THE UNIVERSITY OF CHICAGO

TEICHMULLER DYNAMICS AND HODGE THEORY

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE 2016

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To the memory of my parents, Boris and Eudochia Filip

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ABSTRACT

This thesis is concerned with applications of Hodge theory in Teichmüller dynamics. The moduli space pairs (X, ω) of Riemann surfaces with a holomorphic 1-form carries a natural action of the group $\mathrm{SL}_2\,\mathbb{R}$. The diagonal subgroup gives the Teichmüller geodesic flow, while full $\mathrm{SL}_2\,\mathbb{R}$-orbits give Teichmüller disks. The work of Eskin, Mirzakhani, and Mohammadi shows that the closure of a Teichmüller disk is always an immersed submanifold, usually called an "affine invariant submanifold" since it carries an affine structure.

The first part of the thesis studies the Variation of Hodge Structures (VHS) over an affine manifold, and more generally over a Teichmüller disk. The affine manifold carries a finite measure and this allows one to extend many of the results in the ordinary theory of VHS to this setting.

The second part of the thesis studies the Variation of *Mixed* Hodge Structures that arises in this setting. It shows that a certain part of it is particularly simple - it is split. This, in turn, allows for an algebraic characterization of affine manifolds.

ACKNOWLEDGMENTS

I am extremely grateful to my advisor, Alex Eskin, for his constant support and help throughout graduate school. He suggested the circle of problems treated in this thesis and helped a lot with many aspects of the work. The support and encouragement that he provided at times when I was completely stuck were crucial for all subsequent progress. I learned a great deal of mathematics from him and benefited from his generosity with ideas. Thank you!

I have also learned a great deal from conversations and lectures by professors at UChicago. The community in the math department is fantastic and I can't think of a better place where I could have learned more. The mentorship and advice that I have received, mathematical and otherwise, is very much appreciated. Conversations with Madhav Nori were crucial for work presented here, and his lectures and points of view have shaped my (modest) knowledge of the subject. I am grateful to Benson Farb for his advice and support, as well as insightful lectures. The lectures on geometry, Teichmüller theory, and dynamics by Danny Calegari, Howard Masur, Leonid Polterovich, and Amie Wilkinson provided a wonderful introduction to these topics and have opened the doors to them. I hope to continue learning. I am also grateful to Matt Emerton, Victor Ginzburg, and Kazuya Kato for their courses – they served as a continuous source of inspiration and provided glimpses into other parts of mathematics. Conversations with David Aulicino, Jon Chaika, Ronen Mukamel, and Aaron Silberstein have also been tremendously helpful throughout my years here.

The student community has been a joy to be around and I have learned a lot from my peers. Conversations with Alex Wright have contributed a lot to my development. My office mate Max Engelstein has always been helpful with anything analysis-related, and I've learned a lot from Paul Apisa, Asilata Bapat, Sean Howe, Dan Le, Wouter van Limbeek, Nick Salter, Vaidehee Thatee, and Bena Tshishiku.

Outside UChicago, the support, encouragement, and mathematics of Giovanni Forni and Anton Zorich have been crucial for all my work.

Talia has brought light and happiness to my life and I cannot thank her enough for that.

I wouldn't have been here without my family. I owe everything to my parents and their lives will always be an example to follow. My sister Irina and my nephews Simion and Alexandru have been a source of joy and comfort and their support has been keeping me afloat.

CHAPTER 1

SEMISIMPLICITY OF THE KONTSEVICH–ZORICH COCYCLE

This chapter contains the results obtained in [Fil16a].

1.1 Introduction

1.1.1 Background

On a Riemann surface X giving a holomorphic 1-form ω is the same as giving charts, away from the zeroes of ω, where the transition maps are of the form $z \mapsto z + c$. These are the charts in which $\omega = dz$. The pair (X, ω) is called a "flat surface" and the group $\mathrm{SL}_2 \mathbb{R}$ naturally acts on the set of flat surfaces. The action is on the charts and transition maps, after the identification of $\mathbb{C}$ with $\mathbb{R}^2$. See the survey of Zorich [Zor06] for lots of context and motivation.

Flat surfaces with the same combinatorics of zeroes of the holomorphic 1-form have a moduli space called a stratum and denoted $\mathcal{H}(\kappa)$, where κ is the multi-index encoding the zeroes. The action of $\mathrm{SL}_2 \mathbb{R}$ preserves the Masur-Veech probability measure on such a stratum ([Mas82, Vee82]) and one is interested in other possible invariant measures.

Recent work of Eskin and Mirzakhani in [EM13] shows that such measures must be of a very particular geometric form. Further work by Eskin, Mirzakhani, and Mohammadi [EMM15] shows that these measures share properties with the homogeneous setting and unipotent actions. In particular, all $\mathrm{SL}_2 \mathbb{R}$-orbit closures must be (affine invariant) manifolds.

To describe the local form of the measures, recall that on the stratum $\mathcal{H}(\kappa)$ there are natural period coordinates (see [Zor06, Section 3.3]). Given a flat surface X with zeroes of ω denoted S, local period coordinates are given by the relative cohomology group $H^1(X, S; \mathbb{Z}) \otimes \mathbb{R}^2$. The action of $\mathrm{SL}_2 \mathbb{R}$ is on the $\mathbb{R}^2$ factor.

An *affine invariant manifold* $\mathcal{M}$ is an immersed closed submanifold of the stratum which in local period coordinates has an associated subspace $T_{\mathcal{M}} \subset H^1(X, S; \mathbb{R})$. The manifold $\mathcal{M}$ must equal $T_{\mathcal{M}} \otimes \mathbb{R}^2$, and it then carries a natural invariant probability measure. The results in [EM13] imply that any ergodic $\mathrm{SL}_2\,\mathbb{R}$-invariant measure has to be of this form.

By work of McMullen (see for instance [McM03, McM07]) in genus 2 a much more detailed description is available. Some of those results have also been independently obtained by Calta [Cal04].

In the case of Teichmüller curves (affine manifolds of minimal possible dimension), many results have been obtained by Möller (see for example [Möl06b]). They use techniques from variations of Hodge structures, but are on the algebro-geometric side. In that context, for dimension reasons $\mathrm{SL}_2\,\mathbb{R}$-invariant bundles (see below) are globally flat. Moreover, the Teichmüller curve is automatically algebraic and Deligne's semisimplicity results are available.

Part of this chapter is concerned with extending the above results to affine manifolds. The results from the global theory of variations of Hodge structures cannot be applied directly, because the structure at infinity of the affine manifolds is not clear. This difficulty is bypassed using ergodic theory and the $\mathrm{SL}_2\,\mathbb{R}$-action. In fact, the methods in this chapter are used in [Fil16b] (and the next chapter) to prove that affine invariant manifolds are quasi-projective varieties.

Our methods also provide an alternative route to some of the global consequences of Schmid's work [Sch73]. The main tools come from ergodic theory, rather than a local analysis of variations of Hodge structures on punctured discs. The appendix contains a discussion of this application.

We also obtain rigidity results for $\mathrm{SL}_{\mathbb{R}}$-invariant bundles. In particular, any such measurable bundle has to be real-analytic. This is used in the work of Chaika–Eskin [CE15] on Oseledets regularity.

1.1.2 Main results

Recall that over a stratum we have the local system $E_{\mathbb{Z}}$ corresponding to the absolute cohomology groups $H^1(X;\mathbb{Z})$ giving the Gauss–Manin connection. The corresponding cocycle for the $\mathrm{SL}_2\mathbb{R}$-action is called the Kontsevich-Zorich cocycle (see §1.2.1 for definitions).

Our theorems concern $\mathrm{SL}_2\mathbb{R}$-invariant *subbundles* of $E_{\mathbb{R}} := E_{\mathbb{Z}} \otimes \mathbb{R}$ or $E_{\mathbb{C}} = E_{\mathbb{Z}} \otimes \mathbb{C}$ (either of them denoted E). To define these, fix a finite ergodic $\mathrm{SL}_2\mathbb{R}$-invariant measure μ. An $\mathrm{SL}_2\mathbb{R}$-invariant bundle is any measurable subbundle of E which is invariant under parallel transport along a.e. $\mathrm{SL}_2\mathbb{R}$-orbit. It is defined μ-a.e. The results apply to subbundles of any tensor power of the Hodge bundle (still denoted E).

We have the Hodge decomposition $E_{\mathbb{C}} = \oplus E^{p,q}$ and the bundles $E^{p,q}$ are typically not $\mathrm{SL}_2\mathbb{R}$-invariant. Thus, the Hodge metric on E is not flat for the Gauss–Manin connection. A brief review of facts from Hodge theory is in Section 1.3.

The main result (see Theorem 1.6.8), stated below, is that nevertheless any invariant subbundle must respect the Hodge structure. It holds for any $\mathrm{SL}_2(\mathbb{R})$-invariant measure μ, and the proof does not make use of the Eskin–Mirzakhani–Mohammadi measure rigidity theorems [EM13, EMM15] which implies that μ is Lebesgue on an affine manifold.

Theorem 1.1.1 (Deligne semisimplicity) *The bundle E decomposes as a direct sum of $\mathrm{SL}_2\mathbb{R}$-invariant bundles:*

$$E = \bigoplus_i E_i \tag{1.1.1}$$

The decomposition is Hodge orthogonal and respects the Hodge structure, namely $E_i \otimes \mathbb{C}$ is the direct sum of $(E_i \otimes \mathbb{C}) \cap E^{p,q}$.

The decomposition (1.1.1) *is into "isotypical components", in the following sense. There exist $\mathrm{SL}_2\mathbb{R}$-invariant bundles $V_i \subset E_i$ and vector spaces W_i, each equipped with Hodge* *structures and compatible actions of division algebras A_i, such that we have the isomorphism*

$$E_i \cong V_i \otimes_{A_i} W_i \tag{1.1.2}$$

Moreover, the isomorphism is compatible with the Hodge structures on the terms involved. Typically, A_i is $\mathbb{R}$ and W_i is one-dimensional, thus $E_i \cong V_i$. However, see Remark 1.6.10 for a discussion of how the isotypical components W_i and symmetries A_i can arise.

Any $\mathrm{SL}_2 \mathbb{R}$-invariant bundle $V' \subset E$ is of the form

$$V' = \bigoplus_i V_i \otimes_{A_i} W_i' \tag{1.1.3}$$

where $W_i' \subset W_i$ are A_i-submodules.

For the case of the complexified bundle $E_{\mathbb{C}}$ the subbundles V_i have a Hodge structure in the following sense. They have components $V_i^{j,w-j}$ and the filtrations $V_i^p := \oplus_{p\leq j} V_i^{j,w-j}$ vary holomorphically on Teichmüller disks. The bundles V_i are $\mathrm{SL}_2 \mathbb{R}$-invariant and carry a flat indefinite hermitian metric.

The above theorem is a consequence of the following one (proved as Theorem 1.6.5 in the main text). Informally stated, it means that any invariant bundle has to respect the definite and indefinite Hodge metrics. See Section 1.3 for an introduction to the notions below, and in particular Definition 1.3.4 for the Weil operator C and Definition 1.3.5 for how it interacts with the Hodge metric. Remark 1.3.6 explains why the Weil and Hodge star operators agree for holomorphic 1-forms. Remark 1.3.14 explains the relevance of the Weil operator for semisimplicity.

Theorem 1.1.2 *Recall that the Weil operator C acts as $(\sqrt{-1})^{p-q}$ on $E^{p,q}$. Suppose $V \subset E$ is an $\mathrm{SL}_2 \mathbb{R}$-invariant subbundle.*

Then $C \cdot V$ is also an $\mathrm{SL}_2 \mathbb{R}$-invariant subbundle. In fact, C can be any element of the

Deligne torus (see Definition 1.3.2).

The theorems concern $\mathrm{SL}_2\mathbb{R}$-invariant bundles, which need not be flat in other directions. Their proofs do not use the local structure of invariant measures obtained by Eskin and Mirzakhani [EM13]. In particular, these semisimplicity results are not implied by the usual Deligne semisimplicity (even assuming algebraicity of affine manifolds).

An instance of an $\mathrm{SL}_2\mathbb{R}$-invariant but not flat bundle is the tautological plane. By definition, at (X, ω) it is the span $\langle Re(\omega)\rangle\, Im(\omega) \subset H^1(X; \mathbb{R})$. This bundle is not flat unless the manifold is a Teichmüller curve, or "rank 1" (see [Wri14]). Results of Wright [Wri14] show that the projection of the tangent bundle to absolute homology has no flat subbundles. As remarked, it always has the tautological $\mathrm{SL}_2\mathbb{R}$-invariant subbundle.

Flat bundles on affine manifolds Consider now flat (i.e. locally constant) subbundles on an affine manifold $\mathcal{M}$. By the results of Eskin, Mirzakhani, and Mohammadi [EM13, EMM15] all $\mathrm{SL}_2\mathbb{R}$-invariant measures are Lebesgue on such an $\mathcal{M}$. A flat subbundle is one which is invariant under parallel transport along any path on the manifold. It therefore corresponds to an invariant subspace in the monodromy representation. The same kind of semisimplicity results as above hold in this setting.

Theorem 1.1.3 (see Theorem 1.8.2)

Fixed Part *Suppose ϕ is a flat section of the Hodge bundle (or any tensor power). Then $C \cdot \phi$ is also flat, where C is the Weil operator. This is equivalent to saying that each (p, q)-component of ϕ is also flat.*

Semisimplicity *Suppose $V \subset E$ is an irreducible flat subbundle of the Hodge bundle (or any tensor power). Then so is $C \cdot V$. Moreover, the same kind of decomposition as in Theorem 1.1.1 holds, but with flat subbundles instead of $\mathrm{SL}_2\mathbb{R}$-invariant ones. In particular, the decomposition respects the Hodge structures.*

An instance where a non-trivial decomposition occurs in the above theorem is when the field of affine definition of $\mathcal{M}$ (see [Wri14]) is not $\mathbb{Q}$. The field of affine definition is the number field generated by the coefficients of linear equations defining $\mathcal{M}$ in period coordinates. Then the projection of the tangent space of $\mathcal{M}$ determines a non-trivial subbundle in H^1. Moreover, taking Galois conjugates gives further subbundles. Theorem 1.1.3 implies that these bundles are *Hodge-orthogonal*.

The decomposition into flat subbundles provided by the above theorem need not agree with the one from Eq. (1.1.1) for $\mathrm{SL}_2\,\mathbb{R}$-invariant ones. The latter is a refinement of the former, however.

Rigidity of bundles On affine invariant manifolds, $\mathrm{SL}_2\,\mathbb{R}$-invariant bundles are even more rigid. We prove (see Theorem 1.7.7) that measurable subbundles have to depend, in fact, polynomially on the period coordinates.

Theorem 1.1.4 *Suppose $V \subset E$ is a measurable $\mathrm{SL}_2\,\mathbb{R}$-invariant subbundle on $\mathcal{M}$.*

Then it has a complement for which in local period coordinates on $\mathcal{M}$ the operator of projection to it is polynomial (relative to a fixed flat basis).

A similar statement is proved in [AEM14] for the Forni subspace (see definition there). They prove that it must be flat along the affine invariant manifold (i.e. locally constant).

The above results apply to the Hodge bundle and its tensor powers. This implies the measurable and real-analytic algebraic hulls of the Kontsevich-Zorich cocycle have to agree. For definitions see Section 1.2.

The next result (proved in Theorem 1.8.1) also applies to any tensor power of the cocycle.

Theorem 1.1.5 *Suppose we have a measurable reduction of the structure group (see Definition 1.2.4) of the Kontsevich-Zorich cocycle over some affine invariant manifold $\mathcal{M}$.*

Then in local period coordinates this reduction must be real-analytic. Locally the measurable reduction is a map to a quasi-projective variety (see Remark 1.2.6), and this map is *polynomial.*

In most cases, a reduction of the structure group from Sp_g to a subgroup H means a choice, for each fiber E_x, of a conjugate of H in $\mathrm{Sp}(E_x)$. For example, given an invariant subbundle $V \subset E$, one can take at a point $x \in \mathcal{M}$ its stabilizer $H_x := Stab(V_x \subset E_x)$. For the general situation, see Remark 1.2.5.

Real multiplication Another application (proved in Theorem 1.8.3), suggested by Alex Wright, is that affine manifolds parametrize Riemann surfaces whose Jacobians have real multiplication.

Theorem 1.1.6 *Let $k(\mathcal{M})$ be the field of (affine) definition of the affine manifold $\mathcal{M}$. It is defined in [Wri14, Theorem 1.5].*

Then this field is totally real and the Riemann surfaces parametrized by $\mathcal{M}$ have Jacobians whose rational endomorphism ring contains $k(\mathcal{M})$. Moreover, the 1-forms on $\mathcal{M}$ giving the flat structure are eigenforms for the action of $k(\mathcal{M})$.

In the case when $k(\mathcal{M}) = \mathbb{Q}$ the Jacobians have a factor which contains the 1-forms from $\mathcal{M}$. The factor is nontrivial (i.e. a proper abelian subvariety of the Jacobian) as soon as the projection of the tangent space of $\mathcal{M}$ to H^1 is not all of H^1 (in the terminology of [Wri14], the rank of $\mathcal{M}$ is smaller than g).

Mirzakhani and Wright announced [MW15] that an affine manifold $\mathcal{M}$ which is of rank g is either a full stratum, or a hyperelliptic sublocus in a stratum. This, combined with Theorem 1.1.6, implies that a non-trivial factor occurs, as soon as $\mathcal{M}$ is not a stratum or a hyperelliptic sublocus.

Some examples To illustrate Theorem 1.1.6, consider the Hilbert modular surfaces discovered by McMullen [McM03] in genus 2. These are affine manifolds which are 3-dimensional inside $\mathcal{H}(1, 1)$.

Suppose that one is given some orbit closure $\mathcal{M}$ in $\mathcal{H}(1,1)$ which is 3-dimensional. By results of Wright and Avila–Eskin–Möller it follows that the tangent bundle $T\mathcal{M}$ projects to H^1 as a 2-dimensional (symplectic) bundle, defined over $\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$.

Theorem 1.1.6 implies that there are two possibilities for $\mathcal{M}$. If the field of definition $k(\mathcal{M})$ is $\mathbb{Q}$, then $\mathcal{M}$ parametrizes torus covers, with the 1-form pulled back from the torus. If the field of definition is a quadratic extension of $\mathbb{Q}$, then $\mathcal{M}$ must be a Hilbert modular surface.

As an illustration of the Semisimplicity Theorem 1.1.1, consider some known orbit closure $\mathcal{M}$. Performing some covering construction, e.g. fixing a topological type of Galois cover of the underlying Riemann surfaces, gives another orbit closure $\mathcal{M}'$ in a different stratum. The Galois symmetry of the covering construction will typically force the decomposition of H^1 over $\mathcal{M}'$ to be non-trivial. Theorem 1.1.1 implies that this decomposition respects the Hodge structures, and in particular is Hodge-orthogonal. Thus, one can study Lyapunov exponents, following Eskin–Kontsevich–Zorich [EKZ14], using the special properties of the Hodge metric.

1.1.3 Remarks and references

A general introduction to the subject is available in the survey of Zorich [Zor06]. More recent introductions are in the survey of Wright [Wri15] and Forni–Matheus [FM14].

The question of invariant subbundles and their behavior has been extensively studied. Starting with the work of Forni [For02], the geometry of the Hodge metric has received a lot of attention. The idea of studying the variation of Hodge structure which is available in this context goes back to Kontsevich [Kon97]. The work of Möller [Möl06b, Möl06a] on Teichmüller curves has introduced to the setting of flat surfaces many of the concepts used in this chapter.

Our work is inspired in part by the questions raised by Forni, Matheus and Zorich

[FMZ14b, FMZ14a]. A question they asked concerning zero Lyapunov exponents is addressed, using again techniques from Hodge theory, in [Fil14]. A central ingredient in the current chapter, reductivity of the algebraic hull, is from the paper [EM13, Appendix A] (but follows directly from earlier results of Forni). The expansion-contraction argument from Section 1.7 is standard, see for example [AEM14]. The curvature calculations for the Hodge bundle are also standard, see for example [Sch73].

Throughout, we work in an appropriate finite cover of some connected component of a stratum. This does not affect the statement or conclusions of any of the theorems, but has the advantage of avoiding orbifold issues. In the appropriate finite cover, period coordinates exist locally and the Kontsevich-Zorich cocycle is well-defined.

1.1.4 Outline of the chapter

Section 1.2 contains background on cocycles and their properties. It proves a result about the image of the algebraic hull of a cocycle. This is needed to extend the arguments to tensor powers of the Hodge bundle and in only necessary for the applications to the algebraic hull. The result is likely known to experts.

Section 1.3 contains background from Hodge theory and variations of Hodge structures. In particular, it discusses the point of view on Hodge structures as representations of the Deligne torus.

Section 1.4 collects standard properties of Hodge bundles in a general variation of Hodge structure. It computes the curvature of Hodge bundles, as well as a formula for the Laplacian of the log of the norm of a holomorphic section. In favorable circumstances, the log of the norm is a subharmonic function. This material is again classical and included because not all of it is readily available in the literature.

Section 1.5 contains some results about harmonic and subharmonic functions for random walks on groups. To apply the standard techniques in variations of Hodge structure, we need control over such objects.

Section 1.6 brings together the developed material to prove the Theorem of the Fixed Part. This is the first step, used then to deduce the semisimplicity result.

Section 1.7 establishes the polynomial dependence of invariant bundles. First, Hodge orthgonality of bundles is used to prove real-analyticity along stable and unstable leaves. This, augmented with an expansion-contraction argument, gives polynomiality.

Section 1.8 collects some applications. The first one is that the measurable and analytic algebraic hulls of the cocycle coincide. Then, the semisimplicity theorem is extended to flat, not just $\mathrm{SL}_2\,\mathbb{R}$-invariant, bundles. This version of semisimplicity combined with results of Wright [Wri14] shows that Jacobians over the affine invariant manifold contain the field of (affine) definition in their endomorphism ring.

Acknowledgments I am very grateful to my advisor Alex Eskin for suggesting this circle of problems, as well as numerous encouragements and suggestions throughout the work. His advice and help were invaluable at all stages. I have also benefited a lot from conversations with Madhav Nori and Anton Zorich. Giovanni Forni, Julien Grivaux, Pascal Hubert, and Barak Weiss provided useful feedback on the exposition.

It was suggested to me by Alex Wright that the methods of this chapter could yield the results about real multiplication in §1.8.3. I am very grateful to him for that.

I am also very grateful to the referee of the original [Fil16a] for a thorough reading and detailed feedback, which significantly improved readability.

1.2 Preliminaries on cocycles

This section recalls the notions of cocycle, algebraic hull, and some of their properties. These concepts are presented in more detail in the book of Zimmer [Zim84, Sections 4.2, 9.2]. One proposition about the image of the algebraic hull is not available there (but likely known to experts) and for completeness is proved in this section. In §1.2.1 we introduce some of the same notions, but in an invariant context, i.e. without fixing a trivialization of the bundles.

Throughout, (X, μ) is a standard Borel probability space equipped with an ergodic measure-preserving left action of a Polish group A. One can assume that X is a manifold and A is a Lie group.

1.2.1 Invariant definitions

With the setup as above, suppose that $V \to X$ is an n-dimensional vector bundle (depending on the qualities of X, measurable, smooth, analytic, etc.).

Definition 1.2.1 *A* linear cocycle *on V is the lift of the action of A from X to V. This is a collection of linear maps (varying with appropriate degree of regularity) for each $a \in A$*

$$a_x : V_x \to V_{a \cdot x}$$

Above V_p denotes the fiber of V (a vector space) above $p \in X$.

Example 1.2.2 Suppose that X is a manifold and V is a local system. By definition, for any path between $x, y \in X$ denoted γ this gives a linear *monodromy* map $V_x \xrightarrow{M_\gamma} V_y$. The map M_γ only depends on the homotopy class (with fixed endpoints) of γ.

Suppose that the acting group A is connected, and for any non-trivial loop $\gamma \in \pi_1(A, \mathbb{1})$ and any $x \in X$, the image loop of γ, denoted $\gamma \cdot x \subset X$, has trivial monodromy: $M_{\gamma \cdot x} = \mathbb{1}$. This leads to a cocycle on V for the A-action on X as follows.

Given $a \in A$ pick a path γ_a in A from $\mathbb{1}$ to a. Then for all $x \in X$ the linear map $a_x : V_x \to V_{ax}$ is defined by taking the monodromy of the path $\gamma_a \cdot x$ from x to $a \cdot x$. By assumption, this is independent of the choice of path γ_a.

To define the Kontsevich-Zorich cocycle, let X be a stratum $\mathcal{H}$ or an affine manifold $\mathcal{M}$. The bundle V is the one associated to H^1 - the first cohomology of the surfaces. The group A

is $\mathrm{GL}_2^+(\mathbb{R})$ and $\pi_1 A = \mathbb{Z}$ coming from rotations. But rotations act trivially on cohomology, and so the assumptions above are satisfied.

Remark 1.2.3 Any vector bundle $V \to X$ can be measurably trivialized, i.e. there is a measurable map $V \to X \times \mathbb{R}^n$ mapping linearly fibers V_x to $\{x\} \times \mathbb{R}^n$. Any two such trivializations differ by a map $X \to \mathrm{GL}(\mathbb{R}^n)$, and this leads to the notion of cohomologous cocycles in the sense of §1.2.2 below. We continue to use the invariant definitions in this section, but revert in the next one to the more common definitions following Zimmer [Zim84].

Reducing the structure group Fix some model $\mathbb{R}^n$ (recall $\dim V = n$). For each $x \in X$ define $P_x := \mathrm{Isom}(\mathbb{R}^n, V_x)$ – the space of linear isomorphisms from V_x to $\mathbb{R}^n$. This has a natural right action of $\mathrm{GL}(\mathbb{R}^n)$ and glues to a "principal bundle" $P \to X$. It carries a left action of A, i.e. $a_x : P_x \to P_{ax}$ which commutes with the right $\mathrm{GL}(\mathbb{R}^n)$-action.

Fix $H \subset \mathrm{GL}(\mathbb{R}^n)$. Taking the quotient on the right by H gives a bundle $P/H \to X$ which still carries a left action of A.

Definition 1.2.4 *A* reduction of the structure group *of the bundle V from $\mathrm{GL}(\mathbb{R}^n)$ to H is an A-equivariant section of the bundle $P/H \xrightarrow{\pi} X$. Namely, this is a map $\sigma : X \to P/H$ which is equivariant for the A-action, and such that $\pi \circ \sigma = \mathbb{1}|_X$.*

Remark 1.2.5 Reducing the structure group to H is not the same as specifying an A-equivariant choice of conjugacy class of H in each $\mathrm{GL}(V_x)$.

For instance, scalings (i.e. the multiplicative group $\mathbb{G}_m$) is a canonical subgroup of $\mathrm{GL}(V_x)$ and so is A-equivariant. However, reducing the structure group to $\mathbb{G}_m$ means that effectively, all the linear maps of the cocycle are scalar. This is very rarely possible.

The reason for this difference is that for $H \subset G$, the space of conjugacy classes of H in G is G/N, where N is the *normalizer* of H in G. In many situations $N = H$ and so specifying a conjugacy class is the same as specifying a reduction of the structure group. But, this is not always the case.

Remark 1.2.6 For two algebraic groups $H \subset G$, the quotient G/H is a quasi-projective variety. This follows from a theorem of Chevalley [Bor91, Thm. 5.1], which says that H can be described as the stabilizer of a line in some representation of G. Thus, G/H is a G-orbit in a projective space.

Moreover, the same result reduces questions about algebraic hulls (see next section) to understanding invariant subspaces of cocycles.

1.2.2 Classical definitions

Let $G(k)$ denote the k-points of an algebraic group G, where k is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.2.7 *A cocycle valued in $G(k)$ for the action of A on X is a map*

$$\alpha : A \times X \to G(k)$$

satisfying the compatibility condition

$$\alpha(a_1 a_2, x) = \alpha(a_1, a_2 x)\alpha(a_2, x)$$

Two cocycles α and β are said to be cohomologous (or conjugate) if there exists a function

$$C : X \to G(k)$$

such that

$$\alpha(a, x) = C(ax)^{-1}\beta(a, x)C(x)$$

Remark 1.2.8 All maps in the definition are assumed measurable.

