lemma 3.3.3 to $\hat{\Psi}_i'$ to obtain

$$\hat{\Psi}_i^* : \hat{X}_i \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2),$$

and a homotopy map

$$\hat{H}_i^{\mathrm{DEF}} : [0,1] \times \hat{X}_i \to \mathcal{Z}_n(M; \mathbf{M}_g; \mathbb{Z}_2).$$

Analogously, We define the space

$$\hat{\hat{X}} := ([0,2] \times W_i) \cup \hat{X}_i$$

where $\{2\} \times W_i \subset [0,2] \times W_i$ and $\hat{X}_i^{\alpha} \subset \hat{X}_i$ are identified. On $\hat{\hat{X}}$, we define the map

$$\hat{\hat{\Psi}} : \hat{\hat{X}} \to \mathcal{Z}_n(M; \mathbf{F}_{g_i}; \mathbb{Z}_2)$$

in $\widetilde{H}^i$, as follows

$$\hat{\hat{\Psi}}|_{[0,1]\times W_i}(t,x) \coloneqq \hat{H}_i|_{[0,1]\times \hat{X}_i^\alpha}(t,x)$$

$$\hat{\hat{\Psi}}|_{[1,2]\times W_i}(t,x) \coloneqq \hat{H}_i^{\mathrm{DEF}}|_{[0,1]\times \hat{X}_i^\alpha}(t-1,x)$$

$$\hat{\hat{\Psi}}|_{\hat{X}_i} \coloneqq \Psi_i^*\,.$$

Note that $\hat{\hat{X}}^\alpha = W_i$ and $\hat{\hat{X}}^\omega = [0,2]\times Z_i \cup \hat{X}_i^\omega$.

Moreover, there exists $\varepsilon_4 > 0$ such that

$$\forall x \in \hat{\hat{X}},\ \mathbf{M}_{g_i}(\hat{\hat{\Psi}}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 \implies |\hat{\hat{\Psi}}(x)| \in \hat{\mathcal{G}}_{\bar{g}} \cup \hat{\mathcal{B}}_{\bar{g}}\,,$$

$$\forall x \in \hat{\hat{X}}^\alpha,\ \mathbf{M}_{g_i}(\hat{\hat{\Psi}}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 \implies |\hat{\hat{\Psi}}(x)| \in \hat{\mathcal{B}}_{\bar{g}}\,,$$

$$\forall x \in \hat{\hat{X}}^\omega,\ \mathbf{M}_{g_i}(\hat{\hat{\Psi}}(x)) \geq \mathbf{L}(\mathcal{H}_i) - \varepsilon_4 \implies |\hat{\hat{\Psi}}(x)| \in \hat{\mathcal{G}}_{\bar{g}}\,,$$

Here, we use the inequality on $\hat{\hat{X}}^\omega$,

$$\sup_{[0,2]\times Z_i} \mathbf{M}_{g_i} \circ \hat{\hat{\Psi}}^\omega \leq \mathbf{L}(\mathcal{H}_i) - \bar{\varepsilon}_0/4\,.$$

Constructing a map Ξ^ω homologous to $(\hat{\hat{\Psi}})^\omega$. Similarly, using the fact that $\mathbf{F}_{g_i}(\hat{\mathcal{G}}_{\bar{g}}, \hat{\mathcal{B}}_{\bar{g}}) > 0$, we can construct a map $\Xi \in \widetilde{\mathcal{H}}_i$ from $\hat{\hat{\Psi}}$ such that

$$\mathbf{M}_{g_i} \circ \Xi^\omega < \mathbf{L}(\mathcal{H}_i),$$

thereby arriving at a contradiction. This completes the proof of Theorem 1.2.2.

4. A free boundary minimal surface via a 6-sweepout

4.1. **Preliminaries.** In this section, we first discuss more about the space $\mathcal{E}$ introduced in §1, and then state some preliminaries about min-max theory, in both the Almgren-Pitts setting [Alm62, Alm65, Pit81] and the Simon-Smith setting [CDL03, Smi82].

About the space $\mathcal{E}$. A smooth embedded surface S in $\mathbb{B}^3$ is said to be *properly embedded* if $\partial S = S \cap \partial\mathbb{B}^3$ and S meets $\partial\mathbb{B}^3$ transversely along ∂S. By definition, $\mathcal{E}$ consists of closed sets $S \subset \mathbb{B}^3$ such that there exists a finite set P such that $S\backslash P$ is a smooth and properly embedded surface. (Note that $\partial(S\backslash P)$ does not include P.) Now, for any open set $U \subset\subset \mathbb{B}^3\backslash P$ (meaning $\overline{U} \subset \mathbb{B}^3\backslash P$), $\epsilon > 0$, and non-negative integer k, denote by $B_{P,U,\epsilon,k}(S) \subset \mathcal{E}$ the subset of all surface $S' \in \mathcal{E}$ such that $S'\backslash P$ is smooth and properly embedded and is ϵ-close to S in the graphical C^k-distance within U. Then the following proposition tells us that the topology on $\mathcal{E}$ introduced in §1.3 is well-defined.

Proposition 4.1.1. *The subsets $B_{P,U,\epsilon,k}(S) \subset \mathcal{E}$ form a base.*

Proof. First, these subsets clearly cover $\mathcal{E}$. So it suffices to show that if $B_{P_1,U_1,\epsilon_1,k_1}(S_1) \cap B_{P_2,U_2,\epsilon_2,k_2}(S_2)$ contains some element S, then it contains some subset $B_{P,U,\epsilon,k}(S)$. Indeed, one can just take $P := P_1 \cap P_2$, $U := U_1 \cup U_2$, $k := \max\{k_1, k_2\}$, and $\epsilon > 0$ to be sufficiently small. □

We will mention some mostly obvious remarks. First, $\mathcal{E}$ contains disconnected surfaces, and also the empty surface $\emptyset$ and any finite sets of points tautologically. Taking $P = \emptyset$ in (1.2), we know $\{\emptyset\}$ is an open subset of $\mathcal{E}$. However, $\emptyset \in \mathcal{E}$ is not an isolated point as for any $p \in \mathbb{B}^3$, all open neighborhoods of $\{p\}$ in $\mathcal{E}$ has $\emptyset$ as an element tautologically. Similarly, for any distinct points $p_1, p_2 \in \mathbb{B}^3$, all open neighborhoods of $\{p_1, p_2\}$ in $\mathcal{E}$ has $\{p_1\}$ as an element, but not vice versa. Moreover, for any $p \in \mathbb{B}^3$, let $B_r(p) \subset \mathbb{B}^3$ be the ball centered at p with radius r. Then for $n \geq 2$, $\partial B_{1/n}(p) \in \mathcal{E}$ and converge to $\{p\}$ (not $\emptyset$) as $n \to \infty$. Furthermore, the path $r \mapsto \partial B_r(0)$ of spheres in $\mathcal{E}$ for $r \in (0, 2)$ is not well-defined at $r = 1$, but by perturbing the spheres to ellipsoids, the path becomes well-defined and continuous.

Simon-Smith min-max theory. Let M be a compact oriented Riemannian 3-manifold with strictly mean convex boundary.

Definition 4.1.2. Let X be a compact k-dimensional cubical complex, called the *parameter space*. Suppose we have a map Φ assigning to each $x \in X$ a closed subset $\Phi(x)$ of M such that:

(1) There exists a dense subset $Y \subset X$ of parameters such that:

- For each $x \in Y$, $\Phi(x)$ is an oriented, smooth, and properly embedded surface with boundary.
- For each $x \in X \backslash Y$, there exists a finite set $P(x)$ such that $\Phi(x) \backslash P(x)$ is a smooth and properly embedded surface with boundary.

Moreover, we require that $|P(x)|$ is bounded independent of x. (We can say $P(x) = \emptyset$ for $x \in Y$ for convenience.)

(2) Φ is continuous in the varifold topology.

(3) For any $x_0 \in X$ and open set $U \subset\subset M \backslash P(x_0)$ (i.e. $\overline{U} \subset M \backslash P(x_0)$), $\Phi(x) \to \Phi(x_0)$ in the graphical C^∞-topology in U whenever $x \to x_0$.

(4) $\Phi(x)$ has genus at most g for each $x \in Y$.

Then we call Φ a *smooth family of surfaces with genus at most* g, or in brief, a *genus* $\leq g$ *smooth family.*

Note that when $\Phi(x)$ is disconnected, its genus is defined as the sum of the genus of each of its connected components. For (3), $\Phi(x_0)$ meets $\partial \mathbb{B}^3$ transversely in U, thus the graphical convergence makes sense even near the boundary $\partial\Phi(x_0)$. Moreover, we required continuity in the varifold topology (see [CFS20, Fra21]) instead of the Hausdorff topology because we want to allow a smooth family to contain empty sets: We will explain more about the minor variations between our definition of a smooth family and others' later in the proof of Theorem 4.1.3.

Two smooth families Φ and Φ' parametrized by X are said to be *homotopic* if there exists a map $\psi \in C^\infty(X \times M, M)$ such that $\psi(x, \cdot) \in \mathrm{Diff}_0(M)$ for each x (meaning each

$\psi(x,\cdot)$ is homotopic via diffeomorphisms to the identity map), and $\psi(x, \Phi(x)) = \Phi'(x)$ for each x. Given a homotopy class Λ, its *width* is defined by

$$\mathbf{L}_{\mathrm{SS}}(\Lambda) := \inf_{\Phi\in\Lambda} \max_{x\in X} \operatorname{area}(\Phi(x)).$$

A sequence $\{\Phi_i\}$ in Λ is said to be *minimizing* if

$$\lim_{i\to\infty} \max_{x\in X} \operatorname{area}(\Phi_i(x)) = \mathbf{L}_{\mathrm{SS}}(\Lambda).$$

If $\{\Phi_i\}$ is a minimizing sequence and we pick x_i such that

$$\lim_{i\to\infty} \operatorname{area}(\Phi_i(x_i)) = \mathbf{L}_{\mathrm{SS}}(\Lambda),$$

then $\{\Phi_i(x_i)\}$ is called a *min-max sequence*. Furthermore, a minimizing sequence is *pulled-tight* if all its min-max sequences approach the set of stationary varifolds in the varifold topology.

Theorem 4.1.3. *Let Λ be a homotopy class of genus $\leq g$ smooth families parametrized by X. Then there exists a pulled-tight minimizing sequence in Λ, which contains a min-max sequence converging in the varifold topology to some varifold $V = \sum_{i=1}^{N} n_i\Gamma^i$, in which Γ^i are disjoint embedded free boundary minimal surfaces and n_i are positive integers, such that:*

- $\|V\| = \mathbf{L}_{\mathrm{SS}}(\Lambda)$.
- $\operatorname{index}(\operatorname{spt}(V)) \leq \dim(X)$.
- $\displaystyle\sum_{\Gamma^i \text{ orientable}} \operatorname{genus}(\Gamma^i) + \frac{1}{2}\sum_{\Gamma^i \text{ non-orientable}} (\operatorname{genus}(\Gamma^i) - 1) \leq g.$

Proof. It suffices to prove the following statements:

(1) There exists in Λ a pulled-tight minimizing sequence $\{\Phi_n\}$.

(2) There exists a function $r : M \to \mathbb{R}_{>0}$ and a min-max sequence $\{\Phi_n(x_n)\}$ of the minimizing sequence above such that:

- For every $p \in M$, in every annulus centered at p with outer radius at most $r(p)$, $\Phi_n(x_n)$ is $1/n$–*almost minimizing* (see [CDL03, Definition 3.2]) when n is large enough.
- In any such annulus, $\Phi_n(x_n)$ is smooth when n is large enough.
- $\Phi_n(x_n)$ converges to a stationary varifold V.

(3) V has the desired form $\sum_{i=1}^{N} n_i \Gamma^i$ mentioned above.

(4) The index bound.

(5) The genus bound.

Item (1) follows from the pull-tight procedure in [CDL03, §4] of Colding-De Lellis. Item (2) follows from [CDL03, Proposition 5.1] and its multi-parameter version [CGK18, Appendix] by Colding-Gabai-Ketover. For the adaptation to the case of manifold with boundary, see [Li15] by Li and [Fra21] by Franz. Note the following differences between our setting and previous ones. First, our parameter space X is a cubical complex instead of a cube, but we can embed it into some cube of high dimension so that the same proofs work. Second, even though unlike in [CDL03] we are doing *non-relative* min-max theory, as we allow a homotopy to vary a smooth family on the *boundary* of its parameter space, the same argument of [CDL03] is still applicable (in the Almgren-Pitts setting, the non-relative version was carried out in [MN17]). Third, in our definition of a smooth family Φ, we allow the set $P(x)$ of singularities of $\Phi(x)$ to vary as x varies. However, we can still ensure each $\Phi_n(x_n)$ to be smooth in any small annulus described in (2). This is because, by passing to a subsequence, we can assume that $P(x_n)$ converges as $n \to \infty$ to some finite set P in the Hausdorff topology, so that our claim follows immediately by choosing $r(p)$ to be small enough (see the last paragraph of [CDL03, §5]).

As for item (3), the regularity of V is due to [CDL03, Theorem 7.1] for the closed case and [Li15, Proposition 4.11] for the free boundary case: Notice that we have assumed ∂M to be strictly mean convex, which via the maximum principle guarantees that the interior of V does not touch ∂M. As for the index bound, it was first proven in the Almgren-Pitts setting, by Marques-Neves [MN16] in the closed case and Guang-Li-Wang-Zhou

[GLWZ21] in the free boundary case, and then adapted to the Simon-Smith setting by Franz [Fra21]. Finally, the genus bound is due to [Li15, Theorem 9.1] by Li, based on [DLP10, Theorem 1.6] by De Lellis-Pellandini. We note that although the set $X\backslash Y$ of parameters that give non-smooth surfaces may not be finite, the proof of the genus bound (in [DLP10, §2.3]) using Simon's lifting lemma [DLP10, Proposition 2.1] is still valid using the fact that the complement $(X\backslash Y)^c = Y$ is dense by assumption. (See also [Ket16b, Ket19] by Ketover, which provide a stronger genus bound for limits of min-max sequences of smooth surfaces.) □

Almgren-Pitts min-max theory. Let M be a compact $(n+1)$-dimensional Riemannian manifold with boundary. Let $\mathcal{R}_k(M;\mathbb{Z}_2)$ (resp. $\mathcal{R}_k(\partial M;\mathbb{Z}_2)$) be the set of k-dimensional rectifiable currents in M (resp. ∂M) with $\mathbb{Z}_2$-coefficients. For any $T \in \mathcal{R}_k(M;\mathbb{Z}_2)$ such that its support lies in ∂M, we define an equivalence relation by $T \sim S$ if $T - S \in \mathcal{R}_k(\partial M;\mathbb{Z}_2)$, and then denote by $\mathcal{Z}_k(M,\partial M;\mathbb{Z}_2)$ the set of such equivalence classes. The three common topologies on $\mathcal{Z}_k(M,\partial M;\mathbb{Z}_2)$ are given by the *flat* metric $\mathcal{F}$, the $\mathbf{F}$-metric, and the *mass* $\mathbf{M}$ respectively: Since the definitions are standard, we refer the reader to, for example, [GLWZ21, §3] for the details. Note that by [GLWZ21, §3.3], under the metric $\mathcal{F}$ or $\mathbf{M}$, $\mathcal{Z}_k(M,\partial M;\mathbb{Z}_2)$ is homeomorphic and isometric to the space of *relative k-cycles* considered in [LMN18, §2.2].

