THE UNIVERSITY OF CHICAGO

STABLE SOLITON RESOLUTION FOR WAVE MAPS ON A CURVED SPACETIME

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To Cinthia and our epsilons.

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ABSTRACT

In this thesis we study finite energy equivariant wave maps posed on the (1+3)–dimensional spherically symmetric static spacetime $\mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2) \to \mathbb{S}^3$ where the metric on $\mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2)$ is given by

$$ds^2 = -dt^2 + dr^2 + (r^2 + 1)\big(d\theta^2 + \sin^2\theta d\varphi^2\big), \quad t, r \in \mathbb{R}, (\theta, \varphi) \in \mathbb{S}^2.$$

The metric is asymptotically flat with two ends at $r = \pm\infty$ which are connected by a spherical "throat" of area $4\pi^2$ at $r = 0$. The above spacetime is often cited as a simple example of a wormhole geometry in general relativity but is not expected to exist in nature due to the negative energy density required to obtain it.

We consider equivariant wave maps from the previously described spacetime into the 3–sphere, $\mathbb{S}^3$. Each equivariant wave map can be indexed by its equivariance class $\ell \in \mathbb{N}$ and topological degree $n \in \mathbb{N} \cup \{0\}$. For each ℓ and n, we prove that there exists a unique energy minimizing ℓ–equivariant harmonic map $Q_{\ell,n} : \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2) \to \mathbb{S}^3$ of degree n. Based on mixed numerical and analytic evidence, Bizon and Kahl [3] conjectured that all equivariant wave maps settle down to the harmonic map in the same equivariance and degree class by radiating off excess energy. In this thesis, we prove this conjecture rigorously and establish stable soliton resolution for this model; first for $\ell = 1$ (*corotational* maps) in Chapter 2, and then for general $\ell > 1$ in Chapter 3. More precisely, we show that modulo a free radiation term, every ℓ–equivariant wave map of degree n converges strongly to $Q_{\ell,n}$.

CHAPTER 1

INTRODUCTION

1.1 Background

For several decades there has been a great interest in the study of geometric nonlinear wave equations. One of the fundamental models considered is the *wave map* model. Let (M, g) be a $(1{+}d)$–dimensional Lorentzian spacetime, and let (N, h) be a Riemannian manifold. A wave map $U : M \to N$ is a formal critical point of the action functional

$$\mathcal{S}(U, \partial U) = \frac{1}{2} \int_M g^{\mu\nu} \langle \partial_\mu U, \partial_\nu U \rangle_h dg. \tag{1.1.1}$$

In local coordinates, the Euler–Lagrange equations associated to $\mathcal{S}$ is the following system of semilinear wave equations

$$\Box_g U^i + \Gamma^i_{jk}(U) \partial_\mu U^j \partial_\nu U^k g^{\mu\nu} = 0, \tag{1.1.2}$$

where $\Box_g := \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu)$ is the D'Alembertian associated to the background spacetime (M, g) and the Γ^i_{jk} are the Christoffel symbols associated to the target (N, h). The system is collectively referred to as the *wave map system* and is also known in the physics literature as the classical nonlinear σ–model. A particular case that has been intensely studied is the case when M is $(1{+}d)$–dimensional Minkowski space $\mathbb{R}^{1+d}$ with the flat metric $g = \mathrm{diag}(-1, 1, \ldots, 1)$ (see the classical reference [29] and the recent review [26]). From a physical point of view, wave maps $U : \mathbb{R}^{1+3} \to \mathbb{S}^3$ arose as a way to describe fields which approximate a low energy regime of QCD (see [9] and [10] for nice introductions to this perspective). From a mathematical point of view, a wave map $U : \mathbb{R}^{1+d} \to N$ can be considered as a geometric generalization of the free wave equation on Minkowski space. Indeed, if we take $N = \mathbb{R}$ then the wave map equations (1.1.2) reduce to the well known free wave equation

$$\partial_t^2 U - \Delta U = 0, \quad (t, x) \in \mathbb{R}^{1+d}.$$

It is known that for reasonable target manifolds N, wave maps $U : \mathbb{R}^{1+d} \to N$ evolving from small localized initial data (within a certain smoothness space) are global and behave, in a sense, like solutions to the free wave equation on Minkowski space. Recently, researchers have turned to the problem of describing the long time dynamics of generic large data solutions. The guiding principle is the so called *soliton resolution conjecture*. This belief asserts that for most nonlinear dispersive PDE, generic globally defined solutions asymptotically decouple into a superposition of nonlinear bulk terms (traveling waves, rescaled solitons, etc.) and radiation (a solution to the underlying linear equation). However, when trying to verify this conjecture for wave maps on Minkowski space, complications arise due to the scaling symmetry:

$$U(t, x) : \mathbb{R}^{1+d} \to N \text{ solves (1.1.2)} \implies U_\lambda(t, x) = U(\lambda t, \lambda x) \text{ solves (1.1.2)}.$$

Due to this scaling symmetry, the long–time dynamics of large data wave maps on $\mathbb{R}^{1+d}$ can be very complex and one can have (depending on the geometry of the target and dimension) self–similar solutions, finite time break down via energy concentration, dynamic 'towers' of solitons, and other interesting scenarios (which, however, are often unstable). Therefore, to gain better insight on the role of the soliton resolution conjecture it is instructive to consider models when the background metric does not admit such a scaling symmetry or when stable solitons are present. Moreover, the study of wave maps on a curved background (rather than $\mathbb{R}^{1+d}$) is still relatively unexplored. These reasons motivated the following model introduced by Bizon and Kahl [3] which we consider in this thesis.

In this work, we study wave maps posed on a *curved* background spacetime. In particular, we consider wave maps $U : \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2) \to \mathbb{S}^3$ where the background metric is given by

$$ds^2 = -dt^2 + dr^2 + (r^2+1)\big(d\theta^2 + \sin^2\theta d\varphi^2\big), \quad t, r \in \mathbb{R}, (\theta, \varphi) \in \mathbb{S}^2. \tag{1.1.3}$$

The time slices correspond to the Riemannian manifold $\mathcal{M} := \mathbb{R} \times \mathbb{S}^2$ with metric

$$ds^2 = dr^2 + (r^2+1)\big(d\theta^2 + \sin^2\theta d\varphi^2\big), \quad r \in \mathbb{R}, (\theta, \varphi) \in \mathbb{S}^2. \tag{1.1.4}$$

Heuristically, $\mathcal{M}$ has two asymptotically Euclidean ends at $r = \pm\infty$ connected by a spherical throat at $r = 0$. Because of this, the above spacetime has appeared as a prototype 'wormhole' geometry in the general relativity literature since its introduction by Ellis in the 1970's and popularization by Morris and Thorne in the 1980's (see [22], [8] and the references therein). Due to the rotational symmetry of the background and target, it is natural to consider a subclass of wave maps $U : \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2) \to \mathbb{S}^3$ such that

$$\exists \ell \in \mathbb{N}, \quad U \circ \rho = \rho^\ell \circ U, \quad \forall \rho \in SO(3). \tag{1.1.5}$$

Here, the rotation group $SO(3)$ acts on the background and target in the natural way. The integer ℓ is commonly referred to as the *equivariance class* and can be thought of as parametrizing a fixed amount of angular momentum for the wave map. If we fix spherical coordinates (ψ, ϑ, ϕ) on $\mathbb{S}^3$, then from (1.1.5) it can be shown that U is completely determined by the associated azimuth angle $\psi = \psi(t, r)$ and the wave map equation (1.1.2) reduces to the single scalar semilinear wave equation for ψ:

$$\begin{aligned} &\partial_t^2\psi - \partial_r^2\psi - \frac{2r}{r^2+1}\partial_r\psi + \frac{\ell(\ell+1)}{2(r^2+1)}\sin 2\psi = 0, \quad (t, r) \in \mathbb{R} \times \mathbb{R}, \\ &\vec{\psi}(0) = (\psi_0, \psi_1). \end{aligned} \tag{1.1.6}$$

This is most easily seen in the 1–equivariant class (also known as the *corotational* class) in which we must have $U(t,r,\theta,\varphi) = (\psi(t,r),\theta,\varphi)$ where, as before, (θ,φ) are spherical coordinates on $\mathbb{S}^2$. Throughout this work we use the notation $\vec{\psi}(t) = (\psi(t,\cdot),\partial_t\psi(t,\cdot))$.

Solutions ψ to (1.1.6) will be referred to as ℓ–equivariant *wave maps on a wormhole.* The equation (1.1.6) has the following conserved energy along the flow:

$$\begin{aligned}\mathcal{E}_\ell(\vec{\psi}(t)) &:= \frac{1}{2}\int_{\mathbb{R}} \left[|\partial_t\psi(t,r)|^2 + |\partial_r\psi(t,r)|^2 + \frac{\ell(\ell+1)}{r^2+1}\sin^2\psi(t,r)\right](r^2+1)dr \\ &= \mathcal{E}_\ell(\vec{\psi}(0)).\end{aligned}$$

In order for the initial data to have finite energy, we must have for some $m,n\in\mathbb{Z}$,

$$\psi_0(-\infty) = m\pi \quad \text{and} \quad \psi_0(\infty) = n\pi.$$

For a finite energy solution $\vec{\psi}(t)$ to (1.1.6) to depend continuously on t, we must have that $\psi(t,-\infty) = m\pi$ and $\psi(t,\infty) = n\pi$ for all t. Due to the symmetries $\psi \mapsto m\pi + \psi$ and $\psi \mapsto -\psi$ of (1.1.6), we will, without loss of generality, fix $m = 0$ and assume $n \in \mathbb{N}\cup\{0\}$. Thus, we only consider wave maps which send the left Euclidean end at $r = -\infty$ to the north pole of $\mathbb{S}^3$. The integer n is referred to as the topological degree of the map ψ and, heuristically, represents the minimal number of times $\mathcal{M}$ gets wrapped around $\mathbb{S}^3$ by $\psi(t,\cdot)$. For each $n\in\mathbb{N}\cup\{0\}$, we denote the set of finite energy pairs of degree n by

$$\mathcal{E}_{\ell,n} := \big\{(\psi_0,\psi_1) : \mathcal{E}_\ell(\psi_0,\psi_1) < \infty, \quad \psi_0(-\infty) = 0, \quad \psi_0(\infty) = n\pi\big\}.$$

There are features of the wave maps on a wormhole equation that make it an attractive model in which to study the soliton resolution conjecture. The first feature is that we have global well–posedness for arbitrary solutions to (1.1.6) trivially. The geometry of the wormhole breaks the scaling invariance that the equation has in the flat case and removes the singularity at the origin. By a simple contraction mapping argument, conservation of energy and time stepping we easily deduce that every solution to (1.1.6) is globally defined in time (see Section 2.5 and Section 3.3 for more details). Another feature of (1.1.6) is that there is an abundance of static solutions to (1.1.6). Such solutions are more commonly referred to as *harmonic maps*. More precisely, it can be shown that for every $\ell \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$ there exists a unique solution $Q_{\ell,n} \in \mathcal{E}_{\ell,n}$ to

$$\begin{aligned} &Q'' + \frac{2r}{r^2+1}Q' - \frac{\ell(\ell+1)}{2(r^2+1)}\sin 2Q = 0, \quad r \in \mathbb{R}, \\ &Q(-\infty) = 0, \quad Q(\infty) = n\pi. \end{aligned} \tag{1.1.7}$$

See Section 2.2 and Section 3.2 for more details.

In [3] Bizon and Kahl gave mixed numerical and analytic evidence for the following formulation of the soliton resolution conjecture for this model: for every $\ell \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, and for any $(\psi_0, \psi_1) \in \mathcal{E}_{\ell,n}$ there exist a unique global solution ψ to the initial value problem (3.1.1) and solutions $\varphi_L^{\pm}$ to the linearized equation

$$\partial_t^2 \varphi - \partial_r^2 \varphi - \frac{2r}{r^2+1}\partial_r \varphi + \frac{\ell(\ell+1)}{r^2+1}\varphi = 0,$$

such that

$$\vec{\psi}(t) = (Q_{\ell,n}, 0) + \vec{\varphi}_L^{\pm}(t) + o(1),$$

as $t \to \pm\infty$. In this work, we verify this conjecture for all equivariance classes: first for the corotational class of solutions ($\ell = 1$) in Chapter 2 and for all higher equivariance classes $\ell > 1$ in Chapter 3.

We note here that a model with features similar to wave maps on a wormhole was previously studied in [20], [16] and [18] which served as further motivation and as a road map for the work carried out here. In these works, the authors studied ℓ–equivariant wave maps $U : \mathbb{R} \times (\mathbb{R}\backslash B(0,1)) \to \mathbb{S}^3$. In their work, an ℓ–equivariant wave map U is determined by the associated azimuth angle on the 3–sphere $\psi(t,r)$ which satisfies the equation

$$\begin{aligned} &\partial_t^2\psi - \partial_r^2\psi - \frac{2}{r}\partial_r\psi + \frac{\ell(\ell+1)}{2(r^2+1)}\sin 2\psi = 0, \quad t \in \mathbb{R}, r \geq 1, \\ &\psi(t,1) = 0, \quad \psi(t,\infty) = n\pi, \quad \forall t. \end{aligned} \tag{1.1.8}$$

Such wave maps were called ℓ–equivariant *exterior wave maps*. Similar to wave maps on a wormhole, global well–posedness and an abundance of harmonic maps hold for the exterior wave map equation (1.1.8). In the works [20], [16], and [18], the authors proved the soliton resolution conjecture for ℓ–equivariant exterior wave maps for arbitrary $\ell \geq 1$. We point out that the geometry of the background $\mathbb{R} \times (\mathbb{R}\backslash B(0,1))$ is still flat and could be considered as artificially removing the scaling symmetry present when the domain is $\mathbb{R}^{1+3}$. On the other hand, the curved geometry of the background considered in this work is what removes scaling invariance. This makes wave maps on a wormhole more geometric in nature while still retaining the properties that make them attractive for studying the soliton resolution conjecture. Due to the asymptotically Euclidean nature of the wormhole geometry, we are able to use techniques developed for the flat case for this curved geometry. However, the curved nature of $\mathcal{M}$ presents several obstacles in the proof.

1.2 Main Result

We now state our main result. In what follows we use the following notation. If $r_0 \geq -\infty$ and $w(r)$ is a positive continuous function on $[r_0, \infty)$, then we denote

$$\|(\psi_0,\psi_1)\|^2_{\mathcal{H}([r_0,\infty);w(r)dr)} := \int_{r_0}^{\infty} \left[|\psi_0(r)|^2 + |\psi_1(r)|^2 dr\right] w(r)dr.$$

The Hilbert space $\mathcal{H}([r_0,\infty); w(r)dr)$ is then defined to be the completion of the vector space of $C_0^\infty(r_0,\infty)$ pairs with respect to the norm $\|\cdot\|_{\mathcal{H}([r_0,\infty);w(r)dr)}$. Let $\ell \in \mathbb{N}$ be a fixed equivariance class, and let $n \in \mathbb{N} \cup \{0\}$ be a fixed topological degree. In the $n = 0$ case, the natural space to place the solution $\vec{\psi}(t)$ to (3.1.1) in is the *energy space* $\mathcal{H}_0 := \mathcal{H}((-\infty, \infty); (r^2 + 1)dr)$. Indeed, it is easy to show that $\|\vec{\psi}\|_{\mathcal{E}_{\ell,0}} \simeq \|\vec{\psi}\|_{\mathcal{H}_0}$. For $n \geq 1$, we measure distance relative to $(Q_{\ell,n}, 0)$ and define $\mathcal{H}_{\ell,n} := \mathcal{E}_{\ell,n} - (Q_{\ell,n}, 0)$ with 'norm'

$$\|\vec{\psi}\|_{\mathcal{H}_{\ell,n}} := \|\vec{\psi} - (Q_{\ell,n}, 0)\|_{\mathcal{H}_0}.$$

Note that $\psi(r) - Q_{\ell,n}(r) \to 0$ as $r \to \pm\infty$. The main result of this thesis is the following.

Theorem 1.2.1. *Let $\ell \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}$. For all $(\psi_0, \psi_1) \in \mathcal{E}_{\ell,n}$, there exists a unique global solution $\vec{\psi}(t) \in C(\mathbb{R}; \mathcal{H}_{\ell,n})$ to (3.1.1) which scatters forwards and backwards in time to the harmonic map $(Q_{\ell,n}, 0)$, i.e. there exist solutions $\varphi_L^{\pm}$ to the linearized equation*

$$\partial_t^2 \varphi - \partial_r^2 \varphi - \frac{2r}{r^2 + 1} \partial_r \varphi + \frac{\ell(\ell + 1)}{r^2 + 1} \varphi = 0,$$

such that

$$\vec{\psi}(t) = (Q_{\ell,n}, 0) + \vec{\varphi}_L^{\pm}(t) + o_{\mathcal{H}_0}(1),$$

as $t \to \pm\infty$.

We remark here that, to the author's knowledge, Theorem 1.2.1 is the first result that establishes the soliton resolution conjecture for arbitrary corotational finite energy wave maps on a curved background. See [19] for soliton resolution for corotational wave maps from $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$ with a restriction on the behavior at infinity.

1.3 Outline

We now give a brief and broad outline of the proof of Theorem 1.2.1. To prove Theorem 1.2.1, we use the celebrated concentration–compactness/rigidity theorem method pioneered by Kenig and Merle in [14] and [15]. The proof is by contradiction and is split into three main steps. In the first step, we establish a small data theory for (1.1.6), i.e. if $\|\vec{\psi}(0)-(Q_{\ell,n},0)\|_{\mathcal{H}_0}$ is sufficiently small, then the solution ψ to (1.1.6) is global and scatters to $(Q_{\ell,n},0)$. In the second step, using concentration–compactness arguments and the first step we show that if Theorem 1.2.1 *fails*, then there exists a solution $\vec{\psi}_* \neq (Q_{\ell,n},0)$ to (1.1.6) such that the trajectory $\{\vec{\psi}_*(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{H}_{\ell,n}$. In the third and final step, we establish the following rigidity theorem: if ψ is a solution to (1.1.6) such that $\{\vec{\psi}(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{H}_{\ell,n}$, then $\vec{\psi} = (Q_{\ell,n},0)$. This rigidity theorem contradicts the second step which implies that Theorem 1.2.1 must hold.

An outline of this thesis is as follows. The proof of Theorem 1.2.1 using the Kenig–Merle method in the corotatonal case ($\ell = 1$) is contained in Chapter 2, and the case $\ell > 1$ is treated in Chapter 3. In Section 2.1 and Section 3.1, more detailed outlines of the individual chapters and of how we establish the steps in the Kenig–Merle method can be found. Both chapters can be read independent of each other (although Chapter 3 uses facts proved in Chapter 2) and are largely based on the two works [24] ($\ell = 1$) and [25] ($\ell > 1$) where Theorem 1.2.1 was first established.

CHAPTER 2

SOLITON RESOLUTION FOR COROTATIONAL WAVE MAPS ON A WORMHOLE

2.1 Main Result and Outline of the Chapter

In this chapter, we classify the long time dynamics of all finite energy corotational (1–equivariant) wave maps on a wormhole. We recall from the discussion in Chapter 1 that a corotational wave map $U : \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2) \to \mathbb{S}^3$ is given by the ansatz

$$U(t, r, \theta, \varphi) = (\psi(t, r), \theta, \varphi) \in \mathbb{S}^3, \tag{2.1.1}$$

where ψ is the azimuth angle on $\mathbb{S}^3$. The wave map equation (1.1.2) for U reduces to the single semilinear wave equation for ψ

$$\begin{aligned}
&\partial_t^2\psi - \partial_r^2\psi - \frac{2r}{r^2+1}\partial_r\psi + \frac{\sin 2\psi}{r^2+1} = 0, \quad (t, r) \in \mathbb{R} \times \mathbb{R}, \\
&\vec{\psi}(0) = (\psi_0, \psi_1), \\
&\psi(t, -\infty) = 0, \quad \psi(t, \infty) = n\pi \quad \forall t.
\end{aligned} \tag{2.1.2}$$

The value $n \in \mathbb{N} \cup \{0\}$ is the degree of the wave map, and the fact that it is an integer is required if the solution $\vec{\psi}(t)$ is to have finite (and conserved) energy

$$\mathcal{E}(\vec{\psi}(t)) := \frac{1}{2}\int_{\mathbb{R}} \left[|\partial_t\psi(t, r)|^2 + |\partial_r\psi(t, r)|^2 + \frac{2\sin^2\psi(t, r)}{r^2+1} \right] (r^2+1)dr = \mathcal{E}(\vec{\psi}(0)).$$

We denote the space of finite energy pairs of degree n by

$$\mathcal{E}_n := \left\{ (\psi_0, \psi_1) : \mathcal{E}(\psi_0, \psi_1) < \infty, \quad \psi_0(-\infty) = 0, \quad \psi_0(\infty) = n\pi \right\}.$$

For all $n \in \mathbb{N} \cup \{0\}$, there exists a unique stationary solution $(Q_n, 0) \in \mathcal{E}_n$, a harmonic map, which minimizes the energy (see Section 2.2).

We now turn to stating our main result for this chapter. Let $n \in \mathbb{N} \cup \{0\}$ be a fixed topological degree. In the $n = 0$ case, the natural space to place the solution $\vec{\psi}(t)$ to (2.1.2) in is the *energy space* $\mathcal{H}_0 := \mathcal{H}((-\infty, \infty); (r^2+1)dr)$. Indeed, it is easy to show that $\|\vec{\psi}\|_{\mathcal{E}_0} \simeq \|\vec{\psi}\|_{\mathcal{H}_0}$. For $n \geq 1$, we measure distance relative to $(Q_n, 0)$ and define $\mathcal{H}_n := \mathcal{E}_n - (Q_n, 0)$ with 'norm'

$$\|\vec{\psi}\|_{\mathcal{H}_n} := \|\vec{\psi} - (Q_n, 0)\|_{\mathcal{H}_0}.$$

Note that $\psi(r) - Q_n(r) \to 0$ as $r \to \pm\infty$. The main result of this chapter is the following.

Theorem 2.1.1. *For any energy data $(\psi_0, \psi_1) \in \mathcal{E}_n$, there exists a unique globally defined solution $\vec{\psi}(t) \in C(\mathbb{R}; \mathcal{H}_n)$ which scatters forwards and backwards in time to the harmonic map $(Q_n, 0)$, i.e. there exist solutions $\varphi_L^{\pm}$ to the linear equation*

$$\partial_t^2 \varphi - \partial_r^2 \varphi - \frac{2r}{r^2+1} \partial_r \varphi + \frac{2}{r^2+1} \varphi = 0,$$

such that

$$\vec{\psi}(t) = (Q_n, 0) + \vec{\varphi}_L^{\pm}(t) + o_{\mathcal{H}_0}(1),$$

as $t \to \pm\infty$.

We now give an outline of the chapter and provide an overview of the proof. Section 2.2, Section 2.3, and Section 2.4 contain preliminaries necessary to carry out the concentration–compactness/ rigidity theorem methodology for wave maps on a wormhole. In Section 2.2, we establish various properties of the harmonic maps Q_n needed throughout the work. In particular, we establish existence, uniqueness, and asymptotics. Establishing these properties in the exterior wave map model discussed in Chapter 1 is considerably simpler since the static solutions to the exterior wave maps equation (1.1.8) (in the corotational case) are governed by the well–known equation for a damped pendulum

$$\frac{d^2F}{dx^2} + \frac{dF}{dx} = \sin 2F, \quad x = \log r.$$

The properties needed can then be derived from a simple phase plane analysis. However, in our setting there is no such change of variables that renders the static ODE (1.1.7) (with $\ell = 1$) autonomous. We instead use classical ODE arguments inspired by the work on corotational Skyrmions [21] to derive the properties we need. In Section 2.3 and Section 2.4, we establish results needed to carry out the first two steps in the concentration–compactness /rigidity theorem methodology. We first reformulate Theorem 2.1.1 as the statement that all radial solutions to a certain semilinear wave equation of the form

$$\partial_t^2 u - \Delta_g u + V(r)u = N(r, u), \quad (t, r) \in \mathbb{R} \times \mathbb{R}. \tag{2.1.3}$$

scatter to free waves as $t \to \pm\infty$ (see Theorem 2.4.1 for the exact statement). Here u is related to ψ by $u = \frac{1}{(r^2+1)^{1/2}}(\psi - Q_n)$, $V(r)$ is a smooth potential arising from linearizing about Q_n, and $-\Delta_g$ is the Laplace operator on the $5d$ wormhole $\mathcal{M}^5 = \mathbb{R} \times \mathbb{S}^4$ with metric

$$ds^2 = dr^2 + (r^2 + 1)d\Omega^2_{\mathbb{S}^4},$$

where $d\Omega^2_{\mathbb{S}^4}$ is the round metric on the sphere $\mathbb{S}^4$. In the remainder of the paper we carry out the concentration–compactness/ rigidity theorem method in the equivalent 'u–formulation.' Establishing the first two steps in the u–formulation follows from fairly standard arguments once Strichartz estimates for radial solutions to the free wave equation $\partial_t^2 u - \Delta_g u = 0$ are established. In the exterior wave map model, these estimates follow from previously known results on Strichartz estimates for free waves on Riemannian manifolds. However, Strichartz estimates for free waves on a wormhole fall outside of the literature devoted to free waves on Riemannian manifolds because of the trapping that occurs at the throat $r = 0$. In the works [27] and [28], the authors established dispersive estimates in geometries with trapping which are asymptotic to wormholes as $r \to \pm\infty$ as long as the initial data is localized to a fixed spherical harmonic (i.e. angular momentum). Since we are only interested in radial free waves on a wormhole, in Section 2.3 we are able to refine the dispersive estimates from [27] and [28] in the zero angular momentum case and obtain the Strichartz estimates we need. In fact, we establish Strichartz estimates for radial free waves on d–dimensional wormholes for arbitrary $d \geq 3$. This section is independent of all other sections and may be of interest in its own right. In Section 4, we make the reduction previously described and transfer the Strichartz estimates established in Section 2.3 for $\partial_t^2 - \Delta_g$ to the perturbed operator $\partial_t^2 - \Delta_g + V$. The fact that the Strichartz estimates for the free wave operator carry over to the perturbed operator hinges on spectral information for the Schrödinger operator $-\Delta_g + V$. In Section 2.5 and Section 2.6, we use the concentration–compactness/rigidity theorem method to prove our main result. In Section 2.5 we carry out the first two steps of the concentration–compactness/rigidity theorem methodology in the u–formulation. The main result of this section is that if Theorem 2.1.1 fails so that not all solutions to (2.1.3) scatter, then there exists a nonzero solution u_* to (2.1.3) such that $\{\vec{u}_*(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{H} := \mathcal{H}((-\infty, \infty); (r^2 + 1)^2 dr)$. In Section 2.6, we show that a solution u to (2.1.3) such that $\{\vec{u}(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{H}$ must be identically 0 which completes the proof. In particular, we show that u is zero by showing it must be a static solution to (2.1.3) with finite energy. This is achieved using a change of variables valid in the exterior regions $|r| \gtrsim 1$ that transforms (2.1.3) into an 'exterior wave map equation'. We then use channels of energy arguments similar to those used in [16] and [18] to show that u is a static solution to (2.1.3). This then implies that $\psi = Q_n + (r^2 + 1)^{1/2} u$ satisfies (1.1.8) with $\ell = 1$.

By the uniqueness of harmonic maps, we deduce that $u \equiv 0$ and conclude the proof.

2.2 Harmonic Maps

For the remainder of the paper, we fix a topological degree $n \in \mathbb{N} \cup \{0\}$. In this section, we study static solutions to (2.1.2). In particular, we prove the following.

Proposition 2.2.1. *There exists a unique smooth solution Q_n to the equation*

$$\begin{aligned} &F'' + \frac{2r}{r^2+1}F' - \frac{\sin 2F}{r^2+1} = 0, \quad r \in \mathbb{R}, \\ &F(-\infty) = 0, \quad F(\infty) = n\pi. \end{aligned}$$

In the case $n = 0$, $Q_0 = 0$. For $n \in \mathbb{N}$, Q_n is increasing on $\mathbb{R}$, satisfies $Q(r) + Q(-r) = n\pi$ for all r and there exists $\alpha_n > 0$ such that,

$$\begin{aligned} &Q_n(r) = n\pi - \alpha_n r^{-2} + O(r^{-4}), \quad \text{as } r \to \infty, \\ &Q_n(r) = \alpha_n r^{-2} + O(r^{-4}), \quad \text{as } r \to -\infty. \end{aligned}$$

The $O(\cdot)$ terms also satisfy the natural derivative bounds.

The proof of existence follows from a simple shooting argument sketched in [3]. The proof of uniqueness and properties needed are inspired by the work on the equivariant Skyrme equation [21]. The proof of Proposition 2.2.1 will be contained in the following various lemmas.

2.2.1 Existence of Harmonic Maps

In this section we prove the existence part of Proposition 2.2.1. In order to achieve this and, in fact, uniqueness of the harmonic map constructed, we will need to study general solutions to

$$F'' + \frac{2r}{r^2+1}F' - \frac{\sin 2F}{r^2+1} = 0, \quad r \in \mathbb{R}. \tag{2.2.1}$$

We begin with the following simple lemma.

Lemma 2.2.2. *If F is a solution to* (2.2.1)*, then F exists on all of $\mathbb{R}$. Moreover, F has limits at $\pm\infty$ in $\mathbb{Z}\pi \cup \left(\mathbb{Z} + \frac{1}{2}\right)\pi$.*

Proof. Suppose that F solves (2.2.1). Due to the sublinear growth in F, F' in (2.2.1), it follows from standard ODE theory that F is globally defined. Because of the invariance of the equation under the change $r \leftrightarrow -r$, we need only show that F has a limit at ∞.

Define the following auxiliary function

$$Q(r) = (r^2+1)\frac{(F')^2}{2} - \sin^2 F.$$

Using that F solves (2.2.1), we have that

$$Q'(r) = -r(F')^2. \tag{2.2.2}$$

Thus, Q is nonincreasing on $r \geq 0$ and by definition is also bounded below. Thus, $Q(r) \to c \in [-1, \infty)$ as $r \to \infty$. Moreover, we note that

$$((r^2+1)Q)' = -2r\sin^2 F \leq 0,$$

so that

$$Q(r) \leq \frac{Q(0)}{r^2+1}, \quad r \geq 0.$$

This implies $c \leq 0$.

The previous bound on Q implies that

$$F'(r)^2 = \frac{Q(r)}{r^2+1} + \frac{\sin^2 F(r)}{r^2+1}\sin^2 F(r) = O\left(\frac{1}{r^2}\right).$$

We now claim that F' isn't just $O(r^{-1})$ but in fact satisfies

$$F'(r) = o\left(\frac{1}{r}\right).$$

Suppose towards a contradiction that this is not the case. Then there exist $\delta > 0$ and a sequence $r_n \to \infty$ with the property

$$\frac{\delta}{r_n} \leq |F'(r_n)|.$$

Since F solves (2.2.1), we have that

$$|F''(r)| \leq \frac{K}{r^2}.$$

Thus, for $r_n \leq r \leq (1+\delta/2K)r_n$, we have

$$|F'(r) - F'(r_n)| \leq K\int_{r_n}^{r} \rho^{-2} d\rho \leq K\left(\frac{1}{r_n} - \frac{1}{r}\right) \leq \frac{\delta}{2r_n},$$

so that

$$|F'(r)| \geq \frac{\delta}{2r_n}, \quad r_n \leq r \leq (1+\delta/2K)r_n.$$

Hence

$$-Q'(r) = rF'(r)^2 \geq \frac{\delta^2}{4r_n}, \quad r_n \leq r \leq (1+\delta/2K)r_n.$$

The previous estimate implies that

$$|Q(r_n) - Q((1+\delta/2K)r_n)| = \int_{r_n}^{(1+\delta/2K)r_n} -Q(r)dr \geq \frac{\delta^3}{8K},$$

which contradicts the fact that $\lim_{r\to\infty} Q(r)$ exists in $[-1,0]$. Thus, the claim $F'(r) = o(r^{-1})$ holds.

