Supplementary Information for "Approximation of outcome probabilities of linear optical circuit"

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SUPPLEMENTARY TABLES

	Additive-error	Multiplicative-error
$\lvert\mathrm{Haf}(R)\rvert^2$	$\epsilon \prod_i^M \frac{\lambda_{\max}^2}{\sqrt{\lambda_{\max}^2(W(1/e)-1)^2-\lambda_i^2 W(1/e)^2}}$ (∗) (Th. 1)	#P-hard [1]
$\mathrm{Per}(B)$	$\lambda_{\min}=0 : \epsilon \prod_i^M \frac{4\lambda_{\max}^2}{e(2\lambda_{\max}-\lambda_i)}$ (∗) (Th. 2) $\lambda_{\min}>0 : \epsilon \prod_i^M H_i^B(\lambda_i)$ (∗) Eq. (63)	$\lambda_{\min}=0$: NP-hard [2] $\lambda_{\min}>0 : \frac{\lambda_{\max}}{\lambda_{\min}} \leq 2$ [3] (∗) Eq. (98)
$\mathrm{Haf}(A)$	$\epsilon \prod_i^M H_i^A(n,r_i)$ (∗) Eq. (35)	$n \geq \frac{\left(6\sinh(2r_{\max})+\sqrt{18\cosh(4r_{\max})-14}-2\right)}{4}$ (∗) (Th. 3)
$\mathrm{Tor}(R')$	$\epsilon \prod_i^M T_i(\lambda_i)$ (∗) Eq. (74)	?
$\mathrm{Tor}(B')$	$\epsilon \prod_i^M T_i^B(\lambda_i)$ (∗) Eq. (80)	$\lambda_{\min} \geq \frac{1}{2}$ and $\lambda_{\max} \leq \frac{-\lambda_{\min}^2+3\lambda_{\min}-1}{\lambda_{\min}}$ (∗) Eq. (109)
$\mathrm{Tor}(A')$	$\epsilon \prod_i^M T_i^A(n,r_i)$ (∗) Eq. (92)	$n \geq \frac{1}{2}\left(e^{2r_{\max}}\sqrt{e^{8r_{\max}}+3}+e^{6r_{\max}}-1\right)$ (∗) Eq. (111)

Supplementary Table I: Precision and conditions of efficient algorithms for estimating various matrix functions. R: complex symmetric matrices, B: HPSD matrices, $R' = \begin{pmatrix} 0 & R^* \\ R & 0 \end{pmatrix}$, $B' = \begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix}$, $A = \begin{pmatrix} R & B \\ B^T & R^* \end{pmatrix}$, $A' = \begin{pmatrix} B^T & R^* \\ R & B \end{pmatrix}$. (∗) indicates the results in the present work. A question mark represents unknown.

	Upper bound	Lower bound
$\lvert\mathrm{Haf}(R)\rvert^2$	?	?
$\mathrm{Per}(B)$	$\prod_i^M G_i(\lambda_i)$ (∗) Eq. (116)	$\prod_i^M \frac{\lambda_{\min}^2}{\lambda_i}$ [4] (∗)
$\mathrm{Haf}(A)$	$\prod_i^M G_i^H(r_i,n)$ (∗) Eq. (127)	$\prod_i^M L_i^H(r_i,n)$ (∗) Eq. (121)
$\mathrm{Tor}(R')$	?	?
$\mathrm{Tor}(B')$	$\prod_i^M \frac{\lambda_{\max}^2}{\lambda_i(1-\lambda_{\max})}$ (∗) Eq. (137)	$\prod_i^M \frac{\lambda_{\min}^2}{\lambda_i(1-\lambda_{\min})}$ (∗) Eq. (132)
$\mathrm{Tor}(A')$	$\prod_{i=1}^M G_i^T(r_i,n)$ (∗) Eq. (147)	$\prod_i^M L_i^T(r_i,n)$ (∗) Eq. (142)

Supplementary Table II: Upper and lower bounds on various matrix functions.

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SUPPLENTARY NOTE 1 (ESTIMATION OF MATRIX FUNCTIONS WITHIN ADDITIVE-ERRORS)

Here we give a detailed proof of Theorem 1 in the main text. We restate the theorem for the readability.

Theorem 1. (Estimating hafnian) For an $M \times M$ complex symmetric matrix R, one can approximate $|\text{Haf}(R)|^2$ with a success probability $1-\delta$ using the number of samples $O(\log \delta^{-1}/\epsilon^2)$ within the additive-error

$$\epsilon \left(\frac{\lambda_{\max}}{\sqrt{1-2W(1/e)}} \right)^M \simeq \epsilon (1.502\lambda_{\max})^M, \tag{1}$$

where $\lambda_{\max}$ is the largest singular value of R.

Proof. Consider an input of M-mode product of pure squeezed vacuum states $\{r_i\}_{i=1}^M$ and the all single-photon outcomes $\boldsymbol{m} = (1, ..., 1)$. The outcome probability is given by

$$p_{\text{sq}} = \frac{1}{\mathcal{Z}} \left| \text{Haf}(R') \right|^2, \tag{2}$$

where $\mathcal{Z} = \prod_{i=1}^M \cosh r_i$, $R' = UDU^T$, and $D = \oplus_{i=1}^M \tanh r_i$. Since any complex symmetric matrix can be decomposed by Takagi decomposition as UDU^T [5], the only restriction is the magnitude of singular values $\lambda_i = \tanh r_i \in [0, 1)$.

For a given complex symmetric matrix R, we can construct a quantum circuit, the probability of which is expressed as its hafnian. To do that, first rescale the matrix with the largest singular value $\lambda_{\max}$ as $R' = R/(a\lambda_{\max})$ with $a > 1$, and find the Takagi decomposition of R' as $R' = UDU^T$ so that a GBS probability is matched to $|\text{Haf}(R')|^2$. From Eq. (2),

$$|\text{Haf}(R)|^2 = (a\lambda_{\max})^M |\text{Haf}(R')|^2 = (a\lambda_{\max})^M \mathcal{Z} p_{\text{sq}}. \tag{3}$$

If an estimator of $|\text{Haf}(R)|^2$ lies in the interval $[-C^M, C^M]$, by Hoeffding inequality [6],

$$\Pr(||\text{Haf}(R)|^2 - (a\lambda_{\max})^M \mathcal{Z}\mu| \geq (a\lambda_{\max})^M \mathcal{Z}\epsilon) \leq 2 \exp\Big(-\frac{N\epsilon^2}{2C^{2M}} \Big), \tag{4}$$

where μ is the sample mean of p_{sq}. With a success probability $1-\delta$, a sufficient number of samples for the estimation of $|\text{Haf}(R)|^2$ within an additive-error ϵ is given by

$$N = \frac{2(a\lambda_{\max})^{2M} \mathcal{Z}^2 C^{2M}}{\epsilon^2} \ln \frac{2}{\delta}. \tag{5}$$

In other words, if we fix the sample size as $N = O(\log \delta^{-1}/\epsilon^2)$, one can estimate $|\text{Haf}(R)|^2$ with a success probability $1-\delta$ within an additive-error $2\epsilon(a\lambda_{\max} C \mathcal{Z}^{1/M})^M$. To obtain the bound C, we express the probability using s-PQDs as

$$p_{\text{sq}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^M \frac{1}{\pi \sqrt{\det{(V_{\text{sq},i} - s/2)}}} e^{-\boldsymbol{\alpha}_i (V_{\text{sq},i}-s/2)^{-1} \boldsymbol{\alpha}_i^T} \prod_{j=1}^M \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\frac{2|\beta_j|^2}{s+1}} \tag{6}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^M \frac{2}{\pi\sqrt{(e^{2r_i}-s)(e^{-2r_i}-s)}} e^{-\frac{2\alpha_{ix}^2}{e^{2r_i}-s} - \frac{2\alpha_{iy}^2}{e^{-2r_i}-s}} \prod_{j=1}^M \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\frac{2|\beta_j|^2}{s+1}} \tag{7}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^M \frac{2}{\pi\sqrt{(e^{2r_i}-s)(e^{-2r_i}-s)}} e^{-\left(\frac{2}{e^{2r_i}-s} - \gamma\frac{2}{e^{2r_{\max}}-s}\right)\alpha_{ix}^2 - \left(\frac{2}{e^{-2r_i}-s} - \gamma\frac{2}{e^{2r_{\max}}-s}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^M \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\left(\frac{2}{s+1} + \gamma\frac{2}{e^{2r_{\max}}-s}\right)|\beta_j|^2} \tag{8}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^M \frac{2}{\pi} \sqrt{\frac{e^{2r_{\max}} - s - \gamma(e^{2r_i}-s)}{(e^{2r_i}-s)(e^{2r_{\max}}-s)}} \sqrt{\frac{e^{2r_{\max}} - s - \gamma(e^{-2r_i}-s)}{(e^{-2r_i}-s)(e^{2r_{\max}}-s)}} e^{-\left(\frac{2}{e^{2r_i}-s} - \gamma\frac{2}{e^{2r_{\max}}-s}\right)\alpha_{ix}^2 - \left(\frac{2}{e^{-2r_i}-s} - \gamma\frac{2}{e^{2r_{\max}}-s}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^M \sqrt{\frac{e^{2r_{\max}}-s}{e^{2r_{\max}} - s - \gamma(e^{2r_j}-s)}} \sqrt{\frac{e^{2r_{\max}}-s}{e^{2r_{\max}} - s - \gamma(e^{-2r_j}-s)}} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\left(\frac{2}{s+1} + \gamma\frac{2}{e^{2r_{\max}}-s}\right)|\beta_j|^2} \tag{9}$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^M P_{\text{sq},i}(\alpha_i, r_i, \gamma, s) \prod_{j=1}^M f_{\text{sq},j}(\beta_j, r_j, \gamma, s), \tag{10}$$

where $V_{\text{sq},i}$ is the covariance matrix of a squeezed vacuum state in mode i and $\gamma \in [0,1)$ is the (normalized) parameter shifting the Gaussian factor such that $\gamma \to 1$ ($\gamma = 0$) means maximum (no) shifting. To obtain a bound on $|f_{\text{sq},j}(\beta_j, r_j, \gamma, s)|$, let us set $s = s_{\max} = e^{-2r_{\max}}$ and $r_j = \tanh^{-1}\lambda_j$. The extreme points of $f_{\text{sq},j}$ are $0, \pm\beta^*$ with

$$\beta^* = \sqrt{\frac{\lambda_{\max}(\gamma(\lambda_{\max}-1) - 4\lambda_{\max} - 2)}{(\lambda_{\max}+1)^2(\gamma(\lambda_{\max}-1) - 2\lambda_{\max})}}. \tag{11}$$

Note that for $s_{\max} < 1$, $f_{\text{sq},j}(0, \lambda_j, \gamma, s_{\max}) < 0$ and $f_{\text{sq},j}(\beta^*, \lambda_j, \gamma, s_{\max}) > 0$. Since the extreme values are changing monotonically as γ, we choose γ satisfying the condition $-f_{\text{sq},j}(0, \lambda_j, \gamma, s_{\max}) = f_{\text{sq},j}(\beta^*, \lambda_j, \gamma, s_{\max})$, which is given by

$$\gamma^* = \frac{2(1+\lambda_{\max})W(1/e) - 2\lambda_{\max}}{1-\lambda_{\max}} \quad \text{for } 0 \le \lambda_{\max} < \frac{W(1/e)}{1-W(1/e)}, \tag{12}$$

where $W(x)$ is the Lambert W function, and $\frac{W(1/e)}{1-W(1/e)} \simeq 0.386$. Then an upper bound is obtained by substituting γ^* into $f_{\text{sq},j}(\beta^*, \lambda_j, \gamma, s_{\max})$, such that

$$\min_{s,\gamma}\max_{\beta_j} |f_{\text{sq},j}(\beta_j, \lambda_j, \gamma, s)| \le \frac{\lambda_{\max}^2\sqrt{1-\lambda_j^2}}{\sqrt{\lambda_{\max}^2(1-W(1/e))^2 - \lambda_j^2 W(1/e)^2}} \quad \text{for } 0 \le \lambda_{\max} < \frac{W(1/e)}{1-W(1/e)}, \tag{13}$$

