— Supplementary Material —

Noncommutative Field Theory of the Tkachenko Mode: Symmetries and Decay Rate

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REDUNDANCIES OF SPONTANEOUS BROKEN SYMMETRIES

We believe that the most transparent understanding of spontaneous symmetry breaking and the Goldstone boson counting for a two-dimensional superfluid vortex crystal was obtained by Watanabe and Murayama in Ref. [S1]. Here we summarize their explanation adopted to the lowest Landau level regime.

Emergence of a vortex lattice ground state in a rotating superfluid breaks spontaneously global particle number $U(1)$ symmetry, magnetic translation symmetry, and magnetic rotation symmetry. Notwithstanding, the generators of all these symmetries are linearly related to each other. In particular, the momentum density T^{0i} is given by

$$T^{0i} = mj^i - B\epsilon^{ij}x_j n \tag{S1}$$

where j^i and n are the boson current and particle densities, respectively, and B is the effective magnetic field originating from the rotation [S2]. In the massless regime $m \to 0$, where only the lowest Landau level states survive, we can ignore the first term in Eq. (S1), so the momentum density operator and the particle density operator are proportional to each other. Furthermore, we can define the angular momentum density as $\mathscr{J} = \epsilon^{ij}x^i T^{0j}$ which is also simply related to the boson density operator $\mathscr{J} = B\vec{x}^2 n$. As a result, the densities of all symmetries that are spontaneously broken are not independent, but are linearly related to each other. Therefore we only have a single Goldstone boson, which is the Tkachenko mode.

LINEARIZED THEORY OF THE TKACHENKO MODE

Linearized effective Lagrangian

Our departure point is the low-energy linearized effective theory of a two-dimensional superfluid vortex lattice introduced in Refs. [S3, S4]. We consider a system of bosons with density n_0 placed in a constant magnetic field B. This magnetic field may be effectively created by rotating the system with angular frequency $B/(2m)$, at the same time putting it in a harmonic trap with the trap frequency fine-tuned to cancel the centrifugal force. The lattice is parametrized by the displacement field u^i, $i = x, y$, while the superfluid is characterized by the dual $u(1)$ gauge field a_μ. The Tkachenko mode emerges as the result of the mixing between the of elastic waves on the vortex lattice and the superfluid fluctuations. We start from the leading-order (LO) quadratic Lagrangian linearized around the vortex crystal ground state [S5]

$$\mathscr{L}^{(2)} = -\frac{Bn_0}{2}\epsilon_{ij}u^i\dot{u}^j + \frac{B}{2\pi}e_i u^i - \frac{\lambda}{2}\frac{\delta b^2}{(2\pi)^2} + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\mathscr{A}_\mu\partial_\nu a_\rho - \mathscr{E}^{(2)}_{\text{el}}(\partial u). \tag{S2}$$

The formula for the particle number spacetime current in terms of the gauge field, $j^\mu = \delta S/\delta\mathscr{A}_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$, relates the particle number density n with the magnetic field b and the particle number current with the electric field $e_i = \partial_t a_i - \partial_i a_t$. The term with one time derivative, proportional to $\epsilon_{ij}u^i\dot{u}^j$, encodes the Berry phase that a vortex acquires when moving in a superfluid. This term gives rise to the "Magnus force" acting on the vortex. The elastic energy density $\mathscr{E}^{(2)}_{\text{el}}(\partial u)$ is a function of the linearized strain tensor $u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$. The superfluid internal energy is a function of the superfluid density $n = \frac{1}{2\pi}b$ and here is expanded around the ground state value n_0 to quadratic order in fluctuations $\delta b = b - 2\pi n_0$. Finally, we included the coupling to an external $U(1)$ source $\mathscr{A}_\mu$ which is set to vanish in the ground state [S3].

The quadratic Lagrangian (S2) can be easily obtained from the Lagrangian (3) in the main text [S6]. To this end following [S3], we substitute into Eq. (3) the definition of the vortex current (2), and the definition of X^a in terms of u^a, see Fig. S1. We then expand the action up to the quadratic order in the field fluctuations and arrive at the linearized action (S2) with the fluctuation of the vector potential source $\mathscr{A}_\mu = A_\mu - \bar{A}_\mu$ on top of the background $\bar{A}_\mu$ that produces the constant background magnetic field B.

