Loop celestial amplitudes for gauge theory and gravity

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Scattering amplitudes of massless particles in Minkowski space can be expressed in a conformal basis by Mellin transforming the momentum space amplitudes to correlation functions on the celestial sphere at null infinity. In this paper, we study celestial amplitudes of loop-level gluons and gravitons. We focus on the rational amplitudes that carry all-plus and single-minus external helicities. Because these amplitudes are finite, they provide a concrete example of celestial amplitudes of Yang- Mills and gravity theory beyond tree level. We give explicit examples of four- and five-point functions and comment on higher point amplitudes.

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I. INTRODUCTION

The scattering amplitudes in Minkowski space can be mapped to the celestial sphere at lightlike infinity, where they are encoded in terms of conformal correlators [1]. These correlation functions go by the ethereal name of *celestial amplitudes* and they exhibit conformal symmetry at the boundary for bulk observables. This observation provides a complementary representation of scattering amplitudes where they are beheld as a holographically dual conformal field theory residing in the celestial sphere. Thus, the holographic nature of celestial amplitudes in principle can shine light on an outstanding problem, i.e., what is a concrete holographic formulation for flat spacetime?[1] More terrestrially, one can view celestial amplitude much in the same way as twistors, momentum twistors, and scattering equations, which may help in illuminating hidden mathematical structures in quantum field theory that were not previously accessible from traditional calculations [11].

In the last couple years, celestial amplitudes have garnered a lot of interest. Conformal primary wave function bases for various spins in different dimensions were constructed by Pasterski and Shao in [12]. Soft theorems were connected to the conserved currents on the celestial sphere in [13,14]. Explicit examples of tree-level celestial amplitudes of gluons were computed in [15,16]. Examples of scalar scattering have been shown in [17–19]. Other variant maps have been constructed recently in [20–22]. The investigation of the factorization singularities of celestial amplitudes was done in [23]. Progress has also been made in the celestial four-point superstring amplitudes as well as graviton tree amplitudes [24–26]. Recently, conformal soft theorems have been studied in [19,25–32]. The authors in [33] construct the generators of Poincaré and conformal groups in the celestial representation. Translating an optical theorem in the conformal basis was addressed in [17] and this work was followed by [19] where conformal partial wave decomposition of celestial amplitudes was further discussed. The operator product expansion of the celestial sphere has been carried out in [34–38].

While scalar loops have been studied in [18], most construction of celestial amplitudes have occurred at tree level. In this work, we provide the first explicit construction of loop-level celestial transform for external gluons and gravitons. More concretely, we focus on loop amplitudes where all external gluons and gravitons carry positive helicity (*all-plus*) and the ones where all but one external particles carry positive helicity (*one-minus*). These loop amplitudes are interesting for many reasons. It is a well known result that gluon as well as graviton amplitudes at tree level vanish for all-positive and one-minus external states [39]. This statement can be proved using supersymmetric Ward identity, but nonetheless holds for quantum field theory with or without supersymmetry at tree level. However, the story of loops is different and very interesting: for supersymmetric field theories, such vanishing occurs for

[1]Flat space holography was already proposed in [2] where Minkowski space was foliated along the Euclidean anti–de Sitter and de Sitter slices. For other related approach, please refer to [3–10].

all-plus and one-minus amplitudes at any loop order; in contrast, for pure Yang-Mills and Einstein gravity, such amplitudes receive leading contributions at one loop (for gluons see [40–42] and gravitons [43–48]).[2] Similarly, for one loop, one cannot construct such amplitudes using unitarity cuts in four dimensions as a two particle cut leads to tree-level expressions with at least one vanishing piece. Furthermore, these one-loop amplitudes when integrated are relatively simple rational functions and contain no logarithmic divergences in four dimensions.[3] The simplicity and subtleties of these amplitudes make them ideal candidates to study spinning celestial amplitudes beyond tree level.

We have organized this paper in the following way. In Sec. II we present a detailed review of conformal primary wave functions and celestial amplitudes, which is followed by the rederivation of tree-level celestial amplitudes in our conventions. Thereafter, we switch to computation of loop-level amplitudes in Sec. III where we considered explicit four- and five-point results and discussed the structure of the higher point amplitudes. We end with future directions and a conclusion, and collected several technical details in the Appendix A.

II. REVIEW

A. Conformal primary wave functions and celestial amplitudes

We know that scattering amplitudes in $4d$ have the Poincaré symmetry $\mathrm{ISO}(1,3)$ which can be written as the semidirect product of translation and special orthogonal groups: $\mathrm{ISO}(1,3) = \mathrm{T}(1,3) \otimes_s \mathrm{SO}(1,3)$. The standard momentum space enables us to work with irreducible representations of the translation group, which means translations act only by phases. $\mathrm{SO}(1,3)$ on the other hand acts in a quite complicated manner.

We can try to relate the $4d$ momentum space to some basis of $\mathrm{SO}(1,3)$, or its universal covering group $\mathrm{SL}(2,\mathbb{C})$. As $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to the global conformal group in $2d$, it is intriguing to expand the $4d$ amplitudes in terms of *conformal primary wave functions*, objects that transform covariantly as $2d$ conformal primary operators and satisfy relevant $4d$ equations of motion.

Before analyzing conformal primary wave functions in more detail, let us first set our notations. We will use the standard coordinates z, $\bar{z}$ with $\bar{z} = z^*$ to parametrize $\mathbb{R}^2$, and the $2d$ conformal field theories (CFT) lives at the compactification of this space, i.e., the Riemann sphere $\mathbb{C}_\infty$. We can then view this Riemann sphere as the boundary of a hyperbolic space H^3 via AdS holography. We parametrize H^3 with the coordinates y_i with $i = 0, 1, 2$ for $y_0 > 0$ where the H^3 metric is

$$ds^2_{\mathrm{H}^3} = \frac{dy_0^2 + dy_1^2 + dy_2^2}{y_0^2}. \tag{2.1}$$

We can then embed H^3 as the upper branch of the unit hyperboloid in $\mathbb{R}^{1,3}$ for which we will use lightcone coordinates $x^\mu = (x^+, x^-, x^1, x^2)$ with the metric

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

where we define the lightcone coordinates in terms of the Cartesian ones as $x^\pm = x^3 \pm x^0$. We then embed $y_i \in H^3$ in $y^\mu \in \mathbb{R}^{1,3}$ as

$$y^\mu \equiv \frac{1}{y_0}(1, -y_i y_i, y_1, y_2) \tag{2.3}$$

where $y_i y_i \equiv y_0^2 + y_1^2 + y_2^2$ and where we see that $y_\mu y^\mu = -1$ as required.

Just as we embedded $y_i \to y^\mu$, we can embed $z \to x^\mu(z)$ because d-dimensional conformal groups can be parameterized with the null rays in $d+2$ dimensions.[4] In other words, *the celestial sphere* can be parametrized in $\mathbb{R}^{1,3}$ as $\{x^\mu(z) \in \mathbb{R}^{1,3} | x^\mu(z) x_\mu(z) = 0, x^\mu(z) \sim \lambda x^\mu(z), \lambda \in \mathbb{R}^+\}$ where we choose

$$x^\mu(z) \equiv 2\left(1, -z\bar{z}, \frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \tag{2.4}$$

for later convenience.

Below, we will write y^μ to denote a point in $\mathbb{R}^{1,3}$ constrained to lie on the upper branch of the unit hyperboloid, and $x^\mu(z)$ to denote a null vector in $\mathbb{R}^{1,3}$ whereas x^μ denotes any point in $\mathbb{R}^{1,3}$. In summary, $y_\mu y^\mu = -1$, $x_\mu(z) x^\mu(z) = 0$ with $x^+(z) = 2$, and $x_\mu x^\mu \in \mathbb{R}$.

With our notations set, we can view conformal primary wave functions as maps from $x^\mu \in \mathbb{R}^{1,3}$ to $z \in \mathbb{C}_\infty$, where these particular maps satisfy two conditions:

(i) They satisfy the equation of motion in $\mathbb{R}^{1,3}$

(ii) They transform as conformal primary operators under the action of $\mathrm{SL}(2,\mathbb{C})$

[2]While we will restrict ourselves to one loop in this work, there has been a number of works on higher loops for the external states we are considering. Please see [49–56].

[3]Moreover, these loop amplitudes have interesting factorization properties and have been studied using Britto-Cachazo-Feng-Witten (BCFW) recursion relation [40,57,58] and using a Berends-Giele type of recursion [59] and more recently conformally invariant structure was investigated in [60].

[4]The idea goes back to Dirac who realized that a conformal group in $\mathbb{R}^{p,q}$ dimensions being $\mathrm{SO}(q+1,q+1)$ can most naturally be described in the embedding $\mathbb{R}^{p+1,q+1}$ space [61].

A particularly transparent way to construct these objects for massive scalars can be roughly described as follows: we decompose the map $\mathbb{R}^{1,3} \to \mathbb{C}_\infty$ into $\mathbb{R}^{1,3} \to \mathrm{H}^3$ and $\mathrm{H}^3 \to \mathbb{C}_\infty$, compose these maps, and integrate over whole H^3. Indeed, the necessary ingredients for each map are relatively straightforward: the first map is simply a restriction to the paraboloid whereas the second map is the familiar bulk to the boundary propagator in H^3. Hence, we can immediately define the *massive scalar conformal primary wave function* $\phi^\pm_{\Delta,m}(x^\mu, z)$ as

$$\phi^\pm_{\Delta,m}(x^\mu, z) = \int_{\mathrm{H}^3} [dy_i] G_\Delta(y_i, z) e^{\pm i m x_\mu y^\mu}. \tag{2.5}$$

Here, $e^{\pm i m x_\mu y^\mu}$ makes sure that the equation of motion is satisfied, i.e., $(\partial_\mu \partial^\mu - m^2)\phi^\pm_{\Delta,m}(x^\mu, z) = 0$, whereas $G_\Delta(y_i, z)$ ensures the correct transformation under the $\mathrm{SL}(2,\mathbb{C})$ action [note that $e^{\pm i m x_\mu y^\mu}$ is invariant under $\mathrm{SL}(2,\mathbb{C})$].

The closed form expression for $\phi^\pm_{\Delta,m}(x^\mu, z)$ in $\mathbb{R}^{1,d+1}$ reads as

$$\phi^\pm_{\Delta,m}(x^\mu, z) = \frac{2^{\frac{d+2}{2}} \pi^{\frac{d}{2}}}{(im)^{\frac{d}{2}}} \frac{(\sqrt{-x_\mu x^\mu})^{\Delta - \frac{d}{2}}}{(-x_\mu x^\mu(z) \mp i\epsilon)^\Delta} K_{\Delta - \frac{d}{2}}(m\sqrt{x_\mu x^\mu}). \tag{2.6}$$

One can similarly write down the *massless* spin-0,1,2 conformal primary wave functions $\phi^\pm_\Delta(x^\mu, z)$, $A^{\Delta,\pm}_{\mu,a}(x^\mu, z)$ and $h^{\Delta,\pm}_{\mu_1,\mu_2,a_1,a_2}(x_\mu, z)$ as

$$\phi^\pm_\Delta(x^\mu, z) = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-x_\mu x^\mu(z) \mp i\epsilon)^\Delta}, \tag{2.7a}$$

$$A^{\Delta,\pm}_{\mu,a}(x^\mu, z) = -\frac{1}{(-x_\mu x^\mu(z) \mp i\epsilon)^{\Delta-1}} T^\pm_{\mu,a}(x^\mu, z), \tag{2.7b}$$

$$h^{\Delta,\pm}_{\mu_1,\mu_2,a_1,a_2}(x_\mu, z) = \frac{\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2)} - \frac{1}{d}\delta_{a_1 a_2}\delta^{b_1 b_2}}{(-x_\mu x^\mu(z) \mp i\epsilon)^{\Delta-2}} T^\pm_{\mu_1,b_1}(x^\mu, z, \bar{z}) T^\pm_{\mu_2,b_2}(x^\mu, z) \tag{2.7c}$$

for

$$T^\pm_{\mu,a}(x^\mu, z) \equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial z_a} \log(-x_\mu x^\mu(z) \mp i\epsilon) \tag{2.8}$$

where $z_a \equiv z_{1,2}$ for $z_1 \equiv \frac{z+\bar{z}}{2}$ and $z_2 \equiv \frac{z-\bar{z}}{2i}$.

Conformal primary wave functions, as interesting as they may be, would not be so much of usage if they did not form a complete basis for on-shell wave functions in $\mathbb{R}^{1,d+1}$. Indeed, as shown in [12], we can identify on-shell momenta p_μ as $p_\mu = m y_\mu$ which satisfies $p_\mu p^\mu = -m^2$ as required. With this identification, we can rewrite Eq. (2.5) as

$$\phi^\pm_{\Delta,m}(x^\mu, z, \bar{z}) = \int_{\mathrm{H}^3} [dy_i] G_\Delta(y_i, z, \bar{z}) e^{\pm i x_\mu p^\mu} \tag{2.9}$$

which can be seen as a basis transformation from momentum space spanned by $\{e^{\pm i x_\mu p^\mu}\}$ to a new basis spanned by $\phi^\pm_{\Delta,m}(x^\mu, z)$. The original basis was labeled with $\{p_\mu \in \mathbb{R}^{1,3} | p_\mu p^\mu = -m^2\}$ whereas the new basis is labeled with $\{(\Delta, z) \in (\mathbb{C}, \mathbb{C}_\infty) | \Delta = 1 + i\mathbb{R}\}$.[5,6]

The basis transformation for massless scalars is far more intuitive. To see that, we first write the momentum vector as

$$p^\mu = \epsilon\omega x^\mu(z) \tag{2.10}$$

where $\epsilon = 1(-1)$ for outgoing (incoming) momentum. We can interpret the z coordinate trivially: it just parametrizes the *direction* of the momentum vector on the celestial sphere. Now, we only need to relate the *magnitude* of the momentum, ω, to the scaling dimension of the conformal primary wave function, i.e., Δ. We can see this relation between the basis vectors in the form of a Mellin transformation

$$\phi^\pm_\Delta(x^\mu, z, \bar{z}) = \int_0^\infty d\omega \omega^{\Delta-1} e^{\pm i(x_\mu p^\mu \pm i\epsilon)} \tag{2.11}$$

which follows from Eq. (2.7a) and $p^\mu = \omega x^\mu(z)$. Physically, this means that the Lorentz boosts, which act as $x_\mu p^\mu \to \lambda x_\mu p^\mu$, become the dilatation in conformal primary wave function bases, i.e., $\phi^\pm_\Delta(x^\mu, z) \to \lambda^{-\Delta}\phi^\pm_\Delta(x^\mu, z)$.