We now show that as $r \to \infty$, $F(r)$ tends to $k\pi$ or $\left(k+\frac{1}{2}\right)\pi$ for some $k \in \mathbb{Z}$. Since $F'(r) = o(r^{-1})$ and $Q(r) \to c \in [-1,0]$, we have that

$$\sin^2 F(r) \to \tilde{c} \in [0,1], \quad r \to \infty.$$

Thus, $F(r)$ tends to some limit $F_\infty \in \mathbb{R}$ as $r \to \infty$. Since F solves (2.2.1) and satisfies $F'(r) = o(r^{-1})$, we see that

$$(r^2+1)F''(r) \to \sin 2F_\infty, \quad \text{as } r \to \infty.$$

If $\sin 2F_\infty \neq 0$, then for large r we have

$$F'(r) = \int_r^\infty F''(\rho)d\rho \sim \sin 2F_\infty \int_r^\infty \frac{1}{\rho^2+1}d\rho \sim \sin 2F_\infty \frac{1}{r},$$

which contradicts $F'(r) = o(r^{-1})$. Thus, we must have that $\sin 2F_\infty = 0$ as desired. □

We remark that we will now be interested solely in solutions to (2.2.1) which satisfy $F(\pm\infty) \in \mathbb{Z}\pi$. This is because these solutions are the only solutions that have the potential to have finite energy $\mathcal{E}(F,0) < \infty$. Using Lemma 2.2.2 we can establish the following asymptotics for solutions to (2.2.1).

Lemma 2.2.3. *Suppose that F solves* (2.2.1) *and there exists $k \in \mathbb{N}\cup\{0\}$ such that $F(\infty) =$*

$k\pi$. Then there exists $\alpha \in \mathbb{R}$ such that

$$F(r) = k\pi + \alpha r^{-2} + O(r^{-4}), \tag{2.2.3}$$

as $r \to \infty$ where the $O(\cdot)$ term satisfies the natural derivative bounds. A similar statement holds as $r \to -\infty$ if $F(-\infty) = k\pi$.

We note that Lemma 2.2.3 provides the asymptotics stated in Proposition 2.2.1.

Proof. The proof of Lemma 2.2.3 follows in almost exactly the same way as the proof of Case 1 of Theorem 2.3 in [21]. The idea is to make the change of variables $x = \operatorname{arcsinh} r$ and use the fact that $dF/dx = r dF/dr = o(1)$ to write (2.2.1) as

$$\frac{d^2F}{dx^2} + \frac{dF}{dx} - \sin 2F + O(e^{-2x}) = 0. \tag{2.2.4}$$

The ODE (2.2.4) is asymptotically the autonomous ODE $F'' + F' - \sin 2F = 0$ (the damped pendulum) near $x = \infty$ for which the desired expansion (2.2.3) holds in the x variable. We omit the details and refer the reader to the proof of Case 1 in Theorem 2.3 in [21] for the full details of the argument. □

A fact that will be useful in Section 2.6 is that one can obtain a solution to (2.2.1) with prescribed asymptotics as $r \to \infty$.

Proposition 2.2.4. *Let $k \in \mathbb{N} \cup \{0\}$, and let $\alpha \in \mathbb{R}$. Then there exists a unique solution F_α to (2.2.1) such that*

$$F_\alpha(r) = k\pi + \alpha r^{-2} + O(r^{-4}) \tag{2.2.5}$$

as $r \to \infty$ where the $O(\cdot)$ term satisfies the natural derivative bounds.

Before giving the proof, we note that the symmetry $r \mapsto -r$ of (2.2.1) allows us to conclude from Proposition 2.2.4 that given $k \in \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{R}$, there exists a solution $F_\beta(r)$ to (2.2.1) such that

$$F_\beta(r) = k\pi + \beta r^{-2} + O(r^{-4})$$

as $r \to -\infty$.

Proof. We seek a solution F to (2.2.1) with the stated asymptotics (2.2.5). We first make the change of variables $x = \operatorname{arcsinh} r$ so that (2.2.1) becomes

$$F'' + \tanh x F' - \sin 2F = 0, \quad x \in \mathbb{R}, \tag{2.2.6}$$

where $F' = \frac{dF}{dx}$. We now rewrite (2.2.6) as

$$F'' + F' - 2F = [\sin 2F - 2F] + (1 - \tanh x)\, F'. \tag{2.2.7}$$

Define $G = e^{x/2}(F - k\pi)$. Then G satisfies

$$G'' + \frac{9}{4}G = N(x, G, G'), \tag{2.2.8}$$

where

$$N(x, G, G') = e^{x/2}\left[\sin(2e^{-x/2}G) - 2e^{-x/2}G\right] + (1 - \tanh x)\left[G' - \frac{1}{2}G\right]. \tag{2.2.9}$$

A fundamental system to the underlying linear equation $G'' - \frac{9}{4}G = 0$ is given by

$$G_1(x) = e^{-3x/2}, \quad G_2(x) = e^{3x/2}.$$

The Wronskian $W(G_1, G_2) = G_1' G_2 - G_1 G_2'$ is given by -3. By the variation of constants formula, we seek a solution $G = G_\alpha$ to the integral equation

$$G = \alpha G_1(x) + \frac{1}{3} \int_x^\infty \left[G_1(x) G_2(y) - G_1(y) G_2(x) \right] N(y, G, G') dy, \tag{2.2.10}$$

for $x \geq R$ for some R. For $R > 0$, define the Banach space

$$X_R = \left\{ G \in C^1([R, \infty)) : \|G\|_{X_R} < \infty \right\}$$

where

$$\|G\|_{X_R} := \sup_{x \geq R} e^{3x/2} \left[|G(x)| + |G'(x)| \right].$$

Denote the right side of (2.2.10) by $\Phi(G)$. From (2.2.9), it is easy to see that

$$|N(y, G, G')| \leq e^{-y} |G|^3 + e^{-2y} \left[|G| + |G'| \right].$$

Thus,

$$\|\Phi(G)\|_{X_R} \leq |\alpha| + 3|\alpha|/2 + C \left[e^{-R} \|G\|_{X_R}^3 + e^{-2R} \|G\|_{X_R} \right].$$

For R sufficiently large, a fixed point argument yields the existence of a unique solution G_α to (2.2.10). Moreover, G_α satisfies

$$G_\alpha(x) = \alpha e^{-3x/2} + O(e^{-7x/2})$$

as $x \to \infty$. This means $F_\alpha(x) = k\pi + e^{-x/2}G_\alpha(x)$ satisfies (2.2.6) and

$$F_\alpha(x) = k\pi + \alpha e^{-2x} + O(e^{-4x})$$

as $x \to \infty$. This is the same as (2.2.5) under the change of variables $r = \sinh x$. This concludes the proof of existence of F_α. Uniqueness follows from the fixed point argument and Lemma 2.2.3. □

Using Lemma 2.2.3 and monotonicity of the auxiliary function $Q(r)$, we deduce the following monotonicity result for solutions to (2.2.1).

Lemma 2.2.5. *Suppose that F solves* (2.2.1) *and*

$$F(-\infty) = l\pi, \quad F(\infty) = k\pi.$$

Then F is monotonic on $\mathbb{R}$. In particular, if $l = k$, then F is the constant solution.

Proof. Recall from the proof of Lemma 2.2.2 that the function

$$Q(r) = (r^2+1)\frac{(F')^2}{2} - \sin^2 F,$$

satisfies $Q'(r) = -r(F')^2$. In particular the function Q is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, \infty)$.

By Lemma 2.2.3, there exist $\beta_\pm \in \mathbb{R}$ such that as $r \to -\infty$

$$Q(r) = \beta^2 r^{-4} + O(r^{-6}).$$

Moreover, the case $\beta_+ = 0$ or $\beta_- = 0$ corresponds to the constant solution (which is trivially monotonic). We will assume that $\beta_\pm \neq 0$, and therefore, F is not the constant solution. Thus, if $|r|$ large, then $Q(r)$ is positive.

We now conclude that F has no critical points. If not, and there exists $r_0 \in \mathbb{R}$, a critical point for F, then $Q(r_0) \leq 0$. In particular, since $Q(r)$ is nondecreasing on $(-\infty, 0)$ from a positive value near $r = -\infty$, we must have that $r_0 > 0$. However, since F is nonconstant, $Q(r)$ is strictly decreasing on $[0, \infty)$ since $Q'(r) = -r(F')^2$. Thus, we have $Q(r) < Q(r_0) \leq 0$ for all $r > r_0$. This contradicts the fact that $Q(r) > 0$ for large positive r. Thus, F has no critical points so that F is monotonic on $\mathbb{R}$. □

We now prove the existence part of Proposition 2.2.1.

Lemma 2.2.6. *For each $n \in \mathbb{N}$, there exists a solution Q_n to* (2.2.1) *that satisfies*

$$Q_n(-\infty) = 0, \quad Q_n(\infty) = n\pi,$$
$$\forall r, \quad Q_n(r) + Q_n(-r) = n\pi.$$

The proof of Lemma 2.2.6 now follows from the previous lemmas and a classical shooting argument sketched in [3]. For every $\alpha \in (0, \infty)$, define $F(r, \alpha)$ to be the solution to (2.2.1) such that

$$F(0, \alpha) = \frac{n\pi}{2},$$
$$F'(0, \alpha) = \alpha.$$

The variable α is referred to as the shooting variable. We will show that we can choose α so that $F(\infty, \alpha) = n\pi$. Note that if $F(\infty, \alpha) = n\pi$ for some α, then the symmetry $F \mapsto n\pi - F$ of the equation yields $F(-r, \alpha) + F(r, \alpha) = n\pi$ so that $F(-\infty, \alpha) = 0$. Thus, to prove Lemma 2.2.6, it suffices to show there exists $\alpha^* \in (0, \infty)$ such that

$$F(\infty, \alpha^*) = n\pi.$$

We then set $Q_n(r) = F(r, \alpha^*)$.

Define

$$A := \left\{ \alpha \in (0,\infty) : \lim_{r\to\infty} F(r,\alpha) < n\pi \right\}$$

The proof of Lemma 2.2.6 requires a few claims.

Claim 2.2.7. *There exists* $\alpha_0 > 0$ *so that* $(0,\alpha_0) \subset A$.

Proof. For $\alpha \in (0,\infty)$, we denote

$$Q(r,\alpha) = (r^2+1)\frac{(F'(r,\alpha))^2}{2} - \sin^2 F(r,\alpha).$$

The proof is split into two cases depending on whether n is odd or even.

Case 1. We first consider the case that n is odd. Then we may take $\alpha_0 = \sqrt{2}$. Indeed, if $\alpha \in (0,\sqrt{2})$, then

$$Q(0,\alpha) = \frac{\alpha^2}{2} - \sin^2\left(\frac{n\pi}{2}\right) = \frac{\alpha^2}{2} - 1 < 0.$$

Since $Q(r,\alpha)$ is decreasing on $(0,\infty)$, we must have $Q(r,\alpha) < 0$ for all $r > 0$. This implies that $F(r_0,\alpha) \neq n\pi$ for all $r_0 \in (0,\infty)$. The case that $F(r,\alpha) \to n\pi$ as $r \to \infty$ is also impossible since then $Q(r,\alpha) > 0$ for r sufficiently large (see the proof of Lemma 2.2.5). Thus, if n is odd, we have $(0,\sqrt{2}) \subset A$.

Case 2. We now consider the case that n is even. In particular, $\frac{n\pi}{2} = l\pi$ for some $l \in \mathbb{N}$. We first note that for every $\alpha \in (0,\infty)$, $F(\cdot,\alpha)$ is increasing until F leaves the strip $\left(l\pi, \left(l+\frac{1}{2}\right)\pi\right)$. Indeed, if F attains a local maximum for some r_0 with $F(r_0,\alpha) \in \left(l\pi, \left(l+\frac{1}{2}\right)\pi\right)$, then (2.2.1) implies

$$F''(r_0,\alpha) = \frac{\sin 2F(r_0,\alpha)}{r_0^2+1} > 0.$$

Thus, $F(\cdot, a)$ is increasing as long as $F \in \left(l\pi, \left(l + \frac{1}{2}\right)\pi\right)$.

Note that since $\frac{n\pi}{2} = l\pi$ for some integer l, we have

$$Q(0, \alpha) = \frac{\alpha^2}{2}.$$

We recall that $[(r^2 + 1)Q(r, \alpha)]' = -2r \sin^2 F(r, \alpha)$ so that

$$Q(r, \alpha) \leq \frac{Q(0, \alpha)}{r^2 + 1} = \frac{\alpha^2}{2(r^2 + 1)}. \tag{2.2.11}$$

Thus, for all α sufficiently small, we have

$$F'(r, \alpha)^2 = \frac{2Q(r, \alpha)}{(r^2 + 1)^2} + \frac{2}{r^2 + 1} \sin^2 F(r, \alpha) < \frac{4}{r^2}. \tag{2.2.12}$$

Moreover, by continuity of the initial value problem, for α sufficiently small, we can also ensure that

$$F(r, \alpha) < \left(l + \frac{1}{6}\right)\pi, \quad r \in [0, 1].$$

Fix $\alpha \in (0, \alpha_0)$ with α_0 small to be chosen, and suppose that $F(r, \alpha)$ leaves the strip $\left(l\pi, \left(l + \frac{1}{2}\right)\pi\right)$ (if not then $\alpha \in A$ trivially). Since $F(\cdot, \alpha)$ is increasing until it reaches $\left(l + \frac{1}{2}\right)\pi$, there exist $1 < r_1 < r_2$ such that

$$F(r_1, \alpha) = \left(l + \frac{1}{6}\right)\pi,$$
$$F(r_2, \alpha) = \left(l + \frac{1}{4}\right)\pi.$$

Then the fundamental theorem of calculus and (2.2.12) imply that

$$\frac{\pi}{4} - \frac{\pi}{6} = \int_{r_1}^{r_2} F'(r,\alpha)dr < 2\log(r_2/r_1),$$

so that

$$r_2 - r_1 > \left(e^{\pi/24} - 1\right) r_1 > e^{\pi/24} - 1.$$

By (2.2.11)

$$\begin{aligned}(r_2^2+1)Q(r_2,\alpha) &= (r_1^2+1)Q(r_1,\alpha) - 2\int_{r_1}^{r_2} r\sin^2 F(r,\alpha)dr \\ &\leq Q(0,\alpha) - \frac{1}{2}\int_{r_1}^{r_2} r dr \\ &= \frac{\alpha^2}{2} - \frac{1}{4}(r_2^2 - r_1^2) \\ &< \frac{\alpha_0^2}{2} - \frac{e^{\pi/24}-1}{8}.\end{aligned}$$

Thus, if we choose α_0 so that $\alpha_0^2 < \frac{e^{\pi/24}-1}{8}$, we have, for all $\alpha \in (0,\alpha_0)$, $Q(r_2,\alpha) < 0$. Since $Q(r,\alpha)$ is decreasing on $(0,\infty)$, it follows that $Q(r,\alpha) < 0$ for all $r > r_2$. Thus, we cannot have $F(r,\alpha) = (l+1)\pi$ for any $r \in (0,\infty]$ so that

$$F(\infty,\alpha) < n\pi.$$

Thus, if α_0 is sufficiently small, $\alpha \in A$ for all $\alpha \in (0,\alpha_0)$. □

Claim 2.2.8. *The set A is open.*

We recall that

Proof. Let $\alpha_0 \in A$. We consider two cases.

Case 1. In this case, we assume that there exists $m < n$ such that

$$F(\infty, \alpha_0) = \left(m + \frac{1}{2}\right)\pi.$$

We first note that for all $r \geq 0$

$$F(r, \alpha_0) < (m+1)\pi. \tag{2.2.13}$$

Indeed, if this were not the case, then, since $F(r, \alpha_0)$ is not constant and $F(\infty, \alpha_0) < (m+1)\pi$, there exist $r_1 < r_2 < r_3$ such that

$$F(r_1, \alpha_0) = F(r_3, \alpha_0) = (m+1)\pi,$$
$$F'(r_1, \alpha_0) \neq 0, \quad F'(r_2, \alpha_0) = 0, \quad F'(r_3, \alpha_0) \neq 0.$$

In particular, $Q(r_2, \alpha_0) \leq 0$. But since $Q(r, \alpha)$ is decreasing on $[0, \infty)$, it follows that $Q(r_3, \alpha_0) < 0$ which is a contradiction to our choice of r_3. Thus, for all $r \geq 0$

$$F(r, \alpha_0) < (m+1)\pi.$$

Since $F(\infty, \alpha_0) = \left(m + \frac{1}{2}\right)\pi$ and $F'(r, \alpha_0) = o(r^{-1})$ (see Lemma 2.2.2), there exists $R_0 = R_0(\alpha_0)$ large so that

$$Q(R_0, \alpha_0) < 0.$$

By continuous dependence of $F(\cdot, \alpha)$ on α, we can ensure that for all α is a small neighborhood of α_0 we have

$$F(r,\alpha) < (m+1)\pi, \quad r \in [0, R_0], \tag{2.2.14}$$

$$Q(R_0,\alpha) < 0. \tag{2.2.15}$$

Since $Q(r,\alpha)$ is decreasing on $[0,\infty)$, (2.2.15) implies for all α sufficiently close to α_0, $Q(r,\alpha) < 0$ for all $r \geq R_0$. In particular, $F(r,\alpha) \neq l\pi$ for any $l \in \mathbb{N}$ and all $r \in [R_0,\infty]$. This along with (2.2.14) implies that $F(\infty,\alpha) < (m+1)\pi$. Thus, for all α sufficiently close to α_0, we have $\alpha \in A$ as desired.

Case 2. In this case we assume that there exists $m < n$ such that

$$F(\infty,\alpha_0) = m\pi.$$

We first note that in this case, we have $Q(r,\alpha_0) = O(r^{-4})$ (see the proof of Lemma 2.2.5), so that,

$$\lim_{r\to\infty}(r^2+1)Q(r,\alpha_0) = 0. \tag{2.2.16}$$

Let $\epsilon_0 > 0$ to be chosen later. Then by (2.2.16), there exists $R_0 = R_0(\epsilon_0) > 1$ such that

$$(r^2+1)Q(r,\alpha_0) < \epsilon_0, \quad r \geq R_0.$$

For α in a small (depending on ϵ_0) neighborhood of α_0, we have

$$Q(0,\alpha) < 2Q(0,\alpha_0), \tag{2.2.17}$$

$$(R_0^2+1)Q(R_0,\alpha) < 2\epsilon_0, \tag{2.2.18}$$

$$\frac{n\pi}{2} \leq F(r,\alpha) < m\pi, \quad r \in [0,R_0]. \tag{2.2.19}$$

We now claim that for each such α, we have

$$F(\infty, \alpha) \leq \left(m + \frac{1}{2}\right)\pi. \tag{2.2.20}$$

Let α be sufficiently close to α_0 so that (2.2.17), (2.2.18), and (2.2.19) are satisfied, and assume that

$$F(\infty, \alpha) > m\pi.$$

Then by (2.2.19), there exists $r_0 \geq R_0$ such that $F(r_0, \alpha) = m\pi$. Since $F(\cdot, \alpha)$ is increasing as long as $F(\cdot, \alpha)$ is in the strip $\left(m\pi, \left(m + \frac{1}{2}\right)\pi\right)$ (see the proof of Claim 2.2.7), there exist $r_1, r_2 > R_0$ such that $r_1 < r_2$ and

$$F(r_1, \alpha) = \left(m + \frac{1}{6}\right)\pi,$$
$$F(r_2, \alpha) = \left(m + \frac{1}{4}\right)\pi.$$

As in the proof of Claim 2.2.7, by (2.2.17) we have

$$F'(r, \alpha)^2 \leq \frac{2Q(0, \alpha)}{(r^2 + 1)^2} + \frac{2}{r^2 + 1}\sin^2 F(r, \alpha) \leq \frac{C^2(\alpha_0)}{r^2}, \tag{2.2.21}$$

for some positive constant $C(\alpha_0)$. By our choice of r_1, r_2, (2.2.21), and the fundamental theorem of calculus, we deduce that

$$\frac{\pi}{4} - \frac{\pi}{6} = \int_{r_1}^{r_2} F'(r, \alpha)dr \geq C(\alpha_0)\log(r_2/r_1),$$

whence for some (possibly small) constant $c(\alpha_0) > 0$

$$r_2 - r_1 \geq c(\alpha_0).$$

By the relation $[(r^2+1)Q(r,\alpha)]' = -2r\sin^2 F(r,\alpha)$ and (2.2.18), we have

$$\begin{aligned}(r_2^2+1)Q(r_2,\alpha) &= (r_1^2+1)Q(r_1,\alpha) - 2\int_{r_1}^{r_2} r\sin^2 F(r,\alpha)dr \\ &< 2\epsilon_0 - \frac{1}{2}\int_{r_1}^{r_2} rdr \\ &< 2\epsilon_0 - \frac{1}{2}(r_2^2 - r_1^2) \\ &< 2\epsilon_0 - \frac{1}{2}c(\alpha_0).\end{aligned}$$

By initially choosing ϵ_0 sufficiently small (depending only on α_0), we see that if α is sufficiently close to α_0 so that (2.2.17), (2.2.18), and (2.2.19) are satisfied, we have $Q(r_2,\alpha) < 0$. Thus, $Q(r,\alpha) < 0$ for all $r \geq r_2$. Hence, for any $l > m$, $F(r,\alpha) \neq l\pi$ for all $r \in [R_0,\infty]$. This along with (2.2.19) proves that $F(r,\alpha) \leq \left(m + \frac{1}{2}\right)\pi$ for all $r \geq 0$ which establishes (2.2.20). Thus, all α sufficiently close to α_0 are in A which finishes the proof of Claim 2.2.8. □

Claim 2.2.9. *There exists $\alpha_1 > 0$ such that $(\alpha_1,\infty) \subseteq A^c$.*

Proof. We first note that if $\alpha > 0$ and if $F(r,\alpha) = n\pi$ for some $r > 0$, then $F(\infty,\alpha) > n\pi$. Indeed, suppose $F(r_0,\alpha) = n\pi$ for some $r_0 > 0$ and $F(\infty,\alpha) \leq n\pi$. Since $F(r,\alpha)$ is not the constant function, there exist $r_0 < r_1 < r_2 \leq \infty$ such that $F'(r_1,\alpha) = 0$ and $F(r_2,\alpha) = n\pi$. We then have that $Q(r_1,\alpha) \leq 0$ and $Q(r_2,\alpha) > 0$. This contradicts the fact that $Q(r,\alpha)$ is decreasing on $[0,\infty)$. Thus, if $F(r,\alpha) = n\pi$ for some $r > 0$, then $F(\infty,\alpha) > n\pi$. In particular, we have shown that

$$\{a > 0 : F(r_0,a) = n\pi \text{ for some } r_0 > 0\} \subset A^c.$$

Thus, the proof of Claim 2.2.9 is reduced to showing that there exists $\alpha_1 > 0$ such that

$$(\alpha_1,\infty) \subseteq \{a > 0 : F(r_0,a) = n\pi \text{ for some } r_0 > 0\}.$$

The idea of the proof is now simple. If the initial velocity α is large enough, then $F(r, \alpha) = n\pi$ for some $r > 0$ so that $\alpha \in A^c$. To make this argument precise, we need the precise asymptotics of $F(r, \alpha)$ for r near $r = 0$. First we change variables and set $x = \operatorname{arcsinh} r$. Then $F(x, \alpha) := F(r(x), \alpha)$ satisfies $F(0, \alpha) = n\pi/2$, $F'(0, \alpha) = \alpha$ and

$$F'' + \tanh x F' - \sin 2F = 0. \tag{2.2.22}$$

We first claim there exists $x_0 > 0$ small such that for all $\alpha > 0$

$$\|F(\cdot, \alpha)\|_{C^1([0,x_0])} \leq n\pi + 4\alpha. \tag{2.2.23}$$

Indeed, we solve (2.2.22) near $x = 0$ by a contraction mapping argument. Let $X = C^1([0, x_0])$ where x_0 is to be chosen later. Define $\Phi : X \to X$ by

$$\Phi F(x) = \frac{n\pi}{2} + \alpha x + \int_0^x (x - y) \left[\sin 2F(y) - \tanh y F'(y)\right] dy$$

If x_0 is chosen so small so that $\tanh y \leq 2y$ for $y \in [0, x_0]$, then it is easy to verify that for all $F, G \in X$ and for some absolute constant $C > 0$

$$\begin{aligned} \|\Phi F\|_X &\leq \frac{n\pi}{2} + 2\alpha + Cx_0\|F\|_X, \\ \|\Phi F - \Phi G\|_X &\leq Cx_0\|F - G\|_X. \end{aligned}$$

Now fix x_0 smaller if necessary so that $x_0 < 1/(8C)$. Then, we may contract in the ball $B_X(0, n\pi + 4\alpha)$ and find a unique fixed point (namely $F(x, \alpha)$) of Φ. This shows that there exists x_0 small and independent of α such that $\|F(\cdot, \alpha)\|_{C^1([0,x_0])} \leq n\pi + 4\alpha$ as desired.

We now conclude that if α is sufficiently large (depending on x_0), then in fact $F(x_0, \alpha) \geq$

$n\pi$ where x_0 was defined previously. We write for $x \in [0, x_0]$

$$F(x,\alpha) = \frac{n\pi}{2} + \alpha x + \int_0^x (x-y)\left[\sin 2F(y,\alpha) - \tanh y F'(y,\alpha)\right] dy.$$

Then by (2.2.23), for some constant $C > 0$ and by choosing x_0 smaller if necessary, we have

$$\begin{aligned} |F(x_0,\alpha)| &\geq \frac{n\pi}{2} + \alpha x_0 - Cx_0^2 \|F(\cdot,\alpha)\|_{C^1([0,x_0])} \\ &\geq \frac{n\pi}{2}(1 - Cx_0^2) + \alpha x_0(1 - 4x_0 C) \\ &\geq \frac{n\pi}{4} + \frac{\alpha x_0}{2}. \end{aligned}$$

This shows that for all $\alpha \geq 2n\pi/x_0$, $F(x_0,\alpha) \geq n\pi$, i.e.

$$\alpha \in \{a > 0 : F(r_0, a) = n\pi \text{ for some } r_0 > 0\} \subset A^c$$

which concludes the proof. □

Proof of Lemma 2.2.6. By Claim 1 and Claim 3,

$$\alpha^* := \sup A \in (0, \infty).$$

By Claim 2, $\alpha^* \notin A$. Suppose that $\alpha^* \in \{\alpha \in (0,\infty) : F(\infty,\alpha) > n\pi\}$. Then by continuous dependence of initial data, all α near α^* are also in $\{\alpha \in (0,\infty) : F(\infty,\alpha) > n\pi\}$. This, however, contradicts the facts that $\alpha^* = \sup A$ and that A is open (by Claim 2). Thus, $F(\infty, \alpha^*) = n\pi$, and we are done. □

2.2.2 Uniqueness of the Harmonic Map

In this section we show uniqueness of the harmonic map constructed in the previous section which concludes the proof of Proposition 2.2.1.

Lemma 2.2.10. *Let F_1 and F_2 solve* (2.2.1) *and assume that for $j = 1, 2$*

$$F_j(-\infty) = 0, \quad F_j(\infty) = n\pi.$$

Then $F_1 = F_2$.

Proof. Since any F that solves (2.2.1) and connects 0 to $n\pi$ must be increasing, we may make a change of variables and consider F as the dependent variable and $p = \frac{dF}{dx}$ as the dependent variable, where $x = \operatorname{arcsinh} r$. Thus, the equation solved by p is

$$p\frac{dp}{dF} + (\tanh x)p - \sin 2F = 0. \tag{2.2.24}$$

Suppose towards a contradiction, that we have two different solutions F_1, F_2. These determine two C^∞ diffeomorphisms $x_1, x_2 : (0, n\pi) \to (-\infty, \infty)$ by the condition $F_j \circ x_j$ is the identity on $(0, n\pi)$. Then we have

$$p_j(F)\frac{dp_j}{dF} + (\tanh x_j(F))p_j(F) - \sin 2F = 0, \quad j = 1, 2. \tag{2.2.25}$$

Set $\phi(F) = p_2(F) - p_1(F)$. Subtracting the equation satisfied by p_1 from the equation satisfied by p_2 and rearranging, we have

$$\begin{aligned} 0 &= p_2\frac{dp_2}{dF} - p_1\frac{dp_1}{dF} + \tanh x_2 p_2 - \tanh x_1 p_1 \\ &= p_2\frac{d\phi}{dF} + \left(\frac{dp_1}{dF} + \tanh x_2\right)\phi - (\tanh x_1 - \tanh x_2)\, p_1. \end{aligned}$$

Define $q = p_2^{-1}\left(\frac{dp_1}{dF} + \tanh x_2\right)$, $f = (\tanh x_2 - \tanh x_1)\, p_1 p_2^{-1}$. Then ϕ satisfies

$$\phi' + q\phi = -f \implies (-\phi e^{-Q})' = f,$$

where $Q(F) = \int_F^{F_0} q(\bar{F})d\bar{F}$ for any choice of $F_0 \in (0, n\pi)$. Hence, we have that

$$\phi(F) = e^{Q(F)}\phi(F_0) + \int_F^{F_0} e^{Q(F)-Q(\bar{F})} f(\bar{F})d\bar{F}. \tag{2.2.26}$$

We now make an observation based on (2.2.26). Note that if $p_2(F_0) > p_1(F_0)$ and $x_2(F_0) > x_2(F)$ imply that $p_2(F) > p_1(F)$ and $x_2(F) > x_1(F)$ for all $F \leq F_0$. Indeed, suppose $F_1 < F_0$ and $p_2(F) \geq p_1(F)$ for all $F_1 \leq F \leq F_0$. Then from the definition of p_j, we have for all $F_1 \leq F \leq F_0$

$$p_2(F) \geq p_1(F) \implies (x_2(F) - x_1(F))' \leq 0.$$

This implies upon integrating that

$$0 < x_2(F_0) - x_1(F_0) \leq x_2(F) - x_1(F), \quad F_1 \leq F \leq F_0.$$

Since $\tanh x$ is increasing on $(-\infty, \infty)$,

$$x_2(F) > x_1(F), \quad F_1 \leq F \leq F_0 \implies f(F) > 0, \quad F_1 \leq F \leq F_0.$$

Hence by (2.2.26)

$$p_2(F) > p_1(F), \quad F_1 \leq F \leq F_0.$$

Thus, if $p_2(F) \geq p_1(F)$ for all $F_1 \leq F \leq F_0$, we must in fact have the strict inequalities $p_2(F) > p_1(F)$ and $x_2(F) > x_1(F)$ for $F_1 \leq F \leq F_0$. By continuity, we see that $p_2(F) > p_1(F)$ and $x_2(F) > x_1(F)$ for all $F \leq F_0$.