Then by the Almgren isomorphism theorem [Alm62] (see also [LMN18, §2.5]), if $H_n(M,\partial M;\mathbb{Z}_2) = 0$, then when equipped with the flat topology, $\mathcal{Z}_n(M,\partial M;\mathbb{Z}_2)$ is connected and weakly homotopic equivalent to $\mathbb{RP}^\infty$. Thus we can denote its cohomology ring by $\mathbb{Z}_2[\bar{\lambda}]$. Then an $\mathcal{F}$-continuous map $\Phi : X \to \mathcal{Z}_n(M,\partial M;\mathbb{Z}_2)$, where X is some cubical complex, is said to be a *k-sweepout* if $\Phi^*(\bar{\lambda}^k) \neq 0$. Let $\mathcal{P}_k$ be the set of all $\mathbf{F}$-continuous k-sweepouts. Then, denoting by $\mathrm{dmn}(\Phi)$ the domain of Φ, the *k-width* of M is defined by

$$\omega_k(M) := \inf_{\Phi\in\mathcal{P}_k} \max_{x\in\mathrm{dmn}(\Phi)} \mathbf{M}(\Phi(x)).$$

Remark 4.1.4. There is an equivalent characterization of k-sweepouts (see [MN17, Definition 4.1]): An $\mathcal{F}$-continuous map $\Phi : X \to \mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ is a k-sweepout if there exists an $\lambda \in H^1(X; \mathbb{Z}_2)$ such that:

- λ *detects the 1-sweepouts*, i.e. for any cycle $\gamma : S^1 \to X$, we have $\lambda(\gamma) \neq 0$ if and only if $\Phi \circ \gamma : S^1 \to \mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ is a 1-sweepout.
- The cup product $\lambda^k \in H^k(X; \mathbb{Z}_2)$ is non-zero.

4.2. **Proofs of Main results.** In this section, we prove the results stated in §1.3. From now on, we denote by $\mathcal{Z}$ the space $\mathcal{Z}_2(\mathbb{B}^3, \partial\mathbb{B}^3; \mathbb{Z}_2)$ with the flat topology.

Proof of Theorem 1.3.2. We will construct the desired family Ψ that satisfies condition (A), (B), and (C) of Theorem 1.3.2 in two steps: In step 1 we construct a 6-sweepout (condition (A)). Then in step 2, we modify it such that it becomes, in addition, a genus ≤ 1 smooth family (condition (B)) with maximal area less than 2π (condition (C)).

Step 1. We consider all scalings and translations of the saddle surface $\{x^2 - y^2 + z = 0\}$ in $\mathbb{R}^3$, and then intersect them with $\mathbb{B}^3$. Namely, we define a map $\Phi_4 : \mathbb{RP}^4 \to \mathcal{Z}$ by assign to each $a = [a_0 : a_1 : a_2 : a_3 : a_4] \in \mathbb{RP}^4$ the zero set of the polynomial

$$p_{a_0,a_1,a_2,a_3,a_4}(x, y, z) := a_0(x^2 - y^2) + a_1 x + a_2 y + a_3 z + a_4$$

in $\mathbb{B}^3$. And then we add in rotations. Namely, we define $\widetilde{\Phi}_7 : \mathbb{RP}^4 \times SO(3) \to \mathcal{Z}$ by assigning each (a, Q) to the surface "$\Phi_4(a)$ rotated by Q^{-1}", i.e. the zero set of the polynomial $p_{a_0,a_1,a_2,a_3,a_4}(Q(x, y, z))$ in $\mathbb{B}^3$.

However, a loop in the $SO(3)$ factor does not produce a 1-sweepout (e.g. consider a disk rotating for 360°), and $\widetilde{\Phi}_7$ is not yet a 6-sweepout. To get a 6-sweepout, one needs to take a quotient on the space $\mathbb{RP}^4 \times SO(3)$ as follows.

We first observe $\{x^2 - y^2 + z = 0\}$ has a dihedral symmetry: Let

$$g_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } g_2 := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

They are the 180°-rotation about the line $\{z = 0, x = y\}$ and $\{z = 0, x = -y\}$ respectively. Then $D_2 := \{\text{id}, g_1, g_2, g_1g_2\}$ is a dihedral group, which preserves $\{x^2 - y^2 + z = 0\}$. Motivated by this, we define a D_2-action on $\mathbb{RP}^4$ by

$$g_1[a_0 : a_1 : a_2 : a_3 : a_4] := [-a_0 : a_2 : a_1 : -a_3 : a_4],$$

$$g_2[a_0 : a_1 : a_2 : a_3 : a_4] := [-a_0 : -a_2 : -a_1 : -a_3 : a_4],$$

and then a D_2-action on $\mathbb{RP}^4 \times SO(3)$ by

$$\begin{aligned} g_1(a, Q) &:= (g_1 a, g_1 Q), \\ g_2(a, Q) &:= (g_2 a, g_2 Q). \end{aligned} \tag{4.1}$$

The whole reason we define the action by (4.1) is to ensure the following:

Proposition 4.2.1. *For each $g \in D_2$ and $a \in \mathbb{RP}^4$, $g^{-1}(\Phi_4(g(a))) = \Phi_4(a)$.*

Proof. The proof is straightforward: Let $a = [a_0 : a_1 : a_2 : a_3 : a_4]$. Then

$$g_1^{-1}(\Phi_4(g_1(a))) = \{-a_0(y^2 - x^2) + a_2 y + a_1 x - a_3(-z) + a_4 = 0\} \cap \mathbb{B}^3,$$

$$g_2^{-1}(\Phi_4(g_2(a))) = \{-a_0((-y)^2 - (-x)^2) - a_2(-y) - a_1(-x) - a_3(-z) + a_4 = 0\} \cap \mathbb{B}^3,$$

which are both the same as $\Phi_4(a)$. □

As an immediate result, $\widetilde{\Phi}_7(g(a, Q)) = \widetilde{\Phi}_7(a, Q)$ for all g, a, and Q, and hence one can pass to the quotient space to define a new collection $\Phi_7 : \frac{\mathbb{RP}^4 \times SO(3)}{D_2} \to \mathcal{Z}$. Note that Φ_7 is $\mathcal{F}$-continuous clearly because it parametrizes the collection of all scalings, translations, and rotations of the saddle surface $\{x^2 - y^2 + z = 0\}$, intersected with $\mathbb{B}^3$. Then a crucial fact is:

Proposition 4.2.2. *Φ_7 is a 6-sweepout.*

The proof of Proposition 4.2.2 is a lengthy calculation of algebraic topology. We postpone it to §4.3. Hence, Φ_7 satisfies condition (A).

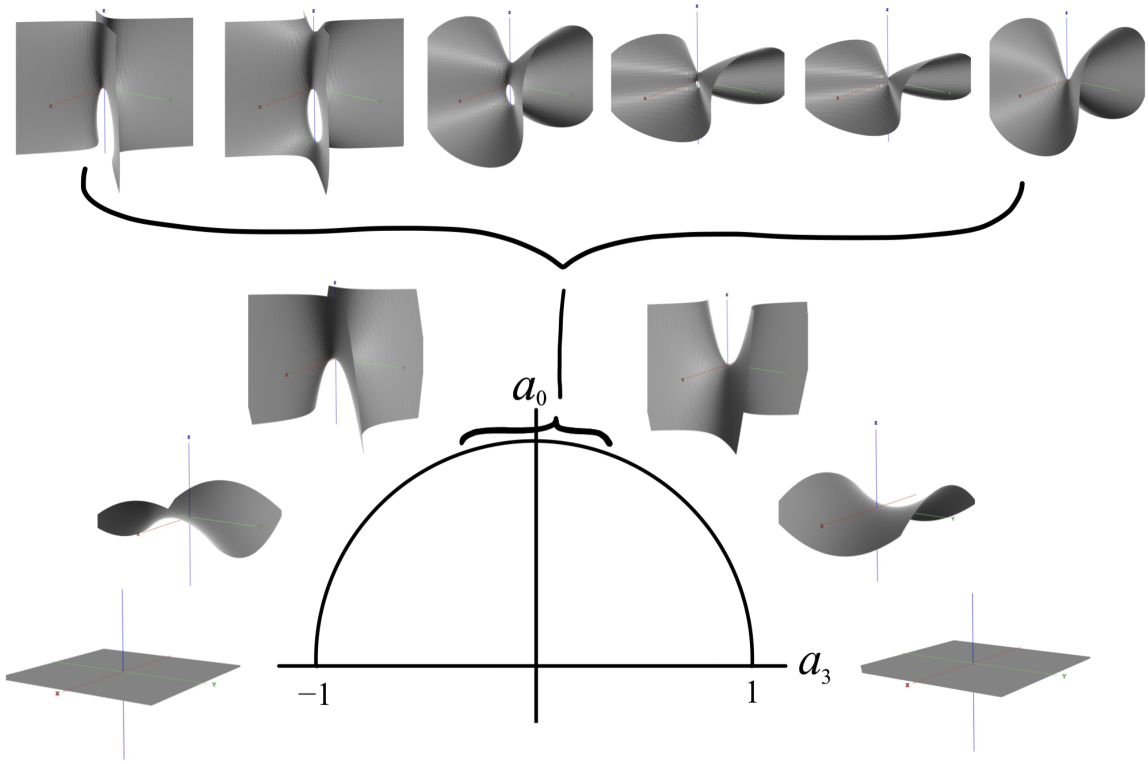


FIGURE 16. The surface $\{a_0(x^2 - y^2 + a_5 z^3) + a_3 z = 0\}$ for various a_0 and a_3, with $a_5 > 0$ small and fixed.

Step 2. However, Φ_7 is not a smooth family, as it contains intersecting disks of the form $\{(x - x_0)^2 - (y - y_0)^2 = 0\} \cap \mathbb{B}^3$. We are going to desingularize them.

For each fixed number $a_5 \geq 0$, let us define a collection $\Phi_4^{a_5} : \mathbb{RP}^4 \to \mathcal{Z}$ by assigning $[a_0 : a_1 : a_2 : a_3 : a_4]$ to the zero set of the polynomial

$$p_{a_0,a_1,a_2,a_3,a_4,a_5}(x, y, z) := a_0(x^2 - y^2 + a_5 z^3) + a_1 x + a_2 y + a_3 z + a_4 \tag{4.2}$$

in $\mathbb{B}^3$. So for small $a_5 > 0$, $\Phi_4^{a_5}$ is slight modification of Φ_4. In fact, as we will see, *the effect of the z^3 term is three-fold: Desingularizing the intersecting disks, creating genus 1 surfaces (Figure 16), and lowering the area to strictly below 2π.*

Remark 4.2.3. Let us geometrically describe the family $\Phi_4^{a_5}$. We will focus on the cubic surfaces, so without loss of generality we put $a_0 = 1$. Since a_1 and a_2 just contribute to translation, let us assume they are both 0. Now, fix some $(a_3, a_4, a_5) \in \mathbb{R}^2 \times (0, \infty)$, and let $s > 0$ varies. We claim that as s increases from 0, the surfaces

$$\Phi_4^{a_5 s}([1:0:0:a_3 s:a_4 s]) = \{x^2 - y^2 + s(a_3 z + a_4 + a_5 z^3) = 0\} \cap \mathbb{B}^3 \tag{4.3}$$

desingularize the intersecting disks $\{x^2 - y^2 = 0\} \cap \mathbb{B}^3$ *along the singular line.* Indeed, we consider the three cases: $a_3 z + a_4 + a_5 z^3$ having 1, 2, or 3 roots; let z_i's be the roots. Then in each case, as s increases from 0, the surfaces $\Phi_4^{a_5 s}([1:0:0:a_3 s:a_4 s])$ stay fixed on the coordinate planes $\{z = z_i\}$, and "open up" smoothly above, below, and in between. This is because by (4.3), at each fixed height $z \neq z_i$, the cross-section of the surface is a hyperbola, which dilates as s increases and has a distance of $\sqrt{s|a_3 z + a_4 + a_5 z^3|}$ between the two branches. In Table 1, we show some examples.

Moreover, we can study for which (a_3, a_4, a_5) the surface $\Phi_4^{a_5}([1:0:0:a_3:a_4])$ has singularities, and where they are: By solving

$$\begin{cases} p_{1,0,0,a_3,a_4,a_5} = x^2 - y^2 + a_3 z + a_4 + a_5 z^3 = 0 \\ \nabla p_{1,0,0,a_3,a_4,a_5} = (2x, -2y, a_3 + 3a_5 z^2) = (0,0,0) \end{cases}$$

we know when $a_3 z + a_4 + a_5 z^3$ has some double or triple root z_1, the surface has a singularity at $(0, 0, z_1)$.

But our goal is to modify Φ_7, not just Φ_4. To do that, first notice by a straightforward calculation that $\Phi_4^{a_5}$ satisfies a property similar to Proposition 4.2.1, namely $g^{-1}(\Phi_4^{a_5}(g(a))) = \Phi_4^{a_5}(a)$. The key idea behind is that *the polynomial* $x^2 - y^2 + a_5 z^3$ *is invariant under the* D_2*-action on* (x, y, z). As a result, we can construct a map $\Phi_7^{a_5} : \frac{\mathbb{RP}^4 \times SO(3)}{D_2} \to \mathcal{Z}$ by rotating all elements in $\Phi_4^{a_5}$, just like how we constructed Φ_7 from Φ_4 in step 1. Hence we have obtained a modification $\Phi_7^{a_5}$ of Φ_7.

