Supplemental Information for "Generalized Matching Condition for Unity Efficiency Quantum Transduction"

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CONVERSION EFFICIENCY DERIVATION

We characterize the dynamics of N-stage transduction using the standard input-output theory [1] and derive the conversion efficiency. The Heisenberg Langevin equations of motion for the system modes in the frequency domain are

$$\begin{aligned}
\chi_a^{-1}\hat{a}^\dagger[\omega] &= ig_1\hat{m}_2^\dagger[\omega] + \sqrt{\kappa_{a,\text{ex}}}\hat{a}^\dagger_{\text{in,ex}}[\omega] + \sqrt{\kappa_{a,\text{i}}}\hat{A}^\dagger_{\text{in}}[\omega],\\
\chi_j^{-1}\hat{m}_j^\dagger[\omega] &= ig_j\hat{m}_{j+1}^\dagger[\omega] + ig_{j-1}\hat{m}_{j-1}^\dagger[\omega] + \sqrt{\kappa_j}\hat{M}^\dagger_{\text{j,in}}[\omega],\\
\chi_b^{-1}\hat{b}^\dagger[\omega] &= ig_{N+1}\hat{m}_{N+1}^\dagger[\omega] + \sqrt{\kappa_{b,\text{ex}}}\hat{b}^\dagger_{\text{in,ex}}[\omega] + \sqrt{\kappa_{b,\text{i}}}\hat{B}^\dagger_{\text{in}}[\omega],
\end{aligned} \tag{S1}$$

where $\kappa_{a(b)\text{ex}}$ is the coupling rate to the external channel with an input signal field $\hat{a}^\dagger(\hat{b}^\dagger)_{\text{in,ex}}[\omega]$, $\kappa_{a(b),\text{i}}$ is the intrinsic loss rate of mode $\hat{a}^\dagger(\hat{b}^\dagger)$ due to the coupling with input noise $\hat{A}(\hat{B})^\dagger_{\text{in}}[\omega]$ from the dissipative environment, and κ_j is the intrinsic loss rate of the intermediate mode $\hat{m}_j^\dagger$ due to the coupling with input noise $\hat{M}^\dagger_{\text{j,in}}[\omega]$. The end mode $\hat{a}^\dagger(\hat{b}^\dagger)[\omega]$ is subject to a total dissipation rate $\kappa_{a(b)} = \kappa_{a(b),\text{ex}} + \kappa_{a(b),\text{i}}$, and χ_j's are the mode susceptibilities defined as $\chi_j^{-1} \equiv i(\omega + \Delta_j) + \kappa_j/2$. For notational simplicity, we will drop the $[\omega]$ symbol for all the frequency domain mode operators from now on.

The input and output fields are connected to the bosonic modes long the chain according to the input-output relations

$$\begin{aligned}
&\hat{a}^\dagger_{\text{out,ex}} = \hat{a}^\dagger_{\text{in,ex}} - \sqrt{\kappa_{a,\text{ex}}}\hat{a}^\dagger, \hat{A}^\dagger_{\text{out}} = \hat{A}^\dagger_{\text{in}} - \sqrt{\kappa_{a,\text{i}}}\hat{a}^\dagger,\\
&\hat{M}^\dagger_{\text{j,out}} = \hat{M}^\dagger_{\text{j,in}} - \sqrt{\kappa_j}\hat{m}_j^\dagger,\\
&\hat{b}^\dagger_{\text{out,ex}} = \hat{b}^\dagger_{\text{in,ex}} - \sqrt{\kappa_{b,\text{ex}}}\hat{b}^\dagger, \hat{B}^\dagger_{\text{out}} = \hat{B}^\dagger_{\text{in}} - \sqrt{\kappa_{b,\text{i}}}\hat{b}^\dagger.
\end{aligned} \tag{S2}$$

Consider an input mode $\hat{a}^\dagger_{\text{in,ex}}$ only, we can iteratively solve the equations of motion Eq. (S1) along with the input-output relations Eq. (S2),

$$\begin{aligned}
\hat{a}^\dagger &= \chi_a(ig_1\hat{m}_2^\dagger + \sqrt{\kappa_{a,\text{ex}}}\hat{a}^\dagger_{\text{in,ex}}),\\
\hat{m}_2^\dagger &= \frac{ig_2\chi_a^{-1}\hat{m}_3^\dagger[\omega] + ig_a\sqrt{\kappa_{a,\text{ex}}}\hat{a}^\dagger_{\text{in,ex}}}{\left(\chi_2^{-1} + \dfrac{g_a^2}{\chi_a^{-1}}\right)\chi_a^{-1}},\\
&\vdots\\
\hat{b}^\dagger_{\text{out,ex}} &= -\frac{\sqrt{\kappa_{a,\text{ex}}\kappa_{b,\text{ex}}}\prod_{j=1}^{N+1}(ig_j)}{\prod_{j=1}^{N+2}(\chi^{-1}_{j,\text{eff,r}})}\hat{a}^\dagger_{\text{in,ex}} = -\frac{\sqrt{\kappa_{a,\text{ex}}\kappa_{b,\text{ex}}}\prod_{j=1}^{N+1}(ig_j)}{\prod_{j=1}^{N+2}(\chi^{-1}_{j,\text{eff}})}\hat{a}^\dagger_{\text{in,ex}},
\end{aligned} \tag{S3}$$

where $\chi_{j,\text{eff}}$ is the effective mode susceptibility viewed from the left due to the couplings