We can see that the relation between plane waves and massless conformal primary wave functions is implemented by a Mellin transformation for spin 1 and 2 cases as well, though there are subtleties of gauge and diffeomorphism invariances.[7] We refer the reader to [12] for further details.

With the bases of conformal primary wave functions set up, we can now construct *celestial amplitudes*. The traditional amplitudes with external particles being momentum eigenstates can be cast into the form

[5]The restriction of Δ to *principal series*, i.e., $\Delta = \frac{d}{2} + i\mathbb{R}$, is a necessary condition for the conformal quadratic Casimir operator to be self-adjoint, which ensures by the spectral theorem that it has an orthonormal basis of eigenvectors; thus $\phi^\pm_{1+i\mathbb{R},m}$ form an orthonormal basis. For $m \neq 0$, shadow symmetry[6] makes $\frac{d}{2} \pm i\nu$ linearly dependent, hence we only take half of the principal series, i.e., $\Delta = \frac{d}{2} + i\mathbb{R}_{\geq 0}$.

[6]Shadow transformation is an interwining map from an operator in representation (Δ, ρ) to another operator in representation $(d - \Delta, \rho^R)$ where ρ is an $\mathrm{SO}(d)$ irrep and where ρ^R denotes the reflected representation. In odd dimensions, one can take $\rho^R \simeq \rho$ hence shadow transformation amounts to $\Delta \to d - \Delta$ which relates the principal series representations $\Delta = \frac{d}{2} \pm i\nu$.

[7]For example, $A^{\Delta,\pm}_{\mu,a}(x^\mu, z, \bar{z})$ can be related to plane waves with a Mellin transformation only if $\Delta \neq 1$ in $d = 4$. This follows from the fact that $A^{1,\pm}_{\mu,a}(x^\mu, z, \bar{z})$ is simply *a pure gauge term* in $d = 4$, hence it cannot be related to the physical plane wave solution.

$$\mathcal{A}(p_1, \ldots, p_n) = \int \prod_{i=1}^{n} d^4x_i e^{i(p_i)_\mu x_i^\mu} a(x_1, \ldots x_n) \tag{2.12}$$

where p_i are outgoing momenta of external scalars and where $a(x_1, \ldots x_n)$ is the rest of the amplitude. The celestial amplitude $\tilde{\mathcal{A}}$ requires all external wave functions to be conformal primary wave functions instead, hence

$$\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}(z_1, \ldots, z_n) = \int \prod_{i=1}^{n} d^4x_i \phi^{+}_{\Delta_i}(x_i^\mu, z_i) a(x_1, \ldots x_n). \tag{2.13}$$

For example, for cubic vertex interaction $\mathcal{L} \sim \lambda\phi_1\phi_2\phi_3$, we can compare the three-point scattering amplitudes as follows:

$$\mathcal{A}(p_1, p_2, p_3) = \int \prod_{i=1}^{3} d^4x_i e^{i(p_i)_\mu x_i^\mu}(i\lambda) = i\lambda(2\pi)^4\delta^4(\vec{p}_1 + \vec{p}_2 + \vec{p}_3), \tag{2.14a}$$

$$\tilde{\mathcal{A}}^{\Delta_1,\Delta_2,\Delta_3}(z_1, z_2, z_3) = \int \prod_{i=1}^{3} d^4x_i \phi^{+}_{\Delta_i}(x_i^\mu, z_i)(i\lambda) \sim \frac{\lambda}{|z_{12}|^{\Delta_{123}}|z_{23}|^{\Delta_{231}}|z_{31}|^{\Delta_{312}}} \tag{2.14b}$$

for $z_{ij} \equiv z_i - z_j$ and $\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k$. As expected by the conformal covariance of $\tilde{\mathcal{A}}$, the three-point celestial amplitude in $4d$ takes the form of the three-point CFT correlator in $2d$.

By using the basis change in Eq. (2.9) for massive scalars or Eq. (2.11) for massless scalars, we can relate $\mathcal{A}(p_i)$ and $\tilde{\mathcal{A}}^{\Delta_i}(z_i)$. In fact, for all massless spin-0,1,2 external particles, the transition from $\mathcal{A}$ to $\tilde{\mathcal{A}}$ takes the form of a Mellin transformation, which implements the map from momentum eigenstates to boost eigenstates:

$$\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(z_1, \ldots, z_n) = \left(\prod_{i=1}^{n}\int_0^\infty d\omega_i \omega_i^{\Delta_i-1}\right)\mathcal{A}_{j_1\ldots j_n}(\omega_1, \ldots, \omega_n; z_1, \ldots, z_n) \tag{2.15}$$

where we used $p^\mu = \omega x^\mu(z,\bar{z}) = \omega(2, -2z\bar{z}, z+\bar{z}, -i(z-\bar{z}))$ and where $2d$ spin J_i is identified with $4d$ helicity j_i: $J_i = j_i$.

n-successive Mellin transforms are relatively straightforward; however, we can simplify it further via the exploitation of the covariance of $\tilde{\mathcal{A}}$ under boosts (dilations in the celestial sphere) by switching to simplex variables

$$s \equiv \sum_{i=1}^{n}\omega_i, \qquad \sigma_i \equiv \frac{\omega_i}{s} \tag{2.16}$$

under which Eq. (2.15) becomes

$$\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(z_1, \ldots, z_n) = 2\pi\delta\left(i(\kappa - n) + \sum_{i=1}^{n}\lambda_i\right)\prod_{k=1}^{n}\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k}\right)\delta^4\left(\sum_{i=1}^{n}q_i^\mu\sigma_i\right)\delta\left(\sum_{i=1}^{n}\sigma_i - 1\right)\mathcal{A}_{j_1\ldots j_n}\begin{bmatrix}\sigma_1\ldots\sigma_n\\ z_1\ldots z_n\end{bmatrix} \tag{2.17}$$

where we wrote down Δ on the principal series as

$$\Delta = 1 + i\lambda, \qquad \lambda \in \mathbb{R} \tag{2.18}$$

and where we define the stripped amplitude $\mathcal{A}_{j_1\ldots j_n}\left[\begin{smallmatrix}\sigma_1\ldots\sigma_n\\ z_1\ldots z_n\end{smallmatrix}\right]$ as

$$\mathcal{A}_{j_1\ldots j_n}(\omega_1, \ldots, \omega_n; z_i, \ldots, z_n) = s^{-\kappa}\mathcal{A}_{j_1\ldots j_n}\begin{bmatrix}\sigma_1\ldots\sigma_n\\ z_1\ldots z_n\end{bmatrix}\delta^4\left(\sum_{i=1}^{n}q_i^\mu\sigma_i\right). \tag{2.19}$$

Here κ is the overall momentum scaling of the amplitude,[8] i.e.,

[8]For example, for tree-level Maximal-Helicity-Violating (MHV) amplitudes, $\kappa = n$ as we can easily see from Eq. (2.32).

$$\mathcal{A}_{j_1\ldots j_n}(\Lambda\omega_1,\ldots,\Lambda\omega_n;z_i,\ldots,z_n)=\Lambda^{-\kappa}\mathcal{A}_{j_1\ldots j_n}(\omega_1,\ldots,\omega_n;z_i,\ldots,z_n) \tag{2.20}$$

and we defined

$$q_i^\mu\equiv\epsilon_i x^\mu(z_i) \tag{2.21}$$

for brevity.

One can leverage the covariance of celestial amplitudes under the conformal group by going to a *conformal frame* where we choose[9]

$$z_1=0,\qquad z_2=\infty,\qquad z_3=1. \tag{2.22}$$

As the dilation acts inversely at infinity, a correct procedure to put an operator at $z=\infty$ is by the limit

$$\mathcal{O}(\infty)=\lim_{L\to\infty}L^{2\Delta_\mathcal{O}}\mathcal{O}(L), \tag{2.23}$$

hence we define the celestial amplitude in this conformal frame as[10]

$$\begin{aligned}\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(0,\infty,1,z_4,\ldots,z_n)={}&\frac{\pi}{2}\delta\left(i(\kappa-n)+\sum_{i=1}^n\lambda_i\right)\prod_{k=3}^n\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k}\right)\left(1-\sum_{i=3}^n\sigma_i\right)^{i\lambda_1}\\&\times\left(-\sum_{i=3}^n\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right)^{i\lambda_2}\delta\left(\sum_{i=3}^n\epsilon_i\sigma_i z_i\right)\delta\left(\sum_{i=3}^n\epsilon_i\sigma_i\bar{z}_i\right)\\&\times\delta\left(1+\sum_{i=3}^n(\epsilon_1\epsilon_i-1)\sigma_i\right)\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_n}\left[\begin{matrix}1-\sum_{i=3}^n\sigma_i,-L^{-2}\sum_{i=3}^n\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i),\sigma_3,\ldots,\sigma_n\\0,L,1,z_4,\ldots,z_n\end{matrix}\right]\end{aligned} \tag{2.25}$$

where delta functions of momentum conservation along the lightcone coordinates were immediately employed to remove $\sigma_{1,2}$ integrations via the use of Eq. (2.4) in Eq. (2.17), hence the delta functions above are due to the momentum conservation along transverse directions and due to the normalization condition of the simplex variables, i.e., $\sum_{i=1}^n\sigma_i=1$.

[9]Given any three points, we can first use translations to fix $z_1=0$, then special conformal transformation to take $z_2\to\infty$, then dilation to bring z_3 to a unit circle, and finally rotation to get $z_3=1$. As this exhausts all conformal transformations, $z_{n>3}$ remains unfixed. By applying these transformations in reverse, we can get any amplitude $\tilde{\mathcal{A}}(z_1,z_2,z_3,z_4,\ldots,z_n)$ from $\tilde{\mathcal{A}}(0,\infty,1,z_4',\ldots,z_n')$. See Appendix A 1 for further details.

[10]We use

$$\begin{aligned}&\lim_{L\to\infty}\prod_{k=1}^n\left(\int_0^1 d\sigma_k\right)\delta^4\left(\sum_{i=1}^n q_i^\mu\sigma_i\right)\delta\left(\sum_{i=1}^n\sigma_i-1\right)f(\sigma_1,\sigma_2,\sigma_3,\ldots,\sigma_n;z_i)\\&=\prod_{k=3}^n\left(\int_0^1 d\sigma_k\right)\delta\left(1+\sum_{i=3}^n(\epsilon_1\epsilon_i-1)\sigma_i\right)\\&\quad\times\delta\left(\sum_{i=3}^n\epsilon_i\sigma_i z_i\right)\delta\left(\sum_{i=3}^n\epsilon_i\sigma_i\bar{z}_i\right)\lim_{L\to\infty}\frac{1}{4L^2}f\left(1-\sum_{i=3}^n\sigma_i,-\frac{1}{L^2}\sum_{i=3}^n\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i),\sigma_3,\ldots,\sigma_n;z_i\right)\end{aligned} \tag{2.24}$$

which follows from the conformal frame we choose and $q_i^\mu=\epsilon_i x^\mu(z_i)$ with our choice of $x^\mu(z)$ in Eq. (2.4).