Although this bound is valid only for a certain range of $\lambda_{\max}$, we can also find the same bound out of the range by shifting the Gaussian factor in the reverse direction. Specifically,

$$p_{\text{sq}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(e^{2r_i}-s)(e^{-2r_i}-s)}} e^{-\frac{2\alpha_{ix}^2}{e^{2r_i}-s} - \frac{2\alpha_{iy}^2}{e^{-2r_i}-s}} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\frac{2|\beta_j|^2}{s+1}} \tag{14}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(e^{2r_i}-s)(e^{-2r_i}-s)}} e^{-\left(\frac{2}{e^{2r_i}-s} + \gamma'\frac{2}{s+1}\right)\alpha_{ix}^2} e^{-\left(\frac{2}{e^{-2r_i}-s} + \gamma'\frac{2}{s+1}\right)\alpha_{iy}^2} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-(1-\gamma')\frac{2|\beta_j|^2}{s+1}} \tag{15}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi} \sqrt{\frac{s+1+\gamma'(e^{2r_i}-s)}{(e^{2r_i}-s)(s+1)}} \sqrt{\frac{s+1+\gamma'(e^{-2r_i}-s)}{(e^{-2r_i}-s)(s+1)}} e^{-\left(\frac{2}{e^{2r_i}-s} + \gamma'\frac{2}{s+1}\right)\alpha_{ix}^2} e^{-\left(\frac{2}{e^{-2r_i}-s} + \gamma'\frac{2}{s+1}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^{M} \sqrt{\frac{s+1}{s+1+\gamma'(e^{2r_j}-s)}} \sqrt{\frac{s+1}{s+1+\gamma'(e^{-2r_j}-s)}} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-(1-\gamma')\frac{2|\beta_j|^2}{s+1}} \tag{16}$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} P'_{\text{sq},i}(\alpha_i, r_i, \gamma', s) \prod_{j=1}^{M} f'_{\text{sq},j}(\beta_j, r_j, \gamma', s), \tag{17}$$

where $\gamma' \in [0,1)$ is the parameter shifting the exponential term in the reverse direction such that $\gamma' \to 1$ ($\gamma' = 0$) means maximum (no) shifting. Similarly, the extreme points are $0, \pm\beta'^*$ with

$$\beta'^* = \sqrt{\frac{(\gamma'-2)\lambda_{\max} - 1}{(\gamma'-1)(\lambda_{\max}+1)^2}}. \tag{18}$$

Then the γ' satisfying the condition $-f'_{\text{sq},j}(0, \lambda_j, \gamma', s_{\max}) = f_{\text{sq},j}(\beta'^*, \lambda_j, \gamma', s_{\max})$ is given by

$$\gamma'^* = \frac{\lambda_{\max} - (1+\lambda_{\max})W(1/e)}{\lambda_{\max}} \quad \text{for } \frac{W(1/e)}{1-W(1/e)} \le \lambda_{\max} < 1. \tag{19}$$

Finally we obtain an upper bound such that

$$\min_{s,\gamma'}\max_{\beta_j} \left|f'_j(\beta_j, \lambda_j, \gamma', s)\right| \le \frac{\lambda_{\max}^2\sqrt{1-\lambda_j^2}}{\sqrt{\lambda_{\max}^2(1-W(1/e))^2 - \lambda_j^2 W(1/e)^2}} \quad \text{for } \frac{W(1/e)}{1-W(1/e)} \le \lambda_{\max} < 1, \tag{20}$$

which covers the remaining range of $\lambda_{\max}$.

Note that

$$\mathcal{Z} = \prod_{i=1}^{M} \cosh r_i = \frac{1}{\prod_{i=1}^{M} \sqrt{1-\lambda_i^2}}. \tag{21}$$

Consequently, by Eq. (4), for a given complex $M \times M$ matrix R, and for the number of samples $N = O(\log \delta^{-1}/\epsilon^2)$, we can estimate $|\mathrm{Haf}(R)|^2$ with a success probability $1-\delta$ within an additive-error as

$$\epsilon \prod_{i=1}^{M} \frac{a\lambda_{\max}\lambda_{\max}'^2}{\sqrt{\lambda_{\max}'^2(W(1/e)-1)^2 - \lambda_i'^2 W(1/e)^2}} = \epsilon \prod_{i=1}^{M} \frac{\lambda_{\max}^2}{\sqrt{\lambda_{\max}^2(W(1/e)-1)^2 - \lambda_i^2 W(1/e)^2}}, \tag{22}$$

where $\lambda_i' = \frac{\lambda_i}{a\lambda_{\max}}$. We find upper and lower bound by setting $\lambda_i = \lambda_{\max}$ and $\lambda_i = 0$, respectively, such as

$$\epsilon(1.386\lambda_{\max})^M \simeq \left(\epsilon \frac{\lambda_{\max}}{1-W(1/e)}\right)^M \le \epsilon \prod_{i=1}^{M} \frac{\lambda_{\max}^2}{\sqrt{\lambda_{\max}^2(W(1/e)-1)^2 - \lambda_i^2 W(1/e)^2}} \le \epsilon \left(\frac{\lambda_{\max}}{\sqrt{1-2W(1/e)}}\right)^M \simeq \epsilon(1.502\lambda_{\max})^M. \tag{23}$$

□

Next, we consider a squeezed thermal input state $\{r_i, n\}_{i=1}^M$ to allow $s_{\max} \ge 1$. For simplicity, the average thermal photon number n is fixed. Then the probability of all single-photon outcomes is written by hafnian as

$$p_{\mathrm{st}} = \frac{1}{\sqrt{|V_Q^{\mathrm{st}}|}} \mathrm{Haf}(A), \tag{24}$$

where $V_Q^{\mathrm{st}} = V_{\mathrm{st}} + \mathbb{I}_{2M}/2$ with V_{st} the covariance matrix of a squeezed thermal state and $A = \begin{pmatrix} R & B \\ B^T & R^* \end{pmatrix}$ with a symmetric matrix R and an HPSD matrix B decomposed by a unitary matrix U as UDU^T and $UD'U^\dagger$, respectively, with

$$D = \bigoplus_{i=1}^{M} \frac{(1+2n)\sinh 2r_i}{1+2n(1+n)+(1+2n)\cosh 2r_i}, \tag{25}$$

$$D' = \bigoplus_{i=1}^{M} \frac{2n(1+n)}{1+2n(1+n)+(1+2n)\cosh 2r_i}. \tag{26}$$

Let us define $a_\pm(r_i,n) = (2n+1)e^{\pm 2r_i}$, $a_{\min} = (2n+1)e^{-2r_{\max}}$, and $a_{\max} = (2n+1)e^{2r_{\max}}$. Then the probability p_{st} is written by s-PQDs as

$$p_{\mathrm{st}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\frac{2\alpha_{ix}^2}{a_+(r_i,n)-s} - \frac{2\alpha_{iy}^2}{a_-(r_i,n)-s}} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\frac{2|\beta_j|^2}{s+1}} \tag{27}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\left(\frac{2}{a_+(r_i,n)-s} + \gamma'\frac{2}{s+1}\right)\alpha_{ix}^2} e^{-\left(\frac{2}{a_-(r_i,n)-s} + \gamma'\frac{2}{s+1}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-(1-\gamma')\frac{2|\beta_j|^2}{s+1}} \tag{28}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi} \sqrt{\frac{s+1+\gamma'(a_+(r_i,n)-s)}{(a_+(r_i,n)-s)(s+1)}} \sqrt{\frac{s+1+\gamma'(a_-(r_i,n)-s)}{(a_-(r_i,n)-s)(s+1)}} e^{-\left(\frac{2}{a_+(r_i,n)-s} + \gamma'\frac{2}{s+1}\right)\alpha_{ix}^2} e^{-\left(\frac{2}{a_-(r_i,n)-s} + \gamma'\frac{2}{s+1}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^{M} \sqrt{\frac{s+1}{s+1+\gamma'(a_+(r_j,n)-s)}} \sqrt{\frac{s+1}{s+1+\gamma'(a_-(r_j,n)-s)}} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-(1-\gamma')\frac{2|\beta_j|^2}{s+1}} \tag{29}$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} P_{\mathrm{st},i}'(\alpha_i, r_i, n, \gamma', s) \prod_{j=1}^{M} f_{\mathrm{st},j}'(\beta_j, r_j, n, \gamma', s), \tag{30}$$

where we use the reverse shifting of the Gaussian factor. To obtain an upper bound on $|f'_{\text{st},j}(\beta_j, r_j, n, \gamma', s)|$, let us set $s = s_{\max} = a_{\min}$. The extreme points are $0, \pm\beta'^*$ with

$$\beta'^* = \frac{1}{2}\sqrt{\frac{e^{-4r_{\max}}\left(2n + e^{2r_{\max}} + 1\right)\left\{(\gamma'-3)e^{2r_{\max}} - (\gamma'-1)(2n+1)\right\}}{\gamma'-1}}. \tag{31}$$

After choosing $\gamma'^* = e^{-\tanh r_{\max}} \frac{n}{n+1}$,

$$\min_{s,\gamma'} \max_{\beta_j} |f'_{\text{st},j}(\beta_j, r_j, n, \gamma', s)| \le |f'_{\text{st},j}(\beta'^*, r_j, n, \gamma'^*, a_{\min})| \tag{32}$$

$$= \exp\left[\frac{1}{2}\left\{-3 + \frac{ne^{-\tanh r_{\max}}}{1+n} - (2n+1)e^{-2r_{\max}}\left(-1 + \frac{ne^{-\tanh r_{\max}}}{1+n}\right) + 2r_j + 8r_{\max} + 3\tanh r_{\max}\right\}\right]$$
$$\times \frac{4(1+n)^2}{(1+e^{2r_{\max}}+2n)\{(1+n)e^{\tanh r_{\max}} - n\}}\sqrt{\frac{1}{1+e^{2r_{\max}}+3n+ne^{2r_{\max}}+2n^2+n(2n+1)(e^{2(r_j+r_{\max})}-1)e^{-\tanh r_{\max}}}}$$
$$\times \sqrt{\frac{1}{(2n+1)e^{2r_j}\left\{-n+(n+1)e^{\tanh r_{\max}}\right\} + e^{2r_{\max}}\left\{n+2n^2+(n+1)e^{2r_j+\tanh r_{\max}}\right\}}}. \tag{33}$$

Meanwhile,

$$\sqrt{|V_Q^{\text{st}}|} = \prod_{i=1}^{M}\sqrt{\frac{1}{2} + n(n+1) + (n+\frac{1}{2})\cosh 2r_i}. \tag{34}$$

Finally, we can estimate $\text{Haf}(A)$ with a success probability $1-\delta$ using number of samples $N = O(\log\delta^{-1}/\epsilon^2)$ within the additive-error given by

$$\epsilon\prod_{i=1}^{M}\sqrt{\frac{1}{2} + n(n+1) + (n+\frac{1}{2})\cosh 2r_i}\; f'_{\text{st},i}(\beta'^*, r_i, n, \gamma'^*, a_{\min}) \coloneqq \epsilon\prod_{i=1}^{M} H_i^A(n, r_i). \tag{35}$$

Next, we give a detailed proof of Theorem 2 in the main text.

