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ABSTRACT

This thesis deals with two problems in percolation theory.

In the first part, we consider accessibility percolation on hypercubes, i.e., we place i.i.d. uniform $[0,1]$ random variables on vertices of a hypercube, and study whether there is a path connecting two vertices such that the values of these random variables increase along the path. We establish a sharp phase transition depending on the difference of the values at the two endpoints, and determine the critical window of the phase transition. Our result completely resolves a conjecture of Berestycki, Brunet and Shi (2014).

Our work on accessibility percolation is motivated by the NK fitness model in biological evolution. We also establish the asymptotics for the global maximum of the NK fitness model, by proving that the maximum is asymptotically equivalent to the case when $K = N$ if and only if $K \to \infty$ as $N \to \infty$.

In the second part, we initiate the study on chemical distances of percolation clusters for level sets of two-dimensional discrete Gaussian free fields as well as loop clusters generated by two-dimensional critical random walk loop soups. We show that in both cases the chemical distance between two macroscopic annuli away from the boundary is of dimension 1 with positive probability. Our proof method is based on an interesting combination of a theorem of Makarov, isomorphism theory and an entropic repulsion estimate for Gaussian free fields in the presence of a hard wall.

CHAPTER 1

PHASE TRANSITION FOR ACCESSIBILITY PERCOLATION ON HYPERCUBES

1.1 Introduction

For $N \in \mathbb{N}$, let $H_N = \{0,1\}^N$ be a hypercube where two vertices are connected by an *undirected* edge if their Hamming distance, i.e. the number of coordinates at which they differ, is precisely 1. Let $\{X_v : v \in H_N\}$ be i.i.d. random variables uniformly distributed in $[0,1]$. We say that a path in H_N is accessible if the associated random variables X_v's are increasing along the path. For $u, w \in H_N$, we say that w is accessible from u if there exists at least one accessible path from u to w. In this paper, we show that the conditional accessible probability (from u to w) given that $X_u = a$ and $X_w = b$ $(0 \le a < b \le 1)$ admits a sharp phase transition, in a sense made precise in Theorem 1.1 below. By symmetry, the conditional accessible probability with fixed a and b depends only on the Hamming distance between u and w. Therefore, we fix $0 < \beta \le 1$ and without loss of generality consider the case when $u = (0,0,\cdots,0)$ and $w = (1,1,\cdots,1,0,0,\cdots,0)$ (here the number of 1's in w is $[\beta N]$). Furthermore, since subtracting a from all X_v's does not change the accessibility from u to w, we can also assume without loss of generality that $a = 0$ and $b = x$ (where x may depend on N). Our main result is summarized in the following theorem.

Theorem 1.1. *Let $f(x) = (\sinh x)^\beta (\cosh x)^{1-\beta}$, and let x_0 be the unique number such that $f(x_0) = 1$. Define $x_c(N) = x_0 - \frac{1}{f'(x_0)} \frac{\ln N}{N}$. For any sequence ϵ_N such that $N\epsilon_N \to \infty$, we have*

$$\lim_{N\to\infty} \mathbb{P}(w \text{ is accessible from } u \mid X_u = 0, X_w = x_c - \epsilon_N) = 0\,, \tag{1.1}$$

$$\lim_{N\to\infty} \mathbb{P}(w \text{ is accessible from } u \mid X_u = 0, X_w = x_c + \epsilon_N) = 1\,. \tag{1.2}$$

In addition, for all $\Delta > 0$, there exist $0 < c_1 < c_2 < 1$ (where c_1 and c_2 depend only on Δ)

such that for all $N \in \mathbb{N}$

$$c_1 \leq \mathbb{P}(w \text{ is accessible from } u \mid X_u = 0, X_w = x_c + \epsilon_N) \leq c_2\,, \text{ if } |N\epsilon_N| \leq \Delta\,. \tag{1.3}$$

Remark 1.1. *A few days before the post of this article, we noted that a paper [16] was posted in January 2015, which proved the version of* (1.2) *(without analyzing the critical window for the phase transition) for the case of* $\beta \geq 0.002$*. While we acknowledge the priority of [16], we emphasize that our work was carried out independently; our method is rather different and allows us to derive the result for all* $0 < \beta \leq 1$*.*

Accessibility percolation on hypercubes with backsteps (i.e., when the hypercube graph is undirected as we have assumed at the beginning) was studied in [1], where they proved (1.1) and conjectured (1.2) (both in a slightly weaker form). Our Theorem 1.1 completes the picture and describes a sharp phase transition for this problem.

An analogue of Theorem 1.1 on accessibility percolation on hypercubes without backsteps (i.e., when the edges of the hypercube are directed toward the vertex with the greater number of ones) was established by [10]. Under the same setting, [2] gives the asymptotic distribution of the number of accessible paths when x is in a different regime. Accessibility percolation has also been studied on N-ary trees [17, 19, 5] and on spherically symmetric trees [7]. In addition, the Hamiltonian increasing path on the complete graph was studied in [14].

Our study on accessibility percolation is motivated by the NK fitness landscapes, which were introduced in [12, 13] as a class of models for biological evolution. In the NK fitness model, we consider H_N corresponding to, e.g., nucleobases in a DNA sequence. Let F be a distribution. Given $K \leq N$, let $Y_{i,\tau}$ be i.i.d. random variables with distribution F for all $1 \leq i \leq N$ and $\tau \in H_K$. For $\sigma \in H_N$, the fitness of σ is then defined to be $X_\sigma = \sum_{i=1}^{N} Y_{i,(\sigma_i,\ldots,\sigma_{i+K-1})}$ (where the addition in the subscript is understood as modulo of N). Since the gene favors better fitness, it is natural to consider an adaptive walk on space H_N such that the corresponding fitness increases until the walk is frozen at a local maximum.

Theorem 1.1 is a preliminary step toward understanding the adaptive walk on the NK fitness model. Indeed, our model (with i.i.d. fitness for each vertex in H_N) corresponds to the case when $K = N$ (the distribution F does not play a role when considering increasing paths as long as F is continuous).

We note that despite intensive research in theoretical biology as well as physics, there were few mathematical results [9, 8, 15, 4] on NK fitness models. In [9], some asymptotic features of NK fitness landscapes are reduced to questions about eigenvalues and Lyapunov exponents; in [8, 15], estimates on the cardinality of local maxima was provided; in [4], certain structural properties of the maxima for NK fitness model was given. We establish the asymptotics for the global maximum of NK fitness model, by proving that the maximum is asymptotically equivalent to the case when $K = N$ if and only if $K \to \infty$ as $N \to \infty$.

Theorem 1.2. *Let Y be a random variable with distribution F. Assume that F possesses super-exponential tails: $\mathbb{E}(e^{\lambda Y}) =: e^{\Lambda(\lambda)} < \infty, \lambda \in \mathbb{R}$. Let $I(x) = \sup_{\lambda\in\mathbb{R}}(\lambda x - \Lambda(\lambda))$. Set x^* to be the unique point so that $x^* > \mathbb{E}(Y)$ and $I(x^*) = \log 2$. See e.g. [20] for the above assumption. Let $M_{N,K} := \max_{\sigma\in H_N} X_\sigma$ be the global maximum of NK fitness model.*

(a) If $K \to \infty$ as $N \to \infty$, then we have $\lim_{N\to\infty} \frac{\mathbb{E}M_{N,K}}{N} = x^*$.

(b) If $K \le K_0 < \infty$ for all N, then we have $\limsup_{N\to\infty} \frac{\mathbb{E}M_{N,K}}{N} < x^*$.

1.2 Accessibility percolation: antipodal case

For clarity of presentation, in the current section we give a proof of Theorem 1.1 in the antipodal case when $\beta = 1$, i.e., when $u = \vec{0} = (0, 0, \cdots, 0)$ and $w = \vec{1} = (1, 1, \cdots, 1)$. In Section 1.3, we modify the arguments and give a proof of Theorem 1.1 in the general case when $0 < \beta < 1$. In both sections, the probability measure $\mathbb{P}$ stands for the conditional probability given $X_u = 0$ and $X_w = x$, unless otherwise specified. Recall that a path from u to w is accessible if the X_v's (including X_u and X_w) along the path are increasing. Denote by $Z_{N,x}$ the number of such accessible paths. Throughout the paper, we sometimes write *with high probability* for brevity to mean with probability tending to 1 as $N \to \infty$.

1.2.1 Proof of the upper bound

In this subsection we give a proof of (1.1) in the antipodal case (the general case is similar). Note that Lemma 1.2 below (which implies (1.9) in Corollary 1.1 and therefore (1.1) in the general case) has already been proved in [1]. Here we give a different proof of Lemma 1.2, by relating the original model to a more tractable one (i.e. $\mu_{k,n}$), and this connection will also be useful in later proofs. We start with a number of definitions.

Definition 1.1. *We say that a path (not necessarily self-avoiding) in H_N has length ℓ if it visits $(\ell - 1)$ inner vertices (a vertex is counted each time it is visited, starting and ending points are excluded). For $n, \ell \in \mathbb{N}$, let $\mathcal{M}(n, \ell)$ be the collection of paths (not necessarily self-avoiding) of length ℓ from $\vec{0}_N = (0, 0, \cdots, 0)$ to $(\vec{1}_n, \vec{0}_{N-n}) = (1, 1, \cdots, 1, 0, 0, \cdots, 0)$ (where there are n 1's in $(\vec{1}_n, \vec{0}_{N-n})$). Write $M(n, \ell) = |\mathcal{M}(n, \ell)|$.*

Definition 1.2. *For $n, \ell \in \mathbb{N}$, let $\mathcal{S}(n, \ell)$ be the collection of integer sequences $(a_1, \ldots, a_\ell) \in \{1, \ldots, N\}^\ell$ such that $|\{1 \le i \le \ell : a_i = k\}|$ is odd for $1 \le k \le n$ and even for $n+1 \le k \le N$. In addition, for $1 \le k \le N$, let $\mathcal{S}_k(n, \ell) \subseteq \mathcal{S}(n, \ell)$ contain all sequences in $\mathcal{S}(n, \ell)$ such that the last number a_ℓ is k and let $\mathcal{S}_k(n) = \cup_{\ell \in \mathbb{N}} \mathcal{S}_k(n, \ell)$.*

For each path (not necessarily self-avoiding) $v_0, v_1, \ldots, v_\ell$ in H_N of length ℓ, we associate a sequence of integers $(a_1, \ldots, a_\ell)$ where a_i is the coordinate at which v_{i-1} and v_i differ. We observe that the association is a bijection between $\mathcal{M}(n, \ell)$ and $\mathcal{S}(n, \ell)$.

Remark 1.2. *In the following we will sometimes call the sequence $(a_1, \ldots, a_\ell)$ an* update sequence*, and each of the $a_i (1 \le i \le \ell)$ an* update *(so that there are ℓ updates in the update sequence $(a_1, \ldots, a_\ell)$).*

Let F_1 be a distribution supported on odd integers such that $F_1(2j+1) = \frac{x^{2j+1}}{(2j+1)! \sinh x}$ for all $j \ge 0$, and let F_2 be a distribution supported on even integers such that $F_2(2j) =$

$\frac{x^{2j}}{(2j)!\cosh x}$ for all $j \geq 0$. For a fixed $1 \leq k \leq N$, let U_i be i.i.d. random variables distributed as F_1 for $i \in \{1, \ldots, n\} \setminus \{k\}$ and independently let U_i be i.i.d. random variables distributed as F_2 for $i \in \{n+1, \ldots, N\} \setminus \{k\}$, and let U_k be another independent random variable with distribution F_2 if $1 \leq k \leq n$ and with distribution F_1 if $n+1 \leq k \leq N$. Given the values of $U_1, \ldots, U_N$, we let $(A_1, \ldots, A_{L-1}, k) \in \{1, \ldots, N\}^L$ (where $L - 1 = \sum_{i=1}^N U_i$) be a sequence uniformly at random subject to $|\{1 \leq j \leq L-1 : A_j = i\}| = U_i$. We denote by $\mu_{k,n}$ the probability measure of the random sequence $(A_1, \ldots, A_{L-1}, k)$.

Lemma 1.1. *For $1 \leq k \leq n \leq \ell$ and any sequence $(a_1, \ldots, a_{\ell-1}, k) \in \mathcal{S}_k(n, \ell)$, we have*

$$\mu_{k,n}((a_1, \ldots, a_{\ell-1}, k)) = \frac{x^{\ell-1}}{(\ell-1)!} \frac{1}{(\sinh x)^{n-1}} \frac{1}{(\cosh x)^{N-n+1}} \,. \tag{1.4}$$

Similarly, for $n+1 \leq k \leq N$ and $\ell \geq n+2$, and any sequence $(a_1, \ldots, a_{\ell-1}, k) \in \mathcal{S}_k(n, \ell)$, we have

$$\mu_{k,n}((a_1, \ldots, a_{\ell-1}, k)) = \frac{x^{\ell-1}}{(\ell-1)!} \frac{1}{(\sinh x)^{n+1}} \frac{1}{(\cosh x)^{N-n-1}} \,. \tag{1.5}$$

Proof. We only prove the first case. Let $n_i = |\{1 \leq j \leq \ell - 1 : a_j = i\}|$. Then we have

$$\mu_{k,n}((a_1, \ldots, a_{\ell-1}, k)) = \mu_{k,n}(U_i = n_i \text{ for all } 1 \leq i \leq N) \cdot \frac{\prod_{i=1}^N n_i!}{(\ell-1)!} \,, \tag{1.6}$$

where the second term on the right hand side counts the conditional probability of sampling $(a_1, \ldots, a_{\ell-1}, k)$ given $U_i = n_i$ for all $1 \leq i \leq N$. By independence of U_i's, we see that

$$\begin{aligned}
\mu_{k,n}(U_i = n_i \text{ for all } 1 \leq i \leq N) &= \prod_{i=1}^N \mu_{k,n}(U_i = n_i) \\
&= \prod_{1 \leq i \neq k \leq n} F_1(n_i) \cdot \prod_{n+1 \leq i \leq N} F_2(n_i) \cdot F_2(n_k) \\
&= \prod_{1 \leq i \neq k \leq n} \frac{x^{n_i}}{n_i! \sinh x} \cdot \prod_{n+1 \leq i \leq N} \frac{x^{n_i}}{n_i! \cosh x} \cdot \frac{x^{n_k}}{n_k! \cosh x} \\
&= x^{\ell-1} \frac{1}{\prod_{i=1}^N n_i!} \frac{1}{(\sinh x)^{n-1}} \frac{1}{(\cosh x)^{N-n+1}} \,.
\end{aligned}$$

Combined with (1.6), this completes the proof of the first part of the lemma. The second part is similar. □

Lemma 1.2. *We have*

$$\sum_{\ell=1}^{\infty} M(n,\ell)\frac{x^\ell}{\ell!} = (\sinh x)^n(\cosh x)^{N-n}\,. \tag{1.7}$$

In addition, we have

$$\begin{aligned}\sum_{\ell=1}^{\infty} M(n,\ell)\frac{x^{\ell-1}}{(\ell-1)!} &= ((\sinh x)^n(\cosh x)^{N-n})' \\ &= (\sinh x)^{n-1}(\cosh x)^{N-n-1}(n(\cosh x)^2 + (N-n)(\sinh x)^2)\,.\end{aligned} \tag{1.8}$$

Proof. We give a proof of the second equality. The first equality can be obtained by integrating the second equality with respect to x.

Since $\mu_{k,n}$ is a probability measure on $\mathcal{S}_k(n)$, we see that $\sum_{\vec{a}\in\mathcal{S}_k(n)}\mu_{k,n}(\vec{a}) = 1$. Combined with Lemma 1.1, it yields that when $1 \le k \le n$

$$1 = \sum_{\ell=n}^{\infty}\sum_{\vec{a}\in\mathcal{S}_k(n,\ell)}\mu_{k,n}(\vec{a}) = \sum_{\ell=n}^{\infty}|\mathcal{S}_k(n,\ell)|\frac{x^{\ell-1}}{(\ell-1)!}\frac{1}{(\sinh x)^{n-1}}\frac{1}{(\cosh x)^{N-n+1}}\,,$$

and when $n+1 \le k \le N$

$$1 = \sum_{\ell=n+2}^{\infty}\sum_{\vec{a}\in\mathcal{S}_k(n,\ell)}\mu_{k,n}(\vec{a}) = \sum_{\ell=n+2}^{\infty}|\mathcal{S}_k(n,\ell)|\frac{x^{\ell-1}}{(\ell-1)!}\frac{1}{(\sinh x)^{n+1}}\frac{1}{(\cosh x)^{N-n-1}}\,.$$

This tells us that when $1 \le k \le n$

$$\sum_{\ell=n}^{\infty}|\mathcal{S}_k(n,\ell)|\frac{x^{\ell-1}}{(\ell-1)!} = (\sinh x)^{n-1}(\cosh x)^{N-n+1}\,,$$

and when $n+1 \le k \le N$

$$\sum_{\ell=n+2}^{\infty} |\mathcal{S}_k(n,\ell)| \tfrac{x^{\ell-1}}{(\ell-1)!} = (\sinh x)^{n+1}(\cosh x)^{N-n-1}.$$

Summing these N equalities (combined with the fact that $M(n,\ell) = |\mathcal{M}(n,\ell)| = |\mathcal{S}(n,\ell)| = \sum_{1\le k\le N} |\mathcal{S}_k(n,\ell)|$) completes the proof of (1.8) and hence the lemma. □

Corollary 1.1. $\mathbb{E}Z_{N,x} \le N(\sinh x)^{N-1}\cosh x$.

Proof. Here we will derive an upper bound for $\mathbb{E}Z_{N,x}$ in the general (not necessarily antipodal) case. Suppose the Hamming distance between u and w is n. Let $\mathcal{M}'(n,\ell)$ be the subset of self-avoiding paths in $\mathcal{M}(n,\ell)$ and write $M'(n,\ell) = |\mathcal{M}'(n,\ell)|$. Since for each path $P \in \mathcal{M}'(n,\ell)$, the probability that P is accessible is $\frac{x^{\ell-1}}{(\ell-1)!}$, we have

$$\begin{aligned}\mathbb{E}Z_{N,x} = \mathbb{E}\sum_{\ell=1}^{\infty}\sum_{P\in\mathcal{M}'(n,\ell)} 1_{P \text{ is accessible}} &= \sum_{\ell=1}^{\infty} M'(n,\ell)\frac{x^{\ell-1}}{(\ell-1)!} \le \sum_{\ell=1}^{\infty} M(n,\ell)\frac{x^{\ell-1}}{(\ell-1)!} \\ &= (\sinh x)^{n-1}(\cosh x)^{N-n-1}(n(\cosh x)^2 + (N-n)(\sinh x)^2), \qquad (1.9)\end{aligned}$$

where the last equality follows from (1.8). In the antipodal case, substituting $n = N$ in (1.9) gives the desired bound. □

Proof of (1.1)**: antipodal case** In this case, $\beta = 1$ so we have $f(x) = \sinh x$, $x_0 = \sinh^{-1}(1) = \ln(\sqrt{2}+1)$, $\sinh x_0 = 1$ and $\cosh x_0 = \sqrt{2}$. We can without loss of generality assume that $\epsilon_N \le N^{-2/3}$ since $\mathbb{P}(Z_{N,x} > 0)$ is increasing in x. By Corollary 1.1, we have (recall that $x_c = x_0 - \frac{1}{f'(x_0)}\frac{\ln N}{N} = x_0 - \frac{\sqrt{2}}{2}\frac{\ln N}{N}$)

$$\begin{aligned}\mathbb{P}(Z_{N,x_c-\epsilon_N} > 0) &\le \mathbb{E}Z_{N,x_c-\epsilon_N} \le N(\sinh(x_c-\epsilon_N))^{N-1}\cosh(x_c-\epsilon_N) \\ &= N(\sinh(x_0) - \cosh(x_0)(\frac{\sqrt{2}}{2}\frac{\ln N}{N} + \epsilon_N) + o(1/N))^{N-1}\cosh(x_c-\epsilon_N) \\ &\le N(1 - \frac{\ln N}{N} - \sqrt{2}\epsilon_N + o(1/N))^{N-1}\sqrt{2} \to 0 \text{ as } N\to\infty.\end{aligned}$$

□

Remark 1.3. *Similarly we can show that for* $x = x_c + \epsilon_N$ *and* $N\epsilon_N \to \infty$*, we have* $N(\sinh x)^{N-1}\cosh x = N(\sinh(x_c+\epsilon_N))^{N-1}\cosh(x_c+\epsilon_N) \to \infty$ *as* $N \to \infty$*, and that for all* $x = x_c + \epsilon_N$ *such that* $|N\epsilon_N| \le \Delta$*, we have* $m_1(\Delta) \le N(\sinh x)^{N-1}\cosh x \le m_2(\Delta)$ *where* $m_1(\Delta), m_2(\Delta) > 0$ *depend only on* Δ*. Combined with Lemma 1.3 below, this suggests (at least in expectation) that* x_c *is the critical value.*

1.2.2 Proof of the lower bound

In order to prove the lower bound, we restrict our attention to certain good paths, i.e., those with desirable properties on the growth of Hamming distances (in particular, a good path needs to be self-avoiding). We will define precisely what we mean by a good path in Definition 1.3 below. Denote by $Z_{N,x,*}$ the number of good accessible paths. Crucially, we demonstrate that with our definition of good paths, we have $\mathbb{E}Z_{N,x,*} \asymp \mathbb{E}Z_{N,x}$ and $\mathbb{E}Z^2_{N,x,*} \asymp (\mathbb{E}Z_{N,x,*})^2$ (where $\asymp$ means that the left and right hand sides are within a constant multiplicative factor) as long as $x = x_c + \epsilon_N$ ($N\epsilon_N \to \infty$) and x stays in a fixed neighborhood of x_0. Thus, an application of the second moment method already yields the existence of an accessible path with probability bounded away from 0. Finally, we use the augmenting method as employed in [10] to deduce the existence of an accessible path with probability tending to 1 as $N \to \infty$.

Recall that $x_0 = \sinh^{-1}(1) = \ln(\sqrt{2}+1) \approx 0.88137$. Let $\alpha = x_0 \coth x_0 \approx 1.24645$.

For any $0 < \epsilon < 1$, we set $\iota, \epsilon_1, \epsilon_2$ and ϵ_3 throughout the rest of the paper as

$$\iota = \epsilon^2\,, \epsilon_1 = \epsilon^{1/2}\,, \epsilon_2 = \epsilon^{1/4} \text{ and } \epsilon_3 = \epsilon^{1/8}\,. \tag{1.10}$$

Let us say a few words about these “infinitesimals” $\iota, \epsilon_1, \epsilon_2, \epsilon_3$ and ϵ. They are not really infinitesimals because throughout the paper we only need to fix each of them to be a certain sufficiently small number (and thanks to (1.10) we only need to fix ϵ). However, to guarantee a viable choice of these numbers, we need $\iota, \epsilon, \epsilon_1, \epsilon_2, \epsilon_3$ to be decreasing in terms of the orders of the infinitesimals, hence our definition (1.10). Our specific choices of the relations between $\iota, \epsilon_1, \epsilon_2, \epsilon_3$ and ϵ though are rather arbitrary.

For $u, v \in H_N$, we denote by $H(u,v)$ the Hamming distance between u and v.

Definition 1.3. *Let $\epsilon > 0$ be a sufficiently small fixed number to be selected. We say a path (or the associated update sequence) $v_0 = \vec{0}, v_1, \ldots, v_{L-1}, v_L = \vec{1}$ is* good *if $L \in [\alpha(1-\epsilon)N, \alpha(1+\epsilon)N]$ and the following holds:*

$H(v_i, v_j) = |i-j|$, *if* $|i-j| = 1, 2, 3$;

$H(v_i, v_j) = |i-j|$ *or* $|i-j| - 2$, *if* $4 \leq |i-j| \leq N^{\frac{1}{5}}$;

$H(v_i, v_j) \leq (1/2 + \epsilon_1)N$, *if* $N^{\frac{1}{5}} \leq |i-j| \leq \alpha(1/2+\epsilon)N$;

$H(v_i, v_j) > (1/2 + \epsilon_1)N$, *if* $|i-j| > \alpha(1/2+\epsilon_2)N$;

$H(v_i, v_j) \geq \frac{|i-j|}{\alpha+\epsilon_3}$, *if* $N^{\frac{1}{5}} \leq |i-j| \leq \alpha(1/2+\epsilon_2)N$.

It is clear from the definition that a good path is self-avoiding.

Lemma 1.3. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_1 > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \leq \iota$ and $N > N'$ we have*

$$\mathbb{E} Z_{N,x,*} \geq C_1 N \sinh^{N-1} x \cosh x \, . \tag{1.11}$$

Proof. We keep all the definitions and notations in the previous subsection 1.2.1. Since we are working in the antipodal case where $\beta = 1$, we have substituted n by N in the following without further notice. Recall that as stated in Definition 1.3, an update sequence is good if its corresponding path is good. For each $1 \leq k \leq N$, we let $\mathcal{S}_{k,*}(N) \subseteq \mathcal{S}_k(N)$ contain all the good sequences ending in k, and let $\mathcal{M}_{k,*}(N)$ be the collection of the corresponding good paths. We claim that in order to show (1.11), it suffices to show that for each $1 \leq k \leq N$

$$\mu_{k,N}(\mathcal{S}_{k,*}(N)) \geq C_1 \, . \tag{1.12}$$

Indeed, summing equation (1.4) over all $(a_1, \ldots, a_{\ell-1}, k) \in \mathcal{S}_{k,*}(N)$ gives that

$$\begin{aligned}\mu_{k,N}(\mathcal{S}_{k,*}(N)) &= \tfrac{1}{(\sinh x)^{N-1}\cosh x} \sum_{\substack{P\in\mathcal{M}_{k,*}(N)\\ P \text{ is of length } \ell}} \tfrac{x^{\ell-1}}{(\ell-1)!} \\ &= \tfrac{1}{(\sinh x)^{N-1}\cosh x} \sum_{P\in\mathcal{M}_{k,*}(N)} \mathbb{P}(P \text{ is accessible})\,,\end{aligned}$$

where the last equality is because any good path is necessarily self-avoiding. If (1.12) holds true, then summing the above equation over $1 \le k \le N$ yields (1.11).

For ease of elaboration we make a slight modification to (1.12), that is, we will show instead that

$$\tilde{\mu}_N(\mathcal{S}_*(N)) \ge \tilde{C}_1\,, \tag{1.13}$$

where $\tilde{\mu}_N$ differs from $\mu_{k,N}$ in that we also let U_k be chosen according to F_1 instead of F_2 (in other words, for each $1 \le i \le N$, the U_i's are now i.i.d. random variables distributed as F_1), and consider the random sequence $(A_1, \ldots, A_{L-1})$ instead of $(A_1, \ldots, A_{L-1}, k)$. See also Case 1 below for the definition of $\tilde{\mu}_{N,\beta}$, the generalization of $\tilde{\mu}_N$ to general β; we use $S_*(N)$ to denote the collection of all the good sequences (not necessarily ending in k).

There are a number of ways to justify our replacement of (1.12) by (1.13). For example, one may argue that if $\tilde{\mu}_{N-1}(\mathcal{S}_*(N-1)) \ge \tilde{C}_1$ holds, then (possibly with a slight change of $N^{\frac{1}{5}}, \epsilon, \epsilon_1, \epsilon_2$ and ϵ_3 in the definition of good paths) $\mu_{k,N}(\mathcal{S}_{k,*}(N)) = \mu_{N,N}(\mathcal{S}_{N,*}(N)) \ge \frac{1}{\cosh x}\tilde{C}_1$ holds, since

$$\begin{aligned}\mu_{N,N}(\mathcal{S}_{N,*}(N)) &\ge \mu_{N,N}(\{(A_1, \ldots, A_{L-1}, N) : U_N = 0, (A_1, \ldots, A_{L-1}) \in S_*(N-1)\}) \\ &= \frac{1}{\cosh x}\tilde{\mu}_{N-1}(\mathcal{S}_*(N-1))\,.\end{aligned}$$

In the rest of the proof, $\mathbb{P}$ and $\mathbb{E}$ refer to $\tilde{\mu}_N$ unless otherwise specified. Note that $\mathbb{P}$ depends on both x and N. Under this probability space (or the more general $\tilde{\mu}_{N,\beta}$), we say an event $\mathcal{E}_N$ happens with probability tending to 1 as $N \to \infty$ (or with high probability for brevity) if $1 - \mathbb{P}(\mathcal{E}_N) \le p(\epsilon, N)$ where $p(\epsilon, N) > 0$ only depends on ϵ and N, and (when ϵ is fixed) goes to 0 as $N \to \infty$. Similarly, we say a quantity (possibly random) Q_N is $o(1)$ if $|Q_N| \le q_N$ where $q_N > 0$ is fixed, only depends on N and goes to 0 as $N \to \infty$.

By a simple calculation, for $U \sim F_1$, we have $\mathbb{E}U = x \coth x$, and $\operatorname{Var} U$ is bounded by an absolute constant (since $|x - x_0| \le \iota$). Therefore it is immediate from, say, Chebyshev's inequality (as used in proving the weak law of large numbers) that with probability tending to 1 as $N \to \infty$ we have $L \in [\alpha(1-\epsilon)N, \alpha(1+\epsilon)N]$ (recall that $\alpha = x_0 \coth x_0$ and $\iota = \epsilon^2$). It now remains to consider the requirements on Hamming distances in the definition of good paths, for which purpose we split into three cases as follows.

Case 1: $H(v_i, v_j) = |i - j|$, **if** $|i - j| = 1, 2, 3$.

We show that this requirement can be satisfied by a sequence generated from $\tilde{\mu}_N$ with probability bounded from below by a constant. We prove the following statement (1.15) for general β.

Fix a $\beta \in (0, 1]$. For $i \in \{1, \ldots, \beta N\}$, let U_i be i.i.d. random variables distributed as F_1, and independently for $i \in \{\beta N + 1, \ldots, N\}$, let U_i be i.i.d. random variables distributed as F_2. Given the values of $U_1, \ldots, U_N$, we let $(A_1, \ldots, A_L)$ (where $L = \sum_{i=1}^N U_i$) be a sequence uniformly at random subject to $|\{1 \le j \le L : A_j = i\}| = U_i$. Let $\tilde{\mu}_{N,\beta}$ be the probability measure of the random sequence $(A_1, \ldots, A_L)$ thus obtained.

