Hamilton principle for chiral anomalies in hydrodynamics

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We developed the spacetime-covariant Hamilton principle for barotropic flows of a perfect fluid in the external axial-vector potential conjugate to the helicity current. Such flows carry helicity, a chiral imbalance, controlled by the axial potential. The interest in such a setting is motivated by the recent observation that the axial-current anomaly of quantum field theories with Dirac fermions appears as a kinematic property of classical hydrodynamics. Especially interesting effects occur under the simultaneous actions of the electromagnetic field and the axial-vector potential. With the help of the Hamilton principle, we obtain the extension of the Euler equations by the axial potential and derive anomalies in the divergence of the axial and vector current. Our approach provides a hydrodynamic expression for vector and axial currents and lays down a platform for studying flows with a chiral imbalance and their anomalies.

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I. INTRODUCTION

Flows of an electrically charged barotropic perfect fluid feature two conserved currents. One is the electric current whose spacetime components we denote by J^α

$$\partial \cdot J = 0. \tag{1}$$

Electric current is a conjugate to the electromagnetic vector potential A_α. The coupling with the electromagnetic field amounts to adding the term

$$\mathcal{S}^{(0)} \to \mathcal{S}^{(0)} + \int_{\mathbb{R}^4} A \cdot J \tag{2}$$

to the action of the perfect fluid $\mathcal{S}^{(0)}$. The integral goes over spacetime $\mathbb{R}^4$ and we omit indices of contraction of spacetime vectors and covectors.

Another conserved current is the helicity current [1]. In the neutral fluid helicity current is expressed through the fluid 4-momentum as

$$j_{\mathrm{A}}^\alpha = \epsilon^{\alpha\beta\gamma\delta} p_\beta \partial_\gamma p_\delta, \qquad \partial \cdot j_A = 0. \tag{3}$$

Helicity current does not depend on the spacetime metric and the expression (3) holds for a relativistic neutral fluid as well as for a Galilean fluid. For a relativistic fluid p_α is the conventional spacetime momentum. We define the 4-momentum for a Galilean fluid later in the text. The temporal component of the helicity current is the familiar helicity density

$$j_A^0 = \boldsymbol{p} \cdot (\nabla \times \boldsymbol{p}), \tag{4}$$

where $\boldsymbol{p}$ is the usual kinematic momentum (or, simply, momentum).

The electric current is a spacetime vector and the helicity current is a pseudovector (or axial vector). Similar to the electric current one could introduce a background axial-vector potential A_α^{A} conjugate to the helicity current by adding the axial coupling to the action [2]

$$\mathcal{S}^{(0)} \to \mathcal{S}^{(0)} - \int_{\mathbb{R}^4} A^{\mathrm{A}} \cdot j_{\mathrm{A}}. \tag{5}$$

The axial-vector potential imposes a *chiral imbalance* by generating flows with helicity. Such flows become especially interesting in the electromagnetic field background when the vector potential A and the axial potential A^{A} interfere. In this case, the action reads

$$\mathcal{S} = \mathcal{S}^{(0)} + \int_{\mathbb{R}^4} [A \cdot J - A^{\mathrm{A}} \cdot j_{\mathrm{A}}]. \tag{6}$$

This is the setting we study in this paper. The paper aims to shed some light on the hydrodynamic meaning of the important kinematic phenomena commonly referred to as *chiral anomalies*.

The chiral anomalies are fundamental phenomena of quantum field theory with fermions, such as QED. In the last decade it had been important developments in a hydrodynamic description of relativistic quantum field theories of chiral (Weyl) fermions consistent with chiral anomalies (see, [3–6] and references therein). Recently in [7,8] it was found that the chiral anomaly of Dirac fermions is a property of the Euler equation for barotropic fluid.

Because the helicity current (3) involves derivatives of the momentum the axial coupling changes the traditional relations between electric current J, the momentum p, and the velocity held in a perfect fluid. Say, in a nonrelativistic perfect fluid $\boldsymbol{p} = m\mathbf{v}$ and $\boldsymbol{J} = en\mathbf{v} = (en/m)\boldsymbol{p}$. These relations will be deformed by background potentials A_α and A^{A}_α (see, (38) below). Furthermore, and most importantly, the electromagnetic potential in turn gives a feedback to the axial current (3). In Ref. [7] it was shown that the axial current is modified by the electromagnetic field as

$$j^\alpha_{\mathrm{A}} = \epsilon^{\alpha\beta\gamma\delta} p_\beta (\partial_\gamma p_\delta + F_{\gamma\delta}). \tag{7}$$

The temporal component of the axial current now reads

$$j^0_{\mathrm{A}} = \boldsymbol{p} \cdot (\nabla \times \boldsymbol{p} + 2\boldsymbol{B}), \tag{8}$$

where $\boldsymbol{B}$ is the magnetic field and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the electromagnetic field tensor.

Now, in the electromagnetic background, the axial current is no longer divergence-free. In [7] it was shown that although the axial current is not conserved its divergence does not depend on dynamic fields. Rather, it is solely controlled by the electromagnetic field as

$$\boxed{\partial \cdot j_A = \frac{1}{2} {}^\star F^{\alpha\beta} F_{\beta\alpha}} \tag{9}$$

or

$$\partial \cdot j_A = 2\boldsymbol{E} \cdot \boldsymbol{B} \tag{10}$$

where $\boldsymbol{E}$ is the electric field, and ${}^\star F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\delta\gamma} F_{\delta\gamma}$ is the dual field tensor.

The formula (9) is identical to the celebrated *axial-current anomaly* of QED. It appears that among two independent gauge symmetries of the classical Lagrangian of QED

$$A_\alpha \to A_\alpha + \partial_\alpha \varphi, \tag{11}$$

$$A^{\mathrm{A}}_\alpha \to A^{\mathrm{A}}_\alpha + \partial_\alpha \varphi^{\mathrm{A}} \tag{12}$$

only the vector gauge transformation (11), gives rise to conserved current, that is the electric current (1). The axial gauge invariance (12) does not. In this paper, we describe a similar situation in hydrodynamics. We will see that the Euler equation deformed by the axial coupling is invariant under both transformations, however, solutions of the equation which describe physical relevant flows are not.

The conflict between the vector and the axial gauge symmetries in quantum field theory with Dirac fermions had been discovered in 1969 by Adler [9] and Bell and Jackiw [10].

We recall that the Dirac multiplet consists of components of the left and right-handed chiral particles. The quantum states of a massless Dirac theory are characterized by the vector current j and the axial current j_{A}, the sum and the difference of the currents of left and right components. Both currents are conserved and could be coupled with a vector and an axial-vector potential. However, due to subtle quantum effects the electromagnetic background field prevents the conservation of the axial current. Adler and Bell and Jackiw found that in units of the flux quantum $\Phi_0 = he/c$, where h is the Planck constant the divergence of the axial current is given precisely by the Eq. (9). Furthermore, this result is largely insensitive to interaction and holds with or without axial potential.

We encounter a remarkable correspondence between the kinematic properties of the classical fluid and the properties of quantum states of Dirac fermions. This must not come as a surprise. It is broadly known that in one spatial dimension the Hilbert space of the Dirac fermions is identified with that of a Bose scalar field, whose dynamic is equivalent to that of a compressible fluid. We see that the part of this correspondence that is based on geometry extends to higher even spacetime dimensions.

