SUPPLEMENTAL MATERIAL

1. Normalization of the Weight-Shifting Operator

An integral representation of the scalar three-point function in momentum space is given by

$$\langle\Phi_{\Delta_1}\Phi_{\Delta_2}\Phi_{\Delta_3}\rangle = -\frac{2^{3/2}}{\pi^{3/2}\Gamma(\frac{d-3}{2})}\int_0^\infty \frac{dz}{z^{d+1}}\mathcal{K}_{\Delta_1}(k_1,z)\mathcal{K}_{\Delta_2}(k_2,z)\mathcal{K}_{\Delta_3}(k_3,z)\,, \tag{1}$$

where $\mathcal{K}_\Delta(k,z) \equiv k^{\Delta-\frac{d}{2}}z^{\frac{d}{2}}K_{\Delta-\frac{d}{2}}(kz)$ is the bulk-to-boundary propagator and $K_{\Delta-\frac{d}{2}}$ is the modified Bessel function of the second kind. The flat-space limit $K\to 0$ corresponds to taking the large z limit of the integrand, in which case the Bessel-K function reduces to

$$\lim_{z\to\infty} K_{\Delta-\frac{d}{2}}(kz) = \sqrt{\frac{\pi}{2kz}}e^{-kz}\,, \tag{2}$$

independent of the weight. The overall normalization in (1) is chosen to cancel the constant factors that arise in this limit, so that the three-point function becomes

$$\lim_{K\to 0}\langle\Phi_{\Delta_1}\Phi_{\Delta_2}\Phi_{\Delta_3}\rangle = k_1^{\Delta_1-\frac{d+1}{2}}k_2^{\Delta_2-\frac{d+1}{2}}k_3^{\Delta_3-\frac{d+1}{2}}K^{\frac{3-d}{2}}\,. \tag{3}$$

In terms of the energy variables, the weight operator W_{12} takes the form

$$W_{12} = \frac{1}{2}\left[\partial_{k_1}^2 + \partial_{k_2}^2 + \frac{k_1^2+k_2^2-k_3^2}{k_1k_2}\partial_{k_1}\partial_{k_2} + \frac{d-1}{k_1}\partial_{k_1} + \frac{d-1}{k_2}\partial_{k_2}\right], \tag{4}$$

when acting on the scalar three-point function. To see how this operator affects the normalization of the three-point function, we express its action on a product of the bulk-to-boundary propagators as [27]

$$W_{12}\Big(\mathcal{K}_{\Delta_1}(k_1,z)\mathcal{K}_{\Delta_2}(k_2,z)\Big) = \frac{1}{2}\Big(\Box_z - (\Delta_1+\Delta_2-2)(\Delta_1+\Delta_2-2-d)\Big)\Big(\mathcal{K}_{\Delta_1-1}(k_1,z)\mathcal{K}_{\Delta_2-1}(k_2,z)\Big)\,, \tag{5}$$

where $\Box_z = z^2\partial_z^2 - (d-1)z\partial_z - k_3^2z^2$ is the AdS$_{d+1}$ box operator. Integrating by part the z derivatives and using the equation of motion for the bulk-to-boundary propagator

$$(\Box_z - \Delta_3(\Delta_3-d))\mathcal{K}_{\Delta_3}(k_3,z) = 0\,, \tag{6}$$

we find that

$$W_{12}\langle\Phi_{\Delta_1}\Phi_{\Delta_2}\Phi_{\Delta_3}\rangle = -\frac{(\Delta_1+\Delta_2-\Delta_3-2)(\Delta_1+\Delta_2-\tilde{\Delta}_3-2)}{2}\langle\Phi_{\Delta_1-1}\Phi_{\Delta_2-1}\Phi_{\Delta_3}\rangle\,. \tag{7}$$

Absorbing this prefactor into the weight operator leads to the normalization choice for $\widehat{W}_{12}$.