Now, from Remark 4.2.3, it follows easily that $\Phi_7^{a_5}$ is $\mathbf{F}$-continuous and is homotopic in the $\mathcal{F}$-topology to Φ_7, so that it is a 6-sweepout (condition (A)). Moreover, we show that condition (B) can be satisfied:

Proposition 4.2.4. *For almost every* $a_5 \in [0, 1]$, $\Phi_7^{a_5}$ *is a genus* ≤ 1 *smooth family.*

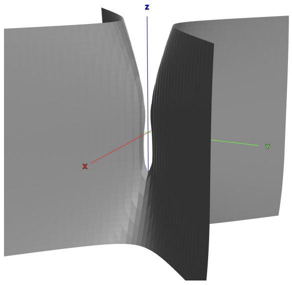
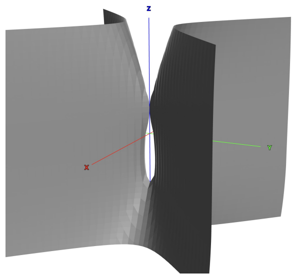
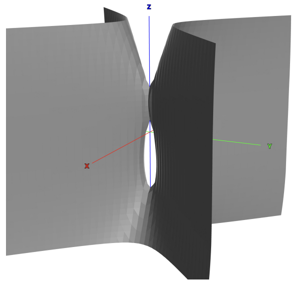
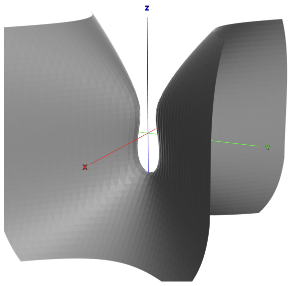
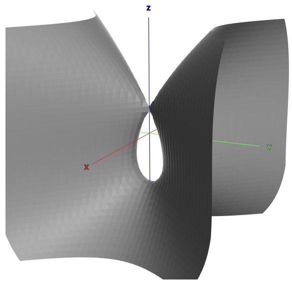
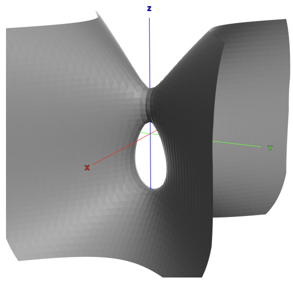
s \ b_3	0.1	$-3\sqrt[3]{\frac{1}{400}} \approx 0.407$	0.6
0.05			
0.3			

TABLE 1. The surface $\{x^2 - y^2 + s(b_3 z + 0.1 + z^3) = 0\}$ for various b_3 and s is shown above. They illustrate the three cases where the polynomial $a_3 z + a_4 + a_5 z^3$ has 1, 2, and 3 roots respectively.

Proof. By Remark 4.2.3, the subset of parameters $a \in \frac{\mathbb{RP}^4 \times SO(3)}{D_2}$ such that $\Phi_7^{a_5}(a)$ has singularities is 1-codimensional, and all such surfaces have at most one singularity. Moreover, it is straightforward to check that for all a, $\Phi_7^{a_5}(a)$ and $\partial\mathbb{B}^3$ touch (i.e. have coinciding tangent planes) at finitely many points. Also, by the transversality theorem and Remark 4.2.3, for a.e. $a_5 \in [0, 1]$, the algebraic surfaces in $\mathbb{R}^3$ that define $\Phi_7^{a_5}(a)$ (i.e. the rotations of the zero set of (4.2) in $\mathbb{R}^3$) intersect $\partial\mathbb{B}^3$ transversely for a.e. $a \in \frac{\mathbb{RP}^4 \times SO(3)}{D_2}$. So for such a_5, $\Phi_7^{a_5}$ satisfies Definition 4.1.2 (1).

Note that each smooth surface of $\Phi_4^{a_5}$ has genus 0 or 1 because they are obtained from opening up the intersecting disks $\{x^2 - y^2 = 0\} \cap \mathbb{B}^3$ above, below, and in between *at most three horizontal planes*, by Remark 4.2.3. So each smooth surface of $\Phi_7^{a_5}$ has genus 0 or 1. Now, using Remark 4.2.3, one can show that Definition 4.1.2 (2) and (3) are also satisfied by $\Phi_7^{a_5}$. So $\Phi_7^{a_5}$ is a genus ≤ 1 smooth family for a.e. a_5. $\square$

Next, we claim that for small $a_5 > 0$, $\Phi_7^{a_5}$ also satisfies condition (C), i.e. $\text{area} \circ \Phi_7^{a_5} < 2\pi$. Indeed, it suffices to show that $\text{area} \circ \Phi_4^{a_5} < 2\pi$ for small $a_5 > 0$, which follows straightforwardly from the following two propositions:

Proposition 4.2.5. *The area function* $\mathrm{area} \circ \Phi_4 : \mathbb{RP}^4 \to \mathbb{R}$ *attains a strict global maximum at* $[1:0:0:0:0]$.

Note that $\Phi_4([1:0:0:0:0])$ is $\{x^2 - y^2 = 0\} \cap \mathbb{B}^3$, which has area 2π.

Proposition 4.2.6. *Define* $\Phi_5 : \mathbb{RP}^4 \times [0,1] \to \mathcal{Z}$ *by* $\Phi_5(a, a_5) = \Phi_4^{a_5}(a)$. *Then the area function* $\mathrm{area} \circ \Phi_5$ *attains a strict local maximum at* $([1:0:0:0:0], 0)$.

The proof of Proposition 4.2.5 is due to the MathOverflow user fedja [Fed22]. We include it in Appendix A. The proof of Proposition 4.2.6 is postponed to §4.3: It uses mainly calculus but is quite technical. However, intuitively Proposition 4.2.6 makes sense because Remark 4.2.3 tells us that Φ_5 gives *a desingularization of the intersecting equatorial disks, and desingularization should lower the area as the sharp bend along the singular line is smoothed.*

Thus, for a.e. sufficiently small $a_5 > 0$, by Proposition 4.2.5 and 4.2.6, $\Phi_7^{a_5}$ satisfies condition (C) also. Defining Ψ as one such $\Phi_7^{a_5}$, we finish the proof of Theorem 1.3.2.

Proof of Theorem 1.3.1. We will first do the case of $\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$. By Theorem 1.3.2 (B) and (C), we can view the family Ψ in Theorem 1.3.2 as the composition of a map $\Psi' : \frac{\mathbb{RP}^4 \times SO(3)}{D_2} \to \overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$ and the natural map i from $\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$ to the space of 2-cycles $\mathcal{Z}_2(\mathbb{B}^3, \partial\mathbb{B}^3; \mathbb{Z}_2)$ equipped with the flat topology (see §4.1). Note that Ψ' is continuous (under the topology of $\mathcal{E}$ defined in §1.3) since Ψ is a smooth family by Theorem 1.3.2 (B), and i is continuous by the fact that the smooth convergence of surfaces is stronger than the flat convergence.

By Almgren isomorphism theorem (see §4.1), we denote the cohomology ring of $\mathcal{Z}_2(\mathbb{B}^3, \partial\mathbb{B}^3; \mathbb{Z}_2)$ in $\mathbb{Z}_2$-coefficients as $\mathbb{Z}_2[\bar{\lambda}]$. To prove Theorem 1.3.1 for the space $\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$, it suffices to show that $(i^*\bar{\lambda})^6 \neq 0$. Thus, it suffices to show $(i \circ \Psi')^*(\bar{\lambda}^6) \neq 0$, i.e. Ψ is a 6-sweepout, which is true by Theorem 1.3.2.

Reusing the argument above with $\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$ replaced by any subspace of $\mathcal{E}^\infty$ that contains $\overline{\mathcal{E}_0 \cup \mathcal{E}_1} \cap \mathcal{E}^{2\pi}$, we finish the proof of Theorem 1.3.1. (Note that elements in $\mathcal{E}^\infty$ have finite area and thus belong to $\mathcal{Z}_2(\mathbb{B}^3, \partial\mathbb{B}^3; \mathbb{Z}_2)$.)

Proof of Corollary 1.3.3. Let Ψ be the smooth family in Theorem 1.3.2. Since area $\circ\, \Psi < 2\pi$ and Ψ is a 6-sweepout by Theorem 1.3.2, $\omega_6(\mathbb{B}^3) < 2\pi$.

Proof of Theorem 1.3.4. Let Ψ be the family satisfying condition (A), (B), and (C) of Theorem 1.3.2, and $\Psi^{(5)}$ be the subfamily of Ψ parametrized by a 5-skeleton of the parameter space of Ψ. Without loss of generality, we can assume that $\Psi^{(5)}$ is also a smooth family. Now, since Ψ is a 5-sweepout by (A), so is $\Psi^{(5)}$ (see the proof of [MN21, Proposition 7.1]). It follows that the width $L := \mathbf{L}_{\mathrm{SS}}(\Lambda(\Psi^{(5)}))$ is positive.

Now, we apply the Simon-Smith min-max theory to $\Psi^{(5)}$. Let $\{\Phi_i\}$ be a pulled-tight minimizing sequence of $\Lambda(\Psi^{(5)})$. Denote by $\mathcal{W}$ the set of all stationary integral varifolds in $\mathbb{B}^3$ whose support is a smooth embedded free boundary minimal hypersurface, and by $\mathbf{C}(\{\Phi_i\})$ the set of subsequential varifold limits of min-max sequences of $\{\Phi_i\}$. Then by Theorem 4.1.3, $\mathbf{C}(\{\Phi_i\}) \cap \mathcal{W}$ is non-empty. Now, there are three cases: $\mathbf{C}(\{\Phi_i\}) \cap \mathcal{W}$ contains (1) some element Γ that is not the equatorial disk or the critical catenoid; (2) only critical catenoids; or (3) only equatorial disks (note that critical catenoids and equatorial disks cannot appear together in $\mathbf{C}(\{\Phi_i\}) \cap \mathcal{W}$ as they have different area). We will consider each case individually in the following.

Case (1). We will show that Γ has the desired property stated in Theorem 1.3.4:

Proposition 4.2.7. *Γ has multiplicity 1, genus 0 or 1, Morse index 4 or 5, and area in the range $(\pi, 2\pi)$.*

Proof. First, by Theorem 4.1.3, area(Γ) when counted with possible multiplicities is equal to $\mathbf{L}_{\mathrm{SS}}(\Lambda(\Psi^{(5)}))$, which is less than 2π by (C). Then, since the least possible area of a free boundary minimal surface in $\mathbb{B}^3$ is π by a result of Fraser-Schoen [FS11, Theorem 5.4], Γ must have multiplicity 1. Moreover, by Theorem 4.1.3, (B) implies genus(Γ) ≤ 1, and index(Γ) ≤ 5 since the parameter space of $\Psi^{(5)}$ is 5-dimensional. Lastly, since Γ is not the equatorial disk, we have area(Γ) $> \pi$ again from [FS11] (and also [Bre12] by Brendle), and index(Γ) ≥ 4 from [Dev19, §5] by Devyver or [Tra20, §3.1] by Tran. □

Hence, case (1) is done.

Case (2). Now we turn to case (2). We will use a technique called *splitting of domains.* First, let $\mathcal{C}$ denote the set of critical catenoids: Note that $\mathcal{C}$ is homeomorphic to $\mathbb{RP}^2$. Fixing a small $\epsilon > 0$, and denoting by W the parameter space of $\Psi^{(5)}$, we consider the set of parameters

$$\{x \in W : \mathbf{F}(\Phi_i(x), \mathcal{C}) \leq \epsilon\}.$$

This subset, after a slight thickening, can be assumed to be a cubical complex; we will denote it by Z_i, and then $\overline{W \backslash Z_i}$ by Y_i. Now, as we mentioned $\Psi^{(5)}$ is a 5-sweepout, hence so is Φ_i. Then using a topological argument by Lusternik-Schnirelmann [LS47] (see also [MN17, Claim 6.3]), we know that either $\Phi_i|_{Z_i}$ is a 1-sweepout or $\Phi_i|_{Y_i}$ is a 4-sweepout. Now note that, if ϵ is small enough, $\Phi_i|_{Z_i}$ lies near $\mathcal{C}$ and thus is homotopic in the $\mathcal{F}$-topology to some $\mathbf{M}$-continuous map into $\mathcal{C}$ (using [Nur16, §3.3.6] by Nurser, together with discretization and interpolation theorems in the free boundary setting by Li-Zhou [LZ21, §4.2]). But no map into $\mathcal{C}$ can be a 1-sweepout as $\mathcal{C}$ can be contracted to just $\{\emptyset\}$, by shrinking each critical catenoid to its axis, which has no mass. Hence, $\Phi_i|_{Z_i}$ cannot be a 1-sweepout, and so *each $\Phi_i|_{Y_i}$ must be a 4-sweepout.*

Now, we claim that *for some i, $\Phi_i|_{Y_i}$ is homotopic (in the Simon-Smith setting) to another smooth family $\widetilde{\Psi}$ with maximal area less than L.* Indeed, if not, then by standard Simon-Smith min-max theory, there exists y_i such that $\Phi_i|_{Y_i}(y_i)$ converges subsequentially to some smooth embedded free boundary minimal surface V with mass L (see [CDL03, §5] and [CGK18, Appendix]: Their arguments apply here because even though our parameter spaces Y_i depend on i, they can all be embedded into some $\mathbb{R}^N$ with N independent of i). Then note two facts: V cannot be the critical catenoid by the definition of Y_i, and $V \in \mathbf{C}(\{\Phi_i\}) \cap \mathcal{W}$ clearly. However, these two facts are contradictory because we are in case (2). Thus, the desired smooth family $\widetilde{\Psi}$ exists.

We then apply the Theorem 4.1.3 to $\Lambda(\widetilde{\Psi})$, and repeat the argument above. Namely, letting $\{\widetilde{\Phi}_i\}$ be a pulled-tight minimizing sequence of $\Lambda(\widetilde{\Psi})$, there are two cases: $\mathbf{C}(\{\widetilde{\Phi}_i\}) \cap$

$\mathcal{W}$ either contains (2a) some element $\widetilde{\Gamma}$ that is not the equatorial disk or the critical catenoid, or (2b) only equatorial disks. The critical catenoid, having area L, cannot appear because area $\circ\, \widetilde{\Psi} < L$.