For convenience we set $A_{i+L} = A_i$ for $i \ge 1$. Let

$$\begin{aligned} &I_i = 1_{\{A_i = A_{i+1}\}} \text{ and } \mathcal{N}_i = \{i, i+1\}, && \text{if } i = 1, 2, \cdots, L; \\ &I_i = 1_{\{A_{i-L} = A_{i+2-L}\}} \text{ and } \mathcal{N}_i = \{i - L, i + 2 - L\}, && \text{if } i = L+1, L+2, \cdots, 2L. \end{aligned} \tag{1.14}$$

Let x_0 be given as in Theorem 1.1, and let $\gamma = \beta x_0 \coth x_0 + (1-\beta) x_0 \tanh x_0$. For any $\iota = \epsilon^2 > 0$, there exists a constant $c^* > 0$ and an integer $N' > 0$ which both depend only on

ι, such that for all $|x - x_0| \le \iota$ and $N > N'$ we have

$$\tilde{\mu}_{N,\beta}(\sum_{i=1}^{2L} I_i = 0) \ge c^* \,. \tag{1.15}$$

Remark 1.4. *In fact, as can be seen from our proof, x_0 could be any fixed positive number (not necessarily given by Theorem 1.1). Moreover, we have $c^* \to e^{-\frac{2x_0^2}{\gamma}}$ as $\iota \to 0$, and if $x \to x_0$ as $N \to \infty$, then $\sum_{i=1}^{2L} I_i$ converges to the Poisson distribution with mean $\frac{2x_0^2}{\gamma}$ as $N \to \infty$. However, we don't need any of these facts.*

Proof of (1.15) In this proof, $\mathbb{P}$ and $\mathbb{E}$ refer to $\tilde{\mu}_{N,\beta}$. Let

$$D_j := |\{1 \le i \le N : U_i = j\}|$$

for $j \in \mathbb{N}$ and

$$\Lambda := L^{-1} \sum_{j=2}^{\infty} D_j j(j-1).$$

By a simple calculation, for $U \sim F_1$, we have $\mathbb{E}U = x \coth x$ and $\mathbb{E}U(U-1) = x^2$, and the variances of U and $U(U-1)$ are both bounded by an absolute constant, as long as x stays in a fixed neighborhood of x_0. Similarly, for $U \sim F_2$, we have $\mathbb{E}U = x \tanh x$ and $\mathbb{E}U(U-1) = x^2$, and the variances of U and $U(U-1)$ are both bounded by an absolute constant. By Chebyshev's inequality, we have with probability tending to 1 as $N \to \infty$,

$$L = \sum_{i=1}^{N} U_i \in [\gamma(1-\epsilon)N, \gamma(1+\epsilon)N] \tag{1.16}$$

and

$$\sum_{j=2}^{\infty} D_j j(j-1) = \sum_{i=1}^{N} U_i(U_i - 1) \in [x_0^2(1-\epsilon)N, x_0^2(1+\epsilon)N] \,. \tag{1.17}$$

(1.16) and (1.17) combined give

$$\Lambda \in [(1-3\epsilon)\frac{x_0^2}{\gamma}, (1+3\epsilon)\frac{x_0^2}{\gamma}]\,.$$

By the uniform convergence of $\sum_{k=1}^{K}(-1)^{k+1}\frac{(2\Lambda)^k}{k!}$ to $1-e^{-2\Lambda}$ on $[(1-3\epsilon)\frac{x_0^2}{\gamma}, (1+3\epsilon)\frac{x_0^2}{\gamma}]$, there exists a finite odd number K and $0 < c^{**} < 1$ (c^{**} may depend on K and ϵ) such that for all $\Lambda \in [(1-3\epsilon)\frac{x_0^2}{\gamma}, (1+3\epsilon)\frac{x_0^2}{\gamma}]$, we have

$$\sum_{k=1}^{K}(-1)^{k+1}\frac{(2\Lambda)^k}{k!} < c^{**}\,. \tag{1.18}$$

Again, by Chebyshev's inequality, we have with probability tending to 1 as $N \to \infty$,

$$\sum_{j=0}^{\infty} D_j j^{2k} = \sum_{i=1}^{N} U_i^{2k} \le C_K N\,,\ \text{for all } 1 \le k \le K \tag{1.19}$$

where $C_K > 0$ is a constant which only depends on K. Also, by a rather loose bound on $\mathbb{P}(U_i \ge 10\log N)$ (directly from the definition of U_i), we have with probability tending to 1 as $N \to \infty$,

$$\max_{1\le i\le N} U_i \le 10\log N\,. \tag{1.20}$$

We will assume (1.16), (1.17), (1.19) and (1.20) without mention in what follows.

Write $\mathcal{F} = \sigma(U_1, U_2, \ldots, U_N)$. By Bonferroni's inequalities [3], we have

$$\mathbb{P}\Big(\sum_{i=1}^{2L} I_i \ge 1 \mid \mathcal{F}\Big) \le \sum_{k=1}^{K}(-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le 2L} \mathbb{P}(I_{i_1}=1, I_{i_2}=1, \cdots, I_{i_k}=1 \mid \mathcal{F})\,. \tag{1.21}$$

In order to prove (1.15), it suffices to show that each summand (of $\sum_{k=1}^{K}$) on the right hand side of (1.21) is asymptotic to the corresponding summand on the left hand side of (1.18).

That is to say, we want to show that for each $1 \le k \le K$,

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2L} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \cdots, I_{i_k} = 1 \mid \mathcal{F}) - \frac{(2\Lambda)^k}{k!} = o(1)\,. \tag{1.22}$$

For this purpose, we will split $\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2L} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \cdots, I_{i_k} = 1 \mid \mathcal{F})$ into two parts according to whether or not any A_i is involved in the definition of more than one I_{i_j}'s ($1 \le j \le k$). More precisely, for a pair of integers $(i_j, i_{j'})$ (or equivalently $(I_{i_j}, I_{i_{j'}})$) where $i_j \ne i_{j'}$ we say it is *intersecting* if $\mathcal{N}_{i_j} \cap \mathcal{N}_{i'_j} \ne \emptyset$ (see (1.14) for the definition of $\mathcal{N}_i$). Let $\mathcal{I}^{k,1}$ ($\mathcal{I}^{k,2}$) denote the set of all sequences $(i_1, i_2, \cdots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_k \le 2L$ and it contains no (at least 1) intersecting pair, respectively. We can write

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2L} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \cdots, I_{i_k} = 1 \mid \mathcal{F}) = \mathcal{J}_1 + \mathcal{J}_2$$

where

$$\mathcal{J}_1 = \sum_{\mathcal{I}^{k,1}} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \cdots, I_{i_k} = 1 \mid \mathcal{F})$$

and

$$\mathcal{J}_2 = \sum_{\mathcal{I}^{k,2}} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \cdots, I_{i_k} = 1 \mid \mathcal{F}).$$

We first bound the term $\mathcal{J}_1$. For any $(i_1, i_2, \cdots, i_k) \in \mathcal{I}^{k,1}$, the neighborhoods $\mathcal{N}_{i_1}, \mathcal{N}_{i_2}, \cdots, \mathcal{N}_{i_k}$ are disjoint by definition. Now given $\mathcal{F}$, for each $r = 1, \ldots, k$, there are at most $\sum_{j=2}^{\infty} D_j \cdot j \cdot (j-1)$ ways of choosing two matching updates for the two slots in $\mathcal{N}_{i_r}$, and there are at most $(L-2k)!$ ways of arranging the remaining $(L-2k)$ updates, therefore we have

$$\begin{aligned} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \cdots, I_{i_k} = 1 \mid \mathcal{F}) &\le \frac{(L-2k)!}{L!}\Big(\sum_{j=2}^{\infty} D_j \cdot j \cdot (j-1)\Big)^k \\ &= (\frac{1}{L})^k (1 + o(1))\Lambda^k. \end{aligned} \tag{1.23}$$

Combined with the simple fact that $|\mathcal{I}^{k,1}| \le (2L)^k/k!$, this gives that $\mathcal{J}_1 \le (2\Lambda)^k(1+o(1))/k!$. On the other hand, by a similar reasoning

$$\begin{aligned}\mathbb{P}(I_{i_1}=1, I_{i_2}=1, \cdots, I_{i_k}=1 \mid \mathcal{F}) &\ge \frac{(L-2k)!}{L!} \prod_{1\le r\le k} \Big(\sum_{j=2}^{10\log N} (D_j - (r-1)) \cdot j \cdot (j-1) \Big) \\ &\ge (\frac{1}{L})^k (1+o(1))(\Lambda + o(1))^k.\end{aligned}$$

Moreover, we have $|\mathcal{I}^{k,1}| \ge (1+o(1))(2L)^k/k!$ since $|\mathcal{I}^{k,1}| \ge \prod_{1\le r\le k}(2L-7(r-1))/k!$ (each $\mathcal{N}_i$ intersects 6 other $\mathcal{N}_i$'s). Hence, we obtain that $\mathcal{J}_1 \ge (2\Lambda)^k(1+o(1))/k!$. Altogether, we get

$$\mathcal{J}_1 = (2\Lambda)^k(1+o(1))/k!\,. \tag{1.24}$$

It remains to control $\mathcal{J}_2$. For any $(i_1, i_2, \cdots, i_k) \in \mathcal{I}^{k,2}$, denote by $\mathcal{E}_{i_1,\dots,i_k} = \{(A_1, A_2, \dots, A_L) : I_{i_1}=1, I_{i_2}=1, \cdots, I_{i_k}=1\}$. Observe that $I_{i_1}=1, I_{i_2}=1, \cdots, I_{i_k}=1$ (the criteria for $\mathcal{E}_{i_1,\dots,i_k}$) can be rewritten (or simplified) uniquely as a set of equalities

$$\begin{aligned}&A_{j_1} = A_{j_1+n_{1,1}} = A_{j_1+n_{1,1}+n_{1,2}} = \cdots = A_{j_1+n_{1,1}+n_{1,2}+\cdots+n_{1,a_1-1}}\\ &A_{j_2} = A_{j_2+n_{2,1}} = A_{j_2+n_{2,1}+n_{2,2}} = \cdots = A_{j_2+n_{2,1}+n_{2,2}+\cdots+n_{2,a_2-1}}\\ &\cdots\\ &A_{j_\ell} = A_{j_\ell+n_{\ell,1}} = A_{j_\ell+n_{\ell,1}+n_{\ell,2}} = \cdots = A_{j_\ell+n_{\ell,1}+n_{\ell,2}+\cdots+n_{\ell,a_\ell-1}}\end{aligned}$$

where $n_{1,1}, \dots, n_{1,a_1-1}, n_{2,1}, \dots, n_{2,a_2-1}, \dots, n_{\ell,1}, \dots, n_{\ell,a_\ell-1}$ are either 1 or 2, $a_1, a_2, \dots, a_\ell$ are integers ≥ 2 and $a_1 + a_2 + \cdots + a_\ell \le 2k$ (in particular each a_i is $\le 2k$). Also, since $(i_1, i_2, \cdots, i_k) \in \mathcal{I}^{k,2}$, i.e. there is at least one intersecting pair in $I_{i_1}, \cdots, I_{i_k}$, at least one of the $a_1, a_2, \dots, a_\ell$ must be strictly larger than 2, so that $a_1 + a_2 + \cdots + a_\ell > 2\ell$. Denote by $\mathcal{A}$ the preceding set of equalities (so $\mathcal{A}$ can also be viewed as an event). By a rather loose bound, $|\{(i_1,\ldots,i_k):\mathcal{E}_{i_1,\ldots,i_k}=\mathcal{A}\}|\le (a_1+a_2+\cdots+a_\ell)^{2k}\le (2k)^{2k}$. Therefore we have

$$\sum_{\mathcal{I}^{k,2}}\mathbb{P}(\mathcal{E}_{i_1,\ldots,i_k}\mid\mathcal{F}) \;\le\; (2k)^{2k}\sum_{\ell}\sum_{\mathcal{D}_1}\sum_{\mathcal{D}_2}\sum_{\mathcal{D}_3}\mathbb{P}(\mathcal{A}\mid\mathcal{F}), \tag{1.25}$$

where $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3$ respectively denote the collections of all valid choices of $(a_1,a_2,\ldots,a_\ell)$, $(n_{1,1},\ldots,n_{1,a_1-1},n_{2,1},\ldots,n_{2,a_2-1},\ldots,n_{\ell,1},\ldots,n_{\ell,a_\ell-1})$ and $(j_1,j_2,\ldots,j_\ell)$. Now similar to (1.23), we have

$$\mathbb{P}(\mathcal{A}\mid\mathcal{F})\le\frac{(L-(a_1+a_2+\cdots+a_\ell))!}{L!}\prod_{r=1}^{\ell}\Big(\sum_{i=a_r}^{\infty}D_i\cdot i\cdot(i-1)\cdots(i-a_r+1)\Big).$$

Therefore, by (1.19) we have

$$\sum_{\mathcal{D}_3}\mathbb{P}(\mathcal{A}\mid\mathcal{F})\le C'_K N^{2\ell-(a_1+a_2+\cdots+a_\ell)}\le C'_K/N\,, \tag{1.26}$$

where C'_K is another constant depending on K, and the second inequality follows from the fact that $a_1+a_2+\cdots+a_\ell>2\ell$. Since $|\mathcal{D}_1|$, $|\mathcal{D}_2|$ and ℓ are all bounded by a number that depends only on K, we combine (1.25) and (1.26) and obtain

$$\sum_{\mathcal{I}^{k,2}}\mathbb{P}(\mathcal{E}_{i_1,\ldots,i_k}\mid\mathcal{F})\le C^*_K/N\,,$$

where $C^*_K>0$ depends only on K. Combined with (1.24), this yields (1.22) and therefore (1.15). □

Case 2 : $H(v_i,v_j)=|i-j|$ **or** $|i-j|-2,$ **if** $4\le|i-j|\le N^{\frac{1}{5}}$.

We show that this requirement is satisfied by a sequence generated from $\tilde{\mu}_N$ with probability tending to 1 as $N\to\infty$. Denote by W_k the event that in some k consecutive updates there are at least two coordinates such that all of them occur at least twice. It suffices to show that $W_{N^{1/5}}$ happens with probability tending to 0 as $N\to\infty$. Given $\mathcal{F}=\sigma(U_1,U_2,\ldots,U_N)$, the conditional probability that the coordinates 1 and 2 both occur at least twice in the first k updates is less than $\binom{U_1}{2}(\frac{k}{L})^2\binom{U_2}{2}(\frac{k}{L})^2$, by a union bound. Therefore,

$$\mathbb{P}(W_k) = \mathbb{E}(\mathbb{P}(W_k \mid \mathcal{F})) \leq \sum_{1\leq i<j\leq N} \mathbb{E}\Big(\binom{U_i}{2}(\frac{k}{L})^2\binom{U_j}{2}(\frac{k}{L})^2 L\Big) \leq \frac{C'k^4}{N} = o(1) \qquad (1.27)$$

for $k = N^{1/5}$ (here C' is an absolute constant).

Case 3:

$H(v_i, v_j) \leq (1/2 + \epsilon_1)N,$ **if** $N^{\frac{1}{5}} \leq |i-j| \leq \alpha(1/2+\epsilon)N;$

$H(v_i, v_j) > (1/2 + \epsilon_1)N,$ **if** $|i-j| > \alpha(1/2+\epsilon_2)N;$

$H(v_i, v_j) \geq \frac{|i-j|}{\alpha+\epsilon_3},$ **if** $N^{\frac{1}{5}} \leq |i-j| \leq \alpha(1/2+\epsilon_2)N.$

We show that these three requirements are satisfied by a sequence generated from $\tilde{\mu}_N$ with probability tending to 1 as $N \to \infty$. Let $\mathcal{R}$ be the collection of all sequences satisfying these three requirements.

Before we proceed, let us first give a hint on why this may be true (i.e. what these three requirements are trying to say). For $t \in [0,1]$, we define

$$g(t) := \frac{\sinh(x_0 t)\cosh(x_0(1-t))}{\sinh x_0} = \sinh(x_0 t)\cosh(x_0(1-t))\,. \qquad (1.28)$$

Vaguely (and roughly) speaking, $g(t)N$ is the "expected Hamming distance traveled by a path in time t" (if the whole path uses a unit time). We will make this precise below. For a derivation of the formula (1.28), see equation (1.32). By plotting $g(t)$ (or an easy calculus), one can easily see that

- $g(t) \leq \frac{1}{2},$ if $0 \leq t \leq \frac{1}{2}$
- $g(t) \geq \frac{1}{2},$ if $\frac{1}{2} \leq t \leq 1$
- $g(t) \geq t,$ if $0 \leq t \leq \frac{1}{2}$

which correspond to the three requirements, respectively. We now carry out the idea above fully and rigorously as follows.

We will consider the following continuous version of $\tilde{\mu}_N$, namely $\hat{\mu}_N$: As in $\tilde{\mu}_N$, we first let $U_i, 1 \leq i \leq N$ be i.i.d. random variables distributed as F_1. Now given the values of $U_1, \ldots, U_N$, we denote $\mathcal{L} = \{(i,j) : 1 \leq i \leq N, 1 \leq j \leq U_i\}$ and $L = |\mathcal{L}| = \sum_{i=1}^N U_i$, and let $\{r_{i,j} : (i,j) \in \mathcal{L}\}$ be L i.i.d. uniform $[0,1]$ random variables. Let $\hat{\mu}_N$ be the underlying probability measure $F_1^N \times U[0,1]^\infty$.

For each $1 \leq i \leq N$, we attach the label "i" to each real number $r_{i,j}, (i,j) \in \mathcal{L}$. Since almost surely under $\hat{\mu}_N$, L is finite and $r_{i,j}$'s are distinct, we can (without ambiguity) let $r_1 < r_2 < \cdots < r_L$ be the reordering of the reals $r_{i,j}, (i,j) \in \mathcal{L}$ in increasing order, and for $1 \leq \ell \leq L$ let $\hat{A}_\ell$ be the unique label of r_ℓ. We have thus formed a random integer sequence $(\hat{A}_1, \ldots, \hat{A}_L)$ under $\hat{\mu}_N$.

It is clear that $(\hat{A}_1, \ldots, \hat{A}_L)$ under $\hat{\mu}_N$ has the same distribution as $(A_1, \ldots, A_L)$ under $\tilde{\mu}_N$, i.e., for any integer sequence $(a_1, \ldots, a_L)$, we have

$$\hat{\mu}_N((\hat{A}_1, \ldots, \hat{A}_L) = (a_1, \ldots, a_L)) = \tilde{\mu}_N((A_1, \ldots, A_L) = (a_1, \ldots, a_L)) .$$

Therefore

$$\hat{\mu}_N((\hat{A}_1, \ldots, \hat{A}_L) \in \mathcal{R}) = \tilde{\mu}_N((A_1, \ldots, A_L) \in \mathcal{R}) . \tag{1.29}$$

For any interval $I \subseteq [0,1]$ and any $1 \leq i \leq N$, we let $N_{I,i}$ be the number of labels "i" in I, i.e., $N_{I,i} = |\{1 \leq j \leq U_i : r_{i,j} \in I\}|$. Let

$$T_I = \sum_{i=1}^N N_{I,i} = |\{(i,j) \in \mathcal{L} : r_{i,j} \in I\}|$$

be the total number of labels in I and

$$O_I = \sum_{i=1}^N 1_{\{N_{I,i} \text{ is an odd number}\}}$$

count all the i's ($1 \le i \le N$) that appear an odd number of times as a label in I. Let $\hat{\mathcal{R}}$ be the following event: for all intervals $I \subseteq [0,1]$, we have

$$O_I \le (1/2+\epsilon_1)N, \text{ if } N^{\frac{1}{5}} \le T_I \le \alpha(1/2+\epsilon)N;$$
$$O_I > (1/2+\epsilon_1)N, \text{ if } T_I > \alpha(1/2+\epsilon_2)N;$$
$$O_I \ge \frac{T_I}{\alpha+\epsilon_3}, \text{ if } N^{\frac{1}{5}} \le T_I \le \alpha(1/2+\epsilon_2)N.$$

We see that

$$\hat{\mu}_N(\hat{\mathcal{R}}) = \hat{\mu}_N((\hat{A}_1,\ldots,\hat{A}_L) \in \mathcal{R})\,. \tag{1.30}$$

In light of equalities (1.29) and (1.30), it suffices to show that under $\hat{\mu}_N$, $\hat{\mathcal{R}}$ happens with probability tending to 1 as $N \to \infty$. In the following $\mathbb{P}$ and $\mathbb{E}$ refer to $\hat{\mu}_N$. To this end, our strategy is to first show that with high probability, for all intervals $I \subseteq [0,1]$ such that $|I| \ge N^{-5/6}$, both T_I and O_I are concentrated around their means respectively.

For any interval $I \subseteq [0,1]$ of length t, conditioning on $T_{[0,1]} = L$, T_I is the sum of L i.i.d. Bernoulli random variables with mean t, thus by Chernoff's bound [6],

$$\mathbb{P}(|T_I - Lt| \ge \epsilon Lt | L) \le 2\exp(-\epsilon^2 Lt/3). \tag{1.31}$$

For O_I, by definition $O_I = \sum_{i=1}^{N} 1_{\{N_{I,i} \text{ is an odd number}\}}$ where $1_{\{N_{I,i} \text{ is an odd number}\}}$ for $1 \le i \le N$ are N i.i.d. Bernoulli random variables with mean $p_I =$

$\mathbb{P}(N_{I,1}$ is an odd number). We can compute p_I as follows:

$$\begin{aligned} p_I &= \mathbb{P}(N_{I,1} \text{ is an odd number}) \\ &= \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!\sinh x} \sum_{j=0}^{i} \binom{2i+1}{2j+1} t^{2j+1}(1-t)^{2i-2j} \\ &= \frac{1}{\sinh x}\Big(\sum_{j=0}^{\infty} \frac{(xt)^{2j+1}}{(2j+1)!}\Big)\Big(\sum_{i-j=0}^{\infty} \frac{(x(1-t))^{2(i-j)}}{(2i-2j)!}\Big) \\ &= \frac{\sinh(xt)\cosh(x(1-t))}{\sinh x}. \end{aligned} \tag{1.32}$$

By Chernoff's bound again, we have

$$\mathbb{P}(|O_I - \mathbb{E}O_I| \geq 3\epsilon \mathbb{E}O_I) \leq 2\exp\Big(-3\epsilon^2 N \frac{\sinh(xt)\cosh(x(1-t))}{\sinh x}\Big). \tag{1.33}$$

Now let us divide $[0,1]$ into N non-overlapping intervals of equal length $1/N$. We say an interval is *integral* if it is of the form $[n_1/N, n_2/N]$, where $n_1, n_2 \in \mathbb{N}, 0 \leq n_1 < n_2 \leq N$ and $n_2 - n_1 \geq N^{1/6}$ (so that its length is at least $N^{-5/6}$). Denote by E_L the event $\{\frac{L}{(x\coth x)N} \in [1-\epsilon, 1+\epsilon]\}$. Since on E_L, $Lt \geq cN^{1/6}$ when $t \geq N^{-5/6}$ for a constant $c > 0$, we can apply (1.31) and a union bound over all integral intervals to obtain that

$$\mathbb{P}\Big(\max_{I \text{ is integral}} | T_I - Lt_I | \geq \epsilon Lt_I \mid L\Big) \leq 2(N+1)^2 \exp(-\epsilon^2 cN^{1/6}/3), \text{ on } E_L.$$

Since $\mathbb{E}T_I = \mathbb{E}Lt_I = (x\coth x)Nt_I$ and therefore $Lt_I \in [(1-\epsilon)\mathbb{E}T_I, (1+\epsilon)\mathbb{E}T_I]$ on E_L, we have

$$\mathbb{P}\Big(\max_{I \text{ is integral}} | T_I - \mathbb{E}T_I | \geq 3\epsilon \mathbb{E}(T_I) \mid L\Big) \leq 2(N+1)^2 \exp(-\epsilon^2 cN^{1/6}/3), \text{ on } E_L.$$

Since E_L happens with probability tending to 1 as $N \to \infty$, we thus have that $\mathcal{E}_T$ happens with probability tending to 1 as $N \to \infty$, where

$$\mathcal{E}_T = \bigcap_{I \text{ is integral}} \left\{ T_I \in [(1-3\epsilon)\mathbb{E}T_I, (1+3\epsilon)\mathbb{E}T_I] \right\}.$$

From (1.33), since $\sinh x \geq x$ for $x \geq 0$, we have $Np_I \geq cN^{1/6}$ when $t \geq N^{-5/6}$ for a constant $c > 0$, we can simply do a union bound over all integral I and deduce that $\mathcal{E}_O$ happens with probability tending to 1 as $N \to \infty$, where

$$\mathcal{E}_O = \bigcap_{I \text{ is integral}} \left\{ O_I \in [(1-3\epsilon)\mathbb{E}O_I, (1+3\epsilon)\mathbb{E}O_I] \right\}.$$

So we may assume without loss that both $\mathcal{E}_T$ and $\mathcal{E}_O$ occur, i.e., both T_I and O_I are within $[1-3\epsilon, 1+3\epsilon]$ times their respective means for any integral interval I.

We will now argue that with high probability, both T_I and O_I are within $[1-4\epsilon, 1+4\epsilon]$ times their respective means for *any* interval I such that $|I| \geq N^{-5/6}$. For convenience we call any interval $[i/N, (i+1)/N]$ (where $0 \leq i \leq N-1$) a small interval. For any small interval, the probability that there are at least $100 \log N$ labels in it is bounded by $\mathbb{E}\binom{L}{100 \log N}/N^{100 \log N}$, which is at most $1/N^2$ for all large N. Therefore by applying a union bound over all N small intervals, we have that the probability that some small interval contains at least $100 \log N$ labels is $o(1)$. Without loss of generality we assume this event does not occur (i.e., any small interval contains less than $100 \log N$ labels) in what follows. Now we can approximate any interval I of length $t \geq N^{-5/6}$ by an integral interval I' with an error of at most two small intervals, so that $|T_I - T_{I'}|, |O_I - O_{I'}| \leq 200 \log N$. Also, from $\mathbb{E}T_I = (x \coth x)Nt$ and $\mathbb{E}O_I = Np_I = N\frac{\sinh(xt)\cosh(x(1-t))}{\sinh x}$ we see that $\mathbb{E}T_{I'}, \mathbb{E}O_{I'} \geq cN^{1/6}$ for a constant $c > 0$ and $\frac{\mathbb{E}T_I}{\mathbb{E}T_{I'}}, \frac{\mathbb{E}O_I}{\mathbb{E}O_{I'}} = 1 + o(1)$. Therefore, $T_I' \in [(1-3\epsilon)\mathbb{E}T_I', (1+3\epsilon)\mathbb{E}T_I']$ and $O_I' \in [(1-3\epsilon)\mathbb{E}O_I', (1+3\epsilon)\mathbb{E}O_I']$ will imply (respectively) $T_I \in [(1-4\epsilon)\mathbb{E}T_I, (1+4\epsilon)\mathbb{E}T_I]$ and $O_I \in [(1-4\epsilon)\mathbb{E}O_I, (1+4\epsilon)\mathbb{E}O_I]$, as desired.

Now if $|I| > (1/2 + 6\epsilon)$, by the concentration of T_I discussed above, we have

$$T_I \geq (1 - 4\epsilon)\mathbb{E}T_I = (1 - 4\epsilon)(x \coth x)N|I| > \alpha(1/2 + \epsilon)N$$

for all sufficiently small but fixed ϵ. And if $|I| < N^{-5/6}$, then

$$T_I \leq T_{I^*} \leq (1 + 4\epsilon)\mathbb{E}T_{I^*} < N^{\frac{1}{5}}$$

where $I^* \supseteq I$ is an interval of length $N^{-5/6}$. Therefore, we have $N^{\frac{1}{5}} \leq T_I \leq \alpha(1/2 + \epsilon)N$ implies $|I| \in [N^{-5/6}, (1/2 + 6\epsilon)]$. However, if $|I| \in [N^{-5/6}, (1/2 + 6\epsilon)]$, then by the concentration of O_I, we have $O_I \leq (1 + 4\epsilon)\mathbb{E}O_I = (1 + 4\epsilon)Np_I \leq (1/2 + \epsilon_1)N$ for $\epsilon_1 = \epsilon^{1/2}$. Therefore, we see that

$$O_I \leq (1/2 + \epsilon_1)N, \text{ if } N^{\frac{1}{5}} \leq T_I \leq \alpha(1/2 + \epsilon)N\,. \tag{1.34}$$

A similar argument shows that for $\epsilon_2 = \epsilon^{1/4}$, $T_I > \alpha(1/2 + \epsilon_2)N$ implies $|I| > (1/2 + 6\epsilon_1)$, which in turn implies $O_I > (1/2 + \epsilon_1)N$. Therefore

$$O_I > (1/2 + \epsilon_1)N, \text{ if } T_I > \alpha(1/2 + \epsilon_2)N\,. \tag{1.35}$$

Finally, $N^{\frac{1}{5}} \leq T_I \leq \alpha(1/2 + \epsilon_2)N$ implies $|I| \in [N^{-5/6}, (1/2 + 6\epsilon_2)]$. But for $|I| \in [N^{-5/6}, (1/2 + 6\epsilon_2)]$ we have $p_I = \frac{\sinh(x|I|)\cosh(x(1-|I|))}{\sinh x} \geq (x \coth x)|I|\frac{1}{\alpha + \epsilon_3'}$ for $\epsilon_3' = 0.1\epsilon^{1/8}$, i.e.,

$$\mathbb{E}O_I \geq \frac{1}{\alpha + \epsilon_3'}\mathbb{E}T_I\,.$$

By our assumptions on the concentration of O_I and T_I again, we deduce that $O_I \geq \frac{1}{\alpha + \epsilon_3}T_I$ for $\epsilon_3 = \epsilon^{1/8}$. In other words

$$O_I \geq \frac{T_I}{\alpha + \epsilon_3}, \text{ if } N^{\frac{1}{5}} \leq T_I \leq \alpha(1/2 + \epsilon_2)N\,. \tag{1.36}$$

By (1.34), (1.35) and (1.36) we have completed the task of Case 3.

Combining the above three cases, we have completed the proof of (1.13), and thus the proof of the lemma. □

Let $\mathcal{P}$ be the collection of good paths. For any path $P \in \mathcal{P}$, let A_P be the event that P is accessible. So we have $Z_{N,x,*} = \sum_{P \in \mathcal{P}} 1_{A_P}$. Notice that

$$\begin{aligned} \mathbb{E} Z_{N,x,*}^2 &= \sum_{P \in \mathcal{P}} \sum_{P' \in \mathcal{P}} \mathbb{P}(A_P \cap A_{P'}) \\ &= \sum_{P \in \mathcal{P}} \mathbb{P}(A_P) \sum_{P' \in \mathcal{P}} \mathbb{P}(A_{P'} \mid A_P) \\ &= \sum_{P \in \mathcal{P}} \mathbb{P}(A_P) \mathbb{E}(Z_{N,x,*} \mid A_P) \,. \end{aligned} \tag{1.37}$$

So in order to estimate $\mathbb{E} Z_{N,x,*}^2$, a key step is to estimate $\mathbb{E}(Z_{N,x,*} \mid A_P)$. For any good path P of length L, let $v_0 = \vec{0}$, v_1, v_2, $\ldots$, $v_L = \vec{1}$ be the $(L+1)$ vertices it passes through. Let X_i be the (random) value at v_i (recall that $X_0 = 0$ and $X_L = x$). We denote the successive differences of X_i's by $\delta_1 = X_1$, $\delta_2 = X_2 - X_1$, $\cdots$, $\delta_L = x - X_{L-1}$. It is clear that conditioning on P to be accessible, the X_i's are distributed as the order statistics of $(L-1)$ i.i.d. uniform $[0, x]$ random variables, so that the conditional distribution of $(\delta_1/x, \delta_2/x, \cdots, \delta_L/x)$ given A_P is the Dirichlet distribution $\mathrm{Dir}(1, 1, \cdots, 1)$. Recall that a Dirichlet distribution $\mathrm{Dir}(\alpha_1, \alpha_2, \cdots, \alpha_K)$ is supported on $(x_1, x_2, \cdots, x_K)$ where $x_i \in [0, 1]$ for all $i = 1, \ldots, K$ and $\sum_{i=1}^K x_i = 1$, and has a density $\frac{\Gamma(\sum_{i=1}^K \alpha_i)}{\prod_{i=1}^K \Gamma(\alpha_i)} \prod_{i=1}^K x_i^{\alpha_i - 1}$.

We first state some properties of $(\delta_1, \delta_2, \cdots, \delta_L)$ conditioning on A_P (they are also known as *the spacings* of the order statistics).

Proposition 1.1. *For* $0 = i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1} = L$ *and nonnegative integers* $\beta_1, \beta_2, \cdots, \beta_{k+1}$, *(i) Conditional on the event A_P, the distribution of*

$$\frac{1}{x}(X_{i_1} - X_0, X_{i_2} - X_{i_1}, \cdots, X_L - X_{i_k}) = \frac{1}{x}\big(\sum_{i=1}^{i_1} \delta_i, \sum_{i=i_1+1}^{i_2} \delta_i, \cdots, \sum_{i=i_k+1}^{L} \delta_i\big)$$

is the Dirichlet distribution $\mathrm{Dir}(i_1, i_2 - i_1, \cdots, L - i_k)$.

(ii) $\mathbb{E}(\prod_{j=1}^{k+1}(X_{i_j} - X_{i_j-1})^{\beta_j} \mid A_P) \leq \prod_{j=1}^{k+1} \mathbb{E}((X_{i_j} - X_{i_j-1})^{\beta_j} \mid A_P)$.

(iii) $\mathbb{E}((X_{i_1} - X_0)^{\beta_1} \mid A_P) \leq C\sqrt{1+t}(x\frac{i_1-1}{L-1}\frac{(1+t)^{1+1/t}}{e})^{\beta_1}$ *for* $\beta_1 \leq t(i_1 - 1)$*, where* $C > 0$ *is an absolute constant.*

Proof. (i) This follows from the aggregation property of the Dirichlet distribution.

(ii) This follows from the moments of Dirichlet-distributed random variables. That is, for $Y \sim \mathrm{Dir}(\alpha_1, \alpha_2, \cdots, \alpha_K)$, we have

$$\begin{aligned}
\mathbb{E}(\prod_{j=1}^{K} Y_j^{\beta_j}) &= \frac{\Gamma(\sum_{j=1}^{K} \alpha_j)}{\Gamma(\sum_{j=1}^{K} \alpha_j + \beta_j)} \prod_{j=1}^{K} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)} \\
&\leq \prod_{i=1}^{K} \frac{\Gamma(\sum_{j=1}^{K} \alpha_j)}{\Gamma(\beta_i + \sum_{j=1}^{K} \alpha_j)} \prod_{j=1}^{K} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)} \\
&= \prod_{j=1}^{K} \mathbb{E}(Y_j^{\beta_j})
\end{aligned}$$

where the inequality follows from the convexity of $\log \Gamma(x)$ for $x > 0$ and induction.

