THE UNIVERSITY OF CHICAGO

ON VANISHING CONDITIONS OF MINIMAL VOLUME OF CLOSED LOCALLY HOMOGENEOUS RIEMANNIAN MANIFOLDS

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ABSTRACT

Our article is motivated by the vanishing condition of minimal volume of closed smooth manifolds M that admit locally homogeneous Riemannian metric. We give a complete criterion of vanishing of $\|M\|$ (a lower bound of $\mathrm{Minvol}(M)$, up to a multiple of dimension-dependent constant) for general M, and a complete criterion of the vanishing of $\mathrm{Minvol}(M)$ of closed homogeneous Riemannian manifolds. We also present a technical fibering lemma, which is useful throughout the article.

CHAPTER 1

INTRODUCTION

In Gromov [1982], Gromov introduced Minvol, a non-negative diffeomorphism invariant for a smooth manifold of dimension ≥ 2.

Definition 1.0.1. Let M be a smooth manifold of dimension ≥ 2. The *minimal volume* of M is defined to be

$$\mathrm{Minvol}(M) = \inf_{g:|K_g|\leq 1} \mathrm{vol}_g(M),$$

where g denotes a complete Riemannian metric, and $|K_g|$ denotes the maximum absolute value of sectional curvature of g, across all 2-subspaces of $T_p(M)$, across all points $p \in M$.

Naturally, given a smooth manifold M, one is interested in the value of $\mathrm{Minvol}(M)$, and what Riemannian metric(s) realize(s) it (if any). Currently, for many classes of M, we do not even know when $\mathrm{Minvol}(M) = 0$.

In this document, we study the vanishing condition of $\mathrm{Minvol}(M)$, for closed (compact, without boundary) smooth manifolds M that admit a locally homogeneous Riemannian metric. Among other reasons,

1. such M's serve as "building block"s for closed smooth manifolds in small dimensions (in the sense of Thurston geometrization conjecture)

2. such M admits a "preferred smooth structure" (the one that descends from "the covering" Lie group G: the underlying topological space of G admits a unique smooth structure that is compatible to its Lie group structure).

Before we proceed with our main results, we recall two definitions, in their bare minimal forms. For a more conceptual and comprehensive formulation, see Helgason [1979] and [Eberlein, 1996, § 2.2].

Definition 1.0.2. Let M be a complete connected Riemannian manifold.

1. We say that M is *locally homogeneous* if for all p, q in M, there exists open neighborhoods $U \ni p$, $V \ni q$, and $f \in \mathrm{Isom}(U, V)$ such that $f(U) = V$ and $f(p) = q$. If U (resp. V) can be made the entire M, then we say that M is *homogeneous*.

2. We say that M is a *locally symmetric space of non-compact type* if it is diffeomorphic to $\Gamma\backslash G/K$ where G is a centerless, semisimple Lie group, K is a maximal compact subgroup in G, and Γ is a discrete subgroup in G that acts freely on G/K via left translation action. In which case, the universal cover of M is G/K. Note that such M is locally homogeneous.

In the future, without further explanation, M is a connected, closed, smooth manifold of dimension ≥ 2.

1.1 The Main Results

In this section, we present two partial results on the vanishing conditions of minimal volume of (the underlying smooth manifolds of) closed locally homogeneous Riemannian manifolds:

1. Theorem 1, which gives a criterion on the vanishing of the simplicial volume of (the underlying smooth manifolds of) closed locally homogeneous Riemannian manifolds.

2. Theorem 2, which gives a criterion on the vanishing of minimal volume of (the underlying smooth manifolds of) closed homogeneous Riemannian manifolds.

As a (pedantic) remark, the minimal volume of (the underlying smooth manifold of) a closed locally homogeneous Riemannian manifold need not be realized by a locally homogeneous Riemannian metric.

1.1.1 *Simplicial Volume of Closed Locally Homogeneous Riemannian Manifolds*

In Gromov [1982], Gromov introduced the following non-negative topological invariant for a connected, closed manifold:

Definition 1.1.1. Let M be a connected, closed, n-manifold.

If M is oriented, the *simplicial volume* of M is the L^1-norm of its fundamental class $[M] \in H_n(M; \mathbb{R})$

$$\|M\| = \inf_{\substack{c \in C_n(M;\mathbb{R}) \\ \partial c = 0 \\ [c]=[M]}} \|c\|_1$$

where $\|c\|_1$ is the L^1-norm on $C_k(M; \mathbb{R})$, i.e.

$$\left\| \sum_i c_i \sigma_i \right\|_1 = \sum_i |c_i| \,,$$

where $c_i \in \mathbb{R}$, $\sigma_i \in C(\Delta^k, M)$.

If M is not oriented, we define $\|M\|$ to be $\frac{1}{2}\left\|\overline{M}\right\|$ where $\overline{M}$ is its orientation double cover.

Among other reasons, a criterion of $\|M\| > 0$ is of interest, as it serves as a lower bound of $\mathrm{Minvol}(M)$, up to a dimension-dependent constant:

Proposition 1.1.1 ([Gromov, 1982, §0.5, Corollary (A)])**.** Let M be a smooth manifold of dimension n. Then

$$\mathrm{Minvol}(M) \geq \frac{1}{(n-1)^n n!} \|M\| \,.$$

In Lafont and Schmidt [2006], applying estimates of Connell-Farb in Connell and Farb [2003] on the barycenter map in higher rank, Lafont-Schmidt showed that every closed, locally symmetric space of non-compact type has positive simplicial volume. In this paper, we prove the converse:

Theorem 1. Let M be a closed, locally homogeneous Riemannian manifold. Then $\|M\| > 0$ if and only if M is diffeomorphic to a closed, locally symmetric space of non-compact type.

A notion closely related to local homogeneity is Thurston model geometry; see [Thurston, 1997, §3]. In this language, Theorem 1 immediately implies the following:

Corollary 1.1.1. Let M be a closed smooth manifold modeled on some Thurston geometry. Then $\|M\| > 0$ if and only if M is diffeomorphic to a closed locally symmetric space of non-compact type.

Remark 1.1.1. Let $n = \dim(M)$. The special cases of $n = 2, 3$ of Corollary 1.1.1 were essentially solved by Gromov. The special case of $n = 4$ (resp. $n = 5$) was done by Zhang in Zhang [2017] (resp. Neofytidis-Zhang in Neofytidis and Zhang [2021]), based on work on classification of dim-4 (resp. 5) geometry by Filipkiewicz (resp. Geng).

1.1.2 Minimal Volume of Closed Homogeneous Riemannian Manifolds

Before we state the theorem, we state a few definitions.

Definition 1.1.2. A closed smooth manifold M is called a *compact homogeneous manifold*, if it is diffeomorphic to G/K, where G is a connected compact Lie group, and K a closed subgroup in G.

Remark 1.1.2. Such manifold admits a homogeneous Riemannian metric, i.e. it is the underlying smooth manifold of a homogeneous Riemannian manifold.

Definition 1.1.3. Suppose G is a Lie group, and X a smooth manifold. A smooth action G on X is said to be *locally free*, if there exists an open neighborhood U that contains the identity $e_G \in G$, such that for any $x \in X$, $g \in U - \{e_G\}$ we have $g \cdot x \neq x$.

Remark 1.1.3. Any free action is locally free; a deck transformation for a non-trivial finite-covering is locally-free, but not free.

Recall the following fact about a connected compact Lie group.

Definition 1.1.4. Let G be a compact Lie group. Then, it has a maximal connected, abelian, closed subgroup T_G (known as the *maximal torus* of G), which is unique up to conjugation. The *rank* of G, denoted as $\mathrm{rank}(G)$, is defined to be the dimension of T_G.

We give a (topological) criterion of the vanishing of Minvol of compact homogeneous manifolds.

Theorem 2. Suppose G be a connected compact Lie group, and K a closed subgroup in G. The following are equivalent:

1. $\mathrm{Minvol}(G/K) = 0$.

2. $\mathrm{rank}(G) > \mathrm{rank}(K)$.

3. $\chi(G/K) = 0$.

4. G/K admits a locally-free S^1-action given by the left translation action of a closed $SO(2)$ subgroup in G.

5. G/K admits a locally-free S^1-action.

Remark 1.1.4. In particular, if $\mathrm{Minvol}(G/K) > 0$ under the standard smooth structure, then $\chi(G/K) > 0$, and thus $\mathrm{Minvol}_\sigma(G/K) > 0$ for any smooth structure σ on the underlying topological manifold of G/K.

1.2 The Document Outline

In chapter 2, we lay some foundations on (the underlying smooth manifolds of) locally homogeneous Riemannian manifolds, showing that they are precisely smooth double coset manifolds of the form $\Gamma\backslash G/K$, where

1. G is a Lie group,
2. K is a compact subgroup in G,
3. Γ is a discrete subgroup in G whose left action on G/K is free.

In particular, we prove the main fibering lemma of $\Gamma\backslash G/K$ (Lemma 2.2.2), without assuming that the discrete subgroup Γ is a lattice. Even though the main fibering lemma is useful, it remains a lemma in our paper, as the conditions appear slightly clunky. Then, using Lemma 2.2.2, by carefully splitting the Lie group G (so that it behaves well with respect to (Γ, K)), we prove Theorem 3 (the fibering theorem of closed locally homogeneous Riemannian manifolds), which is used to prove Thereom 1. Besides, we follows through the proof of Lemma 2.2.2 on a case of non-compact total space ($\widetilde{Sp(2n;\mathbb{R})}$), demonstrating its potential for generalization despite technical caveats.

In chapter 3 (resp. chapter 4), we present the proofs of Theorem 1 (resp. Theorem 2), and discuss some related examples.

Finally, in chapter 5, we discuss the gap between simplicial volume and minimal volume of closed locally homogeneous Riemannian manifolds, present a few open problems in this area, and mention some related technical caveats.

CHAPTER 2

STRUCTURE THEORY OF LOCALLY HOMOGENEOUS (RIEMANNIAN) MANIFOLDS

In this chapter, we lay some foundations on the smooth double coset manifolds $\Gamma\backslash G/K$. In particular, we show that they are precisely the smooth manifolds that admit locally homogeneous Riemannian metrics (via Lemma 2.1.1 and Lemma 2.1.2), which allows us to use the two notions interchangeably.

In particular, we prove the main fibering lemma of $\Gamma\backslash G/K$ (Lemma 2.2.2), without any conditions on Γ. Even though the main fibering lemma is useful, it remains a lemma in our paper, as the conditions appear slightly clunky. Then, using Lemma 2.2.2, by carefully splitting the Lie group G (so that it behaves well with respect to (Γ, K)), we prove Theorem 3 (the fibering theorem of closed locally homogeneous Riemannian manifolds), which is used to prove Thereom 1 (in section 3.1).

2.1 Locally Homogeneous Riemannian Manifolds and $\Gamma\backslash G/K$'s

We make a remark, which follows from an easy set-theoretic exercise.

Remark 2.1.1 (Smooth Double Coset Manifold)**.** Let G be a Lie group, and K a compact subgroup in G. Then, G/K is a smooth manifold, with L_g denoting the left translation by any $g \in G$, we have

$$p : G \to \mathrm{Diff}(G/K)$$
$$g \mapsto L_g$$

being a Lie group homomorphism, with $\ker(p) = \bigcap_{g\in G} gKg^{-1} =: N_K$, where N_K is a compact normal subgroup of G that lies in K.