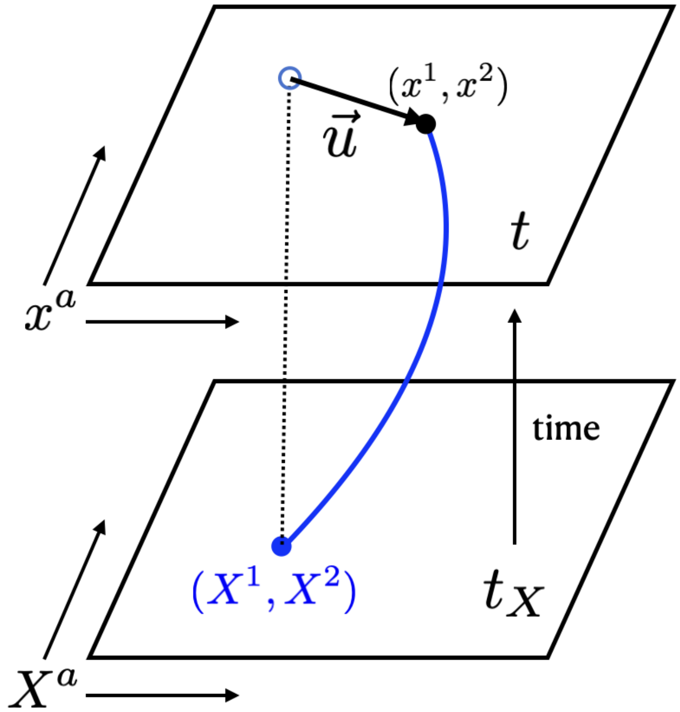

FIG. S1. The cartesian coordinates X^1 and X^2 label vortices at the initial time t_X. These coordinates are frozen into the vortex system and thus are fixed along each vortex (blue) worldline during time evolution. The displacement $u^a(t,x^i) = x^a - X^a(t,x^i)$.

In the absence of the $U(1)$ source, one finds after integrating out a_i (see Appendix of Ref. [S7] for details)

$$S_{\text{eff}}\left[u^i, a_t\right] = \int dt d\mathbf{x} \left(\mathscr{L}_{\text{el}}^{(2)} + \frac{B}{2\pi} a_t \partial_i u^i\right) + \frac{B^2}{2\lambda} \int \frac{dt d\mathbf{k}}{(2\pi)^2} \frac{\dot{u}^i_{-\mathbf{k}} \dot{u}^i_{\mathbf{k}}}{\mathbf{k}^2}, \tag{S3}$$

where the first two terms in the Lagrangian (S2) constitute $\mathscr{L}_{\text{el}}^{(2)}$. Now one can integrate out the temporal component a_t which gives rise to the Gauss law constraint $\partial_i u^i = 0$. In the low-frequency and long-distance domain, the vortex crystal thus appears to be incompressible. This constraint can be resolved explicitly by introducing a dimensionless scalar field ϕ in terms of which $u^i = -\theta \epsilon^{ij} \partial_j \phi$, where following the main text we defined $\theta = -l^2 = -1/B$. The transverse phonon ϕ, known as the Tkachenko mode, is the only low-energy dynamical excitation mode of the vortex lattice. In terms of the field ϕ, the effective theory reduces to the local form

$$\mathscr{L}_\phi = \frac{1}{2\lambda} \dot{\phi}^2 - \frac{C_2}{B_0^2} (\Delta \phi)^2. \tag{S4}$$

In contrast to phonons in ordinary crystals, the Tkachneko mode has a soft quadratic dispersion relation at low momenta.

Coupling to $U(1)$ source and linear response

After including the coupling to the $U(1)$ source via the mixed Chern-Simons term, we find that after integrating out a_i one ends up with the action (S3) with the replacement

$$\dot{u}^i_{\mathbf{k}} \to \dot{u}^i_{\mathbf{k}} + \frac{\epsilon^{ij} \mathscr{E}^j_{\mathbf{k}}}{B} \tag{S5}$$

in the second integral. So the displacement velocity is measured with respect to the LLL drift that has the velocity $-\epsilon^{ij}\mathscr{E}_j/B$. As a result, the action is invariant under Galilean boosts. We thus have the effective action

$$S_{\text{eff}}\left[u^i, a_t\right] = \int dt d\mathbf{x} \left(\mathscr{L}^{(2)}_{\text{el}} + \frac{a_t}{2\pi}[B\partial_i u^i + \mathscr{B}]\right) + \frac{B^2}{2\lambda}\int \frac{dt d\mathbf{k}}{(2\pi)^2}\left(\dot{u}^i_{-\mathbf{k}} + \frac{\epsilon^{ij}\mathscr{E}^j_{-\mathbf{k}}}{B_0}\right)\frac{1}{\mathbf{k}^2}\left(\dot{u}^i_{\mathbf{k}} + \frac{\epsilon^{ij}\mathscr{E}^j_{\mathbf{k}}}{B_0}\right), \tag{S6}$$