2. Scalar Seed Functions in Momentum Space

In the main text, the differential representations (17), (20), and (27) involve scalar seed functions of weights d, $d+2$, and $d+2\ell-2$, respectively. For example, the calculation of the graviton three-point function for $d=5,7$, as presented in Eqs. (24) and (25), uses scalar seeds with $\Delta_\Phi = 7,9$ as the initial input. Strictly speaking, the integral in (1) that correspond to these weight choices diverges and therefore requires renormalization. (In general, the integral is divergent for weights satisfying $\frac{d}{2}\pm(\Delta_1-\frac{d}{2})\pm(\Delta_2-\frac{d}{2})\pm(\Delta_3-\frac{d}{2}) = -2n$ with $n=0,1,2,\cdots$, and any independent choice of the signs.) This can be done by regulating the z integral with a finite cutoff and deforming the seed weights with infinitesimal parameters. The divergences that arise in the small deformation limit can then be absorbed by local counterterms. The precise details of this renormalization procedure can be found in [69,70].

An alternative approach for deriving general seed functions is to start from a known expression for renormalized scalar correlators of lower weights. In this case, one can repeatedly act on the initial expression with the "weight-raising" operator

$$W_{ab}^+ = \frac{1}{2}k_a^2k_b^2\vec{K}_{ab}\cdot\vec{K}_{ab} - (2\Delta_a-d)(2\Delta_b-d)\vec{k}_a\cdot\vec{k}_b + \left[k_a^2(\Delta_b-d)(\vec{k}_b\cdot\vec{K}_{ab}+\Delta_b-d+1)+(a\leftrightarrow b)\right], \tag{8}$$

which raises each of the weights Δ_a and Δ_b by one unit. It is convenient to use the rescaled version that retains the normalization choice of (1), given by

$$\widehat{W}_{ab}^{+} = \frac{2W_{ab}^{+}}{(\Delta_a+\Delta_b-\Delta_c+2-d)(\Delta_a+\Delta_b-\tilde{\Delta}_c+2-d)}\,, \tag{9}$$

where a,b,c are field labels with $c \neq a,b$, and the weights appearing on the right-hand side are those before acting with the weight-raising operator. For example, the scalar correlator of weight $\Delta_\Phi = 3$ in $d = 3$ has the well-known expression [70]

$$\langle \Phi_\Delta\Phi_\Delta\Phi_\Delta\rangle_{\Delta=3}^{d=3} = \frac{1}{3}\left[\log(K/\mu)\sum_a k_a^3 - \sum_{a\neq b} k_a^2 k_b + k_1k_2k_3\right], \tag{10}$$

where μ plays the role of a renormalization scale.

It is straightforward to derive the desired odd-d momentum-space seed functions in the differential representations (17), (20), and (27) in the main text, where the seed weight can take on the values d, $d+2$, or $d+2\ell-2$. For concreteness, let us provide the explicit expressions for the seed functions, with $\Delta_\Phi = 7,9$ and $d = 5,7$, that were used to obtain (24) and (25):