$g_j, \cdots, g_b$,

$$\chi_{j,\text{eff}}^{-1} \equiv \chi_j^{-1} + \cfrac{g_j^2}{\chi_{j+1}^{-1} + \cfrac{g_{j+1}^2}{\chi_{j+2}^{-1} + \cfrac{\ddots}{\ddots + \cfrac{g_b^2}{\chi_b^{-1}}}}}, \tag{S4}$$

and $\chi_{j,\text{eff,r}}$ is the effective mode susceptibility viewed from the reverse direction due to the couplings $g_{j-1}, \cdots, g_a$,

$$\chi_{j,\text{eff,r}}^{-1} \equiv \chi_j^{-1} + \cfrac{g_{j-1}^2}{\chi_{j-1}^{-1} + \cfrac{g_{j-2}^2}{\chi_{j-2}^{-1} + \cfrac{\ddots}{\ddots + \cfrac{g_a^2}{\chi_a^{-1}}}}}. \tag{S5}$$

The photon(boson) number conversion efficiency is defined as

$$\eta_N[\omega] = \frac{\left\langle \hat{b}_{\text{out,ex}}^\dagger \hat{b}_{\text{out,ex}} \right\rangle}{\left\langle \hat{a}_{\text{in,ex}}^\dagger \hat{a}_{\text{in,ex}} \right\rangle} = \left| \frac{\sqrt{\kappa_{a,\text{ex}}\kappa_{b,\text{ex}}} \prod_{j=1}^{N+1} (ig_j)}{\prod_{j=1}^{N+2} \chi_{j,\text{eff}}^{-1}} \right|^2 = \frac{\left\langle \hat{a}_{\text{out,ex}}^\dagger \hat{a}_{\text{out,ex}} \right\rangle}{\left\langle \hat{b}_{\text{in,ex}}^\dagger \hat{b}_{\text{in,ex}} \right\rangle}. \tag{S6}$$

The transduction process is reciprocal with identical conversion efficiency for $\hat{a}^\dagger \to \hat{b}^\dagger$ and for $\hat{b}^\dagger \to \hat{a}^\dagger$.

Alternatively, one can find a scattering matrix $\mathcal{S}_N[\omega]$ between the input and output modes such that $\vec{a}_{\text{out}}^\dagger[\omega] = \mathcal{S}_N[\omega]\vec{a}_{\text{in}}^\dagger[\omega]$, where $\vec{a}_{\text{in}}^\dagger[\omega] = (\hat{a}_{\text{in,ex}}^\dagger, \hat{A}_{\text{in}}^\dagger, \hat{M}_{2,\text{in}}^\dagger, \cdots, \hat{B}_{\text{in}}^\dagger, \hat{b}_{\text{in,ex}}^\dagger)^T$ and $\vec{a}_{\text{out}}^\dagger[\omega] = (\hat{a}_{\text{out,ex}}^\dagger, \hat{A}_{\text{out}}^\dagger, \hat{M}_{2,\text{out}}^\dagger, \cdots, \hat{B}_{\text{out}}^\dagger, \hat{b}_{\text{out,ex}}^\dagger)^T$. By directly solving for the matrix, the conversion efficiency is given by

$$\eta_N[\omega] \equiv \left| \mathcal{S}_{N,\hat{b}_{\text{out,ex}}^\dagger \hat{a}_{\text{in,ex}}^\dagger}[\omega] \right|^2 = \left| \frac{\sqrt{\kappa_{a,\text{ex}}\kappa_{b,\text{ex}}} \prod_{j=1}^{N+1} (ig_j)}{D_N[\omega]} \right|^2 = \left| \mathcal{S}_{N,\hat{a}_{\text{out,ex}}^\dagger \hat{b}_{\text{in,ex}}^\dagger}[\omega] \right|^2, \tag{S7}$$

where $D_N[\omega]$ is the determinant of a $(N+2)$-dimensional tridiagonal matrix

$$D_N[\omega] \equiv \begin{vmatrix} \chi_a^{-1} & ig_a & 0 & \cdots & \cdots & 0 \\ ig_a & \chi_2^{-1} & ig_2 & \ddots & & \vdots \\ 0 & ig_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & ig_b \\ 0 & \cdots & \cdots & 0 & ig_b & \chi_b^{-1} \end{vmatrix}. \tag{S8}$$