If it is case (2a), one can reapply the proof of Proposition 4.2.7 to $\widetilde{\Gamma}$ to show that $\widetilde{\Gamma}$ has the desired properties in Theorem 1.3.4 (this time index$(\widetilde{\Gamma})$ is actually 4), and we are done. If it is case (2b), we *split the domains* again to arrive at a contradiction. This time the key ideas are: There is no 3-sweepout near the set of equatorial disks, which is merely an $\mathbb{RP}^2$; and there is no 1-sweepout with maximal area less than π, which is the least possible area for a free boundary minimal surface. Therefore case (2b) is impossible. Now case (2) is also done.

Case (3). Case (3) is entirely analogous to case (2b).

So we have finished the proof of Theorem 1.3.4.

Explanation of Remark 1.3.5. Letting $a_5 > 0$ be sufficiently small, we define the family $\Phi_1 : \mathbb{RP}^1 \to \mathcal{Z}$ of D_2-symmetric surfaces:

$$\Phi_1([a_0 : a_3]) := \Phi_4^{a_5}([a_0 : 0 : 0 : a_3 : 0]) = \{a_0(x^2 - y^2 + a_5 z^3) + a_3 z = 0\} \cap \mathbb{B}^3$$

(see Figure 16). Note that area $\circ\, \Phi_1 < 2$ as area $\circ\, \Phi_4^{a_5} < 2$ by Theorem 1.3.2 (C). Then applying the equivariant Simon-Smith min-max theorem to Φ_1, we obtain a free boundary minimal surface. To show it has the desired properties mentioned in Remark 1.3.5, we just proceed in a way similar to [CFS20, §4]. In particular, we can use the proof of [CFS20, Lemma 4.1] to show that the number of boundary component of the minimal surface obtained is one. Indeed, we first note that for each surface $\Phi_1(a)$, the complement of the three axes of rotations (the z-axis, $\{z = 0, x = y\}$, and $\{z = 0, x = -y\}$) in $\Phi_1(a)$ are topological disks. And this fact is what one need to carry out the proof of [CFS20, Lemma 4.1].

4.3. **Technical Ingredients.** In this section, we prove Proposition 4.2.2 and 4.2.6.

Proof of Proposition 4.2.2. Throughout §4.3, we write $X := \frac{\mathbb{RP}^4 \times SO(3)}{D_2}$. The proof has four steps. In step 1 we compute $H_1(X;\mathbb{Z}_2)$, and understand which first homology classes give 1-sweepouts under Φ_7. In step 2 we find the cohomology class $\lambda \in H^1(X;\mathbb{Z}_2)$ that *detects the 1-sweepouts* (as explained Remark 4.1.4), and understand its Poincaré dual. In step 3 we show that $\lambda^6 \neq 0$. And in step 4, we prove a technical lemma used in step 3. By Remark 4.1.4, we immediately obtain the desired claim that Φ_7 is a 6-sweepout.

Step 1. Let us first find $\pi_1(X)$. Let Q_8 be the *quaternion group* $\{\pm 1, \pm i, \pm j, \pm k\}$, contained in the group S^3 of unit quaternions.

Lemma 4.3.1. $\pi_1(X) = \mathbb{Z}_2 \times Q_8$.

Proof. First, the universal cover of $\mathbb{RP}^4 \times SO(3)$ is $S^4 \times S^3$; in fact, without loss of generality one may assume the double covering $S^3 \to SO(3)$ maps $\pm i$ to g_1 and $\pm j$ to g_2, and thus Q_8 to D_2. Then to prove the lemma, it suffices to construct a $\mathbb{Z}_2 \times Q_8$-action on $S^4 \times S^3$ that descends, under the projections $S^4 \times S^3 \to \mathbb{RP}^4 \times SO(3)$ and $\mathbb{Z}_2 \times Q_8 \to 1 \times D_2$, to the D_2-action on $\mathbb{RP}^4 \times SO(3)$ defining X.

First, we define a Q_8-action on $S^4 \times S^3$ by

$$(\pm i) \cdot ((a_0, a_1, a_2, a_3, a_4), q) := ((-a_0, a_2, a_1, -a_3, a_4), \pm iq),$$

$$(\pm j) \cdot ((a_0, a_1, a_2, a_3, a_4), q) := ((-a_0, -a_2, -a_1, -a_3, a_4), \pm jq).$$

Then, let $\mathbb{Z}_2$ act on $S^4 \times S^3$ by acting antipodally on *only the S^4 factor*. After checking these two actions commute, we obtain a $\mathbb{Z}_2 \times Q_8$-action on $S^4 \times S^3$, and it is straightforward to check that this action has the desired property. □

Now, abelianizing $\pi_1(X) = \mathbb{Z}_2 \times Q_8$, we have $H_1(X;\mathbb{Z}) = \mathbb{Z}_2 \times D_2$, which then by the universal coefficient theorem gives $H_1(X;\mathbb{Z}_2) = \mathbb{Z}_2 \times D_2$. In fact, some first homology classes can be described explicitly as follows. Denote $e_0 := (1, 0, 0, 0, 0)$, and let $\widetilde{x}_0 :=$

$(e_0, 1)$ be the base point in $S^4 \times S^3$. Then consider the path

$$\{((a_0, 0, 0, \sqrt{1 - a_0^2}, 0), 1) : -1 \leq a_0 \leq 1\}$$

in $S^4 \times S^3$ joining $\widetilde{x}_0$ to $(-e_0, 1)$, a path joining $\widetilde{x}_0$ to (e_0, i), and a path joining $\widetilde{x}_0$ to (e_0, j). Call the projection of these three paths onto X, which are actually loops, c_1, c_2 and c_3 respectively. Then $[c_1] = (1, \mathrm{id})$, $[c_2] = (0, g_1)$, and $[c_3] = (0, g_2)$ in $H_1(X; \mathbb{Z}_2) = \mathbb{Z}_2 \times D_2$, and hence they form a base.

Lemma 4.3.2. $(1, \mathrm{id}), (0, g_1), (0, g_2), (1, g_1 g_2)$ *are exactly the homology classes that give 1-sweepouts under* Φ_7.

Proof. It suffices to show that $(1, \mathrm{id}), (0, g_1), (0, g_2)$ give 1-sweepouts. To show that $(1, \mathrm{id})$ gives a 1-sweepout, note that

$$\Phi_7 \circ c_1 = \Phi_4(\{[a_0 : 0 : 0 : a_3 : 0] : a_0^2 + a_3^2 = 1\}).$$

But in $\mathbb{RP}^4$ the loop $\{a_0^2 + a_3^2 = 1\}$ is homotopic to $\{a_3^2 + a_4^2 = 1\}$, which under Φ_4 gives the collection of all horizontal planes together with the empty set, and that certainly is a 1-sweepout. So $\Phi_7 \circ c_1$ is a 1-sweepout.

To show that $(0, g_1)$ gives a 1-sweepout, note that $\Phi_7 \circ c_2$ gives the motion of rotating the intersecting planes $\{x^2 - y^2 = 0\}$ about the axis $\{z = 0, x = y\}$ by $180°$. Let us call $\{|y| > |x|\}$ the inside region of $\{x^2 - y^2 = 0\}$, and $\{|y| < |x|\}$ the outside. Then the rotation *switches the inside and the outside*. So $\Phi_7 \circ c_2$ is a 1-sweepout.

The proof that $(0, g_2)$ gives a 1-sweepout is similar. □

Step 2. By the universal coefficient theorem, $H^1(X; \mathbb{Z}_2) = \mathrm{Hom}(H_1(X; \mathbb{Z}_2), \mathbb{Z}_2)$. Since $[c_1], [c_2], [c_3]$ form a base of $H_1(X; \mathbb{Z}_2)$, we can define respectively their Hom-duals $\lambda_i := [c_i]^* \in H^1(X; \mathbb{Z}_2)$ for $i = 1, 2, 3$. Let $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, then by Lemma 4.3.2, λ is the cohomology class that detects exactly the 1-sweepouts. Hence, to prove Proposition 4.2.2, we need $\lambda^6 \neq 0$. We are going to prove this by considering the Poincaré dual of λ^6, so let us first understand the Poincaré dual $PD(\lambda_i) \in H_6(X;\mathbb{Z}_2)$ of λ_i.

In the remaining of §4.3, *we will view X as an $\mathbb{RP}^4$-bundle over the base $B := SO(3)/D_2$*, and let $p : X \to B$ be the projection. Let A_0 be the 6-dimensional sub-bundle of X over B on which $a_0 = 0$. Note that A_0 is well-defined because the subset $\{a_0 = 0\}$ of $\mathbb{RP}^4 \times SO(3)$ is D_2-invariant.

Lemma 4.3.3. *$PD(\lambda_1) = [A_0]$ in $H_6(X;\mathbb{Z}_2)$.*

Proof. This is because the loop c_1 intersects A_0 at only one point in X, $D_2 \cdot ([0:0:0:1:0], \mathrm{id})$. □

To construct $PD(\lambda_2 + \lambda_3)$, we will need to know the cohomology groups of B, which is S^3/Q_8. We quote the result [TZ08, Theorem 2.2 (1)] of Tomoda-Zvengrowsk:

Proposition 4.3.4. *The cohomology ring $H^*(S^3/Q_8;\mathbb{Z}_2)$ is given by*

$$\mathbb{Z}_2[\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3]/\sim,$$

in which the subscript of each generator denotes its degree, and $\sim$ denotes the following equivalence:

$$\alpha_1^2 = \alpha_2 + \alpha_2',\ \alpha_1\alpha_1' = \alpha_2',\ (\alpha_1')^2 = \alpha_2,$$

$$\alpha_1\alpha_2 = \alpha_1\alpha_2' = \alpha_1'\alpha_2' = \alpha_3,\ \alpha_1'\alpha_2 = 0,$$

products of cohomology classes with total degree greater than 3 is 0.

Now, from the definition of c_2 and c_3, we know $p \circ c_2$ and $p \circ c_3$ form a base in $H_1(B;\mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Remembering * denotes the Hom-dual, we let b_2, b_3 be 2-dimensional submanifolds in B such that

$$[b_2] = PD([p \circ c_2]^*), [b_3] = PD([p \circ c_3]^*) \in H_2(B;\mathbb{Z}_2).$$

We moreover assume b_2 and b_3 intersect transversely, and define $b := b_2 \cup b_3$. Let $X|_{b_2}$ be the restriction of the $\mathbb{RP}^4$-bundle X over the base b_2, and similarly for $X|_{b_3}$ and $X|_b$. Note that they are 6-dimensional.

Lemma 4.3.5. *In $H_6(X; \mathbb{Z}_2)$ we have:*

(1) $PD(\lambda_2) = [X|_{b_2}]$.

(2) $PD(\lambda_3) = [X|_{b_3}]$.

(3) $PD(\lambda_2 + \lambda_3) = [X|_b]$.

Proof. Since λ_2 is the Hom-dual of $[c_2]$, to show that $[X|_{b_2}] = PD(\lambda_2)$, it suffices to show that the intersections number of $X|_{b_2}$ with c_1, c_2 and c_3 respectively are $0, 1$, and 0. Indeed, this is true because, respectively, b_2 can be perturbed to avoid the point $p \circ c_1$ (in B), the intersection number of b_2 with $p \circ c_2$ is 1, and the intersection number of b_2 with $p \circ c_3$ is 0.

Similarly, one can show $PD(\lambda_3) = [X|_{b_3}]$, and thus $PD(\lambda_2 + \lambda_3) = [X|_{b_2}] + [X|_{b_3}] = [X|_b]$. □

Step 3. Note that

$$\lambda^6 = \lambda_1^6 + \lambda_1^4(\lambda_2 + \lambda_3)^2 + \lambda_1^2(\lambda_2 + \lambda_3)^4 + (\lambda_2 + \lambda_3)^6.$$

To show $\lambda^6 \neq 0$, it suffices to show the following lemma.

Lemma 4.3.6. *In the cohomology ring $H^*(X; \mathbb{Z}_2)$ we have:*

(1) $(\lambda_2 + \lambda_3)^3 = 0$.

(2) $\lambda_1^4(\lambda_2 + \lambda_3)^2 = 1$.

(3) $\lambda_1^5 = 0$.

Proof. To prove (1), it suffices to perturb three copies of $X|_b$ and show that their intersection is empty. To achieve this, we can just perturb three copies of the base $b \subset B$. But their intersection number will be 0, because by Proposition 4.3.4 any element of $H^1(B; \mathbb{Z}_2)$ cubes to 0. Hence we have proven (1).

To prove (2) and (3), we need to find different representatives of $[A_0]$. Let A_1, A_2, A_3, and A_4 be the subbundle of X over B on which $a_1 = a_2$, $a_1 = -a_2$, $a_3 = 0$, and $a_4 = 0$ respectively.

Lemma 4.3.7. *In $H_6(X;\mathbb{Z}_2)$ we have:*

(1) $[A_1] = [A_0] + [X|_{b_3}]$.

(2) $[A_2] = [A_0] + [X|_{b_2}]$.

(3) $[A_3] = [A_0]$.

(4) $[A_4] = [A_0] + [X|_b]$.

We postpone the proof of Lemma 4.3.7 to step 4.

To prove (2) of Lemma 4.3.6, first note that $A_0 \cap A_1 \cap A_2 \cap A_4$ is the $[0:0:0:1:0]$-bundle over B. Also, b can be perturbed to some $\widetilde{b}$ such that $b \cap \widetilde{b}$ is non-trivial in $H_1(B;\mathbb{Z}_2)$, because Proposition 4.3.4 says the square of any element in $H^1(B;\mathbb{Z}_2)$ is non-trivial. As a result, $A_0 \cap A_1 \cap A_2 \cap A_4 \cap X|_b \cap X|_{\widetilde{b}}$ is non-trivial in $H_1(X;\mathbb{Z}_2)$. Hence, writing $\mu := \lambda_2 + \lambda_3$ for simplicity, we have

$$
\begin{aligned}
(4.4)\qquad 1 &= PD(A_0)PD(A_1)PD(A_2)PD(A_4)PD(X|_b)^2 \\
&= \lambda_1(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)(\lambda_1+\mu)\mu^2 \\
&= \lambda_1^4\mu^2 + \lambda_1^3(\lambda_2+\lambda_3+\mu)\mu^2 + \lambda_1^2(\lambda_2\lambda_3+\lambda_2\mu+\lambda_3\mu)\mu^2 + \lambda_1\lambda_2\lambda_3\mu^3 \\
&= \lambda_1^4\mu^2.
\end{aligned}
$$

The first equality above is from Lemma 4.3.7. The last equality holds because $\lambda_2+\lambda_3+\mu$, $\lambda_2\lambda_3+\lambda_2\mu+\lambda_3\mu$, and $\lambda_2\lambda_3\mu$ all are zero: This is straightforward to check by considering how the bases b_2, b_3, and b intersect using Proposition 4.3.4. Hence, we have proven (2) of Lemma 4.3.6.