where $\mathscr{B} = \epsilon^{ij}\partial_i \mathscr{A}_j$ is a variation of the magnetic field on top of the constant background B. In the presence of a such inhomogenity, the Gauss law is $\partial_i u^i = -\mathscr{B}/B$, so the crystal becomes compressible. In momentum space (our conventions: $\partial_i \to ik_i$ and $\partial_t \to -i\omega$), this Gauss law can be resolved as $u^i_{\mathbf{k}} = -\theta\epsilon^i{}_j(ik^j\phi_{\mathbf{k}} - \mathscr{A}^j_{\mathbf{k}})$. The way the dimensionless scalar ϕ couples to the $U(1)$ source suggest that in addition to fixing the transverse displacement of the vortex crystal, ϕ also represents the regular part of the superfluid phase of the Bose-Einstein condensate. The latter interpretation was the key point for Watanabe and Murayama, who developed the effective theory of the superfluid vortex crystal in Ref. [S1].

Now we are ready to write the generalization of the effective theory (S4) in the presence of the $U(1)$ source. The simplest result is obtained, when one considers a special type of the source with vanishing $\mathscr{B}$ which thus does not violate the incompressibility condition. To this end, we will set $\mathscr{A}_i = 0$. In that case, after resolving the Gauss law, we find the quadratic effective action for the ϕ fluctuation

$$S_{\text{eff}}[\phi] = \frac{1}{2\lambda}\int \frac{d\omega d\mathbf{k}}{(2\pi)^3}\left((-i\omega\phi_{-k} + \mathscr{A}^0_{-k})(i\omega\phi_k + \mathscr{A}^0_k) - \frac{2C_2\lambda}{B^2}\phi_{-k}\mathbf{k}^4\phi_k\right), \tag{S7}$$

where $k = (\omega, \mathbf{k})$.

We will extract now the density susceptibility χ_k that (up to a sign) is just the correlation function $\langle n_{-k} n_k\rangle$. To this end, we first compute the superfuid density

$$n_k = \frac{\delta S}{\delta \mathscr{A}^0_{-k}} = \frac{1}{\lambda}\left(i\omega\phi_k + \mathscr{A}^0_k\right) \tag{S8}$$

substitute into it the solution of the equation of motion for ϕ_{-k}

$$\phi_k = \frac{i\omega\mathscr{A}^0_k}{\omega^2 - \frac{2C_2\lambda}{B^2}\mathbf{k}^4} \tag{S9}$$

and get

$$n_k = -\frac{2C_2\mathbf{k}^4\mathscr{A}^0_k}{B^2(\omega^2 - \frac{2C_2\lambda}{B^2}\mathbf{k}^4)}. \tag{S10}$$

Finally, we differentiate the result with respect to $\mathscr{A}^0_k$ to get

$$\chi_k = -\frac{\partial n_k}{\mathscr{A}^0_k} = \frac{2C_2\mathbf{k}^4}{B^2(\omega^2 - \frac{2C_2\varepsilon''}{B^2}\mathbf{k}^4)}. \tag{S11}$$

This agrees with the result of Ref. [S3].

More generally, the linear electromagnetic responses extracted from the effective action for the Tkachenko field ϕ are expected to agree with the massless LLL limit of the results derived in Ref. [S3]. This amounts to discarding the contribution originating from the Kohn's mode [S8].

Hamiltonian formulation

Starting from the Lagrangian (S4), the canonical momentum conjugate to ϕ is

$$\pi_\phi = \frac{\partial \mathscr{L}_\phi}{\partial \dot{\phi}} = \frac{1}{\lambda}\dot{\phi} \tag{S12}$$

which according to Eq. (S8) is (minus) the superfluid density $\pi_\phi = -n$. So ϕ is indeed the superfluid phase field. The Hamiltonian density can now be computed to be

$$\mathscr{H} = \pi_\phi \dot{\phi} - \mathscr{L}_\phi = \frac{1}{2}\lambda \pi_\phi^2 + \frac{C_2}{B^2}(\Delta\phi)^2. \tag{S13}$$

Using now the canonical Poisson bracket $[\phi(\mathbf{x}), \pi_\phi(\mathbf{y})] = \delta(\mathbf{x}-\mathbf{y})$, we end up with the Hamiltonian equations of motion

$$\begin{aligned} \partial_t \phi &= [\phi, H] = \lambda \pi_\phi = -\lambda n, \\ \partial_t \pi_\phi &= -\partial_t n = [\pi_\phi, H] = -2\frac{C_2}{B^2}\Delta^2\phi. \end{aligned} \tag{S14}$$

We observe that the time-evolution of the Tkachenko field ϕ is fixed by the superfluid density, while the time-evolution of the latter is fixed by the fourth-spacial derivative of ϕ.

