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To my parents

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ABSTRACT

If G is a simply-connected semisimple complex algebraic group, its adjoint form admits a particularly nice equivariant completion called the De Concini–Procesi or wonderful compactification. The complement of the adjoint group in its wonderful compactification $\mathbf{X}$ is a divisor $\mathbf{Y}$ with normal crossings and smooth irreducible components, so it makes sense to consider the sheaf of logarithmic differential operators $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ on the pair $(\mathbf{X}, \mathbf{Y})$. After reviewing the construction of $\mathbf{X}$ and $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$, we relate the latter object to a canonical $(G \times G)$-action on $\mathbf{X}$. In particular, this gives rise to a homomorphism from the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ to the global logarithmic vector fields on $\mathbf{X}$, which extends to a homomorphism from the universal enveloping algebra to the global logarithmic differential operators on $\mathbf{X}$. We show that the latter homomorphism is surjective, compute its kernel, and relate the result to global differential operators on the adjoint group. We also demonstrate analogous results in the setting of logarithmic differential operators twisted by an invertible sheaf on $\mathbf{X}$. We end with a short application of the results to certain modules over $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ with support on the closed orbit in $\mathbf{X}$, relating them to the now classical Beilinson-Bernstein theory of differential operators on flag varieties.

CHAPTER 1

INTRODUCTION

The flag variety associated to a connected semisimple complex Lie group G is an important and well-studied object whose geometry relates deeply to the representation theory of G [CG]. There are several ways to realize this variety. We will view it as the homogeneous space G/B, where $B \subset G$ is a Borel subgroup. When $G = SL_n$, B is the stabilizer of the standard flag in the natural representation $\mathbb{C}^n$. It follows, in this situation, that G/B is the set of complete flags in $\mathbb{C}^n$, whence the term *flag variety*.

One example of the relationship between the geometry of G/B and the representation theory of G is the Borel-Weil theorem. The theorem describes the G-module structure of the spaces of global sections of certain G-equivariant line bundles on G/B. The total space of any such line bundle is of the form $L_\lambda := G \times_B \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional B-module of weight λ. If we fix a maximal torus $T \subset B$ and choose λ to be dominant with respect to the pair (T, B), then the irreducible G-module V_λ with highest weight λ can be realized as the space of global sections of $L_{w\lambda}$, where w is the longest element of the Weyl group $W := N_G(T)/T$. This is a basic result in representation theory, and it is significant because it explicitly constructs all the irreducible G-modules as geometric objects. A simple proof follows from Frobenius reciprocity.

In the early 1980s, a vast generalization of the Borel-Weil theorem was established in the work of Beilinson and Bernstein [BB]. To state this result, we must introduce the notion of twisted differential operators on an algebraic variety. Just like the structure sheaf $\mathcal{O}$ of the variety forms a natural module for the sheaf of differential operators $\mathcal{D}$, so too do the sections of an invertible sheaf $\mathcal{L}$ on the variety form a natural module for a ring of *twisted* differential operators. The simplest way to define this object is to set

$$\mathcal{D}^{\mathcal{L}} := \mathcal{L} \otimes_{\mathcal{O}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{L}^{\vee},$$

which also makes clear the natural action of $\mathcal{D}^{\mathcal{L}}$ on $\mathcal{L}$. From this point of view, the invertible sheaf $\mathcal{L}_{w\lambda}$ on G/B is a $\mathcal{D}^{\mathcal{L}_{w\lambda}}$-module. By the Borel-Weil theorem, the space of global sections of $\mathcal{L}_{w\lambda}$ is the G-module V_λ. It is also a module for the Lie algebra $\mathfrak{g}$ of G and its universal enveloping algebra $U(\mathfrak{g})$. The characters of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ are indexed by the weights of T, and irreducibility of V_λ implies that $Z(\mathfrak{g})$ acts by the central character $\chi_{\lambda+\rho}$, where ρ is the half-sum of positive roots [Hum, sec. 23.2].

In general, the *Beilinson-Bernstein localization theorem* states that for a dominant weight λ, there is an equivalence of categories between the category $\mathrm{Mod}(\mathfrak{g}, \chi_{\lambda+\rho})$ of $U(\mathfrak{g})$-modules with central character $\chi_{\lambda+\rho}$ and the category $\mathrm{Mod}_{qc}(\mathcal{D}^{\mathcal{L}_{w\lambda}})$ of $\mathcal{O}_{G/B}$-quasicoherent $\mathcal{D}^{\mathcal{L}_{w\lambda}}$-modules on G/B. The equivalence is given by the *localization functor* $\mathcal{D}^{\mathcal{L}_{w\lambda}} \otimes_{U(\mathfrak{g})} (\,\cdot\,)$, which makes sense after we define a certain canonical homomorphism

$$\xi^{\mathcal{L}_{w\lambda}} : U(\mathfrak{g}) \longrightarrow \Gamma(G/B, \mathcal{D}^{\mathcal{L}_{w\lambda}}).$$

The inverse equivalence is given by the global sections functor. The Borel-Weil theorem shows that these two functors identify V_λ and $\mathcal{L}_{w\lambda}$. These results are proved in [BB] and exposited in [HTT, ch. 11] and the article [Ka] of Kashiwara. See also the work [BK] of Brylinski and Kashiwara for applications of the localization theorem to the proof of the famous Kazhdan-Lusztig conjectures.

Around the same time as the work of Beilinson and Bernstein on twisted D-modules on flag varieties, a class of remarkable objects was introduced in the work of De Concini and Procesi [DP], called *complete symmetric varieties*. This is a class of particularly nice equivariant embeddings of *symmetric spaces*: homogeneous spaces G/K, where $K \subset G$ is the fixed subgroup of an involution on G. The complete symmetric variety $\mathbf{X}$ associated to G/K is a smooth projective variety, and the complement of the open orbit G/K in $\mathbf{X}$ is a divisor with normal crossings and smooth irreducible components. There are finitely many G-orbits on $\mathbf{X}$, and their closures are smooth and coincide with partial intersections of the set of irreducible components of the boundary divisor. In particular, the intersection of all the irreducible components is the unique closed orbit, isomorphic to a certain partial flag variety G/P. Today, the variety $\mathbf{X}$ is often called the *wonderful compactification* of G/K, as part of a broader class of so-called "wonderful" varieties [Tim, ch. 30], and further generalizations have been established, for example Brion's work on "log homogeneous" varieties [Br1].

The very special geometry of complete symmetric varieties has representation-theoretic consequences. For example, Ginzburg used a certain class of "admissible" D-modules on a symmetric space and its wonderful compactification to give a simple description of Lusztig character sheaves [Gin]. In a different direction, the group of isomorphism classes of G-equivariant line bundles on $\mathbf{X}$ is well-understood, as is the G-module structure of the spaces of global sections of such line bundles [DP, sec. 8], establishing an analogue of the Borel-Weil theorem for $\mathbf{X}$. The G-modules obtained in this way are in general not irreducible, in contrast to the situation on the flag variety G/B.

In our discussion, we will focus on the wonderful compactification $\mathbf{X}$ of the adjoint form $G_{\mathrm{ad}} := G/Z(G)$, which can be viewed as a symmetric space for $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$, the involution being simply a flip of the two factors. The geometry of $\mathbf{X}$ is summarized in some amount of detail in sections 2.1 and 2.2 of this thesis, while line bundles on $\mathbf{X}$ are discussed in section 2.4. In this setting, the universal enveloping algebra $U(\mathfrak{g} \oplus \mathfrak{g})$ acts on the structure sheaf $\mathcal{O}_{\mathbf{X}}$ by differential operators and, more generally, on any $(G \times G)$-equivariant invertible sheaf $\mathcal{L}$ on $\mathbf{X}$ by twisted differential operators. However, the differential operators obtained in this way are special: the vector fields coming from $\mathfrak{g} \oplus \mathfrak{g}$ will be tangent to the $(G \times G)$-orbits on $\mathbf{X}$, and thus tangent to the irreducible components of the boundary divisor $\mathbf{Y}$, the complement of the open orbit G_{ad} in $\mathbf{X}$. Such vector fields generate the sheaf of *logarithmic differential operators* $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ on $\mathbf{X}$, a subsheaf of the full sheaf of differential operators $\mathcal{D}_{\mathbf{X}}$. The twisted analogue is the sheaf

$$\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}} := \mathcal{L} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{D}_{\mathbf{X},\mathbf{Y}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{L}^{\vee}$$

of *twisted logarithmic differential operators* on $\mathcal{L}$. We give a summary of the construction of $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ and some of its properties in sections 3.1 and 3.3.

The sheaf $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ is quite interesting. It of course has a natural action on $\mathcal{O}_{\mathbf{X}}$, obtained by restricting the natural action of $\mathcal{D}_{\mathbf{X}}$. However, there turn out to be more examples of natural $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-modules. Since the logarithmic vector fields are tangent to the irreducible components of $\mathbf{Y}$, they have a natural action on any invertible sheaf $\mathcal{L}$ on $\mathbf{X}$ corresponding to an integral linear combination of these irreducible components, and this action extends to $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$. Thus, we obtain a set of invertible sheaves on $\mathbf{X}$ which are natural $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-modules, and the $U(\mathfrak{g}\oplus\mathfrak{g})$-module structure of the spaces of global sections of these sheaves is known. It is then natural to ask, by analogy with the Beilinson-Bernstein localization theorem on the flag variety, about the $U(\mathfrak{g}\oplus\mathfrak{g})$-module structure of the space of global sections of any $\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}$-module.

In this thesis, theorem 3.6.1 will give a description of the global differential operators on $\mathbf{X}$ coming from $U(\mathfrak{g}\oplus\mathfrak{g}) = U(\mathfrak{g})\otimes U(\mathfrak{g})$. It turns out to be the full space of global logarithmic differential operators. On the adjoint group, the first factor of $U(\mathfrak{g})$ will map to the right invariant differential operators, while the second factor will map to the left invariant differential operators. The centre $Z(\mathfrak{g})$ of either factor will then map to the bi-invariant differential operators, and, in particular, the element $1\otimes z - z\otimes 1$ for any $z\in Z(\mathfrak{g})$ will map to the zero operator. Thus, there is a homomorphism from $U(\mathfrak{g})\otimes_{Z(\mathfrak{g})} U(\mathfrak{g})$ to the space of global differential operators on G_{ad}. The image is precisely the space of those differential operators that can be written as sums of products of left and right invariant ones. However, since the adjoint group is affine, it admits many more global differential operators. Remarkably, it turns out that the ones we have described so far are the only ones that extend to global logarithmic differential operators $\mathbf{X}$. In fact, this is true even for twisted differential operators: there is an isomorphism of filtered algebras,

$$\gamma^{\mathcal{L}} : U(\mathfrak{g})\otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathbf{X}, \mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}).$$

At the time of investigating this question, we were not aware that the result for the case $\mathcal{L} = \mathcal{O}_{\mathbf{X}}$ had already appeared in Brion's work [Br2], although the overall focus of that paper is quite different. To prove the surjectivity of $\gamma^{\mathcal{L}}$, our techniques are not essentially different from Brion's. Actually, both proofs are essentially parallels to the proof of the analogous fact for the flag variety, which is the first step in establishing the Beilinson-Bernstein correspondence: the canonical morphism

$$\xi^{\mathcal{L}} : U(\mathfrak{g}) \longrightarrow \Gamma(G/B, \mathcal{D}^{\mathcal{L}}_{G/B})$$

is surjective. An essential ingredient of the latter proof is the *moment map*

$$\mu : T^*_{G/B} \longrightarrow \mathfrak{g}^*,$$

which has a close relationship with the associated graded morphism of $\xi^{\mathcal{L}}$. The geometry of μ has been studied extensively, in particular by Kostant [Kos]. Most importantly, μ is proper and birational, with scheme-theoretic image the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$, making it a resolution of singularities of $\mathcal{N}$. This is the famous *Springer resolution.* Normality of $\mathcal{N}$, also due to Kostant [loc. cit.], plays an important role in the argument.

In the situation of the wonderful compactification $\mathbf{X}$, there is a logarithmic analogue of the moment map, which was considered already in [Gin]. This logarithmic moment map has a more complicated geometry, which we discuss extensively in sections 3.4 and 3.5. Nevertheless, it is proper, its fibres are connected, and its scheme-theoretic image is normal, so the added ingredient of a Stein factorization allows us to use an argument that is largely the same as in the case of the flag variety. Once we have deduced the surjectivity of $\operatorname{gr} \gamma^{\mathcal{L}}$ and computed its kernel in proposition 3.6.1, we deduce the surjectivity of $\gamma^{\mathcal{L}}$ and compute its kernel in proposition 3.6.2 by an induction on the degree of the filtration of $U(\mathfrak{g} \oplus \mathfrak{g})$.

We end the thesis with two examples demonstrating the action of $Z(\mathfrak{g})$ from either copy of $U(\mathfrak{g})$ in some special situations. The first, example 3.6.1, is simply the aforementioned case of an invertible sheaf $\mathcal{L}$ on $\mathbf{X}$ corresponding to a linear combination of the irreducible components of $\mathbf{Y}$. The second, example 3.6.2, is more involved. Here we consider the pullback of $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ along the inclusion $i : \mathbf{Z} \hookrightarrow \mathbf{X}$ of the closed orbit, which turns out to be the flag variety $(G/B) \times (G/B^-)$. The sheaf $i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ admits some canonical global sections, the *Euler vector fields*, which we discuss in sections 3.2 and 3.3. It turns out that the images of the elements of $Z(\mathfrak{g})$ in $i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ are certain polynomials in the Euler vector fields. These polynomials are determined by the Harish-Chandra homomorphism [Hum, sec. 23.3]. If $\mathcal{M}$ is a sheaf of $i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-modules such that the Euler vector fields act locally finitely, then the action of $Z(\mathfrak{g})$ on $\mathcal{M}$ is also locally finite, with certain central characters as generalized eigenvalues. In particular, this provides a link between two natural actions of $U(\mathfrak{g} \oplus \mathfrak{g})$ on $\Gamma(\mathbf{Z}, i^*\mathcal{L})$: the action by twisted differential operators on $\mathbf{Z}$ as in the Beilinson-Bernstein theory, and the action by restrictions of logarithmic differential operators on $\mathbf{X}$ to $\mathbf{Z}$.

CHAPTER 2

THE WONDERFUL COMPACTIFICATION

Let G be a simply-connected semisimple linear algebraic group over $\mathbb{C}$, $Z(G)$ the centre of G, and $G_{\mathrm{ad}} := G/Z(G)$ the adjoint form. We summarize here some well-known results on the De Concini–Procesi ("wonderful") compactification of G_{ad}. For proofs of these results, consult [EJ], or see [DP] for a more general treatment that includes them as special cases.

2.1 Construction

Fix a pair (T, B) of maximal torus and Borel subgroup of G. Let B^- be the Borel subgroup opposite B relative to T, and let $U \subset B$, $U^- \subset B^-$ be the unipotent radicals. (T, B) determines a choice of positive roots R_+, a base of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq R_+$, and a cone of dominant weights in the weight lattice $\mathfrak{X}(T)$. Let $R := R_+ \cup (-R_+)$ be the full set of roots of T in G. Let $W := N_G(T)/T$ be the Weyl group.

Fix a *regular*, dominant weight $\lambda \in \mathfrak{X}(T)$. Let V_λ be the irreducible representation of G with this highest weight. $\operatorname{End} V_\lambda$ is canonically a $(G\times G)$–module, and the variety $\mathbb{P}(\operatorname{End} V_\lambda)$ carries a canonical $(G \times G)$–structure

$$(g_1, g_2)[M] = [g_1 M g_2^{-1}].$$

The stabilizer of the line through the identity operator $\mathbf{1} \in \operatorname{End} V_\lambda$ is the subgroup

$$K := \left\{(g_1, g_2) \in G \times G : g_1 g_2^{-1} \in Z(G)\right\},$$

so

$$\begin{aligned}(G \times G)[\mathbf{1}] \cong (G \times G)/K &\xrightarrow{\sim} G_{\mathrm{ad}} \\ (g_1, g_2)K &\longmapsto g_1 g_2^{-1} Z(G).\end{aligned}$$

Definition 2.1.1. The *wonderful compactification* $\mathbf{X}$ of G_{ad} is the closure of $(G \times G)[\mathbf{1}]$ in $\mathbb{P}(\operatorname{End} V_\lambda)$. ✓

It turns out that the isomorphism class of $\mathbf{X}$ as a $(G \times G)$-variety does not depend on λ, so long as λ is regular. If λ were not regular, this construction would produce an equivariant completion of G_{ad} or some quotient thereof, which necessarily would not have the very special orbit structure of the wonderful compactification. For example, $\mathbf{X}$ has a unique closed orbit: the orbit of the highest weight line in $\operatorname{End} V_\lambda$ with respect to the pair $(T \times T, B \times B^-)$. In general, the stabilizer of that line is the Borel subgroup $B \times B^-$ (and the orbit is a complete flag variety) if and only if λ is regular.

Example 2.1.1. Let $G = SL_2$, $B \subset G$ the subgroup of upper triangular matrices, $T \subset B$ the subgroup of diagonal matrices. In this case, the cone of dominant weights is $\mathbb{N}\rho$, where

$$\rho \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a$$

is the fundamental weight, and the only simple root is $\alpha = 2\rho$. ρ is regular, and $V_\rho = \mathbb{C}^2$ is the natural representation of G. Then $\operatorname{End} V_\rho \cong M_{2\times 2}$, and the image of $G_{\text{ad}} = PSL_2 = PGL_2$ in $\mathbb{P}(\operatorname{End} V_\rho)$ is the nonvanishing set of the homogeneous form det. It follows that $\mathbf{X} = \mathbb{P}(\operatorname{End} V_\rho) = \mathbb{P}^3$.