(iii) As a special case of the moments of Dirichlet-distributed random variables, we have

$$\mathbb{E}((X_{i_1} - X_0)^{\beta_1} \mid A_P) = x^{\beta_1} \frac{\Gamma(L)}{\Gamma(L + \beta_1)} \frac{\Gamma(i_1 + \beta_1)}{\Gamma(i_1)} = x^{\beta_1} \frac{(L-1)!}{(L + \beta_1 - 1)!} \frac{(i_1 + \beta_1 - 1)!}{(i_1 - 1)!}. \tag{1.38}$$

By Stirling's formula, we have for an absolute constant $C > 0$

$$\begin{aligned}
&\mathbb{E}((X_{i_1} - X_0)^{\beta_1} \mid A_P) \\
\leq\ & Cx^{\beta_1} \frac{\sqrt{(L-1)}(\frac{L-1}{e})^{L-1}}{\sqrt{(L+\beta_1-1)}(\frac{L+\beta_1-1}{e})^{L+\beta_1-1}} \frac{\sqrt{(i_1+\beta_1-1)}(\frac{i_1+\beta_1-1}{e})^{i_1+\beta_1-1}}{\sqrt{(i_1-1)}(\frac{i_1-1}{e})^{i_1-1}} \\
=\ & C\big(x\frac{i_1-1}{L-1}\big)^{\beta_1} \frac{\sqrt{(L-1)(i_1+\beta_1-1)}}{\sqrt{(L+\beta_1-1)(i_1-1)}} \left(\frac{(1+\frac{\beta_1}{i_1-1})^{1+\frac{i_1-1}{\beta_1}}}{(1+\frac{\beta_1}{L-1})^{1+\frac{L-1}{\beta_1}}}\right)^{\beta_1}.
\end{aligned}$$

Now by our assumption, we have $\frac{(L-1)(i_1+\beta_1-1)}{(L+\beta_1-1)(i_1-1)} \leq \frac{i_1+\beta_1-1}{i_1-1} \leq 1+t$. In addition, since the function $(1+z)^{1+1/z}$ is increasing in z and tends to e as $z \to 0$, we have $\frac{(1+\frac{\beta_1}{i_1-1})^{1+\frac{i_1-1}{\beta_1}}}{(1+\frac{\beta_1}{L-1})^{1+\frac{L-1}{\beta_1}}} \leq \frac{(1+t)^{1+1/t}}{e}$. Substituting these bounds into the preceding display completes the proof. □

In order to compute $\mathbb{E}(Z_{N,x,*} \mid A_P)$, we first calculate $\mathbb{E}(Z_{N,x,*}(\vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1}) \mid A_P)$, where $\vec{0}$, v_{i_1}, v_{i_2}, ... , v_{i_k} , $\vec{1}$ $(0 = i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1} = L)$ are vertices on path P and $Z_{N,x,*}(\vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1})$ counts the number of good accessible paths P' that intersect P (vertex wise) at $\vec{0}$, v_{i_1}, v_{i_2}, ... , v_{i_k} , $\vec{1}$. For ease of notation we let $v_{i_0} = \vec{0}$ and $v_{i_{k+1}} = \vec{1}$. Naturally these $(k+2)$ common vertices divide both P and P' into $(k+1)$ segments. The lengths of these segments on P are i_1, $(i_2 - i_1)$, ... , $(L - i_k)$. Suppose that P' visits these $(k+2)$ common vertices at its $j_0 = 0$-th, j_1-th, ... , j_{k+1}-th steps. Then on A_P we have

$$\mathbb{P}(A_{P'} \mid X_0, X_1, \cdots, X_L) = \frac{{X_{i_1}}^{j_1-1}}{(j_1-1)!} \frac{(X_{i_2} - X_{i_1})^{j_2-j_1-1}}{(j_2-j_1-1)!} \cdots \frac{(x - X_{i_k})^{j_{k+1}-j_k-1}}{(j_{k+1}-j_k-1)!}.$$

By Part (ii) of Proposition 1.1 we have

$$\begin{aligned}\mathbb{P}(A_{P'} \mid A_P) &= \mathbb{E}(\mathbb{P}(A_{P'} \mid X_0, X_1, \cdots, X_L) \mid A_P) \\ &= \mathbb{E}\Big[\frac{{Y_{i_1}}^{j_1-1}}{(j_1-1)!} \cdot \frac{(Y_{i_2}-Y_{i_1})^{j_2-j_1-1}}{(j_2-j_1-1)!} \cdots \frac{(x-Y_{i_k})^{j_{k+1}-j_k-1}}{(j_{k+1}-j_k-1)!}\Big] \\ &\le \mathbb{E}\frac{{Y_{i_1}}^{j_1-1}}{(j_1-1)!}\mathbb{E}\frac{(Y_{i_2}-Y_{i_1})^{j_2-j_1-1}}{(j_2-j_1-1)!} \cdots \mathbb{E}\frac{(x-Y_{i_k})^{j_{k+1}-j_k-1}}{(j_{k+1}-j_k-1)!}\end{aligned}$$

where $Y_0 = 0, Y_1, \cdots, Y_{L-1}, Y_L = x$ are distributed as the order statistics of $(L-1)$ i.i.d. uniform $[0, x]$ random variables. Therefore, we have

$$\begin{aligned}&\mathbb{E}(Z_{N,x,*}(\vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1}) \mid A_P) \\ = &\sum_{\substack{P' \in \mathcal{P}, \\ P' \text{ intersects } P \text{ at } \vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1}}} \mathbb{P}(A_{P'} \mid A_P) \\ \le &\sum_{\substack{P' \in \mathcal{P}, \\ P' \text{ intersects } P \text{ at } \vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1}}} \prod_{\ell=1}^{k+1} \mathbb{E}\frac{(Y_{i_\ell} - Y_{i_{\ell-1}})^{j_\ell - j_{\ell-1}-1}}{(j_\ell - j_{\ell-1}-1)!} \\ \le &\prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell}) \end{aligned} \tag{1.39}$$

where $F(v_{i_{\ell-1}}, v_{i_\ell})$ is defined as follows.

Definition 1.4. *For $u, v \in H_N$, we say a path P^* connecting u to v is a good segment from u to v, if there exists at least one good path whose subpath from u to v is P^*. For any good path $P = v_0, v_1, \ldots, v_L$ and $0 \le i < j \le L$, let $F(v_i, v_j) = \mathbb{E}G(v_i, v_j, Y_i, Y_j)$ where $G(v_i, v_j, y_i, y_j)$ is the conditional expectation of the number of good accessible segments from v_i to v_j, given that $X_i = y_i$ and $X_j = y_j$.*

Now summing inequality (1.39) over $i_1, i_2, \ldots, i_k$ and k, we have

$$\mathbb{E}(Z_{N,x,*} \mid A_P) \le \sum_{k, i_1, i_2, \ldots, i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell}). \tag{1.40}$$

We can further split the sum on the right hand side into two parts, according to whether $\max\{i_1, (i_2 - i_1), \ldots, (L - i_k)\} > L/2$ (i.e. whether the longest segment on P is longer than $L/2$). That is,

$$\begin{aligned}
&\sum_{k,i_1,i_2,\ldots,i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell}) \\
= &\left(\sum_{\substack{k,i_1,i_2,\ldots,i_k, \\ \max\{i_1,(i_2-i_1),\ldots,(L-i_k)\}>L/2}} + \sum_{\substack{k,i_1,i_2,\ldots,i_k, \\ \max\{i_1,(i_2-i_1),\ldots,(L-i_k)\}\le L/2}} \right) \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell}) \\
\le &\Big(\sum_{d=0}^{L/2} \sum_{d_1+d_2=d} F(v_{d_1}, v_{L-d_2}) \Big) \prod_{j=0}^{L-1} (\sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i})) + \prod_{j=0}^{L-1} (\sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i})) . \quad (1.41)
\end{aligned}$$

To justify the last inequality, we first point out that $F(v_j, v_{j+1})$ is always 1 because the Hamming distance between a pair of vertices on a good path is 1 if and only if these two vertices are neighboring each other on the path. Given any k and $0 < i_1 < i_2 < \cdots < i_k < L$, we define $u_j(k, i_1, i_2, \ldots, i_k)$ for $j = 0, 1, \ldots, L-1$ as:

$$u_j(k, i_1, i_2, \ldots, i_k) = \begin{cases} v_{i_{\ell+1}} , & \text{if } j = i_\ell \text{ for some } 1 \le \ell \le k \text{ and } i_{\ell+1} - i_\ell > 1 \\ v_{j+1} , & \text{otherwise} \end{cases}$$

Thus for any k and $0 < i_1 < i_2 < \cdots < i_k < L$

$$\prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell}) = \prod_{j=0}^{L-1} F(v_j, u_j) .$$

Moreover, it is not hard to verify that $\vec{u} := (u_0, u_1, \cdots, u_{L-1})$ is an injective function of $(k, i_1, i_2, \ldots, i_k)$, i.e., for any $(k, i_1, i_2, \ldots, i_k) \neq (k', i'_1, i'_2, \ldots, i'_{k'})$ such that $0 < i_1 < i_2 < \cdots < i_k < L$ and $0 < i'_1 < i'_2 < \cdots < i'_{k'} < L$, $u_j(k, i_1, i_2, \ldots, i_k) = u_j(k', i'_1, i'_2, \ldots, i'_{k'})$

cannot hold for all $j = 0, 1, \ldots, L-1$. Therefore

$$\sum_{\substack{k, i_1, i_2, \ldots, i_k, \\ \max\{i_1, (i_2-i_1), \ldots, (L-i_k)\} \le L/2}} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell}) = \sum_{\substack{k, i_1, i_2, \ldots, i_k, \\ \max\{i_1, (i_2-i_1), \ldots, (L-i_k)\} \le L/2}} \prod_{j=0}^{L-1} F(v_j, u_j)$$
$$\le \prod_{j=0}^{L-1} (\sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i}))\,.$$

The other part of the inequality can be obtained similarly.

The following two lemmas are useful for bounding $\mathbb{E}(Z_{N,x,*} \mid A_P)$.

Lemma 1.4. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_2 > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \le \iota$, $N > N'$ and any good path P we have $\sum_{d=0}^{L/2} \sum_{d_1+d_2=d} F(v_{d_1}, v_{L-d_2}) \le C_2 N (\sinh x)^{N-1} \cosh x$.*

Lemma 1.5. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_3 > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \le \iota$, $N > N'$, any good path P and any j we have $\sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i}) \le 1 + \frac{C_3}{N}$.*

Corollary 1.2. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_4 > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \le \iota$ and $N > N'$*

$$\mathbb{E} Z_{N,x,*}^2 \le (C_4 N \sinh^{N-1} x \cosh x + C_4) N \sinh^{N-1} x \cosh x\,.$$

Proof. Substituting the bounds from Lemmas 1.4 and 1.5 into (1.41) and using (1.40), we see that

$$\mathbb{E}(Z_{N,x,*} \mid A_P) \le \sum_{k, i_1, i_2, \ldots, i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_\ell})$$
$$\le (C_2 N (\sinh x)^{N-1} \cosh x + 1)(1 + \frac{C_3}{N})^{(1+\epsilon)\alpha N}$$
$$\le (C_2 N (\sinh x)^{N-1} \cosh x + 1) e^{C_3 (1+\epsilon)\alpha}\,.$$

Substituting the above inequality into (1.37) and applying the inequality

$$\sum_{P\in\mathcal{P}} \mathbb{P}(A_P) = \mathbb{E}Z_{N,x,*} \leq \mathbb{E}Z_{N,x} \leq N(\sinh x)^{N-1}\cosh x$$

(here the last inequality follows from Corollary 1.1), we complete the proof of the corollary.

□

In order to prove Lemmas 1.4 and 1.5, we need the following lemma.

Lemma 1.6. *Suppose that $N \geq 7$, $s \geq 1$. Let $g(y,s) = (\sinh y)^s(\cosh y)^{N-s}$. Then $\frac{\partial g}{\partial y}(y,s)$ is decreasing in s for all fixed $y > 0$.*

Proof. By a direct calculation

$$\begin{aligned}\frac{\partial g}{\partial y}(y,s) &= (\sinh y)^s(\cosh y)^{N-s}(s\coth y + (N-s)\tanh y)\\ &= (\sinh y)^{-1}(\cosh y)^{N-1}(\tanh y)^s(s + N(\sinh y)^2)\,.\end{aligned}$$

Therefore it suffices to show that $(\tanh y)^s(s + N(\sinh y)^2)$ is decreasing in s. Taking the partial derivative with respect to s we get

$$\frac{\partial}{\partial s}[(\tanh y)^s(s+N(\sinh y)^2)] = (\tanh y)^s + (\ln\tanh y)(\tanh y)^s(s+N(\sinh y)^2)\,,$$

so we only need to show that $(\coth y)^{(s+N(\sinh y)^2)} \geq e$. If $\coth y \geq e$, then plainly we have $(\coth y)^{(s+N(\sinh y)^2)} \geq (\coth y)^s \geq \coth y \geq e$. On the other hand, if $\coth y < e$, then $y > \operatorname{arccoth} e := y_0$. Since $(\coth y)^{(\sinh y)^2}$ is increasing in y, we have $(\coth y)^{(\sinh y)^2} \geq (\coth y_0)^{(\sinh y_0)^2} = e^{\frac{1}{e^2-1}} \approx 1.17$. Therefore we have $(\coth y)^{(s+N(\sinh y)^2)} \geq (\coth y)^{7(\sinh y)^2} \geq (\coth y_0)^{7(\sinh y_0)^2} > e$ in this case. □

Proof of Lemma 1.4 For d_1 and d_2 such that $d_1 + d_2 = d$, it is clear that the Hamming distance $H(v_{d_1}, v_{L-d_2})$ between v_{d_1} and v_{L-d_2} is greater than or equal to $N-d$. Therefore, by (1.9) and Lemma 1.6, we have

$$\begin{aligned}
F(v_{d_1}, v_{L-d_2}) &= \mathbb{E}G(v_{d_1}, v_{L-d_2}, Y_{d_1}, Y_{L-d_2}) \\
&\leq \mathbb{E}((\sinh y)^{H(v_{d_1}, v_{L-d_2})}(\cosh y)^{N-H(v_{d_1}, v_{L-d_2})})'|_{y=Y_{L-d_2}-Y_{d_1}} \\
&\leq \mathbb{E}((\sinh y)^{N-d}(\cosh y)^{d})'|_{y=Y_{L-d_2}-Y_{d_1}} \\
&= \mathbb{E}((\sinh y)^{N-d}(\cosh y)^{d})'|_{y=x-Y_d}
\end{aligned}$$

where the last equality is because the distribution of $Y_{L-d_2} - Y_{d_1}$ does not depend on (d_1, d_2) provided the value of $d = d_1 + d_2$. Writing out the derivative in the last step, we have the following estimate

$$\begin{aligned}
F(v_{d_1}, v_{L-d_2}) &\leq \mathbb{E}((\sinh y)^{N-d-1}(\cosh y)^{d-1}((N-d)(\cosh y)^2 + d(\sinh y)^2))|_{y=x-Y_d} \\
&\leq \mathbb{E}((\sinh y)^{N-d-1}(\cosh y)^{d-1}N(\cosh y)^2)|_{y=x-Y_d} \\
&\leq N(\cosh x)^2\mathbb{E}(\sinh(x-Y_d))^{N-d-1}(\cosh x)^{d-1}.
\end{aligned}$$

Since $\sinh(x-y) \leq \sinh x - \frac{\sinh x}{x}y$ for $0 \leq y \leq x$, we have further

$$\begin{aligned}
F(v_{d_1}, v_{L-d_2}) &\leq N(\cosh x)^2\mathbb{E}(\sinh x - \frac{\sinh x}{x}Y_d)^{N-d-1}(\cosh x)^{d-1} \\
&= N(\cosh x)^2(\sinh x)^{N-d-1}(\cosh x)^{d-1}\mathbb{E}(1-\frac{Y_d}{x})^{N-d-1}. \qquad (1.42)
\end{aligned}$$

It remains to bound $\mathbb{E}(1-\frac{Y_d}{x})^{N-d-1}$. Since $1-\frac{Y_d}{x}$ is the $(L-d)$th order statistic of $(L-1)$ i.i.d. uniform $[0,1]$ random variables, it has a Beta$(L-d, d)$ distribution. Thus by the moments of Beta-distributed random variables (or applying (1.38)) we have

$$\mathbb{E}(1-\frac{Y_d}{x})^{N-d-1} = \prod_{r=0}^{N-d-2}\frac{L-d+r}{L+r} \qquad (1.43)$$

which can be further bounded by

$$\prod_{r=0}^{N-d-2} \frac{L-d+r}{L+r} \leq (1 - \frac{d}{L+N-d-2})^{N-d-1}$$
$$\leq (e^{-\frac{N-d-1}{L+N-d-2}})^d \leq (\frac{0.995}{\coth x})^d \tag{1.44}$$

for $d \leq 0.32N$, ϵ (and therefore ι) sufficiently small and N sufficiently large (recall that $L \in [\alpha(1-\epsilon)N, \alpha(1+\epsilon)N]$ for a good path). Here we used the inequality $e^{-\frac{1-0.32}{\alpha+1-0.32}} \leq \frac{0.994}{\coth x_0}$ (by brute force calculation).

For $0.32N \leq d \leq \alpha(1/2+\epsilon)N$, set $t = d/N$ and $s = L/N$. Then by Stirling's formula

$$\prod_{r=0}^{N-d-2} \frac{L-d+r}{L+r} \leq C_5 \prod_{r=1}^{N-d} \frac{L-d+r}{L+r} \leq C_6 \frac{(L+N-2d)^{L+N-2d} L^L}{(L-d)^{L-d}(L+N-d)^{L+N-d}}$$
$$= C_6 \Big((\frac{(1+s-2t)^{1+s-2t} s^s}{(s-t)^{s-t}(1+s-t)^{1+s-t}})^{\frac{1}{t}} \Big)^d.$$

Another brute force calculation gives

$$\Big(\frac{(1+\alpha-2t)^{1+\alpha-2t} \alpha^\alpha}{(\alpha-t)^{\alpha-t}(1+\alpha-t)^{1+\alpha-t}} \Big)^{\frac{1}{t}} \leq \frac{0.999}{\coth x_0}$$

for $t \leq \alpha(1/2+\epsilon)$ and ϵ sufficiently small. Since the function $h(y,t)$ given by

$$h(y,t) = \Big(\frac{(1+y-2t)^{1+y-2t} y^y}{(y-t)^{y-t}(1+y-t)^{1+y-t}} \Big)^{\frac{1}{t}}$$

is uniformly continuous with respect to (y,t) on $[1.0, 1.5] \times [0.2, 0.8]$, we have for ϵ sufficiently small (so that s is sufficiently close to α) and $0.32 \leq t \leq \alpha(1/2+\epsilon)$

$$\Big(\frac{(1+s-2t)^{1+s-2t} s^s}{(s-t)^{s-t}(1+s-t)^{1+s-t}} \Big)^{\frac{1}{t}} \leq \frac{0.9999}{\coth x_0}.$$

In addition, for ϵ (and therefore ι) sufficiently small, the right hand side of the above inequality is at most $0.99999/\coth x$. So we get $\prod_{r=0}^{N-d-2} \frac{L-d+r}{L+r} \leq C_6(\frac{0.99999}{\coth x})^d$ in this case. Combined with (1.42), (1.43) and (1.44), this completes the proof of the lemma. $\square$

Proof of Lemma 1.5 Recall that $P = v_0, v_1, \ldots, v_L$ is a good path of length L. For an arbitrary j, we will bound $F(v_j, v_{j+i})$ in a number of regimes depending on the value of i, as follows.

Case (a): $i = 1$**.** Since for any good path (or good segment), the Hamming distance between a pair of vertices on the path is 1 if and only if these two vertices are neighboring each other on the path, we have $F(v_j, v_{j+1}) = 1$.

Case (b): $i = 2$**.** The Hamming distance between v_j and v_{j+2} is precisely 2 (since P is good), and thus the length of any good segment connecting v_j to v_{j+2} is either 2 or 4. There are at most 2 such segments of length 2, and the probability for each of them to be accessible given $X_j = y_j$ and $X_{j+2} = y_{j+2}$ is $(y_{j+2} - y_j)$. Similarly, there are at most $(N\binom{4}{2}2!)$ such segments of length 4, and the probability for each of them to be accessible given $X_j = y_j$ and $X_{j+2} = y_{j+2}$ is $\frac{(y_{j+2}-y_j)^3}{3!}$. Therefore,

$$\begin{aligned} G(v_j, v_{j+2}, y_j, y_{j+2}) &\leq 2(y_{j+2} - y_j) + (N\binom{4}{2}2!)\frac{(y_{j+2} - y_j)^3}{3!} \\ &= 2(y_{j+2} - y_j) + 2N(y_{j+2} - y_j)^3. \end{aligned}$$

Combined with (1.38), this yields that

$$F(v_j, v_{j+2}) \leq 20/N \text{ for sufficiently large } N\,.$$

Case (c): $i = 3$**.** The Hamming distance between v_j and v_{j+3} is precisely 3 (since P is good), and thus the length of any good segment connecting v_j to v_{j+3} is either 3 or 5.

Similar to the previous case, we have

$$G(v_j, v_{j+3}, y_j, y_{j+3}) \le 3(y_{j+3} - y_j)^2 + (N\binom{5}{2}3!)\frac{(y_{j+3} - y_j)^4}{4!}$$
$$= 3(y_{j+3} - y_j)^2 + (\frac{5}{2}N)(y_{j+3} - y_j)^4.$$

Combined with (1.38), this yields that

$$F(v_j, v_{j+3}) \le 1000 \cdot N^{-2} \text{ for sufficiently large } N\,.$$

Case (d): $4 \le i \le N^{\frac{1}{5}}$. By the definition of good path and good segment again, we see that all the possible values of the pair $(H(v_j, v_{j+i}), L(v_j, v_{j+i}))$ (where $L(v_j, v_{j+i})$ is the length of a good segment connecting v_j to v_{j+i}) are (i, i), $(i, i+2)$, $(i-2, i-2)$ and $(i-2, i)$. Therefore $G(v_j, v_{j+i}, y_j, y_{j+i})$ is at most

$$i(y_{j+i}-y_j)^{i-1} + \frac{N\binom{i+2}{2}i!}{(i+1)!}(y_{j+i}-y_j)^{i+1} + (i-2)(y_{j+i}-y_j)^{i-3} + \frac{N\binom{i}{2}(i-2)!}{(i-1)!}(y_{j+i}-y_j)^{i-1}\,.$$

Combined with (1.38), this yields that

$$F(v_j, v_{j+4}) \le 10^4 \cdot N^{-1} \text{ for sufficiently large } N\,,$$

$$F(v_j, v_{j+5}) \le 10^4 \cdot N^{-1} \text{ for sufficiently large } N\,,$$

$$F(v_j, v_{j+6}) \le 10^4 \cdot N^{-1} \text{ for sufficiently large } N$$

and

$$F(v_j, v_{j+i}) \le 10^4 \cdot (i(\frac{i}{N})^4 + Ni(\frac{i}{N})^6) \le 10^4 \cdot N^{-2} \text{ for sufficiently large } N$$

when $7 \le i \le N^{\frac{1}{5}}$.

Case (e): $N^{\frac{1}{5}} \le i \le L/2$. Recall the definitions of $\epsilon_1, \epsilon_2, \epsilon_3$ in (1.10). By the definition of good path, we have $\frac{i}{\alpha+\epsilon_3} \leq H(v_j, v_{j+i}) \leq (1/2+\epsilon_1)N$. Therefore (by the definition of good path again) any good segment that connects v_j to v_{j+i} must have length $L(v_j, v_{j+i}) \leq \alpha(1/2+\epsilon_2)N$, so that $L(v_j, v_{j+i})$ also satisfies $L(v_j, v_{j+i}) \leq (\alpha+\epsilon_3)H(v_j, v_{j+i}) \leq (\alpha+\epsilon_3)i$. By Part (iii) of Proposition 1.1, we have

$$\mathbb{E}(Y_{j+i}-Y_j)^{\ell-1} \leq C\sqrt{1+\alpha+\epsilon_3}(x\frac{i-1}{L-1}\frac{(1+(\alpha+\epsilon_3))^{1+1/(\alpha+\epsilon_3)}}{e})^{\ell-1}$$

for $\ell \leq (\alpha+\epsilon_3)(i-1)+1$. Therefore by (1.8) and Lemma 1.6, we have

$$\begin{aligned}
& F(v_j, v_{j+i}) \\
= & \sum_{\substack{P^* \text{ is a good segment of length } \ell \\ \text{connecting } v_j \text{ to } v_{j+i}}} \frac{\mathbb{E}(Y_{j+i}-Y_j)^{\ell-1}}{(\ell-1)!} \\
\leq & C\sqrt{1+\alpha+\epsilon_3}((\sinh y)^{H(v_j,v_{j+i})}(\cosh y)^{N-H(v_j,v_{j+i})})'|_{y=x\frac{i-1}{L-1}\frac{(1+(\alpha+\epsilon_3))^{1+1/(\alpha+\epsilon_3)}}{e}} \\
\leq & C\sqrt{1+\alpha+\epsilon_3}((\sinh y)^{\frac{i}{\alpha+\epsilon_3}}(\cosh y)^{N-\frac{i}{\alpha+\epsilon_3}})'|_{y=x\frac{i-1}{L-1}\frac{(1+(\alpha+\epsilon_3))^{1+1/(\alpha+\epsilon_3)}}{e}} \\
\leq & C_7N^2(\sinh y)^{\frac{i}{\alpha+\epsilon_3}}(\cosh y)^{N-\frac{i}{\alpha+\epsilon_3}}|_{y=x\frac{i}{L-1}\frac{(1+(\alpha+\epsilon_3))^{1+1/(\alpha+\epsilon_3)}}{e}}.
\end{aligned}$$

Set $a = \frac{N(\alpha+\epsilon_3)}{L-1}$, $c = x\frac{(1+(\alpha+\epsilon_3))^{1+1/(\alpha+\epsilon_3)}}{e}$, and $c_0 = x_0\frac{(1+\alpha)^{1+1/\alpha}}{e} \approx 1.39$. Clearly c will be sufficiently close to c_0 if ϵ (and therefore ι) is sufficiently small. Let $t = \frac{i}{L-1}$ (so that $\frac{N^{\frac{1}{5}}}{L} \leq t \leq 1/2$) and $h(t) := (\sinh(ct))^{\frac{t}{\alpha+\epsilon_3}}(\cosh(ct))^{\frac{N}{L-1}-\frac{t}{\alpha+\epsilon_3}}$. Then the preceding inequality can be rewritten as $F(v_j, v_{j+i}) \leq C_7N^2(h(t))^{L-1}$. In order to estimate $F(v_j, v_{j+i})$, we analyze the behavior of the function $h(t)$ as follows. By straightforward computation, we have

$$(\alpha+\epsilon_3)\ln h(t) = t\ln\sinh(ct) + (a-t)\ln\cosh(ct)\,,$$

$$((\alpha+\epsilon_3)\ln h(t))' = \ln\sinh(ct) - \ln\cosh(ct) + ct\coth(ct) + c(a-t)\tanh(ct)$$

and

$$
\begin{aligned}
((\alpha+\epsilon_3)\ln h(t))'' &= c\coth(ct) - c\tanh(ct) + c\coth(ct) - c\tanh(ct) \\
&\quad - \frac{c^2 t}{(\sinh(ct))^2} + \frac{c^2(a-t)}{(\cosh(ct))^2} \\
&\geq 2c(\coth(ct) - \tanh(ct)) - c^2 t\left(\frac{1}{(\sinh(ct))^2} + \frac{1}{(\cosh(ct))^2}\right) \\
&= \frac{c}{(\sinh(ct))^2(\cosh(ct))^2}(\sinh(2ct) - ct\cosh(2ct)) > 0
\end{aligned}
$$

for $t \leq 1/2$ (since $ct \leq c/2 < 0.8$).

Therefore $(\alpha+\epsilon_3)\ln h(t)$, and consequently $h(t)$ is convex up to $t = 1/2$. Thus we have $h(t) \leq \max(h(\frac{N^{\frac{1}{5}}}{L}), h(1/2))$, and so $F(v_j, v_{j+i}) \leq C_7 N^2 \max((h(\frac{N^{\frac{1}{5}}}{L}))^{L-1}, (h(1/2))^{L-1})$. However, since $(h(1/2))^{2(\alpha+\epsilon_3)} = \sinh(\frac{c}{2})(\cosh(\frac{c}{2}))^{2(\alpha+\epsilon_3)\frac{N}{L-1}-1}$ which is sufficiently close to $\sinh(\frac{c_0}{2})\cosh(\frac{c_0}{2}) = \frac{1}{2}\sinh(c_0) < 1$ if ϵ is sufficiently small and N is sufficiently large, we have $h(1/2) \leq p$ where p is a constant strictly less than 1. Thus, $(h(1/2))^{L-1} \leq p^{L-1}$. On the other hand, $(h(\frac{N^{\frac{1}{5}}}{L}))^{L-1} \leq (N^{-\frac{3}{5}})^{N^{\frac{1}{5}}}(1+N^{-\frac{8}{5}})^N$ for sufficiently large N. Thus we have for N sufficiently large,

$$
F(v_j, v_{j+i}) \leq C_7 N^2 \max(p^{L-1}, (N^{-\frac{3}{5}})^{N^{\frac{1}{5}}}(1+N^{-\frac{8}{5}})^N)\,.
$$

Conclusion. Summing $F(v_j, v_{j+i})$ over $1 \leq i \leq L/2$ and applying the bounds we obtained in Cases (a), (b), (c), (d) and (e), we see that $\sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i}) \leq 1 + \frac{C_3}{N}$ for some $C_3 > 0$, completing the proof of the lemma. □

Proposition 1.2. *There exists* $0 \leq K < 1$ *such that, if* $\liminf_{N\to\infty} \mathbb{P}(Z_{N,x_c+\epsilon_N} > 0) \geq C$ *for some constant* $C \geq 0$ *whenever* $N\epsilon_N \to \infty$*, then whenever* $N\epsilon_N \to \infty$ *we have*

$$
\liminf_{N\to\infty} \mathbb{P}(Z_{N,x_c+\epsilon_N} > 0) \geq 1 - (1-C)K\,.
$$

Proof. Our strategy basically follows that of [10]. First we pick four vertices a_1, a_2, b_1, b_2

satisfying: a_1 and a_2 are neighbors of $\vec{0}$ and have a value in $[0, \epsilon_N/3]$, b_1 and b_2 are neighbors of $\vec{1}$ and have a value in $[x - \epsilon_N/3, x]$, and none of the four pairs (a_i, b_j) are antipodal. Since $N\epsilon_N \to \infty$, this can be achieved with probability $1 - o_N(1)$.

Without loss of generality assume that the only coordinates of a_1, a_2, b_1 and b_2 that are different from $\vec{0}$ or $\vec{1}$ are $1, 2, 3$ and 4, respectively. Let $\tilde{H}_1$ and $\tilde{H}_2$ be the $(N-2)$ dimensional sub-hypercubes of $\{0,1\}^N$ formed by a_1, b_1 and a_2, b_2, respectively. That is, $\tilde{H}_1$ is the sub-hypercube with the first coordinate being 1 and the third coordinate being 0, and $\tilde{H}_2$ is the sub-hypercube with the second coordinate being 1 and the fourth coordinate being 0. Let H_2' be $\tilde{H}_2 \setminus \tilde{H}_1$. Denote by $p_{\tilde{H}_1}$ and $p_{H_2'}$ the probabilities that there is an accessible path in $\tilde{H}_1$ (from a_1 to b_1) and H_2' (from a_2 to b_2) respectively. From the disjointness (and hence independence) of $\tilde{H}_1$ and H_2' we have $\mathbb{P}(Z_{N,x_c+\epsilon_N} > 0) \geq 1 - (1 - p_{\tilde{H}_1})(1 - p_{H_2'}) - o_N(1)$. Clearly $p_{\tilde{H}_1} \geq \mathbb{P}(Z_{N-2,x_c+\epsilon_N/3} > 0) \geq C - o_N(1)$.