By Lemma 2.2.3, a solution F to (2.2.1) such that $F(-\infty) = 0$, $F(\infty) = n\pi$ satisfies for unique $a, b > 0$,

$$F(x) - n\pi \sim -ae^{-2x} + a\frac{2}{5}e^{-4x}, \quad \text{as } x \to \infty,$$
$$F(x) \sim be^{2x} - b\frac{2}{5}e^{4x}, \quad \text{as } x \to -\infty.$$

It follows that p satisfies

$$\begin{aligned} p &\sim 2ae^{-2x} - a\frac{8}{5}e^{-4x} \\ &\sim 2(n\pi - F) - a\frac{4}{5}e^{-4x} \\ &\sim 2(n\pi - F) - a^{-1}\frac{4}{5}(n\pi - F)^2, \end{aligned}$$

as $F \to n\pi^-$. Similarly, we have

$$p \sim 2F - b^{-1}\frac{4}{5}F^2,$$

as $F \to 0^+$. Suppose F_2 has coefficients $a_2, b_2 > 0$ and F_1 has coefficients $a_1, b_1 > 0$ where (without loss of generality) $a_2 > a_1$. Then clearly $x_2(F) > x_1(F)$ for all F sufficiently close to $n\pi$ since for x large

$$F_2(x) \sim n\pi - a_2e^{-2x} < n\pi - a_1e^{-2x} \sim F_1(x).$$

Moreover, we have $p_2(F) > p_1(F)$ for F sufficiently close to $n\pi$ by our previous calculation

$$\begin{aligned} p_2(F) &\sim 2(n\pi - F) - a_2^{-1}\frac{4}{5}(n\pi - F)^2 \\ &> 2(n\pi - F) - a_1^{-1}\frac{4}{5}(n\pi - F)^2 \sim p_1(F). \end{aligned}$$

Thus, by our observation following (2.2.26), we have $p_2(F) > p_1(F)$ and $x_2(F) > x_1(F)$ for all $F \in (0, n\pi)$. In particular, the constraint $x_2(F) > x_1(F)$ for all $F \in (0, n\pi)$ implies that $b_1 > b_2$. But then for F near 0

$$\begin{aligned} p_1(F) &\sim 2F - b_1^{-1}\frac{4}{5}F^2 \\ &> 2F - b_2^{-1}\frac{4}{5}F^2 \sim p_2(F), \end{aligned}$$

which contradicts $p_2(F) > p_1(F)$ for all $F \in (0, n\pi)$. Thus, no two distinct solutions F_1, F_2 exist. This completes the proof. □

2.3 Strichartz Estimates for the Free Wave Equation on Wormholes

In this section we establish Strichartz estimates for radial solutions to the free wave equation on the $(d+1)$–dimensional wormhole $\mathcal{M}^{d+1} = \{(r, \omega) : r \in \mathbb{R}, \omega \in \mathbb{S}^d\}$ with metric g satisfying

$$ds^2 = dr^2 + (r^2 + 1)d\Omega^2_{\mathbb{S}^d}(\omega).$$

Here $d\Omega^2_{\mathbb{S}^d}$ is the line element on $\mathbb{S}^d$ corresponding to the usual round metric. When we say radial functions we mean functions $f : \mathcal{M}^{d+1} \to \mathbb{R}$ with $f = f(r)$. These Strichartz estimates will be used in Section 2.4 and Section 2.5 to establish a small data theory for (2.1.2). However, the results and methods of this section are independent of all other sections in this work and may be of interest in their own right.

For the remainder of the section, we fix $d \geq 2$ and drop the superscript by writing $\mathcal{M}$ instead of $\mathcal{M}^{d+1}$. We denote $\mathcal{H}(\mathbb{R}; (r^2+1)^{d/2}dr)$ simply by $\mathcal{H}$. For an interval I, we denote the spatial norms on $\mathcal{M}$ and spacetime norms on $I \times \mathcal{M}$ by

$$\|f\|_{L^p} := \left(\int |f(r)|^p (r^2+1)^{d/2} dr \right)^{1/p},$$
$$\|u\|_{L^p_t L^q_x(I)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(t,r)|^q (r^2+1)^{d/2} dr \right)^{p/q} dt \right)^{1/p}.$$

Since we only consider radial functions on $\mathcal{M}$, we abuse notation slightly and let Δ_g denote the radial part of the Laplace operator on $\mathcal{M}$,

$$\Delta_g f(r) = \partial_r^2 f + \frac{dr}{r^2+1} \partial_r f.$$

Let I be an interval with $0 \in I$. Let $F : I \times \mathbb{R} \to \mathbb{R}$, and let $u = u(t,r)$ solve the inhomogeneous wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta_g u = F, \quad (t,r) \in I \times \mathbb{R}. \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H}. \end{aligned} \tag{2.3.1}$$

We say that a triple (p,q,γ) is *admissible* if

$$p > 2, q \geq 2, \quad \frac{1}{p} + \frac{d+1}{q} = \frac{d+1}{2} - \gamma, \quad \frac{1}{p} \leq \frac{d}{2}\left(\frac{1}{2} - \frac{1}{q} \right).$$

The main result of this section is the following family of Strichartz estimates for (2.3.1).

Proposition 2.3.1. *Let (p,q,γ) and (a,b,ρ) be admissible triples. Then any solution u to* (2.3.1) *satisfies*

$$\||\nabla|^{1-\gamma} u\|_{L^p_t L^q_x(I)} + \||\nabla|^{-\gamma} \partial_t u\|_{L^p_t L^q_x(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \||\nabla|^{\rho} F\|_{L^{a'}_t L^{b'}_x(I)}.$$

It is well known (see for example [13] [30] [31]) that by a standard argument using Littlewood–Paley theory (for our wormhole geometry see [32]) and TT^* arguments, establishing Proposition 2.3.1 can be reduced to proving the following frequency localized dispersive estimate: let E denote the spectral measure for $-\Delta_g$ (restricted to radial functions). For a standard Littlewood–Paley cutoff $\varphi \in C_0^\infty(\mathbb{R})$ with support in $(1/2, 2)$, define (via the functional calculus)

$$\varphi\left(2^{-j}\sqrt{-\Delta_g}\right) = \int_0^\infty \varphi(2^{-j}\sqrt{\lambda})E(d\lambda).$$

Then for all $f \in C_0^\infty(\mathbb{R})$,

$$\left\| e^{\pm it\sqrt{-\Delta_g}} \varphi\left(2^{-j}\sqrt{-\Delta_g}\right) f \right\|_{L^\infty} \lesssim 2^{\frac{d+2}{2}} (2^{-j} + |t|)^{-\frac{d}{2}} \|f\|_{L^1}. \tag{2.3.2}$$

The proof of (2.3.2) draws heavily from the works [27] [28]. In these works, the authors prove dispersive estimates for free waves on a manifold with metric of the form

$$ds^2 = dr^2 + R^2(r) ds^2_\Omega(\omega), \quad r \in \mathbb{R},$$

where $ds^2_\Omega(\omega)$ is the metric on a compact embedded Riemannian manifold $\Omega \subset \mathbb{R}^N$ with dimension $d \geq 1$. The function $R(r)$ is assumed to be asymptotically conic:

$$R(r) = |r|\left(1 + O(r^{-1})\right), \quad \text{as } r \to \pm\infty.$$

Note that in the case of the wormhole geometry, $\Omega = \mathbb{S}^d$ and $R(r) = \langle r \rangle$. In particular, the authors proved weighted $L^1 \to L^\infty$ type estimates for data of the form $f(r)Y_n(\omega)$ where Y_n are eigenfunctions of $-\Delta_\Omega$. For the $n = 0$ case (i.e. a radial solution), they established the dispersive estimate

$$\left\| e^{\pm it\sqrt{-\Delta_g}} f(r) \right\|_{L^\infty} \lesssim |t|^{-d/2} \left(\|f\|_{L^1} + \|f'\|_{L^1} \right).$$

In our proof of 2.3.2, we refine their methods for the case of frequency localized data.

In what follows, we use the standard Japanese bracket notation $\langle r\rangle = (r^2+1)^{1/2}$. One readily verifies that

$$-\Delta_g f(r) = \left(\langle r\rangle^{-d/2} H \langle r\rangle^{d/2}\right) f(r),$$

where H is the Schrodinger operator on $\mathbb{R}$ given by

$$H = -\frac{d^2}{dr^2} + V, \quad V(r) = \frac{d(d-4)}{4} r^2 \langle r\rangle^{-4} + \frac{d}{2}\langle r\rangle^{-2}.$$

Note that the potential V satisfies

$$V(r) = \frac{d(d-2)}{2r^2} + O(r^{-3}),$$

as $r \to \pm\infty$ with natural derivative bounds. We denote the following resolvents $R(z) = (-\Delta_g - z)^{-1}$ and $R_H(z) = (H-z)^{-1}$ for $z \notin \sigma(-\Delta_g) = \sigma(H) = [0,\infty)$. We note that the decay of V implies that the spectrum of H in $(0,\infty)$ is purely absolutely continuous (in fact, absolute continuity follows from the following explicit formula for the spectral measure).

Via Stone's theorem, we can write (as an identity of Schwartz kernels)

$$\begin{aligned} E(d\lambda^2)(r,\rho) &= \frac{\lambda}{\pi i} \lim_{\epsilon\to 0^+} (R(\lambda^2+i\epsilon) - R(\lambda^2 - i\epsilon))(r,\rho) d\lambda \\ &= \frac{\lambda}{\pi i} \lim_{\epsilon\to 0^+} \langle r\rangle^{-d/2} (R_H(\lambda^2+i\epsilon) - R_H(\lambda^2-i\epsilon))(r,\rho)\langle\rho\rangle^{d/2} d\lambda. \end{aligned}$$

The final limit may be evaluated 'explicitly' by using the fact that

$$\begin{aligned} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}(R_H(\lambda^2+i\epsilon) - R_H(\lambda^2-i\epsilon))(r,\rho) =& \Im\left[\frac{f_+(r,\lambda) f_-(\rho,\lambda)}{W(\lambda)}\right]\chi_{[r>\rho]} \\ &+ \Im\left[\frac{f_-(r,\lambda) f_+(\rho,\lambda)}{W(\lambda)}\right]\chi_{[r<\rho]}, \end{aligned}$$

where $f_\pm(\cdot,\lambda)$ are the *Jost solutions* which satisfy

$$Hf_\pm(r,\lambda) = \lambda^2 f_\pm(r,\lambda),$$
$$f_\pm(r,\lambda) \sim e^{\pm ir\lambda} \quad \text{as } r \to \pm\infty,$$

and

$$W(\lambda) = W(f_-(\cdot,\lambda), f_+(\cdot,\lambda)) = f_+'(\cdot,\lambda)f_-(\cdot,\lambda) - f_+(\cdot,\lambda)f_-'(\cdot,\lambda),$$

is their Wronskian. It is easy to see via a standard contraction argument that $f_\pm(\cdot,\lambda)$ exist provided $V \in L^1(\mathbb{R})$. In summary, we see that the spectral measure for $-\Delta_g$ satisfies

$$\begin{aligned} E(d\lambda^2)(r,\rho) = 2\lambda\langle r\rangle^{-d/2}\Bigg\{ &\Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W_\nu(\lambda)}\right]\chi_{[r>\rho]} \\ &+\Im\left[\frac{f_-(r,\lambda)f_+(\rho,\lambda)}{W_\nu(\lambda)}\right]\chi_{[r<\rho]}\Bigg\}\langle\rho\rangle^{d/2}d\lambda. \end{aligned}$$

Therefore, the estimate (2.3.2) (and thus, Proposition 2.3.1) reduces to proving the following oscillatory integral estimate uniformly in $r > \rho$ (the case $r < \rho$ is analagous) which we state as a proposition.

Proposition 2.3.2. *For all $\rho < r$ and $t \in \mathbb{R}$ we have the estimate*

$$\left|\int_0^\infty e^{\pm it\lambda}\varphi(2^{-j}\lambda)\lambda\Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right]d\lambda\right| \lesssim (\langle r\rangle\langle\rho\rangle)^{d/2}2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

The implied constant depends only on φ and d.

Note that we absorbed the volume form $(r^2+1)^{d/2}dr$ implicit in the right hand side of (2.3.2) into the left hand side in order to conclude that proving the estimate 2.3.2 reduces to proving Proposition 2.3.2. To prove Proposition 2.3.2, we will need asymptotics for $f_\pm(\cdot,\lambda)$ and $W(\lambda)$ for λ small. The asymptotics that we require are contained in the following subsection.

2.3.1 Scattering Theory for Schrodinger Operators

In this section, we briefly summarize the scattering theory developed in Section 3 of [28] for the Schrödinger operator $H = -\frac{d^2}{dr^2} + V$ on $\mathbb{R}$ where $V \in C^\infty(\mathbb{R})$ is real–valued and such that

$$V(r) = \frac{d(d-2)}{4} r^{-2} + U(r), \quad U \in C^\infty(\mathbb{R}\backslash\{0\}),$$

with

$$|U^{(k)}(r)| \leq C_k |r|^{-3-k}, \quad |r| \geq 1.$$

In particular, we summarize the asymptotics for $f_\pm(\cdot, \lambda)$ and $W(\lambda)$ as $\lambda \to 0$ under a condition on the point spectrum of H. This condition will be elaborated on below. In what follows, we assume, as before, that $d \geq 2$.

First, solutions to the zero energy equation with slow decay at $\pm\infty$ were constructed.

Lemma 2.3.3 (Lemma 3.2 [28])**.** *For $j = 0, 1$, there exist real–valued solutions $u_j^\pm(\cdot)$ to the zero energy equation*

$$-u_j^\pm(r)'' + V(r) u_j^\pm(r) = 0, \quad r \in \mathbb{R},$$

such that $W(u_0^\pm(\cdot), u_1^\pm(\cdot)) =$constant, and $u_j^\pm$ have the asymptotics

$$u_0^\pm(r) = |r|^{d/2}(1 + O(|r|^{-1})), \quad \text{as } r \to \pm\infty,$$
$$u_1^\pm(r) = |r|^{-(d-2)/2}(1 + O(|r|^{-1})), \quad \text{as } r \to \pm\infty.$$

The $O(\cdot)$ terms behave like symbols under differentiation in r.

Definition 2.3.4. *We say that the Schrödinger operator H has 0 as a resonance if*

$$W(u_1^+(\cdot), u_1^-(\cdot)) = 0,$$

where $u_1^{\pm}(\cdot)$ are the solutions constructed in Lemma 2.3.3. This condition is equivalent to the existence of a nonzero solution f to $-f'' + Vf = 0$ such that f is asymptotic to $|r|^{-(d-2)/2}$ at $\pm\infty$.

The previously mentioned condition on the point spectrum of H is that 0 is not a resonance. Next, perturbing in small λ, for $j = 0, 1$ a basis of real–valued solutions $u_j^{\pm}(\cdot, \lambda)$ to

$$-u_j^{\pm}(r,\lambda)'' + V(r)u_j^{\pm}(r,\lambda) = \lambda^2 u_j^{\pm}(r,\lambda), \quad r \in \mathbb{R},$$

was constructed which are well approximated by $u_j^{\pm}$ when $|r\lambda| \ll 1$.

Lemma 2.3.5 (Corollary 3.5 [28])**.** *Let $u_j^+(\cdot)$ be as in Lemma 2.3.3. There exist solutions $u_j^+(\cdot, \lambda)$ of $Hf = \lambda^2 f$ with*

$$W(u_1^+(\cdot,\lambda), u_0^+(\cdot,\lambda)) = 1,$$

such that for $j = 0, 1$ and $r_0 \leq r \ll \lambda^{-1}$, we have

$$u_j^+(r,\lambda) = u_j^+(r)(1 + a_j^+(r,\lambda)).$$

The functions $a_j^+(\cdot,\lambda)$ *satisfy the bounds*

$$\left|\partial_r^l \partial_\lambda^k a_j^+(r,\lambda)\right| \lesssim_{k,l} \begin{cases} \lambda^{2-k}\langle r\rangle^{2-l}\log|\lambda r| & \text{if } d=2, \\ \lambda^{2-k}\langle r\rangle^{2-l} & \text{if } d>2. \end{cases}$$

A similar statement holds with $u_0^+(\cdot,\lambda)$ *replaced by* $u_0^-(\cdot,\lambda)$ *for* $r \leq 0$.

In what follows, $\beta_d = \sqrt{\frac{\pi}{2}}e^{id\pi/4}$. The outgoing Jost solution for

$$H_0 = -\frac{d^2}{dr^2} + \frac{d(d-2)}{2r^2}$$

is known explicitly. In particular, we have that the solution to $H_0 f_0(\cdot,\lambda) = \lambda^2 f_0(\cdot,\lambda)$ with $f_0(r,\lambda) \sim e^{i\lambda r}$ as $r \to \infty$ is given by

$$f_0(r,\lambda) = \beta_d \sqrt{r\lambda} H^+_{(d-1)/2}(r\lambda),$$

where $H^+_{(d-1)/2}(z) = J_{(d-1)/2}(z) + iY_{(d-1)/2}(z)$ is the Hankel function. Perturbing off of this explicit solution, we obtain the following asymptotic form for the Jost function $f_+(\cdot,\lambda)$. Similar asymptotics hold for $f_-(\cdot,\lambda)$.

Lemma 2.3.6 (Corollary 3.10 [28])**.** *For* $\lambda \neq 0$, $\lambda \ll 1$, *and in the range* $1 \ll r \ll \lambda^{-1}$, *we have*

$$\begin{aligned} f_+(r,\lambda) = \beta_d\sqrt{\lambda r}\Big[&J_{(d-1)/2}(r\lambda)(1+O(\lambda))(1+O(r^{-1})) \\ &+ Y_{(d-1)/2}(r\lambda)O(\lambda)(1+O(r^{-1}))\Big] \\ + i\beta_d\sqrt{\lambda r}\Big[&Y_{(d-1)/2}(r\lambda)(1+O(\lambda))(1+O(r^{-1})) \\ &+ J_{(d-1)/2}(r\lambda)O(\lambda)(1+O(r^{-1}))\Big]. \end{aligned}$$

In the range $r\lambda \gtrsim 1$*, we have*

$$f_+(r,\lambda) = e^{ir\lambda} m_+(r,\lambda),$$

where

$$m_+(r,\lambda) = 1 + O_{\mathbb{C}}(r^{-1}\lambda^{-1})$$

The $O(\cdot)$ *terms are real–valued, the* $O_{\mathbb{C}}(\cdot)$ *term is complex–valued, and all terms obey the natural bounds with respect to differentiation in* λ *and* r*.*

Using the previous lemmas, the following expansions were obtained.

Lemma 2.3.7 (Corollary 3.6 and Proposition 3.12 [28])**.** *We have the expansions*

$$f_\pm(r,\lambda) = a_\pm(\lambda) u_0^\pm(r,\lambda) + b_\pm(\lambda) u_1^\pm(r,\lambda),$$

where the coefficients satisfy with some small $\epsilon > 0$ *depending on* d *and with some real constants* $\alpha_0^\pm, \beta_0^\pm$*,*

$$\begin{aligned} a_\pm(\lambda) &= \lambda^{d/2}\beta_d\left(\alpha_0^\pm + O(\lambda^\epsilon) + iO(\lambda^{-(d-2)\epsilon})\right), \\ b_\pm(\lambda) &= i\lambda^{-(d-2)/2}\beta_d\left(\beta_0^\pm + O(\lambda^\epsilon) + iO(\lambda^{d\epsilon})\right). \end{aligned}$$

The $O(\cdot)$ *terms are real–valued and satisfy the natural derivative bounds.*

Using the expansions in Lemma 2.3.7, an asymptotic expansion for $W(\lambda)$ for small λ under the nonresonant condition was obtained.

Lemma 2.3.8 (Corollary 3.13)**.** *If 0 is not a resonance for* H*, then for all* $0 < \epsilon < \epsilon_0(d)$*,* *we have*

$$W(\lambda) = ie^{i\pi(d-1)/2}\lambda^{-(d-2)}(W_0 + O_{\mathbb{C}}(\lambda^{\epsilon})). \tag{2.3.3}$$

Here W_0 is a nonzero real constant and $O_{\mathbb{C}}(\lambda^{\epsilon})$ is complex valued, and all terms satisfy the natural derivative bounds. We remark that the nonresonant condition is what guarantees that the constant W_0 is nonzero.

Finally, the following asymptotic expansion for the spectral measure corresponding to H for small λ was obtained.

Lemma 2.3.9 (Corollary 5.1 [28])**.** *If 0 is not a resonance for H, then for $0 < \lambda \ll 1$ and any $r, \rho \in \mathbb{R}$,*

$$\begin{aligned} \Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right] &= O(\lambda^{d-1})u_0^+(r,\lambda)u_1^-(\rho,\lambda) \\ +O(\lambda^{d-1})u_1^+(r,\lambda)u_0^-(\rho,\lambda) &+ O(\lambda^{d-1})u_0^+(r,\lambda)u_0^-(\rho,\lambda) \\ &+O(\lambda^{d-1})u_1^+(r,\lambda)u_1^-(\rho,\lambda), \end{aligned}$$

where the $O(\cdot)$ terms are real–valued and satisfy the natural derivative bounds.

We now turn to proving the oscillatory integral estimate Proposition 2.3.2.

2.3.2 Proof of Proposition 2.3.2

We recall that we wish to prove the oscillatory integral estimate

$$\left|\int_0^{\infty} e^{\pm it\lambda}\varphi(2^{-j}\lambda)\lambda\Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right]d\lambda\right| \lesssim (\langle r\rangle\langle\rho\rangle)^{d/2}2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2},$$

for all $r > \rho$ and $t \in \mathbb{R}$. Here H is the Schrödinger operator on $\mathbb{R}$

$$H = -\frac{d^2}{dr^2} + V, \quad V(r) = \frac{d(d-4)}{4} r^2 \langle r \rangle^{-4} + \frac{d}{2} \langle r \rangle^{-2},$$

and $f_\pm(\cdot, \lambda)$ are the Jost functions associated to H. We distinguish the cases $j \ll 0$ and $j \gtrsim 0$. The case $j \ll 0$ will rely heavily on the scattering theory summarized in the previous subsection.

We first consider the case $j \ll 0$ so that the integrand in the oscillatory integral is localized to small λ. We first claim that H is nonresonant so that the results summarized in the previous section apply. Indeed, if 0 is a resonance of H, then there exists a nonzero function f such that $Hf = 0$ and $f(r) = O(\langle r \rangle^{-(d-2)/2})$ as $|r| \to \infty$. This implies by the relation $-\Delta_g(\langle r \rangle^{-d/2} f) = \langle r \rangle^{-d/2} Hf$ that there exists a nonzero function u such that $\Delta_g u = 0$ and $u(r) = O(\langle r \rangle^{-(d-1)})$ as $|r| \to \infty$. Since $d \geq 2$, the maximum principle on $\mathcal{M}$ implies that $u \equiv 0$, a contradiction. Thus, 0 is not a resonance of the Schrodinger operator H.

The proof of Proposition 2.3.2 for $j \ll 0$ is split up into several lemmas. In what follows, we differentiate between the oscillatory regime and the exponential regime for the Jost solutions $f_\pm(\cdot, \lambda)$. This transition occurs at $|r\lambda| = 1$. Let $\chi \in C_0^\infty(\mathbb{R})$ be even with $\chi(r) = 1$ for $|r| \leq 1$ and supp $\chi \subset \{|r| < 2\}$. We denote the smooth cutoff $\chi(r\lambda)$ by $\chi_{[|r\lambda|<1]}$ and the smooth cutoff $(1 - \chi(r\lambda))$ by $\chi_{[|r\lambda|>1]}$.

Lemma 2.3.10. *For all $t \in \mathbb{R}$ and $r, \rho \in \mathbb{R}$,*

$$\left| \int_0^\infty e^{\pm it\lambda} \lambda \varphi(2^{-j}\lambda) \chi_{[|r\lambda|<1]} \chi_{[|\rho\lambda|<1]} (\langle r \rangle \langle \rho \rangle)^{-d/2} \Im \left[\frac{f_+(r,\lambda) f_-(\rho,\lambda)}{W(\lambda)} \right] d\lambda \right| \tag{2.3.4}$$

$$\lesssim 2^{j(d+2)/2} (2^{-j} + |t|)^{-d/2}. \tag{2.3.5}$$

Proof. By Lemma 2.3.9 we may write

$$\Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right] = O(\lambda^{d-1})O\left((\langle r\rangle\langle\rho\rangle)^{d/2}\right), \tag{2.3.6}$$

where the $O(\cdot)$ terms satisfy natural derivative bounds. We write (2.3.5) as

$$\left|\int_0^\infty e^{\pm it\lambda}a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}, \tag{2.3.7}$$

where

$$a_j(r,\rho,\lambda) = \lambda\varphi(2^{-j}\lambda)\chi_{[|r\lambda|<1]}\chi_{[|\rho\lambda|<1]}(\langle r\rangle\langle\rho\rangle)^{-d/2}\Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right].$$

By (2.3.6) the function $a_j(r,\rho,\lambda)$ satisfies

$$a_j(r,\rho,\lambda) = \varphi(2^{-j}\lambda)O(\lambda^d), \tag{2.3.8}$$

with natural derivative bounds.

First note that if $|t| \le 2^{-j}$, then by (2.3.8)

$$\left|\int_0^\infty e^{\pm it\lambda}a_j(r,\rho,\lambda)d\lambda\right| \lesssim \int_{[\lambda\simeq 2^j]}\lambda^d d\lambda \lesssim 2^{j(d+1)} \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

We now assume that $|t| \ge 2^{-j}$. Integration by parts d times and (2.3.8) yield

$$\begin{aligned}\left|\int_0^\infty e^{\pm it\lambda}a_j(r,\rho,\lambda)d\lambda\right| &= |t|^{-d}\left|\int_0^\infty e^{\pm it\lambda}\partial_\lambda^d a_j(r,\rho,\lambda)d\lambda\right| \\ &\lesssim |t|^{-d}\int_{[\lambda\sim 2^j]}d\lambda \\ &\lesssim |t|^{-d}2^j \\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

This concludes the proof. □

We now consider the case when the integrand is supported in $|r\lambda| > 1$ and $|\rho\lambda| > 1$. With the convention that $f_{\pm}(\cdot, -\lambda) = \overline{f_{\pm}(\cdot, \lambda)}$, we remove the taking of an imaginary part in the integrand and write

$$\int_0^{\infty} e^{\pm it\lambda} \lambda \varphi(2^{-j}\lambda)\chi(\lambda r)\chi(\lambda\rho)(\langle r\rangle\langle\rho\rangle)^{-d/2} \Im\left[\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right] d\lambda$$
$$= \int_{-\infty}^{\infty} e^{\pm it|\lambda|} \lambda \varphi(2^{-j}\lambda)\chi(\lambda r)\chi(\lambda\rho)(\langle r\rangle\langle\rho\rangle)^{-d/2} \frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)} d\lambda.$$

We first consider the case $\rho < 0 < r$.

Lemma 2.3.11. *For all $t \in \mathbb{R}$ and $\rho < 0 < r$*

$$\left|\int_{-\infty}^{\infty} e^{\pm it|\lambda|} \lambda \varphi(2^{-j}\lambda)\chi_{[|r\lambda|>1]}\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2} \frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)} d\lambda\right| \tag{2.3.9}$$
$$\lesssim 2^{j(d+2)/2}(2^{-j} + |t|)^{-d/2}. \tag{2.3.10}$$

Proof. We first note that by Lemma 2.3.6 and Lemma 2.3.8,

$$\sup_{|r\lambda|>1, |\rho\lambda|>1} |\lambda| \left|\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}\right| \lesssim 1.$$

This implies that

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} e^{\pm it|\lambda|} \lambda \varphi(2^{-j}\lambda)\chi_{[|r\lambda|>1]}\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2} \frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)} d\lambda\right| \\
&\lesssim \int_{-\infty}^{\infty} \varphi(2^{-j}\lambda)\chi_{[|r\lambda|>1]}\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2} d\lambda \\
&\lesssim \int_{-\infty}^{\infty} \varphi(2^{-j}\lambda)\lambda^d d\lambda \\
&\lesssim 2^{j(d+1)}.
\end{aligned}$$

Thus, we only need to consider the case $|t| \geq 2^{-j}$.

Assume that $|t| \geq 2^{-j}$. By Lemma 2.3.6, we write

$$f_+(r,\lambda) = e^{i\lambda r} m_+(r,\lambda), \quad f_-(\rho,\lambda) = e^{-i\lambda\rho} m_-(\rho,\lambda),$$

where

$$m_+(r,\lambda) = 1 + O(\lambda^{-1} r^{-1}), \quad r|\lambda| > 1, \tag{2.3.11}$$

with natural derivative bounds. A similar expression holds for $m_-(\rho,\lambda)$. We express (2.3.10) as

$$\left| \int_{-\infty}^{\infty} e^{i|\lambda|\left(\pm t + \frac{\lambda}{|\lambda|}(r-\rho)\right)} a_j(r,\rho,\lambda) d\lambda \right| \lesssim 2^{j(d+2)/2} (2^{-j} + |t|)^{-d/2},$$

where

$$a_j(r,\rho,\lambda) = \lambda \varphi(2^{-j}\lambda) \chi_{[|r\lambda|>1]} \chi_{[|\rho\lambda|>1]} (\langle r \rangle \langle \rho \rangle)^{-d/2} \frac{m_+(r,\lambda) m_-(\rho,\lambda)}{W(\lambda)}.$$

By Lemma 2.3.8

$$\frac{\lambda}{W(\lambda)} = O(\lambda^{d-1}),$$

with natural derivative bounds. This fact and (2.3.11) imply that

$$a_j(r,\rho,\lambda) = \varphi(2^{-j}\lambda) \chi_{[|r\lambda|>1]} \chi_{[|\rho\lambda|>1]} O(\lambda^{d-1}) (\langle r \rangle \langle \rho \rangle)^{-d/2}. \tag{2.3.12}$$

Note that if $|\lambda|$ is small, $|r\lambda| > 1$, and $|\rho\lambda| > 1$, then we have

$$\begin{aligned}(\langle r\rangle\langle\rho\rangle)^{-d/2} &\lesssim \langle r-\rho\rangle^{-d/2},\\ (\langle r\rangle\langle\rho\rangle)^{-d/2} &\leq \lambda^d.\end{aligned}$$

If $|t| \lesssim |r-\rho|$, then since $j \ll 0$, we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{i|\lambda|\left(\pm t+\frac{\lambda}{|\lambda|}(r-\rho)\right)} a_j(r,\rho,\lambda)d\lambda\right| &\lesssim \int_{[|\lambda|\sim 2^j]} |\lambda|^{d-1} d\lambda \langle r-\rho\rangle^{-d/2}\\ &\lesssim 2^{dj}|t|^{-d/2}\\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

Now suppose $|t| \gg |r-\rho|$. By (2.3.12) and integration by parts

$$\begin{aligned}\left|\int_0^{\infty} e^{i\lambda(\pm t+(r-\rho))} a_j(r,\rho,\lambda)d\lambda\right| &= |\pm t+(r-\rho)|^{-d}\left|\int_0^{\infty} e^{i\lambda(\pm t+(r-\rho))}\partial_\lambda^d a_j(r,\rho,\lambda)d\lambda\right|\\ &\lesssim |t|^{-d}\int_{[\lambda\sim 2^j]} \lambda^{d-1}d\lambda\\ &\lesssim |t|^{-d}2^{dj}\\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

A similar argument shows that

$$\left|\int_{-\infty}^{0} e^{-i\lambda(\pm t-(r-\rho))} a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

This concludes the proof. □

We now consider the case when $|r\lambda| > 1$ and $|\rho\lambda| < 1$ in the integrand. The case $|r\lambda| < 1$ and $|\rho\lambda| > 1$ can be handled similarly.

Lemma 2.3.12. *For all $t, r \in \mathbb{R}$ and $\rho < r$*

$$\left| \int_{-\infty}^{\infty} e^{\pm it|\lambda|} \lambda \varphi(2^{-j}\lambda) \chi_{[|r\lambda|>1]} \chi_{[|\rho\lambda|<1]} (\langle r \rangle \langle \rho \rangle)^{-2} \frac{f_+(r,\lambda) f_-(\rho,\lambda)}{W(\lambda)} d\lambda \right| \tag{2.3.13}$$

$$\lesssim 2^{j(d+2)/2} (2^{-j} + |t|)^{-d/2}. \tag{2.3.14}$$

Proof. We write $f_+(r,\lambda) = e^{ir\lambda} m_+(r,\lambda)$ as before, but since $|\rho\lambda| < 1$, we use the representation

$$f_-(\rho,\lambda) = a_-(\lambda) u_0^-(\rho,\lambda) + b_-(\lambda) u_1^-(\rho,\lambda). \tag{2.3.15}$$

In particular, we have that

$$f_-(\rho,\lambda) = O(\lambda^{-(d-2)/2}) O(\langle \rho \rangle^{d/2}).$$

Now we write (2.3.14) as

$$\left| \int_{-\infty}^{\infty} e^{i|\lambda|(\pm t + \frac{\lambda}{|\lambda|} r)} a_j(r,\rho,\lambda) d\lambda \right| \lesssim 2^{j(d+2)/2} (2^{-j} + |t|)^{-d/2},$$

where

$$\begin{aligned} a_j(r,\rho,\lambda) &= \lambda \varphi(2^{-j}\lambda) \chi_{[|r\lambda|>1]} \chi_{[|\rho\lambda|<1]} (\langle r \rangle \langle \rho \rangle)^{-d/2} \frac{m_+(r,\lambda) f_-(\rho,\lambda)}{W(\lambda)} \\ &= \varphi(2^{-j}) \chi_{[|r\lambda|>1]} \chi_{[|\rho\lambda|<1]} O(\lambda^{d/2}) \langle r \rangle^{-d/2}, \end{aligned}$$

with natural derivative bounds. As before, in the case $|t| \geq 2^{-j}$ we have

$$\begin{aligned}\left|\int_0^\infty e^{i\lambda(\pm t+r)}a_j(r,\rho,\lambda)d\lambda\right| &\lesssim \int_{[\lambda\sim 2^j]}\lambda^d d\lambda\\ &\lesssim 2^{j(d+1)}\\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

Thus, we need only consider the case that $|t| \geq 2^{-j}$.