RESOLUTION OF THE NONLINEAR CONSTRAINT

Constant magnetic field

As argued in the main text, in a constant effective magnetic field $B = 2m\Omega$, the scalar fields X^a must satisfy the non-linear constraint

$$\frac{1}{2}\epsilon^{ij}\epsilon^{ab}\partial_i X^a \partial_j X^b = 1. \tag{S15}$$

To resolve the constraint, introduce an *auxiliary Poisson bracket*

$$\{f, g\} = \epsilon^{ij}\partial_i f \partial_j g \tag{S16}$$

such that

$$\{x^a, x^b\} = \epsilon^{ab}. \tag{S17}$$

As the consequence of the constraint (S15), the fields X^a also satisfy

$$\{X^a, X^b\} = \epsilon^{ij}\frac{\partial X^a}{\partial x^i}\frac{\partial X^b}{\partial x^j} = \epsilon^{ab}. \tag{S18}$$

Thus the transformation from x^i to X^a belongs to the group of canonical transformation and can be generated by a scalar function ϕ [S9]

$$\begin{aligned} X^a &= x^a - \theta\{\phi, x^a\} + \frac{\theta^2}{2!}\{\phi, \{\phi, x^a\}\} + \dots \\ &= x^a + \theta\epsilon^{ab}\partial_b\phi - \frac{\theta^2}{2}\epsilon^{ab}\epsilon^{cd}\partial_c\phi\partial_b\partial_d\phi + \dots \end{aligned} \tag{S19}$$

Notice that this expression of X^a is identical with Eq. (7) in the main text. As a superfluid order phase, the scalar generator ϕ is even under 2d parity $x \leftrightarrow y$ and odd under time reversal $t \to -t$.

Here we comment on two natural ways how to organize the derivative expansion in terms of ϕ: If we scale $\phi \sim O(\epsilon^{-1})$ and $\partial_i \sim O(\epsilon)$, higher order non-linearities in the expansion (S19) are systematically suppressed. On the other hand, in order to include all non-linearities on equal footing, we can scale $\phi \sim O(\epsilon^{-2})$ and $\partial_i \sim O(\epsilon)$. In this way, all non-linear terms in Eq. (S19) are of the same order.

Inhomogeneous magnetic field

It is possible to generalize the above construction to the case, where the effective magnetic field is not constant. The constraint to be resolved is now

$$\frac{1}{2}\epsilon^{ij}\epsilon^{ab}\frac{\partial X^a}{\partial x^i}\frac{\partial X^b}{\partial x^j} = \frac{B + \mathscr{B}}{B} = 1 - \theta\mathscr{B}, \tag{S20}$$

where $\mathscr{B}$ is a magnetic perturbation on top of a constant background B. We resolve the constraint (S20) by the following ansatz

$$X^a = x^a + y^a(x) \tag{S21}$$

y^a is a perturbation in the same order as the perturbed background fields. We write the shift y^a in perturbative orders

$$y^a = y_1^a + y_2^a + \cdots \tag{S22}$$

From the previous derivation, we can easily guess the first order

$$y_1^a = -\theta[\{\phi(x), x^a\} + \epsilon^{ab}\mathscr{A}_b] = \theta\epsilon^{ab}\underbrace{\left(\partial_b\phi - \mathscr{A}_b\right)}_{D_b\phi}, \tag{S23}$$

where the field $\mathscr{A}_b$ satisfies

$$\epsilon^{ab}\partial_a\mathscr{A}_b = \mathscr{B}. \tag{S24}$$

In the perturbative approach, we consider that ϕ and $\mathscr{B}$ are of the same order. Given that ϕ is the superfluid phase, the perturbation y_i^a is invariant under a $U(1)$ gauge transformation

$$\phi \to \phi + \beta, \quad \mathscr{A}_i \to \mathscr{A}_i + \partial_i\beta. \tag{S25}$$

One can check that the constraint (S20) can be satisfied up to all orders in perturbation if we choose the recurrence relation for y_n^a as follows

$$y_n^a = -\sum_{0<m<n}\frac{1}{2}\epsilon^{ab}\epsilon^{cd}y_m^c\partial_b y_{n-m}^d. \tag{S26}$$

Explicitly, the second order term y_2^a is

$$y_2^a = -\frac{1}{2}\epsilon^{ab}\epsilon^{cd}y_1^c\partial_b y_1^d = -\frac{\theta^2}{2}\epsilon^{ab}\epsilon^{cd}D_c\phi\partial_b D_d\phi. \tag{S27}$$

So given y_1^a from Eq. (S23), we can, in principle, obtain y_n^a for all orders n using repeatedly the recurrence relation (S26).