The following holds:

1. Let Γ be any discrete subgroup in G whose left translation on G/K is free (equivalently $\Gamma \cap \bigcup_{g \in G} gKg^{-1} = \{e_G\}$). Then, $\Gamma \cap \ker(p) = \{e_G\}$ (i.e. $p|_\Gamma$ is injective), and thus the canonical projection $G/K \to \Gamma\backslash G/K$ is a covering map of smooth manifolds with deck transformation group $p(\Gamma) = \Gamma/(\Gamma \cap \ker(p)) \cong \Gamma$.

2. In general, $G/K = (G/N_K)/(K/N_K)$, and $G/N_K \leq \mathrm{Diff}(G/K) = \mathrm{Diff}((G/N_K)/(K/N_K))$.

   Moreover, the following are equivalent ((b) $\Leftrightarrow$ (c) can be checked easily):

   (a) The left G-translation on G/K is an effective action.

   (b) $\ker(p) = \{e_G\}$.

   (c) K does not contain a nontrivial normal subgroup of G.

   In which case, we have $p : G \hookrightarrow \mathrm{Diff}(G/K)$.

3. For any left G-invariant Riemannian metric on G/K (if exists), we have $p(G) \leq I(G/K, h)$. In particular, when $\ker(p) = \{e_G\}$, then $p : G \hookrightarrow I(G/K, h)$.

With Remark 2.1.1 in mind, from now on, unless there is further explanation, (G, K, Γ) is given by

- G a Lie group
- K a compact subgroup in G
- Γ a discrete subgroup in G who left translation acts freely on G/K.

2.1.1 $\Gamma\backslash G/K$ *as a Locally Homogeneous Riemannian Manifold*

In the following lemma, we show that the smooth coset manifold G/K (resp. smooth double coset manifold $\Gamma\backslash G/K$) admits a homogeneous (resp. locally homogeneous) Riemannian metric.

Lemma 2.1.1. The following holds:

1. The smooth coset manifold G/K admits a left G-invariant Riemannian metric.
2. The smooth double coset manifold $\Gamma\backslash G/K$ admits a locally homogeneous Riemannian metric.

Proof. By Remark 2.1.1.2, we note that 1. implies 2. immediately. Thus, it suffices to prove 1. To show 1., note that the following is a bijection:

$$\left\{\begin{array}{c}\text{left } G\text{-invariant Riemannian metric}\\ \text{on } G/K\end{array}\right\} \to \left\{\begin{array}{c}\mathrm{Ad}_G(K)\text{-invariant scalar product}\\ \text{on } \mathfrak{g}\end{array}\right\}$$

$$h \mapsto h|_{e_G K}$$

$$\text{left } G\text{-translation of } x \leftarrow\!\shortmid x$$

Since K is compact, thus $\mathrm{Ad}_G(K)$ is compact. Thus, by averaging argument (i.e. start with inner product, take average with respect to $\mathrm{Ad}_G(K)$), the set above is non-empty, and this proves 1. □

2.1.2 Locally Homogeneous Riemannian Manifold as $\Gamma\backslash G/K$

It is crucial to note that a locally homogeneous Riemannian manifold is a smooth double coset manifold. We will use this repeatedly later.

Lemma 2.1.2. Let M be a locally homogeneous Riemannian manifold. Then, there exists

- G a Lie group with finitely many connected components
- K a compact subgroup of G
- Γ a discrete subgroup of G that acts freely on G/K

where

1. The universal cover $\widetilde{M}$ of M is diffeomorphic to G/K, and

2. M is diffeomorphic to $\Gamma\backslash G/K$.

Proof. Let M be a locally homogeneous Riemannian manifold. By Singer [1960], its universal cover $\widetilde{M}$, equipped with the pullback metric, is a homogeneous Riemannian manifold. Let G be the isometry group of $\widetilde{M}$, and K the isotropy subgroup of any arbitrary point in $\widetilde{M}$. By [Helgason, 1962, §4 Theorem 2.5, pp.169], K is compact.

We claim that $\widetilde{M}$ is diffeomorphic to G/K. To see this, note that by assumption, G acts transitively on $\widetilde{M}$. If G acts on $\widetilde{M}$ on the left (resp. right), then, as Riemannian manifolds, $\widetilde{M} \cong G/K$ (resp. $\widetilde{M} \cong K\backslash G$), where the latter is equipped with a Riemannian metric descended from a left-invariant (resp. right-invariant) Riemannian metric on G. In particular, as smooth manifolds, $\widetilde{M} \cong G/K$, as desired.

We claim that G has finitely many connected components. To see this, note that since $\widetilde{M}$ is simply-connected, it follows from the long exact sequence of the fibration of $K \to G \to \widetilde{M}$ that $\pi_0(G) \cong \pi_0(K)$. Since K is compact, thus $\pi_0(K)$ is finite, and so is $\pi_0(G)$, as desired.

With the above in mind, let $\Gamma = \pi_1(M)$. Since it is the deck transformation group of $\widetilde{M} \to M$, which is a local isometry, thus $\pi_1(M)$ is a discrete subgroup of G that acts freely on $\widetilde{M} \cong G/K$. This completes the proof. □

Remark 2.1.2 (of Lemma 2.1.2)**.** A discrete subgroup Γ in G is said to be a lattice (resp. cocompact lattice) in G if $\mathrm{vol}(\Gamma\backslash G)$ (under the Haar measure of G) is of finite volume (resp. compact). When M is of finite volume (resp. compact), Γ in the lemma is a lattice (resp. cocompact lattice).

For technical convenience in various situations, we would like to consider $\Gamma\backslash G/K$ for connected G. Thus, we establish the following lemma, as a corollary of Lemma 2.1.2.

Lemma 2.1.3. Let M be a closed locally homogeneous Riemannian manifold. Then M has a finite-sheeted cover given by a double coset manifold $\Gamma\backslash G/K$ where

- G is a connected Lie group
- K is a compact subgroup of G
- Γ is a cocompact lattice in G

Proof. Let M be a closed locally homogeneous Riemannian manifold. By Lemma 2.1.2, there exists

- G' a Lie group with finitely many connected components
- K' a compact subgroup of G
- Γ' a discrete subgroup of G' that acts freely on G'/K'

where

1. The universal cover $\widetilde{M}$ of M is diffeomorphic to G'/K', and
2. M is diffeomorphic to $\Gamma'\backslash G'/K'$.

With above in mind, let

- G be the identity connected component of G'.
- $K = K' \cap G$ (K is a compact subgroup in G).
- $\Gamma = \Gamma' \cap G$.

Since G is a finite index subgroup of G', thus Γ is a finite index subgroup of Γ'. Thus, $\Gamma\backslash G/K = \Gamma\backslash G'/K'$ is a finite-sheeted cover of M.

It remains to justify that Γ is a cocompact lattice in G, equivalently, $\Gamma\backslash G$ is compact. Since K is compact, this is equivalent to $\Gamma\backslash G/K$ being compact. This holds, since $\Gamma\backslash G/K$ is a finite-sheeted cover of the compact M. This completes the proof. $\square$

2.1.3 A Few Caveats about $\Gamma\backslash G/K$

In the previous subsections, we showed that the double coset manifolds $\Gamma\backslash G/K$ are precisely the smooth manifolds that admits locally homogeneous Riemannian metric. In this subsections, we note two subtleties.

1. A smooth double coset manifold might admit non-isometric locally homogeneous Riemannian metrics. As a result, its presentation as $\Gamma\backslash G/K$ is not unique, even after trivial reductions.

2. Let g be a left G-invariant metric on G/K. Tautologically, the isometry group of the Riemannian manifold $(G/K, g)$, denoted as $I := I(G/K, g)$, acts on G/K effectively and transitively. However, often, there are strict subgroups of H of I that acts effectively and transitively on G/K, in which case, $G/K = H/(H \cap K)$.

Plenty of examples can be found in standard references.

2.2 The Main Fibering Lemma of $\Gamma\backslash G/K$ (Lemma 2.2.2)

Let G be a conencted Lie group, and Γ is a discrete subgroup that acts freely on G/K. Γ need not be a lattice.

In this section, we state and prove the main fibering lemma of $\Gamma\backslash G/K$ (Lemma 2.2.2), which specify a set of delicate technical conditions for a short exact sequence of Lie groups of G can descend (with respect to (Γ, K)) to give rise to a fiber bundle structure of (a finite cover of) $\Gamma\backslash G/K$.

In addition, we give several applications of (the statement and proof of) Lemma 2.2.2. In particular,

1. In section 2.3, we use Lemma 2.2.2 to prove the structure theorem of closed locally homogeneous Riemannian manifolds (Theorem 3), that is, in the special case when Γ is a cocompact lattice.

2. In subsection 2.2.2, we follow through the proof of Lemma 2.2.2 to give partial evidence to a fibering of $\widetilde{Sp(2n;\mathbb{R})}$ (which is a non-compact, non-linear Lie group).

Before we state the lemma, we give a few definitions.

Definition 2.2.1. Let G be a connected Lie group, and let Γ be a discrete subgroup in G. The short exact sequence of Lie groups given by

$$1 \to H \to G \xrightarrow{p} \overline{G} \to 1,$$

denoted by $(p, H, \overline{G})$ in short, is said to be Γ-*good* if the following holds:

1. $\overline{G}$ is connected.
2. The subgroup $p(\Gamma)$ in $\overline{G}$ is
   (a) discrete and closed
   (b) virtually torsion-free.

We also state technical proposition, which will be used repeatedly throughout this chapter.

Proposition 2.2.1 ([Geng, 2015, Theorem 2.6], Onishchik and Vinberg [2000], [Raghunathan, 1972, Theorem 1.13 (statement and proof)])**.** Let G be a Lie group. Let $H < G$ be a closed subgroup, and let $\Gamma < G$ be lattice. If Γ is cocompact or H is normal, then the following are equivalent:

1. $\Gamma \cap H$ is a lattice in H (in this case, we say that H is Γ-*hereditary*).
2. $\Gamma \cap H$ is a cocompact lattice in H (when Γ is cocompact).
3. The image of Γ in G/H is a lattice (when H is normal).
4. The image of Γ in G/H is a cocompact lattice (when H is normal and Γ is cocompact).

Before we continue, we note a family of $(p, H, \overline{G})$ that is Γ-good.

Lemma 2.2.1. Let G be a connected Lie group, if

$$1 \to H \to G \xrightarrow{p} \overline{G} \to 1$$

where

1. H is a closed subgroup in G that is lattice hereditary (i.e. any lattice Γ in G yields a lattice $\Gamma \cap H$ in H).

2. $\overline{G}$ is connected and linear.

Then, $(p, H, \overline{G})$ is Γ-good for any lattice Γ in G.

Proof. Let Γ be any lattice in G. Since H is lattice hereditary, thus $\Gamma \cap H$ is a lattice in H. By Proposition 2.2.1, $p(\Gamma)$ is a lattice in $\overline{G}$ (and thus, trivially a closed, discrete subgroup of $\overline{G}$). To prove the lemma, it remains to prove the following claim.