Theorem 2. (Estimating permanent of HPSD matrices) For an $M \times M$ HPSD matrix B, one can approximate $\text{Per}(B)$ with a success probability $1-\delta$ using the number of samples $O(\log\delta^{-1}/\epsilon^2)$ within the error

$$\epsilon\prod_{i=1}^{M}\frac{4\lambda_{\max}^2}{e(2\lambda_{\max}-\lambda_i)}, \tag{36}$$

where λ_i are singular values of the matrix B and $\lambda_{\max}$ is the largest one.

Proof. When a thermal state input with average photon numbers $\{n_i\}_{i=1}^M$ goes through a linear optical circuit instead of a squeezed vacuum state, the probability of all single-photon outcomes corresponds to the permanent of HPSD matrices [4]. In Ref. [4], an algorithm for estimating the permanent of an HPSD matrix is proposed. Here, we improve the precision of the estimation using s-PQDs and shifting Gaussian factors. The probability of all single-photon outcomes is connected to the permanent of an HPSD matrix such that [4]

$$p_{\text{th}} = \frac{1}{\mathcal{Z}'}\text{Per}(B'), \tag{37}$$

where $\mathcal{Z}' = \prod_{i=1}^M(1+n_i)$, $B' = UDU^\dagger$, and $D = \text{diag}\{\frac{n_1}{n_1+1}, ..., \frac{n_M}{n_M+1}\}$. Thus if we have an $M \times M$ HPSD matrix B, firstly we rescale the matrix with the largest eigenvalue $\lambda_{\max}$ as $B' = B/(a\lambda_{\max})$ with $a > 1$, and find its unitary diagonalization such as $B' = UDU^\dagger$ so that we can find a GBS circuit U with thermal input state $\{n_i\}_{i=1}^M$, whose probability matches $\text{Per}(B')$. Then,

$$\text{Per}(B) = (a\lambda_{\max})^M\text{Per}(B') = (a\lambda_{\max})^M\mathcal{Z}'p_{\text{th}}. \tag{38}$$

If an estimator of $\text{Per}(B)$ lies in the interval $[-C^M, C^M]$, by Hoeffding's inequality [6],

$$\Pr(|\text{Per}(B) - (a\lambda_{\max})^M\mathcal{Z}'\mu| \ge (a\lambda_{\max})^M\mathcal{Z}'\epsilon) \le 2\exp\left(-\frac{N\epsilon^2}{2C^{2M}}\right), \tag{39}$$

where μ is the sample mean of $p_{\rm th}$. Thus for the number of samples $N = O(\log \delta^{-1}/\epsilon^2)$, we can estimate $\mathrm{Per}(B)$ with a success probability $1-\delta$ within an additive-error $\epsilon(a\lambda_{\max} C\mathcal{Z}'^{1/M})^M$.

Now we use the same method as in the hafnian case by introducing the shifting parameter $\gamma \in [0,1)$, such as

$$p_{\rm th} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi(2n_i+1-s)} e^{-\left(\frac{2}{2n_i+1-s} - \gamma\frac{2}{2n_{\max}+1-s}\right)|\alpha_i|^2} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\left(\frac{2}{s+1}+\gamma\frac{2}{2n_{\max}+1-s}\right)|\beta_j|^2} \tag{40}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi} \frac{2n_{\max}+1-s-\gamma(2n_i+1-s)}{(2n_i+1-s)(2n_{\max}+1-s)} e^{-\left(\frac{2}{2n_i+1-s} - \gamma\frac{2}{2n_{\max}+1-s}\right)|\alpha_i|^2}$$

$$\times \prod_{j=1}^{M} \frac{2n_{\max}+1-s}{2n_{\max}+1-s-\gamma(2n_j+1-s)} \frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3} e^{-\left(\frac{2}{s+1}+\gamma\frac{2}{2n_{\max}+1-s}\right)|\beta_j|^2} \tag{41}$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} P_{{\rm th},i}(\alpha_i, n_i, \gamma, s) \prod_{j=1}^{M} f_{{\rm th},j}(\beta_j, n_j, \gamma, s). \tag{42}$$

One can compute the upper bound of $|f_{{\rm th},j}(\beta_j, n_j, \gamma, s)|$ in three different regions of $\lambda_{\min} = \frac{n_{\min}}{1+n_{\min}} \in [0,1)$, as $\lambda_{\min} = 0$, $0 < \lambda_{\min} < 1/2$, and $1/2 \le \lambda_{\min} < 1$. We set $s = s_{\max} = 2n_{\min}+1$, $n_j = \frac{\lambda_j}{1-\lambda_j}$ and note that the extreme points of $f_{{\rm th},j}(\beta_j, \lambda_j, \gamma, s)$ are at 0, $\pm\beta^*$ with

$$\beta^* = \sqrt{\frac{\lambda_{\min}(\gamma - 2\lambda_{\min}+1) - \lambda_{\max}((\gamma-2)\lambda_{\min}+1)}{(\lambda_{\min}-1)^2(\gamma(\lambda_{\max}-1) - \lambda_{\max}+\lambda_{\min})}}. \tag{43}$$

For $\lambda_{\min} = 0$, we can find γ satisfying $\frac{\partial f_j(\beta^*, \lambda_{\max}, \gamma, s_{\max})}{\partial \gamma} = 0$ as

$$\gamma^* = \frac{1-2\lambda_{\max}}{2(1-\lambda_{\max})} \quad \text{for} \quad 0 \le \lambda_{\max} < \frac{1}{2}. \tag{44}$$

Then an upper bound on the $|f_{{\rm th},j}(\beta_j, \lambda_j, \gamma, s)|$ is obtained as

$$\min_{s,\gamma} \max_{\beta_j} |f_{{\rm th},j}(\beta_j, \lambda_j, \gamma, s)| \le \frac{4\lambda_{\max}^2(1-\lambda_j)}{e(2\lambda_{\max}-\lambda_j)} \quad \text{for} \quad 0 \le \lambda_j < \frac{1}{2}. \tag{45}$$

For $0 < \lambda_{\min} < 1/2$, we similarly obtain γ as

$$\gamma^* = \frac{\lambda_{\min} + \lambda_{\max}(4\lambda_{\min}-2) + D - \lambda_{\min}(3\lambda_{\min}+D)}{2\lambda_{\min}(\lambda_{\max}-1)} \quad \text{for} \quad \lambda_j \le \frac{\lambda_{\min}^2 + \lambda_{\min} - 1}{3\lambda_{\min}-2}, \tag{46}$$

where $D = \sqrt{4\lambda_{\max}^2 - 8\lambda_{\max}\lambda_{\min} + 5\lambda_{\min}^2}$. Then an upper bound on the $|f_{{\rm th},j}(\beta_j, \lambda_j, \gamma, s)|$ is given by

$$\min_{s,\gamma} \max_{\beta_j} |f_{{\rm th},j}(\beta_j, \lambda_j, \gamma, s)| \le \frac{4(1-\lambda_j)\lambda_{\min}^2 e^{\frac{\lambda_{\min}-D}{2\lambda_{\max}-2\lambda_{\min}}}(\lambda_{\max}-\lambda_{\min})^2}{(D-2\lambda_{\max}+\lambda_{\min})\left(\lambda_{\min}(D-4\lambda_{\max}+3\lambda_{\min}) - \lambda_j(D-2\lambda_{\max}+\lambda_{\min})\right)} \tag{47}$$

$$\text{for} \quad \lambda_j \le \frac{\lambda_{\min}^2+\lambda_{\min}-1}{3\lambda_{\min}-2}. \tag{48}$$

We will consider the case of $1/2 \le \lambda_{\min} < 1$ later, in which we achieve a multiplicative-error estimation scheme. To cover the full range of λ_j, we consider the reverse shifting such that

$$p_{\rm th} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi(2n_i+1-s)} e^{-\left(\frac{2}{2n_i+1-s}+\gamma'\frac{2}{s+1}\right)|\alpha_i|^2} \prod_{j=1}^{M} \frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3} e^{-(1-\gamma')\frac{2}{s+1}|\beta_j|^2} \tag{49}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi} \frac{s+1+\gamma'(2n_i+1-s)}{(2n_i+1-s)(s+1)} e^{-\left(\frac{2}{2n_i+1-s}+\gamma'\frac{2}{s+1}\right)|\alpha_i|^2}$$

$$\times \prod_{j=1}^{M} \frac{s+1}{s+1+\gamma'(2n_j+1-s)} \frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3} e^{-(1-\gamma')\frac{2}{s+1}|\beta_j|^2} \tag{50}$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} P'_{{\rm th},i}(\alpha_i, n_i, \gamma', s) \prod_{j=1}^{M} f'_{{\rm th},j}(\beta_j, n_j, \gamma', s). \tag{51}$$

where $\gamma' \in (0,1]$. When we put $s = 2n_{\min}+1$, the extreme points of $f'_{\mathrm{th},j}(\beta_j, \lambda_j, \gamma', s)$ are at 0, $\pm\beta'^*$ with

$$\beta'^* = \sqrt{\frac{-\gamma'\lambda_{\min} + 2\lambda_{\min} - 1}{\gamma'\lambda_{\min}^2 - 2\gamma'\lambda_{\min} + \gamma' - \lambda_{\min}^2 + 2\lambda_{\min} - 1}}. \tag{52}$$

For $\lambda_{\min} = 0$, the condition $\frac{\partial f_j(\beta'^*, \lambda_{\max}, \gamma', s_{\max})}{\partial \gamma'} = 0$ yields

$$\gamma'^* = \frac{2\lambda_{\max} - 1}{2\lambda_{\max}}, \quad \text{for} \quad \frac{1}{2} \le \lambda_{\max} < 1. \tag{53}$$

Then an upper bound of $|f'_{\mathrm{th},j}(\beta_j, \lambda_j, \gamma', s)|$ is given by

$$\min_{s,\gamma'} \max_{\beta_j} |f'_{\mathrm{th},j}(\beta_j, \lambda_j, \gamma', s)| \le \frac{4\lambda_{\max}^2(1-\lambda_j)}{e(2\lambda_{\max} - \lambda_j)}, \quad \text{for} \quad \frac{1}{2} \le \lambda_j < 1, \tag{54}$$

which is consistent with the bound for $0 < \lambda_j < \frac{1}{2}$.