NONLINEAR THEORY OF THE TKACHENKO MODE FROM THE EFT OF REF. [S3]

We begin with the non-linear effective theory of the vortex lattice introduced in [S3]

$$\mathscr{L} = -\frac{B}{4\pi}\epsilon^{\mu\nu\lambda}\epsilon^{ab}a_\mu\partial_\nu X^a\partial_\lambda X^b - \varepsilon(b) - \varepsilon_{el}(U_{ab}) + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda. \tag{S28}$$

In this formulation, positions of vortices are encoded in two scalar fields $X^1(t,\mathbf{x})$ and $X^2(t,\mathbf{x})$ that are the Lagrange coordinates frozen into the vortex lattice. The first term in the Lagrangian (S28) encodes the coupling of the vortex current to the dual $u(1)$ gauge field. Given that in the following we want to integrate out superfluid density fluctuations, we will expand the energy density $\varepsilon(b)$ around its minimum $b = b_0$ and keep track only of the quadratic term

$$\varepsilon(b) = \varepsilon_0 + \frac{\lambda}{2}\frac{\delta b^2}{(2\pi)^2} + \ldots, \tag{S29}$$

with $\delta b = b - b_0$. The elastic energy density ε_{el} depends on X^a fields via the combination $U^{ab} = \delta^{ij}\partial_i X^a\partial_j X^b$ [S10, S11]. Finally, the superfluid current $j^\mu = \delta S/\delta A_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$ is coupled minimally to the external $U(1)$ source A_μ that includes the background magnetic field B.

Following [S3], one can introduce a derivative expansion with the power-counting scheme

$$a_i, X^a, A_i \sim O\left(\epsilon^{-1}\right), \quad a_t, A_t \sim O\left(\epsilon^0\right), \quad \partial_i \sim O\left(\epsilon^1\right), \quad \partial_t \sim O\left(\epsilon^2\right), \tag{S30}$$

where ϵ is a small parameter. The difference in scaling of time and spatial derivatives has its root in the quadratic dispersion of the collective Tkachenko mode. Within this derivative expansion, all terms in the Lagrangian (S28) are of order ϵ^0 and will be called leading-order (LO) in the following. Higher derivative next-to-leading (NLO) corrections to these terms have been considered in the literature [S3, S4] and we will discuss them briefly in Sec. .

Effective action

We use the resolution of the non-linear constraint discussed in Sec. and formulate the LO non-linear effective theory in terms of the scalar field ϕ. Here we will derive explicitly only leading order non-linearities that involve $\dot\phi$, a more general discussion (based on symmetries) of allowed non-linear terms can be found in the main text. We will turn off $\mathscr{A}_i$ perturbation on top of the constant magnetic background, but will keep arbitrary A_0. The vortex current which couples minimally to the dual gauge field a_i is

$$j_v^i = \frac{B}{4\pi}\epsilon^{ij}\epsilon^{ab}\left(-\partial_t X^a \partial_j X^b + \partial_j X^a \partial_t X^b\right). \tag{S31}$$

We then substitute the expression of X^a (S19) into it and obtain the vortex current up to second order in ϕ

$$j_v^i = \frac{1}{2\pi}\epsilon^{ij}\partial_j(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi) + O(\phi^3). \tag{S32}$$

Now we will rewrite the theory (S28) in terms of the field ϕ

$$\mathscr{L} = -\frac{1}{2\pi}a_i\epsilon^{ij}\partial_j(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi) - \frac{\lambda}{2}\frac{(\epsilon^{ij}\partial_i\delta a_j)(\epsilon^{kl}\partial_k\delta a_l)}{(2\pi)^2} - \varepsilon_{el}(U^{ab}) + \frac{1}{2\pi}A_0\epsilon^{ij}\partial_i a_j, \tag{S33}$$

with the fluctuation of the emergent gauge field defined by $\epsilon^{ij}\partial_i\delta a_j = \delta b$. The equation of motion of δa_i is the constraint

$$-\epsilon^{ij}\partial_j(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi) - \frac{\lambda}{2\pi}\epsilon^{ij}\partial_j\delta b + \epsilon^{ij}\partial_j A_0 = 0 \tag{S34}$$

with the solution

$$\delta b = -\frac{2\pi}{\lambda}\left(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi - A_0\right). \tag{S35}$$

Substituting this into the Lagrangian gives us

$$\mathscr{L} = \frac{1}{2\lambda}\left(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi - A_0\right)^2 - \frac{b_0}{2\pi}\left(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi - A_0\right) - \varepsilon_{el}(U^{ab}). \tag{S36}$$

The form of the coupling between the scalar ϕ and A_0 is fixed by Galilean symmetry [S12]. This Lagrangian is the leading non-linear generalization of the effective theory (S4), i.e., it captures reliably the cubic non-linear dynamical term $\frac{\theta}{2\lambda}\dot\phi\epsilon^{kl}\partial_k\dot\phi\partial_l\phi$ and modifies the coupling of the field ϕ to the potential A_0. While the former is a total derivative and does not change the equations of motion, the latter modifies the expression of the $U(1)$ particle number density in terms of ϕ and gives rise to the celebrated GMP algebra, see Sec. . Furthermore, one can see that the dynamical terms in the Lagrangian (S36) that are proportional to b_0 can be rewritten as total derivatives and thus do not affect the equations of motion.