Claim 2.2.1. Every lattice in a connected linear Lie group is virtually torsion-free.

Proof. Let $\overline{\Gamma}$ be a lattice in $\overline{G}$. Since $\overline{G}$ is linear, thus $\overline{\Gamma}$ is linear. Then, by [Gelander and Slutsky, 2020, Corollary 1.6], $\overline{\Gamma}$ is finitely-generated. By a classical lemma of Selberg, any finitely-generated linear subgroup over characteristic-0 is virtually torsion-free. Thus, $\overline{\Gamma}$ is virtually torsion-free, as required. □

□

We are now ready to state and prove the main fibering lemma of $\Gamma \backslash G/K$ (Lemma 2.2.2).

Lemma 2.2.2 (The Main Fibering Lemma of $\Gamma \backslash G/K$)**.** Let G be a connected Lie group, and Γ a discrete subgroup in G, K a compact subgroup in G.

Let $(p, H, \overline{G})$ be a Γ-good sequence, which is (Γ, K)-*good*, i.e. there exists a maximal compact subgroup $\overline{K}^{\max}$ in $\overline{G}$ that contains $p(K)$, with $P := p^{-1}(\overline{K}^{\max})$, such that Γ is a cocompact lattice in P.

Then, there exists a finite-index subgroup $\overline{\Gamma}'$ of $p(\Gamma)$, such that with $\Gamma' = p^{-1}(\overline{\Gamma}') \cap \Gamma$, the p-induced map

$$f : \underbrace{\Gamma' \backslash G/K}_{=:M'} \to \underbrace{\overline{\Gamma}' \backslash \overline{G}/\overline{K}^{\max}}_{=:B}$$

is a fiber bundle map, with compact, connected fiber. Further,

1. The total space $\Gamma' \backslash G/K$ is a finite covering of $\Gamma \backslash G/K$.

2. The fiber is given by

   $$F := (\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P) \backslash P/K,$$

   where $P := p^{-1}(\overline{K})$. (Recall that by assumption F is compact and connected.)

3. The base is compact if and only if the total space is compact.

Remark 2.2.1. A natural candidate for $\overline{\Gamma}'$ is $\overline{\Gamma} := p(\Gamma) = \Gamma/(\Gamma \cap H)$. However, a priori, there are two potential problems:

1. $\overline{\Gamma}$ might not be a cocompact lattice (or even a discrete subgroup) in $\overline{G}$.

2. Even if we assume that $\overline{\Gamma}$ is discrete, we can only guarantee that $\overline{\Gamma}$ acts on $\overline{G}/\overline{K}$ with finite point-stabilizer (since the stabilizer of the point $\overline{g}\overline{K} \in \overline{G}/\overline{K}$ is $\overline{\Gamma}_{\overline{g}\overline{K}} = \Gamma \cap \overline{g}\overline{K}^{\max}\overline{g}^{-1}$, which is discrete and compact, hence finite). In particular, $\overline{\Gamma}$ might not act freely on $\overline{G}/\overline{K}$.

With the requirements specified in the hypothesis in Lemma 2.2.2, $\overline{\Gamma}$ has a finite-index subgroup $\overline{\Gamma}'$ that acts on $\overline{G}/\overline{K}$, which guarantees that $\overline{\Gamma}' \backslash \overline{G}/\overline{K}$ is a smooth manifold.

Remark 2.2.2. When $p(\Gamma)$ is torsion-free, then the original $\Gamma\backslash G/K$ can be written as the total space in the fiber bundle in Lemma 2.2.2 (i.e. no need to lift to finite cover). When this holds, the resulting fiber bundle is

$$(\Gamma \cap p^{-1}(\overline{\Gamma}') \cap p^{-1}(\overline{K}^{\max}))\backslash p^{-1}(\overline{K}^{\max})/K \hookrightarrow \Gamma\backslash G/K \overset{f}{\twoheadrightarrow} p(\Gamma)\backslash\overline{G}/\overline{K}^{\max}.$$

For example, this is clearly the case when $\Gamma \leq H$ (which yields $p(\Gamma) = \{e_{\overline{G}}\} \in \overline{G}$).

Remark 2.2.3. In the special case when $\Gamma = K = \{e_G\}$, all sequences of $(p, H, \overline{G})$ such that $(p^{-1}(\Gamma) \cap p^{-1}(\overline{K}^{\max}))\backslash p^{-1}(\overline{K}^{\max})$ is compact is (Γ, K)-good; the resulting fiber bundle is

$$(\Gamma \cap p^{-1}(\overline{K}^{\max}))\backslash p^{-1}(\overline{K}^{\max}) \hookrightarrow G \overset{f}{\twoheadrightarrow} \overline{G}/\overline{K}^{\max}.$$

2.2.1 Proof of The Main Fibering Lemma of $\Gamma\backslash G/K$ *(Lemma 2.2.2)*

In this section, let G be a connected Lie group, and Γ a discrete subgroup in G that acts freely on G/K. Let $(p, H, \overline{G})$ be a Γ-good sequence.

Then, the following holds.

Claim 2.2.2. There exists a finite-index subgroup $\overline{\Gamma}'$ of $p(\Gamma)$ that acts freely on $\overline{G}/\overline{K}$ for every compact subgroup $\overline{K}$ in $\overline{G}$.

Proof. By assumption $(p, H, \overline{G})$ be a Γ-good sequence. In particular, this means that $p(\Gamma)$ is discrete, virtually torsion-free. Thus, there exists a torsion-free, discrete subgroup $\overline{\Gamma}'$ that is of finite-index in $p(\Gamma)$.

A priori, since $\overline{\Gamma}'$ is discrete, the stabilizer of $\overline{\Gamma}'$ on any $\overline{g}\overline{K} \in \overline{G}/\overline{K}$ is

$$\overline{\Gamma}'_{\overline{g}} := \overline{\Gamma}' \cap \overline{g}\overline{K}\overline{g}^{-1},$$

which is discrete and compact, and hence finite. Since $\overline{\Gamma}'$ is torsion-free, thus $\overline{\Gamma}'_{\overline{g}}$ is trivial, and thus $\overline{\Gamma}'$ acts freely on $\overline{G}/\overline{K}$. $\square$

Pick any $\overline{K}^{\max}$ a maximal compact subgroup in $\overline{G}$ that contains $p(K)$. By Claim 2.2.2, there exists a finite-index subgroup $\overline{\Gamma}'$ of $p(\Gamma)$ that acts freely on $\overline{G}/\overline{K}^{\max}$. Fix any choice of $\overline{\Gamma}'$. Let $\Gamma' = p^{-1}(\overline{\Gamma}') \cap \Gamma$.

To complete the proof of Lemma 2.2.2, it suffices to prove the following claims accordingly.

First, we deal with the total space $\Gamma' \backslash G/K$, showing that it is a finite covering of $\Gamma' \backslash G/K$.

Claim 2.2.3. (Recall that $\Gamma' := p^{-1}(\overline{\Gamma}) \cap \Gamma$.)

$\Gamma' \backslash G/K$ is a finite covering of $\Gamma \backslash G/K$.

Proof. By assumption, $\overline{\Gamma}'$ is a finite-index subgroup of $\overline{\Gamma}$ that acts freely on $\overline{G}/\overline{K}$.

Since $[p(\Gamma) : \overline{\Gamma}'] < \infty$, thus $n := [\Gamma : p^{-1}(\overline{\Gamma}') \cap \Gamma] < \infty$, By construction, $\Gamma' \backslash G/K$ is a n-sheeted covering of $\Gamma' \backslash G/K$, as desired. $\square$

Next, we consider the p-induced map

$$f : \Gamma' \backslash G/K \to \overline{\Gamma}' \backslash \overline{G}/\overline{K},$$

a (well-defined) set-theoretic double-coset map between smooth manifolds.

Claim 2.2.4. The following holds:

1. The preimage of the identity double coset in $\overline{\Gamma}' \backslash \overline{G}/\overline{K}$ under f is given by

$$F := f^{-1}(\mathrm{Id}) = (\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P) \backslash P/K.$$

2. F is compact.

3. F is connected.

Proof. To complete the proof, we check that each condition holds.

First, by set-theoretic checks, 1. holds.

Next, we show that 3. holds (i.e. F is connected), by showing that P is connected. To see this, recall that P is $p^{-1}(\overline{K}^{\max})$. Since $\overline{K}^{\max}$ is a maximal compact subgroup of $\overline{G}$, it is homotopy equivalent to $\overline{G}$. Since $\overline{G}$ is connected, thus $\overline{K}^{\max}$ is connected. Since $\overline{K}^{\max}$ is connected, thus P is connected.

Finally, we show that 2. holds (i.e. F is compact). Since K is compact, the compactness of $F(= (\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P)\backslash P/K)$ is equivalent to the compactness of $(\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P)\backslash P$. To show the compactness of $(\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P)\backslash P$, it suffices to show that it is a finite cover over $(\Gamma \cap P)\backslash P$ (which is compact by assumption). This holds, since $[p(\Gamma) : \overline{\Gamma}'] < \infty$, and thus $[\Gamma \cap P : \Gamma \cap p^{-1}(\overline{\Gamma}') \cap P] = [\Gamma \cap p^{-1}(p(\Gamma)) \cap P : \Gamma \cap p^{-1}(\overline{\Gamma}') \cap P] < \infty$. Thus, 2. holds. □

Claim 2.2.5. f is a fiber bundle map, with a compact, connected fiber given by F given in Claim 2.2.4.

Proof. By Claim 2.2.4, the set-theoretic preimage F of $\mathrm{Id} \in B$ under f is compact, connected.

To complete the proof, it remains to show that f is a fiber bundle. By the Ehresmann fibration theorem, it suffices to prove that f is a proper submersion. Since F is compact, it suffices to show that f is a surjective submersion.

To complete the proof, we prove f is a surjective submersion, which follows from the following, in logical order:

1. $p : G \to \overline{G}$ is a surjective submersion: this holds by the fact that H is a closed subgroup of G and Cartan's closed subgroup theorem

2. $\overline{p} : G/K \to \overline{G}/\overline{K}^{\max}$ (with $\overline{p}$ being the canonical coset map) is a surjective submersion: this follows from the following facts:

(a) $\overline{p}$ is smooth (as it descends from the smooth p) with respect to proper (right) actions by K (resp. $\overline{K}^{\max}$)

(b) $\overline{p}$ is equivariant with respective to the transitive left G-action (c.f. [Lee, 2012, Theorem 7.25 (equivariant rank theorem)])

3. $f : \Gamma' \backslash G/K \to \overline{\Gamma}' \backslash \overline{G}/\overline{K}^{\max}$ is a surjective submersion: this follows from the following facts:

(a) $\overline{p}$ is p-equivariant (i.e. $l_{p(g)} \circ \overline{p} = \overline{p} \circ l_g$ for all $g \in G$)

(b) Γ' (resp. $\overline{\Gamma}' = p(\Gamma')$) acts freely and properly discontinuously on G/K (resp. $\overline{G}/\overline{K}^{\max}$)

This completes the proof. $\square$

Now that we have the fiber bundle

$$f : M \twoheadrightarrow B,$$

it remains to show that M is compact if and only if B is compact. This is trivial by the compacness of F.