$$\begin{aligned}
\langle \Phi_\Delta\Phi_\Delta\Phi_\Delta\rangle_{\Delta=7}^{d=5} &= \frac{e_3^4}{K} - 45e_2^2e_3^2K + \frac{5}{4}K^2(63e_2^3e_3 - 29e_3^3) + \frac{5}{12}e_2K^3(375e_3^2-77e_2^3) - \frac{385}{2}e_2^2e_3K^4 \\
&+ \frac{1}{6}K^5(427e_2^3-339e_3^2) + \frac{427}{4}e_2e_3K^6 - \frac{1371}{28}e_2^2K^7 - \frac{457e_3K^8}{28} + \frac{1441e_2K^9}{108} - \frac{16481K^{11}}{13068} \\
&- 5\bigg[2e_2e_3^3 - 9e_2^2e_3^2K - \frac{1}{2}e_3K^2(11e_3^2-21e_2^3) - \frac{1}{2}K^3(7e_2^4-39e_2e_3^2) - 21e_2^2e_3K^4 \\
&- \frac{1}{5}K^5(30e_3^2-35e_2^3) + \frac{21}{2}e_2e_3K^6 - \frac{9}{2}e_2^2K^7 - \frac{3e_3K^8}{2} + \frac{7e_2K^9}{6} - \frac{7K^{11}}{66}\bigg]\log(K/\mu)\,, \qquad (11)\\
\langle \Phi_\Delta\Phi_\Delta\Phi_\Delta\rangle_{\Delta=9}^{d=7} &= \frac{e_3^5}{K^2} + \frac{15e_2e_3^4}{K} - 30e_3^2K(14e_2^3-3e_3^2) + \frac{315}{4}e_2e_3K^2(9e_2^3-13e_3^2) - \frac{35}{4}e_2^2K^3(33e_2^3-326e_3^2) \\
&- \frac{35}{2}e_3K^4(165e_2^3-41e_3^2) + \frac{7}{4}e_2K^5(549e_2^3-1466e_3^2) + \frac{11529}{4}e_2^2e_3K^6 + \frac{1}{4}K^7(2165e_3^2-4113e_2^3) \\
&- \frac{4113}{4}e_2e_3K^8 + \frac{1441}{12}(4e_2^2+e_3K)K^9 - \frac{49443e_2K^{11}}{484} + \frac{663549K^{13}}{81796} - 105\bigg[e_2^2e_3^3 - e_3^2K(4e_2^3-e_3^2) \\
&+ \frac{3}{2}e_2e_3K^2(3e_2^3-5e_3^2) - \frac{1}{2}e_2^2K^3(3e_2^3-34e_3^2) - e_3K^4(15e_2^3-4e_3^2) + \frac{1}{2}e_2K^5(9e_2^3-26e_3^2) \qquad (12)\\
&+ \frac{27}{2}e_2^2e_3K^6 - \frac{1}{2}(9e_2^3-5e_3^2)K^7 - \frac{9}{2}e_2e_3K^8 + 2e_2^2K^9 + \frac{e_3K^{10}}{2} - \frac{9e_2K^{11}}{22} + \frac{9K^{13}}{286}\bigg]\log(K/\mu)\,,
\end{aligned}$$

where $e_2 \equiv k_1k_2 + k_2k_3 + k_3k_1$ and $e_3 \equiv k_1k_2k_3$. It is manifest that these have the desired normalization in the flat-space limit $K \to 0$ from the leading singular term. It can be checked that these satisfy the anomalous conformal Ward identities

$$\sum_{a=1}^{3}\left(3\Delta - 2d - k_a\frac{\partial}{\partial k_a}\right)\langle\Phi_\Delta\Phi_\Delta\Phi_\Delta\rangle = \text{local}\,, \tag{13}$$

$$\sum_{a=1}^{3}\vec{k}_a\left(\frac{\partial^2}{\partial k_a^2} - \frac{2\Delta-d-1}{k_1}\frac{\partial}{\partial k_a}\right)\langle\Phi_\Delta\Phi_\Delta\Phi_\Delta\rangle = 0\,, \tag{14}$$

where 'local' refers to terms analytic in two of the three momenta k_1, k_2, k_3.

3. Three-Point Function from Higher-Derivative Interactions

Here we prove the formula for the three-point function arising from higher-derivative interactions

$$\langle J_\ell J_\ell J_\ell\rangle_{\text{h.d.}} = \widehat{F}_{12}^{\ell}\widehat{F}_{23}^{\ell}\widehat{F}_{31}^{\ell}\langle\Phi\Phi\Phi\rangle_{\Delta_\Phi=d+2\ell-2}\,. \tag{15}$$