Throughout this section, cocycles will be strict (in the sense of [Zim84, Section 4.2]). Whether certain identities hold a.e. or everywhere can be addressed as it is done in that section (and Appendix B, *loc. cit.*).

Cohomologous cocycles have essentially equivalent dynamical properties, so one is interested in conjugating a given cocycle into a minimal subgroup of $G(k)$.

Theorem 1.2.9 ([Zim84, Prop. 9.2.1, Def. 9.2.2]) *With the setup as in the beginning of the section, there exists a k-algebraic subgroup $L \subset G$ such that the cocycle α can be conjugated into $L(k)$, but cannot be conjugated into the k-points of a smaller k-subgroup of G.*

The (equivalence class of) L is called the algebraic hull *of α.*

1.2.3 Algebraic hull under homomorphisms

We consider the behavior of algebraic hulls under homomorphisms.

Proposition 1.2.10 *Suppose α is a $G(k)$-valued cocycle and $\rho : G \to H$ is an algebraic representation. Then the algebraic hull of the cocycle $\rho \circ \alpha$ coincides with the image under ρ of the algebraic hull of the cocycle α.*

Proof. Without loss of generality, we can assume that the algebraic hull of α is G itself. Suppose, by contradiction, that the algebraic hull of $\rho \circ \alpha$ is a subgroup $F \subset H$ which is not $\rho(G)$.

First, we can assume that $F \subsetneq \rho(G)$. This follows from Zimmer's proof of the uniqueness (up to conjugation) of the algebraic hull. Indeed, ordering by inclusion subgroups into which the cocycle can be conjugated, he shows that any two minimal subgroups are conjugate. In particular, we can take a minimal element contained in $\rho(G)$.

Next, recall ([Zim84, 4.2.18(b)]) that reducing a cocycle to a subgroup $F(k)$ is the same as giving a $\rho \circ \alpha$ equivariant map

$$\sigma : X \to H(k)/F(k)$$

The action of $\rho(G)(k)$ on $H(k)/F(k)$ has locally closed orbits, in particular the quotient is a T_0 topological space and its Borel σ-algebra separates points. Because the action of A on X

is ergodic, we conclude the image of σ must lie in a single $\rho(G)(k)$ orbit. One can therefore pick $t_0 \in H(k)$ and a measurable section

$$s : X \to \rho(G)(k)$$

such that

$$\sigma(x) = s(x) \cdot t_0 F(k)$$

Writing out the equivariance condition for σ, which reads

$$\sigma(ax) = \rho(\alpha(a, x))\sigma(x)$$

we obtain

$$s(ax)t_0 F(k) = \rho(\alpha(a, x))s(x)t_0 F(k)$$

This implies

$$t_0 F(k) = s(ax)^{-1} \cdot \rho(\alpha(a, x)) \cdot s(x) \cdot t_0 F(k)$$

Multiplying on the right by t_0^{-1}, we deduce that

$$s(ax)^{-1}\rho(\alpha(a, x))s(x) \in t_0 F(k)t_0^{-1} \tag{1.2.1}$$

These elements also lie in $\rho(G)(k)$. Because $F \subsetneq \rho(G)$, it follows that $t_0 F t_0^{-1} \cap \rho(G) \subsetneq \rho(G)$. So, in fact, we could have reduced the cocycle to this latter subgroup, and could have done so using the coboundary

$$s : X \to \rho(G)(k)$$

We denote by F the group $t_0 F t_0^{-1}$ and will show we could have conjugated the original cocycle into its preimage in G. To do so, we must lift the map s to $G(k)$.

By the remarks following Theorem 3.1.3 in [Zim84] the set $\rho(G(k))\backslash\rho(G)(k)$ is finite. We rewrite Eq. (1.2.1) as

$$\rho(\alpha(a,x))s(x) \in s(ax)F(k)$$

This implies the equality in the double coset space

$$[s(ax)] = [s(x)] \text{ in } \rho(G(k)) \setminus \rho(G)(k)/F(k)$$

Because the action of A is ergodic, this must land in a single double coset. After choosing measurable sections for the corresponding actions, we find that

$$s(x) = \rho(\tilde{s}(x)) \cdot f(x)$$

where $\tilde{s}(x) \in G(k)$ and $f(x) \in F(k)$. In particular, we have

$$\rho(\tilde{s}(x))^{-1}\rho(\alpha(a,x))\rho(\tilde{s}(x)) \in F(k)$$

We can thus use $\tilde{s}$ to make a change of basis for α to find that it lands in $\tilde{F} := \rho^{-1}(F) \subsetneq G$. This is a contradiction. $\square$

Corollary 1.2.11 *If the algebraic hull of a cocycle is reductive, then it stays reductive under any algebraic representation, in particular under considering various tensor operations.*

Proof. This follows from the above proposition, since images of reductive groups stay reductive. $\square$

Remark 1.2.12 We use *reductive* to mean that any representation is semisimple, i.e. any invariant subspace has a complement.

The Kontsevich-Zorich cocycle (for the $\mathrm{SL}_2 \mathbb{R}$-action) is reductive by the results in [EM13, Appendix A] (see also Theorem 1.5 in [AEM14]), so the corollary applies to it. We can extend scalars from $\mathbb{R}$ to $\mathbb{C}$ and it will stay reductive.

1.3 Preliminaries from Hodge theory

In this section, we recall some basic facts from Hodge theory. Definitions and properties of Hodge structures are in §1.3.1. Variations of Hodge structures are defined in §1.3.2. Textbook introductions to this topic are the monographs of Peters–Steenbrink [PS08] and Carlson–Müller-Stach–Peters [CMSP03]. An economical introduction is also given by Deligne [Del71].

1.3.1 Hodge structures

Throughout this section H denotes a finite-dimensional vector space over $\mathbb{R}$. In §1.8.3 the real vector space H will arise as an extension of scalars from a $\mathbb{Z}$, $\mathbb{Q}$, or k-module (where k is a number field) and so will have this extra data. Until then, this extra structure is not coming into play and can be ignored. The extension of scalars to complex numbers is denoted $H_{\mathbb{C}} := H \otimes_{\mathbb{R}} \mathbb{C}$. Sometimes, to emphasize that a certain structure is defined over $\mathbb{R}$, we write $H_{\mathbb{R}}$ instead of H.

Definition 1.3.1 *A* Hodge structure of weight w *on H is a decomposition of the complexification*

$$H_{\mathbb{C}} = \bigoplus_{p+q=w} H^{p,q} \tag{1.3.1}$$

satisfying $H^{p,q} = \overline{H^{q,p}}$.

Definition 1.3.2 ([Del71, Definition 1.4]) *The* Deligne torus $\mathbb{S}$ *is the real algebraic group*

$$\mathbb{S} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in \mathrm{Mat}_{2\times 2} \middle| a^2 + b^2 \neq 0 \right\} \tag{1.3.2}$$

Its real points are naturally isomorphic to the (non-zero) complex numbers $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^{\times}$.

Its complex points are isomorphic to two such copies: $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$*. The complex conjugation action on* $\mathbb{S}(\mathbb{C})$ *exchanges the two factors.*

Remark 1.3.3 A Hodge structure on $H_\mathbb{R}$ is the same as an algebraic representation of the Deligne torus $\mathbb{S}$ on $H_\mathbb{R}$. Given a Hodge decomposition as in Eq. (1.3.1), let $z \in \mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$ act on $H^{p,q}$ by $z^p\overline{z}^q$. This action commutes with complex conjugation, so it is defined on $H_\mathbb{R}$.

Conversely, given a representation of $\mathbb{S}$ on $H_\mathbb{R}$, after complexification this gives a representation of $\mathbb{C}^\times \times \mathbb{C}^\times$. The condition that it comes from a *real* representation of $\mathbb{S}$ implies that the eigenspaces of the two factors must be exchanged by complex conjugation.

The weight of a Hodge structure on $H_\mathbb{R}$ is recovered as the degree with which $\mathbb{R}^\times \subset \mathbb{S}(\mathbb{R})$ acts on $H_\mathbb{R}$.

When talking about Hodge structure on H, we shall freely pass between the definition via the decomposition (1.3.1) and as a representation of $\mathbb{S}$.

Definition 1.3.4 *The* Weil operator C *associated to a Hodge structure on* H *acts by* $(\sqrt{-1})^{p-q}$ *on* $H^{p,q}$ *in the decomposition* (1.3.1)*. This is a real operator, i.e. well-defined at the level of* $H_\mathbb{R}$*.*

Equivalently, if the Hodge structure on H *is defined as a representation* $\rho : \mathbb{S} \to \mathrm{GL}(H)$*, then the Weil operator is* $C := \rho(\sqrt{-1})$*.*

Definition 1.3.5 *A* polarization *on a Hodge structure* $H_\mathbb{R}$ *is a non-degenerate bilinear form* $I(-,-)$ *on* $H_\mathbb{R}$*, which is invariant under* $\{|z| = 1\} \subset \mathbb{S}(\mathbb{R})$ *and such that the form* $Q(x, y) := I(Cx, y)$ *is symmetric and positive-definite.*

Equivalently, defining on the complexification $Q(x, y) := I(Cx, \overline{y})$*, this gives a positive-definite hermitian inner product on* $H_\mathbb{C}$*, such that the decomposition* (1.3.1) *is orthogonal.*

The Hodge norm *on* $H_\mathbb{R}$ *or* $H_\mathbb{C}$ *is the norm obtained from* $Q(-,-)$ *on the corresponding spaces.*

Remark 1.3.6

(i) On $H_{\mathbb{R}}$ since $I(Cx, y)$ is symmetric in x and y, and I is preserved by C, it follows that I is $(-1)^w$-symmetric, where w is the weight of the Hodge structure. This follows since $C^2 = (-1)^w$.

(ii) In the case of Hodge structures of weight 1, i.e. $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$, the Weil operator C is the same as the Hodge-star operator. Recall that if H is the first cohomology of a Riemann surface, the Hodge-star operator $* : H_{\mathbb{R}} \to H_{\mathbb{R}}$ is defined as follows. For $u \in H_{\mathbb{R}}$, let $\alpha = u + \sqrt{-1}v$ be the holomorphic 1-form with real part equal to u. Then $*u := -v$. It can then be checked that $(*)^2 = -1$.

The Hodge decomposition of $2u$ is $2u = \alpha \oplus \overline{\alpha}$. Then $C \cdot (2u) = \sqrt{-1}\alpha \oplus -\sqrt{-1}\overline{\alpha} = -2v$, from which the equivalence of the Weil operator C and the Hodge star operator $*$ follows.

Definition 1.3.7 *A* morphism *between two Hodge structures H_1, H_2 is an $\mathbb{R}$-linear map $\phi : H_1 \to H_2$, such that its complexification preserves the grading: $\phi_{\mathbb{C}}(H_1^{p,q}) \subset H_2^{p,q}$. In particular, the Hodge structures must have the same weight.*

Equivalently, a morphism of Hodge structures is a morphism between representations of $\mathbb{S}$. Similarly, tensor products and duals of Hodge structures are defined at the level of representations of $\mathbb{S}$. Note that a morphism from H_1 to H_2 is an element $\phi \in \mathrm{Hom}(H_1, H_2)^{0,0}$.

Remark 1.3.8

(i) Suppose that A is an $\mathbb{R}$-algebra, i.e. equipped with a multiplication map $m : A \otimes A \to A$ with the usual properties. A Hodge structure on A is *compatible with the algebra structure* if the multiplication map is a morphism of Hodge structures. In particular, the weight of the Hodge structure on A must be 0.

(ii) For a given Hodge structure H of any weight, $\mathrm{End}(H)$ is an algebra with a Hodge structure. The $(0,0)$ part, i.e. $\mathrm{End}(H)^{0,0}$ is the algebra of endomorphisms of H which preserve the Hodge structure.

Lemma 1.3.9 *Let H be a finite-dimensional real vector space. A Hodge structure on* $\mathrm{End}(H)$ *which is compatible with the algebra structure comes from a Hodge structure on H, unique up to shift of weight.*

The same statement is true for H a module over a division algebra A. Namely, a Hodge structure on $\mathrm{End}_A(H)$ *compatible with the algebra structure gives a Hodge structure on H, compatible with the A-action (i.e. such that A acts by endomorphisms of the Hodge structure on H).*

Proof. Because $\mathrm{End}(H)$ is a simple algebra, its automorphisms group is $\mathrm{PGL}(H)$. To give a Hodge structure on a space is the same as to give an action of the Deligne torus $\mathbb{S}$ on it, thus we have a homomorphism $\mathbb{S} \to \mathrm{PGL}(H)$.

Consider the exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}(H) \to \mathrm{PGL}(H) \to 1$$

We see that a homomorphism of $\mathbb{S}$ to $\mathrm{PGL}(H)$ must lift to $\mathrm{GL}(H)$ because any extension of $\mathbb{S}$ by $\mathbb{G}_m$ splits (non-uniquely; see proof below). This lift gives H a Hodge structure.

When a division algebra A acts on H, the same argument applies but with $\mathrm{GL}(H)$ replaced by $GL_A(H)$, and $\mathrm{PGL}(H)$ by $\mathrm{PGL}_A(H)$.

To prove that a splitting always exists, suppose given an exact sequence of real algebraic groups

$$1 \to \mathbb{G}_m \to G \to \mathbb{S} \to 1$$

Consider the dual exact sequence of character lattices

$$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}^3 \leftarrow \mathbb{Z}^2 \leftarrow 0 \tag{1.3.3}$$

Each group above has an action of the group $\mathbb{Z}/2 = \langle \sigma | \sigma^2 = 1 \rangle$. On $\mathbb{Z}$ it is trivial, on $\mathbb{Z}^2$ it is by $\sigma(x, y) = (y, x)$.

Let now e_1, e_2, e_3 be a basic of $\mathbb{Z}^3$ adapted to Eq. (1.3.3), with e_1 projecting to a generator of $\mathbb{Z}$ and e_2, e_3 restricting to a basis of $\mathbb{Z}^2$. Then we have, for some values of a and b, that:

$$\sigma(e_1) = e_1 + ae_2 + be_3 \qquad \sigma(e_2) = e_3 \qquad \sigma(e_3) = e_2.$$

The condition $\sigma^2 = \mathbb{1}$ implies that $a = -b$. Redefining the basis by

$$e_1' := e_1 + ae_2 \qquad e_2' = e_2 \qquad e_3' = e_3$$

it follows that Eq. (1.3.3) splits with respect to the action of σ. Indeed, $\sigma(e_1') = e_1'$ and so we can construct a splitting. $\square$

Remark 1.3.10 The formalism above also makes sense for "complex Hodge structures" (see [PS08, Def. 2.6]). By definition, on a complex vector space H this is just a bigrading $H = \oplus H^{p,q}$. This is the same as a representation of $\mathbb{C}^\times \times \mathbb{C}^\times$ on H. Lemma 1.3.9 holds as well.

Finally, the same properties also hold for polarized Hodge structures.

1.3.2 Variations of Hodge structures

Throughout this section, M is a fixed complex manifold.

Definition 1.3.11 *A* variation of Hodge structures of weight w *on* M *is a flat vector bundle* $H_\mathbb{R}$ *(equivalently: a local system) equipped with a connection* ∇*, called the Gauss–Manin connection. Additionally, the complexification* $H_\mathbb{C}$ *is equipped with holomorphic subbundles* $F^p \subseteq F^{p-1} \subseteq \ldots \subset H_\mathbb{C}$ *satisfying:*

(i) The bundles F^p satisfy the Griffiths transversality condition:

$$\nabla_X F^p \subset F^{p-1} \qquad \text{for } X \text{ a vector field} \tag{1.3.4}$$

(ii) On each fiber of H, the bundles F^p determine a Hodge structure by the recipe

$$H^{p,q} := F^p \cap \overline{F^q} \text{ with } H_{\mathbb{C}} = \bigoplus_{p+q=w} H^{p,q} \tag{1.3.5}$$

A variation of Hodge structures as above is polarized, *if in addition the bundle $H_{\mathbb{R}}$ carries a bilinear pairing $I(-,-)$ satisfying*

(i) The pairing is flat, i.e. preserved by the Gauss–Manin connection ∇.

(ii) On each fiber of H, the pairing induces a polarization of the Hodge structure, in the sense of Definition 1.3.5.

Remark 1.3.12 A local system, or flat bundle, such as H above, is determined by a representation of $\pi_1(M)$. A typical representation need not be semisimple, since $\pi_1(M)$ can be for example a free group (e.g. if M is a punctured compact Riemann surface). The key property of polarized variations of Hodge structures is that nevertheless this semisimplicity holds. Moreover, *any* flat subbundle must respect all the structures in Definition 1.3.11.

Remark 1.3.13 In the case of weight 1 variations, we only have one non-trivial holomorphic subbundle $F^1 \subset H_{\mathbb{C}}$. The Griffiths transversality condition is empty in this case. All bundles obtained from $H_{\mathbb{C}}$ using natural operations (e.g. tensors, duals) will be of other weights, but will automatically satisfy Griffiths transversality because $H_{\mathbb{C}}$ satisfies it.

Remark 1.3.14 The intersection pairing $I(-,-)$ on a polarized variation of Hodge structures is flat for the Gauss–Manin connection, but the Hodge metric $Q(-,-)$ is not. Since Q is expressed in terms of I and the Weil operator C, compatibility of subbundles with the Weil operator C implies compatibility with the Hodge metric. This, in turn, has consequences for semisimplicity.

1.4 Differential geometry of Hodge bundles

In this section we work with a holomorphic vector bundle E equipped with a (pseudo-)hermitian metric denoted $\langle - \rangle -$. The bundle is over some unspecified complex manifold. On E-valued smooth differential forms $\mathcal{A}^\bullet(E)$ we have the canonical Chern connection $\nabla : \mathcal{A}^\bullet(E) \to \mathcal{A}^{\bullet+1}(E)$. It is uniquely defined by two conditions. The first is compatibility with the metric, which requires for $\alpha, \beta \in \mathcal{A}^\bullet(E)$ (with $|\alpha| := \deg \alpha$ as a differential form)

$$d \langle \alpha \rangle \beta = \langle \nabla \alpha \rangle \beta + (-1)^{|\alpha|} \langle \alpha \rangle \nabla \beta \tag{1.4.1}$$

The second requires ∇ to take a holomorphic section s of E to a form of type $(1,0)$:

$$\nabla s \in \mathcal{A}^{1,0}(E)$$

The Chern connection is explicit once we fix a local holomorhpic basis of E. The hermitian metric is given by the matrix h and the connection then has the form $\nabla = d + A$, where A is a matrix of 1-forms (in fact, $(1,0)$-forms by the second requirement on ∇). The relation between them becomes

$$A = \partial h \cdot h^{-1}$$

Throughout this section Ω denotes the curvature of the connection ∇. In the local trivialization $\nabla = d + A$ we have $\Omega = dA + A \wedge A$ (see Remark 1.4.6 for the sign).

First, we provide a formula for what can be considered the second variation of the log of the norm of a holomorphic section. Next, we prove formulas for the curvature of quotient and subbundles. These are used to derive the formulas for the curvature of the Hodge bundles.

Finally, positivity for differential forms is recalled.

The material in this section is standard. It is included here because different sources use different conventions and it is hard to refer to a single place. Similar calculations can be found, for instance, in [Gri84, pg. 32, Prop. 3] or [CMSP03, Prop. 11.1.8]. The first two chapters of [Gri84] provide a clear introduction to the circle of questions discussed here.

1.4.1 Second variation formula

Lemma 1.4.1 *Suppose ϕ is a holomorphic section of E. Then we have the formula*

$$\overline{\partial}\partial \log \|\phi\|^2 = \frac{\langle \Omega\phi\rangle \phi}{\|\phi\|^2} + \frac{\langle \nabla\phi\rangle \phi \langle \phi\rangle \nabla\phi - \|\phi\|^2 \langle \nabla\phi\rangle \nabla\phi}{\|\phi\|^4}$$

Proof. We shall use throughout the fact that the connection respects the metric (Eq. (1.4.1)) and that

$$d \langle \alpha\rangle \beta = \partial \langle \alpha\rangle \beta + \overline{\partial} \langle \alpha\rangle \beta$$

Because ϕ is holomorphic, we have

$$\partial \|\phi\|^2 = (1,0)\text{-part of } d \|\phi\|^2 = \langle \nabla\phi\rangle \phi \tag{1.4.2}$$

Therefore

$$\partial \log \|\phi\|^2 = \frac{1}{\|\phi\|^2} \cdot \partial \|\phi\|^2 = \frac{\langle \nabla\phi\rangle \phi}{\|\phi\|^2} \tag{1.4.3}$$

Next, we apply the product rule

$$\overline{\partial}\left(\partial \log \|\phi\|^2\right) = \left(\overline{\partial}\frac{1}{\|\phi\|^2}\right) \langle \nabla\phi\rangle \phi + \frac{1}{\|\phi\|^2}\overline{\partial} \langle \nabla\phi\rangle \phi \tag{1.4.4}$$

For the first term, we have

$$\overline{\partial} \frac{1}{\|\phi\|^2} = \frac{-1}{\|\phi\|^4} \cdot \overline{\partial} \|\phi\|^2 = \frac{-\langle \phi \rangle \nabla \phi}{\|\phi\|^4} \tag{1.4.5}$$

For the second term, we have

$$\overline{\partial} \langle \nabla \phi \rangle \phi = (1,1) - \text{part of } d \langle \nabla \phi \rangle \phi \tag{1.4.6}$$

However, $d \langle \nabla \phi \rangle \phi$ has only a $(1,1)$-part. Indeed, its $(2,0)$-part can be identified with $\partial\partial \langle \phi \rangle \phi$ and this vanishes always.

By applying the product rule for the Chern connection, we then have

$$d \langle \nabla \phi \rangle \phi = \langle \Omega \phi \rangle \phi - \langle \nabla \phi \rangle \nabla \phi \tag{1.4.7}$$

Combining the above, we at last have

$$\begin{aligned}\overline{\partial}\partial \log \|\phi\|^2 &= \frac{-\langle \phi \rangle \nabla \phi \cdot \langle \nabla \phi \rangle \phi - \|\phi\|^2 \cdot \langle \nabla \phi \rangle \nabla \phi}{\|\phi\|^4} + \frac{\langle \Omega \phi \rangle \phi}{\|\phi\|^2} = \\ &= \frac{\langle \Omega \phi \rangle \phi}{\|\phi\|^2} + \frac{\langle \nabla \phi \rangle \phi \cdot \langle \phi \rangle \nabla \phi - \|\phi\|^2 \cdot \langle \nabla \phi \rangle \nabla \phi}{\|\phi\|^4}\end{aligned}$$

Note that $\nabla \phi$ is a $(1,0)$-form (the reason for a sign switch in Eq. (1.4.7)) and moreover, the second term is always a negative $(1,1)$-form by the Cauchy-Schwartz inequality (see §1.4.3 for notions of positivity). □

Remark 1.4.2

(i) In the case of Hodge bundles, the curvature term in the above calculation will be arranged to be negative as well, yielding a subharmonic function.

(ii) In the situation when ϕ has zeroes, the above equation has to be interpreted. In the distributional sense, we get a current corresponding to the zero-divisor. This will not affect the discussion as we'll be concerned with subharmonic functions, and adding this current does not affect the conclusion.

Indeed, our considerations will be in complex dimension one on the base. We have that $\log|z|$ is subharmonic in $\mathbb{C}$, owing to the distributional identity

$$\overline{\partial}\partial \log|z| = 2\pi\sqrt{-1}\delta_0 \text{ with } \delta_0 \text{ a Dirac mass at } 0$$

If ϕ is a holomorphic section with zero of order k at the origin, it can be written as $\phi = z^k v(z)$ where $v(z)$ is also a holomorphic section, now without any zeroes. It then satisfies

$$\log\|\phi\|^2 = 2k\log|z| + \log\|v(z)\|^2$$

This implies that $\log\|\phi\|^2$ is a subharmonic function, even near the zeroes of ϕ.

Remark 1.4.3 The following formula is also useful (see Equation (7.13) in [Sch73]):

$$\overline{\partial}\partial\,\|\phi\|^2 = \langle\Omega\phi\rangle\,\phi - \langle\nabla\phi\rangle\,\nabla\phi \tag{1.4.8}$$

For this, recall that by Eq. (1.4.2) we have $\partial\,\|\phi\|^2 = \langle\nabla\phi\rangle\,\phi$. However $\overline{\partial}\,\langle\nabla\phi\rangle\,\phi$ was computed in Eq. (1.4.6) and Eq. (1.4.7) to be exactly the quantity above.

Suppose now that $\|\phi\|^2$ is constant, and the curvature term is negative (see §1.4.3 for conventions). Then each of the terms in Eq. (1.4.8) must vanish, so we find

$$\nabla\phi = 0 \qquad \text{and} \qquad \Omega\phi = 0$$

In the context of Hodge bundles, this will give that certain sections are flat and holomorphic (see Lemma 7.19 in [Sch73] for the exact analogue).

1.4.2 Quotients and Subbundles

Consider a short exact sequence of holomorphic vector bundles

$$0 \to S \hookrightarrow E \overset{\pi}{\twoheadrightarrow} Q \to 0 \tag{1.4.9}$$

Suppose that E is equipped with a non-degenerate hermitian form $\langle - \rangle -$, not necessarily positive-definite. Assume however that its restriction to S is non-degenerate. That is, we assume that S is disjoint from $S^{\perp}$, which we identify with Q by the natural map.

Remark 1.4.4 The condition on the hermitian form not being positive definite is relaxed because it is not true in the case of the Hodge bundles. The indefinite metric there plays a crucial role and corresponding signs are essential.

The second fundamental form is now defined using the connection ∇ on E and the projection from Eq. (1.4.9):

$$B : S \xrightarrow{\nabla} E \otimes \mathcal{A}^{1,0} \xrightarrow{\pi \otimes \mathbb{1}} Q \otimes \mathcal{A}^{1,0} \tag{1.4.10}$$

Note that $B : S \to Q \otimes \mathcal{A}^{1,0}$ is a linear tensor, i.e. $B(f \cdot s) = f \cdot B(s)$ where f is a function and s is a section of S. This follows since $\nabla(f \cdot s) = df \cdot s + f \cdot \nabla(s)$ and the first term vanishes after projecting to Q.

By the above discussion, the second fundamental form is a tensor $B \in \mathrm{Hom}(S, Q) \otimes \mathcal{A}^{1,0}$. Define its adjoint, $B^{\dagger} \in \mathrm{Hom}(Q, S) \otimes \mathcal{A}^{0,1}$ by taking the adjoint in the Hom-space using the hermitian form, and complex conjugation on the second factor.

Proposition 1.4.5 *The curvature of the bundles S and Q is given by*

$$\Omega_S = \Omega_E|_S + B^\dagger \wedge B$$
$$\Omega_Q = \Omega_E|_Q + B \wedge B^\dagger$$

Here $\Omega_E|_?$ represents the restriction of the curvature of E to the corresponding subbundle.

Proof. If we denote the connection matrix on E by A_E, we have

$$A_E = \begin{bmatrix} A_S & -B^\dagger \\ B & A_Q \end{bmatrix} \tag{1.4.11}$$

where A_S and A_Q are the connection matrices for the bundles S and Q. Recall now that the curvature is $dA + A \wedge A$, so we have

$$\Omega_E = \begin{bmatrix} \Omega_S - B^\dagger \wedge B & * \\ * & \Omega_Q - B \wedge B^\dagger \end{bmatrix}$$

This yields the claimed formula. $\square$

Remark 1.4.6 In various places in the literature, the formula for the curvature appears as either $dA - A \wedge A$ or $dA + A \wedge A$. The issue is that once a trivialization of the bundle is fixed, we can write $\nabla = d + A$, where A is an operator. Then we have $\nabla^2 f = (dA)f - A(Af)$ (due to the sign rule), but if we write out A as a matrix of 1-forms, then $\nabla^2 f = (dA)f + (A \wedge A)f$.

1.4.3 Positivity

Below are the conventions about which forms are positive and which ones are negative (see also [CMSP03, Sec. 11.1]). Note that the complex vector space of $(1,1)$-forms is preserved by complex conjugation, so it has a notion of real and imaginary $(1,1)$-forms.