It remains to show that $p_{H_2'}$ is bounded from below by a positive constant $1 - K$. To this end, we note that if we only consider the good path in $\tilde{H}_2$ (from a_2 to b_2) which only updates Coordinate 1 and Coordinate 3 once and Coordinate 3 is updated before Coordinate 1 (that is, in the associated sequence the numbers 1 and 3 occur precisely once each and 3 occurs ahead of 1), such path must be contained in H_2'. Clearly, the number of such accessible paths has second moment less than $\mathbb{E}Z^2_{N-2,x_c+\epsilon_N/3,*}$ and first moment within an absolute multiplicative constant of $\mathbb{E}Z_{N-2,x_c+\epsilon_N/3,*}$ (indeed, the first moment is at least $C_1(N-2)\sinh^{N-3}(x)\cosh x \cdot (\frac{x}{\sinh x})^2 \cdot \frac{1}{2}$ where $x = x_c + \epsilon_N/3$). Combined with Lemma 1.3 and Corollary 1.2, this yields that $p_{H_2'} \geq 1 - K - o_N(1)$ for some constant $K < 1$. This completes the proof of the proposition. □

Proof of (1.2): antipodal case Applying Proposition 1.2 recursively (starting from $C = 0$) completes the proof of (1.2). □

At the end of this section, we provide

Proof of (1.3)**: antipodal case** For the lower bound, it suffices to consider $x = x_c - \Delta/N$. By Remark 1.3, we have in this case $N(\sinh x)^{N-1}\cosh x \geq m_1(\Delta)$ where $m_1(\Delta) > 0$ depends only on Δ. Applying the second moment method and using Lemma 1.3 and Corollary 1.2, we obtain that (for sufficiently large N)

$$\mathbb{P}(Z_{N,x} > 0) \geq \mathbb{P}(Z_{N,x,*} > 0) \geq \frac{(\mathbb{E}Z_{N,x,*})^2}{\mathbb{E}Z^2_{N,x,*}} \geq c_1(\Delta)\,,$$

where $c_1(\Delta) > 0$ depends only on Δ.

For the upper bound, it suffices to consider $x = x_c + \Delta/N$. Let $K > 0$ be a large number depending on Δ that we specify later. The idea is to condition on the values of the neighbors of $\vec{0}$. Let $u_1, u_2, \ldots, u_N$ be these neighbors. For $1 \leq i \leq N$ and $\frac{K}{N} \leq y_i \leq x$, we upper bound the conditional probability that $\vec{1}$ is accessible from u_i given $X_{u_i} = y_i$ by the corresponding first moment, which by (1.9) can be further bounded by $((\sinh t)^{N-1}\cosh t)'|_{t=x-y_i} \leq 2N(\sinh(x-y_i))^{N-2}$. Therefore

$$\begin{aligned}
\mathbb{P}(Z_{N,x} = 0) \;\geq\; & \int_{\frac{K}{N}}^1 \int_{\frac{K}{N}}^1 \cdots \int_{\frac{K}{N}}^1 [1 - 2N(\sinh(x-y_1))^{N-2} 1_{y_1 \leq x} - \cdots \\
& -2N(\sinh(x-y_N))^{N-2} 1_{y_N \leq x}]\, dy_1\, dy_2 \cdots dy_N \\
=\; & (1 - \frac{K}{N})^N - (1 - \frac{K}{N})^{N-1} \int_{\frac{K}{N}}^1 2N^2 (\sinh(x-y_1))^{N-2} 1_{y_1 \leq x}\, dy_1\,,
\end{aligned}$$

where

$$\begin{aligned}
\int_{\frac{K}{N}}^1 2N^2(\sinh(x-y_1))^{N-2} 1_{y_1 \leq x}\, dy_1 &= \int_{\frac{K}{N}}^x 2N^2(\sinh(x-y_1))^{N-2}\, dy_1 \\
&= \int_0^{x_0 - \frac{\sqrt{2}}{2}\frac{\ln N}{N} + \frac{\Delta}{N} - \frac{K}{N}} 2N^2(\sinh y)^{N-2}\, dy \\
&\to \sqrt{2} e^{\sqrt{2}(\Delta - K)}
\end{aligned}$$

Here the last step follows from [18, problem 213 (in Part Two Chapter 5 section 2)] by setting

$\varphi(x) = 1, h(x) = \ln \sinh x, a = 0, \xi = x_0, \alpha = -\frac{\sqrt{2}}{2}, \beta = \Delta - K$. Therefore $\liminf_{N\to\infty} \mathbb{P}(Z_{N,x} = 0) \geq e^{-K}(1-\sqrt{2}e^{\sqrt{2}(\Delta-K)})$, and we are done by choosing K to be a large number depending on Δ. $\square$

1.3 Accessibility percolation: general case

Since most of our proof in the antipodal case carries over to the general case, in the following proof for the general case we will emphasize the parts that require nontrivial modification.

Fix $0 < \beta < 1$ throughout this section. Recall from the statement of Theorem 1.1 that $f(x) = (\sinh x)^\beta (\cosh x)^{1-\beta}$, that x_0 is the unique root of $f(x) = 1$ and that $x_c = x_0 - \frac{1}{f'(x_0)} \frac{\ln N}{N}$. We have

$$f'(x) = (\beta \coth x + (1-\beta)\tanh x)(\sinh x)^\beta (\cosh x)^{1-\beta},$$

so that $f'(x_0) = \beta \coth x_0 + (1-\beta)\tanh x_0$. In addition, it is straightforward to check that $0 < f''(x_0) < \infty$. The proof of (1.1) resembles that in the antipodal case.

Proof of (1.1)**: general case** In light of (1.8) we denote by

$$M_{N,\beta,x} := \big((\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}\big)' = ((f(x))^N)' = N(f(x))^{N-1}f'(x)\,.$$

We have $M_{N,\beta,x} \asymp N(f(x))^N$ for, say $|x - x_0| \leq 1/10$. Since $\mathbb{P}(Z_{N,x} > 0)$ is monotone in x, we can assume without loss of generality that $\epsilon_N \leq N^{-2/3}$. With this assumption, we have for $x = x_c \pm \epsilon_N = x_0 - \frac{1}{f'(x_0)}\frac{\ln N}{N} \pm \epsilon_N$,

$$(x - x_0)^2 = (\frac{1}{f'(x_0)}\frac{\ln N}{N} \pm \epsilon_N)^2 = o(1/N)$$

and thus

$$
\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x - x_0) + o(1/N) \\
&= 1 - \frac{\ln N}{N} \pm f'(x_0)\epsilon_N + o(1/N)\,.
\end{aligned}
$$

Therefore, $M_{N,\beta,x_c-\epsilon_N} \to 0$ and $M_{N,\beta,x_c+\epsilon_N} \to \infty$ as $N \to \infty$. Combined with (1.9), it gives that $\mathbb{E}Z_{N,x_c-\epsilon_N} \to 0$ as $N \to \infty$, yielding (1.1). $\square$

We next turn to prove (1.2). To this end, we first need to revise the definition of good path. Let

$$
\gamma = \beta x_0 \coth x_0 + (1-\beta)x_0 \tanh x_0 = x_0 f'(x_0)
$$

as in statement (1.15) (it will play the role of α). Also in the general case, by a similar calculation as equation (1.32), we see that the definition of $g(t)$ in (1.28) should be modified as

$$
g(t) := \beta \frac{\sinh(x_0 t)\cosh(x_0(1-t))}{\sinh x_0} + (1-\beta)\frac{\sinh(x_0(1-t))\sinh(x_0 t)}{\cosh x_0}
$$

so that $g(t)N$ still means the "expected Hamming distance traveled by a path in time t". In addition, for a pair of vertices u and v, we let $H'(u,v)$ be their Hamming distance restricted to the first βN coordinates (i.e., the number of the first βN coordinates at which u differs from v).

Definition 1.5 (general case)**.** *Let $\epsilon > 0$ be a sufficiently small fixed number to be selected and set $\epsilon_4 = \epsilon^{1/8}$. We say a path (or the associated update sequence) $v_0 = \vec{0}_N = (0,0,\cdots,0), v_1, \ldots, v_{L-1}, v_L = (\vec{1}_{\beta N}, \vec{0}_{N-\beta N}) = (1,\cdots,1,0,\cdots,0)$ is good if the following holds:*

(a) The total number of updates of the first βN coordinates lies within

$$
[\beta x_0 \coth x_0 (1-\epsilon)N, \beta x_0 \coth x_0 (1+\epsilon)N]
$$

and the total number of updates of the last $(1-\beta)N$ *coordinates lies within*

$$[(1-\beta)x_0 \tanh x_0(1-\epsilon)N, (1-\beta)x_0 \tanh x_0(1+\epsilon)N]\,.$$

(b) $H(v_i, v_j) = |i-j|$, *if* $|i-j| = 1, 2, 3.$

(c) *For* $|i-j| > 3$ *we have*

$H(v_i, v_j) = |i-j|$ *or* $|i-j|-2$, *if* $4 \le |i-j| \le N^{\frac{1}{5}}$;

$H'(v_i, v_j) \le (1/2+\epsilon_1)\beta N$, *if* $|i-j| \le \gamma(1/2+\epsilon)N$;

$H'(v_i, v_j) > (1/2+\epsilon_1)\beta N$, *if* $|i-j| > \gamma(1/2+\epsilon_2)N$;

$H(v_i, v_j) \ge \frac{2g(1/2)|i-j|}{\gamma+\epsilon_3}$, *if* $N^{\frac{1}{5}} \le |i-j| \le \gamma(1/2+\epsilon_2)N.$

(d) *Let* $D(v_0, v_i)$ *be the number of updates of the first* βN *coordinates among the first* i *updates, and* $D(v_{L-i}, v_L)$ *be the number of updates of the first* βN *coordinates among the last* i *updates. Then both* $D(v_0, v_i)$ *and* $D(v_{L-i}, v_L)$ *are less than or equal to* δi *for any* $i \le L/2$, *where* $\delta := \frac{\beta \coth x_0}{\beta \coth x_0 + (1-\beta)\tanh x_0} + \epsilon_4$.

As in the antipodal case, it is clear that a good path is self-avoiding. In addition, we have $L \in [\gamma(1-\epsilon)N, \gamma(1+\epsilon)N]$ by Property (a).

Lemma 1.7. *For any sufficiently small but fixed number* $\epsilon > 0$, *there exist* $C_1' > 0$ *and an integer* $N' > 0$ *which both depend only on* ϵ, *such that for all* $|x - x_0| \le \iota$ *and* $N > N'$ *we have*

$$\mathbb{E}Z_{N,x,*} \ge C_1' M_{N,\beta,x} = C_1' N(f(x))^{N-1} f'(x)\,. \tag{1.45}$$

Proof. Recall the definition of $\tilde{\mu}_{N,\beta}$ introduced in the statement (1.15): For $i \in \{1, \ldots, \beta N\}$, let U_i be i.i.d. random variables distributed as F_1, and independently for $i \in \{\beta N+1, \ldots, N\}$, let U_i be i.i.d. random variables distributed as F_2. Given the values of $U_1, \ldots, U_N$, we let $(A_1, \ldots, A_L)$ (where $L = \sum_{i=1}^N U_i$) be a sequence uniformly at random subject to $|\{1 \le j \le L : A_j = i\}| = U_i$. Let $\tilde{\mu}_{N,\beta}$ be the probability measure of the random sequence $(A_1, \ldots, A_L)$ thus obtained.

Following a similar argument given at the beginning of the proof of Lemma 1.3, we see that it suffices to show that under $\tilde{\mu}_{N,\beta}$ the set of good sequences has probability bounded from below by a constant.

We first observe that Properties (a) and (c) in Definition 1.5 can be satisfied by a random sequence under $\tilde{\mu}_{N,\beta}$ with probability tending to 1 as $N \to \infty$. This can be derived quite similarly as Case 2 and Case 3 in the proof of Lemma 1.3, with the last requirement in Property (c) hinted by the following inequality

$$\frac{g(t)}{t} \geq \frac{g(1/2)}{1/2}, \text{ if } 0 \leq t \leq \frac{1}{2}.$$

In addition, we claim that Properties (b) and (d) in Definition 1.5 can be satisfied simultaneously by a random sequence under $\tilde{\mu}_{N,\beta}$ with probability bounded from below. Altogether, this would imply the desired bound in the lemma.

To verify this claim, we show that the update sequence $(A_1, \ldots, A_L)$ can be obtained by the following two-step procedure, where in each step one property can be satisfied with probability bounded from below. Let us recall the notation that $\mathcal{F} = \sigma(U_1, U_2, \ldots, U_N)$. For convenience, we write $L_1 = \sum_{i=1}^{\beta N} U_i$ and $L_2 = \sum_{i=\beta N+1}^{N} U_i$.

As the first step, conditioning on $\mathcal{F}$, we choose L_1 indices $i_1 < i_2 < \cdots < i_{L_1}$ uniformly from $\{1, 2, \ldots, L\}$ and call them type 1 (they represent updates of the first βN coordinates). Denote by $\mathcal{I} = \{i_1, i_2, \cdots, i_{L_1}\}$ the collection of these type 1 indices. Let $j_1 < j_2 < \cdots < j_{L_2}$ be the rest of the indices and call them type 2 (they represent updates of the last $(1-\beta)N$ coordinates). In the following $\mathbb{P}$ refers to this (conditional) probability space (so that L_1 and L_2 should be seen as constants).

Denote by $\mathcal{E}$ the following event:

$$|\{1, \cdots, i\} \cap \mathcal{I}|, |\{L-i+1, \cdots, L\} \cap \mathcal{I}| \leq \Big(\frac{\beta \coth x_0}{\beta \coth x_0 + (1-\beta) \tanh x_0} + \epsilon_4\Big) i \text{ for all } 1 \leq i \leq L/2$$

and by $\mathcal{E}'$ the following event:

$$|\{1,\cdots,i\}\cap\mathcal{I}|,|\{L-i+1,\cdots,L\}\cap\mathcal{I}|\le(\frac{L_1}{L_1+L_2}+\epsilon)i \text{ for all } 1\le i\le L/2\,.$$

We want to show that Property (d) can be satisfied with probability bounded from below in this step, that is $\mathbb{P}(\mathcal{E})\ge c$ for a constant $c>0$. Without loss we can assume that Property (a) holds (since it is $\mathcal{F}$-measurable and can be satisfied with high probability), so that we have

$$\tfrac{L_1}{L_1+L_2}+\epsilon\le\tfrac{\beta x_0\coth x_0(1+\epsilon)N}{\beta x_0\coth x_0(1-\epsilon)N+(1-\beta)x_0\tanh x_0(1-\epsilon)N}+\epsilon\le\tfrac{\beta\coth x_0}{\beta\coth x_0+(1-\beta)\tanh x_0}+\epsilon_4$$

for sufficiently small ϵ and therefore $\mathcal{E}'\subseteq\mathcal{E}$. It thus remains to lower bound $\mathbb{P}(\mathcal{E}')$.

To this end, for each $1\le i\le L$, we let $T_i=1_{\{i \text{ is of type } 1\}}$. Then $T_1,T_2,\dots,T_L$ can be viewed as a sample without replacement from L_1 1's and L_2 0's. By Hoeffding's inequality in the case of sampling without replacement [11, Theorem 4], we have for any n,

$$\mathbb{P}(\frac{\sum_{i=1}^n T_i}{n}\ge\frac{L_1}{L_1+L_2}+\epsilon)\le\exp(-2n\epsilon^2)$$

and

$$\mathbb{P}(\frac{\sum_{i=L-n+1}^L T_i}{n}\ge\frac{L_1}{L_1+L_2}+\epsilon)\le\exp(-2n\epsilon^2)\,.$$

By a union bound over $M\le n\le\frac{L}{2}$ (where M depending only on ϵ is chosen later), we have $\mathbb{P}(\mathcal{E}_1)\ge 1-\frac{2\exp(-2\epsilon^2M)}{1-\exp(-2\epsilon^2)}$, where

$$\mathcal{E}_1=\Big\{\frac{\sum_{i=1}^n T_i}{n}\le\frac{L_1}{L_1+L_2}+\epsilon \text{ and } \frac{\sum_{i=L-n+1}^L T_i}{n}\le\frac{L_1}{L_1+L_2}+\epsilon \text{ for all } M\le n\le\frac{L}{2}\Big\}\,.$$

Let $\mathcal{K}$ be the set of all positive integer pairs (k_1,k_2) such that $\frac{M-k_1}{M}\le\frac{L_1}{L_1+L_2}+\epsilon$ and

$\frac{M-k_2}{M} \leq \frac{L_1}{L_1+L_2} + \epsilon$. It is clear that

$$\mathcal{E}_1 = \bigsqcup_{(k_1,k_2)\in\mathcal{K}} \mathcal{E}_1 \cap \{\sum_{i=1}^{M} T_i = M - k_1\} \cap \{\sum_{i=L-M+1}^{L} T_i = M - k_2\}\,. \tag{1.46}$$

For $(k_1, k_2) \in \mathcal{K}$, define

$$\mathcal{E}_2(k_1) = \{T_i = 0 \text{ for } 1 \leq i \leq k_1\} \cap \{T_i = 1 \text{ for } k_1 + 1 \leq i \leq M\}\,,$$
$$\mathcal{E}_3(k_2) = \{T_i = 0 \text{ for } L - k_2 + 1 \leq i \leq L\} \cap \{T_i = 1 \text{ for } L - M + 1 \leq i \leq L - k_2\}\,.$$

Then for all $(k_1, k_2) \in \mathcal{K}$, on the event $\mathcal{E}_2(k_1) \cap \mathcal{E}_3(k_2)$ we have $\frac{\sum_{i=1}^{n} T_i}{n} \leq \frac{L_1}{L_1+L_2} + \epsilon$ and $\frac{\sum_{i=L-n+1}^{L} T_i}{n} \leq \frac{L_1}{L_1+L_2} + \epsilon$ for all $1 \leq n \leq M$. Therefore, we have

$$\mathcal{E}' \supseteq \bigsqcup_{(k_1,k_2)\in\mathcal{K}} \mathcal{E}_1 \cap \mathcal{E}_2(k_1) \cap \mathcal{E}_3(k_2)\,. \tag{1.47}$$

However,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2(k_1) \cap \mathcal{E}_3(k_2)) &= \binom{M}{k_1}^{-1}\binom{M}{k_2}^{-1}\mathbb{P}(\mathcal{E}_1, \sum_{i=1}^{M} T_i = M - k_1, \sum_{i=L-M+1}^{L} T_i = M - k_2) \\ &\geq 2^{-2M}\mathbb{P}(\mathcal{E}_1, \sum_{i=1}^{M} T_i = M - k_1, \sum_{i=L-M+1}^{L} T_i = M - k_2)\,. \end{aligned} \tag{1.48}$$

Summing (1.48) over all $(k_1, k_2) \in \mathcal{K}$ and using (1.46) and (1.47), we deduce that

$$\mathbb{P}(\mathcal{E}') \geq 2^{-2M}\mathbb{P}(\mathcal{E}_1) \geq 2^{-2M}(1 - \frac{2\exp(-2\epsilon^2 M)}{1 - \exp(-2\epsilon^2)})\,. \tag{1.49}$$

By (1.49) and choosing M depending on ϵ (e.g. $M = -\frac{10}{\epsilon^2}\ln\epsilon$), we have proved that in the first step, Property (d) can be satisfied with probability bounded from below by a number depending only on ϵ.

Now, conditioning on the previous step, let $(B_1, B_2, \ldots, B_{L_1})$ be a sequence uniformly at random subject to $|\{1 \le j \le L_1 : B_j = i\}| = U_i$ for $i = 1, 2, \ldots, \beta N$, and *independently* let $(C_1, C_2, \ldots, C_{L_2})$ be a sequence uniformly at random subject to $|\{1 \le j \le L_2 : C_j = i\}| = U_i$ for $i = \beta N + 1, \beta N + 2, \ldots, N$. Let $A_{i_k} = B_k$ for $1 \le k \le L_1$ and $A_{j_k} = C_k$ for $1 \le k \le L_2$ (recall that i_k's and j_k's are sampled in the previous step). Thanks to the general proof of Case 1 in Lemma 1.3, we have that with high probability (with respect to the U_i's), we have $B_i \neq B_{i+1}$ and $B_i \neq B_{i+2}$ hold for all $1 \le i \le L_1$ with at least constant probability; and with high probability (with respect to the U_i's), we have $C_i \neq C_{i+1}$ and $C_i \neq C_{i+2}$ hold for all $1 \le i \le L_2$ with at least constant probability. However, note that $B_i \neq B_{i+1}$, $B_i \neq B_{i+2}$ for all $1 \le i \le L_1$ and $C_i \neq C_{i+1}$, $C_i \neq C_{i+2}$ for all $1 \le i \le L_2$ together would imply $A_i \neq A_{i+1}$ and $A_i \neq A_{i+2}$ for all $1 \le i \le L$, which corresponds to Property (b). By the (conditional) independence of $(B_1, B_2, \ldots, B_{L_1})$ and $(C_1, C_2, \ldots, C_{L_2})$, we see that in the second step, Property (b) can be satisfied with probability bounded from below by a constant.

Finally, it is clear that the sequence $(A_1, \ldots, A_L)$ obtained by this two-step procedure has the same distribution as under $\tilde{\mu}_{N,\beta}$ originally. This completes the verification of our claim and therefore the lemma. $\square$

Lemma 1.8. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_2' > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \le \iota$, $N > N'$ and any good path $P = v_0, v_1, \ldots, v_L$ we have*

$$\sum_{d=0}^{L/2} \sum_{d_1+d_2=d} F(v_{d_1}, v_{L-d_2}) \le C_2' N f(x)^N \asymp C_2' N (f(x))^{N-1} f'(x)\,.$$

Proof. We continue to let $Y_0 = 0, Y_1, \ldots, Y_{L-1}, Y_L = x$ be distributed as the order statistics of $(L-1)$ i.i.d. uniform $[0, x]$ random variables. For d_1 and d_2 such that $d_1 + d_2 = d$, by Property (d) of Definition 1.5 we have that the Hamming distance $H(v_{d_1}, v_{L-d_2})$ between v_{d_1} and v_{L-d_2} is at least $\beta N - D(v_0, v_{d_1}) - D(v_{L-d_2}, v_L)$, which is at least $\beta N - \delta d$. Therefore, by (1.9) and Lemma 1.6, we have

$$\begin{aligned} F(v_{d_1}, v_{L-d_2}) &= \mathbb{E}G(v_{d_1}, v_{L-d_2}, Y_{d_1}, Y_{L-d_2}) \\ &\le \mathbb{E}((\sinh y)^{H(v_{d_1}, v_{L-d_2})}(\cosh y)^{N-H(v_{d_1}, v_{L-d_2})})'|_{y=Y_{L-d_2}-Y_{d_1}} \\ &\le \mathbb{E}((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})'|_{y=Y_{L-d_2}-Y_{d_1}} \\ &= \mathbb{E}((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})'|_{y=x-Y_d} \end{aligned} \tag{1.50}$$

where the last equality is because the distribution of $Y_{L-d_2} - Y_{d_1}$ does not depend on (d_1, d_2) provided the value of $d = d_1 + d_2$. Since $x - Y_d$ is the $(L-d)$th order statistic of $(L-1)$ i.i.d. uniform $[0, x]$ random variables, $\frac{x-Y_d}{x}$ has a Beta$(L-d, d)$ distribution. Thus, the density of $x - Y_d$ is $\frac{1}{x}(\frac{y}{x})^{L-d-1}(1-\frac{y}{x})^{d-1}\frac{(L-1)!}{(L-d-1)!(d-1)!}$ for $y \in [0, x]$. Therefore

$$\begin{aligned} &\mathbb{E}((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})'|_{y=x-Y_d} \\ = &\int_0^x ((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})'\frac{1}{x}(\frac{y}{x})^{L-d-1}(1-\frac{y}{x})^{d-1}\frac{(L-1)!}{(L-d-1)!(d-1)!}\,dy\,. \end{aligned} \tag{1.51}$$

We will split the above integral into two parts according to whether y is smaller or greater than $\frac{x}{2}$, and denote by $\mathcal{J}_1(d)$ the integral over $[0, \frac{x}{2}]$ and by $\mathcal{J}_2(d)$ the integral over $[\frac{x}{2}, x]$. On one hand, for $y \in [0, \frac{x}{2}]$, by Lemma 1.6 we have

$$\begin{aligned} &((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})' \le ((\sinh y)^{\beta N-\delta\frac{L}{2}}(\cosh y)^{(1-\beta)N+\delta\frac{L}{2}})' \\ = \; &(\sinh y)^{\beta N-\delta\frac{L}{2}-1}(\cosh y)^{(1-\beta)N+\delta\frac{L}{2}-1}((\beta N-\delta\frac{L}{2})\cosh y + ((1-\beta)N+\delta\frac{L}{2})\sinh y)\,. \end{aligned}$$

Since $\cosh y \le \cosh(\frac{x}{2})$ and $\sinh y \le \sinh(\frac{x}{2})$ for $y \in [0, \frac{x}{2}]$, we have

$$\begin{aligned} ((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})' &\le C_8 N(\sinh(\frac{x}{2}))^{\beta N-\delta\frac{L}{2}-1}(\cosh(\frac{x}{2}))^{(1-\beta)N+\delta\frac{L}{2}-1} \\ &\le C_8 N(\sinh(\frac{x}{2}))^{\beta N-\delta\frac{\gamma(1+2\epsilon)N}{2}}(\cosh(\frac{x}{2}))^{(1-\beta)N+\delta\frac{\gamma(1+\epsilon)N}{2}} \end{aligned}$$

where the last inequality follows from Property (a) of Definition 1.5. Therefore

$$\begin{aligned}\sum_{d=1}^{\frac{L}{2}}(d+1)\mathcal{J}_1(d) &\le C_9N^3(\sinh(\frac{x}{2}))^{\beta N-\delta\frac{\gamma(1+2\epsilon)N}{2}}(\cosh(\frac{x}{2}))^{(1-\beta)N+\delta\frac{\gamma(1+\epsilon)N}{2}} \\ &\le C_9N^3r^N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}\,, \end{aligned} \tag{1.52}$$

where $0 < r < 1$ is a constant that depends only on β. Here we used the fact (by brute force computation) that

$$\frac{(\sinh(\frac{x}{2}))^{\beta-\delta\frac{\gamma(1+2\epsilon)}{2}}(\cosh(\frac{x}{2}))^{(1-\beta)+\delta\frac{\gamma(1+\epsilon)}{2}}}{(\sinh x)^{\beta}(\cosh x)^{1-\beta}} \le r < 1\,.$$

On the other hand, for $y \in [\frac{x}{2}, x]$, we have $\coth y \le \coth(\frac{x}{2})$ and $\tanh y \le \tanh(x)$. Thus

$$\begin{aligned} & ((\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d})'\frac{1}{x} \\ = \; & (\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d}((\beta N-\delta d)\coth y+((1-\beta)N+\delta d)\tanh y)\tfrac{1}{x} \\ \le \; & C_{10}N(\sinh y)^{\beta N-\delta d}(\cosh y)^{(1-\beta)N+\delta d} \\ \le \; & C_{11}N((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N})(\coth y)^{\delta(d-2)}\,. \end{aligned}$$

Therefore, the integrand of (1.51) is smaller than
$C_{11}N((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N})\varphi(x,y,d,\beta,N,L)$ for $y \in [\frac{x}{2}, x]$, where

$$\varphi(x,y,d,\beta,N,L) = (\coth y)^{\delta(d-2)}(\frac{y}{x})^{L-d-1}(1-\frac{y}{x})^{d-1}\frac{(L-1)!}{(L-d-1)!(d-1)!}\,,$$

and thus

$$\sum_{d=1}^{\frac{L}{2}}(d+1)\mathcal{J}_2(d) \le C_{11}N\int_{\frac{x}{2}}^{x}((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N})\sum_{d=1}^{\frac{L}{2}}(d+1)\varphi(x,y,d,\beta,N,L)\,dy \tag{1.53}$$

Now for $d = 1$, we have

$$(d+1)\varphi(x,y,d,\beta,N,L) = 2(\tanh y)^{\delta}(\tfrac{y}{x})^{L-2}(L-1) \leq C_{12}(\tfrac{y}{x})^{L-2}(L-1)\,. \qquad (1.54)$$

In addition, for $d \geq 2$, we have

$$\begin{aligned}
& (d+1)\varphi(x,y,d,\beta,N,L) \\
= \;& (1-\frac{y}{x})(\coth y)^{\delta(d-2)}(1-\frac{y}{x})^{d-2}(\frac{y}{x})^{L-d-1}\frac{(L-3)!}{(L-d-1)!(d-2)!}(L-1)(L-2)\frac{(d+1)}{(d-1)} \\
\leq \;& 3(1-\frac{y}{x})L^2 \cdot [((\coth y)^{\delta})(1-\frac{y}{x})]^{d-2}(\frac{y}{x})^{L-d-1}\frac{(L-3)!}{(L-d-1)!(d-2)!}\,.
\end{aligned}$$

Observing that the second factor of the product in the previous line is a binomial term, we have

$$\sum_{d=2}^{\frac{L}{2}}(d+1)\varphi(x,y,d,\beta,N,L) \leq 3(1-\tfrac{y}{x})L^2 \cdot ((\coth y)^{\delta}(1-\tfrac{y}{x}) + \tfrac{y}{x})^{L-3}\,. \qquad (1.55)$$

Combining (1.54) and (1.55) and using Property (a) of Definition 1.5, we have

$$\sum_{d=1}^{\frac{L}{2}}(d+1)\varphi(x,y,d,\beta,N,L) \leq C_{12}(\frac{y}{x})^{\gamma(1-2\epsilon)N}(L-1)+3L^2(1-\frac{y}{x})((\coth y)^{\delta}(1-\frac{y}{x})+\frac{y}{x})^{\gamma(1+\epsilon)N}\,.$$

Therefore (1.53) translates to

$$\begin{aligned}
\sum_{d=1}^{\frac{L}{2}}(d+1)\mathcal{J}_2(d) \leq & C_{11}N\int_{\frac{x}{2}}^{x}((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N})\cdot \\
& (C_{12}(\tfrac{y}{x})^{\gamma(1-2\epsilon)N}(L-1) + 3L^2(1-\tfrac{y}{x})((\coth y)^{\delta}(1-\tfrac{y}{x}) + \tfrac{y}{x})^{\gamma(1+\epsilon)N})\,dy\,.
\end{aligned}$$

For convenience, let

$$\begin{aligned}\psi_1(y) &= \frac{((\sinh y)^{\beta}(\cosh y)^{1-\beta})\cdot(\frac{y}{x})^{\gamma(1-2\epsilon)}}{(\sinh x)^{\beta}(\cosh x)^{1-\beta}}\,,\\ \psi_2(y) &= \frac{((\sinh y)^{\beta}(\cosh y)^{1-\beta})\cdot((\coth y)^{\delta}(1-\frac{y}{x})+\frac{y}{x})^{\gamma(1+\epsilon)}}{(\sinh x)^{\beta}(\cosh x)^{1-\beta}}\,,\\ \psi_3(y) &= \beta\ln\sinh y+(1-\beta)\ln\cosh y\,.\end{aligned}$$

We will show that for $i\in\{1,2\}$, we have $\psi_i(x)=1$ (which is trivial) and $\ln\psi_i(y)\le -K(x-y)$ for $y\in[\frac{x}{2},x]$, where $K>0$ is a constant that only depends on β. These conditions on $\psi_1(y)$ and $\psi_2(y)$ will guarantee that both $\int_{\frac{x}{2}}^{x}N(\psi_1(y))^N\,dy$ and $\int_{\frac{x}{2}}^{x}N^2(x-y)(\psi_2(y))^N\,dy$ are bounded as $N\to\infty$, so that (**??**) is bounded by $C_{13}N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}$.