By using Eq. (A13), we can rewrite this equation as

$$\begin{aligned}&\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(0,\infty,1,z_4,\ldots,z_n)\\&=\frac{\pi}{2}\frac{\mathcal{U}(\beta_i)}{|M_{1,2,3}|}\delta\left(i(\kappa-n)+\sum_{i=1}^{n}\lambda_i\right)\prod_{k=6}^{n}\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k}\right)\beta_1^{i\lambda_3}\beta_2^{i\lambda_4}\beta_3^{i\lambda_5}\\&\quad\times\left(1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i\right)^{i\lambda_1}\left(-\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})-\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right)^{i\lambda_2}\\&\quad\times\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_n}\left[\begin{matrix}1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i,-L^{-2}\left(\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})+\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right),\beta_1,\beta_2,\beta_3,\sigma_6,\ldots,\sigma_n\\0,L,1,z_4,\ldots,z_n\end{matrix}\right]\end{aligned}\tag{2.26a}$$

for $n>4$; in particular, we do not have any integrals left for $n=5$:

$$\begin{aligned}&\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_5}_{J_1\ldots J_5}(0,\infty,1,z_4,z_5)\\&\quad=\frac{\pi}{2}\frac{\mathcal{U}(\beta_i)}{|M_{1,2,3}|}\delta\left(i(\kappa-5)+\sum_{i=1}^{5}\lambda_i\right)\beta_1^{i\lambda_3}\beta_2^{i\lambda_4}\beta_3^{i\lambda_5}\left(1-\sum_{i=1}^{3}\beta_i\right)^{i\lambda_1}\\&\quad\times\left(-\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})\right)^{i\lambda_2}\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_n}\left[\begin{matrix}1-\sum_{i=1}^{3}\beta_i,-L^{-2}\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2}),\beta_1,\beta_2,\beta_3\\0,L,1,z_4,z_5\end{matrix}\right].\end{aligned}\tag{2.26b}$$

For $n=4$, we instead use Eq. (A18), with which Eq. (2.25) becomes

$$\begin{aligned}\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_4}_{J_1\ldots J_4}(0,\infty,1,z_4)=&\frac{\pi}{2}\mathcal{U}(\beta_i)\delta(\bar{z}_4-z_4)\delta\left(i(\kappa-4)+\sum_{i=1}^{4}\lambda_i\right)\beta_1^{i\lambda_3}\beta_2^{i\lambda_4}\left(1-\sum_{i=1}^{2}\beta_i\right)^{i\lambda_1}\left(-\sum_{i=1}^{2}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})\right)^{i\lambda_2}\\&\times\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_4}\left[\begin{matrix}1-\sum_{i=1}^{2}\beta_i,-L^{-2}\sum_{i=1}^{2}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2}),\beta_1,\beta_2\\0,L,1,z_4\end{matrix}\right].\end{aligned}\tag{2.26c}$$

For details regarding the coefficients β, $M_{1,2,3}$, and the function $\mathcal{U}$ in Eq. (2.26), please see Appendix A 2. One curious observation regarding Eq. (2.26) is that the celestial amplitudes are on the principal series (i.e., $\lambda\in\mathbb{R}$) only if $\kappa=n$; in other words, we need to analytically continue off the principal series if the amplitude $\mathcal{A}_{j_1\ldots j_n}(p_i)$ does not have the mass dimension $-n$. The celestial amplitude $\tilde{\mathcal{A}}$ being off the principal series means that the CFT operators are no longer in the unitary representation of the group[11]; and in particular, it means that the conformal primary wave functions would not constitute an orthonormal basis for these amplitudes.[12] Nevertheless, the procedure of doing harmonic analysis for the CFT correlators on the principal series and then analytically continuing them to the regions of interest is relatively well known and has been extensively used to extract CFT data through a Euclidean inversion formula [63–66].[13] We should also note that *the generalized soft limit* on a celestial sphere may relate amplitudes on principal series to amplitudes off principal series as we have

[11]One should not confuse the unitarity of the group representation, which has to do with the self-adjointness of the Casimir operator, with the unitarity of the field theory, which is the requirement that norms of the states in Hilbert space are non-negative. Indeed, unitarity of the CFT actually requires other conditions for Δ than it being on the principal series (i.e., $\Delta=1+i\mathbb{R}$); for example, we need $\Delta\geq l+d-1-\frac{d}{2}\delta_{l,0}$ for a CFT$_d$ operator in symmetric traceless tensor representation for the CFT to be unitary.

[12]Principal series representations are actually not the only unitary representations for conformal groups, but they are the only tempered unitary representation that appears in $2d$ [62].

[13]For possible subtleties regarding the analytic continuation, see [67] and references therein.

$$\lim_{\lambda_m\to 0}\lambda_m\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(z_1,\ldots,z_n)=-2i\sum_a f^{\rm soft}_{a,J_m}(z_i)\tilde{\mathcal{A}}^{\Delta_1^{(a)},\ldots,\Delta_{m-1}^{(a)},\Delta_{m+1}^{(a)},\ldots,\Delta_n^{(a)}}_{J_1,\ldots,J_{m-1},J_{m+1},\ldots,J_n}(z_1,\ldots,z_{m-1},z_{m+1},\ldots,z_n) \tag{2.27a}$$

for

$$\Delta_k=1+i\lambda_k,\qquad \Delta_k^{(a)}=1+i\lambda_k+n_k^{(a)}; \tag{2.27b}$$

thus in the rest of the paper we will not dwell on the appearance of $\delta(i(\kappa-n)+\sum_{i=1}^n\lambda_i)$ in Eq. (2.26).[14]

B. Tree-level gluon celestial amplitudes

In this section, we will review some tree-level results before we move on to computing loop-level gluon celestial amplitudes in the body.

One of the simplest tree-level examples that we can consider is a color-ordered MHV amplitude, which in spinor helicity notation takes the form

$$\mathcal{A}^{\rm MHV}_{--+\cdots+}(p_i)=\frac{\langle 12\rangle^3}{\langle 23\rangle\ldots\langle n1\rangle}\delta^4\left(\sum_{i=1}^n p_i^\mu\right) \tag{2.30}$$

where we are following the notation of [15] for spinor helicity formalism. In particular,

$$[ij]=2\sqrt{\omega_i\omega_j}\bar{z}_{ij},\qquad \langle ij\rangle=-2\epsilon_i\epsilon_j\sqrt{\omega_i\omega_j}z_{ij}, \tag{2.31}$$

hence

$$\mathcal{A}^{\rm MHV}_{--+\cdots+}(\omega_1,\ldots,\omega_n;z_1,\ldots,z_n)=(-2)^{4-n}\frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}\frac{\omega_1^2\omega_2^2}{\omega_1\omega_2\ldots\omega_n}\delta^4\left(\sum_{i=1}^n\epsilon_i\omega_i x^\mu(z_i)\right). \tag{2.32}$$

Clearly, $\kappa=n$ for this amplitude and we can write down

$$\mathcal{A}^{\rm MHV}_{--+\cdots+}\begin{bmatrix}\sigma_1\ldots\sigma_n\\ z_1\ldots z_n\end{bmatrix}=(-2)^{4-n}\frac{z_{12}^4}{z_{12}z_{23}\ldots z_{n1}}\frac{\sigma_1^2\sigma_2^2}{\sigma_1\sigma_2\ldots\sigma_n} \tag{2.33}$$

for which we have

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[14] We can take the soft limit in momentum space as

$$\lim_{\omega_m\to 0}\omega_m\mathcal{A}_{J_1\ldots J_n}(\omega_1,\ldots,\omega_n;z_1,\ldots,z_n)=\left(\sum_a f^{\rm soft}_{a,J_m}(z_i)\prod_{k=1,\ldots,m-1,m+1,\ldots,n}(\omega_k)^{n_k^{(a)}}\right)\times\mathcal{A}_{J_1\ldots J_{m-1},J_{m+1},\ldots,J_n}(\omega_1,\ldots,\omega_{m-1},\omega_{m+1},\ldots,\omega_n;z_1,\ldots,z_{m-1},z_{m+1},\ldots,z_n) \tag{2.28}$$

for the soft factor $f^{\rm soft}_{a,J_m}(z_i)$ and the coefficients $n_k^{(a)}$ which depend on the amplitude under consideration. We can then derive Eq. (2.27) with this equation and the representation of delta distribution as

$$\delta(x)=\frac{i}{2}\lim_{\lambda_m\to 0}\lambda_m\omega_m^{i\lambda_m-1}. \tag{2.29}$$

For example, in the case of an n point graviton amplitude with the choice of $m=n$, we have $f^{\rm soft}_{a,J_n=+}(z_i)=\frac{1}{\epsilon_a\epsilon_n}\frac{\bar{z}_{na}z_{xa}z_{ya}}{z_{na}z_{xn}z_{yn}}$ and $n_k^{(a)}=\delta_k^a$ for $a=1,\ldots,n-1$ where x and y are properly chosen reference points [25].

$$\lim_{L\to\infty}\mathcal{A}^{\mathrm{MHV}}_{--+\cdots+}\left[\begin{matrix}1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i,-L^{-2}\left(\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})+\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right),\beta_1,\beta_2,\beta_3,\sigma_6,\ldots,\sigma_n\\ 0,L,1,z_4,\ldots,z_n\end{matrix}\right]$$
$$=(-2)^{4-n}\frac{1}{(1-z_4)z_{45}z_{56}\ldots z_{(n-1)n}z_n}\frac{1}{\beta_1\beta_2\beta_3}\frac{1}{\sigma_6\ldots\sigma_n}\left(1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i\right)$$
$$\times\left(\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})+\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right). \tag{2.34}$$

Therefore, Eq. (2.26a) becomes

$$(\tilde{\mathcal{A}}^{\mathrm{MHV}})^{\Delta_1,\ldots,\Delta_n}_{--+\cdots+}(0,\infty,1,z_4,\ldots,z_n)$$
$$=\frac{\pi(-2)^{3-n}}{(1-z_4)z_{45}z_{56}\ldots z_{(n-1)n}z_n}\frac{\mathcal{U}(\beta_i)}{|M_{1,2,3}|}\delta\left(\sum_{i=1}^{n}\lambda_i\right)\prod_{k=6}^{n}\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k-1}\right)$$
$$\times\beta_1^{i\lambda_3-1}\beta_2^{i\lambda_4-1}\beta_3^{i\lambda_5-1}\left(1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i\right)^{i\lambda_1+1}\left(-\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})-\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right)^{i\lambda_2+1} \tag{2.35}$$

where $n=5$ case can be straightforwardly written as

$$(\tilde{\mathcal{A}}^{\mathrm{MHV}})^{\Delta_1,\ldots,\Delta_n}_{--+\cdots+}(0,\infty,1,z_4,z_5)=\frac{\pi}{4(1-z_4)z_{45}z_5}\frac{\mathcal{U}(\beta_i)}{|M_{1,2,3}|}\delta\left(\sum_{i=1}^{5}\lambda_i\right)\beta_1^{i\lambda_3-1}\beta_2^{i\lambda_4-1}\beta_3^{i\lambda_5-1}$$
$$\times\left(1-\sum_{i=1}^{3}\beta_i\right)^{i\lambda_1+1}\left(-\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})\right)^{i\lambda_2+1}. \tag{2.36}$$

By using the prescription detailed in Appendix A 2, we can compute β_i and write down the explicit expression for any given momenta; for example, we have

$$(\tilde{\mathcal{A}}^{\mathrm{MHV}})^{\Delta_1,\ldots,\Delta_n}_{--+\cdots+}(0,\infty,1,z_4,\ldots,z_n)\Big|_{\substack{p_{2,3}:\ \text{incoming}\\ p_{1,k\geq4}:\ \text{outgoing}}}$$
$$=\frac{\pi(-2)^{3-n}}{(1-z_4)z_{45}z_{56}\ldots z_{(n-1)n}z_n}\frac{\mathcal{U}(\beta_i)}{|\chi_{45}|}\delta\left(\sum_{i=1}^{n}\lambda_i\right)$$
$$\times\prod_{k=6}^{n}\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k-1}\right)\left(\frac{\chi_{34}+\chi_{53}+\chi_{45}(1-2\sum_{k=6}^{n}\sigma_k)-2\sum_{k=6}^{n}(\chi_{4k}-\chi_{5k})\sigma_k}{\chi_{45}}\right)^{i\lambda_1+1}$$
$$\times\left(\frac{\chi_{34}\chi_5+\chi_{45}(\chi_3-2\sum_{k=6}^{n}\chi_k\sigma_k)+\chi_{53}\chi_4}{\chi_{54}}\right)^{i\lambda_2+1}\left(\frac{\chi_{53}+2\sum_{k=6}^{n}\chi_{5k}\sigma_k}{\chi_{54}}\right)^{i\lambda_4-1}\left(\frac{\chi_{43}+2\sum_{k=6}^{n}\chi_{4k}\sigma_k}{\chi_{45}}\right)^{i\lambda_5-1} \tag{2.37}$$

for

$$\chi_i\equiv1+z_i\bar{z}_i,\qquad\chi_{ij}\equiv z_i\bar{z}_j-z_j\bar{z}_i\quad\text{for } z_3=\bar{z}_3=1 \tag{2.38}$$

where $\mathcal{U}(\beta_i)=0,1$ and it should be understood as a reminder that the expression is nonzero only for certain regions of z_i, regions whose explicit description we will not provide for the most generic case.

For $n = 5$, the expression significantly simplifies; in particular,

$$(\tilde{\mathcal{A}}^{\mathrm{MHV}})^{\Delta_1,\ldots,\Delta_n}_{--+++}(0,\infty,1,z_4,z_5)\big|_{\substack{p_{2,3}:\,\text{incoming}\\ p_{1,4,5}:\,\text{outgoing}}}$$
$$= -\frac{\pi\mathcal{U}(\beta_i)}{4(1-z_4)z_{45}z_5}\delta\left(\sum_{i=1}^{5}\lambda_i\right)\times(\chi_{43}+\chi_{35}+\chi_{54})^{i\lambda_1+1}(\chi_{34}\chi_5+\chi_{45}\chi_3+\chi_{53}\chi_4)^{i\lambda_2+1}(\chi_{54})^{i\lambda_3-1}(\chi_{53})^{i\lambda_4-1}(\chi_{34})^{i\lambda_5-1} \quad (2.39)$$

if we restrict to $\chi_{54} \in \mathbb{R}^+$.

Of course, there is nothing specific about choosing second and third momenta to be outgoing and the rest incoming; for example,

$$(\tilde{\mathcal{A}}^{\mathrm{MHV}})^{\Delta_1,\ldots,\Delta_n}_{--+++}(0,\infty,1,z_4,z_5)\big|_{\substack{p_{4,5}:\,\text{incoming}\\ p_{1,2,3}:\,\text{outgoing}}}$$
$$= -\frac{\pi\mathcal{U}(\beta_i)}{4(1-z_4)z_{45}z_5}\delta\left(\sum_{i=1}^{5}\lambda_i\right)(-\chi_{43}-\chi_{35}-\chi_{54})^{i\lambda_1+1}(\chi_{34}\chi_5+\chi_{45}\chi_3+\chi_{53}\chi_4)^{i\lambda_2+1}(\chi_{54})^{i\lambda_3-1}(\chi_{53})^{i\lambda_4-1}(\chi_{34})^{i\lambda_5-1} \quad (2.40)$$

if we restrict to $\chi_{53} + \chi_{34} \in \mathbb{R}^+$.