Similarly, when $0 < \lambda_{\min} < 1/2$,

$$\gamma'^* = \frac{\lambda_{\min} + \lambda_{\max}(4\lambda_{\min} - 2) + D - \lambda_{\min}(3\lambda_{\min} + D)}{2\lambda_{\min}(\lambda_{\max} - \lambda_{\min})}, \quad \text{for} \quad \frac{\lambda_{\min}^2 + \lambda_{\min} - 1}{3\lambda_{\min} - 2} \le \lambda_j < 1 \tag{55}$$

and the corresponding bound is

$$\min_{s,\gamma'} \max_{\beta_j} |f'_{\mathrm{th},j}(\beta_j, \lambda_j, \gamma', s)| \le \frac{4(1-\lambda_j)\lambda_{\min}^2 e^{\frac{\lambda_{\min} - D}{2\lambda_{\max} - 2\lambda_{\min}}} (\lambda_{\max} - \lambda_{\min})^2}{(D - 2\lambda_{\max} + \lambda_{\min})\left(\lambda_{\min}(D - 4\lambda_{\max} + 3\lambda_{\min}) - \lambda_j (D - 2\lambda_{\max} + \lambda_{\min})\right)}, \tag{56}$$

$$\text{for} \quad \frac{\lambda_{\min}^2 + \lambda_{\min} - 1}{3\lambda_{\min} - 2} \le \lambda_j < 1. \tag{57}$$

Meanwhile,

$$\mathcal{Z}' = \prod_{i=1}^{M}(1 + n_i) = \frac{1}{\prod_{i=1}^{M}(1 - \lambda_i)}. \tag{58}$$

Consequently, for the number of samples $N = O(\log \delta^{-1}/\epsilon^2)$ and success probability $1-\delta$, we can estimate $\mathrm{Per}(B)$ when the minimum eigenvalue $\lambda_{\min} = 0$ within an additive-error as

$$\epsilon \prod_{i=1}^{M} \frac{4a\lambda_{\max}\lambda_{\max}'^2}{e(2\lambda'_{\max} - \lambda'_i)} = \epsilon \prod_{i=1}^{M} \frac{4\lambda_{\max}^2}{e(2\lambda_{\max} - \lambda_i)}, \tag{59}$$

where $\lambda'_i = \frac{\lambda_i}{a\lambda_{\max}}$. □

Let us first compare our result with the existing result [4], where $\gamma = 0$ and $s = 1$. In the latter case, the upper bound on the estimator is such as $\max|f_j| \le e^{-1}$, so thus corresponding additive-error is given by

$$\epsilon \prod_{i=1}^{M} \frac{1}{e(1-\lambda_i)}, \tag{60}$$

where we assume $\lambda_i \in [0,1)$. Note that Eq. (59) is smaller than Eq. (60) when $\lambda_{\max} \in (0, 1/2)$, so we have a better precision.

Next, we compare our algorithm's precision with Gurvits' randomized algorithm for the permanent of a general complex matrix A giving the additive-error as $\epsilon||A||^M = \epsilon\lambda_{\max}^M$ with samples $N = O(M^2/\epsilon^2)$. Thus from Eq. (59), the necessary and sufficient condition for beating Gurvits' precision is written as

$$\prod_{i=1}^{M} \frac{4\lambda_{\max}^2}{e(2\lambda_{\max} - \lambda_i)} < \lambda_{\max}^M. \tag{61}$$

To get lower and upper bounds, we put $\lambda_i = 0$ and $\lambda_i = \lambda_{\max}$ for all i, respectively, such as

$$\epsilon(0.736\lambda_{\max})^M \simeq \epsilon\prod_{i=1}^{M}\frac{2\lambda_{\max}}{e} \leq \epsilon\prod_{i=1}^{M}\frac{4\lambda_{\max}^2}{e(2\lambda_{\max}-\lambda_i)} \leq \epsilon\prod_{i=1}^{M}\frac{4\lambda_{\max}}{e} \simeq \epsilon(1.472\lambda_{\max})^M. \tag{62}$$

Also for $0 < \lambda_{\min} < 1/2$, the additive-error for $\mathrm{Per}(B)$ is given by

$$\epsilon\prod_{i=1}^{M}\frac{4a\lambda_{\max}\lambda'^2_{\min}e^{\frac{\lambda'_{\min}-D'}{2\lambda'_{\max}-2\lambda'_{\min}}}(\lambda'_{\max}-\lambda'_{\min})^2}{(D'-2\lambda'_{\max}+\lambda'_{\min})\left(\lambda'_{\min}\left(D'-4\lambda'_{\max}+3\lambda'_{\min}\right)-\lambda'_i\left(D'-2\lambda'_{\max}+\lambda'_{\min}\right)\right)} \tag{63}$$

$$= \epsilon\prod_{i=1}^{M}\frac{4\lambda^2_{\min}e^{\frac{\lambda_{\min}-D}{2\lambda_{\max}-2\lambda_{\min}}}(\lambda_{\max}-\lambda_{\min})^2}{(D-2\lambda_{\max}+\lambda_{\min})\left(\lambda_{\min}\left(D-4\lambda_{\max}+3\lambda_{\min}\right)-\lambda_i\left(D-2\lambda_{\max}+\lambda_{\min}\right)\right)} := \epsilon\prod_{i=1}^{M}H_i^B(\lambda_i), \tag{64}$$

where $D' = \sqrt{4\lambda'^2_{\max} - 8\lambda'_{\max}\lambda'_{\min} + 5\lambda'^2_{\min}}$. Similarly, the necessary and sufficient condition for beating Gurvits' is given by

$$\prod_{i=1}^{M}H_i^B(\lambda_i) < \lambda_{\max}^M. \tag{65}$$

Our method is also applicable to another matrix function, called Torontonian. The Torontonian of a $2M \times 2M$ complex matrix A' is defined as [7]

$$\mathrm{Tor}(A') = \sum_{Z\in P([M])}(-1)^{|Z|}\frac{1}{\sqrt{\det(\mathbb{I}-A'_{(Z)})}}, \tag{66}$$

where $P([M])$ is the power set of $[M] := \{1, 2, ..., M\}$ and the matrix A' has block structure such that

$$A' = \begin{pmatrix} B^T & R^* \\ R & B \end{pmatrix}, \tag{67}$$

where B is HPSD and R is symmetric. Let us first consider a special case $B = 0$, which corresponds to a GBS with pure squeezed input state $\{r_i\}_{i=1}^M$. The probability of all threshold detectors "click" in a GBS circuit is related to the Torontonian as

$$p_{\mathrm{on|sq}} = \frac{1}{\sqrt{|V_Q|}}\mathrm{Tor}(\mathbb{I}_{2M} - V_Q^{-1}) = \frac{1}{\mathcal{Z}}\mathrm{Tor}\begin{pmatrix} 0 & R^* \\ R & 0 \end{pmatrix}, \tag{68}$$

where $V_Q = V + \mathbb{I}/2$ with the covariance matrix V, $\Pi_0 = |0\rangle\langle 0|$, $\mathcal{Z} = \prod_i^M \cosh r_i$, $R = UDU^T$, and $D = \oplus_i^M \tanh r_i$. Meanwhile, the probability $p_{\mathrm{on|sq}}$ can be written in terms of s-PQDs as

$$p_{\mathrm{on|sq}} = \int d^{2M}\boldsymbol{\alpha}\prod_{i=1}^{M}P_{\mathrm{sq},i}(\alpha_i, r_i, \gamma, s)\prod_{j=1}^{M}f_{\mathrm{on|sq},j}(\beta_j, r_j, \gamma, s), \tag{69}$$

where $P_{\mathrm{sq},i}(\alpha_i, r_i, \gamma, s)$ is given in Eq. (10) and $f_{\mathrm{on|sq},j}(\beta_j, r_j, \gamma, s)$ is defined as

$$f_{\mathrm{on|sq},j}(\beta_j, r_j, \gamma, s) = \sqrt{\frac{e^{2r_{\max}}-s}{e^{2r_{\max}}-s-\gamma(e^{2r_j}-s)}}\sqrt{\frac{e^{2r_{\max}}-s}{e^{2r_{\max}}-s-\gamma(e^{-2r_j}-s)}}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right)e^{-\gamma\frac{2}{e^{2r_{\max}}-s}|\beta_j|^2}. \tag{70}$$

We set $s = s_{\max} = e^{-2r_{\max}}$ and $r_j = \tanh^{-1}\lambda_j$. The extreme points are at $0, \pm\beta^*$ with

$$\beta^* = \sqrt{\frac{1}{\lambda_{\max}+1}\log\left(\frac{(\lambda_{\max}+1)(\gamma(\lambda_{\max}-1)-2\lambda_{\max})}{\gamma(\lambda_{\max}-1)}\right)}. \tag{71}$$

After we choose $\gamma^* = \frac{1}{2}(1-\lambda_{\max})$, the corresponding upper bound is given by

$$\min_{s,\gamma}\max_{\beta_j} |f_{\text{on},j}(\beta_j,\lambda_j,\gamma,s)| \tag{72}$$

$$\leq \frac{16\sqrt{1-\lambda_i^2}\lambda_{\max}^2}{(1+\lambda_{\max}^3)\sqrt{\{\lambda_i+(\lambda_i+3)\lambda_{\max}-\lambda_{\max}^2\}\{-\lambda_i-(\lambda_i-3)\lambda_{\max}-\lambda_{\max}^2\}}}\left[\frac{(1+\lambda_{\max})^3}{(1-\lambda_{\max})^2}\right]^{-\frac{(1-\lambda_{\max})^2}{4\lambda_{\max}}}, \tag{73}$$

Therefore, with success probability $1-\delta$, we can estimate the Torontonian of a matrix $\begin{pmatrix} 0 & R^* \\ R & 0 \end{pmatrix}$ using the number of samples $O(\log\delta^{-1}/\epsilon^2)$ within an additive-error

$$\epsilon\prod_{i=1}^M \frac{16\lambda_{\max}^2}{(1+\lambda_{\max}^3)\sqrt{\{\lambda_i+(\lambda_i+3)\lambda_{\max}-\lambda_{\max}^2\}\{-\lambda_i-(\lambda_i-3)\lambda_{\max}-\lambda_{\max}^2\}}}\left[\frac{(1+\lambda_{\max})^3}{(1-\lambda_{\max})^2}\right]^{-\frac{(1-\lambda_{\max})^2}{4\lambda_{\max}}} := \epsilon\prod_{i=1}^M T_i(\lambda_i). \tag{74}$$

Next, we consider a special case where $R=0$ corresponds to a thermal input state $\{n_i\}_{i=1}^M$. In that case,

$$p_{\text{on}|\text{th}} = \frac{1}{\mathcal{Z}'}\text{Tor}\begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix}, \tag{75}$$

where $\mathcal{Z}' = \prod_{i=1}^M(1+n_i)$, $B = UDU^\dagger$, and $D = \text{diag}\{\frac{n_1}{n_1+1},...,\frac{n_M}{n_M+1}\}$. Meanwhile, the probability $p_{\text{on}|\text{th}}$ can also be written as

$$p_{\text{on}|\text{th}} = \int d^{2M}\boldsymbol{\alpha}\prod_{i=1}^M P_{\text{th},i}(\alpha_i,n_i,\gamma,s)\prod_{j=1}^M f_{\text{on}|\text{th},j}(\beta_j,n_j,\gamma,s), \tag{76}$$

where $P_{\text{th},i}(\alpha_i,n_i,\gamma,s)$ is given in Eq. (42) and $f_{\text{on}|\text{th},j}(\beta_j,n_j,\gamma,s)$ is defined as

$$f_{\text{on}|\text{th},j}(\beta_j,n_j,\gamma,s) = \frac{2n_{\max}+1-s}{2n_{\max}+1-s-\gamma(2n_j+1-s)}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right)e^{-\gamma\frac{2}{2n_{\max}+1-s}|\beta_j|^2}. \tag{77}$$

We set $s = s_{\max} = 2n_{\min}+1$ and $n_j = \frac{\lambda_j}{1-\lambda_j}$. The extreme points are at $0, \pm\beta^*$ with

$$\beta^* = \sqrt{\frac{1}{\lambda_{\min}-1}\log\left(\frac{\gamma-\gamma\lambda_{\max}}{(\lambda_{\min}-1)(\gamma\lambda_{\max}-\gamma-\lambda_{\max}+\lambda_{\min})}\right)}. \tag{78}$$