The boson particle number density is

$$n = \frac{\delta S}{\delta A_0} = \frac{b_0}{2\pi} - \frac{1}{\lambda}\left[\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi - A_0\right] \tag{S37}$$

with the background bosonic charge $n_0 = b_0/(2\pi)$ and the fluctuation δn fixed by the field ϕ. Notably, the first two dynamical terms of the action (S36) originate from the short-range interaction between the elementary bosons, namely the first term is just $\frac{\lambda}{2}\delta n^2$, while the second term of (S36) is $\lambda n_0\delta n$.

Finally, we work out the canonical structure of the theory (S36). The canonical conjugate momentum of ϕ is

$$\pi_\phi = \frac{\delta\mathscr{L}}{\delta\dot\phi} = \frac{1}{\lambda}\dot\phi + \mathscr{O}(\phi^2). \tag{S38}$$

Given the canonical commutation relation

$$[\phi(\mathbf{x}), \pi_\phi(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}), \tag{S39}$$

at second order we end up with the leading order canonical commutation relation [S13]

$$[\phi(\mathbf{x}), \dot\phi(\mathbf{y})] = i\lambda\delta(\mathbf{x}-\mathbf{y}) \tag{S40}$$

which is identical to the linearized theory, see Eq. (S12).

GMP algebra

The particle number density, in the absence of the background A_0, extracted from the non-linear effective theory (S36) is given by

$$n = \frac{b_0}{2\pi} - \frac{1}{\lambda}(\dot\phi + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi\partial_l\phi) + O(\phi^3). \tag{S41}$$

In the derivation of the GMP algebra, we can ignore the background part of the charge density $n_0 = b_0/(2\pi)$ since it only contributes to the zero momentum density, which is absent in the GMP algebra. Here we will compute explicitly the commutation relation of particle number density operator

$$[n(\mathbf{x}), n(\mathbf{y})] = \frac{1}{\lambda^2}\left[\dot\phi(\mathbf{x}) + \frac{\theta}{2}\epsilon^{kl}\partial_k\dot\phi(\mathbf{x})\partial_l\phi(\mathbf{x}), \dot\phi(\mathbf{y}) + \frac{\theta}{2}\epsilon^{mn}\partial_m\dot\phi(\mathbf{y})\partial_n\phi(\mathbf{y})\right]. \tag{S42}$$

To this end, we use the commutation relation (S40) and obtain

$$\begin{aligned}[n(\mathbf{x}), n(\mathbf{y})] = \frac{1}{\lambda}\Bigg[&\frac{\theta}{2}\epsilon^{kl}\frac{\partial}{\partial x^k}\dot\phi(\mathbf{x})\frac{\partial}{\partial x^l}i\delta(\mathbf{x}-\mathbf{y}) - \frac{\theta}{2}\epsilon^{mn}\frac{\partial}{\partial y^m}\dot\phi(\mathbf{y})\frac{\partial}{\partial y^n}i\delta(\mathbf{y}-\mathbf{x})\\ &+\frac{\theta^2}{4}\epsilon^{kl}\epsilon^{mn}\frac{\partial}{\partial x^k}\dot\phi(\mathbf{x})\frac{\partial}{\partial y^n}\phi(\mathbf{y})\frac{\partial}{\partial x^l}\frac{\partial}{\partial y^m}i\delta(\mathbf{x}-\mathbf{y}) - \frac{\theta^2}{4}\epsilon^{kl}\epsilon^{mn}\frac{\partial}{\partial y^m}\dot\phi(\mathbf{y})\frac{\partial}{\partial x^l}\phi(\mathbf{x})\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^n}i\delta(\mathbf{y}-\mathbf{x})\Bigg].\end{aligned} \tag{S43}$$

Now we do Fourier transformation in both $\mathbf{x}$ and $\mathbf{y}$ by taking the integral $\int d^2\mathbf{x}d^2\mathbf{y}\, e^{i\mathbf{kx}}e^{i\mathbf{qy}}[\cdots]$. The first term of (S43) gives us $\frac{i\theta}{2\lambda}\mathbf{k}\times\mathbf{q}\dot\phi(\mathbf{k}+\mathbf{q})$. The second term gives rise to an identical result. The summation of the third and the fourth terms produces

$$-\frac{i\theta^2}{2\lambda}(\mathbf{k}\times\mathbf{q})\int d^2\mathbf{x}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}}\epsilon^{ij}\frac{\partial}{\partial x^i}\dot\phi(\mathbf{x})\frac{\partial}{\partial x^j}\phi(\mathbf{x}). \tag{S44}$$