It is relatively easy to check that $\psi_1(y)$ satisfies the second condition (i.e. $\ln\psi_1(y)\le -K(x-y)$), so we focus on verifying it for $\psi_2(y)$. To start with, we have

$$\ln\psi_2(y)=(\psi_3(y)-\psi_3(x))+\gamma(1+\epsilon)\ln((\coth y)^{\delta}(1-\frac{y}{x})+\frac{y}{x})\,.$$

For the first part of the sum on the right hand side, i.e. $(\psi_3(y)-\psi_3(x))$, we can first compute the derivatives of $\psi_3(y)$ as follows:

$$\begin{aligned}\psi_3'(y) &= \beta\coth y+(1-\beta)\tanh y\,,\\ \psi_3''(y) &= \beta(1-(\coth y)^2)+(1-\beta)(1-(\tanh y)^2)\,.\end{aligned}$$

Since $\coth y\ge\coth x\ge(\frac{1-\beta}{\beta})^{\frac{1}{4}}$ for $y\le x$, we have $\psi_3''(y)$ is increasing in y. Therefore, by Taylor's theorem (Lagrange form of the remainder) we find that for some $\xi\in[y,x]$,

$$\begin{aligned}\psi_3(y) &= \psi_3(x)+\psi_3'(x)(y-x)+\frac{\psi_3''(\xi)}{2}(y-x)^2\\ &\le \psi_3(x)+\psi_3'(x)(y-x)+\frac{\psi_3''(x)}{2}(y-x)^2\,. \qquad (1.56)\end{aligned}$$

For the second part of the sum, i.e. $\gamma(1+\epsilon)\ln((\coth y)^\delta(1-\frac{y}{x})+\frac{y}{x})$, we set $C(y) := (\coth y)^\delta - 1$ and $\theta(y) := 1-\frac{y}{x}$. Then

$$(\coth y)^\delta(1-\frac{y}{x})+\frac{y}{x} = 1+(1-\frac{y}{x})C(y) = 1+\theta(y)C(y)\,.$$

Clearly $0 \le \theta(y)C(y) \le \theta(\frac{x}{2})C(\frac{x}{2}) = \frac{1}{2}((\coth\frac{x}{2})^\delta - 1) \le 0.75$. Since $\ln(1+t) \le t - \frac{t^2}{3}$ for $0 \le t \le 0.75$, we have

$$\ln((\coth y)^\delta(1-\frac{y}{x})+\frac{y}{x}) \le \theta(y)C(y) - \frac{(\theta(y))^2(C(y))^2}{3}\,. \tag{1.57}$$

Combining (1.56) and (1.57), we have

$$\begin{aligned}\ln\psi_2(y) &\le -\psi_3'(x)x(1-\frac{y}{x}) + \frac{\psi_3''(x)x^2}{2}(1-\frac{y}{x})^2 \\ &\quad + \gamma(1+\epsilon)\theta(y)C(y) - \gamma(1+\epsilon)\frac{(\theta(y))^2(C(y))^2}{3} \\ &= \theta(y)\gamma(1+\epsilon)[-\frac{\theta(y)}{3}(C(y))^2 + C(y) - \frac{1}{\gamma(1+\epsilon)}(\psi_3'(x)x - \frac{\psi_3''(x)x^2}{2}\theta(y))]\,.\end{aligned}$$

We wish to show that the factor in the square bracket above is less than some constant $-\eta$, where $\eta > 0$ only depends on β, i.e. for any $y \in [\frac{x}{2}, x]$

$$-\frac{\theta(y)}{3}(C(y))^2 + C(y) - \frac{1}{\gamma(1+\epsilon)}(\psi_3'(x)x - \frac{\psi_3''(x)x^2}{2}\theta(y)) \le -\eta\,.$$

Set $c := \frac{\psi_3''(x_0)x_0^2}{2\gamma}$. Since $|x-x_0| < \iota = \epsilon^2$, and ϵ can be made arbitrarily small, we only need to show that for some constant $\eta_1 > 0$ which only depends on β, for any $y \in [\frac{x}{2}, x]$

$$-\frac{\theta(y)}{3}(C(y))^2 + C(y) - 1 + c\theta(y) \le -\eta_1\,.$$

To do this, we let $q(s) := -\frac{\theta(y)}{3}s^2 + s - 1 + c\theta(y)$. Solving the quadratic equation $q(s) = 0$ with respect to s, we get the smaller root (since $\theta(y) \in [0, 1/2]$ for $y \in [\frac{x}{2}, x]$ and $c > -1/3$,

$q(s)$ always has two roots)

$$r(y) := \frac{-1 + \sqrt{1 - \frac{4\theta(y)}{3}(1 - c\theta(y))}}{-\frac{2\theta(y)}{3}}, \text{ for } y \in [\frac{x}{2}, x].$$

We claim that we only need to show that for any $\frac{x}{2} \le y \le x$, $C(y) \le r(y) - \eta_2$ for some constant $\eta_2 > 0$ which only depends on β. Indeed, if this holds true, then from $q'(s) = -\frac{2\theta(y)}{3}s + 1$, we see that $q'(C(y)) = -\frac{2}{3}\theta(y)C(y) + 1 \ge 0.5$ and $q'(C(y) + \eta_2) = q'(C(y)) - \frac{2\theta(y)}{3}\eta_2 \ge 0.5 - \frac{2}{3}\eta_2$. Consequently $0 = q(r(y)) \ge q(C(y) + \eta_2) \ge q(C(y)) + \eta_2(0.5 - \frac{2}{3}\eta_2)$ and we can take $\eta_1 = \eta_2(0.5 - \frac{2}{3}\eta_2)$.

To this end, we first point out that $r(y)$ is convex in y if $c < \frac{1}{3}$, $r(y)$ is concave in y if $c > \frac{1}{3}$ and $r(y) \equiv 1$ if $c = \frac{1}{3}$. This can be seen by observing that $r = \frac{-1+\sqrt{1-\frac{4\theta}{3}(1-c\theta)}}{-\frac{2\theta}{3}}$ is the inverse function of $\theta = \frac{r-1}{\frac{r^2}{3}-c}$, whose properties such as monotonicity and convexity are not hard to justify. Now if $c < \frac{1}{3}$, then by convexity of $r(y)$, we have

$$r(y) \ge r'(\frac{3x}{4})(y - \frac{3x}{4}) + r(\frac{3x}{4}) := t(y)$$

where $t(y)$ can be computed as

$$t(y) = -\frac{1}{x}(\frac{120}{\sqrt{3c+24}} - 24)(y - \frac{3x}{4}) + 6 - \sqrt{3c+24}.$$

Since $C(y)$ is convex in y, we only need to have $t(x) \ge C(x) + \eta_2$ and $t(\frac{x}{2}) \ge C(\frac{x}{2}) + \eta_2$, i.e.,

$$-\frac{30}{\sqrt{3c+24}} + 12 - \sqrt{3c+24} \ge (\coth x)^\delta - 1 + \eta_2 \tag{1.58}$$

and

$$\frac{30}{\sqrt{3c+24}} - \sqrt{3c+24} \ge (\coth\frac{x}{2})^\delta - 1 + \eta_2. \tag{1.59}$$

If $c = \frac{1}{3}$, then $r(y) \equiv 1$, which is a degenerate case. If $c > \frac{1}{3}$, then since $r(y)$ is concave in y, we only need to have $r(x) \geq C(x) + \eta_2$ and $r(\frac{x}{2}) \geq C(\frac{x}{2}) + \eta_2$, i.e.,

$$1 \geq (\coth x)^{\delta} - 1 + \eta_2 \tag{1.60}$$

and

$$3 - \sqrt{3(c+1)} \geq (\coth \frac{x}{2})^{\delta} - 1 + \eta_2\,. \tag{1.61}$$

All of the inequalities (1.58), (1.59), (1.60) and (1.61) boil down to comparisons of constants which only involve x_0 (since $|x - x_0| < \iota = \epsilon^2$ and ϵ can be made arbitrarily small), so we have finally shown that (**??**) is bounded by $C_{13}N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}$.

Combining (1.50), (1.51), (1.52), (**??**) and the fact that $F(v_0, v_L) \leq N(f(x))^{N-1}f'(x)$ when $d = 0$, we conclude that $\sum_{d=0}^{\frac{L}{2}} \sum_{d_1+d_2=d} F(v_{d_1}, v_{L-d_2}) \leq C_2' N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}$ for some $C_2' > 0$. □

Lemma 1.9. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_3' > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \leq \iota$, $N > N'$, any good path P and any j we have $\sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i}) \leq 1 + \frac{C_3'}{N}$.*

Proof. The proof can be carried out in the same manner as that of Lemma 1.5, except that the role of $\alpha + \epsilon_3$ in Case (e) there is now played by $\frac{\gamma+\epsilon_3}{2g(1/2)}$. We thus omit the details. □

Corollary 1.3. *For any sufficiently small but fixed number $\epsilon > 0$, there exist $C_4' > 0$ and an integer $N' > 0$ which both depend only on ϵ, such that for all $|x - x_0| \leq \iota$ and $N > N'$*

$$\mathbb{E}Z_{N,x,*}^2 \leq (C_4' N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N} + C_4')N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}\,.$$

Proof. This follows from Lemmas 1.8 and 1.9 in the same manner as Corollary 1.2 follows from Lemmas 1.4 and 1.5. □

Proposition 1.3. *There exists $0 \leq K' < 1$ such that, if $\liminf_{N\to\infty} \mathbb{P}(Z_{N,x_c+\epsilon_N} > 0) \geq C$ for* *some constant* $C \geq 0$ *whenever* $N\epsilon_N \to \infty$*, then whenever* $N\epsilon_N \to \infty$ *we have*

$$\liminf_{N\to\infty} \mathbb{P}(Z_{N,x_c+\epsilon_N} > 0) \geq 1 - (1-C)K'\,.$$

Proof. The basic idea is the same as Proposition 1.2. Fix a large integer M. We first choose vertices $A_1, \ldots A_M, B_1, \ldots, B_M$ and $C_1, \ldots, C_M, D_1, \ldots, D_M$ such that for $1 \leq i \leq M$:

- The only coordinate at which A_{i-1} and A_i differ is a_i. The only coordinate at which B_{i-1} and B_i differ is b_i. The only coordinate at which C_{i-1} and C_i differ is c_i. The only coordinate at which D_{i-1} and D_i differ is d_i (set $A_0 = C_0 = \vec{0}_N$ and $B_0 = D_0 = (\vec{1}_{\beta N}, \vec{0}_{N-\beta N})$ for convenience).
- All of the $4M$ coordinates a_i, b_i, c_i and d_i are different and are among the first βN coordinates.
- $X(A_i), X(C_i) \in [\frac{(i-1)\epsilon_N}{4M}, \frac{i\epsilon_N}{4M}]$ and $X(B_i), X(D_i) \in [x - \frac{i\epsilon_N}{4M}, x - \frac{(i-1)\epsilon_N}{4M}]$.

Since $N\epsilon_N \to \infty$, this can be achieved with probability $1 - o_N(1)$.

Now let $M_2 = \frac{(1-\beta)}{\beta}2M$, and select distinct coordinates $e_1, e_2, \cdots, e_{M_2}$ and $f_1, f_2, \cdots, f_{M_2}$ arbitrarily among the last $(1-\beta)N$ coordinates. Let $\tilde{H}_1$ be the $(N-2M-M_2)$ dimensional sub-hypercube formed by A_M and B_M with the coordinates $e_1, e_2, \cdots, e_{M_2}$ being 0, i.e.,

$$\begin{aligned}\tilde{H}_1 = \{\sigma \in H_N : &\sigma_{e_i} = 0 \text{ for all } 1 \leq i \leq M_2, \sigma_{a_i} = 1 \text{ for all } 1 \leq i \leq M,\\ &\sigma_{b_i} = 0 \text{ for all } 1 \leq i \leq M\}\,.\end{aligned}$$

Similarly, let $\tilde{H}_2$ be the $(N-2M-M_2)$ dimensional sub-hypercube formed by C_M and D_M

with the coordinates $f_1, f_2, \cdots, f_{M_2}$ being 0, i.e.,

$$\tilde{H}_2 = \{\sigma \in H_N : \sigma_{f_i} = 0 \text{ for all } 1 \le i \le M_2, \sigma_{c_i} = 1 \text{ for all } 1 \le i \le M,$$
$$\sigma_{d_i} = 0 \text{ for all } 1 \le i \le M\}\,.$$

Let $H_2' = \tilde{H}_2 \setminus \tilde{H}_1$. Denote by $p_{\tilde{H}_1}$ and $p_{H_2'}$ the probabilities that there is an accessible path in $\tilde{H}_1$ (from A_M to B_M) and H_2' (from C_M to D_M) respectively. Since $\tilde{H}_1$ and H_2' are disjoint, by independence we have $\mathbb{P}(Z_{N,x_c+\epsilon_N} > 0) \ge 1-(1-p_{H_1})(1-p_{H_2'}) - o_N(1)$. From the construction above it is clear that we are reduced to accessibility percolation of dimension $(N-2M-M_2)$ (with the same β) with $x \ge x_c + \epsilon_N/2$, in either $\tilde{H}_1$ (from A_M to B_M) or $\tilde{H}_2$ (from C_M to D_M). Thus,

$$p_{\tilde{H}_1} \ge \mathbb{P}(Z_{N-2M-M_2,x_c+\epsilon_N/2} > 0) \ge C - o_N(1)\,.$$

To show that $p_{H_2'}$ is bounded from below by a positive constant $1-K'$, we only consider the good path in $\tilde{H}_2$ (from C_M to D_M) which updates each of coordinates a_1 and b_1 precisely once and b_1 is updated before a_1. Such paths must be contained in H_2'. Clearly, the number of such accessible paths has second moment less than $\mathbb{E}Z^2_{N-2M-M_2,x_c+\epsilon_N/2,*}$ and first moment within an absolute multiplicative constant of $\mathbb{E}Z_{N-2M-M_2,x_c+\epsilon_N/2,*}$ (or $M_{N-2M-M_2,\beta,x_c+\epsilon_N/2}$). Combined with Lemma 1.7 and Corollary 1.3, this yields that $p_{H_2'} \ge 1-K' - o_N(1)$ for some constant $K' < 1$. This completes the proof of the proposition. □

Proof of (1.2): general case Applying Proposition 1.3 recursively (starting from $C=0$) completes the proof of (1.2). □

Proof of (1.3): general case The proof is basically the same as in the antipodal case except that for the upper bound, the role of $\sinh(x)$ is now played by $f(x) =$

$(\sinh x)^{\beta}(\cosh x)^{1-\beta}$. □

1.4 Asymptotics for the global maximum of the NK fitness model

In this section we give a proof of Theorem 1.2. Our proof relies on the following observation, which can be proved in the same way as part of the proof of Proposition 3 in [9].

Proposition 1.4. *For fixed N, the global maximum of the NK fitness model $M_{N,K}$ is stochastically nondecreasing in K.*

Proof of Theorem 1.2

(a) Clearly we have $\lim_{N\to\infty} \frac{\mathbb{E}M_{N,N}}{N} = x^*$. Thanks to Proposition 1.4, we only need to show that if $K \to \infty$ and $\frac{K}{N} \to 0$ as $N \to \infty$, we have $\liminf_{N\to\infty} \frac{\mathbb{E}M_{N,K}}{N} \geq x^*$.

We divide the N coordinates into $[\frac{N}{K}]$ blocks of length K. We use the following algorithm to find a $\hat{\sigma} \in \{0,1\}^N$ such that $X_{\hat{\sigma}}$ is large. First we set the coordinates of $\hat{\sigma}$ in the first block to be 0: $\hat{\sigma}_1 = \hat{\sigma}_2 = \cdots = \hat{\sigma}_K = 0$. For each stage $j = 1, \ldots, [\frac{N}{K}] - 1$, we set $(\hat{\sigma}_{jK+1}, \hat{\sigma}_{jK+2}, \cdots, \hat{\sigma}_{(j+1)K})$ to be the maximizer of

$$\max_{(\sigma_{jK+1},\sigma_{jK+2},\cdots,\sigma_{(j+1)K})\in\{0,1\}^K} \sum_{i=2}^{K+1} Y_{i+(j-1)K,(\sigma_{i+(j-1)K},\cdots,\sigma_{i+jK-1})}.$$

Finally we set $\hat{\sigma}_i = 0$ for $[\frac{N}{K}]K+1 \leq i \leq N$. Note that for each $j = 1, \ldots, [\frac{N}{K}]-1$, we have $\{\sum_{i=2}^{K+1} Y_{i+(j-1)K,(\sigma_{i+(j-1)K},\cdots,\sigma_{i+jK-1})} : (\sigma_{jK+1}, \sigma_{jK+2}, \cdots, \sigma_{(j+1)K}) \in \{0,1\}^K\}$ behaves exactly as a binary branching random walk (BRW) of depth K. Furthermore, this BRW is independent of all BRWs in the previous stages. By Theorem 4 of [20] we have

$$\mathbb{E}\sum_{i=2}^{K+1} Y_{i+(j-1)K,(\hat{\sigma}_{i+(j-1)K},\cdots,\hat{\sigma}_{i+jK-1})} = Kx^* - \frac{3}{2I'(x^*)}\log K + O_K(1).$$

Summing this over $j = 1, \ldots, [\frac{N}{K}] - 1$, we have

$$\mathbb{E}M_{N,K} \geq \mathbb{E}X_{\hat{\sigma}} = ([\frac{N}{K}] - 1)(Kx^* - \frac{3}{2I'(x^*)} \log K + O_K(1)) + (N - ([\frac{N}{K}] - 1)K)\mathbb{E}(Y),$$

which gives us $\liminf_{N\to\infty} \frac{\mathbb{E}M_{N,K}}{N} \geq x^*$.

(b) By Proposition 1.4 again, we have $\limsup_{N\to\infty} \frac{\mathbb{E}M_{N,K}}{N} \leq \limsup_{N\to\infty} \frac{\mathbb{E}M_{N,K_0}}{N} < x^*$.

□

CHAPTER 2

CHEMICAL DISTANCES FOR PERCOLATION OF PLANAR GAUSSIAN FREE FIELDS AND CRITICAL RANDOM WALK LOOP SOUPS

2.1 Introduction

For an $N \times N$ box $V_N \subseteq \mathbb{Z}^2$ centered at the origin, we let the discrete Gaussian free field (GFF) $\{\eta_{N,v} : v \in V_N\}$ with Dirichlet boundary condition be a mean zero Gaussian process which takes value 0 on ∂V_N and has covariances given by

$$\mathbb{E}\eta_{N,v}\eta_{N,u} = \frac{1}{4} G_{V_N}(u,v)\,,$$

where $G_{V_N}(u,v)$ is the Green's function for simple random walk, i.e., the expected number of visits to v before exiting V_N for a simple random walk started at u. The first goal of the present paper is to study chemical distances (i.e., graph distances) on the percolation cluster for level sets of GFF. Precisely, for any $\lambda \in \mathbb{R}$, we let $\mathcal{H}_{N,\lambda} = \{v \in V_N : \eta_{N,v} \leq \lambda\}$ be the λ-level sets, i.e., the collection of all vertices with values no more than λ. In the context of no confusion, we also denote by $\mathcal{H}_{N,\lambda}$ the induced subgraph on $\mathcal{H}_{N,\lambda}$. For $u, v \in V_N$, we let $D_{N,\lambda}(u,v)$ be the graph distance between u and v if u, v are in the same connected component of $\mathcal{H}_{N,\lambda}$, and let $D_{N,\lambda}(u,v) = \infty$ otherwise. For $A, B \subseteq V_N$, we denote $D_{N,\lambda}(A,B) = \min_{u\in A, v\in B} D_{N,\lambda}(u,v)$.

Theorem 2.1. *For any $0 < \alpha < \beta < 1$, there exist constants $c > 0, \lambda_0 > 0$ such that for all N*

$$\mathbb{P}(D_{N,\lambda}(\partial V_{\alpha N}, \partial V_{\beta N}) \geq N\mathrm{e}^{(\log N)^{9/10}}) \leq c^{-1}(\mathrm{e}^{-c\lambda^2} + N^{-20}),\ \textit{for all } \lambda \geq \lambda_0\,.$$

Remark 2.1. *Note that even for any fixed $\lambda < 0$, the event $D_{N,\lambda}(\partial V_{\alpha N}, \partial V_{\beta N}) \leq N\mathrm{e}^{(\log N)^{9/10}}$ in Theorem 2.1 occurs with non-vanishing probability; see Corollary 2.1. In* *addition, we expect that for any fixed* λ*, the probability for* $D_{N,\lambda}(\partial V_{\alpha N}, \partial V_{\beta N}) < \infty$ *is strictly less than 1; we do not study this in the present paper so as not to dilute the focus.*

We next consider the random walk loop soup introduced in [51], which is a discrete analogue of the Brownian loop soup [52]. For convenience, we follow [53] where the loops are endowed with a continuous-time parametrization. Formally, let (X_t) be a continuous-time sub-Markovian jump process on V_N which is killed at the boundary. Given two neighboring vertices x and y, let the transition rate from x to y be 1. Let $(\mathbb{P}^t_{x,y}(\cdot))_{x,y\in V_N, t>0}$ be the bridge probability measures of X conditioned on not killed until time t, and let $(p_t(x,y))_{x,y\in V_N,t\geq 0}$ be the transition probabilities of X. Then the measure μ on time-parametrized loops associated to X is, as defined in [53],

$$\mu(\cdot) = \sum_{x\in V_N} \int_0^\infty \mathbb{P}^t_{x,x}(\cdot) \frac{p_t(x,x)}{t} dt\,. \tag{2.1}$$

For $\alpha > 0$, the random walk loop soup with intensity α on V_N, denoted as $\mathcal{L}_{\alpha,N}$, is defined to be the Poisson point process on the space of loops with intensity $\alpha\mu$. Naturally $\mathcal{L}_{\alpha,N}$ induces a subgraph (which we also denote as $\mathcal{L}_{\alpha,N}$) of $\mathcal{G}_N$ where an edge is open if it is contained in a loop in $\mathcal{L}_{\alpha,N}$. Our next theorem is on chemical distances (which we denote by $D_{\mathcal{L}_{1/2,N}}(\cdot,\cdot)$) of such loop clusters at its critical intensity $\alpha_c = 1/2$.

Theorem 2.2. *For any* $0 < \alpha < \beta < 1$*, there exists a constant* $c > 0$ *such that for all* N

$$\mathbb{P}(D_{\mathcal{L}_{1/2,N}}(\partial V_{\alpha N}, \partial V_{\beta N}) \leq N \mathrm{e}^{(\log N)^{9/10}}) \geq c\,.$$

Remark 2.2. *We expect that the probability for* $D_{\mathcal{L}_{1/2,N}}(\partial V_{\alpha N}, \partial V_{\beta N}) < \infty$ *is strictly less than 1; see Remark 2.1.*

2.1.1 Backgrounds and related works

Chemical distances for percolation models is a substantially more challenging problem than the question on connectivities. For instance, it is a major challenge to compute the exponent on the chemical distance between (say) the left and right boundaries for the critical planar percolation, conditioning on the existence of an open crossing. It was proved in [21] that the dimension is strictly larger than 1, and it was shown in recent works [33, 34] that the chemical distance is substantially smaller than the length of the lowest open crossing. Let us remark that the current result does not imply that the exponent for the chemical distance is strictly less than that of the lowest open crossing, despite that it was strongly believed so.

Due to the strong correlation and hierarchical nature of the two-dimensional GFF as well as the random walk loop soup, our model is perhaps in spirit more closely related to the fractal percolation process (see [31] for a survey). For fractal percolation process, it was proved [32, 61] that the dimension of the chemical distance is strictly larger than 1 (which suggests an interesting dichotomy in view of our dimension 1 results for the GFF and the random walk loop soup).

As for loop soups, in two-dimensions the connectivity of the loop clusters has been studied recently. In [65], it was shown that there is a phase transition around the critical intensity $\alpha_c = \frac{1}{2}$ for percolation of the Brownian loop soup, below which there are only bounded clusters and above which the loops forms a single cluster. In recent works of [54, 55], analogous results were proved for the random walk loop soup.

In three-dimensions or higher, there has been an intensive study on percolation of level sets for GFF, random walks, random interlacements as well as random walk loop soups; see, e.g., [66, 67, 63, 30]. In fact, much on the chemical distances for these percolation models has been studied; see [28, 40, 29]. We remark that there is a drastic difference between two-dimensions and higher dimensions.

Besides chemical distances, other metric aspects of two-dimensional GFF has been studied recently: see [56] on the random pseudo-metric defined via the zero-set, and see

[39, 37, 36, 38] for some progress on the first passage percolation on the exponential of these underlying fields.

Finally, the random walk loop soup percolation is naturally related to the following percolation dimension question for planar random walks (Brownian motion) proposed in [42, 26]. Run the random walk until it exits the boundary of a box and declare a vertex to be open if it is visited and closed otherwise. Then what is the dimension of the minimal open crossing from the origin to the boundary? We are currently not able to prove something for this question, for the crucial reason that we are not able to construct a coupling between GFFs and random walks under which an event on GFFs will certify "small" chemical distances for random walk percolation models.

2.1.2 Discussions on main proof ingredients

Our proofs of Theorems 2.1 and 2.2 are based on an interesting combination of a theorem of Makarov, isomorphism theory and an entropic repulsion estimate for GFF in the presence of hard wall. In this subsection, we will provide a brief review on these three ingredients.

A theorem of Makarov. A fundamental ingredient for our proofs, is a classical theorem of Makarov [58] which states that the dimension of the support for the harmonic measure on simply connected domain in $\mathbb{R}^2$ is 1. In this article, we will use a discrete analogue of Makarov's theorem which was proved in [48] by approximating Brownian motions with random walks (and then using [58]). Previous to [58], the Beurling's projection theorem (see, e.g., [23, Theorem V.4.1.], and see [45, 49] for its discrete analogue) was established, which gives an (achievable) upper bound on the maximal local expansion of the harmonic measure compared with 1-dimensional Hausdorff measure (in the language of simple random walk, it states that the harmonic measure at a lattice point on a simply connected curve of diameter n is bounded by $O(1/\sqrt{n})$). In a sense, Makarov's theorem states that the upper bound in Berling's estimate cannot be achieved globally, and thus providing a much better control (than that guaranteed by Beurling's projecting theorem) on the global expansion and compression of harmonic measure. Finally, we remark that examples have been given in [60, 27], in which the harmonic measure is *singular* to the 1-dimensional Hausdorff measure. In our opinion, this suggests that Question 2.4 below could be of serious challenge.

Isomorphism theory. The distribution of the occupation times for random walks can be fully characterized by Gaussian free fields; results of this flavor go by the name of isomorphism theorems (see [59, 53, 68, 64] for an excellent account on this topic). Of significance to the present article is the following version of isomorphism theorem between occupation times for random walk loop soups and Gaussian free fields shown in [53].

Recall the definition of random walk loop soups $\mathcal{L}_{\alpha,N}$. We define the associated occupation time field $(\hat{\mathcal{L}}^x_\alpha)_{x\in V_N}$ by

$$\hat{\mathcal{L}}^x_\alpha = \sum_{\gamma\in\mathcal{L}_{\alpha,N}} \int_0^{T(\gamma)} \mathbf{1}_{\gamma(t)=x} dt$$

where $T(\gamma)$ is the duration of the loop γ. The isomorphism theorem in [53] states that

$$\{\hat{\mathcal{L}}^x_{1/2} : x \in V_N\} \overset{law}{=} \{\frac{1}{2}\eta^2_{N,x} : x \in V_N\} \tag{2.2}$$

(Note that this holds for random walks in general graphs). Recently, couplings between random walks/random walk loop soups and Gaussian free fields have been developed in [54], where the signs of GFFs are incorporated in the coupling in order to provide certificate for vertices/edges not visited by random walks/random walks loop soups. The paper [54] was motivated by connectivity of the loop soup clusters as well as random interlacement. Independent of [54], such coupling was established for random walks in [71] with the application of deriving an exponential concentration for cover times. The work [71] was motivated by [35], where such coupling was proved for general trees and questioned for general graphs; the advance in [54] was independent of [35].

In fact, using the coupling derived in [54] only allows us to prove a version of Theorem 2.2

for Lupu's loop soup on the metric graph introduced in [54]; see Section 2.3 and in particular Theorem 2.3. In order to deal with the random walk loop soup, we will use a recent advancing on the *random current model* for random walk loop soups. A random current model on a graph, say $\mathcal{G}_N = (V_N, E_N)$ in our case, is the probability measure $\mathbb{P}$ with

$$\mathbb{P}((n_e)_{e\in E_N}) \propto \prod_{e\in E_N} \frac{(\beta_e)^{n_e}}{n_e!}\,, \tag{2.3}$$

where $(n_e)_{e\in E_N}$ are nonnegative integers such that $\sum_{v\in e} n_e$ is even for any $v \in V_N$, and $(\beta_e)_{e\in E_N}$ are positive parameters on E_N. Conditioning on $\{\hat{\mathcal{L}}^v_{1/2} = \ell_v\}_{v\in V_N}$, let $(n_e)_{e\in E_N}$ be a random current model with parameters $\beta_e = 2\sqrt{\ell_x \ell_y}$ on edge $e = (x, y)$.

It was shown in [70, 57, 46] (see [46, Theorem 1] for a formal statement) that conditioning on the local times the distribution of $(n_e)_{e\in E_N}$ is the same as that of the number of jumps of the random walk loop soup $\mathcal{L}_{1/2,N}$ along each $e \in E_N$, and therefore $(1_{n_e>0})_{e\in E_N}$ has the same distribution as the graph induced by $\mathcal{L}_{1/2,N}$ on V_N.

We remark that the random current representation played a crucial role in a recent work [22] which proved the continuity of spontaneous magnetization for the three-dimensional Ising model at the critical temperature. Finally, we remark that the random Eulerian graph model considered in [35] (which was used to reconstruct the number of visits to vertices from the continuous occupation times) was of high resemblance of the random current model.

Entropic repulsions. Unlike the Lupu's loop soup, the clusters for the critical random walk loop soup is strictly dominated by the sign clusters of the GFF on the metric graph. In order to address this, we apply the aforementioned random current model and see that the loop clusters dominates a generalized sign cluster on the metric graph, where we replace each original edge (which can be viewed as a unit resistor) by two edges and assign the conductances so that it sums to 1. This is summarized in Lemma 2.3. When employing the proof idea of Theorem 2.3, we encounter a problem which amounts to bound the typical value of a GFF under the conditioning of staying positive in a subset. Results of this type, on such entropic repulsions for two-dimensional GFFs under the presence of hard wall, has been obtained in [41, 24]. Our set up is slightly more complicated (and somewhat non-standard), and dealing with it forms the main technical ingredient in Section 2.4. As standard in this type of problems, our proof crucially relies on the FKG inequality [43, 62] and the Brascamp-Lieb inequality [25].

2.1.3 Open problems

Our results motivate a number of interesting questions, as we list below.

Question 2.1. *For the random walk loop soup in the supercritical regime (i.e., with intensity strictly larger than $\frac{1}{2}$), is the dimension of the chemical distance 1 with high probability?*

Question 2.2. *Can one prove an analogous result for Brownian loop soups?*

Next, we will ask a number of questions in the context of level set percolation for GFF, but one can ask natural analogous questions for loop soups as well as random walks. We feel that, perhaps the questions regarding to GFF may be answered before that on random walks.

Question 2.3. *Under assumptions of Theorem 2.1, is the dimension of chemical distance 1 with high probability conditioned on the existence of an open crossing?*

Question 2.4. *Under assumptions of Theorem 2.1, is the length of minimal open crossing $O(n)$ with positive probability?*

Question 2.5. *Under assumptions of Theorem 2.1, is the number of disjoint open crossings tight?*

Finally, we pose a question regarding to universality of Theorem 2.1, whose difficulty is due to the crucial role of Makarov's theorem (which seems to only apply for GFF) in the proof of Theorem 2.1. In fact, we choose to keep an open mind on whether such universality holds, in light of a non-universality result in [39].

Question 2.6. *Does an analogous result to Theorem 2.1 hold for all log-correlated Gaussian fields?*

2.2 Percolation for Gaussian free fields

This section is devoted to the proof of Theorem 2.1. For notation convenience, we say a vertex v is λ-open (or open if no risk of confusion) if $v \in \mathcal{H}_{N,\lambda}$, and λ-closed (or closed) otherwise. For any $A, B \subseteq V_N$, we denote by $A \xleftrightarrow{\leq \lambda} B$ the event that there exists a λ-open path P connecting A and B, i.e., $D_{N,\lambda}(A, B) < \infty$.

2.2.1 One-arm estimate: a warm up argument

In this subsection, we give a warm up argument on level set percolation for GFF. Despite being rather simple, the argument is a clear demonstration of the fundamental idea of the paper, which allows to take advantage of the Markov field property in studying percolations. We remark that a similar argument was employed in [69, Section 3].

Proposition 2.1. *For any $0 < \alpha < \beta < 1$, there exists a constant $c > 0$ such that for all $\lambda > 0$*

$$\mathbb{P}(\partial V_{\alpha N} \xleftrightarrow{\leq \lambda} \partial V_{\beta N}) \geq 1 - 2\mathrm{e}^{-c\lambda^2} .$$

Proof. By duality, the complement of the event $\{\partial V_{\alpha N} \xleftrightarrow{\leq \lambda} \partial V_{\beta N}\}$ is the same as the event that there exists a λ-closed contour $\mathcal{C} \subseteq V_{\beta N}$ surrounding $V_{\alpha N}$. We let $\mathfrak{C}$ be the collection of all such contours. It suffices to estimate $\mathbb{P}(\mathfrak{C} \neq \emptyset)$.