Definition 1.4.7 *A purely imaginary* $(1,1)$*-form* ω *is* positive *if it can be written*

$$\omega = \sum h_{ij} dz^i \wedge \overline{dz^j}$$

where h_{ij} *is a positive hermitian matrix. For example,* $dz \wedge \overline{dz}$ *is positive.*

A purely real $(1,1)$*-form* ω *is* positive *if* $\sqrt{-1}\omega$ *is a positive form. For example,* $dx \wedge dy = \sqrt{-1} dz \wedge \overline{dz}$ *is positive.*

A form $\Omega \in \mathcal{A}^{1,1} \otimes \mathrm{End}(E)$ *is* positive *if for any section* $e \in \Gamma(E)$ *we have that* $\langle \Omega e \rangle e$ *is positive.*

Finally, a form ω *is* negative *if* $-\omega$ *is positive. Thus* $\overline{dz} \wedge dz$ *is negative.*

Remark 1.4.8 Note that an equivalent definition of positivity for imaginary $(1,1)$-forms is that for any tangent vector ξ we have $\omega(\xi, \overline{\xi}) \geq 0$.

Proposition 1.4.9 *Consider the setting of the previous section, where curvatures of quotients and subbundles were computed.* Assume that the metric is positive-definite. *Then* $B \wedge B^\dagger$ *is positive and* $B^\dagger \wedge B$ *is negative.*

Proof. Recall from Eq. (1.4.10) that B is of type $(1,0)$, i.e. $B \in \mathrm{Hom}(S,Q) \otimes \mathcal{A}^{1,0}$. This can be seen by choosing a holomorphic trivialization for S and computing the matrix B. A change of frame will not affect the type of B. Similarly $B^\dagger \in \mathrm{Hom}(Q,S) \otimes \mathcal{A}^{0,1}$.

According to Definition 1.4.7, to check the signs of the forms we need to consider arbitrary sections s, q of S and Q respectively. Note that when exchanging the position of two matrices of 1-forms inside a hermitian product, a negative sign appears. We now have:

$$\left\langle B \wedge B^\dagger q \right\rangle q = - \left\langle B^\dagger q \right\rangle B^\dagger q$$

This is a *positive* form. Indeed, $\xi := B^\dagger q$ is an S-valued $(0,1)$-form, and $\langle \xi \rangle \xi$ will be a negative $(1,1)$-form (of type $-\overline{dz} \wedge dz$, see Definition 1.4.7). Similarly

$$\left\langle B^\dagger \wedge Bs \right\rangle s = -\langle Bs\rangle\, Bs$$

which is a *negative* form. $\square$

Remark 1.4.10 The claim above concerning the positivity used the definiteness of the hermitian form. But the curvature calculation remains valid without this assumption.

1.4.4 Curvature of Hodge Bundles

Setup Consider a variation of polarized Hodge structures of weight w over some fixed complex manifold. This is the data of a flat bundle $H_{\mathbb{C}}$ equipped with the Gauss–Manin connection ∇^{GM}. We further have a filtration by holomorphic subbundles

$$\ldots \subset \mathcal{F}^p \subset \mathcal{F}^{p-1} \subset \ldots \subset H_{\mathbb{C}}$$

Denote the quotient subbundles by

$$\mathcal{H}^{p,q} := \mathcal{F}^p / \mathcal{F}^{p+1}$$

The polarization provides the indefinite form $\langle\cdot\rangle\cdot_i$ which is flat for the Gauss–Manin connection. By assumption, we also have the definite metric

$$\langle\cdot\rangle\cdot := \langle C\cdot\rangle\cdot_i$$

Here C is the Weil operator (note the indefinite metric already has a conjugation in the definition; see Section 1.3). We also view $\mathcal{H}^{p,q}$ as subbundles of $H_{\mathbb{C}}$, but note that they are not holomorphically embedded (for the holomorphic structure coming from the Gauss–

Manin connection). Restricted to $\mathcal{H}^{p,q}$, the definite and indefinite metrics agree up to a sign.

Note that ∇^{GM} is the Chern connection on $H_{\mathbb{C}}$ equipped with the indefinite metric and complex structure coming from the flat structure. Viewing $H_{\mathbb{C}}$ as the direct sum of the holomorphic bundles $\mathcal{H}^{p,q}$, each equipped with the definite metric, we also have the Hodge connection ∇^{Hg}. It is defined as the Chern connection of $\oplus\mathcal{H}^{p,q}$ equipped with the definite metric (and taking direct sums).

Remark 1.4.11 The bundle $H_{\mathbb{C}}$ carries two different holomorphic structures and metrics. On the one hand, we have the flat structure (inducing a holomorphic one) and indefinite metric. On the other, we have a direct sum of holomorphic bundles, the $\mathcal{H}^{p,q}$, each equipped with a definite metric.

Consider also the second fundamental form (for the indefinite metric)

$$\sigma_p : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \mathcal{A}^{1,0}$$

Note that it is at first defined as $\sigma_p : \mathcal{F}^p \to H_{\mathbb{C}}/\mathcal{F}^p \otimes \mathcal{A}^{1,0}$, but the Griffiths transversality condition implies it must in fact map subspaces as above.

Finally, let $\sigma_p^\dagger$ denote the adjoint of σ_p for the indefinite metric. It differs from the adjoint for the definite metric by exactly one minus sign. We then have the equality of connections (see [CMSP03, Sec. 13.1, pg. 337])

$$\nabla^{GM} = \nabla^{Hg} + \sigma_\bullet + \sigma_\bullet^\dagger \tag{1.4.12}$$

Here $\sigma_\bullet$ denotes $\oplus_p \sigma_p$ and similarly for $\sigma_\bullet^\dagger$, and this formula can also be deduced by looking at Eq. (1.4.11). Note that the curvature of $\mathcal{H}^{p,q}$ for either metric is the same, since the metrics agree up to a sign.

Proposition 1.4.12 *We have the formula for the curvature*

$$\Omega_{\mathcal{H}^{p,q}} = \sigma_p^\dagger \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^\dagger$$

Proof. By the remark above, it suffices to compute the curvature for the indefinite metric. From the exact sequence of bundles

$$0 \to \mathcal{F}^p \hookrightarrow H_{\mathbb{C}} \twoheadrightarrow H_{\mathbb{C}}/\mathcal{F}^p \to 0$$

we find using Proposition 1.4.5 that

$$\Omega_{\mathcal{F}^p} = \sigma_p^\dagger \wedge \sigma_p$$

Next, consider the exact sequence

$$0 \to \mathcal{F}^{p+1} \hookrightarrow \mathcal{F}^p \twoheadrightarrow \mathcal{H}^{p,q} \to 0$$

Again Proposition 1.4.5 yields

$$\begin{aligned} \Omega_{\mathcal{H}^{p,q}} &= \Omega_{\mathcal{F}^p} + \sigma_{p+1} \wedge \sigma_{p+1}^\dagger \\ &= \sigma_p^\dagger \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^\dagger \end{aligned}$$

This is the claimed formula. □

Remark 1.4.13 This formula agrees with that in Lemma 7.18 of [Sch73]. Note that in *loc. cit.* adjoints are for the *definite* metric, so formulas differ by a minus sign everywhere.

For future use, we also record the following result.

Proposition 1.4.14 *Suppose e, e' are two smooth sections of $\mathcal{H}^{p,q}$. Then for the definite* *metric, we have the formula*

$$\langle \Omega_{\mathcal{H}^{p,q}} e \rangle \, e' = \left\langle \sigma_p e \right\rangle \sigma_p e' + \left\langle \sigma_{p+1}^{\dagger} e \right\rangle \sigma_{p+1}^{\dagger} e' \tag{1.4.13}$$

Proof. This will follow from the fact that on $\mathcal{H}^{p,q}$, we have

$$\langle - \rangle \, -_i = (-1)^p \, \langle - \rangle \, -$$

Note that whenever we exchange two 1-forms, a sign gets switched. We abbreviate $\Omega_{\mathcal{H}^{p,q}}$ by Ω.

$$\begin{aligned}
\langle \Omega e \rangle \, e' &= (-1)^p \, \langle \Omega e \rangle \, {e'}_i = \\
&= (-1)^p \left(\left\langle \sigma_p^{\dagger} \wedge \sigma_p e \right\rangle {e'}_i + \left\langle \sigma_{p+1} \wedge \sigma_{p+1}^{\dagger} e \right\rangle {e'}_i \right) \\
&= (-1)^{p+1} \left(\left\langle \sigma_p e \right\rangle \sigma_p {e'}_i + \left\langle \sigma_{p+1}^{\dagger} e \right\rangle \sigma_{p+1}^{\dagger} {e'}_i \right) \\
&= (-1)^{p+1} \left((-1)^{p-1} \left\langle \sigma_p e \right\rangle \sigma_p e' + (-1)^{p+1} \left\langle \sigma_{p+1}^{\dagger} e \right\rangle \sigma_{p+1}^{\dagger} e' \right)
\end{aligned}$$

The desired formula then follows. □

Corollary 1.4.15 *The "rightmost" bundle $\mathcal{H}^{0,w}$ has negative curvature.*

Proof. The second fundamental form σ_0 vanishes in this case, so the only curvature term in Eq. (1.4.13) involves $\sigma_1^{\dagger}$. The corresponding term is negative-definite. □

Remark 1.4.16 The above calculations are standard, and presented in detail for example in Section 7 of [Sch73]. But in order to apply the same techniques as in Lemma 7.19 and Theorem 7.22 of [Sch73], one needs control over subharmonic functions on Teichmüller disks. This is addressed in the next section.

1.5 Random Walks

Setup Suppose $G := \mathrm{SL}_2\,\mathbb{R}$ acts (on the left) on a measure space X, preserving a probability measure μ. Let also ν be a measure on G with compact support. For this section, only ν-stationarity of μ is required.

We also assume that the action of G on (X, μ) is ergodic and that the support of ν generates G. This suffices for the Furstenberg Random Ergodic Theorem to hold. The survey of Furman [Fur02] (see Section 3) provides a discussion of the needed facts.

We shall need the following form of the Random Ergodic Theorem.

Theorem 1.5.1 (Furstenberg) *With the setup as above, consider a function $f \in L^1(X, \mu)$. Then for a.e. $(x, \omega) \in X \times G^{\mathbb{N}}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(g_i(\omega)\cdots g_0(\omega)x) = \int_X f\, d\mu$$

Moreover, suppose that $f : X \to \mathbb{R}_{\geq 0}$ takes only positive values. Then the same conclusion holds, even if the integral is $+\infty$.

The second statement is not usually part of the Random Ergodic Theorem, but clearly follows by applying the first part to the truncated above function.

1.5.1 Harmonic functions

Definition 1.5.2 *For a measurable function $f : X \to \mathbb{R}$ define*

$$(\nu * f)(x) := \int_G f(gx)\, d\nu(g)$$

The function $f : X \to \mathbb{R}$ is said to be ν-harmonic *if we have for a.e. $x \in X$*

$$f(x) = (\nu * f)(x)$$

It is said to be ν-subharmonic *if we have for a.e.* $x \in X$

$$f(x) \leq (\nu * f)(x)$$

Part of the definition is that $\nu * f$ *is well-defined.*

Define also the analogue of the Laplacian

$$Lf := \nu * f - f$$

Now, assume G is endowed with some non-trivial norm $\|-\|$ satisfying the triangle inequality. Assume it gives a left-invariant distance inducing the same topology. For $\mathrm{SL}_2 \mathbb{R}$ the operator or matrix norm will do.

Definition 1.5.3 *A measurable function* $f : G \to \mathbb{R}$ *is* tame *if it satisfies the bound*

$$|f(g)| = O(\log \|g\|)$$

A measurable function $f : X \to \mathbb{R}$ *is* tame *if for* μ*-a.e.* $x \in X$*, the function* f_x *defined by*

$$f_x(g) = f(gx)$$

is a tame function on G*.*

The next result puts some restrictions on (sub)harmonic functions on X.

Proposition 1.5.4

(i) Suppose that $f \in L^1(X, \mu)$ *is* ν*-subharmonic. Then* f *is a.e. constant.*

(ii) Suppose that $f : X \to \mathbb{R}$ *is positive, tame, and* ν*-harmonic. Then* f *is a.e. constant.*

Proof. Consider a random walk on G, sampled by the measure ν. For part (i) note that by subharmonicity we have

$$f(x) \leq \frac{1}{N}\sum_{1}^{N}\mathbb{E}[f(g_n\cdots g_1 x)]$$

By the Furstenberg Random Ergodic theorem, the right-hand side converges a.e. to $\int_X f\,d\mu$. We thus have

$$f(x) \leq \int_X f\,d\mu$$

Integrating the above inequality over X for the measure μ, we see that equality must occur μ-a.e.

For (ii) note that the tameness of f implies that $\nu * f$ is well-defined and still tame. Now, because f is harmonic, we have

$$f(x) = \frac{1}{N}\sum_{1}^{N}\mathbb{E}[f(g_n\cdots g_1 x)]$$

By the Furstenberg Random Ergodic theorem, the right hand side converges to $\int_X f\,d\mu$. If the integral is finite, we conclude as before.

If this integral is $+\infty$, then f must also be infinite a.e. This contradicts the tameness of f. □

1.5.2 Subharmonic functions with sublinear growth

We keep the setup from the previous section.

Definition 1.5.5 *A function $f : X \to \mathbb{R}$ is of* sublinear growth *if for a random walk sampled from ν we have for μ-a.e. x that ω-almost surely*

$$|f(g_n(\omega)\cdots g_1(\omega)x)| = o(n)$$

The estimate is allowed to depend on x and ω. Here, ω denotes the point in the (unspecified) *probability space modeling the random walk.*

Remark 1.5.6 It is possible for a function to be of sublinear growth, yet not be tame.

Proposition 1.5.7

(i) Suppose that f is positive, tame, of sublinear growth, and ν-subharmonic. Then f is constant.

(ii) Suppose that f is ν-subharmonic. Let $f^+ := \max(0, f)$ be its positive part. Assume f^+ is tame and of sublinear growth (it automatically is ν-subharmonic, as the max *of two such).*

Then f is constant.

Proof. Part (ii) is of course stronger, but Part (i) is needed to deduce it. We prove it first.

Consider $Lf := \nu * f - f$. Because f is subharmonic, this function is non-negative. We shall prove that it must be zero, thus reducing this statement to Proposition 1.5.4, part (ii).

We shall prove the integral $I := \int_X (Lf)\, d\mu$ which is non-negative (perhaps $+\infty$) must in fact be zero. By the Furstenberg Random Ergodic Theorem the functions

$$A_N(x, \omega) := \frac{1}{N} \sum_{k=0}^{N-1} Lf(g_k(\omega) \cdots g_1(\omega) x)$$

converge (x, ω)-a.e. to I (for $k = 0$, we take $Lf(x)$ in the sum). Note also that $A_N(x, \omega)$ is always non-negative.

We now rewrite the expression for A_N using the definition of Lf:

$$A_N(x, \omega) = \frac{1}{N} \sum_{k=0}^{N-1} \int_G [f(gg_k(\omega) \cdots g_1(\omega)x)\, d\nu(g) - \\ - f(g_k(\omega) \cdots g_1(\omega)x)]\, d\nu(g)$$

Taking expectations over ω we find a telescoping sum

$$A_N(x) = \frac{1}{N}\left[\int_{G^N} f(g_N \cdots g_1 x)\, d\nu^{\otimes N} - f(x)\right]$$

Now consider the functions

$$B_N(x,\omega) := \frac{1}{N}\left[f(g_{N+1}(\omega)\cdots g_1(\omega)x) - f(x)\right]$$

For the moment, freeze x. Because f is tame, $B_N(x,\omega)$ is uniformly bounded in N by a constant $C(x)$ (since ν has compact support; note the $\frac{1}{N}$ factor in B_N). But f is also of sublinear growth, so ω-pointwise $B_N(x,\omega)$ goes to zero as $N \to \infty$. From the Dominated Convergence Theorem applied to $|B_N(x,\omega)| < C(x)$, it follows that

$$\lim_N \mathbb{E}_\omega[B_N(x,\omega)] = \lim_N \int B_N(x,\omega) d\omega \to \int \lim_N B_N(x,\omega) = 0 \tag{1.5.1}$$

Above, $\mathbb{E}_\omega$ denotes expectation w.r.t. ω. Now unfreeze x and note that

$$A_N(x) = \mathbb{E}_\omega[B_N(x,\omega)]$$

Moreover, the above quantity is non-negative. From Eq. (1.5.1) it follows that $A_N(x)$ also converges pointwise to zero. By the Furstenberg Random Ergodic theorem, the integral $I := \int_X Lf\, d\mu$ must also be zero.

For Part (ii), fix $A \in \mathbb{R}$ and consider the function

$$f_A := A + \max(-A, f)$$

This is still subharmonic, since the maximum of two subharmonic functions is subharmonic. But it satisfies the assumptions of part (i) and is thus constant. Sending A to $+\infty$, we conclude f itself must be constant. $\square$

1.6 Semisimplicity

In this section we consider an ergodic $\mathrm{SL}_2\,\mathbb{R}$-invariant measure μ on some stratum $\mathcal{H}$. First we consider an integrable cocycle $H_{\mathbb{C}}$ over μ which gives a variation of Hodge structures on every Teichmüller disk. In particular, it is invariant under $K = SO(2)$.

Moreover, we make the boundedness assumption on the cocycle matrix $A(g, x)$ for $g \in \mathrm{SL}_2\,\mathbb{R}, x \in X$

$$\log \|A(g, x)\| \leq C_1 \log \|g\| + C_2 \tag{1.6.1}$$

This is satisfied in the case of the Kontsevich-Zorich cocycle and cocycles obtained from it by tensor operations. This was first proved by Forni in [For02] but see also [FMZ14a, Lemma 2.3].

Using the results on random walks and curvature of Hodge bundles, we prove the Theorem of the Fixed Part. It states that a section of this cocycle flat along every Teichmüller disk must have each (p, q) component flat as well.

This theorem applies to endomorphism bundles of the Kontsevich-Zorich cocycle (i.e. the Hodge bundle) or tensor powers thereof. To use this, we need the reductivity of the algebraic hull (see Remark 1.2.12).

The above discussion gives a semisimplicity theorem similar to Deligne's in the case of usual variations of Hodge structure.

1.6.1 Theorem of the Fixed Part

We need some preliminary results.

Lemma 1.6.1 *Let g_t be an ergodic measure-preserving flow on a space (X, μ) and let H be* *some integrable linear cocycle over the flow. Suppose that ϕ is a measurable section of the cocycle which is invariant under the flow. Then ϕ must a.e. lie in the central Lyapunov subspace (i.e. it has Lyapunov exponent zero).*

Proof. If not, then $\|\phi\|$ would grow along a.e. trajectory. But the flow recurs to sets where the norm of ϕ is bounded. □

Lemma 1.6.2 *Suppose f is a μ-measurable function, invariant under $K := \mathrm{SO}(2)$. It descends to Teichmüller disks, and assume it is subharmonic on μ-almost all of them. This means that $\partial\overline{\partial} f \geq 0$ in the sense of §1.4.3 (note the change from $\overline{\partial}\partial$ to $\partial\overline{\partial}$).*

Denote by $f^+ := \max(0, f)$ the positive part of f, also subharmonic. Suppose that f^+ grows sublinearly along a.e. Teichmüller geodesic (non-uniformly in the geodesic). Finally, suppose that $|f^+(x) - f^+(gx)| \leq C\,\|g\|$ for some fixed C and for every $g \in \mathrm{SL}_2\,\mathbb{R}$.

Then f must be μ-a.e. constant.

Proof. Pick a K-bi-invariant measure ν on $\mathrm{SL}_2\,\mathbb{R}$, with compact support which generates the group. Then f is also ν-subharmonic in the sense of Definition 1.5.2.

Consider the random walk generated by νon $\mathrm{SL}_2\,\mathbb{R}$. In Kaimanovich's interpretation [Kaĭ87] of the Oseledets Multiplicative ergodic theorem, random walk trajectories track geodesics with sublinear error. Namely, considering the image of a random walk trajectory in $K\backslash\mathrm{SL}_2\,\mathbb{R}$ (which is the hyperbolic plane), there is a hyperbolic geodesic which approximates it well.

Concretely, this means we have a rate of drift $\delta > 0$ and a random geodesic $\gamma_\bullet(\omega)$ on $K\backslash\mathrm{SL}_2\,\mathbb{R}$ such that

$$d\Big(\,[g_n(\omega)\cdots g_1(\omega)x]\,,[\gamma_{\delta n}(\omega)x]\Big) = o(n)$$

Here $[-]$ denotes the projection or equivalence class in $K\backslash\mathcal{H}$, i.e. the stratum divided by the action of K. The hyperbolic geodesic $\gamma_\bullet$ becomes a Teichmüller geodesic, and the drift δ is strictly positive because the measure ν generates $\mathrm{SL}_2\,\mathbb{R}$.

Note that the function f satisfies the assumptions of Proposition 1.5.7 part (ii). Tameness and subharmonicity are part of the current assumptions and the sublinear tracking of Teichmüller geodesics gives the sublinear growth along paths of the random walk. We conclude f must be a.e. constant. $\square$

Theorem 1.6.3 (Theorem of the Fixed Part) *Let $H_{\mathbb{C}}$ be a variation of Hodge structures satisfying the boundedness assumption from the beginning of the section (see Eq. (1.6.1)). Suppose that ϕ is a measurable section of $H_{\mathbb{C}}$, flat along a.e. $\mathrm{SL}_2\,\mathbb{R}$-orbit.*

Then each (p,q)-component of ϕ is also flat along a.e. $\mathrm{SL}_2\,\mathbb{R}$-orbit.

Proof. Write $\phi = \phi^{w,0} + \cdots + \phi^{p,w-p}$. Recall (equation 1.4.12) that we have the relation

$$\nabla^{GM} = \nabla^{Hg} + \sigma_\bullet + \sigma_\bullet^\dagger$$

Because ϕ is flat, by inspecting the $(p-1, w-p+1)$ component of $\nabla^{GM}\phi$ we see that $\sigma_p\phi^{p,w-p} = 0$ (along Teichmüller disks).

Consider now the projection of ϕ to the bundle $\mathcal{H}^{p,w-p}$. Since $\mathcal{H}^{p,w-p} := \mathcal{F}^p/\mathcal{F}^{p+1}$ is a quotient of holomorphic bundles and ϕ is a holomorphic section of $\mathcal{F}^p$, the projection is also holomorphic. It also equals $\phi^{p,w-p}$ and we denote it by ψ for simplicity.

Applying Lemma 1.4.1 (note the switch in order of $\overline{\partial}$ and ∂), we find

$$\partial\overline{\partial}\log\|\psi\|^2 = -\frac{\langle\Omega\psi\rangle\,\psi}{\|\psi\|^2} + \frac{\|\psi\|^2\cdot\langle\nabla\psi\rangle\,\nabla\psi - \langle\nabla\psi\rangle\,\psi\cdot\langle\psi\rangle\,\nabla\psi}{\|\psi\|^4}$$

Note that the second term is positive by Cauchy-Schwartz. For the first one, recall that by Proposition 1.4.12

$$\Omega_{\mathcal{H}^{p,w-p}} = \sigma_p^\dagger\wedge\sigma_p + \sigma_{p+1}\wedge\sigma_{p+1}^\dagger$$

Because $\sigma_p \psi = 0$, we apply Proposition 1.4.14 to find

$$\langle \Omega_{\mathcal{H}^{p,w-p}} \psi \rangle \, \psi = \left\langle \sigma^\dagger_{p+1} \psi \right\rangle \sigma^\dagger_{p+1} \psi$$

The term above is negative, so the function is subharmonic along a.e. Teichmüller disk. Note that we might have δ-masses coming from zeroes of ψ, but the function will stay subharmonic (see Remark 1.4.2 (ii)).

By Lemma 1.6.1 we have that the positive part of $\log \|\phi\|^2$ grows sublinearly along a.e. Teichmüller geodesic. The same must be true of the positive part of each of its (p,q)-components, in particular of $\log \|\psi\|^2$.

We can thus apply Proposition 1.6.2 to conclude that $\log \|\psi\|^2$ must be constant along a.e. Teichmüller disk.

Looking at $0 = \overline{\partial}\partial \|\psi\|^2$ (see Remark 1.4.3) we find that $\sigma^\dagger_{p+1}\psi = 0$ and $\nabla^{Hg}\psi = 0$. By looking at the relationship between the Gauss–Manin and Hodge connections (see Eq. (1.4.12)) we conclude ψ is flat for the Gauss–Manin connection.

Subtracting $\psi = \phi^{p,w-p}$ from ϕ, the above argument can be iterated. □

We also record for future use the next result. Note that it is also used in [EM13] to compare the volume forms coming from the symplectic pairing and the Hodge norm.

Corollary 1.6.4 *Suppose H is a cocycle which induces a variation of Hodge structure on Teichmüller disks with appropriate boundedness conditions (e.g. a tensor power of the Kontsevich-Zorich cocycle). Suppose that ϕ is a measurable global section of H, flat along a.e.* $\mathrm{SL}_2 \mathbb{R}$*-orbit.*

Then the Hodge norm of ϕ is a.e. constant, and each (p,q)-component of ϕ is flat along a.e. $\mathrm{SL}_2 \mathbb{R}$*-orbit. Moreover, each (p,q)-component also has constant Hodge norm.*

1.6.2 Deligne semisimplicity

Setup In this section we denote by E the Hodge bundle or some tensor power, defined over an $\mathrm{SL}_2\,\mathbb{R}$-invariant measure μ in some stratum. For the real and complex bundles, we use the notation $E_{\mathbb{R}}$ and $E_{\mathbb{C}}$.

Theorem 1.6.5 *Suppose that $V \subset E_{\mathbb{C}}$ is an $\mathrm{SL}_2\,\mathbb{R}$-invariant subbundle. Then $C \cdot V$ is also $\mathrm{SL}_2\,\mathbb{R}$-invariant, where C is the Weil operator.*

Corollary 1.6.6 *Suppose $V \subset E_{\mathbb{R}}$ is $\mathrm{SL}_2\,\mathbb{R}$-invariant. Then so is $C \cdot V$.*

Proof of Corollary. Apply the previous theorem to $V_{\mathbb{C}} := V \otimes \mathbb{C}$ and note that C is an operator defined over $\mathbb{R}$. □

Proof of Theorem 1.6.5. By Remark 1.2.12, we know that the cocycle corresponding to E has reductive algebraic hull. In particular, any $\mathrm{SL}_2\,\mathbb{R}$-invariant subbundle has an invariant complement. Denote this complement by $V^{\perp}$ and let $\pi_V \in \mathrm{End}(E_{\mathbb{C}})$ be the projection to V along $V^{\perp}$. This projection operator is $\mathrm{SL}_2\,\mathbb{R}$-invariant, because the bundles are.

We apply the Theorem of the Fixed Part 1.6.3 to conclude that $C \cdot \pi_V$ is also $\mathrm{SL}_2\,\mathbb{R}$-invariant. But this last operator is projection to $C \cdot V$ along $C \cdot V^{\perp}$, so we conclude $C \cdot V$ must be $\mathrm{SL}_2\,\mathbb{R}$-invariant. □

Remark 1.6.7 It is clear that Theorem 1.6.5 is valid if we replace the Weil operator by any other element of the Deligne torus $\mathbb{S}$. This is relevant in the case of higher-weight variations.

The proof of the following result is along the lines presented by Deligne in [Del87].

Theorem 1.6.8 (Deligne semisimplicity) *There exist $\mathrm{SL}_2\,\mathbb{R}$-invariant bundles $V_i \subset E$ and vector spaces W_i equipped with Hodge structures and an isomorphism*

$$E \cong \bigoplus_i E_i \quad \textit{with} \quad E_i \cong V_i \otimes_{A_i} W_i$$

Moreover, each V_i carries a variation of Hodge structure making the above isomorphism compatible. The A_i are division algebras which act on V_i, compatible with Hodge structures. They also act compatibly on W_i (see Remark 1.6.10 for a discussion of these conditions).

Any $\mathrm{SL}_2 \mathbb{R}$-invariant bundle $V' \subset E$ is of the form

$$V' = \bigoplus_i V_i \otimes_{A_i} W_i'$$

where $W_i' \subset W_i$ are A_i-submodules. In the case of complexified bundles $E_{\mathbb{C}}$ a Hodge structure is understood as defined in Theorem 1.1.1.