We combine the results of the Fourier transformation of (S43) and obtain

$$[n(\mathbf{k}), n(\mathbf{q})] = -i\theta(\mathbf{k}\times\mathbf{q})n(\mathbf{k}+\mathbf{q}) = i\ell^2(\mathbf{k}\times\mathbf{q})n(\mathbf{k}+\mathbf{q}) \tag{S45}$$

with the definition $\ell = 1/\sqrt{B}$. We thus end up with the long wavelength version of the celebrated GMP algebra [S14] that indicate that our starting point (S28) is a theory operating purely in the lowest Landau level. The long wavelength limit of GMP algebra was also obtained in the composite fermion theories [S15, S16] and the bi-metric theory of fractional quantum Hall [S17, S18].

LLL volume-preserving diffeomorphisms

Given that we work in the LLL regime, we consider here a combination of an infinitesimal two-dimensional volume-preserving diffeomorphism generated by

$$x^i \to x^i + \xi^i, \quad \xi^i = -\theta\epsilon^{ij}\partial_j\alpha \tag{S46}$$

and a $U(1)$ gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\vartheta, \quad \vartheta = \alpha - \theta\epsilon^{ij}A_i\partial_j\alpha. \tag{S47}$$

Under these transformations, in the constant magnetic field and in the LLL approximation, the background A_0 transforms as

$$\delta_\alpha A_0 = \dot\alpha + \theta\epsilon^{ij}\partial_i A_0\partial_j\alpha = \dot\alpha + \theta\{A_0, \alpha\}, \tag{S48}$$

while A_i is unchanged [S19]. The corresponding transformation of ϕ is given by

$$\delta_\alpha\phi = \alpha - \frac{\theta}{2}\{\phi, \alpha\} + \cdots, \tag{S49}$$

where the non-linear terms originate from the non-commutativity of the area-preserving diffeomorphisms[S20]. It is straightforward to check that these transformations realize the canonical w_∞ algebra on the Tkachenko field ϕ, namely $[\delta_\alpha, \delta_\beta]\phi = \delta_{\{\alpha,\beta\}_\theta}\phi$.

One can check that up and including the leading order non-linearity, the action built from the Lagrangian (S36) is invariant under the combination of (S46) and (S47). Indeed, the variation of the first and second terms of (S36) is a total derivative [S21], while the elasticity energy density is invariant on its own. Notably, this invariance automatically insures the emergence of the LLL GMP algebra [S19] that we derived explicitly above. Moreover, it implies that in the LLL limit the charge current density can be expressed as a derivative of the stress tensor and the charge conservation law has a higher-rank form that arises naturally in higher-rank tensor gauge theories coupled to fractons [S22, S23].

Beyond leading-order theory (S28)

Following the power-counting scheme (S30), one can systematically add sub-leading symmetry-allowed terms to the LO effective theory Lagrangian (S28). In fact, some next-to-leading (NLO) terms have already been investigated before.

In particular, already in Ref. [S3], the non-linear NLO term $me_i^2/(4\pi b)$ (whose form is fixed by Galilean invariance) has been incorporated. Here $e_i = \partial_t a_i - \partial_i a_t$ is the dual electric field that encodes the superfluid current, and m is the mass of the elementary boson. This term allows us to go beyond the LLL approximation and incorporate some effects of higher Landau levels into the low-energy description of the superfluid vortex lattice. In particular, the inclusion of this term gives rise to the finite-frequency Kohn mode in the EFT excitation spectrum, which correspondingly modifies the $U(1)$ linear response [S3].

Another NLO term that breaks time-reversal symmetry and has the form $e^i\partial_i b/(4\pi b)$ has been discussed in Ref. [S4]. Given that this term does not depend on the mass m of the boson, it survives in the LLL limit and incorporates higher-order corrections to the LLL coarse-grained description developed above. In particular, we expect that this term is responsible for leading-order non-linear corrections to the low-momentum GMP algebra (S45).

We expect that the NLO terms discussed above will generate corrections to the decay rate Γ of the Tkachenko mode. Those however will disappear faster in the limit $E \to 0$ than the leading-order result $\Gamma \sim E^3$ that we discovered in this paper.

Notice that either of the NLO terms mentioned here modify the Gauss law constraint (S15) and make the vortex crystal compressible. As a result, the resolution of the constraint by a canonical transformation (S19) is not applicable anymore. Nevertheless, it should be possible to resolve the modified constraint by generalizing the method used in Sec. .