Since Claims 1-4 form the statement of Lemma 2.2.2, this completes the proof of Lemma 2.2.2.

2.2.2 A Technical Caveat of Lemma 2.2.2 and a Related Example

In the proof of Lemma 2.2.2, we notice that $\Gamma \cap P$ being a cocompact lattice in P is equivalent to F (:= the preimage of the identity double coset, which is diffeomorphic the preimage of any point in the base) being compact. The only place where the requirement of $\Gamma \cap P$ being a cocompact lattice in P is used, is to prove that the constructed smooth map $f : M' \to B$ is actually a fiber bundle. More specifically, the proof appeals to the Ehresmann fibration theorem to justify that f is actually a fiber bundle, and this requires that f is proper, which is where the compactness of $(\Gamma \cap P)\backslash P$ is used.

We believe this is not a hard requirement. In case f have non-compact preimage, if we can find other ways (e.g. directly by definition) to prove that f is a fiber bundle, we can extend the statement to that case. For $n \in \mathbb{Z}^+$, we consider the case of $\widetilde{Sp(2n;\mathbb{R})}$ (a simply-connected, non-compact Lie group), constructing the fibering map (following through the proof of Lemma 2.2.2) that makes it a $(\mathbb{R} \times SU(n))$-bundle over $PSp(2n;\mathbb{R})/PU(n)$. (In the case $n = 1$, this is the familiar picture that $\widetilde{SL(2;\mathbb{R})}$ being a line bundle over $\mathbb{H}^2$).

Note that $\widetilde{Sp(2n;\mathbb{R})} = \Gamma\backslash G/K$ where $(G, K, \Gamma) = (\widetilde{Sp(2n;\mathbb{R})}, \{e_G\}, \{e_G\})$. Consider the following is a short exact sequence of $\widetilde{SL(2;\mathbb{R})}$:

$$1 \to H \to G \xrightarrow{p} \overline{G} \to 1,$$

where

1. $H = Z(\widetilde{Sp(2n;\mathbb{R})}$ $(\cong \mathbb{Z})$
2. $\overline{G} = PSp(2n;\mathbb{R})$.

Take $\overline{K}^{\max} = PU(n)$. Then, $P = p^{-1}(\overline{K}^{\max})$ is the universal cover of $PU(n)$, which is as a smooth manifold $\mathbb{R} \times SU(n)$.

Without appealing to the Ehresmann fibration theorem, one can show (directly from definition) that f is a fiber bundle. This gives the line bundle structure

$$F \hookrightarrow \widetilde{G} \overset{f}{\twoheadrightarrow} B,$$

where as smooth manifolds,

- the fiber is $F = P = p^{-1}(PU(n)) = \mathbb{R} \times SU(n)$
- the base $B = PSp(2n;\mathbb{R})/PU(n)$,

as anticipated.

2.3 The Structure Theorem for Closed Locally Homogeneous Riemannian Manifolds

In this section, we state and prove the structure theorem for closed $\Gamma\backslash G/K$, which establishes a nice fibering structure for a finite cover of such manifolds. Recall that this means that Γ is a cocompact lattice in G that whose left action on G/K is free.

Before we state the theorem, we give the following definition.

Definition 2.3.1. We say that a group G is X*-by-*Y if there exists a short exact sequence

$$1 \to N \to G \to Q \to 1$$

where N satisfies property X and Q satisfies property Y. For example, G is said to be solvable-by-compact if there exists a short exact sequence as above where N is solvable and Q is compact.

Remark 2.3.1. The binary operation given by "-by-" need not be associative, i.e. a (X-by-Y)-by-Z group need a not be a X-by-(Y-by-Z) group. However, in the special case when G is the kernel of the composition of a sequence of surjective group homomorphisms, i.e.

$$G = \ker(H \overset{p_1}{\twoheadrightarrow} H_1 \overset{p_2}{\twoheadrightarrow} H_2 \overset{p_3}{\twoheadrightarrow} H_3),$$

where $\ker(p_1)$ (resp. $\ker(p_2)$, $\ker(p_3)$) satisfies property X (resp. Y, Z), then $G = \ker(p_3 \circ p_2 \circ p_1) = \ker(p_3 \circ (p_2 \circ p_1)) = \ker((p_3 \circ p_2) \circ p_1)$, and thus the both following short exact sequences hold:

1. $1 \to \ker(p_2 \circ p_1) \to G \to \ker(p_3) \to 1$, which implies that G is (X-by-Y)-by-Z.

2. $1 \to \ker(p_1) \to G \to \ker(p_3 \circ p_2) \to 1$, which implies that G is X-by-(Y-by-Z).

In this case, since dropping the parenthesis does not introduce ambiguity, we would simply call G a *X-by-Y-by-Z* group.

We are now ready to state the theorem.

Theorem 3 (Structure theorem 1 for closed, locally homogeneous Riemannian manifolds)**.** Let M be (the underlying smooth manifold of) a closed locally homogeneous Riemannian manifold. Then M admits a finite cover M' such that there exists a smooth fiber bundle

$$F \to M' \to B$$

where up to diffeomorphism:

1. F is a closed locally homogeneous Riemannian manifold $\Gamma\backslash G/K$ given by a (solvable-by-compact-by-abelian)-by-compact Lie group G.

2. B is a closed locally symmetric space of non-compact type.

We call such M' as a *fibered cover* of M.

Remark 2.3.2. Theorem 3 is likely known to experts, but I could not find it in this form in the literature. A similar result (in a more general context, with a longer proof and more indirect specification of the fiber to take care of higher generality) was proved by van Limbeek in van Limbeek [2014].

2.3.1 Proof of the Structure Theorem (Theorem 3)

Our goal in this section is to prove Theorem 3 (the structure theorem for closed locally homogeneous Riemannian manifolds).

By Lemma 2.1.3, a closed locally Riemannian manifold M has a finite cover that is $\Gamma\backslash G/K$ where G is connected, Γ is a cocompact lattice in G, and K a compact subgroup in G. Thus, since Theorem 3 is a statement that holds up to finite covers, we can reduce our consideration to the case when M is a double coset manifold $\Gamma\backslash G/K$ where G is a connected Lie group, Γ is a cocompact lattice in G, and K a compact subgroup in G.

We would like to use Lemma 2.2.2 to prove Theorem 3. To do so, we establish Lemma 2.3.1, a nice splitting of a connected Lie group.

Lemma 2.3.1. Let G be a connected Lie group. Then there is a short exact sequence of connected Lie groups

$$1 \to H \to G \xrightarrow{p} \overline{G} \to 1$$

where

1. The sequence behaves well with lattices, i.e.

   (a) H is lattice hereditary (i.e. every (cocompact) lattice Γ in G yields a (cocompact) lattice $\Gamma \cap H$ as a lattice in H)

   (b) Every (cocompact) lattice Γ in G yields a (cocompact) lattice $p(\Gamma)$ in $\overline{G}$

2. H is solvable-by-compact-by-abelian

3. $\overline{G}$ is a semisimple Lie group without compact factors and without center.

Proof. We will define a sequence of surjective Lie group homomorphisms.

$$p : G =: G_0 \overset{p_1}{\twoheadrightarrow} G_1 \overset{p_2}{\twoheadrightarrow} G_2 \overset{p_3}{\twoheadrightarrow} G_3 =: \overline{G}$$

Let G^{sol} be a solvradical (maximal connected solvable normal Lie subgroup) of G. By Raghunathan [1972], G^{sol} exists, and is unique up to conjugation. Fix a choice of G^{sol}. A classical result (Levi decomposition) states that $G_1 := G/G^{sol}$ is semisimple. Let $p_1 : G \to G_1$ be the canonical projection map, so that $\ker(p_1) = G^{sol}$ is solvable.

Let G_1^K be a compact radical (maximal connected compact normal Lie subgroup) of G_1. Then, $G_2 := G_1/G_1^K$ is a semisimple Lie group without compact factor. Let $p_2 : G_1 \to G_2$ be the canonical projection map, so that $\ker(p_2) = G_1^K$ is compact.

Let G_2^Z be the center of G_2. By [Knapp, 1996, § 3, Proposition 6.30], $G_3 := G_2/G_2^Z$ is a semisimple Lie group without center and without compact factors. Let $p_3 : G_2 \to G_3$ be the canonical projection map, so that $\ker(p_3) = G_2^Z$ is abelian.

Now, let $p := p_3 \circ p_2 \circ p_1$, and $H := \ker(p)$. Then, by construction (c.f. Remark 2.3.1), $\ker(p)$ is solvable-by-compact-by-abelian. Thus, combining the above, this proves 1 and 3.

It remains to prove 1.1, i.e. $H = \ker(p)$ is lattice hereditary. (By Proposition 2.2.1, 1.2 is equivalent to 1.2.)

To see that H is lattice hereditary, let Λ be any lattice in G. By the Bieberbach-Auslander-Wang Theorem[Baues and Kamishima, 2018, Proposition 2.5], $G^N := \ker(p_2 \circ p_1)$ is lattice hereditary in G. Thus, $\Lambda \cap G^N$ is a lattice in G^N. By Proposition 2.2.1, $\Lambda_2 := p_2(p_1(\Lambda)) = \Lambda/(\Lambda \cap G^N)$ is a lattice in G_2. By [Raghunathan, 1972, Proposition 5.18], $\ker(p_3) = G_2^Z$ is lattice hereditary in G_2. Thus, $\Lambda_2 \cap G_2^Z$ is a lattice in G_2^Z. By Proposition 2.2.1, $p(\Lambda) = p_3(\Lambda_2)$ is a lattice in G_3. By Proposition 2.2.1 again, Λ is a lattice in H, as desired.

This completes the proof. □

From now on, we assume that $M = \Gamma \backslash G/K$, where G is a connected Lie group, and

- K is a compact subgroup of G
- Γ is a cocompact lattice of G

Remark 2.3.3. It might be tempting to use the Levi decomposition to justify Lemma 2.3.1 directly. Unfortunately, a solvradical need not be lattice hereditary, even when Γ is cocompact. To see this, consider $G = \mathbb{R} \times SO(3)$, so that $G^{sol} = \mathbb{R}$. Let A be an infinite order element in $SO(3)$ and $\Gamma := \rho(\mathbb{Z})$ where

$$\rho : \mathbb{Z} \to G$$
$$n \mapsto (n, A^n).$$

Then Γ is a cocompact lattice in G where $\Gamma \cap G^{sol} = \{0\}$, which is clearly not cocompact in G^{sol}.

We are now ready to use Lemma 2.2.2 to prove Theorem 3.

Let $(p, H, \overline{G})$ be the short exact sequence of Lie groups as in Lemma 2.3.1.

To complete the prove of Theorem 3, we first show that $(p, H, \overline{G})$ is (Γ, K)-good, so that we can apply Lemma 2.2.2.

Claim 2.3.1. The sequence $(p, H, \overline{G})$ is (Γ, K)-good.