Remark 1.6.9 When E is the Hodge bundle, we know that every invariant subbundle is either symplectic or inside the Forni bundle (see [AEM14]). This means that in the decomposition above, besides the Forni subspace, the only other possibility is to have W_i a vector space with positive-definite inner product (i.e. a polarized Hodge structure of weight 0) and V_i some weight 1 polarized variation of Hodge structure.

Proof. Let V be an $\mathrm{SL}_2 \mathbb{R}$-invariant subbundle of minimal dimension. Because it is of minimal dimension its endomorphism algebra, denoted A, is a division algebra. Let W denote the space of morphisms of $\mathrm{SL}_2 \mathbb{R}$-invariant bundles from V to E (not required to respect the Hodge structures).

$$W := \mathrm{Hom}_{\mathrm{SL}_2 \mathbb{R}}(V, E)$$

Because A acts by endomorphisms on V, it also acts on the left on W by precomposition. We can then define the natural evaluation map

$$ev : V \otimes_A W \to E$$

Denote its image by E'. By Theorem 1.6.5 (see also Remark 1.6.7) it follows that E' is a sub-variation of Hodge structure. By considering the orthogonal to E' (the definite or indefinite metric give the same complement) we reduce to applying the argument below by induction to this complement.

Any $\phi \in W$ is either injective or zero because the dimension of V is smallest possible. Therefore, we have an isomorphism $V \otimes_A W \to E'$. Because V has no invariant subbundles, we have

$$\mathrm{End}_{\mathrm{SL}_2 \mathbb{R}}(E') \cong \mathrm{End}_{\mathrm{SL}_2 \mathbb{R}}(V \otimes_A W) \cong \mathrm{End}_A(W)$$

Note that the first object has a natural Hodge structure inherited from the underlying variation, thus it induces one on $\mathrm{End}_A(W)$. Lemma 1.3.9 provides W with a Hodge structure, such that A acts on it compatibly.

We want to endow V with a Hodge structure such that the isomorphism $V \otimes_A W \to E'$ is compatible. The bundle V is naturally identified with the subbundle of $\mathrm{Hom}(W, E')$ which is equivariant for the action of the algebra $\mathrm{End}_A(W)$ (acting by $\mathrm{End}(E')$ on the second factor). Namely, every $v \in V$ gives an evaluation map $W \to E'$ (recall that W itself is a Hom-space). The subbundle thus-obtained is characterized by the equivariance property for the action of $\mathrm{End}_A(W)$.

This provides V with the required Hodge structure. Note that the structures on V and W are unique up to a simultaneous shift (in opposite directions).

The proof of the first part is now complete, as we have endowed the required spaces with Hodge structures.

Consider now a general invariant subbundle $V' \subset E$ and the given direct sum decomposition

$$E = \oplus V_i \otimes_{A_i} W_i$$

Let π_i be the projection onto the factor with index i. We claim $V' = \pi_1 V' \oplus (1 - \pi_1)V'$. If this is proved, then we can iterate the argument to $(1 - \pi_1)V'$. It is also clear that any invariant subbundle of $V_i \otimes_{A_i} W_i$ has to be of the form $V_i \otimes_{A_i} W_i'$ for some A_i-submodule

$W_i' \subset W_i$.

To prove the claimed decomposition of V', suppose that $\ker \pi_1$ and $\ker(1 - \pi_1)$ don't span V'. Their span has some non-trivial invariant complement V''. But the image of V'' under π_1 and $1 - \pi_1$ is isomorphic to V'' (and $\mathrm{SL}_2 \mathbb{R}$-invariant) and reversing one of the arrows, we get an embedding of V_1 into E which was not accounted for by W_1. This is a contradiction. $\square$

Remark 1.6.10

(i) The above proof (and statement) applies to both the real and complex Hodge bundles, so this specification is omitted from the notation. The spaces W_i correspond to possible isotypical components of the bundles, but formulated in an invariant way.

(ii) The algebras A_i can only arise over $\mathbb{R}$, since there are no nontrivial division algebras over $\mathbb{C}$. In this case, they take into account possible symmetries of the real decomposition. The possibilities are the complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$. Example 1.6.11 contains a further discussion.

(iii) The above proof is also compatible with the underlying polarizations, since all the constructions were natural. The subvariation E' carries a polarization and it extends to $\mathrm{End}(W)$ and then lifts to W. It automatically gives one on V by construction.

The next example is meant to illustrate how some of the structures in Theorem 1.6.8 can arise.

Example 1.6.11 Let us start with a situation which has been considered many times (e.g. [FMZ14a, FMZ14b]). Suppose that $E_{\mathbb{R}}$ is an irreducible *real* piece of the cocycle, and suppose that its complexification splits as $E_{\mathbb{C}} := E_{\mathbb{R}} \otimes \mathbb{C} = E_1 \oplus \overline{E_1}$. The indefinite hermitian pairing on E_1 has signature (p, q) and on $\overline{E_1}$ signature (q, p).

The splitting of $E_{\mathbb{C}}$ can be seen at the level of $E_{\mathbb{R}}$ as follows. Let $z \in \mathbb{C}$ act on $e \oplus f \in E_1 \oplus \overline{E_1} = E_{\mathbb{C}}$ by

$$z \cdot (e \oplus f) := ze \oplus \overline{z} f$$

This action commutes with complex conjugation on $E_{\mathbb{C}}$, so it is defined by a real operator on $E_{\mathbb{R}}$. In other words, we exhibited $\mathbb{C}$ inside the endomorphisms of the bundle $E_{\mathbb{R}}$.

A bit more generally, suppose that a finite group G acts on the real bundle E. This is the situation considered in detail by Matheus–Yoccoz–Zmiaikou [MYZ14] and also in [FFM15].

Fix an irreducible representation of G on a vector space V_ρ. This representation could be of real, complex, or quaternionic type, i.e. the algebra of endomorphisms of V_ρ commuting with G, denoted A_ρ, could be $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Inside E the representation V_ρ appears potentially with some multiplicity. Let E_ρ denote the subbundle where G acts as on V_ρ, perhaps on several copies thereof. Note that the multiplicity is constant, but in general there is no way to isolate an individual representation. We have that $E = \oplus_\rho E_\rho$ as ρ ranges over all representations.

Consider now the local system of isotypical components: $W_\rho := \operatorname{Hom}_G(V_\rho, E)$. Note that W_ρ carries a left action of A_ρ, since V_ρ carries a right action of A_ρ. It now follows that the natural evaluation map $V_\rho \otimes_{A_\rho} W_\rho \to E_\rho$ is an isomorphism.

1.7 Rigidity

Consider an affine invariant manifold $\mathcal{M}$ as defined in §1.1.1. Fix some tensor construction on the Kontsevich-Zorich cocycle and call it E. According to Theorem 1.6.8, it has a decomposition into $\mathrm{SL}_2 \mathbb{R}$-invariant bundles

$$E = \bigoplus_i E_i \tag{1.7.1}$$

A priori, the bundles E_i are allowed to vary *measurably* in directions transverse to the $\mathrm{SL}_2\,\mathbb{R}$-action. The goal of this section is to show that they vary, in fact, polynomially.

Note that in the notation of Theorem 1.6.8 we have $E_i = V_i \otimes_{A_i} W_i$. To show regularity of any $\mathrm{SL}_2\,\mathbb{R}$-invariant bundle, it suffices to show it for the E_i.

This is achieved in three steps. In §1.7.1 using Hodge-orthgonality of the decomposition (1.7.1) we establish real-analyticity along a.e. stable or unstable leaf for each E_i. In fact, the forward Lyapunov bundles of E_i are analytic on a.e. stable leaf, and the backward Lyapunov bundles are analytic on unstable leaves.

Next, §1.7.2 shows that the dependence along leaves must be polynomial. This is based on the contraction properties of the Teichmüller flow. Finally, these statements are assembled in §1.7.3 to give joint polynomiality in all directions.

1.7.1 Leafwise analyticity

Notation For a bundle F (e.g. E or one of the E_i from (1.7.1)) denote its forward Lyapunov filtration by

$$F = F^{\leq \lambda_1} \supsetneq F^{\leq \lambda_2} \supsetneq \cdots \supsetneq F^{\leq \lambda_n} \supsetneq \{0\} \tag{1.7.2}$$

By definition

$$f \in F^{\leq \lambda_i} \text{ if and only if } \limsup_{t\to+\infty} \frac{1}{t} \log \|g_t f\| \leq \lambda_i \tag{1.7.3}$$

Similarly, one can define the backward filtration $F^{\geq \lambda_\bullet}$ using $T \to -\infty$. This gives the subbundles $F^{\lambda_i} := F^{\leq \lambda_i} \cap F^{\geq \lambda_i}$ and the g_t-invariant decomposition

$$F = \bigoplus_i F^{\lambda_i} \tag{1.7.4}$$

Period coordinates Recall that on a stratum $\mathcal{H}$, local coordinates near (X, ω) are given by the relative cohomology group $H^1(X, \Sigma; \mathbb{C})$ where Σ denotes the zeros of ω, which will be abbreviated $H^1_{rel}(\mathbb{C})$. The unstable foliation is given by the real planes $H^1_{rel}(\mathbb{R})$ and the stable foliation by $\sqrt{-1}H^1_{rel}(\mathbb{R})$. The affine manifold $\mathcal{M}$ is cut out by real-linear equations, so it inherits the same foliations. For simplicity, we denote the (local) stable leaf through $x \in \mathcal{M}$ by $W^-(x)$, and the unstable by $W^+(x)$.

Flatness Recall that E (see (1.7.1)) is equipped with the Gauss–Manin connection. Since the Lyapunov filtration $E^{\leq \lambda_i}$ defined in (1.7.3) depends only on the future trajectory, this filtration is flat (for the Gauss–Manin connection) along stable leaves. In other words, $E^{\leq \lambda_i}(x) = E^{\leq \lambda_i}(y)$ if $x \in W^-(y)$.

Note that $E^{\leq \lambda_i}(x)$ exists whenever the same space exists for y, and $x \in W^-(y)$. In other words, the forward Lyapunov filtration exists for a.e. unstable leaf, and is locally constant on such an unstable leaf.

The following lemma, stated as Corollary 4.5 in [EM13], will be useful when two bundles in the decomposition (1.7.2) have overlap in the Lyapunov spectrum. This invariance principle goes back to Ledrappier [Led86].

Proposition 1.7.1 *Suppose M is a g_t-invariant subbundle and $E^{\leq \lambda_k} \subseteq M \subseteq E^{\leq \lambda_{k-1}}$. Then M is locally flat along the stable leaves.*

The next statement is the main result of this section.

Proposition 1.7.2 *Consider the decomposition*

$$E = \oplus_j E_j$$

into $\mathrm{SL}_2\,\mathbb{R}$-invariant bundles. Let $E_j^{\leq \lambda_i}$ be the corresponding forward Lyapunov filtrations for each E_j. Then the bundles $E_j^{\leq \lambda_i}$ vary real-analytically on a.e. stable leaf.

The same holds for the backward Lyapunov filtration and a.e. unstable leaf. In particular, each $E_j = E_j^{\leq \lambda_1} = E_j^{\geq \lambda_1}$ varies real-analytically on a.e. stable or a.e. unstable leaf.

Proof. It suffices the prove the claim for the forward filtration, the backward being analogous. We have the decomposition of Lyapunov bundles

$$E^{\leq \lambda_i} = \oplus_j E_j^{\leq \lambda_i} \text{ and } E^{\lambda_i} = \oplus_j E_j^{\lambda_i}$$

Recall that we ordered the Lyapunov exponents as $\lambda_1 > \cdots > \lambda_n$. The proof that $E_j^{\leq \lambda_i}$ varies real-analytically is by descending induction on the index of the exponent i, starting from the smallest exponent λ_n.

Consider the first step $i = n$. Then $E^{\leq \lambda_n} = E^{\lambda_n}$ and the same holds for all the E_j. Note that typically, e.g. if the spectrum is simple such as in the work of Avila–Viana [AV07], all but one of the $E_j^{\lambda_n}$ are empty. Since $E^{\leq \lambda_n}$ is flat along unstables, the claim follows.

In the general case, we still have for all j a g_t-invariant sequence of bundles

$$\{0\} \subseteq E_j^{\lambda_n} \subseteq E^{\lambda_n}$$

By Proposition 1.7.1, it follows that $E_j^{\lambda_n}$ is flat along unstables, for all j.

To perform the induction step, assume that for all j we know that $E_j^{\leq \lambda_{i+1}}$ vary real analytically. We want to deduce the same statement for $E_j^{\leq \lambda_i}$.

Fix some index k and consider the g_t-invariant bundle

$$B := span\left(E_k^{\lambda_i} + E^{\leq \lambda_{i+1}}\right)$$

Then B is sandwiched as in the assumption of Proposition 1.7.1:

$$E^{\leq \lambda_{i+1}} \subseteq B \subseteq E^{\leq \lambda_i}$$

Therefore B is flat along unstable leaves. Another way to write B is

$$B = \left(\bigoplus_{j \neq k} E_j^{\leq \lambda_{i+1}} \right) \oplus E_k^{\leq \lambda_i}$$

By induction, the summands with $j \neq k$ all vary real-analytically on unstable leaves. By Hodge orthogonality of the bundles E_i this direct sum is Hodge orthogonal. Therefore $E_k^{\leq \lambda_i}$ is the Hodge-orthgonal of something real-analytic inside something flat. Thus, it is itself real-analytic. This completes the induction step. □

1.7.2 Analyticity to polynomiality. Leafwise.

The previous section established analyticity of invariant bundles on a.e. stable or unstable leaf. This section proves they must vary polynomially on a.e. leaf.

Remark 1.7.3 We will work below with two cocycles. One is a tensor power of the Kontsevich-Zorich cocycle, which is the Gauss–Manin connection on a tensor power of the Hodge bundle $H^1_{\mathbb{R}}$. By abuse of notation, we continue to denote it by g_t (when considering the Teichmüller geodesic flow).

The positive part of the Lyapunov spectrum of the Kontsevich-Zorich cocycle is

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \cdots \geq \lambda_g \geq 0$$

The spectral gap inequality $1 > \lambda_2$ is due to Forni [For02] and is key to the argument below.

The second cocycle comes from the action of g_t on the stratum and the induced cocycle on the tangent space. We denote this cocycle by dg_t. The positive part of its Lyapunov spectrum is

$$1 + \lambda_1 > 1 + \lambda_2 \geq \cdots \geq 1 + \lambda_g \geq 1 \geq \cdots \geq 1 \geq 1 - \lambda_g \geq \cdots \geq 1 - \lambda_1$$

Although the last term above is $0 = 1 - \lambda_1$, the above quoted spectral gap result of Forni implies that on unstable leaves, the cocycle dg_t is expanding at rate at least $1 - \lambda_2$. The reason is that the subspace corresponding to the zero exponent comes from the centralizer of g_t inside $\mathrm{GL}_2 \mathbb{R}$ and does not appear when restricted to area one surfaces. A discussion of these questions is available in section 5.8 of the survey [Zor06].

Let us return to the invariant subbundles which by Proposition 1.7.2 vary real-analytically on a.e. leaf. From now on, fix a bundle E_i from the decomposition (1.7.1).

Definition 1.7.4 *For a point x in the affine manifold $\mathcal{M}$ and vector v in the unstable direction on $\mathcal{M}$ define the operator*

$$\pi(x, v) : E_x \to E_x$$

It is the operator of projection onto E_i at the point $x + v$ (using the decomposition (1.7.1)), transported to the point x by the Gauss–Manin connection.

We view $\pi(x, v)$ as a section of the bundle of endomorphisms of the Hodge bundle, thus g_t acts on it by the Gauss–Manin connection. From the equivariance properties of the bundles, we deduce that

$$g_{-t}\pi(x, v) = \pi(g_{-t}x, dg_{-t}v) \tag{1.7.5}$$

Note that the vector v is moved by the cocycle dg_{-t} because it lives in the ambient manifold. This will be crucial.

Proposition 1.7.5 *Suppose that for a.e. x the operator $\pi(x, v)$ varies real-analytically in v, where v is in some small neighborhood of x along the unstable leaf. Then for a.e. x we have that $\pi(x, v)$ varies polynomially in v, and the degrees of the polynomials are uniformly bounded.*

Proof. Equation (1.7.5) is equivalent to

$$\pi(x, v) = g_t(\pi(g_{-t}x, dg_{-t}v)) \tag{1.7.6}$$

Fix a coordinate chart. At a point x where the dependence is real analytic (and the Oseledets theorem holds) we have

$$\pi(x, v) = \sum_\alpha c_\alpha(x) v^\alpha$$

Here α is a multi-index and c_α are (measurably varying in x) endomorphism of E_x.

Fix a set of positive measure (on $\mathcal{M}$) where the above equality holds. By Poincaré recurrence, a.e. trajectory in this set will return to it for arbitrarily large times. At such times, using the equivariance properties of π under the flow given by Eq. (1.7.6) we have two different Taylor expansions

$$\begin{aligned}\pi(x, v) &= \sum_\alpha g_t c_\alpha(g_{-t}x)(dg_{-t}v)^\alpha \\ &= \sum_\alpha c_\alpha(x) v^\alpha\end{aligned}$$

Let Λ be the largest Lyapunov exponent of $\mathrm{End}(E)$. We will show that for $|\alpha| > \frac{\Lambda}{1-\lambda_2}$ we must have $c_\alpha = 0$. For this, fix a further subset set K of positive measure on which c_α is bounded above.

Considering times t such that $g_{-t}x \in K$ we have for any $\epsilon > 0$ (as $t \to +\infty$)

$$\|g_t c_\alpha(g_{-t}x)\| = o(e^{(\Lambda+\epsilon)t})$$

On the other hand for any $\epsilon_1 < 1 - \lambda_2$

$$\|dg_{-t}v\| = o(e^{-t\epsilon_1})$$

So we have that

$$\|g_t c_\alpha(g_{-t}x)(dg_{-t}v)^\alpha\| = o(e^{t(\Lambda+\epsilon-|\alpha|\epsilon_1)})$$

We conclude that whenever $|\alpha| > \frac{\Lambda}{1-\lambda_2}$, the corresponding terms in the Taylor expansion of $\pi(x, v)$ must vanish. $\square$

1.7.3 Joint polynomiality

To continue, we record the following observation which goes back at least to Margulis (see the appendix to the Russian translation of Raghunathan's book [Rag77] or [Zim84, Thm. 3.4.4]).

Lemma 1.7.6 *Let $U_i \subset \mathbb{R}^{n_i}$ with $i = 1..2$ be connected open "boxes", i.e. of the form product of intervals. Let $f : U_1 \times U_2 \to \mathbb{R}$ be a measurable function. Assume that for a.e. $x_1 \in U_1$, the function $f(x_1, -)$ agrees a.e. with a polynomial in the variable $x_2 \in U_2$. Assume that the same holds for the two variables swapped.*

Then f agrees a.e. with a polynomial in x_1 and x_2.

Proof. **Step 1:** Assume that two polynomials $p_1, p_2 : \mathbb{R}^n \to \mathbb{R}$ agree on a set E of positive Lebesgue measure. Then they coincide.

We show this by induction. In dimension 1, this is immediate.

Consider now on $\mathbb{R}^n \times \mathbb{R}$ with coordinates (x, t) the decomposition into polynomials

$$p_i(x, t) = \sum_k c_k^i(x) t^k$$

By Fubini, there is a positive measure set of x such that a positive measure set of t satisfy that $(x, t) \in E$. For such x, it must be that $c_k^1(x) = c_k^2(x)$. By induction, $c_k^1 = c_k^2$ as polynomials.

Step 2: We now proceed by induction. In fact, it suffices to check the claim when $U_2 \subset \mathbb{R}$

is an interval. Applying iteratively this simpler case by specializing all but one of the coordinates in $\mathbb{R}^{n_2}$, the general claim follows.

Note that there exist positive measure sets $E_i \subset U_{n_i}$ such that the degrees of the polynomials in the hypothesis on f are bounded (by some N). We then have a.e.

$$f(x_1, x_2) = \sum_{|\alpha|<N} c_\alpha(x_2) x_1^\alpha$$
$$f(x_1, x_2) = \sum_{n<N} d_n(x_1) x_2^n$$

where $x_i \in E_i$ and $c_\alpha(x_2), d_n(x_1)$ are measurable. Here α denotes a multi-index, while n a positive integer.

By assumption, for a.e. value of $x_2 \in E_2$, the two sides agree a.e. in x_1. We can thus pick $N+1$ distinct values for x_2 and solve to find that $d_n(x_1)$ are a.e. equal to polynomials in x_1. Note that the determinant of the system to solve is of Vandermonde type, so non-zero.

We conclude that f on $E_1 \times E_2$ is a polynomial function. Enlarge now E_1 and E_2 but such that the degrees of the polynomials in the assumption stay bounded. Then the same argument shows f is polynomial on the larger set. By Step 1, it must be the same polynomial. Exhausting $\mathbb{R}^{n_i}$ by such sets we find that f is a.e. equal to a polynomial. $\square$

Combining the above Lemma with Proposition 1.7.5, we prove the next result.

Theorem 1.7.7 *On an affine invariant manifold $\mathcal{M}$, an $\mathrm{SL}_2\,\mathbb{R}$-invariant measurable subbundle of the Hodge bundle (or its tensor powers) must in fact be polynomial in linear coordinates.*

Specifically, let the affine coordinates be $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $A(x, y)$ be the area function, quadratic in x and y. Assign homogeneous degree 1 to each of x and y, and degree -2 to $\frac{1}{A(x,y)}$. Then the projection operator, in a flat trivialization, is a matrix with entries polynomials of homogeneous degree zero in the variables $x, y, \frac{1}{A(x,y)}$.

For an illustration of how non-trivial projection operators can arise, see Example 1.7.8 discussing the tautological plane.

Proof of Theorem 1.7.7. By abuse of notation, denote by $\pi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ one of the matrix coefficients of the projection map. For a point $x \in \mathbb{R}^N$, denote the "symplectic orthogonal" subspace by $x^\perp = \{y \in \mathbb{R}^N | A(x,y) = 0\}$.

The function π satisfies $\pi(\lambda x, \mu y) = \pi(x,y)$. By Proposition 1.7.5 for a.e. (x,y) we have that $\pi(x+v,y)$ is a polynomial in $v \in y^\perp$, and $\pi(x, y+w)$ is a polynomial in $w \in x^\perp$.

The first step of the proof will reduce to an affine subspace S which will be a transverse section for the $\mathbb{R}^\times \times \mathbb{R}^\times$ action under which π is invariant. Using Lemma 1.7.6 will give polynomiality on S, and the next step will be to restrict the properties further.

The notation for the local stable and unstable leaves will be:

$$
\begin{aligned}
W^+(x,y) &= \{(x+v,y) | v \in y^\perp\} \\
W^-(x,y) &= \{(x,y+w) | w \in x^\perp\}
\end{aligned}
$$

Fix a point $(x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

- $A(x_0, y_0) = 1$
- For a.e. $(x, y_0) \in W^+(x_0, y_0)$ we have polynomiality for π on the leaf $W^-(x, y_0)$
- For a.e. $(x_0, y) \in W^-(x_0, y_0)$ we have polynomiality for π on the leaf $W^+(x_0, y)$

Define now a transverse set for the scaling action:

$$
S = \{(x_0 + v, y_0 + w) | v \in y_0^\perp, w \in x_0^\perp\}
$$

Note that S is not contained in the $A(x,y) = 1$ locus, but $(x,y) \in S$ if and only if $A(x, y_0) = 1$ and $A(x_0, y) = 1$.

Consider now the retraction map

$$
\begin{aligned}
retr :&\mathbb{R}^N \times \mathbb{R}^N \to S \\
&(x, y) \mapsto \left(\frac{x}{A(x, y_0)}, \frac{y}{A(x_0, y)} \right)
\end{aligned}
$$

Because π is invariant by scaling of either variable, we have $\pi(x, y) = \pi\big(retr(x, y)\big)$.

Next, we examine how stable and unstable leaves map under retraction to S. First, we introduce notation for points on S:

$$
\begin{aligned}
W_S^-(p) &:= \{(p, q) \in S\} = \{(p, q) | A(x_0, q) = 1\} \text{ for } p \in x_0 + y_0^\perp \\
W_S^+(q) &:= \{(p, q) \in S\} = \{(p, q) | A(p, y_0) = 1\} \text{ for } q \in y_0 + x_0^\perp
\end{aligned}
$$

Choose now some $q_0 \in y_0 + x_0^\perp$ (note that $A(x_0, q_0) = 1$). It determines the unstable leaf $W^+(x_0, q_0) = \{(x_0 + v, q_0) | A(v, q_0) = 0\}$. Under the retraction map we have

$$
retr\big(W^+(x_0, q_0)\big) = \left\{ \left(\frac{x_0 + v}{A(x_0 + v, y_0)}, q_0 \right) \middle| A(v, q_0) = 0 \right\} = W_S^+(q_0)
$$

We also have the inverse maps

$$
\begin{aligned}
inv :&W_S^+(q_0) \to W^+(x_0, q_0) \\
&(p, q_0) \mapsto \left(\frac{p}{A(p, q_0)}, q_0 \right)
\end{aligned}
$$

The polynomiality of π on $W^+(x_0, q_0)$ now implies that on $W_S^+(q_0)$, π is a polynomial in the variable $\frac{p}{A(p,q_0)}$. The degree of the polynomial is uniformly bounded by Proposition 1.7.5.

Similarly we obtain for a.e. $p_0 \in x_0 + y_0^\perp$ that π is polynomial on $W_S^-(p_0)$ in the variables $\frac{q}{A(p_0,q)}$. Therefore multiplying π by a sufficiently high power of $A(p, q)$ (depending on the uniform bound of the degree from Proposition 1.7.5) and applying Lemma 1.7.6 gives a polynomial function on S.

To finish, note that π is invariant by the scalings of each variable, so it must be a ratio $\frac{P(x,y)}{A(x,y)^k}$ which is homogeneous in each variable. Finally, because π is also invariant by rotation, it follows that x and y have the same homogeneous degree in P. $\square$

Example 1.7.8 Let us describe the operator of projection onto the tautological plane, which at (x, y) is the 2-plane spanned by x and y in H^1. We denote the corresponding cohomology classes by $[x]$ and $[y]$.

Explicitly, with notation as in the above theorem, projection onto the tautological plane is:

$$\pi\big(x, y\big)(v) = \frac{A(v, y)}{A(x, y)}[x] + \frac{A(x, v)}{A(x, y)}[y] \tag{1.7.7}$$

Note that $[x]$ and $[y]$ are *non-constant* sections of the cohomology bundle; in fact, they have degree of homogeneity 1 for the scaling action. This makes $\pi(x, y)$ a map from H^1 to H^1 which invariant under scaling of either variable.

To see why Eq. (1.7.7) holds, note that π acts as the identity on $[x]$ and $[y]$, and annihilates anything symplectically orthogonal to them.

1.8 Applications

In this section we collect some applications. First we consider the algebraic hulls of the Kontsevich-Zorich cocycle. Then we prove semisimplicity for flat bundles. Finally, we prove that affine invariant manifolds parametrize Jacobians with non-trivial endomorphisms.

1.8.1 Algebraic Hulls

We show here that the real-analytic and measurable algebraic hulls of the Kontsevich-Zorich cocycle over an affine invariant manifold have to coincide. For the bundle E (which is the

Hodge bundle or a tensor power thereof) we have the associated principal G-bundle P of automorphism of the fibers. In the case of the Hodge bundle, this is a principal Sp_{2g}-bundle.

Given an algebraic subgroup $H \subset G$ to measurably (resp. analytically) reduce the structure group to H is the same as to give an $\mathrm{SL}_2\,\mathbb{R}$-equivariant measurable (resp. real-analytic) section σ of the bundle P/H (whose fiber is G/H).

Theorem 1.8.1 *Given a measurable section $\sigma : \mathcal{M} \to P/H$ as above, in local affine coordinates on $\mathcal{M}$ it must agree a.e. with a real-analytic section.*

In fact, since the fibers of P/H are varieties (see Remark 1.2.6) and it makes sense to speak of polynomials maps, we have that σ is polynomial.