III. LOOP AMPLITUDES ON THE CELESTIAL SPHERE

In this section, we will consider gluon and graviton loop amplitudes with all-plus helicity (all external particles have positive helicity) and one-minus helicity (all but one external particles have positive helicity). These helicity configurations are particularly interesting choices as their tree-level counterparts vanish as shown in [39]. These *rational* amplitudes are also interesting for other reasons; for instance, they are not cut-constructible through unitarity cuts in four dimensions. In addition their expressions take surprisingly compact forms, reminiscent of tree-level amplitudes. Finally, they are free of logarithmic divergences, which make them ideal candidates for celestial amplitudes beyond tree level.

A. Four-point amplitudes

Let us start with the four-point celestial amplitudes for gluons and gravitons. We have seen that a generic four-point celestial amplitude can be computed from Eq. (2.26c), which becomes

$$\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_4}_{J_1\ldots J_4}(0,\infty,1,z_4) = \frac{\pi}{2}\mathcal{U}(\beta_i)\delta(\bar{z}_4-z_4)\delta\left(i(\kappa-4)+\sum_{i=1}^{4}\lambda_i\right)\left(\frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}\right)^{i\lambda_1}$$
$$\times\left(-\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}\right)^{i\lambda_2}\left(\frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}\right)^{i\lambda_3}\left(\frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1}\right)^{i\lambda_4}$$
$$\times\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_4}\begin{bmatrix}\frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}, -L^{-2}\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}, \frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}, \frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1}\\ 0, L, 1, z_4\end{bmatrix} \quad (3.1)$$

where we defined the shorthand notation

$$\epsilon_{i_1,i_2\ldots i_n} \equiv \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_n} \quad (3.2)$$

and where we have used the prescription detailed in Appendix A 2 to compute β_i. With β_i, we can also compute $\mathcal{U}(\beta_i)$ explicitly as can be seen in Table I.

The computation of the last term in Eq. (3.1) is straightforward, but we can simplify it even further with the following prescription. Given *any* four-point amplitude of the form

TABLE I. The breakdown of the support of the four-point celestial amplitude on $\mathbb{R}$ depending on which momenta lie on future lightcone ($\epsilon = 1$) and which momenta lie on past lightcone ($\epsilon = -1$).

Case	Region $\mathcal{U}(\beta_i)$ is 1
$\epsilon_1 = \epsilon_4 = -\epsilon_3$	$z_4 > \frac{1}{2}$
$\epsilon_1 = \epsilon_3 = -\epsilon_4$	$2 \geq z_4 \geq 0$
$\epsilon_3 = \epsilon_4 = -\epsilon_1$	$0 > z_4$

$$\mathcal{A}_{j_1\ldots j_4}(p_i) = \sum_{1\le i<j\le 4} \langle ij\rangle^{m_{ij}}[ij]^{n_{ij}}\delta^4\left(\sum_{i=1}^{n} p_i^{\mu}\right) \tag{3.3}$$

we can immediately write

$$\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_4}\begin{bmatrix} \frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}, -L^{-2}\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}, \frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}, \frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1} \\ 0, L, 1, z_4 \end{bmatrix}$$
$$= \sum_{1\le i<j\le 4}(-\epsilon_{i,j})^{m_{ij}}(2a_{ij}^{(4)})^{m_{ij}+n_{ij}} \tag{3.4}$$

for

$$a_{12}^{(4)} = -\sqrt{-\frac{(z_4-1)^3\epsilon_{1,2}}{((z_4-1)\epsilon_{4,1,3}-z_4\epsilon_4+\epsilon_3)^2}}, \qquad a_{13}^{(4)} = -\sqrt{-\frac{(z_4-1)z_4\epsilon_{1,3}}{((z_4-1)\epsilon_{4,1,3}-z_4\epsilon_4+\epsilon_3)^2}},$$
$$a_{23}^{(4)} = \sqrt{\frac{(z_4-1)^2 z_4\epsilon_{2,3}}{((z_4-1)\epsilon_{4,1,3}-z_4\epsilon_4+\epsilon_3)^2}}, \qquad a_{14}^{(4)} = -z_4\sqrt{\frac{(z_4-1)\epsilon_{1,4}}{((z_4-1)\epsilon_{4,1,3}-z_4\epsilon_4+\epsilon_3)^2}},$$
$$a_{24}^{(4)} = \sqrt{-\frac{(z_4-1)^2\epsilon_{2,4}}{((z_4-1)\epsilon_{4,1,3}-z_4\epsilon_4+\epsilon_3)^2}}, \qquad a_{34}^{(4)} = (1-z_4)\sqrt{-\frac{z_4\epsilon_{3,4}}{((z_4-1)\epsilon_{4,1,3}-z_4\epsilon_4+\epsilon_3)^2}}. \tag{3.5}$$

This prescription is valid for any amplitude as it simply follows from the kinematics. For example, the color-ordered gluon four-point one-loop amplitudes for all-plus and single-minus helicities in pure Yang-Mills theory can be written as

$$\mathcal{A}_{++++}^{\texttt{gluon}}(p_i) = -c\frac{[23][41]}{\langle 23\rangle\langle 41\rangle}\delta^4\left(\sum_{i=1}^{n} p_i^{\mu}\right), \qquad \mathcal{A}_{-+++}^{\texttt{gluon}}(p_i) = c\frac{\langle 24\rangle[24]^3}{[12]\langle 23\rangle\langle 34\rangle[41]}\delta^4\left(\sum_{i=1}^{n} p_i^{\mu}\right) \tag{3.6}$$

as seen in [40–42].[15] With the prescription above, we get

$$\lim_{L\to\infty}\mathcal{A}_{++++}^{\texttt{gluon}}\begin{bmatrix} \frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}, -L^{-2}\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}, \frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}, \frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1} \\ 0, L, 1, z_4 \end{bmatrix} = -\frac{c}{\epsilon_{1,2,3,4}}, \tag{3.7a}$$

$$\lim_{L\to\infty}\mathcal{A}_{-+++}^{\texttt{gluon}}\begin{bmatrix} \frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}, -L^{-2}\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}, \frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}, \frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1} \\ 0, L, 1, z_4 \end{bmatrix} = \frac{c\,\mathrm{sgn}(z_4(1-z_4))}{z_4^2}. \tag{3.7b}$$

We can also consider one-loop gravity amplitudes for both all-positive and one-minus cases. These amplitudes have been computed using string based methods [43–45]. Also, it is interesting to note that the four-point all-plus one-loop gravity amplitude can be calculated using the Bern-Carrasco-Johansson (BCJ) double copy construction and we refer the readers to [48] for more details. In spinor helicity formalism, they can be written in a compact form [46,48][16]:

$$\mathcal{A}_{++++}^{\texttt{graviton}}(p_i) = -\frac{i}{(4\pi)^2}\frac{s^2+t^2+u^2}{120}\left(\frac{st}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)^2\delta^4\left(\sum_{i=1}^{n} p_i^{\mu}\right),$$
$$\mathcal{A}_{-+++}^{\texttt{graviton}}(p_i) = 4\left(\frac{st}{u}\right)^2\left(\frac{s^2+st+t^2}{5760}\right)^2\left(\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\right)^2\delta^4\left(\sum_{i=1}^{n} p_i^{\mu}\right). \tag{3.8}$$

With the prescription above, we get

[15] Here, the coefficient $c = i\frac{N_p}{96\pi^2}$ where N_p is the net number of states circulating in the loop.
[16] The parameters s, t, and u are the standard Mandelstam variables, e.g., $s = \langle 12\rangle[12]$.

$$\lim_{L\to\infty}\mathcal{A}^{\texttt{graviton}}_{++++}\left[\begin{matrix}\frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}, -L^{-2}\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}, \frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}, \frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1}\\ 0, L, 1, z_4\end{matrix}\right]$$
$$= -\frac{2i}{15(4\pi)^2}\frac{(z_4-1)^2(z_4(z_4(2(z_4-2)z_4+7)-4)+1)}{z_4^2((z_4-1)\epsilon_{1,3,4}-z_4\epsilon_4+\epsilon_3)^4}, \tag{3.9a}$$

$$\lim_{L\to\infty}\mathcal{A}^{\texttt{graviton}}_{-+++}\left[\begin{matrix}\frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}, -L^{-2}\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}, \frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}, \frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1}\\ 0, L, 1, z_4\end{matrix}\right]$$
$$= \frac{25}{60^4}\frac{(z_4-1)^4((z_4-1)^2\epsilon_{2,3}|z_4|+z_4((z_4-3)z_4+4)(z_4-1)+1)}{z_4^6((z_4-1)\epsilon_{1,3,4}-z_4\epsilon_4+\epsilon_3)^4}. \tag{3.9b}$$

We obtain the full celestial amplitudes by inserting Eq. (3.7) and Eq. (3.9) into Eq. (3.1). For example,

$$(\tilde{\mathcal{A}}^{\texttt{gluon}})^{\Delta_1,\dots,\Delta_4}_{++++}(0,\infty,1,z_4) = -\frac{\pi c}{2\epsilon_{1,2,3,4}}\mathcal{U}(\beta_i)\delta(\bar{z}_4-z_4)\delta\left(\sum_{i=1}^4\lambda_i\right)$$
$$\times\left(\frac{z_4-1}{-z_4(\epsilon_{1,3}-1)+\epsilon_{1,4}-1}\right)^{i\lambda_1}\left(-\frac{z_4(z_4-2)+1}{(z_4-1)\epsilon_{1,2}-z_4\epsilon_{2,3}+\epsilon_{2,4}}\right)^{i\lambda_2}$$
$$\times\left(\frac{z_4}{z_4(-\epsilon_{1,3})+\epsilon_{1,3}-\epsilon_{3,4}+z_4}\right)^{i\lambda_3}\left(\frac{1}{(z_4-1)\epsilon_{1,4}-z_4\epsilon_{3,4}+1}\right)^{i\lambda_4}. \tag{3.10}$$

For specific choices of incoming/outgoing momenta, the expression simplifies significantly; i.e.,

$$(\tilde{\mathcal{A}}^{\texttt{gluon}})^{\Delta_1,\dots,\Delta_4}_{++++}(0,\infty,1,z_4)|_{\substack{p_{2,3}:\,\text{incoming}\\ p_{1,4}:\,\text{outgoing}}} = \begin{cases} -\frac{\pi c}{2}(z_4-1)^{i(\lambda_1+2\lambda_2)}z_4^{i\lambda_3}\delta(\bar{z}_4-z_4)\delta\left(\sum_{i=1}^4\lambda_i\right) & z_4\geq 1\\ -\frac{\pi c}{2}(z_4-1)^{i(\lambda_1+2\lambda_2)}z_4^{i\lambda_3}\delta(\bar{z}_4-z_4)\delta\left(\sum_{i=1}^4\lambda_i\right)e^{2\pi\lambda_2} & 1>z_4\geq\frac{1}{2}.\end{cases} \tag{3.11}$$

We would like to remind the reader that one can get the standard form of the amplitude, i.e., $(\tilde{\mathcal{A}}^{\texttt{gluon}})^{\Delta_1,\dots,\Delta_4}_{++++}(\chi_1,\chi_2,\chi_3,\chi_4)$, from its form in the conformal frame using Eq. (A6).

B. Five-point amplitudes

After considering four points, we would like to extend the computation of celestial amplitudes to five points. Again, we will focus our attention to all-plus and single-minus results. We have seen that a generic five-point celestial amplitude can be computed from Eq. (2.26b), which becomes

$$\tilde{\mathcal{A}}^{\Delta_1,\dots,\Delta_5}_{J_1\dots J_5}(0,\infty,1,z_4,z_5)$$
$$=\frac{\pi}{2}\frac{\mathcal{U}(\beta_i)}{|\varphi|}\delta\left(i(\kappa-5)+\sum_{i=1}^5\lambda_i\right)\left(\frac{(-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5)\epsilon_{1,3,4,5}}{\varphi}\right)^{i\lambda_1}$$
$$\times\left(\frac{(\bar{z}_4+z_5(\bar{z}_4(\bar{z}_5-2)+1)-\bar{z}_5+z_4(z_5\bar{z}_4-(\bar{z}_4+z_5-2)\bar{z}_5-1))\epsilon_{2,3,4,5}}{\varphi}\right)^{i\lambda_2}\left(\frac{(z_5\bar{z}_4-z_4\bar{z}_5)\epsilon_{4,5}}{\varphi}\right)^{i\lambda_3}$$
$$\times\left(\frac{(\bar{z}_5-z_5)\epsilon_{3,5}}{\varphi}\right)^{i\lambda_4}\left(\frac{(z_4-\bar{z}_4)\epsilon_{3,4}}{\varphi}\right)^{i\lambda_5}\lim_{L\to\infty}\mathcal{A}_{j_1\dots j_n}\left[\begin{matrix}1-\sum_{i=1}^3\beta_i, -L^{-2}\sum_{i=1}^3\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2}),\beta_1,\beta_2,\beta_3\\ 0,L,1,z_4,z_5\end{matrix}\right] \tag{3.12}$$

for

$$\varphi\equiv(\bar{z}_5-z_5)(\epsilon_{3,5}-\epsilon_{1,3,4,5})+\bar{z}_4(z_5\epsilon_{4,5}-(z_5-1)\epsilon_{1,3,4,5}-\epsilon_{3,4})+z_4(-\bar{z}_5\epsilon_{4,5}+(\bar{z}_5-1)\epsilon_{1,3,4,5}+\epsilon_{3,4}) \tag{3.13}$$

TABLE II. The breakdown of the support of the five-point celestial amplitude on $\{(x_4, y_4, x_5, y_z) \in \mathbb{R}^4 | z_4 = x_4 + iy_4, z_5 = x_5 + iy_5\}$ depending on which momenta lie on future lightcone ($\epsilon = 1$) and which momenta lie on past lightcone ($\epsilon = -1$).