Choosing $\gamma^* = \frac{1}{2}(1-\lambda_{\max})$, an upper bound on $|f_{\text{on}|\text{th},j}(\beta_j,\lambda_j,\gamma,s)|$ is given by

$$\min_{s,\gamma}\max_{\beta_j}\left|f_{\text{on}|\text{th},j}(\beta_j,\lambda_j,\gamma,s)\right| \leq \frac{4(1-\lambda_j)(\lambda_{\max}-\lambda_{\min})^2\left(\frac{(1-\lambda_{\max})^2}{(1-\lambda_{\min})(1+\lambda_{\max}^2-2\lambda_{\min})}\right)^{\frac{1+\lambda_{\max}^2-2\lambda_{\min}}{2\lambda_{\max}-2\lambda_{\min}}}(\lambda_{\min}-1)}{(1-\lambda_{\max})^2\{\lambda_j(1+\lambda_{\max}^2-2\lambda_{\min})+\lambda_{\min}-\lambda_{\max}(2+(\lambda_{\max}-2)\lambda_{\min}))\}}. \tag{79}$$

Therefore, we can estimate Torontonian of a matrix $\begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix}$ with success probability $1-\delta$ using number of samples $O(\log\delta^{-1}/\epsilon^2)$ within an additive-error

$$\epsilon\prod_{i=1}^M \frac{4(\lambda_{\max}-\lambda_{\min})^2\left[\frac{(1-\lambda_{\max})^2}{(1-\lambda_{\min})(1+\lambda_{\max}^2-2\lambda_{\min})}\right]^{\frac{1+\lambda_{\max}^2-2\lambda_{\min}}{2\lambda_{\max}-2\lambda_{\min}}}(\lambda_{\min}-1)}{(1-\lambda_{\max})^2\{\lambda_i(1+\lambda_{\max}^2-2\lambda_{\min})+\lambda_{\min}-\lambda_{\max}(2+(\lambda_{\max}-2)\lambda_{\min}))\}} := \epsilon\prod_{i=1}^M T_i^B(\lambda_i). \tag{80}$$

Finally, we consider a matrix A' defined by Eq. (67), satisfying Eqs. (25) and (26). Then the Torontonian of matrix A' is related with a squeezed thermal input state $\{r_i,n\}_{i=1}^M$ such that

$$p_{\text{on}|\text{st}} = \frac{\text{Tor}(A')}{\sqrt{|V_Q|}}. \tag{81}$$

Meanwhile, $p_{\mathrm{on|st}}$ is also can be written as

$$p_{\mathrm{on|st}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\frac{2\alpha_{ix}^2}{a_+(r_i,n)-s}-\frac{2\alpha_{iy}^2}{a_-(r_i,n)-s}} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{82}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\left(\frac{2}{a_+(r_i,n)-s}-\gamma\frac{2}{a_{\max}-s}\right)\alpha_{ix}^2-\left(\frac{2}{a_-(r_i,n)-s}-\gamma\frac{2}{a_{\max}-s}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) e^{-\gamma\frac{2}{a_{\max}-s}|\beta_j|^2} \tag{83}$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi}\sqrt{\frac{a_{\max}-s-\gamma(a_+(r_i,n)-s)}{(a_+(r_i,n)-s)(a_{\max}-s)}}\sqrt{\frac{a_{\max}-s-\gamma(a_-(r_i,n)-s)}{(a_-(r_i,n)-s)(a_{\max}-s)}} e^{-\left(\frac{2}{a_+(r_i,n)-s}-\gamma\frac{2}{a_{\max}-s}\right)\alpha_{ix}^2-\left(\frac{2}{a_-(r_i,n)-s}-\gamma\frac{2}{a_{\max}-s}\right)\alpha_{iy}^2}$$

$$\times \prod_{j=1}^{M}\sqrt{\frac{a_{\max}-s}{a_{\max}-s-\gamma(a_+(r_j,n)-s)}}\sqrt{\frac{a_{\max}-s}{a_{\max}-s-\gamma(a_-(r_j,n)-s)}}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) e^{-\gamma\frac{2}{a_{\max}-s}|\beta_j|^2} \tag{84}$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} P_{\mathrm{st},i}(\alpha_i, r_i, n, \gamma, s) \prod_{j=1}^{M} f_{\mathrm{on|st},j}(\beta_j, r_j, n, \gamma, s). \tag{85}$$

where $a_\pm(r_i,n) = (2n+1)e^{\pm 2r_i}$, $a_{\max} = (2n+1)e^{2r_{\max}}$, and $a_{\min} = (2n+1)e^{-2r_{\max}}$. We set $s = s_{\max} = a_{\min}$ and note the extreme points are $0, \pm\beta^*$ with

$$\beta^* = \sqrt{\frac{1}{2}\left((2n+1)e^{-2r_{\max}}+1\right)\log\left(\frac{2e^{2r_{\max}}\left((\gamma-1)(2n+1)+(2n+1)e^{4r_{\max}}+\gamma e^{2r_{\max}}\right)}{\gamma\left(2n+e^{2r_{\max}}+1\right)^2}\right)} \tag{86}$$

By choosing $\gamma^* = \frac{e^{-\tanh r_{\max}}}{n+1}$, an upper bound on $|f_{\mathrm{on|st},j}(\beta_j, r_j, n, \gamma, s)|$ is given by

$$|f_{\mathrm{on|st},j}(\beta_j, r_j, n, \gamma, s)| \le f_{\mathrm{on|st},j}(\beta_j^*, r_j, n, \gamma^*, a_{\min}) \tag{87}$$

$$= \frac{(n+1)^2(2n+1)\left(e^{4r_{\max}}-1\right)^2 e^{r_j+2r_{\max}+2\tanh(r_{\max})}}{\left(2n+e^{2r_{\max}}+1\right)^2}\sqrt{\frac{1}{(n+1)\left(-e^{\tanh(r_{\max})}\right)+(n+1)e^{4r_{\max}+\tanh(r_{\max})}-e^{2(r_j+r_{\max})}+1}} \tag{88}$$

$$\times\sqrt{\frac{1}{(n+1)\left(-e^{2r_j+\tanh(r_{\max})}\right)+(n+1)e^{2r_j+4r_{\max}+\tanh(r_{\max})}+e^{2r_j}-e^{2r_{\max}}}}\,2^F \tag{89}$$

$$\times\left(\frac{e^{2r_{\max}}\left(-\left(2n^2+3n+1\right)e^{\tanh(r_{\max})}+\left(2n^2+3n+1\right)e^{4r_{\max}+\tanh(r_{\max})}+2n+e^{2r_{\max}}+1\right)}{\left(2n+e^{2r_{\max}}+1\right)^2}\right)^F, \tag{90}$$

$$F = -\frac{e^{-\tanh(r_{\max})}\left(n\tanh(r_{\max})+(n+1)\coth(r_{\max})-2n-1\right)}{2(n+1)(2n+1)}. \tag{91}$$

Therefore, we can estimate the Torontonian of a matrix $\begin{pmatrix} C^T & R^* \\ R & C \end{pmatrix}$ with a success probability $1-\delta$ for number of samples $O(\log\delta^{-1}/\epsilon^2)$ within an additive-error as

$$\epsilon\prod_{i=1}^{M}\sqrt{\frac{1}{2}+n(n+1)+(n+\frac{1}{2})\cosh 2r_i}\; f_{\mathrm{on|st},i}(\beta^*, r_i, n, \gamma^*, a_{\min}) := \epsilon\prod_{i=1}^{M} T_i^A(n, r_i). \tag{92}$$

SUPPLEMENTARY NOTE 2 (ESTIMATION OF MATRIX FUNCTIONS WITHIN MULTIPLICATIVE-ERRORS)

The previous section investigates the efficient estimation schemes for various matrix functions within additive-errors. This section proposes a much stronger scheme, such as estimations within multiplicative-errors. We show that for highly classical input states, the s-PQDs representations of the outcome probabilities corresponding to those matrix functions can be written as integrals of log-concave functions. Then we can use an FPRAS to approximate the matrix functions.

Recently, it has been proven that the multiplicative-error estimation of an HPSD matrix is NP-hard [2]. The case of a Hermitian positive definite matrix, i.e., $\lambda_{\min} > 0$, is still unknown yet, but introduced an FPRAS for $1 \leq \lambda_i \leq 2$ [3]. In this section, we reproduce the same result by showing the log-concavity for $\lambda_{\max}/\lambda_{\min} \leq 2$. By a slight generalization of the result in Ref. [3], we give a following lemma:

Lemma 1. *Let $a, b, c \geq 0$ and $q : \mathbb{R}^M \to \mathbb{R}_+$ is a positive semidefinite quadratic form. Then the function $(a + bq(x))e^{-cq(x)}$ is log-concave when $ca \geq b$.*

Proof. It is enough to show the function

$$h(x) = \log(a + bq(x)) - cq(x) \tag{93}$$

is concave when $ca \geq b > 0$. This leads to check that h onto any affine line $x(\tau) = \alpha\tau + \beta$ with $\alpha, \beta \in \mathbb{R}^M$ is concave. Via the affine substitution $\tau \coloneqq (\tau - \beta)/\alpha$, what we need to check is $g''(\tau) \leq 0$ for all $\tau \in \mathbb{R}$, where

$$g(\tau) = \log(a + b(\tau^2 + \gamma^2)) - c(\tau^2 + \gamma^2). \tag{94}$$

Then by straightforward calculation,

$$\begin{aligned} g''(\tau) &= -2c - \frac{4b^2\tau^2}{(a + b(\gamma^2 + \tau^2))^2} + \frac{2b}{a + b(\gamma^2 + \tau^2)} \\ &= \frac{2b(a + b(\gamma^2 + \tau^2)) - 2c(a + b(\gamma^2 + \tau^2))^2 - 4b^2\tau^2}{(a + b(\gamma^2 + \tau^2))^2} \\ &= \frac{-2a(ca - b) - 2b(\gamma^2 + \tau^2)(ca - b) - 2abc(\gamma^2 + \tau^2) - 4b^2\tau^2 - 2b^2(\gamma^2 + \tau^2)^2}{(a + b(\gamma^2 + \tau^2))^2} \\ &\leq 0 \quad \text{for } ca \geq b. \end{aligned}$$

□

From Eq. (40),

$$f_{\mathrm{th},j}(\beta_j, n_i, \gamma, s) \propto \frac{8|\beta_j|^2 + 2(s^2 - 1)}{(s + 1)^3} e^{-\left(\frac{2}{s+1} + \gamma\frac{2}{2n_{\max}+1-s}\right)|\beta_j|^2}. \tag{95}$$

Therefore, the condition for log-concavity of $f_{\mathrm{th},j}(\beta_j, n_i, \gamma, s)$ is written as

$$\left(\frac{2}{s+1} + \gamma\frac{2}{2n_{\max} + 1 - s}\right) 2(s^2 - 1) \geq 8. \tag{96}$$

After putting $\gamma = 1$ and $s = s_{\max} = 2n_{\min} + 1$,

$$\frac{n_{\max}}{n_{\max} + 2} \leq n_{\min}. \tag{97}$$

Substituting $n_i = \frac{\lambda_i}{1-\lambda_i}$,

$$\frac{\lambda_{\max}}{\lambda_{\min}} \leq 2, \tag{98}$$

as desired.

When the input is a squeezed thermal state $\{r_i, n\}_{i=1}^M$, we can find the condition for multiplicative estimation of the hafnian of a particular matrix, which proves Theorem 3 in the main text.