Suppose that $|t| \geq 2^{-j}$. If $|t| \lesssim |r|$ then

$$\begin{aligned}\left|\int_0^\infty e^{i\lambda(\pm t+r)}a_j(r,\rho,\lambda)d\lambda\right| &\lesssim \int_{[\lambda\simeq 2^j]}\lambda^{d/2}|t|^{-d/2}d\lambda\\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

If $|t| \gg |r|$, then by integration by parts

$$\begin{aligned}\left|\int_0^\infty e^{i\lambda(\pm t+r)}a_j(r,\rho,\lambda)d\lambda\right| &= |\pm t+r|^{-d}\left|\int_0^\infty e^{i\lambda(\pm t+r)}\partial_\lambda^d a_j(r,\rho,\lambda)d\lambda\right|\\ &\lesssim |t|^{-d}\int_{[\lambda\sim 2^j]}dr\\ &\lesssim |t|^{-d}2^j\\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2},\end{aligned}$$

as desired. Similarly,

$$\left|\int_{-\infty}^0 e^{i\lambda(\pm t-r)}a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

This concludes the proof. □

To finish proving Proposition 2.3.2 in the case $j \ll 0$, we need only consider the case when the integrand is supported in $|\lambda|^{-1} < \rho < r$. The case $\rho < r < -|\lambda|^{-1}$ can be dealt with in a similar fashion. We consider *reflection and transmission coefficients* $\alpha_-(\lambda), \beta_-(\lambda)$ defined by the relation

$$f_-(\rho,\lambda) = \alpha_-(\lambda) f_+(\rho,\lambda) + \beta_-(\lambda)\overline{f_+(\rho,\lambda)}.$$

Then

$$\begin{aligned}
W(\lambda) &= W(f_-(\cdot,\lambda), f_+(\cdot,\lambda)) \\
&= -\beta_-(\lambda) W(f_+(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) \\
&= -\beta_-(\lambda) \lim_{r\to\infty} W(f_+(r,\lambda), \overline{f_+(r,\lambda)}) \\
&= -\beta_-(\lambda) \lim_{r\to\infty} W(e^{i\lambda r}, e^{-i\lambda r}) \\
&= 2i\lambda\beta_-(\lambda).
\end{aligned}$$

Let $\widetilde{W}(\lambda) = W(f_-(\cdot,\lambda), \overline{f_+(\cdot,\lambda)})$. Then similar to $W(\lambda)$ we have

$$\begin{aligned}
\widetilde{W}(\lambda) &= \alpha_-(\lambda) W(f_+(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) \\
&= -2i\lambda\alpha_-(\lambda).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\lambda\frac{\beta_-(\lambda)}{W(\lambda)} &= \frac{1}{2i}, \\
\lambda\frac{\alpha_-(\lambda)}{W(\lambda)} &= -\frac{1}{2i}\frac{\widetilde{W}(\lambda)}{W(\lambda)} = \text{constant} + O(\lambda^\epsilon),
\end{aligned}$$

where the $O(\lambda^\epsilon)$ term is complex valued and satisfies natural derivative bounds. The second equality in the second line above follows from Lemma 2.3.7.

Lemma 2.3.13. *For all $t \in \mathbb{R}$ and $0 < \rho < r$*

$$\left|\int_{-\infty}^{\infty} e^{\pm it|\lambda|}\lambda\varphi(2^{-j}\lambda)\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-2}\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}. \tag{2.3.16}$$

Proof. We write $f_+(r,\lambda) = e^{i\lambda r}m_+(r,\lambda)$. Then

$$\begin{aligned}\lambda\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)} &= e^{i(r+\rho)\lambda}\lambda\frac{\alpha_-(\lambda)}{W(\lambda)}m_+(r,\lambda)m_+(\rho,\lambda) + e^{i(r-\rho)\lambda}\lambda\frac{\beta_-(\lambda)}{W(\lambda)}m_+(r,\lambda)\overline{m_+(\rho,\lambda)}\\ &= e^{i(r+\rho)\lambda}O(1)m_+(r,\lambda)m_+(\rho,\lambda) + \frac{1}{2i}e^{i(r-\rho)\lambda}m_+(r,\lambda)\overline{m_+(\rho,\lambda)}\end{aligned}$$

where the $O(1)$ term is complex valued and satisfies natural derivative bounds. We are thus reduced to proving the following two estimates

$$\left|\int_{-\infty}^{\infty} e^{\pm it|\lambda|(\pm t+\frac{\lambda}{|\lambda|}(r+\rho))}\varphi(2^{-j}\lambda)\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2}O(1)m_+(r,\lambda)m_+(\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}, \tag{2.3.17}$$

$$\left|\int_{-\infty}^{\infty} e^{\pm it|\lambda|(\pm t+\frac{\lambda}{|\lambda|}(r-\rho))}\varphi(2^{-j}\lambda)\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2}m_+(r,\lambda)\overline{m_+(\rho,\lambda)}d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}. \tag{2.3.18}$$

We now prove (2.3.17).

We write (2.3.17) as

$$\left|\int_{-\infty}^{\infty} e^{i|\lambda|(\pm t+\frac{\lambda}{|\lambda|}(r+\rho))}a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

where

$$
\begin{aligned}
a_j(r,\rho,\lambda) &= \varphi(2^{-j}\lambda)\chi_{[|\rho\lambda|>1]})(\langle r\rangle\langle\rho\rangle)^{-d/2}O(1)m_+(r,\lambda)m_+(\rho,\lambda) \\
&= \varphi(2^{-j}\lambda)\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2}O(1),
\end{aligned}
$$

with the $O(\cdot)$ term behaving like a symbol under differentiation in λ. Note that if $|t| \leq 2^{-j}$ then $|r\lambda| > 1$ and $|\rho\lambda| > 1$ imply that

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} e^{i|\lambda|(\pm t+\frac{\lambda}{|\lambda|}(r+\rho))}a_j(r,\rho,\lambda)d\lambda\right| &\lesssim \int_{[\lambda\sim 2^j]} \lambda^d d\lambda \\
&\lesssim 2^{j(d+1)} \\
&\lesssim 2^{j(d+2)/2)}(2^{-j}+|t|)^{-d/2}.
\end{aligned}
$$

Thus, we need only consider $|t| \geq 2^{-j}$.

Suppose that $|t| \leq 2(r+\rho)$. Then $0 < \rho < r$ implies that $r \geq |t|/4$ so that

$$
\chi_{[|\rho\lambda|>1]}(\langle r\rangle\langle\rho\rangle)^{-d/2} \lesssim \lambda^{d/2}|t|^{-d/2}.
$$

Thus,

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} e^{i|\lambda|(\pm t+\frac{\lambda}{|\lambda|}(r+\rho))}a_j(r,\rho,\lambda)d\lambda\right| &\lesssim \int_{[\lambda\simeq 2^j]} \lambda^{d/2}|t|^{-d/2}d\lambda \\
&\lesssim 2^{j(d+2)/2}|t|^{-d/2}.
\end{aligned}
$$

as desired. Suppose now that $|t| \geq 2(r+\rho)$. Integration by parts yields

$$\begin{aligned}\left|\int_0^\infty e^{i\lambda(\pm t+(r+\rho))} a_j(r,\rho,\lambda)d\lambda\right| &= |\pm t+(r+\rho)|^{-d}\left|\int_0^\infty e^{i\lambda(\pm t+(r+\rho))}\partial_\lambda^d a_j(r,\rho,\lambda)d\lambda\right| \\ &\lesssim |t|^{-d}\int_{[\lambda\simeq 2^j]} \lambda^{-d}\lambda^d d\lambda \\ &\lesssim |t|^{-d}2^j \\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

In a similar fashion, we obtain

$$\left|\int_{-\infty}^0 e^{-i\lambda(\pm t-(r+\rho))} a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}|t|^{-d/2}.$$

This proves (2.3.17). The proof of (2.3.18) is similar and is omitted. □

We now prove Proposition 2.3.2 in the case $j \gtrsim 0$. This case is considerably simpler than the case $j \ll 0$ since the Jost functions $f_\pm(\cdot,\lambda)$ and their Wronskian $W(\lambda)$ are to given by the free case $H = -\frac{d^2}{dr^2}$ to leading order. Indeed, we write

$$f_+(r,\lambda) = e^{ir\lambda}m_+(r,\lambda), \quad f_-(\rho,\lambda) = e^{-i\rho\lambda}m_-(\rho,\lambda).$$

From [27], we have the estimates

$$\begin{aligned} m_+(r,\lambda) &= 1 + O(\lambda^{-1}\langle r\rangle^{-1}), \\ \left|\partial_\lambda^l \partial_r^k m_+(r,\lambda)\right| &\lesssim_{l,k} \lambda^{-1-l}\langle r\rangle^{-1-k}\end{aligned}$$

for $r \geq 0$ and $l+k>0$. Similar estimates hold for $m_-(\rho,\lambda)$ with $\rho \leq 0$. It is well known that $|W(\lambda)| \geq |\lambda|$ for all λ. Using the asymptotics for $m_\pm(\cdot,\lambda)$, we compute the Wronskian

$$\begin{aligned}
W(\lambda) &= W(f_-(\cdot,\lambda), f_+(\cdot,\lambda)) \\
&= m_+(0,\lambda)(m_-'(0,\lambda) - i\lambda m_-(0,\lambda)) - m_-(0,\lambda)(m_+'(0,\lambda) + i\lambda m_+(0,\lambda)) \\
&= -2i\lambda + O(\lambda^{-1}),
\end{aligned}$$

with natural derivative bounds. We also compute the Wronskian

$$\begin{aligned}
W(f_-(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) &= m_-(0,\lambda)(\bar{m}_+'(0,\lambda) - 2i\lambda \bar{m}_+(0,\lambda)) \\
&\quad - \bar{m}_+(0,\lambda)(m_-'(0,\lambda) - 2i\lambda m_-(0,\lambda)) \\
&= m_-(0,\lambda)\bar{m}_+'(0,\lambda) - m_-'(0,\lambda)\bar{m}_+(0,\lambda) \\
&= O(\lambda^{-1})
\end{aligned}$$

with symbol character in λ. We now prove Proposition 2.3.2 in the case $j \gtrsim 0$.

Lemma 2.3.14. *For all $\rho < r$*

$$\left| \int_{-\infty}^{\infty} e^{\pm it|\lambda|} \lambda \varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2} \frac{f_+(r,\lambda) f_-(\rho,\lambda)}{W(\lambda)} d\lambda \right| \lesssim 2^{j(d+2)/2}(2^{-j} + |t|)^{-d/2}. \tag{2.3.19}$$

Proof. We first note that the fact that

$$\sup_{r,\rho} |\lambda| \left| \frac{f_+(r,\lambda) f_-(\rho,\lambda)}{W(\lambda)} \right| \lesssim 1,$$

implies that

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{\pm it|\lambda|}\lambda\varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d}\frac{f_+(r,\lambda)f_-(\rho,\lambda)}{W(\lambda)}d\lambda\right| &\lesssim \int_{-\infty}^{\infty}\varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d}d\lambda\\ &\lesssim 2^j\\ &\lesssim 2^{j(d+2)/2}.\end{aligned}$$

In the last line we used $j \gtrsim 0$. Thus, we only need to consider the case $|t| \geq 2^{-j}$. We split the remainder of the proof into cases: $\rho < 0 < r$, $0 < \rho < r$, and $\rho < r < 0$. By symmetry we consider only the first two.

Assume $\rho < 0 < r$. We write

$$f_+(r,\lambda) = e^{i\lambda r}m_+(r,\lambda), \quad f_-(\rho,\lambda) = e^{-i\lambda\rho}m_-(\rho,\lambda),$$

where we have for all $r \geq 0$

$$|m_+(r,\lambda)| \lesssim 1, \tag{2.3.20}$$

$$\left|\partial_\lambda^l m_+(r,\lambda)\right| \lesssim \lambda^{-1-l}, \quad l > 0, \tag{2.3.21}$$

with similar estimates holding for $m_-(\rho,\lambda)$ for $\rho \leq 0$. We express (2.3.19) as

$$\left|\int_{-\infty}^{\infty} e^{i|\lambda|\left(\pm t+\frac{\lambda}{|\lambda|}(r-\rho)\right)}a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2},$$

where

$$a_j(r,\rho,\lambda) = \lambda\varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2}\frac{m_+(r,\lambda)m_-(\rho,\lambda)}{W(\lambda)}.$$

By (2.3.20) and (2.3.21) we have

$$a_j(r,\rho,\lambda) = \varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2}O(1), \tag{2.3.22}$$

with natural derivative bounds.

Suppose that $|t| \le 2|r-\rho|$. Then either $|r| \ge |t|/4$ or $|\rho| \ge |t|/4$. Suppose, without loss of generality, $|r| \ge |t|/4$. Then by (2.3.22) we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{i|\lambda|\left(\pm t + \frac{\lambda}{|\lambda|}(r-\rho)\right)} a_j(r,\rho,\lambda)d\lambda\right| &\lesssim \int_{[\lambda\sim 2^j]} d\lambda(\langle r\rangle\langle\rho\rangle)^{-d/2} \\ &\lesssim 2^j|t|^{-d/2} \\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.\end{aligned}$$

Suppose now that $|t| \ge 2|r-\rho|$. Then by (2.3.22) and integration by parts

$$\begin{aligned}\left|\int_0^{\infty} e^{i\lambda(\pm t+(r-\rho))}a_j(r,\rho,\lambda)d\lambda\right| &= |\pm t+(r-\rho)|^{-d}\left|\int_0^{\infty} e^{i\lambda(\pm t+(r-\rho))}\partial_\lambda^d a_j(r,\rho,\lambda)d\lambda\right| \\ &\lesssim |t|^{-d}\int_{[\lambda\sim 2^j]} d\lambda \\ &\lesssim |t|^{-d}2^j \\ &\lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2},\end{aligned}$$

as desired. Similarly,

$$\left|\int_{-\infty}^{0} e^{i|\lambda|(\pm t-(r-\rho))}a_j(r,\rho,\lambda)d\lambda\right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

This concludes the case $\rho < 0 < r$.

We now consider the case $0 < \rho < r$. In this case, we use transmission and reflection coefficients and write

$$f_-(\rho,\lambda) = \alpha_-(\lambda) f_+(\rho,\lambda) + \beta_-(\lambda)\overline{f_+(\rho,\lambda)},$$

where

$$\begin{aligned} \alpha_-(\lambda) &= \frac{W(f_-(\cdot,\lambda)\overline{f_+(\cdot,\lambda)})}{-2i\lambda}, \\ \beta_-(\lambda) &= \frac{W(\lambda)}{2i\lambda}. \end{aligned}$$

Then using our high energy asymptotics for $W(\lambda)$ and $W(f_-(\cdot,\lambda),\overline{f_+(\cdot,\lambda)})$, we have for $\lambda \gtrsim 1$

$$\lambda\frac{\alpha_-(\lambda)}{W(\lambda)} = O(\lambda^{-2}) = O(1), \quad \lambda\frac{\beta_-(\lambda)}{W(\lambda)} = O(1),$$

with natural derivative bounds. Thus, to prove (2.3.19) for the case $0 < \rho < r$, we are reduced to proving the bounds

$$\left| \int_{-\infty}^{\infty} e^{\pm it|\lambda|(\pm t + \frac{\lambda}{|\lambda|}(r+\rho))} \varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2} O(1) m_+(r,\lambda) m_+(\rho,\lambda) d\lambda \right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}, \tag{2.3.23}$$

$$\left| \int_{-\infty}^{\infty} e^{\pm it|\lambda|(\pm t + \frac{\lambda}{|\lambda|}(r-\rho))} \varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2} O(1) m_+(r,\lambda) \overline{m_+(\rho,\lambda)} d\lambda \right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}. \tag{2.3.24}$$

We write (2.3.23) as

$$\left| \int_{-\infty}^{\infty} e^{i|\lambda|(\pm t + \frac{\lambda}{|\lambda|}(r+\rho))} a_j(r,\rho,\lambda) d\lambda \right| \lesssim 2^{j(d+2)/2}(2^{-j}+|t|)^{-d/2}.$$

where

$$a_j(r,\rho,\lambda) = \varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2}O(1)m_+(r,\lambda)m_+(\rho,\lambda).$$

Then $a_j(r,\rho,\lambda)$ satisfies

$$a_j(r,\rho,\lambda) = \varphi(2^{-j}\lambda)(\langle r\rangle\langle\rho\rangle)^{-d/2}O(1), \tag{2.3.25}$$

with natural derivative bounds. But now we are in the same situation as in the case $\rho < 0 < r$ with (2.3.25) replacing (2.3.22) and we obtain (2.3.23) in a similar fashion. The estimate (2.3.24) is obtained similarly and we omit the details. This concludes the proof of Lemma 2.3.14 and also Proposition 2.3.2. □

2.4 Reduction to Higher Dimensions and the Linearized Equation

In this section, we initiate the study of the evolution (2.1.2). In the first subsection, we linearize degree n solutions to (2.1.2) around the harmonic map Q_n and make a reduction that incorporates the extra dispersion inherent in (2.1.2). Our main result, Theorem 2.1.1, is then restated in an equivalent form which we devote the rest of this work to proving. The remaining subsections establish Strichartz estimates for the linear part of the new equation which will be used in Section 2.5. In what follows we use the notation from the previous section and denote the d–dimensional wormhole by $\mathcal{M}^d$.

2.4.1 Reduction to a Wave Equation on a 5*d Wormhole*

We recall from the introduction that a corotational wave map on a wormhole $U : \mathbb{R}\times\mathcal{M}^3 \to \mathbb{S}^3$ with topological degree n is a map $U(t,r,\theta,\varphi) = (\psi(t,r),\theta,\varphi)$ such the azimuth angle

$\psi = \psi(t,r)$ satisfies the Cauchy problem

$$\begin{aligned}
&\partial_t^2 \psi - \partial_r^2 \psi - \frac{2r}{r^2+1}\partial_r \psi + \frac{\sin 2\psi}{r^2+1} = 0, \\
&\psi(t,-\infty) = 0, \quad \psi(t,\infty) = n\pi, \quad \forall t, \\
&\vec{\psi}(0) = (\psi_0, \psi_1).
\end{aligned} \tag{2.4.1}$$

The following energy is conserved along the flow

$$\mathcal{E}(\psi) = \frac{1}{2}\int \left[|\partial_t \psi|^2 + |\partial_r \psi|^2 + \frac{2\sin^2\psi}{r^2+1} \right](r^2+1)dr,$$

and so, it is natural to take initial data (ψ_0, ψ_1) in the metric space

$$\mathcal{E}_n = \left\{ (\psi_0,\psi_1) : \mathcal{E}(\psi_0,\psi_1) < \infty, \quad \psi_0(-\infty) = 0, \quad \psi_0(\infty) = n\pi \right\}.$$

For the remainder of this work, we fix the topological degree $n \in \mathbb{N} \cup \{0\}$. We now reduce the study of the large data solutions to (2.4.1) to the study of large data solutions to a semilinear wave equation on a $5d$ wormhole.

By Proposition 2.2.1, there exists a unique finite energy static solution Q_n to (2.4.1), i.e. a solution $Q_n \in \mathcal{E}_n$ such that

$$\partial_r^2 Q_n + \frac{2r}{r^2+1}\partial_r Q_n - \frac{\sin 2Q_n}{r^2+1} = 0. \tag{2.4.2}$$

To simplify notation, we write Q instead of Q_n. For a solution ψ to (2.4.1), define φ by

$$\psi(t,r) = Q(r) + \varphi(t,r).$$

Then (2.4.1) and (2.4.2) imply that φ satisfies

$$\begin{aligned} &\partial_t^2\varphi - \partial_r^2\varphi - \frac{2r}{r^2+1}\partial_r\varphi + \frac{2\cos 2Q}{r^2+1}\varphi = Z(r,\varphi), \\ &\varphi(t,-\infty) = \varphi(t,\infty) = 0, \quad \forall t, \\ &\vec{\varphi}(0) = (\psi_0 - Q, \psi_1), \end{aligned} \tag{2.4.3}$$

where

$$Z(r,\phi) = \frac{1}{r^2+1}\left[2\varphi - \sin 2\varphi\right]\cos 2Q + (1-\cos 2\varphi)\sin 2Q.$$

The left–hand side of (2.4.3) has more dispersion than a free wave on $\mathcal{M}^3$ due to the repulsive potential

$$\frac{2\cos 2Q}{r^2+1} = \frac{2}{r^2+1} + O(\langle r\rangle^{-6})$$

as $r \to \pm\infty$. The $O(\langle r\rangle^{-6})$ term is due to the asymptotics of Q at $\pm\infty$ (see Proposition 2.2.1). We now make a standard reduction that incorporates this extra dispersion. Set $\varphi = \langle r\rangle u$. Then u satisfies the radial semilinear wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + V(r)u = N(r,u), \\ &u(t,-\infty) = u(t,\infty) = 0, \quad \forall t, \\ &\vec{u}(0) = (u_0, u_1), \end{aligned} \tag{2.4.4}$$

where $-\Delta_g$ is the (radial) Laplace operator on $\mathcal{M}^5$

$$-\Delta_g u = -\partial_r^2 u - \frac{4r}{r^2+1}\partial_r u,$$

the potential is

$$V(r) = \langle r\rangle^{-4} + 2\langle r\rangle^{-2}(\cos 2Q - 1), \tag{2.4.5}$$

and $N(r,u) = F(r,u) + G(r,u)$ with

$$\begin{aligned} F(r,u) &= 2\langle r\rangle^{-3}\sin^2(\langle r\rangle u)\sin 2Q, \\ G(r,u) &= \langle r\rangle^{-3}\left[2\langle r\rangle u - \sin(2\langle r\rangle u)\right]\cos 2Q. \end{aligned} \tag{2.4.6}$$

By Proposition 2.2.1, the potential V is smooth and satisfies

$$V(r) = \langle r\rangle^{-4} + O(\langle r\rangle^{-6}). \tag{2.4.7}$$

Moreover, since $Q(-r) + Q(r) = n\pi$, $V(r)$ is an even function. The nonlinearities F and G satisfy

$$F(r,u) = \left(2\sin 2Q\langle r\rangle^{-1}\right)u^2 + F_0(r,u), \tag{2.4.8}$$

$$|F_0(r,u)| \lesssim \langle r\rangle^{-1}u^4, \tag{2.4.9}$$

$$|G(r,u)| \lesssim |u|^3, \tag{2.4.10}$$

where the implied constants are absolute. Based on our definition of u in terms of the original azimuth function ψ, we consider radial initial data $(u_0, u_1) \in \mathcal{H}(\mathbb{R}; (r^2+1)^2 dr)$ in (2.4.4). For the remainder of this section, we denote $\mathcal{H}_0 := \mathcal{H}(\mathbb{R}; (r^2+1)dr)$ and $\mathcal{H} := \mathcal{H}(\mathbb{R}; (r^2+1)^2 dr)$ by $\mathcal{H}$. We note that $\mathcal{H}_0$ is simply the space of radial functions in $\dot{H}^1 \times L^2(\mathcal{M}^3)$ and $\mathcal{H}$ is the space of radial functions in $\dot{H}^1 \times L^2(\mathcal{M}^5)$.

In the remainder of the paper, we work only with the 'u–formulation' rather than with the original azimuth angle ψ. The reason that a solution $\vec{\psi}(t) \in C(\mathbb{R}; \mathcal{H}_n)$ to (2.4.1) with initial data $(\psi_0, \psi_1) \in \mathcal{E}_n$ yields a solution $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H})$ with initial data $(u_0, u_1) =$

$\langle r\rangle^{-1}(\psi_0 - Q, \psi_1) \in \mathcal{H}$ and vice versa is as follows. The only fact that needs to be checked is that

$$\|\vec{u}\|_{\mathcal{H}} \simeq \|\vec{\psi} - (Q,0)\|_{\mathcal{H}_0}. \tag{2.4.11}$$

Set $\varphi = \psi - Q = \langle r\rangle u$. Then

$$\partial_r \varphi = \langle r\rangle \partial_r u + \frac{r}{\langle r\rangle^2} u. \tag{2.4.12}$$

We note that we have the following Hardy's inequalities

$$\int |\varphi|^2 dr \lesssim \int |\partial_r \varphi|^2 (r^2+1) dr,$$
$$\int |u|^2 (r^2+1) dr \lesssim \int |\partial_r u|^2 (r^2+1)^2 dr.$$

These estimates follow easily from integration by parts and the Strauss estimates

$$|\varphi(r)| \lesssim \langle r\rangle^{-1/2} \left(\int |\partial_r \varphi|^2 (r^2+1) dr \right)^{1/2},$$
$$|u(r)| \lesssim \langle r\rangle^{-3/2} \left(\int |\partial_r u|^2 (r^2+1)^2 dr \right)^{1/2}. \tag{2.4.13}$$

The Strauss estimates are a simple consequence of the fundamental theorem of calculus. The two Hardy's inequalities and (2.4.12) imply (2.4.11). Hence, the two Cauchy problems (2.4.1) and (2.4.4) are equivalent.

The equivalent u–formulation of our main result, Theorem 2.1.1, is the following.

Theorem 2.4.1. *For any initial data $(u_0, u_1) \in \mathcal{H}$, there exists a unique global solution $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H})$ to (2.4.4) which scatters to free waves on $\mathcal{M}^5$, i.e. there exist solutions $v_L^{\pm}$*

to

$$\partial_t^2 v - \partial_r^2 v - \frac{4r}{r^2+1}\partial_r v = 0, \quad (t,r) \in \mathbb{R} \times \mathbb{R},$$

such that

$$\lim_{t \to \pm\infty} \|\vec{u}(t) - \vec{v}_L^{\pm}(t)\|_{\mathcal{H}} = 0.$$

The remainder of this work is devoted to proving Theorem 2.4.1. In order to study the nonlinear evolution (2.4.4), we will need Strichartz estimates for the linear operator $\partial_t^2 - \Delta_g + V$ where V is as in (2.4.5).

2.4.2 Strichartz Estimates for the Linearized Operator

The goal of this subsection is to prove Strichartz estimates for radial solutions to the free wave equation on $\mathcal{M}^5$ perturbed by a radial potential $V = V(r)$

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + Vu = F, \quad (t,r) \in I \times \mathbb{R}, \\ &\vec{u}(0) = (u_0, u_1). \end{aligned} \tag{2.4.14}$$

The particular case we are interested in is the case that the potential V is given by

$$V(r) = \langle r \rangle^{-4} + 2\langle r \rangle^{-2}(\cos 2Q - 1),$$

where Q is the unique harmonic map of degree n. The Strichartz estimates we establish will be used in the next section to study the nonlinear evolution (2.4.4). We recall from Section 2.3 that we say that a triple (a, b, γ) is admissible for M^5 if

$$p > 2, q \geq 2, \quad \frac{1}{p} + \frac{5}{q} = \frac{5}{2} - \gamma, \quad \frac{1}{p} \leq 1 - \frac{2}{q}.$$

The main result of this subsection is the following.