Case	Region $\mathcal{U}(\beta_i)$ is 1
$\epsilon_1 = \epsilon_3 = \epsilon_4 = -\epsilon_5$	$x_4 < \frac{x_5 y_4}{y_5} \wedge \frac{(x_5-2)y_4}{y_5} < x_4 \wedge ((2y_4 < y_5 \wedge y_5 < 0) \vee (2y_4 > y_5 \wedge y_5 > 0))$
$\epsilon_1 = \epsilon_3 = -\epsilon_4 = \epsilon_5$	$\frac{x_5 y_4}{y_5} < x_4 \wedge x_4 < \frac{x_5 y_4}{y_5} + 2 \wedge ((y_4 > 0 \wedge y_4 < 2y_5) \vee (y_4 < 0 \wedge 2y_5 < y_4))$
$\epsilon_1 = \epsilon_3 = -\epsilon_4 = -\epsilon_5$	$x_4 < \frac{(x_5-2)y_4}{y_5} + 2 \wedge \frac{x_5 y_4}{y_5} < x_4 \wedge ((y_5 > 0 \wedge y_4 < 0) \vee (y_4 > 0 \wedge y_5 < 0))$
$\epsilon_1 = -\epsilon_3 = \epsilon_4 = \epsilon_5$	$(x_4 > \frac{x_5 y_4}{y_5} + \frac{1}{2} \wedge ((y_4 > 0 \wedge y_4 + y_5 \leq 0) \vee (y_4 < 0 \wedge y_4 + y_5 \geq 0)))$ $\vee (y_5(-2x_5 y_4 + 2x_4 y_5 + y_4) > 0 \wedge ((y_4 + y_5 > 0 \wedge y_5 < 0) \vee (y_5 > 0 \wedge y_4 + y_5 < 0)))$
$\epsilon_1 = -\epsilon_3 = \epsilon_4 = -\epsilon_5$	$((y_4 < 0 \vee 2y_4 \leq y_5) \wedge x_4 > \frac{(x_5-1)y_4}{y_5} + \frac{1}{2} \wedge (y_4 > 0 \vee 2y_4 > y_5))$ $\vee ((y_5 < 0 \vee 2y_4 > y_5) \wedge x_4 > \frac{x_5 y_4}{y_5} \wedge (2y_4 \leq y_5 \vee y_5 > 0))$
$\epsilon_1 = -\epsilon_3 = -\epsilon_4 = \epsilon_5$	$(x_4 < \frac{x_5 y_4}{y_5} \wedge ((y_4 > 0 \wedge y_4 < 2y_5) \vee (y_4 < 0 \wedge 2y_5 < y_4)))$ $\vee (x_4 < \frac{(2x_5-1)y_4}{2y_5} + 1 \wedge ((y_5 > 0 \wedge y_4 \geq 2y_5) \vee (y_5 < 0 \wedge y_4 \leq 2y_5)))$
$\epsilon_1 = -\epsilon_3 = -\epsilon_4 = -\epsilon_5$	$x_4 < \frac{x_5 y_4}{y_5} \wedge ((y_5 > 0 \wedge y_4 < 0) \vee (y_4 > 0 \wedge y_5 < 0))$

where we have used the prescription detailed in Appendix A 2 to compute β_i. With β_i, we can also compute $\mathcal{U}(\beta_i)$ explicitly as can be seen in Table II.

We can provide a prescription to compute the last term in Eq. (3.12), similar to what we did for four-point amplitudes in Eq. (3.5). Given *any* five-point amplitude of the form

$$\mathcal{A}_{j_1 \ldots j_5}(p_i) = \sum_{1 \leq i < j \leq 5} \langle ij \rangle^{m_{ij}} [ij]^{n_{ij}} \delta^4 \left(\sum_{i=1}^{n} p_i^{\mu} \right) \tag{3.14}$$

we have

$$\lim_{L \to \infty} \mathcal{A}_{j_1 \ldots j_n} \begin{bmatrix} 1 - \sum_{i=1}^{3} \beta_i, -L^{-2} \sum_{i=1}^{3} \epsilon_2 \epsilon_{i+2} \beta_i (1 + z_{i+2} \bar{z}_{i+2}), \beta_1, \beta_2, \beta_3 \\ 0, L, 1, z_4, z_5 \end{bmatrix} = \sum_{1 \leq i < j \leq 5} \left(-\frac{\epsilon_{i,j} z_{ij}}{\bar{z}_{ij}} \right)^{m_{ij}} (2a_{ij}^{(5)})^{m_{ij} + n_{ij}} \tag{3.15}$$

for the coefficients $a_{ij}^{(5)}$ given in Eq. (A20).

By inserting $\lim_{L \to \infty} \mathcal{A}_{j_1 \ldots j_n} [\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix}]$ into Eq. (3.12), we can obtain the celestial form of *any* amplitude. As an example, we know that the color-ordered gluon five-point amplitude in pure Yang-Mills theory reads in spinor helicity variables as

$$\begin{aligned} \mathcal{A}_{+++++}^{\texttt{gluon}}(p_i) &= -c \frac{\sum_{1 \leq i < j < k < l \leq 5} \langle ij \rangle [jk] \langle kl \rangle [li]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \delta^4 \left(\sum_{i=1}^{n} p_i^{\mu} \right), \\ \mathcal{A}_{-++++}^{\texttt{gluon}}(p_i) &= \frac{c}{\langle 34 \rangle^2} \left(-\frac{[25]^3}{[12][51]} + \frac{\langle 14 \rangle^3 [45] \langle 35 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} - \frac{\langle 13 \rangle^3 [32] \langle 42 \rangle}{\langle 15 \rangle \langle 54 \rangle \langle 32 \rangle^2} \right) \delta^4 \left(\sum_{i=1}^{n} p_i^{\mu} \right), \end{aligned} \tag{3.16}$$

where c is given in footnote 15. With the prescription above, we can find the full expression as given in Eq. (A21).

For special configurations, the expressions simplify; for example,

$$\begin{aligned} &\lim_{L \to \infty} \mathcal{A}_{-++++}^{\texttt{gluon}} \begin{bmatrix} 1 - \sum_{i=1}^{3} \beta_i, -L^{-2} \sum_{i=1}^{3} \epsilon_2 \epsilon_{i+2} \beta_i (1 + z_{i+2} \bar{z}_{i+2}), \beta_1, \beta_2, \beta_3 \\ 0, L, 1, z_4, z_5 \end{bmatrix}_{\substack{p_{1,2}:\text{ incoming} \\ p_{3,4,5}:\text{ outgoing}}} \\ &= \frac{2 \left| \frac{z_5 \bar{z}_4 - z_4 \bar{z}_5}{2z_4 - 2\bar{z}_4} \right|}{z_5 (z_5 - z_4)} + \frac{\left| \frac{(z_4 - \bar{z}_4)(\bar{z}_4 (\bar{z}_5 - 2) z_5 + z_5 + \bar{z}_4 - \bar{z}_5 + z_4 (z_5 (\bar{z}_4 - \bar{z}_5) - (\bar{z}_4 - 2) \bar{z}_5 - 1))}{(-\bar{z}_4 z_5 + z_5 + \bar{z}_4 + z_4 (\bar{z}_5 - 1) - \bar{z}_5)^2} \right|}{\bar{z}_5} \\ &\quad - \frac{(z_5 - 1) z_4^3 (\bar{z}_4 - \bar{z}_5) \left| \frac{\bar{z}_5 - z_5}{\bar{z}_4 (\bar{z}_5 - 2) z_5 + z_5 + \bar{z}_4 - \bar{z}_5 + z_4 (z_5 (\bar{z}_4 - \bar{z}_5) - (\bar{z}_4 - 2) \bar{z}_5 - 1)} \right|}{(z_4 - z_5)^2} \end{aligned} \tag{3.17}$$

with which the full result in Eq. (3.12) becomes

$$
\begin{aligned}
&(\tilde{\mathcal{A}}^{\mathtt{gluon}})^{\Delta_1,\ldots,\Delta_5}_{-++++}(0,\infty,1,z_4,z_5)\big|_{\substack{p_{1,2}:\ \text{incoming}\\ p_{3,4,5}:\ \text{outgoing}}}\\
&\quad=\frac{\pi}{4}\frac{\mathcal{U}(\beta_i)}{|z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5|}\delta\left(\sum_{i=1}^{5}\lambda_i\right)\\
&\qquad\times\left(\frac{\bar{z}_4+z_5(\bar{z}_4(\bar{z}_5-2)+1)-\bar{z}_5+z_4(z_5\bar{z}_4-(\bar{z}_4+z_5-2)\bar{z}_5-1)}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_2}\left(\frac{z_4\bar{z}_5-z_5\bar{z}_4}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_3}\\
&\qquad\times\left(\frac{z_5-\bar{z}_5}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_4}\left(\frac{\bar{z}_4-z_4}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_5}\\
&\qquad\times\left(\frac{2\left|\frac{z_5\bar{z}_4-z_4\bar{z}_5}{2z_4-2\bar{z}_4}\right|}{z_5(z_5-z_4)}+\frac{\left|\frac{(z_4-\bar{z}_4)(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))}{(-\bar{z}_4z_5+z_5+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5)^2}\right|}{\bar{z}_5}\right.\\
&\qquad\left.-\frac{(z_5-1)z_4^3(\bar{z}_4-\bar{z}_5)\left|\frac{\bar{z}_5-z_5}{\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1)}\right|}{(z_4-z_5)^2}\right).
\end{aligned}
\tag{3.18}
$$

As another explicit example, we can consider a five-point graviton amplitude. For the all-plus rational-loop amplitude, we have

$$
\begin{aligned}
&\mathcal{A}^{\mathtt{graviton}}_{+++++}(p_i)\\
&\quad=\frac{i}{960(4\pi)^2}\left(\frac{[45]\left(\langle 12\rangle[12]\langle 34\rangle[34]-(2\leftrightarrow 3)+(2\leftrightarrow 4)+(4\leftrightarrow 5)-\begin{pmatrix}4\leftrightarrow 5\\ 2\leftrightarrow 3\end{pmatrix}+\begin{pmatrix}2\to 5\\ 4\to 2\end{pmatrix}\right)^3}{\langle 12\rangle^2\langle 23\rangle^2\langle 45\rangle\langle 34\rangle\langle 41\rangle\langle 35\rangle\langle 51\rangle}+\text{permutations}\right)\\
&\qquad\times\delta^4\left(\sum_{i=1}^{n}p_i^{\mu}\right)
\end{aligned}
\tag{3.19}
$$

where there are 30 distinct permutations in total. Below, we will consider this term alone; one can analogously repeat the computation for permuted terms; for details of the derivation of Eq. (3.19) and for further information on the other permutations, see Eq. (4.23) of [46].

For generic ϵ_i, the expression $\lim_{L\to\infty}\mathcal{A}^{\mathtt{graviton}}_{+++++}[{\cdots \atop \cdots}]$ takes a rather complicated form; however, it becomes manageable if we switch to real parameters $\{(x_k,y_k)\in\mathbb{R}^2|z_i=x_k+iy_k\}$:

$$
\begin{aligned}
&\lim_{L\to\infty}\mathcal{A}^{\mathtt{graviton}}_{-++++}\begin{bmatrix}1-\sum_{i=1}^{3}\beta_i,-L^{-2}\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+x_{i+2}^2+y_{i+2}^2),\beta_1,\beta_2,\beta_3\\ 0,L,1,x_4+iy_4,x_5+iy_5\end{bmatrix}\\
&\quad=-\frac{128\epsilon_{4,5}(x_4-x_5-i(y_4-y_5))(y_5(x_4(\epsilon_{4,5}-\epsilon_{1,3,4,5})-\epsilon_{3,5}+\epsilon_{1,3,4,5})+y_4(x_5(\epsilon_{1,3,4,5}-\epsilon_{4,5})+\epsilon_{3,4}-\epsilon_{1,3,4,5}))^8}{|y_4y_5|(x_4+iy_4-1)(x_4+iy_4)(x_4-x_5+i(y_4-y_5))(x_5+iy_5-1)(x_5+iy_5)((x_5-1)y_4-(x_4-1)y_5)^2(x_5y_4-x_4y_5)^2}\\
&\qquad\times\frac{\left(\frac{\epsilon_{1,2,3,4}(x_4^2-x_4+y_4^2)|y_5((x_5-1)y_4-(x_4-1)y_5)(x_5y_4-x_4y_5)(y_5(x_4-1)^2+y_4^2y_5-y_4(x_5^2-2x_5+y_5^2+1))|}{(y_5(x_4(\epsilon_{4,5}-\epsilon_{1,3,4,5})-\epsilon_{3,5}+\epsilon_{1,3,4,5})+y_4(x_5(\epsilon_{1,3,4,5}-\epsilon_{4,5})+\epsilon_{3,4}-\epsilon_{1,3,4,5}))^4}+\frac{\epsilon_{1,2,3,5}(x_5^2-x_5+y_5^2)|y_4((x_5-1)y_4-(x_4-1)y_5)(x_5y_4-x_4y_5)(y_5(x_4-1)^2+y_4^2y_5-y_4(x_5^2-2x_5+y_5^2+1))|}{(y_5(x_4(\epsilon_{4,5}-\epsilon_{1,3,4,5})-\epsilon_{3,5}+\epsilon_{1,3,4,5})+y_4(x_5(\epsilon_{1,3,4,5}-\epsilon_{4,5})+\epsilon_{3,4}-\epsilon_{1,3,4,5}))^4}\right)^3}{((x_4-1)^2y_5-y_4(x_5^2-2x_5+y_5^2+1)+y_4^2y_5)^2}\\
&\qquad+\text{contributions due to permuted terms.}
\end{aligned}
\tag{3.20}
$$