It is not so easy to describe $\mathbf{X}$ explicitly for other groups, even SL_n for $n > 2$, since the condition that λ be regular forces the dimension of the ambient projective space to be quite large. In particular, the natural representation of SL_n does not have regular highest weight when $n > 2$, and $\mathbf{X}$ is *not* $\mathbb{P}^{n^2-1}$. ✓

There is also another construction of the wonderful compactification, which does not make any of the above choices. Let $\mathfrak{g} = \operatorname{Lie} G$, $n = \dim \mathfrak{g}$. The variety $\operatorname{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ of

n-dimensional subspaces of $\mathfrak{g} \oplus \mathfrak{g}$ carries a canonical $(G \times G)$-structure

$$(g_1, g_2)[S] = [(\mathrm{Ad}\, g_1 \oplus \mathrm{Ad}\, g_2)(S)],$$

and K is the stabilizer of the subspace

$$\mathfrak{g}_{\mathrm{diag}} := \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g}\}.$$

Thus

$$(G \times G)[\mathfrak{g}_{\mathrm{diag}}] \cong (G \times G)/K \cong G_{\mathrm{ad}},$$

and one can prove that the closure of this orbit in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ is again a $(G \times G)$-variety isomorphic to $\mathbf{X}$. The unique isomorphism with the previous construction maps $[\mathfrak{g}_{\mathrm{diag}}]$ to $[\mathbf{1}]$, as both points correspond to the identity element e of the open orbit G_{ad}.

2.2 Orbit Structure

In what follows, we make a choice of homogeneous coordinates on $\mathbb{P}(\mathrm{End}\, V_\lambda)$. This is done only to facilitate the description of the objects involved, and none of the results will depend on this choice. Choose a basis of weight vectors $\{v_i\}_{0 \le i \le m}$ in V_λ with weights $\{\lambda_i\}_{0 \le i \le m}$ respectively, such that $\lambda_0 = \lambda$, and $\lambda_i = \lambda - \alpha_i$ for each $1 \le i \le r$. For each $0 \le i \le m$, write

$$\lambda_i = \lambda - \sum_{j=1}^{r} n_{ij} \alpha_j.$$

Let $\{v_i^*\}_{0 \le i \le m}$ be the dual basis of V_λ^*. By the natural isomorphisms

$$\mathrm{End}\, V_\lambda = V_\lambda \boxtimes V_\lambda^*,$$

$$(V_\lambda \boxtimes V_\lambda^*)^* = V_\lambda^* \boxtimes V_\lambda,$$

$\{v_i^* \otimes v_j\}_{0 \le i,j \le m}$ is a set of homogeneous coordinates on $\mathbb{P}(\operatorname{End} V_\lambda)$. By restriction, these are also homogeneous coordinates on $\mathbf{X}$.

Definition 2.2.1. The *big cell* $\mathbf{X}_o$ is the nonvanishing locus of the homogeneous form $v_0^* \otimes v_0$ on $\mathbf{X}$. ✓

The morphism

$$\begin{aligned} \alpha : T &\longrightarrow \mathbb{G}_m^r \\ t &\longmapsto (\alpha_1(t)^{-1}, \ldots, \alpha_r(t)^{-1}) \end{aligned}$$

descends to an isomorphism of the adjoint maximal torus $T_{\mathrm{ad}} := T/Z(G)$ with $\mathbb{G}_m^r$. $\mathbb{G}_m^r \times \mathbb{G}_m^r$ has an obvious action on $\mathbb{C}^r$,

$$((a_1, \ldots, a_r), (b_1, \ldots, b_r))(z_1, \ldots, z_r) = (a_1 b_1^{-1} z_1, \ldots, a_r b_r^{-1} z_r) \quad (a_i, b_i \in \mathbb{G}_m, z_i \in \mathbb{C}),$$

and the action pulls back to $T \times T$ via $\alpha \times \alpha$. The closed immersion

$$\begin{aligned} \mathbb{C}^r &\longrightarrow \mathbf{X}_o \\ (z_1, \ldots, z_r) &\longmapsto \left[\sum_{i=0}^{m} \prod_{j=1}^{r} z_j^{n_{ij}} v_i \otimes v_i^* \right] \end{aligned}$$

is then $(T \times T)$-equivariant. Denote its image by $\mathbf{T}$, and the image of $z = (z_1, \ldots, z_r)$ by $[z]$.

Informally, we have constructed an equivariant embedding $T_{\mathrm{ad}} \hookrightarrow \mathbf{T}$ and extended the simple roots to allow the value ∞ on $\mathbf{T}$. Thus, any positive root takes the value ∞ at certain points of $\mathbf{T}$, while its negative takes the value 0 at the same points. It turns out that the closure of $\mathbf{T}$ in $\mathbf{X}$ is

$$\overline{\mathbf{T}} = \bigcup_{wT \in W} (w, w)\mathbf{T}.$$

This is a smooth, complete toric variety corresponding to the fan of Weyl chambers in the vector space $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$ (see [EJ, rmk. 4.5], and [Ful] for an introduction to toric varieties). For example, the antidominant Weyl chamber is dual to the cone generated by the negatives of the simple roots, which clearly form a set of algebraically independent generators of the coordinate ring $\mathbb{C}[\mathbf{T}]$. Any root is conjugate to a simple root by an element of W. Thus, $\mathbf{X}$ contains a "maximal torus" $\overline{\mathbf{T}}$, and all the roots extend to allow the values 0 and ∞ on $\overline{\mathbf{T}}$.

The name *big cell* for $\mathbf{X}_o$ is inspired by the following important result:

Theorem 2.2.1. The morphism

$$\begin{aligned} U^- \times \mathbf{T} \times U &\longrightarrow \mathbf{X} \\ (u, [z], v) &\longmapsto u[z]v^{-1} \end{aligned}$$

is an isomorphism onto $\mathbf{X}_o$. Moreover,

$$\mathbf{X} = \bigcup_{g_1, g_2 \in G} (g_1, g_2)\mathbf{X}_o. \qquad \square$$

Thus, $\mathbf{X}_o$ is the analogue in $\mathbf{X}$ of the big Bruhat cell $U^- T_{\text{ad}} U$ in G_{ad}. The first statement in the theorem implies that $\mathbf{X}_o$ is an affine space. The second statement then implies that $\mathbf{X}$ has an open cover by affine spaces. In particular, $\mathbf{X}$ is smooth. Moreover, $\mathbf{X}_o$ intersects every orbit in $\mathbf{X}$.

Using the previously constructed coordinates on $\mathbf{T}$, we can give the following description of the orbit structure of $\mathbf{X}$. The orbits are indexed by subsets of Δ. For such a subset S, choose the point $z_S = (z_1, \ldots, z_r) \in \mathbb{C}^r$ such that

$$z_i = \begin{cases} 0 & , \quad \alpha_i \in S \\ 1 & , \quad \alpha_i \notin S \end{cases} .$$

Let P_S be the parabolic subgroup of G generated by B and the root subgroups U_β for $\beta \in -(\Delta \backslash S)$. Let P_S^- be the parabolic subgroup opposite P_S relative to T, and $P_S = L_S U_S$,

$P_S^- = L_S U_S^-$ the Levi decompositions.

Theorem 2.2.2.

1. There are 2^r $(G \times G)$-orbits in $\mathbf{X}$: $\mathbf{O}_S := (G \times G)[z_S]$ for each $S \subseteq \Delta$.

2. The stabilizer of $[z_S]$ is

$$K_S := \left\{(xu, yv) : x, y \in L_S, xy^{-1} \in Z(L_S), u \in U_S, v \in U_S^-\right\}.$$

3. The closure of $\mathbf{O}_S$ in $\mathbf{X}$ is $\bigcup_{S' \supseteq S} \mathbf{O}_{S'}$. □

It particular, $\mathbf{X}$ has a unique closed orbit,

$$\mathbf{O}_\Delta = (G \times G)[z_\Delta] = (G \times G)[v_0 \otimes v_0^*] \cong (G/B) \times (G/B^-).$$

Combining this description of the orbits with theorem 2.2.1, it follows that each orbit closure is smooth, and the complement of the open orbit

$$\mathbf{O}_\varnothing = (G \times G)[z_\varnothing] = (G \times G)[\mathbf{1}] \cong G_{\mathrm{ad}}$$

in $\mathbf{X}$ is a $(G \times G)$-stable normal crossings divisor with r smooth irreducible components.

Informally, there is a bijective correspondence between orbits and subsets of the set of simple roots: each orbit intersects $\mathbf{T}$ precisely in the subvariety on which the the corresponding simple roots have the value ∞. The closure of an orbit is the union of all the orbits for which the same simple roots have the value ∞.

Example 2.2.1. We continue example 2.1.1 and describe explicitly the big cell and the orbits in $\mathbf{X}$ for $G = SL_2$. We use the standard weight basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of V_ρ and the associated weight basis of $\operatorname{End} V_\rho \cong M_{2\times 2}$,

$$e_1 \otimes e_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 \otimes e_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$e_2 \otimes e_1^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_2 \otimes e_2^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The big cell is the nonvanishing locus of the homogeneous form $e_1^* \otimes e_1$ on $\mathbf{X}$, which is the principal affine open subset

$$\mathbf{X}_o = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \neq 0 \right\}.$$

The image of T_{ad} in $\mathbf{X}$ is

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t \in \mathbb{C}^\times \right\},$$

so $\mathbf{T}$ is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} : z \in \mathbb{C} \right\},$$

and

$$z = t^{-2} = \alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{-1}$$

on T_{ad}. Thus, $\mathbf{T}$ is obtained by attaching to T_{ad} the point $\{z = 0\}$ on which α takes the value ∞. Moreover, the map

$$(u, z, v) \longmapsto \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}^{-1} = \begin{bmatrix} 1 & -v \\ u & z - uv \end{bmatrix}$$

is easily seen to be an isomorphism onto $\mathbf{X}_o$, as predicted by theorem 2.2.1.

There are two orbits in $\mathbf{X}$, the vanishing and nonvanishing sets of det. This is to be expected, since there is only one simple root α. We make standard identifications

$$G/B \xrightarrow{\sim} G[e_1] = \mathbb{P}^1,$$
$$G/B^- \xrightarrow{\sim} G[e_1^*] = \mathbb{P}^1$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} B \longmapsto \begin{bmatrix} a \\ c \end{bmatrix}, \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} B^- \longmapsto [d' : -b'].$$

Then, $(G/B) \times (G/B^-) \cong \mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the closed orbit in $\mathbf{X}$, the orbit of

$$[e_1 \otimes e_1^*] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

by a variant of the Segre embedding,

$$\begin{bmatrix} a \\ c \end{bmatrix} \times [d' : -b'] \longmapsto \begin{bmatrix} ad' & -ab' \\ cd' & -cb' \end{bmatrix}. \qquad \checkmark$$

2.3 The Stabilizer Bundle

In what follows, we denote the Lie algebra of any algebraic group by the corresponding Fraktur letter, with the exception $\mathfrak{h} := \operatorname{Lie} T$, as is standard.

Definition 2.3.1. The *stabilizer bundle* $\pi : E \to \mathbf{X}$ is the pullback of the tautological vector bundle on $\operatorname{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ with respect to the canonical immersion $\mathbf{X} \hookrightarrow \operatorname{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. ✓

E has a canonical $(G\times G)$-equivariant structure, and its fibre over $[\mathfrak{g}_{\text{diag}}]$ is $\mathfrak{g}_{\text{diag}} = \mathfrak{k} = \operatorname{Lie} K$. Thus

$$E|_{\mathbf{O}_\varnothing} \cong (G \times G) \times_K \mathfrak{k}.$$

It immediately follows that the fibre of E over any point of $\mathbf{O}_\varnothing \cong G_{\mathrm{ad}}$ is the Lie algebra of the stabilizer of that point, justifying the name *stabilizer bundle*. Of course, by dimensional considerations, the fibres over points in other orbits will not be the full Lie algebras of their stabilizers, but rather certain ideals thereof. Indeed, consider a point

$$[S] \in \mathbf{X} \subset \mathrm{Gr}\,(n, \mathfrak{g} \oplus \mathfrak{g}).$$

First, we note that $S \subset \mathfrak{g} \oplus \mathfrak{g}$ is a Lie subalgebra. Indeed, the set of n-dimensional Lie subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ is closed in $\mathrm{Gr}\,(n, \mathfrak{g} \oplus \mathfrak{g})$, and the (dense) open orbit of $\mathbf{X}$ is contained in this set, whence so is $\mathbf{X}$. Let $G_{[S]} \subset G \times G$ be the stabilizer of $[S]$. Then $\mathfrak{g}_{[S]} := \mathrm{Lie}\, G_{[S]}$ is the stabilizer of $[S]$ under the adjoint action of $\mathfrak{g} \oplus \mathfrak{g}$ on itself. It follows that $S \subseteq \mathfrak{g}_{[S]}$, and moreover is an ideal.

Lemma 2.3.1. For each $S \subseteq \Delta$, the subspace

$$\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^- := \left\{(X + Z, Y + Z) : X \in \mathfrak{u}_S, Y \in \mathfrak{u}_S^-, Z \in \mathfrak{l}_S\right\}$$

of $\mathfrak{p}_S \oplus \mathfrak{p}_S^-$ is stable under the adjoint action of $K_S \subseteq P_S \times P_S^-$. In particular, it is an ideal of $\mathfrak{k}_S \subseteq \mathfrak{p}_S \oplus \mathfrak{p}_S^-$.

Proof. Since $\mathfrak{u}_S \subset \mathfrak{p}_S$ is an ideal, it is preserved by the adjoint action of P_S. Likewise, P_S^- preserves $\mathfrak{u}_S^-$. It remains to show that K_S transports

$$\mathfrak{l}_{S,\mathrm{diag}} := \{(Z, Z) \in \mathfrak{l}_S \oplus \mathfrak{l}_S\}$$

into $\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-$.

Let $Z \in \mathfrak{l}_S$, $(g_1, g_2) \in K_S$. Then $g_1 = xu, g_2 = yv$ for some $x, y \in L_S$ such that $xy^{-1} \in Z(L_S)$, $u \in U_S$, $v \in U_S^-$. Since U_S is unipotent, $u = \exp X$ for some $X \in \mathfrak{u}_S$. Since

$\mathfrak{u}_S$ is nilpotent, $\operatorname{ad} X$ is a nilpotent operator on $\mathfrak{g}$, so $(\operatorname{ad} X)^N = 0$ for some $N \gg 0$. Then

$$\begin{aligned} uZu^{-1} &= (\exp X)Z(\exp X)^{-1} \\ &= \exp(\operatorname{ad} X)Z \\ &= Z + Z', \end{aligned}$$

where

$$Z' := \sum_{i=1}^{N} \frac{1}{i!}(\operatorname{ad} X)^i Z.$$

In particular, $Z' \in \mathfrak{u}_S$, since $\mathfrak{u}_S \subset \mathfrak{p}_S$ is an ideal. Likewise,

$$vZv^{-1} = Z + Z''$$

for some $Z'' \in \mathfrak{u}_S^-$. Then

$$\begin{aligned} (g_1, g_2)(Z, Z) &= (g_1 Z g_1^{-1}, g_2 Z g_2^{-1}) \\ &= (x(uZu^{-1})x^{-1}, y(vZv^{-1})y^{-1}) \\ &= (x(Z + Z')x^{-1}, y(Z + Z'')y^{-1}) \\ &= (xZx^{-1} + xZ'x^{-1}, yZy^{-1} + yZ''y^{-1}). \end{aligned}$$

Since $xy^{-1} \in Z(L_S)$,

$$xZx^{-1} = yZy^{-1},$$

and $xZ'x^{-1} \in \mathfrak{u}_S$, $yZ''y^{-1} \in \mathfrak{u}_S^-$ again because $\mathfrak{u}_S \subset \mathfrak{p}_S$, $\mathfrak{u}_S^- \subset \mathfrak{p}_S^-$ are ideals. Thus, $(g_1, g_2)(Z, Z) \in \mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-$. □

Proposition 2.3.1. For each $S \subseteq \Delta$, the fibre of E over $[z_S]$ is $\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-$. Thus, the pullback of E to $\mathbf{O}_S$ is isomorphic to the homogeneous vector bundle

$$(G \times G) \times_{K_S} (\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-).$$

Proof. For each $S \subseteq \Delta$, let

$$E'_{[z_S]} := \mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-.$$

By theorem 2.2.2 (1), any $p \in \mathbf{X}$ is of the form $(g_1, g_2)[z_S]$ for some $g_1, g_2 \in G, S \subseteq \Delta$. Let

$$E'_p = E'_{(g_1,g_2)[z_S]} := (\operatorname{Ad} g_1 \oplus \operatorname{Ad} g_2)(E'_{[z_S]}),$$

which is well-defined by lemma 2.3.1. Let

$$E' := \left\{(p, X, Y) : p \in \mathbf{X}, (X, Y) \in E'_p\right\} \subset \mathbf{X} \times (\mathfrak{g} \oplus \mathfrak{g}),$$

and let

$$\pi' : E' \longrightarrow \mathbf{X}$$

be the first projection. Then π' is $(G \times G)$-equivariant, and its fibre over any $p \in \mathbf{X}$ is the n-dimensional vector space E'_p.

Consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Fix a basis of weight vectors of $\mathfrak{g}$, for example a Chevalley basis

$$\{X_\alpha, Y_\alpha : \alpha \in R_+\} \cup \{H_i : 1 \leq i \leq r\},$$

where X_α, Y_α are basis vectors for $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ respectively, and H_i is a basis vector for $\operatorname{Lie}(\alpha_i^\vee(\mathbb{G}_m))$.

For each $\alpha \in R_+$, write

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i.$$

Using the isomorphism $U^- \times \mathbf{T} \times U \cong \mathbf{X}_o$ of theorem 2.2.1, define the following sections of the trivial vector bundle $\mathbf{X}_o \times (\mathfrak{g} \oplus \mathfrak{g}) \to \mathbf{X}_o$:

$$\begin{aligned} s_\alpha(u[z]v^{-1}) &:= (u,v)\left([z], X_\alpha, \left(\prod_{i=1}^{r} z_i^{n_i}\right) X_\alpha\right), \\ s_{-\alpha}(u[z]v^{-1}) &:= (u,v)\left([z], \left(\prod_{i=1}^{r} z_i^{n_i}\right) Y_\alpha, Y_\alpha\right), \\ s_i(u[z]v^{-1}) &:= (u,v)([z], H_i, H_i) \quad (u \in U^-, [z] \in \mathbf{T}, v \in U) \end{aligned}$$

for each $\alpha \in R^+$, $1 \le i \le r$.

For each $S \subseteq \Delta$, these sections span $E'_{[z_S]}$ at $[z_S]$. A computation shows that

$$\begin{aligned} (t_1,t_2)(s_\alpha[z]) &= \alpha(t_1) s_\alpha((t_1,t_2)[z]), \\ (t_1,t_2)(s_{-\alpha}[z]) &= \alpha(t_2)^{-1} s_{-\alpha}((t_1,t_2)[z]), \\ (t_1,t_2)s_i([z]) &= s_i((t_1,t_2)[z]) \quad (t_1, t_2 \in T, [z] \in \mathbf{T}), \end{aligned}$$

so in fact these sections also span $E'_{(t_1,t_2)[z_S]}$ at $(t_1,t_2)[z_S]$. It now follows that they span $E'_{u[z]v^{-1}} = (u,v)E'_{[z]}$ for any $u \in U^-, [z] \in \mathbf{T}, v \in U$.

We conclude that the above sections define a trivialization of $\pi' : E' \to \mathbf{X}$ over $\mathbf{X}_o$. Using the $(G \times G)$-equivariance of π' and the second half of theorem 2.2.1, it follows that $\pi' : E' \to \mathbf{X}$ is locally trivial, hence a vector bundle. Since E' and E are both $(G \times G)$-equivariant and have the same fibres over $[z_\varnothing]$, they agree on the dense open orbit $\mathbf{O}_\varnothing \subset \mathbf{X}$. Thus, they are equal. □

A slightly different approach to computing the fibres $E_{[z_S]}$ is outlined in [EJ, rmk. 3.9].

2.4 Picard Group

We discuss the Picard groups of $\mathbf{X}$ and the closed orbit $\mathbf{O}_\Delta \cong (G/B) \times (G/B^-)$, a flag variety. Both $\mathbf{X}$ and $\mathbf{O}_\Delta$ are smooth $(G \times G)$-varieties, so we may canonically realize their Picard groups as either divisor class groups or groups of isomorphism classes of line bundles. Moreover, since G is semisimple and simply-connected, any line bundle on either variety admits a unique $(G \times G)$-linearization, so there is no need to distinguish the equivariant Picard groups [Tim, app. C].

For any $\mu \in \mathfrak{X}(T)$, let $\mathbb{C}_\mu$ denote the one-dimensional T-module of weight μ. Then $B = T \ltimes U$ acts on $\mathbb{C}_\mu$ by its projection to T, and all one-dimensional B-modules arise in this way. Thus, there is a natural isomorphism of weight lattices $\mathfrak{X}(B) = \mathfrak{X}(T)$. Likewise, $\mathfrak{X}(B^-) = \mathfrak{X}(T)$.

Since $\mathbf{O}_\Delta \cong (G/B) \times (G/B^-)$ is a homogeneous space, the map

$$\begin{aligned} \mathfrak{X}(B \times B^-) = \mathfrak{X}(T) \times \mathfrak{X}(T) &\longrightarrow \operatorname{Pic} \mathbf{O}_\Delta \\ (\mu, \nu) &\longmapsto (G \times G) \times_{B \times B^-} (\mathbb{C}_\mu \boxtimes \mathbb{C}_\nu) =: L_{\mu,\nu} \end{aligned}$$

is an isomorphism [Tim, thm. 2.5]. The following theorem relating $\operatorname{Pic} \mathbf{X}$ to $\operatorname{Pic} \mathbf{O}_\Delta$ is due to [DP, sec. 8], where an analogue is proved in a more general setting:

Theorem 2.4.1. The pullback map

$$\operatorname{Pic} \mathbf{X} \longrightarrow \operatorname{Pic} \mathbf{O}_\Delta$$

is injective, and its image corresponds under the isomorphism $\operatorname{Pic} \mathbf{O}_\Delta \cong \mathfrak{X}(T) \times \mathfrak{X}(T)$ to the antidiagonal

$$\{(\mu, -\mu) \in \mathfrak{X}(T) \times \mathfrak{X}(T)\}. \qquad \square$$

We denote by L_μ the unique line bundle on $\mathbf{X}$ that pulls back to $L_{\mu,-\mu}$ on $\mathbf{O}_\Delta$. The following useful result, also due to [loc. cit.], identifies the line bundles in $\operatorname{Pic} \mathbf{X}$ corresponding to the components of the complement of G_{ad} in $\mathbf{X}$ (the "boundary divisor"):

Theorem 2.4.2. For each $\alpha \in \Delta$, there is a global section of $L_{-\alpha}$ with scheme of zeroes the closure of $\mathbf{O}_{\{\alpha\}}$ in $\mathbf{X}$. $\square$

Since λ is regular and dominant, the line bundle L_λ admits a very simple description. The restriction of the canonical immersion $\mathbf{X} \hookrightarrow \operatorname{End} V_\lambda$ to $\mathbf{O}_\Delta$ is just the Plücker embedding

$$
\begin{aligned}
(G/B) \times (G/B^-) &\longhookrightarrow \mathbb{P}(\operatorname{End} V_\lambda) \\
(g_1 B, g_2 B^-) &\longmapsto (g_1, g_2)[v_0 \otimes v_0^*],
\end{aligned}
$$

since $[v_0 \otimes v_0^*]$ is the highest weight line in $\operatorname{End} V_\lambda$ with respect to the pair $(T \times T, B \times B^-)$, and its weight $(\lambda, -\lambda)$ is regular (see [CG, sec. 3.1.13]). This extends to a closed immersion

$$
\begin{aligned}
L_{\lambda,-\lambda} = (G \times G) \times_{B \times B^-} (\mathbb{C}_\lambda \boxtimes \mathbb{C}_{-\lambda}) &\longhookrightarrow \mathbb{P}(\operatorname{End} V_\lambda) \times \operatorname{End} V_\lambda \\
((g_1, z_1)B, (g_2, z_2)B^-) &\longmapsto ((g_1, g_2)[v_0 \otimes v_0^*], z_1 g_1 v_0, z_2 g_2 v_0^*),
\end{aligned}
$$

whose image is the tautological line bundle $O_{\mathbb{P}(\operatorname{End} V_\lambda)}(-1)$. It follows from this description of $L_{\lambda,-\lambda}$ and theorem 2.4.1 that L_λ is the pullback of $O_{\mathbb{P}(\operatorname{End} V_\lambda)}(-1)$ to $\mathbf{X}$. In particular, the dual bundles $L_\lambda^* \cong L_\lambda$ and $L_{\lambda,-\lambda}^* \cong L_{-\lambda,\lambda}$ are very ample. It follows also that for $\mu \in \mathfrak{X}(T)$, L_μ is (very) ample if and only if $L_{\mu,-\mu}$ is (very) ample, that is, if and only if μ is regular and antidominant [HTT, thm. 9.11.2 (ii)].

Example 2.4.1. If $G = SL_2$, we have already noted the isomorphisms

$$
\mathbf{O}_\Delta \cong (G/B) \times (G/B^-) \cong G[e_1] \times G[e_1^*] = \mathbb{P}^1 \times \mathbb{P}^1.
$$

On $G[e_1] = \mathbb{P}^1$, we use coordinates v and u on the affine open subsets

$$
\left\{ \begin{bmatrix} v \\ 1 \end{bmatrix} : v \in \mathbb{C} \right\}, \left\{ \begin{bmatrix} 1 \\ u \end{bmatrix} : u \in \mathbb{C} \right\}.
$$

Then $v = u^{-1}$ on the intersection of the two subsets, and the G-action in coordinates is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \frac{av+b}{cv+d},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} u = \frac{c+du}{a+bu}.$$

$\operatorname{Pic}\mathbb{P}^1 \cong \mathbb{Z}$ is generated by the hyperplane bundle $O_{\mathbb{P}^1}(1)$, and the line bundle $O_{\mathbb{P}^1}(n)$ is constructed by gluing together two copies of $\mathbb{C}^2$ with coordinates (v, ζ) and (u, ξ):

$$u = v^{-1}, \qquad v = u^{-1},$$
$$\xi = v^{-n}\zeta, \qquad \zeta = u^{-n}\xi.$$

The G-action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (v, \zeta) = \left(\frac{av+b}{cv+d}, (cv+d)^{-n}\zeta\right),$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (u, \xi) = \left(\frac{c+du}{a+bu}, (a+bu)^{-n}\xi\right)$$

then gives $O_{\mathbb{P}^1}(n)$ the structure of a G-equivariant line bundle. B acts by the character $-n\rho$ on the B-fixed point $u = 0$, so we have an isomorphism

$$\begin{array}{ccc} G \times_B \mathbb{C}_{-n\rho} & \xrightarrow{\sim} & O(n) \\ \downarrow & & \downarrow \\ G/B & \xrightarrow{\sim} & \mathbb{P}^1 \end{array}$$

At the same time, B^- acts by the character $n\rho$ on the B^--fixed point $v = 0$, so we also have an isomorphism

$$\begin{array}{ccc} G\times_{B^-}\mathbb{C}_{n\rho} & \xrightarrow{\sim} & O(n) \\ \downarrow & & \downarrow \\ G/B^- & \xrightarrow{\sim} & \mathbb{P}^1 \end{array}$$

Since $\mathbf{X}\cong\mathbb{P}^3$, $\operatorname{Pic}\mathbb{X}\cong\mathbb{Z}$ with generator the hyperplane bundle $O_{\mathbb{P}^3}(1)$. We use the coordinates (u,z,v) constructed in example 2.2.1 on the affine open subset $\mathbf{X}_o$. Here the tautological line bundle $O_{\mathbb{P}^3}(-1)$ has an obvious trivializing section

$$\sigma\left(\begin{bmatrix} 1 & -v \\ u & z-uv \end{bmatrix}\right)=\begin{bmatrix} 1 & -v \\ u & z-uv \end{bmatrix}\times\begin{pmatrix} 1 & -v \\ u & z-uv \end{pmatrix}.$$

Let ς be the coordinate on the fibres of $O_{\mathbb{P}^3}(-1)$ over $\mathbf{X}_o$ with respect to the trivialization by σ. Then the $(G\times G)$-action on ς is

$$\begin{aligned}(g_1,g_2)(\varsigma(u,z,v))&=\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)\varsigma(u,z,v)\\ &=(a_1+b_1u)(d_2+c_2v)\varsigma((g_1,g_2)(u,z,v)).\end{aligned}$$

Since $O_{\mathbb{P}^3}(n)=O_{\mathbb{P}^3}(1)^{\otimes n}$ for $n\in\mathbb{Z}$, the section $\sigma^{\otimes(-n)}$ trivializes $O_{\mathbb{P}^3}(n)$ on $\mathbf{X}_o$ with coordinate ς^{-n} along the fibres, and the $(G\times G)$-action is

$$(g_1,g_2)(\varsigma^{-n}(u,z,v))=(a_1+b_1u)^{-n}(d_2+c_2v)^{-n}\varsigma^n((g_1,g_2)(u,z,v)).$$

It follows now that the immersion $\mathbf{O}_\Delta\hookrightarrow\mathbf{X}$, identified with the Segre embedding $\mathbb{P}^1\times\mathbb{P}^1\hookrightarrow\mathbb{P}^3$, extends to a $(G\times G)$-equivariant immersion of line bundles

$$O_{\mathbb{P}^1}(n)\otimes O_{\mathbb{P}^1}(n)\hookrightarrow O_{\mathbb{P}^3}(n).$$

Thus, $O_{\mathbb{P}^3}(n)$ pulls back to $O_{\mathbb{P}^1}(n)\otimes O_{\mathbb{P}^1}(n)=L_{-n\rho,n\rho}$, whence $O_{\mathbb{P}^3}(n)=L_{-n\rho}$.

We note that the divisor $\mathbf{O}_\Delta$ in $\mathbf{X}$ is the vanishing set of the homogeneous form det of degree 2, so its associated line bundle is $O_{\mathbb{P}^3}(2) = L_{-2\rho} = L_{-\alpha}$, as predicted by theorem 2.4.2. ✓

There exists an interesting relationship between L_λ and the stabilizer bundle E. $\operatorname{End} V_\lambda$ has a canonical $(\mathfrak{g} \oplus \mathfrak{g})$-module structure, obtained by differentiating the $(G \times G)$-action:

$$(X, Y)M = XM - MY.$$

For each $p \in \mathbf{X}$, $E_p \subset \mathfrak{g} \oplus \mathfrak{g}$ is a Lie subalgebra. Thus, $\operatorname{End} V_\lambda$ is an E_p-module by restriction. Let

$$(\operatorname{End} V_\lambda)^{E_p} := \{M \in \operatorname{End} V_\lambda : (X, Y)M = 0 \text{ for all } (X, Y) \in E_p\}$$

be the subspace of E_p-invariants. Let

$$L := \{(p, M) : p \in \mathbf{X}, M \in (\operatorname{End} V_\lambda)^{E_p}\} \subseteq \mathbf{X} \times \operatorname{End} V_\lambda.$$

The first projection $L \to \mathbf{X}$ is a $(G \times G)$-equivariant fibration.

Proposition 2.4.1. L is a line subbundle of the trivial bundle $\mathbf{X} \times \operatorname{End} V_\lambda \to \mathbf{X}$ and is isomorphic to L_λ. In particular, $\dim (\operatorname{End} V_\lambda)^{E_p} = 1$ for each $p \in \mathbf{X}$.

Proof. We have to check only that the fibre of $O_{\mathbb{P}(\operatorname{End} V_\lambda)}(-1)$ over any $p \in \mathbf{X}$ is $(\operatorname{End} V_\lambda)^{E_p}$. By $(G \times G)$-equivariance, it suffices to check this at the points $[z_S]$ for each $S \subseteq \Delta$.

Since $\mathfrak{u}_S \oplus \mathfrak{u}_S^- \subset \mathfrak{p}_S \oplus \mathfrak{p}_S^-$ is an ideal, $(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}$ is $(\mathfrak{p}_S \oplus \mathfrak{p}_S^-)$-stable and, in particular, $(\mathfrak{l}_S \oplus \mathfrak{l}_S)$-stable. Since $\mathfrak{l}_S$ is reductive, we may write

$$\mathfrak{l}_S = \mathfrak{z}_S \oplus \mathfrak{s}_S,$$

where $\mathfrak{z}_S$ is the centre of $\mathfrak{l}_S$, and

$$\mathfrak{s}_S := [\mathfrak{l}_S, \mathfrak{l}_S]$$

is semisimple. Using the Cartan decomposition of $\mathfrak{g}$, we see that $\mathfrak{z}_S \subseteq \mathfrak{h}$, so the action of $\mathfrak{z}_S \oplus \mathfrak{z}_S$ on $(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}$ is diagonalizable. It follows that there exists a decomposition of $(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}$ into irreducible $(\mathfrak{s}_S \oplus \mathfrak{s}_S)$-modules that are also irreducible $(\mathfrak{l}_S \oplus \mathfrak{l}_S)$-modules. The highest weight line of any such module (with respect to the pair $(\mathfrak{h} \oplus \mathfrak{h}, (\mathfrak{s}_S \cap \mathfrak{b}) \oplus (\mathfrak{s}_S \cap \mathfrak{b}^-))$ must then be annihilated by both $\mathfrak{u}_S \oplus \mathfrak{u}_S^-$ and $(\mathfrak{s}_S \cap \mathfrak{u}) \oplus (\mathfrak{s}_S \cap \mathfrak{u}^-)$, hence also by $\mathfrak{u} \oplus \mathfrak{u}^-$. But there is a unique such line in $\operatorname{End} V_\lambda$ (an irreducible $(\mathfrak{g} \oplus \mathfrak{g})$-module): the highest weight line $[v_0 \otimes v_0^*]$. We conclude that $(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}$ is an irreducible $(\mathfrak{l}_S \oplus \mathfrak{l}_S)$-module generated by $v_0 \otimes v_0^*$, that is,

$$(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-} = U(\mathfrak{l}_S \oplus \mathfrak{l}_S)(v_0 \otimes v_0^*).$$

Recall that P_S is generated by B and the root subgroups $\{U_\alpha : -\alpha \in \Delta \backslash S\}$. Then L_S is generated by T and the root subgroups $\{U_\alpha : \alpha \in \Delta \backslash S \text{ or } -\alpha \in \Delta \backslash S\}$. It follows that the set

$$\{v_\mu \otimes v_\nu^* : \mu, \nu \in \Lambda\},$$

where Λ is the set of all weights of V_λ of the form

$$\lambda - \sum_{\alpha \notin \Delta \backslash S} n_\alpha \alpha,$$

is a basis of $(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}$ (in fact, it is a weight basis). By Schur's lemma, there is a unique line in $(\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}$ annihilated by the diagonal subalgebra

$$\mathfrak{l}_{S,\text{diag}} := \{(X, X) \in \mathfrak{l}_S \oplus \mathfrak{l}_S\},$$

the line

$$\left[\sum_{\mu \in \Lambda} v_\mu \otimes v_\mu^*\right] = [z_S].$$

It follows that

$$\begin{aligned}(\operatorname{End} V_\lambda)^{E_{[z_S]}} &= (\operatorname{End} V_\lambda)^{\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-} \\ &= \left((\operatorname{End} V_\lambda)^{\mathfrak{u}_S \oplus \mathfrak{u}_S^-}\right)^{\mathfrak{l}_{S,\mathrm{diag}}} \\ &= \mathbb{C} z_S,\end{aligned}$$

the fibre of $O_{\mathbb{P}(\operatorname{End} V_\lambda)}(-1)$ over $[z_S]$.