Theorem 3. (FPRAS for hafnian) *Suppose we have a block matrix $A = \begin{pmatrix} R & B \\ B^T & R^* \end{pmatrix}$ with an $M \times M$ complex symmetric matrix R and an $M \times M$ HPSD matrix B, which have decompositions Eqs. (25, 26), then $\mathrm{Haf}(A)$ can be approximated by FPRAS when the parameters satisfy a condition as*

$$n \geq \frac{1}{4}\left(6\sinh(2r_{\max}) + \sqrt{18\cosh(4r_{\max}) - 14} - 2\right), \tag{99}$$

where $r_{\max} = \max_i r_i$.

Proof. Note that the estimated function in the case of all single-photon outcomes is given by

$$f_{\mathrm{st},j}(\beta_j, r_j, n, \gamma, s) \propto \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\left(\frac{2}{s+1}+\gamma\frac{2}{a_{\max}-s}\right)|\beta_j|^2}, \tag{100}$$

where $a_{\max} = (2n+1)e^{2r_{\max}}$. Then from Lemma. 1, the condition for log-concavity of $f_{\mathrm{st},j}(\beta_j, r_j, n, \gamma, s)$ is

$$n \geq \frac{1}{4}\left(6\sinh(2r_{\max}) + \sqrt{18\cosh(4r_{\max}) - 14} - 2\right), \tag{101}$$

where we put $\gamma = 1$ and $s = s_{\max} = (2n+1)e^{-2r_{\max}}$. Thus for a matrix $A = \begin{pmatrix} R & B \\ B^T & R^* \end{pmatrix}$ satisfying conditions Eqs. (25) and (26), Haf(A) can be estimated within a multiplicative-error efficiently when Eq. (101) holds. □

Furthermore, we can apply the same method to the Torontonian. Here, we use the following lemma:

Lemma 2. Let $a, b, c \geq 0$ and $q : \mathbb{R}^M \to \mathbb{R}_+$ is a positive semidefinite quadratic form. Then the function $(a - be^{-bq(x)})e^{-cq(x)}$ is log-concave when $a \geq \frac{b^2+2bc}{c}$.

Proof. By the same argument in Lemma. 1, what we need to check is $g''(\tau) \leq 0$ for all $\tau \in \mathbb{R}$, where

$$g(\tau) = \log(a - be^{-b(\tau^2+\gamma^2)}) - c(\tau^2+\gamma^2). \tag{102}$$

By a straightforward calculation,

$$\begin{aligned} g''(\tau) &= -2c + \frac{-2b^3 - 2ab^2(2b\tau^2-1)e^{b(\gamma^2+\tau^2)}}{\left(b - ae^{b(\gamma^2+\tau^2)}\right)^2} && (103)\\ &= \frac{-2c\left(b^2 - 2abe^{b(\gamma^2+\tau^2)} + a^2e^{2b(\gamma^2+\tau^2)}\right) - 2b^3 - 4ab^3\tau^2e^{b(\gamma^2+\tau^2)} + 2ab^2e^{b(\gamma^2+\tau^2)}}{\left(b - ae^{b(\gamma^2+\tau^2)}\right)^2} && (104)\\ &= \frac{-2cb^2e^{-b(\gamma^2+\tau^2)} + 4abc - 2a^2ce^{b(\gamma^2+\tau^2)} - 2b^3e^{-b(\gamma^2+\tau^2)} - 4ab^3\tau^2 + 2ab^2}{e^{-b(\gamma^2+\tau^2)}\left(b - ae^{b(\gamma^2+\tau^2)}\right)^2} && (105)\\ &\leq \frac{-2cb^2e^{-b(\gamma^2+\tau^2)} + 4abc - 2a^2c - 2b^3e^{-b(\gamma^2+\tau^2)} - 4ab^3\tau^2 + 2ab^2}{e^{-b(\gamma^2+\tau^2)}\left(b - ae^{b(\gamma^2+\tau^2)}\right)^2} && (106)\\ &\leq 0 \quad \text{for } a \geq \frac{b^2+2bc}{c}. \end{aligned}$$

□

First we consider thermal input state $\{n_i\}_{i=1}^M$. From Eq. (77),

$$f_{\mathrm{on|th},j}(\beta_j, n_j, \gamma, s) \propto \left(1 - \frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right)e^{-\gamma\frac{2}{2n_{\max}+1-s}|\beta_j|^2}. \tag{107}$$

By Lemma. 2, the function $(a - be^{-b|\beta|^2})e^{-c|\beta|^2}$ is log-concave when $a \geq \frac{b^2+2bc}{c}$. Consequently, the condition for log-concavity of $f_{\mathrm{on|th},j}(\beta_j, n_j, \gamma, s)$ is written as

$$\frac{2(3 + 2n_{\max} + s)}{(1+s)^2} \leq 1. \tag{108}$$

When $\gamma = 1$ and $s = s_{\max} = 2n_{\min} + 1$, this condition yields

$$\frac{1}{2} \leq \lambda_{\min} \leq \lambda_{\max} \leq \frac{-\lambda_{\min}^2 + 3\lambda_{\min} - 1}{\lambda_{\min}}, \tag{109}$$

where $\lambda_j = \frac{n_j}{1+n_j}$. Thus for an HPSD matrix B, Tor $\begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix}$ can be estimated within a multiplicative-error when eigenvalues $\{\lambda_i\}$ of C satisfy the condition Eq. (109). Note that this condition is more stringent than the permanent case in which there is no restriction for $\lambda_{\max}$ when $\lambda_{\min} \geq \frac{1}{2}$.

Next, when the input is a squeezed thermal state $\{r_i,n\}_{i=1}^M$, $f_{\text{on}|\text{st},j}(\beta_j,r_j,n,\gamma,s)$ is given by

$$f_{\text{on}|\text{st},j}(\beta_j,r_j,n,\gamma,s)\propto\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right)e^{-\gamma\frac{2}{a_{\max}-s}|\beta_j|^2}, \tag{110}$$

where $a_{\max}=(2n+1)e^{2r_{\max}}$. If we set $\gamma=1$ and $s=s_{\max}=(2n+1)e^{-2r_{\max}}$, the condition for log-concavity of $f_{\text{on}|\text{st},j}(\beta_j,r_j,n,\gamma,s)$ is given by using Lemma. 2

$$n\geq\frac{1}{2}\left(e^{2r_{\max}}\sqrt{e^{8r_{\max}}+3}+e^{6r_{\max}}-1\right). \tag{111}$$

Thus for a matrix $A'=\begin{pmatrix}B^T & R^*\\ R & B\end{pmatrix}$ satisfying Eqs. (25) and (26), we can estimate the Torontonian with multiplicative-error when the above condition is satisfied.

SUPPLEMENTARY NOTE 3 (LOWER AND UPPER BOUNDS ON MATRIX FUNCTIONS)

So far, we have suggested polynomial-time randomized algorithms for estimating various matrix functions. Here, we provide lower and upper bounds on the matrix functions by using s-PQD and appropriately choosing s, which is summarized in Supplementary Table II.

From Eq. (37), the permanent of an HPSD matrix is connected to a GBS circuit with a thermal state input. The probability of all single-photon measurements is written as

$$p_{\text{th}}=\int d^{2M}\boldsymbol{\alpha}\prod_{i=1}^M\frac{2}{\pi(2n_i+1-s)}e^{-\frac{2}{(2n_i+1-s)}|\alpha_i|^2}\prod_{j=1}^M\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\frac{2}{s+1}|\beta_j|^2} \tag{112}$$

$$\leq\frac{2}{\prod_{i=1}^M\pi(2n_i+1-s)}\int d^{2M}\boldsymbol{\alpha}\prod_{i=1}^M e^{-\frac{2}{2n_{\max}+1-s}|\alpha_i|^2}\prod_{j=1}^M\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\frac{2}{s+1}|\beta_j|^2} \tag{113}$$

$$=\frac{2}{\prod_{i=1}^M\pi(2n_i+1-s)}\int d^{2M}\boldsymbol{\beta}\prod_{j=1}^M\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\left(\frac{2}{s+1}+\frac{2}{2n_{\max}+1-s}\right)|\beta_j|^2} \tag{114}$$

$$=\prod_{i=1}^M(1-\lambda_i)\frac{\lambda_{\max}\{\lambda_{\max}(2+\lambda_{\min})-2-3\lambda_{\min}\}}{\lambda_i(2+\lambda_{\min})-2-3\lambda_{\min}}, \tag{115}$$

where the inequality is valid when $s\geq1$. Note that we take $s=2n_{\min}+1$ and $n_i=\frac{\lambda_i}{1-\lambda_i}$ in the last equality.

Thus the permanent of an HPSD matrix B are bounded from above as

$$\text{Per}(B)=p_{\text{th}}\mathcal{Z}'\leq\prod_{i=1}^M\frac{\lambda_{\max}\{\lambda_{\max}(2+\lambda_{\min})-2-3\lambda_{\min}\}}{\lambda_i(2+\lambda_{\min})-2-3\lambda_{\min}}:=\prod_{i=1}^M G_i(\lambda_i). \tag{116}$$

For the lower bound, our method gives the same result in Ref. [4].

Let us set $a_{i,\pm}(r_i,n)=(2n+1)e^{\pm2r_i}$, $a_{\min}=(2n+1)e^{-2r_{\max}}$, and $a_{\max}=(2n+1)e^{2r_{\max}}$. When the input state is a squeezed thermal state $\{r_i,n\}_{i=1}^M$ with a fixed n, the probability of all single-photon detection is

$$p_{\text{st}}=\int d^{2M}\boldsymbol{\alpha}\prod_{i=1}^M\frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}}e^{-\frac{2\alpha_{ix}^2}{a_+(r_i,n)-s}-\frac{2\alpha_{iy}^2}{a_-(r_i,n)-s}}\prod_{j=1}^M\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\frac{2|\beta_j|^2}{s+1}} \tag{117}$$

$$\geq\frac{2}{\prod_{i=1}^M\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}}\int d^{2M}\boldsymbol{\alpha}\prod_{i=1}^M e^{-\frac{2|\alpha_i|^2}{a_{\min}-s}}\prod_{j=1}^M\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\frac{2|\beta_j|^2}{s+1}} \tag{118}$$

$$=\frac{2}{\prod_{i=1}^M\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}}\int d^{2M}\boldsymbol{\beta}\prod_{j=1}^M\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\left(\frac{2}{s+1}+\frac{2}{a_{\min}-s}\right)|\beta_j|^2} \tag{119}$$

$$=\prod_{i=1}^M\frac{2\left(-2n+e^{2r_{\max}}-1\right)^2}{\left(2n+e^{2r_{\max}}+1\right)^2\sqrt{-2(2n+1)\cosh(2r_i)+4n(n+1)+2}}, \tag{120}$$

where the inequality holds for $s \geq 1$, and we put $s = 1$ for the last equality. Here, assume $a_{\min} \geq 1$. Then for a matrix $A = \begin{pmatrix} R & B \\ B^T & R^* \end{pmatrix}$ satisfying Eqs. (25) and (26), a lower bound of the Hafnian is written as

$$\mathrm{Haf}(A) = p_{\mathrm{st}} \mathcal{Z}'' \geq \prod_{i=1}^{M} \left(\frac{-2n + e^{2r_{\max}} - 1}{2n + e^{2r_{\max}} + 1} \right)^2 \frac{\sqrt{\frac{1}{2} + n(n+1) + (n+\frac{1}{2})\cosh 2r_i}}{\sqrt{\frac{1}{2} + n(n+1) - (n+\frac{1}{2})\cosh 2r_i}} := \prod_{i=1}^{M} L_i^H(r_i, n), \tag{121}$$

$$\mathcal{Z}'' = \sqrt{|V_Q|} = \prod_{i=1}^{M} \sqrt{\frac{1}{2} + n(n+1) + (n + \frac{1}{2}) \cosh 2r_i}. \tag{122}$$