Proof. Suppose we are given an algebraic group G with a faithful linear representation ρ. Then for any algebraic subgroup $H \subset G$ there exists a tensor power T of ρ and a subspace $R \subset T$ (can take it one-dimensional) such that H coincides with the stabilizer of the subspace. This fact is classical and due to Chevalley (see e.g. [Bor91, Thm. 5.1]).

We can apply this to our situation and find corresponding to H an invariant subbundle in some tensor power. From Theorem 1.7.7 we see that the corresponding subbundle has to vary polynomially in affine coordinates. The conclusion about its stabilizer follows. $\square$

1.8.2 Flat bundles

Theorems 1.6.3 and 1.6.5 refer to $\mathrm{SL}_2\,\mathbb{R}$-invariant subbundles. Flat subbundles are $\mathrm{SL}_2\,\mathbb{R}$-invariant, but not necessarily the other way around. Therefore, the assumptions of these theorems are weaker than their classical analogues, but so are the conclusions.

In this section, we note that the theorems extend to the flat situation as well. An object is flat if it is locally constant on the affine manifold. The theorem of the fixed part extends to all tensor power and so does the semisimplicity result.

Theorem 1.8.2 *Suppose $\mathcal{M}$ is an affine invariant manifold and let E denote the Hodge* *bundle (or any tensor power). Denote by C the Weil operator (or any other element of the Deligne torus $\mathbb{S}$).*

If ϕ is a global flat section of E, then so is $C \cdot \phi$.

If $V \subset E$ is a flat subbundle, then so is $C \cdot V$.

Also, the flat analogue of the decomposition provided by Theorem 1.6.8 is valid.

Proof. First we prove the theorem of the fixed part. Suppose given a flat section ϕ over the entire affine manifold $\mathcal{M}$. We apply the same argument as in Theorem 1.6.3.

Since ϕ if flat on all of $\mathcal{M}$, it is in particular flat along $\mathrm{SL}_2 \mathbb{R}$-orbits. We can apply Corollary 1.6.4 to find that the Hodge norm of ϕ is constant.

If we decompose $\phi = \phi^{w,0} + \cdots + \phi^{p,w-p}$ into its Hodge components, the same corollary gives that each component has constant Hodge norm. Because ϕ is flat on $\mathcal{M}$ and ∇^{GM} can be expressed via Eq. (1.4.12), we see (by inspecting the $(p, w-p)$-component of $\nabla^{GM}\phi$) that $\sigma_p \phi^{p,w-p} = 0$. This holds everywhere on $\mathcal{M}$, not just along $\mathrm{SL}_2 \mathbb{R}$-orbits.

By Remark 1.4.3 applied to $\phi^{p,w-p}$ viewed as a holomorphic section of $\mathcal{H}^{p,q}$ over $\mathcal{M}$, we see that $\phi^{p,w-p}$ is flat on $\mathcal{M}$. We can now consider $\phi - \phi^{p,w-p}$ and iterate.

Once the theorem of the fixed part is available, the proof of semisimplicity and invariance of bundles is as before. We only sketch the argument (see [Del87] and [Sch73]).

In the context of $\mathrm{SL}_2 \mathbb{R}$-invariant bundles, we did not have the monodromy available. But for flat bundles we do, and we consider some rational invariant subspace $V \subset E$. The monodromy preserves a lattice inside V, namely $V \cap E_{\mathbb{Z}}$. Now take a wedge power such that V becomes one-dimensional, so the monodromy acts by ± 1. On a double cover the bundle can now be flatly trivialized by a section ϕ. The theorem of the fixed part applies to ϕ and we conclude that $C \cdot V$ must be flat.

Once we have semisimplicity of the monodromy representation over $\mathbb{Q}$, it follows by standard arguments over the field extensions. This gives the claimed results. $\square$

1.8.3 Real Multiplication

The applications in this section have been suggested by Alex Wright.

Recall that according to Theorem 1.5 in [Wri14], over an affine invariant manifold $\mathcal{M}$ we have a decomposition

$$H^1_{\mathbb{C}} = \left(\bigoplus_{\iota \in I} \mathbb{V}_\iota \right) \bigoplus \mathbb{W} \tag{1.8.1}$$

This is a decomposition into pairwise non-isomorphic local systems on $\mathcal{M}$. Moreover the $\mathbb{V}_\iota$ have no local subsystems.

The results of the same paper associate to the affine manifold $\mathcal{M}$ its field of (affine) definition $k(\mathcal{M})$. This is the minimal field such that in affine coordinates, the affine manifold is defined by linear equations with coefficients in that field.

The summation in Eq. (1.8.1) is over the set I of all complex embeddings of the field $k(\mathcal{M})$. We also have a distinguished real embedding ι_0 because $k(\mathcal{M})$ can be viewed as the trace field of a representation (see [Wri14, Theorem 1.5]). With these preliminaries, we can now state the main result of this section.

Theorem 1.8.3 *An affine invariant manifold $\mathcal{M}$ parametrizes Riemann surfaces whose Jacobians have real multiplication by its field of (affine) definition $k(\mathcal{M})$. In particular, this field is totally real.*

Moreover, the 1*-forms giving the flat structure are eigenforms for the action.*

Proof. We combine the decompositions from Eq. (1.8.1) and Theorem 1.8.2. This implies that each summand $\mathbb{V}_\iota$ underlies a variation of Hodge structure. Let an element $a \in k(\mathcal{M})$

act on the Hodge bundle according to the decomposition from Eq. (1.8.1)

$$\rho(a) := \left(\bigoplus_{\iota \in I} \iota(a)\right) \oplus 0 \tag{1.8.2}$$

So a acts by the scalar $\iota(a)$ on the summand corresponding to the embedding ι, and by zero on the remaining part.

Let us make explicit the Galois action on the decomposition (1.8.1). Note that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set I of embeddings of k into $\overline{\mathbb{Q}}$, and it also gives maps $\sigma : \mathbb{V}_\iota \to \mathbb{V}_{\sigma\iota}$. The action on $\oplus_\iota \mathbb{V}_\iota$ is then

$$\sigma(v_{\iota_0} \oplus \cdots \oplus v_{\iota_r}) = \sigma(v_{\sigma^{-1}\iota_0}) \oplus \cdots \oplus \sigma(v_{\sigma^{-1}\iota_r}) \tag{1.8.3}$$

In other words, σ moves both the embeddings and the vectors accordingly. It then follows that $\rho(a)$ as defined in Eq. (1.8.2) is a map defined over $\mathbb{Q}$, i.e. it is invariant under Galois conjugation: $\sigma(\rho(a)) = \rho(a)$. Indeed, by the definition of the Galois action in (1.8.3)

$$\begin{aligned}\sigma\Big(\iota_1(a) \oplus \cdots \oplus \iota_r(a)\Big) &= \sigma\big(\sigma^{-1}\iota_1(a)\big) \oplus \cdots \oplus \sigma\big(\sigma^{-1}\iota_r(a)\big) \\ &= \iota_1(a) \oplus \cdots \oplus \iota_r(a)\end{aligned}$$

Because the individual local systems underlie Hodge structures the operator $\rho(a)$ is a rational endomorphism of type $(0,0)$. This gives the desired action of the field $k(\mathcal{M})$.

It is known that a field in the endomorphism ring of an abelian variety must be CM or totally real (see [BL04, Theorem 5.5.3]). Because $k(\mathcal{M})$ has one real embedding, the latter possibility must occur.

Finally, the 1-forms giving the flat structure belong to the space $H^1_{\iota_0}$ and are thus eigenforms for real multiplication. $\square$

1.9 Connection to Schmid's work

In this appendix, we explain the connection of our methods to Schmid's work [Sch73]. Namely, we show that methods from ergodic theory yield some of the global consequences of his results.

Setup. Consider a variation of Hodge structures E over a smooth quasi-projective base B. We do not assume B is compact. Since it is quasi-projective, through every point $b \in B$ there is at least one Riemann surface of finite type contained in B. In this setup, Theorems 7.22 through 7.25 from [Sch73] hold. The main one, which implies the rest, is the Theorem of the Fixed Part - Theorem 7.22 in loc. cit. We explain how to prove it using ergodic theory.

Theorem 1.9.1 (Theorem of the Fixed Part) *With notation as above, if ϕ is a flat global section over B of $E_{\mathbb{C}}$, then each (p,q)-component of ϕ is also flat.*

The case of curves. We first assume that B is one-dimensional, i.e. a compact Riemann surface with finitely many punctures. Recall that the universal cover of B maps to the classifying space of the variation. The appropriate version of the Schwartz lemma implies that if E is non-trivial, then B is necessarily a hyperbolic Riemann surface, of finite area by assumption. Moreover, the classifying map is a contraction for the appropriate metrics.

We can now consider the geodesic flow on B for the hyperbolic metric and the induced cocycle from the local system of E. Because the classifying map is a contraction, the boundedness assumption in the Oseledets theorem is satisfied. Moreover, we have not only the geodesic flow but also the full $\mathrm{SL}_2\,\mathbb{R}$-action. The claims from Section 1.5 and Section 1.6 therefore apply verbatim.

This proves the claim in the case when the base is a finite-type Riemann surface.

The general case To prove the general case, note we assumed the base is quasi-projective. In particular, it has lots of 1-dimensional subvarieties, to which the previous step applies. The main step in proving the theorem of the fixed part is showing that the subharmonic function coming from the norm of the section must be constant.

In the previous step, we established this when the base is 1-dimensional. Since B has such 1-dimensional subvarieties through every point, the subharmonic function is constant on all of them, so on all of B. The proof then proceeds as in Section 1.6, or [Sch73, Section 7].

1.10 The Kontsevich-Forni formula

In this appendix, we provide a derivation of the Kontsevich-Forni formula. This was first stated by Kontsevich in [Kon97], then proved by Forni in [For02] (see also [FMZ14a]). This appendix contains a proof in the formalism used in this chapter.

Setup. Consider some complex manifold B of unspecified dimension, and consider over it a variation of weight-1 Hodge structure H^1. We have the decomposition $H^1_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$. Inside we have the real (flat) subbundle $H^1_{\mathbb{R}}$ with elements of the form $\overline{\alpha} \oplus \alpha$. We have the positive-definite Hodge norm, and all statements below are with respect to it. In a change of convention from Section 1.4, we take adjoints for the positive-definite metric now.

Proposition 1.10.1 *Suppose $c_1, \ldots, c_k \in H^1_{\mathbb{R}}$ is a basis, at some point of B, of an isotropic subspace of $H^1_{\mathbb{R}}$. Extend c_i using the Gauss–Manin connection to flat sections in a neighborhood. Denoting the second fundamental form of the Hodge bundle*

$$\sigma : H^{1,0} \to \Omega^1 \otimes H^{0,1}$$

we have the formula (notation explained below)

$$\partial\overline{\partial}\log\left\|\bigwedge_{i=1}^{k} c_i\right\|^2 = \operatorname{tr}(\sigma\wedge\sigma^\dagger) - \operatorname{tr}\left(\sigma\wedge\pi_{\overline{\mathcal{C}}^\perp}\sigma^\dagger\pi_{\mathcal{C}^\perp}\right) \tag{1.10.1}$$

We view c_i as flat sections of $H^1_{\mathbb{C}}$ and project them to $H^{0,1}$ to get holomorphic sections ϕ_i. Then the ϕ_i span a k-dimensional subbundle which we denote $\mathcal{C}$, and $\mathcal{C}^\perp$ is its Hodge-orthogonal inside $H^{0,1}$. The operator $\pi_{\mathcal{C}^\perp}$ is orthogonal projection to the space $\mathcal{C}^\perp$ and $\pi_{\overline{\mathcal{C}}^\perp}$ is orthogonal projection to its complex-conjugate.

Remark 1.10.2 (i) The equation 1.10.1 is an equality of $(1,1)$-forms. The left-hand side can be interpreted as a Laplacian once a metric is introduced on B. For example, the hyperbolic metric on Teichmüller disks recovers the usual Kontsevich-Forni formula.

(ii) The right-hand side of the formula is always a non-negative $(1,1)$-form. This is because adjoints are for a *positive-definite* hermitian inner-product.

Notation. Write the c_i in their Hodge decomposition

$$c_i = \overline{\phi_i}\oplus\phi_i \quad \text{where } \phi_i \text{ holomorphic section of } H^{0,1} = H^1_{\mathbb{C}}/H^{1,0}$$

Proposition 1.10.3 *The isotropy condition on c_i gives the pointwise on B equality of Hodge norms*

$$\left\|\bigwedge_1^k c_i\right\| = 2^k\left\|\bigwedge_1^k \phi_i\right\|$$

Proof. If we apply a fixed real $k\times k$ matrix to the c_i everywhere on B, then the claimed equality is not affected - both sides are rescaled by the determinant of the matrix. To check the equality at some given point of B, we can choose a real linear change of variables for the c_i such that at the considered point, the c_i are also Hodge-orthogonal.

Combined with the isotropy condition on c_i we find that the ϕ_i must also be Hodge-orthogonal. Indeed, the c_i being Hodge-orthogonal implies the real part of $\langle \phi_i \rangle \phi_j$ has to vanish. The isotropy condition implies vanishing of the imaginary part.

But in this situation, the formula can be checked directly. Therefore, the asserted equality holds everywhere. □

Proof of Proposition 1.10.1. By the previous result, we need to compute $\overline{\partial}\partial \log \|\wedge\phi_i\|^2$. Recall $\bigwedge_{i=1}^k \phi_i$ is a holomorhpic section of $\bigwedge^k H^{0,1}$ and so we shall use Lemma 1.4.1 to compute the desired expression.

Recall from Section 1.4 the relation between the Gauss–Manin and Hodge connections on H^1

$$\nabla^{GM} = \nabla^{Hg} + \sigma - \sigma^\dagger$$

Because the c_i are flat for ∇^{GM}, looking at the component in $H^{0,1}$ we find

$$\nabla^{Hg}\phi_i = -\sigma\overline{\phi_i}$$

From now on, ∇ denotes ∇^{Hg} and we focus on the bundle $H^{0,1}$. We shall use the Leibniz rule for the connection and curvature

$$\nabla \bigwedge_{i=1}^{k} \phi_i = \sum_{i=1}^{k} \phi_1 \wedge \cdots \wedge \nabla\phi_i \wedge \cdots \wedge \phi_k$$
$$\Omega_{\wedge^k H^{0,1}} \bigwedge_{i=1}^{k} \phi_i = \sum_{i=1}^{k} \phi_1 \wedge \cdots \wedge \Omega\phi_i \wedge \cdots \wedge \phi_k$$

We abuse notation and denote by ∇ the connection on both $H^{0,1}$ and its wedge powers, but we distinguish the curvatures. Denoting by $\phi := \phi_1 \wedge \cdots \wedge \phi_k$ Lemma 1.4.1 reduces the proof to evaluating

$$-\frac{\langle \Omega_{\wedge^k}\phi \rangle \phi}{\|\phi\|^2} - \frac{\langle \nabla\phi \rangle \phi \cdot \langle \phi \rangle \nabla\phi - \|\phi\|^2 \cdot \langle \nabla\phi \rangle \nabla\phi}{\|\phi\|^4} \tag{1.10.2}$$

We need to check the pointwise equality of the above $(1,1)$-form and the right-hand side of Proposition 1.10.1. The minus sign comes from the switched order of ∂ and $\overline{\partial}$ in Lemma 1.4.1 and the proposition we are proving.

Remark that we are proving a pointwise equality. In particular, if we apply any fixed $k \times k$ complex matrix to the sections ϕ_i the value given by 1.10.2 does not change. Thus, to prove the claimed equality at a point of B we can apply a matrix to assume that the ϕ_i are mutually orthogonal and of unit norm at the considered point. For the calculation, we also complete them to an orthonormal basis $\{\phi_i\}_{i=1}^g$ of the fiber considered.

Finally, Eq. (1.4.13) gives $\Omega_{H^{0,1}} = -\sigma \wedge \sigma^\dagger$ (recall we are taking adjoints for the positive-definite metric now, hence the minus sign). Denote the entries of 1-form valued maps σ and $\sigma^\dagger$ by

$$\sigma\overline{\phi_i} = \sum_{j=1}^{g} \sigma_i^j \phi_j$$

$$\sigma^\dagger \phi_k = \sum_{l=1}^{g} (\sigma^\dagger)_k^l \overline{\phi}_l$$

We then have, using orthonormality of $\{\phi_i\}_{i=1}^g$

$$\begin{aligned} -\left\langle \Omega_{\wedge^k} \phi \right\rangle \phi &= \left\langle \sum_{i=1}^{k} \phi_1 \wedge \cdots \wedge (\sigma\sigma^\dagger \phi_i) \wedge \cdots \wedge \phi_k \right\rangle \phi_1 \wedge \cdots \wedge \phi_k = \\ &= \sum_{i=1}^{k} \sum_{j=1}^{g} \sigma_j^i \wedge \left(\sigma^\dagger\right)_i^j \end{aligned}$$

We next have

$$\begin{aligned} \left\langle \nabla \phi \right\rangle \phi &= \left\langle \sum_{i=1}^{k} \phi_1 \wedge \cdots \wedge (-\sigma\overline{\phi}_i) \wedge \cdots \wedge \phi_k \right\rangle \phi_1 \wedge \cdots \wedge \phi_k = \\ &= -\sum_{i=1}^{k} \sigma_i^i \end{aligned}$$

We then find

$$\langle \nabla\phi \rangle \phi \cdot \langle \phi \rangle \nabla\phi = \left(\sum_{i=1}^{k} \sigma_i^i \right) \cdot \left(\sum_{i=1}^{k} \overline{\sigma}_i^i \right)$$

We also have

$$\begin{aligned} \langle \nabla\phi \rangle \nabla\phi = & \left\langle \sum_{i=1}^{k} \phi_1 \wedge \cdots \sigma_i^i \phi_i \cdots \wedge \phi_k + \sum_{k<l} \phi_1 \wedge \cdots \sigma_i^l \phi_l \cdots \wedge \phi_k, \right. \\ & \left. \sum_{i=1}^{k} \phi_1 \wedge \cdots \sigma_i^i \phi_i \cdots \wedge \phi_k + \sum_{k<l} \phi_1 \wedge \cdots \sigma_i^l \phi_l \cdots \wedge \phi_k \right\rangle = \\ & = \left(\sum_{i=1}^{k} \sigma_i^i \right) \left(\sum_{i=1}^{k} \overline{\sigma}_i^i \right) + \sum_{i=1}^{k} \sum_{l=k+1}^{g} \sigma_i^l \overline{\sigma}_i^l \end{aligned} \tag{1.10.3}$$

Note that $\|\phi\| = 1$ by our normalization and we also have that $\left(\sigma^\dagger\right)_i^j = \overline{\sigma}_j^i$. We can now combine all three terms to get the claimed formula. The first term in Eq. (1.10.3) cancels the $\langle \nabla\phi \rangle \phi$ term. The second term of the same equation provides the needed contribution to the desired formula. Indeed, summing up all the terms we obtain

$$\sum_{i=1}^{g} \sum_{l=1}^{g} \sigma_i^l \overline{\sigma}_i^l - \sum_{i=k+1}^{g} \sum_{l=k+1}^{g} \sigma_i^l \overline{\sigma}_i^l$$

and this corresponds to the right-hand side of (1.10.1) written out explicitly. □

CHAPTER 2

MIXED HODGE STRUCTURES AND ALGEBRAICITY

This chapter contains the results obtained in [Fil16b].

2.1 Introduction

Let (X, ω) be a Riemann surface with a holomorphic 1-form on it. The set of all such pairs forms an algebraic variety $\mathcal{H}(\kappa)$ called a stratum, where κ encodes the multiplicities of the zeros of ω. The stratum carries a natural action of the group $\mathrm{SL}_2 \mathbb{R}$ which is of transcendental nature.

A stratum $\mathcal{H}(\kappa)$ has natural charts to complex affine spaces. The coordinates are the periods of ω on X, thus in $\mathbb{C}$. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the action of $\mathrm{SL}_2 \mathbb{R}$ is the standard one on each coordinate individually. In particular, the stratum carries a natural Lebesgue-class measure which is invariant under $\mathrm{SL}_2 \mathbb{R}$. The finiteness of the measure was proved by Masur [Mas82] and Veech [Vee82].

The action of $\mathrm{SL}_2 \mathbb{R}$ and knowledge of invariant measures can be applied to study other dynamical systems. For interval exchange transformations this started in the work of Masur and Veech [Mas82, Vee82]. A starting point for applications to polygonal billiards was by Kerckhoff-Masur-Smillie in [KMS86]. Some recent applications involve a detailed analysis of the wind-tree model by Hubert-Lelièvre-Troubetzkoy [HLT11]. For a comprehensive introduction to the subject, see the survey of Zorich [Zor06].

For more precise applications, especially to concrete examples, one needs to understand all possible invariant measures. For instance, polygonal billiards correspond to a set of Masur-Veech measure zero.

Recent results of Eskin and Mirzakhani [EM13] show that finite ergodic invariant measures are rigid and in particular are of Lebesgue class and supported on smooth manifolds.

In further work with Mohammadi [EMM15] they show that many other analogies with the homogeneous setting and Ratner's theorems hold.

The $\mathrm{SL}_2\,\mathbb{R}$-invariant measures give rise to *affine invariant manifolds*. These are complex manifolds which are given in local period coordinates by linear equations. It was shown by Wright [Wri14] that the linear equations can be taken with coefficients in a number field.

Note that *finite* $\mathrm{SL}_2\,\mathbb{R}$-invariant measures are supported on real codimension 1 hypersurfaces inside affine manifolds, the issue arising from the action of scaling by $\mathbb{R}^\times$. The affine invariant manifolds are then closed $\mathrm{GL}_2\,\mathbb{R}$-invariant sets. In fact, by [EMM15] the closure of any $\mathrm{GL}_2\,\mathbb{R}$-orbit is an affine manifold.

Period coordinates are transcendental and so apriori affine manifolds, which are given by linear equations, are only complex-analytic submanifolds. In this chapter, we prove the following result (see Theorem 2.5.4).

Theorem 2.1.1 *Affine invariant manifolds are algebraic subvarieties of the stratum $\mathcal{H}(\kappa)$, defined over $\overline{\mathbb{Q}}$.*

I am grateful to Curtis McMullen for suggesting the next result (see also [McM07, Theorem 1.1] for a much more precise result in genus 2).

Corollary 2.1.2 (see Remark 2.1.6(ii)) *Let $\mathbb{H} \to \mathcal{M}_g$ be a Teichmüller disk. Then its closure (in the standard topology) inside $\mathcal{M}_g$ is an algebraic subvariety of $\mathcal{M}_g$.*

The lowest-dimensional affine manifolds are Teichmüller curves. It was proved they are algebraic by Smillie and Weiss [SW04, Proposition 8]; a different sketch of proof (attributed to Smillie) is in [Vee95]. That they are defined over $\overline{\mathbb{Q}}$ was proved by McMullen [McM09]. It was proved by Möller in [Möl06b] that they are defined over $\overline{\mathbb{Q}}$ after embedding into a moduli space of abelian varieties.

Teichmüller curves and higher-dimensional $\mathrm{SL}_2\,\mathbb{R}$-invariant loci also have interesting arithmetic properties. McMullen has related in [McM03] Teichmüller curves in genus 2 with real multiplication. He also gave further constructions using Prym loci [McM06]. In genus 2 algebraicity follows from a complete classification of invariant loci by McMullen [McM07]. In the stratum $\mathcal{H}(4)$ algebraicity is known by results of Aulicino, Nguyen and Wright [ANW13, NW14]. Lanneau and Nguyen have also done extensive work on Prym loci in genus 3 and 4 [LN, LN14].

Techniques from variations of Hodge structures were introduced by Möller, starting in [Möl06b]. In particular, he showed that Teichmüller curves always parametrize surfaces with Jacobians admitting real multiplication on a factor. He also showed that over a Teichmüller curve, the Mordell-Weil group of the corresponding factor is finite [Möl06a]. In particular, zeros of the 1-form are torsion under the Abel-Jacobi map. See also [Möl08] for further results.

The results in [Fil16a] show that on affine manifolds, the topological decomposition of cohomology (e.g. the local systems from [Wri14]) are compatible with the Hodge structures. As a consequence, affine manifolds also parametrize Riemann surfaces with non-trivial endormophisms, typically real multiplication on a factor (see [Fil16a, Thm. 1.6]).

This chapter extends Möller's torsion result to affine manifolds. The precise statement and definitions are in §2.5.1 and Theorem 2.5.2.

Theorem 2.1.3 *Let $\mathcal{M}$ be an affine invariant manifold, parametrizing Riemann surfaces with real multiplication by the order $\mathcal{O}$ on a factor of the Jacobians. Then $\mathcal{M}$ carries a natural local system Λ of $\mathcal{O}$-linear combinations of the zeros of the* 1*-form (see Eq. (2.5.2)) and a twisted Abel-Jacobi map (see Definition 2.5.1)*

$$\nu : \Lambda \to \operatorname{Jac}_{\mathbb{Z}} \left(\oplus_\iota H^1_\iota \right)$$

The range is the factor of Jacobians admitting real multiplication (see Eq. (2.2.2)).

Then the image of ν always lies in the torsion of the abelian varieties.

Remark 2.1.4

(i) The expression "real multiplication by $\mathcal{O}$" is used in a rather loose sense. It means that the ring $\mathcal{O}$ maps to the endomorphisms of a factor of the Jacobian. The factor of the Jacobian is always nontrivial, as it contains at least the part coming from the 1-form ω. The ring $\mathcal{O}$ could be $\mathbb{Z}$, and the factor could be the entire Jacobian.

(ii) The local system Λ can be trivialized on a finite cover of the stratum, and is defined as follows. The tangent space to the stratum contains the relative cohomology classes that vanish on absolute homology, denoted W_0. The tangent space $T\mathcal{M}$ of the affine manifold intersects it in a sublocal system (over $\mathcal{M}$), denoted $W_0\mathcal{M}$. The dual of W_0, denoted $\check{W}_0$, is canonically identified with linear combinations of the zeros of the 1-form with zero total weight. Then Λ is an $\mathcal{O}$-submodule of $(W_0\mathcal{M})^{\perp} \subset \check{W}_0$, i.e. of the annihilator of $W_0\mathcal{M}$.

Indeed, by results of Wright [Wri14], $W_0\mathcal{M}$ and thus $(W_0\mathcal{M})^{\perp}$ are defined over k - the field giving real multiplication. Since $\check{W}_0$ carries a $\mathbb{Z}$-structure, extending scalars to $\mathcal{O}$, define $\Lambda := \check{W}_0(\mathcal{O}) \cap (W_0\mathcal{M})^{\perp}(k)$, where $A(R)$ denotes the R-points of A.

(iii) When $W_0\mathcal{M}$ is empty, e.g. for Teichmüller curves, Λ coincides with $\check{W}_0(\mathcal{O})$. In particular, it contains (up to finite index) all the $\mathbb{Z}$-linear combinations of zeroes of the 1-form, with total weight zero; on them ν is the usual Abel-Jacobi map. The extension of ν to $\mathcal{O}$-linear combinations of zeros uses the $\mathcal{O}$-action on the Jacobian factor.

Remark 2.1.5 The result on torsion can be described concretely using periods of 1-forms. Note that it refers, in particular, to 1-forms other than ω; describing them using the flat structure does not seem immediate.

Let $\{r_j\}_j$ be formal integral combinations of the zeroes of ω, such that the coefficients of each r_j add up to 0. They can be lifted to actual relative cycles on the surface, denoted r_j' (these are now actual curves that connect zeroes of ω). Let also $\{a_i, b_i\}_{i=1}^{g}$ denote an integral basis of the first homology of the surface.

Suppose $\sum_j c^j r_j$ is an element of Λ, where c_j are elements of $\mathcal{O}$. The condition that its image is torsion under the twisted cycle map ν is equivalent to the following: There exist $\alpha_i, \beta_i \in \mathbb{Q}$ such that whenever $\omega_l \in H^{1,0}_{\iota_l}$ is a holomorphic 1-form, we have

$$\sum_j \iota_l(c^j) \int_{r'_j} \omega_l = \sum_i \left(\alpha_i \int_{a_i} \omega_l + \beta_i \int_{b_i} \omega_l \right) \tag{2.1.1}$$

In other words, the absolute and relative periods of ω_l satisfy some linear relations. The coefficients $\iota_l(c^j)$ vary with the embedding ι_l corresponding to the subspace in which ω_l lives.