We do this by showing that its divergence manifestly vanishes in embedding space. The first step is to prove the analogous statement for the mixed three-point function $\langle J_\ell \Phi\Phi\rangle = \widehat{F}_{12}^\ell \langle\Phi\Phi\Phi\rangle$. To do so, let us write down the definition of the operator $\widehat{F}_{ab}$ in embedding space:

$$F_{ab} = H_a^{AB} L_{b,AB} = X_a^{[A} Z_a^{B]} (X_{b,[A} \partial_{X_b^{B]}} + Z_{b,[A} \partial_{Z_b^{B]}}) \,, \tag{16}$$

where H_a^{AB} is an operator that raises the spin and lowers the weight by one unit at point a, whereas L_b^{AB} is the conformal generator that acts on point b without changing any quantum numbers, and we consider an unnormalized operator F_{ab} for simplicity. Acting with this operator on an equal-weight scalar seed, we get

$$F_{12}\langle\Phi\Phi\Phi\rangle = \Delta_\Phi V_1 \langle\Phi\Phi\Phi\rangle \,, \quad V_1 \equiv \frac{X_{13} Z_1 \cdot X_2 - X_{12} Z_1 \cdot X_3}{X_{23}} \,, \tag{17}$$

where $X_{ab} \equiv X_a \cdot X_b$. We have $F_{12} V_1 = V_1^2$, and applying the operator ℓ times on the seed function gives

$$F_{12}^\ell \langle\Phi\Phi\Phi\rangle = 2^\ell (\tfrac{\Delta_\Phi}{2})_\ell V_1^\ell \langle\Phi\Phi\Phi\rangle \,, \tag{18}$$

where $(a)_\ell$ is the Pochhammer symbol. Taking the divergence, it is straightforward to show that

$$\mathrm{div}_1 V_1^\ell \langle\Phi\Phi\Phi\rangle = \frac{\ell(\ell-1)(d+2\ell-2-\Delta_\Phi)}{2} \frac{V_1^{\ell-2}(X_{13} Z_1 \cdot X_2 + X_{12} Z_1 \cdot X_3)}{X_{23}} \langle\Phi\Phi\Phi\rangle \,. \tag{19}$$

For $\ell = 1$, this vanishes for any scalar weights, while for $\ell \geq 2$ this vanishes for $\Delta_\Phi = d + 2\ell - 2$.

The next step is to consider the mixed three-point function with two conserved currents

$$\langle J_\ell J_\ell \Phi\rangle = F_{12}^\ell F_{23}^\ell \langle\Phi\Phi\Phi\rangle = (-1)^\ell F_{13}^\ell F_{23}^\ell \langle\Phi\Phi\Phi\rangle = (-1)^\ell F_{23}^\ell F_{13}^\ell \langle\Phi\Phi\Phi\rangle = F_{23}^\ell F_{12}^\ell \langle\Phi\Phi\Phi\rangle \,. \tag{20}$$

Note that we are free to permute the second index of F_{ab} (up to a minus sign) using the conformal Ward identity $(L_1^{AB} + L_2^{AB} + L_3^{AB})\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = 0$ for the three-point function of any operators $\mathcal{O}_a$ and the fact that $F_{aa} = 0$. This then allows us to commute the F_{ab} operators as shown above. Since div_1 and div_2 simply pass through L_3^{AB}, we have

$$\mathrm{div}_1 F_{12}^\ell F_{23}^\ell \langle\Phi\Phi\Phi\rangle_{\Delta_\Phi = d+2\ell-2} = F_{23}^\ell \, \mathrm{div}_1 F_{12}^\ell \langle\Phi\Phi\Phi\rangle_{\Delta_\Phi = d+2\ell-2} = 0 \,, \tag{21}$$

by virtue of (19), and similarly for div_2. Finally for the three-spinning case, we can also commute the operators so that the divergence operator div_a acts purely on $F_{ab}^\ell\langle\Phi\Phi\Phi\rangle$, which brings down the same prefactor $(d + 2\ell - 2 - \Delta_\Phi)$ as in (19). It then follows that

$$\mathrm{div}_1 F_{12}^\ell F_{23}^\ell F_{31}^\ell \langle\Phi\Phi\Phi\rangle_{\Delta_\Phi = d+2\ell-2} = F_{23}^\ell F_{32}^\ell \, \mathrm{div}_1 F_{12}^\ell \langle\Phi\Phi\Phi\rangle_{\Delta_\Phi = d+2\ell-2} = 0 \,, \tag{22}$$

and similarly for div_2 and div_3. This proves the desired formula (15).