The kinematic properties of hydrodynamics are closely related to the geometry of flows [11]. Likewise, anomalies of quantum field theory, too are related to the geometry of quantum states. Both are largely decoupled from the dynamics. An apparent relation between the geometric properties of classical flows and quantum states revealed by the anomalies is noteworthy. In this paper, we further develop this subject. Our motivation comes from the fermionic quantum field theory, but the paper is written from the side of fluid mechanics with a minimal appeal to the vast literature on anomalies in quantum field theory.

Part of the results of this paper appeared in [8], where we discussed the coupling with the time-independent axial chemical potential $\mu^{\mathrm{A}} = A^{\mathrm{A}}_0$. Inclusion the spatial components $\boldsymbol{A}^{\mathrm{A}}$ of the axial potential requires a spacetime covariant approach we developed here.

We use the Hamilton principle on the Lie group of spacetime diffeomorphisms $\mathrm{Diff}(\mathbb{R}^4)$. With the help of the Hamilton principle, we obtain the spacetime-covariant equations of motion, the conservation laws, and the hydrodynamic representation of various currents by Eulerian fields. A brief account of essential results presented here was reported in [12], where we used the spacetime-covariant Hamilton principle in the dual space. See, also a related paper [13]. For references, we collect the major formulas in the last section of the paper.

The paper is organized as follows. In the first few sections, we assume that the axial anomaly equation (9) holds and on this basis draw general consequences. Then we describe the class of fluid systems which feature the axial-current anomaly. Such systems are referred to as *uniformly canonical*. The simplest system in this class is perfect barotropic fluid and it remains uniformly canonical under the axial coupling. We describe the Hamiltion principle for this class of fluids and with its help obtain the equations of motion.

The notion of uniformly canonical fluid systems was introduced by Carter [1]. These fluids are Hamiltonian systems, that do not possess any advected local scalar fields, such as, e.g., entropy. In this case, the four components of the spacetime canonical momentum

$$\pi_\alpha = p_\alpha + A_\alpha. \tag{13}$$

are the only dynamical fields. The configuration space of such systems is identified with the group of spacetime diffeomorphisms $\mathrm{Diff}(\mathbb{R}^4)$ and its dynamics follow from the Hamilton principle with the action defined on the Lie algebra of vector field (flow field) $\mathrm{Diff}(\mathbb{R}^4)$ or in its dual space of the canonical momentum.

II. PERFECT BAROTROPIC FLUID

This section aims to establish notations and to introduce the 4-momentum for a Galilean fluid.

Flows of a perfect charged fluid in the electromagnetic field are governed by the Euler equation and the continuity equation

$$\begin{cases} \dot{\mathbf{v}} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + \rho^{-1}\boldsymbol{\nabla}P = \boldsymbol{E} + \mathbf{v}\times\boldsymbol{B}, \\ \dot{n} + \boldsymbol{\nabla}\cdot\boldsymbol{n} = 0, \end{cases} \tag{14}$$

where $\rho = mn$ is the mass density, n is the particle number density, $\mathbf{v}$ is the velocity of the fluid, P is pressure and we denote $n^\alpha = (n, n\mathbf{v})$ the particle number 4-current. The same equations hold in a neutral fluid rotating with the Larmor frequency $\boldsymbol{B}/m$ placed in the potential A_0. A barotropic fluid is singled out by the condition that pressure is a function of the particle number n. Then $\boldsymbol{\nabla}P$ and $\boldsymbol{\nabla}n$ are collinear (this property is the origin of the term barotropic) and $\boldsymbol{\nabla}P/\rho = \boldsymbol{\nabla}\mu$, where $\mu = \partial\varepsilon/\partial n$ is the chemical potential and ε is the fluid internal energy.

We will use the spacetime covariant formalism, based on the notion of the spacetime fluid 4-momentum. In the case of the relativistic fluid, this is a usual 4-momentum. It is a spacetime covector constrained by the condition $p_\alpha p^\alpha = -(mc + c^{-1}\mu)^2$. In the nonrelativistic (Galilean) fluid the notion of the 4-momentum may be less familiar. It could be obtained as a non-relativistic limit of the above formula and dropping mc^2. We find that p_0 is (minus) Bernoulli function, that is the energy per particle

$$p_\alpha = (p_0, \boldsymbol{p}): \; p_0 = -\mu - \boldsymbol{p}^2/2m, \qquad \boldsymbol{p} = m\mathbf{v}. \tag{15}$$

Alternatively, p_0 could be introduced as a canonical conjugate to the particle number n. Consider the action of a Galilean perfect fluid and treat it as a function of particle number 4-current

$$n^\alpha = (n, n\mathbf{v}). \tag{16}$$

The action reads

$$\mathcal{S}^{(0)} = \int_{\mathbb{R}^4}\left(\frac{m}{2n}\boldsymbol{n}^2 - \varepsilon[n]\right). \tag{17}$$

Then the 4-momentum is defined through the variation

$$\delta\mathcal{S}^{(0)} = \int_{\mathbb{R}^4} p_\alpha \delta n^\alpha. \tag{18}$$

Because the axial current (7) depends only on the momentum, the axial coupling does not affect (15).

The Legendre transform on the action defines the spacetime version of the "Hamiltonian" $\mathcal{H}^{(0)} = \int_{\mathbb{R}^4}(p\cdot n) - \mathcal{S}^{(0)}$. That gives

$$\mathcal{H}^{(0)} = -\int_{\mathbb{R}^4} P, \tag{19}$$

where $P = n\mu - \varepsilon$ is the pressure treated as a function of the 4-momentum under the relation (15).The integral of the pressure (19) treated as a functional of 4-momentum was used as a variational functional for the perfect fluid in [14,15].

An equivalent form of equations of motion (14) is the conservation laws of energy and momentum

$$\partial_\alpha T^\alpha_\beta = F_{\beta\gamma}J^\gamma, \tag{20}$$

where T^β_α is the momentum-stress-energy tensor (or, simply, stress tensor) and J^α is conserved electric current. The right-hand side (rhs) in (20) is the Lorentz force. In perfect fluid (in units of the electric charge) the electric current equals the particle number current $J^\alpha = en^\alpha$. Then, the stress tensor of the perfect fluid is

$$T^\beta_\alpha = n^\beta p_\alpha + P\delta^\beta_\alpha. \tag{21}$$

The formulas (20), (21) hold in the relativistic case with the particle number n replaced by $n/\sqrt{1-\mathbf{v}^2/c^2}$. In the relativistic case, the stress tensor is symmetric.

Four equations (20) fully describe the barotropic fluid as the continuity equation in (14) follows from (20). This would not be the case for a baroclinic fluid.

In the rest of the paper, we seek a deformation of the two equivalent forms of equations of motions (14) and (20) by the axial vector potential A^{A}.

III. SYMMETRIES OF EQUATIONS OF MOTION

According to basic principles, equations of motion and their solutions are invariant under the gauge transformations (11). The axial gauge transformation (12) is more subtle. We will see that equations of motion are locally expressed through the Eulerian fields (the 4-momentum p^α) and the field tensors $F_{\alpha\beta}$ and $F^{\mathrm{A}}_{\alpha\beta}$. Then at given p^α, the equations of motion do not change under the axial gauge transformations (12). However, as a consequence of the anomaly solutions of the equations, the flows, could be different at gauge equivalent values of the axial potential A^{A} and $A^{\mathrm{A}} + \partial\varphi^{\mathrm{A}}$. From this perspective, the axial-gauge transformations are not true symmetry.

We assume that under the action of the axial transformations the Lagrangian density changes only by a full spacetime derivative and explore the consequences. Later we justify this assumption.