Remark 2.1.6

(i) The algebraicity result also applies to strata of quadratic differentials. Indeed, these embed via the double-covering construction to strata of holomorphic 1-forms. An affine invariant submanifold of a stratum of quadratic differentials thus gives one in a stratum of 1-forms. This construction is described analytically e.g. in [Lan04, Construction 1] and algebraically as follows.

If (X, q) is an algebraic curve with a quadratic differential (over a base field k), let TX denote the tangent bundle over X and view $q : TX \to k$ as a quadratic function. In the cotangent bundle $T^\vee X$ consider the locus $\tilde{X} := \{\lambda \in T^\vee X | \lambda^2 = q\}$. Away from the zeros of q, the projection $\tilde{X} \xrightarrow{\pi} X$ is $2:1$ and $\tilde{X}$ naturally carries a 1-form ω such that $\omega^{\otimes 2} = \pi^* q$.

Note that if q has a zero of order more than 1, then $\tilde{X}$ is not smooth and one has to argue on the normalization of $\tilde{X}$. On it the 1-form ω has no poles. The construction works in families, and so applies to strata, embedding algebraically (over $\mathbb{Q}$) a stratum of quadratic differentials into a stratum of 1-forms.

(i) To establish Corollary 2.1.2, consider $\mathcal{Q}_g \to \mathcal{M}_g$ the cotangent bundle of the moduli space of curves, with fiber over X the space of quadratic differentials on X. Taking a quotient by the scaling action on $\mathcal{Q}_g$ gives the proper projection $\mathbb{P}\mathcal{Q}_g \to \mathcal{M}_g$.

A Teichmüller disk $f : \mathbb{H} \to \mathcal{M}_g$ as in Corollary 2.1.2 lifts, essentially by definition, to $\tilde{f} : \mathbb{H} \to \mathbb{P}\mathcal{Q}_g$. Note that $\tilde{f}(\mathbb{H})$ is contained in some stratum $\mathbb{P}\mathcal{Q}_g(\kappa) \subset \mathbb{P}\mathcal{Q}_g$. Combining Theorem 2.1 from [EMM15] and Theorem 2.5.4 we find that the (Zariski and usual) closure of the Teichmüller disk in $\mathbb{P}\mathcal{Q}_g(\kappa)$ is an algebraic variety $\mathbb{P}\mathcal{M}$. We can further take its (Zariski and usual) closure inside $\mathbb{P}\mathcal{Q}_g$ to find that it is also an algebraic variety $\mathbb{P}\overline{\mathcal{M}} \subseteq \mathbb{P}\mathcal{Q}_g$. Note that the Zariski and usual closure of a quasi-projective set coincide.

Finally, recall that the map $\mathbb{P}\mathcal{Q}_g \to \mathcal{M}_g$ is proper. Therefore, the topological closure of $f(\mathbb{H})$ will agree with the projection of the topological closure of $\tilde{f}(\mathbb{H})$ inside $\mathbb{P}\mathcal{Q}_g$, which is $\mathbb{P}\overline{\mathcal{M}}$. Properness also ensures the projection of an algebraic variety is still a variety, so Corollary 2.1.2 follows.

Outline of the chapter Section 2.2 proves in a special case that affine invariant manifolds are algebraic. This special case occurs when the tangent space of the affine manifold contains all relative cohomology classes. The proof only uses results from [Fil16a].

Section 2.3 contains basic definitions and constructions about mixed Hodge structures. We only describe the small part of the theory that is necessary for our arguments. The proofs in later sections use this formalism, and we also include some concrete examples throughout. One does not need to be acquainted with the full theory to follow the arguments.

Section 2.4 contains the main technical part. It proves that certain sequences of mixed Hodge structures are *split*, i.e. as simple as possible. This uses the negative curvature properties of Hodge bundles.

Section 2.5 combines the previous results to deduce the Torsion Theorem 2.5.2. This is then used to prove the Algebraicity Theorem 2.5.4.

Remark 2.1.7 (**Self-intersections**) Affine manifolds are only immersed in a stratum (see [EM13, Def. 1.1]), and could have locally finitely many self-intersecting sheets. Thus, any affine manifold $\mathcal{M}$ can be written as the union $\mathcal{M} = \mathcal{M}_0 \coprod \mathcal{M}'$ where $\mathcal{M}_0$ is a smooth open subset of $\mathcal{M}$ and $\mathcal{M}'$ is a lower-dimensional proper closed $\mathrm{GL}_2^+(\mathbb{R})$-invariant affine manifold (possibly disconnected or with self-intersections). Moreover the topological closure of $\mathcal{M}_0$ contains $\mathcal{M}'$. Since $\dim_{\mathbb{C}} \mathcal{M}' < \dim_{\mathbb{C}} \mathcal{M}$, it follows that $\mathcal{M}_0$ is connected if $\mathcal{M}$ is.

The arguments in Section 2.2 and Section 2.5 about algebraicity apply locally on $\mathcal{M}_0$ and identify it (locally) with a quasi-projective variety. Thus, assuming by induction that $\mathcal{M}'$ is quasi-projective, they show that $\mathcal{M}_0$ is quasi-projective inside $\mathcal{H} \setminus \mathcal{M}'$. Again, since $\mathcal{M}'$ is quasi-projective and contained in the (topological) closure of $\mathcal{M}_0$, it follows that $\mathcal{M}$ is quasi-projective inside $\mathcal{H}$.

Orbifolds. All the arguments are made in some finite cover of a stratum, to avoid orbifold issues. In particular, period coordinates are well-defined and the zeros of the 1-form are labeled. The results are invariant under passing to finite covers.

Acknowledgments. I would like to thank my advisor Alex Eskin, who was very helpful and supportive at various stages of this work, and in particular about the paper [Fil16a] whose methods are used here. I have also benefited a lot from conversations with Madhav Nori, especially on the topic of mixed Hodge structures.

I also had several conversations on the topic of algebraicity with Alex Eskin and Alex Wright. In particular, Alex Wright explained his result ([Wri15, Prop. 2.18]) that the torsion and real multiplication theorems of Möller characterize Teichmüller curves and suggested that finding and proving some generalization of the torsion theorem could imply algebraicity. I have also discussed and received very useful feedback on an earlier draft of the paper [Fil16b] from both Alex Eskin and Alex Wright. I am very grateful for their feedback and the numerous insights they shared with me.

I am also grateful to Curtis McMullen for suggesting Corollary 2.1.2.

2.2 Algebraicity in a particular situation

In this section, we prove a special case of algebraicity. It only requires results of [Fil16a]. In this special case the location of the tangent space to an affine manifold can be described precisely.

Setup. Consider an affine invariant manifold $\mathcal{M}$ in some stratum $\mathcal{H}(\kappa)$. We omit κ from the notation, and refer to the stratum as $\mathcal{H}$. Let $T\mathcal{H}$ be the tangent bundle of the stratum, and let $W_0 \subset T\mathcal{H}$ be the subbundle corresponding to the purely relative cohomology classes. The survey of Zorich [Zor06, Section 3] provides a clear and detailed exposition of these objects.

The purpose of this section is to prove the following result.

Proposition 2.2.1 *Suppose that everywhere on $\mathcal{M}$ we have $W_0 \subset T\mathcal{M}$, where $T\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$.*

Then $\mathcal{M}$ is a quasi-projective algebraic subvariety of $\mathcal{H}$.

Before proceeding to the proof, we recall some facts about the local structure on a stratum $\mathcal{H}$. In particular, we discuss the way in which the relative cohomology groups $H^1(X, (\omega)_{red}; \mathbb{C})$ provide local coordinates. These arise from the Gauss-Manin connection and the tautological section, which assigns to (X, ω) the cohomology class of ω.

Some preliminaries. As explained in Remark 2.1.7, it suffices to argue locally in the open part of $\mathcal{M}$ where there are no self-intersections.

Focus on a small neighborhood in $\mathcal{H}$ of some $(X_0, \omega_0) \in \mathcal{M}$, denoted $N_\epsilon(X_0, \omega_0)$. Introduce the identification coming from parallel transport (i.e. the Gauss-Manin connection) on relative cohomology

$$GM_{(X,\omega)} : H^1(X, (\omega)_{red}; \mathbb{C}) \xrightarrow{\sim} H^1(X_0, (\omega_0)_{red}; \mathbb{C})$$

This is defined for all $(X, \omega) \in N_\epsilon(X_0, \omega_0)$ and $(\omega)_{red}$ denotes the zeroes of ω, forgetting the multiplicities (i.e. the reduced divisor).

Recall that for (X, ω) we have the natural element $\omega \in H^1(X, (\omega)_{red}; \mathbb{C})$ viewing ω as a relative cohomology class. Call this the *tautological section* $\omega : \mathcal{H} \to T\mathcal{H}$.

Period coordinates are then described by the tautological section:

$$\begin{aligned} \Pi : N_\epsilon(X_0, \omega_0) &\to H^1(X_0, (\omega_0)_{red}; \mathbb{C}) \\ (X, \omega) &\mapsto GM_{(X,\omega)}\omega \end{aligned} \tag{2.2.1}$$

Recall that we have a short exact sequence of vector bundles over $\mathcal{H}$

$$0 \to W_0 \to T\mathcal{H} \xrightarrow{p} H^1 \to 0$$

Here H^1 is the bundle of absolute first cohomology of the underlying family of Riemann surfaces, and W_0 is the purely relative part as before.

It is proved by Wright in [Wri14, Theorem 1.5] that over $\mathcal{M}$ the local system of cohomology decomposes as

$$H^1 = \left(\bigoplus_\iota H^1_\iota \right) \oplus V \tag{2.2.2}$$

The summation is over embeddings ι of a number field k in $\mathbb{C}$, and there is a distinguished real embedding ι_0. We then have $p(T\mathcal{M}) = H^1_{\iota_0}$.

Proof of Proposition 2.2.1. In period coordinates in the neighborhood $N_\epsilon(X_0, \omega_0)$ the property $p(T\mathcal{M}) = H^1_{\iota_0}$ translates to the statement

$$\Pi(\mathcal{M} \cap N_\epsilon) \subseteq p^{-1}(H^1_{\iota_0})_{(X_0,\omega_0)} \subseteq T\mathcal{H}_{(X_0,\omega_0)} \tag{2.2.3}$$

Because $T\mathcal{M}$ contains all the purely relative cohomology classes, locally $\mathcal{M}$ coincides with an open subset of the middle space above.

An equivalent way to state the above local description of $\mathcal{M}$ is to say

$$\mathcal{M} \cap N_\epsilon = \{(X,\omega) \mid p(GM_{(X,\omega)}\omega) \in H^1_{\iota_0}(X_0,\omega_0)\} \tag{2.2.4}$$

So over $\mathcal{M}$ the section $p(\omega)$ must belong to the local system $H^1_{\iota_0}$.

Note the local system $H^1_{\iota_0}$ cannot be globally defined on $\mathcal{H}$. However, in the neighborhood N_ϵ one can still define it using the Gauss-Manin connection.

Checking algebraicity. We also know by [Fil16a, Thm. 1.6] that each H^1_ι carries a Hodge structure. Moreover, for each $a \in k$ we have the operator

$$\rho(a) = \left(\bigoplus_\iota \iota(a)\right) \oplus 0 \tag{2.2.5}$$

which acts by the corresponding scalar on each factor of the decomposition (2.2.2). These operators give real multiplication on the Jacobians, with ω as an eigenform.

As a consequence, there is an order $\mathcal{O} \subset k$ which acts by genuine endomorphisms of the Jacobian factor obtained via (2.2.2). Moreover, the isomorphism class of the corresponding $\mathbb{Z}$-lattice, viewed as an $\mathcal{O}$-module, is constant on $\mathcal{M}$. Recall that we are working on the open subset of $\mathcal{M}$ where there are no self-intersections, and this is still connected if $\mathcal{M}$ is (see Remark 2.1.7).

Define $\mathcal{N}'$ to be the locus in $\mathcal{H}$ of (X,ω) which admit real multiplication by $\mathcal{O}$ on a factor of the Jacobian, with the same $\mathcal{O}$-module structure on the integral lattice of the factor, and with ω as an eigenform. This is a countable union of algebraic subvarieties of $\mathcal{H}$, since it is the preimage of such a collection under the period map to $\mathbb{P}(H^{1,0})$, which is the projectivization of the Hodge bundle over $\mathcal{A}_g$. See Remark 2.2.3 for an explanation why, in fact, it is a finite union.

The discussion above, in particular Eq. (2.2.4), gives that $\mathcal{M} \subseteq \mathcal{N}'$. In the neighborhood N_ϵ of (X_0, ω_0) let $\mathcal{N}$ be one of the irreducible components of $\mathcal{N}'$ which contains $\mathcal{M}$. We will check this component is unique, and coincides with $\mathcal{M}$.

Recall the defining conditions of $\mathcal{N}'$ in N_ϵ:

$$N_\epsilon \cap \mathcal{N}' = \{(X,\omega) | \forall a \in k \text{ we have } \rho(a) \text{ is of Hodge type } (0,0)$$
$$\text{and } \rho(a)\omega = \iota_0(a)\omega\}$$

The local systems H^1_ι are defined on $\mathcal{N}$, but are extended to N_ϵ using the flat connection. They serve as "eigen-systems" for the action of $\rho(a)$, which itself is defined in N_ϵ using the flat connection. We then have the containment

$$N_\epsilon \cap \mathcal{N}' \subseteq \{(X,\omega) \mid \forall a \in k \text{ we have } \rho(a)\omega = \iota_0(a)\omega\}$$

However, we also have the (local) equality which follows from the identifications via parallel transport

$$\{(X,\omega)|\forall a \in k, \rho(a)\omega = \iota_0(a)\omega\} = \{(X,\omega)|p\left(\Pi(X,\omega)\right) \in H^1_{\iota_0,(X_0,\omega_0)}\}$$

Looking back at the local definition of $\mathcal{M}$ given by the inclusions (2.2.3), we see that this locus is exactly $\mathcal{M}$. So we found that locally near (X_0, ω_0) we have

$$\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}$$

This finishes the proof that $\mathcal{M}$ is an algebraic subvariety of $\mathcal{H}$. □

Remark 2.2.2 In local period coordinates, requiring the section $\omega : \mathcal{H} \to T\mathcal{H}$ to be in some local system is the same as restricting to a (local) affine manifold. Algebraicity in the above theorem followed because we could identify the tangent space to $\mathcal{M}$ with $p^{-1}(H^1_{\iota_0})$. In general, we need to know the precise location of the tangent space in relative cohomology. The next few sections deal with this question.

Remark 2.2.3 We now discuss the finiteness of the components of the eigenform locus for real multiplication. Assume the type of real multiplication is fixed, in other words the order $\mathcal{O}$ and the isomorphism of $\mathcal{O}$-lattice with symplectic form corresponding to the factor of the Jacobian.

To prove finiteness of the eigenform locus, it suffices to prove finiteness of the real multiplication locus in $\mathcal{A}_g$. Indeed, the eigenform locus is a projective space bundle over the real multiplication locus.

Finiteness of the real multiplication locus will follow from the following general theorem of Borel and Harish-Chandra [BHC62]. Let Γ be an arithmetic lattice in a $\mathbb{Q}$-algebraic group G. Consider some representation V of G, with a $\mathbb{Z}$-structure compatible with Γ, and an integral vector $v \in V(\mathbb{Z})$ such that $G(\mathbb{R}) \cdot v \subset V(\mathbb{R})$ is closed (equivalently, the stabilizer of v is reductive). Then the set of integral points in the orbit $G(\mathbb{R}) \cdot v$ form *finitely many* classes under the action of Γ.

Consider the decomposition of absolute cohomology given in Eq. (2.2.2), and abbreviate it as $H^1 = M \oplus V$ where M is the factor with real multiplication. Each factor contains a lattice, denoted $M(\mathbb{Z})$ and $V(\mathbb{Z})$ respectively.

Now consider the abstract $\mathcal{O} \oplus \mathbb{Z}$-module $M \oplus V$, equipped with the compatible symplectic form (the $\mathbb{Z}$-factor in $\mathcal{O} \oplus \mathbb{Z}$ acts on V only). Associated to it is the period domain $\mathbb{H}_{M,V}$ parametrizing pairs of abelian varieties (A_M, A_V), with markings $M \xrightarrow{\sim} H^1(A_M)$ and $V \xrightarrow{\sim} H^1(A_V)$. Moreover, the markings should respect the symplectic forms and $\mathcal{O}$ should act by endormorphisms of A_M.

Consider possible embeddings $\phi : M(\mathbb{Z}) \oplus V(\mathbb{Z}) \hookrightarrow H^1_{\mathbb{Z}}$ respecting the symplectic form. Note that $M(\mathbb{Z})$ and $V(\mathbb{Z})$ are free $\mathbb{Z}$-modules, equipped with non-degenerate symplectic pairings, and the map ϕ should respect this for the symplectic pairing on the target. Choose bases $\{m_i\}$ and $\{v_j\}$ for $M(\mathbb{Z})$ and $V(\mathbb{Z})$ respectively. Then ϕ is determined by a collection of vectors $\{m'_i := \phi(m_i), v'_j := \phi(v_j)\}$ in $H^1_{\mathbb{Z}}$ such that $\forall a, b$ $\langle m_a, m_b \rangle = \langle m'_a, m'_b \rangle$ and same for v_j, v'_j where $\langle -, - \rangle$ denotes the appropriate symplectic form. Additionally, require that $\left\langle m'_i, v'_j \right\rangle = 0$. Equivalently, ϕ is determined by a single vector in the direct sum of several $H^1_{\mathbb{Z}}$'s with the appropriate constraints.

The set of real vectors with the same constraints forms a single orbit under $\mathrm{Sp}(H^1)(\mathbb{R})$. Therefore, by the Borel–Harish-Chandra theorem there are finitely many possible ϕ up to the action of $\mathrm{Sp}(H^1)(\mathbb{Z})$ on the target.

Finally, each ϕ determines an embedding I_ϕ of the period domain $\mathbb{H}_{M,V}$ into the Siegel space corresponding to H^1. The stabilizer of the image of I_ϕ acts with co-finite volume on this image. Since there are only finitely many $\mathrm{Sp}(H^1)(\mathbb{Z})$-equivalence classes of ϕ's, there are finitely many corresponding subvarieties in $\mathcal{A}_g$. These subvarieties of $\mathcal{A}_g$ parametrize abelian varieties with real multiplication by $\mathcal{O}$ on a factor and $\mathcal{O}$-module structure as the one corresponding to the affine manifold $\mathcal{M}$.

2.3 Mixed Hodge structures and their splittings

This section contains background on mixed Hodge structures and their properties. The monograph of Peters and Steenbrink [PS08] provides a thorough treatment. We include examples relevant to our situation. The full machinery, as developed e.g. by Carlson in [Car80], is not strictly necessary. However, using this language streamlines some of the arguments. Throughout this section, we fix a ring k such that $\mathbb{Z} \subseteq k \subseteq \mathbb{R}$. Most often, k will be a field.

2.3.1 Definitions

First recall some standard definitions.

Definition 2.3.1 *A k-Hodge structure of weight w is a k-module V_k and the data of a decreasing filtration by complex subspaces $F^\bullet$ on $V_{\mathbb{C}} := V_k \otimes_k \mathbb{C}$*

$$\cdots \subseteq F^p \subseteq F^{p-1} \subseteq \cdots \subseteq V_{\mathbb{C}}$$

The filtration is called the Hodge filtration and is required to satisfy

$$F^p \oplus \overline{F^{w+1-p}} = V_{\mathbb{C}}$$

Definition 2.3.2 *A k-Mixed Hodge structure is a k-module V_k with an increasing filtration $W_\bullet$ defined over $k \otimes_{\mathbb{Z}} \mathbb{Q}$*

$$\cdots W_n \subseteq W_{n+1} \subseteq \cdots \subseteq (V_k \otimes_{\mathbb{Z}} \mathbb{Q})$$

and a decreasing filtration $F^\bullet$ on $V_{\mathbb{C}}$ such that $F^\bullet$ induces a k-Hodge structure of weight n on

$$\operatorname{gr}_n^W V = W_n / W_{n-1}$$

The filtration $W_\bullet$ is called the weight filtration.

Remark 2.3.3 We can take duals of (mixed) Hodge structures and overall, they form an abelian category. Negative indexing in the filtration is allowed. See [PS08, Section 3.1] for more background.

For future use, we recall the definition of the dual Hodge structure. To describe the Hodge and weight filtrations on the dual of V, denoted $V^\vee$, let

$$\begin{aligned} F^{p+1}V^\vee &= \{\xi \in V^\vee | \xi(F^{-p}V) = 0\} \\ W_{n-1}V^\vee &= \{\xi \in V^\vee | \xi(W_{-n}) = 0\} \end{aligned} \tag{2.3.1}$$

Remark 2.3.4 In the definition of mixed Hodge structures, the weight filtration was defined only after allowing $\mathbb{Q}$-coefficients. However, if it came from a $\mathbb{Z}$-module, the position of the lattice will be relevant.

Example 2.3.5 Let C be a compact Riemann surface and $S \subset C$ a finite set of points. On the relative cohomology group $H^1(C, S)$ we have a natural mixed Hodge structure with weights 0 and 1. This is the same as the compactly supported cohomology of the punctured surface $C \setminus S$.

We have the exact sequence

$$0 \to W_0 \hookrightarrow H^1(C, S) \twoheadrightarrow H^1(C) \to 0$$

The sequence is valid with any coefficients, so we consider it over $\mathbb{Z}$. We have the canonical identification $W_0 = \widetilde{H^0(S)}$ which is the reduced cohomology of the set S.

Here is the mixed Hodge structure on $H^1(C, S)$. The weight filtration has W_0 defined by the exact sequence, and W_1 is the entire space. The holomorphic 1-forms on the Riemann surface C give also relative cohomology classes, so form a subspace

$$F^1 \subset H^1_{\mathbb{C}}(C, S)$$

This subspace maps isomorphically onto the holomorphic 1-forms on $H^1_{\mathbb{C}}(C)$. This describes the mixed Hodge structure, and according to Carlson [Car80, Theorem A] it recovers the punctured curve in many cases.

The dual picture. We shall often work with duals, because the constructions are more natural. Dualizing the above sequence we find

$$0 \leftarrow W_0^\vee \leftarrow \check{H}^1(C, S) \leftarrow \check{H}^1(C) \leftarrow 0$$

On the space $\check{H}^1(C,S)$ we have a mixed Hodge structure of weights -1 and 0 (see Eq. (2.3.1)).

The space W_{-1} is the image of $\check{H}^1(C)$ which is pure of weight -1. The space W_0 is everything. The Hodge filtration has $F^0\check{H}^1(C,S)$ equal to the annihilator of $F^1H^1(C,S)$. In particular it contains the image of $F^0\check{H}^1(C)$, which is the annihilator of $F^1H^1(C)$. Moreover, we have the (natural) isomorphism over $\mathbb{C}$

$$W_0^\vee \leftarrow F^0\check{H}^1(C,S)/F^0\check{H}^1(C)$$

2.3.2 Splittings

The concepts below were first analyzed by Carlson [Car80], which provides more details. We work exclusively with mixed Hodge structures of weights $\{0,1\}$ and their duals, with weights $\{-1,0\}$. They are viewed as extensions of pure Hodge structures of corresponding weights. Example 2.3.5 describes the mixed Hodge structures that occur in later sections.

Definition 2.3.6 *Fix a ring L with $k \subseteq L \subseteq \mathbb{R}$. A k-mixed Hodge structure*

$$0 \to W_0 \to E \to H^1 \to 0$$

is L-split *if the sequence, after extending scalars to L, is isomorphic as a sequence of L-mixed Hodge structures to*

$$0 \to W_0 \to W_0 \oplus H^1 \to H^1 \to 0$$

The mixed Hodge structure in this sequence is the direct sum of the pure structures. The isomorphism is required to be defined over L, but it must map the weight and Hodge filtrations isomorphically.

Remark 2.3.7

(i) The definition for splittings of duals is analogous. A mixed Hodge structure is L-split if and only if its dual is.

(ii) Giving a splitting is the same as giving a map defined over L

$$\sigma : H^1 \to E$$

which is the identity when composed with projection back to H^1. It is required to map F^1H^1 isomorphically to F^1E.

(iii) Mixed Hodge structures as above are always $\mathbb{R}$-split. In the dual picture, we have the sequence

$$0 \leftarrow W_0^\vee \leftarrow E^\vee \leftarrow \check{H}^1 \leftarrow 0$$

Then $F^0E^\vee \cap \overline{F^0E^\vee}$ is a real subspace which maps isomorphically onto $W_0^\vee(\mathbb{R})$. Over $\mathbb{R}$ we can thus lift $W_0^\vee$ inside $E^\vee$ using this subspace and this provides the splitting.

Example 2.3.8 It is more convenient to describe splittings of dual sequences, and below is the simplest example. Consider a pure weight -1 Hodge structure $\check{H}^1_{\mathbb{Z}} = \langle a, b\rangle$ with filtration $F^0\check{H}^1 = \langle a + \tau b\rangle$, where $\operatorname{Im}\tau > 0$. This defines an elliptic curve

$$\operatorname{Jac}(\check{H}^1) := \check{H}^1_{\mathbb{C}} / \left(F^0\check{H}^1 + \check{H}^1_{\mathbb{Z}}\right) \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

Note that $\check{H}^1_{\mathbb{C}} = F^0\check{H}^1 \oplus \overline{F^0\check{H}^1}$ (since $\tau \neq \overline{\tau}$) and in particular $\check{H}^1_{\mathbb{Z}} \cap F^0\check{H}^1 = \{0\}$ (since integral elements are invariant under complex conjugation).

Consider now possible extensions of the form

$$0 \leftarrow W_0 \leftarrow E \leftarrow \check{H}^1 \leftarrow 0$$

Assume W_0 is of $\mathbb{Z}$-rank 1, generated by c. Lift it to some $c_1 \in E_{\mathbb{Z}}$. It gives a map

$$\sigma_{\mathbb{Z}} : W_0(\mathbb{Z}) \to E_{\mathbb{Z}}$$

Note that the lift c_1 is ambiguous, up to addition of terms $xa + yb$ with $x, y \in \mathbb{Z}$. Here, the generators of $\check{H}^1$ are identified with their image inside E.

The extra data on E is a subspace F^0E which contains $F^0\check{H}^1$, is complex two-dimensional, and maps surjectively onto W_0. Pick a vector $v \in F^0E$ which maps to c. It must be of the form

$$v = c_1 + \lambda a + \mu b$$

where $\lambda, \mu \in \mathbb{C}$. Note that the lift v is ambiguous, up to the addition of complex multiples of $a + \tau b$ (which generate $F^0\check{H}^1 = F^0E \cap \ker(E \to W_0)$). This provides a second lift

$$\sigma_{\mathbb{R}} : W_0(\mathbb{Z}) \to E_{\mathbb{C}}/F^0\check{H}^1$$

We can take the difference of $\sigma_{\mathbb{Z}}$ and $\sigma_{\mathbb{R}}$. Because projecting $\sigma_{\mathbb{Z}} - \sigma_{\mathbb{R}}$ back to W_0 is the zero map, their image must land in $\check{H}^1_{\mathbb{C}}$. Taking into account also the ambiguity in the definition of $\sigma_{\mathbb{Z}}$, we get a *canonical* map

$$\sigma_{\mathbb{Z}} - \sigma_{\mathbb{R}} : W_0(\mathbb{Z}) \to \check{H}^1_{\mathbb{C}}/\left(F^0\check{H}^1 + \check{H}^1_{\mathbb{Z}}\right)$$

So we get a canonical element of the elliptic curve associated to the Hodge structure $\check{H}^1$. This element is zero if and only if the sequence is $\mathbb{Z}$-split. Indeed, the vanishing of this element means we could choose the lift v above to be integral.

The element is torsion in the elliptic curve if and only if the sequence is $\mathbb{Q}$-split. It means we could choose v above with rational coefficients in a and b.

2.3.3 Extension classes and field changes

This section contains a discussion of algebraic facts needed later. The details of the constructions are available in [Car80] and [PS08, Chapter 3.5].

