Supplemental Material
to
"Neutron stars as photon double-lenses: constraining resonant conversion into ALPs"

Kyrylo Bondarenko,[1,2,3,*] Alexey Boyarsky,[4,†] Josef Pradler,[5,6,7,‡] and Anastasia Sokolenko[8,9,§]

[1]*IFPU, Institute for Fundamental Physics of the Universe, via Beirut 2, I-34014 Trieste, Italy*
[2]*SISSA, via Bonomea 265, I-34132 Trieste, Italy*
[3]*INFN, Sezione di Trieste, SISSA, Via Bonomea 265, 34136, Trieste, Italy*
[4]*Institute Lorentz, Leiden University, Niels Bohrweg 2, Leiden, NL-2333 CA, the Netherlands*
[5]*Institute of High Energy Physics, Austrian Academy of Sciences, Georg-Coch-Platz 2, 1010 Vienna, Austria*
[6]*University of Vienna, Faculty of Physics, Boltzmanngasse 5, A-1090 Vienna, Austria*
[7]*CERN, Theoretical Physics Department, 1211 Geneva 23, Switzerland*
[8]*Theoretical Astrophysics Department, Fermi National Accelerator Laboratory, Batavia, Illinois, 60510, USA*
[9]*Kavli Institute for Cosmological Physics, The University of Chicago, Chicago, IL 60637, USA*

I. RESONANCE CONDITION IN STRONG MAGNETIC FIELD

We start from Maxwell's equations supplied with the external current $\vec{J} = g_{a\gamma}\vec{B}_0\partial_t a$ where $\vec{B}_0$ is the external magnetic field,

$$\nabla \times \vec{E} + \partial_t \vec{B} = 0, \quad \vec{D} = \boldsymbol{\epsilon}\cdot\vec{E}, \tag{1}$$

$$\nabla \times \vec{H} - \partial_t \vec{D} = g_{a\gamma}\vec{B}_0\partial_t a, \quad \vec{B} = \boldsymbol{\mu}\vec{H}. \tag{2}$$

Here, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the dielectric and magnetic permeability tensors. In a coordinate system in which the external $\vec{B}_0$ field is in z-direction, the dielectric and magnetic permeability tensors take the following form,

$$\boldsymbol{\epsilon}_{\vec{B}_0\|\hat{z}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\omega_p^2/\omega^2 \end{pmatrix} - A\begin{pmatrix} 2\vec{B}_0^2 & 0 & 0 \\ 0 & 2\vec{B}_0^2 & 0 \\ 0 & 0 & -5\vec{B}_0^2 \end{pmatrix}. \tag{3}$$

The first contribution is the plasma effect, where we neglect relativistic corrections. As we shall see below, our results are robust to restricting ourselves to spatial regions that are not in the immediate vicinity of the neutron star surface where relativistic electron populations occur. The second contribution is the vacuum polarization contribution to the dielectric tensor with $A = 4\alpha^2/45m_e^4$.

In addition, the magnetic permeability tensor reads $\boldsymbol{\mu} = (1 + 2AB_0^2)\mathbf{1}_{3\times3}$.

Taking the time dependence of the axion and electric field as $e^{-i\omega t}$, and neglecting any time derivatives of $\vec{B}_0$, $\boldsymbol{\epsilon}$ and μ the Maxwell equations can be combined. Supplied with the Klein-Gordan equation for the axion, the equations to solve are,

$$-\omega^2(\boldsymbol{\epsilon}\cdot\vec{E}) + \nabla\times[\boldsymbol{\mu}^{-1}(\nabla\times\vec{E})] = g_{a\gamma}\omega^2\vec{B}_0 a, \tag{4}$$

$$\left(-\omega^2 - \nabla^2 + m_a^2\right)a = g_{a\gamma}\vec{E}\cdot\vec{B}_0, \tag{5}$$

If we now switch to a coordinate system where the photon propagates in the z-direction, magnetic field has angle θ to the photon direction and is in xz-plane, that requires a rotation of the magnetic field and the dielectric tensor as

$$\vec{B}_0 = B_0 R(\theta)\hat{z} = B_0\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}, \quad \boldsymbol{\epsilon} = R(\theta)\cdot\boldsymbol{\epsilon}_{\vec{B}_0\|\hat{z}}\cdot R(-\theta),$$