A. Vector current and chiral anomaly

Under the axial gauge transformation (12) and due to the axial-current anomaly (9) the axial coupling taken at a fixed momentum transforms as

$$\begin{aligned}\delta^{\mathrm{A}}\left(\int_{\mathbb{R}^4} A^{\mathrm{A}}\cdot j_{\mathrm{A}}\right) &= \int_{\mathbb{R}^4} \partial\varphi^{\mathrm{A}}\cdot j_{\mathrm{A}} \\ &= -\int_{\mathbb{R}^4} \varphi^{\mathrm{A}}(\partial\cdot j_{\mathrm{A}}) \\ &= -\frac{1}{2}\int_{\mathbb{R}^4} \varphi^{\mathrm{A}} F\cdot {}^\star F \\ &= 2\int_{\mathbb{R}^4} A_\alpha {}^\star F^{\alpha\beta}\partial_\beta\varphi^{\mathrm{A}}. \qquad (22)\end{aligned}$$

If we demand that the action (up to boundary terms) is invariant under (12), then (22) must be compensated by the transformation of the electric current

$$\delta^{\mathrm{A}} J^\alpha|_p = 2{}^\star F^{\alpha\beta}\partial_\beta\varphi^{\mathrm{A}}. \qquad (23)$$

We see that as a result of the anomaly the electric current, an observable quantity, is not an axial-gauge invariant. We introduce the axial-gauge invariant *vector current*

$$j^\alpha := J^\alpha - 2{}^\star F^{\alpha\beta} A^{\mathrm{A}}_\beta. \qquad (24)$$

Because the electric current conserves, the vector current does not

$$\boxed{\partial\cdot j = F\cdot {}^\star F_{\mathrm{A}}} \qquad (25)$$

Equation (25) is the consequence of the axial-current anomaly In this context the anomaly says that the conserved electric current J is not axial-gauge invariant, and the axial-gauge invariant vector current j is not conserved. Together Eqs. (9), (25) are sometimes referred to as *chiral anomalies*.

Both currents J and j are physical observables. The difference between them depends only on the background fields. It stays the same for different flows occurring at the same external fields. This property suggests that under the axial coupling, the fluid is necessarily connected with a reservoir, a spectator medium whose motion is solely determined by the background fields. Then, the term $2{}^\star F^{\alpha\beta} A^{\mathrm{A}}_\beta$ in (24) should be associated with the electric current run through the reservoir and the vector current j is associated with the electric current conducted by the fluid. Since the fluid and the reservoir are open systems their currents taken independently do not conserve. The total electric current does.

We illustrate this interpretation by the experimentally established Landauer-Sharvin effect of electronic ballistic transport (see, [16,17]). In the Landauer-Sharvin setting, the electric current ballistically runs through a wire whose open ends are connected to metallic leads (the reservoir). Then the one-dimensional version of the formula (25) $\partial\cdot j = 2E$, written in the static case as $j = 2U$, where U is the voltage drop between the leads ($E = \nabla U$) describes the universal conductance. Restoring the units the universal conductance reads $2e^2/h$.

B. Spacetime diffeomorphisms: Equations of motions in the form of conservation laws

Next, we draw the consequences of the invariance under spacetime diffeomorphisms. On a general ground, a system coupled to vector potentials features the relation between the stress tensor of the perfect fluid (21) and external potentials.

$$\partial_\alpha T^\alpha_\beta = F_{\beta\alpha}J^\alpha - A_\beta(\partial\cdot J) + F^{\mathrm{A}}_{\beta\alpha} j^\alpha_A - A^{\mathrm{A}}_\beta(\partial\cdot j_{\mathrm{A}}). \qquad (26)$$

At the end of this section, we recall the derivation of (26).

If the currents are conserved the terms with divergence drop and the rhs of (26) is a sum of two Lorentz forces. In our case the electric current is conserved but the axial current is not. Given the axial-current anomaly (9) and with the help of the identity valid for any two antisymmetric tensors

$$2({}^\star m^{\beta\gamma} l_{\gamma\alpha} + {}^\star l^{\beta\gamma} m_{\gamma\alpha}) = \delta^\beta_\alpha({}^\star m^{\delta\gamma} l_{\gamma\delta}) \qquad (27)$$

we write the last term in (26) as $-A^{\mathrm{A}}_\beta\partial\cdot j_{\mathrm{A}} = -\frac{1}{2}A^{\mathrm{A}}_\beta F\cdot {}^\star F = -2F_{\beta\alpha}{}^\star F^{\alpha\gamma} A^{\mathrm{A}}_\gamma$ [we applied the identity (27) by setting both tensors m, l to be equal to the field tensor F. See Appendix for the origin of the identity]. This term joins the first term in (26) bringing the equation to the form

$$\partial_\alpha T^\alpha_\beta = F_{\beta\alpha} j^\alpha + F^{\mathrm{A}}_{\beta\alpha} j^\alpha_A, \qquad (28)$$

where j is the axial-gauge invariant vector current introduced in Sec. III A, Eq. (24). We observe that this equation is expressed through the axial field tensor F^{A} and the 4-momentum. This justifies the assumption made at the beginning of the section.

The axial-current anomaly gives a different meaning to the Lorentz force as the currents entered the Lorentz force in (28) do not conserve. Their divergence is determined by the chiral anomaly (9), (25).

These equations are valid for the fermionic quantum field theory and the classical fluid alike. In the case of the fluid we know the stress tensor [see, Eq. (21)] and the axial current (7). This gives us a set of six equations for four components of the vector current which remains undetermined. It is quite remarkable that the overcomplete set is consistent. An observation that the conservation laws and an imposed anomaly completely determine the currents was first made by Son and Surowka in Ref. [3] for a related problem. We do not pursue this avenue. In the next section, we obtain the vector current differently (see, Sec. IV A). Also, it remains to be seen that Eq. (28) follows from the action (6). We will do this in the Sec. IX B.

Now we recall the standard arguments leading to Eq. (26). The equation follows from the invariance of the action $\mathcal{S}$ under the group of spacetime diffeomorphisms $\mathrm{Diff}(\mathbb{R}^4)$. The $\mathcal{S}^{(0)}$ part of the action depends on the spacetime metric, and the coupling depends on the external potentials. External potentials are spacetime covectors. This is obvious for the vector potential A_α. It is also true for the axial potential because the components of the axial current constitute the 3-form $j_A = p\wedge d(p+2A)$. Then the coupling $\int_{\mathbb{R}^4} A^{\mathrm{A}}\cdot j_{\mathrm{A}}$ be a spacetime scalar if $A^{\mathrm{A}}_\alpha dx^\alpha$ be 1-form and A^{A}_α is a spacetime covector.

Under a diffeomorphisms $x^\alpha \to x^\alpha + \epsilon^\alpha(x)$ the transformation of the spacetime, covectors are given by the action of the Lie derivative along the vector field with components ϵ^α. They are [18]

$$\delta_\epsilon A^{\mathrm{A}}_\alpha = \epsilon^\beta F^{\mathrm{A}}_{\beta\alpha} + \partial_\alpha(\epsilon\cdot A^{\mathrm{A}}), \tag{29}$$

$$\delta_\epsilon A_\alpha = \epsilon^\beta F_{\beta\alpha} + \partial_\alpha(\epsilon\cdot A). \tag{30}$$

Then the transformation of the action under combined transformations of the external potentials, the spacetime frame, and dynamical fields vanishes. The contribution of the transformations of the dynamical fields will be dropped if we evaluate the result on the equations of motion (the Hamilton principle).