Jacobians and extensions For a $\mathbb{Z}$-Hodge structure H^1 of weight 1, its Jacobian is defined using the dual Hodge structure $\check{H}^1$ by

$$\operatorname{Jac}_{\mathbb{Z}} H^1 := \check{H}^1(\mathbb{C}) / \left(F^0 \check{H}^1 + \check{H}^1(\mathbb{Z}) \right)$$

This is a compact complex torus, again since $\check{H}^1_{\mathbb{C}} = F^0\check{H}^1 \oplus \overline{F^0\check{H}^1}$ and the $\mathbb{Z}$-lattice doesn't intersect F^0. As in Example 2.3.8, extensions of H^1 by a weight 0 Hodge structure W_0

$$0 \to W_0 \to E \to H^1 \to 0$$

are classified by elements in $\operatorname{Hom}_{\mathbb{Z}}(\check{W}_0(\mathbb{Z}), \operatorname{Jac}_{\mathbb{Z}} H^1)$. Rather, it is dual extensions that are classified by such elements.

Let now K be a larger field or ring $\mathbb{Z} \subseteq K \subseteq \mathbb{R}$ and H^1 be a K-Hodge structure of weight 1. We can also define a Jacobian

$$\operatorname{Jac}_K H^1 := \check{H}^1(\mathbb{C}) / \left(F^0 \check{H}^1 + \check{H}^1(K) \right)$$

It is an abelian group (even a K-vector space, usually of infinite dimension) with no structure of manifold. Extensions are still classified by elements of $\operatorname{Hom}_K(\check{W}_0(K), \operatorname{Jac}_K H^1)$, where the Jacobian is viewed as a K-vector space.

Morphisms If H^1 has a $\mathbb{Z}$-structure and we extend scalars to K, then we have a natural map of abelian groups

$$\operatorname{Jac}_{\mathbb{Z}} H^1 \to \operatorname{Jac}_K H^1$$

For example $\operatorname{Jac}_{\mathbb{Q}} H^1 = \operatorname{Jac}_{\mathbb{Z}} H^1/\langle \text{torsion} \rangle$ and an extension class is torsion in the usual Jacobian if and only if the extension splits over $\mathbb{Q}$.

More generally, suppose we have (after extending scalars to K) an inclusion of Hodge structures $H^1_\iota \hookrightarrow H^1$. The dual map becomes $\check{H}^1 \twoheadrightarrow \check{H}^1_\iota$ and we have an induced surjection on Jacobians

$$\operatorname{Jac}_{\mathbb{Z}} H^1 \twoheadrightarrow \operatorname{Jac}_K H^1_\iota$$

From an extension class $\xi \in \operatorname{Hom}_{\mathbb{Z}}(\check{W}_0(\mathbb{Z}), \operatorname{Jac}_{\mathbb{Z}} H^1)$ we get another one $\xi_\iota \in \operatorname{Hom}_K(\check{W}_0(K), \operatorname{Jac}_K H^1)$ by composing with the above map. This gives a corresponding extension of K-Hodge structures. Given a K-subspace $S \subseteq \check{W}_0(K)$ we can restrict the extension class ξ_ι to it and get another such extension. Note that a subspace of $\check{W}_0(K)$ corresponds to a quotient of $W_0(K)$.

2.4 Splittings over Affine Invariant Manifolds

In this section we identify variations of mixed Hodge structure that naturally exist on affine invariant manifolds. The main result is that they split after an appropriate extension of scalars, in the sense of the previous section.

2.4.1 Setup.

Consider an affine invariant manifold $\mathcal{M}$ inside a stratum of flat surfaces $\mathcal{H}$. It was proved in [Fil16a, Thm. 1.6] that the variation of Hodge structure over $\mathcal{M}$ given by the first cohomology H^1 splits as

$$H^1 = \left(H^1_{\iota_0} \oplus \cdots \oplus H^1_{\iota_{r-1}}\right) \oplus V$$

Each term above gives a variation of Hodge structure. Moreover, the direct sum in the parenthesis comes from a $\mathbb{Q}$-local system. Each summand H^1_ι corresponds to an embedding ι of a totally real number field k, and comes from a local system defined over that embedding. The embedding ι_0 is the distinguished embedding.

Recall the tangent space $T\mathcal{H}$ to the stratum is given, via period coordinates, by the relative cohomology of the underlying surfaces. Restrict the tangent bundle $T\mathcal{H}$ to $\mathcal{M}$. It projects to H^1 and we can take the preimage of the summands coming from the totally real field. This yields an exact sequence of $\mathbb{Q}$-local systems

$$0 \to W_0 \to E \xrightarrow{p} \left(H^1_{\iota_0} \oplus \cdots \oplus H^1_{\iota_{r-1}}\right) \to 0 \tag{2.4.1}$$

Here W_0 is the local system of purely relative cohomology classes. It coincides with the (reduced) cohomology of the marked points (i.e. zeroes of the 1-form).

This provides a variation of $\mathbb{Q}$-mixed Hodge structures in the following sense. Above each point in $\mathcal{M}$ we have an induced mixed Hodge structure. The Hodge filtrations $F^\bullet$ give holomorphic subbundles of the corresponding local systems. The Griffiths transversality conditions are empty in this case.

Remark 2.4.1 Below we discuss local systems defined over a particular field, for example $\iota_0(k)$ from above. To define this notion, fix a normal closure K containing all the embeddings of k, with Galois group over $\mathbb{Q}$ denoted G. Given a $\mathbb{Q}$-local system $V_\mathbb{Q}$, we can extend scalars to K and denote it V_K. A sublocal system $W \subset V_K$ is "defined over $\iota_l(k)$" if it is invariant by the subgroup $G_{\iota_l} \subset G$ stabilizing $\iota_l(k)$.

We will omit the explicit extension of scalars to K below, and just say that $W \subset V$ is a sublocal system defined over $\iota_l(k)$.

The tangent space of the affine manifold. According to results of Wright [Wri14, Thm. 1.5] the tangent space $T\mathcal{M}$ to the affine submanifold gives a local subsystem $T\mathcal{M} \subset E$, which is defined over k (rather, $\iota_0(k)$). It has the property that $p(T\mathcal{M}) = H^1_{\iota_0}$. Define the kernel of the map p (see Eq. (2.4.1)) by

$$W_0\mathcal{M} := W_0 \cap T\mathcal{M}$$

Define also $T\mathcal{M}_{\iota_l}$ to be the Galois-conjugate of the local system $T\mathcal{M}$ corresponding to the embedding ι_l. Analogously define

$$W_0\mathcal{M}_{\iota_l} := W_0 \cap T\mathcal{M}_{\iota_l}$$

Note that $T\mathcal{M}_{\iota_l}$ surjects onto $H^1_{\iota_l}$ with kernel $W_0\mathcal{M}_{\iota_l}$.

2.4.2 The splitting

The space $H^1_{\iota_l}$ contains holomorphic 1-forms denoted $H^{1,0}_{\iota_l}$. These also give *relative* cohomology classes, i.e. a natural map $H^{1,0}_{\iota_l} \to H^1_{rel}$. The main theorem of this section (below) is the compatibility of this map with $T\mathcal{M}_{\iota_l}$. Note that in the case when W_0 is contained in $T\mathcal{M}$ (i.e. the setup of Section 2.2) the statement below holds trivially.

Theorem 2.4.2 *Consider the variation of $\iota_l(k)$-mixed Hodge structures*

$$0 \to W_0/W_0\mathcal{M}_{\iota_l} \hookrightarrow p^{-1}(H^1_{\iota_l})/W_0\mathcal{M}_{\iota_l} \twoheadrightarrow H^1_{\iota_l} \to 0 \tag{2.4.2}$$

It is an exact sequence of local systems defined over $\iota_l(k)$, and each space carries compatible (mixed) Hodge structures.

Then this sequence is (pointwise) split over $\iota_l(k)$ in the sense of Definition 2.3.6.

The splitting is provided by the local system $T\mathcal{M}_{\iota_l}/W_0\mathcal{M}_{\iota_l}$, i.e. the surjection in the *sequence* (2.4.2) *can be split by an* $\iota_l(k)$*-isomorphism*

$$H^1_{\iota_l} \xrightarrow{\sim} T\mathcal{M}_{\iota_l}/W_0\mathcal{M}_{\iota_l} \tag{2.4.3}$$

Note that there are two ways to lift a cohomology class in $H^{1,0}_{\iota_l}$ to H^1_{rel}. One uses the map (2.4.3), and the other is by viewing a holomorphic 1-form as a relative cohomology class. This theorem claims that these two ways in fact agree.

Proof. The proof is in three steps. In Step 1 we dualize the sequence (2.4.2) and use the Galois-conjugate tangent space to produce a flat splitting of the local systems. We also find the $\mathbb{R}$-splitting coming from the underlying Hodge structures. Their difference is a (holomorphic) section of a bundle with negative curvature.

In Step 2, we show the section must have constant Hodge norm. In Step 3 we use this to show that the section must come from a flat one, and thus must in fact be zero. This concludes the proof, since it shows that the $\mathbb{R}$-splitting was in fact defined over $\iota_l(k)$. The next three sections deal with each step. □

2.4.3 Proof of Step 1

Because in the exact sequence (2.4.2) we quotient out $W_0\mathcal{M}\iota_l$ we deduce that the bundle $T\mathcal{M}_{\iota_l}/W_0\mathcal{M}_{\iota_l}$ maps isomorphically onto $H^1_{\iota_l}$. Dualizing the sequence (2.4.2) we obtain

$$0 \leftarrow \left(W_0/W_0\mathcal{M}_{\iota_l}\right)^{\vee} \twoheadleftarrow \left(p^{-1}\left(H^1_{\iota_l}\right)/W_0\mathcal{M}_{\iota_l}\right)^{\vee} \hookleftarrow \check{H}^1_{\iota_l} \leftarrow 0 \tag{2.4.4}$$

Denote the annihilator of $T\mathcal{M}_{\iota_l}/W_0\mathcal{M}_{\iota_l}$ by $\left(T\mathcal{M}_{\iota_l}/W_0\mathcal{M}_{\iota_l}\right)^{\perp}$. By the remark above, it maps isomorphically onto $\left(W_0/W_0\mathcal{M}_{\iota_l}\right)^{\vee}$. The inverse isomorphism defines a canonical flat map of $\iota_l(k)$ local systems, which is a splitting of the left surjection in the exact sequence

(2.4.4)

$$\sigma_{\iota_l} : \left(W_0/W_0\mathcal{M}_{\iota_l}\right)^\vee \to \left(p^{-1}\left(H^1_{\iota_l}\right)/W_0\mathcal{M}_{\iota_l}\right)^\vee$$

We now construct another splitting using the Hodge structures (see Remark 2.3.7 (iii)). Consider the F^0 piece of the middle term in the sequence (2.4.4). It maps surjectively onto $\left(W_0/W_0\mathcal{M}_{\iota_l}\right)^\vee$ with kernel $F^0\check{H}^1_{\iota_l}$. This gives a canonical splitting

$$\sigma_{\mathbb{R}} : \left(W_0/W_0\mathcal{M}_{\iota_l}\right)^\vee_{\mathbb{C}} \to \left(p^{-1}\left(H^1_{\iota_l}\right)/W_0\mathcal{M}_{\iota_l}\right)^\vee_{\mathbb{C}}/F^0\check{H}^1_{\iota_l}$$

Note that because it really comes from an isomorphism of vector bundles, it is in fact holomorphic over the affine manifold.

Note that both maps $\sigma_{\mathbb{R}}$ and σ_{ι_l} are splittings. This means that composing either with the surjection onto the left term of the sequence (2.4.4) gives the identity. So their difference has image in the kernel of the surjection in (2.4.4):

$$\sigma_{\iota_l} - \sigma_{\mathbb{R}} : \left(W_0/W_0\mathcal{M}_{\iota_l}\right)^\vee_{\mathbb{C}} \to \check{H}^1_{\iota_l}/F^0\check{H}^1_{\iota_l}$$

Next, we assume that we are in some finite cover of the stratum where the local system W_0 is trivial; labeling the zeroes of the 1-form suffices. Then we can choose an element (same as a global section) of $\left(W_0/W_0\mathcal{M}_{\iota_l}\right)^\vee_{\mathbb{C}}$ denoted by e. By taking its image under the above map, we obtain a global over $\mathcal{M}$ holomorphic section

$$\psi_e := (\sigma_{\iota_l} - \sigma_{\mathbb{R}})(e) \in \Gamma(\mathcal{M}; \check{H}^1_{\iota_l}/F^0\check{H}^1_{\iota_l})$$

2.4.4 Proof of Step 2

Given the holomorphic section ψ_e produced in Step 1, we now show it has constant Hodge norm.

Notation. Denote the Hodge decomposition of $\check{H}^1$ by

$$\check{H}^1 = \check{H}^{0,-1} \oplus \check{H}^{-1,0}$$

We keep the same notation for indices involving the embeddings ι_l.

Note that $H^{0,-1} = F^0\check{H}^1$ gives a holomorphic subbundle, being identified with the annihilator of $H^{1,0} = F^1H^1$ (which is the bundle of holomorphic 1-forms). Using the polarization of H^1, we see that $\check{H}^1$ is isomorphic to H^1, up to a shift of weight by $(1,1)$.

The section ψ_e produced above is a holomorphic section of $\check{H}^{-1,0}$, endowed with the holomorphic structure when viewed as a quotient $\check{H}^{-1,0} = \check{H}^1/F^0\check{H}^1$. This bundle has negative curvature (see [Fil16a, Corollary 3.15], with a weight shift).

We want to apply Lemma 5.2 from [Fil16a] to conclude the function $\log\|\psi_e\|$ is constant. For this, we need to check the boundedness and sublinear growth assumptions. This is done below.

Note that this function is subharmonic by the calculation in [Fil16a, Lemma 3.1] and the negative curvature of the bundle. If $\|\psi_e\| \neq 0$ identically, then it can vanish only on a lower-dimensional analytic subset. Therefore, one can define the functions $f_N := \max(-N, \log\|\psi_e\|)$ and apply [Fil16a, Lemma 5.2] to them (and let $N \to +\infty$). As the maximum of two subharmonic functions, each f_N is itself subharmonic.

Checking assumptions. First, we examine how ψ_e was defined. We have the exact sequence

$$0 \to W_0 \to E \to (H^1_{\iota_0} \oplus \cdots \oplus H^1_{\iota_{r-1}}) \to 0 \tag{2.4.5}$$

and its dual

$$0 \leftarrow W_0^\vee \leftarrow E^\vee \leftarrow (\check{H}^1_{\iota_0} \oplus \cdots \oplus \check{H}^1_{\iota_{r-1}}) \leftarrow 0$$

We have the $\mathbb{R}$-splitting coming from Hodge theory (see Remark 2.3.7 (iii))

$$W_0^\vee(\mathbb{R}) \to E_\mathbb{R}^\vee$$

This gives a direct sum decomposition

$$E^\vee(\mathbb{R}) \cong W_0^\vee(\mathbb{R}) \oplus \check{H}^1(\mathbb{R}) \tag{2.4.6}$$

and an induced metric from each factor. At the level of the exact sequence (2.4.5) this is the same as using the harmonic representatives of absolute cohomology classes to obtain relative cohomology classes.

Cocycles and Norms. Recall that if $V \to \mathcal{M}$ is a vector bundle over $\mathcal{M}$, and $\mathcal{M}$ carries an action of a (connected) group G (for instance $G = \mathrm{SL}_2\,\mathbb{R}$), a cocycle on V is the lift of the action of G to V by linear maps on the fibers. If V has a flat connection, every path γ in $\mathcal{M}$ from x to y gives a monodromy matrix $V_x \to V_y$ between the fibers of V. Suppose that any non-trivial loop in G has trivial monodromy. Namely, assume for any $x \in \mathcal{M}$ and a loop $\gamma' \subset G$, the loop $\gamma' \cdot x \subset \mathcal{M}$ has trivial monodromy. Then the flat connection on V gives a cocycle for the G-action. For any $g \in G, x \in \mathcal{M}$ the linear map $A(x, g) : V_x \to V_{gx}$ is defined by connecting g to the identity in G, and parallel-transporting along the corresponding path in $\mathcal{M}$.

Suppose now the vector bundle V carries a metric. A cocycle is *bounded* if there is a constant $C > 0$ such that

$$\|A(x, g)\|_{op} \leq C\,\|g\| \text{ where } \|-\|_{op} \text{ is the operator norm of } A(g, x) : V_x \to V_{gx}$$

For $g \in \mathrm{SL}_2\,\mathbb{R}$ the right-hand side above, $\|g\|$, denotes the matrix norm.

In our case, we have three cocycles (coming from vector bundles with flat connections) which sit in an exact sequence:

$$0 \to W_0 \to H^1_{rel} \to H^1 \to 0 \tag{2.4.7}$$

The bundles H^1 and W_0 have natural norms for which the cocycle is bounded (for H^1, this is due to Forni [For02]; for W_0, we can assume the cocycle is trivial). Moreover, the exact sequence (2.4.7) has a natural splitting, as vector bundles, but *not as cocycles*.

For the induced metric coming from the splitting of the vector bundles, the cocycle on H^1_{rel} is not (apriori) bounded. Using the splitting, we can write the cocycle on H^1_{rel} as

$$\begin{bmatrix} 1 & U(x,g) \\ 0 & A(x,g) \end{bmatrix} \text{ where } A(x,g) : H^1_x \to H^1_{gx} \text{ and } U(x,g) : H^1_x \to (W_0)_{gx} \tag{2.4.8}$$

The issue is that in general, there is no universal bound on the operator norm of $U(x,g)$ in terms of the matrix norm of g. However, the operator norm of $U(x,g)$ can be bounded in terms of $\|g\|$ and how far x is in the cusp of the stratum. Using this, one can increase the metric on H^1 to achieve two things: first, keep the cocycle on H^1 bounded, and second, make $U(x,g)$ bounded. This is achieved by the modified Hodge norm from [EMM15], defined on H^1, and is discussed below.

Note that to make $U(x,g)$ bounded, we have to *increase* the metric on H^1. Note also that H^1 is the *quotient* of H^1_{rel}. When we dualize (see Eq. (2.4.14) and the discussion before), we have to again *increase* the metric on the *quotient* of $\check{H}^1_{rel}$, which will be $W_0^\vee$ (compare also Eq. (2.4.8) and Eq. (2.4.15) describing the cocycles on H^1_{rel} and its dual $\check{H}^1_{rel}$).

Terminology for norms. Using the splitting from (2.4.6) and more generally the same for H^1_{rel}, we can put norms on the cocycle by putting a norm on each factor. Several norms will appear below, and we explain now the terminology.

On H^1 and its dual $\check{H}^1$ one has the Hodge norms, coming from the Hodge structure (which are preserved by duality). These Hodge norms will be denoted $\|-\|$ in the sequel. Eskin, Mirzakhani and Mohammadi [EMM15, Sec. 7] introduced a modified Hodge norm on H^1, obtained by increasing the usual Hodge norm. Note that the dual of this modified Hodge norm, on $\check{H}^1$, is *less than* the Hodge norm on $\check{H}^1$. The modified Hodge norms on H^1 and $\check{H}^1$ will be denoted $\|-\|'$.

Finally, we have the modified mixed Hodge norms on H^1_{rel} and $\check{H}^1_{rel}$. These will be denoted $|||-|||$ (note that they will not be dual to each other). On H^1_{rel} the modified mixed Hodge norm is defined in [EMM15] by putting the constant norm on W_0 and the modified Hodge norm on H^1. On $\check{H}^1_{rel}$ the modified mixed Hodge norm is defined in Eq. (2.4.14) by changing the norm on the $W_0^\vee$ factor.

Below, the adjective *dual* means that the norm is on the dual space, and the adjective *mixed* means that it is in relative (co)homology. We now proceed to the details.

Modified Hodge norm. The modified Hodge norm $\|-\|'$ on H^1 is defined in [EMM15, eqn. (33) and below]) and the cocycle on H^1 is integrable for it ([EMM15, Lemma 7.4]).

Next, recall the splitting $H^1_{rel} = W_0 \oplus H^1$ coming from Hodge theory using holomorphic lifts (in their language – harmonic representatives). Using it, the modified Hodge norm $|||-|||$ on H^1_{rel} is defined using the modified Hodge norm on H^1 and the constant norm on W_0 (see [EMM15, eq. (40) and above]).

For this modified Hodge norm on H^1_{rel} the cocycle is bounded [EMM15, Lemma 7.5]. The main consequence [EMM15, Lemma 7.6] is the following. For the splitting $H^1_{rel} = W_0 \oplus H^1$, write the cocycle matrix as

$$\begin{bmatrix} 1 & U(x,s) \\ 0 & A(x,s) \end{bmatrix} \text{ where } U(x,s) : H^1_x \to (W_0)_{g_s x} \tag{2.4.9}$$

Then $\|U(x,s)\|_{op} \leq e^{m's}$ for some m', where H^1 is viewed with the modified Hodge norm

$\|-\|'$ and W_0 with the constant norm; here $\|-\|_{op}$ denotes the operator norm. Explictly, this says

$$\|U(x,s)v\| \le e^{m's}\,\|v\|' \text{ where } v \in H^1_x \text{ and so } U(x,s)v \in (W_0)_{g_s x} \tag{2.4.10}$$

Dual modified Hodge norm. For a point x in the stratum, let $r(x) \in [1, \infty)$ be the ratio of the modified to the usual Hodge norms on H^1. In other words

$$r(x) := \sup_{0 \neq v \in H^1_x} \frac{\|v\|'}{\|v\|} \tag{2.4.11}$$

Note that $r(x) \ge 1$ since $\|-\|'$ is always at least the usual Hodge norm (see before eq. (40) in [EMM15]). By definition $r(x) \cdot \|v\| \ge \|v\|'$ for any $v \in H^1_x$.

On $\check{H}^1$ there are two norms – the dual of the usual Hodge norm, denoted $\|-\|$, and the dual of the modified Hodge norm, denoted $\|-\|'$. The inequality involving $r(x)$ is now reversed, i.e. letting $\xi \in \check{H}^1_x$, we have

$$\|\xi\| \le r(x) \cdot \|\xi\|' \tag{2.4.12}$$

To see this, using $\frac{1}{r(x)}\,\|v\|' \le \|v\|$ for the second step, we have

$$\|\xi\| = \sup_{\|v\|=1} |\xi(v)| \le \sup_{\frac{1}{r}\|v\|'=1} |\xi(v)| = r(x)\,\|\xi\|' \tag{2.4.13}$$

Dual modified mixed Hodge norm. We now explain how to modify the Hodge norm on the dual cocycle $\check{H}^1_{rel}$ which is relevant to our arguments. Recall the splitting $\check{H}^1_{rel} = W_0^\vee \oplus \check{H}^1$ coming from Hodge theory. Given a vector $w \oplus h \in H^1_{rel}$ its ordinary mixed Hodge norm is $\|w \oplus h\|^2 = \|w_0\|^2 + \|h\|^2$ (using the usual Hodge norm for h and constant norm for w_0). Define now its dual modified mixed Hodge norm via

$$|||w \oplus h|||^2 := \|w_0\|^2 \cdot r(x)^2 + \|h\|^2 \tag{2.4.14}$$

Note that the modification affects only the $W_0^\vee$ part of the norm, not the $\check{H}^1$.

Proposition 2.4.3 *For the modified dual Hodge norm defined in Eq. (2.4.14), the Kontsevich–Zorich cocycle is bounded.*

Proof. Given the explicit form of the KZ cocycle in Eq. (2.4.9), its dual cocycle (for the dual splitting) will be given by the inverse transpose of that matrix. This reads

$$\left(\begin{bmatrix} 1 & U(x,s) \\ 0 & A(x,s) \end{bmatrix}^t\right)^{-1} = \begin{bmatrix} 1 & 0 \\ -(A(x,s)^t)^{-1} \cdot U(x,s)^t & (A(x,s)^t)^{-1} \end{bmatrix} \tag{2.4.15}$$

Next, recall that the cocycle $A(x,s)$ is bounded in both forward and backwards time, since it corresponds to H^1, and we have the usual Hodge norm on that part. To show boundedness of the full cocycle, it suffices to prove that $|||U(x,s)^t|||_{op} \leq e^{m's}$ for some m'. Note that $U(x,s)^t : (W_0^\vee)_{g_s x} \to \check{H}^1_x$, since $U(x,s)$ goes the other way between dual spaces (see (2.4.9)). The bound on the operator norm of $U(x,s)^t$ needs to hold when $W_0^\vee$ is viewed with the norm $r(x)\,\|-\|$, and $\check{H}^1$ is with the usual Hodge norm.

The operator $U(x,s)$ is bounded for the modified Hodge norm on H^1 and usual norm on W_0 (see (2.4.10)). Therefore $U(x,s)^t$ satisfies the same bound for the dual modified Hodge norm on $\check{H}^1$ and usual norm on $W_0^\vee$. This reads

$$\left\|U(x,s)^t \check{w}\right\|' \leq e^{m's}\,\|\check{w}\| \text{ where } \check{w} \in (W_0^\vee)_x \tag{2.4.16}$$

However, using Eq. (2.4.12) which relates the usual and modified Hodge norms in the dual

$\check{H}^1$ we find

$$\left\| U(x,s)^t \check{w} \right\| \cdot \frac{1}{r(x)} \leq \left\| U(x,s)^t \check{w} \right\|' \leq e^{m's} \left\| \check{w} \right\| \tag{2.4.17}$$

Moving $r(x)$ to the right side, this exactly says that $U(x,s)^t$ is bounded when $W_0^\vee$ carries the dual modified mixed Hodge norm.

Finally, note that the identity map on $W_0^\vee$ no longer acts by isometries, since the norm on that factor depends now on the basepoint. However, from the boundedness of the cocycle on H^1 with the usual, as well as the modified Hodge norm, it follows that $r(g_s x) \leq e^{m'' s} r(x)$ for all s. Thus, the Kontsevich-Zorich cocycle on $\check{H}^1_{rel}$ is bounded for the dual modified mixed Hodge norm. □

Properties of the dual modified mixed Hodge norm. First, note that the boundedness properties of the cocycle descend to the various pieces of $\check{H}^1$ and $\check{H}^1_{rel}$. Returning to the notation in Eq. (2.4.5), (2.4.6), we constructed a norm $|||-|||$ satisfying

(i) The Kontsevich-Zorich cocycle on E and $E^\vee$ is integrable for this norm. Moreover, it satisfies the absolute bound for some universal constant $C > 0$

$$-C \cdot T \leq \log |||g_T||| \leq C \cdot T \tag{2.4.18}$$

(ii) The projection $E \to H^1$ is norm non-increasing and dually the embedding $\check{H}^1 \hookrightarrow E^\vee$ is norm non-decreasing (where H^1 and $\check{H}^1$ have the usual Hodge norms). In other words, if $\psi \in \check{H}^1$ is a section, then

$$\|\psi\| \leq |||\psi||| \text{ (in fact, we have equality in this case)} \tag{2.4.19}$$

Moreover, if ϕ if a section of H^1_{rel} and ψ is its $\check{H}^1$-component, then $|||\phi||| \geq \|\psi\|$. In other words, the dual modified mixed Hodge norm of ϕ dominates the usual Hodge norm of ψ, since the decomposition $\check{H}^1_{rel} = W_0^\vee \oplus \check{H}^1$ is orthogonal for the dual modified mixed Hodge norm.

Consider now

$$0 \to W_0 \to p^{-1}(H^1_{\iota_l}) \to H^1_{\iota_l} \to 0$$

In the dual picture, with the dual modified mixed Hodge norm, we have

$$0 \leftarrow W_0^\vee \twoheadleftarrow p^{-1}(H^1_{\iota_l})^\vee \hookleftarrow \check{H}^1_{\iota_l} \leftarrow 0 \tag{2.4.20}$$

Corresponding to $W_0\mathcal{M}_{\iota_l}$ in the first sequence is its annihilator $W_0\mathcal{M}_{\iota_l}^\perp$ in the dual. It is naturally identified with $\big(W_0/W_0\mathcal{M}_{\iota_l}\big)^\vee$.

Now $p^{-1}(H^1_{\iota_l})^\vee$ also contains the annihilator of $T\mathcal{M}_{\iota_l}$ denoted $T\mathcal{M}_{\iota_l}^\perp$. The surjection in the dual sequence (2.4.20) gives an isomorphism

$$W_0\mathcal{M}_{\iota_l}^\perp \leftarrow T\mathcal{M}_{\iota_l}^\perp$$

The section σ_{ι_l} from Step 1 is the inverse of this isomorphism.