Proof. We first prove that $(p, H, \overline{G})$ is Γ-good. Since H is lattice hereditary, by Lemma 2.2.1, it suffices to prove that $\overline{G}$ is linear. This holds, since $\overline{G}$ is a centerless, thus its adjoint map is injective, and thus it is linear.

To complete the proof, it remains to show that $(p^{-1}(p(\Gamma)) \cap P)\backslash P$ is compact. Since K is compact, this follows from the compactness of $(\Gamma' \cap P)\backslash P/K$, which follows from the compactness of $\Gamma'\backslash G/K$, which follows from the compactness of $\Gamma\backslash G/K$. □

Since $(p, H, \overline{G})$ is (Γ, K)-good, by Lemma 2.2.2, there exists a finite-index subgroup $\overline{\Gamma}'$ of $p(\Gamma)$, such that with $\Gamma' = p^{-1}(\overline{\Gamma}') \cap \Gamma$, the p-induced map

$$f : \Gamma'\backslash G/K \to \overline{\Gamma}'\backslash\overline{G}/\overline{K}^{\max}$$

is a fiber bundle map, with compact, connected fiber. Further,

1. The base is compact if and only if the total space is compact (if and only if the lattice Γ is cocompact in G).

2. The fiber is given by

$$F := (\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P)\backslash P/K.$$

where $P := p^{-1}(\overline{K})$, and is compact and connected.

3. $\Gamma'\backslash G/K$ is a finite covering of $\Gamma\backslash G/K$.

With this in mind, we pick any finite-index subgroup $\overline{\Gamma}'$ of $p(\Gamma)$ that satisfies the aforementioned conditions. To complete the proof, it remains to justify the following claims.

First, about the nature of the base B.

Claim 2.3.2.

$$B := \overline{\Gamma}'\backslash\overline{G}/\overline{K}^{\max}$$

is a (closed) locally symmetric space of non-compact type.

Proof. By construction, $\overline{G}/\overline{K}^{\max}$ is a symmetric space of non-compact type, and by Lemma 2.3.1, $\overline{\Gamma}'$ is a (cocompact) lattice that acts freely on $\overline{G}/\overline{K}^{\max}$. □

Next, we specify a finite-sheeted cover of M that covers B, that will serve as the total space in Theorem 3.

Claim 2.3.3. Let $\Gamma' := p^{-1}(\overline{\Gamma}') \cap \Gamma$. The double coset manifold $M' := \Gamma'\backslash G/K$ is a finite cover of M.

Proof. By construction, Γ' is a finite index subgroup of Γ, thus $\Gamma'\backslash G/K$ is a finite cover of $\Gamma\backslash G/K = M$. □

Finally, we show that the fiber of f is presented as a double coset manifold as in the statement of Theorem 3.

Claim 2.3.4. Let F be the fiber of f. Then F is diffeomorphic to $(\Gamma' \cap P)\backslash P/K$ where P is a connected, (solvable-by-compact-by-abelian)-by-compact subgroup of G. (The connectedness of P immediately implies the connectedness of F.)

Proof. To complete the proof, note that the fiber F is diffeomorphic to the preimage at the identity double coset point in the base B, i.e.

$$F = f^{-1}(\mathrm{Id}) = (\Gamma \cap p^{-1}(\overline{\Gamma}') \cap P)\backslash P/K. = (\Gamma' \cap P)\backslash P/K.$$

where $P = p^{-1}(\overline{K}^{\max})$.

Note that P is the total space of the fiber bundle given by restricting the base of the fiber bundle (given by the short exact sequence of the underlying Lie groups)

$$H \to G \to \overline{G}$$

to $\overline{K}^{\max}$. Thus,

1. there exists a short exact sequence of Lie groups

$$1 \to H \to P \to \overline{K}^{\max} \to 1.$$

   By Lemma 2.3.1, H is solvable-by-compact-by-abelian. Since $\overline{K}^{\max}$ is compact, thus P is (solvable-by-compact-by-abelian)-by-compact.

2. Since $\overline{G}$ is homotopy equivalent to $\overline{K}^{\max}$, thus P is homotopy equivalent to G, and hence is connected.

This completes the proof. □

Since Claims 1-4 form the statement of Theorem 3, this completes the proof of Theorem 3.

CHAPTER 3

SIMPLICIAL VOLUME OF CLOSED LOCALLY HOMOGENEOUS RIEMANNIAN MANIFOLDS

3.1 Proof of Theorem 1

In the previous chapter, we proved Theorem 3. Now, we use it to prove Theorem 1.

Remark 3.1.1. Since $\|M\|$ is a homotopy invariant of M, to answer whether or not $\|M\| = 0$, it suffices to analyze its homotopy type. In particular, it suffice to analyze its diffeomorphism type. In Lemma 2.1.2 (resp. Remark 2.1.2), we showed that a locally homomgeneous Riemannian manifold (resp. closed locally homogeneous Riemannian manifold) M is diffeomorphic to a double coset manifold $\Gamma\backslash G/K$ where

- G is a Lie group with finitely many connected components.
- K is a compact subgroup of G.
- Γ is a discrete subgroup (resp. cocompact lattice) of G that acts freely on G/K.

Conversely, for any (G, K, Γ) that satisfies the above conditions, since K is compact, by basic Lie theory, $\Gamma\backslash G/K$ can be equipped with a locally homogeneous Riemannian metric. This allows us to treat locally homogeneous Riemannian manifolds as double coset manifolds interchangably. Since we only focus on the case when M is closed, thus Γ in this chapter is always a cocompact lattice.

3.1.1 Theorem 3 implies Theorem 1

Our goal in this subsection is to prove Theorem 1 using Theorem 3. An immediate corollary of Theorem 3 is the following, which we will use to prove Theorem 1.

Corollary 3.1.1. Let M be a closed locally homogeneous Riemannian manifold. Then M admits a smooth finite-sheeted cover M' such that there exists a smooth fiber bundle

$$F \to M' \to B$$

where:

1. F is a closed connected (possibly trivial) manifold with amenable $\pi_1(F)$. (Recall that a locally compact topological group G is said to be *amenable* if G has a fixed point for every continuous affine action on a nonempty G-space that is a compact, convex subset of a locally convex topological vector space.)

2. B is a closed locally symmetric space of non-compact type.

We call such M' as a *fibered cover* of M.

Proof. We use the notations in Theorem 3. The only thing to be proven is that $\pi_1(F)$ is amenable.

To do so, we recall the following:

Proposition 3.1.1 ([Morris, 2001, §12.1,§12.2])**.** 1. The following types of groups are amenable: compact, solvable.

2. The amenable property is preserved under: group extension, quotients, taking closed subgroups, taking finite index subgroups.

Continuing with the proof of Corollary 3.1.1, since G is (solvable-by-compact-by-abelian)-by-compact, by Proposition 3.1.1, G is amenable.

To prove that $\pi_1(F)$ is amenable, note that basic covering space theory yields the short exact sequence of groups

$$1 \to \pi_1(G/K) \to \pi_1(F) \to \Gamma \to 1.$$

Since Γ is a closed subgroup of G, by Proposition 3.1.1, Γ is amenable. By Proposition 3.1.1, since the amenable property is closed under group extension, it suffices to show that $\pi_1(G/K)$ is abelian. This holds, since the long exact sequence of fibration for $K \to G \to G/K$ yields the short exact sequence of groups

$$1 \to \pi_1(K) \to \pi_1(G) \to \pi_1(G/K) \to 1,$$

By elementary algebraic topology, the fundamental group of any topological group is abelian. Thus, in particular, $\pi_1(G)$ is abelian, and thus $\pi_1(G/K)$ is also abelian. Since $\pi_1(G/K)$ is abelian, by Proposition 3.1.1, it is amenable, as required. □

Now, still assuming Theorem 3, which as we just showed implies Corollary 3.1.1 as a special case, we prove the following, which contains Theorem 1.

Lemma 3.1.1. Let M be a closed locally homogeneous Riemannian manifold, and M' a fibered cover (as in Theorem 3) of M. Then the following are equivalent:

1. M' is a closed locally symmetric space of non-compact type.
2. M is a closed locally symmetric space of non-compact type.
3. $\|M'\| > 0$.
4. $\|M\| > 0$.

Proof. Since M' is a finite cover of M, the following holds:

1. (1) $\Leftrightarrow$ (2): This is an immediate consequence of the following facts:
    (a) Their universal covers agree, i.e. $\widetilde{M'} = \widetilde{M}$.
    (b) The following are equivalent for a Riemannian manifold N:
        i. N is a locally symmetric space of non-compact type.

ii. its universal cover $\widetilde{N}$ is a symmetric space of non-compact type.

(c) The compactness property is preserved under finite covering.

2. (3) $\Leftrightarrow$ (4): This is an immediate consequence of the following remark:

Remark 3.1.2 ([Gromov, 1982, § 0.2])**.** Let $N \twoheadrightarrow M$ be a finite d-sheeted covering. Then $\|N\| = d\|M\|$. In particular, $\|N\| = 0 \Leftrightarrow \|M\| = 0$.

As stated in the introduction, (1) $\Rightarrow$ (3) was proven in [Lafont and Schmidt, 2006, Main Theorem]. It remains to prove (3) $\Rightarrow$ (1).

By Corollary 3.1.1, there is a fiber bundle

$$F \to M' \to B$$

where:

- $\pi_1(F)$ is amenable.
- B is a closed locally symmetric space of non-compact type.

According to [Lück, 2002, Exercise 14.13], if there is a fiber bundle

$$F \hookrightarrow E \twoheadrightarrow B$$

with $\pi_1(F)$ amenable and F nontrivial (i.e. F connected with $\dim(F) > 0$), then $\|E\| = 0$.

This implies that if $\|M\| > 0$, F must be a point, i.e. M' is a closed locally symmetric space of non-compact type. This completes the proof of Lemma 3.1.1. $\square$

CHAPTER 4

MINIMAL VOLUME OF COMPACT HOMOGENEOUS MANIFOLDS AND THEIR FINITE-INDEX QUOTIENTS

In this chapter, we prove Theorem 2, make some remarks on the full case $\Gamma\backslash G/K$, and give some examples to illustrate the theorem.

Here, G is a connected compact Lie group, and K a closed subgroup in G.

Remark 4.0.1. By assumption, K is automatically compact, and thus $|\pi_0(K)| < \infty$. However, K need not be connected. For example, all finite subgroups of G are examples of K.

4.1 Proof of Theorem 2

Most of the implications are known:

1. (iv) $\Rightarrow$ (v) trivially.

2. By Wang [1949], we have (ii) $\Leftrightarrow$ (iii).

3. By [Paternain and Petean, 2003, Page 17, top] or Cheeger and Gromov [1986], we have (v) $\Rightarrow$ (i).

4. As stated in Gromov [1982], we have (i) $\Rightarrow$ (iii).

   Remark 4.1.1. Since this is only stated, and the proof is not spelled out anywhere in the literature, we decide to spell it out in subsection 4.1.1.

Thus, to prove the theorem, it suffices to prove (ii) $\Rightarrow$ (iv), which we do in subsection 4.1.2.

4.1.1 An Argument for (i) ⇒ (iii)

The desired implication (i.e. (i) ⇒ (iii)) is a direct consequence of the following proposition.