To prove (3) of Lemma 4.3.6, one note that $A_0 \cap A_1 \cap A_2 \cap A_3 \cap A_4$ is empty, and then $\lambda_1^5 = 0$ would follow from a calculation analogous to (4.4). □

Lemma 4.3.6 implies $\lambda^6 \neq 0$, finishing the proof of Proposition 4.2.2.

Step 4. Finally, let us prove Lemma 4.3.7.

Proof of Lemma 4.3.7. For Lemma 4.3.7 (3): $[A_3] = [A_0]$ because we can homotope A_0 to A_3 using the $\{(1-s)a_0 = sa_3\}$-bundles over B, for $0 \le s \le 1$. (Note that we cannot prove, say, $[A_2] = [A_0]$ this way because the $\{(1-s)a_0 = sa_2\}$-bundles over B is not well-defined.)

We now prove Lemma 4.3.7 (1), by acting on the basic elements $[c_1], [c_2]$, and $[c_3]$ of $H_1(X; \mathbb{Z}_2)$. More precisely, recall that λ_i is by definition the Hom-dual of c_i. So by Lemma 4.3.3 and 4.3.5, $PD([A_0]+[X|_{b_3}])$ acts on $[c_1], [c_2]$, and $[c_3]$ to give 1+0, 0+0, and 0+1 respectively. Therefore, to prove (1) it suffices to show that

- $PD([A_1])[c_1] = 1$.
- $PD([A_1])[c_2] = 0$.
- $PD([A_1])[c_3] = 1$.

To show that $PD([A_1])[c_1] = 1$, just observe that we can homotope c_1 to the loop $\{[0 : a_1 : 0 : 0 : a_4] : a_1^2 + a_4^2 = 1\}$ within the same fiber $X|_{D_2 \cdot \mathrm{id}}$ as $\pi_1(\mathbb{RP}^4) = \mathbb{Z}_2$, and this loop intersects A_1 only once.

To show that $PD([A_1])[c_2] = 0$, we will perturb c_2 to another loop $\widetilde{c}_2$ as follows. Let $d_2 : [0,1] \to SO(3)$ be the path that lifts $p \circ c_2 \subset B = SO(3)/D_2$, starts at id, and ends at g_1. Fix a small constant $\epsilon_0 > 0$. We define $\widetilde{c}_2 : [0,1] \to X$ to be such that it is over the same base $p \circ c_2$ as c_2, but has different fibers:

$$\widetilde{c}_2(s) := D_2 \cdot ([1 : \epsilon_0 : -\epsilon_0 : 0 : 0], d_2(s)) \tag{4.5}$$

Then one can check that $\widetilde{c}_2(0) = \widetilde{c}_2(1)$ so that $\widetilde{c}_2$ is a loop, and $\widetilde{c}_2$ does not intersect A_1. Thus $PD([A_1])[c_2] = 0$.

To show that $PD([A_1])[c_3] = 1$, we will perturb c_3 to $\widetilde{c}_3$ as follows. Let $d_3 : [0,1] \to SO(3)$ be the path that lifts $p \circ c_3$, starts at id, and ends at g_2. This time, we let ϵ be a function from $[0,1]$ to $\mathbb{R}$ that strictly decreases from ϵ_0 to $-\epsilon_0$, for some fixed small

$\epsilon_0 > 0$. We define $\widetilde{c}_3 : [0,1] \to X$ by:

$$\widetilde{c}_3(s) := D_2 \cdot ([1 : \epsilon(s) : -\epsilon(s) : 0 : 0], d_3(s)) \tag{4.6}$$

One can again check $\widetilde{c}_3$ is indeed a loop, but $\widetilde{c}_3$ intersects A_1 at one point: where $\epsilon(s) = 0$. Thus $PD([A_1])[c_3] = 1$. This finishes the proof of Lemma 4.3.7 (1).

The proof of Lemma 4.3.7 (2) is similar. We only state the modifications needed: To prove $PD([A_2])[c_2] = 1$ and $PD([A_2])[c_3] = 0$, instead of (4.5) and (4.6), respectively, we use

$$D_2 \cdot ([1 : \epsilon(s) : \epsilon(s) : 0 : 0], d_2(s)) \text{ and } D_2 \cdot ([1 : \epsilon_0 : \epsilon_0 : 0 : 0], d_3(s)).$$

The proof of Lemma 4.3.7 (4) is also similar. We only state the modifications needed: To prove $PD([A_4])[c_2] = 1$ and $PD([A_4])[c_3] = 1$, instead of (4.5) and (4.6), respectively, we use

$$D_2 \cdot ([1 : 0 : 0 : 0 : \epsilon(s)], d_2(s)) \text{ and } D_2 \cdot ([1 : 0 : 0 : 0 : \epsilon(s)], d_3(s)).$$

□

Proof of Proposition 4.2.6.

Step 1. For convenience, let us reparametrize the family $\Phi_5 : \mathbb{RP}^4 \times [0,1] \to \mathcal{Z}$ as follows. First, we write $\mathbb{RP}^4$ as $\mathbb{R}^4 \sqcup \mathbb{RP}^3$ in which $\mathbb{RP}^3$ is where $a_0 = 0$. Then on $\mathbb{R}^4 \times [0,1]$, we reparametrize the family Φ_5 by

$$\Phi_5(b_1, b_2, b_3, b_4, b_5) := \{(x - b_1)^2 - (y - b_2)^2 + b_3 z + b_4 + b_5 z^3 = 0\} \cap \mathbb{B}^3.$$

Throughout this section *we will adopt this new parametrization.* And then our goal is to show that area $\circ\, \Phi_5$ has a strict local maximum at $(0,0,0,0,0)$. In fact, it suffices to prove:

Proposition 4.3.8. *There exists $\epsilon_1, \epsilon_2 > 0$ such that for any $(b_1, b_2) \in (-\epsilon_1, \epsilon_1)^2$, $(b_3, b_4, b_5) \in \mathbb{R}^2 \times [0, 1]$ such that $b_3^2 + b_4^2 + b_5^2 = 1$, and $t \in (0, \epsilon_2)$,*

$$\text{area}(\Phi_5(b_1, b_2, b_3t, b_4t, b_5t)) < \text{area}(\Phi_5(0, 0, 0, 0, 0)). \tag{4.7}$$

Geometrically, t governs how much the surface opens up (see Remark 4.2.3). Namely, it is elementary to show that the width of the hole opened up in Σ_t is at most $\sqrt{3t}$.

Step 2. We begin to prove Proposition 4.3.8. Let b_1, b_2, b_3, b_4, b_5 satisfy the assumptions — we will explain how small ϵ_1, ϵ_2 need to be later. For each $t \in (0, \epsilon_2)$, denote $\widetilde{\Sigma}_t := \Phi_5(b_1, b_2, b_3t, b_4t, b_5t)$. Then by Lemma B.0.1,

$$\frac{d}{dt}\text{area}(\widetilde{\Sigma}_t) = -\int_{\widetilde{\Sigma}_t} \mathbf{H} \cdot V - \int_{\partial\widetilde{\Sigma}_t} \frac{\mathbf{n} \cdot w}{\nu \cdot w} V \cdot \mathbf{n},$$

which one would hope to show to be negative in order to prove Proposition 4.2.6. However, the second integral is difficult to bound, since $\nu \cdot w$ can be zero on $\partial\mathbb{B}^3$. *To prevent $\nu \cdot w = 0$ on $\partial\mathbb{B}^3$, we will slightly enlarge $\mathbb{B}^3$ to some domain* Ω, which we will soon define. Then for each $s \in (0, t]$, we let

$$\Sigma_s := \{(x - b_1)^2 - (y - b_2)^2 + s(b_3z + b_4 + b_5z^3) = 0\} \cap \Omega. \tag{4.8}$$

(Here Σ_s has a boundary, so the notation is different from Lemma B.0.1.) Then

$$\begin{aligned} &\text{area}(\Phi_5(b_1, b_2, b_3t, b_4t, b_5t)) - \text{area}(\Phi_5(0, 0, 0, 0, 0)) \\ &< \text{area}(\Sigma_t) - \text{area}(\Phi_5(0, 0, 0, 0, 0)) \\ &= \big(\text{area}(\Sigma_0) - \text{area}(\Phi_5(0, 0, 0, 0, 0))\big) + \int_0^t \frac{d}{ds}\text{area}(\Sigma_s)ds \end{aligned} \tag{4.9}$$

Therefore to prove Proposition 4.3.8, it suffices to show that expression (4.9) is negative. We will achieve this by showing *the initial area added by enlarging $\mathbb{B}^3$ to Ω, which is the first term of (4.9), is dominated by the area decrease as s increases from 0 to t, which is the second term of (4.9).*

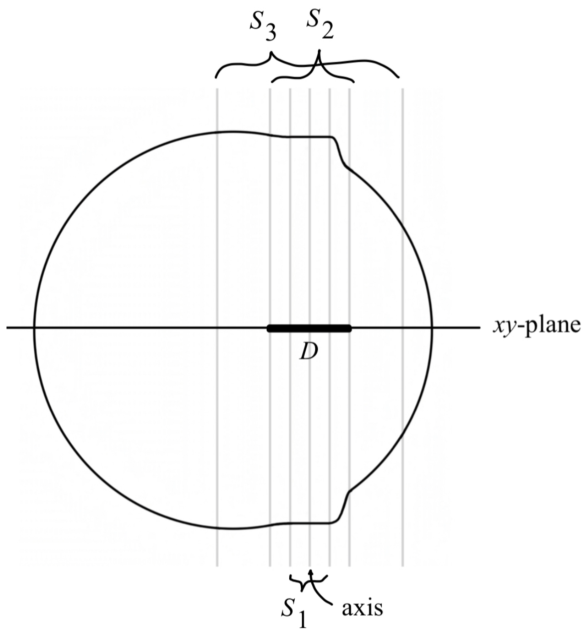


FIGURE 17. A schematic picture of Ω.

But let us first define Ω. We now fix $t \in (0, \epsilon_2)$ also. The new region Ω *will depend on* t as follows. Let

$$R := 20\sqrt{t}. \tag{4.10}$$

Let S_1, S_2, S_3 be the *solid* cylinders in $\mathbb{R}^3$ with axis $\{x = b_1, y = b_2\}$ and radius $R, 2R, \frac{1}{4}$ respectively. By letting ϵ_2 be small we can assume $S_2 \subset S_3$. Let Ω be the unit 3-ball with a bump within S_2, such that $\partial\Omega$ *becomes horizontal in* S_1 (see Figure 17). Moreover, let us view $\partial\mathbb{B}^3 \cap S_2$ (resp. $\partial\Omega \cap S_2$) as the 2-sheeted graph of some function $\pm f$ (resp. $\pm g$) over a disk D on the xy-plane. Then since $\text{dist}(p, (0,0)) < 2\epsilon_1 + 40\sqrt{\epsilon_2}$ for all $p \in D$, we know that $|\nabla f| \lesssim (2\epsilon_1 + 40\sqrt{\epsilon_2})^2$ (here $\lesssim$ means the inequality holds up to a multiplicative constant that is universal), thus *we can also assume* $|\nabla g| \lesssim (2\epsilon_1 + 40\sqrt{\epsilon_2})^2$. As a result, if we write the outward unit normal w of Ω as (w_1, w_2, w_3), then for ϵ_1, ϵ_2 sufficiently small, we have in S_2

$$|\nabla g|, |w_1|, |w_2| < C''(\epsilon_1 + \epsilon_2) \tag{4.11}$$

for some universal constant C''.

Now we have defined Ω, *it suffices to show that expression (4.9) is negative.* The second term of (4.9) can be computed using the first variation formula in Lemma B.0.1. In order to estimate, let us derive some preliminary results in the next step.

Step 3. We are interested in surfaces Σ_s defined in (4.8), which is the zero set in Ω of the polynomial

$$p(x, y, z) := (x - b_1)^2 - (y - b_2)^2 + b_3 sz + b_4 s + b_5 sz^3,$$

for s increases from 0 to t, where $t \in (0, \epsilon_2)$ is fixed. Note that

$$\partial_s p = b_3 z + b_4 + b_5 z^3, \ \nabla p = (2(x - b_1), -2(y - b_2), b_3 s + 3b_5 sz^2),$$

$$\text{Hess}(p) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6b_5 sz \end{pmatrix}, \ \Delta p = 6b_5 sz.$$

Moreover, denoting $\mathbf{H} = H\mathbf{n}$,

$$\begin{aligned} H &= \frac{\nabla p \ \text{Hess}(p) \nabla p^T - |\nabla p|^2 \Delta p}{|\nabla p|^3} \\ &= \frac{1}{|\nabla p|^3}[8(x - b_1)^2 - 8(y - b_2)^2 + 6b_5 sz(b_3 s + 3b_5 sz^2)^2 \\ &\quad - (4(x - b_1)^2 + 4(y - b_2)^2 + (b_3 s + 3b_5 sz^2)^2)(6b_5 sz)] \\ &= \frac{1}{|\nabla p|^3}[-8s\partial_s p - 24b_5 sz((x - b_1)^2 + (y - b_2)^2)], \end{aligned} \tag{4.12}$$

in which in the third equality we used $p = 0$ on Σ_s. We can choose $\frac{\nabla p}{|\nabla p|}$ as the normal vector field $\mathbf{n}$, and $V := -\frac{\partial_s p}{|\nabla p|^2}\nabla p$ as the deformation vector field of Σ_s (because by differentiating $p(s, \mathbf{x}(s)) = 0$ with respect to s, one has $\partial_s p + \nabla p \cdot \mathbf{x}' = 0$). As a result, by (4.12) and Lemma B.0.1,

$$\frac{d}{ds}\text{area}(\Sigma_s) = -\int_{\Sigma_s} \mathbf{H}\cdot V - \int_{\partial\Sigma_s} \frac{\mathbf{n}\cdot w}{\nu\cdot w}\, V\cdot\mathbf{n}$$

$$(4.13)\qquad = \int_{\Sigma_s} -\frac{8s(\partial_s p)^2}{|\nabla p|^4} - \frac{24b_5 sz\partial_s p((x-b_1)^2+(y-b_2)^2)}{|\nabla p|^4}$$

$$-\left(\int_{\partial\Sigma_s\cap S_1} + \int_{\partial\Sigma_s\cap(S_2\backslash S_1)} + \int_{\partial\Sigma_s\cap(S_3\backslash S_2)} + \int_{\partial\Sigma_s\cap(\mathbb{R}^3\backslash S_3)}\right)\frac{\mathbf{n}\cdot w}{\nu\cdot w}\, V\cdot\mathbf{n}.$$

We can write this as a *sum* of six integrals, which we will denote in the above order as $I_1, I_2, ..., I_6$.