If we consider a specific configuration of incoming/outgoing momenta, the expression simplifies enough so that we can write it again in terms of z_i and $\bar{z}_i$; for instance,

$$
\begin{aligned}
&\lim_{L\to\infty}\mathcal{A}^{\mathtt{graviton}}_{-++++}\begin{bmatrix}1-\sum_{i=1}^{3}\beta_i,-L^{-2}\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2}),\beta_1,\beta_2,\beta_3\\ 0,L,1,z_4,z_5\end{bmatrix}_{\substack{p_{1,2}:\ \text{incoming}\\ p_{3,4,5}:\ \text{outgoing}}}\\
&\quad=-\frac{(\bar{z}_4-\bar{z}_5)\left|\frac{1}{(z_4-\bar{z}_4)(z_5-\bar{z}_5)}\right|}{(z_4-1)z_4(z_4-z_5)(z_5-1)z_5(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)^3(z_5\bar{z}_4-z_4\bar{z}_5)^2(z_5\bar{z}_4(\bar{z}_5-2)+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1)+z_5)^2}\\
&\qquad\times((z_4(2\bar{z}_4-1)-\bar{z}_4)|(z_5-\bar{z}_5)(z_5\bar{z}_4-z_4\bar{z}_5)(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))|\\
&\qquad+(z_5(2\bar{z}_5-1)-\bar{z}_5)|(z_4-\bar{z}_4)(z_5\bar{z}_4-z_4\bar{z}_5)(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))|)^3\\
&\qquad+\text{contributions due to permuted terms}
\end{aligned}
\tag{3.21}
$$

with which the celestial amplitude becomes

$$
\begin{aligned}
&(\tilde{\mathcal{A}}^{\mathtt{graviton}})^{\Delta_1,\ldots,\Delta_5}_{-++++}(0,\infty,1,z_4,z_5)|_{\substack{p_{1,2}:\ \text{incoming}\\ p_{3,4,5}:\ \text{outgoing}}}\\
&\quad= -\frac{\pi}{4}\frac{\mathcal{U}(\beta_i)}{|z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5|}\delta\left(\sum_{i=1}^{5}\lambda_i\right)\left(\frac{z_5-\bar{z}_5}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_4}\\
&\qquad\times\left(\frac{\bar{z}_4+z_5(\bar{z}_4(\bar{z}_5-2)+1)-\bar{z}_5+z_4(z_5\bar{z}_4-(\bar{z}_4+z_5-2)\bar{z}_5-1)}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_2}\left(\frac{z_4\bar{z}_5-z_5\bar{z}_4}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_3}\left(\frac{\bar{z}_4-z_4}{-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5}\right)^{i\lambda_5}\\
&\qquad\times\frac{(\bar{z}_4-\bar{z}_5)|\frac{1}{(z_4-\bar{z}_4)(z_5-\bar{z}_5)}|}{(z_4-1)z_4(z_4-z_5)(z_5-1)z_5(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)^3(z_5\bar{z}_4-z_4\bar{z}_5)^2(z_5\bar{z}_4(\bar{z}_5-2)+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1)+z_5)^2}\\
&\qquad\times((z_4(2\bar{z}_4-1)-\bar{z}_4)|(z_5-\bar{z}_5)(z_5\bar{z}_4-z_4\bar{z}_5)(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))|\\
&\qquad+(z_5(2\bar{z}_5-1)-\bar{z}_5)|(z_4-\bar{z}_4)(z_5\bar{z}_4-z_4\bar{z}_5)(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))|)^3\\
&\quad+\text{contributions due to permuted terms.}
\end{aligned}
\tag{3.22}
$$

With Eq. (3.22), we have concluded our series of explicit gluon and graviton celestial amplitude results. As we can see in Eq. (2.26a), there are complicated integrations that need to be carried out beyond five points, hence it is not practical to provide the full explicit answers for higher point amplitudes. Nevertheless, in the next section, we will discuss their generic forms and provide an explicit integrand for an all-plus one-loop gluon amplitude.

C. Higher point amplitudes

We have seen that a generic higher point celestial amplitude can be computed from Eq. (2.26a), which becomes

$$
\begin{aligned}
&\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(0,\infty,1,z_4,\ldots,z_n)\\
&\quad=\frac{\pi}{2}\frac{\mathcal{U}(\beta_i)}{|\varphi|}\delta\left(i(\kappa-n)+\sum_{i=1}^{n}\lambda_i\right)\prod_{k=6}^{n}\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k}\right)\beta_1^{i\lambda_3}\beta_2^{i\lambda_4}\beta_3^{i\lambda_5}\\
&\qquad\times\left(1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i\right)^{i\lambda_1}\left(-\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})-\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right)^{i\lambda_2}\\
&\qquad\times\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_n}\begin{bmatrix}1-\sum_{i=1}^{3}\beta_i-\sum_{i=6}^{n}\sigma_i,-L^{-2}\left(\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})+\sum_{i=6}^{n}\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right),\beta_1,\beta_2,\beta_3,\sigma_6,\ldots,\sigma_n\\ 0,L,1,z_4,\ldots,z_n\end{bmatrix}
\end{aligned}
\tag{3.23}
$$

where φ is defined in Eq. (3.13). One can also compute β_i straightforwardly as explained in Appendix A 2; for the reader's convenience, we provide the explicit results:

$$
\begin{aligned}
\varphi\beta_1 &= (z_5\bar{z}_4-z_4\bar{z}_5)\epsilon_{4,5}+\sum_{i=6}^{n}\sigma_i(z_4(\bar{z}_i(\epsilon_{4,i}-\epsilon_{1,4,5,i})+\bar{z}_5(\epsilon_{1,4,5,i}-\epsilon_{4,5}))+z_i(\bar{z}_5(\epsilon_{5,i}-\epsilon_{1,4,5,i})\\
&\quad+\bar{z}_4(\epsilon_{1,4,5,i}-\epsilon_{4,i}))+z_5(\bar{z}_4(\epsilon_{4,5}-\epsilon_{1,4,5,i})+\bar{z}_i(\epsilon_{1,4,5,i}-\epsilon_{5,i}))),
\end{aligned}
\tag{3.24a}
$$

$$
\begin{aligned}
\varphi\beta_2 &= (\bar{z}_5-z_5)\epsilon_{3,5}+\sum_{i=6}^{n}\sigma_i(\bar{z}_5(\epsilon_{3,5}-\epsilon_{1,3,5,i})+\bar{z}_i(\epsilon_{1,3,5,i}-\epsilon_{3,i})+z_5(\bar{z}_i(\epsilon_{5,i}-\epsilon_{1,3,5,i})+\epsilon_{1,3,5,i}-\epsilon_{3,5})\\
&\quad+z_i(\bar{z}_5(\epsilon_{1,3,5,i}-\epsilon_{5,i})+\epsilon_{3,i}-\epsilon_{1,3,5,i})),
\end{aligned}
\tag{3.24b}
$$

$$
\begin{aligned}
\varphi\beta_3 &= (z_4-\bar{z}_4)\epsilon_{3,4}+\sum_{i=6}^{n}\sigma_i(\bar{z}_i(\epsilon_{3,i}-\epsilon_{1,3,4,i})+\bar{z}_4(\epsilon_{1,3,4,i}-\epsilon_{3,4})+z_i(\bar{z}_4(\epsilon_{4,i}-\epsilon_{1,3,4,i})-\epsilon_{3,i}+\epsilon_{1,3,4,i})\\
&\quad+z_4(\bar{z}_i(\epsilon_{1,3,4,i}-\epsilon_{4,i})-\epsilon_{1,3,4,i}+\epsilon_{3,4})).
\end{aligned}
\tag{3.24c}
$$

By inserting these into Eq. (3.23), we obtain the most generic form with the kinematic constraints applied.

As the most generic form is rather complicated, let us specialize into the situation where the first two momenta are incoming and the rest are outgoing. Indeed, we can show that

$$
\begin{aligned}
&\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}(0,\infty,1,z_4,\ldots,z_n)\Big|_{\substack{p_{1,2}:\ \text{incoming}\\ p_{3,\ldots,n}:\ \text{outgoing}}}\\
&\quad=\frac{\pi}{2}\frac{\mathcal{U}(\beta_i)}{|-2(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)|}\delta\left(i(\kappa-n)+\sum_{i=1}^n\lambda_i\right)\\
&\qquad\times\prod_{k=6}^n\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k}\right)\left(\frac{z_5\bar{z}_4-z_4\bar{z}_5+\sum_{i=6}^n 2\sigma_i((\bar{z}_5-\bar{z}_4)z_i+(z_4-z_5)\bar{z}_i+z_5\bar{z}_4-z_4\bar{z}_5)}{2(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}\right)^{i\lambda_3}\\
&\qquad\times\left(\frac{\bar{z}_5-z_5+\sum_{i=6}^n 2\sigma_i(-\bar{z}_5z_i+(z_5-1)\bar{z}_i+\bar{z}_5+z_i-z_5)}{2(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}\right)^{i\lambda_4}\left(-\frac{-\bar{z}_4+z_4+\sum_{i=6}^n 2\sigma_i(z_4(-\bar{z}_i)+(\bar{z}_4-1)z_i+\bar{z}_i-\bar{z}_4+z_4)}{2(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)}\right)^{i\lambda_5}\\
&\qquad\times\left(\frac{1}{2}-\frac{\sum_{i=6}^n 4(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)\sigma_i}{2(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)}\right)^{i\lambda_1}\\
&\qquad\times\left(\frac{1}{2(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}\Big[-\bar{z}_4-z_5(\bar{z}_4(\bar{z}_5-2)+1)+\bar{z}_5+z_4(-z_5\bar{z}_4+(\bar{z}_4+z_5-2)\bar{z}_5+1)\right.\\
&\qquad+\sum_{i=6}^n 2\sigma_i(((z_4-1)\bar{z}_4+(z_5(\bar{z}_4-1)-z_4\bar{z}_4+1)\bar{z}_5)z_i+(z_5(\bar{z}_5-1)+z_4((z_5-1)\bar{z}_4-z_5\bar{z}_5+1))\bar{z}_i\\
&\qquad\left.-2\bar{z}_4+2\bar{z}_5+z_5(\bar{z}_5^2-z_5\bar{z}_5+\bar{z}_4((z_5-2)\bar{z}_5+3)-2)+z_4((\bar{z}_4-3)\bar{z}_5-z_5(\bar{z}_4+(\bar{z}_5-2)\bar{z}_5)+2))\Big]\right)^{i\lambda_2}\\
&\qquad\times\lim_{L\to\infty}\mathcal{A}_{j_1\ldots j_n}\left[\begin{matrix}1-\sum_{i=1}^3\beta_i-\sum_{i=6}^n\sigma_i,-L^{-2}\left(\sum_{i=1}^3\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2})+\sum_{i=6}^n\epsilon_2\epsilon_i\sigma_i(1+z_i\bar{z}_i)\right),\beta_1,\beta_2,\beta_3,\sigma_6,\ldots,\sigma_n\\ 0,L,1,z_4,\ldots,z_n\end{matrix}\right].
\end{aligned}
\tag{3.25}
$$

With the equation above, we can consider several higher point amplitudes for this momentum configuration. Below, we will only focus on one simple case: all-plus gluon rational amplitude:

$$
\mathcal{A}^{\mathtt{gluon}}_{+\cdots+}(p_i)=-c\frac{\sum_{1\le i<j<k<l\le n}\langle ij\rangle[jk]\langle kl\rangle[li]}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\delta^4\left(\sum_{i=1}^n p_i^\mu\right)
\tag{3.26}
$$

where c is given in footnote 15 [40]. This leads to

$$
\mathcal{A}^{\mathtt{gluon}}_{+\cdots+}(\omega_1,\ldots,\omega_n;z_1,\ldots,z_n)=-c(-2)^{4-n}\frac{\sum_{1\le i<j<k<l\le n}\epsilon_{i,j,k,l}z_{ij}\bar{z}_{jk}z_{kl}\bar{z}_{li}\omega_i\omega_j\omega_k\omega_l}{z_{12}z_{23}\cdots z_{n1}\omega_1\omega_2\ldots\omega_n}\delta^4\left(\sum_{i=1}^n p_i^\mu\right)
\tag{3.27}
$$

for which we can immediately write

$$
\mathcal{A}^{\mathtt{gluon}}_{+\cdots+}\left[\begin{matrix}\sigma_1\ldots\sigma_n\\ z_1\ldots z_n\end{matrix}\right]=-c(-2)^{4-n}\frac{\sum_{1\le i<j<k<l\le n}\epsilon_{i,j,k,l}z_{ij}\bar{z}_{jk}z_{kl}\bar{z}_{li}\sigma_i\sigma_j\sigma_k\sigma_l}{z_{12}z_{23}\cdots z_{n1}\sigma_1\sigma_2\ldots\sigma_n}
\tag{3.28}
$$

where we also see that $\kappa=n$.