Proposition 2.4.2. *Let $V \in C^\infty(\mathbb{R})$ be even such that*

$$|V^{(j)}(r)| \lesssim_j \langle r \rangle^{-4-j} \tag{2.4.15}$$

for all $r \in \mathbb{R}$. Assume that $-\Delta_g + V$ has no point spectrum (when restricted to radial functions) and that 0 is not a resonance of the Schrödinger operator on the line given by $-\frac{d^2}{dr^2} + 2\langle r \rangle^{-2} + V(r)$. Let (p, q, γ) and (a, b, ρ) be two admissible triples for $\mathcal{M}^5$. Then any radial solution u to (2.4.14) *satisfies*

$$\||\nabla|^{1-\gamma} u\|_{L_t^p L_x^q(I)} + \||\nabla|^{-\gamma} \partial_t u\|_{L_t^p L_x^q(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \||\nabla|^{\rho} F\|_{L_t^{a'} L_x^{b'}(I)}. \tag{2.4.16}$$

Proof. The proof is based on arguments in Section 5 of [20]. By standard TT^* arguments and Minkowski's inequality (c.f. [30] or [31]), we only need to consider the case $F = 0$. As we will see, the proof of Proposition 2.4.2 reduces to proving certain local energy estimates. Indeed, define

$$A = \sqrt{-\Delta_g}.$$

Note that

$$\|Af\|_{L^2}^2 = (A^2 f, f)_{L^2} = (-\Delta_g f, f)_{L^2} = \|\nabla f\|_{L^2}^2. \tag{2.4.17}$$

For a solution u to (2.4.14), define

$$w(t) = Au(t) + i\partial_t u(t). \tag{2.4.18}$$

Then by (2.4.17),

$$\|w(t)\|_{L^2} = \|\vec{u}(t)\|_{\mathcal{H}}, \tag{2.4.19}$$

and w satisfies

$$\begin{aligned} &i\partial_t w = Aw + Vu, \quad (t,r) \in I \times \mathbb{R}, \\ &w(0) = Au_0 + iu_1. \end{aligned} \tag{2.4.20}$$

By Duhamel's principle, (2.4.20) implies that

$$w(t) = e^{-itA}w(0) - i\int_0^t e^{-i(t-s)A}Vu(s)ds.$$

The Strichartz estimates (2.4.16) can be restated as

$$\|Pw\|_X \leq \|w(0)\|_{L^2}, \tag{2.4.21}$$

where $P := A^{-1}\Re$ and $\|\cdot\|_X := \||\nabla|^{-\gamma}\nabla_{t,x}\cdot\|_{L_t^p L_x^q(I)}$. By Proposition 2.3.1,

$$\|Pe^{-itA}w(0)\|_X \lesssim \|w(0)\|_{L^2}. \tag{2.4.22}$$

Thus,

$$\|Pw\|_X \lesssim \|w(0)\|_{L^2} + \left\| P\int_0^t e^{-i(t-s)A}Vu(s)ds \right\|_X.$$

By the Christ–Kiselev lemma, to bound the second term above, it suffices to show that

$$\left\| P\int_{-\infty}^{\infty} e^{-i(t-s)A}Vu(s)ds \right\|_X \lesssim \|w(0)\|_{L^2}. \tag{2.4.23}$$

To prove (2.4.23), we write $V = V_1 V_2$ were each factor V_j is even and satisfies $|V_j(r)| \lesssim \langle r \rangle^{-2}$. Then

$$\left\| P \int_{-\infty}^{\infty} e^{-i(t-s)A} V u(s) ds \right\|_X \lesssim \|K\|_{L^2_{t,x} \to X} \|V_2 u\|_X \tag{2.4.24}$$

where

$$KF(t) := P \int_{-\infty}^{\infty} e^{-i(t-s)A} V_1 F(s) ds.$$

If $F \in L^2_{t,x}$, then by (2.4.22)

$$\begin{aligned} \|KF\|_X &\leq \|Pe^{-itA}\|_{L^2_x \to X} \left\| \int_{-\infty}^{\infty} e^{isA} V_1 F(s) ds \right\|_{L^2_x} \\ &\lesssim \left\| \int_{-\infty}^{\infty} e^{isA} V_1 F(s) ds \right\|_{L^2_x}. \end{aligned}$$

We now wish to show that

$$\left\| \int_{-\infty}^{\infty} e^{isA} V_1 F(s) ds \right\|_{L^2_x} \lesssim \|F\|_{L^2_{t,x}}.$$

By duality, this estimate is equivalent to the local energy estimate

$$\|V_1 e^{-itA} \varphi\|_{L^2_{t,x}} \lesssim \|\varphi\|_{L^2_x}.$$

Thus, by (2.4.24), the proof of Proposition 2.4.2 is reduced to proving the local energy estimates

$$\|V_1 e^{-itA} \varphi\|_{L^2_{t,x}} \lesssim \|\varphi\|_{L^2_x}, \tag{2.4.25}$$

$$\|V_2 u\|_{L^2_{t,x}} \lesssim \|\vec{u}(0)\|_{\mathcal{H}}. \tag{2.4.26}$$

To prove (2.4.25) and (2.4.26), we first eliminate the weight $\langle r\rangle^4$ inherent in them. Consider the isomorphism $\phi \mapsto f := \langle r\rangle^2\varphi$ from $L^2(\mathcal{M}^5)$ (restricted to radial functions) to $L^2(\mathbb{R})$. Define the following Schrödinger operators on $\mathbb{R}$ by

$$\begin{aligned} H_0 &:= -\frac{d^2}{dr^2} + \frac{2}{r^2+1}, \\ H := H_0 + V &:= -\frac{d^2}{dr^2} + \frac{2}{r^2+1} + V(r). \end{aligned} \tag{2.4.27}$$

Then

$$\begin{aligned} H_0 &= \langle r\rangle^2(-\Delta_g)\langle r\rangle^{-2}, \\ H &= \langle r\rangle^2(-\Delta_g + V)\langle r\rangle^{-2}. \end{aligned} \tag{2.4.28}$$

Thus, from (2.4.28), we see that (2.4.25) is equivalent to the estimate

$$\|V_1 e^{-it\sqrt{H_0}} f\|_{L^2_{t,r}(\mathbb{R}\times\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}. \tag{2.4.29}$$

We claim that there exist a distorted Fourier basis $\{\theta_0(r,\lambda^2), \phi_0(r,\lambda^2)\}$ that satisfies

$$\begin{aligned} H_0\theta_0(r,\lambda^2) = \lambda^2\theta_0(r,\lambda^2), &\quad H_0\phi_0(r,\lambda^2) = \lambda^2\phi_0(r,\lambda^2), \\ \theta_0(0,\lambda^2) = 1, &\quad \phi_0(0,\lambda^2) = 0, \\ \theta_0'(0,\lambda^2) = 0, &\quad \phi_0'(0,\lambda^2) = 1, \end{aligned} \tag{2.4.30}$$

and positive measures $\rho_{0,1}(d\lambda) = \omega_{0,1}(\lambda)d\lambda$ and $\rho_{0,2}(d\lambda) = \omega_{0,2}(\lambda)d\lambda$ such that if we define

$$\hat{f}_{0,1}(\lambda) := \int \theta_0(r,\lambda^2)f(r)dr, \quad \hat{f}_{0,2}(\lambda) := \int \phi_0(r,\lambda^2)f(r)dr, \quad f \in L^2(\mathbb{R}),$$

then

$$f(r) = \int_0^\infty \theta_0(r,\lambda^2)\hat{f}_{0,1}(\lambda)\rho_{0,1}(d\lambda) + \int_0^\infty \phi_0(r,\lambda^2)\hat{f}_{0,2}(\lambda)\rho_{0,2}(d\lambda), \tag{2.4.31}$$

$$\|f\|^2_{L^2(\mathbb{R})} = \int_0^\infty |\hat{f}_{0,1}(\lambda)|^2\rho_{0,1}(d\lambda) + \int_0^\infty |\hat{f}_{0,2}(\lambda)|^2\rho_{0,2}(d\lambda), \tag{2.4.32}$$

$$\sup_{r\in\mathbb{R},\lambda>0}\left(\frac{1+\lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right)\Big[|\theta_0(r,\lambda^2)|^2\omega_{0,1}(\lambda) + |\phi_0(r,\lambda^2)|^2\omega_{0,2}(\lambda)\Big] < \infty. \tag{2.4.33}$$

The proof of this claim is postponed until the next subsection. Assuming the claim, we can easily establish (2.4.29). Indeed, since $H_0 \mapsto \lambda^2$ on the Fourier side, (2.4.29) can be rewritten as

$$\int \left\| V_1(r)\left[\int_0^\infty e^{-it\lambda}\theta_0(r,\lambda^2)\hat{f}_{0,1}(\lambda)\rho_{0,1}(d\lambda) + \int_0^\infty e^{-it\lambda}\phi_0(r,\lambda^2)\hat{f}_{0,2}(\lambda)\rho_{0,2}(d\lambda)\right]\right\|^2_{L^2(\mathbb{R})} dt \lesssim \|f\|^2_{L^2(\mathbb{R})}. \tag{2.4.34}$$

Expanding and carrying out the t–integration, the left hand side of (2.4.34) becomes

$$\begin{aligned}\int V_1^2(r)\Big[&\int_0^\infty\int_0^\infty \delta(\lambda-\mu)\theta_0(r,\lambda^2)\theta_0(r,\mu^2)\hat{f}_{0,1}(\lambda)\overline{\hat{f}_{0,1}(\mu)}\rho_{0,1}(d\lambda)\rho_{0,1}(d\mu)\\ &+\int_0^\infty\int_0^\infty \delta(\lambda-\mu)\phi_0(r,\lambda^2)\phi_0(r,\mu^2)\hat{f}_{0,2}(\lambda)\overline{\hat{f}_{0,2}(\mu)}\rho_{0,2}(d\lambda)\rho_{0,2}(d\mu)\Big]dr\\ &= \int V_1^2(r)\Big[\int_0^\infty |\hat{f}_{0,1}(\lambda)|^2|\theta_0(r,\lambda^2)|^2\omega_{0,1}^2(\lambda)\,d\lambda\\ &\qquad+\int_0^\infty |\hat{f}_{0,2}(\lambda)|^2|\phi_0(r,\lambda^2)|^2\omega_{0,2}^2(\lambda)\,d\lambda\Big]dr.\end{aligned} \tag{2.4.35}$$

We remark here that no cross terms involving $\theta_0(r,\lambda^2)\phi_0(r,\mu^2)$ appeared when expanding since

$V_1^2(r)\theta_0(r,\lambda^2)\phi_0(r,\mu^2)$ is an odd function of r by (2.4.30) and our assumption that $V(r)$ is even. By (2.4.33) and (2.4.32), we conclude that

$$
\begin{aligned}
(2.4.35) &\lesssim \int V_1^2(r)\Big[\int_0^\infty |\hat{f}_{0,1}(\lambda)|^2\omega_{0,1}(\lambda)\, d\lambda + \int_0^\infty |\hat{f}_{0,2}(\lambda)|^2\omega_{0,2}(\lambda)\, d\lambda\Big] dr \\
&= \|f\|_{L^2(\mathbb{R})}^2 \int V_1^2(r) dr \\
&\lesssim \|f\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

This proves (2.4.34) which proves (2.4.29) as desired.

The proof of (2.4.26) is very similar and we sketch the details. As in the case for H_0, we claim that there exist a distorted Fourier basis $\{\theta(r,\lambda^2), \phi(r,\lambda^2)\}$ that satisfies

$$
\begin{aligned}
H\theta(r,\lambda^2) = \lambda^2\theta(r,\lambda^2), &\quad H\phi(r,\lambda^2) = \lambda^2\phi(r,\lambda^2), \\
\theta(0,\lambda^2) = 1, &\quad \phi(0,\lambda^2) = 0, \\
\theta'(0,\lambda^2) = 0, &\quad \phi'(0,\lambda^2) = 1,
\end{aligned}
$$

and positive measures $\rho_1(d\lambda) = \omega_1(\lambda)d\lambda$ and $\rho_2(d\lambda) = \omega_2(\lambda)d\lambda$ such that if we define

$$
\hat{f}_1(\lambda) := \int \theta(r,\lambda^2) f(r) dr, \quad \hat{f}_2(\lambda) := \int \phi(r,\lambda^2) f(r) dr, \quad f \in L^2(\mathbb{R}),
$$

then

$$
f(r) = \int_0^\infty \theta(r,\lambda^2)\hat{f}_1(\lambda)\rho_1(d\lambda) + \int_0^\infty \phi(r,\lambda^2)\hat{f}_2(\lambda)\rho_2(d\lambda), \tag{2.4.36}
$$

$$
\|f\|_{L^2(\mathbb{R})}^2 = \int_0^\infty |\hat{f}_1(\lambda)|^2\rho_1(d\lambda) + \int_0^\infty |\hat{f}_2(\lambda)|^2\rho_2(d\lambda), \tag{2.4.37}
$$

$$
\sup_{r\in\mathbb{R},\lambda>0} \left(\frac{1+\lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right) \Big[|\theta(r,\lambda^2)|^2\omega_1(\lambda) + |\phi(r,\lambda^2)|^2\omega_2(\lambda)\Big] < \infty. \tag{2.4.38}
$$

Again, the proof of this claim is postponed until the next subsection. We remark that it is in proving (2.4.36), (2.4.37), and especially (2.4.38) that the spectral assumptions are crucial.

By (2.4.28) and (2.4.36), we see that (2.4.26) follows from showing

$$\begin{aligned}
\int \Big\| V_2(r)\Big[\int_0^\infty &\big(\cos(t\lambda)\hat{f}_1(\lambda) + \lambda^{-1}\sin(t\lambda)\hat{g}_1(\lambda)\big)\theta(r,\lambda^2)\rho_{0,1}(d\lambda) \\
&+ \int_0^\infty \big(\cos(t\lambda)\hat{f}_2(\lambda) + \lambda^{-1}\sin(t\lambda)\hat{g}_2(\lambda)\big)\phi(r,\lambda^2)\rho_2(d\lambda)\Big]\Big\|^2_{L^2(\mathbb{R})} dt \qquad (2.4.39) \\
&\lesssim \|(\sqrt{H}f, g)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Assume that $g = 0$. Then, as in the case for H_0, the left side of (2.4.39) becomes after expanding and integrating in t

$$\begin{aligned}
&\int V_2^2(r)\Big[\int_0^\infty |\hat{f}_1(\lambda)|^2\cos^2(t\lambda)|\theta(r,\lambda^2)|^2\omega_1^2(\lambda)\, d\lambda \\
&\qquad + \int_0^\infty |\hat{f}_2(\lambda)|^2\cos^2(t\lambda)|\phi(r,\lambda^2)|^2\omega_2^2(\lambda)\, d\lambda\Big] dr \\
&\lesssim \int V_2^2(r)\langle r\rangle^2\Big[\int_0^\infty \lambda^2|\hat{f}_1(\lambda)|^2\omega_1(\lambda)\, d\lambda + \int_0^\infty \lambda^2|\hat{f}_2(\lambda)|^2\omega_2(\lambda)\, d\lambda\Big] dr \\
&= \|\sqrt{H}f\|^2_{L^2(\mathbb{R})}\int V_2^2(r)\langle r\rangle^2 dr \\
&\lesssim \|\sqrt{(H)}f\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

The case $g = 0$ is handled similarly. This establishes (2.4.39) which proves (2.4.26). This completes the proof of Proposition 2.4.2 modulo the proofs of the claims about the distorted Fourier bases. We address this in the next subsection. □

2.4.3 The Distorted Fourier Transform

In this subsection, we prove the technical statements about the distorted Fourier bases for H_0 and H used in the previous section.

Proposition 2.4.3. *Let* $H = -\frac{d^2}{dr^2} + V(r)$ *be a Schrödinger operator on the line where*

$V \in C^\infty(\mathbb{R})$ is even and

$$V(r) = \frac{2}{r^2} + O(r^{-3}), \tag{2.4.40}$$

as $r \to \pm\infty$ with natural derivative bounds. Assume that H has no point spectrum and that 0 is not a resonance of H. Then there exist a distorted Fourier basis $\{\theta(r, \lambda^2), \phi(r, \lambda^2)\}$ that satisfies

$$\begin{aligned} H\theta(r, \lambda^2) = \lambda^2\theta(r, \lambda^2), &\quad H\phi(r, \lambda^2) = \lambda^2\phi(r, \lambda^2), \\ \theta(0, \lambda^2) = 1, &\quad \phi(0, \lambda^2) = 0, \\ \theta'(0, \lambda^2) = 0, &\quad \phi'(0, \lambda^2) = 1, \end{aligned} \tag{2.4.41}$$

and positive measures $\rho_1(d\lambda) = \omega_1(\lambda)d\lambda$ and $\rho_2(d\lambda) = \omega_2(\lambda)d\lambda$ such that if we define

$$\hat{f}_1(\lambda) := \int \theta(r, \lambda^2) f(r) dr, \quad \hat{f}_2(\lambda) := \int \phi(r, \lambda^2) f(r) dr, \quad f \in L^2(\mathbb{R}),$$

then

$$f(r) = \int_0^\infty \theta(r, \lambda^2)\hat{f}_1(\lambda)\rho_1(d\lambda) + \int_0^\infty \phi(r, \lambda^2)\hat{f}_2(\lambda)\rho_2(d\lambda), \tag{2.4.42}$$

$$\|f\|^2_{L^2(\mathbb{R})} = \int_0^\infty |\hat{f}_1(\lambda)|^2\rho_1(d\lambda) + \int_0^\infty |\hat{f}_2(\lambda)|^2\rho_2(d\lambda), \tag{2.4.43}$$

$$\sup_{r\in\mathbb{R}, \lambda>0} \left(\frac{1 + \lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right)\left[|\theta(r, \lambda^2)|^2\omega_1(\lambda) + |\phi(r, \lambda^2)|^2\omega_2(\lambda)\right] < \infty. \tag{2.4.44}$$

Many of the statements made in Proposition 2.4.3 follow from basic Weyl–Titchmarsch theory for

Schrödinger operators on the line. We recall these basic facts now (see Section 2 of [11] for a thorough discussion). Let $H = -\frac{d}{dr^2} + V$ with $V \in L^\infty(\mathbb{R})$ (much less is needed) such that H is in the limit point case at $\pm\infty$. We define $\theta(r, z), \phi(r, z)$ to be the fundamental system of solutions to

$$Hf(r) = zf(r), \quad z \in \mathbb{C},$$

such that

$$\begin{aligned} \theta(0,z) &= 1, \quad \phi(0,z) = 0, \\ \theta'(0,z) &= 0, \quad \phi'(0,z) = 1. \end{aligned} \tag{2.4.45}$$

By (2.4.45), the Wronskian is computed

$$W(\theta(\cdot,z),\phi(\cdot,z)) = 1.$$

The condition that H is in the limit point case at $\pm\infty$ implies that for $z \in \mathbb{C}\backslash\mathbb{R}$ there exist unique solutions $\psi_{\pm}(r,z)$ to $Hf = zf$ that satisfy

$$\begin{aligned} &\psi_{\pm}(\cdot,z) \in L^2([0,\pm\infty)), \\ &\psi_{\pm}(0,z) = 1. \end{aligned}$$

The condition at $r = 0$ implies that

$$\psi_{\pm}(r,z) = \theta(r,z) + m_{\pm}(z)\phi(r,z) \tag{2.4.46}$$

where $m_{\pm}(z) = W(\theta(\cdot,z),\psi_{\pm}(\cdot,z))$ and

$$W(\psi_{+}(\cdot,z),\psi_{-}(\cdot,z)) = m_{-}(z) - m_{+}(z).$$

The functions $m_{\pm}(z)$ can be shown to be Herglotz functions ($\Im z > 0 \implies \Im m_{\pm}(z) > 0$) and are referred to as the *Weyl–Titchmarsch functions.* The associated *Weyl–Titchmarsch* *matrix*

$$M(z) := \begin{bmatrix} \frac{1}{m_-(z)-m_+(z)} & \frac{1}{2}\frac{m_-(z)+m_+(z)}{m_-(z)-m_+(z)} \\ \frac{1}{2}\frac{m_-(z)+m_+(z)}{m_-(z)-m_+(z)} & \frac{m_-(z)m_+(z)}{m_-(z)-m_+(z)} \end{bmatrix} \tag{2.4.47}$$

is a Herglotz matrix. Thus, there exists a nonnegative 2×2 matrix–valued measure $\Omega(d\lambda)$ such that

$$M(z) = C + \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right] \Omega(d\lambda),$$

where

$$C^* = C, \quad \int \frac{\|\Omega(d\lambda)\|}{1+\lambda^2} < \infty.$$

The measure $\Omega(d\lambda)$ is computed via

$$\Omega((\lambda_1, \lambda_2]) = \frac{1}{\pi} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \Im M(\lambda + i\epsilon)\, d\lambda.$$

A consequence of Weyl–Titchmarsch theory is that we have the following distorted Fourier representation for H.

Proposition 2.4.4. *Let $f, g \in C_0^\infty(\mathbb{R}), F \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $E(d\lambda)$ denote the spectral measure for H. Define*

$$\hat{f}_1(\lambda) := \int \theta(r,\lambda) f(r) dr, \quad \hat{f}_2(\lambda) := \int \phi(r,\lambda) f(r) dr,$$

and

$$\hat{f}(\lambda) = (\hat{f}_1(\lambda), \hat{f}_2(\lambda))^T.$$

Then

$$(f, F(H)E((\lambda_1, \lambda_2])g)_{L^2(\mathbb{R})} = \int_{(\lambda_1,\lambda_2]} \hat{f}(\lambda)^T \Omega(d\lambda)\overline{\hat{g}(\lambda)}F(\lambda).$$

For the free case $V = 0$, we have the following explicit expressions:

$$\begin{aligned} \theta(r,z) &= \cos(rz^{1/2}), \quad \phi(r,z) = \frac{\sin(rz^{1/2})}{z^{1/2}}, \\ \psi_{\pm}(r,z) &= e^{\pm irz^{1/2}}, \quad m_{\pm}(z) = \pm iz^{1/2} \\ \Omega(d\lambda) &= \frac{1}{2\pi}\chi_{(0,\infty)}(\lambda) \begin{bmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{bmatrix}. \end{aligned} \tag{2.4.48}$$

This leads to the usual Fourier transform on the line.

Proof of Proposition 2.4.3. The decay of V at $\pm\infty$ implies that $H = -\frac{d^2}{dr^2} + V$ is in the limit point case at $\pm\infty$ (see [11]). The decay of V and the assumption that H has no point spectrum imply that $\sigma(H) = [0, \infty)$ and that the spectrum is purely absolutely continuous. By Proposition 2.4.4, this implies that the matrix valued measure $\Omega(d\lambda)$ is supported in $[0, \infty)$. Since V is even, we have by (2.4.41)

$$\theta(-r, \lambda) = \theta(r, \lambda), \quad \phi(-r, \lambda) = -\phi(r, \lambda), \quad \psi_-(r, \lambda) = \psi_+(-r, \lambda),$$

so that $m_-(\lambda) = -m_+(\lambda)$. We recall from the previous section that the Jost solutions $f_{\pm}(r, \lambda)$ are the unique solutions to $Hf = \lambda^2 f$ such that $f_{\pm}(r, \lambda) \sim e^{\pm ir\lambda}$ as $r \to \pm\infty$, and that for $\lambda \neq 0$, $W(f_+(\cdot,\lambda), f_-(\cdot,\lambda)) \neq 0$. Then

$$\begin{aligned} f_-(r,\lambda) &= f_+(-r,\lambda), \\ W(f_+(\cdot,\lambda), f_-(\cdot,\lambda)) &= -2f_+(0,\lambda)f_+'(0,\lambda), \\ \psi_+(r,\lambda^2) &= \frac{f_+(r,\lambda)}{f_+(0,\lambda)}, \\ m_+(\lambda^2) &= \frac{f_+'(0,\lambda)}{f_+(0,\lambda)}. \end{aligned}$$

The matrix (2.4.47) satisfies

$$\begin{aligned} M(\lambda^2) &= \begin{bmatrix} -\frac{1}{2}\frac{f_+(0,\lambda)}{f_+'(0,\lambda)} & 0 \\ 0 & -\frac{1}{2}\frac{f_+'(0,\lambda)}{f_+(0,\lambda)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\frac{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))} & 0 \\ 0 & -\frac{1}{2}\frac{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}. \end{bmatrix} \end{aligned}$$

Thus,

$$\Omega(d\lambda^2) = \begin{bmatrix} \rho_1(d\lambda) & 0 \\ 0 & \rho_2(d\lambda) \end{bmatrix} \tag{2.4.49}$$

where

$$\begin{aligned} \rho_1(d\lambda) &:= \frac{1}{\pi}\lambda\Im\left[\frac{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}\right]d\lambda, \\ \rho_2(d\lambda) &:= -\frac{1}{\pi}\lambda\Im\left[\frac{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}\right]d\lambda. \end{aligned} \tag{2.4.50}$$

By Proposition 2.4.4, (2.4.49) and (2.4.50) imply (2.4.42) and (2.4.43). It remains to prove (2.4.44). As in Section 2.3, the main difficulty is encountered when considering $0 < \lambda \ll 1$. Indeed, it is not hard to show that if λ is bounded away from 0, $\lambda \gtrsim 1$, then the distorted Fourier basis $\theta(r,\lambda^2), \phi(r,\lambda^2)$ and measure $\Omega(d\lambda^2)$ in (2.4.49) are approximated to leading order by the free case (2.4.48). For the free case, (2.4.44) holds (for $\lambda \gtrsim 1$). Thus, (2.4.44) holds in the perturbed case for $\lambda \gtrsim 1$. We omit the details, and instead focus on establishing (2.4.44) in the case $0 < \lambda \ll 1$. To establish (2.4.44) in the small λ regime, we use the scattering theory summarized in Section 3 to derive asymptotic expansions for $\theta(r,\lambda^2), \phi(r,\lambda^2), \rho_1(d\lambda)$, and $\rho_2(d\lambda)$. The upcoming calculations will freely use the notation from Section 2.3.

We first consider the zero energy equation. Let $\theta_0(r), \phi_0(r)$ be the fundamental system for $Hf = 0$ such that

$$\begin{aligned} \theta_0(0) &= 1, \quad \phi_0(0) = 0, \\ \theta_0'(0) &= 0, \quad \phi_0'(0) = 1. \end{aligned} \tag{2.4.51}$$

Then

$$\begin{aligned} \phi_0(r) &= a_0 u_0^+(r) + a_1 u_1^+(r), \\ \theta_0(r) &= b_0 u_0^+(r) + b_1 u_1^+(r), \end{aligned} \tag{2.4.52}$$

where, we recall that, the solutions $u_j^+(r)$ satisfy $Hu_j^+(r) = 0$ and

$$\begin{aligned} u_0^+(r) &= \frac{1}{3}r^2 + O(r), \\ u_1^+(r) &= r^{-1} + O(r^{-2}), \end{aligned} \tag{2.4.53}$$

as $r \to \infty$ (see Lemma 2.3.3). Since $W(\theta_0, \phi_0) = 1 = W(u_1^+, u_0^+)$, we conclude that

$$a_0 b_1 - a_1 b_0 = 1. \tag{2.4.54}$$

Since ϕ_0 and θ_0 are odd and even respectively, the assumption that 0 is not a resonance implies the crucial condition that

$$a_0 \neq 0 \quad \text{and} \quad b_0 \neq 0. \tag{2.4.55}$$

We now perturb in small λ. We claim that the smooth fundamental system $\theta(r, \lambda^2), \phi(r, \lambda^2)$ that satisfies (2.4.41) also satisfies

$$\begin{aligned} \phi(r, \lambda^2) &= \phi_0(r) + O(\lambda^2 \langle r \rangle^2 r^2), \\ \theta(r, \lambda^2) &= \theta_0(r) + O(\lambda^2 \langle r \rangle^2 r^2), \end{aligned} \tag{2.4.56}$$

for $0 \leq r \leq \lambda^{-1}$. The $O(\cdot)$ terms are real–valued and satisfy natural derivative bounds. Indeed, by variation of constants, we can write $\phi(r, \lambda^2)$ as a solution to

$$\phi(r, \lambda^2) = \phi_0(r) + \lambda^2 \int_0^r \left[u_0^+(r) u_1^+(\rho) - u_0^+(\rho) u_1^+(r) \right] \phi(\rho, \lambda^2) d\rho. \tag{2.4.57}$$

If we define $\tilde{\phi}(r, \lambda^2) = \langle r \rangle^{-2} \phi(r, \lambda^2)$ and

$$K(r, \rho, \lambda) = \lambda^2 \langle \rho \rangle^2 \langle r \rangle^{-2} \left[u_0^+(r) u_1^+(\rho) - u_0^+(\rho) u_1^+(r) \right]$$

then (2.4.57) takes the form of the Volterra equation

$$\tilde{\phi}(r, \lambda^2) = \langle r \rangle^{-2} \phi_0(r) + \int_0^r K(r, \rho, \lambda) \tilde{\phi}(\rho, \lambda^2) d\rho. \tag{2.4.58}$$

By (2.4.53), if $0 < \rho < r$, then the kernel satisfies

$$|K(r, \rho, \lambda)| \lesssim \lambda^2 \langle \rho \rangle.$$

Thus,

$$\int_0^{\lambda^{-1}} \sup_{r>\rho} |K(r,\rho,\lambda)| d\rho \lesssim 1,$$

which implies that the Volterra iterates for (2.4.58) converge on $[0, \lambda^{-1}]$ to a unique solution $\tilde{\phi}(r, \lambda^2)$ satisfying

$$\tilde{\phi}(r, \lambda^2) = \langle r \rangle^{-2} \phi_0(r) + O(\lambda^2 r^2).$$

This proves (2.4.56) for $\phi(r, \lambda^2)$. An identical argument proves (2.4.56) for $\theta(r, \lambda^2)$ as well. By Lemma 2.3.5 there exists a fundamental system $u_0^+(r, \lambda), u_1^+(r, \lambda)$ for $Hf = \lambda^2 f$ such that $W(u_1^+(\cdot, \lambda), u_0^+(\cdot, \lambda)) = 1$ and for $j = 0, 1$

$$u_j^+(r, \lambda) = u_j^+(r)(1 + O(\langle r \rangle^2 \lambda^2)), \quad r \in [r_0, \epsilon_0 \lambda^{-1}], \tag{2.4.59}$$

for some fixed $r_0, \epsilon_0 > 0$. Similar to (2.4.52), we can write

$$\begin{aligned} \phi(r, \lambda^2) &= a_0(\lambda) u_0^+(r, \lambda) + a_1(\lambda) u_1^+(r, \lambda), \\ \theta(r, \lambda^2) &= b_0(\lambda) u_0^+(r, \lambda) + b_1(\lambda) u_1^+(r, \lambda), \end{aligned} \tag{2.4.60}$$

with $a_0(\lambda) b_1(\lambda) - a_1(\lambda) b_0(\lambda) = 1$. We claim that

$$\begin{aligned} a_0(\lambda) &= a_0 + O(\lambda^2), \quad a_1(\lambda) = a_1 + O(\lambda^2), \\ b_0(\lambda) &= b_0 + O(\lambda^2), \quad b_1(\lambda) = b_1 + O(\lambda^2), \end{aligned} \tag{2.4.61}$$

as $\lambda \to 0$ where a_0, a_1, b_0, and b_1 are as in (2.4.52). Indeed, using (2.4.56) and (2.4.59), we evaluate the Wronskian at $r = r_0$ and deduce that

$$\begin{aligned} a_0(\lambda) &= W(u_1^+(r,\lambda), \phi(r,\lambda^2) \\ &= W\left(u_1^+(r)(1+O(\lambda^2\langle r\rangle^2), a_0 u_0^+(r) + a_1 u_+^1(r) + O(\lambda^2\langle r\rangle^2 r^2)\right) \\ &= a_0 + O(\lambda^2). \end{aligned}$$

The computation for a_1, b_0, and b_1 are similar so that (2.4.61) follows. We are now in a position to derive asymptotics for

$$\begin{aligned} \omega_1(\lambda) &:= \frac{1}{\pi}\lambda\Im\left[\frac{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}\right], \\ \omega_2(\lambda) &:= -\frac{1}{\pi}\lambda\Im\left[\frac{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}\right]. \end{aligned}$$

By Lemma 2.3.7, we have

$$\begin{aligned} W(f_+(\cdot,\lambda), u_1^+(\cdot,\lambda)) &= \alpha_0^+(\lambda) + i\alpha_1^+(\lambda), \\ W(f_+(\cdot,\lambda), u_0^+(\cdot,\lambda)) &= \beta_0^+(\lambda) + i\beta_1^+(\lambda), \end{aligned} \tag{2.4.62}$$

where

$$\begin{aligned} \alpha_0^+(\lambda) = \lambda^2(\alpha_0 + O(\lambda^\epsilon)), &\quad \alpha_1^+(\lambda) = O(\lambda^{2-2\epsilon}), \\ \beta_0^+(\lambda) = O(\lambda^{-1+4\epsilon}), &\quad \beta_1^+(\lambda) = \lambda^{-1}(\beta_0 + O(\lambda^\epsilon)), \end{aligned} \tag{2.4.63}$$

for all $0 < \epsilon < \epsilon_0$. The constants α_0 and β_0 in (2.4.63) are positive. From (2.4.60) and (2.4.62), we conclude that

$$\begin{aligned} W(f_+(\cdot,\lambda), \phi(\cdot,\lambda^2)) &= A_0(\lambda) + iA_1(\lambda), \\ W(f_+(\cdot,\lambda), \theta(\cdot,\lambda^2)) &= B_0(\lambda) + iB_1(\lambda), \end{aligned}$$

where

$$\begin{aligned}
A_0(\lambda) &= a_0(\lambda)\beta_0^+(\lambda) + a_1(\lambda)\alpha_0^+(\lambda), \\
A_1(\lambda) &= a_0(\lambda)\beta_1^+(\lambda) + a_1(\lambda)\alpha_1^+(\lambda), \\
B_0(\lambda) &= b_0(\lambda)\beta_0^+(\lambda) + b_1(\lambda)\alpha_0^+(\lambda), \\
B_1(\lambda) &= b_0(\lambda)\beta_1^+(\lambda) + b_1(\lambda)\alpha_1^+(\lambda).
\end{aligned} \tag{2.4.64}$$

Then

$$\Im\left[\frac{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}\right] = \frac{A_1B_0 - A_0B_1}{B_0^2 + B_1^2}. \tag{2.4.65}$$

By (2.4.63) and the condition that $a_0(\lambda)b_1(\lambda) - a_1(\lambda)b_0(\lambda) = 1$, we conclude that

$$\begin{aligned}
A_1B_0 - A_0B_1 &= \beta_1^+(\lambda)\alpha_0^+(\lambda) - \alpha_1^+(\lambda)\beta_0^+(\lambda) \\
&= \lambda\left(\alpha_0\beta_0 + O(\lambda^\epsilon)\right).
\end{aligned} \tag{2.4.66}$$

By (2.4.63) and (2.4.61)

$$B_0^2 + B_1^2 = \lambda^{-2}\left(b_0^2\beta_0^2 + O(\lambda^\epsilon)\right). \tag{2.4.67}$$

Thus, (2.4.65), (2.4.66), and (2.4.67) yield

$$\Im\left[\frac{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}\right] = \lambda^3\frac{\alpha_0\beta_0 + O(\lambda^\epsilon)}{b_0^2\beta_0^2 + O(\lambda^\epsilon)}, \tag{2.4.68}$$

as $\lambda \to 0^+$. Similarly,

$$-\Im\left[\frac{W(f_+(\cdot,\lambda),\theta(\cdot,\lambda^2))}{W(f_+(\cdot,\lambda),\phi(\cdot,\lambda^2))}\right] = \lambda^3\frac{\alpha_0\beta_0 + O(\lambda^\epsilon)}{a_0^2\beta_0^2 + O(\lambda^\epsilon)}. \tag{2.4.69}$$

The crucial nonresonant condition (2.4.55) implies that (2.4.68) and (2.4.69) are both $O(\lambda^3)$. In summary, we have shown that the measures $\rho_1(d\lambda) = \omega_1(\lambda)d\lambda$ and $\rho_2(d\lambda) = \omega_2(\lambda)d\lambda$ in (2.4.50) have weights that satisfy

$$\omega_1(\lambda) = O(\lambda^4), \quad \omega_2(\lambda) = O(\lambda^4). \tag{2.4.70}$$

We now prove (2.4.44) using the asymptotics from the previous paragraph. The expressions (2.4.56), (2.4.52), (2.4.53), and (2.4.70) imply that

$$\left(\frac{1+\lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right)\left[|\theta(r,\lambda^2)|^2\omega_1(\lambda) + |\phi(r,\lambda^2)|^2\omega_2(\lambda)\right] \lesssim 1, \quad r \in [0,\lambda^{-1}]. \tag{2.4.71}$$

We now consider the case $r \geq \lambda^{-1}$. We first recall that

$$W(f_+(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) = -2i\lambda \neq 0,$$

for $\lambda > 0$. Thus, we can write

$$\phi(r,\lambda^2) = c(\lambda)f_+(r,\lambda) + d(\lambda)\overline{f_+(r,\lambda)}. \tag{2.4.72}$$

Since $\phi(r,\lambda^2)$ is real–valued, $d(\lambda) = \overline{c(\lambda)}$. Note that

$$W(\phi(\cdot,\lambda^2), \overline{f_+(\cdot,\lambda)}) = -2i\lambda c(\lambda),$$

so that by (2.4.62)–(2.4.64), we conclude that

$$c(\lambda) = \frac{1}{2i\lambda}W(f_+(\cdot,\lambda), \phi(\cdot,\lambda^2)) = O_{\mathbb{C}}(\lambda^{-2}). \tag{2.4.73}$$

By Lemma 2.3.6 we have

$$f_+(r,\lambda) = e^{ir\lambda}(1 + O(\lambda^{-1}\langle r\rangle^{-1})), \quad r \geq \lambda^{-1}. \tag{2.4.74}$$

From (2.4.70)–(2.4.74), we conclude that

$$\left(\frac{1+\lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right) |\phi(r,\lambda^2)|^2\omega_2(\lambda) \lesssim 1, \quad r \geq \lambda^{-1}. \tag{2.4.75}$$

By the exact same arguments,

$$\left(\frac{1+\lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right) |\theta(r,\lambda^2)|^2\omega_1(\lambda) \lesssim 1, \quad r \geq \lambda^{-1}. \tag{2.4.76}$$

In summary, we have shown that for $0 < \lambda \ll 1$,

$$\left(\frac{1+\lambda^2\langle r\rangle^2}{\lambda^2\langle r\rangle^2}\right) \left[|\theta(r,\lambda^2)|^2\omega_1(\lambda) + |\phi(r,\lambda^2)|^2\omega_2(\lambda)\right] \lesssim 1, \quad r \in \mathbb{R}.$$

This proves (2.4.44) and concludes the proof of Proposition 2.4.3. □

2.5 Small Data Theory and Concentration–Compactness

In this section we use the tools developed in the previous sections to initiate the study of the nonlinear evolution introduced in the previous section:

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + V(r)u = N(r,u), \quad (t,r) \in \mathbb{R}\times\mathbb{R}, \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H}, \end{aligned} \tag{2.5.1}$$

where $\mathcal{H} := \mathcal{H}(\mathbb{R};(r^2+1)^2dr)$, $-\Delta_g$ is the (radial) Laplace operator on the $5d$ wormhole $\mathcal{M}^5$, and $V(r)$ and $N(r,u)$ are given in (2.4.5) and (2.4.6). In particular, we begin our proof of Theorem 2.4.1, i.e. every solution to (2.5.1) is global and scatters to free waves on $\mathcal{M}^5$.