Given that under coordinate transformation at fixed momentum $\delta_\epsilon \mathcal{S}^{(0)} = \int_{\mathbb{R}^4} T^\beta_\alpha \partial_\beta \epsilon^\alpha$, we have

$$\delta_\epsilon \mathcal{S} = \int_{\mathbb{R}^4} [T^\beta_\alpha \partial_\beta \epsilon^\alpha + J\cdot\delta_\epsilon A + j_{\mathrm{A}}\cdot\delta_\epsilon A^{\mathrm{A}}], \tag{31}$$

Reporting (29)–(30) in Eq. (31) and setting $\delta_\epsilon\mathcal{S}=0$ on equations of motions (EOM) we obtain

$$0 = \int_{\mathbb{R}^4} \epsilon^\alpha[-\partial_\beta T^\beta_\alpha + F_{\beta\alpha}J^\alpha - A_\beta\partial\cdot J + F^{\mathrm{A}}_{\beta\alpha} j^\alpha_A - A^{\mathrm{A}}_\beta \partial\cdot j_{\mathrm{A}}]$$

and (26).

C. Energy flow driven by electric current

This section aims to clarify the relationship between the electric current J and the vector current j.

Consider the temporal component of Eq. (28), that is the equation for the rate of change of the energy density $e = n\frac{\boldsymbol{p}^2}{2m} + \varepsilon$ and the energy flux $\boldsymbol{j}_e = n\mathbf{v}(\frac{\boldsymbol{p}^2}{2m}+\mu)$ of the perfect fluid

$$\dot{e} + \boldsymbol{\nabla}\boldsymbol{j}_e = \boldsymbol{E}\cdot\boldsymbol{j} + \boldsymbol{E}^{\mathrm{A}}\cdot\boldsymbol{j}^{\mathrm{A}}. \tag{32}$$

Let us assume that the axial potential is time-independent. Then we may speak about the energy density and the energy flux of the combined system, the reservoir and the fluid

$$\mathcal{E} = e - A^{\mathrm{A}}_0 j^0_{\mathrm{A}}, \qquad \boldsymbol{J}_{\mathcal{E}} = \boldsymbol{j}_e - A^{\mathrm{A}}_0 \boldsymbol{j}_{\mathrm{A}}. \tag{33}$$

and write (32) in terms of the combined system

$$\dot{\mathcal{E}} + \boldsymbol{\nabla}\cdot\boldsymbol{J}_{\mathcal{E}} = \boldsymbol{E}\cdot\boldsymbol{J}. \tag{34}$$

We see that the energy of the combined system is driven by the current $\boldsymbol{J}$. This suggests identifying $\boldsymbol{J}$ with the electric current of the combined system.

IV. HYDRODYNAMIC REPRESENTATION OF THE CURRENTS

In this section, we find the explicit expression for the electric current in terms of Eulerian observables. The reciprocal consistency relation offers an economic approach.

A. Reciprocal relation

The reciprocal relation states that the cross-variations of the currents evaluated on the equations of motion are equal. That is

$$\int_{\mathbb{R}^4} (\delta J^\alpha)\delta A_\alpha = \int_{\mathbb{R}^4} (\delta j^\alpha_{\mathrm{A}})\delta A^{\mathrm{A}}_\alpha. \tag{35}$$

Here the variation of J is taken over A^{A} at a fixed A and the variation of j_{A} is taken over A at a fixed A^{A} and both variations are set on equations of motions (EOM).

The reciprocal principle follows from the Hamilton principle and the definitions of currents as the first variation of the action (evaluated on EOM) [19]

$$\delta\mathcal{S}|_{\mathrm{EOM}} = \int_{\mathbb{R}^4} [J\cdot\delta A + j_{\mathrm{A}}\cdot\delta A^{\mathrm{A}}]. \tag{36}$$

Now, given the explicit form of the axial current (7), the reciprocal relation determines the electric current. Given (7) we vary it over A at a fixed canonical momentum π [The variation at a fixed canonical momentum π sets the variation on EOM]. We obtain (with the help (23))

$$\delta^{\mathrm{A}} J^{\alpha}|_{\pi} = -\delta^{\star}F_{\mathrm{A}}^{\alpha\beta}p_{\beta} + 2^{\star}F^{\alpha\beta}\delta A_{\beta}^{\mathrm{A}}. \tag{37}$$

Now we can integrate (37) with a condition that in a perfect fluid the electric current is equal to the particle number current n^{α}. This gives

$$J^{\alpha} = n^{\alpha} + {}^{\star}F_{\mathrm{A}}^{\alpha\beta}p_{\beta} + 2^{\star}F^{\alpha\beta}A_{\beta}^{\mathrm{A}},, \tag{38}$$

$$j^{\alpha} = n^{\alpha} + {}^{\star}F_{\mathrm{A}}^{\alpha\beta}p_{\beta}. \tag{39}$$

These are the hydrodynamic representation of the electric current (and the vector current). Given that, we obtain the explicit form of the fluid action (6). We write it in terms of the vector and the axial currents

$$\mathcal{S} = \mathcal{S}^{(0)} + \int_{\mathbb{R}^4}(A\cdot j - A^{\mathrm{A}}\cdot j_{\mathrm{A}}) + \int_{\mathbb{R}^4} 2A_{\alpha}{}^{\star}F^{\alpha\beta}A_{\beta}^{\mathrm{A}}, \tag{40}$$

and in terms of the Eulerian fields

$$\begin{aligned}\mathcal{S} = \mathcal{S}^{(0)} &+ \int_{\mathbb{R}^4}(A_{\alpha}n^{\alpha} - A_{\alpha}^{\mathrm{A}}\epsilon^{\alpha\beta\gamma\delta}p_{\beta}\partial_{\gamma}p_{\delta}) \\ &+ \int_{\mathbb{R}^4} -(A_{\alpha}{}^{\star}F_{\mathrm{A}}^{\alpha\beta} + 2A_{\alpha}^{\mathrm{A}\star}F^{\alpha\beta})p_{\beta} \\ &+ \int_{\mathbb{R}^4} 2A_{\alpha}{}^{\star}F^{\alpha\beta}A_{\beta}^{\mathrm{A}}.\end{aligned} \tag{41}$$

We wrote the action in the form which separates orders of the external field.

V. UNIFORMLY CANONICAL FLUID SYSTEMS

A barotropic perfect fluid is the simplest example of a general class of fluids referred to as *uniformly canonical* [1]. These fluids do not possess advected scalar field (a Lagrangian scalar), such as entropy, or if there is one, it must be chosen to be uniformly the same on all streamlines (if the Lagrangian scalar is entropy such flows are called homentropic). In this case, the number of independent Eulerian fields equals the dimension of spacetime, and the 4-momentum could be chosen to characterize the flow. The barotropic fluid under axial coupling belongs to this class.

Equivalently, the uniformly canonical systems could be defined by identifying their configuration space with the group of space-time diffeomorphisms $\mathrm{Diff}(\mathbb{R}^4)$ and its Eulerian observables (currents) with the Lie algebra $\mathrm{Diff}(\mathbb{R}^4)$, or the dual to the Lie algebra (the momentum).

Uniformly canonical systems possess remarkable geometric properties our analysis relays upon. Vorticity surfaces of such flows are integral and form a foliation of spacetime. A major consequence of this property is the conservation of helicity current [1] which we discussed in the next section (also, see Appendix).