To this end, we consider a natural partial order on all contours. For any contour $\mathcal{C}$, we let $\bar{\mathcal{C}}$ be the collection of vertices that are surrounded by $\mathcal{C}$. For two contours $\mathcal{C}_1$ and $\mathcal{C}_2$, we say $\mathcal{C}_1 \leq \mathcal{C}_2$ if $\bar{\mathcal{C}_1} \subseteq \bar{\mathcal{C}_2}$. A key observation is that this partial order generates a well-defined (unique) global minimum on $\mathfrak{C}$, which we denote by $\mathcal{C}^*$. Furthermore, for any contour $\mathcal{C} \subseteq V_{\beta N}$ surrounding $V_{\alpha N}$, we have

$$\{\mathcal{C}^* = \mathcal{C}\} \in \mathcal{F}_{\bar{\mathcal{C}}} \triangleq \sigma(\{\eta_{N,v} : v \in \bar{\mathcal{C}}\})\,. \tag{2.4}$$

Define

$$X = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{v \in \partial V_{(1+\beta)N/2}} \eta_{N,v}\,. \tag{2.5}$$

We will need the following standard estimates on simple random walks; we include a proof merely for completeness.

Lemma 2.1. *For any fixed $0 < r < 1$, there exist constants $c_1, c_2 > 0$ which depend on r such that*

$$\sum_{v \in \partial V_{rN}} G_{V_N}(u, v) \leq c_1 N, \quad \forall u \in \partial V_{rN} \tag{2.6}$$

and

$$G_{V_N}(u, v) \geq c_2, \quad \forall u, v \in V_{rN}\,. \tag{2.7}$$

Furthermore, for any $0 < \alpha < \beta < 1$, there exists a constant $c_3 > 0$ such that for all $u \in \partial V_{\beta N}$, the simple random walk started at u will hit $\partial V_{\alpha N}$ before ∂V_N with probability at least c_3.

Proof. For convenience we assume that V_N is centered at the origin. Let $S_n = (S_{1,n}, S_{2,n})$ be a simple random walk on $\mathbb{Z}^2$. It is clear that if S is on ∂V_{rN} at some point, in the next step it will move to some vertex on ∂V_{rN+1} with probability at least $1/4$, and after that, it will hit ∂V_N before ∂V_{rN} with probability at least $\frac{1}{(1-r)N}$ (since $\max\{|S_{1,n}|, |S_{2,n}|\}$ is a submartingale). Therefore, a simple random walk started at any $u \in \partial V_{rN}$ will in expectation visit ∂V_{rN} at most $4(1-r)N$ times before hitting ∂V_N. This proves our first bound (2.6).

For the second bound (2.7), let $\epsilon = \frac{1-r}{100}$. Denote by $u = (u_1, u_2)$ and $v = (v_1, v_2)$. By independence of the simple random walks in x and y-coordinates, there exists $c' = c'(r)$ such that with probability at least c' the simple random walk started at u will hit some point v^* in the vertical line $x = v_1$ before exiting V_N or before the y-coordinate deviates by more than ϵN; started from v^*, there is again probability at least c' for the simple random walk to hit the horizontal line $y = v_2$ before the horizontal coordinate deviates by more than ϵN. Altogether, there is probability at least $(c')^2$ for the random walk to hit the ℓ_∞-ball of radius ϵN around v before exiting V_N. At this point, an application of [50, Proposition 4.6.2, Theorem 4.4.4.] completes the verification of (2.7).

The last statement of lemma was implicitly proved in the above derivation of (2.7). $\square$

As a simple corollary of (2.6), there exists a constant $c_4 > 0$ which depends on β such that

$$\operatorname{Var} X = \frac{1}{4} \frac{1}{|\partial V_{(1+\beta)N/2}|^2} \sum_{u,v \in \partial V_{(1+\beta)N/2}} G_{V_N}(u,v) \le c_4 \tag{2.8}$$

and thus we also have $\operatorname{Var}(X \mid \mathcal{F}_{\bar{\mathcal{C}}}) \le c_4$.

By the Markov property of the GFF, we have for each $v \in \partial V_{(1+\beta)N/2}$

$$\mathbb{E}(\eta_{N,v} \mid \mathcal{F}_{\bar{\mathcal{C}}}) = \sum_{u \in \mathcal{C}} \operatorname{Hm}(v, u; \mathcal{C} \cup \partial V_N) \cdot \eta_{N,u} \,. \tag{2.9}$$

Here for a set A, we use $\operatorname{Hm}(v, u; A)$ to denote the harmonic measure at u with respect to starting point v and the target set A (i.e., $\operatorname{Hm}(v, u; A) = \mathbb{P}_v(S_{\tau_A} = u)$, where (S_n) is a simple random walk on $\mathbb{Z}^2$ and τ_A is the first time it hits set A). We further denote by $\operatorname{Hm}(v, B; A) = \sum_{u \in B} \operatorname{Hm}(v, u; A)$. Now on the event $\{\mathcal{C}^* = \mathcal{C}\}$, we have $\eta_{N,u} \ge \lambda$ for all $u \in \mathcal{C}$. Combined with Lemma 2.1, it gives that

$$\mathbb{E}(\eta_{N,v} \mid \mathcal{F}_{\bar{\mathcal{C}}}) \ge \lambda \operatorname{Hm}(v, \mathcal{C}; \mathcal{C} \cup \partial V_N) \ge \lambda \operatorname{Hm}(v, \partial V_{\alpha N}; \partial V_{\alpha N} \cup \partial V_N) \ge c_3 \lambda \,. \tag{2.10}$$

Therefore, we have $\mathbb{E}(X \mid \mathcal{F}_{\bar{\mathcal{C}}}) \ge c_3 \lambda$ on the event $\{\mathcal{C}^* = \mathcal{C}\}$. Thus,

$$\mathbb{P}(X \ge c_3 \lambda/2 \mid \mathcal{F}_{\bar{\mathcal{C}}}) \ge 1 - \mathbb{P}(Z(c_4) \ge c_3 \lambda/2) \text{ on the event } \{\mathcal{C}^* = \mathcal{C}\} \,,$$

where we denote by $Z(c_4)$ a mean zero Gaussian variable with variance c_4. Since $\{\mathcal{C}^* = \mathcal{C}\} \in \mathcal{F}_{\bar{\mathcal{C}}}$, we have

$$\mathbb{P}(X \geq c_3\lambda/2 \mid \mathcal{C}^* = \mathcal{C}) \geq 1 - \mathbb{P}(Z(c_4) \geq c_3\lambda/2)\,.$$

Summing this over all possible contours $\mathcal{C} \subseteq V_{\beta N}$ surrounding $V_{\alpha N}$, we obtain that

$$\mathbb{P}(X \geq c_3\lambda/2 \mid \mathfrak{C} \neq \emptyset) \geq 1 - \mathbb{P}(Z(c_4) \geq c_3\lambda/2)\,.$$

Combined with the simple fact that

$$\mathbb{P}(X \geq c_3\lambda/2) \leq \mathbb{P}(Z(c_4) \geq c_3\lambda/2) \tag{2.11}$$

it follows that

$$\mathbb{P}(\mathfrak{C} \neq \emptyset) \leq \frac{\mathbb{P}(X \geq c_3\lambda/2)}{\mathbb{P}(X \geq c_3\lambda/2 \mid \mathfrak{C} \neq \emptyset)} \leq \frac{\mathbb{P}(Z(c_4) \geq c_3\lambda/2)}{1 - \mathbb{P}(Z(c_4) \geq c_3\lambda/2)}\,.$$

This completes the proof of the proposition. □

2.2.2 Proof of Theorem 2.1

Consider $\lambda > 0$. Our goal is to provide a lower bound on the probability that there exists a λ-open path with length less than $N\mathrm{e}^{(\log N)^{9/10}}$ connecting $\partial V_{\alpha N}$ and $\partial V_{\beta N}$. Therefore we may restrict our attention to λ-open paths connecting $\partial V_{\alpha N}$ and $\partial V_{\beta N}$ that do not touch the interior of $V_{\alpha N}$ or $V_N \setminus V_{\beta N}$. This motivates the following exploration procedure. We set $\mathcal{A}_0 = \partial V_{\alpha N} \cap \mathcal{H}_{N,\lambda}$, $\mathcal{B}_0 = \partial V_{\alpha N} \setminus \mathcal{H}_{N,\lambda}$, $\mathcal{C}_0 = \partial V_{\alpha N}$, and for $i = 0, 1, 2, \ldots$, we define inductively

$$\begin{aligned}\mathcal{A}_{i+1} &= \{v \in ((V_N \setminus V_{\alpha N}) \setminus \mathcal{C}_i) \cap H_{N,\lambda} : v \sim u \text{ for some } u \in \mathcal{A}_i\}\,,\\ \mathcal{B}_{i+1} &= \{v \in ((V_N \setminus V_{\alpha N}) \setminus \mathcal{C}_i) \setminus H_{N,\lambda} : v \sim u \text{ for some } u \in \mathcal{A}_i\} \cup \mathcal{B}_i\,,\\ \mathcal{C}_{i+1} &= \cup_{j=0}^{i+1} \mathcal{A}_j \cup \mathcal{B}_{i+1}\,.\end{aligned}$$

In other words, $\mathcal{C}_i$ records all the vertices that have been explored before (or at) stage i. At stage $i+1$, we check all the neighbors of $\mathcal{A}_i$ that is in $V_N \setminus V_{\alpha N}$ and has not been explored: if the vertex is in $\mathcal{H}_{N,\lambda}$ then we put it to $\mathcal{A}_{i+1}$, otherwise we put it to $\mathcal{B}_{i+1}$. It is clear that $\mathcal{A}_i$ records all the vertices in $V_N \setminus V_{\alpha N}$ that are of chemical distance i to $\partial V_{\alpha N}$, and $\mathcal{B}_i$ records all the closed vertices we have encountered. Furthermore, we observe that

- $\mathcal{A}_i$'s are disjoint from each other.
- $\mathcal{C}_i$ is a connected set in $V_N \setminus V_{\alpha N}$.
- $\partial \mathcal{C}_i$ (the boundary points of $\mathcal{C}_i$) is a subset of $\mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{A}_0$.
- Let $\mathcal{C}'_i \triangleq \{u : \mathrm{Hm}(\infty, u; \mathcal{C}_i) > 0\}$. Then $\mathcal{C}'_i \subseteq \partial \mathcal{C}_i$ and $\mathcal{C}'_i \cap \mathcal{A}_0 = \emptyset$, so that $\mathcal{C}'_i \subseteq \mathcal{A}_i \cup \mathcal{B}_i$.

Now suppose that the event $\mathcal{E} \triangleq \{D_{N,\lambda}(\partial V_{\alpha N}, \partial V_{\beta N}) \geq N \mathrm{e}^{(\log N)^{9/10}}\}$ occurs, then we must have that $\mathcal{C}_i$ is disjoint from $V_N \setminus V_{\beta N}$ for all $0 \leq i < N \mathrm{e}^{(\log N)^{9/10}}$. Further, since $\mathcal{A}_i$'s are disjoint from each other, we see (from a simple volume consideration) that there exists at least an $i_0 < N \mathrm{e}^{(\log N)^{9/10}}$ such that

$$|\mathcal{A}_{i_0}| \leq N \mathrm{e}^{-(\log N)^{9/10}}\,. \tag{2.12}$$

We let τ be the minimal number i_0 which satisfies (2.12). In summary, we have

$$\mathcal{E} \subseteq \mathcal{E}' \triangleq \bigsqcup_{\substack{0 \leq k < N \mathrm{e}^{(\log N)^{9/10}} \\ (A_0, \ldots A_k, B_0, \ldots B_k) \in \mathcal{P}_k}} \{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \leq i \leq k\} \tag{2.13}$$

where $\mathcal{P}_k$ indicates all $(A_0, \ldots A_k, B_0, \ldots B_k)$ that are compatible with $\mathcal{E}$ and $\{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k\}$. In particular, they should satisfy:

- Denote $C_k \triangleq \cup_{i=0}^k A_i \cup B_k$. Then C_k is a connected set in $V_{\beta N} \setminus V_{\alpha N}$.
- Denote $C'_k \triangleq \{u : \mathrm{Hm}(\infty, u; C_k) > 0\}$. Then $C'_k \subseteq A_k \cup B_k$.
- $|A_k| \le N\mathrm{e}^{-(\log N)^{9/10}}$.

Now we fix any $0 \le k < N\mathrm{e}^{(\log N)^{9/10}}$ and any $(A_0, \ldots A_k, B_0, \ldots B_k) \in \mathcal{P}_k$. It is not hard to verify that

$$\{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k\} \in \mathcal{F}_{C_k}\,. \tag{2.14}$$

Conditioning on $\mathcal{F}_{C_k}$, the field $\{\eta_{N,v} : v \in V_N \setminus C_k\}$ is again distributed as a GFF. In particular, for each $v \in \partial V_{(1+\beta)N/2}$, we have

$$\mathbb{E}(\eta_{N,v} \mid \mathcal{F}_{C_k}) = \sum_{u \in C_k} \mathrm{Hm}(v, u; C_k \cup \partial V_N) \cdot \eta_{N,u}\,. \tag{2.15}$$

Since $C_k \subseteq V_{\beta N} \setminus V_{\alpha N}$, we have from Lemma 2.1

$$\mathrm{Hm}(v, C_k; C_k \cup \partial V_N) \ge \mathrm{Hm}(v, \partial V_{\alpha N}; \partial V_{\alpha N} \cup \partial V_N) \ge c_3\,. \tag{2.16}$$

In addition, we have that

$$\{u \in C_k : \mathrm{Hm}(v, u; C_k \cup \partial V_N) > 0\} = \{u : \mathrm{Hm}(\infty, u; C_k) > 0\} = C'_k \subseteq A_k \cup B_k\,. \tag{2.17}$$

We want to show that $\mathrm{Hm}(v, A_k; C_k \cup \partial V_N)$ is small. First, we note that

$$\mathrm{Hm}(v, A_k; C_k \cup \partial V_N) \le \mathrm{Hm}(v, A_k; C_k)\,. \tag{2.18}$$

By a combination of Theorem 1.7.6 (Harnack principle), Theorem 2.1.3 and Exercise 2.1.4 in [47], we have for constants $c_6, c_7, c_8 > 0$ which depend on β, any $u \in C_k$ and arbitrary $w \in \partial V_{20N}$

$$\mathrm{Hm}(v, u; C_k) \le c_6 \mathrm{Hm}(w, u; C_k) \le c_7 \mathrm{Hm}(8N, u; C_k) \le c_8 \mathrm{Hm}(\infty, u; C_k)\,,$$

where $\mathrm{Hm}(8N, u; C_k)$ corresponds to the $H_A^m(y)$ in [47, Theorem 2.1.3] with $A = C_k$, $m = 8N$ and $y = u$. Therefore,

$$\mathrm{Hm}(v, A_k; C_k) \le c_8 \mathrm{Hm}(\infty, A_k; C_k)\,. \tag{2.19}$$

Since C_k is a connected set of radius between $(\alpha/2)N$ and $2N$, by [48, Proposition 4.1] we deduce that for constants $c_9, c_{10} > 0$ depending only on α

$$\mathrm{Hm}(\infty, \{u \in C_k : \mathrm{Hm}(\infty, u; C_k) > c_9 N^{-1} \mathrm{e}^{(\log N)^{4/5}}\}; C_k) \le c_{10} (\log N)^{-20}\,.$$

Therefore,

$$\begin{aligned}
\mathrm{Hm}(\infty, A_k; C_k) &= \mathrm{Hm}(\infty, A_k \cap \{u \in C_k : \mathrm{Hm}(\infty, u; C_k) \le c_9 N^{-1} \mathrm{e}^{(\log N)^{4/5}}\}; C_k) \\
&\quad + \mathrm{Hm}(\infty, A_k \cap \{u \in C_k : \mathrm{Hm}(\infty, u; C_k) > c_9 N^{-1} \mathrm{e}^{(\log N)^{4/5}}\}; C_k) \\
&\le (c_9 N^{-1} \mathrm{e}^{(\log N)^{4/5}}) \cdot N \mathrm{e}^{-(\log N)^{9/10}} + c_{10} (\log N)^{-20} \\
&= o(\log N)^{-10}\,.
\end{aligned} \tag{2.20}$$

Combining (2.18), (2.19) and (2.20), we finally have

$$\mathrm{Hm}(v, A_k; C_k \cup \partial V_N) = o(\log N)^{-10}\,. \tag{2.21}$$

Combined with (2.16) and (2.17), it yields that

$$\mathrm{Hm}(v, B_k; C_k \cup \partial V_N) \ge c_3 - o(\log N)^{-10}\,.$$

Now on the event $\{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k\}$, we have by definition that $\eta_{N,u} \ge \lambda$ for all $u \in B_k$, so we can derive from (2.15) that

$$\begin{aligned}\mathbb{E}(\eta_{N,v} \mid \mathcal{F}_{C_k}) &= \sum_{u \in C_k} \mathrm{Hm}(v, u; C_k \cup \partial V_N) \cdot \eta_{N,u} \\ &= \sum_{u \in A_k} \mathrm{Hm}(v, u; C_k \cup \partial V_N) \cdot \eta_{N,u} + \sum_{u \in B_k} \mathrm{Hm}(v, u; C_k \cup \partial V_N) \cdot \eta_{N,u} \\ &\ge (c_3 - o((\log N)^{-10}))\lambda - o((\log N)^{-10}) \sup_{u \in V_N} |\eta_{N,u}| \,.\end{aligned}$$

Define $\Lambda_{\mathrm{bad}} = \{\sup_{u \in V_N} |\eta_{N,u}| \ge 100 \log N\}$. By a straightforward computation, we have

$$\mathbb{P}(\Lambda_{\mathrm{bad}}) \le N^{-20} \,. \tag{2.22}$$

We can assume without loss that Λ_{bad} does not occur. To be precise, for sufficiently large N, on the event $\{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k\} \setminus \Lambda_{\mathrm{bad}}$, we have

$$\mathbb{E}(\eta_{N,v} \mid \mathcal{F}_{C_k}) \ge 9c_3\lambda/10 \,.$$

Recall the definition of X in (2.5). Then on the same event, we have $\mathbb{E}(X \mid \mathcal{F}_{C_k}) \ge 9c_3\lambda/10$ and $\mathrm{Var}(X \mid \mathcal{F}_{C_k}) \le \mathrm{Var}\, X \le c_4$. Thus (still on the same event),

$$\mathbb{P}(X \ge c_3\lambda/2 \mid \mathcal{F}_{C_k}) \ge 1 - \mathbb{P}(Z(c_4) \ge 2c_3\lambda/5) \,,$$

where $Z(c_4)$ is a mean zero Gaussian variable with variance c_4. By (2.14), this gives

$$\begin{aligned}& \mathbb{P}(X \ge c_3\lambda/2, \tau = k, \mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k) \\ = \;& \mathbb{E}(\mathbb{P}(X \ge c_3\lambda/2 \mid \mathcal{F}_{C_k}) \mathbf{1}_{\{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k\}}) \\ \ge \;& (1 - \mathbb{P}(Z(c_4) \ge 2c_3\lambda/5)) \mathbb{P}(\{\tau = k\} \cap \{\mathcal{A}_i = A_i, \mathcal{B}_i = B_i \text{ for } 0 \le i \le k\} \setminus \Lambda_{\mathrm{bad}}) \,.\end{aligned}$$

Summing this over all $0 \le k < N e^{(\log N)^{9/10}}$ and all $(A_0, \dots A_k, B_0, \dots B_k) \in \mathcal{P}_k$ and using (2.13), we have

$$\mathbb{P}(X \ge c_3\lambda/2) \ge (1 - \mathbb{P}(Z(c_4) \ge 2c_3\lambda/5))\mathbb{P}(\mathcal{E}' \setminus \Lambda_{\text{bad}}) .$$

Therefore,

$$\begin{aligned}\mathbb{P}(\mathcal{E}) \le \mathbb{P}(\mathcal{E}') &\le \mathbb{P}(\mathcal{E}' \setminus \Lambda_{\text{bad}}) + \mathbb{P}(\Lambda_{\text{bad}}) \\ &\le \frac{\mathbb{P}(X \ge c_3\lambda/2)}{1 - \mathbb{P}(Z(c_4) \ge 2c_3\lambda/5)} + \mathbb{P}(\Lambda_{\text{bad}}) \\ &\le \frac{\mathbb{P}(Z(c_4) \ge c_3\lambda/2)}{1 - \mathbb{P}(Z(c_4) \ge 2c_3\lambda/5)} + \mathbb{P}(\Lambda_{\text{bad}}) .\end{aligned}$$

Combined with (2.22), this completes the proof of Theorem 2.1.

2.3 Percolation of the continuous loop soup

In this section we prove an analogous result to Theorem 2.2 for the continuous loop soups defined on the *metric graph* of $\mathcal{G}_N = (V_N, E_N)$ at critical intensity 1/2. The result in this section will not be used in the derivation of Theorem 2.2. However, our proof method of Theorem 2.2 is hugely inspired by the consideration of the continuous loop soup. Therefore, we include the present section, with the hope of conveying the source of insight.

The continuous loop soup as well as the Gaussian free field on the metric graph were considered in [54]. We follow the setup and definitions there. We let $\tilde{\mathcal{G}}_N$ be the metric graph (or the cable system) of $\mathcal{G}_N$ where each edge in $\tilde{\mathcal{G}}_N$ has length $\frac{1}{2}$. On $\tilde{\mathcal{G}}_N$ we can define a standard Brownian motion $B^{\tilde{\mathcal{G}}_N}$, so that $B^{\tilde{\mathcal{G}}_N}$ when restricted to V_N is the same as the aforementioned continuous-time sub-Markovian jump process (X_t). Let $G_{\tilde{\mathcal{G}}_N}(u,v)$ be the Green's function of $B^{\tilde{\mathcal{G}}_N}$, so that for $u, v \in V_N$, $G_{\tilde{\mathcal{G}}_N}(u,v) = \frac{1}{4} G_{V_N}(u,v)$, the Green's function of (X_t). Let $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N\}$ be the Gaussian free field on $\tilde{\mathcal{G}}_N$ with covariance function $G_{\tilde{\mathcal{G}}_N}(u,v)$. Then the restriction of $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N\}$ to V_N is the same as the

Gaussian free field $\{\eta_{N,v} : v \in V_N\}$. Moreover, $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N\}$ can be obtained from $\{\eta_{N,v} : v \in V_N\}$ by (for each edge $e = (u, v)$) independently sampling a variance 2 Brownian bridge of length $\frac{1}{2}$ with values $\eta_{N,u}$ and $\eta_{N,v}$ at the endpoints. In particular, as shown in [54] $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N\}$ has a continuous realization.

Now we can associate to $B^{\tilde{\mathcal{G}}_N}$ a measure $\tilde{\mu}_N$ on continuous loops in $\tilde{\mathcal{G}}_N$, and for each $\alpha > 0$ consider the continuous loop soup $\tilde{\mathcal{L}}_{\alpha,N}$ which is a Poisson point process with intensity $\alpha\tilde{\mu}_N$. The loops of $\tilde{\mathcal{L}}_{\alpha,N}$ may be partitioned into clusters. For $u, v \in \tilde{\mathcal{G}}_N$, we define the chemical distance of $\tilde{\mathcal{L}}_{\alpha,N}$ between u and v by

$$D_{\tilde{\mathcal{L}}_{\alpha,N}}(u, v) = \min_{\gamma} |\gamma| ,$$

where the minimum is over all path $\gamma \subseteq \tilde{\mathcal{G}}_N$ joining u and v that stays within a cluster of $\tilde{\mathcal{L}}_{\alpha,N}$.

Theorem 2.3. *For any $0 < \alpha < \beta < 1$, there exists a constant $c > 0$ such that for all N*

$$\mathbb{P}(D_{\tilde{\mathcal{L}}_{1/2,N}}(\partial V_{\alpha N}, \partial V_{\beta N}) \leq N \mathrm{e}^{(\log N)^{9/10}}) \geq c .$$

Remark 2.3. *We expect that the probability for $D_{\tilde{\mathcal{L}}_{1/2,N}}(\partial V_{\alpha N}, \partial V_{\beta N}) < \infty$ is strictly less than 1; see Remark 2.1.*

By [54, Proposition 2.1], there is a coupling between $\tilde{\mathcal{L}}_{1/2,N}$ and a continuous version of $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N\}$ such that the clusters of loops of $\tilde{\mathcal{L}}_{1/2,N}$ are exactly the sign clusters of $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N\}$. In light of this, define

$$D_{\tilde{\eta}_N}(u, v) = \min_{\gamma} |\gamma| ,$$

where the minimum is over all path $\gamma \subseteq \tilde{\mathcal{G}}_N$ joining u and v such that $\tilde{\eta}_{N,\gamma}$ are of the same sign (plus or minus). In order to prove Theorem 2.3, it suffices to prove the following proposition.

Proposition 2.2. *For any $0 < \alpha < \beta < 1$, there exists a constant $c > 0$ such that for all N*

$$\mathbb{P}(D_{\tilde{\eta}_N}(\partial V_{\alpha N}, \partial V_{\beta N}) \leq N\mathrm{e}^{(\log N)^{9/10}}) \geq c\,.$$

Proof. For a compact connected set $K \subseteq \tilde{\mathcal{G}}_N$ and any $v \in \tilde{\mathcal{G}}_N \setminus K$, let $T_{K\cup\partial V_N, v}$ be the first time the Brownian motion $B^{\tilde{\mathcal{G}}_N}$ started at v hits $K \cup \partial V_N$. It is clear that $B^{\tilde{\mathcal{G}}_N}_{T_{K\cup\partial V_N,v}}$ can only take on finitely many values u in $K \cup \partial V_N$. Therefore for such $u \in K \cup \partial V_N$ we can define

$$\widetilde{\mathrm{Hm}}(v, u; K \cup \partial V_N) \triangleq \mathbb{P}^v(B^{\tilde{\mathcal{G}}_N}_{T_{K\cup\partial V_N,v}} = u)$$

(and set $\widetilde{\mathrm{Hm}}(v, u; K \cup \partial V_N) = 0$ otherwise) to be the harmonic measure of $B^{\tilde{\mathcal{G}}_N}$ at u with respect to starting point v and target set $K \cup \partial V_N$.

Our proof strategy is similar to that of Theorem 2.1. Consider the following exploration procedure on $\tilde{\mathcal{G}}_N$. Let $\mathcal{A}_0 = \partial V_{\alpha N}$, $\mathcal{B}_0 = \emptyset$ and $\tilde{\mathcal{I}}_0 = \partial\tilde{\mathcal{G}}_{\alpha N}$. For $i = 0, 1, 2, \ldots$, at stage $(i + 1)$

- We set initially $\mathcal{A}_{i+1} = \emptyset$, $\mathcal{B}_{i+1} = \mathcal{B}_i$ and $\tilde{\mathcal{I}}_{i+1} = \tilde{\mathcal{I}}_i$.
- If $\mathcal{A}_i = \emptyset$, stop. Otherwise, for each $v \in \mathcal{A}_i$ and every edge $e = (v, u)$ incident to v, if $u \in V_N \setminus V_{\alpha N}$ and the neighborhood of v along e does not belong to $\tilde{\mathcal{I}}_i$, we go (explore) from v along e to u until we reach a zero for $\{\tilde{\eta}_N\}$. In the case no zero is reached, we add all the points in e into $\tilde{\mathcal{I}}_{i+1}$ and add u, if it is not already in $\cup_{j=0}^{i}\mathcal{A}_j$, into $\mathcal{A}_{i+1}$; in the case that the first zero is reached at $w \in e$, we add all the points between v and w into $\tilde{\mathcal{I}}_{i+1}$ and add w into $\mathcal{B}_{i+1}$.

In summary, $\mathcal{A}_i$ records all the lattice points in $V_N \setminus V_{\alpha N}$ that are of chemical distance (under $D_{\tilde{\eta}_N}$) i to $\partial V_{\alpha N}$; $\mathcal{B}_i$ records all the zeros reached in the exploration procedure up to stage i; $\tilde{\mathcal{I}}_i$ records all the points that have been explored (including the internal points of edges) up to stage i.

It is clear from the construction that

- $\tilde{\mathcal{I}}_i$ is a compact connected set in $\tilde{\mathcal{G}}_N \setminus \tilde{\mathcal{G}}_{\alpha N}$.

- $\mathcal{A}_i$'s are disjoint from each other. $\cup_{j=0}^{i} \mathcal{A}_j$ is exactly the set of all the lattice points in $\tilde{\mathcal{I}}_i$.

- $\partial \tilde{\mathcal{I}}_i$ (the boundary points of $\tilde{\mathcal{I}}_i$) is a subset of $\mathcal{A}_i \cup \mathcal{B}_i \cup \tilde{\mathcal{I}}_0$.

- Let $\mathcal{C}_i \triangleq \{u : \widetilde{\mathrm{Hm}}(\infty, u; \tilde{\mathcal{I}}_i) > 0\}$. Then $\mathcal{C}_i \subseteq \partial \tilde{\mathcal{I}}_i$ and $\mathcal{C}_i \cap \tilde{\mathcal{I}}_0 = \emptyset$, so that $\mathcal{C}_i \subseteq \mathcal{A}_i \cup \mathcal{B}_i$.

Suppose that the event $\mathcal{E} \triangleq \{D_{\tilde{\eta}_N}(\partial V_{\alpha N}, \partial V_{\beta N}) > N\mathrm{e}^{(\log N)^{9/10}}\}$ occurs, then $\tilde{\mathcal{I}}_i$ must be a subset of $\tilde{\mathcal{G}}_{\beta N} \setminus \tilde{\mathcal{G}}_{\alpha N}$ for all $0 \leq i < N\mathrm{e}^{(\log N)^{9/10}}$. Further, let τ be the minimal number i_0 which satisfies

$$|\mathcal{A}_{i_0}| \leq N\mathrm{e}^{-(\log N)^{9/10}}.$$

Then since $\mathcal{A}_i$'s are disjoint from each other, we have $\tau < N\mathrm{e}^{(\log N)^{9/10}}$ on the event $\mathcal{E}$.