Proposition 4.1.1. Let n be any positive integer. Then there is a constant $c_n \in \mathbb{R}^+$, such that for any closed smooth manifold M of dimension $2n$, we have $\text{Minvol}(M) \geq c_n \cdot \chi(M)$

Proof. To complete the proof, we recall (a consequence of) the classical Chern-Gauss-Bonnet theorem.

Before we state the proposition, we state a definition.

Definition 4.1.1 (Pfaffian)**.** Let n be a positive integer.

1. Let A be a skew-symmetric $2n$-by-$2n$ matrix. The *Pfaffian* of A is defined to be

$$\text{Pf}(A) = \frac{1}{2^n \cdot n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}$$

2. Let M be a smooth manifold of dimension-$2n$, and α a 2-form on it. Then, under any choice of local coordinates, we have $\alpha = \sum_{1 \leq i < j \leq 2n} \alpha_{i,j} dx_i \wedge dx_j = \sum_{1 \leq i,j \leq 2n} \alpha_{i,j} dx_i \otimes dx_j$, where $\alpha_{i,j}$ is a $\mathbb{R}$-valued smooth function on the corresponding patch, and $\alpha_{j,i} = -\alpha_{i,j}$ (i.e. $[\alpha_{i,j}]_{1 \leq i,j \leq 2n}$) is a $\mathfrak{so}(2n)$-valued smooth function. The *Pfaffian* of α is defined to be

$$\text{Pf}(\alpha) = \text{Pf}(A) dx_1 \wedge \ldots \wedge dx_{2n}.$$

   One can check that this definition is well-defined.

Now, we state the proposition. For our purpose, we present it using coordinate; nevertheless, note that there exists a more elegant, intrinsic (coordinate-free) presentation, which can be found in standard charactersitic class texts.

Proposition 4.1.2. For any positive integer n, there exists a $v_n \in \mathbb{R}^+$ such that for any (M, g) be a closed Riemannian manifold of dimension $2n$, we have

$$\int_M \mathrm{Pf}(\omega_g) = v_n \cdot \chi(M),$$

where ω_g is the corresponding sectional curvature 2-form of g.

We are now ready to complete the proof. To do so, let g be any complete metric such that $\left|K_g\right| \leq 1$, and let $\omega = \omega_g$. Suppose $\{x_1, \ldots, x_{2n}\}$ is a g-orthonormal frame on M. Then in local coordinates, we have

$$\omega = \sum_{1 \leq i < j \leq 2n} a_{ij} dx_i \wedge dx_j,$$

where $\left|a_{ij}\right| \leq 1$.

Thus, with $c_n := \frac{(2n)!}{n! \cdot 2^n}$, we have

$$\begin{aligned}
c_n \cdot \mathrm{vol}_g(M) &= \sum_{\substack{s \in S_{2n} \\ s(1)<s(3)<\ldots<s(2n-1)}} \mathrm{vol}_g(M) \\
&= \sum_{\substack{s \in S_{2n} \\ s(1)<s(3)<\ldots<s(2n-1)}} \int_M dx_1 \wedge \ldots \wedge dx_{2n} \\
&\geq \sum_{\substack{s \in S_{2n} \\ s(1)<s(3)<\ldots<s(2n-1)}} \int_M \left(\prod_{j=1}^{n} \left| a_{s(2j-1),s(2j)} \right| \right) dx_1 \wedge \ldots \wedge dx_{2n} \\
&\geq \left| \int_M \sum_{\substack{s \in S_{2n} \\ s(1)<s(3)<\ldots<s(2n-1)}} \left(\prod_{j=1}^{n} a_{s(2j-1),s(2j)} \right) dx_1 \wedge \ldots \wedge dx_{2n} \right| \\
&= \left| \int_M \mathrm{Pf}(\omega) \right| \\
&\overset{(\because \text{Proposition 4.1.2})}{=} v_n \cdot |\chi(M)|.
\end{aligned}$$

To summarize, let $C_n := \frac{v_n}{c_n}$ (> 0). Then, we have

$$\mathrm{vol}_g(M) \geq C_n \cdot |\chi(M)|.$$

Since g is arbitrary, take infimum across all g such that $\left|K_g\right| \leq 1$, we have

$$\mathrm{Minvol}(M) \geq C_n \cdot |\chi(M)|,$$

as required. □

Remark 4.1.2 (Case $n = 1$)**.** When $n = 1$, we have $(v_n, c_n, C_n) = (4\pi, 2, 2\pi)$. In particular, the desired implication simply reads

$$\mathrm{Minvol}(M) \geq 2\pi \cdot |\chi(M)|.$$

Moreover, equality holds. We spell out the case when M is oriented:

1. In case $\chi(M) \neq 0$ ($\Leftrightarrow$ M is a closed surface of genus-$g \neq 1$), the minimal volume is realized by the respective constant-curvature-± 1 metric.

2. In case $\chi(M) = 0$ ($\Leftrightarrow$ M is a torus), M admits a flat metric and thus minimal volume is never realized.

4.1.2 Proof of (ii) $\Rightarrow$ (iv)

Suppose $\mathrm{rank}(G) > \mathrm{rank}(K)$, we want to show that G/K admits a locally-free S^1-action given by the left-translation of a closed circle subgroup in G.

Before delving into the proof, we make a simple remark.

Remark 4.1.3. Suppose S is a closed circle subgroup in G, and let $\rho : S \to \mathrm{Diff}(G/K)$ denote the action by left-translation (by design, it is smooth). The stabilizer subgroup of $gK \in G/K$ is given by

$$S_{gK} := S \cap gKg^{-1}.$$

In this context, ρ is locally-free precisely when there exists a open neighborhood U of the identity e_S ($= e_G$) in S such that

$$U \cap S_{gK} = U \cap gKg^{-1} = \{e_G\}$$

for all $g \in G$.

By the Remark 4.1.3, to prove the theorem, it remains to prove the following lemma.

Lemma 4.1.1. Let G be a connected compact Lie group, and K a closed subgroup in it, such that $\mathrm{rank}(G) > \mathrm{rank}(K)$. Then there exists a closed circle subgroup S in G with an open neighborhood U of the identity e_S $(= e_G)$ in S where

$$U \cap gKg^{-1} = \{e_G\}$$

for all $g \in G$.

Proof. To complete the proof of Lemma 4.1.1, we pick

1. a $v \notin [\mathfrak{g}, \mathfrak{k}]$ such that its one-parameter subgroup $S^1_v := \exp(\mathbb{R}\{v\})$ is closed. (By Claim 4.1.3, such v exists).

2. a normal neighborhood U (i.e. an open $U \ni e_G$ such that $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism $\exp : V \to U$) that avoids the non-identity components of $\bigcup_{g\in G} gKg^{-1}$ (such U exists, as exp is a local diffeomorphism). By Claim 4.1.1, for such U, we have $U \cap S^1_v \cap \bigcup_{g\in G} gKg^{-1} = \{e_G\}$.

With S^1_v as the desired closed circle subgroup (with $U \cap S^1_v$ as the required open neighborhood), we are done proving Lemma 4.1.1.

To complete the proof of Lemma 4.1.1, it remains to prove Claim 4.1.1 and Claim 4.1.3.

Claim 4.1.1. Let U be a normal neighborhood that avoids all non-identity components of $\bigcup_{g\in G} gKg^{-1}$. Then, if $v \notin [\mathfrak{g}, \mathfrak{k}]$, we have $U \cap S^1_v \cap \bigcup_{g\in G} gKg^{-1} = \{e_G\}$.

Proof. Since U avoids all non-identity components of $\bigcup_{g\in G} gKg^{-1}$, thus

$$U \cap S^1_v \cap \bigcup_{g\in G} gKg^{-1} = U \cap S^1_v \cap \left(\bigcup_{g\in G} gKg^{-1}\right)^0.$$

Thus, to prove the claim, it suffices to prove that the RHS above is $\{e_G\}$. To show this, the key is to note that

$$\left(\bigcup_{g\in G} gKg^{-1}\right)^0 = \bigcup_{g\in G} gK^0g^{-1} = \bigcup_{X\in\mathfrak{g}} \exp(\mathrm{ad}(X)(\mathfrak{k})) \subseteq \exp([\mathfrak{g},\mathfrak{k}]).$$

Thus, to prove the claim, it suffices to prove that $U \cap S_v \cap \exp([\mathfrak{g},\mathfrak{k}]) = \{e_G\}$. This follows from the following (recall that $\exp|_V$ and $V = \exp^{-1}(U)$):

$$\begin{aligned}
U \cap S_v^1 \cap \exp([\mathfrak{g},\mathfrak{k}]) &= (U \cap S_v^1) \cap (U \cap \exp([\mathfrak{g},\mathfrak{k}])) \\
&= \exp(\exp^{-1}(U \cap S_v^1) \cap \exp^{-1}(U \cap \exp([\mathfrak{g},\mathfrak{k}])) \\
&= \exp((V \cap \mathbb{R}v) \cap (V \cap [\mathfrak{g},\mathfrak{k}])) \\
&= \exp(\{0\}) \\
&= \{e_G\}.
\end{aligned}$$

Thus, we are done proving Claim 4.1.1. □

Before proving Claim 4.1.3, we prove Claim 4.1.2.

Claim 4.1.2. Let dim denote the dimension of the underlying (real) vector space. Then, $\dim([\mathfrak{g},\mathfrak{k}]) < \dim(\mathfrak{g})$.

Proof. Since $\bigcup_{g\in G} gKg^{-1} = \bigcup_{g\in G} gT_K g^{-1}$, thus $[\mathfrak{g}, \mathfrak{k}] = [\mathfrak{g}, \mathfrak{t}_{\mathfrak{k}}]$. Thus,

$$\begin{aligned}
\dim([\mathfrak{g}, \mathfrak{k}]) &= \dim([\mathfrak{g}, \mathfrak{t}_{\mathfrak{k}}]) \\
&= \dim(\mathfrak{g}/\mathfrak{n}_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{k}})) + \dim(\mathfrak{t}_{\mathfrak{k}}) \\
&= \dim(\mathfrak{g}) - \dim(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{k}})) + \dim(\mathfrak{t}_{\mathfrak{k}}) \\
&\le \dim(\mathfrak{g}) - \dim(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{g}})) + \dim(\mathfrak{t}_{\mathfrak{k}}) \\
&= \dim(\mathfrak{g}) - \dim(\mathfrak{t}_{\mathfrak{g}}) + \dim(\mathfrak{t}_{\mathfrak{k}}) \\
&< \dim(\mathfrak{g}),
\end{aligned}$$

where the last equality follows from the fact that $[N_G(T_G) : T_G] < \infty$, and the last inequality comes from the fact that $\dim(\mathfrak{t}_{\mathfrak{g}}) = \text{rank}(G)$ and $\dim(\mathfrak{t}_{\mathfrak{k}}) = \text{rank}(K)$, and the assumption that $\text{rank}(G) > \text{rank}(K)$. □

With Claim 4.1.2 in mind, we prove Claim 4.1.3.

Claim 4.1.3. There exists $v \notin [\mathfrak{g}, \mathfrak{k}]$ such that its one-parameter S_v is closed.