* kyrylo.bondarenko@cern.ch
† boyarsky@lorentz.leidenuniv.nl
‡ josef.pradler@univie.ac.at
§ sokolenko@kicp.uchicago.edu

with the rotation matrix given by

$$R(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Since $\boldsymbol{\mu}$ is diagonal, it remains invariant under the rotation and its inverse is simply given by $\boldsymbol{\mu}^{-1} = (1+2AB_0^2)^{-1}\mathbf{1}_{3\times3}$. The expression for the dielectric tensor becomes,

$$\boldsymbol{\epsilon} = \begin{pmatrix} 1-\sin^2\theta(\omega_p^2/\omega^2) & 0 & -\sin\theta\cos\theta(\omega_p^2/\omega^2) \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta(\omega_p^2/\omega^2) & 0 & 1-\cos^2\theta(\omega_p^2/\omega^2) \end{pmatrix} + \frac{1}{2}AB_0^2 \begin{pmatrix} 3-7\cos 2\theta & 0 & 14\sin\theta\cos\theta \\ 0 & -4 & 0 \\ 14\sin\theta\cos\theta & 0 & 3+7\cos 2\theta \end{pmatrix}.$$

We see that E_y is not directly coupled to the axion since $B_{0y} = 0$ and it is only generated through further spatial derivatives, which makes it suppressed. In the following we hence treat the reduced problem with the y-components neglected. If we are in addition to neglecting E_y also neglect any second order derivatives or products of first order derivatives that do not include ∂_z, we arrive at the following equations

$$-\omega^2(\epsilon_{xx}E_x + \epsilon_{xz}E_z) - \mu_{yy}^{-1}(\partial_z^2 E_x - \partial_x\partial_z E_z) - (\partial_z\mu_{yy}^{-1})(\partial_z E_x - \partial_x E_z) = g_{a\gamma}\omega^2 B_{0x} a, \tag{6}$$

$$-\omega^2(\epsilon_{zx}E_x + \epsilon_{zz}E_z) + \mu_{yy}^{-1}\partial_x\partial_z E_x + \partial_x\mu_{yy}^{-1}\partial_z E_x = g_{a\gamma}\omega^2 B_{0z} a, \tag{7}$$

where $\mu_{xx} = (\boldsymbol{\mu}^{-1})_{xx}$ and so forth. We may now use the second equation to express E_z

$$E_z = -\frac{\epsilon_{zx}}{\epsilon_{zz}}E_x + \frac{1}{\epsilon_{zz}\omega^2}\left[-g_{a\gamma}\omega^2 B_{0z} a + \mu_{yy}^{-1}\partial_x\partial_z E_x + (\partial_x\mu_{yy}^{-1})\partial_z E_x\right]. \tag{8}$$

The resulting expression is rather cumbersome. Following [1] in a number of approximations, neglecting any external electric fields as well as any derivatives of B_0^2 (which also amounts to neglecting derivatives of μ) then we arrive at the following expression in a generalization of [1],

$$-\partial_z^2 E_x + \frac{(\omega_p^2 - 7AB_0^2\omega^2)\sin 2\theta}{\omega^2 - \omega_p^2\cos^2\theta + \frac{1}{2}AB_0^2\omega^2(7\cos 2\theta + 3)}\partial_x\partial_z E_x + \frac{\omega^2\omega_p(1-2AB_0^2)\sin 2\theta}{\left[\omega^2 - \omega_p^2\cos^2\theta + \frac{1}{2}AB_0^2\omega^2(7\cos 2\theta+3)\right]^2}(\partial_z E_x\partial_x\omega_p + \partial_x E_x\partial_z\omega_p) \tag{9}$$

$$= \frac{(1-4A^2B_0^4)\omega^2\left(\omega^2 - \omega_p^2 + 5AB_0^2\omega^2\right)}{\omega^2 - \omega_p^2\cos^2\theta + \frac{1}{2}AB_0^2\omega^2(7\cos 2\theta+3)}E_x + B_0 g_{a\gamma} a \frac{(1-4A^2B_0^4)\omega^4\sin\theta}{\omega^2 - \omega_p^2\cos^2\theta + \frac{1}{2}AB_0^2\omega^2(7\cos 2\theta+3)}. \tag{10}$$