Let us focus on the first term, i.e.,

$$
\mathcal{A}^{\mathtt{gluon}}_{+\cdots+}\left[\begin{matrix}\sigma_1\ldots\sigma_n\\ z_1\ldots z_n\end{matrix}\right]=-\epsilon_{1,2,3,4}c(-2)^{4-n}\frac{\bar{z}_{23}\bar{z}_{41}}{z_{23}z_{45}\cdots z_{n1}}\prod_{i=5}^n\sigma_i^{-1}+\text{other terms}.
\tag{3.29}
$$

We can immediately insert this into Eq. (3.30) and obtain

$$
\begin{aligned}
&(\tilde{\mathcal{A}}^{\mathtt{gluon}})^{\Delta_1,\ldots,\Delta_n}_{+\cdots+}(0,\infty,1,z_4,\ldots,z_n)\big|_{\substack{p_{1,2}:\text{ incoming}\\ p_{3,\ldots,n}:\text{ outgoing}}}\\
&\quad=\pi c(-2)^{3-n}\frac{\mathcal{U}(\beta_i)}{|-2(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)|}\frac{\bar{z}_{23}\bar{z}_{41}}{z_{23}z_{45}\cdots z_{n1}}\\
&\qquad\times\delta\left(\sum_{i=1}^n\lambda_i\right)\prod_{k=6}^n\left(\int_0^1 d\sigma_k\sigma_k^{i\lambda_k-1}\right)\left(\frac{z_5\bar{z}_4-z_4\bar{z}_5+\sum_{i=6}^n 2\sigma_i((\bar{z}_5-\bar{z}_4)z_i+(z_4-z_5)\bar{z}_i+z_5\bar{z}_4-z_4\bar{z}_5)}{2(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}\right)^{i\lambda_3}\\
&\qquad\times\left(\frac{\bar{z}_5-z_5+\sum_{i=6}^n 2\sigma_i(-\bar{z}_5z_i+(z_5-1)\bar{z}_i+\bar{z}_5+z_i-z_5)}{2(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}\right)^{i\lambda_4}\\
&\qquad\times\left(-\frac{-\bar{z}_4+z_4+\sum_{i=6}^n 2\sigma_i(z_4(-\bar{z}_i)+(\bar{z}_4-1)z_i+\bar{z}_i-\bar{z}_4+z_4)}{2(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)}\right)^{i\lambda_5-1}\\
&\qquad\times\left(\frac{1}{2}-\frac{\sum_{i=6}^n 4(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)\sigma_i}{2(z_5(-\bar{z}_4)+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5+z_5)}\right)^{i\lambda_1}\\
&\qquad\times\left(\frac{1}{2(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}\Big[-\bar{z}_4-z_5(\bar{z}_4(\bar{z}_5-2)+1)+\bar{z}_5+z_4(-z_5\bar{z}_4+(\bar{z}_4+z_5-2)\bar{z}_5+1)\right.\\
&\qquad+\sum_{i=6}^n 2\sigma_i(((z_4-1)\bar{z}_4+(z_5(\bar{z}_4-1)-z_4\bar{z}_4+1)\bar{z}_5)z_i+(z_5(\bar{z}_5-1)+z_4((z_5-1)\bar{z}_4-z_5\bar{z}_5+1))\bar{z}_i\\
&\qquad\left.-2\bar{z}_4+2\bar{z}_5+z_5(\bar{z}_5^2-z_5\bar{z}_5+\bar{z}_4((z_5-2)\bar{z}_5+3)-2)+z_4((\bar{z}_4-3)\bar{z}_5-z_5(\bar{z}_4+(\bar{z}_5-2)\bar{z}_5)+2))\Big]\right)^{i\lambda_2}\\
&\qquad+\text{other terms.}
\end{aligned}\tag{3.30}
$$

As we can see, insertion of the first term in Eq. (3.29) into Eq. (3.30) simply shifted $\lambda_{k\geq 5}$ by i and included an overall prefactor $f(z,\bar{z})$. The other terms in the final result have the same property: all but four of λ_k are shifted by i and they have relative factors $f(z,\bar{z})$ which can be read from Eq. (3.29). Therefore, for all-plus one-loop gluon amplitudes, we have a summation of $\frac{n!}{4!(n-4)!}$ terms where the first term is given above and the others are *almost* the same, the difference being shifts in different λ_k and the overall factor of z_k's in the first line which can be extracted from Eq. (3.29).

IV. CONCLUSION

In this paper, we have provided explicit construction of loop-level celestial amplitudes for gluons and gravitons. We believe examples of celestial scattering amplitudes at loop level is of deep theoretical interest.

As they are rational and without any divergences, the one-loop all-plus and single-minus amplitudes for Yang-Mills and gravity are natural candidates that will help in understanding the holographic properties of scattering amplitudes beyond tree level. The simplicity and subtleties of these amplitudes made them excellent candidates to study spinning celestial amplitudes beyond tree level. We computed explicit examples of four and five points of such amplitudes, and provided the integrand for a particular n-point amplitude.

There are many interesting future directions that one can consider. The study of pure Yang-Mills and gravity theory at one loop may have interesting implications for $\mathcal{N}=4$ Yang-Mills and $\mathcal{N}=8$ supergravity theories. In particular, all positive helicity amplitudes that we considered in pure Yang-Mills is related to MHV amplitude in $\mathcal{N}=4$ super-Yang-Mills and similarly, all positive helicity amplitudes in Einstein gravity is related to $\mathcal{N}=8$ supergravity theories [68]. It would be interesting to investigate these connections with the usage of celestial amplitudes technology. On a related note, it is known that there are interesting relations between scattering amplitudes of gravity and of gauge theories (see [69]). Such dualities have been checked for many cases but a fundamental origin of this relation is still lacking. From a practical point of view, such dualities' most powerful applications are expected at loop-level computations and it is intriguing to study such relations using celestial technology.

Another specific goal is to generalize our one-loop amplitudes in pure Yang-Mills theory and gravity beyond the cases we have considered in this paper to provide more concrete examples of the celestial CFTs. We leave all of these exciting investigations to future work.

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APPENDIX A: TECHNICAL DETAILS

1. Conventions for conformal frame

It is well known that a conformal correlator can be rewritten in terms of conformally invariant cross ratios; for example, one can write the four-point correlator as[17]

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{(x_{12}^2)^{\Delta_{31}+\Delta_{42}}(x_{23}^2)^{\Delta_{12}-(\Delta_3+\Delta_4)/2}(x_{31}^2)^{\Delta_{21}+\Delta_{43}}}{(x_{14}^2)^{\Delta_4}}g(u,v) \tag{A1a}$$

where Δ_i are the scaling dimensions of the operators $\mathcal{O}_i$ and where the conformal cross ratios are given as

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} \tag{A1b}$$

for $x_{ij}\equiv x_i-x_j$ and $2\Delta_{ij}\equiv\Delta_i-\Delta_j$. Here, the function $g(u,v)$ is not fixed by the conformal symmetry.[18]

For higher point correlators there are multiple cross ratios; the conformal moduli space of n points in d dimensions is given as

$$\#\text{ of cross ratios}=\frac{m(m-3)}{2}+d(n-m)\quad\text{for } m=\min(n,d+2) \tag{A2}$$

which becomes

$$\#\text{ of cross ratios for }\tilde{\mathcal{A}}^{\Delta_1,\ldots,\Delta_n}_{J_1\ldots J_n}=2(n-3) \tag{A3}$$

as we are interested in celestial amplitudes of $n\geq 4$ gluons. We can intuitively understand this by the following argument: given any three points, we can first use translations to fix $z_1=0$, then special conformal transformation to take $z_2\to\infty$, then dilation to bring z_3 to unit circle, and finally rotation to get $z_3=1$. As this exhausts all conformal transformations, $z_{n>3}$ remains unfixed, hence we have $2(n-3)$ real degrees of freedom.[19]

For higher point correlators, we can generalize Eq. (A1) as

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \frac{(x_{12}^2)^{\sigma_{12}}(x_{23}^2)^{\sigma_{23}}(x_{31}^2)^{\sigma_{31}}}{\prod_{k=4}^n(x_{1k}^2)^{\Delta_k}}g(u_4,v_4;u_5,v_5;\ldots;u_n;v_n) \tag{A4a}$$

for

$$\sigma_{12}\equiv\frac{1}{2}\left(-\Delta_1-\Delta_2+\sum_{i=3}^n\Delta_i\right),\quad \sigma_{23}\equiv\frac{1}{2}\left(\Delta_1-\sum_{i=2}^n\Delta_i\right),\quad \sigma_{31}\equiv\frac{1}{2}\left(-\Delta_1+\Delta_2-\Delta_3+\sum_{i=4}^n\Delta_i\right) \tag{A4b}$$

where the conformal cross ratios are given as

$$u_k\equiv\frac{x_{1k}^2x_{23}^2}{x_{13}^2x_{2k}^2},\qquad v_k\equiv\frac{x_{12}^2x_{3k}^2}{x_{13}^2x_{2k}^2}. \tag{A4c}$$

We note that, for $n=4$, we get back Eq. (A1) from Eq. (A4) with the identification $u_4=v$ and $v_4=u$.[20]

As we mentioned above, conformal transformations allow us to fix $\{x_1,x_2,x_3\}\to\{0,\infty,1\}$. In higher dimensions, we can further constrain remaining points; in $2d$, they remain as unfixed variables. Thus Eq. (A4a) becomes

$$\langle\mathcal{O}_1(0)\mathcal{O}_2(\infty)\mathcal{O}_2(1)\mathcal{O}_4(\omega_4)\cdots\mathcal{O}_n(\omega_n)\rangle = \prod_{k=4}^n|\omega_k|^{-2\Delta_k}g(u_4,v_4;u_5,v_5;\ldots;u_n;v_n) \tag{A5}$$

where we implicitly used Eq. (2.23). Extracting the function g from the equation above and inserting it back into Eq. (A4a) we obtain

$$\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\mathcal{O}_n(z_n)\rangle = |z_{12}|^{2\sigma_{12}}|z_{23}|^{2\sigma_{23}}|z_{31}|^{2\sigma_{31}} \times\prod_{k=4}^n\frac{|\omega_k|^{2\Delta_k}}{|z_{1k}|^{2\Delta_k}}\langle\mathcal{O}_1(0)\mathcal{O}_2(\infty)\mathcal{O}_2(1)\mathcal{O}_4(\omega_4)\cdots\mathcal{O}_n(\omega_n)\rangle \tag{A6}$$

for

[17]This follows from the homogeneity of the correlator in the embedding space, i.e., $\langle\mathcal{O}_1(X_1)\cdots\mathcal{O}_k(\lambda X_k)\cdots\mathcal{O}_n(X_n)\rangle=\lambda^{-\Delta_k}\langle\mathcal{O}_1(X_1)\cdots\mathcal{O}_k(X_k)\cdots\mathcal{O}_n(X_n)\rangle$.

[18]Other information about the theory or general assumptions does constrain $g(u,v)$; for example, the whole program of conformal bootstrap is based on determining/constraining this function using (among other ingredients) operator product expansion associativity and unitarity [70].

[19]To understand where Eq. (A2) comes from, we recommend the nice discussion in [71].

[20]The reason for this inverted notation is our choice of conformal frame: as we will see below, we put the second operator at infinity whereas the fourth operator is put at infinity for the standard conformal frame of four points. Our choice of u_k, v_k in our conformal frame matches the form of u, v in the standard conformal frame.

$$\omega_k\omega_k^* = \frac{|z_{1k}|^2|z_{23}|^2}{|z_{13}|^2|z_{2k}|^2}, \qquad (1-\omega_k)(1-\omega_k^*) = \frac{|z_{12}|^2|z_{3k}|^2}{|z_{13}|^2|z_{2k}|^2}. \tag{A7}$$

2. Generalized Cramer's rule

In this section, we will review the generalized Cramer's rule as derived in [72]. Let us consider a system of equations of the form

$$\left\{\sum_{i=1}^{n} a_{1,i}x_i = a_{1,n+1}, \sum_{i=1}^{n} a_{2,i}x_i = a_{2,n+1}, \ldots, \sum_{i=1}^{n} a_{m,i}x_i = a_{m,n+1}\right\}, \quad n \geq m \tag{A8}$$

for which we can define the order-m minors of the augmented matrix as

$$M_{j_1,j_2,\ldots,j_m} \equiv \det\begin{pmatrix} a_{1,j_1} & a_{1,j_2} & \ldots & a_{1,j_m} \\ a_{2,j_1} & a_{2,j_2} & \ldots & a_{2,j_m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,j_1} & a_{m,j_2} & \ldots & a_{m,j_m} \end{pmatrix} \tag{A9}$$

we can then write down $x_{1,\ldots,m}$ in terms of a_{ij} and $x_{m+1,\ldots,n}$ as

$$x_i = \alpha_{i,n+1} + \sum_{j=m+1}^{n} \alpha_{i,j}x_j, \quad 1 \leq i \leq m \tag{A10}$$

for

$$\alpha_{i,j} \equiv \frac{M_{1,2\ldots,i-1,j,i+1,\ldots,m-1,m}}{M_{1,2,\ldots,m-1,m}}. \tag{A11}$$

We hence have

$$\prod_{k=1}^{m} \delta\left(\sum_{i=1}^{n} a_{k,i}x_i - a_{k,n+1}\right) = \frac{\prod_{i=1}^{m} \delta(x_i - \alpha_{i,n+1} - \sum_{j=m+1}^{n} \alpha_{i,j}x_j)}{|M_{1,2,\ldots,m}|}. \tag{A12}$$