Similarly, an upper bound can be obtained as

$$p_{\mathrm{st}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\frac{2\alpha_{ix}^2}{a_+(r_i,n)-s} - \frac{2\alpha_{iy}^2}{a_-(r_i,n)-s}} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\frac{2|\beta_j|^2}{s+1}} \tag{123}$$

$$\leq \frac{2}{\prod_{i=1}^{M} \pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} e^{-\frac{2|\alpha_i|^2}{a_{\max}-s}} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-\frac{2|\beta_j|^2}{s+1}} \tag{124}$$

$$= \frac{2}{\prod_{i=1}^{M} \pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} \int d^{2M}\boldsymbol{\beta} \prod_{j=1}^{M} \frac{8|\beta_j|^2 + 2(s^2-1)}{(s+1)^3} e^{-(\frac{2}{s+1} + \frac{2}{a_{\max}-s})|\beta_j|^2} \tag{125}$$

$$= \prod_{i=1}^{M} \left(\frac{e^{r_{\max}} n + \sinh r_{\max}}{(1+n)\cosh r_{\max} + n \sinh r_{\max}} \right)^2 \frac{1}{\sqrt{\frac{1}{2} + n(n+1) - (n+\frac{1}{2})\cosh 2r_i}}, \tag{126}$$

where we put $s = 1$ for the last inequality. Consequently, an upper bound is obtained as

$$\mathrm{Haf}(A) = p_{\mathrm{st}} \mathcal{Z}'' \leq \prod_{i=1}^{M} \left(\frac{e^{r_{\max}} n + \sinh r_{\max}}{(1+n)\cosh r_{\max} + n \sinh r_{\max}} \right)^2 \frac{\sqrt{\frac{1}{2} + n(n+1) + (n+\frac{1}{2})\cosh 2r_i}}{\sqrt{\frac{1}{2} + n(n+1) - (n+\frac{1}{2})\cosh 2r_i}} := \prod_{i=1}^{M} G_i^H(r_i, n). \tag{127}$$

Moreover, our approach can apply to bounds on the Torontonian. First, we consider thermal state input $\{n_i\}_{i=1}^M$ and the probability of all "click" outcomes is written as

$$p_{\mathrm{on|th}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi(2n_i+1-s)} e^{-\frac{2}{2n_i+1-s}|\alpha_i|^2} \prod_{j=1}^{M} \left(1 - \frac{2}{s+1} e^{-\frac{2}{s+1}|\beta_j|^2} \right) \tag{128}$$

$$\geq \frac{2}{\prod_{i=1}^{M} \pi(2n_i+1-s)} \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} e^{-\frac{2}{2n_{\min}+1-s}|\alpha_i|^2} \prod_{j=1}^{M} \left(1 - \frac{2}{s+1} e^{-\frac{2}{s+1}|\beta_j|^2} \right) \tag{129}$$

$$= \frac{2}{\prod_{i=1}^{M} \pi(2n_i+1-s)} \int d^{2M}\boldsymbol{\beta} \prod_{j=1}^{M} \left(1 - \frac{2}{s+1} e^{-\frac{2}{s+1}|\beta_j|^2} \right) e^{-\frac{2}{2n_{\min}+1-s}|\beta_j|^2} \tag{130}$$

$$= \prod_{i=1}^{M} \frac{(1-\lambda_i)\lambda_{\min}^2}{\lambda_i(1-\lambda_{\min})}, \tag{131}$$

where the inequality is valid when $s \geq 1$, and we take $s = 1$ for the last equality. Thus for an $M \times M$ HPSD matrix B, the Torontonian of $\begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix}$ is

$$\mathrm{Tor} \begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix} = p_{\mathrm{on|th}} \mathcal{Z}' \geq \prod_{i=1}^{M} \frac{\lambda_{\min}^2}{\lambda_i(1-\lambda_{\min})}. \tag{132}$$

Similarly, we also obtain an upper bound as

$$p_{\text{on}|\text{th}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi(2n_i+1-s)} e^{-\frac{2}{2n_i+1-s}|\alpha_i|^2} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{133}$$

$$\leq \frac{2}{\prod_{i=1}^{M}\pi(2n_i+1-s)} \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} e^{-\frac{2}{2n_{\max}+1-s}|\alpha_i|^2} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{134}$$

$$= \frac{2}{\prod_{i=1}^{M}\pi(2n_i+1-s)} \int d^{2M}\boldsymbol{\beta} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) e^{-\frac{2}{2n_{\max}+1-s}|\beta_j|^2} \tag{135}$$

$$= \prod_{i=1}^{M} \frac{(1-\lambda_i)\lambda_{\max}^2}{\lambda_i(1-\lambda_{\max})}, \tag{136}$$

where we take $s=1$ for the last equality. Consequently,

$$\text{Tor}\begin{pmatrix} B^T & 0 \\ 0 & B \end{pmatrix} = p_{\text{on}|\text{th}}\mathcal{Z}' \leq \prod_{i=1}^{M} \frac{\lambda_{\max}^2}{\lambda_i(1-\lambda_{\max})}. \tag{137}$$

Next, when the input state is a squeezed thermal state $\{r_i, n\}_{i=1}^{M}$. Then the probability of all "click" detection is

$$p_{\text{on}|\text{st}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\frac{2\alpha_{ix}^2}{a_+(r_i,n)-s}-\frac{2\alpha_{iy}^2}{a_-(r_i,n)-s}} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{138}$$

$$\geq \frac{2}{\prod_{i=1}^{M}\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} e^{-\frac{2|\alpha_i|^2}{a_{\min}-s}} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{139}$$

$$= \frac{2}{\prod_{i=1}^{M}\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} \int d^{2M}\boldsymbol{\beta} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) e^{-\frac{2|\beta_j|^2}{a_{\min}-s}} \tag{140}$$

$$= \prod_{i=1}^{M} \frac{e^{-2r_{\max}}(1-e^{2r_{\max}}+2n)^2}{2(1+e^{2r_{\max}}+2n)} \frac{1}{\sqrt{\frac{1}{2}+n(n+1)-(n+\frac{1}{2})\cosh 2r_i}}, \tag{141}$$

where $a_{i,\pm}(r_i,n) = (2n+1)e^{\pm 2r_i}$, $a_{\min} = (2n+1)e^{-2r_{\max}}$, $a_{\max} = (2n+1)e^{2r_{\max}}$, and we put $s=1$ for the last inequality. Assume $a_{\min} \geq 1$. Then for a matrix $A' = \begin{pmatrix} B^T & R^* \\ R & B \end{pmatrix}$ satisfying Eqs. (25) and (26), a lower bound of the Torontonian is given by

$$\text{Tor}(A') = p_{\text{on}|\text{st}}\mathcal{Z}'' \geq \prod_{i=1}^{M} \frac{e^{-2r_{\max}}(1-e^{2r_{\max}}+2n)^2}{2(1+e^{2r_{\max}}+2n)} \frac{\sqrt{\frac{1}{2}+n(n+1)+(n+\frac{1}{2})\cosh 2r_i}}{\sqrt{\frac{1}{2}+n(n+1)-(n+\frac{1}{2})\cosh 2r_i}} := \prod_{i=1}^{M} L_i^T(r_i,n). \tag{142}$$

An upper bound can be obtained by a similar method, such as

$$p_{\text{on}|\text{st}} = \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} e^{-\frac{2\alpha_{ix}^2}{a_+(r_i,n)-s}-\frac{2\alpha_{iy}^2}{a_-(r_i,n)-s}} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{143}$$

$$\leq \frac{2}{\prod_{i=1}^{M}\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} e^{-\frac{2|\alpha_i|^2}{a_{\max}-s}} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) \tag{144}$$

$$= \frac{2}{\prod_{i=1}^{M}\pi\sqrt{(a_+(r_i,n)-s)(a_-(r_i,n)-s)}} \int d^{2M}\boldsymbol{\beta} \prod_{j=1}^{M}\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right) e^{-\frac{2|\beta_j|^2}{a_{\max}-s}} \tag{145}$$

$$= \prod_{i=1}^{M} \frac{e^{r_{\max}}(e^{r_{\max}}n+\sinh r_{\max})^2}{(1+n)\cosh r_{\max}+n\sinh r_{\max}} \frac{1}{\sqrt{\frac{1}{2}+n(n+1)-(n+\frac{1}{2})\cosh 2r_i}}, \tag{146}$$

where we put $s=1$ for the last inequality. Consequently, an upper bound of the $\mathrm{Tor}(A')$ is obtained as

$$\mathrm{Tor}(A') = p_{\mathrm{on|st}}\mathcal{Z}'' \leq \prod_{i=1}^{M} \frac{e^{r_{\max}}(e^{r_{\max}}n+\sinh r_{\max})^2}{(1+n)\cosh r_{\max}+n\sinh r_{\max}} \frac{\sqrt{\frac{1}{2}+n(n+1)+(n+\frac{1}{2})\cosh 2r_i}}{\sqrt{\frac{1}{2}+n(n+1)-(n+\frac{1}{2})\cosh 2r_i}} := \prod_{i=1}^{M} G_i^T(r_i,n). \quad (147)$$

SUPPLEMENTARY NOTE 4 (SIMULABILITY OF GAUSSIAN BOSON SAMPLING)

Our approximation algorithm for outcome probabilities of a linear optical circuit have applications not only for the matrix functions, but also for Gaussian boson sampling, which is crucial for the demonstration of quantum supremacy [8]. From the results in Refs. [9], we have three level of a hierarchy of notions of classical simulation as following:

1. Poly-box : Inverse-polynomial additive-error approximation of any outcome probabilities including any marginals.

2. ϵ-simulation : Approximate sampling simulation of probability distributions with ϵ-error in total variation distance.

3. Multiplicative precision estimation : multiplicative-error approximation of any outcome probabilities including any marginals.