Conditioning on $\mathcal{F}_\tau \triangleq \sigma(\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{I}}_\tau\})$, by the strong Markov property in [54, Section 3], $\{\tilde{\eta}_{N,v} : v \in \tilde{\mathcal{G}}_N \setminus \tilde{\mathcal{I}}_\tau\}$ is distributed as a mean zero GFF in $\tilde{\mathcal{G}}_N \setminus \tilde{\mathcal{I}}_\tau$ plus the harmonic extension of $\tilde{\eta}_{N,v}$ from $\tilde{\mathcal{I}}_\tau$ to $\tilde{\mathcal{G}}_N \setminus \tilde{\mathcal{I}}_\tau$. In particular, on the event $\mathcal{E}$ where $\tilde{\mathcal{I}}_\tau$ is contained in $\tilde{\mathcal{G}}_{\beta N} \setminus \tilde{\mathcal{G}}_{\alpha N}$, we have for each $v \in \partial V_{(1+\beta)N/2}$

$$\mathbb{E}(\tilde{\eta}_{N,v} \mid \mathcal{F}_\tau) = \sum_{u \in \tilde{\mathcal{I}}_\tau} \widetilde{\mathrm{Hm}}(v, u; \tilde{\mathcal{I}}_\tau \cup \partial V_N) \cdot \tilde{\eta}_{N,u} = \sum_{u \in \mathcal{C}_\tau} \widetilde{\mathrm{Hm}}(v, u; \tilde{\mathcal{I}}_\tau \cup \partial V_N) \cdot \tilde{\eta}_{N,u}. \quad (2.23)$$

By definition we know that $\tilde{\eta}_{N,u} = 0$ for all $u \in \mathcal{B}_\tau$. Thus

$$\sum_{u \in \mathcal{B}_\tau} \widetilde{\mathrm{Hm}}(v, u; \tilde{\mathcal{I}}_\tau \cup \partial V_N) \cdot \tilde{\eta}_{N,u} = 0. \quad (2.24)$$

We want to show that $\widetilde{\mathrm{Hm}}(v, \mathcal{A}_\tau; \tilde{\mathcal{I}}_\tau \cup \partial V_N)$ is small. Let $\mathcal{D}_\tau = \cup_{j=0}^{\tau} \mathcal{A}_j$ be the set of all the lattice points in $\tilde{\mathcal{I}}_\tau$. Then $\mathcal{D}_\tau$ contains $\mathcal{A}_\tau$ and $\mathcal{A}_0 (= \partial V_{\alpha N})$ by definition, and it is a connected subgraph of $\mathcal{G}_N$. Since $\mathcal{D}_\tau$ is a subset of $\tilde{\mathcal{I}}_\tau$, and the print of $B^{\tilde{\mathcal{G}}_N}$ on V_N is the same as the simple random walk on V_N, we have

$$\widetilde{\mathrm{Hm}}(v, \mathcal{A}_\tau; \tilde{\mathcal{I}}_\tau \cup \partial V_N) \le \mathrm{Hm}(v, \mathcal{A}_\tau; \mathcal{D}_\tau \cup \partial V_N)\,. \tag{2.25}$$

On the event $\mathcal{E}$, $\tilde{\mathcal{I}}_\tau$ is contained in $\tilde{\mathcal{G}}_{\beta N} \setminus \tilde{\mathcal{G}}_{\alpha N}$. Therefore $\mathcal{D}_\tau$ is a connected set in $V_{\beta N} \setminus V_{\alpha N}$ of radius between $(\alpha/2)N$ and $2N$. Moreover $|\mathcal{A}_\tau| \le N\mathrm{e}^{-(\log N)^{9/10}}$. Therefore by (2.21), we have

$$\mathrm{Hm}(v, \mathcal{A}_\tau; \mathcal{D}_\tau \cup \partial V_N) = o(\log N)^{-10}\,. \tag{2.26}$$

Let $\Lambda_{\mathrm{bad}} = \{\sup_{u \in V_N} |\tilde{\eta}_{N,u}| \ge 100 \log N\}$ be as before. Then on the event $\mathcal{E} \setminus \Lambda_{\mathrm{bad}}$, combining (2.23), (2.24), (2.25) and (2.26) gives

$$\mathbb{E}(\tilde{\eta}_{N,v} \mid \mathcal{F}_\tau) = o(\log N)^{-8}$$

and therefore

$$|\mathbb{E}(X \mid \mathcal{F}_\tau)| = o(\log N)^{-8} < \epsilon$$

for some $\epsilon > 0$ and sufficiently large N. Now

$$\mathrm{Var}\, X = \frac{1}{4} \frac{1}{|\partial V_{(1+\beta)N/2}|^2} \sum_{u,v \in \partial V_{(1+\beta)N/2}} G_{V_N}(u,v)$$

and on the event $\mathcal{E}$,

$$\begin{aligned} \mathrm{Var}(X \mid \mathcal{F}_\tau) &= \frac{1}{|\partial V_{(1+\beta)N/2}|^2} \sum_{u,v \in \partial V_{(1+\beta)N/2}} G_{\tilde{\mathcal{G}}_N \setminus \tilde{\mathcal{I}}_\tau}(u,v) \\ &\le \frac{1}{4} \frac{1}{|\partial V_{(1+\beta)N/2}|^2} \sum_{u,v \in \partial V_{(1+\beta)N/2}} G_{V_N \setminus V_{\alpha N}}(u,v)\,. \end{aligned}$$

Lemma 2.2. *There exists a constant $c_{11} > 0$ which depends on α and β such that*

$$\frac{1}{4}\frac{1}{|\partial V_{(1+\beta)N/2}|^2}\Big(\sum_{u,v\in\partial V_{(1+\beta)N/2}} G_{V_N}(u,v) - \sum_{u,v\in\partial V_{(1+\beta)N/2}} G_{V_N\setminus V_{\alpha N}}(u,v)\Big) \geq c_{11}\,. \quad (2.27)$$

Proof. We consider two scenarios where in scenario (1) we kill the random walk upon hitting ∂V_N and in scenario (2) we kill the random walk upon hitting $\partial V_{\alpha N} \cup \partial V_N$. For any $u \in \partial V_{(1+\beta)N/2}$, we will compare the expected number of visits to $\partial V_{(1+\beta)N/2}$ of a simple random walk started at u in these two scenarios. On the event E that it hits ∂V_N before $\partial V_{\alpha N}$, it will visit $\partial V_{(1+\beta)N/2}$ the same number of times in both scenarios (1) and (2). On the complement of E (i.e. it hits $\partial V_{\alpha N}$ before ∂V_N), however, it will come back to some $w \in \partial V_{(1+\beta)N/2}$, and will then (conditionally in expectation) visit $\partial V_{(1+\beta)N/2}$ exactly $\sum_{v\in\partial V_{(1+\beta)N/2}} G_{V_N}(w,v)$ more times in scenario (1) than in scenario (2). Since we have a uniform lower bound of $\sum_{v\in\partial V_{(1+\beta)N/2}} G_{V_N}(w,v)$ (by (2.7)) and that $\mathbb{P}(E^c) \geq c_3$ by Lemma 2.1, we see that

$$\sum_{v\in\partial V_{(1+\beta)N/2}} G_{V_N}(u,v) - \sum_{v\in\partial V_{(1+\beta)N/2}} G_{V_N\setminus V_{\alpha N}}(u,v) \geq c_3 c_2 |\partial V_{(1+\beta)N/2}|\,,$$

and (2.27) follows by summing this over all $u \in \partial V_{(1+\beta)N/2}$. □

Therefore we have $\operatorname{Var} X - \operatorname{Var}(X \mid \mathcal{F}_\tau) \geq c_{11}$ on the event $\mathcal{E}$ by (2.27) and also recall that $\operatorname{Var} X \leq c_4$ by (2.8). Now let $t = \epsilon + s\sqrt{\operatorname{Var} X - c_{11}}$ where $s > 0$ is a constant to be chosen later. Since given $\mathcal{F}_\tau$, X is Gaussian, we have on the event $\mathcal{E} \setminus \Lambda_{\mathrm{bad}}$

$$\mathbb{P}(X \leq t \mid \mathcal{F}_\tau) \geq \mathbb{P}(Z(\epsilon, \operatorname{Var} X - c_{11}) \leq t) = \mathbb{P}(Z \leq s)\,,$$

where $Z(\epsilon, \operatorname{Var} X - c_{11})$ is a Gaussian variable with mean ϵ and variance $\operatorname{Var} X - c_{11}$, and

Z is a standard Gaussian variable. In addition, we have

$$\mathbb{P}(X \le t) = \mathbb{P}(Z \le \frac{\epsilon + s\sqrt{\operatorname{Var} X - c_{11}}}{\sqrt{\operatorname{Var} X}}).$$

Since $c_{11} \le \operatorname{Var} X \le c_4$, we have $0 \le \frac{\sqrt{\operatorname{Var} X - c_{11}}}{\sqrt{\operatorname{Var} X}} \le \frac{\sqrt{c_4 - c_{11}}}{\sqrt{c_4}} < 1$, so

$$\frac{\epsilon + s\sqrt{\operatorname{Var} X - c_{11}}}{\sqrt{\operatorname{Var} X}} \le \frac{\epsilon}{\sqrt{c_{11}}} + s\frac{\sqrt{c_4 - c_{11}}}{\sqrt{c_4}} \le s(\frac{\frac{\sqrt{c_4 - c_{11}}}{\sqrt{c_4}} + 1}{2})$$

for a sufficiently large constant $s > 0$. Therefore (recalling $\frac{\sqrt{c_4 - c_{11}}}{\sqrt{c_4}} < 1$)

$$\mathbb{P}(\mathcal{E} \setminus \Lambda_{\text{bad}}) \le \frac{\mathbb{P}(Z \le \frac{\epsilon + s\sqrt{\operatorname{Var} X - c_{11}}}{\sqrt{\operatorname{Var} X}})}{\mathbb{P}(Z \le s)} \le \frac{\mathbb{P}(Z \le s(\frac{\frac{\sqrt{c_4 - c_{11}}}{\sqrt{c_4}} + 1}{2}))}{\mathbb{P}(Z \le s)} < 1\,.$$

Combined with (2.22), this completes the proof of the proposition. □

Finally, we remark that, using an almost identical proof of Proposition 2.2, we can prove the following result.

Corollary 2.1. *For all $0 < \alpha < \beta < 1$ and $\lambda > 0$, there exists a constant $c > 0$ such that for all N*

$$\mathbb{P}(D_{N,-\lambda}(\partial V_{\alpha N}, \partial V_{\beta N}) \le N\mathrm{e}^{(\log N)^{9/10}}) \ge c\,.$$

2.4 Percolation of the random walk loop soup

This section is devoted to the proof of Theorem 2.2, into which the three main proof ingredients merge. Recall that as stated in (2.2), the occupation time field $\{\hat{\mathcal{L}}^v_{1/2}\}_{v \in V_N}$ of $\mathcal{L}_{1/2,N}$ has the same law as $\{\frac{1}{2}\eta^2_{N,v}\}_{v \in V_N}$; and as stated around (2.3), conditioning on $\{\hat{\mathcal{L}}^v_{1/2} = \ell_v\}_{v \in V_N}$, the graph of $\mathcal{L}_{1/2,N}$ has the same law as $(1_{n_e > 0})_{e \in E_N}$, where $(n_e)_{e \in E_N}$ follows the random current model (see (2.3)) with parameters $\beta_e = 2\sqrt{\ell_x \ell_y}$ on edge $e = (x, y)$.

Each edge of $\mathcal{G}_N = (V_N, E_N)$ has conductance 1. In this section we will consider a graph $\mathcal{G}'_N = (V_N, E_N(1) \cup E_N(2))$, where we replace each edge $e \in E_N$ in the graph $\mathcal{G}_N$ by two multiple edges $e(1)$ and $e(2)$ and assign conductance $1/8$ to $e(1)$ and conductance $7/8$ to $e(2)$, and we denote $E_N(1) = \{e(1) : e \in E_N\}$ and $E_N(2) = \{e(2) : e \in E_N\}$. The graph $\mathcal{G}'_N$ is equivalent to $\mathcal{G}_N$ in the sense that the Gaussian free fields on $\mathcal{G}_N$ and $\mathcal{G}'_N$ have the same law.

As in [54], we will consider the Gaussian free field on the metric graph $\tilde{\mathcal{G}}'_N$ of $\mathcal{G}'_N$. The metric graph $\tilde{\mathcal{G}}'_N$ can be obtained from $\mathcal{G}'_N$ by assigning each edge $e(1) \in E_N(1)$ length 4 and each edge $e(2) \in E_N(2)$ length $7/4$. Let $B^{\tilde{\mathcal{G}}'_N}$ be a standard Brownian motion on $\tilde{\mathcal{G}}'_N$, let $G_{\tilde{\mathcal{G}}'_N}(u, v)$ be the Green's function of $B^{\tilde{\mathcal{G}}'_N}$, and let $\{\tilde{\eta}'_{N,v} : v \in \tilde{\mathcal{G}}'_N\}$ be a continuous realization of the Gaussian free field on $\tilde{\mathcal{G}}'_N$ with covariances given by $G_{\tilde{\mathcal{G}}'_N}(u, v)$. The restriction of $\{\tilde{\eta}'_{N,v} : v \in \tilde{\mathcal{G}}'_N\}$ to V_N is the same as the Gaussian free field $\{\eta_{N,v} : v \in V_N\}$. Moreover, $\{\tilde{\eta}'_{N,v} : v \in \tilde{\mathcal{G}}'_N\}$ can be obtained from $\{\eta_{N,v} : v \in V_N\}$ by, for each edge $e' = (x, y) \in E_N(1) \cup E_N(2)$, independently sampling a variance 2 Brownian bridge of the same length as e' with values $\eta_{N,x}$ and $\eta_{N,y}$ at the endpoints.

We now describe a coupling between the random walk loop soup cluster $\mathcal{L}_{1/2,N}$ and a graph $\mathcal{O}$ obtained from $\{\tilde{\eta}'_{N,v} : v \in \tilde{\mathcal{G}}'_N\}$. Fix some $\lambda > 0$. We say an edge $e \in E_N$ is *open* if $\tilde{\eta}'_{N,v} > \lambda$ for all $v \in e(1)$ (which we denote as $\tilde{\eta}'_{N,e(1)} > \lambda$ for notation convenience, and the similar applies to the case of $< -\lambda$) or $\tilde{\eta}'_{N,v} < -\lambda$ for all $v \in e(1)$. Let $\mathcal{O}$ be the graph (seen as a subgraph of $\mathcal{G}_N = (V_N, E_N)$) induced by these open edges.

Lemma 2.3. *For any $\lambda \geq 2$, the graph $\mathcal{O}$ is stochastically dominated by $\mathcal{L}_{1/2,N}$. That is to say, we have $(1_{\tilde{\eta}'_{N,e(1)} > \lambda \text{ or } \tilde{\eta}'_{N,e(1)} < -\lambda})_{e \in E_N}$ is stochastically dominated by $(1_{n_e > 0})_{e \in E_N}$.*

Proof. Since $\{\frac{1}{2}(\tilde{\eta}'_{N,v})^2\}_{v \in V_N}$ and $\{\hat{\mathcal{L}}^v_{1/2}\}_{v \in V_N}$ both have the same law as $\{\frac{1}{2}\eta^2_{N,v}\}_{v \in V_N}$, we only need to show the stochastic dominance of $\mathcal{O}$ when conditioned on the former, by $\mathcal{L}_{1/2,N}$ when conditioned on the latter (with the same realization).

On one hand, conditioning on $\{\tilde{\eta}'_{N,v}\}_{v \in V_N}$ we see that $1_{\tilde{\eta}'_{N,e(1)} > \lambda \text{ or } \tilde{\eta}'_{N,e(1)} < -\lambda}$'s are independent Bernoulli variables with mean p_e's, where (we let (B_t) be a Brownian motion with variance 2 at time 1)

$$p_e = \begin{cases} \mathbb{P}(B_t > \lambda, \forall t \in [0,4] \mid B_0 = \tilde{\eta}'_{N,x}, B_4 = \tilde{\eta}'_{N,y}), & \text{if } \tilde{\eta}'_{N,x}, \tilde{\eta}'_{N,y} > \lambda\,, \\ \mathbb{P}(B_t < -\lambda, \forall t \in [0,4] \mid B_0 = \tilde{\eta}'_{N,x}, B_4 = \tilde{\eta}'_{N,y}), & \text{if } \tilde{\eta}'_{N,x}, \tilde{\eta}'_{N,y} < -\lambda\,, \\ 0, & \text{otherwise.} \end{cases}$$

In the case when $\tilde{\eta}'_{N,x}, \tilde{\eta}'_{N,y} > \lambda$ (the other case is essentially the same), by the reflection principle we get that

$$1 - p_e = \mathrm{e}^{-\frac{(\tilde{\eta}'_{N,x}+\tilde{\eta}'_{N,y}-2\lambda)^2}{16}} / \mathrm{e}^{-\frac{(\tilde{\eta}'_{N,x}-\tilde{\eta}'_{N,y})^2}{16}} = \mathrm{e}^{-\frac{1}{4}(\tilde{\eta}'_{N,x}-\lambda)(\tilde{\eta}'_{N,y}-\lambda)}\,.$$

Therefore, conditioning on $\{\frac{1}{2}(\tilde{\eta}'_{N,v})^2 = \ell_v\}_{v\in V_N}$, we have $(1_{\tilde{\eta}'_{N,e(1)}>\lambda \text{ or } \tilde{\eta}'_{N,e(1)}<-\lambda})_{e\in E_N}$ is stochastically dominated by independent Bernoulli's with mean p'_e's, where

$$p'_e = \begin{cases} 1 - \mathrm{e}^{-\frac{1}{4}(\sqrt{2\ell_x}-\lambda)(\sqrt{2\ell_y}-\lambda)}, & \text{if } \ell_x, \ell_y > \frac{\lambda^2}{2}\,, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, conditioning on $\{\hat{\mathcal{L}}^v_{1/2} = \ell_v\}_{v\in V_N}$, the graph of $\mathcal{L}_{1/2,N}$ has the same law as $(1_{n_e>0})_{e\in E_N}$, where $(n_e)_{e\in E_N}$ follows the random current model (as in (2.3)) with parameters $\beta_e = 2\sqrt{\ell_x\ell_y}$ on edge $e = (x,y)$. Note that if we further condition on the parities of $(n_e)_{e\in E_N}$, then $n(e)$'s are independent with distribution F_{1,β_e} if $n(e)$ is odd and distribution F_{2,β_e} if $n(e)$ is even. Here F_{1,β_e} and F_{2,β_e} are both probability distributions on nonnegative integers such that

$$F_{1,\beta_e}(n) = \frac{(\beta_e)^n}{n!\sinh\beta_e} \text{ for } n = 1,3,5,\dots \text{ and } F_{2,\beta_e}(n) = \frac{(\beta_e)^n}{n!\cosh\beta_e} \text{ for } n = 0,2,4,\dots$$

This implies that, conditioning on $\{\hat{\mathcal{L}}^v_{1/2} = \ell_v\}_{v\in V_N}$ and the parities of $(n_e)_{e\in E_N}$, $1_{n_e>0}$'s are independent Bernoulli variables with mean p''_e's, where $p''_e = 1$ if $n(e)$ is odd and $p''_e =$

$1 - 1/\cosh \beta_e$ if $n(e)$ is even. However, for $\ell_x, \ell_y > \frac{\lambda^2}{2}$ and $\lambda \geq 2$, we have

$$
\begin{aligned}
1/\cosh \beta_e \leq 2/\mathrm{e}^{\beta_e} &= 2/\mathrm{e}^{2\sqrt{\ell_x \ell_y}} \\
&\leq \mathrm{e}^{-\sqrt{\ell_x \ell_y}} \\
&\leq \mathrm{e}^{-\frac{1}{4}(\sqrt{2\ell_x}-\lambda)(\sqrt{2\ell_y}-\lambda)} = 1 - p'_e \,.
\end{aligned}
$$

Therefore, conditioning on $\{\hat{\mathcal{L}}^v_{1/2} = \ell_v\}_{v \in V_N}$, we have $(1_{n_e > 0})_{e \in E_N}$ stochastically dominates independent Bernoulli's with mean p'_e's.

Combining the above two parts completes the proof of the lemma. □

Remark 2.4. *It is worth pointing out that for our proof strategy to go through, it suffices as long as the law of the edge visits conditioned on vertex local times dominates the random current model as in* (2.3) *for* $\beta_e \geq c\sqrt{\ell_x \ell_y}$ *for some fixed positive constant* c *(since we can tune the resistance on* $e(1)$ *and* $e(2)$*). The fact that* $c = 2$ *is of no importance to us. This flexibility may be useful when attempting to extend our proof strategy to some other contexts.*

In light of Lemma 2.3, define (for $> \lambda$, the definition for $< -\lambda$ is similar)

$$
D_{\tilde{\eta}'_N, E_N(1), > \lambda}(u, v) = \min_{\gamma} |\gamma| \,,
$$

where the minimum is over all path $\gamma \subseteq V_N \cup E_N(1) \subseteq \tilde{\mathcal{G}}'_N$ joining u and v such that $\tilde{\eta}'_{N,x} > \lambda$ for all $x \in \gamma$. In order to prove Theorem 2.2, it suffices to prove the following proposition.

Proposition 2.3. *For any* $0 < \alpha < \beta < 1$*, there exists a constant* $c > 0$ *such that for all* N

$$
\mathbb{P}(\min\{D_{\tilde{\eta}'_N, E_N(1), > \lambda}(\partial V_{\alpha N}, \partial V_{\beta N}), D_{\tilde{\eta}'_N, E_N(1), < -\lambda}(\partial V_{\alpha N}, \partial V_{\beta N})\} \leq N \mathrm{e}^{(\log N)^{9/10}}) \geq c \,.
$$

Proof. For the rigor of proof (when applying e.g., FKG inequality later), we will consider the following discrete approximation of the exploration procedure. We let $\Pi_N = (V_N \cup E_N(1)) \cap$

$\frac{1}{N^3}\mathbb{Z}^2$ be an N^3-discretization of $V_N \cup E_N(1)$. In particular, we will only examine the values of $\tilde{\eta}'_{N,v}$ for $v \in \Pi_N$ in the following exploration procedure. Initially we set

$$\mathcal{A}_0^{>\lambda} = \{v \in \partial V_{\alpha N} : \tilde{\eta}'_{N,v} > \lambda + \frac{1}{N}\}, \quad \mathcal{A}_0^{<-\lambda} = \{v \in \partial V_{\alpha N} : \tilde{\eta}'_{N,v} < -\lambda - \frac{1}{N}\},$$
$$\mathcal{B}_0 = \{v \in \partial V_{\alpha N} : |\tilde{\eta}'_{N,v}| \leq \lambda + \frac{1}{N}\}, \quad \mathcal{C}_0^{>\lambda} = \mathcal{A}_0^{>\lambda}, \quad \mathcal{C}_0^{<-\lambda} = \mathcal{A}_0^{<-\lambda}.$$

For $i = 0, 1, 2, \ldots$, at stage $(i+1)$, we run the exploration procedure as follows:

- We set initially $\mathcal{A}_{i+1}^{>\lambda} = \mathcal{A}_{i+1}^{<-\lambda} = \emptyset$, $\mathcal{B}_{i+1}^{=\lambda} = \mathcal{B}_{i+1}^{=-\lambda} = \emptyset$ and $\mathcal{C}_{i+1}^{>\lambda} = \mathcal{C}_i^{>\lambda}$, $\mathcal{C}_{i+1}^{<-\lambda} = \mathcal{C}_i^{<-\lambda}$.
- If $\mathcal{A}_i^{>\lambda} = \emptyset$, stop. Otherwise, for each $v \in \mathcal{A}_i^{>\lambda}$ and every edge $e(1) = (v, u) \in E_N(1)$ incident to v, if $u \in V_N \setminus V_{\alpha N}$ and the neighborhood of v along $e(1)$ does not belong to $\mathcal{C}_i^{>\lambda}$, we go (explore) from v along $e(1)$ to u until we reach a point $w \in \Pi_N$ with $\tilde{\eta}'_{N,w} \leq \lambda + \frac{1}{N}$. In the case no such w is reached, we add all the points in $e(1) \cap \Pi_N$ (including v and u) into $\mathcal{C}_{i+1}^{>\lambda}$ and add u, if it is not already in $\cup_{j=0}^{i}\mathcal{A}_j^{>\lambda}$, into $\mathcal{A}_{i+1}^{>\lambda}$; in the case that the first such $w \in \Pi_N$ is reached on $e(1)$, we add w into $\mathcal{B}_{i+1}^{=\lambda}$, and add all the points in $e(1) \cap \Pi_N$ between v and w (but not w) into $\mathcal{C}_{i+1}^{>\lambda}$.
- We employ a similar procedure for the version of $< -\lambda$.

Let $\Lambda_{\text{bad}} = \{\sup_{v \in V_N} |\tilde{\eta}'_{N,v}| \geq 100 \log N\}$ be as before. Define

$$\Lambda_c = \{|\tilde{\eta}'_{N,u} - \tilde{\eta}'_{N,v}| \leq \frac{1}{N}, \forall e(1) \in E_N(1) \text{ and } u, v \in e(1) \text{ such that } |u - v| \leq \frac{1}{N^3}\}. \quad (2.28)$$

Since conditioning on $\{\tilde{\eta}'_{N,v} : v \in V_N\}$, we have $\{\tilde{\eta}'_{N,e(1)} : e(1) = (x, y) \in E_N(1)\}$ are independent variance 2 Brownian bridges (which are Hölder continuous of any order less than 1/2) of length 4 with values $\tilde{\eta}'_{N,x}$ and $\tilde{\eta}'_{N,y}$ at the endpoints, we see that

$$\mathbb{P}(\Lambda_c) \geq \mathbb{P}(\Lambda_{\text{bad}}^c)(1 - o_N(1)) = 1 - o_N(1). \quad (2.29)$$

Denote

$$\mathcal{E} \triangleq \{\min\{D_{\tilde{\eta}'_N, E_N(1), >\lambda}(\partial V_{\alpha N}, \partial V_{\beta N}), D_{\tilde{\eta}'_N, E_N(1), <-\lambda}(\partial V_{\alpha N}, \partial V_{\beta N})\} > N\mathrm{e}^{(\log N)^{9/10}}\}\,.$$

Suppose that both events $\mathcal{E}$ and Λ_c occur, then by (2.28) we must have that $\mathcal{C}_i^{>\lambda}$ and $\mathcal{C}_i^{<-\lambda}$ are both disjoint from $V_N \setminus V_{\beta N}$ for all $0 \le i < N\mathrm{e}^{(\log N)^{9/10}}$. Further, since all of the $\mathcal{A}_i^{>\lambda}$ and $\mathcal{A}_i^{<-\lambda}$ where $0 \le i < N\mathrm{e}^{(\log N)^{9/10}}$ are disjoint from each other, we see that there exists at least an $i_0 < N\mathrm{e}^{(\log N)^{9/10}}$ such that

$$|\mathcal{A}_{i_0}^{>\lambda} \cup \mathcal{A}_{i_0}^{<-\lambda}| \le N\mathrm{e}^{-(\log N)^{9/10}}\,. \tag{2.30}$$

Moreover (still on the event $\mathcal{E} \cap \Lambda_c$), for all $1 \le i < N\mathrm{e}^{(\log N)^{9/10}}$, we have by (2.28) again

$$\lambda < \tilde{\eta}'_{N,v} \le \lambda + \frac{1}{N} \text{ for all } v \in \mathcal{B}_i^{=\lambda} \quad \text{and} \quad -\lambda - \frac{1}{N} \le \tilde{\eta}'_{N,v} < -\lambda \text{ for all } v \in \mathcal{B}_i^{=-\lambda}\,.$$

It is clear that for any $k \ge 0$, from $\mathcal{C}_k^{>\lambda} = C_k^{>\lambda}$ and $\mathcal{C}_k^{<-\lambda} = C_k^{<-\lambda}$ we can determine (uniquely) the sets $\mathcal{A}_i^{>\lambda}, \mathcal{A}_i^{<-\lambda}, \mathcal{B}_i^{=\lambda}, \mathcal{B}_i^{=-\lambda}$ for all $1 \le i \le k$ as well as $\mathcal{B}_0$. We denote them as $A_i^{>\lambda}, A_i^{<-\lambda}, B_i^{=\lambda}, B_i^{=-\lambda}$ for all $1 \le i \le k$ and B_0, respectively. They are all functions of $C_k^{>\lambda}$ and $C_k^{<-\lambda}$. We let $\mathcal{P}_k$ denote all $(C_k^{>\lambda}, C_k^{<-\lambda})$ such that $C_k^{>\lambda}$ and $C_k^{<-\lambda}$ are both disjoint from $V_N \setminus V_{\beta N}$, and such that

$$\min\{i_0 : |A_{i_0}^{>\lambda} \cup A_{i_0}^{<-\lambda}| \le N\mathrm{e}^{-(\log N)^{9/10}}\} = k\,. \tag{2.31}$$

In summary of the discussions above, we have

$$\mathcal{E} \cap \Lambda_c \subseteq \bigsqcup_{\substack{0 \le k < N\mathrm{e}^{(\log N)^{9/10}} \\ (C_k^{>\lambda}, C_k^{<-\lambda}) \in \mathcal{P}_k}} \mathcal{E}_{C_k^{>\lambda}, C_k^{<-\lambda}} \tag{2.32}$$

where

$$
\begin{aligned}
&\mathcal{E}_{C_k^{>\lambda}, C_k^{<-\lambda}} = \\
&\{\tilde{\eta}'_{N,v} > \lambda + \frac{1}{N} \text{ for all } v \in C_k^{>\lambda}, \quad \tilde{\eta}'_{N,v} < -\lambda - \frac{1}{N} \text{ for all } v \in C_k^{<-\lambda}, \\
&\lambda < \tilde{\eta}'_{N,v} \le \lambda + \frac{1}{N} \text{ for all } v \in \cup_{i=1}^k B_i^{=\lambda}, \; -\lambda - \frac{1}{N} \le \tilde{\eta}'_{N,v} < -\lambda \text{ for all } v \in \cup_{i=1}^k B_i^{=-\lambda}, \\
&|\tilde{\eta}'_{N,v}| \le \lambda + \frac{1}{N} \text{ for all } v \in B_0\}\,. \qquad (2.33)
\end{aligned}
$$

Denote $I_k \overset{\triangle}{=} B_0 \cup (\cup_{i=1}^k B_i^{=\lambda}) \cup (\cup_{i=1}^k B_i^{=-\lambda}) \cup C_k^{>\lambda} \cup C_k^{<-\lambda}$. Then (2.33) above is $\mathcal{F}_{I_k}$ measurable. Conditioning on $\mathcal{F}_{I_k}$, the field $\{\tilde{\eta}'_{N,u} : u \in \tilde{\mathcal{G}}'_N \setminus I_k\}$ is distributed as a GFF with boundary condition $\{\tilde{\eta}'_{N,v} : v \in I_k\}$ and zero on ∂V_N. In particular, for each $u \in \partial V_{(1+\beta)N/2}$, we have

$$
\mathbb{E}(\tilde{\eta}'_{N,u} \mid \mathcal{F}_{I_k}) = \sum_{v \in I_k} \widetilde{\mathrm{Hm}}'(u, v; I_k \cup \partial V_N) \cdot \tilde{\eta}'_{N,v}\,. \qquad (2.34)
$$

Here $\widetilde{\mathrm{Hm}}'(u, v; K)$ denotes the harmonic measure of $B^{\tilde{\mathcal{G}}'_N}$ at v with respect to starting point u and target set K.

From the definition of our exploration procedure, we know that for any $u \in \partial V_{(1+\beta)N/2}$, we have $\{v \in I_k : \widetilde{\mathrm{Hm}}'(u, v; I_k \cup \partial V_N) \neq 0\} \subseteq J_1 \cup J_2 \cup J_3 \cup J_4$, which can be described as follows.

- $J_1 = B_0$.
- $J_2 = (\cup_{i=1}^k B_i^{=\lambda}) \cup (\cup_{i=1}^k B_i^{=-\lambda})$.
- For each $v \in J_3$, we have $v \in A_i^{>\lambda}$ (or $v \in A_i^{<-\lambda}$ respectively) for some $0 \le i \le k-1$, and on an edge $e(1) = (v, v') \in E_N(1)$ there is a $w \in B_{i+1}^{=\lambda}$ (or $w \in B_{i+1}^{=-\lambda}$ respectively). In particular, each $v \in J_3$ must satisfy that $v \in V_N$ and that v has Euclidean distance less than 1 to a point $w \in J_2$.
- $J_4 = A_k^{>\lambda} \cup A_k^{<-\lambda}$.

Define

$$X = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{u \in \partial V_{(1+\beta)N/2}} \tilde{\eta}'_{N,u}\,.$$

Further define

$$X_1 = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{u \in \partial V_{(1+\beta)N/2}} \sum_{v \in J_1} \widetilde{\mathrm{Hm}}'(u, v; I_k \cup \partial V_N) \cdot \tilde{\eta}'_{N,v}\,,$$

$$X_2 = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{u \in \partial V_{(1+\beta)N/2}} \sum_{v \in J_2} \widetilde{\mathrm{Hm}}'(u, v; I_k \cup \partial V_N) \cdot \tilde{\eta}'_{N,v}\,,$$

$$X_3 = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{u \in \partial V_{(1+\beta)N/2}} \sum_{v \in J_3} \widetilde{\mathrm{Hm}}'(u, v; I_k \cup \partial V_N) \cdot \tilde{\eta}'_{N,v}\,,$$

$$X_4 = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{u \in \partial V_{(1+\beta)N/2}} \sum_{v \in J_4} \widetilde{\mathrm{Hm}}'(u, v; I_k \cup \partial V_N) \cdot \tilde{\eta}'_{N,v}\,.$$

Then by (2.34), we have

$$\mathbb{E}(X \mid \mathcal{F}_{I_k}) = X_1 + X_2 + X_3 + X_4\,.$$

It is clear that $|X_1| \leq \lambda + \frac{1}{N}$ and $|X_2| \leq \lambda + \frac{1}{N}$ always hold. Let $D_k = (\cup_{i=0}^k A_i^{>\lambda}) \cup (\cup_{i=0}^k A_i^{<-\lambda}) \cup B_0$ be the set of all the lattice points in I_k. Then $J_4 \subseteq D_k \subseteq I_k$, so that

$$\widetilde{\mathrm{Hm}}'(u, J_4; I_k \cup \partial V_N) \leq \widetilde{\mathrm{Hm}}'(u, J_4; D_k \cup \partial V_N)\,. \tag{2.35}$$

Since $|J_4| \leq N\mathrm{e}^{-(\log N)^{9/10}}$ by (2.31), we have by (2.21)

$$\widetilde{\mathrm{Hm}}'(u, J_4; D_k \cup \partial V_N) = \mathrm{Hm}(u, J_4; D_k \cup \partial V_N) = o(\log N)^{-10}\,,$$

and therefore $\widetilde{\mathrm{Hm}}'(u, J_4; I_k \cup \partial V_N) = o(\log N)^{-10}$. Recall that $\Lambda_{\mathrm{bad}} = \{\sup_{v \in V_N} |\tilde{\eta}'_{N,v}| \geq 100 \log N\}$. Then if the event Λ_{bad} does not occur, we have $|X_4| = o(\log N)^{-8}$.