We can show that the two results agree with each other by using a mathematical theorem for tridiagonal matrices [2],

$$\begin{vmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & & \vdots \\ 0 & c_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & \cdots & 0 & c_{n-1} & a_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & & \vdots \\ 0 & c_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & \cdots & 0 & c_{n-1} & a_n \end{vmatrix}$$

$$= \prod_{j=1}^{n} a_j + \cfrac{-b_{j-1}c_{j-1}}{a_{j-1} + \cfrac{-b_{j-2}c_{j-2}}{a_{j-2} + \cfrac{\ddots}{\ddots + \cfrac{-b_1c_1}{a_1}}}} = \prod_{j=1}^{n} a_j + \cfrac{-b_jc_j}{a_{j+1} + \cfrac{-b_{j+1}c_{j+1}}{a_{j+2} + \cfrac{\ddots}{\ddots + \cfrac{-b_{n-1}c_{n-1}}{a_n}}}}. \tag{S9}$$

EFFECTIVE ELECTRIC CIRCUIT MODEL

Consider a two-port network circuit in Fig. S1(a) with inductance $L_j \in \mathbb{R}$, capacitance $C_j \in \mathbb{R}$, resistance $R_j = 1/G_j \in \mathbb{R}$, and generalized resistance $\mathcal{R}_j = 1/\mathcal{G}_j \in i\mathbb{R}$. The internal transmission coefficient for power waves can be calculated as [3, 4]

$$t^p_{N,\mathrm{int}}[\omega] = \mathcal{S}^p_{N,\mathrm{int}\ \beta_{\mathrm{out}}\alpha_{\mathrm{in}}} = 2\sqrt{\frac{R_1}{R_{N+2}}}\frac{V_{N+2}[\omega]}{V_g[\omega]} = \begin{cases} 2\sqrt{R_1 R_{N+2}}\prod_{j=1}^{N+2} X^{-1}_{j,\mathrm{eff}} & \text{odd } N \\ 2\sqrt{R_1/R_{N+2}}\prod_{j=1}^{N+2} X^{-1}_{j,\mathrm{eff}} & \text{even } N \end{cases}. \tag{S10}$$

where $X_{j,\mathrm{eff}}$ is the effective impedance $Z_{j,\mathrm{eff}}$ (for odd j) or admittance $Y_{j,\mathrm{eff}}$(for even j) viewed from the $j-1$ element [2],

$$X^{-1}_{j,\mathrm{eff}} \equiv X_j + \cfrac{1}{X_{j-1} + \cfrac{1}{X_{j-2} + \cfrac{\ddots}{\ddots + X_{N+2}}}}, \tag{S11}$$

where X_j is the bare impedance or admittance, $X_j \equiv i\omega L_j + R_j + \mathcal{R}_j = Z_j$ for odd j and $X_j \equiv i\omega C_j + G_j + \mathcal{G}_j = Y_j$ for even j. By setting

$$\begin{cases} L_j^{-1}C_{j+1}^{-1} = g_j^2,\ R_jL_j^{-1} = \kappa_j/2,\ \mathcal{R}_jL_j^{-1} = i\Delta_j & \text{odd } j \\ C_j^{-1}L_{j+1}^{-1} = g_j^2,\ G_jC_j^{-1} = \kappa_j/2,\ \mathcal{G}_jC_j^{-1} = i\Delta_j & \text{even } j \end{cases}, \tag{S12}$$

one can show that $X_{j,\mathrm{eff}}/L_j = \chi^{-1}_{j,\mathrm{eff}}$ for odd j, $X_{j,\mathrm{eff}}/C_j = \chi^{-1}_{j,\mathrm{eff}}$ for even j, and that

$$t^p_{N,\mathrm{int}}[\omega] = \frac{\sqrt{\kappa_a\kappa_b}\prod_{\mathrm{j}=1}^{N+1} g_j}{\prod_{\mathrm{j}=1}^{N+2}\chi^{-1}_{j,\mathrm{eff}}}. \tag{S13}$$

TABLE S1. Analogy between the transducer system and the effective electric circuit

Transducer System	Electric Circuit
$\hat{a}^\dagger_{\text{in,ex}}$	c_s[a]$\alpha_{\text{in,ex}} = \frac{c_s V_g}{2\sqrt{R_{1,\text{ex}}}}$
$\hat{b}^\dagger_{\text{out,ex}}$	$c_s\beta_{\text{out,ex}} = \frac{c_s V_{N+2,\text{ex}}}{\sqrt{R_{N+2,\text{ex}}}}$
$\hat{a}^\dagger_{\text{in}}$	$c_s\alpha_{\text{in}} = \frac{c_s V_g}{2\sqrt{R_1}}$
$\hat{b}^\dagger_{\text{out}}$	$c_s\beta_{\text{out}} = \frac{c_s V_{N+2}}{\sqrt{R_{N+2}}}$
$\hat{a}^\dagger$	$\frac{c_s I_1\sqrt{R_1}}{\sqrt{\kappa_a}}$
$\hat{b}^\dagger$	$\frac{c_s I_{N+2}\sqrt{R_{N+2}}}{\sqrt{\kappa_b}}$