The factors $(1-4A^2B_0^4)$ are of higher order, originating from the the product of elements of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}^{-1}$ and we will neglect them. We now consider the propagation in the z-direction and an ansatz for the envelopes of the electric and axion fields (time dependence as introduced above)

$$E_x \equiv \tilde{E}_x(x,z)e^{i(\omega t - kz)}, \quad a \equiv \tilde{a}(z)e^{i(\omega t - kz)}. \tag{11}$$

Following the WKB approximation detailed in [1] we arrive at

$$2ik\partial_z\tilde{E}_x - 2ik\frac{(\omega_p^2 - 7A\omega^2 B_0^2)\xi}{\omega^2\tan\theta}\partial_x\tilde{E}_x \simeq \left[m_a^2 - \omega^2 + \xi\csc^2\theta(\omega^2 - \omega_p^2 + 5A\omega^2B_0^2) + ik\mathcal{D}\right]\tilde{E}_x + \frac{\omega^2\xi}{\sin\theta}g_{a\gamma}\tilde{a}B_{\rm NS} \tag{12}$$

Here we defined,

$$\xi \equiv \frac{\omega^2\sin^2\theta}{\omega^2 - \omega_p^2\cos^2\theta + \frac{1}{2}AB_0^2\omega^2(7\cos 2\theta+3)}, \quad \mathcal{D} = \frac{2\omega_p\xi^2(1-2AB_0^2)}{\omega^2\tan\theta\sin^2\theta}\partial_x\omega_p. \tag{13}$$

Defining the differential operator

$$\partial_s = \partial_z - \frac{(\omega_p^2 - 7A\omega^2B_0^2)\xi}{\omega^2\tan\theta}\partial_x, \tag{14}$$

we obtain the Schroedinger type equation,

$$i\partial_s \tilde{E}_x = \frac{1}{2k}\left\{\left[m_a^2 - \omega^2 + \xi \csc^2\theta(\omega^2 - \omega_p^2 + 5A\omega^2 B_0^2) + ik\mathcal{D}\right]\tilde{E}_x + \frac{\omega^2\xi}{\sin\theta} g_{a\gamma}\tilde{a}B_{\rm NS}\right\} \tag{15}$$

This equation can be solved in the stationary phase approximation which yields the resonance condition. Neglecting an overall phase, the solution reads,

$$i\tilde{E}_x(s) = \frac{1}{2k}\int_0^s ds' \,\frac{\omega^2\xi}{\sin\theta} g_{a\gamma}\tilde{a}B_{\rm NS}\exp\left\{\frac{1}{2}\int_0^{s'} ds''\mathcal{D}\right\}\exp\left\{-i\int_0^{s'} ds''\,\frac{1}{2k}\left[m_a^2 - \omega^2 + \xi\csc^2\theta(\omega^2 - \omega_p^2 + 5A\omega^2 B_0^2)\right]\right\} \tag{16}$$

Let us first solve for $A = 0$, i.e., without vacuum polarization. Then

$$f(s) = -\int_0^s \frac{1}{2k}\left[m_a^2 - \xi\omega_p^2\right] \quad\Rightarrow\quad \partial_s f(s) = -\frac{1}{2k}\left[m_a^2 - \xi\omega_p^2\right] \overset{s=s_0}{=} 0 \quad (A=0) \tag{17}$$

The latter relation gives the stationary point s_0 with the resonance condition, $m_a^2 = \xi\omega_p^2$ reproducing the expression [1],

$$\omega_p^2 = \frac{m_a^2\omega^2}{m_a^2\cos^2\theta + \omega^2\sin^2\theta} = \frac{2m_a^2\omega^2}{m_a^2 + \omega^2 + (m_a^2 - \omega^2)\cos 2\theta} \quad (A=0) \tag{18}$$

In the current context, this is being replaced by the following resonance condition,

$$m_a^2 \simeq \omega^2 - \left[\frac{\cos^2\theta}{\omega^2\,(1 - 2AB_0^2)} + \frac{\sin^2\theta}{\omega^2(1 + 5AB_0^2) - \omega_p^2}\right]^{-1} \tag{19}$$

For the special case of $\theta = \pi/2$ this becomes,

$$m_a^2 \simeq \omega_p^2 - \frac{4\alpha^2\omega^2 B_0^2}{9m_e^4}. \tag{20}$$

Comparing the negative contribution on the right-hand side with the known correction to the effective photon mass [2] $m^2_{\rm eff}|_{\rm vac.pol.} = -(44/135)\alpha^2\omega^2 B_0^2/m_e^4$ we see that the numerical factor is different by 35%. This difference is expected, because the negative contribution to the photon mass in [2] was calculated for the isotropic case, where electrons are free to move in all three spatial directions, while in the anisotropic case considered here, electrons only move along magnetic field lines. Because of this, we retain an angular dependence in Eq. (19) for the resonant condition.