□

CHAPTER 3

LOGARITHMIC DIFFERENTIAL OPERATORS ON THE WONDERFUL COMPACTIFICATION

3.1 Logarithmic Differential Operators

Although we continue to work with the wonderful compactification of G_{ad}, the objects discussed in this section can be constructed on any smooth variety with a normal crossings divisor with smooth irreducible components, and much of the theory carries over to this more general case. In particular, we make no reference here to the $(G \times G)$-action on $\mathbf{X}$ beyond using it to construct some local coordinates. Some of the properties of logarithmic differential operators are summarized already in [Gin].

Let $\mathbf{Y} \subset \mathbf{X}$ be the complement of the open orbit, the "boundary divisor" or "divisor at infinity". The results of section 2.2 imply that $\mathbf{Y}$ is a normal crossings divisor with $r = \operatorname{rk} G$ smooth irreducible components. Indeed, U and U^- are unipotent, hence each is an affine space of dimension

$$d := \frac{1}{2}(\dim G - \operatorname{rk} G) = \frac{1}{2}(n - r);$$

if $u_1, \ldots, u_d$ are coordinates on U, $v_1, \ldots, v_d$ coordinates on U^-, then theorem 2.2.1 implies that

$$z_i, u_j, v_k \quad (1 \leq i \leq r \text{ and } 1 \leq j, k \leq d)$$

is a system of étale coordinates on $\mathbf{X}_o \subset \mathbf{X}$, and $\mathbf{Y} \cap \mathbf{X}_o$ is cut out by the equation

$$z_1 \ldots z_r = 0.$$

By the same theorem, the $(G \times G)$-translates of $\mathbf{X}_o$ form an open cover of $\mathbf{X}$; by taking an appropriate translate of the above coordinates, we can obtain a system of étale local coordinates on $(\mathbf{X}, \mathbf{Y})$ centred at any $y \in \mathbf{Y}$.

Let $\mathcal{O}_{\mathbf{X}}$ be the structure sheaf of $\mathbf{X}$, $\mathcal{I}_{\mathbf{Y}}$ the ideal sheaf of $\mathbf{Y}$, and $\mathcal{D}_{\mathbf{X}}$ the sheaf of (algebraic) differential operators on $\mathbf{X}$. Of course, $\mathcal{O}_{\mathbf{X}}$ is naturally a $\mathcal{D}_{\mathbf{X}}$-module.

Definition 3.1.1. The sheaf of *logarithmic differential operators* on $\mathbf{X}$ is the subsheaf $\mathcal{D}_{\mathbf{X},\mathbf{Y}} \subset \mathcal{D}_{\mathbf{X}}$ whose sections are the operators preserving the $\mathcal{I}_{\mathbf{Y}}$-adic filtration

$$\mathcal{O}_{\mathbf{X}} \supseteq \mathcal{I}_{\mathbf{Y}} \supseteq \mathcal{I}_{\mathbf{Y}}^2 \supseteq \mathcal{I}_{\mathbf{Y}}^3 \supseteq \cdots. \qquad \checkmark$$

$\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ has a canonical order filtration $F_{\bullet}$ induced by the order filtration on $\mathcal{D}_{\mathbf{X}}$. In particular, $F_0\mathcal{D}_{\mathbf{X},\mathbf{Y}} = \mathcal{O}_{\mathbf{X}}$ is a direct summand of $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$. Its complement in $F_1\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ can be taken to be the sheaf of *logarithmic derivations*, that is, operators $\delta \in F_1\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ such that

$$\delta(fg) = g\delta f + f\delta g \quad (f, g \in \mathcal{O}_{\mathbf{X}}).$$

This is the sheaf of those derivations on $\mathbf{X}$ that preserve $\mathcal{I}_{\mathbf{Y}}$. It also has another name:

Definition 3.1.2. The *logarithmic tangent sheaf* $\mathcal{T}_{\mathbf{X},\mathbf{Y}}$ is the sheaf of logarithmic derivations on $(\mathbf{X}, \mathbf{Y})$. ✓

With respect to the coordinates constructed above, $\mathcal{T}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$ is a free left $\mathcal{O}_{\mathbf{X}}(\mathbf{X}_o)$-module generated by

$$z_i\frac{\partial}{\partial z_i}, \frac{\partial}{\partial u_j}, \frac{\partial}{\partial v_k} \quad (1 \le i \le r \text{ and } 1 \le j, k \le d).$$

In particular, $\mathcal{T}_{\mathbf{X},\mathbf{Y}}$ is locally free (of rank n), hence is the sheaf of sections of a vector bundle on $\mathbf{X}$.

Definition 3.1.3. The *logarithmic tangent bundle* is the vector bundle

$$T_{\mathbf{X},\mathbf{Y}} := \mathbf{Spec}_{\mathbf{X}}(\mathrm{Sym}\,\mathcal{T}_{\mathbf{X},\mathbf{Y}}^{\vee}) \longrightarrow \mathbf{X}$$

with sheaf of sections $\mathcal{T}_{\mathbf{X},\mathbf{Y}}$. The dual bundle

$$\pi : T^*_{\mathbf{X},\mathbf{Y}} \longrightarrow \mathbf{X}$$

is the *logarithmic cotangent bundle* with sheaf of sections $\mathcal{T}^\vee_{\mathbf{X},\mathbf{Y}}$, the sheaf of *logarithmic differentials.* ✓

$\mathcal{T}^\vee_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$ is generated over $\mathcal{O}_{\mathbf{X}}(\mathbf{X}_o)$ by

$$\frac{dz_i}{z_i}, du_j, dv_k \quad (1 \leq i \leq r \text{ and } 1 \leq j, k \leq d).$$

This explains the term *logarithmic*, since formally

$$\frac{dz_i}{z_i} = d(\log z_i).$$

A computation in local coordinates proves the following logarithmic analogue of a standard result on differential operators:

$$\operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}} = \pi_* \mathcal{O}_{T^*_{\mathbf{X},\mathbf{Y}}},$$

where the associated graded is taken with respect to the order filtration.

Definition 3.1.4. For any locally free $\mathcal{O}_{\mathbf{X}}$-module $\mathcal{E}$ of finite rank, the associated sheaf of *twisted differential operators* is

$$\mathcal{D}^{\mathcal{E}}_{\mathbf{X}} := \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{D}_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{E}^\vee,$$

and the sheaf of *twisted logarithmic differential operators* is the subsheaf

$$\mathcal{D}^{\mathcal{E}}_{\mathbf{X},\mathbf{Y}} := \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{D}_{\mathbf{X},\mathbf{Y}} \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{E}^\vee \subset \mathcal{D}^{\mathcal{E}}_{\mathbf{X}}.$$ ✓

While $\mathcal{D}_{\mathbf{X}}$ and $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ act naturally on $\mathcal{O}_{\mathbf{X}}$, $\mathcal{D}_{\mathbf{X}}$ and $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ act naturally on $\mathcal{E}$:

$$(\sigma \otimes \xi \otimes \phi)(s) := \xi(\phi(s))\sigma \quad (\sigma, s \in \mathcal{E}, \xi \in \mathcal{D}, \phi \in \mathcal{E}^{\vee}).$$

We will be especially interested in the case where $\mathcal{E} =: \mathcal{L}$ is an invertible sheaf on $\mathbf{X}$ (and we described all of them up to isomorphism in section 2.4). If $A \subset \mathbf{X}$ is an open set such that $\mathcal{L}(A)$ is free over $\mathcal{O}_{\mathbf{X}}(A)$, then a nowhere vanishing section $\sigma \in \mathcal{L}(A)$ gives rise to a trivialization

$$\begin{aligned} \mathcal{O}_{\mathbf{X}}(A) &\xrightarrow{\sim} \mathcal{L}(A) \\ f &\longmapsto f\sigma. \end{aligned}$$

In a similar manner, we may use σ to identify $\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}(A)$ with $\mathcal{D}_{\mathbf{X},\mathbf{Y}}(A)$. There is a unique dual section $\sigma^* \in \mathcal{L}^{\vee}(A)$ such that $\sigma^*(\sigma) = 1$. If $\xi \in \mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}(A)$, $s \in \mathcal{L}(A)$, then s corresponds to the element

$$f := \sigma^*(s) \in \mathcal{O}_{\mathbf{X}}(A)$$

under the above isomorphism, while

$$\xi s = (\delta f)\sigma$$

for some $\delta \in \mathcal{D}_{\mathbf{X},\mathbf{Y}}(A)$. Thus,

$$\xi = \sigma \otimes \delta \otimes \sigma^*,$$

and we identify ξ with δ.

In what follows, we will sometimes use this approach to explicitly write down twisted differential operators in coordinates. It should be noted that the transition maps for gluing these operators will not be the usual ones for differential operators, since the choice of σ

plays a role. If τ is another trivializing section of $\mathcal{L}(A)$, then

$$\tau = g\sigma$$

for some $g \in \mathcal{O}_{\mathbf{X}}^{\times}(A)$. If δ above turns out to be a *derivation*, then with respect to the trivialization by τ, ξ corresponds to the operator

$$\begin{aligned}
f \mapsto \tau^*(\xi(f\tau)) &= \frac{1}{g}\sigma^*(\xi(fg\sigma)) \\
&= \frac{1}{g}\sigma^*(\delta(\sigma^*(fg\sigma))\sigma) \\
&= \frac{1}{g}\delta(fg) \\
&= \frac{1}{g}(g\delta f + f\delta g) \\
&= \delta f + \frac{\delta g}{g} f \\
&= \left(\delta + \frac{\delta g}{g}\right) f,
\end{aligned}$$

which is not necessarily a derivation! Nevertheless, this does show that $\mathcal{D}_{\mathbf{X},\mathbf{Y}}^{\mathcal{L}}$ has an order filtration induced by that of $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$, and any two local descriptions as above for an order 1 operator differ by a function. In particular, it remains true in the twisted setting that

$$\operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}}^{\mathcal{L}} = \pi_* \mathcal{O}_{T^*_{\mathbf{X},\mathbf{Y}}}.$$

3.2 Euler Vector Fields

Although it is not strictly essential to what follows, we will use this section to demonstrate a special construction in logarithmic geometry. The boundary divisor $\mathbf{Y}$ has irreducible components $\mathbf{Y}_1, \ldots, \mathbf{Y}_r$, the closures in $\mathbf{X}$ of $\mathbf{O}_{\{\alpha_1\}}, \ldots, \mathbf{O}_{\{\alpha_r\}}$ respectively. Their intersections with the big cell are the vanishing loci of the coordinates $z_1, \ldots, z_r$ respectively. In particular, each divisor $\mathbf{Y}_k$ is locally principal. We call the derivation

$$z_k \frac{\partial}{\partial z_k} \in \mathcal{T}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$$

a *local Euler vector field* associated to $\mathbf{Y}_k$. It does not, in general, extend to a global section of $\mathcal{T}_{\mathbf{X},\mathbf{Y}}$.

Instead, let $i_k : \mathbf{Y}_k \hookrightarrow \mathbf{X}$ denote the inclusion, a closed immersion, and consider the sheaf

$$i_k^* \mathcal{T}_{\mathbf{X},\mathbf{Y}} = i_k^{-1} \left(\mathcal{T}_{\mathbf{X},\mathbf{Y}} / \mathcal{I}_{\mathbf{Y}_k} \mathcal{T}_{\mathbf{X},\mathbf{Y}} \right)$$

of $\mathcal{O}_{\mathbf{Y}_k}$-modules. Let $\mathbf{X}_o'$ denote some $(G \times G)$-translate of $\mathbf{X}_o$ (alternatively, any open subset of $\mathbf{X}$ that intersects each $\mathbf{Y}_k$ will do). We denote by $u_1', \ldots, u_d'$, $z_1', \ldots, z_r'$, $v_1', \ldots, v_d'$ the translates of the coordinates $u_1, \ldots, u_d$, $z_1, \ldots, z_r$, $v_1, \ldots, v_d$ respectively. Then for each $1 \leq j \leq r$, $\mathbf{Y}_j \cap \mathbf{X}_o'$ is the vanishing locus of $z_j' \in \mathcal{O}_{\mathbf{X}}(\mathbf{X}_o')$, and

$$z_j = f_j z_j'$$

for some $f_j \in \mathcal{O}_{\mathbf{X}}^{\times}(\mathbf{X}_o \cap \mathbf{X}_o')$. It follows that

$$\begin{aligned}
z_k' \frac{\partial}{\partial z_k'} &= z_k' \left(\sum_{j=1}^{r} \frac{\partial z_j}{\partial z_k'} \frac{\partial}{\partial z_j} + \sum_{l=1}^{d} \left(\frac{\partial u_l}{\partial z_k'} \frac{\partial}{\partial u_l} + \frac{\partial v_l}{\partial z_k'} \frac{\partial}{\partial v_l} \right) \right) \\
&= z_k' \left(\left(f_k + z_k' \frac{\partial f_k}{\partial z_k'} \right) \frac{\partial}{\partial z_k} + \sum_{j \neq k} \frac{\partial f_j}{\partial z_k'} z_j' \frac{\partial}{\partial z_j} \right) + z_k' \sum_{l=1}^{d} \left(\frac{\partial u_l}{\partial z_k'} \frac{\partial}{\partial u_l} + \frac{\partial v_l}{\partial z_k'} \frac{\partial}{\partial v_l} \right) \\
&= z_k \frac{\partial}{\partial z_k} + z_k' \frac{1}{f_k} \frac{\partial f_k}{\partial z_k'} z_k \frac{\partial}{\partial z_k} + z_k' \sum_{j \neq k} \frac{1}{f_j} \frac{\partial f_j}{\partial z_k'} z_j \frac{\partial}{\partial z_j} + z_k' \sum_{l=1}^{d} \left(\frac{\partial u_l}{\partial z_k'} \frac{\partial}{\partial u_l} + \frac{\partial v_l}{\partial z_k'} \frac{\partial}{\partial v_l} \right),
\end{aligned}$$

whence

$$z_k' \frac{\partial}{\partial z_k'} - z_k \frac{\partial}{\partial z_k} \in (\mathcal{I}_{\mathbf{Y}_k} \mathcal{T}_{\mathbf{X},\mathbf{Y}})(\mathbf{X}_o \cap \mathbf{X}_o').$$

We conclude that the local Euler vector field associated to $\mathbf{Y}_k$ canonically determines a global section of $i_k^*\mathcal{T}_{\mathbf{X},\mathbf{Y}}$. We will call this section the *(global) Euler vector field* associated to $\mathbf{Y}_k$ and denote it eu_k. In particular, if we let

$$i : \mathbf{O}_\Delta = \bigcap_{k=1}^{r} \mathbf{Y}_k \hookrightarrow \mathbf{X}$$

be the inclusion of the closed orbit, then each of the Euler vector fields $\mathrm{eu}_1, \ldots, \mathrm{eu}_r$ determines a global section of the sheaf $i^*\mathcal{T}_{\mathbf{X},\mathbf{Y}}$ on $\mathbf{O}_\Delta$, which we denote in the same way.

We remark that in this section too, we have not used the geometry of or the $(G \times G)$-structure on $\mathbf{X}$ in any essential way. The Euler vector fields can be constructed, in the same way, on any smooth variety with a normal crossings divisor with smooth irreducible components. In the next section, we will give equivariant descriptions of logarithmic differential operators on $\mathbf{X}$ and of the Euler vector fields on the irreducible components of $\mathbf{Y}$.

3.3 The Equivariant Picture

We now relate the objects introduced in section 3.1 to the $(G \times G)$-action on $\mathbf{X}$. Let $E \to \mathbf{X}$ be a $(G \times G)$-equivariant vector bundle, then its sheaf of sections $\mathcal{E}$ is a $(G \times G)$-equivariant locally free $\mathcal{O}_{\mathbf{X}}$-module of finite rank. For any open $U \subseteq \mathbf{X}$ and $s \in \mathcal{E}(U)$,

$$((g_1, g_2)s)(p) := (g_1, g_2)s((g_1, g_2)^{-1}p) \quad (g_1, g_2 \in G, p \in (g_1, g_2)U)$$

is a section in $\mathcal{E}((g_1, g_2)U)$. In particular, the tangent bundle $T_{\mathbf{X}}$ has a canonical $(G \times G)$-equivariant structure

$$(g_1, g_2)(p, v) = ((g_1, g_2)p, d_p(g_1, g_2)v) \quad (p \in \mathbf{X}, v \in T_{\mathbf{X},p}),$$

and the equivariant structure on the associated tangent sheaf extends to an equivariant structure on $\mathcal{D}_{\mathbf{X}}$:

$$\begin{aligned}((g_1, g_2)\xi)f &= (g_1, g_2)(\xi((g_1, g_2)^{-1} f)) \\ &= \xi(f \circ (g_1, g_2)) \circ (g_1, g_2)^{-1} \quad (\xi \in \mathcal{D}_{\mathbf{X}}, f \in \mathcal{O}_{\mathbf{X}}).\end{aligned}$$

Since $\mathbf{Y}$ is $(G \times G)$-stable, so are $\mathcal{T}_{\mathbf{X},\mathbf{Y}}$ and $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$.