Transducer System	Electric Circuit (odd j)	Electric Circuit (even j)
g_j	$\sqrt{L_j^{-1}C_{j+1}^{-1}}$	$\sqrt{C_j^{-1}L_{j+1}^{-1}}$
$\kappa_j/2$	$R_jL_j^{-1}$	$G_jC_j^{-1}$
$i\Delta_j$	$\mathcal{R}_jL_j^{-1}$	$\mathcal{G}_jC_j^{-1}$
χ_j^{-1}	Z_j/L_j	Y_j/C_j
$\chi_{j,\text{eff}}^{-1}$	$Z_{j,\text{eff}}/L_j$	$Y_{j,\text{eff}}/C_j$
$\chi_{j,\text{eff}}^{-1} - \chi_j^{-1} = g_j^2\chi_{j+1,\text{eff}}$	$(Z_{j,\text{eff}} - Z_j)/L_j = Y_{j+1,\text{eff}}^{-1}/L_j$	$(Y_{j,\text{eff}} - Y_j)/C_j = Z_{j+1,\text{eff}}^{-1}/C_j$

[a] c_s is some dimensional scaling factor.

The effective circuit renders a transmission coefficient that coincides with the transducer scattering matrix element $\mathcal{S}_{N,\hat{b}^\dagger_{\text{out,ex}}\hat{a}^\dagger_{\text{in,ex}}}$ up to a prefactor $-(i)^{N+1}\sqrt{\eta_{\text{ext}}}$. The input and output modes are analogous to the incident and reflected power waves of the circuit. The explicit analogy between the transducer system and the effective electric circuit are summarized in table S1.

We conclude that the internal power efficiency of the circuit is $\eta^p_{N,\text{int}}[\omega] \equiv \left|t^p_{N,\text{int}}[\omega]\right|^2 = \eta_{N,\text{int}}[\omega]$. One may also separate the external coupling rates from the total dissipation rates of the end modes to replicate the full conversion efficiency $\eta^p_N[\omega] \equiv |t^p_N[\omega]|^2 = \eta_N[\omega]$. Specifically,

$$R_{1,\text{ex}}L_1^{-1} = \kappa_{a,\text{ex}}/2,\ R_{1,\text{i}}L_1^{-1} = \kappa_{a,\text{i}}/2,\ R_1 = R_{1,\text{ex}} + R_{1,\text{i}}, \tag{S14}$$

$$\begin{cases} R_{N+2,\text{ex}}L_{N+2}^{-1} = \kappa_{b,\text{ex}}/2,\ R_{N+2,\text{i}}L_{N+2}^{-1} = \kappa_{b,\text{i}}/2,\ R_{N+2} = R_{N+2,\text{ex}} + R_{N+2,\text{i}} & \text{odd } N \\ G_{N+2,\text{ex}}C_{N+2}^{-1} = \kappa_{b,\text{ex}}/2,\ G_{N+2,\text{i}}C_{N+2}^{-1} = \kappa_{b,\text{i}}/2,\ G_{N+2} = G_{N+2,\text{ex}} + G_{N+2,\text{i}} & \text{even } N \end{cases}, \tag{S15}$$

$$t^p_N[\omega] = S^p_{N,\beta_{\text{out,ex}}\alpha_{\text{in,ex}}} = 2\sqrt{\frac{R_{1,\text{ex}}}{R_{N+2,\text{ex}}}}\frac{V_{N+2,\text{ex}}[\omega]}{V_g[\omega]} = \begin{cases} \sqrt{\frac{R_{1,\text{ex}}}{R_1}\frac{R_{N+2,\text{ex}}}{R_{N+2}}}\,t^p_{N,\text{int}}[\omega] & \text{odd } N \\ \sqrt{\frac{R_{1,\text{ex}}}{R_1}\frac{R_{N+2}}{R_{N+2,\text{ex}}}}\,t^p_{N,\text{int}}[\omega] & \text{even } N \end{cases}. \tag{S16}$$

One can also verify that the effective circuits of type 2 form in Fig. S1(b) possess identical transmission coefficients as type 1 circuits (Fig. S1(a)) by interchanging the values of L_j with C_j, $\mathcal{R}_j$ with $\mathcal{G}_j$, and R_j with G_j.