An important question in relation to our work is to what extend a detailed treatment such as the one above modifies the simple assumptions taking in the paper. For obtaining an answer, let us consider $m_a \ll \omega_p$, such that resonant conversion is only possible if a cancellation between plasma frequency and the negative contribution from the magnetic field happens (termed "double lens effect" in the main paper). Neglecting the m_a^2 term in (19), one can rewrite the equation as

$$\omega_p^2 \simeq (5 - 2\cot^2\theta)A\omega^2 B_0^2. \tag{21}$$

We see that such cancellation is possible if $\cot^2\theta < 5/2$ or

$$\theta_{\rm cr} < \theta < 180^\circ - \theta_{\rm cr}, \qquad \theta_{\rm cr} = \arctan\left(\sqrt{2/5}\right) \approx 32.3^\circ. \tag{22}$$

Hence, in a wide range of θ the resonance condition is approximately given by (20). For example, the condition (21) coincides with (20) with a precision of 20% if $55^\circ < \theta < 125^\circ$.

II. INFLUENCE OF THE PLACE OF PHOTON CREATION

One of the important assumptions of our paper was the creation of the radio photon close to the surface of the magnetar. In this section, we analyze the influence of this assumption on the sensitivity region of the axion-photon conversion (Fig. 3 in the main text) and find that the dependence of the lower bound of this region on the radius of the radio signal creation is surprisingly weak (for $m_a^2 \ll \omega_p^2$).

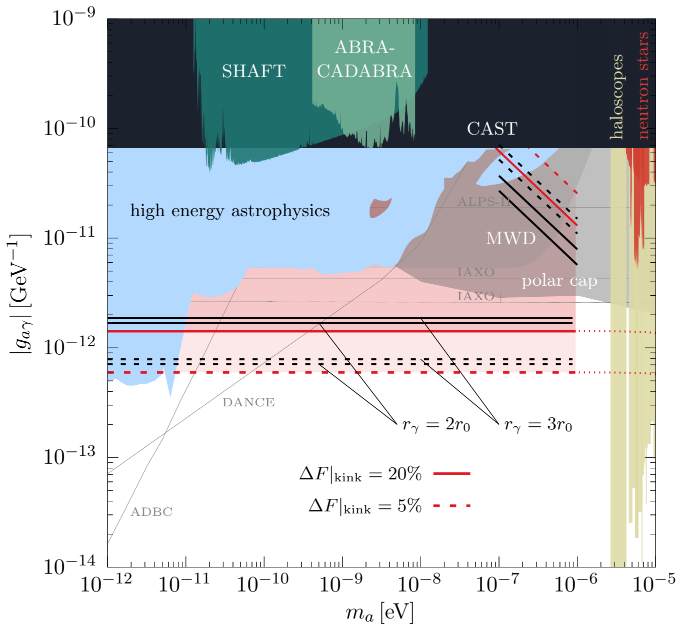

FIG. 1. Dependence of sensitivity on the assumed location of radio-wave creation. In the main text it is assumed that it happens close to the surface. If the radio signal is instead created at a significant distance $r_\gamma = 2r_0$ or $r_\gamma = 3r_0$ from the NS surface (as labeled), the lower boundaries of the sensitivities change only very mildly. We also show an estimate for the change of the upper bound.

In this section we will use the same assumption as in the main paper: the magnitude of the magnetic field decays as a power law,

$$B(r) = B_0 \left(\frac{r_0}{r}\right)^3, \tag{23}$$

we will use average values for trigonometric function (assuming isotropic distribution), and electron number density is given by Goldreich-Julian model, $n_e(r) = \Omega B(r)/e$.