For any $p \in U$, $((g_1, g_2)s)(p)$ is defined for all g_1, g_2 in some open $W_p \subseteq G \times G$ containing (e, e). Applying a local trivialization at p, we obtain $m := \operatorname{rk} E$ functions $f_1, \ldots, f_m$ on W_p. Differentiating these functions in the direction of $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ at (e, e) and applying the inverse of the local trivialization produces an element of E_p. Thus, we obtain a new section $(X, Y)s \in \mathcal{E}(U)$. Moreover, if s vanishes along $\mathbf{Y}$, then $(X, Y)s$ does as well. It follows that we have a canonical map

$$\gamma^{\mathcal{E}} : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow \Gamma(\mathbf{X}, \mathcal{T}^{\mathcal{E}}_{\mathbf{X},\mathbf{Y}}).$$

This map is easily seen to be $(G \times G)$-equivariant and a Lie algebra homomorphism, so it extends to an equivariant homomorphism of filtered associative algebras

$$\gamma^{\mathcal{E}} : U(\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \Gamma(\mathbf{X}, \mathcal{D}^{\mathcal{E}}_{\mathbf{X},\mathbf{Y}}).$$

We let $\gamma := \gamma^{\mathcal{O}_{\mathbf{X}}}$.

Example 3.3.1. Since $\gamma^{\mathcal{E}}$ is a Lie algebra homomorphism, it suffices to compute it on a basis of $\mathfrak{g} \oplus \mathfrak{g}$. And since $\mathfrak{g}$ is semisimple, this can be done very easily for a Chevalley basis. For each $\alpha \in R$, let X_α be a basis vector for the root space $\mathfrak{g}_\alpha$, and let

$$\begin{aligned}u_\alpha : \mathbb{G}_\alpha &\longrightarrow G \\ t &\longmapsto \exp(tX_\alpha)\end{aligned}$$

be the associated root subgroup. For each $1 \leq i \leq r$, consider the coroot

$$\alpha_i^\vee : \mathbb{G}_m \longrightarrow T \hookrightarrow G,$$

and let

$$H_i := d\alpha_i^\vee(1).$$

Then

$$\{X_\alpha : \alpha \in R\} \cup \{H_i : 1 \leq i \leq r\}$$

is a basis of $\mathfrak{g}$, and

$$\gamma^{\mathcal{E}}(X_\alpha, 0) : \sigma \longmapsto \frac{d}{dt}\bigg|_{t=0} (u_\alpha(t), e)\sigma((u_\alpha(-t), e)(\,\cdot\,)),$$
$$\gamma^{\mathcal{E}}(H_i, 0) : \sigma \longmapsto \frac{d}{dt}\bigg|_{t=1} (\alpha_i^\vee(t), e)\sigma((\alpha_i^\vee(t^{-1}), e)(\,\cdot\,)) \quad (\sigma \in \mathcal{E}),$$

and likewise for $(0, X_\alpha)$ and $(0, H_i)$.

If $G = SL_2$, we take the standard Chevalley basis of $\mathfrak{g} = \mathfrak{sl}_2$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Suppose $\mathcal{E}$ is a $(G \times G)$-equivariant invertible sheaf on $\mathbf{X} = \mathbb{P}^3$. From example 2.4.1, this means $\mathcal{E} = \mathcal{L}_{-n\rho} = \mathcal{O}_{\mathbb{P}^3}(n)$ for some $n \in \mathbb{Z}$. The same example describes the $(G \times G)$-action on $\mathcal{E}$ and a trivialization of $\mathcal{E}$ over $\mathbf{X}_o$. We will use this trivialization to write down twisted differential operators, as explained at the end of section 3.1, using coordinates on $\mathbf{X}_o$ from example 2.2.1. Computing $\gamma^{\mathcal{E}}$ as above, we have

$$
\begin{aligned}
(E,0) &\longmapsto z\frac{\partial}{\partial v} + u^2\frac{\partial}{\partial u} + 2uz\frac{\partial}{\partial z} - nu, & (0,E) &\longmapsto -\frac{\partial}{\partial v}, \\
(H,0) &\longmapsto 2u\frac{\partial}{\partial u} + 2z\frac{\partial}{\partial z} - n, & (0,H) &\longmapsto -2v\frac{\partial}{\partial v} - 2z\frac{\partial}{\partial z} + n, \\
(F,0) &\longmapsto -\frac{\partial}{\partial u}, & (0,F) &\longmapsto z\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v} + 2vz\frac{\partial}{\partial z} - nv.
\end{aligned}
$$

We will see later that these maps, for each $n \in \mathbb{Z}$, are isomorphisms realizing $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as spaces of global twisted derivations on $\mathbb{P}^3$ logarithmic with respect to $\mathbb{P}^1 \times \mathbb{P}^1$.

It is well-known that the centre $U(\mathfrak{sl}_2)$ is a polynomial ring in the Casimir element

$$C := EF + \frac{1}{2}H^2 + FE.$$

By the Poincaré-Birkhoff-Witt theorem,

$$U(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2) = U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2).$$

It follows by another computation that both $1 \otimes C$ and $C \otimes 1$ act by the same differential operator,

$$-2z\frac{\partial^2}{\partial u \partial v} + 2z^2\frac{\partial^2}{\partial z^2} - 2nz\frac{\partial}{\partial z} + \left(\frac{n^2}{2} + n\right).$$

This will end up being true for any $\mathfrak{g}$: $1 \otimes x - x \otimes 1$ maps to the zero operator for any $x \in Z(\mathfrak{g})$. ✓

An important geometric property of $\mathbf{X}$ is the so-called $(G \times G)$*-regularity* [Gin, def. 4.2.1 and prop. 6.1]: the map

$$\gamma : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow \Gamma(\mathbf{X}, \mathcal{T}_{\mathbf{X},\mathbf{Y}})$$

composed with the restriction

$$\Gamma(\mathbf{X}, \mathcal{T}_{\mathbf{X},\mathbf{Y}}) \longrightarrow T_{\mathbf{X},\mathbf{Y},p} := \mathcal{T}_{\mathbf{X},\mathbf{Y},p} / \mathfrak{m}_p \mathcal{T}_{\mathbf{X},\mathbf{Y},p}$$

for any $p \in \mathbf{X}$ is surjective. It follows by Nakayama's lemma that the induced morphism of $\mathcal{O}_{\mathbf{X}}$-modules

$$\gamma : \mathcal{O}_{\mathbf{X}} \otimes_{\mathbb{C}} (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \mathcal{T}_{\mathbf{X},\mathbf{Y}}$$

is surjective. Equivalently,

$$\begin{aligned} \gamma : \mathbf{X} \times (\mathfrak{g} \oplus \mathfrak{g}) &\longrightarrow T_{\mathbf{X},\mathbf{Y}} \\ (p, X, Y) &\longmapsto \gamma(X, Y)(p) \end{aligned}$$

is a surjection of vector bundles. After restricting the vector bundles to G_{ad}, we see that the kernel of the restricted morphism

$$\gamma|_{G_{\mathrm{ad}}} : G_{\mathrm{ad}} \times (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow T_{G_{\mathrm{ad}}}$$

is the restriction to G_{ad} of the stabilizer bundle E introduced in section 2.3. Since γ is a surjection of locally-free $\mathcal{O}_{\mathbf{X}}$-modules of finite rank, its kernel is also a locally-free $\mathcal{O}_{\mathbf{X}}$-module of finite rank. Thus $\ker \gamma$ is vector bundle on $\mathbf{X}$, and the restrictions of $\ker \gamma$ and E to G_{ad} coincide. It follows that the kernel of γ is E. Thus

$$0 \longrightarrow E \longrightarrow \mathbf{X} \times (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow T_{\mathbf{X},\mathbf{Y}} \longrightarrow 0$$

is an exact sequence of vector bundles on $\mathbf{X}$.

This exact sequence pulls back via i to an exact sequence of vector bundles on $\mathbf{O}_{\Delta}$,

$$0 \longrightarrow i^*E \longrightarrow \mathbf{O}_{\Delta} \times (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow i^*T_{\mathbf{X},\mathbf{Y}} \longrightarrow 0.$$

From the description of E given in section 2.3, we have

$$i^*E = (G \times G) \times_{B \times B^-} (\mathfrak{b} \oplus_{\mathfrak{h}} \mathfrak{b}^-),$$

whence

$$i^*T_{\mathbf{X},\mathbf{Y}} \cong (G \times G) \times_{B\times B^-} \left((\mathfrak{g}\oplus\mathfrak{g})/(\mathfrak{b}\oplus_{\mathfrak{h}}\mathfrak{b}^-)\right).$$

For each $X, Y \in \mathfrak{g}$, let $\overline{(X,Y)}$ denote the image of (X,Y) in $(\mathfrak{g}\oplus\mathfrak{g})/(\mathfrak{b}\oplus_{\mathfrak{h}}\mathfrak{b}^-)$. Then for each $H \in \mathfrak{h}$, we obtain a canonical $(G\times G)$-equivariant global section of $i^*T_{\mathbf{X},\mathbf{Y}}$,

$$s_H(g_1B, g_2B^-) := \left[(g_1,g_2), \overline{(H,0)}\right].$$

Let $\alpha_{r+1},\ldots,\alpha_d$ be some enumeration of the elements of $R_+\backslash\Delta$. We use the notation and approach of example 3.3.1 to compute the differential operators corresponding to the sections s_H for $H \in \mathfrak{h}$. We have isomorphisms of algebraic varieties

$$U^- \cong u_{-\alpha_1}(\mathbb{C})\ldots u_{-\alpha_d}(\mathbb{C}),$$
$$U \cong u_{\alpha_1}(\mathbb{C})\ldots u_{\alpha_d}(\mathbb{C}),$$

whence we have coordinates

$$(y_1,\ldots,y_d,z_1,\ldots,z_r,x_1,\ldots,x_d) \longmapsto u_{-\alpha_1}(y_1)\ldots u_{-\alpha_d}(y_d)[z]u_{\alpha_d}(x_d)^{-1}\ldots u_{\alpha_1}(x_1)^{-1}$$

on $\mathbf{X}_o$ (c.f. thm. 2.2.1). Note that

$$\alpha_i^\vee(t)u_{\alpha_j}(x_j)\alpha_i^\vee(t)^{-1} = u_{\alpha_j}\left(t^{\langle\alpha_j,\alpha_i^\vee\rangle}x_j\right),$$
$$\alpha_i^\vee(t)u_{-\alpha_j}(y_j)\alpha_i^\vee(t)^{-1} = u_{-\alpha_j}\left(t^{-\langle\alpha_j,\alpha_i^\vee\rangle}y_j\right),$$
$$\alpha_i^\vee(t)[z] = \left[z_1t^{-\langle\alpha_1,\alpha_i^\vee\rangle},\ldots,z_rt^{-\langle\alpha_r,\alpha_i^\vee\rangle}\right],$$
$$[z]\alpha_i^\vee(t)^{-1} = \left[z_1t^{\langle\alpha_1,\alpha_i^\vee\rangle},\ldots,z_rt^{\langle\alpha_r,\alpha_i^\vee\rangle}\right].$$

We compute the action of $(H_i, 0)$ on a function f on $\mathbf{X}_o$, suppressing some of the notation:

$$\begin{aligned}
\gamma(H_i,0)(f)\left(u_{-\alpha}(y)[z]u_\alpha(x)^{-1}\right) &= \left.\frac{d}{dt}\right|_{t=1} \left(\alpha_i^\vee(t)f\right)\left(u_{-\alpha}(y)[z]u_\alpha(x)^{-1}\right) \\
&= \left.\frac{d}{dt}\right|_{t=1} f\left(\alpha_i^\vee(t)^{-1}\left(u_{-\alpha}(y)[z]u_\alpha(x)^{-1}\right)\right) \\
&= \left.\frac{d}{dt}\right|_{t=1} f\left(u_{-\alpha}(t^{\langle\alpha,\alpha_i^\vee\rangle}y)\left[zt^{\langle\alpha,\alpha_i^\vee\rangle}\right]u_\alpha(x)^{-1}\right) \\
&= \left\langle\alpha,\alpha_i^\vee\right\rangle y\frac{\partial f}{\partial y} + \left\langle\alpha,\alpha_i^\vee\right\rangle z\frac{\partial f}{\partial z}.
\end{aligned}$$

After a similar computation for the action of $(0, H_i)$, we conclude that

$$\begin{aligned}
\gamma(H_i,0) &= \sum_{j=1}^{d}\alpha_j(H_i)y_j\frac{\partial}{\partial y_j} + \sum_{k=1}^{r}\alpha_k(H_i)z_k\frac{\partial}{\partial z_k}, \\
\gamma(0,H_i) &= -\sum_{j=1}^{d}\alpha_j(H_i)x_j\frac{\partial}{\partial x_j} - \sum_{k=1}^{r}\alpha_k(H_i)z_k\frac{\partial}{\partial z_k}.
\end{aligned}$$

It follows now that $s_H([z_\Delta])$ is equal to

$$\sum_{k=1}^{r}\alpha_k(H)z_k\frac{\partial}{\partial z_k}$$

viewed as an element in the fibre $T_{\mathbf{X},\mathbf{Y},[z_\Delta]}$. Since s_H and $\mathrm{eu}_1,\ldots,\mathrm{eu}_r$ are $(U^-\times U)$-invariant, we have

$$s_H = \sum_{k=1}^{r}\alpha_k(H)\mathrm{eu}_k$$

on $U^-\times[z_\Delta]\times U\cong\mathbf{X}_o\cap\mathbf{O}_\Delta$, a dense open subset of $\mathbf{O}_\Delta$. Thus, the equality holds on all of $\mathbf{O}_\Delta$. Let $\varpi_1,\ldots,\varpi_r\in\mathfrak{h}$ be the *fundamental coweights*, that is, elements satisfying

$$\alpha_i(\varpi_j) = \delta_{ij}.$$

It follows that s_{ϖ_k} is the Euler vector field eu_k (restricted to $\mathbf{O}_\Delta$).

3.4 The Moment Map

We consider now the graded homomorphism associated to

$$\gamma^{\mathcal{L}} : U(\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \Gamma(\mathbf{X}, \mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}})$$

for an invertible sheaf $\mathcal{L}$ on $\mathbf{X}$. If a $(G \times G)$-equivariant local trivialization is chosen for $\mathcal{L}$ (over the open orbit, say), then the local descriptions of $\gamma = \gamma^{\mathcal{O}_{\mathbf{X}}}$ and $\gamma^{\mathcal{L}}$ are the same. Such an equivariant trivialization exists, that is, $\mathcal{L}$ is equivariantly trivial over $\mathbf{O}_{\varnothing} \cong G_{\text{ad}}$, because the stabilizer K of $e \in G_{\text{ad}}$ is semisimple, and therefore has no nontrivial characters. Moreover, as noted at the end of section 3.1, $\operatorname{gr} \mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}$ and $\operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}}$ coincide, both being canonically isomorphic to $\pi_* \mathcal{O}_{T^*_{\mathbf{X},\mathbf{Y}}}$. It follows then that $\operatorname{gr} \gamma^{\mathcal{L}}$ coincides with $\operatorname{gr} \gamma$.