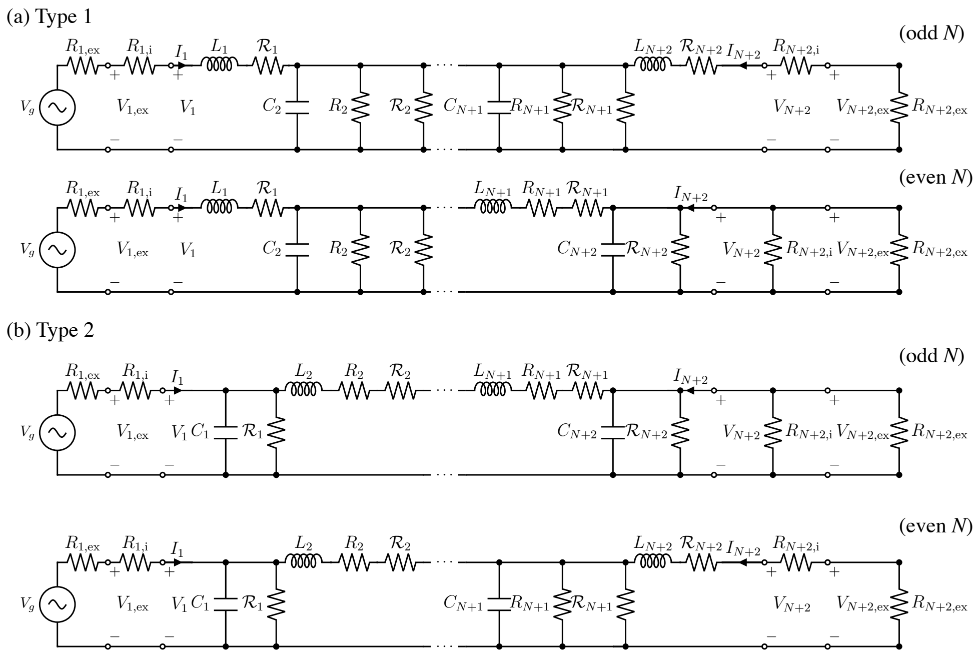

FIG. S1. Schematic of effective network circuits that give rise to identical frequency response as a N-stage quantum transducer. (a) Type 1 network circuit models that start with a series inductor. (b) Type 2 network circuit models that start with a parallel capacitor.

GENERALIZED MATCHING CONDITION

The matching condition viewed at the $\hat{a}^\dagger$ mode is

$$(\chi_a^{-1})^* = g_a^2 \chi_{2,\mathrm{eff}}. \tag{S17}$$

Reorganizing the condition into $-(\chi_a^{-1})^* + g_a^2 \chi_{2,\mathrm{eff}} = 0$ and multiply the expressions by $\prod_{j=2}^{N+2} \chi_{j,\mathrm{eff}}^{-1}$, where all $\chi_{j,\mathrm{eff}}$'s are all nonzero with κ_a, κ_b, g_j's $\neq 0$, we can again use the theorem for tridiagonal matrices S9 to show that

$$(-(\chi_a^{-1})^* + g_a^2 \chi_{2,\mathrm{eff}}) \prod_{j=2}^{N+2} \chi_{j,\mathrm{eff}}^{-1} = \begin{vmatrix} -(\chi_a^{-1})^* & i g_a & 0 & \cdots & \cdots & 0 \\ i g_a & \chi_2^{-1} & i g_2 & \ddots & & \vdots \\ 0 & i g_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i g_b \\ 0 & \cdots & \cdots & 0 & i g_b & \chi_b^{-1} \end{vmatrix}. \tag{S18}$$

Combining with the condition of no intermediate loss, we arrive at the generalized matching condition in a matrix form

$$M_N \equiv \left|\begin{matrix} -(\chi_a^{-1})^* & ig_a & 0 & \cdots & \cdots & 0 \\ ig_a & \chi_2^{-1} & ig_2 & \ddots & & \vdots \\ 0 & ig_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & ig_b \\ 0 & \cdots & \cdots & 0 & ig_b & \chi_b^{-1} \end{matrix}\right|\Bigg|_{\kappa_2=\cdots=\kappa_{N+1}=0} = 0. \tag{S19}$$

When the generalized matching condition is satisfied, the added noise for the conversion can be expressed in a simple form. The added noise for conversion from $\hat{a}^\dagger$ to $\hat{b}^\dagger$, relative to the signal, is quantified as

$$\left\langle \hat{a}^\dagger_{\text{out,ex}}\hat{a}_{\text{out,ex}}\right\rangle = \eta_N \left(\left\langle \hat{b}^\dagger_{\text{in,ex}}\hat{b}_{\text{in,ex}}\right\rangle + n_{\text{add}}^{a\to b}\right), \tag{S20}$$

where the angular bracket represents the quantum statistical average over a density matrix. For the impedance-matched N-stage tranduction with no intermediate loss, the added noise from $\hat{a}^\dagger$ to $\hat{b}^\dagger$ is

$$n_{\text{add}}^{a\to b}[\omega] = \frac{\kappa_{a,\text{i}}}{\kappa_{a,\text{ex}}}\left\langle \hat{B}^\dagger_{\text{in}}\hat{B}_{\text{in}}\right\rangle + \frac{\kappa_a}{\kappa_{a,\text{ex}}}\frac{\kappa_{b,\text{i}}}{\kappa_b}\left\langle \hat{A}^\dagger_{\text{in}}\hat{A}_{\text{in}}\right\rangle + \frac{\kappa_a}{\kappa_{a,\text{ex}}}\frac{\kappa_b}{\kappa_{b,\text{ex}}}\frac{\kappa_{b,\text{i}}^2}{\kappa_b^2}\left\langle \hat{a}^\dagger_{\text{in,ex}}\hat{a}_{\text{in,ex}}\right\rangle. \tag{S21}$$