2.5.1 Small Data Theory

As summarized in the introduction, the proof of Theorem 2.4.1 (which we have shown in Section 2.4 is equivalent to Theorem (2.1.1)) uses the powerful concentration compactness/rigidity methodology introduced by Kenig and Merle in their study of energy–critical dispersive equations [14] [15]. The methodology is split up into three main steps and proceeds by contradiction. In the first step, we establish small data global well–posedness and scattering for (2.4.4). In particular, we establish Theorem 2.4.1 for small data (u_0, u_1). In the second step, the first step and a concentration–compactness argument shows that the *failure* of Theorem 2.4.1 implies that that there exists a nonzero 'critical element' u_*; a minimal non–scattering global solution to (2.4.4). The minimality of u_* imposes the following compactness property on u_*: the trajectory

$$K = \{\vec{u}_*(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H}$. In the third and final step, we establish the following rigidity theorem: every solution u with $\{\vec{u}(t) : t \in \mathbb{R}\}$ precompact in $\mathcal{H}$ must identically 0. This contradicts the second step which implies that Theorem 2.4.1 holds. In this section we complete the first two steps in the program: small data theory and concentration–compactness. These steps follow from, by now, standard arguments using the Strichartz estimates for $\partial_t^2 - \Delta_g + V$ established in Section 2.4.

We first establish a global well–posedness and small data theory for (2.4.4). This follows from a contraction mapping argument using Strichartz estimates established in Proposition 2.4.2 for the inhomogeneous wave equation with potential

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + V(r)u = h(t,r), \quad (t,r) \in \mathbb{R} \times \mathbb{R}, \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H}. \end{aligned} \tag{2.5.2}$$

Here, as in the previous section, the potential V is given by

$$V(r) = \langle r\rangle^{-4} + 2\langle r\rangle^{-2}\left(\cos 2Q - 1\right),$$

where Q is the unique harmonic map of degree n. To see that V satisfies the hypotheses in Proposition 2.4.2, we note that by Proposition 2.2.1, we only need to verify the spectral assumptions are satisfied. This was shown in [3], and we recall the argument. We have the relation

$$\langle r\rangle^2(-\Delta_g + V)\langle r\rangle^{-2} = H, \tag{2.5.3}$$

where H is the Schrödinger operator on $L^2(\mathbb{R})$ given by

$$H = -\frac{d^2}{dr^2} + \frac{2}{r^2+1} + V(r).$$

We need to check that H has no point spectrum and that 0 is not a resonance for H. First, we note that the decay of the potential $\frac{2}{r^2+1} + V(r)$ implies that $\sigma_{ac}(H) = [0, \infty)$ and there are no embedded eigenvalues. If $Q \equiv 0$ (the $n = 0$ case), the fact that H has no eigenvalues in $(-\infty, 0]$ follows from the fact that the potential term $2\langle r\rangle^{-2} + V(r)$ is nonnegative. For the case $n \in \mathbb{N}$, multiply the equation

$$\partial_r^2 Q + \frac{2r}{r^2+1}\partial_r Q - \frac{\sin 2Q}{r^2+1} = 0$$

by $r^2 + 1$ and differentiate to conclude that

$$\tilde{H}(\langle r\rangle^2 Q'(r)) = 0,$$

where $\tilde{H} = H - \langle r\rangle^4$. By Proposition 2.2.1 the harmonic map Q is strictly increasing on $\mathbb{R}$

so that $\langle r\rangle^2 Q'(r) > 0$ for all $r \in \mathbb{R}$. By Sturm oscillation theory we conclude that $\tilde{H}$ has no negative eigenvalues and that $\sigma(\tilde{H}) = [0,\infty)$. In particular, we have for all $h \in C_0^\infty(\mathbb{R})$

$$(Hh,h)_{L^2(\mathbb{R})} = (\tilde{H}h,h)_{L^2(\mathbb{R})} + \int |h|^2 \langle r\rangle^{-4} dr \geq \int |h|^2 \langle r\rangle^{-4} dr. \tag{2.5.4}$$

By a variational principle, the previous implies that H has no eigenvalues in $(-\infty,0]$, and thus, H has no point spectrum. We now check that 0 is not a resonance of H. The asymptotics of the potential $2\langle r\rangle^{-2} + V(r)$ imply that 0 is a resonance if and only if 0 is an eigenvalue (see Lemma 2.3.3 and Definition 2.3.4). Thus, 0 is not a resonance of H. We conclude that V satisfies the hypotheses of Proposition 2.4.2.

For $I \subseteq \mathbb{R}$, we denote the following spacetime norms

$$\|u\|_{S(I)} := \|u\|_{L_t^3 L_x^6(I)}, \quad \|u\|_{W(I)} := \|u\|_{L_t^3 \dot{W}_x^{1/2,L^3}(I)},$$
$$\|h\|_{N(I)} := \|F\|_{L_t^1 L_x^2(I) + L_t^{3/2} \dot{W}_x^{1/2,3/2}(I)}.$$

By the previous discussion and Proposition 2.4.2, a solution u to (2.5.2) satisfies the estimate

$$\|u\|_{W(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{N(I)}. \tag{2.5.5}$$

We claim that if $f \in C_0^\infty(\mathcal{M}^5)$ is radial, then

$$\|f\|_{L_x^6} \lesssim \||\nabla|^{1/2} f\|_{L_x^3}.$$

Indeed, by the fundamental theorem of calculus, we have

$$|f(r)| \lesssim \langle r\rangle^{-2/3} \left(\int |f'(r)|^3 (r^2+1) dr \right)^{1/3} = \langle r\rangle^{-2/3} \|\nabla f\|_{L_x^3}.$$

Thus, $\|f\|_{L_x^\infty} \lesssim \|\nabla f\|_{L_x^3}$. Interpolating this estimate with the trivial embedding $L_x^3 \hookrightarrow L_x^3$

yields the desired bound $\|f\|_{L^6_x} \lesssim \||\nabla|^{1/2} f\|_{L^3_x}$. Thus, we have that the 'scattering norm' $\|\cdot\|_{S(I)}$ is weaker than the norm $\|\cdot\|_{W(I)}$. This fact and (2.5.5) imply that a solution to (2.5.2) satisfies the Strichartz estimate

$$\|u\|_{S(\mathbb{R})} + \|u\|_{W(\mathbb{R})} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{N(\mathbb{R})}. \tag{2.5.6}$$

We now use (2.5.6) and standard contraction mapping arguments to establish the following global well–posedness and small data theory. We remark here that it will be important in later applications to use the weaker norm $\|\cdot\|_{S(I)}$ along with the norm $\|\cdot\|_{W(\mathbb{R})}$ when establishing the small data scattering.

Proposition 2.5.1. *For every $(u_0, u_1) \in \mathcal{H}$, there exists a unique global solution u to* (2.5.1) *such that $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H}) \cap L^\infty(\mathbb{R}; \mathcal{H})$. A solution u scatters to a free wave on $\mathcal{M}^5$ as $t \to \infty$ if and only if*

$$\|u\|_{S(\mathbb{R})} < \infty.$$

Here, scattering to a free wave on $\mathcal{M}^5$ as $t \to \infty$ means that there exists a solution v_L to (2.5.2) *with $V \equiv h \equiv 0$ such that*

$$\lim_{t \to \infty} \|\vec{u}(t) - \vec{v}_L(t)\|_{\mathcal{H}} = 0.$$

A similar characterization of u scattering to a free wave on $\mathcal{M}^5$ as $t \to -\infty$ also holds. Moreover, there exists $\delta > 0$ such that if $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, then

$$\|\vec{u}\|_{L^\infty_t \mathcal{H}} + \|u\|_{S(\mathbb{R})} + \|u\|_{W(\mathbb{R})} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} < \delta. \tag{2.5.7}$$

Proof. We first show that for every $(u_0, u_1) \in \mathcal{H}$, there exists a unique global solution $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H}) \cap L^\infty(\mathbb{R}; \mathcal{H})$ to (2.5.1) with $\vec{u}(0) = (u_0, u_1)$. Denote the propagator for the free wave equation on $\mathcal{M}^5$ by $S(t)$, i.e. $S(t)(u_0, u_1)$ solves (2.5.2) with $V \equiv h \equiv 0$. Denote the propagator for the free wave equation on $\mathcal{M}^5$ with potential V by $S_V(t)$, i.e. $S_V(t)(u_0, u_1)$ solves (2.5.2) with $h \equiv 0$. Let

$$\mathcal{E}_V(f, g) := \frac{1}{2} \int \left(|g|^2 + |\partial_r f|^2 + V|f|^2 dr \right) (r^2+1)^2 dr \tag{2.5.8}$$

denote the conserved energy associated to S_V. Using the coercivity bound (2.5.4) it is not hard to conclude that

$$\|\partial_r f\|_{L^2(\mathbb{R};(r^2+1)^2)} \simeq \left\| \sqrt{-\Delta_g + V} f \right\|_{L^2(\mathbb{R};(r^2+1)^2)} \tag{2.5.9}$$

for all radial f so that

$$\|(f, g)\|_{\mathcal{H}}^2 \simeq \mathcal{E}_V(f, g) \tag{2.5.10}$$

for all radial f, g. Indeed, by the decay of V and the Strauss estimate (2.4.13) we have

$$\begin{aligned} \left\| \sqrt{-\Delta_g + V} f \right\|_{L^2(\mathbb{R};(r^2+1)^2)}^2 &= \int \left((-\Delta_g f) f + V|f|^2 \right) (r^2+1)^2 dr \\ &= \|\partial_r f\|_{L^2(\mathbb{R};(r^2+1)^2 dr)}^2 + \int V|f|^2 (r^2+1)^2 dr \\ &\lesssim \|\partial_r f\|_{L^2(\mathbb{R};(r^2+1)^2 dr)}^2. \end{aligned}$$

We now note that by the second equality above and the decay of V we have

$$\|\partial_r f\|_{L^2(\mathbb{R};(r^2+1)^2 dr)}^2 \lesssim \left\| \sqrt{-\Delta_g + V} f \right\|_{L^2(\mathbb{R};(r^2+1)^2)}^2 + \int |f|^2 dr.$$

By (2.5.3) and (2.5.4) (applied to $h = (r^2+1)f$) we see that

$$\int |f|^2 dr \lesssim \left\| \sqrt{-\Delta_g + V} f \right\|^2_{L^2(\mathbb{R};(r^2+1)^2)}$$

whence

$$\|\partial_r f\|^2_{L^2(\mathbb{R};(r^2+1)^2 dr)} \lesssim \left\| \sqrt{-\Delta_g + V} f \right\|^2_{L^2(\mathbb{R};(r^2+1)^2)}.$$

This proves (2.5.9).

We write the nonlinear equation (2.5.1) in Duhamel form as

$$u(t) = S_V(t)(u_0, u_1) + \int_0^t S_V(t-s)\left(0, F(\cdot, u(s)) + G(\cdot, u(s))\right) ds.$$

Using a simple energy estimate, (2.4.8), (2.4.9), (2.4.10), (2.4.13), and (2.5.10), we obtain the following a–priori estimate for a solution $\vec{u}(t) \in C([0,T];\mathcal{H})$ to (2.5.1): for $t \in [0,T]$

$$\begin{aligned} \|\vec{u}(t)\|_{\mathcal{H}} &\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \int_0^T \|F(\cdot, u(s)) + G(\cdot, u(s)\|_{L^2} ds \\ &\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + T\left(\|\vec{u}\|^2_{L^\infty_t([0,T];\mathcal{H})} + \|\vec{u}\|^3_{L^\infty_t([0,T];\mathcal{H})} \right). \end{aligned} \tag{2.5.11}$$

By a contraction mapping argument based on (2.5.11) and the conservation of energy for (2.4.1), we conclude that there exists a unique global solution $\vec{u}(t) \in C(\mathbb{R};\mathcal{H}) \cap L^\infty(\mathbb{R};\mathcal{H})$ to (2.5.1).

We now prove the scattering criterion and small data scattering. Note that every solution $\vec{u}(t) \in C(\mathbb{R};\mathcal{H})$ to (2.5.1) satisfies $\|u\|_{S(I)} + \|u\|_{W(I)} < \infty$ for all $I \Subset \mathbb{R}$. Indeed, by (2.4.13) we have $\|u\|_{L^6_x} \lesssim \|\nabla u\|_{L^2_x}$ whence by interpolation we have $\|u\|_{\dot{W}^{1/2,3}_x} \lesssim \|\nabla u\|_{L^2_x}$. We first prove the small data scattering estimate (2.5.7) as this will also illustrate the validity of the scattering criterion. Let u be a solution to (2.5.1) and let $I \subset \mathbb{R}$. We first note that by the

Leibniz rule for Sobolev spaces (see [5] for asymptotically conic manifolds), we have

$$\begin{aligned}\|(\langle r\rangle^{-1}\sin 2Q)u^2\|_{\dot W_x^{1/2,3/2}} &\lesssim \|\langle r\rangle^{-1}\sin 2Q\|_{\dot W_x^{1/2,3}}\|u^2\|_{L_x^3} + \|\langle r\rangle^{-1}\sin 2Q\|_{L_x^6}\|u^2\|_{\dot W_x^{1/2,2}} \\ &\lesssim \|u\|_{L_x^6}^2 + \|u\|_{L_x^6}\|u\|_{\dot W_x^{1/2,3}},\end{aligned}$$

whence by Hölder's inequality in time we have

$$\|(\langle r\rangle^{-1}\sin 2Q)u^2\|_{L_t^{3/2}\dot W_x^{1/2,3/2}(I)} \lesssim \|u\|_{S(I)}^2 + \|u\|_{S(I)}\|u\|_{W(I)}. \tag{2.5.12}$$

Then by (2.4.8), (2.4.9), (2.4.10), the Strichartz estimate (2.5.6), and (2.5.12) we have

$$\begin{aligned}\|u\|_{S(I)} + \|u\|_{W(I)} &\lesssim \|\vec u(0)\|_{\mathcal H} + \|N(\cdot,u)\|_{N(I)} \\ &\lesssim \|\vec u(0)\|_{\mathcal H} + \|F(\cdot,u)\|_{N(I)} + \|G(\cdot,u)\|_{N(I)} \\ &\lesssim \|\vec u(0)\|_{\mathcal H} + +\|(\langle r\rangle^{-1}\sin 2Q)u^2\|_{L_t^{3/2}\dot W_x^{1/2,3/2}(I)} + \|\langle r\rangle^{-1}u^4\|_{L_t^1L_x^2(I)} \\ &\quad + \||u|^3\|_{L_t^1L_x^2(I)} \\ &\lesssim \|\vec u(0)\|_{\mathcal H} + \|u\|_{S(I)}\|u\|_{W(I)} + \|u\|_{S(I)}^2 + \|\vec u\|_{L_t^\infty\mathcal H}\|u\|_{S(I)}^3 + \|u\|_{S(I)}^3.\end{aligned}$$

By a standard continuity argument, there exists $\delta > 0$ such that if $\|\vec u(0)\|_{\mathcal H} < \delta$ then $\|\vec u\|_{L_t^\infty\mathcal H} + \|u\|_{S(\mathbb R)} + \|u\|_{W(\mathbb R)} \lesssim \|\vec u(0)\|_{\mathcal H}$ as desired. A simple variant of the above argument also shows that if $\|u\|_{S(0,\infty)} < \infty$, then

$$w_L(0) = \vec u(0) + \int_0^\infty S_V(-s)(0, N(\cdot, u(s)))ds$$

converges in $\mathcal H$. Thus, by Duhamel we conclude that

$$\vec u(t) = \vec S_V(t)w_L(0) + o_{\mathcal H}(1), \tag{2.5.13}$$

as $t \to \infty$. To extract a free wave $v_L(t) = S(t)\vec{v}_L(0)$ from the perturbed wave $w_L(t) = S_V(t)\vec{w}_L(0)$, we write, via Duhamel,

$$\begin{aligned} w_L(t) &= S(t)\vec{w}_L(0) + \int_0^t S(t-s)(0, V w_L(s))ds \\ &= S(t)\left[\vec{w}_L(0) + \int_0^t S(-s)(0, V w_L(s))ds\right]. \end{aligned}$$

We then take

$$\vec{v}_L(0) = \vec{w}_L(0) + \int_0^\infty S(-s)(0, V w_L(s))ds$$

which converges in $\mathcal{H}$ by (2.4.23) with $X = L_t^\infty \mathcal{H}$. Then $\vec{w}_L(t) = \vec{v}_L(t) + o_{\mathcal{H}}(1)$ as $t \to \infty$. This along with (2.5.13) allow us to conclude that if $\|u\|_{S(0,\infty)} < \infty$, then u scatters to a free wave on $\mathcal{M}^5$ as $t \to \infty$. The fact that the finiteness of $\|u\|_{S(0,\infty)}$ is necessary if u scatters as $t \to \infty$ follows from similar arguments using the fact that $\|v_L\|_{S(0,\infty)} < \infty$ holds for any free wave v_L on $\mathcal{M}^5$. This concludes the proof. □

A tool that will be essential in establishing the second step of the concentration compactness/rigidity theorem method is the following long–time perturbation theory for (2.5.1).

Proposition 2.5.2 (Long–time perturbation theory)**.** *Let $A > 0$. Then there exists $\epsilon_0 = \epsilon_0(A) > 0$ and $C = C(A) > 0$ such that the following holds. Let $0 < \epsilon < \epsilon_0$, $(u_0, u_1) \in \mathcal{H}$, and $I \subseteq \mathbb{R}$ with $0 \in I$. Assume that $\vec{U}(t) \in C(I; \mathcal{H})$ satisfies on I*

$$\partial_t^2 U - \Delta_g U + VU = N(\cdot, U) + e, \tag{2.5.14}$$

such that

$$\sup_{t\in I}\|\vec{U}(t)\|_{\mathcal{H}} + \|U\|_{S(I)} \leq A, \tag{2.5.15}$$

$$\|\vec{U}(0) - (u_0, u_1)\|_{\mathcal{H}} + \|e\|_{N(I)} \leq \epsilon. \tag{2.5.16}$$

Then the unique global solution u *to* (2.5.1) *with initial data* $\vec{u}(0) = (u_0, u_1)$ *satisfies*

$$\sup_{t\in I}\|\vec{u}(t) - \vec{U}(t)\|_{\mathcal{H}} + \|u - U\|_{S(I)} \leq C(A)\epsilon. \tag{2.5.17}$$

Proof. We establish the estimate (2.5.17) with $I_+ := I \cap [0, \infty)$ in place of I. Establishing (2.5.17) with $I_- := I \cap (-\infty, 0]$ in place of I is similar, and these two estimates yield (2.5.17). We first make some preliminary observations. The bounds (2.5.15) and (2.5.16) along with conservation of energy imply that

$$\|\vec{u}(t)\|_{\mathcal{H}} \leq C_0(A). \tag{2.5.18}$$

Also, by interpolation and (2.5.15), $\|U\|_{W(J)} < \infty$ for all $J \Subset I$. We claim that

$$\|U\|_{W(I)} \leq C_1(A). \tag{2.5.19}$$

To see this, let $\eta > 0$ to be chosen later, and partition I_+ into subintervals $I_+ = \cup_{j=1}^{J_0(A)} I_j$ such that $\forall j$, $\|U\|_{S(I_j)} < \eta$. Then via (2.5.14) and Duhamel, we have on $I_j := [t_j, t_{j+1}]$

$$U(t) = S_V(t - t_j)\vec{U}(t_j) + \int_{t_j}^{t} S_V(t - s)\,(0, N(\cdot, U(s)) + e)\, ds.$$

By arguing as in the proof of Proposition 2.5.1 and Strichartz estimates, we have

$$\begin{aligned}\|U\|_{W(I_j)} &\leq C\|\vec{U}(t_j)\|_{\mathcal{H}} + C\|N(\cdot,U)\|_{N(I_j)} + C\|e\|_{N(I_j)} \\ &\leq CA + C\|U\|_{W(I_j)}\|U\|_{S(I_j)} + C\|U\|_{S(I_j)}^2 \\ &\quad + C\|\vec{U}\|_{L_t^\infty \mathcal{H}(I_j)}\|U\|_{S(I_j)}^3 + C\|U\|_{S(I_j)}^3 + C\epsilon \\ &\leq C\eta\|U\|_{W(I_j)} + C\epsilon + C\cdot(A+1)^4.\end{aligned}$$

If we choose $\eta = (2C)^{-1}$, then we obtain (2.5.19).

We now establish (2.5.17). Define $w = u - U$. Then w solves on I

$$\begin{aligned}&\partial_t^2 w - \Delta_g w + Vw = N(\cdot, U+w) - N(\cdot, U) - e, \\ &\vec{w}(0) = (u_0, u_1) - \vec{U}(0).\end{aligned} \tag{2.5.20}$$

By (2.5.15) and (2.5.18), w satisfies

$$\sup_{t\in I}\|\vec{w}(t)\|_{\mathcal{H}} \leq A + C_0(A). \tag{2.5.21}$$

Let $\eta > 0$ to be chosen later. Partition I_+ into subintervals $I = \cup_{j=1}^{J_1(A)} I_j$ such that

$$\forall j, \quad \|U\|_{S(I_j)} + \|U\|_{W(I_j)} \leq \eta. \tag{2.5.22}$$

On $I_j := [t_j, t_{j+1}]$, we have via (2.5.20) and Duhamel

$$w(t) = S_V(t-t_j)\vec{w}(t_j) + \int_{t_j}^t S_V(t-s)\,(0, N(\cdot, U(s)+w(s)) - N(\cdot, U(s)) - e)\,ds. \tag{2.5.23}$$

By arguing as in the proof of Proposition 2.5.1 and Strichartz estimates, we have

$$\begin{aligned}
\|w\|_{S(I_j)} + \|w\|_{W(I_j)} &\le \|S_V(t-t_j)\vec{w}(t_j)\|_{S(\mathbb{R})} + \|S_V(t-t_j)\vec{w}(t_j)\|_{W(\mathbb{R})} + C\|e\|_{N(I_j)} \\
&+ C\|N(\cdot, U+w) - N(\cdot, U)\|_{N(I_j)} \\
&\le \|S_V(t-t_j)\vec{w}(t_j)\|_{S(\mathbb{R})} + \|S_V(t-t_j)\vec{w}(t_j)\|_{W(\mathbb{R})} + C\|e\|_{N(I_j)} \\
&+ C\Big[\|w\|_{W(I_j)}\|U\|_{S(I_j)} + \|w\|_{S(I_j)}\|U\|_{W(I_j)} + \|w\|_{S(I_j)}\|w\|_{W(I_j)} \\
&+ \|w\|_{S(I_j)}^2 + \|w\|_{S(I_j)}\|U\|_{S(I_j)} \\
&+ \|w\|_{S(I_j)}\|U\|_{S(I_j)}^2(\|U\|_{L_t^\infty \mathcal{H}(I_j)} + 1) \\
&+ \|w\|_{S(I_j)}^2\|U\|_{S(I_j)}(\|U\|_{L_t^\infty \mathcal{H}(I_j)} + 1) \\
&+ \|w\|_{S(I_j)}^3(\|U\|_{L_t^\infty \mathcal{H}} + \|w\|_{L_t^\infty \mathcal{H}(I_j)} + 1)\Big] \\
&\le \|S_V(t-t_j)\vec{w}(t_j)\|_{S(\mathbb{R})} + \|S_V(t-t_j)\vec{w}(t_j)\|_{W(\mathbb{R})} + C\epsilon \\
&+ (\eta + \eta^2)(A+1)C\Big[\|w\|_{S(I_j)} + \|w\|_{W(I_j)}\Big] \\
&+ C_2(A)\Big[(\|w\|_{S(I_j)} + \|w\|_{W(I_j)})^2 + (\|w\|_{S(I_j)} + \|w\|_{W(I_j)})^3\Big].
\end{aligned}$$

Here $C_2 = C_2(A)$ is a constant which depends only A. Define

$$\gamma_j := \|S_V(t-t_j)\vec{w}(t_j)\|_{S(\mathbb{R})} + \|S_V(t-t_j)\vec{w}(t_j)\|_{W(\mathbb{R})} + C\epsilon.$$

If we fix η so small so that $\eta + \eta^2 < (2(A+1)C)^{-1}$, then we obtain

$$\|w\|_{S(I_j)} + \|w\|_{W(I_j)} \le 2\gamma_j + 2C_1(A)\Big[(\|w\|_{S(I_j)} + \|w\|_{W(I_j)})^2 + (\|w\|_{S(I_j)} + \|w\|_{W(I_j)})^3\Big]. \tag{2.5.24}$$

In particular, by a standard continuity argument there exists $\delta_0 = \delta_0(C_1(A))$ such that if

$\gamma_j < \delta_0$, then

$$\|w\|_{S(I_j)} + \|w\|_{W(I_j)} \le 4\gamma_j, \tag{2.5.25}$$

$$2C_2(A)\Big[(\|w\|_{S(I_j)} + \|w\|_{W(I_j)})^2 + (\|w\|_{S(I_j)} + \|w\|_{W(I_j)})^3\Big] \le 4\gamma_j. \tag{2.5.26}$$

We now iterate, and insert t_{j+1} into (2.5.23). Applying $S_V(t-t_{j+1})$ to both sides, we obtain

$$S(t-t_{j+1})\vec{w}(t_{j+1}) = S(t-t_j)\vec{w}(t_j) + \int_{t_j}^{t_{j+1}} S(t-s)\,(0, N(\cdot, U(s)+w(s)) - N(\cdot, U(s)) - e)\, ds.$$

By (2.5.25) and (2.5.26) and the previous arguments, we deduce that

$$\gamma_{j+1} \le 10\gamma_j,$$

provided that $\gamma_j < \delta_0$. By Strichartz estimates and (2.5.16), we have for some absolute constant C_3

$$\gamma_1 := \|S_V(t)\vec{w}(0)\|_{S(\mathbb{R})} + \|S_V(t)\vec{w}(0)\|_{S(\mathbb{R})} + C\epsilon \le C_3\epsilon < C_3\epsilon_0.$$

Iterating, we have that $\gamma_{j+1} \le 10^j C_3\epsilon$ as long as $\gamma_j < \delta_0$. If we choose $\epsilon_0 = \epsilon_0(A)$ so small so that $10^J C_3\epsilon_0 < \delta_0$, then the condition $\gamma_j < \delta_0$ is always satisfied. This along with (2.5.25) imply that

$$\|w\|_{S(I_+)} + \|w\|_{W(I_+)} \le C(A)\epsilon$$

as desired. The estimate for $\|w\|_{L_t^\infty \mathcal{H}(I_+)}$ follows a posteriori from (2.5.20), (2.5.15), (2.5.16), the estimate for $\|w\|_{S(I_+)} + \|w\|_{W(I_+)}$, and Strichartz estimates. This completes the proof. $\square$

2.5.2 Concentration–compactness

In this, the second step of the concentration–compactness methodology, we show that if our main result Theorem 2.4.1 (or equivalently Theorem 2.1.1) fails, then there exists a nonzero 'critical element.' More precisely, we prove the following.

Proposition 2.5.3. *Suppose that Theorem 2.4.1 fails. Then there exists a nonzero global solution u_* to* (2.5.1) *such that the set*

$$K = \{\vec{u}_*(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H}$.

Essential tools for proving Proposition 2.5.3 are the following linear and nonlinear *profile decompositions.*

Lemma 2.5.4 (Linear Profile Decomposition). *Let $\{(u_{0,n}, u_{1,n})\}_n$ be a bounded sequence in $\mathcal{H}$. Then after extraction of subsequences and relabeling, there exist a sequence of solutions $\left\{U_L^j\right\}_{j\geq 1}$ to* (2.5.2) *with $h \equiv 0$ which are bounded in $\mathcal{H}$ and a sequence of times $\{t_{j,n}\}_n$ for $j \geq 1$ that satisfy the orthogonality condition*

$$\forall j \neq k, \quad \lim_{n\to\infty} |t_{j,n} - t_{k,n}| = \infty,$$

such that for all $J \geq 1$,

$$(u_{0,n}, u_{1,n}) = \sum_{j=1}^{J} \vec{U}_L^j(-t_{j,n}) + (w_{0,n}^J, w_{1,n}^J),$$

where the error $w_n^J(t) := S_V(t)(w_{0,n}^J, w_{1,n}^J)$ satisfies

$$\lim_{J\to\infty} \overline{\lim_{n\to\infty}} \|w_n^J\|_{L_t^\infty L_x^p(\mathbb{R})\cap S(\mathbb{R})} = 0, \quad \forall \frac{10}{3} < p \leq \infty. \tag{2.5.27}$$

Moreover, we have the following Pythagorean expansion of the energy

$$\mathcal{E}_V(\vec{u}_n) = \sum_{j=1}^{J} \mathcal{E}_V(\vec{U}_L^j) + \mathcal{E}_V(\vec{w}_n^J) + o(1), \tag{2.5.28}$$

as $n \to \infty$.

The proof of of Lemma 2.5.4 is identical to the proof of Lemma 3.2 in [20] and we omit it. The sequence $\{(u_{0,n}, u_{1,n})\}_n$ in Lemma 2.5.4 is said to have a *profile decomposition* with profiles $\{U_L^j\}_j$ and parameters $\{t_{j,n}\}_{j,n}$. We note that after passing to a further subsequence if necessary, we may assume that for all $j \geq 1$, either $t_{j,n} = 0$ $\forall n$ or $\lim_n t_{j,n} = \pm\infty$.