By the Poincaré-Birkhoff-Witt theorem ([Hum, sec. 17.3]),

$$\operatorname{gr} U(\mathfrak{g} \oplus \mathfrak{g}) \cong \operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{g}) = \mathbb{C}[(\mathfrak{g} \oplus \mathfrak{g})^*].$$

On the other hand, applying the global sections functor to the exact sequence

$$0 \longrightarrow F_{n-1}\mathcal{D}_{\mathbf{X},\mathbf{Y}} \longrightarrow F_n\mathcal{D}_{\mathbf{X},\mathbf{Y}} \longrightarrow \operatorname{gr}_n\mathcal{D}_{\mathbf{X},\mathbf{Y}} \longrightarrow 0$$

produces the exact sequence

$$0 \longrightarrow \Gamma(\mathbf{X}, F_{n-1}\mathcal{D}_{\mathbf{X},\mathbf{Y}}) \longrightarrow \Gamma(\mathbf{X}, F_n\mathcal{D}_{\mathbf{X},\mathbf{Y}}) \longrightarrow \Gamma(\mathbf{X}, \operatorname{gr}_n\mathcal{D}_{\mathbf{X},\mathbf{Y}}),$$

whence we have an injection

$$\operatorname{gr}_n\Gamma(\mathbf{X}, \mathcal{D}_{\mathbf{X},\mathbf{Y}}) := \Gamma(\mathbf{X}, F_n\mathcal{D}_{\mathbf{X},\mathbf{Y}})/\Gamma(\mathbf{X}, F_{n-1}\mathcal{D}_{\mathbf{X},\mathbf{Y}}) \hookrightarrow \Gamma(\mathbf{X}, \operatorname{gr}_n\mathcal{D}_{\mathbf{X},\mathbf{Y}}).$$

Also,

$$\Gamma(\mathbf{X}, \operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}}) \cong \Gamma(\mathbf{X}, \pi_* \mathcal{O}_{T^*_{\mathbf{X},\mathbf{Y}}}) = \Gamma(T^*_{\mathbf{X},\mathbf{Y}}, \mathcal{O}_{T^*_{\mathbf{X},\mathbf{Y}}}) =: \mathbb{C}[T^*_{\mathbf{X},\mathbf{Y}}].$$

Putting everything together, we have a commutative diagram

$$\begin{array}{ccc}
\operatorname{gr} U(\mathfrak{g} \oplus \mathfrak{g}) & \xrightarrow{\operatorname{gr}\gamma} & \operatorname{gr} \Gamma(\mathbf{X}, \mathcal{D}_{\mathbf{X},\mathbf{Y}}) \\
\wr \downarrow & & \downarrow \\
S(\mathfrak{g} \oplus \mathfrak{g}) & & \Gamma(\mathbf{X}, \operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}}) \\
\| & & \downarrow \wr \\
\mathbb{C}[(\mathfrak{g} \oplus \mathfrak{g})^*] & \xrightarrow[\mu^*]{} & \mathbb{C}[T^*_{\mathbf{X},\mathbf{Y}}]
\end{array}$$

It turns out that the bottom arrow in the diagram is actually the comorphism induced by a certain morphism

$$\mu : T^*_{\mathbf{X},\mathbf{Y}} \longrightarrow (\mathfrak{g} \oplus \mathfrak{g})^*$$

called the *(logarithmic) moment map* (c.f. prop. 3.4.1). This morphism was introduced already in [Gin], and much of the subsequent discussion appears there. We will now define μ and describe it in some detail.

Recall the exact sequence

$$0 \longrightarrow E \longrightarrow \mathbf{X} \times (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow T_{\mathbf{X},\mathbf{Y}} \longrightarrow 0$$

of vector bundles on $\mathbf{X}$. From this sequence, it follows that $T_{\mathbf{X},\mathbf{Y}}$ is the vector bundle quotient of $\mathbf{X} \times (\mathfrak{g} \oplus \mathfrak{g})$ by E. This gives a description of the logarithmic cotangent bundle:

$$T^*_{\mathbf{X},\mathbf{Y}} = E^\perp := \{(p, \phi) \in \mathbf{X} \times (\mathfrak{g} \oplus \mathfrak{g})^* : E_p \subseteq \ker \phi\}.$$

Definition 3.4.1. The *moment map*

$$\mu : T^*_{\mathbf{X},\mathbf{Y}} = E^\perp \longrightarrow (\mathfrak{g} \oplus \mathfrak{g})^*$$

is the second projection. ✓

For any $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$, we have

$$\mu(p, \phi)(X, Y) = \phi(\gamma(X, Y)(p))$$

for $p \in \mathbf{X}$, $\phi \in T^*_{\mathbf{X},\mathbf{Y},p}$. An essential property of μ, clear from the definition, is that it is proper, even projective, since $\mathbf{X}$ is a projective variety.

Proposition 3.4.1. The diagram

$$\begin{array}{ccc} \operatorname{gr} U(\mathfrak{g} \oplus \mathfrak{g}) & \xrightarrow{\operatorname{gr}\gamma} & \operatorname{gr} \Gamma(\mathbf{X}, \mathcal{D}_{\mathbf{X},\mathbf{Y}}) \\ \wr\downarrow & & \downarrow \\ S(\mathfrak{g} \oplus \mathfrak{g}) & & \Gamma(\mathbf{X}, \operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}}) \\ \| & & \downarrow\wr \\ \mathbb{C}[(\mathfrak{g} \oplus \mathfrak{g})^*] & \xrightarrow[\mu^*]{} & \mathbb{C}[T^*_{\mathbf{X},\mathbf{Y}}] \end{array}$$

commutes.

Proof. It suffices to write down all the maps explicitly. Suppose $X_1, \ldots, X_m \in \mathfrak{g} \oplus \mathfrak{g}$. Then

$$X_1 \ldots X_m + U_{m-1}(\mathfrak{g} \oplus \mathfrak{g}) \in \operatorname{gr}_m U(\mathfrak{g} \oplus \mathfrak{g})$$

is mapped to $X_1 \ldots X_m \in S(\mathfrak{g} \oplus \mathfrak{g})$. We view it as an element $f \in \mathbb{C}[(\mathfrak{g} \oplus \mathfrak{g})^*]$, given by

$$f(\phi) = \phi(X_1) \ldots \phi(X_m)$$

for $\phi \in (\mathfrak{g} \oplus \mathfrak{g})^*$. If $p \in \mathbf{X}$, and $\phi \in T^*_{\mathbf{X},\mathbf{Y},p}$, then

$$\begin{aligned} \mu^* f(p, \phi) &= f(\mu(p, \phi)) \\ &= f([(X, Y) \longmapsto \phi(\gamma(X, Y)(p))]) \\ &= \phi(\gamma(X_1)(p)) \ldots \phi(\gamma(X_m)(p)). \end{aligned}$$

On the other hand,

$$\operatorname{gr}\gamma(X_1 \ldots X_m + U_{m-1}(\mathfrak{g}\oplus\mathfrak{g})) = \gamma(X_1)\ldots\gamma(X_m) + \Gamma(\mathbf{X}, F_{m-1}\mathcal{D}_{\mathbf{X},\mathbf{Y}}),$$

which defines the same function on $T^*_{\mathbf{X},\mathbf{Y}}$. □

Remark 3.4.1. Definition 3.4.1 is very simple and is sufficient for our purposes. Indeed, it makes the above proposition almost tautological. However, it is not quite standard, although it appears already in [Gin]. The concept of a "moment map" is very general, and can be defined for any smooth variety with a *Hamiltonian* action by a Lie group. In particular, the natural associated action on the cotangent bundle of any smooth variety with Lie group action is Hamiltonian [CG, sec. 1.4], and in our situation we obtain a moment map

$$\mu' : T^*_{\mathbf{X}} \longrightarrow (\mathfrak{g}\oplus\mathfrak{g})^*.$$

The inclusion of sheaves

$$\mathcal{T}_{\mathbf{X},\mathbf{Y}} \longrightarrow \mathcal{T}_{\mathbf{X}}$$

induces a morphism of vector bundles (the so-called "anchor map")

$$\alpha : T_{\mathbf{X},\mathbf{Y}} \longrightarrow T_{\mathbf{X}}.$$

We consider the dual morphism

$$\alpha^* : T^*_{\mathbf{X}} \longrightarrow T^*_{\mathbf{X},\mathbf{Y}},$$

which is an isomorphism over the open orbit $(G\times G)/K$. There is a standard description of the moment map for the cotangent bundle of a homogeneous space [loc. cit.], from which it follows that μ' and $\mu\circ\alpha^*$ agree on the open orbit. Thus, they must be equal. We conclude that μ is actually a logarithmic analogue of and factors through the usual moment map. ✓

Let κ be the Killing form of $\mathfrak{g}$. It is nondegenerate, since $\mathfrak{g}$ is semisimple, so it can be viewed as a G-equivariant isomorphism

$$\begin{aligned} \kappa : \mathfrak{g} &\xrightarrow{\sim} \mathfrak{g}^* \\ X &\longmapsto [Y \longmapsto \kappa(X,Y)]. \end{aligned}$$

Then $\chi := \kappa \oplus (-\kappa)$ identifies $\mathfrak{g} \oplus \mathfrak{g}$ with $\mathfrak{g}^* \oplus \mathfrak{g}^* = (\mathfrak{g} \oplus \mathfrak{g})^*$. Thus, we can identify $E^{\perp}$, a subbundle of $X \times (\mathfrak{g} \oplus \mathfrak{g})^*$, with a subbundle of $X \times (\mathfrak{g} \oplus \mathfrak{g})$. By $(G \times G)$-equivariance, it suffices to compute $\chi^{-1}(E^{\perp}_{[z_S]})$ for each $S \subseteq \Delta$.

Lemma 3.4.1. For each $S \subseteq \Delta$, $\chi(E_{[z_S]}) = E^{\perp}_{[z_S]}$.

Proof. If $\beta_1, \beta_2 \in \mathfrak{X}(T)$ and $\beta_1 + \beta_2 \neq 0$, then $\mathfrak{g}_{\beta_1}$ and $\mathfrak{g}_{\beta_2}$ are orthogonal with respect to κ [Hum, prop. 8.1]. It follows that $\mathfrak{u}_S$ and $\mathfrak{p}_S$ are orthogonal, as are $\mathfrak{u}_S^-$ and $\mathfrak{p}_S^-$. In particular, $\mathfrak{l}_S = \mathfrak{p}_S \cap \mathfrak{p}_S^-$ is orthogonal to both $\mathfrak{u}_S$ and $\mathfrak{u}_S^-$. Since $\mathfrak{l}_{S,\mathrm{diag}}$ is clearly orthogonal to itself with respect to χ, we conclude that so is $E_{[z_S]} = \mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-$. But $\dim(\mathfrak{g} \oplus \mathfrak{g}) = 2\dim(\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-)$, so nondegeneracy of χ implies that $\chi(E_{[z_S]}) = E^{\perp}_{[z_S]}$. □

We conclude that

$$T^*_{\mathbf{X},\mathbf{Y}} \cong E.$$

In this description, the moment map

$$\mu : E \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$$

is again just the second projection.

3.5 Geometry of the Moment Map

In what follows, we will make use of the morphism

$$q : \mathfrak{g} \longrightarrow \mathfrak{g}//G := \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$$

induced by the inclusion

$$\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}].$$

We will state a number of results about q, which are due to Kostant, who studied this morphism extensively in [Kos]. q is surjective, since it is the structure morphism of an affine GIT quotient (see [Mu, sec. 5.1] for an introduction to GIT). Set-theoretically, its fibres are the closures of G-orbits of maximal dimension $2|R^+|$ in $\mathfrak{g}$, which by definition are the orbits of regular elements. By the famous Chevalley restriction theorem [HTT, thm. 10.1.1], the restriction map

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]$$

is injective, and its image is $\mathbb{C}[\mathfrak{h}]^W$. Thus,

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W.$$

Geometrically,

$$\mathfrak{g}//G \cong \mathfrak{h}/W.$$

By another theorem of Chevalley [Bou, thm. VIII.8.3.1], $\mathbb{C}[\mathfrak{h}]^W$ is a $\mathbb{C}$-algebra generated by algebraically independent homogeneous polynomials $f_1, \ldots, f_r$. Identifying them with elements of $\mathbb{C}[\mathfrak{g}]^G$ via the above Chevalley isomorphism, we see that the morphism

$$\begin{aligned}(f_1, \ldots, f_r) : \mathfrak{g} &\longrightarrow \mathbb{C}^r \\ X &\longmapsto (f_1(X), \ldots, f_r(X))\end{aligned}$$

factors as a composition of q with an isomorphism

$$\mathfrak{g}//G \cong \mathbb{C}^r.$$

Thus, we identify q with the morphism $(f_1, \dots, f_r)$.

We summarize some important properties of the fibres of q, again due to Kostant [Kos]. These properties are also exposited in [HTT, sec. 10.3]. Any G-orbit in $\mathfrak{g}$ has a canonical symplectic structure due to Kirillov-Kostant-Souriau (see, for example, [CG, prop. 1.1.5]), and therefore is necessarily even-dimensional. The fibres of q, being closures of orbits of regular elements, are thus even-dimensional. Since all the fibres also have the same dimension, and q is a morphism between smooth varieties (affine spaces), q is flat [GW, thm. 14.126]. Moreover, by a theorem of Kostant, the scheme-theoretic fibres of q are reduced and normal, and q is smooth on the open subset of regular elements in $\mathfrak{g}$.

Proposition 3.5.1. The set-theoretic image of μ is the algebraic set

$$V := \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g} := \{(X, Y) \in \mathfrak{g} \oplus \mathfrak{g} : q(X) = q(Y)\}.$$

Proof. Recall from section 2.3 that

$$E|_{\mathbf{O}_\varnothing} \cong (G \times G) \times_K \mathfrak{k},$$

where

$$K := \left\{(g_1, g_2) : g_1 g_2^{-1} \in Z(G)\right\},$$

and $\mathfrak{k} = \mathfrak{g}_{\mathrm{diag}}$. It follows that

$$\mu(E_\varnothing) = \{(X, Y) \in \mathfrak{g} \oplus \mathfrak{g} : X \text{ and } Y \text{ are } G\text{-conjugate}\} \subseteq V.$$

Since $E_\varnothing \subset E$ is open and dense, $\mu(E)$ is contained in the closure of $\mu(E_\varnothing)$ in V. But μ is proper, hence closed. Thus, $\mu(E)$ *is* the closure of $\mu(E_\varnothing)$ in V.

Recall that q is a flat morphism, that is, $\mathbb{C}[\mathfrak{g}]$ is a flat $\mathbb{C}[\mathfrak{g}]^G$-module. It follows that $\mathbb{C}[V] = \mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{g}]^G} \mathbb{C}[\mathfrak{g}]$ is a flat $\mathbb{C}[\mathfrak{g}]$-module. Thus the first (or second) projection $V \to \mathfrak{g}$ is a flat morphism. Moreover, its fibre over some $X \in \mathfrak{g}$ is the fibre of q over $q(X)$, an irreducible variety. It follows that V is irreducible.

Now consider

$$i^*E \cong (G \times G) \times_{B \times B^-} (\mathfrak{b} \oplus_{\mathfrak{h}} \mathfrak{b}^-),$$

where i, as before, is the inclusion of $\mathbf{O}_\Delta$ into $\mathbf{X}$. Let $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}$ be the dense open subset of regular semisimple elements (consult [CG, lem. 3.1.4] for its basic properties). Then

$$V^{\mathrm{rs}} := V \cap (\mathfrak{g}^{\mathrm{rs}} \times \mathfrak{g}^{\mathrm{rs}})$$

is a dense open subset of V, since V is irreducible. Suppose $(X, Y) \in V^{\mathrm{rs}}$. Then X and Y are regular, and $q(X) = q(Y)$. It follows that X and Y are conjugate, since each fibre of q contains a unique regular orbit. Since X and Y are also semisimple, there exist $g_1, g_2 \in G$ such that

$$g_1 X g_1^{-1} = g_2 Y g_2^{-1} \in \mathfrak{h}.$$

Then

$$[(g_1, g_2), (g_1^{-1} X g_1, g_2^{-1} Y g_2)] \in i^*E,$$

and

$$\mu([(g_1, g_2), (g_1^{-1} X g_1, g_2^{-1} Y g_2)]) = (X, Y).$$

It follows that

$$V^{\mathrm{rs}} \subseteq \mu(E_\Delta) \subseteq V.$$

Since $E_\Delta \subset E$ is closed, using properness of μ again, $\mu(E_\Delta) = V$. Thus $\mu(E) = V$. $\square$

Corollary 3.5.1.

$$\dim V \leq \dim(E_\Delta) = \dim((G/B) \times (G/B^-)) + \dim(\mathfrak{b} \oplus_{\mathfrak{h}} \mathfrak{b}^-) = 2\dim \mathfrak{g} - r. \qquad \square$$

Let $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$ be the subset of regular elements [Gin, def. 3.1.3]. It is a dense open subset of $\mathfrak{g}$ containing $\mathfrak{g}^{\mathrm{rs}}$. Since the latter is a principal open subset of $\mathfrak{g}$, and there exist regular nilpotent elements, the codimension of $\mathfrak{g}^{\mathrm{reg}}$ is greater than 1.

Proposition 3.5.2. *V is an integral normal variety of dimension $4|R^+| + r$.*

Proof. Consider the morphism

$$\begin{aligned} \eta : \mathfrak{g} \oplus \mathfrak{g} &\longrightarrow \mathbb{C}^r \\ (X, Y) &\longmapsto q(X) - q(Y). \end{aligned}$$

V is the fibre of η over $0 \in \mathbb{C}^r$, so

$$\dim V \geq 2\dim \mathfrak{g} - r.$$

Combining this estimate with that of the preceeding corollary, we conclude that

$$\dim V = 2\dim \mathfrak{g} - r.$$

Since q is flat, η is also flat. It follows that every fibre of η has dimension $2\dim \mathfrak{g} - r$. In particular, this means that each fibre is a complete intersection, and therefore Cohen-Macaulay [Har, prop. II.8.23]. By the same proposition, normality of V will follow once we show that its singular locus has codimension at least 2.

The fibres of η, being Cohen-Macaulay, are equidimensional [GW, prop. 14.124], hence the subset of regular elements is dense in each fibre. Moreover, η is smooth on $\mathfrak{g}^{\mathrm{reg}} \times \mathfrak{g}^{\mathrm{reg}}$, so the general points of each fibre of η are reduced. Each fibre, being Cohen-Macaulay and reduced at general points, is therefore reduced [loc. cit.]. In particular, V is reduced. Irreducibility of V was proved in proposition 3.5.1.