The added noise for the impedance-matched N-stage transduction from $\hat{b}^\dagger$ to $\hat{a}^\dagger$ can be found analogously,

$$n_{\text{add}}^{b\to a}[\omega] = \frac{\kappa_{b,\text{i}}}{\kappa_{b,\text{ex}}}\left\langle \hat{A}^\dagger_{\text{in}}\hat{A}_{\text{in}}\right\rangle + \frac{\kappa_b}{\kappa_{b,\text{ex}}}\frac{\kappa_{a,\text{i}}}{\kappa_a}\left\langle \hat{B}^\dagger_{\text{in}}\hat{B}_{\text{in}}\right\rangle + \frac{\kappa_b}{\kappa_{b,\text{ex}}}\frac{\kappa_a}{\kappa_{a,\text{ex}}}\frac{\kappa_{a,\text{i}}^2}{\kappa_a^2}\left\langle \hat{b}^\dagger_{\text{in,ex}}\hat{b}_{\text{in,ex}}\right\rangle. \tag{S22}$$

Large added noise from the thermal environment will decrease the fidelity of the transferred quantum state.

IMPEDANCE MATCHING FOR LOSSY 1-STAGE TRANSDUCTION

In this section we discuss the detailed impedance matching interpretation for the optimal parameters of lossy 1-stage conversion (Fig. S2(a)). An effective circuit mode for 1-stage transduction is presented in Fig. S2(b). Here we consider at the Thevenin equivalent circuit for the type 2 form in Fig. S1(b) and take the perspective from mode $\hat{m}_2^\dagger$ by taking $L_2 = 1$ such that the circuit interpretation will be symmetric in $\hat{a}^\dagger$ and $\hat{b}^\dagger$. In particular, with this choice of circuit, the middle mode $\hat{m}_2^\dagger$ has an impedance $Z_2 = \chi_2^{-1}$, and the first mode $\hat{a}^\dagger$ has an effective impedance $Z_1 = g_a^2\chi_a$ while the third mode $\hat{b}^\dagger$ has $Z_3 = g_b^2\chi_b$.

When one of the mode is over-coupled, say if $\mathcal{C}_{a,2} > \mathcal{C}_{2,b} + 1$, the maximal internal efficiency is achieved by the choice of optimal frequencies satisfying $g_a^2\chi_a^* = \chi_2^{-1} + g_b^2\chi_b$ and $\nu_b = \omega + \Delta_b = 0$. This condition can be understood in analogy to an impedance matched circuit with internal loss in Fig. S2(c). Treating the middle mode and the third mode altogether as the load, the system is impedance matched when $g_a^2\chi_a^* = Z_1^* = Z_2 + Z_3 =$

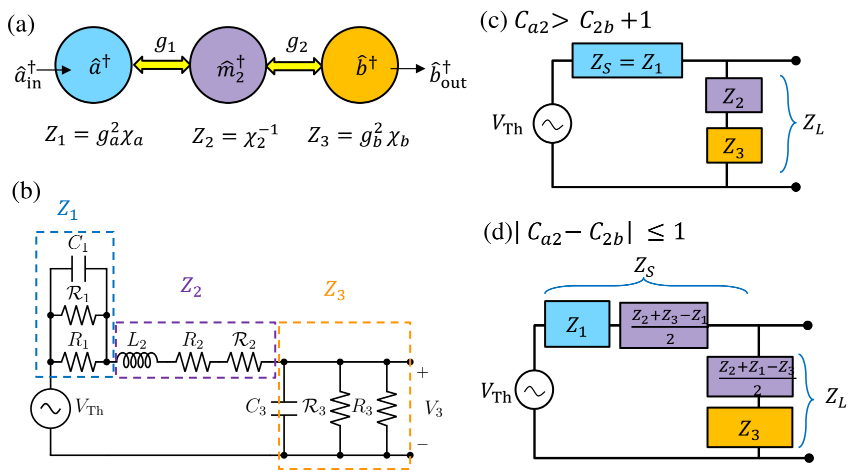

FIG. S2. (a) Schematic of 1-stage quantum transduction system. (b) Effective circuit that reproduces the 1-stage internal conversion efficiency by setting $C_1 = 1/g_a^2$, $\mathcal{R}_1 = g_a^2/(i\Delta_a)$, $R_1 = 2g_a^2/\kappa_a$, $L_2 = 1$, $R_2 = \kappa_2/2$, $\mathcal{R}_2 = i\Delta_2$, $C_3 = 1/g_b^2$, $\mathcal{R}_3 = g_b^2/(i\Delta_b)$, $R_3 = 2g_b^2/\kappa_b$, and $V_{\rm Th} = V_g/R_1/(i\omega C_1 + \mathcal{G}_1 + G_1)$. This is the Thevenin equivalent circuit of the type 2 effective circuit mode for 1-stge transduction. (c) Impedance matched lossy circuit configuration if $\mathcal{C}_{a2} > \mathcal{C}_{2b} + 1$. (d) Impedance matched lossy circuit configuration if $|\mathcal{C}_{a2} - \mathcal{C}_{2b}| \le 1$.