It now remains to control X_3 on the event $\mathcal{E}_{C_k^{>\lambda}, C_k^{<-\lambda}}$. To be more precise, we consider any fixed $\{x_{N,v}\}_{v \in B_0 \cup (\cup_{i=1}^k B_i^{=\lambda}) \cup (\cup_{i=1}^k B_i^{=-\lambda})}$ such that

$$|x_{N,v}| \le \lambda + \frac{1}{N} \text{ for all } v \in B_0, \quad \lambda < x_{N,v} \le \lambda + \frac{1}{N} \text{ for all } v \in \bigcup_{i=1}^k B_i^{=\lambda},$$
$$-\lambda - \frac{1}{N} \le x_{N,v} < -\lambda \text{ for all } v \in \bigcup_{i=1}^k B_i^{=-\lambda}.$$

We define three events $H_=, H_+, H_-$ as follows:

- $H_= = \{\tilde{\eta}'_{N,v} = x_{N,v} \text{ for all } v \in B_0 \cup (\cup_{i=1}^k B_i^{=\lambda}) \cup (\cup_{i=1}^k B_i^{=-\lambda})\}$;
- $H_+ = \{\tilde{\eta}'_{N,v} > \lambda + \frac{1}{N} \text{ for all } v \in C_k^{>\lambda}\}$;
- $H_- = \{\tilde{\eta}'_{N,v} < -\lambda - \frac{1}{N} \text{ for all } v \in C_k^{<-\lambda}\}$.

We will show that conditioning on $H_= \cap H_- \cap H_+$, we have

(a) $\mathbb{E}(X_3 \mid H_=, H_+, H_-) \le \lambda + \frac{1}{N} + C_0$ where C_0 is a constant;

(b) $\mathrm{Var}(X_3 \mid H_=, H_+, H_-) = o_N(1)$ (where we use $o_N(1)$ to denote a quantity that only depends on N and tends to 0 as $N \to \infty$).

As a corollary of (a) and (b), conditioning on $\{H_=, H_+, H_-\}$, we have X_3 itself is bounded from above by $\lambda + C_0 + 1$ with probability $(1 - o_N(1))$. By integrating over all $\{x_{N,v}\}_{v \in B_0 \cup (\cup_{i=1}^k B_i^{=\lambda}) \cup (\cup_{i=1}^k B_i^{=-\lambda})}$, we see that there exists a $\mathcal{F}_{I_k}$ measurable event $\mathcal{E}'_{C_k^{>\lambda}, C_k^{<-\lambda}} \subseteq \mathcal{E}_{C_k^{>\lambda}, C_k^{<-\lambda}}$ such that on the event $\mathcal{E}'_{C_k^{>\lambda}, C_k^{<-\lambda}} \setminus \Lambda_{\mathrm{bad}}$, we have $\mathbb{E}(X \mid \mathcal{F}_{I_k})$ is bounded from above by a constant Δ, and moreover

$$\mathbb{P}(\mathcal{E}'_{C_k^{>\lambda}, C_k^{<-\lambda}}) \ge (1 - o_N(1)) \mathbb{P}(\mathcal{E}_{C_k^{>\lambda}, C_k^{<-\lambda}}). \tag{2.36}$$

Now on the event $\mathcal{E}'_{C_k^{>\lambda}, C_k^{<-\lambda}} \setminus \Lambda_{\mathrm{bad}}$, we have $\mathbb{E}(X \mid \mathcal{F}_{I_k}) \le \Delta$ and $\mathrm{Var}\, X - \mathrm{Var}(X \mid \mathcal{F}_{I_k}) \ge c_{11} > 0$ by (2.27) (also recall that $\mathrm{Var}\, X \le c_4$ by (2.8)). Let $t = \Delta + s\sqrt{\mathrm{Var}\, X - c_{11}}$.

Then on the event $\mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}} \setminus \Lambda_{\mathrm{bad}}$, we have

$$\mathbb{P}(X \le t \mid \mathcal{F}_{I_k}) \ge \mathbb{P}(Z(\Delta, \operatorname{Var} X - c_{11}) \le t) = \mathbb{P}(Z \le s)\,,$$

where $Z(\Delta, \operatorname{Var} X - c_{11})$ is a Gaussian variable with mean Δ and variance $\operatorname{Var} X - c_{11}$, and Z is a standard Gaussian variable. Therefore (since $\mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}}$ is $\mathcal{F}_{I_k}$ measurable)

$$\mathbb{P}(X \le t, \mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}}) = \mathbb{E}(\mathbb{P}(X \le t \mid \mathcal{F}_{I_k})\mathbf{1}_{\mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}}}) \ge \mathbb{P}(Z \le s)\mathbb{P}(\mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}} \setminus \Lambda_{\mathrm{bad}})\,.$$

Summing this over all $0 \le k < N\mathrm{e}^{(\log N)^{9/10}}$ and all $(C_k^{>\lambda}, C_k^{<-\lambda}) \in \mathcal{P}_k$, we have

$$\mathbb{P}(X \le t) \ge \mathbb{P}(Z \le s)\mathbb{P}\Big(\Big(\bigsqcup_{\substack{0\le k<N\mathrm{e}^{(\log N)^{9/10}} \\ (C_k^{>\lambda},C_k^{<-\lambda})\in\mathcal{P}_k}} \mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}}\Big) \setminus \Lambda_{\mathrm{bad}}\Big)\,.$$

Therefore, for a sufficiently large constant $s > 0$ and a constant $c' > 0$, we have

$$\mathbb{P}\Big(\bigsqcup_{\substack{0\le k<N\mathrm{e}^{(\log N)^{9/10}} \\ (C_k^{>\lambda},C_k^{<-\lambda})\in\mathcal{P}_k}} \mathcal{E}'_{C_k^{>\lambda},C_k^{<-\lambda}}\Big) \le \frac{\mathbb{P}(X \le t)}{\mathbb{P}(Z \le s)} + \mathbb{P}(\Lambda_{\mathrm{bad}}) \le 1 - c'\,.$$

Combined with (2.36), (2.32) and (2.29), this gives us the result of the proposition. □

It remains to prove (a) and (b), which are incorporated in Lemmas 2.4 and 2.5 below.

Lemma 2.4. *There exists a constant $C_0 > 0$ such that for any $v \in J_3$, we have*

$$\mathbb{E}(\tilde{\eta}'_{N,v} \mid H_=, H_+, H_-) \le \lambda + \frac{1}{N} + C_0\,.$$

Proof. If for some $0 \le i \le k-1$, $v \in A_i^{<-\lambda}$, then clearly we have $\mathbb{E}(\tilde{\eta}'_{N,v} \mid H_=, H_+, H_-) \le -\lambda - \frac{1}{N} \le \lambda + \frac{1}{N} + C_0$. So in what follows we assume that for some $0 \le i \le k-1$, $v \in A_i^{>\lambda}$ and on an edge $e(1) = (v, v') \in E_N(1)$ there is a $w \in B_{i+1}^{=\lambda}$. This type of argument is known as the entropic repulsion estimates in the presence of a hard wall [41, 24]. Our context is close to [41] with some slight complication, and our proof essentially follows from the same line of arguments.

First of all, we claim that there exist a function $f(u)$ defined on $\tilde{\mathcal{G}}'_N$ and absolute constants $C, C_1 > 0$ that can be taken to be arbitrarily large, such that $f(u)$ is harmonic on $\tilde{\mathcal{G}}'_N \setminus \{w\}$, i.e., $f(u)|_{u \in V_N \cup \{w\}}$ is harmonic on the discrete graph $(V_N \cup \{w\}, E'_{N,w})$ except at w, and $f(u)$ is linear on each segment $e' \in E'_{N,w}$, where $E'_{N,w} = \{e' : e' \in E_N(1) \cup E_N(2) \text{ and } w \notin e'\} \cup \{(v, w), (w, v')\}$. In addition, we have that

$$|f(u) - C\log(|u - w| + 2) - C_1| \leq L(C) \quad \text{for all } u \in \tilde{\mathcal{G}}'_N\,, \tag{2.37}$$

where $L(C)$ is a function that only depends on C. In particular, we take $C_1 > L(C)$ so that $f(u) > 0$ for all $u \in \tilde{\mathcal{G}}'_N$.

Indeed, by [41, (B17)] or [50, Theorem 4.4.4], there exist a function $g(u)$ defined on $\mathbb{Z}^2$ and absolute constants $C, C'_1 > 0$ that can be taken to be arbitrarily large, such that $g(u)$ is harmonic on $\mathbb{Z}^2 \setminus \{(0,0)\}$ and $|g(u) - C\log(|u| + 2) - C'_1| \leq L'(C)$ for all $u \in \mathbb{Z}^2$ (where $L'(C)$ is a function that only depends on C). Now let us define for $u \in \tilde{\mathcal{G}}'_N$

$$f(u) = \begin{cases} |v' - w|g(u - v) + |v - w|g(u - v'), & \text{if } u \in V_N\,, \\ f(w), & \text{if } u = w\,, \\ \text{linear interpolation between } f(x) \text{ and } f(y), & \text{if } u \in e' = (x, y) \in E'_{N,w}\,, \end{cases}$$

where

$$f(w) = (|v-w|^2+|v'-w|^2)g((0,0))-8|v-w||v'-w|Dg((0,0))+|v-w||v'-w|(g(v'-v)+g(v-v'))$$

and

$$Dg((0,0)) = g((0,1)) + g((0,-1)) + g((1,0)) + g((-1,0)) - 4g((0,0))\,.$$

Then by definition, $f(u)$ is clearly harmonic on $\tilde{\mathcal{G}}'_N \setminus \{v, v', w\}$. To show that it is also harmonic at v and v', we have to verify that

$$(3 + \frac{7}{8} + \frac{1}{8|v-w|})f(v) = \sum_{i=1}^{3} f(v_i) + \frac{7}{8}f(v') + \frac{1}{8|v-w|}f(w)$$

and

$$(3 + \frac{7}{8} + \frac{1}{8|v'-w|})f(v') = \sum_{i=1}^{3} f(v'_i) + \frac{7}{8}f(v) + \frac{1}{8|v'-w|}f(w)$$

where $v_1, v_2, v_3 \in V_N$ are the three neighbors of v other than v', and $v'_1, v'_2, v'_3 \in V_N$ are the three neighbors of v' other than v. We give the details for verification of the first identity (the second one is similar) as follows:

$$\begin{aligned}
& \sum_{i=1}^{3} f(v_i) + \frac{7}{8}f(v') + \frac{1}{8|v-w|}f(w) \\
= \ & \sum_{i=1}^{3}(|v'-w|g(v_i - v) + |v-w|g(v_i - v')) + \frac{7}{8}(|v'-w|g(v'-v) + |v-w|g((0,0))) \\
& + \frac{1}{8|v-w|}f(w) \\
= \ & |v'-w|(Dg((0,0)) + 4g((0,0)) - g(v'-v)) + |v-w|(4g(v-v') - g((0,0))) \\
& + \frac{7}{8}(|v'-w|g(v'-v) + |v-w|g((0,0))) + \frac{1}{8|v-w|}f(w) \\
= \ & (3 + \frac{7}{8} + \frac{1}{8|v-w|})(|v'-w|g((0,0)) + |v-w|g(v-v')) \\
= \ & (3 + \frac{7}{8} + \frac{1}{8|v-w|})f(v)\,,
\end{aligned}$$

where the penultimate equality follows by comparing the coefficients of $g((0,0))$, $Dg((0,0))$, $g(v'-v)$ and $g(v-v')$. For completeness, we record the detailed computations on these coefficients here:

$$\begin{aligned}
g((0,0)): &\quad 4|v'-w| + \frac{1}{8|v-w|}|v'-w|^2 - (3+\frac{7}{8}+\frac{1}{8|v-w|})|v'-w| \\
&= |v'-w|(4+\frac{1-|v-w|}{8|v-w|} - (3+\frac{7}{8}+\frac{1}{8|v-w|})) = 0 \\
&\quad \text{and } -|v-w| + \frac{7}{8}|v-w| + \frac{1}{8|v-w|}|v-w|^2 = 0\,; \\
Dg((0,0)): &\quad |v'-w| + \frac{1}{8|v-w|}(-8|v-w||v'-w|) = 0\,; \\
g(v'-v): &\quad -|v'-w| + \frac{7}{8}|v'-w| + \frac{1}{8|v-w|}|v-w||v'-w| = 0\,; \\
g(v-v'): &\quad 4|v-w| + \frac{1}{8|v-w|}|v-w||v'-w| - (3+\frac{7}{8}+\frac{1}{8|v-w|})|v-w| \\
&= |v-w|(4+\frac{1-|v-w|}{8|v-w|} - (3+\frac{7}{8}+\frac{1}{8|v-w|})) \\
&= 0\,.
\end{aligned}$$

Therefore we completed the verification that $f(u)$ is harmonic on $\tilde{\mathcal{G}}'_N\setminus\{w\}$, and (2.37) follows easily from our definition of $f(u)$.

We now claim that

$$\mathbb{E}(\tilde{\eta}'_{N,v} \mid H_=, H_+, H_-) \le \mathbb{E}(\tilde{\eta}'_{N,v} \mid \tilde{\eta}'_{N,w} = f(w)+\lambda+\frac{1}{N},\ \tilde{\eta}'_{N,u} = f(u)+\lambda+\frac{1}{N}\ \forall u\in\partial V_N, \\ \tilde{\eta}'_{N,u} > \lambda+\frac{1}{N}\ \forall u \in \Pi_N\setminus\{w\})\,. \quad (2.38)$$

To show this, we follow [44, Appendix B.1]. We observe that $\{\tilde{\eta}'_{N,u} : u \in \Pi_N \cup \partial V_N\}$ is a Gaussian free field indexed on a finite set (here we have actually added an artificial site o, connected it to ∂V_N, and conditioned on $\tilde{\eta}'_{N,o} = 0$). In particular, its law μ has density $\mu(dr) = \exp(-H(r))\,dr$ (here $r = (r_u)_{u\in\Pi_N\cup\partial V_N}$ denotes a general $|\Pi_N| + |\partial V_N|$ dimensional vector) such that for every $r, r' \in \mathbb{R}^{|\Pi_N|+|\partial V_N|}$

$$H(r\vee r') + H(r\wedge r') \le H(r) + H(r')\,.$$

For $k > 0$, we define

$$U^{(k)}(t) = \begin{cases} kt^4, & \text{if } t < 0 \\ 0, & \text{if } t \geq 0 \end{cases}, \quad V^{(k)}(t) = \begin{cases} 0, & \text{if } t < 0 \\ kt^4, & \text{if } t \geq 0 \end{cases}, \quad W^{(k)}(t) = kt^4 \tag{2.39}$$

and

$$\begin{aligned} \mu_1^{(k)}(\,dr) \propto \exp(-& \sum_{u \in B_0 \cup (\cup_{i=1}^k B_i^{=\lambda}) \cup (\cup_{i=1}^k B_i^{=-\lambda})} W^{(k)}(r_u - x_{N,u}) - \sum_{u \in C_k^{>\lambda}} U^{(k)}(r_u - \lambda - \frac{1}{N}) \\ &- \sum_{u \in C_k^{<-\lambda}} V^{(k)}(r_u + \lambda + \frac{1}{N}) - \sum_{u \in \partial V_N} W^{(k)}(r_u))\mu(\,dr)\,, \\ \mu_2^{(k)}(\,dr) \propto \exp(-& \sum_{u \in \{w\} \cup \partial V_N} W^{(k)}(r_u - f(u) - \lambda - \frac{1}{N}) \\ &- \sum_{u \in \Pi_N \setminus \{w\}} U^{(k)}(r_u - \lambda - \frac{1}{N}))\mu(\,dr)\,. \end{aligned}$$

It is not hard to verify that for any real numbers $t_0 < t_1$ and any pair of functions

$$\begin{aligned} (h_1(t), h_2(t)) \in \{&(W^{(k)}(t - t_0), W^{(k)}(t - t_1)), (W^{(k)}(t - t_0), U^{(k)}(t - t_0)), \\ &(V^{(k)}(t - t_0), U^{(k)}(t - t_0)), (0, U^{(k)}(t - t_0)), (U^{(k)}(t - t_0), U^{(k)}(t - t_1))\}\,, \end{aligned}$$

we have for every $t, t' \in \mathbb{R}$,

$$h_2(t \vee t') + h_1(t \wedge t') \leq h_2(t) + h_1(t')\,,$$

and therefore for any $k > 0$, $\mu_2^{(k)}$ dominates $\mu_1^{(k)}$ in the strong FKG sense ($\mu_1^{(k)} \overset{S}{\prec} \mu_2^{(k)}$), i.e., we have $\mu_1^{(k)}(\,dr) = \exp(-H_1^{(k)}(r))\,dr$, $\mu_2^{(k)}(\,dr) = \exp(-H_2^{(k)}(r))\,dr$ and for every r, r'

$$H_2^{(k)}(r \vee r') + H_1^{(k)}(r \wedge r') \leq H_2^{(k)}(r) + H_1^{(k)}(r')\,.$$

It follows that $\mu_2^{(k)}$ dominates $\mu_1^{(k)}$ in the FKG sense ($\mu_1^{(k)} \prec \mu_2^{(k)}$), i.e., $\mu_1^{(k)}$ is stochastically smaller than $\mu_2^{(k)}$. As $k \to \infty$, $\mu_1^{(k)}$ and $\mu_2^{(k)}$ will converge weakly to the conditional laws on the left and right hand sides of (2.38), respectively. Therefore (2.38) is verified.

Clearly, the right hand side of (2.38) equals

$$\lambda + \frac{1}{N} + \mathbb{E}_{\tilde{\eta}'_{N,w}=f(w),\ \tilde{\eta}'_{N,u}=f(u)\ \forall u \in \partial V_N}(\tilde{\eta}'_{N,v} \mid \tilde{\eta}'_{N,u} > 0 \text{ for all } u \in \Pi_N \setminus \{w\}) .$$

Denote by M the boundary condition $\tilde{\eta}'_{N,w} = f(w)$, $\tilde{\eta}'_{N,u} = f(u)\ \forall u \in \partial V_N$. Now under M, for any $u \in \tilde{\mathcal{G}}'_N$, we have $\tilde{\eta}'_{N,u}$ is Gaussian with mean $\mathbb{E}_M(\tilde{\eta}'_{N,u}) = f(u)$ and variance $\mathrm{Var}_M(\tilde{\eta}'_{N,u}) = G_{\tilde{\mathcal{G}}'_N \setminus \{w\}}(u,u)$. It is well known that there exists a constant $C_2 > 0$ such that $G_{V_N \setminus \{v\}}(u,u) \le C_2 \log(|u-v|+2)$ for all $u \in V_N$. Therefore we have for all $u \in V_N$,

$$\mathrm{Var}_M(\tilde{\eta}'_{N,u}) = G_{\tilde{\mathcal{G}}'_N \setminus \{w\}}(u,u) \le 32(1+G_{V_N \setminus \{v\}}(u,u)) \le 32(1+C_2 \log(|u-v|+2)) . \quad (2.40)$$

In particular, for $u = v$ we have the following bound (using (2.37) and (2.40))

$$\mathbb{E}_M(\tilde{\eta}'_{N,v} \mathbf{1}_{\tilde{\eta}'_{N,u}>0 \text{ for all } u \in \Pi_N \setminus \{w\}}) \le \mathbb{E}_M(|\tilde{\eta}'_{N,v}|) \le C_3 , \quad (2.41)$$

where C_3 is a positive constant which only depends on C and C_1.

It now remains to lower bound $\mathbb{P}_M(\tilde{\eta}'_{N,u} > 0$ for all $u \in \Pi_N \setminus \{w\})$. We will do this by giving a lower bound of $\mathbb{P}_M(\tilde{\eta}'_{N,u} > 0$ for all $u \in \tilde{\mathcal{G}}'_N \setminus \{w\})$. First, by a union bound over all $u \in V_N$ and using the bounds in (2.37) and (2.40), we have (first take C, then C_1 to be sufficiently large)

$$\mathbb{P}_M(\tilde{\eta}'_{N,u} \ge f(u)/2 \text{ for all } u \in V_N) \ge 1/2 . \quad (2.42)$$

Conditioning on the values $\tilde{\eta}'_{N,u}$ for all $u \in V_N$, for each segment $e' = (x,y) \in E'_{N,w}$, we have (here $d(x,y)$ denotes the distance between x and y in the metric graph $\tilde{\mathcal{G}}'_N$)

$$\mathbb{P}(\tilde{\eta}'_{N,u} = 0 \text{ for some } u \in e' \mid M, \mathcal{F}_{V_N}) = \mathrm{e}^{-\tilde{\eta}'_{N,x}\tilde{\eta}'_{N,y} \cdot \frac{1}{d(x,y)}} \le \mathrm{e}^{-\frac{1}{16} f(x) f(y)}$$

on the event $\{\tilde{\eta}'_{N,u} \geq f(u)/2 \text{ for all } u \in V_N\}$. By another union bound over all segments $e' \in E'_{N,w}$ and using (2.37), we have on the same event,

$$\mathbb{P}(\tilde{\eta}'_{N,u} > 0 \text{ for all } u \in e' \text{ and all } e' \in E'_{N,w} \mid M, \mathcal{F}_{V_N}) \geq 1/2\,. \tag{2.43}$$

Combining (2.42) and (2.43) we have $\mathbb{P}_M(\tilde{\eta}'_{N,u} > 0 \text{ for all } u \in \tilde{\mathcal{G}}'_N \setminus \{w\}) \geq 1/4$, and therefore

$$\mathbb{P}_M(\tilde{\eta}'_{N,u} > 0 \text{ for all } u \in \Pi_N \setminus \{w\}) \geq 1/4\,. \tag{2.44}$$

Combining (2.38), (2.41) and (2.44), we see that we can take $C_0 = 4C_3$ and we completed the proof of the lemma. □

Lemma 2.5. *There exists a constant $C_4 > 0$ such that*

$$\mathrm{Var}(X_3 \mid H_=, H_+, H_-) \leq \frac{C_4}{\log N}\,.$$

Proof. We first claim that

$$\mathrm{Var}(X_3 \mid H_=, H_+, H_-) \leq \mathrm{Var}(X_3 \mid H_=) = \mathrm{Var}(X_3 \mid \mathcal{F}_{J_1 \cup J_2})\,. \tag{2.45}$$

To show this, we use the Brascamp-Lieb inequalities (see, e.g., [44, Appendix B.2]). Denote by μ the law of Z where Z is distributed as $\{\tilde{\eta}'_{N,u} : u \in \Pi_N \setminus (J_1 \cup J_2)\}$ conditioned on $H_=$. Then Z is a finite dimensional Gaussian vector. Let m and A be its mean vector and covariance matrix, respectively. The density of μ is of the form $\mu(\,dr) \propto \exp(-\frac{1}{2}(r-m) \cdot A^{-1}(r-m))\,dr$. For any $k > 0$, consider the measure

$$\mu^{(k)}(\,dr) \propto \exp(-\sum_{u \in C_k^{>\lambda}} U^{(k)}(r_u - \lambda - \frac{1}{N}) - \sum_{u \in C_k^{<-\lambda}} V^{(k)}(r_u + \lambda + \frac{1}{N}))\mu(\,dr)\,,$$

where $U^{(k)}$ and $V^{(k)}$ are as defined in (2.39). Since the second order derivatives of $U^{(k)}$

and $V^{(k)}$ are both nonnegative, we see that the density of $\mu^{(k)}$ is of the form $\mu^{(k)}(\,dr) = \exp(-H(r))\,dr$ where $\inf_r \mathrm{Hess}(H)(r) \geq \frac{1}{2}A^{-1}$. Therefore, by the Brascamp-Lieb inequality, for the random vector $Y^{(k)} \sim \mu^{(k)}$ and for every $l \in \mathbb{R}^{|\Pi_N \setminus (J_1 \cup J_2)|}$, we have

$$\mathrm{Var}(l \cdot Y^{(k)}) \leq \mathrm{Var}(l \cdot Z)\,.$$

Since as $k \to \infty$, the law of $Y^{(k)}$ (i.e. $\mu^{(k)}$) converges weakly to the law of Z conditioned on H_+ and H_-, we see that

$$\mathrm{Var}(l \cdot Z \mid H_+, H_-) \leq \mathrm{Var}(l \cdot Z)\,.$$

Note that

$$\begin{aligned} & \mathrm{Var}(X_3 \mid H_=, H_+, H_-) \\ = \; & \mathrm{Var}\Big(\frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{v \in \partial V_{(1+\beta)N/2}} \sum_{u \in J_3} \widetilde{\mathrm{Hm}}'(v, u; I_k \cup \partial V_N) \cdot \tilde{\eta}'_{N,u} \mid H_=, H_+, H_-\Big)\,. \end{aligned}$$

Thus, by setting $l_u = \frac{1}{|\partial V_{(1+\beta)N/2}|} \sum_{v \in \partial V_{(1+\beta)N/2}} \widetilde{\mathrm{Hm}}'(v, u; I_k \cup \partial V_N)$ for $u \in J_3$ and 0 otherwise, this gives the inequality (2.45).

Now let us define

$$U_1 = \{u_1 \in J_3 : |u_1 - u| \geq (\log N)^{10} \text{ for all } u \in J_4\}$$

and for $u_1 \in U_1$, define

$$U_2(u_1) = \{u_2 \in J_3 : |u_1 - u_2| \geq (\log N)^{10}\}\,.$$

For $u_1, u_2 \in J_3$, we say a pair (u_1, u_2) is good if $u_1 \in U_1$ and $u_2 \in U_2(u_1)$. We can expand the right hand side of (2.45) as follows (where we write $\mathcal{I}_{k,N} = I_k \cup \partial V_N$):

$$\begin{aligned}
&\mathrm{Var}\Big(\sum_{v\in\partial V_{(1+\beta)N/2}} \sum_{u\in J_3} \widetilde{\mathrm{Hm}}'(v,u;\mathcal{I}_{k,N})\cdot \tilde{\eta}'_{N,u} \mid \mathcal{F}_{J_1\cup J_2}\Big) = \\
&\sum_{v_1,v_2\in\partial V_{(1+\beta)N/2}} \sum_{u_1\in J_3\setminus U_1} \sum_{u_2\in J_3} \widetilde{\mathrm{Hm}}'(v_1,u_1;\mathcal{I}_{k,N})\widetilde{\mathrm{Hm}}'(v_2,u_2;\mathcal{I}_{k,N}) G_{\tilde{\mathcal{G}}'_N\setminus(J_1\cup J_2)}(u_1,u_2) + \\
&\sum_{v_1,v_2\in\partial V_{(1+\beta)N/2}} \sum_{u_1\in U_1} \sum_{u_2\in J_3\setminus U_2(u_1)} \widetilde{\mathrm{Hm}}'(v_1,u_1;\mathcal{I}_{k,N})\widetilde{\mathrm{Hm}}'(v_2,u_2;\mathcal{I}_{k,N}) G_{\tilde{\mathcal{G}}'_N\setminus(J_1\cup J_2)}(u_1,u_2) \\
&+ \sum_{v_1,v_2\in\partial V_{(1+\beta)N/2}} \sum_{(u_1,u_2)\text{ is good}} \widetilde{\mathrm{Hm}}'(v_1,u_1;\mathcal{I}_{k,N})\widetilde{\mathrm{Hm}}'(v_2,u_2;\mathcal{I}_{k,N}) G_{\tilde{\mathcal{G}}'_N\setminus(J_1\cup J_2)}(u_1,u_2)\,.
\end{aligned} \tag{2.46}$$

Recall that we have $|J_4| \le N\mathrm{e}^{-(\log N)^{9/10}}$. By a simple volume consideration, we have $|J_3\setminus U_1| \le N\mathrm{e}^{-(\log N)^{9/10}}(\log N)^{21}$ and for $u_1\in U_1$, $|J_3\setminus U_2(u_1)| \le N\mathrm{e}^{-(\log N)^{9/10}}(\log N)^{21}$. Therefore, for any $v_1, v_2 \in \partial V_{(1+\beta)N/2}$, we have

$$\widetilde{\mathrm{Hm}}'(v_1, J_3\setminus U_1; I_k\cup\partial V_N) = o(\log N)^{-10} \tag{2.47}$$

and

$$\widetilde{\mathrm{Hm}}'(v_2, J_3\setminus U_2(u_1); I_k\cup\partial V_N) = o(\log N)^{-10}\,. \tag{2.48}$$

It is well known that for a constant $C_5 > 0$, we have for any $u_1, u_2 \in \tilde{\mathcal{G}}'_N$

$$G_{\tilde{\mathcal{G}}'_N\setminus(J_1\cup J_2)}(u_1,u_2) \le G_{\tilde{\mathcal{G}}'_N}(u_1,u_2) \le C_5\log N\,. \tag{2.49}$$

We claim that there exists a constant $C_6 > 0$, such that if (u_1, u_2) is good, then

$$G_{\tilde{\mathcal{G}}'_N\setminus(J_1\cup J_2)}(u_1,u_2) \le \frac{C_6}{\log N}\,. \tag{2.50}$$

We will show that if $B^{\tilde{\mathcal{G}}'_N}$ is started at u_1, then the probability that it goes $(\log N)^{10}$ away from u_1 before hitting $J_1 \cup J_2$ is, say, less than $\frac{C_7}{(\log N)^2}$ for a constant $C_7 > 0$. Since $|u_1 - u_2| \geq (\log N)^{10}$, and the expected number of visits of u_2 by $B^{\tilde{\mathcal{G}}'_N}$ is by (2.49) at most $C_5 \log N$, we see that (2.50) is valid with $C_6 = C_7 C_5$.

To do this, we use the Beurling's estimate (see, e.g., [50, Theorem 6.8.1]). We observe that $J_1 \cup J_2 \cup J_4$ (as the "outer boundary" of I_k) is a $*$-connected set (where we regard two vertices as neighboring each other if their ℓ_1-distance is at most 1) with diameter of order N, in the sense that

$$V \triangleq \{v \in V_N : |v - u| \leq 1 \text{ for some } u \in J_1 \cup J_2 \cup J_4\}$$

is a connected set with diameter of order N. In particular, by the definition of J_3 we have $J_3 \subseteq V$. By Beurling's estimate, once $B^{\tilde{\mathcal{G}}'_N}$ is at $v \in V$, it will hit V again before going $(\log N)^6$ away from v, with probability at least $1 - \frac{C_8}{(\log N)^3}$ (where $C_8 > 0$ is an absolute constant). Thus, if $B^{\tilde{\mathcal{G}}'_N}$ is started at $u_1 \in J_3$, then with probability at least $1 - \frac{C_8}{(\log N)^2}$, it will hit V at least $\log N$ times, before going $(\log N)^7$ away from u_1. However, it is clear that if $B^{\tilde{\mathcal{G}}'_N}$ is at $v \in V$, then it has at least constant probability ($\geq 1/32$) to hit $J_1 \cup J_2 \cup J_4$ before (or at) hitting a neighbor of v. Therefore, at these $\log N$ times that $B^{\tilde{\mathcal{G}}'_N}$ hits V (before going $(\log N)^7$ away from u_1), it has at least $1 - \frac{1}{N^{\log \frac{32}{31}}}$ probability to hit $J_1 \cup J_2 \cup J_4$ at least once in the following step, and since J_4 is $(\log N)^{10}$ away from u_1, it must hit $J_1 \cup J_2$. That is to say, the probability that $B^{\tilde{\mathcal{G}}'_N}$ hits $J_1 \cup J_2$ before going $(\log N)^{10}$ away from u_1 is at least $(1 - \frac{C_8}{(\log N)^2})(1 - \frac{1}{N^{\log \frac{32}{31}}})$, which is greater than $1 - \frac{C_7}{(\log N)^2}$ for any $C_7 > C_8$.

Now substituting the bounds in (2.47), (2.48), (2.49) and (2.50) into (2.46) completes the proof of the lemma. □

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