[S1] H. Watanabe and H. Murayama, Redundancies in Nambu-Goldstone Bosons, Phys. Rev. Lett. **110**, 181601 (2013), arXiv:1302.4800.

[S2] One can understand the last term of (S1) as the contribution from the Lorentz force. If we take the time derivative of that equation and use the continuity equation $\dot{n} = -\partial_i j^i$, we have the time variation of the total momentum is given by

$$\int d^2x\, \dot{T}^{0i} = \int d^2x \frac{\partial(mj^i)}{\partial t} - \int d^2x\, \epsilon^{ik} j^k B, \tag{S50}$$

the last term is nothing but the total Lorentz force acting on the system.

[S3] S. Moroz, C. Hoyos, C. Benzoni, and D. T. Son, Effective field theory of a vortex lattice in a bosonic superfluid, SciPost Phys. **5**, 39 (2018), arXiv:1803.10934.

[S4] S. Moroz and D. T. Son, Bosonic Superfluid on the Lowest Landau Level, Phys. Rev. Lett. **122**, 235301 (2019), arXiv:1901.06088.

[S5] The normalization of the dual gauge field a_μ differs here by a factor 2π from the normalization used in Refs. [S3, S4]. Moreover, to be consistent with the main text, the sign of the $U(1)$ source $\mathscr{A}$ also differs with Refs. [S3, S4].

[S6] Notice, that the elastic term was for simplicity suppressed in Eq. (3), but can be easily expressed in terms of the fields $X^a(t,x^i)$, see e.g. Appendix B of Ref. [S3].

[S7] B. Jeevanesan, C. Benzoni, and S. Moroz, Surface waves and bulk Ruderman mode of a bosonic superfluid vortex crystal in the lowest Landau level, Phys. Rev. B **106**, 144501 (2022), arXiv:2202.10924.

[S8] Notice that in Ref. [S3], the contribution of the Kohn's mode is higher order in the derivative expansion since the power-counting scheme with $\omega \sim \mathbf{k}^2$ was used there. That counting originates from the dispersion of the low-energy Tkachenko mode.

[S9] The argument we use here is nothing but saying that X^a and x^i are related by a area-preserving diffeomorphism generated by a function $\phi(\mathbf{x})$. Notice that the auxiliary Poisson bracket in Eq. (S17) does not originate from the effective field theory Lagrangian.

[S10] H. Leutwyler, Phonons as Goldstone Bosons, Helv. Phys. Acta **70**, 275 (1997), hep-ph/9609466.

[S11] D. E. Soper, *Classical Field Theory* (Wiley, 1976).
[S12] While the coupling to A_0 source is local, one can check that in this formulation the coupling to the $\mathscr{A}_i$ source is non-local.
[S13] The quadratic in ϕ contribution to the canonical conjugate momentum π_ϕ does not contain temporal derivatives and thus does not affect the canonical commutation relation (S40) up to second order in ϕ.
[S14] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Magneto-roton theory of collective excitations in the fractional quantum Hall effect, Phys. Rev. B **33**, 2481 (1986).
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[S17] A. Gromov and D. T. Son, Bimetric Theory of Fractional Quantum Hall States, Phys. Rev. X **7**, 041032 (2017), arXiv:1705.06739.
[S18] D. X. Nguyen, A. Gromov, and D. T. Son, Fractional quantum Hall systems near nematicity: Bimetric theory, composite fermions, and Dirac brackets, Phys. Rev. B **97**, 195103 (2018), arXiv:1712.08169.
[S19] Y.-H. Du, U. Mehta, D. X. Nguyen, and D. T. Son, Volume-preserving diffeomorphism as nonabelian higher-rank gauge symmetry, SciPost Phys. **12**, 050 (2022), arXiv:2203.05004.
[S20] Note that here we consider ϕ as the dynamical field generating an area-preserving diffeomorphism that captures the dynamical Tkachenko mode. On the other hand, α is the generator of a non-dynamical infinitesimal area-preserving diffeomorphism that parametrizes the local symmetry of the lowest Landau level system [S19].
[S21] Indeed, the first term in the Lagrangian (S36) is just the contact density-density interaction term and the charge density transforms under volume-preserving diffeomorphisms as $\delta n = 1/B_0 \epsilon^{ij} \partial_i \alpha \partial_j n$, which is a total derivative. Similarly, the variation of the second term is also a total derivative.
[S22] M. Pretko, Subdimensional particle structure of higher rank $U(1)$ spin liquids, Phys. Rev. B **95**, 115139 (2017), arXiv:1604.05329.
[S23] M. Pretko, Generalized electromagnetism of subdimensional particles: A spin liquid story, Phys. Rev. B **96**, 035119 (2017), arXiv:1606.08857.