$\chi_2^{-1} + g_b^2\chi_b$, and the maximal internal efficiency is $\eta_{\rm int}^{\rm max} = \mathcal{C}_{2,b}/(\mathcal{C}_{2,b}+1)$ when $\nu_b = 0$, limited by the smaller cooperativity. In other word, we are adding nonzero ν_a and ν_2 to modify the source impedance such that it matches with the load impedance, which effectively reduce the source resistance $\mathrm{Re}[Z_1]$, while keeping the third mode at resonance, which gives rise to the maximal value of $\mathrm{Re}[Z_3]$, to get the largest fraction of power output at the load. Note that the internal reflection coefficient vanishes at the over-coupled port, $r_{1,a} = 0$, while $r_{1,b} \neq 0$ due to the intermediate loss.

If $|\mathcal{C}_{a,2} - \mathcal{C}_{2,b}| \le 1$, the conversion is optimized when the system is all-resonant, $\nu_a = \nu_2 = \nu_b = 0$. This condition can again be understood as an impedance matched lossy circuit (Fig. S2(d)). Treating the middle mode as a lossy component partly in the source and partly in the load, the impedance is matched at a point where $Z_1 + Z_{S,2} = Z_{L,2} + Z_3$, which corresponds to $\mathcal{C}_{a,2} + (\mathcal{C}_{2,b} - \mathcal{C}_{a,2})/2 = (\mathcal{C}_{a,2} - \mathcal{C}_{2,b})/2 + \mathcal{C}_{2,b}$. The maximal achievable internal conversion efficiency in this parameter regime is $\eta_{\rm int}^{\rm max} = 4\mathcal{C}_{a,2}\mathcal{C}_{2,b}/(\mathcal{C}_{a,2} + \mathcal{C}_{2,b} + 1)^2$.

OPTIMAL FREQUENCIES

In this section we summarize the optimal frequencies for 1-stage and 2-stage transduction that give rise to maximal achievable internal efficiency at given cooperativities $\mathcal{C}_{j,j+1} = \frac{4g_j^2}{\kappa_j\kappa_{j+1}}$.

[1] D. F. Walls and G. J. Milburn, *Quantum Opt.* (Springer-Verlag Berlin Heidelberg, Berlin, 2008).

[2] E. Kiliç, Appl. Math. Comput. **197**, 345 (2008).

TABLE S2. Optimal frequencies for 1-stage transduction

N=1	$\kappa_2=0$	$\mathcal{C}_{a,2}+1<\mathcal{C}_{2,b}$	$\lvert\mathcal{C}_{a,2}-\mathcal{C}_{2,b}\rvert\le 1$	$\mathcal{C}_{a,2}>\mathcal{C}_{2,b}$+1
ν_a	Multiple solutions satisfying $M_1=0$	0	0	$\pm\frac{\kappa_a}{2}\sqrt{\frac{\mathcal{C}_{a,2}}{\mathcal{C}_{2,b}+1}-1}$
ν_2		$\pm\frac{\kappa_2}{2}(\mathcal{C}_{a,2}+1)\sqrt{\frac{\mathcal{C}_{2,b}}{\mathcal{C}_{a,2}+1}-1}$	0	$\pm\frac{\kappa_2}{2}(\mathcal{C}_{2,b}+1)\sqrt{\frac{\mathcal{C}_{a,2}}{\mathcal{C}_{2,b}+1}-1}$
ν_b		$\pm\frac{\kappa_b}{2}\sqrt{\frac{\mathcal{C}_{2,b}}{\mathcal{C}_{a,2}+1}-1}$	0	0
$\eta_{1,\mathrm{int}}^{\max}$	1	$\frac{\mathcal{C}_{a,2}}{\mathcal{C}_{a,2}+1}$	$\frac{4\mathcal{C}_{a,2}\mathcal{C}_{2,b}}{(\mathcal{C}_{a,2}+\mathcal{C}_{2,b}+1)^2}$	$\frac{\mathcal{C}_{2,b}}{\mathcal{C}_{2,b}+1}$