Let us start with the perfect barotropic fluid. We may write the Euler equation (14) in the form

$$n(\dot{\boldsymbol{\pi}} - \boldsymbol{\nabla}\pi_0) - \boldsymbol{n}\times(\boldsymbol{\nabla}\times\boldsymbol{\pi}) = 0. \tag{42}$$

This form of the Euler equation emphasizes a different role of the particle number current n^{α} (16) and the canonical momentum (13). Introducing canonical vorticity tensor

$$\Omega_{\alpha\beta} = \partial_{\alpha}\pi_{\beta} - \partial_{\beta}\pi_{\alpha} \tag{43}$$

we write Eq. (42) in a remarkably compact form

$$\mathcal{J}^{\alpha}\Omega_{\alpha\beta} = 0, \tag{44}$$

where we renamed the particle number current n^{α} by $\mathcal{J}^{\alpha}$. This is the Carter-Lichnerowicz equation (see, [17] and [20,21] for a review). It must be complimented by the continuity equation

$$\partial\cdot\mathcal{J} = 0. \tag{45}$$

Equation (44) with a not specified current $\mathcal{J}$ used as a definition of uniformly canonical fluid systems by Carter [1]. Additional information about the relation between the conserved current $\mathcal{J}$ and the momentum defines a specific fluid system. This relation involves the spacetime metric, while the Eq. (44) does not. In the perfect fluid $\mathcal{J}^{\alpha} = n^{\alpha}$ but, generally, it could be different.

An easy consequence of the Carter-Lichnerowicz the equation is the Helmholtz law: vorticity Ω is advected along the vector field defined by $\mathcal{J}$ (see [21] and the Appendix). This property gives the meaning to $\mathcal{J}$: a vector field associated with $\mathcal{J}$ defines the streamlines. We, therefore, refer to $\mathcal{J}$ as a *flow field*.

An example of a nonuniformly canonical system is a baroclinic fluid. In this case, the rhs of (44) equals $n(\partial\varepsilon/\partial S)\partial_{\beta}S$, where S is a Lagrangian scalar, such as the entropy per particle.

VI. SPACETIME COVARIANT HAMILTON PRINCIPLE FOR UNIFORMLY CANONICAL FLUID SYSTEMS

Now we turn to the Hamilton principle which leads to Eq. (44). We consider the action $\mathcal{S}[\mathcal{J}]$ as a functional of the flow field $\mathcal{J}$ in the tangent bundle of $\mathbb{R}^4$ and define the canonical momentum π as a conjugate

$$\delta\mathcal{S} = \int_{\mathbb{R}^4} \pi\cdot\delta\mathcal{J}. \tag{46}$$

in the dual space, cotangent bundle of $\mathbb{R}^4$. Often it is more convenient to operate in a dual space considering a spacetime covariant Hamiltonian, the extension of the functional (19). It is a functional of π defined by the Legendre transform

$$\mathcal{H}[\pi] = \int_{\mathbb{R}^4} \pi\cdot\mathcal{J} - \mathcal{S}. \tag{47}$$

The equations of motions follow from the Hamilton principle

$$\delta\mathcal{S}[\mathcal{J}] = 0, \tag{48}$$

defined in the tangent bundle of $\mathbb{R}^4$, or equivalently from the Hamilton principle defined in the cotangent bundle of $\mathbb{R}^4$

$$\delta\mathcal{H}[\pi] = 0. \tag{49}$$

A. Admissible variations and the Lie algebra of spacetime diffeomorphisms

In 1966 Arnold [22] developed a framework in which a set of possible fluid motions (a configuration space) is identified as an infinite-dimensional Lie group of diffeomorphisms. Within this framework, the configuration space of uniformly canonical systems is the group manifold of spacetime diffeomorphisms $\mathrm{Diff}(\mathbb{R}^4)$. The elements of the Lie algebra $\mathrm{Diff}(\mathbb{R}^4)$ are vector fields $\mathcal{U}^\alpha = \mathcal{J}^\alpha\sqrt{|g|}$, where $|g|$ is the determinant of the spacetime metric.

The Lie algebra $\mathrm{Diff}(\mathbb{R}^4)$ is endowed with the Lie (or Jacobi-Lie) bracket, or the commutator of vector fields. If $X(x) = X^\alpha(x)\partial_\alpha$ and $Y(x) = Y^\alpha(x)\partial_\alpha$ are two vector fields then their commutator is

$$[X, Y] \coloneqq (X^\alpha\partial_\alpha Y^\beta - Y^\alpha\partial_\alpha X^\beta)\partial_\beta. \tag{50}$$

The Lie bracket defines the transformation of the vector field under spacetime diffeomorphism. Denote a spacetime diffeomorphism

$$x^\alpha \to x^\alpha + \epsilon^\alpha(x). \tag{51}$$

Then the change of X along the flow generated by the vector field $\epsilon = \epsilon^\alpha\partial_\alpha$ known as the Lie derivative $\mathcal{L}_\epsilon X = \delta_\epsilon X$, is

$$\delta_\epsilon X = [\epsilon, X]. \tag{52}$$

Explicitly

$$\delta_\epsilon X^\alpha = (\epsilon\cdot\partial)X^\alpha - (X\cdot\partial)\epsilon^\alpha. \tag{53}$$

Taking into account the transformation of the volume element $\delta(\sqrt{|g|})|_{|g|=1} = \partial\cdot\epsilon$ we obtain the transformation of the current

$$\delta\mathcal{J}^\alpha = \partial_\beta(\epsilon^\beta\mathcal{J}^\alpha - \epsilon^\alpha\mathcal{J}^\beta) + \epsilon^\alpha\partial_\beta\mathcal{J}^\beta. \tag{54}$$

Given the transformations (54) we find the transformation law of the momentum, the spacetime covector. Its transformation law. It follows from (54) under condition that $\int_{\mathbb{R}^4}(\pi\cdot\mathcal{J})$ is a scalar

$$\delta\pi_\alpha = \epsilon^\beta(\partial_\beta\pi_\alpha - \partial_\alpha\pi_\beta) + \partial_\alpha(\pi_\beta\epsilon^\beta). \tag{55}$$

Equations (54), (55) are particular cases of the general Cartan formula for transformations of the contravariant tensors [see Appendix (A1)].

The transformation laws determine admissible variations of the Hamilton principle (48), (49). The physical meaning of the admissible variations become clear if we treat the diffeomorphism parameter ϵ in (51) as unconstraint D'Alambertian virtual displacements of fluid parcels. Then the flow field $\mathcal{J}$ and the momentum change as (54), (55).

B. Equation of motions in the Carter-Lichnerowicz form

Let us compute the variation (46) of the action functional, where the variation of the flow field is given by (54). After the integration by parts, we obtain

$$\delta\mathcal{S} = -\int_{\mathbb{R}^4}(\mathcal{J}^\alpha\Omega_{\alpha\beta} + \partial\cdot\mathcal{J}\,\pi_\beta)\epsilon^\beta, \tag{56}$$

where the canonical vorticity is defined by (43). Naturally, the same the result follows from the dual form of the Hamilton principle (49) as $\delta\mathcal{S} = -\delta\mathcal{H} = \int_{\mathbb{R}^4}(\mathcal{J}\cdot\delta\pi)$ and the admissible variations of the momentum (55).