Let $p_1, p_2 : V \to \mathfrak{g}$ be the first and second projections, respectively. Then for any $X \in \mathfrak{g}$, the regular elements in the fibre $p_1^{-1}(X) = q^{-1}(q(X))$ form a single open G-orbit of codimension at least 2 in the fibre, since adjoint orbits have even dimension. The same is true if p_1 is replaced with p_2. Since $V \cap (\mathfrak{g}^{\mathrm{reg}} \times \mathfrak{g}^{\mathrm{reg}})$ is smooth, it follows that the singular locus of V has codimension at least 2 in V. $\square$

Remark 3.5.1. Consider the variety

$$\tilde{\mathfrak{g}} := G \times_B \mathfrak{b}$$

and its associated morphisms

$$\begin{aligned} \tau : G \times_B \mathfrak{b} &\longrightarrow \mathfrak{g} \\ [g, X] &\longmapsto gXg^{-1}, \\ \nu : G \times_B \mathfrak{b} &\longrightarrow \mathfrak{b}/\mathfrak{u} \cong \mathfrak{h} \\ [g, X] &\longmapsto X + \mathfrak{u}. \end{aligned}$$

We can also consider the "opposite" variety

$$\tilde{\mathfrak{g}}^- := G \times_{B^-} \mathfrak{b}^-,$$

and we will denote the corresponding morphisms by τ^- and ν^-. We can then write

$$E_\Delta = (G \times G) \times_{B \times B^-} (\mathfrak{b} \oplus_{\mathfrak{h}} \mathfrak{b}^-) = \tilde{\mathfrak{g}} \times_{\mathfrak{h}} \tilde{\mathfrak{g}}^-.$$

These objects fit into a double Grothendieck-Springer resolution [CG, sec. 3.1]:

$$
\begin{array}{ccccccc}
 & \tilde{\mathfrak{g}} & & & \tilde{\mathfrak{g}}^{-} & & \\
 {\scriptstyle\tau}\swarrow & & \searrow{\scriptstyle\nu} & {\scriptstyle\nu^{-}}\swarrow & & \searrow{\scriptstyle\tau^{-}} & \\
\mathfrak{g} & & & \mathfrak{h} & & & \mathfrak{g}\;, \\
 & \searrow{\scriptstyle q} & & \downarrow & & {\scriptstyle q}\swarrow & \\
 & & \mathfrak{g}/\!/G & \xleftarrow{\sim} \mathfrak{h}/W \xrightarrow{\sim} & \mathfrak{g}/\!/G & &
\end{array}
$$

Commutativity of the diagram implies that $\mu(E_\Delta) \subseteq \mathfrak{g} \times_{\mathfrak{g}/\!/G} \mathfrak{g}$. Since the restriction of τ to $\tau^{-1}(\mathfrak{g}^{\mathrm{rs}})$ is a principal W-bundle on $\mathfrak{g}^{\mathrm{rs}}$ [CG, prop. 3.1.36], and likewise for τ^{-}, it follows that any $X, Y \in \mathfrak{g}^{\mathrm{rs}}$ with $q(X) = q(Y)$ can be lifted to $\tilde{X} \in \tilde{\mathfrak{g}}$, $\tilde{Y} \in \tilde{\mathfrak{g}}^{-}$ with $\nu(\tilde{X}) = \nu^{-}(\tilde{Y})$. Properness of μ implies that actually $\mu(E_\Delta) = \mathfrak{g} \times_{\mathfrak{g}/\!/G} \mathfrak{g}$, so this can be done for *any* $X, Y \in \mathfrak{g}$ with $q(X) = q(Y)$.

Lemma 3.5.1. If

$$(X, Y) \in (\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^{-}) \cap (\mathfrak{g}^{\mathrm{rs}} \times \mathfrak{g}^{\mathrm{rs}})$$

for some $S \subseteq \Delta$, then there exists $(k_1, k_2) \in K_S$ such that

$$k_1 X k_1^{-1} = k_2 Y k_2^{-1} \in \mathfrak{h}^{\mathrm{rs}}.$$

Proof. This is entirely analogous to [CG, lem. 3.1.44]. Suppose $X \in \mathfrak{p}_S \cap \mathfrak{g}^{\mathrm{rs}}$. By the computations as in lemma 2.3.1, $uXu^{-1} \in X + \mathfrak{u}_S$ for any $u \in U_S$. It follows that the affine subspace $X + \mathfrak{u}_S \subset \mathfrak{g}$ is U_S-stable. Since X is regular semisimple, it cannot commute with any nonzero nilpotent. It follows that the linear map

$$[\,\cdot\,, X] : \mathfrak{u}_S \longrightarrow \mathfrak{u}_S$$

has trivial kernel, and is therefore surjective. But this map is the differential at the identity element of the orbit map

$$\begin{aligned} U_S &\longrightarrow X + \mathfrak{u}_S \\ u &\longmapsto uXu^{-1}, \end{aligned}$$

so the image of the latter map is open in $X + \mathfrak{u}_S$. On the other hand, orbits of unipotent groups on affine varieties are closed. It follows that the orbit map is an isomorphism, since $X + \mathfrak{u}_S$ is irreducible. In particular, $X + \mathfrak{u}_S$ is a single U_S-orbit. Likewise, if $Y \in \mathfrak{p}_S^- \cap \mathfrak{g}^{\mathrm{rs}}$, then $Y + \mathfrak{u}_S^-$ is a single U_S^--orbit.

Suppose now that

$$(X, Y) \in (\mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-) \cap (\mathfrak{g}^{\mathrm{rs}} \times \mathfrak{g}^{\mathrm{rs}}).$$

Write

$$X = X' + Z,$$
$$Y = Y' + Z,$$

where $X' \in \mathfrak{u}_S, Y' \in \mathfrak{u}_S^-, Z \in \mathfrak{l}_S$. There exist $u \in U_S, v \in U_S^-$ such that

$$uXu^{-1} = vYv^{-1} = Z.$$

In particular, $Z \in \mathfrak{l}_S \cap \mathfrak{g}^{\mathrm{rs}}$. Then there exists $g \in L_S$ such that $gZg^{-1} \in \mathfrak{h}^{\mathrm{rs}}$, and

$$(k_1, k_2) := (gu, gv) \in K_S$$

satisfies

$$k_1 X k_1^{-1} = k_2 Y k_2^{-1} \in \mathfrak{h}^{\mathrm{rs}}.$$

□

Proposition 3.5.3. The fibre of μ over any point of V^{rs} is isomorphic to $\overline{\mathbf{T}}$. In particular, it is connected.

Proof. Any element of $\mathfrak{g}^{\mathrm{rs}}$ is conjugate to an element of $\mathfrak{h}^{\mathrm{rs}}$, and μ is $(G \times G)$-equivariant. It thus suffices to check the fibre over (Z, Z) for some $Z \in \mathfrak{h}^{\mathrm{rs}}$.

Since $\mathbf{T}$ is the closure of T_{ad} in $\mathbf{X}_o$, the closure of $\mathbf{T}$ in $\mathbf{X}$ coincides with the closure of T_{ad} in $\mathbf{X}$. The latter is known [EJ, rmk. 4.5] to be

$$\overline{\mathbf{T}} = \bigcup_{wT \in W} (w, w)\mathbf{T},$$

as we have already noted in section 2.2.

Using the (first) projection $\pi : E \to \mathbf{X}$, we identify the sets

$$\mu^{-1}(Z, Z) \cong \{p \in \mathbf{X} : (Z, Z) \in E_p\}.$$

If $S \subseteq \Delta$, $g_1, g_2 \in G$, then

$$E_{(g_1,g_2)[z_S]} = (g_1, g_2)(\mathfrak{p}_S \otimes_{\mathfrak{l}_S} \mathfrak{p}_S^-) = (g_1, g_2)K_S(\mathfrak{p}_S \otimes_{\mathfrak{l}_S} \mathfrak{p}_S^-)$$

contains (Z, Z) if and only if there exists $(X, Y) \in \mathfrak{p}_S \oplus_{\mathfrak{l}_S} \mathfrak{p}_S^-$ such that

$$g_1 X g_1^{-1} = g_2 Y g_2^{-1} = Z.$$

It must be the case then that $X, Y \in \mathfrak{g}^{\text{rs}}$, since $Z \in \mathfrak{h}^{\text{rs}}$. By the preceeding lemma, there exists $(k_1, k_2) \in K_S$ such that

$$k_1 X k_1^{-1} = k_2 Y k_2^{-1} =: A \in \mathfrak{h}^{\text{rs}}.$$

Since K_S is the stabilizer of $[z_S]$, we may replace g_1 by $g_1 k_1^{-1}$ and g_2 by $g_2 k_2^{-1}$, so

$$g_1 A g_1^{-1} = g_2 A g_2^{-1} = Z.$$

Then $\mathfrak{h}$ and $g_1 \mathfrak{h} g_1^{-1}$ are Cartan subalgebras of $\mathfrak{g}$ that both contain $Z \in \mathfrak{h}^{\text{rs}}$, so $g_1 \mathfrak{h} g_1^{-1} = \mathfrak{h}$. Thus, $g_1 \in N_G(\mathfrak{h}) = N_G(T)$. Likewise, $g_2 \in N_G(T)$. On the other hand, $g_1 g_2^{-1} \in C_G(A) =$

T, since $A \in \mathfrak{h}^{\mathrm{rs}}$. Thus g_1 and g_2 represent the same element of $W = N_G(T)/T$. It follows that

$$\mu^{-1}(Z, Z) \cap \mathbf{O}_S = \bigcup_{wT \in W} (w, w)(T \times T)[z_S].$$

We conclude that $\mu^{-1}(Z, Z) = \overline{\mathbf{T}}$, since

$$\mathbf{T} = \bigcup_{S \subseteq \Delta} (T \times T)[z_S].$$

□

3.6 Global Logarithmic Differential Operators

We use the geometric properties of the moment map, established in the previous section, to show that

$$\gamma^{\mathcal{L}} : U(\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \Gamma(\mathbf{X}, \mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}})$$

is surjective for an invertible sheaf $\mathcal{L}$ on $\mathbf{X}$. Due to the following general fact about $\mathbb{N}$-filtered abelian groups, it suffices to prove that $\operatorname{gr} \gamma^{\mathcal{L}} = \operatorname{gr} \gamma$ is surjective.

Lemma 3.6.1. Suppose $\phi : A \to B$ is a homomorphism of $\mathbb{N}$-filtered abelian groups. If $\operatorname{gr} \phi$ is surjective, then so is ϕ.

Proof. $\mathrm{gr}_0 A = A_0$ surjects onto $\mathrm{gr}_0 B = B_0$. Suppose A_n surjects onto B_n for some $n \geq 0$. If $b \in B_{n+1}$, then there exists $\overline{a} \in \mathrm{gr}_{n+1} A$ such that $\overline{b} = \operatorname{gr} \phi(\overline{a}) = \overline{\phi(a)}$. Then $b - \phi(a) \in B_n$. By the inductive hypothesis, there exists some $a' \in A_n$ such that $\phi(a') = b - \phi(a)$. Then $b = \phi(a + a')$, so A_{n+1} surjects onto B_{n+1}. The result follows. □

By proposition 3.4.1, to show that $\operatorname{gr} \gamma$ is surjective, it suffices to show that

$$\mu^* : \mathbb{C}[(\mathfrak{g} \oplus \mathfrak{g})^*] \longrightarrow \mathbb{C}[T^*_{\mathbf{X},\mathbf{Y}}]$$

is surjective. Using the identifications established at the end of section 3.4, we look instead at

$$\mu^* : \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}] \longrightarrow \mathbb{C}[E].$$

Proposition 3.6.1. The comorphism

$$\mu^* : \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}] \longrightarrow \mathbb{C}[E]$$

is surjective, with kernel generated by $\{1 \otimes f_i - f_i \otimes 1 : 1 \leq i \leq r\}$.

Proof. By proposition 3.5.2, $V = \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is a reduced closed subvariety of $\mathfrak{g} \oplus \mathfrak{g}$ with defining ideal

$$I(V) := (1 \otimes f_i - f_i \otimes 1 : 1 \leq i \leq r)$$

in $\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}] = \mathbb{C}[\mathfrak{g}] \otimes \mathbb{C}[\mathfrak{g}]$. By proposition 3.5.1, the underlying topological space of V coincides with the image of μ. Since E is reduced too, it follows that V is the scheme-theoretic image of μ. Thus, there is a factorization

$$E \xrightarrow{\ \nu\ } V \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}, \qquad \mu : E \to \mathfrak{g} \oplus \mathfrak{g}$$

where the second map is the closed immersion induced by the quotient map

$$\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]/I(V).$$

It suffices to show that

$$\nu^* : \mathbb{C}[V] \longrightarrow \mathbb{C}[E]$$

is an isomorphism. Note that since μ is proper, so is ν [Har, cor. II.4.8].

Let $V' = \operatorname{Spec} \mathbb{C}[E]$. Then $\mathbb{C}[V'] = \mathbb{C}[E]$, and there is a factorization

$$E \xrightarrow{\ \nu'\ } V' \xrightarrow{\ p\ } V, \qquad \nu : E \to V.$$

Since ν is proper morphism, $\nu_* \mathcal{O}_E$ is a coherent sheaf of $\mathcal{O}_V$-modules ([GD, thm. 3.2.1], or [Har, thm. III.8.8], since ν is projective). In particular, $\Gamma(E, \mathcal{O}_E) = \mathbb{C}[E]$ is a finite $\mathbb{C}[V]$-module. It follows that p is a finite morphism. Also, ν' is a proper morphism, and its fibres are connected by Zariski's connectedness theorem [GD, thm. 4.3.1]. Now, the general fibres (specifically, the fibres over the regular semisimple points) of μ (hence also of ν) are connected by proposition 3.5.3. It follows that the general fibres of p are singletons, that is, p is birational. Since p is also a finite morphism onto the normal variety V, by Zariski's main theorem [Liu, cor. 4.6], it is an isomorphism. It follows that $p^* = \nu^*$ is an isomorphism of algebras. □

Remark 3.6.1. In the above argument, we have, in effect, constructed the Stein factorization of the morphism $\nu : E \to V$ [Har, cor. III.11.5]. The argument shows that

$$V = \operatorname{Spec} \mathbb{C}[E] = \mathbf{Spec}_V \nu_* \mathcal{O}_E.$$

A significant consequence of the proof is that *all* the fibres of ν (and hence of μ) are connected, not only the fibres over the regular semisimple locus. Another consequence (see proposition 3.4.1) is that the injection

$$\operatorname{gr} \Gamma(\mathbf{X}, \mathcal{D}_{\mathbf{X},\mathbf{Y}}) \hookrightarrow \Gamma(\mathbf{X}, \operatorname{gr} \mathcal{D}_{\mathbf{X},\mathbf{Y}})$$

must also be surjective, and therefore an isomorphism. ✓

Having proved that $\gamma^{\mathcal{L}}$ is surjective, we now describe its kernel. We denote by $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$. It is precisely subalgebra of G-invariants of $U(\mathfrak{g})$, as is well-known. By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g} \oplus \mathfrak{g}) = U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Both $U(\mathfrak{g})$ and $U(\mathfrak{g} \oplus \mathfrak{g})$ are filtered algebras, and the filtrations satisfy

$$U_n(\mathfrak{g} \oplus \mathfrak{g}) = \sum_{i+j=n} U_i(\mathfrak{g}) \otimes U_j(\mathfrak{g}).$$

Let

$$I := \sum_{z \in Z(\mathfrak{g})} U(\mathfrak{g} \oplus \mathfrak{g})(1 \otimes z - z \otimes 1),$$

and

$$I_n := I \cap U_n(\mathfrak{g} \oplus \mathfrak{g}).$$

Similarly, $S(\mathfrak{g} \oplus \mathfrak{g}) = S(\mathfrak{g}) \otimes S(\mathfrak{g})$, $S(\mathfrak{g})$ and $S(\mathfrak{g} \oplus \mathfrak{g})$ are graded algebras, and the gradings satisfy

$$S_n(\mathfrak{g} \oplus \mathfrak{g}) = \bigoplus_{i+j=n} S_i(\mathfrak{g}) \otimes S_j(\mathfrak{g}).$$

Let

$$J := \sum_{i=1}^{r} (1 \otimes f_i - f_i \otimes 1) S(\mathfrak{g} \oplus \mathfrak{g}),$$

and

$$J_n := J \cap S_n(\mathfrak{g} \oplus \mathfrak{g}).$$

Lemma 3.6.2. $I_{n+1}/I_n \cong J_{n+1}$ for all $n \in \mathbb{N}$.

Proof. By the Poincaré-Birkhoff-Witt theorem, the natural map

$$U_k(\mathfrak{g}) \longrightarrow S_k(\mathfrak{g})$$

is surjective for any $k \in \mathbb{N}$. It is also a homomorphism of finite-dimensional G-modules, which are completely reducible, because G is semisimple. It follows that the restricted map

$$U_k(\mathfrak{g})^G \longrightarrow S_k(\mathfrak{g})^G$$

on G-invariants is surjective too. Since $U(\mathfrak{g} \oplus \mathfrak{g}) = U(\mathfrak{g}) \boxtimes U(\mathfrak{g})$ as $(G \times G)$-modules, it follows that the natural map

$$I_{n+1} \longrightarrow J_{n+1}$$

is surjective as well, with kernel $I_{n+1} \cap U_n(\mathfrak{g} \oplus \mathfrak{g}) = I_n$. $\square$

Lemma 3.6.3. The kernel of $\gamma^{\mathcal{L}}$ contains I.