TABLE S3. Optimal frequencies for 2-stage transduction

N=2	$\kappa_2=\kappa_3=0$	$\kappa_2=0$, $\kappa_3\neq 0$	$\kappa_2\neq 0$, $\kappa_3=0$	$\mathcal{C}_{2,3}>(\mathcal{C}_{a,2}+1)(\mathcal{C}_{3,b}+1)$	$\left\lvert\mathcal{C}_{a,2}-\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{3,b}+1}\right\rvert\le 1$ and $\left\lvert\mathcal{C}_{3,b}-\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{a,2}+1}\right\rvert\le 1$
ν_a	Solutions satisfying $M_2=0$	Solutions satisfying $r_{2,a}=0$	0	0	0
ν_2			Solutions satisfying $r_{2,b}=0$	$\pm\frac{\kappa_2(\mathcal{C}_{a,2}+1)\sqrt{\frac{\mathcal{C}_{2,3}}{(\mathcal{C}_{a,2}+1)(\mathcal{C}_{3,b}+1)}-1}}{2}$	0
ν_3				$\pm\frac{\kappa_3(\mathcal{C}_{3,b}+1)\sqrt{\frac{\mathcal{C}_{2,3}}{(\mathcal{C}_{a,2}+1)(\mathcal{C}_{3,b}+1)}-1}}{2}$	0
ν_b		0	0	0	0
$\eta_{2,\mathrm{int}}^{\max}$	1	$\frac{\mathcal{C}_{3,b}}{\mathcal{C}_{3,b}+1}$	$\frac{\mathcal{C}_{a,2}}{\mathcal{C}_{a,2}+1}$	$\frac{\mathcal{C}_{a,2}}{\mathcal{C}_{a,2}+1}\frac{\mathcal{C}_{3,b}}{\mathcal{C}_{3,b}+1}$	$\frac{4\mathcal{C}_{a,2}\mathcal{C}_{2,3}\mathcal{C}_{3,b}}{(\mathcal{C}_{a,2}+\mathcal{C}_{2,3}+\mathcal{C}_{3,b}+\mathcal{C}_{a,2}\mathcal{C}_{3,b}+1)^2}$

N=2	$\mathcal{C}_{a,2}\ge\mathcal{C}_{3,b}$ and $\mathcal{C}_{a,2}>\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{3,b}+1}+1$		$\mathcal{C}_{3,b}\ge\mathcal{C}_{a,2}$ and $\mathcal{C}_{3,b}>\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{a,2}+1}+1$	
	$\mathcal{C}_{3,b}^2>\mathcal{C}_{2,3}+1$	$\mathcal{C}_{3,b}^2\le\mathcal{C}_{2,3}+1$	$\mathcal{C}_{a,2}^2>\mathcal{C}_{2,3}+1$	$\mathcal{C}_{a,2}^2\le\mathcal{C}_{2,3}+1$
ν_a	$\pm\frac{\kappa_e}{2}\sqrt{\frac{\mathcal{C}_{a,2}}{\sqrt{\mathcal{C}_{2,3}+1}}-1}$	$\pm\frac{\kappa_e}{2}\sqrt{\frac{\mathcal{C}_{a,2}}{\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{3,b}+1}+1}-1}$	$\pm\frac{\kappa_e}{2}\sqrt{\frac{\mathcal{C}_{a,2}}{\sqrt{\mathcal{C}_{2,3}+1}}-1}$	0
ν_2	$\frac{\kappa_2}{\kappa_a}\sqrt{\mathcal{C}_{2,3}+1}\nu_a$	$\frac{\kappa_2}{\kappa_a}\left(\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{3,b}+1}+1\right)\nu_a$	$\frac{\kappa_2}{\kappa_a}\sqrt{\mathcal{C}_{2,3}+1}\nu_a$	0
ν_3	$\frac{\kappa_3}{\kappa_b}\sqrt{\mathcal{C}_{2,3}+1}\nu_b$	0	$\frac{\kappa_3}{\kappa_b}\sqrt{\mathcal{C}_{2,3}+1}\nu_b$	$\frac{\kappa_3}{\kappa_b}\left(\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{a,2}+1}+1\right)\nu_b$
ν_b	$\pm\mp\frac{\kappa_b}{2}\sqrt{\frac{\mathcal{C}_{3,b}}{\sqrt{\mathcal{C}_{2,3}+1}}-1}$	0	$\pm\mp\frac{\kappa_b}{2}\sqrt{\frac{\mathcal{C}_{3,b}}{\sqrt{\mathcal{C}_{2,3}+1}}-1}$	$\pm\frac{\kappa_b}{2}\sqrt{\frac{\mathcal{C}_{3,b}}{\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{a,2}+1}+1}-1}$
$\eta_{2,\mathrm{int}}^{\max}$	$\frac{\mathcal{C}_{2,3}}{(\sqrt{\mathcal{C}_{2,3}+1}+1)^2}$	$\frac{\mathcal{C}_{3b}}{\mathcal{C}_{3,b}+1}\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{2,3}+\mathcal{C}_{3,b}+1}$	$\frac{\mathcal{C}_{2,3}}{(\sqrt{\mathcal{C}_{2,3}+1}+1)^2}$	$\frac{\mathcal{C}_{a,2}}{\mathcal{C}_{a,2}+1}\frac{\mathcal{C}_{2,3}}{\mathcal{C}_{a,2}+\mathcal{C}_{2,3}+1}$

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