Hamilton's principle requires the integrand to vanish. Hence,

$$\mathcal{J}^\alpha\Omega_{\alpha\beta} + \partial\cdot\mathcal{J}\,\pi_\beta = 0. \tag{57}$$

Furthermore, the second term in (56) vanishes by the continuity equation (45). The remaining part gives the Carter-Lichnerowicz equation (44) for the uniformly canonical fluid systems.

C. Conserved currents: Helicity and the flow field

1. Conservation of the helicity current

Hamilton principle is the convenient platform to obtain the conservation of the helicity current,, a major property of the uniformly canonical system. That is

$$\partial \cdot h = 0, \qquad h^\alpha = \epsilon^{\alpha\beta\delta\gamma}\pi_\beta\partial_\delta\pi_\gamma, \tag{58}$$

Let us compute the variation of the action under a specific diffeomorphism

$$\epsilon^\alpha = 2\eta h^\alpha/(\mathcal{J}\cdot\pi), \tag{59}$$

where η is a scalar. The vector ϵ is normal to π_α. The variation (56) reads $\delta_\epsilon S = -2\int_{\mathbb{R}^4}(\eta/\mathcal{J}\cdot\pi)\mathcal{J}^\alpha\Omega_{\alpha\beta}h^\beta$. Next, with the help of identities

$$2\Omega_{\alpha\beta}{}^\star\Omega^{\beta\gamma} = \frac{1}{2}\delta^\gamma_\alpha\Omega_{\delta\beta}{}^\star\Omega^{\beta\delta} = -(\partial\cdot h) \tag{60}$$

followed from (27) and the definition of the helicity current (58), we obtain

$$\delta\mathcal{S} = \int_{\mathbb{R}^4}\eta(\partial\cdot h). \tag{61}$$

This variation must vanish for any η. This is possible only if the helicity current is conserved.

2. Continuity equation

Another specific diffeomorphism

$$\epsilon^\alpha = -\varphi\mathcal{J}^\alpha/(\mathcal{J}\cdot\pi) \tag{62}$$

where φ is a scalar, yields the variation

$$\delta\mathcal{S} = \int_{\mathbb{R}^4}\varphi(\partial\cdot\mathcal{J}) \tag{63}$$

and the continuity equation.

We observe that in uniformly canonical systems the continuity equation (45) follows from equations of motion (57). It does not need to be imposed separately, as it should, for example, in baroclinic fluids.

VII. AXIAL CURRENT AND THE AXIAL-CURRENT ANOMALY IN EULER FLUID

Given helicity conservation, we obtain the axial-current anomaly. In the absence of electromagnetic field we identify the helicity current with the axial current $j^\alpha_{\rm A} = \epsilon^{\alpha\beta\delta\gamma}p_\beta\partial_\delta p_\gamma$. However, in the electromagnetic field helicity current (58) fails to be a local Eulerian observable as it explicitly depends on the vector potential A. The global helicity $\mathcal{H} = \int_{\mathbb{R}^3} h^0$ is the gauge invariant, but its density is not. The problem had been addressed in [7]. There it was argued that axial current should be defined as green

$$j^\alpha_{\rm A} = \epsilon^{\alpha\beta\gamma\delta}p_\beta(\partial_\gamma p_\delta + F_{\gamma\delta}). \tag{64}$$

Counter to the helicity current (58) the axial current (64) is a local Eulerian observable. However, it is no longer conserved. The relation $j^\alpha_{\rm A} = h^\alpha - \epsilon^{\alpha\beta\gamma\delta}\partial_\beta(A_\delta p_\gamma) - 2{}^\star F^{\alpha\beta}A_\beta$ and the conservation of the helicity (58) shows that the divergence of the axial current does not vanish, but is solely expressed through the external electromagnetic field

$$\partial\cdot j_{\rm A} = \frac{1}{2}F_{\alpha\beta}\cdot{}^\star F^{\beta\alpha}. \tag{65}$$

Equations (58), (64), (65) represent the origin of the axial anomaly: the gauge invariance of observables conflicts with the axial-gauge transformations. The axial current generated by the axial gauge transformations does not conserve.

VIII. THE FLOW FIELD AND SPACETIME COVARIANT HAMILTONIAN UNDER THE AXIAL COUPLING

We return to the axial coupling. Given the explicit form of the action (41), we may now compute the flow field by exploring the defining relation (46). Varying (41) over Eulerian fields we obtain (after some algebra)

$$\delta\mathcal{S} = \int_{\mathbb{R}^4}\pi_\alpha\cdot[\delta n^\alpha + \epsilon^{\alpha\beta\gamma\delta}(\partial_\gamma A^{\rm A}_\beta\delta p_\delta + 2A^{\rm A}_\beta\partial_\gamma\delta p_\delta)].$$

This formula defines the flow field $\mathcal{J}$ up to a term that does not depend on dynamical fields. We chose it such that the difference between $\mathcal{J}$ and the electric current (38) is a curl

$$\begin{aligned}\mathcal{J}^\alpha &= n^\alpha - {}^\star F^{\alpha\beta}_{\rm A}p_\beta + 2\epsilon^{\alpha\beta\gamma\delta}A^{\rm A}_\beta\partial_\gamma\pi_\delta \\ &= J^\alpha - 2\epsilon^{\alpha\beta\gamma\delta}\partial_\beta(A^{\rm A}_\gamma p_\delta).\end{aligned} \tag{66}$$

Then the conservation of the electric charge yields the continuity condition $\partial\cdot\mathcal{J} = 0$.

Now, when we know the explicit form of the flow field we obtain the spacetime Hamiltonian by computing the Legendre transform (47). We obtain a remarkably simple result

$$\mathcal{H} = \mathcal{H}^{(0)} - \int_{\mathbb{R}^4}A^{\rm A}\cdot j_{\rm A}. \tag{67}$$

In Ref. [12] this formula was used as a definition of axial coupling. Advantage of the functional $\mathcal{H}$ is that it is the explicit function of the momentum and its derivative. This is in contrast to the action $\mathcal{S}$ (41) which does not have a simple expression in terms of $\mathcal{J}$.

IX. EQUATIONS OF MOTION

A. Euler equation deformed by the axial coupling

As Eq. (66) shows the flow field $\mathcal{J}$ explicitly depends on the axial potential and like the electric current, it is not invariant under the axial-gauge transformations. At the same time, the equations of motion (28) are axial-gauge invariant. This occurs because the last term in (66) is the null vector of vorticity tensor as $^{\star}\Omega^{\gamma\alpha}\Omega_{\alpha\beta} = 0$ due to helicity conservation (60). It drops from the equations of motions (44).

The remaining part of $\mathcal{J}$

$$I^{\alpha} = n^{\alpha} - {}^{\star}F_{\mathrm{A}}^{\alpha\beta} p_{\beta} \tag{68}$$

is axial gauge-invariant and so the equations of motions. Also, I^{α} does not depend on the electromagnetic field.

In terms of I^{α} the Carter-Lichnerowicz equation (44) and the continuity equation (45) read

$$\begin{cases} I^{\alpha}\Omega_{\beta\alpha} = 0, \\ \partial \cdot I = {}^{\star}F_{\mathrm{A}} \cdot \Omega. \end{cases} \tag{69}$$

We may bring these equations the form close to Eq. (42). Introducing

$$\boldsymbol{V} \coloneqq \frac{\boldsymbol{I}}{I^0} = \frac{\boldsymbol{n} - \boldsymbol{p} \times \boldsymbol{E}^{\mathrm{A}} + p_0 \boldsymbol{B}^{\mathrm{A}}}{n - \boldsymbol{B}^{\mathrm{A}} \cdot \boldsymbol{p}} \tag{70}$$

and after some algebra, we obtain

$$\begin{cases} \dot{\boldsymbol{\pi}} - \boldsymbol{\nabla}\pi_0 - \boldsymbol{V} \times (\boldsymbol{\nabla} \times \boldsymbol{\pi}) = 0, \\ \dot{n} + \boldsymbol{\nabla} \cdot \boldsymbol{n} = 2(\boldsymbol{E}^{\mathrm{A}} + \boldsymbol{V} \times \boldsymbol{B}^{\mathrm{A}}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{\pi}). \end{cases} \tag{71}$$

This form emphasizes the effect of the axial field as a deformation of the Euler equation and the continuity equation of the perfect fluid.