In order to apply Lemma 2.5.4 in the context of the nonlinear problem (2.5.1), we will need the notion of *nonlinear profiles*. For each profile U_L^j with time parameters $\{t_{j,n}\}_n$, we define its associated nonlinear profile U^j to be the unique global solution to (2.5.1) such that

$$\lim_{n\to\infty} \|\vec{U}^j(-t_{j,n}) - \vec{U}_L^j(-t_{j,n})\|_{\mathcal{H}} = 0.$$

It is easy to see that a nonlinear profile always exists. Indeed, if $t_{j,n} = 0$ for all n, then we set U^j to be the solution to (2.5.1) with initial data $\vec{U}^j(0) = \vec{U}_L^j(0)$. If $\lim_n -t_{j,n} = \infty$, say, then we set U^j to be the unique globally defined solution to the integral equation

$$U^j(t) = \vec{U}_L^j(-t_{j,n}) - \int_t^\infty S_V(t-s)(0, N(\cdot, U^j(s)))ds. \tag{2.5.29}$$

A unique global solution to (2.5.29) can be shown to exist using contraction mapping arguments in the spirit of those used in Proposition 2.5.1 and Proposition 2.5.2.

For each nonlinear profile U^j, we denote

$$U_n^j(t) := U^j(t - t_{j,n}).$$

Using Proposition 2.5.2, we obtain the following nonlinear profile decomposition from the linear profile decomposition in Lemma 2.5.4.

Lemma 2.5.5 (Nonlinear Profile Decomposition)**.** *Let* $\{(u_{0,n}, u_{1,n})\}_n$ *be a bounded sequence in* $\mathcal{H}$ *admitting a profile decomposition with profiles* $\{U_L^j\}_j$ *and parameters* $\{t_{j,n}\}_{j,n}$*. Let* $T_n \in [0, +\infty)$*. Assume*

$$\forall j \geq 1, \quad \limsup_{n\to\infty} \|U^j\|_{S(-t_{j,n}, T_n - t_{j,n})} < \infty. \tag{2.5.30}$$

Let u_n *be the unique global solution to* (2.5.1) *with initial data* $\vec{u}(0) = (u_{0,n}, u_{1,n})$*. Then*

$$\limsup_{n\to\infty} \|u_n\|_{S(0,T_n)} < \infty,$$

and for all $t \in [0, T_n]$

$$\vec{u}_n(t) = \sum_{j=1}^{J} \vec{U}_n^j(t) + \vec{w}_n^J(t) + \vec{r}_n^J(t),$$

with

$$\lim_{J\to\infty} \overline{\lim_{n\to\infty}} \left[\|r_n^J\|_{S(0,T_n)} + \sup_{t\in[0,T_n]} \left\| \vec{r}_n^J(t) \right\|_{\mathcal{H}} \right] = 0.$$

An analogous statement holds if $T_n < 0$*.*

Proof. For $J \geq 1$, $n \geq 1$, define

$$U_n^J(t) := \sum_{j=1}^{J} U_n^j(t) + w_n^J(t).$$

We will apply Proposition 2.5.2 with $U = U_n^J$ and $u = u_n$ for n and J large. We first show that

$$\overline{\lim_J}\,\overline{\lim_n}\,\|U_n^J\|_{S(0,T_n)} < \infty. \tag{2.5.31}$$

By assumption, there exists $M > 0$ such that $\forall n$, $\|(u_{0,n}, u_{1,n})\|_{\mathcal{H}}^2 \simeq \mathcal{E}_V(u_{0,n}, u_{1,n}) \leq M$. The Pythagorean expansion of the energy (2.5.28) implies that

$$\overline{\lim_J}\,\overline{\lim_n}\,\mathcal{E}_V(w_n^J) + \sum_{j=1}^{\infty} \mathcal{E}_V(U_L^j) \leq M. \tag{2.5.32}$$

Hence, there exists $J_0 \geq 1$ such that

$$\sum_{j>J_0} \mathcal{E}_V(\vec{U}_L^j) \ll \delta^2,$$

where δ is from Proposition 2.5.1. In particular, this implies by Proposition 2.5.1 that the nonlinear profiles satisfy for all $j > J_0$

$$\|\vec{U}^j\|_{L_t^\infty \mathcal{H}} + \|U^j\|_{S(\mathbb{R})} + \|U^j\|_{W(\mathbb{R})} \lesssim \mathcal{E}_V(\vec{U}_L^j)^{1/2}.$$

Let $J \geq 1$. Then

$$\|U_n^J\|_{S(0,T_n)} \leq \left\| \sum_{j=1}^{J} U_n^j \right\|_{S(0,T_n)} + \|w_n^J\|_{S(\mathbb{R})}.$$

Now

$$\begin{aligned} \left\| \sum_{j=1}^{J} U_n^j \right\|_{S(0,T_n)}^3 &\leq \left\| \sum_{j=1}^{J} \|U_n^j\|_{L_x^6} \right\|_{L_t^3(0,T_n)}^3 \\ &= \sum_{j=1}^{J} \|U_n^j\|_{S(0,T_n)}^3 + \epsilon_n^J, \end{aligned} \tag{2.5.33}$$

where the error ϵ_n^J is a sum of terms of the form

$$\int_0^T \|U_n^j(t)\|_{L_x^6} \|U_n^k(t)\|_{L_x^6} \|U_n^l(t)\|_{L_x^6} dt,$$

with $1 \leq j, k, l \leq J$ and $j \neq k$. We claim that

$$\lim_{n\to\infty} \int_0^{T_n} \|U_n^j(t)\|_{L_x^6} \|U_n^k(t)\|_{L_x^6} \|U_n^l(t)\|_{L_x^6} dt = 0. \tag{2.5.34}$$

Indeed, by the assumption (2.5.30) and an approximation argument, we may assume that the functions U^j, U^k are compactly supported in t. Now

$$\begin{aligned}\int_0^{T_n} \|U_n^j(t)\|_{L_x^6} \|U_n^k(t)\|_{L_x^6} \|U_n^l(t)\|_{L_x^6} dt &\lesssim \Big(\int_0^{T_n} \|U_n^j(t)\|_{L_x^6}^{3/2} \|U_n^k(t)\|_{L_x^6}^{3/2} dt\Big)^{2/3} \|U_n^l\|_{S(0,T_n)} \\ &\lesssim \Big(\int_0^{T_n} \|U_n^j(t)\|_{L_x^6}^{3/2} \|U_n^k(t)\|_{L_x^6}^{3/2} dt\Big)^{2/3}.\end{aligned}$$

Extending the integration over all of $\mathbb{R}$ and changing variables implies that

$$\int_0^{T_n} \|U_n^j(t)\|_{L_x^6}^{3/2} \|U_n^k(t)\|_{L_x^6}^{3/2} dt \leq \int \|U^j(t)\|_{L_x^6}^{3/2} \|U^k(t + t_{j,n} - t_{k,n})\|_{L_x^6}^{3/2} dt$$

The orthogonality of the parameters implies that $|t_{j,n} - t_{k,n}| \to_n \infty$. Thus, the support of $U^j(\cdot)$ and $U^k(\cdot + t_{j,n} - t_{k,n})$ are eventually disjoint whence

$$\lim_{n\to\infty} \int \|U^j(t)\|_{L_x^6}^{3/2} \|U^k(t + t_{j,n} - t_{k,n})\|_{L_x^6}^{3/2} dt = 0.$$

This proves (2.5.34). Returning to (2.5.33) and recalling our choice of J_0, we see that

$$\begin{aligned}
\varlimsup_n \left\| \sum_{j=1}^J U_n^j \right\|_{S(0,T_n)}^3 &\leq \varlimsup_n \sum_{j=1}^J \|U_n^j\|_{S(0,T_n)}^3 \\
&\lesssim \sum_{j=1}^{J_0} \varlimsup_n \|U_n^j\|_{S(0,T_n)}^3 + \sum_{j>J_0} \|U^j\|_{S(\mathbb{R})}^3 \\
&\lesssim 1 + \sum_{j>J_0} \mathcal{E}_V(\vec{U}_L^j)^{3/2} \\
&\lesssim 1 + M,
\end{aligned}$$

where the implied constant is independent of J. Thus,

$$\varlimsup_J \varlimsup_n \|U_n^J\|_{S(0,T_n)} \leq \varlimsup_J \varlimsup_n \left\| \sum_{j=1}^J U_n^j \right\|_{S(0,T_n)} + \varlimsup_J \varlimsup_n \|w_n^J\|_{S(\mathbb{R})} < \infty.$$

Using similar arguments, we also conclude that

$$\varlimsup_J \varlimsup_n \|U_n^J\|_{L_t^\infty \mathcal{H}(0,T_n)} < \infty.$$

We now verify that the following error

$$\begin{aligned}
e_n^J &:= \partial_t^2 U_n^J - \Delta_g U_n^J + V U_n^J - N(\cdot, U_n^J) \\
&= \sum_{j=1}^J N(\cdot, U_n^j) - N\left(\cdot, \sum_{j=1}^J U_n^j + w_n^J\right),
\end{aligned}$$

satisfies

$$\varlimsup_J \varlimsup_n \|e_n^J\|_{N(0,T_n)} = 0. \tag{2.5.35}$$

We focus only on the quadratic part of $N(\cdot, u)$ since the other parts can be handled similarly. More precisely, we show that

$$\overline{\lim_J}\,\overline{\lim_n}\left\|(2\langle r\rangle^{-1}\sin 2Q)\left(\sum_{j=1}^{J}(U_n^j)^2-\left(\sum_{j=1} U_n^j+w_n^J\right)^2\right)\right\|_{L_t^{3/2}\dot{W}_x^{1/2,3/2}(0,T_n)}=0. \quad (2.5.36)$$

To lessen the notation, for $I \subseteq \mathbb{R}$, we denote $W'(I) := L_t^{3/2}\dot{W}_x^{1/2,3/2}(I)$. We observe that

$$\begin{aligned}
&\left\|(2\langle r\rangle^{-1}\sin 2Q)\left(\sum_{j=1}^{J}(U_n^j)^2-\left(\sum_{j=1} U_n^j+w_n^J\right)^2\right)\right\|_{W'(0,T_n)}\\
&\lesssim \left\|(2\langle r\rangle^{-1}\sin 2Q)w_n^J\sum_{j=1}^{J}U_n^j\right\|_{W'(0,T_n)}+\sum_{j\neq k}\left\|(2\langle r\rangle^{-1}\sin 2Q)U_n^j\sum_{j=1}^{J}U_n^k\right\|_{W'(0,T_n)}\\
&\quad+\|(2\langle r\rangle^{-1}\sin 2Q)(w_n^J)^2\|_{W'(0,T_n)}\\
&=: A_n^J+B_n^J+C_n^J.
\end{aligned}$$

Using the orthogonality of the parameters and arguments as in the previous paragraph, it is straightforward to show that

$$\lim_n B_n^J = 0.$$

To estimate C_n^J, we recall that $\overline{\lim}_j\,\overline{\lim}_n\,\|w_n^J\|_{S(\mathbb{R})}=0$ and $\vec{w}_n^J(0)$ is bounded in $\mathcal{H}$. Thus, by the product rule (see the proof of Proposition 2.5.1) and Strichartz estimates, we have

$$C_n^J \lesssim \|w_n^J\|_{S(\mathbb{R})}\|w_n^J\|_{W(\mathbb{R})}+\|w_n^J\|_{S(\mathbb{R})}^2 \lesssim \|w_n^J\|_{S(\mathbb{R})}+\|w_n^J\|_{S(\mathbb{R})}^2,$$

whence $\overline{\lim}_J\,\overline{\lim}_n\,C_n^J = 0$. We now show that $\overline{\lim}_J\,\overline{\lim}_n\,A_n^J = 0$. Let $\epsilon > 0$. By the arguments used to show that $\overline{\lim}_J\,\overline{\lim}_n\,\|U_n^J\|_{S(0,T_n)} < \infty$, there exists $J_1 = J_1(\epsilon) > J_0$ such that for all $J > J_1$

$$\overline{\lim_n} \left(\left\| \sum_{j=J_1+1}^{J} U_n^j \right\|_{S(0,T_n)} + \left\| \sum_{j=J_1+1}^{J} U_n^j \right\|_{S(0,T_n)} \right) < \epsilon. \tag{2.5.37}$$

Thus, by the product rule, we obtain

$$\begin{aligned} &\overline{\lim_n} \left\| (2\langle r \rangle^{-1} \sin 2Q) w_n^J \sum_{j=J_1+1}^{J} U_n^j \right\|_{W'(0,T_n)} \\ &\lesssim \overline{\lim_n} \, \|w_n^J\|_{W(\mathbb{R})} \left\| \sum_{j=J_1+1}^{J} U_n^j \right\|_{S(0,T_n)} + \overline{\lim_n} \, \|w_n^J\|_{S(\mathbb{R})} \left\| \sum_{j=J_1+1}^{J} U_n^j \right\|_{S(0,T_n)} \\ &\quad + \overline{\lim_n} \, \|w_n^J\|_{S(\mathbb{R})} \left\| \sum_{j=J_1+1}^{J} U_n^j \right\|_{W(0,T_n)} \\ &\lesssim \epsilon, \end{aligned} \tag{2.5.38}$$

where the implied constant is independent of J. Thus,

$$\overline{\lim_J} \, \overline{\lim_n} \, A_n^J \lesssim \epsilon + \overline{\lim_J} \, \overline{\lim_n} \sum_{j=1}^{J_1} \left\| (2\langle r \rangle^{-1} \sin 2Q) w_n^J U_n^j \right\|_{W'(0,T_n)}. \tag{2.5.39}$$

Fix $j \in \{1, \dots, J_1\}$. We wish to show that

$$\overline{\lim_J} \, \overline{\lim_n} \sum_{j=1}^{J_1} \left\| (2\langle r \rangle^{-1} \sin 2Q) w_n^J U_n^j \right\|_{W'(0,T_n)} = 0. \tag{2.5.40}$$

By the product rule,

$$\left\| (2\langle r \rangle^{-1} \sin 2Q) w_n^J U_n^j \right\|_{L_x^{3/2}} \lesssim \|w_n^J\|_{L_x^6} \|U_n^j\|_{L_x^6} + \|w_n^J\|_{L_x^6} \|U_n^j\|_{\dot{W}_x^{1/2,3}} + \|w_n^J\|_{\dot{W}_x^{1/2,3}} \|U_n^j\|_{L_x^6} \tag{2.5.41}$$

Arguing as in the proof of Proposition 2.5.2, the assumption (2.5.30) also implies that for all $j \geq 1$,

$$\varlimsup_{n} \|U^j\|_{W(0,T_n)} < \infty. \tag{2.5.42}$$

This fact, (2.5.41), Hölder's inequality, and the fact that $\varlimsup_J \varlimsup_n \|w_n^J\|_{S(0,T_n)} = 0$ imply that

$$\varlimsup_{J} \varlimsup_{n} \left\| (2\langle r\rangle^{-1} \sin 2Q) w_n^J U_n^j \right\|_{W'(0,T_n)} \lesssim \varlimsup_{J} \varlimsup_{n} \left\| \|w_n^J(t)\|_{\dot{W}_x^{1/2,3}} \|U_n^j(t)\|_{L_x^6} \right\|_{L_t^{3/2}(0,T_n)}. \tag{2.5.43}$$

We now show that

$$\varlimsup_{J} \varlimsup_{n} \int_0^{T_n} \|w_n^J(t)\|_{\dot{W}_x^{1/2,3}}^{3/2} \|U_n^j(t)\|_{L_x^6}^{3/2} dt = 0. \tag{2.5.44}$$

By the assumption (2.5.30) and an approximation argument, we can assume that U^j is compactly supported in t. By interpolation, we have the estimate

$$\forall t, \quad \|w^J(t)\|_{\dot{W}_x^{1/2,3}} \lesssim \|\nabla w_n^J(t)\|_{L_x^2}^{1/2} \|w_n^J(t)\|_{L_x^6}^{1/2}. \tag{2.5.45}$$

Thus, by Hölder's inequality

$$\begin{aligned}
\int_0^{T_n} \|w_n^J(t)\|_{\dot{W}_x^{1/2,3}}^{3/2} \|U_n^j(t)\|_{L_x^6}^{3/2} dt &\lesssim \int \|w_n^J(t + t_{j,n})\|_{\dot{W}_x^{1/2,3}}^{3/2} \|U^j(t)\|_{L_x^6}^{3/2} dt \\
&\lesssim \int \|\vec{w}_n^J\|_{L_t^\infty \mathcal{H}}^{3/4} \|w_n^J(t + t_{j,n})\|_{L_x^6}^{3/4} \|U^j(t)\|_{L_x^6}^{3/2} dt \\
&\lesssim \|w_n^J\|_{S(\mathbb{R})}^{3/4},
\end{aligned}$$

where the implied constant depends on U^j. Thus,

$$\overline{\lim_J}\,\overline{\lim_n}\int_0^{T_n}\|w_n^J(t)\|^{3/2}_{\dot W_x^{1/2,3}}\|U_n^j(t)\|^{3/2}_{L_x^6}dt \lesssim \overline{\lim_J}\,\overline{\lim_n}\,\|w_n^J\|^{3/4}_{S(\mathbb{R})} = 0.$$

This proves (2.5.44). By (2.5.43), this also proves (2.5.40). By (2.5.39), this proves

$$\overline{\lim_J}\,\overline{\lim_n}\,A_n^J \lesssim \epsilon,$$

which proves (2.5.36).

We have now demonstrated that the function U_n^J satisfies the hypotheses stated in Proposition 2.5.2 uniformly in J, n large and

$$\overline{\lim_J}\,\overline{\lim_n}\,\|e_n^J\|_{N(0,T_n)} = 0.$$

Since $\vec{U}_n^J(0) = u_n(0) + o_{\mathcal{H}}(1)$ as $n \to \infty$, we have by Proposition 2.5.2, for $t \in [0, T_n]$,

$$\vec{u}_n(t) = \vec{U}_n^J(t) + \vec{r}_n^J(t),$$

with

$$\lim_{J\to\infty}\overline{\lim_{n\to\infty}}\left[\|r_n^J\|_{S(0,T_n)} + \sup_{t\in[0,T_n]}\left\|\vec{r}_n^J(t)\right\|_{\mathcal{H}}\right] = 0.$$

This completes the proof. □

We now prove Proposition 2.5.3.

Proof of Proposition 2.5.3. For $A > 0$, define

$$\mathcal{B}(A) := \Big\{(u_0, u_1) \in \mathcal{H} : \text{if } u \text{ solves (2.5.1) with } \vec{u}(0) = (u_0, u_1) \text{ then} \\ \sup_{t\in[0,\infty)} \mathcal{E}_V(\vec{u}_n(t))^{1/2} \leq A\Big\}.$$

We say that the property $\mathcal{SC}(A)$ holds if for all $(u_0, u_1) \in \mathcal{B}(A)$, the solution u to (2.5.1) satisfies $\|u\|_{S(0,\infty)} < \infty$. Note that by Proposition 2.5.1 and (2.5.10), every solution u to (2.5.1) is in $\mathcal{B}(A)$ for some A and if $0 < A < \delta$, where δ is as in Proposition 2.5.1, then $\mathcal{SC}(A)$ holds. Define

$$A_C := \sup\{A > 0 : \mathcal{SC}(A) \text{ holds.}\} > 0.$$

By the temporal symmetry of (2.5.1) and Proposition 2.5.1, we see that Theorem 2.4.1 is equivalent to the statement

$$A_C = \infty.$$

Suppose not, i.e. $0 < A_C < \infty$. Then there exists a sequence of real numbers $A_n \downarrow A$ and a sequence $\{(u_{0,n}, u_{1,n})\}_n$ in $\mathcal{H}$ such that the corresponding solutions u_n to (2.5.1) with initial data $\vec{u}_n(0) = (u_{0,n}, u_{1,n})$ satisfy

$$\begin{aligned} \exists T_n < 0, T_n \to -\infty, \qquad \sup_{t\in(T_n,\infty)} \mathcal{E}_V(\vec{u}_n(t))^{1/2} &\leq A_n, \\ \|u_n\|_{S(0,\infty)} &= \infty, \\ \lim_{n\to\infty} \|u_n\|_{S(-T_n,0)} &= \infty. \end{aligned} \tag{2.5.46}$$

Note that (2.5.46) and (2.5.10) imply that the sequence $\{\vec{u}_n(0) = (u_{0,n}, u_{1,n})\}_n$ is bounded in $\mathcal{H}$. After passing to a subsequence if necessary, $\vec{u}_n(0)$ admits a profile decomposition

$$\vec{u}_n(0) = \sum_{j=1}^{J} \vec{U}_L^j(-t_{j,n}) + \vec{w}_n^J(0) \tag{2.5.47}$$

with profiles $\{U_L^j\}_j$ and time parameters $\{t_{j,n}\}_{j,n}$ by Lemma 2.5.4. As before we assume, without loss of generality, that for all j either $t_{j,n} = 0$ $\forall n$ or $\lim_n t_{j,n} = \pm\infty$. Let $\{U^j\}_j$ be the sequence of associated nonlinear profiles. By the Pythagorean expansion of the energy, there exists $J_0 > 1$ such that

$$\sum_{j>J_0} \mathcal{E}_V(\vec{U}_L^j) \ll \delta^2,$$

where δ is as in the small data theory, Proposition 2.5.1. Thus, the associated nonlinear profiles satisfy

$$\|U^j\|_{S(\mathbb{R})} \lesssim \mathcal{E}_V(\vec{U}_L^j)^{1/2}.$$

Define

$$\mathcal{J} = \left\{ j \in \{1, \ldots, J_0\} : \|U^j\|_{S(0,\infty)} = \infty \right\}.$$

First, we note that $\mathcal{J} \neq \varnothing$. Otherwise, by the definition of nonlinear profiles and our choice of J_0, we have

$$\forall j \geq 1, \quad \|U^j\|_{S(0,\infty)} < \infty.$$

By Lemma 2.5.5, this would imply that $\|u_n\|_{S(0,\infty)} < \infty$ for large n, a contradiction to (2.5.46). Thus, $\mathcal{J} \neq \varnothing$. Note that if $j \in \mathcal{J}$ and $-t_{j,n} \to_n \infty$, then U^j scatters forward in time, i.e. $\|U^j\|_{S(0,\infty)} < \infty$, a contradiction to our definition of $\mathcal{J}$. Thus, for all $j \in \mathcal{J}$, we have that $-t_{j,n} \to_n -\infty$. By the orthogonality of the parameters and after rearranging the first J_0 profiles if necessary, we may assume that if $j > 1$, then

$$\lim_{n\to\infty} t_{1,n} - t_{j,n} = -\infty$$

We now claim that $\mathcal{J} = \{1\}$ and that for all $j \geq 2$, $\vec{U}_L^j = 0$. Suppose not and, say, $\vec{U}_L^2 \neq 0$. Then for $T \geq 0$ and for all $j \geq 1$,

$$\overline{\lim_n} \|U^j\|_{S(-t_{j,n}, T+t_{1,n}-t_{j,n})} < \infty.$$

By Proposition 2.5.5, the Pythagorean expansion of the energy, and conservation of the energy $\mathcal{E}_V(\cdot)$ we conclude that

$$\begin{aligned}
\mathcal{E}_V(\vec{u}_n(T+t_{1,n})) &= \mathcal{E}_V(\vec{U}^1(T)) + \sum_{j=2}^{J} \mathcal{E}_V(\vec{U}^j(T+t_{1,n}-t_{j,n})) + \mathcal{E}_V(\vec{w}_n^J) \\
&\quad + \mathcal{E}_V(\vec{r}_n^J(T+t_{1,n})) + o_n(1) \\
&= \mathcal{E}_V(\vec{U}^1(T)) + \sum_{j=2}^{J} \mathcal{E}_V(\vec{U}_L^j) + \mathcal{E}_V(\vec{w}_n^J) + \mathcal{E}_V(\vec{r}_n^J(T+t_{1,n})) + o_n(1) \\
&\geq \mathcal{E}_V(\vec{U}^1(T)) + \mathcal{E}_V(\vec{U}_L^2) + o_n(1)
\end{aligned}$$

as $n \to \infty$. In particular,

$$\mathcal{E}_V(\vec{U}^1(T)) \leq A_0^2 < A_C^2.$$

Since $T \geq 0$ was arbitrary, we conclude that $\sup_{t\in[0,\infty)} \mathcal{E}_V(\vec{U}^1(t))^{1/2} \leq A_0 < A_C$. By the minimality of A_C, it follows that $\|U^1\|_{S(0,\infty)} < \infty$, a contradiction to the fact that $1 \in \mathcal{J}$.

Thus, $\vec{U}_L^j = 0$ for all $j \geq 2$. By a similar argument, we also deduce that

$$\lim_{n\to\infty} \mathcal{E}_V(\vec{w}_n^1) = 0,$$

or equivalently $\lim_n \|\vec{w}_n^J(0)\|_{\mathcal{H}} = 0$.

We have now shown that

$$\vec{u}_n(0) = \vec{U}^1(-t_{1,n}) + o_{\mathcal{H}}(1),$$

as $n \to \infty$. We claim that $t_{1,n} = 0$ for all n. If not, then by our initial assumptions on the parameters we have $-t_{1,n} \to_n -\infty$. This implies that $\|U^1\|_{S(-\infty,0)} < \infty$. By Proposition 2.5.2, we deduce that $\overline{\lim}_n \|u_n\|_{S(-\infty,0)} < \infty$, a contradiction to (2.5.46). Thus,

$$\vec{u}_n(0) = \vec{U}^1(0) + o_{\mathcal{H}}(1),$$

as $n \to \infty$. Define $u_* = U^1$. Then by Proposition 2.5.2 and (2.5.46), u_* satisfies

$$\begin{gathered}\sup_{t\in(-\infty,\infty)} \mathcal{E}_V(\vec{u}_*(t)) \leq A_C, \\ \|u_*\|_{S(-\infty,0)} = \|u_*\|_{S(0,\infty)} = \infty.\end{gathered} \tag{2.5.48}$$

Finally, we show that $\{\vec{u}_*(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{H}$. By continuity of the flow, it suffices to show that if $\{t_n\}_n$ is a sequence in $\mathbb{R}$, with $\lim_{n\to\infty} t_n = \pm\infty$, then there exists a subsequence (still denoted by t_n) such that $\vec{u}_*(t_n)$ converges in $\mathcal{H}$. Suppose that $t_n \to_n \infty$. Define $(u_{0,n}, u_{1,n}) := \vec{u}_*(t_n)$. Then the solution u_n to (2.5.1) with initial data $\vec{u}_n(0) = (u_{0,n}, u_{1.n})$ is given by $u_n(t) = u_*(t+t_n)$ whence by (2.5.48), the solutions u_n satisfy the conditions given in (2.5.46). Thus, we may repeat the previous argument to conclude that there exists a subsequence (still indexed by n) and $\vec{U}^1(0) \in \mathcal{H}$ such that

$$u_*(t_n) = \vec{u}_n(0) = \vec{U}^1(0) + o_{\mathcal{H}}(1),$$

as $n \to \infty$. If $t_n \to -\infty$, then we apply the previous argument to $u_*(-t)$ to conclude. Thus, the set

$$K := \{\vec{u}_*(t) : t \in \mathbb{R}\},$$

is precompact in $\mathcal{H}$. This completes the proof.

□

2.6 Rigidity Theorem

In this section, we show that the critical element from Proposition 2.5.3 does not exist and conclude the proof of our main result Theorem 2.4.1 (equivalently Theorem 2.1.1). In particular, we prove the following.

Proposition 2.6.1. *Let u be a global solution of* (2.5.1) *such that the trajectory*

$$K = \{\vec{u}(t) : t \in \mathbb{R}\},$$

is precompact in $\mathcal{H} := \mathcal{H}((-\infty, \infty); (r^2+1)^2 dr)$. Then $\vec{u} = (0, 0)$.

We first note that for a solution u as in Proposition 2.6.1, we have the following uniform control of the energy in exterior regions.

Lemma 2.6.2. *Let u be as in Proposition 2.6.1. Then we have*

$$\begin{aligned} \forall R \geq 0, \quad \lim_{|t|\to\infty} \|\vec{u}(t)\|_{\mathcal{H}(|r|\geq R+|t|;(r^2+1)^2dr)} = 0, \\ \lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}(t)\|_{\mathcal{H}(|r|\geq R+|t|;(r^2+1)^2dr)}\right] = 0. \end{aligned} \tag{2.6.1}$$

To prove that $\vec{u} = (0,0)$, we will show that u is a finite energy static solution to (2.5.1).

Proposition 2.6.3. *Let u be as in Proposition 2.6.1. Then there exists a static solution U to* (2.5.1) *such that $\vec{u} = (U, 0)$.*

We will first show that $\vec{u}$ is equal to static solutions $(U_\pm, 0)$ to (2.5.1) on $\pm r > 0$. Since the proof for $r < 0$ is nearly identical, we only consider the case $r > 0$.

Proposition 2.6.4. *Let u be as in Lemma 2.6.1. Then there exists a static solution U_+ to* (2.5.1) *such that $\vec{u}(t,r) = (U_+(r), 0)$ for all $t \in \mathbb{R}$ and all $r > 0$.*

2.6.1 Proof of Proposition 2.6.4

Let $\eta > 0$ and let u be as in Proposition 2.6.1. We will first show that $\vec{u}(0,r) = (U_+(r), 0)$ on $r \geq \eta$ for some static solution U_+ to (2.5.1).

We now introduce a function that will be integral in the proof. Define

$$u_e(t,r) := \frac{r^2+1}{r^2} u(t,r), \quad (t,r) \in \mathbb{R} \times (0,\infty).$$

If u solves (2.5.1), then u_e solves

$$\partial_t^2 u_e - \partial_r^2 u_e - \frac{4}{r}\partial_r u_e + V_e(r) u_e = N_e(r, u_e), \quad t \in \mathbb{R}, r > 0, \tag{2.6.2}$$

where

$$V_e(r) = V(r) - \frac{2}{r^2(r^2+1)}, \tag{2.6.3}$$

and $N_e(r,u_e) = F_e(r,u_e) + G_e(r,u_e)$ where

$$F_e(r,u_e) = \frac{r^2+1}{r^2} F\left(r, \frac{r^2}{r^2+1}u_e\right), \tag{2.6.4}$$

$$G_e(r,u_e) = \frac{r^2+1}{r^2} G\left(r, \frac{r^2}{r^2+1}u_e\right). \tag{2.6.5}$$

Note that for all $R > 0$, we have

$$\|\vec{u}_e(t)\|_{\mathcal{H}(r\geq R; r^4 dr)} \leq C(R)\|\vec{u}\|_{\mathcal{H}(r\geq R;(r^2+1)^2 dr)}, \tag{2.6.6}$$

so that by Lemma 2.6.2, u_e inherits the compactness properties

$$\begin{aligned} &\forall R > 0, \quad \lim_{|t|\to\infty} \|\vec{u}_e(t)\|_{\mathcal{H}(r\geq R+|t|; r^4 dr)} = 0, \\ &\lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}_e(t)\|_{\mathcal{H}(r\geq R+|t|; r^4 dr)}\right] = 0. \end{aligned} \tag{2.6.7}$$

We also note that due to (2.4.7)–(2.4.10) and the definition of V_e, F_e, and G_e, we have for all $r > 0$,

$$|V_e(r)| \lesssim r^{-4}, \tag{2.6.8}$$

$$|F_e(u_e, r)| \lesssim r^{-3}|u_e|^2, \tag{2.6.9}$$

$$|G_e(u_e, r)| \lesssim |u_e|^3, \tag{2.6.10}$$

where the implied constants depend on the harmonic map Q.