Proof. By Claim 4.1.2, $\dim([\mathfrak{g}, \mathfrak{k}]) < \dim(\mathfrak{g})$. Thus, there exists a $w \in \mathfrak{g}$ such that $w \notin [\mathfrak{g}, \mathfrak{k}]$. Pick any such w. If S^1_w is closed, we are done.

Otherwise, consider the closure $T_w := \overline{S^1_w}$ of S^1_w. Note that T_w is a closed torus subgroup in G, this follows from the following:

1. Since S^1_w is abelian, thus T_w is abelian.

2. Since T_w is an abelian closed subgroup in the compact G, it is compact and abelian, and thus is a closed torus subgroup.

Let $\mathfrak{t}_w$ be the Lie algebra of the torus T_w (it is a Lie subalgbera in $\mathfrak{g}$). We note the following:

1. $\mathfrak{t}_w \subseteq [\mathfrak{g}, \mathfrak{k}]$.

2. The set of $v \in \mathfrak{t}_w$ with closed S_v is dense. (This is a general fact for torus, which can be easily seen: in real coodinates, the vectors $v \in \mathbb{R}^n \cong \text{Lie}(T^n)$ are precisely those whose coordinates are rationally dependent, and the set of rationally dependent vectors in $\mathbb{R}^n$ is dense in $\mathbb{R}^n$.)

Thus, there exists a vector $v \in \mathfrak{t}_w$ which is not in $[\mathfrak{g}, \mathfrak{k}]$ where S_v is closed, as required. □

We are done constructing $S = S_v^1$, and thus completed the proof. □

4.2 A Few Remarks on Theorem 2

First, we consider extending Theorem 2 to the case of all $\Gamma \backslash G/K$ when G is connected, compact (equivalently, the finite-index quotients of G/K).

Remark 4.2.1. 1. When $\text{rank}(G) = \text{rank}(K)$, we have $\chi(G/K) = 0$, and thus $\chi(\Gamma \backslash G/K) > 0$, and thus $\text{Minvol}(\Gamma \backslash G/K) > 0$.

2. When $\text{rank}(G) > \text{rank}(K)$, by Theorem 2, there is a closed circle subgroup S in G whose left action on G/K is locally-free. It would be great if this action could descend to at least, a well-defined action on $\Gamma \backslash G/K$.

   In fact, if there exists a closed circle subgroup S in the $N_G(\Gamma)$ (the normalizer of Γ) that acts locally-freely on G/K, then $(\Gamma \cap S) \backslash S$ acts locally-freely on $\Gamma \backslash G/K$. Note that $\Gamma \cap S$ is finite, and the quotient group $(\Gamma \cap S) \backslash S$ is also a circle.

   (a) Indeed, if Γ is abelian, then the circle generated by any of its element satisfies this condition. This is the case for lens spaces, which is illustrated in Remark 4.3.2.

   (b) If Γ is not abelian, we are not sure what happens.

In light of the following, we have completely resolved the "Minvol-0" problem of closed locally homogeneous Riemannian manifold in the case of finite $\pi_1(M)$, up to finite covering.

Remark 4.2.2. Let M be a closed locally homogeneous Riemannian manifold. Let $\widetilde{M}$ be the universal cover of M equipped with the pullback metric, $I(\widetilde{M})$ as its isometry group, and $I(\widetilde{M})_0$ as the identity connected componetn of $I(\widetilde{M})$. Then, the following are equivalent:

1. $\widetilde{M}$ is compact.

2. $I(\widetilde{M})$ is compact.

3. $I(\widetilde{M})_0$ is compact.

4. $|\pi_1(M)| < \infty$.

Finally, we make a remark about fixed points. Theorem 2 tells us when $\text{rank}(G) > \text{rank}(K)$, G/K admits a free circle action given by the left translation of a closed circle in G. In the case $\text{rank}(G) = \text{rank}(K)$, not only such free circle action does not exist, in fact, every left circle action has a global fixed point. (While this fixed point property could be proven directly (even if we do not restrict our consideration to the left $SO(2)$-actions) using Euler characteristic arguments, we prove this claim directly using Lie theory.)

Remark 4.2.3. When $\text{rank}(G) = \text{rank}(K)$, the left translation action of every closed circle subgroup S in G has a global fixed point.

Proof. Let S be any closed circle subgroup in G. A point $gK \in S$ if and only if the stabilizer subgroup S_{gK} lies in S, i.e. $S \cap gKg^{-1} \leq S$, which holds if and only if $S \leq gKg^{-1}$.

Thus, to prove Remark 4.2.3, it suffices to prove that there exists $g \in G$ such that $S \leq gKg^{-1}$. To see this, consider any maximal torus T in G that contains S. Then, for any maximal torus T_K of K, since $\text{rank}(K) = \text{rank}(G)$ by assumption, T_K is also a maximal torus of G. Thus, there exists $g \in G$ such that $T = gT_Kg^{-1}$. Thus, for such g, we have

$$S \leq T = gT_Kg^{-1} \leq gKg^{-1},$$

as desired. □

4.3 Examples

4.3.1 The Family of Spheres

In this subsection, we give a constructive illustration of Theorem 2 and Remark 4.2.3 via the family of spheres $S^n =$ (i.e. the case $(G, K) = (G_n, K_n) = (SO(n+1), SO(1) \times SO(n))$).

Let $G_n = SO(n+1)$, and $K = SO(1) \times SO(n)$. For any n, S^n admits an effective (linear) circle action, one that comes from the left translation of a closed circle subgroup of $SO(n+1)$. We consider an example that comes from the (diagonal) embedding of $SO(2)$ into a (diagonal) maximal torus, i.e.

$$\rho_n : SO(2) \to \mathrm{Diff}(G_n/K_n)$$
$$s \mapsto L_{\iota(s)},$$

where L_g is the left translation of $g \in G_n$ on G_n/K_n, and ι is given by the following:

$$\iota(s) = \begin{cases} (s, \ldots, s) \in SO(2)^m & n = 2m-1 \quad (n \text{ is odd}) \\ (1, s, \ldots, s) \in SO(1) \times SO(2)^m & n = 2m \quad (n \text{ is even}) \end{cases},$$

where $m \in \mathbb{Z}^+$.

Remark 4.3.1 (Geometric Interpretation)**.** Consider the following diffeomorphism:

$$\varphi_n : SO(n+1)/(SO(1) \times SO(n)) \to S^n$$
$$g(SO(1) \times SO(n)) \mapsto g \cdot e_1,$$

where

- $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$,

- $g\cdot$ is the (left) linear action of $SO(n+1)$ on $\mathbb{R}^{n+1}$ restricted to S^n.

Under φ_n, by viewing the circle $SO(2)$ as $U(1) \in \mathbb{C}^\times$, the action ρ_n specified above is given by the "diagonal scalar multiplication".

- When n is odd, $n = 2m-1$ for $m \in \mathbb{Z}^+$. We have

$$S^n = \{z_1, \ldots, z_m \in \mathbb{C}^m \mid \sum_{i=1}^{m} |z_i|^2 = 1\};$$

  under this, we have

$$\rho_n(s)(z_1, \ldots, z_m) = (sz_1, \ldots, sz_m).$$

- When n is even, $n = 2m$ for $m \in \mathbb{Z}^+$. We have

$$S^n = \{t, z_1, \ldots, z_m \in \mathbb{R} \times \mathbb{C}^m \mid t^2 + \sum_{i=1}^{m} |z_i|^2 = 1\};$$

  under this, we have

$$\rho_n(s)(t, z_1, \ldots, z_m) = (t, sz_1, \ldots, sz_m).$$

Thus, clearly

- When n is odd, ρ_n is free.
- When n is even, ρ_n has 2 global fixed points, given by $\{e_1, -e_1\}$.

4.3.2 On the Lens Spaces (Some Finite-index Quotients of S^{2m+1})

In Remark 4.3.2, we inspect lens spaces, a family of finite-index quotients of S^{2m+1}, via the special case when $m = 1$. We express them as double coset manifolds $\Gamma\backslash G/K$, and construct a free S^1-action on it that arises by (a left quotient of) a closed circle subgroup in G. (This example can be easily generalized to higher dimensional lens spaces, to show that they too admit free S^1-action, by replacing S with the circle subgroup that appropriately embeds in the diagonal maximal torus.)

Remark 4.3.2 (Lens Spaces in Dimension-3)**.** Let (p, q) be a pair of coprime positive integers. The (p, q)-*Lens space*, denoted as $L(p; q)$, is the quotient of S^3 (viewed as the unit sphere in $\mathbb{C}^2$) under the following $\mathbb{Z}/p\mathbb{Z}$-action:

$$(z_1, z_2) \mapsto (e^{\frac{2\pi i}{p}} \cdot z_1, e^{\frac{2\pi i q}{p}} \cdot z_2).$$

Written as a double coset manifold, we have

$$L(p; q) = \Gamma_{p,q} \backslash SO(4)/(SO(1) \times SO(3)),$$

where $\Gamma_{p,q} := \left\langle (\zeta_p, \zeta_p^q) \right\rangle$, where $\zeta_p = \begin{bmatrix} \cos(\frac{2\pi}{p}) & -\sin(\frac{2\pi}{p}) \\ \sin(\frac{2\pi}{p}) & \cos(\frac{2\pi}{p}) \end{bmatrix}$. We show that $L(p; q)$ admits free S^1-action, given by the left translation of (a quotient of) a $SO(2)$-subgroup in $SO(4)$.

To do so, we construct a closed circle subgroup S_q of $SO(4)$ that contains $\Gamma_{p,q}$, given by $S_q = \iota_q(SO(2))$, given by the following embedding into the diagonal maximal torus in $SO(4)$:

$$\iota_q : SO(2) \to SO(2) \times SO(2)$$
$$t \mapsto (t, t^q).$$

The left translation of S_q on $L(p, q)$ is has isotopy group of $\Gamma_{p,q}$ for every point $x \in L(p, q)$, which descends to a free circle action given by the left translation of $\Gamma_{p,q} \backslash S_q$ on $L(p, q)$.

CHAPTER 5

EPILOGUE

5.1 A Few Examples

Even though we have a complete criterion on the vanishing of $\|M\|$ for closed locally homogeneous Riemannian manifolds M, we are far from having a criterion on the vanishing of $\mathrm{Minvol}(M)$. In particular,

Remark. Let G be a connected compact Lie group, and K a strict closed subgroup in G (so that $\dim(K) < \dim(G)$) such that $\mathrm{rank}(K) = \mathrm{rank}(G)$. Then,

1. by Theorem 1, we have $\left\|G/K\right\| = 0$.

2. by Theorem 2, we have $\mathrm{Minvol}(G/K) > 0$.

A notable family of examples is given by $SO(2m+1)/SO(2m)$ where $m \in \mathbb{Z}^+$.

Here, we look at (the vanishing of) a few common invariants on several classes of closed locally homogeneous Riemannian manifolds. Here, we present a few examples, to illustrate the relationship between several manifold invariants. All the manifolds here are equipped with the standard smooth structures, and $n \in \mathbb{Z}^+$.