B. Direct derivation of equations of motion in the form of conservation laws

Although it may not be obvious, the Eqs. (69) are equivalent to Eq. (28). A direct check requires a good deal of uninspiring algebra. More instructive is to obtain (28) directly from the Hamilton principle (48). This will be a prove that the action has the form (6) and is consistent with (31),(36) despite of apparent sign difference of the axial coupling.

We compute the variation of the action (6) (repeated below) under variations of vector fields (54)

$$\mathcal{S} = \mathcal{S}^{(0)} + \int_{\mathbb{R}^4} [A \cdot J - A^{\mathrm{A}} \cdot j_{\mathrm{A}}]. \tag{72}$$

We start from the action $\mathcal{S}^{(0)}$. Its variation follows from (56) under the identification the particle number current with the flow field $\mathcal{J}^{\alpha} = n^{\alpha}$ and $\Omega_{\alpha\beta} = \partial_{\alpha} p_{\beta} - \partial_{\beta} p_{\alpha}$. We obtain the divergence of the momentum-stress-energy tensor plus the divergence of n^{α}

$$\delta\mathcal{S}^{(0)} = \int_{\mathbb{R}^4} [-\partial_{\alpha} T^{\alpha}_{\beta} + p_{\beta}\partial_{\alpha} n^{\alpha}]\epsilon^{\beta}. \tag{73}$$

In the perfect fluid $\partial_{\alpha} n^{\alpha} = 0$, and the last term vanishes. The axial field changes this. With the help of the continuity equation (45), the explicit form of the flow field (66) and the identity (60) we obtain the relation

$$p_{\beta}\partial_{\alpha} n^{\alpha} = 2F^{\mathrm{A}}_{\beta\alpha} j^{\alpha}_{\mathrm{A}}. \tag{74}$$

Therefore,

$$\delta\mathcal{S}^{(0)} = \int_{\mathbb{R}^4} [-\partial_{\alpha} T^{\alpha}_{\beta} + 2F^{\mathrm{A}}_{\beta\alpha} j^{\alpha}_{\mathrm{A}}]\epsilon^{\beta}. \tag{75}$$

The next step is to find the variation of the couplings in (72). We get

$$\begin{aligned} \delta\left(\int_{\mathbb{R}^4} A \cdot J\right) &= \int_{\mathbb{R}^4} \epsilon^{\beta}[F_{\beta\alpha} J^{\alpha} + A_{\beta}(\partial \cdot J)] \\ &= \int_{\mathbb{R}^4} \epsilon^{\beta} F_{\alpha\beta} J^{\alpha} \\ &= \int_{\mathbb{R}^4} \epsilon^{\beta}[F_{\beta\alpha} j^{\alpha} + 2F_{\beta\alpha}{}^{\star}F^{\alpha\gamma} A^{\mathrm{A}}_{\gamma}] \\ &= \int_{\mathbb{R}^4} \epsilon^{\beta}[F_{\beta\alpha} j^{\alpha} + \frac{1}{2} A^{\mathrm{A}}_{\beta} F_{\gamma\alpha}{}^{\star}F^{\alpha\gamma}] \\ &= \int_{\mathbb{R}^4} \epsilon^{\beta}[F_{\beta\alpha} j^{\alpha} + A^{\mathrm{A}}_{\beta}(\partial \cdot j_{\mathrm{A}})], \end{aligned} \tag{76}$$

$$-\delta\left(\int_{\mathbb{R}^4} A^{\mathrm{A}} \cdot j_{\mathrm{A}}\right) = -\int_{\mathbb{R}^4} \epsilon^{\beta}[F^{\mathrm{A}}_{\beta\alpha} j^{\alpha}_{\mathrm{A}} + A^{\mathrm{A}}_{\beta}(\partial \cdot j_{\mathrm{A}})] \tag{77}$$

In these equations, we took into account that the currents are transformed according to the rule (54), that the electric current is divergence-free, the relation between the electric current and the vector current (24), the identity $2F_{\beta\alpha}{}^{\star}F^{\alpha\lambda} = \frac{1}{2}\delta^{\lambda}_{\beta} F_{\gamma\alpha}{}^{\star}F^{\alpha\gamma}$ and the divergence of the axial current (65). Combining the contributions of (75)–(77) we notice that the last terms in (76) and (77) cancel each other and the last term in (75) changes the sign in the first term of (77). The result is

$$\delta\mathcal{S} = \int_{\mathbb{R}^4} \epsilon^{\beta}[-\partial_{\alpha} T^{\alpha}_{\beta} + F_{\beta\alpha} j^{\alpha} + F^{\mathrm{A}}_{\beta\alpha} j^{\alpha}_{\mathrm{A}}]. \tag{78}$$

That prompts Eq. (28) and explains the choice of an unusual sign in the axial coupling in (72).

X. DISCUSSION

We highlight the major points of the paper.

We developed the spacetime covariant Hamilton principle on the Lie algebra Diff$(\mathbb{R}^4)$ and used it to study the effects of the chiral anomaly in hydrodynamics.

In particular, we obtain the set of conservation laws that give the over-complete, but consistent set of equations of motion. For references we collect them here

$$\begin{cases} \partial_\alpha T^\alpha_\beta = F_{\beta\alpha} j^\alpha + F^{\mathrm{A}}_{\beta\alpha} j^\alpha_A, \\ \partial_\alpha j^\alpha_{\mathrm{A}} = \frac{1}{2} {}^\star F^{\alpha\beta} F_{\beta\alpha}, \\ \partial_\alpha j^\alpha = F_{\alpha\beta} {}^\star F^{\beta\alpha}_{\mathrm{A}}. \end{cases} \tag{79}$$

Also, we found the hydrodynamic representations of the vector and the axial current

$$\begin{cases} j^\alpha = n u^\alpha + {}^\star F^{\alpha\beta}_{\mathrm{A}} p_\beta, \\ j^\alpha_{\mathrm{A}} = \epsilon^{\alpha\beta\delta\gamma} p_\beta (\partial_\delta p_\gamma + F_{\delta\gamma}). \end{cases} \tag{80}$$

Given the explicit form of the stress tensor and the currents, the conservation laws provide six equations for four Eulerian fields n and $\mathbf{v}$. Various consistency conditions we used in the text warrant that this overdetermine set is consistent. Equation (80) could be seen as a realization of the overcomplete set of conservation laws (79).

We identified the class of fluids where the chiral anomaly takes place. These are uniformly canonical fluid systems. Such systems could be defined in different ways. One of them is to say that spacetime momentum is the only dynamic variable of the variational functional. This natural property gives rise to the foliation of the spacetime by vorticity surfaces, the geometric root of the conservation of helicity, and the chiral anomaly.