Now, remember that to prove Proposition 4.3.8, it suffices to show that expression (4.9) is negative. We claim that it suffices to prove:

Lemma 4.3.9. *There exist some large universal constant $C > 0$ and small $\epsilon_1, \epsilon_2 > 0$ such that the following is true. For any $(b_1, b_2) \in (-\epsilon_1, \epsilon_1)^2$, $(b_3, b_4, b_5) \in \mathbb{R}^2 \times [0, 1]$ such that $b_3^2 + b_4^2 + b_5^2 = 1$, and $0 < s < t < \epsilon_2$, we have:*

- $I_1 < -\frac{1}{C\sqrt{s}} < 0$.
- $|I_2|, |I_3|, |I_4|, |I_6| < C$.
- $|I_5| < -C\log(400s)$.
- $\text{area}(\Sigma_0) - \text{area}(\Phi_5(0,0,0,0,0)) < Ct$.

Indeed, from this lemma it follows that in (4.9), when ϵ_1, ϵ_2 are small, the dominating term is $\int_0^t I_1 ds$, which is of order $\sqrt{t}$ and is negative. Thus the expression (4.9) is negative, as desired.

Step 4. We now begin to prove Lemma 4.3.9, by bounding the seven quantities listed one by one.

First, $I_1 < 0$ is clear, so we just need to lower bound $|I_1|$. By Lemma C.0.1, there exists a universal constant $h > 0$ and an interval $[z_0, z_0 + \frac{1}{8}] \subset [-\frac{1}{2}, \frac{1}{2}]$ on which $\partial_s p = b_3 z + b_4 + b_5 z^3 > h$ or $\partial_s p < -h$.

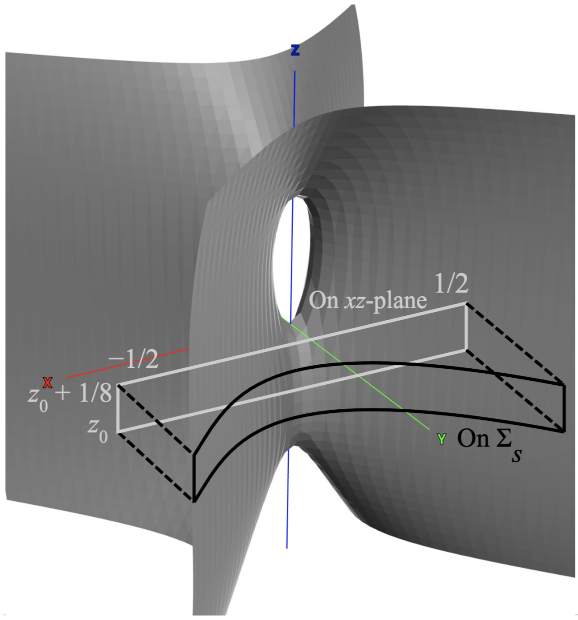


FIGURE 18. A projection onto the xz-plane.

Let us first tackle the case $\partial_s p > h$. Namely, we have

$$|I_1| = \int_{\Sigma_s} \frac{8s(\partial_s p)^2}{|\nabla p|^4} \geq \int_{\Sigma_s \cap \{z_0 < z < z_0 + \frac{1}{8}\}} \frac{8sh^2}{(8(x-b_1)^2 + 4s\partial_s p + (b_3 + 3b_5 z^2)^2 s^2)^2}$$
$$\geq \int_{\substack{\Sigma_s \cap \{z_0 < z < z_0 + \frac{1}{8}\} \\ \cap \{-\frac{1}{2} < x < \frac{1}{2}\}}} \frac{8sh^2}{(8(x-b_1)^2 + 13s)^2}.$$

Note that in the first inequality we rewrote $(y - b_2)^2$ in $|\nabla p|^4$ using $p = 0$, and used that $\partial_s p > h$ for $z \in [z_0, z_0 + \frac{1}{8}]$; while the second inequality holds because $|\partial_s p| \leq 3$ and $(b_3 + 3b_5 z^2)^2 s^2 < s$ if ϵ_2 and thus s is small. Moreover, note that the domain of the last integral is a two-sheeted graph over the rectangle $[-\frac{1}{2}, \frac{1}{2}] \times [z_0, z_0 + \frac{1}{8}]$ on the xz-plane, and clearly the graph has a larger area than the rectangle (see Figure 18). As a result,

$$|I_1| > 2 \cdot \frac{1}{8} \int_{-1/2}^{1/2} \frac{8sh^2}{(8(x-b_1)^2 + 13s)^2} dx \gtrsim \frac{1}{\sqrt{s}}.$$

This finishes proving $I_1 < -\frac{1}{C\sqrt{s}}$ for the case $\partial_s p > h$. The second case $\partial_s p < -h$ is similar: One would integrate with respect to y instead of x in the last step.

To bound I_2, we observe that

$$|\nabla p|^2 \geq 4(x-b_1)^2 + 4(y-b_2)^2 \geq 4|(x-b_1)^2 - (y-b_2)^2| = 4s|\partial_s p|. \tag{4.14}$$

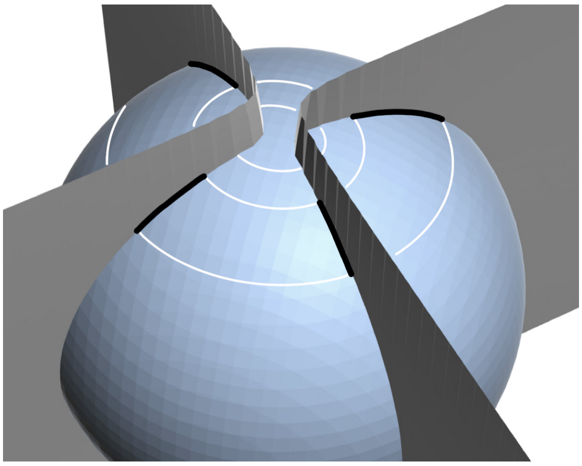


FIGURE 19. This figure (not drawn to scale) shows $\partial\Omega$ intersecting the cubic surface defining Σ_s. The three white circles are $\partial\Omega \cap \partial S_i$, for $i = 1, 2,$ and 3, which have radii $R, 2R,$ and $\frac{1}{4}$ respectively. The black thick segments are $\partial\Sigma_s \cap (S_3 \backslash S_2)$.

So

$$|I_2| \leq \int_{\Sigma_s} |24b_5 z| \frac{|s\partial_s p|}{|\nabla p|^2} \frac{(x-b_1)^2 + (y-b_2)^2}{|\nabla p|^2} \leq \int_{\Sigma_s} 24 \cdot 1 \cdot 1.$$

Now, since when ϵ_1, ϵ_2 are small enough, Σ_s is close to Σ_0, which is two disks, we can assume area$(\Sigma_s) < 3\pi$. Hence, $|I_2| < 24 \cdot 3\pi$.

To bound I_3, note that

(4.15)
$$\frac{(\mathbf{n}\cdot w)(V\cdot \mathbf{n})}{\nu \cdot w} = \frac{(\mathbf{n}\cdot w)(V\cdot \mathbf{n})}{\sqrt{1-(\mathbf{n}\cdot w)^2}} = \frac{(\mathbf{n}\cdot w)(-\partial_s p)}{|\nabla p|\sqrt{1-(\mathbf{n}\cdot w)^2}} = \frac{(\nabla p\cdot w)(-\partial_s p)}{|\nabla p|\sqrt{|\nabla p|^2-(\nabla p\cdot w)^2}}.$$

Now, inside S_1, $\partial\Omega$ is horizontal by definition and so $w = \pm e_3$. Thus

$$\left|\frac{(\mathbf{n}\cdot w)(V\cdot \mathbf{n})}{\nu \cdot w}\right| \leq \frac{|\nabla p \cdot w||\partial_s p|}{|\nabla p|^2 - (\nabla p\cdot w)^2} \leq \frac{|s(b_3 + 3b_5 z^2)||\partial_s p|}{4(x-b_1)^2 + 4(y-b_2)^2} \leq \frac{4s|\partial_s p|}{4s|\partial_s p|} \leq 1,$$

in which the third inequality used (4.14). Now by Remark 4.2.3, $\partial\Sigma_s \cap S_1$ is two hyperbolas near respectively the north and the south pole (see Figure 19). Since the radius of S_1 is R, it is elementary to show that length$(\partial\Sigma_s \cap S_1) < C'R$ for some universal constant C'. Hence, using (4.10), $|I_3| < C'R < 20C'\sqrt{\epsilon_2}$, which is less than some universal constant, assuming, say, $\epsilon_2 < 1$.

To bound I_4, using (4.11), we have

$$(4.16)\qquad |\mathbf{n}\cdot w| \le \frac{\left|2(x-b_1)\right|}{|\nabla p|}|w_1| + \frac{\left|2(y-b_2)\right|}{|\nabla p|}|w_2| + \frac{\left|s(b_3+3b_5z^2)\right|}{|\nabla p|}|w_3|$$

$$\le 1\cdot C''(\epsilon_1+\epsilon_2) + 1\cdot C''(\epsilon_1+\epsilon_2) + \frac{4s}{\sqrt{4(x-b_1)^2+4(y-b_2)^2}}\cdot 1$$

$$\le 2C''(\epsilon_1+\epsilon_2) + \frac{4(R/20)^2}{2R} < \frac{1}{10}.$$

Note that in the second inequality we used that $s \le t = (R/20)^2$ by (4.10), and in the last inequality assumed ϵ_1, ϵ_2 are small.

Then using (4.15),

$$\left|\frac{(\mathbf{n}\cdot w)(V\cdot\mathbf{n})}{\nu\cdot w}\right| = \frac{|\mathbf{n}\cdot w||\partial_s p|}{|\nabla p|\sqrt{1-(\mathbf{n}\cdot w)^2}} \le \frac{(1/10)\cdot 4}{2R\sqrt{1-(1/10)^2}} < \frac{1}{R}.$$

Again, it is elementary to show that the length of $\partial\Sigma_s \cap (S_2\backslash S_1)$ is less than CR for some universal constant C. As a result, $|I_4|$ is less than $CR\cdot\frac{1}{R} = C$.

To bound I_5, using the fact that $p=0$ we can rewrite

$$\nabla p\cdot w = 2(x-b_1)x - 2(y-b_2)y + s(b_3+3b_5z^2)z$$

$$= 2(x-b_1)b_1 - 2(y-b_2)b_2 + s(-b_3z - 2b_4 + b_5z^3).$$

Therefore, using (4.10) and that $|\nabla p| > 2R$, and assuming ϵ_1, ϵ_2 to be small,

$$(4.17)\qquad |\mathbf{n}\cdot w| \le \frac{\left|2(x-b_1)\right|}{|\nabla p|}|b_1| + \frac{\left|2(y-b_2)\right|}{|\nabla p|}|b_2| + \frac{\left|s(-b_3z-2b_4+b_5z^3)\right|}{|\nabla p|}$$

$$\le 1\cdot\epsilon_1 + 1\cdot\epsilon_1 + \frac{4s}{2R} \le 2\epsilon_1 + \frac{4(R/20)^2}{2R} < \frac{1}{10}.$$

Then

$$(4.18)\qquad \left|\frac{(\mathbf{n}\cdot w)(V\cdot\mathbf{n})}{\nu\cdot w}\right| = \frac{|\mathbf{n}\cdot w||\partial_s p|}{|\nabla p|\sqrt{1-(\mathbf{n}\cdot w)^2}} \le \frac{(1/10)\cdot 4}{\sqrt{4(x-b_1)^2}\sqrt{1-(1/10)^2}} < \frac{1}{|x-b_1|}.$$

Now, on $\Sigma_s \cap (S_3\backslash S_2)$, it follows from the definition that $|x-b_1|, |y-b_2| > R$. From this, it is elementary to estimate $\mathbf{n}$ and show that in $S_3\backslash S_2$ the tangent planes of Σ_s are close to that of $\{(x-b_1)^2-(y-b_2)^2=0\}$ (see Figure 19). In this step, the dependence of R on t, namely $R=20\sqrt{t}$, is crucial: We choose $20\sqrt{t}$ because the width of the hole opened up in Σ_s is at most $\sqrt{3s}$, which we want to be small compared to R. As a result, $\partial\Sigma_s\cap(S_3\backslash S_2)$ consists of eight arcs such that if we let ρ be the orthogonal projection map from $\partial\Sigma_s\cap(S_3\backslash S_2)$ to the x-axis, then the norm of the derivative $D\rho$ is lower bounded by some universal constant $C'>0$. In addition, the image J of ρ is contained in $[b_1-\frac{1}{4},b_1-R]\cup[b_1+R,b_1+\frac{1}{4}]$, and the preimage of each $x\in J$ has at most 4 points. Therefore, by (4.18), for ϵ_2 small,

$$|I_5|\leq 4\int_{[b_1-\frac{1}{4},b_1-R]\cup[b_1+R,b_1+\frac{1}{4}]}\frac{1}{C'|x-b_1|}dx\lesssim-\log(R)\lesssim-\log(400s).$$

To bound I_6, note that $|\nabla p|>\frac{1}{2}$ outside S_3. Then as in (4.17), we have $\mathbf{n}\cdot w<\frac{1}{10}$. It follows easily that the expression (4.15) is bounded by some universal constant. Then since length$(\partial\Sigma_s\cap(\mathbb{B}^3\backslash S_3))$ bounded, so is $|I_6|$.