M	$\mathrm{Minvol}(M)$	$\|M\|$	$\chi(M)$
T^n	0	0	0
compact Lie groups	0	0	0
S^{2n}	> 0	0	2
S^{2n+1}	0	0	0
$\dim -(2n+1)$ hyperbolic manifolds	> 0	> 0	0
$\dim -2n$ hyperbolic manifolds	> 0	> 0	0
$(\mathbb{Z}^2 \rtimes_A \mathbb{Z})/(\mathbb{R}^2 \rtimes_A \mathbb{R})$, $A \in SL(2;\mathbb{Z})$	0	0	0
$S^2 \times H_3$ (where H_3 closed hyperbolic 3-manifold	?	0	0

Table 5.1: Table of Minvol and its lower bounds (up to a factor of dimension-dependent constant).

5.2 Open Questions

In subsection 5.2.3, we present a few open questions (and our guesses) about the minimal volume of closed $\Gamma\backslash G/K$'s, highlighting the dependencies on the answers of some questions about in Minvol(M) of general closed smooth manifolds presented in subsection 5.2.2. Before these, in subsection 5.2.1, summarize a few invariants related to Minvol that are related to our problem.

5.2.1 Literature Review

Fact 1. If M admits a locally-free circle action, then Minvol$(M) = 0$. More generally, there is a condition called polarized $\mathcal{F}$-structure (introduced in Cheeger and Gromov [1986]) which generalizes locally-free torus action, such that if M admits a polarized $\mathcal{F}$-structure, then Minvol$(M) = 0$. (For more details, see the notes by Brian Weber Weber [2000].)

Remark. A special case of a polarized $\mathcal{F}$-structure is the following data:

- a $k \in \mathbb{Z}^+$
- an *atlas of locally-free, good T^k-action*, i.e. $\{(U_i, \rho_i)\}_{i\in I}$ where

    1. U_i is an open cover on M
    2. ρ_i is a locally-free T^k-action on U_i

  such that there exists a global Riemannian metric g on M such that each ρ_i is isometric (with respect to $g|_{U_i}$).

In this case, the orbits of ρ_i's coincide on the overlaps of the open neighborhoods. Thus, we can shrink g along the orbits while keeping ρ_i's isometric and $\left|K_g\right|$ uniformly bounded, and thus Minvol$(M) = 0$. The special case when M admits a free S^1-action satisfies this condition (by taking average, we see that it admits a free isometric S^1-action), and a proof that Minvol$(M) = 0$ in this case is detailed in Bessières [2009].

Fact 2. For all positive number $n \geq 2$, there exists a constant $c_n \in \mathbb{R}^+$ such that for any closed smooth manifold M of dimension n, we have

1. $\text{Minvol}(M) \geq c_n \cdot \left|\chi(M)\right|$.

2. $\text{Minvol}(M) \geq c_n \cdot \left|p(M)\right|$ for any Pontryagin number p.

In particular, if any of $\chi(M)$ or $p(M)$ is non-zero, then $\text{Minvol}(M) > 0$.

Remark. The proof uses standard Chern-Weil theory arguments. We spelled this out for case 1. in subsection 4.1.1; the case of 2. can be proved similarly.

Fact 3. As mentioned before, for any positive integer $n \geq 2$, there exists a constant $c_n \in \mathbb{R}^+$, such that

$$\text{Minvol}(M) \geq c_n \|M\| \,.$$

In particular, if $\|M\| > 0$, then $\text{Minvol}(M) > 0$. In Kotschick [2012], a more elaborative chain of inequality is presented, i.e. one with a family of toplogical invariants in between $\text{Minvol}(M)$ and $\|M\|$.

Remark. We think that the invariants in the more elaborative chain (except for Minvol) vanish precisely under the same condition in that of Theorem 1.

Fact 4 (c.f. Fact 2, Fact 3)**.** For any $n \in \mathbb{Z}^+$, there exists a constant $c_n \in \mathbb{R}^+$, such that for any M_1 and M_2 be closed smooth manifolds such that $\dim(M_1 \times M_2) = n$, we have

$$\text{Minvol}(M_1 \times M_2) \geq c_n \cdot p(M_1)\|M_2\| \,.$$

5.2.2 *Questions about Minvol(M) of general closed smooth manifold M*

Question 1 (Minvol under finite cover)**.** Let N be a closed smooth manifold with $\text{Minvol}(N) = 0$. Suppose $N \to M$ is a finite covering, is it guaranteed that $\text{Minvol}(M) = 0$? (What if we restrict the consideration to finite regular covering?)

Question 2 (Minvol of mapping torus)**.** Let N be a closed smooth manifold with $\text{Minvol}(N) = 0$ (resp. $\text{Minvol}(N) > 0$). For $\rho \in \text{Diff}(N)$, let $M_\rho := \frac{N\times\mathbb{R}}{\{(x,r)\sim(\rho(x),r+1)\}}$ be the smooth mapping torus of N with monodromy ρ.

1. Is it true that $\text{Minvol}(M_\rho) = 0$ for all ρ?
2. If not, is there a criterion of ρ for $\text{Minvol}(M_\rho) = 0$ (resp. $\text{Minvol}(M_\rho) > 0$)?

Suppose $\text{Minvol}(N) = 0$, is it true that for any smooth mapping torus M of N, we have $\text{Minvol}(M) = 0$ (resp. $\text{Minvol}(M_\rho) > 0$)? If not, what is the criterion for this implication to hold?

Question 3 (Minvol of products)**.** Let m and n be positive integers ≥ 2.

1. Is it true that there exists $c_{m,n}, C_{m,n} \in \mathbb{R}^+$, such that

$$c_{m,n}\text{Minvol}(M)\text{Minvol}(N) \leq \text{Minvol}(M \times N) \leq C_{m,n}\text{Minvol}(M)\text{Minvol}(N)$$

   for any closed smooth manifolds M (resp. N) of dimension m (resp. n)?

2. Taking a step back, is it true that if $\text{Minvol}(M) > 0$ and $\text{Minvol}(N) > 0$, then we have $\text{Minvol}(M) \times \text{Minvol}(N) > 0$?

Remark. We make 2 comments about Question 3.1.

1. Trivially, $\text{Minvol}(M \times N) \leq \text{Minvol}(M)\text{Minvol}(N)$.
2. The same statement holds for the simplicial volume counterpart, as stated in Gromov [1982]. A constructive version can be showed via the duality between simplicial volume and bounded cohomology:

$$\|M\|\|N\| \leq \|M \times N\| \leq \binom{m+n}{m}\|M\|\|N\|,$$

   see standard references (for example [Chen, 2022, Theorem 3.17]) for more details.

Question 4. Let M be a the total space of a torus bundle over a closed smooth manifold B. Under what condition does $\mathrm{Minvol}(M) = 0$?

5.2.3 Conjectures on *Minvol*$(\Gamma\backslash G/K)$

Conjecture 1. Suppose M is a closed smooth manifold such that $M = \Gamma\backslash G/K$ for some Lie group G that is compact or semisimple. With $\mathrm{rank}(H)$ as the dimension of a maximal torus of a (not necessarily compact) Lie group H (so that $\mathrm{rank}(H)$, even though is well-defined, might be 0 unlike in the case of compact Lie group), we have $\mathrm{Minvol}(M) > 0$ if and only if $\mathrm{rank}(G) = \mathrm{rank}(K)$. (In particular, this result is independent of the presentation of M as a smooth double coset manifold, as long as the G is compact or semisimple).

Remark (About Conjecture 1)**.** We comment about Conjecture 1 regarding its scope, and regarding several special cases.

1. (Scope) The statement is false when G is non-compact and solvable. For example, consider the n-torus $M = \mathbb{Z}^n\backslash\mathbb{R}^n = \Gamma\backslash G/K$, where $(G, K, \Gamma) = (\mathbb{R}^n, \{e\}, \mathbb{Z}^n)$. Then, $\mathrm{rank}(G) = \mathrm{rank}(K) = 0$, yet clearly $\mathrm{Minvol}(M) = 0$. Plenty of similar examples can be produced by considering simply-connected, solvable G with (cocompact) lattices Γ in G.

2. (Case G compact) The case of G compact is precisely the case when Γ is finite. We know that the conjecture is true when Γ is trivial. However, a priori, when $\mathrm{Minvol}(G/K) = 0$, we cannot conclude that $\mathrm{Minvol}(\Gamma\backslash G/K) = 0$ (c.f. Question 1), unless in the special case when Γ is abelian (c.f. Remark 4.2.1).

3. (Case G semisimple, non-compact) Let H be a closed dimension-3 hyperbolic manifold. Then, $S^2 \times H = (\pi_1(H) \times \{e\})\backslash(SO(3) \times SO(3,1))/(SO(2) \times SO(2))$. This conjecture implies that $\mathrm{Minvol}(S^2 \times H) > 0$. However, it is unknown whether or not this is

true. (In particular, none of Fact 2, Fact 3, or Fact 4 helped.) Still, we are hopeful, for the following reasons:

(a) For any positive integer $g \geq 2$, denote Σ_g as the closed oriented surface with genus g. For any Anosov $\rho \in \mathrm{Diff}(\Sigma_g)$, it is known that the mapping torus H_ρ of Σ_g under the monodromy ρ is a hyperbolic manifold. Such H_ρ provides a special family to the hyperbolic manifolds in Question 3. Now, $H_\rho \times S^2$ is the mapping torus of $\Sigma_g \times S^2$ under the monodromy $(\rho, \mathrm{Id}_{S^2})$. Since $\chi(\Sigma_g \times S^2) > 0$, thus $\mathrm{Minvol}(\Sigma_g \times S^2) > 0$. If the answer to Question 2.1 is affirmative, then $\mathrm{Minvol}(H_\rho \times S^2) > 0$.

(b) Both $\mathrm{Minvol}(S^2)$ and $\mathrm{Minvol}(H)$ are positive. So far, we have not found a counterexample to the statement in Question 3.2; if that statement is true, then $\mathrm{Minvol}(S^2 \times H) > 0$.

Conjecture 2. If $M = \Gamma \backslash G/K$ where

- G^{sol} is nontrivial,
- $K \neq G^{\mathrm{sol}}$,
- and (the semisimple Lie group) G/G^{sol} has no compact factor (i.e. no nontrivial connected normal compact subgroup),

then M is the total space of a flat torus bundle, and thus $\mathrm{Minvol}(M) = 0$.

5.3 About the Choice of Smooth Structure

By definition, Minvol is a diffeomorphism-invariant. In fact, there exists a family of closed smooth manifolds in dimension-4, whose (vanishing of) Minvol depends on its choice of smooth structure. Let $M := k(S^2 \times S^2)\#(1+k)(S^1 \times S^3)$. By [Kotschick, 2012, § 5], it is shown that

1. Under the standard smooth structure, for all k, M admits free S^1-action, and thus $\text{Minvol}(M) = 0$.

2. For sufficiently large, odd k's, there are infinitely many smooth structures such that $\text{Minvol}(M) > 0$.

For more details (on the construction and proof), refer to the cited source.

Remark 5.3.1. Nevertheless, this concern is particularly pertinent only to dimension-4. Let M be a closed topological manifold. Then, up to equivalence class, M has infinitely smooth structures if and only if $\dim(M) = 4$. In the special case when $\dim(M) \leq 3$, M has a unique smooth structure.

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