The spacetime covariant Hamilton principle is commonly used in relativistic fluid dynamics. It could be used for nonrelativistic hydrodynamics but often does not bring a new content if the action of the Lie derivative $\mathcal{L}_\epsilon$ which evaluates a change of the spacetime tensors is essentially reduced to the action of $\partial_t + \mathcal{L}_{\boldsymbol{\epsilon}}$, where $\boldsymbol{\epsilon}$ is the spatial $\mathbb{R}^3$ vector field as it happens in the Galilean perfect fluid. This is no longer true if there is the axial "magnetic" $\boldsymbol{B}^{\mathrm{A}}$. In this case, the spacetime covariant approach used in the paper is indispensable.

In the nutshell the chiral anomaly could be formulated as follows: equations of motions, such as (28) are composed of the fields tensors of vector and axial potentials $F_{\alpha\beta}$ and $F^{\mathrm{A}}_{\alpha\beta}$. They are gauge invariant under vector and axial gauge transformations (11), (12). Solutions to these equations being invariant under vector gauge transformation (11) may not be necessarily invariant under the axial transformation (12). This is also true for physical observables such as electric current (23). Important solutions, such as the stationary flow explicitly depend on the value of the axial potential. In [8] it had been shown that the stationary solutions are Beltrami flows. Let us recall this result. To simplify the matter, let us turn the electromagnetic field off. Then if the remaining axial background potential is time-independent, the stationary flow exists and is given by the condition $p_0 = 0$ and the Beltrami condition, that is the "velocity" $\boldsymbol{V}$ and vorticity $\boldsymbol{\nabla}\times\boldsymbol{p}$ are collinear [11]. The Beltrami condition satisfies the first equation (71). Then the second equation (71) reduces to the equation $\boldsymbol{\nabla}\cdot\boldsymbol{n} = 2\boldsymbol{E}^{\mathrm{A}}\cdot(\boldsymbol{\nabla}\times\boldsymbol{p})$ whose solution is Beltrami flow

$$\mathrm{curl}\,\mathbf{v} = \kappa\mathbf{v}. \tag{81}$$

Here $\mathrm{curl}\,\mathbf{v} = \frac{1}{\sqrt{|A_0^{\mathrm{A}}|}}\boldsymbol{\nabla}(\sqrt{|A_0^{\mathrm{A}}|}\times\mathbf{v})$ and the Beltrami coefficient $\kappa = n/2mA_0^{\mathrm{A}}$ is determined by the axial-vector potential.

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APPENDIX: SPACETIME COVARIANT HAMILTON PRINCIPLE AND CONSERVATION LAWS IN TERMS OF DIFFERENTIAL FORMS

Here we recast the equations of motions and conservation laws of the uniformly canonical fluid systems in terms of differential forms emphasizing geometric aspects of the flows.

We describe the fluid configuration space by the flow vector field $\mathcal{U}$ dual to the flow current 3-form $\mathcal{J}$. The formal relation is: the current 3-form is the volume form evaluated on the vector field $\iota_{\mathcal{U}}\mathrm{vol} = \mathcal{J}$. We also consider the momentum 1-form $\pi = \pi_\alpha dx^\alpha$.

Under a diffeomorphism, $x^\alpha \to x^\alpha + \epsilon^\alpha$ the change of a spacetime tensor defines the Lie derivative $\mathcal{L}_\epsilon$ along the vector field $\epsilon^\alpha\partial_\alpha$. The action of the Lie derivative on a differential form is expressed in terms of the interior product and the exterior derivative by the Cartan formula

$$\mathcal{L}_\epsilon = \iota_\epsilon d + d\iota_\epsilon. \tag{A1}$$

We recall that the interior product ι_ϵ acting on a differential form is the mere contraction of the cotensor with the vector ϵ.

Applying the Cartan formula we obtain Lie derivative to the current 3-form $\delta\mathcal{J} = \mathcal{L}_\epsilon\mathcal{J}$ and the momentum 1-form $\delta\pi = \mathcal{L}_\epsilon\pi$. In components they are given by (54), (55).

Using these formulas we obtain the variation of the action. Say, we can choose π to be a natural variable. Then

$$\delta\mathcal{S} = -\int_{\mathbb{R}^4} \iota_{\mathcal{U}}\delta\pi = \int_{\mathbb{R}^4} \iota_\epsilon \iota_{\mathcal{U}}\Omega, \tag{A2}$$

where we used the rule of the interior product $\iota_{\mathcal{U}}\iota_\epsilon = -\iota_\epsilon\iota_{\mathcal{U}}$ and the continuity equation. Then the Carter-Lichnerowicz equation follows [21]

$$\iota_{\mathcal{U}}\Omega = 0. \tag{A3}$$

Lagrange-Cauchy form of Euler equation Let us examine the advection of the fluid momentum. That is the Lie derivative of the momentum along the flow vector field $\mathcal{U}$. Applying the Cartan formula (A1) we write $\mathcal{L}_{\mathcal{U}}\pi = \iota_{\mathcal{U}}\Omega + d(\iota_{\mathcal{U}}\pi)$. Then by the equation of motion (A3), we obtain the Lagrange-Cauchy form of the Euler equation

$$\mathcal{L}_{\mathcal{U}}\pi = d(\iota_{\mathcal{U}}\pi). \tag{A4}$$

It represents the fluid dynamics as a degree to which the acceleration deviates from streamlines.

Helmholtz law Taking the exterior derivative of (A4) and using its commutativity with the Lie derivative we obtain the Helmholtz law: canonical vorticity 2-form $\Omega = d\pi$ is advected by the flow along the fluid streamlines

$$\mathcal{L}_{\mathcal{U}}\Omega = 0. \tag{A5}$$

As a consequence, the vorticity flux through an area encompassed by a fluid contour comoving with the flow does not change in time (Kelvin theorem).

Foliation The Carter-Lichnerowicz equation states that the vorticity matrix is rank-2 degenerate having two zero-eigenvectors $\mathcal{J}^\alpha$ (streamlines) and ${}^\star\Omega^{\alpha\beta}\mathcal{J}_\beta$. This entails a special geometric property of uniformly canonical systems [1]: the spacetime is foliated by a family of integral vorticity 2-surfaces whose tangent vectors are null vectors of vorticity. In other words, streamlines define a family of smooth 2-dimensional vorticity surfaces each passing through a given point of spacetime.

Conservation of helicity The conservation of helicity (58) could be traced to the foliation property with no reference to a relation between the flow field $\mathcal{J}$ and the fluid momentum.

The divergence of the helicity current is the exterior derivative dh of the helicity 3-form $h = \pi \wedge d\pi$. It is the top form

$$dh = \Omega \wedge \Omega.$$

Using the rule for the interior product (see, (27) for this formula written in components)

$$\iota_X(m \wedge l) = (\iota_X m) \wedge l + (-1)^q m \wedge (\iota_X l)$$

where l, m are differential forms and q is the degree of the form m, we obtain $\iota_{\mathcal{U}}(dh) = \iota_{\mathcal{U}}(\Omega \wedge \Omega) = 2\Omega \wedge (\iota_{\mathcal{U}}\Omega)$ which vanishes due to (A3). Hence, $dh = 0$.

Alternatively, we notice that $dh = \Omega \wedge \Omega = \mathrm{Pf}[\Omega_{\alpha\beta}]$, where $\mathrm{Pf} = \sqrt{\det[\Omega_{\alpha\beta}]}$ is the Pffafian of the 4×4 matrix $\Omega_{\alpha\beta}$. Since this matrix has null vectors its determinant and the Pffafian vanishes.

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