Given $e \in (W_0/W_0\mathcal{M}_{\iota_l})^\vee$ (which is naturally $W_0\mathcal{M}_{\iota_l}^\perp$) we have ϕ_e defined as

$$\phi_e := \sigma_{\iota_l}(e) \in T\mathcal{M}_{\iota_l}^\perp \subset p^{-1}(H^1_{\iota_l})^\vee$$

Note that ϕ_e is a flat global section of $p^{-1}(H^1_{\iota_l})^\vee$. This last bundle, equipped with the dual modified mixed Hodge norm, gives rise to an integrable cocycle; the Oseledets theorem thus holds.

Just like in [Fil16a, Lemma 5.1] ϕ_e must be in the zero Lyapunov subspace, otherwise its norm would be unbounded on any set of positive measure. In particular, its dual modified mixed Hodge norm grows subexponentially along a.e. Teichmüller geodesic.

Recall the splitting over $\mathbb{R}$ for the sequence (2.4.20). This comes from the decomposition

$$p^{-1}(H^1_{\iota_l})^\vee(\mathbb{R}) \cong \check{H}^1_{\iota_l}(\mathbb{R}) \oplus \left(F^0 \cap \overline{F^0}\right) \tag{2.4.21}$$

Here F^0 refers to the corresponding piece of the filtration of $p^{-1}(H^1_{\iota_l})$ and the construction was explained in Remark 2.3.7 (iii). Eq. (2.4.21) is a direct sum decomposition of $\mathbb{R}$-vector bundles and $\sigma_{\mathbb{R}}$ comes from inverting the $\mathbb{R}$-isomorphism

$$W_0^\vee(\mathbb{R}) \leftarrow F^0 \cap \overline{F^0}$$

This means that

$$\psi_e = (\sigma_{\iota_l} - \sigma_{\mathbb{R}})(e) \in \check{H}^1_{\iota_l}(\mathbb{R})$$

is just the $\check{H}^1_{\iota_l}(\mathbb{R})$-component of ϕ_e in the decomposition (2.4.21).

Remark 2.4.4 For purposes of comparing metrics, we are using the isomorphism of $\mathbb{R}$-vector bundles

$$\check{H}^1_{\iota_l}(\mathbb{R}) \xrightarrow{\sim} \check{H}^1_{\iota_l}(\mathbb{C})/F^0\check{H}^1_{\iota_l}$$

The section we are considering can be viewed as living in either. In the second one, it is also holomorphic for the natural holomorphic structure.

From the property of the dual modified mixed Hodge norm in (2.4.19), $|||\phi_e|||$ bounds from above the usual Hodge norm of its $\check{H}^1_{\iota_l}$-component (which is ψ_e). This gives the desired subexponential growth for the Hodge norm $\|\psi_e\|$.

We must also verify that upon acting by an element $g \in \mathrm{SL}_2\,\mathbb{R}$, the function has not increased by more than $C\,\|g\|$, for some absolute constant C. This follows from (2.4.18) and the fact that the norms are $\mathrm{SO}_2(\mathbb{R})$-invariant [EMM15, eq. (41)].

To conclude, the conditions of [Fil16a, Lemma 5.2] are satisfied; therefore the Hodge norm $\|\psi_e\|$ is constant.

2.4.5 Proof of Step 3

Step 2 showed that $\|\psi_e\|$ is constant. To conclude, we now show it must be zero. Because $\|\psi_e\|$ is constant, using [Fil16a, Remark 3.3] for the formula for $\overline{\partial}\partial$ it follows that

$$0 = \overline{\partial}\partial \|\psi_e\|^2 = \langle \Omega\psi_e, \psi_e \rangle - \left\langle \nabla^{Hg}\psi_e, \nabla^{Hg}\psi \right\rangle$$

where Ω is the curvature of $\check{H}^{-1,0}_{\iota_l}$, which is negative-definite. We conclude

$$\sigma^{\dagger}\psi_e = 0 \qquad \text{and} \qquad \nabla^{Hg}\psi_e = 0$$

Recall that $\sigma : \Omega^{1,0}(\mathcal{M}) \otimes \check{H}^{0,-1}_{\iota_l} \to \check{H}^{-1,0}_{\iota_l}$ is the second fundamental form of $\check{H}^1_{\iota_l}$ and the above identities hold in all direction on $\mathcal{M}$, not just the $\mathrm{SL}_2\,\mathbb{R}$ direction. The curvature satisfies $\Omega = \sigma\sigma^{\dagger}$.

Define now a section of $\check{H}^1_{\iota_l}$ by $\alpha := \overline{\psi_e} \oplus \psi_e$. Then α is flat for the Gauss-Manin connection (since $\nabla^{GM} = \nabla^{Hg} + \sigma + \sigma^{\dagger}$). But the local system $H^1_{\iota_l}$ is irreducible (see [Wri14, Thm. 1.5]) so it has no flat global sections over the affine manifold $\mathcal{M}$.

Therefore $\alpha = 0$, so $\psi_e = 0$ and thus $\sigma_{\iota_l} = \sigma_{\mathbb{R}}$. This finishes the proof of Theorem 2.4.2.

□

2.5 Algebraicity and torsion

In this section, we prove the theorems stated in the introduction. First we combine the results of the previous section to find that a certain twisted version of the Abel-Jacobi map is torsion. Using this result, we then prove that affine invariant manifolds are algebraic.

2.5.1 Combining the splittings

Setup. In the setting of the previous section, over an affine manifold $\mathcal{M}$ we had an exact sequence of $\mathbb{Z}$-mixed Hodge structures

$$0 \to W_0 \to E \xrightarrow{p} \bigoplus_{\iota} H^1_{\iota} \to 0$$

We are assuming some fixed $\mathbb{Z}$-structure on $\oplus_{\iota} H^1_{\iota}$. As was explained in §2.3.3 this corresponds to a map

$$\xi : \check{W}_0(\mathbb{Z}) \to \operatorname{Jac}_{\mathbb{Z}}\left(\bigoplus_{\iota} H^1_{\iota}\right) \tag{2.5.1}$$

Thus for each element of $\check{W}_0(\mathbb{Z})$ we get a section of the bundle of Jacobians. We are working with a factor of the actual Jacobian, but omit this from the wording.

By Theorem 2.4.2 for each ι the variation of $\iota(k)$-mixed Hodge structures

$$0 \to W_0/W_0\mathcal{M}_{\iota} \to p^{-1}(H^1_{\iota})/W_0\mathcal{M}_{\iota} \to H^1_{\iota} \to 0$$

is split. In the language of §2.3.3 this means that pointwise on $\mathcal{M}$ the induced map of abelian groups (and $\iota(k)$-vector spaces)

$$\xi_{\iota} : (W_0\mathcal{M}_{\iota})^{\perp}(\iota(k)) \to \operatorname{Jac}_{\iota(k)} H^1_{\iota}$$

is in fact the zero map. Recall ξ_{ι} is obtained from ξ by composing with the quotient map

$$\operatorname{Jac}_{\mathbb{Z}}\left(\bigoplus_{\iota} H^1_{\iota}\right) \twoheadrightarrow \operatorname{Jac}_{\iota(k)} H^1_{\iota}$$

and restricting the domain to $(W_0\mathcal{M}_{\iota})^{\perp}$ (after extending scalars).

Another description of ξ_ι is as follows. Given $c^j \in \iota(k)$ and $r_j \in \check{W}_0(\mathbb{Z})$ with $\sum_j c^j r_j \in (W_0\mathcal{M}_\iota)^\perp$ we have

$$\xi_\iota\left(\sum_j c^j r_j\right) = \sum_j c^j \xi(r_j)\Bigg|_{H^1_\iota}$$

The next step will be to combine all the above statements, as ι ranges over all embeddings of k into $\mathbb{R}$.

The twisted cycle map. Since $\oplus_\iota H^1_\iota$ has real multiplication by k, we have an order $\mathcal{O} \subseteq k$ which acts by endomorphisms on the bundle of abelian varieties $\mathrm{Jac}_{\mathbb{Z}}(\oplus_\iota H^1_\iota)$. Inside $(W_0\mathcal{M}_{\iota_0})^\perp$ which is a local system over $\iota_0(k)$, we can choose an $\mathcal{O}$-lattice, i.e. the $\iota_0(\mathcal{O})$-submodule

$$\Lambda_{\iota_0} := \big(\check{W}_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \iota_0(\mathcal{O})\big) \cap (W_0\mathcal{M}_{\iota_0})^\perp \tag{2.5.2}$$

Note that $(W_0\mathcal{M}_{\iota_0})^\perp \subset \check{W}_0$, since this is the annihilator of $W_0\mathcal{M} \subset W_0$. After extending scalars to $\mathbb{Q}$ we have an isomorphism

$$\Lambda_{\iota_0} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} (W_0\mathcal{M}_{\iota_0})^\perp$$

Definition 2.5.1 *Recall that for $a \in k$ we denote by $\rho(a)$ the corresponding endomorphism of the family of Jacobians and the map ξ was defined in Eq. (2.5.1). For $c^j \in \mathcal{O}$ and $r_j \in \check{W}_0(\mathbb{Z})$ we can then define a twisted cycle map*

$$\nu : \Lambda_{\iota_0} \to \mathrm{Jac}_{\mathbb{Z}}\left(\bigoplus_\iota H^1_\iota\right)$$
$$\sum_j c^j r_j \mapsto \sum_j \rho(c^j)\xi(r_j)$$

The following is the main theorem of this section. It is a generalization of Theorem 3.3

of Möller from [Möl06a].

Theorem 2.5.2 *The image of ν is torsion in the Jacobian.*

Proof. The proof is in two steps. First, we show that the image of ν is zero after we apply the quotient map (where K is a normal closure of k)

$$\mathrm{Jac}_{\mathbb{Z}}\left(\bigoplus_{\iota} H^1_\iota\right) \twoheadrightarrow \mathrm{Jac}_K\left(\bigoplus_{\iota} H^1_\iota\right)$$

To finish, we prove that it must have been torsion to begin with.

We have the following spaces and the relation between them which will appear in the proof.

$$\begin{tikzcd}
 & \check{W}_0 \arrow[rrr, "\xi"] & & & \mathrm{Jac}(\oplus_\iota H^1_\iota) \arrow[dl] \arrow[d] \arrow[dr] & \\
(W_0\mathcal{M}_{\iota_0})^\perp \arrow[ur, hook] & \cdots \arrow[u] & (W_0\mathcal{M}_\iota)^\perp \arrow[ul, hook] & \mathrm{Jac}_{\iota_0(k)}(H^1_{\iota_0}) & \cdots & \mathrm{Jac}_{\iota(k)}(H^1_\iota)
\end{tikzcd} \tag{2.5.3}$$

To combine the splittings given by Theorem 2.4.2 we need to extend scalars to the field K which contains all the $\iota(k)$. Consider the classifying map obtained from ξ

$$\xi_K : \check{W}_0(K) \to \mathrm{Jac}_K\left(\bigoplus_{\iota} H^1_\iota\right)$$

Given $x = \sum_j c^j r_j \in \Lambda_{\iota_0}$ the splittings from Theorem 2.4.2 say that for any $g \in \mathrm{Gal}(K/\mathbb{Q})$ we have

$$\xi_K(gx)|_{H^1_{g\iota_0}} = 0$$

Recall that to get the splitting we had to apply the Galois action both on H^1 and on W_0.

Assuming $g\iota_0 = \iota_l$ an explicit way to write the above vanishing is

$$\xi_K \left(\sum_j \iota_l(c^j) r_j \right) \Bigg|_{H^1_{\iota_l}} = 0$$

Using the K-linearity of ξ_K we find

$$\sum_j \iota_l(c^j) \xi_K(r_j) \Bigg|_{H^1_{\iota_l}} = 0$$

Taking a direct sum over the different embeddings ι_l gives that $\nu(x) = 0$ in $\mathrm{Jac}_K(\oplus_\iota H^1_\iota)$. To see this, recall that the real multiplication action of k on the factor H^1_ι was by the embedding ι.

To finish, we want to conclude $\nu(x)$ is torsion in the $\mathbb{Z}$-Jacobian. We know that $\nu(x)$ defines a holomorphic section of the corresponding family of abelian varieties. We just proved its image lies in the (group-theoretic) kernel of the map

$$\mathrm{Jac}_{\mathbb{Z}} \left(\bigoplus_\iota H^1_\iota \right) \twoheadrightarrow \mathrm{Jac}_K \left(\bigoplus_\iota H^1_\iota \right) \tag{2.5.4}$$

Viewing the bundle of Jacobians as a flat family of real tori (flat in the sense of the Gauss-Manin connection) we conclude that $\nu(x)$ is itself flat. For this, the continuity of ν would be sufficient.

However, the monodromy acts by parallel transport on a fixed fiber of this family of tori and $\nu(x)$ is invariant by this action. Since the underlying local system over $\mathbb{Q}$ is irreducible (the monodromy acts irreducibly) we conclude that $\nu(x)$ must be among the rational points.

This implies that $\nu(x)$ is a torsion section of the bundle of Jacobians. $\square$

Remark 2.5.3 The splitting over $\iota(k)$ of the mixed Hodge structure obtained in Theorem 2.4.2 is equivalent to the composition (see diagram (2.5.3))

$$(W_0\mathcal{M}_\iota)^\perp \to \check{W}_0 \xrightarrow{\xi} \mathrm{Jac}(\oplus_\iota H^1_\iota) \to \mathrm{Jac}_{\iota(k)}(H^1_\iota) \tag{2.5.5}$$

being the zero map. The torsion condition obtained in Theorem 2.5.2 implies that the twisted cycle map $\nu : \Lambda_{\iota_0} \to \mathrm{Jac}_{\mathbb{Z}}(\oplus_\iota H^1_\iota)$ is torsion. In particular, $A \cdot \nu$ is the zero map for some integer $A \neq 0$. Note that the twisted cycle map ν involves the maps in (2.5.5) ranging over all embeddings ι.

The condition that $A \cdot \nu$ is the zero map implies, in particular, that the composition defined in (2.5.5) is also the zero map, as ι ranges over all embeddings. Indeed, these maps are the components of ν after an extension of scalars to the appropriate number field.

Thus, the torsion condition in Theorem 2.5.2 implies the splitting property in Theorem 2.4.2.

2.5.2 Algebraicity in the general case

Setup. We keep the notation from the previous section. The affine invariant manifold is denoted $\mathcal{M}$, its tangent bundle is $T\mathcal{M}_{\iota_0}$ and $W_0\mathcal{M}_{\iota_0}$ is its intersection with the purely relative subbundle. We have the decomposition of the Hodge bundle

$$H^1 = \left(\bigoplus_\iota H^1_\iota \right) \oplus V$$

Recall also that $\mathcal{M}$ lives in a stratum $\mathcal{H}$ and we have the tautological section

$$\omega : \mathcal{H} \to T\mathcal{H}$$

This section is algebraic and even defined over $\mathbb{Q}$ (see Remark 2.5.6). Combined with the Gauss-Manin connection it gives the local flat coordinates on the stratum $\mathcal{H}$.

Theorem 2.5.4 *The affine invariant manifold $\mathcal{M}$ is a quasi-projective algebraic subvariety* *of $\mathcal{H}$, defined over $\overline{\mathbb{Q}}$.*

Proof. The proof is similar to that of Proposition 2.2.1. In particular, as per Remark 2.1.7 it is carried out in the open subset of $\mathcal{M}$ where there are no self-intersections. We first define an algebraic variety $\mathcal{N}'''$ which has the same properties as $\mathcal{M}$ and contains it. Then we check that one of its irreducible components is contained in $\mathcal{M}$ and thus has to coincide with it.

First, let $\mathcal{N}' \subseteq \mathcal{H}$ be the locus where (X, ω) admits real multiplication of the same type as on $\mathcal{M}$, with ω as an eigenform. On $\mathcal{N}'$ we also have the map ν described in Definition 2.5.1

$$\nu : \Lambda_{\iota_0} \to \mathrm{Jac}_{\mathbb{Z}}\left(\bigoplus_{\iota} H^1_{\iota}\right)$$

By Theorem 2.5.2, over $\mathcal{M}$ the map lands in the torsion so there exists $A \in \mathbb{N}$ such that $A \cdot \nu \equiv 0$ everywhere on $\mathcal{M}$. Let $\mathcal{N}'' \subseteq \mathcal{N}'$ be the sublocus where the equation $A\nu = 0$ holds on $\mathcal{N}'$. This is again algebraic, defined over $\overline{\mathbb{Q}}$, and contains $\mathcal{M}$.

Recall now that we are working in a finite cover where the local system W_0 is trivial and so $W_0\mathcal{M}_{\iota_0}$ is globally defined. Over $\mathcal{N}''$ the condition $A \cdot \nu = 0$ implies (see Remark 2.5.3) that the exact sequence of mixed Hodge structures splits (over $\iota_0(k)$)

$$0 \to W_0/W_0\mathcal{M}_{\iota_0} \to p^{-1}(H^1_{\iota_0})/W_0\mathcal{M}_{\iota_0} \to H^1_{\iota_0} \to 0$$

The splitting subbundle in $p^{-1}(H^1_{\iota_0})/W_0\mathcal{M}_{\iota_0}$ is unique and itself algebraic. Denote this bundle by T'. It also gives a local system and it is the candidate for the tangent space.

Recall we had the tautological section $\omega : \mathcal{H} \to T\mathcal{H}$ and let $\mathcal{N}''' \subseteq \mathcal{N}''$ be the locus where $\omega \in T'$. This is an algebraic variety over $\overline{\mathbb{Q}}$ by construction, and we claim $\mathcal{M}$ coincides with the irreducible component of $\mathcal{N}'''$ containing it.

To see this, first note that $T' = T\mathcal{M}$ on $\mathcal{M}$. However, locally near a point of $\mathcal{M}$ the condition $\omega \in T'$ is the same as being on $\mathcal{M}$. Indeed, requiring ω to lie in some local subsystem is the same as requiring the flat surface to be in a linear subspace in local period coordinates (see Remark 2.2.2).

Finally, note that T' is a subquotient of $T\mathcal{H}$, but the condition $\omega \in T'$ still makes sense. It is understood as ω belonging to the preimage of T' in $T\mathcal{H}$. This completes the proof of algebraicity. $\square$

Remark 2.5.5 Let us explain how affine manifolds are distinguished among loci of real multiplication, since usually these are not $\mathrm{GL}_2^+\mathbb{R}$-invariant.

Suppose given a subvariety $\mathcal{N}' \subset \mathcal{H}$ parametrizing (X, ω) admitting real multiplication by k with ω an eigenform. Given a k-local system $W_0T \subset W_0$ (thinking of it as $W_0\mathcal{M}_{\iota_0}$), further require that the quotient mixed Hodge structure splits as in the above theorem. Let the splitting be given by a bundle (and also local system) denoted T. Finally, require that ω lies in T.

This provides us with a locus $\mathcal{N} \subseteq \mathcal{H}$. Note that the requirement $\omega \in T$ implies that the Zariski tangent bundle of $\mathcal{N}$, denoted $T\mathcal{N}$, is contained in T. The variety $\mathcal{N}$ will also be $\mathrm{GL}_2^+\mathbb{R}$-invariant if and only if we have the equality $T\mathcal{N} = T$.

Remark 2.5.6 (**On fields of algebraic definition**)

(i) For the purposes of this discussion, a variety is "defined over a field K" if it can be described in a projective space as the zero locus of polynomials with coefficients in K. It is quasi-projective if it can be described in a projective space as the locus where a given collection of polynomials vanish, and another collection doesn't vanish. A map between varieties (in particular a section of a bundle) is "defined over K" if it can be described using polynomials with coefficients in K.

(ii) Suppose given varieties $X \subset Y$, with Y is defined over $\mathbb{Q}$ but X apriori only over a finite extension of $\mathbb{Q}$. To check that X is defined over $\mathbb{Q}$, it suffices to check that X is invariant by the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

(iii) An intersection of two varieties (e.g. imposing the condition that some section of a bundle lies in a subbundle) is defined over a field which contains the defining fields for both varieties. To select connected components, one might need to pass to a further finite field extension.

(iv) Given a variety X defined over k (where $k = \mathbb{Q}$ or a finite extension thereof), to find the topological connected components of the complex points $X(\mathbb{C})$ it suffices to pass to a finite extension of k. This follows from two standard facts. First, if a variety Y over $\mathbb{C}$ is irreducible in the sense of algebraic geometry, then its complex points $Y(\mathbb{C})$ are connected (see [Sha94, Ch. VII.2]). Second, if a variety over $\overline{\mathbb{Q}}$ is irreducible, then it is also irreducible when extending scalars to $\mathbb{C}$ (see [Har77, Ex. 3.15]).

Therefore given a variety X over k, extend scalars to $\overline{\mathbb{Q}}$ to obtain the irreducible (over $\overline{\mathbb{Q}}$) components. Moreover, a finite extension of k suffices, since there are finitely many components, and X is assumed of finite type. Finally, the irreducible components over $\overline{\mathbb{Q}}$ agree with the topological components over $\mathbb{C}$.

We can now collect some properties of the varieties which appear in the proofs above.

Proposition 2.5.7

(i) A stratum $\mathcal{H}$ is a (quasi-projective) algebraic variety defined over $\mathbb{Q}$. Moreover, so is any connected component of a stratum.

(ii) The tautological section ω of the Hodge bundle restricted to a stratum is also defined over $\mathbb{Q}$.

(iii) The locus where a factor of the Jacobian had real multiplication by $\mathcal{O}$ with ω as an eigenform is defined over $\mathbb{Q}$. To select a connected component (e.g. $\mathcal{N}'$ appearing in the proof of Theorem 2.5.4) where the real multiplication is as on $\mathcal{M}$, one might need to pass to some finite extension of $\mathbb{Q}$.

Proof. For (i), recall that the moduli space of genus g Riemann surfaces $\mathcal{M}_g$ is defined over $\mathbb{Q}$, and so is the Hodge bundle $\mathcal{H}_g \xrightarrow{\pi} \mathcal{M}_g$. Moreover, the pullback of the Hodge bundle $\pi^*\mathcal{H}_g$ to $\mathcal{H}_g$ has the tautological section $\omega : \mathcal{H}_g \to \pi^*\mathcal{H}_g$, also defined over $\mathbb{Q}$. Next, letting $\mathcal{C}_g \to \mathcal{M}_g$ be the universal bundle of Riemann surfaces, the Hodge bundle admits a map $Div : \mathcal{H}_g \to \mathrm{Sym}^{2g-2}_{\mathcal{M}_g} \mathcal{C}_g$ which takes a 1-form to its zero locus. The space $\mathrm{Sym}^{2g-2}_{\mathcal{M}_g} \mathcal{C}_g$ parametrizes $2g-2$-tuples of (not necessarily distinct) points on the same fiber of the universal family. This space is defined over $\mathbb{Q}$ and has a stratification, also defined over $\mathbb{Q}$, depending on multiplicities. A stratum of $\mathcal{H}_g$ is then the preimage of one of the strata on $\mathrm{Sym}^{2g-2}_{\mathcal{M}_g} \mathcal{C}_g$.

To distinguish connected components of strata, one might apriori need to extend the base field $\mathbb{Q}$. However, from the classification of connected components due to Kontsevich & Zorich [KZ03], each connected component can be described by an algebraic condition also defined over $\mathbb{Q}$. Indeed, the spin invariant of a square root of the canonical bundle is invariant by Galois conjugation, and so is the property of being hyperelliptic.

For (ii) recall the stratum carries a universal family of Riemann surfaces, and also the canonical set of marked points corresponding to the zeros of ω. The vector bundle H^1_{rel} has a description as the algebraic de Rham cohomology of this family of Riemann surfaces, and is thus itself defined over $\mathbb{Q}$. The natural map from the Hodge bundle $\pi^*\mathcal{H}_g$ to H^1_{rel} is also defined over $\mathbb{Q}$, and so $\omega : \mathcal{H} \to H^1_{rel} \cong T\mathcal{H}$ is defined over $\mathbb{Q}$. See also [Möl08, Section 2] for a detailed discussion of these constructions.

For (iii), note that the condition of having real multiplication by $\mathcal{O}$ (i.e. having a map $\mathcal{O} \to \mathrm{End}(A)$) is invariant by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so the locus is defined over $\mathbb{Q}$. The eigenform condition is also Galois-invariant since the tautological section ω is defined over $\mathbb{Q}$. Note that this argument applies to the locus in $\mathcal{A}_g$, but also to its lift inside $\mathbb{P}H^{1,0}$ – the eigenform locus inside the projectivization of the Hodge bundle over $\mathcal{A}_g$. $\square$

The above discussion explains why affine invariant manifolds are quasi-projective varieties defined over $\overline{\mathbb{Q}}$. Acting by the Galois group of $\mathbb{Q}$ on the defining equations inside $\mathcal{H}$ produces new quasi-projective varieties. These will also be affine invariant manifolds.

Corollary 2.5.8 *The group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on the set of affine invariant manifolds.*

Proof. From the proof of Theorem 2.5.4 (see also the discussion in Remark 2.5.5) an affine manifold $\mathcal{M}$ is defined by the following set of algebraic conditions.

- The parametrized Riemann surfaces have a factor with real multiplication by $\mathcal{O}$, and ω as an eigenform.

- There is a sublocal system $\Lambda_{\iota_0} \subset \check{W}_0$ such that its image under the twisted Abel-Jacobi map is torsion of some fixed degree, i.e. $A \cdot \nu \equiv 0$ for some integer $A \neq 0$.

- There is an upper bound for the dimension of this locus inside the stratum $\mathcal{H}$ (computed from the ranks of the objects above) and the dimension of $\mathcal{M}$ achieves this bound.

Each of the above conditions persists when the Galois group of $\mathbb{Q}$ acts on the defining equations. Note that ω is defined with $\mathbb{Q}$-coefficients, and so if it is an eigenform on one locus, it will also be an eigenform in the Galois-conjugate locus.

Next, to define the locus $\mathcal{N}''$ by imposing the torsion condition $A \cdot \nu \equiv 0$ required a finite cover where the zeros are labeled. This requires another finite extension of the base field. $\square$

Remark 2.5.9

(i) Certain invariants of the affine manifold stay constant under Galois conjugation. For example, the ranks of vector bundles stay the same and therefore so do the rank of $\mathcal{M}$ (i.e. $\frac{1}{2}\dim\ p\,(T\mathcal{M})$), its dimension, and the number of torsion conditions. Moreover, the stratum, and even the connected component of the stratum, do not change, since these are (algebraically) defined over $\mathbb{Q}$.

The order $\mathcal{O}$ giving real multiplication stays the same, since it is the data of a map $\mathcal{O} \to \mathrm{End}\left(\mathrm{Jac}\left(\oplus_\iota H^1_\iota\right)\right)$. In particular, the field of affine definition, cutting out $\mathcal{M}$ in period coordinates, does not change either. However, the $\mathcal{O}$-module structure on the integer $\mathbb{Z}$-lattice inside $\oplus_\iota H^1_\iota$ can change.

(ii) The intersection of connected varieties need not be connected. For example intersecting a stratum with an eigenform locus might lead to several components. This occurs for some of the Calta-McMullen curves, intersecting $\mathcal{H}(2)$ with the locus of real multiplication by $\mathcal{O}_D$ when $D \equiv 1 (\mathrm{mod}\ 8)$. This gives two components, each defined over $\mathbb{Q}(\sqrt{D})$ (see e.g. [MZ15, Thm. 5.4]). Note that the "spin" invariant used by McMullen [McM05] to distinguish them is thus not Galois invariant.

Remark 2.5.10 The results of Wright [Wri14] show that in local period coordinates on H^1_{rel} an affine manifold is described by linear equations with coefficients in the number field k. These equations and field of definition are usually not related to the algebraic equations and respective fields. In particular, acting by the Galois group on the linear equations would typically not produce another affine manifold.

For comparison, the field of affine definition of square-tiled surfaces is $\mathbb{Q}$. However, algebraically these can be quite rich – Möller [Möl05, Thm. 5.4] showed the action of the Galois group is faithful on the corresponding Teichmüller curves.

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