With Eq. (2.25) in mind, we can use the equation above to write down

$$\left(\prod_{k=3}^{n}\int_0^1 d\sigma_k\right)\delta\left(\sum_{i=3}^{n}\epsilon_i\sigma_i z_i\right)\delta\left(\sum_{i=3}^{n}\epsilon_i\sigma_i \bar{z}_i\right)\delta\left(1+\sum_{i=3}^{n}(\epsilon_1\epsilon_i-1)\sigma_i\right)f(\sigma_3,\ldots,\sigma_n) = \frac{\mathcal{U}(\beta_i)}{|M_{1,2,3}|}\left(\prod_{k=6}^{n}\int_0^1 d\sigma_k\right)f(\beta_1,\beta_2,\beta_3,\sigma_6,\ldots,\sigma_n) \tag{A13}$$

for

$$\beta_k \equiv \begin{cases} \alpha_{k,n-1} + \sum_{j=4}^{n-2}\alpha_{k,j}\sigma_j & n \geq 6 \\ \alpha_{k,n-1} & 5 \geq n \geq 3 \end{cases} \tag{A14}$$

and

$$\mathcal{U}(\beta_i) \equiv \begin{cases} 1 & 0 \leq \beta_i \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{A15}$$

where α and M are given in Eqs. (A11) and (A9) for

$$\left\{\begin{matrix} a_{1,i} = \epsilon_{i+2}z_{i+2}, & a_{2,i} = \epsilon_{i+2}\bar{z}_{i+2}, & a_{3,i} = (\epsilon_1\epsilon_{i+2}-1) & \text{for } 1 \leq i \leq n-2 \\ a_{1,n-1} = 0, & a_{2,n-1} = 0, & a_{3,n-1} = -1 & \end{matrix}\right\}, \quad n > 4. \tag{A16}$$

When $n = 5$ all integrals are taken care of by the Dirac-delta functions, so Eq. (A13) becomes quite simple:

$$\left(\prod_{k=3}^{5}\int_0^1 d\sigma_k\right)\delta\left(\sum_{i=3}^{5}\epsilon_i\sigma_i z_i\right)\delta\left(\sum_{i=3}^{5}\epsilon_i\sigma_i \bar{z}_i\right)\delta\left(1+\sum_{i=3}^{5}(\epsilon_1\epsilon_i-1)\sigma_i\right)f(\sigma_3,\ldots,\sigma_n) = \frac{\mathcal{U}(\beta_i)}{|M_{1,2,3}|}f(\beta_1,\beta_2,\beta_3). \tag{A17}$$

Fo $n=4$, we cannot exhaust all delta functions hence we cannot use Eq. (A13).[21] We instead have

$$\left(\prod_{k=3}^{4}\int_0^1 d\sigma_k\right)\delta\left(\sum_{i=3}^{4}\epsilon_i\sigma_i z_i\right)\delta\left(\sum_{i=3}^{4}\epsilon_i\sigma_i\bar{z}_i\right)\delta\left(1+\sum_{i=3}^{4}(\epsilon_1\epsilon_i-1)\sigma_i\right)f(\sigma_3,\sigma_4)=\mathcal{U}(\beta_i)\delta(z_3\bar{z}_4-z_4\bar{z}_3)f(\beta_1,\beta_2) \quad \text{(A18)}$$

where α and M are given in Eqs. (A11) and (A9) for

$$\left\{\begin{matrix} a_{1,i}=\epsilon_{i+2}z_{i+2}, & a_{2,i}=(\epsilon_1\epsilon_{i+2}-1) & \text{for } 1\le i\le 2 \\ a_{1,3}=0, & a_{2,3}=-1 & \end{matrix}\right\},\quad n=4. \quad \text{(A19)}$$

3. Details for the five-point amplitudes

The coefficients of the prescription given in Eq. (3.15) read as

$$\begin{aligned}
a_{12}^{(5)} &= -\sqrt{\frac{(-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5)(\bar{z}_4+z_5(\bar{z}_4(\bar{z}_5-2)+1)-\bar{z}_5+z_4(z_5\bar{z}_4-(\bar{z}_4+z_5-2)\bar{z}_5-1))\epsilon_{1,2}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{13}^{(5)} &= -\sqrt{-\frac{(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)(z_5\bar{z}_4-z_4\bar{z}_5)\epsilon_{1,3}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{23}^{(5)} &= \sqrt{\frac{(z_5\bar{z}_4-z_4\bar{z}_5)(\bar{z}_4+z_5(\bar{z}_4(\bar{z}_5-2)+1)-\bar{z}_5+z_4(z_5\bar{z}_4-(\bar{z}_4+z_5-2)\bar{z}_5-1))\epsilon_{2,3}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{14}^{(5)} &= -\bar{z}_4\sqrt{\frac{(z_5-\bar{z}_5)(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)\epsilon_{1,4}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{24}^{(5)} &= \sqrt{\frac{(z_5-\bar{z}_5)(-\bar{z}_4-z_5(\bar{z}_4(\bar{z}_5-2)+1)+\bar{z}_5+z_4(-z_5\bar{z}_4+(\bar{z}_4+z_5-2)\bar{z}_5+1))\epsilon_{2,4}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{34}^{(5)} &= -(\bar{z}_4-1)\sqrt{-\frac{(z_5-\bar{z}_5)(z_5\bar{z}_4-z_4\bar{z}_5)\epsilon_{3,4}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{15}^{(5)} &= -\bar{z}_5\sqrt{\frac{(z_4-\bar{z}_4)(-z_5\bar{z}_4+\bar{z}_4+(z_4-1)\bar{z}_5-z_4+z_5)\epsilon_{1,5}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{25}^{(5)} &= \sqrt{\frac{(z_4-\bar{z}_4)(\bar{z}_4+z_5(\bar{z}_4(\bar{z}_5-2)+1)-\bar{z}_5+z_4(z_5\bar{z}_4-(\bar{z}_4+z_5-2)\bar{z}_5-1))\epsilon_{2,5}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{35}^{(5)} &= -(\bar{z}_5-1)\sqrt{-\frac{(z_4-\bar{z}_4)(z_4\bar{z}_5-z_5\bar{z}_4)\epsilon_{3,5}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}},\\
a_{45}^{(5)} &= (\bar{z}_4-\bar{z}_5)\sqrt{-\frac{(z_4-\bar{z}_4)(z_5-\bar{z}_5)\epsilon_{4,5}}{(\epsilon_5(\epsilon_4((-z_5(\bar{z}_4-1)+\bar{z}_4+(z_4-1)\bar{z}_5)\epsilon_{1,3}-z_4(\bar{z}_5+\epsilon_{1,3})+z_5\bar{z}_4)+\epsilon_3(\bar{z}_5-z_5))+(z_4-\bar{z}_4)\epsilon_{3,4})^2}}.
\end{aligned} \quad \text{(A20)}$$

[21]The situation is similar in the $n=3$ case; however it needs to be treated separately due to kinematics of massless scattering. As $p_i\cdot p_j=0$, which follows from $(p_i+p_j)^2=p_k^2$ and $p_i^2=0$ for $i\neq j\neq k\in\{1,2,3\}$, we get $\langle ij\rangle[ij]=2p_i\cdot p_j=0$. In the standard $(-,+,+,+)$ metric with real momenta, $[ij]$ and $\langle ij\rangle$ are related to each other by complex conjugation as $[ij]\propto\bar{z}_{ij}$ and $\langle ij\rangle\propto z_{ij}$; therefore the only consistent solution is if $\langle ij\rangle=[ij]=0$. To circumvent this issue, one either complexifies the momenta or uses the metric $(-,+,-,+)$ for which z and $\bar{z}$ are real and independent variables, allowing a nontrivial solution for $\langle ij\rangle[ij]=0$. In this paper, we focus on $n>3$ amplitudes and do not deal with such subtleties.

By using these coefficients in the prescription of Eq. (3.15) for the five-point gluon amplitudes given in Eq. (3.16), we obtain

$$\begin{aligned}
&\lim_{L\to\infty}\mathcal{A}^{\texttt{gluon}}_{+++++}\begin{bmatrix} 1-\sum_{i=1}^{3}\beta_i, -L^{-2}\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2}),\beta_1,\beta_2,\beta_3 \\ 0,L,1,z_4,z_5 \end{bmatrix} \\
&= \frac{\bar{z}_4\epsilon_{1,2,3,4}\left|\frac{(\bar{z}_4-z_4)(\epsilon_{1,5}-1)\epsilon_{3,4}+(z_5-\bar{z}_5)(\epsilon_{1,4}-1)\epsilon_{3,5}+(z_4\bar{z}_5-z_5\bar{z}_4)(\epsilon_{1,3}-1)\epsilon_{4,5}}{z_4-\bar{z}_4}\right|}{2z_4z_5-2z_5^2} \\
&\quad+\frac{(\bar{z}_4-1)\epsilon_{2,3,4,5}\left|\frac{(z_5-\bar{z}_5)(\epsilon_{3,5}-\epsilon_{1,3,4,5})+z_4(-\epsilon_{3,4}+\bar{z}_5(\epsilon_{4,5}-\epsilon_{1,3,4,5})+\epsilon_{1,3,4,5})+\bar{z}_4(\epsilon_{3,4}-\epsilon_{1,3,4,5}+z_5(\epsilon_{1,3,4,5}-\epsilon_{4,5}))}{-\bar{z}_4z_5+z_5+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5}\right|}{2(z_4-1)z_5} \\
&\quad+\frac{(\bar{z}_4-1)\bar{z}_5\epsilon_{1,3,4,5}\left|\frac{(z_5-\bar{z}_5)(\epsilon_{3,5}-\epsilon_{1,3,4,5})+z_4(-\epsilon_{3,4}+\bar{z}_5(\epsilon_{4,5}-\epsilon_{1,3,4,5})+\epsilon_{1,3,4,5})+\bar{z}_4(\epsilon_{3,4}-\epsilon_{1,3,4,5}+z_5(\epsilon_{1,3,4,5}-\epsilon_{4,5}))}{\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1)}\right|}{2(z_4-1)z_5} \\
&\quad+\frac{(z_5-1)\bar{z}_5\epsilon_{1,2,3,5}\left|\frac{(\bar{z}_4-z_4)(\epsilon_{1,5}-1)\epsilon_{3,4}+(z_5-\bar{z}_5)(\epsilon_{1,4}-1)\epsilon_{3,5}+(z_4\bar{z}_5-z_5\bar{z}_4)(\epsilon_{1,3}-1)\epsilon_{4,5}}{z_5-\bar{z}_5}\right|}{2(z_4-1)(z_4-z_5)z_5} \\
&\quad+\frac{\bar{z}_5\epsilon_{1,2,4,5}\left|\frac{(\bar{z}_4-z_4)(\epsilon_{1,5}-1)\epsilon_{3,4}+(z_5-\bar{z}_5)(\epsilon_{1,4}-1)\epsilon_{3,5}+(z_4\bar{z}_5-z_5\bar{z}_4)(\epsilon_{1,3}-1)\epsilon_{4,5}}{z_5\bar{z}_4-z_4\bar{z}_5}\right|}{2z_5-2z_4z_5},
\end{aligned} \tag{A21a}$$

and

$$\begin{aligned}
&\lim_{L\to\infty}\mathcal{A}^{\texttt{gluon}}_{-++++}\begin{bmatrix} 1-\sum_{i=1}^{3}\beta_i, -L^{-2}\sum_{i=1}^{3}\epsilon_2\epsilon_{i+2}\beta_i(1+z_{i+2}\bar{z}_{i+2}),\beta_1,\beta_2,\beta_3 \\ 0,L,1,z_4,z_5 \end{bmatrix} \\
&= -\frac{2\epsilon_{2,3}\left|\frac{(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)(z_5\bar{z}_4-z_4\bar{z}_5)}{(z_4-\bar{z}_4)((\bar{z}_5-z_5)(\epsilon_{3,5}-\epsilon_{1,3,4,5})+\bar{z}_4(-\epsilon_{3,4}+z_5(\epsilon_{4,5}-\epsilon_{1,3,4,5})+\epsilon_{1,3,4,5})+z_4(\epsilon_{3,4}-\epsilon_{1,3,4,5}+\bar{z}_5(\epsilon_{1,3,4,5}-\epsilon_{4,5})))}\right|}{z_5(z_5-z_4)} \\
&\quad+\frac{2\left|\frac{(z_4-\bar{z}_4)(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))}{(-\bar{z}_4z_5+z_5+\bar{z}_4+z_4(\bar{z}_5-1)-\bar{z}_5)((z_5-\bar{z}_5)(\epsilon_{3,5}-\epsilon_{1,3,4,5})+z_4(-\epsilon_{3,4}+\bar{z}_5(\epsilon_{4,5}-\epsilon_{1,3,4,5})+\epsilon_{1,3,4,5})+\bar{z}_4(\epsilon_{3,4}-\epsilon_{1,3,4,5}+z_5(\epsilon_{1,3,4,5}-\epsilon_{4,5})))}\right|}{\bar{z}_5} \\
&\quad-\frac{2z_4^3(z_5-1)(\bar{z}_4-\bar{z}_5)\epsilon_{4,5}\left|\frac{(z_5-\bar{z}_5)(z_5(\bar{z}_4-1)-\bar{z}_4-z_4(\bar{z}_5-1)+\bar{z}_5)}{(\bar{z}_4(\bar{z}_5-2)z_5+z_5+\bar{z}_4-\bar{z}_5+z_4(z_5(\bar{z}_4-\bar{z}_5)-(\bar{z}_4-2)\bar{z}_5-1))((z_5-\bar{z}_5)(\epsilon_{3,5}-\epsilon_{1,3,4,5})+z_4(-\epsilon_{3,4}+\bar{z}_5(\epsilon_{4,5}-\epsilon_{1,3,4,5})+\epsilon_{1,3,4,5})+\bar{z}_4(\epsilon_{3,4}-\epsilon_{1,3,4,5}+z_5(\epsilon_{1,3,4,5}-\epsilon_{4,5})))}\right|}{(z_4-z_5)^2}.
\end{aligned} \tag{A21b}$$

One can insert these expressions into Eq. (3.12) to get the full celestial amplitude.

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