The experimental signature that we found from the double-lens effect is a kink in the spectrum of the radio waves. It appears when B_+ solution of the resonance condition is equal to the maximal value $B_{\rm max}$ of the magnetic field in the region where radio waves are created. Let us assume that radio waves are created at some radius $r_\gamma > r_0$. In this case $B_{\rm max} = B_0(r_0/r_\gamma)^3$, while kink in the spectrum appear at energy

$$\omega_{\rm kink} \approx \sqrt{\frac{C_1\langle|\cos\theta|\rangle}{C_2 B_{\rm max}}}, \qquad m_a^2 \ll \omega_p^2. \tag{24}$$

The bound of the sensitivity is defined by the condition that the probability of the resonant conversion is equal to some minimal value, $P_{\rm tot} = P_{\rm min}$. This corresponds to the condition on the $P_{\rm lin}$,

$$P_{\rm lin} = -\frac{1}{3}\ln\left(1 - 3P_{\rm min}\right). \tag{25}$$

At the lower bound of sensitivity, the contribution of B_+ resonance dominates over B_-, so we can neglect the latter. The maximal linear probability is at $r = r_\gamma$ and it is given by

$$P_{\rm lin} = \frac{\pi g_{a\gamma}^2 \omega_{\rm kink}}{m_a^2}\langle\sin^2\theta\rangle B_{\rm max}^2 R, \tag{26}$$

so the minimal coupling constant the one can probe is

$$|g_{a\gamma}|_{\rm min} = \sqrt{\frac{P_{\rm lin} m_a^2}{\pi\langle\sin^2\theta\rangle\omega_{\rm kink} B_{\rm max}^2 R}}. \tag{27}$$

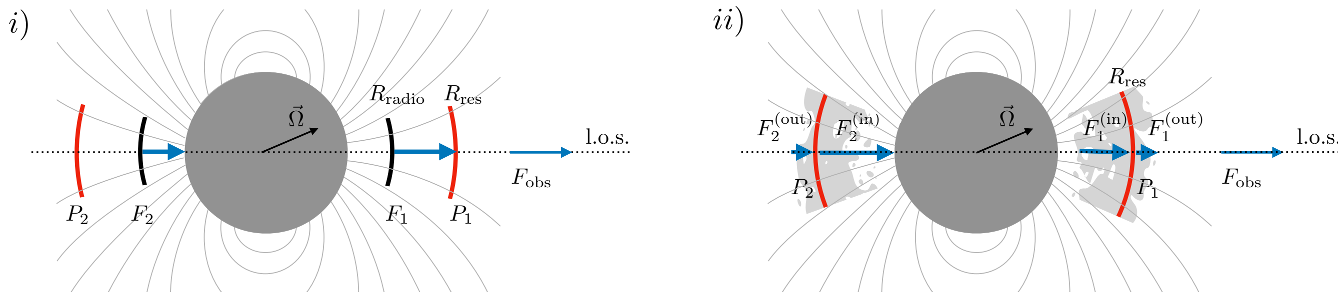

FIG. 2. Two principal options for the location of the resonance surface with respect to the surface of the radio photon creation. The left panel corresponds to our main assumption $R_{\rm radio} < R_{\rm res}$, while on the right panel radio signal is created in the extended gray area.

The dependence of the R-factor on radius is quite nontrivial. We remind that

$$R = \left| \frac{d \ln m_{\rm eff}^2}{dr} \right|^{-1}_{r=r_\gamma}, \qquad m_{\rm eff}^2 = \omega_p^2 - C_2 \omega^2 B^2. \tag{28}$$

Using this definition and condition of the resonance $m_a^2 = m_{\rm eff}^2$ one can obtain the following expression for R-factor,

$$R = \frac{m_a^2}{\omega_p^2} \left| \frac{d \ln B}{dr} \right|^{-1}_{r=r_\gamma}, \qquad m_a^2 \ll \omega_p^2. \tag{29}$$

Taking into account that $\omega_p^2 \propto B_{\rm max}$ we get

$$|g_{a\gamma}|_{\rm min} \propto \frac{1}{B_{\rm max}^{1/4}} \left| \frac{d \ln B}{dr} \right|^{1/2}_{r=r_\gamma} \propto r_\gamma^{1/4}. \tag{30}$$

We see that in our model the dependence of the lower bound on r_γ is quite weak. Even if radio emission is created at $r_\gamma = 10 r_0$ the lower bound is relaxed only by a factor $10^{1/4} \approx 1.8$. To illustrate this point we show in Fig. 1 the change of the region of sensitivity for $r_\gamma = 2 r_0$ and $3 r_0$.