A poly-box can be promoted to ϵ-simulation when the outcomes are poly-sparse, and multiplicative precision estimator implies ϵ-simulation [9]. In our work, we can investigate this hierarchy in the GBS via a degree of classicality of the input state, $s_{\max}$. To do that, we consider a lossy GBS, in which the input state is product of lossy squeezed state having the covariance matrix on ith mode with $V_i = \frac{1}{2}\begin{pmatrix}\eta e^{2r_i}+1-\eta & 0 \\ 0 & \eta e^{-2r_i}+1-\eta\end{pmatrix} := \frac{1}{2}\begin{pmatrix}a_{i+}(\eta,r_i) & 0 \\ 0 & a_{i-}(\eta,r_i)\end{pmatrix}$. Note that $-1<s\leq s_{\max}\leq 1$ for a lossy squeezed state, and $s_{\max}=a_-(\eta,r_{\max})$. Thus $s_{\max}$ goes to 0 as the maximum squeezing parameter $r_{\max}\to\infty$, which is consistent with the fact that a general Gaussian state can be well described by Wigner distribution. Then the outcome probability $p_{\mathrm{GBS}}(\boldsymbol{m})$ is given by

$$p_{\mathrm{GBS}}(\boldsymbol{m}) = \pi^M \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} W_{V_i}^{(s)}(\alpha_i) \prod_{j=1}^{M} W_{\Pi_{m_j}}^{(-s)}(\beta_j) \quad (148)$$

$$= \pi^M \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{1}{\pi\sqrt{\det(V_i-s/2)}} e^{-\boldsymbol{\alpha}_i(V_i-s/2)^{-1}\boldsymbol{\alpha}_i^T} \prod_{j=1}^{M} \frac{2}{\pi(s+1)}\left(\frac{s-1}{s+1}\right)^{m_j} \mathrm{L}_{m_j}\left(\frac{4|\beta_j|^2}{1-s^2}\right) e^{-\frac{2|\beta_j|^2}{s+1}} \quad (149)$$

$$= \pi^M \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi\sqrt{(a_{i+}(\eta,r_i)-s)(a_{i-}(\eta,r_i)-s)}} e^{-\frac{2\alpha_{ix}^2}{a_{i+}(\eta,r_i)-s}-\frac{2\alpha_{iy}^2}{a_{i-}(\eta,r_i)-s}} \prod_{j=1}^{M} \frac{2}{\pi(s+1)}\left(\frac{s-1}{s+1}\right)^{m_j} \mathrm{L}_{m_j}\left(\frac{4|\beta_j|^2}{1-s^2}\right) e^{-\frac{2|\beta_j|^2}{s+1}} \quad (150)$$

$$= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} \frac{2}{\pi} \sqrt{\frac{a_+(\eta,r_{\max})-s-\gamma(a_{i+}(\eta,r_i)-s)}{(a_{i+}(\eta,r_i)-s)(a_+(\eta,r_{\max})-s)}} \sqrt{\frac{a_+(\eta,r_{\max})-s-\gamma(a_{i-}(\eta,r_i)-s)}{(a_{i-}(\eta,r_i)-s)(a_+(\eta,r_{\max})-s)}} \quad (151)$$

$$\times e^{-\left(\frac{2}{a_{i+}(\eta,r_i)-s}-\gamma\frac{2}{a_+(\eta,r_{\max})-s}\right)\alpha_{ix}^2-\left(\frac{2}{a_{i-}(\eta,r_i)-s}-\gamma\frac{2}{a_+(\eta,r_{\max})-s}\right)\alpha_{iy}^2} \quad (152)$$

$$\times \prod_{j=1}^{M} \sqrt{\frac{a_+(\eta,r_{\max})-s}{a_+(\eta,r_{\max})-s-\gamma(a_{i+}(\eta,r_i)-s)}} \sqrt{\frac{a_+(\eta,r_{\max})-s}{a_+(\eta,r_{\max})-s-\gamma(a_{i-}(\eta,r_i)-s)}} \quad (153)$$

$$\times \frac{2}{s+1}\left(\frac{s-1}{s+1}\right)^{m_j} \mathrm{L}_{m_j}\left(\frac{4|\beta_j|^2}{1-s^2}\right) e^{-\left(\frac{2}{s+1}+\gamma\frac{2}{a_+(\eta,r_{\max})-s}\right)|\beta_j|^2} \quad (154)$$

$$:= \int d^{2M}\boldsymbol{\alpha} \prod_{i=1}^{M} P_{\mathrm{GBS},i}(\alpha_i,\eta,r_i,\gamma,s) \prod_{j=1}^{M} f_{\mathrm{GBS},j}(\beta_j,\eta,r_j,m_j,\gamma,s), \quad (155)$$

where $a_\pm(\eta,r_{\max}) = \eta e^{\pm 2r_{\max}}+1-\eta$, and $\gamma\in(0,1]$ is a parameter modulating the Gaussian factor such as $\gamma\to 1$ ($\gamma=0$) means maximum (no) shifting. The maximum shifting is limited by the maximum squeezing parameter

$r_{\max}$. To check the poly-box condition, let us first consider the single-mode estimate for the single-photon outcome. Explicitly,

$$f_{\mathrm{GBS},j}(\beta_j,\eta,r_j,1,\gamma,s)=\sqrt{\frac{a_+(\eta,r_{\max})-s}{a_+(\eta,r_{\max})-s-\gamma(a_{j+}(\eta,r_j)-s)}}\sqrt{\frac{a_+(\eta,r_{\max})-s}{a_+(\eta,r_{\max})-s-\gamma(a_{j-}(\eta,r_j)-s)}} \times\frac{8|\beta_j|^2+2(s^2-1)}{(s+1)^3}e^{-\left(\frac{2}{s+1}+\gamma\frac{2}{a_+(\eta,r_{\max})-s}\right)|\beta_j|^2}. \tag{156}$$

We can efficiently estimate the probability if $\max_{\beta_j}|f_{\mathrm{GBS},j}|\leq 1$ for all j. An upper bound of the absolute value of the estimate is given by

$$\min_{s,\gamma}\max_{\beta_j}|f_{\mathrm{GBS},j}(\beta_j,\eta,r_j,1,\gamma,s)|\leq\max_{\beta_j}|f_{\mathrm{GBS},j}(\beta_j,\eta,r_{\max},1,0,s_{\max})|, \tag{157}$$

for given $\eta,r_{\max}$, and $\gamma=0$, $s=s_{\max}$ for the inequality. Then from the condition $\max_{\beta_j}|f_j(\beta_j,\eta,r_{\max},1,0,s_{\max})|\leq 1$, $s_{\max}$ satisfies $s_{\max}\geq\sqrt{5}-2\simeq 0.236$. This corresponds to $r_{\max}\leq\frac{1}{2}\log(2+\sqrt{5})\simeq 0.722$ for an ideal GBS ($\eta=1$). However, if we allow the photon loss, any squeezed input state is possible when $\eta\leq 3-\sqrt{5}\simeq 0.764$, which is much higher transimissivity than those used in current experiments [10, 11]. Next, we need to check whether this condition is valid for any other outcomes. From the behavior of $f_j(\beta,\eta,r,m,0,s)$, we can find out that

$$\max_{\beta}|f_j(\beta,\eta,r,m,0,s)|\leq\max_{\beta}|f_j(\beta,\eta,r,1,0,s)|, \tag{158}$$

for $m\geq 2$ and $s\geq 0$. Finally, we consider $n=0$ for zero-photon detection and $f_j=1$ for the marginalized probability owing to the normalization of measurement operators. In both cases, the integrals for β_j's can be easily computed because $f_j(\beta_j)$ and β_j components in $P(\boldsymbol{\alpha})$ are just Gaussian distributions. Therefore, we can always perform the integrals including β_j's corresponding to zero-photon or marginalized one, and estimate remaining terms. Furthermore, we examine the case of threshold detectors instead of number resolving measurements [7]. The corresponding $f_{\mathrm{on},j}$ for a 'click' event is written as

$$f_{\mathrm{on},j}(\beta_j,\eta,r_j,\gamma,s)=\sqrt{\frac{a_+(\eta,r_{\max})-s}{a_+(\eta,r_{\max})-s-\gamma(a_{j+}(\eta,r_j)-s)}}\sqrt{\frac{a_+(\eta,r_{\max})-s}{a_+(\eta,r_{\max})-s-\gamma(a_{j-}(\eta,r_j)-s)}} \times\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right)e^{-\gamma\frac{2}{a_+(\eta,r_{\max})-s}|\beta_j|^2}. \tag{159}$$

Then for $\gamma=0$ and $s=0$, the range of $f_{\mathrm{on},j}(\beta_j,\eta,r_j,0,0)=1-2e^{-2|\beta_j|^2}$ is on $[-1,1]$, thus the poly-box condition is satisfied for all input squeezing and loss parameter.

Now we investigate whether an efficient estimation of GBS probability within multiplicative-error is possible. To do that, we consider Gaussian states which can have $s_{\max}>1$, where the covariance matrix of ith mode state is given by

$$V_i=\frac{1}{2}\begin{pmatrix}\eta e^{2r_i}+(2n_{\mathrm{th}}+1)(1-\eta) & 0\\ 0 & \eta e^{-2r_i}+(2n_{\mathrm{th}}+1)(1-\eta)\end{pmatrix}:=\frac{1}{2}\begin{pmatrix}a_{i+}(\eta,n_{\mathrm{th}},r_i) & 0\\ 0 & a_{i-}(\eta,n_{\mathrm{th}},r_i)\end{pmatrix}. \tag{160}$$

These are squeezed thermal states, in which pure squeezed states undergo a thermal noise with average photon number n_{th} instead of the vacuum loss. In this case $-1<s\leq a_-(\eta,n_{\mathrm{th}},r_{\max})$ for given η, n_{th}, and $r_{\max}$. Then we need to check the log-concavity of $f_{\mathrm{on},j}$ such that

$$f_{\mathrm{on},j}(\beta_j,\eta,n_{\mathrm{th}},r_i,\gamma,s)\propto\left(1-\frac{2}{s+1}e^{-\frac{2}{s+1}|\beta_j|^2}\right)e^{-\gamma\frac{2}{a_+(\eta,N,r_{\max})-s}|\beta_j|^2}. \tag{161}$$

From Lemma 2, the condition for log-concavity of $f_{\mathrm{on},j}$ when $\gamma\to 1$ is

$$\sqrt{4e^{r_{\max}}\eta\sinh r_{\max}+4n_{\mathrm{th}}(1-\eta)+5}\leq s\leq a_-(\eta,n_{\mathrm{th}},r_{\max}), \tag{162}$$

where $a_-(\eta,n_{\mathrm{th}},r_{\max})=\eta^{-2r_{\max}}+(2n_{\mathrm{th}}+1)(1-\eta)$. Thus for given $r_{\max}$ and η, the average photon number of thermal noise N satisfies

$$n_{\mathrm{th}}\geq\frac{e^{-r_{\max}}\eta\sinh r+\sqrt{1+\eta\sinh 2r_{\max}}}{1-\eta}>1. \tag{163}$$

For instance, if $\eta=0.5$ and $r_{\max}=1$, then $n_{\mathrm{th}}\geq n_{\mathrm{th}}^*\simeq 3.79$ for the multiplicative-error estimation of the probability, and the minimum value of $s_{\max}$ is 3 when $\eta\to 0$.

SUPPLEMENTARY REFERENCES

[1] Barvinok, A. *Combinatorics and complexity of partition functions*, vol. 9 (Springer, 2016).

[2] Meiburg, A. Inapproximability of positive semidefinite permanents and quantum state tomography. In *63rd Symp. Found. Comput. Sci. (FOCS)*, 58–68 (IEEE, 2022).

[3] Barvinok, A. A remark on approximating permanents of positive definite matrices. *Linear Algebra Appl.* **608**, 399–406 (2021).

[4] Chakhmakhchyan, L., Cerf, N. J. & Garcia-Patron, R. Quantum-inspired algorithm for estimating the permanent of positive semidefinite matrices. *Phys. Rev. A* **96**, 022329 (2017).

[5] Bunse-Gerstner, A. & Gragg, W. B. Singular value decompositions of complex symmetric matrices. *J. Comput. Appl. Math.* **21**, 41–54 (1988).

[6] Hoeffding, W. *Probability Inequalities for sums of Bounded Random Variables*, 409–426 (Springer New York, New York, NY, 1994).

[7] Quesada, N., Arrazola, J. M. & Killoran, N. Gaussian boson sampling using threshold detectors. *Phys. Rev. A* **98**, 062322 (2018).

[8] Hamilton, C. S. *et al.* Gaussian boson sampling. *Phys. Rev. Lett.* **119**, 170501 (2017).

[9] Pashayan, H., Bartlett, S. D. & Gross, D. From estimation of quantum probabilities to simulation of quantum circuits. *Quantum* **4**, 223 (2020).

[10] Zhong, H.-S. *et al.* Phase-programmable Gaussian boson sampling using stimulated squeezed light. *Phys. Rev. Lett.* **127**, 180502 (2021).

[11] Madsen, L. S. *et al.* Quantum computational advantage with a programmable photonic processor. *Nature* **606**, 75–81 (2022).