Proof. We continue to use the isomorphism $U(\mathfrak{g}\oplus\mathfrak{g}) = U(\mathfrak{g})\otimes U(\mathfrak{g})$. The open orbit in $\mathbf{X}$ is isomorphic to G_{ad}, so we can identify $\mathfrak{g} = \operatorname{Lie} G_{\text{ad}}$ in the usual way with left or right invariant derivations. For $X \in \mathfrak{g}$, let λ_X (resp. ρ_X) be the left-invariant (resp. right-invariant) vector field on G_{ad} corresponding to X. Then

$$\gamma|_{G_{\text{ad}}}(1\otimes X) = \lambda_X,$$
$$\gamma|_{G_{\text{ad}}}(X\otimes 1) = \rho_{-X}.$$

In particular, $\gamma(1\otimes U(\mathfrak{g}))$ is the set of left-invariant differential operators on G_{ad}, $\gamma(U(\mathfrak{g})\otimes 1)$ is the set of right-invariant differential operators, and $\gamma(1\otimes Z(\mathfrak{g})) = \gamma(Z(\mathfrak{g})\otimes 1)$ is the set of bi-invariant differential operators. For any $X_1,\ldots,X_k \in \mathfrak{g}$, the differential operators $\gamma|_{G_{\text{ad}}}(1\otimes X_1\ldots X_k)$ and $\gamma|_{G_{\text{ad}}}((-X_k)\ldots(-X_1)\otimes 1)$ coincide at $e \in G_{\text{ad}}$.

Suppose $z \in Z(\mathfrak{g})$. Choose basis vectors $X_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in R$ in such a way that $[X_\alpha, X_{-\alpha}] = H_\alpha := d\alpha^\vee(1)$. There exists an automorphism of $\mathfrak{g}$, extending uniquely to an automorphism of $U(\mathfrak{g})$, that fixes $Z(\mathfrak{g})$ and sends $X_\alpha \in \mathfrak{g}_\alpha$ to $-X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $X_{-\alpha}$ to $-X_\alpha$ for every $\alpha \in R_+$, and H to $-H$ for all $H \in \mathfrak{h}$ [Hum, prop. 14.3, lem. 23.2]. In particular, z is fixed by this involution. Moreover, there is a canonical G-module isomorphism $S(\mathfrak{g}) \to U(\mathfrak{g})$ given by symmetrization. Thus z can be identified with a sum of symmetrized monomials in $U(\mathfrak{g})$. Combining the two results, it follows that the differential operators $\gamma|_{G_{\text{ad}}}(1\otimes z)$ and $\gamma|_{G_{\text{ad}}}(z\otimes 1)$ coincide at e. Since both are bi-invariant, they are equal on G_{ad}.

Since $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ is locally free, it has no nonzero sections supported on $\mathbf{Y}$. Thus, since $\gamma(1\otimes z - z\otimes 1)$ is the zero differential operator on G_{ad}, it is in fact zero on $\mathbf{X}$. This proves that the kernel of γ contains I. As was discussed at the beginning of section 3.4, the local description of twisted differential operators in the image of $\gamma^{\mathcal{L}}$ with respect to an equivariant trivialization of $\mathcal{L}$ over G_{ad} is the same as for γ. By the same reasoning as for γ, it follows that $\gamma^{\mathcal{L}}(1\otimes z - z\otimes 1) = 0$, whence the kernel of $\gamma^{\mathcal{L}}$ contains I. $\square$

Remark 3.6.2. The above lemma admits a different proof. Since G_{ad} is a smooth affine variety, it is also *D-affine* [HTT, def. 1.4.2, prop. 1.4.3]. In particular,

$$\mathcal{O}_{G_{\text{ad}}} \cong \mathcal{D}_{G_{\text{ad}}} \otimes_{\Gamma(G_{\text{ad}}, \mathcal{D}_{G_{\text{ad}}})} \mathbb{C}[G_{\text{ad}}]$$

as $\mathcal{D}_{G_{\text{ad}}}$-modules. Thus, $\gamma(1 \otimes z - z \otimes 1)$ is determined by the action of $1 \otimes z - z \otimes 1$ on $\mathbb{C}[G_{\text{ad}}]$. Since $G_{\text{ad}} \cong (G \times G)/K$, a standard Frobenius reciprocity argument shows that

$$\operatorname{Hom}(W^*, \mathbb{C}[G_{\text{ad}}]) = W^K$$

for every irreducible $(G \times G)$-module W, where $W^K \subseteq W$ is the space of K-invariants. It follows that

$$\mathbb{C}[G_{\text{ad}}] = \bigoplus_{\mu} V_\mu^* \otimes V_\mu,$$

where μ runs over all dominant weights in the root lattice of T. Since z acts as multiplication by $\chi_{\mu+\rho}(z)$ on both V_μ and V_μ^*, it follows, as before, that $\gamma(1 \otimes z - z \otimes 1) = 0$.

Proposition 3.6.2. The kernel of $\gamma^{\mathcal{L}}$ is I.

Proof. It suffices to prove that

$$0 \longrightarrow I_n \longrightarrow U_n(\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \Gamma(\mathbf{X}, F_n\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) \longrightarrow 0 \quad (*)$$

is an exact sequence for all $n \in \mathbb{N}$. The proof is by induction on n.

Note that $U_0(\mathfrak{g} \oplus \mathfrak{g}) = \mathbb{C}$, by construction of $U(\mathfrak{g} \oplus \mathfrak{g})$. Since $\mathbf{X}$ is projective,

$$\Gamma(\mathbf{X}, F_0\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) = \Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = \mathbb{C}.$$

Clearly $I_0 = 0$. It follows that for $n = 0$, the sequence $(*)$ is

$$0 \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}} \mathbb{C} \longrightarrow 0,$$

which is certainly exact.

Suppose now that $(*)$ is exact for some $n \in \mathbb{N}$. We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_n & \longrightarrow & U_n(\mathfrak{g} \oplus \mathfrak{g}) & \longrightarrow & \Gamma(\mathbf{X}, F_n\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_{n+1} & \longrightarrow & U_{n+1}(\mathfrak{g} \oplus \mathfrak{g}) & \longrightarrow & \Gamma(\mathbf{X}, F_{n+1}\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_{n+1} & \longrightarrow & S_{n+1}(\mathfrak{g} \oplus \mathfrak{g}) & \longrightarrow & \Gamma(\mathbf{X}, \mathrm{gr}_{n+1}\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

All the columns are exact: the left column by lemma 3.6.2, the middle column by the Poincaré-Birkhoff-Witt theorem, and the right column by the second half of remark 3.6.1. The top row is exact by the inductive hypothesis, and the bottom row is exact by proposition 3.6.1. We also know that the middle row is a complex, by lemma 3.6.2. Note that the map

$$I_{n+1} \longrightarrow U_{n+1}(\mathfrak{g} \oplus \mathfrak{g})$$

is of course injective, as it is just an inclusion.

We prove that the rightmost arrow of the middle row, the restricted map

$$\gamma^{\mathcal{L}}_{n+1} : U_{n+1}(\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \Gamma(\mathbf{X}, F_{n+1}\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}),$$

is surjective. Suppose $\xi \in \Gamma(\mathbf{X}, F_{n+1}\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}})$. By the exactness of the bottom row, its image

$$\xi + \Gamma(\mathbf{X}, F_n\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) \in \Gamma(\mathbf{X}, \mathrm{gr}_{n+1}\mathcal{D}^{\mathcal{L}}_{\mathbf{X},\mathbf{Y}}) = \mathbb{C}_{n+1}[T^*_{\mathbf{X},\mathbf{Y}}]$$

is the pullback under μ of some $f \in S_{n+1}(\mathfrak{g} \oplus \mathfrak{g})$, which in turn lifts to some $X \in U_{n+1}(\mathfrak{g} \oplus \mathfrak{g})$, by exactness of the middle column. Then

$$\gamma_{n+1}^{\mathcal{L}}(X) - \xi \in \Gamma(\mathbf{X}, F_n \mathcal{D}_{\mathbf{X},\mathbf{Y}}^{\mathcal{L}})$$

by the commutativity of the bottom right square and the exactness of the right column. It follows by exactness of the top row that

$$\gamma_{n+1}^{\mathcal{L}}(X) - \xi = \gamma_n^{\mathcal{L}}(Y)$$

for some $Y \in U_n(\mathfrak{g} \oplus \mathfrak{g})$. Thus $\xi = \gamma_{n+1}^{\mathcal{L}}(X+Y)$, and $\gamma_{n+1}^{\mathcal{L}}$ is indeed surjective.

It remains to show that if $X \in \ker \gamma_{n+1}^{\mathcal{L}}$, then $X \in I_{n+1}$. Let f denote the image of X in $S_{n+1}(\mathfrak{g} \oplus \mathfrak{g})$. By the commutativity of the bottom right square, $f \in J_{n+1}$, and by exactness of the left column, f lifts to some $Y \in I_{n+1}$. Then $X - Y \in U_n(\mathfrak{g} \oplus \mathfrak{g})$ by exactness of the middle column, and

$$\gamma_n^{\mathcal{L}}(X-Y) = \gamma_{n+1}^{\mathcal{L}}(X) - \gamma_{n+1}^{\mathcal{L}}(Y) = 0$$

by lemma 3.6.3. It follows by exactness of the top row that $X - Y \in I_n$, so

$$X = (X-Y) + Y \in I_{n+1}.$$

Thus, the middle row, which is $(*)$ for $n+1$, is exact, completing the induction. □

Note that, by definition, $(U(\mathfrak{g}) \otimes U(\mathfrak{g}))/I = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g})$. Combining the results of lemma 3.6.1 and propositions 3.6.1 and 3.6.2, we have

Theorem 3.6.1. $\gamma^{\mathcal{L}}$ descends to an isomorphism $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \cong \Gamma(\mathbf{X}, \mathcal{D}_{\mathbf{X},\mathbf{Y}}^{\mathcal{L}})$.

Example 3.6.1. Suppose $\lambda \in \mathfrak{X}(T)$ lies in the *root* lattice. Then, by the description of $\operatorname{Pic}\mathbf{X}$ in section 2.4 and theorem 2.4.2, there exist $n_1, \ldots, n_r \in \mathbb{Z}$ and an isomorphism

$$\mathcal{L}_{-\lambda} \cong \mathcal{I}_{\mathbf{Y}_1}^{\otimes n_1} \otimes \ldots \otimes \mathcal{I}_{\mathbf{Y}_r}^{\otimes n_r}.$$

of $(G \times G)$-equivariant sheaves. By definition of $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$, each ideal sheaf $\mathcal{I}_{\mathbf{Y}_i}$ is naturally a $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-module. Thus, $\mathcal{L}_{-\lambda}$ is also a $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-module. Let

$$S_\lambda := \{\mu \in \mathfrak{X}(T) : \mu \prec \lambda \text{ and } \mu \text{ is dominant}\}.$$

Since $\mathcal{L}_{-\lambda}$ is $(G \times G)$-equivariant, its global sections have a natural $(G \times G)$-module structure. The corresponding $(\mathfrak{g} \oplus \mathfrak{g})$-module structure coincides with the $\Gamma(\mathbf{X}, \mathcal{D}_{\mathbf{X},\mathbf{Y}})$-module structure via γ. By [DP, sec. 8.2],

$$\Gamma(\mathbf{X}, \mathcal{L}_{-\lambda}) = \bigoplus_{\mu \in S_\lambda} V_\mu^* \otimes V_\mu.$$

Any $z \in Z(\mathfrak{g})$ acts on V_μ and V_μ^* as multiplication by $\chi_{\mu+\rho}(z)$, whence $1 \otimes z - z \otimes 1$ acts on $\Gamma(\mathbf{X}, \mathcal{L}_{-\lambda})$ by zero. Note that, in contrast to the Borel-Weil theory of line bundles on the flag variety, $\Gamma(\mathbf{X}, \mathcal{L}_{-\lambda})$ is not an irreducible $(G \times G)$-module, and $1 \otimes z$, $z \otimes 1$ do not act by a single central character. ✓

Example 3.6.2. Let $H_1, \ldots, H_r$ be the coroot basis for $\mathfrak{h}$ from example 3.3.1. From the Poincaré-Birkhoff-Witt theorem, we have a direct sum decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{u}\, U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{u}^-).$$

Let

$$\pi : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$$

be the projection with respect to this direct sum decomposition. Since $\mathfrak{h}$ is abelian, there is a canonical algebra isomorphism $U(\mathfrak{h}) = S(\mathfrak{h})$. Suppose $z \in Z(\mathfrak{g})$, and $\pi(z) = p(H_1, \ldots, H_r)$. Recall from section 3.3 that any $H \in \mathfrak{h}$ corresponds to a section $s_H \in \Gamma(\mathbf{O}_\Delta, i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}})$. We claim that the image of $\gamma(z \otimes 1)$ in $\Gamma(\mathbf{O}_\Delta, i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}})$ is $p(s_{H_1}, \ldots, s_{H_r})$. It suffices to check the equality on

$$\mathbf{O}_\Delta \cap \mathbf{X}_o \cong U^- \times [z_\Delta] \times U.$$

For every $w \in U^-, H \in \mathfrak{h}$, define a section

$$s_{w,H}(u[z]v^{-1}) = \gamma(u[z]v^{-1}, w(H,0)w^{-1})$$

in $\mathcal{D}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$. We also define for every $H \in \mathfrak{h}$ a local lift

$$\tilde{s}_H(u[z]v^{-1}) = \gamma(u[z]v^{-1}, u(H,0)u^{-1})$$

of s_H to a section in $\mathcal{D}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$.

First, we show that

$$\gamma(z \otimes 1) - p(s_{w,H_1}, \ldots, s_{w,H_r}) \in \mathfrak{m}_{w[z_\Delta]v^{-1}}\mathcal{D}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$$

for all $w \in U^-$, $v \in U$. Note that $wzw^{-1} = z$, since $z \in Z(\mathfrak{g}) = U(\mathfrak{g})^G$, so it suffices to check this when $w = e$. The claim is true because

$$z - \pi(z) \in \mathfrak{u}\, U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{u}^-$$

[HTT, lem. 9.4.4], and if $E \in \mathfrak{u}$, then $(E,0) \in \mathfrak{b} \oplus_{\mathfrak{h}} \mathfrak{b}^- = E_{[z_\Delta]}$, whence

$$\gamma(E,0) \in \mathfrak{m}_{[z_\Delta]v^{-1}}\mathcal{D}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o).$$

Now, note that

$$s_{w,H}(w[z]v^{-1}) = \tilde{s}_H(w[z]v^{-1})$$

for all $w \in U^-, [z] \in \mathbf{T}, v \in U$. It follows that

$$p(\tilde{s}_{H_1}, \ldots, \tilde{s}_{H_r}) - p(s_{w,H_1}, \ldots, s_{w,H_r}) \in \mathfrak{m}_{w[z_\Delta]v^{-1}}\mathcal{D}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o),$$

because $(H,0)$ preserves $\mathfrak{m}_{[z_\Delta]}$ for any $H \in \mathfrak{h}$.

We conclude that

$$\gamma(z \otimes 1) - p(\tilde{s}_{w,H_1}, \ldots, \tilde{s}_{w,H_r}) \in \mathfrak{m}_{w[z_\Delta]v^{-1}} \mathcal{D}_{\mathbf{X},\mathbf{Y}}(\mathbf{X}_o)$$

for all $w \in U^-$, $v \in U$. Since $\mathcal{D}_{\mathbf{X},\mathbf{Y}}$ is a locally free left $\mathcal{O}_{\mathbf{X}}$-module, and

$$\mathcal{I}_{\mathbf{O}_\Delta}(\mathbf{X}_o) = \bigcap_{w \in U^-, v \in V} \mathfrak{m}_{w[z_\Delta]v^{-1}},$$

our original claim follows.

Applying to $U(\mathfrak{g})$ the automorphism sending $X_\alpha \in \mathfrak{g}_\alpha$ to $-X_{-\alpha}$ for $\alpha \in R$, $H \in \mathfrak{h}$ to $-H$, we have another direct sum decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{u}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{u}).$$

Let

$$\pi' : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$$

be the projection with respect to this new direct sum decomposition. Suppose $\pi'(z) = p'(H_1, \ldots, H_r)$. Then

$$p(H_1, \ldots, H_r) = p'(-H_1, \ldots, -H_r),$$

and a similar argument shows that the image of $\gamma(1 \otimes z)$ in $\Gamma(\mathbf{O}_\Delta, i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}})$ is $p(s_{H_1}, \ldots, s_{H_r})$.

Let $\mathcal{M}$ be an $i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-module such that $\mathrm{eu}_1, \ldots, \mathrm{eu}_r$ act locally finitely. If $e_1, \ldots, e_k$ are their respective generalized eigenvalues on some direct summand of $\mathcal{M}$, then we have $\epsilon \in \mathfrak{h}^*$ such that $\epsilon(\varpi_i) = e_i$ for each $1 \leq i \leq r$. It follows that any $z \in Z(\mathfrak{g})$ acts locally finitely, with generalized eigenvalue $\chi_{\lambda-\rho}(z)$ on this direct summand, where $\lambda = e_1\alpha_1 + \ldots e_r\alpha_r$.

This is somewhat reminiscent of the Beilinson-Bernstein theory for twisted D-modules on flag varieties, and that is not a coincidence. The invertible sheaf $i^*\mathcal{L}_\lambda = \mathcal{L}_{\lambda,-\lambda}$ is both an $i^*\mathcal{D}_{\mathbf{X},\mathbf{Y}}$-module and a $\mathcal{D}_{\mathbf{O}_\Delta}^{\mathcal{L}_{\lambda,-\lambda}}$-module. On global sections, both module structures come from the action of $U(\mathfrak{g} \oplus \mathfrak{g})$, so they coincide. And the Beilinson-Bernstein theorem predicts that $z \in Z(\mathfrak{g})$ will act locally finitely, with generalized eigenvalue $\chi_{\lambda-\rho}(z)$. ✓

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