Finally, we prove the last item of Lemma 4.3.9. Note that the difference between Σ_0 and $\Phi_5(0,0,0,0,0)$ is $\{x^2-y^2=0\}\cap(\Omega\backslash\mathbb{B}^3)$, which is four small planar pieces (two near the north pole and two near the south). Each piece can be contained in a rectangle of width $4R$ and, by (4.11), height $C''(\epsilon_1+\epsilon_2)(4R)$. As a result, using (4.10),

$$\text{area}(\Sigma_0)-\text{area}(\Phi_5(0,0,0,0,0))\leq 4(4R)\cdot C''(\epsilon_1+\epsilon_2)(4R)\lesssim t.$$

This finishes the proof of Lemma 4.3.9. Hence, we have proven Proposition 4.3.8, and thus Proposition 4.2.6.

Appendix A. Proof of Proposition 4.2.5

The following proof is due to the MathOverflow user fedja [Fed22]. Let M denote the saddle in $\mathbb{R}^3$ given by $x^2 - y^2 + z = 0$. Then to prove Proposition 4.2.5, it suffices, by rescaling, to show that for any ball B with center (x_0, y_0, z_0) and radius $R > 0$, the area of $M \cap B$ is less than $2\pi R^2$.

Recall that M is foliated by straight lines: It can be parametrized by $\mathbf{x}(s,t) = (s + t, s - t, 4st)$. Then the Jacobian of $\mathbf{x}$ is $2\sqrt{1 + 8s^2 + 8t^2}$. Thus we have

$$\text{area}(M \cap B) < \iint_{\{(s,t):\mathbf{x}(s,t)\in B\}} (2\sqrt{1+8s^2} + 2\sqrt{1+8t^2})dsdt. \tag{A.1}$$

Now, for each fixed s, let L_s be the corresponding coordinate line segment in $M \cap B$. Letting d be the distance between L_s and the center of B, we have

$$d^2 \geq \min_{t\in\mathbb{R}}[(s + t - x_0)^2 + (s - t - y_0)^2] = 2\left(s - \frac{x_0 + y_0}{2}\right)^2. \tag{A.2}$$

Note that L_s is parameterized by a time interval of length

$$\frac{\text{length}(L_s)}{\|\partial_t \mathbf{x}\|} = \frac{2\sqrt{(R^2 - d^2)^+}}{\sqrt{2(1 + 8s^2)}}, \tag{A.3}$$

where $^+$ denotes the positive part. It follows that, using (A.2) and (A.3),

$$\iint_{\{(s,t):\mathbf{x}(s,t)\in B\}} 2\sqrt{1+8s^2}dtds \leq \int_{s\in\mathbb{R}} 2\sqrt{2}\sqrt{\left[R^2 - 2\left(s - \frac{x_0+y_0}{2}\right)^2\right]^+} ds = \pi R^2.$$

The second integral in (A.1) can be similarly bounded, by integrating with respect to s first. So area$(M \cap B) < 2\pi R^2$, finishing the proof of Proposition 4.2.5.

Appendix B. First Variation Formula

Lemma B.0.1. *Let Ω be a compact $(n+1)$-dimensional region with smooth boundary in $\mathbb{R}^{n+1}$, $\{\Sigma_s\}$ a 1-parameter family of hypersurfaces without boundary in $\mathbb{R}^3$, and V a* *deformation vector field of* $\{\Sigma_s\}$*. Then*

$$\frac{d}{ds}\text{area}(\Sigma_s \cap \Omega) = -\int_{\Sigma_s\cap\Omega} \mathbf{H}\cdot V - \int_{\partial(\Sigma_s\cap\Omega)} \frac{\mathbf{n}\cdot w}{\nu\cdot w}\, V\cdot\mathbf{n},$$

where $\mathbf{n}$ *is a chosen unit normal vector field of* Σ_s*,* w *the outward unit normal of* $\partial\Omega$*, and* ν *the outward unit conormal of* Σ_s *on* $\partial\Omega$*.*

Proof. We first smoothly extend w to a unit vector field on a neighborhood of $\partial\Omega$ in Ω, and ν to a unit *tangent* vector field on a neighborhood of $\partial\Sigma_s$ in Σ_s. Let $\epsilon > 0$, and $\Omega_\epsilon \subset \Omega$ be where the distance from $\partial\Omega$ is at least ϵ. Then by using the function $\text{dist}(\cdot, \partial\Omega)$ on Ω, with suitable smoothening, we can approximate the indicator function χ_Ω by a smooth function χ_Ω^ϵ that is 0 outside Ω and 1 on Ω_ϵ, with $\nabla\chi_\Omega^\epsilon = -|\nabla\chi_\Omega^\epsilon| w$ in between.

Now, using the first variation formula (4.2) in [Eck12, p.49],

$$\frac{d}{ds}\int_{\Sigma_s} \chi_\Omega^\epsilon = \int_{\Sigma_s} -\chi_\Omega^\epsilon \mathbf{H}\cdot V + \nabla\chi_\Omega^\epsilon\cdot\mathbf{n}\, V\cdot\mathbf{n}.$$

Note that

$$\nabla\chi_\Omega^\epsilon\cdot\mathbf{n}\, V\cdot\mathbf{n} = -|\nabla\chi_\Omega^\epsilon|\; w\cdot\mathbf{n}\, V\cdot\mathbf{n} = \nabla\chi_\Omega^\epsilon\cdot\nu\, \frac{w\cdot\mathbf{n}}{w\cdot\nu}\, V\cdot\mathbf{n}.$$

Denoting $g := \frac{w\cdot\mathbf{n}}{w\cdot\nu}\, V\cdot\mathbf{n}$, we then have

$$\int_{\Sigma_s} \nabla\chi_\Omega^\epsilon\cdot\mathbf{n}\, V\cdot\mathbf{n} = \int_{\Sigma_s} \nabla\chi_\Omega^\epsilon\cdot g\nu = -\int_{\Sigma_s} \chi_\Omega^\epsilon \text{div}(g\nu) \xrightarrow{\epsilon\to 0} -\int_{\Sigma_s\cap\Omega} \text{div}(g\nu) = -\int_{\partial(\Sigma_s\cap\Omega)} g,$$

in which the second and the third equality are due to divergence theorem. Hence, $\frac{d}{ds}\text{area}(\Sigma_s\cap\Omega)$ is equal to

$$\lim_{\epsilon\to 0}\frac{d}{ds}\int_{\Sigma_s}\chi_\Omega^\epsilon = \lim_{\epsilon\to 0}\int_{\Sigma_s} -\chi_\Omega^\epsilon\mathbf{H}\cdot V + \nabla\chi_\Omega^\epsilon\cdot\mathbf{n}\, V\cdot\mathbf{n} = -\int_{\Sigma_s\cap\Omega}\mathbf{H}\cdot V - \int_{\partial(\Sigma_s\cap\Omega)} g.$$

□

Appendix C. A Lemma about Cubic Polynomials

Lemma C.0.1. *There exists* $h > 0$ *such that the following is true. For any* a, b, c *such that* $a^2 + b^2 + c^2 = 1$*, define* $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ *by* $f(x) = ax^3 + bx + c$*. Then there exists some interval of length* $\frac{1}{8}$ *in* $[-\frac{1}{2}, \frac{1}{2}]$ *on which* $|f| > h$*.*

Proof. Assume, by contradiction, that for each positive integer n there exists a cubic function $f_n(x) = a_n x^3 + b_n x + c_n$, with $a_n^2 + b_n^2 + c_n^2 = 1$ such that there is no interval of length $\frac{1}{8}$ in $[-\frac{1}{2}, \frac{1}{2}]$ on which $|f| > \frac{1}{n}$. For each n, let x_i, for i runs from 1 to at most 3, be the roots of $f_n(x) = 0$, and $I_i \subset [-\frac{1}{2}, \frac{1}{2}]$ be the maximal interval containing x_i on which $|f_n| < \frac{1}{n}$. Then $[-\frac{1}{2}, \frac{1}{2}] \backslash (I_1 \cup I_2 \cup I_3)$ is a union of at most 4 intervals, each of which has length at most $\frac{1}{8}$. Thus, $I_1 \cup I_2 \cup I_3$ has length at least $1 - 4 \cdot \frac{1}{8} = \frac{1}{2}$, and on it $|f_n| < \frac{1}{n}$. Then it follows easily that $\sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} |f_n'(x)| \to 0$ as $n \to \infty$. Since $f_n'(x) = 3a_n x^2 + b_n$, we must have $a_n \to 0$ and $b_n \to 0$ too, which forces $c_n \to 1$ since $a_n^2 + b_n^2 + c_n^2 = 1$. But then f_n is very close to 1 on $[-\frac{1}{2}, \frac{1}{2}]$, contradicting that $|f_n| < \frac{1}{n}$ on a set of length at least $\frac{1}{2}$. ∎

References

[ACS18a] Lucas Ambrozio, Alessandro Carlotto, and Ben Sharp. Comparing the morse index and the first betti number of minimal hypersurfaces. *Journal of Differential Geometry*, 108(3):379–410, 2018.

[ACS18b] Lucas Ambrozio, Alessandro Carlotto, and Ben Sharp. Index estimates for free boundary minimal hypersurfaces. *Mathematische Annalen*, 370(3):1063–1078, 2018.

[ADS19] Sigurd Angenent, Panagiota Daskalopoulos, and Natasa Sesum. Unique asymptotics of ancient convex mean curvature flow solutions. *J. Differential Geom.*, 111(3):381–455, 2019.

[ADS20] Sigurd Angenent, Panagiota Daskalopoulos, and Natasa Sesum. Uniqueness of two-convex closed ancient solutions to the mean curvature flow. *Ann. of Math. (2)*, 192(2):353–436, 2020.

[Aie19] Nicolau Sarquis Aiex. The width of ellipsoids. *Communications in Analysis and Geometry*, 27(2):251–285, 2019.

[All72] William K. Allard. On the First Variation of a Varifold. *Annals of Mathematics*, 95(3):417–491, 1972.

[Alm62] Frederick Almgren. The homotopy groups of the integral cycle groups. *Topology*, 1(4):257–299, 1962.

[Alm65] Frederick Almgren. The theory of varifolds. *Mimeographed notes*, 1965.

[And12] Ben Andrews. Noncollapsing in mean-convex mean curvature flow. *Geom. Topol.*, 16(3):1413–1418, 2012.

[Ang92] Sigurd B Angenent. Shrinking doughnuts. In *Nonlinear diffusion equations and their equilibrium states, 3*, pages 21–38. Springer, 1992.

[BK21] Yakov Berchenko-Kogan. The entropy of the Angenent torus is approximately 1.85122. *Exp. Math.*, 30(4):587–594, 2021.

[BL22a] Márcio Batista and Anderson Lima. Min-max widths of the real projective 3-space. *Transactions of the American Mathematical Society*, 375(07):5239–5258, 2022.

[BL22b] Márcio Batista and Anderson Lima. Min-max widths of the real projective 3-space. *Transactions of the American Mathematical Society*, 2022.

[BL23] Márcio Batista and Anderson Lima. A short note about 1-width of lens spaces. *Bulletin of Mathematical Sciences*, 13(01):2250005, 2023.

[BP02] V. M. Buchstaber and Taras E. Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* Number v. 24 in University Lecture Series. American Mathematical Society, Providence, R.I, 2002.

[Bra78] Kenneth A. Brakke. *The Motion of a Surface by Its Mean Curvature. (MN-20).* Princeton University Press, Princeton, 1978.

[Bre12] Simon Brendle. A sharp bound for the area of minimal surfaces in the unit ball. *Geometric and Functional Analysis*, 22(3):621–626, 2012.

[Bre15] Simon Brendle. A sharp bound for the inscribed radius under mean curvature flow. *Invent. Math.*, 202(1):217–237, 2015.

[BW17] Jacob Bernstein and Lu Wang. A topological property of asymptotically conical self-shrinkers of small entropy. *Duke Math. J.*, 166(3):403–435, 2017.

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. *arXiv preprint arXiv:2003.14344*, 2020.

[CCMS21] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic low-entropy initial data. *arXiv preprint arXiv:2102.11978*, 2021.

[CCS23] Otis Chodosh, Kyeongsu Choi, and Felix Schulze. Mean curvature flow with generic initial data ii. *arXiv preprint arXiv:2302.08409*, 2023.

[CDL03] Tobias Colding and Camillo De Lellis. The min-max construction of minimal surfaces. *Surveys in Differential Geometry*, 8(1):75–107, 2003.

[CFS20] Alessandro Carlotto, Giada Franz, and Mario B Schulz. Free boundary minimal surfaces with connected boundary and arbitrary genus. *arXiv preprint arXiv:2001.04920*, 2020.

[CGG91] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *Journal of differential geometry*, 33(3):749–786, 1991.

[CGK18] Tobias Holck Colding, David Gabai, and Daniel Ketover. On the classification of heegaard splittings. *Duke Mathematical Journal*, 167(15):2833–2856, 2018.

[CHH22] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits. Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness. *Acta Math.*, 228(2):217–301, 2022.

[CHHW22] Kyeongsu Choi, Robert Haslhofer, Or Hershkovits, and Brian White. Ancient asymptotically cylindrical flows and applications. *Invent. Math.*, 229(1):139–241, 2022.

[Chu22] Adrian Chun-Pong Chu. A free boundary minimal surface via a 6-sweepout. *arXiv preprint arXiv:2208.06577*, 2022.

[CIM15] Tobias Holck Colding, Tom Ilmanen, and William P Minicozzi. Rigidity of generic singularities of mean curvature flow. *Publications mathématiques de l'IHÉS*, 121(1):363–382, 2015.

[CIMIW13] Tobias Holck Colding, Tom Ilmanen, William P Minicozzi II, and Brian White. The round sphere minimizes entropy among closed self-shrinkers. *Journal of Differential Geometry*, 95(1):53–69, 2013.

[CM12] Tobias H Colding and William P Minicozzi. Generic mean curvature flow i; generic singularities. *Annals of mathematics*, pages 755–833, 2012.