III. FAR SIDE CONTRIBUTION TO THE SIGNAL

Here we discuss the dependence of the signal prediction on the typical radius where the original radio signal is created, $R_{\rm radio}$ vs. the location of the B_+ resonance, $R_{\rm res}$ for which we include a discussion on the potential contribution from the far side of the NS. We take a simplistic picture of spherical symmetry. This will make the arguments easily tractable and maximize canceling effects that are rooted in the geometry. There are various configurations possible, and in Fig. 2 we highlight two principial ones. Left panel *i*) corresponds to our principal assumption that $R_{\rm radio} < R_{\rm res}$. The right panel *ii*) highlights a situation where the radio emission occurs in an extended zone indicated by the gray regions. Throughout we assume it to be a localized phenomenon in the sense that we are not required to perform integrals over extended angular regions.

In case *i*) when $R_{\rm radio} < R_{\rm res}$ an emitted radio flux F_1 at radius $R_{\rm radio}$ becomes processed at $R_{\rm res}$ above a threshold frequency with conversion probability P_1. The observed flux is then $F_{\rm obs} = F_1 (1 - P_1)$. As a comparative measure to other cases, we may consider the flux decrement at some characteristic frequency, i.e., by comparing the flux without conversion $F_{\rm obs}^{(P=0)}$ with the flux that would emanate when resonant conversion is possible, $F_{\rm obs}^{(P\neq 0)}$,

$$\Delta F \equiv F_{\rm obs}^{(P\neq 0)} - F_{\rm obs}^{(P=0)} = -P_1 F_1. \tag{31}$$

This is the flux-decrement that we advertise as a signature in the main text. Under the current circumstances there is no contribution from the "far side".

The situation changes when we reverse the roles of both radii, and assume $R_{\rm radio} > R_{\rm res}$. As is evident, no feature can be imprinted from the radio component that reaches us from the near side of the neutron star. However, denoting far side quantities by index “2”, the observed flux is $F_{\rm obs} = F_1 + F_2 P_2 P_1$ where the second term is from axions that traverse the NS before being regenerated as photons and a non-vanishing flux *increment* is present even in this case,

$$\Delta F = P_1 P_2 F_2. \tag{32}$$

We may now turn to case *ii*) with an extended zone of radio emission with the radius of resonance situated within this zone. Then we have two contributions to the unprocessed radio signal on each side. One generated within $r < R_{\rm res}$ denoted by $F^{\rm (in)}$ and one generated in the exterior $r > R_{\rm res}$ of the resonance zone, $F^{\rm (out)}$. The observed flux then becomes $F_{\rm obs} = F_1^{\rm (out)} + F_1^{\rm (in)}\,[1 - P_1] + F_2^{\rm (out)} P_2 P_1$. The change in photon flux then has contributions of both signs,

$$\Delta F = -F_1^{\rm (in)} P_1 + F_2^{\rm (out)} P_2 P_1. \tag{33}$$

The far side contribution may hence “wash out” the signal or, in a fine-tuned situation, cancel it altogether.

Finally, we mention that we assumed a geometric “obscuration limit” for the far side. However, if $R_{\rm NS} \ll R_{\rm radio}$, $R_{\rm res}$ both near and far side contribute on similar footing. This limit is of less interest for the current purposes as we rely on the resonance B_+ that is found in vicinity of the NS surface. Therefore, as stated in the main text, our proposal is sensitive to the assumption that $R_{\rm radio}$ is of similar order than $R_{\rm NS}$.

[1] A. J. Millar, S. Baum, M. Lawson, and M. C. D. Marsh, *Axion-photon conversion in strongly magnetised plasmas*, *JCAP* **11** (2021) 013, [arXiv:2107.07399].

[2] A. Dobrynina, A. Kartavtsev, and G. Raffelt, *Photon-photon dispersion of TeV gamma rays and its role for photon-ALP conversion*, *Phys. Rev. D* **91** (2015) 083003, [arXiv:1412.4777]. [Erratum: Phys.Rev.D 95, 109905 (2017)].