[CM15] Tobias Holck Colding and William P. Minicozzi, II. Uniqueness of blowups and łojasiewicz inequalities. *Ann. of Math. (2)*, 182(1):221–285, 2015.

[CM16] Tobias Holck Colding and William P Minicozzi. The singular set of mean curvature flow with generic singularities. *Inventiones mathematicae*, 204(2):443–471, 2016.

[CM21] Otis Chodosh and Christos Mantoulidis. The p-widths of a surface. *arXiv preprint arXiv:2107.11684*, 2021.

[CM22] Kyeongsu Choi and Christos Mantoulidis. Ancient gradient flows of elliptic functionals and morse index. *American Journal of Mathematics*, 144(2):541–573, 2022.

[CM23] Otis Chodosh and Christos Mantoulidis. The p-widths of a surface. *Publications mathématiques de l'IHÉS*, 137(1):245–342, June 2023.

[CSW22] Alessandro Carlotto, Mario B Schulz, and David Wiygul. Infinitely many pairs of free boundary minimal surfaces with the same topology and symmetry group. *arXiv preprint arXiv:2205.04861*, 2022.

[Dev19] Baptiste Devyver. Index of the critical catenoid. *Geometriae Dedicata*, 199(1):355–371, 2019.

[DLP10] Camillo De Lellis and Filippo Pellandini. Genus bounds for minimal surfaces arising from min-max constructions. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010(644):47–99, 2010.

[DN18] Gregory Drugan and Xuan Hien Nguyen. Shrinking doughnuts via variational methods. *The Journal of Geometric Analysis*, 28:3725–3746, 2018.

[Don22] Sidney Donato. The first p-widths of the unit disk. *The Journal of Geometric Analysis*, 32(6):1–38, 2022.

[Eck12] Klaus Ecker. *Regularity theory for mean curvature flow*, volume 57. Springer Science & Business Media, 2012.

[ES91] Lawrence C Evans and Joel Spruck. Motion of level sets by mean curvature. i. *Journal of Differential Geometry*, 33(3):635–681, 1991.

[Fed22] Fedja. (https://mathoverflow.net/users/1131/fedja). All saddles in the unit ball have area $< 2\pi$? MathOverflow, 2022. https://mathoverflow.net/q/423592 (version: 2022-05-30).

[FPZ17] Abigail Folha, Frank Pacard, and Tatiana Zolotareva. Free boundary minimal surfaces in the unit 3-ball. *manuscripta mathematica*, 154(3):359–409, 2017.

[Fra21] Giada Franz. Equivariant index bound for min-max free boundary minimal surfaces. *arXiv preprint arXiv:2110.01020*, 2021.

[FS11] Ailana Fraser and Richard Schoen. The first steklov eigenvalue, conformal geometry, and minimal surfaces. *Advances in Mathematics*, 226(5):4011–4030, 2011.

[FS16] Ailana Fraser and Richard Schoen. Sharp eigenvalue bounds and minimal surfaces in the ball. *Inventiones mathematicae*, 203(3):823–890, 2016.

[GLWZ21] Qiang Guang, Martin Man-chun Li, Zhichao Wang, and Xin Zhou. Min-max theory for free boundary minimal hypersurfaces ii: general morse index bounds and applications. *Mathematische Annalen*, 379(3):1395–1424, 2021.

[Hat83] Allen Hatcher. A proof of the smale conjecture. *Annals of Mathematics*, pages 553–607, 1983.

[HK17] Robert Haslhofer and Bruce Kleiner. Mean curvature flow of mean convex hypersurfaces. *Communications on Pure and Applied Mathematics*, 70(3):511–546, 2017.

[HS99a] Gerhard Huisken and Carlo Sinestrari. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta mathematica*, 183(1):45–70, 1999.

[HS99b] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow singularities for mean convex surfaces. *Calculus of Variations and Partial Differential Equations*, 8(1):1–14, 1999.

[Hui84] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *Journal of Differential Geometry*, 31(1):285–299, 1990.

[HW20] Or Hershkovits and Brian White. Nonfattening of mean curvature flow at singularities of mean convex type. *Communications on Pure and Applied Mathematics*, 73(3):558–580, 2020.

[Ilm92] Tom Ilmanen. Generalized flow of sets by mean curvature on a manifold. *Indiana University mathematics journal*, pages 671–705, 1992.

[Ilm94] Tom Ilmanen. *Elliptic regularization and partial regularity for motion by mean curvature*, volume 520. American Mathematical Soc., 1994.

[Ilm95] Tom Ilmanen. Singularities of mean curvature flow of surfaces. *preprint*, 1995.

[IMN18] Kei Irie, Fernando Marques, and André Neves. Density of minimal hypersurfaces for generic metrics. *Annals of Mathematics*, 187(3):963–972, 2018.

[INS19] Tom Ilmanen, André Neves, and Felix Schulze. On short time existence for the planar network flow. *J. Differential Geom.*, 111(1):39–89, 2019.

[Ket16a] Daniel Ketover. Equivariant min-max theory. *arXiv preprint arXiv:1612.08692*, 2016.

[Ket16b] Daniel Ketover. Free boundary minimal surfaces of unbounded genus. *arXiv preprint arXiv:1612.08691*, 2016.

[Ket19] Daniel Ketover. Genus bounds for min-max minimal surfaces. *Journal of Differential Geometry*, 112(3):555–590, 2019.

[KL17] Nikolaos Kapouleas and Martin Man-chun Li. Free boundary minimal surfaces in the unit three-ball via desingularization of the critical catenoid and the equatorial disk. *arXiv preprint arXiv:1709.08556*, 2017.

[KM20] Nikolaos Kapouleas and Peter McGrath. Generalizing the linearized doubling approach, i: General theory and new minimal surfaces and self-shrinkers. *arXiv preprint arXiv:2001.04240*, 2020.

[KMl14] Stephen Kleene and Niels Martin Mø ller. Self-shrinkers with a rotational symmetry. *Trans. Amer. Math. Soc.*, 366(8):3943–3963, 2014.

[KW17] Nicolaos Kapouleas and David Wiygul. Free-boundary minimal surfaces with connected boundary in the 3-ball by tripling the equatorial disc. *arXiv preprint arXiv:1711.00818*, 2017.

[KZ21] Nikolaos Kapouleas and Jiahua Zou. Free boundary minimal surfaces in the euclidean three-ball close to the boundary. *arXiv preprint arXiv:2111.11308*, 2021.

[Lee21] Tang-Kai Lee. Compactness and rigidity of self-shrinking surfaces. *arXiv preprint arXiv:2108.03919*, 2021.

[Li15] Martin Man-chun Li. A general existence theorem for embedded minimal surfaces with free boundary. *Communications on Pure and Applied Mathematics*, 68(2):286–331, 2015.

[Li19] Martin Man-Chun Li. Free boundary minimal surfaces in the unit ball: recent advances and open questions. *arXiv preprint arXiv:1907.05053*, 2019.

[Li23a] Yangyang Li. Existence of infinitely many minimal hypersurfaces in higher-dimensional closed manifolds with generic metrics. *J. Differential Geom.*, 124(2):381–395, 2023.

[Li23b] Yangyang Li. An improved Morse index bound of min-max minimal hypersurfaces. *Calc. Var. Partial Differential Equations*, 62(6):Paper No. 179, 32, 2023.

[Liu16] Zihan Hans Liu. The index of shrinkers of the mean curvature flow. *arXiv preprint arXiv:1603.06539*, 2016.

[LMN18] Yevgeny Liokumovich, Fernando Codá Marques, and André Neves. Weyl law for the volume spectrum. *Annals of Mathematics*, 187(3):933–961, 2018.

[LS47] Lazar Aronovich Lyusternik and Lev Genrikhovich Shnirel'man. Topological methods in variational problems and their application to the differential geometry of surfaces. *Uspekhi Matematicheskikh Nauk*, 2(1):166–217, 1947.

[LS22] Zhengjiang Lin and Ao Sun. Bifurcation of perturbations of non-generic closed self-shrinkers. *Journal of Topology and Analysis*, 14(04):979–999, 2022.

[Lun19] Alejandra Ramírez Luna. Compact minimal hypersurfaces of index one and the width of real projective spaces. *arXiv preprint arXiv:1902.08221*, 2019.

[LZ21] Martin Man-Chun Li and Xin Zhou. Min-max theory for free boundary minimal hypersurfaces i: Regularity theory. *Journal of Differential Geometry*, 118(3):487–553, 2021.

[Mar23] Fernando Codá Marques. Personal communication, 2023.

[MF10] John W Morgan and Frederick Tsz-Ho Fong. *Ricci flow and geometrization of 3-manifolds*, volume 53. American Mathematical Soc., 2010.

[MN14] Fernando Codá Marques and André Neves. Min-max theory and the willmore conjecture. *Annals of mathematics*, pages 683–782, 2014.

[MN16] Fernando Codá Marques and André Neves. Morse index and multiplicity of min-max minimal hypersurfaces. *Cambridge Journal of Mathematics*, 2016.

[MN17] Fernando Codá Marques and André Neves. Existence of infinitely many minimal hypersurfaces in positive ricci curvature. *Inventiones mathematicae*, 209(2):577–616, 2017.

[MN20] Fernando Codá Marques and André Neves. Applications of min–max methods to geometry. *Geometric Analysis: Cetraro, Italy 2018*, pages 41–77, 2020.

[MN21] Fernando Codá Marques and André Neves. Morse index of multiplicity one min-max minimal hypersurfaces. *Advances in Mathematics*, 378:107527, 2021.

[Nur16] Charles Arthur George Nurser. *Low min-max widths of the round three-sphere*. PhD thesis, Imperial College London, 2016.

[OS88] Stanley Osher and James A. Sethian. Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. *J. Comput. Phys.*, 79(1):12–49, 1988.

[Pit81] Jon Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds.(MN-27)*. Princeton University Press, 81.

[RS82] Colin Patrick Rourke and Brian Joseph Sanderson. *Introduction to Piecewise-Linear Topology*. Springer Study Edition. Springer, Berlin, rev. printing edition, 1982.

[Sar17] Pam Sargent. Index bounds for free boundary minimal surfaces of convex bodies. *Proceedings of the American Mathematical Society*, 145(6):2467–2480, 2017.

[Sch] Mario B. Schulz. Geometric analysis gallery, free boundary minimal surfaces with connected boundary.

[Smi82] Francis Smith. On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary riemannian metric. *Doctoral dissertation*, 1982.

[Son23] Antoine Song. Existence of infinitely many minimal hypersurfaces in closed manifolds. *Annals of Mathematics*, 197(3):859 – 895, 2023.

[SS81] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Communications on Pure and Applied Mathematics*, 34(6):741–797, 1981.

[SS23] Lorenzo Sarnataro and Douglas Stryker. Optimal regularity for minimizers of the prescribed mean curvature functional over isotopies. *arXiv preprint arXiv:2304.02722*, 2023.

[Sun23] Ao Sun. Local entropy and generic multiplicity one singularities of mean curvature flow of surfaces. *J. Differential Geom.*, 124(1):169–198, 2023.

[SW09] Weimin Sheng and Xu-Jia Wang. Singularity profile in the mean curvature flow. *Methods Appl. Anal.*, 16(2):139–155, 2009.

[SW20] Ao Sun and Zhichao Wang. Compactness of self-shrinkers in r3 with fixed genus. *Advances in Mathematics*, 367:107110, 2020.

[SWZ20] Ao Sun, Zhichao Wang, and Xin Zhou. Multiplicity one for min-max theory in compact manifolds with boundary and its applications, 2020.

[SX21a] Ao Sun and Jinxin Xue. Initial perturbation of the mean curvature flow for asymptotical conical limit shrinker. *arXiv preprint arXiv:2107.05066*, 2021.

[SX21b] Ao Sun and Jinxin Xue. Initial perturbation of the mean curvature flow for closed limit shrinker. *arXiv preprint arXiv:2104.03101*, 2021.

[SZ19] Graham Smith and Detang Zhou. The morse index of the critical catenoid. *Geometriae Dedicata*, 201(1):13–19, 2019.

[Tra20] Hung Tran. Index characterization for free boundary minimal surfaces. *Communications in Analysis and Geometry*, 28(1):189–222, 2020.

[TZ08] Satoshi Tomoda and Peter Zvengrowski. Remarks on the cohomology of finite fundamental groups of 3-manifolds. *Geometry and Topology monographs*, 14:519–556, 2008.

[Wan11] Xu-Jia Wang. Convex solutions to the mean curvature flow. *Annals of mathematics*, pages 1185–1239, 2011.

[Whi95] Brian White. The topology of hypersurfaces moving by mean curvature. *Communications in analysis and geometry*, 3(2):317–333, 1995.

[Whi97] Brian White. Stratification of minimal surfaces, mean curvature flows, and harmonic maps. *Journal für die reine und angewandte Mathematik*, 488:1–36, 1997.

[Whi00] Brian White. The size of the singular set in mean curvature flow of mean-convex sets. *Journal of the American Mathematical Society*, 13(3):665–695, 2000.

[Whi03] Brian White. The nature of singularities in mean curvature flow of mean-convex sets. *Journal of the American Mathematical Society*, 16(1):123–138, 2003.

[Whi05] Brian White. A local regularity theorem for mean curvature flow. *Annals of mathematics*, pages 1487–1519, 2005.

[Whi09] Brian White. Currents and flat chains associated to varifolds, with an application to mean curvature flow. *Duke Mathematical Journal*, 148(1):41–62, 2009.

[WZ22] Zhichao Wang and Xin Zhou. Min-max minimal hypersurfaces with higher multiplicity. *arXiv preprint arXiv:2201.06154*, 2022.

[WZ23] Zhichao Wang and Xin Zhou. Existence of four minimal spheres in S^3 with a bumpy metric. *arXiv preprint arXiv:2305.08755*, 2023.

[Yau82] Shing-Tung Yau. *Seminar on differential geometry*. Number 102. Princeton University Press, 1982.

[Zho20] Xin Zhou. On the multiplicity one conjecture in min-max theory. *Annals of Mathematics*, 192(3):767–820, 2020.

[Zhu22] Jonathan J Zhu. Widths of balls and free boundary minimal submanifolds. *arXiv preprint arXiv:2203.10031*, 2022.

[Zhu23] Jonathan J Zhu. Widths of balls and free boundary minimal submanifolds. *Advanced Nonlinear Studies*, 23(1), 2023.