The proof that $\vec{u} = (U_+, 0)$ on $r \geq \eta$ for some U_+ is split into three main steps. In the first two steps, we determine the precise asymptotics of the associated "Euclidean" solution $u_{e,0}(r) := u_e(0,r), u_{e,1}(r) := \partial_t u_e(0,r)$, as $r \to \infty$. In particular, we show that there exists $\alpha \in \mathbb{R}$ such that

$$r^3 u_{e,0}(r) = \alpha + O(r^{-1}), \tag{2.6.11}$$

$$r \int_r^\infty u_{e,1}(\rho)\rho d\rho = O(r^{-1}), \tag{2.6.12}$$

as $r \to \infty$. In the final step, we use this information to conclude the argument. For the remainder of this subsection we denote $\mathcal{H}(r \geq R; r^4 dr)$ simply by $\mathcal{H}(r \geq R)$ and the exterior region $\mathbb{R}^5 \backslash B(0,\eta)$ by $\mathbb{R}^5_*$.

The key tool for establishing (2.6.11) and (2.6.12) is the following exterior energy estimate for radial free waves on Minkowski space $\mathbb{R}^{1+5}$.

Proposition 2.6.5 (Proposition 4.1 [16])**.** *Let v be a radial solution to the free wave equation in $\mathbb{R}^{1+5}$*

$$\partial_t^2 v - \Delta v = 0, \quad (t,x) \in \mathbb{R}^{1+5},$$
$$\vec{v}(0) = (f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^5).$$

Then for any $R > 0$,

$$\max_{\pm} \inf_{\pm t \geq 0} \int_{r \geq R + |t|} |\nabla_{t,x} v(t,r)|^2 r^4 dr \geq \frac{1}{2} \|\pi_R^\perp (f,g)\|_{\mathcal{H}(r \geq R; r^4 dr)}, \tag{2.6.13}$$

where $\pi_R = I - \pi_R^\perp$ is the orthogonal projection onto the plane

$$P(R) = span\{(r^{-3}, 0), (0, r^{-3})\}$$

in $\mathcal{H}(r \geq R)$. The left–hand side of (2.6.13) is identically 0 for data satisfying $(f,g) =$

$(\alpha r^{-3}, \beta r^{-3})$ *for on* $r \geq R$.

We remark here that Proposition 2.6.5 states, quantitatively, that generic solutions to the free wave equation on $\mathbb{R}^{1+5}$ emit a fixed amount of energy into regions exterior to light cones. However, this property is very sensitive to dimension and in fact fails in the case $R = 0$ for general data (f, g) in even dimensions (see [6]). Proposition 2.6.5 has been generalized to all odd dimensions $d \geq 3$ in the work [17]. We note that the orthogonal projections π_R, $\pi_R^\perp$ are given by

$$\begin{aligned} &\pi_R(f,0) = R^3 r^{-3} f(R), \quad \pi_R(0,g) = R r^{-3} \int_R^\infty g(\rho)\rho \, d\rho, \\ &\pi_R^\perp(f,0) = f(r) - R^3 r^{-3} f(R), \quad \pi_R^\perp(0,g) = g(r) - R r^{-3} \int_R^\infty g(\rho)\rho \, d\rho, \end{aligned} \tag{2.6.14}$$

and thus we have

$$\|\pi_R(f,g)\|^2_{\dot{H}^1 \times L^2(r>R)} = 3R^3 f^2(R) + R \left(\int_R^\infty r g(r) \, dr \right)^2, \tag{2.6.15}$$

$$\begin{aligned} \|\pi_R^\perp(f,g)\|^2_{\dot{H}^1 \times L^2(r>R)} = &\int_R^\infty f_r^2(r) \, r^4 \, dr - 3R^3 f^2(R) \\ &+ \int_R^\infty g^2(r) \, r^4 \, dr - a \left(\int_R^\infty r g(r) \, dr \right)^2. \end{aligned} \tag{2.6.16}$$

We now proceed to the first step in proving Proposition 2.6.3.

Step 1: Estimate for $\pi_R^\perp \vec{u}_e$ in $\mathcal{H}(r \geq R)$

The first step in proving Proposition 2.6.3 is the following decay estimate for $\pi_R^\perp \vec{u}_e(t)$.

Lemma 2.6.6. *There exists* $R_0 > 1$ *such that for all* $R \geq R_0$ *and for all* $t \in \mathbb{R}$ *we have*

$$\begin{aligned} \|\pi_R^\perp \vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-10/3} &\|\pi_R \vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} + R^{-22/6} \|\pi_R \vec{u}_e(t)\|^4_{\mathcal{H}(r\geq R)} \\ &+ \|\pi_R \vec{u}_e(t)\|^6_{\mathcal{H}(r\geq R)}. \end{aligned} \tag{2.6.17}$$

Since we are only interested in the behavior of u in exterior regions $\{r \geq R+|t|\}$, we first consider a modified Cauchy problem. In particular, we can, by finite speed of propagation, alter V_e, F_e, and G_e in (2.6.2) on the interior region $\{1 \leq r \leq R+|t|\}$ without affecting the behavior of $\vec{u}_e$ on the exterior region $\{r \geq R+|t|\}$.

Definition 2.6.7. *For a function $f = f(r,u) : \mathbb{R}^5_* \times \mathbb{R} \to \mathbb{R}$, we define for $R \geq \eta$,*

$$f_R(t,r,u) := \begin{cases} f(R+|t|,u) & \text{if } \eta \leq r \leq R+|t|, \\ f(r,u) & \text{if } r \geq R+|t|. \end{cases}.$$

We now consider solutions to a modified version of (2.6.2):

$$\begin{aligned} &\partial_t^2 h - \partial_r^2 h - \frac{4}{r}\partial_r h = N_R(t,r,h), \quad (t,r) \in \mathbb{R} \times \mathbb{R}^5_*, \\ &\vec{h}(0) = (h_0, h_1) \in \mathcal{H}_0(r \geq \eta), \end{aligned} \tag{2.6.18}$$

where $\mathcal{H}_0(r \geq \eta) = \{(u_0, u_1) \in \mathcal{H}(r \geq \eta) : u_0(\eta) = 0\}$ and

$$N_R(t,r,h) = -V_{e,R}(t,r)h + F_{e,R}(t,r,h) + G_e(t,r,h).$$

We note that from Definition 2.6.7 and (2.6.8), (2.6.9), and (2.6.10), we have that

$$|V_{e,R}(t,r)| \lesssim \begin{cases} (R+|t|)^{-4} & \text{if } \eta \leq r \leq R+|t|, \\ r^{-4} & \text{if } r \geq R+|t|, \end{cases} \tag{2.6.19}$$

$$|F_{e,R}(t,r,h)| \lesssim \begin{cases} (R+|t|)^{-3}|h|^2 & \text{if } \eta \leq r \leq R+|t|, \\ r^{-3}|h|^3 & \text{if } r \geq R+|t|, \end{cases} \tag{2.6.20}$$

$$|G_e(r,h)| \lesssim |h|^3, \tag{2.6.21}$$

Lemma 2.6.8. *There exist $R_0 > 0$ large and $\delta_0 > 0$ small such that for all $R \geq R_0$ and all*

$(h_0, h_1) \in \mathcal{H}_0(r \geq \eta)$ *with*

$$\|(h_0, h_1)\|_{\mathcal{H}_0(r\geq\eta)} \leq \delta_0,$$

there exists a unique globally defined solution h *to* (2.6.18) *such that*

$$\|h\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}_*^5)} \lesssim \|\vec{h}(0)\|_{\mathcal{H}(r\geq\eta)}. \tag{2.6.22}$$

Moreover, if we define h_L *to be the solution to the free equation* $\partial_t^2 h_L - \Delta h_L = 0$, $(t, x) \in \mathbb{R} \times \mathbb{R}_*^5$, $\vec{h}_L(0) = (h_0, h_1)$, *then*

$$\sup_{t\in\mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}(r\geq\eta)} \lesssim R^{-5/3}\|\vec{h}(0)\|_{\mathcal{H}(r\geq\eta)} + R^{-11/6}\|\vec{h}(0)\|^2_{\mathcal{H}(r\geq\eta)} + \|\vec{h}(0)\|^3_{\mathcal{H}(r\geq\eta)}. \tag{2.6.23}$$

Proof. The small data global well–posedness and spacetime estimate (2.6.22) follow from standard contraction mapping and continuity arguments using Strichartz estimates for free waves on $\mathbb{R} \times \mathbb{R}_*^5$ with Dirichlet boundary condition (see [16]). We now prove (2.6.23). By the Duhamel formula and Strichartz estimates we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}(r\geq\eta)} &\lesssim \|N_R(\cdot, \cdot, h)\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \\ &\lesssim \|V_{e,R} h\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} + \|F_{e,R}(\cdot, \cdot, h)\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \\ &\quad + \|G_e(\cdot, h)\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}_*^5)}. \end{aligned}$$

The third term is readily estimated by (2.6.22) and (2.6.23)

$$\|G_e(\cdot, h)\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \lesssim \|h^3\|_{L_t^1 L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \lesssim \|\vec{h}(0)\|^3_{\mathcal{H}(r\geq\eta)}.$$

For the first term we have

$$\|V_{e,R}h\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \leq \|V_{e,R}\|_{L_t^{3/2}L_x^3(\mathbb{R}\times\mathbb{R}_*^5)}\|h\|_{L_t^3L_x^6(\mathbb{R}\times\mathbb{R}_*^5)}$$
$$\lesssim \|V_{e,R}\|_{L_t^{3/2}L_x^3(\mathbb{R}\times\mathbb{R}_*^5)}\|\vec{h}(0)\|_{\mathcal{H}(r\geq\eta)}.$$

By (2.6.19)

$$\|V_{e,R}\|_{L_x^3(\mathbb{R}_*^5)}^3 \lesssim \int_0^{R+|t|}(R+|t|)^{-12}r^4dr + \int_{R+|t|}^{\infty} r^{-12}r^4dr \lesssim (R+|t|)^{-7}.$$

Hence,

$$\|V_{e,R}\|_{L_t^{3/2}L_x^3(\mathbb{R}\times\mathbb{R}_*^5)}^{3/2} \lesssim \int (R+|t|)^{-7/2}dt \lesssim R^{-5/2}.$$

Thus,

$$\|V_{e,R}h\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \lesssim R^{-5/3}\|\vec{h}(0)\|_{\mathcal{H}(r\geq1)}.$$

Similarly, using (2.6.20) and (2.6.22) we conclude that

$$\|F_{e,R}(\cdot,\cdot,h)\|_{L_t^1L_x^2(\mathbb{R}\times\mathbb{R}_*^5)} \lesssim R^{-11/6}\|h(0)\|_{\mathcal{H}(r\geq1)}^2$$

which proves (2.6.23). □

Proof of Lemma 2.6.6. We first prove Lemma 2.6.6 for $t=0$. For $R>\eta$, define the truncated initial data $\vec{u}_R(0) = (u_{0,R}, u_{1,R}) \in \mathcal{H}_0(r \geq \eta)$ via

$$u_{0,R} = \begin{cases} u_e(0,r) & \text{if } r \geq R, \\ \frac{r-\eta}{R-\eta} u_e(0,R) & \text{if } r < R, \end{cases} \tag{2.6.24}$$

$$u_{1,R} = \begin{cases} \partial_t u_e(0,r) & \text{if } r \geq R, \\ 0 & \text{if } r < R. \end{cases} \tag{2.6.25}$$

Note that for R large,

$$\|\vec{u}_R(0)\|_{\mathcal{H}(r\geq\eta)} \lesssim \|\vec{u}_e(0)\|_{\mathcal{H}(r\geq R)}. \tag{2.6.26}$$

In particular, by Lemma 2.6.2, there exists $R_0 > \eta$ such that for all $R \geq R_0$, $\|\vec{u}_R(0)\| \leq \delta_0$ where δ_0 is from Lemma 2.6.8. Let $u_R(t)$ be the solution to (2.6.18) with initial data $(u_{0,R}, u_{1,R})$, and let $\vec{u}_{R,L}(t) \in \mathcal{H}_0(r \geq \eta)$ be the solution to the free wave equation $\partial_t^2 u_{R,L} - \Delta u_{R,L} = 0$, $(t,x) \in \mathbb{R} \times \mathbb{R}^5_*$, $\vec{u}_{R,L}(0) = (u_{0,R}, u_{1,R})$. By finite speed of propagation

$$r \geq R + |t| \implies \vec{u}_R(t,r) = \vec{u}_e(t,r).$$

By Proposition 2.6.5, for all $t \geq 0$ or for all $t \leq 0$,

$$\|\pi_R^\perp \vec{u}_{R,L}(0)\|_{\mathcal{H}(r\geq R)} \lesssim \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)}.$$

Suppose, without loss of generality, that the above bound holds for all $t \geq 0$. By (2.6.23) we conclude that for all $t \geq 0$

$$\begin{aligned}\|\vec{u}_e(t)\|_{\mathcal{H}(r\geq R+|t|)} &\geq \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)} - \|\vec{u}_R(t) - \vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq\eta)} \\ &\geq c\|\pi_R^{\perp}\vec{u}_{R,L}(0)\|_{\mathcal{H}(r\geq R)} - C\Big[R^{-5/3}\|u_R(0)\|_{\mathcal{H}(r\geq\eta)} \\ &\qquad + R^{-11/6}\|\vec{u}_R(0)\|^2_{\mathcal{H}(r\geq\eta)} + \|\vec{u}_R(0)\|^3_{\mathcal{H}(r\geq\eta)}\Big].\end{aligned}$$

Letting $t \to \infty$ and using the decay property (2.6.1) and the definition of $(u_{0,R}, u_{1,R})$, we conclude that

$$\|\pi_R^{\perp}\vec{u}_e(0)\|_{\mathcal{H}(r\geq R)} \lesssim R^{-5/3}\|u_e(0)\|_{\mathcal{H}(r\geq R)} + R^{-11/6}\|\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} + \|\vec{u}_e(0)\|^3_{\mathcal{H}(r\geq R)}.$$

Note that $\|\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} = \|\pi_R^{\perp}\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)}$. Thus, if we take R_0 large enough to absorb terms involving $\|\pi_R^{\perp}\vec{u}_e(0)\|_{\mathcal{H}(r\geq R)}$ into the left hand side in the previous estimate, we obtain for all $R \geq R_0$

$$\begin{aligned}\|\pi_R^{\perp}\vec{u}_e(0)\|_{\mathcal{H}(r\geq R)} &\lesssim R^{-5/3}\|\pi_R u_e(0)\|_{\mathcal{H}(r\geq R)} + R^{-11/6}\|\pi_R\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} \\ &\quad + \|\pi_R\vec{u}_e(0)\|^3_{\mathcal{H}(r\geq R)},\end{aligned}$$

as desired. This proves Lemma 2.6.6 for $t = 0$.

For general $t = t_0$ in (2.6.18), we first set

$$u_{0,R,t_0} = \begin{cases} u_e(t_0, r) & \text{if } r \geq R, \\ \frac{r-\eta}{R-\eta}u(t_0, R) & \text{if } r < R, \end{cases}$$

$$u_{1,R,t_0} = \begin{cases} \partial_t u_e(t_0, r) & \text{if } r \geq R, \\ 0 & \text{if } r < R. \end{cases}$$

By (2.6.7) we can find $R_0 = R_0(\delta_0, \eta)$ independent of t_0 such that for all $R \geq R_0$

$$\|(u_{0,R,t_0}, u_{1,R,t_0})\|_{\mathcal{H}(r\geq\eta)} \lesssim \|\vec{u}_e(t_0)\|_{\mathcal{H}(r\geq R)} \leq \delta_0.$$

The previous argument for $t_0 = 0$ repeated with obvious modifications yield (2.6.17) for $t = t_0$. □

Before proceeding to the next step, it will be useful to reformulate the conclusion of Lemma 2.6.6. Define

$$\lambda(t,r) := r^3 u_e(t,r), \tag{2.6.27}$$

$$\mu(t,r) := r\int_r^\infty \partial_t u_e(t,\rho)\rho d\rho. \tag{2.6.28}$$

We denote $\lambda(r) := \lambda(0,r)$ and $\mu(r) := \mu(0,r)$. By (2.6.15) and (2.6.16) the functions λ and μ arise in the explicit computation of $\pi_R^\perp \vec{u}(t)$ and $\pi_R \vec{u}(t)$ as follows:

$$\|\pi_R^\perp \vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} = \int_R^\infty \left(\frac{1}{r}\partial_r \lambda(t,r)\right)^2 dr + \int_R^\infty (\partial_r \mu(t,r))^2 dr, \tag{2.6.29}$$

$$\|\pi_R \vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} = 3R^{-3}\lambda^2(t,R) + R^{-1}\mu^2(t,R). \tag{2.6.30}$$

Thus, Lemma 2.6.6 can be restated using λ, μ in the following way.

Lemma 2.6.9. *Let μ, λ be as in* (2.6.27) *and* (2.6.28)*. Then there exists $R_0 \geq \eta$ such that for all $R > R_0$ and for all $t \in \mathbb{R}$*

$$\begin{aligned}\int_R^\infty \left(\frac{1}{r}\partial_r \lambda(t,r)\right)^2 dr + \int_R^\infty (\partial_r \mu(t,r))^2 dr \\ \lesssim R^{-19/3}\lambda^2(t,R) + R^{-29/3}\lambda^4(t,R) + R^{-9}\lambda^6(t,R) \\ + R^{-13/3}\mu^2(t,R) + R^{-17/3}\mu^4(t,R) + R^{-3}\mu^6(t,R).\end{aligned}$$

Step 2: Asymptotics for $\vec{u}_e(0)$

In this step, we prove that $\vec{u}_e(0)$ has the asymptotic expansions (2.6.11), (2.6.12) which we now formulate as a lemma.

Lemma 2.6.10. *Let u_e be a solution to* (2.6.2) *which satisfies* (2.6.7)*. Let $\vec{u}_e(0) = (u_{e,0}, u_{e,1})$. Then there exists $\alpha \in \mathbb{R}$ such that*

$$\begin{aligned} r^3 u_{e,0}(r) &= \alpha + O(r^{-1}), \\ r \int_r^\infty u_{e,1}(\rho)\rho d\rho &= O(r^{-1}), \end{aligned}$$

as $r \to \infty$. Equivalently, with λ and μ defined as in (2.6.27) *and* (2.6.28)*, there exists $\alpha \in \mathbb{R}$ such that*

$$\lambda(r) = \alpha + O(r^{-1}), \tag{2.6.31}$$

$$\mu(r) = O(r^{-1}). \tag{2.6.32}$$

The proof of Lemma 2.6.10 is split up into a few further lemmas. First, we use Lemma 2.6.9 to prove the following difference estimate for λ and μ.

Lemma 2.6.11. *Let $\delta_1 \leq \delta_0$ where δ_0 is from Lemma 2.6.8. Let $R_1 \geq R_0 > 1$ be large enough so that for all $R \geq R_1$ and for all $t \in \mathbb{R}$*

$$\begin{aligned} \|\vec{u}_e(t)\|_{\mathcal{H}(r \geq R)} &\leq \delta_1, \\ R^{-5/3} &\leq \delta_1. \end{aligned}$$

Then for all r, r' with $R_1 \le r \le r' \le 2r$ and for all $t \in \mathbb{R}$

$$\begin{aligned}|\lambda(t,r) - \lambda(t,r')| \lesssim r^{-5/3}|\lambda(t,r)| + r^{-10/3}|\lambda(t,r)|^2 + r^{-3}|\lambda(t,r)|^3 \\ + r^{-2/3}|\mu(t,r)| + r^{-4/3}|\mu(t,r)|^2 + |\mu(t,r)|^3,\end{aligned} \tag{2.6.33}$$

$$\begin{aligned}|\mu(t,r) - \mu(t,r')| \lesssim r^{-8/3}|\lambda(t,r)| + r^{-13/3}|\lambda(t,r)|^2 + r^{-4}|\lambda(t,r)|^3 \\ + r^{-5/3}|\mu(t,r)| + r^{-7/3}|\mu(t,r)|^2 + r^{-1}|\mu(t,r)|^3.\end{aligned} \tag{2.6.34}$$

Proof. By the fundamental theorem of calculus and Lemma 2.6.9 we have, for r, r' such that $R_1 \le r \le r' \le 2r$,

$$\begin{aligned}|\lambda(t,r) - \lambda(t,r')|^2 &= \left(\int_r^{r'} \partial_\rho \lambda(t,\rho) d\rho\right)^2 \\ &\le \left(\int_r^{r'} \rho^2 d\rho\right)\left(\int_r^{r'} \left(\frac{1}{\rho}\partial_\rho \lambda(t,\rho)\right)^2 d\rho\right) \\ &\lesssim r^3 \Big(r^{-19/3}\lambda^2(t,r) + r^{-29/3}\lambda^4(t,r) + r^{-9}\lambda^6(t,r) \\ &\quad + r^{-13/3}\mu^2(t,r) + r^{-17/3}\mu^4(t,r) + r^{-3}\mu^6(t,r)\Big)\end{aligned}$$

which proves (2.6.33).

Similarly, we have

$$\begin{aligned}|\mu(t,r) - \mu(t,r')|^2 &\le r\left(\int_r^{r'} (\mu(t,\rho))^2 d\rho\right) \\ &\lesssim r\Big(r^{-19/3}\lambda^2(t,r) + r^{-29/3}\lambda^4(t,r) + r^{-9}\lambda^6(t,r) \\ &\quad + r^{-13/3}\mu^2(t,r) + r^{-17/3}\mu^4(t,r) + r^{-3}\mu^6(t,r)\Big)\end{aligned}$$

which proves (2.6.34). □

A simple consequence of Lemma 2.6.11 is the following.

Corollary 2.6.12. *Let R_1 be as in Lemma 2.6.11. Then for all r, r' with $R_1 \le r \le r' \le 2r$*

and for all $t \in \mathbb{R}$

$$|\lambda(t,r) - \lambda(t,r')| \lesssim \delta_1|\lambda(t,r)| + r\delta_1|\mu(t,r)|, \tag{2.6.35}$$

$$|\mu(t,r) - \mu(t,r')| \lesssim r^{-1}\delta_1|\lambda(t,r)| + \delta_1|\mu(t,r)|. \tag{2.6.36}$$

Next we establish the following improved growth rate for λ and μ.

Lemma 2.6.13. *For all $t \in \mathbb{R}$,*

$$|\lambda(t,r)| \lesssim r^{1/6}, \tag{2.6.37}$$

$$|\mu(t,r)| \lesssim r^{1/18}. \tag{2.6.38}$$

Proof. As in the proof of Lemma 2.6.6, we only consider the case $t = 0$. Fix $r_0 \geq R_1$. By Corollary 2.6.12,

$$|\lambda(2^{n+1}r)| \leq (1 + C_1\delta_1)|\lambda(2^n r_0)| + (2^n r_0)C_1\delta_1|\mu(2^n r_0)|, \tag{2.6.39}$$

$$|\mu(2^{n+1}r)| \leq (1 + C_1\delta_1)|\mu(2^n r_0)| + (2^n r_0)^{-1}C_1\delta_1|\lambda(2^n r_0)|. \tag{2.6.40}$$

If we define $a_n := |\mu(2^n r_0)|$ and $b_n := (2^n r_0)^{-1}|\lambda(2^n r_0)|$, then (2.6.39) and (2.6.40) imply

$$a_{n+1} + b_{n+1} \leq (1 + 2C_1\delta_1)\,(a_n + b_n).$$

By induction

$$a_n + b_n \leq (1 + 2C_1\delta_1)^n\,(a_0 + b_0).$$

Choose δ_1 so small so that $1 + 2C_1\delta_1 < 2^{1/18}$. We conclude that

$$a_n \leq C(2^n r_0)^{1/18}, \tag{2.6.41}$$

where $C = C(r_0)$. This proves (2.6.38) for $r = 2^n r_0$. Define

$$c_n = |\lambda(2^n r_0)| = (2^n r_0) b_n.$$

Using (2.6.41) and (2.6.33) we have, for some $C = C(r_0)$,

$$c_{n+1} \leq (1 + C_1\delta_1)c_n + C\delta_1(2^n r_0)^{1/6}.$$

By induction

$$\begin{aligned} c_n &\leq (1 + C_1\delta_1)^n c_0 + Cr_0^{1/6} \sum_{k=1}^{n} (1 + C_1\delta_1)^{n-k} 2^{(k-1)/6} \\ &\leq C(2^n r_0)^{1/6}. \end{aligned}$$

This proves (2.6.37) for $r = 2^n r_0$.

To establish (2.6.37) and (2.6.38) for general r, let $r \geq r_0$ such that for some $n \geq 0$, $2^n r_0 \leq r \leq 2^{n+1} r_0$. Then applying (2.6.33) to the pair $2^n r_0, r$, we conclude that

$$\begin{aligned} |\lambda(r)| &\leq |\lambda(2^n r_0)| + |\lambda(2^n r_0) - \lambda(r)| \\ &\leq C(2^n r_0)^{1/6} + \left[(2^n r_0)^{-5/3}(2^n r_0)^{1/6} + (2^n r_0)^{-10/3}(2^n r_0)^{1/3} + (2^n r_0)^{-3}(2^n r_0)^{1/2}\right] \\ &\leq C(2^n r_0)^{1/6} \\ &\leq Cr^{1/6}, \end{aligned}$$

where $C = C(r_0)$. This proves (2.6.37). A similar argument also establishes the bound

(2.6.38) for all $r \geq r_0$ as well. $\square$

We now show that for each $t \in \mathbb{R}$, $\mu(t,r)$ has a limit $\beta(t)$ as $r \to \infty$.

Lemma 2.6.14. *For all $t \in \mathbb{R}$, there exists $\beta(t) \in \mathbb{R}$ such that*

$$|\mu(t,r) - \beta(t)| \leq Cr^{-1}. \tag{2.6.42}$$

The constant $C > 0$ is uniform in time.

Proof. We only consider the case $t = 0$. The general case follows, again, by using the decay of the trajectory $\vec{u}_e(t)$ on exterior regions. Let $R_1 > 1$ be as in Lemma 2.6.11, and fix $r_0 \geq R_1$. Then Lemma 2.6.13 and (2.6.34) yield the estimate

$$\begin{aligned} |\mu(2^{n+1}r_0) - \mu(2^n r_0)| &\lesssim (2^n r_0)^{-8/3}(2^n r_0)^{1/6} + (2^n r_0)^{-13/3}(2^n r_0)^{1/3} + (2^n r_0)^{-4}(2^n r_0)^{1/2} \\ &\quad + (2^n r_0)^{-5/3}(2^n r_0)^{1/18} + (2^n r_0)^{-7/3}(2^n r_0)^{1/9} + (2^n r_0)^{-1}(2^n r_0)^{1/6} \\ &\lesssim (2^n r_0)^{-5/6} \\ &\lesssim 2^{-5n/6}, \end{aligned}$$

where the implied constant is uniform in r_0. Thus,

$$\sum_{n \geq 0} |\mu(2^{n+1}r_0) - \mu(2^n r_0)| \lesssim 1,$$

with a constant uniform in r_0. This implies that there exists $\beta = \beta \in \mathbb{R}$ such that $\lim_{n\to\infty} \mu(2^n r_0) = \beta$. Moreover, the sequence $\{\mu(2^n r_0)\}_n$ is bounded by a constant depending only on r_0 since

$$\begin{aligned}
|\mu(2^n r_0)| &\le |\mu(r_0)| + |\mu(2^n r_0) - \mu(r_0)| \\
&= |\mu(r_0)| + \left|\sum_{k=0}^{n-1} (\mu(2^{k+1} r_0) - \mu(2^k r_0))\right| \\
&\le |\mu(r_0)| + C_1 \sum_{n\ge 0} 2^{-5n/6} \le C(r_0).
\end{aligned}$$

Inserting this bound into the difference estimate (2.6.34) improves the previous bound on $|\mu(2^{n+1}r_0) - \mu(2^n r_0)|$ to

$$|\mu(2^{n+1}r_0) - \mu(2^n r_0)| \le C(2^n r_0)^{-1}, \tag{2.6.43}$$

where $C = C(r_0)$. Now let $r \ge r_0$ such that $2^n r_0 \le r \le 2^{n+1} r_0$. By Lemma 2.6.13, (2.6.34), and our improved bound (2.6.43), we have that

$$\begin{aligned}
|\mu(r) - \beta| &\le |\mu(r) - \mu(2^n r_0)| + |\beta - \mu(2^n r_0)| \\
&= |\mu(r) - \mu(2^n r_0)| + \left|\sum_{k\ge n} (\mu(2^{k+1} r_0) - \mu(2^k r_0))\right| \\
&\lesssim (2^n r_0)^{-1} + \sum_{k\ge n} (2^k r_0)^{-1} \\
&\lesssim (2^n r_0)^{-1} \\
&\lesssim r^{-1}.
\end{aligned}$$

This proves (2.6.42). □

We now conclude the proof of the bound (2.6.32) in Proposition 2.6.10.

Lemma 2.6.15. *Let $\beta(t)$ be as in Lemma 2.6.14. Then $\beta(t) \equiv 0$.*

Proof. The proof follows in two steps.

Step 1. We first show that $\beta(t)$ is constant in time. By Lemma 2.6.14 and the definition of μ, we have shown that

$$\beta(t) = r\int_r^\infty \partial_t u_e(t,\rho)\rho d\rho + O(r^{-1}),$$

where the $O(r^{-1})$ is uniform in time. Fix $t_1 < t_2$. Since u_e solves (2.6.2), we have for $R \geq R_1$,

$$\begin{aligned}
\beta(t_2) - \beta(t_1) &= \frac{1}{R}\int_R^{2R} \beta(t_2) - \beta(t_1) ds \\
&= \frac{1}{R}\int_R^{2R} s\int_s^\infty [\partial_t u_e(t_2, r) - \partial_t u_e(t_1, r)] r dr ds + O(R^{-1}) \\
&= \frac{1}{R}\int_R^{2R} s\int_s^\infty \int_{t_1}^{t_2} \partial_t^2 u_e(t,r) dt r dr ds + O(R^{-1}) \\
&= \frac{1}{R}\int_R^{2R} s\int_s^\infty \int_{t_1}^{t_2} r^{-3}\partial_r(r^4\partial_r u_e(t,r)) dt dr ds \\
&+ \frac{1}{R}\int_R^{2R} s\int_s^\infty \int_{t_1}^{t_2} [-rV_e(r)u_e(t,r) + rN_e(r, u_e(t,r))] dt dr ds + O(R^{-1}) \\
&=: A + B + O(R^{-1}).
\end{aligned}$$

We first estimate B. Recall that $\lambda(t,r) = r^3 u_e(t,r)$. By (2.6.37),

$$|u_e(t,r)| \lesssim r^{-17/6}, \tag{2.6.44}$$

uniformly in t. This estimate, (2.6.8), (2.6.9), and (2.6.10), imply that

$$\begin{aligned}
|B| &\lesssim (t_2 - t_1)\frac{1}{R}\int_R^{2R} s\int_s^\infty \left[r^{-35/6} + r^{-23/3} + r^{-15/2}\right] dr ds \\
&\lesssim (t_2 - t_1)\frac{1}{R}\int_R^{2R} s\int_s^\infty r^{-5} dr ds \\
&\lesssim (t_2 - t_1)R^{-3}.
\end{aligned}$$

For A, we repeatedly use integration by parts and use (2.6.44) to conclude that

$$\begin{aligned}
A &= \frac{3}{R}\int_{t_1}^{t_2}\int_R^{2R} s\int_s^{\infty} \partial_r u_e(t,r)drdsdt - \frac{1}{R}\int_{t_1}^{t_2}\int_R^{2R} s^2\partial_s u_e(t,s)dsdt \\
&= -\frac{3}{R}\int_{t_1}^{t_2}\int_R^{2R} s\partial_s u_e(t,s)dsdt - \frac{1}{R}\int_{t_1}^{t_2}\int_R^{2R} s^2\partial_s u_e(t,s)dsdt \\
&= -\frac{1}{R}\int_{t_1}^{t_2}\int_R^{2R} s\partial_s u_e(t,s)dsdt + \int_{t_1}^{t_2}[Ru_e(t,R) - 2Ru_e(t,2R)]dt \\
&= O(t_2 - t_1)O(R^{-11/6}).
\end{aligned}$$

In summary, we have that $|A| + |B| \lesssim R^{-1}(t_2 - t_1)$ so that

$$\beta(t_2) - \beta(t_1) = O(t_2 - t_1)O(R^{-1}) + O(R^{-1}).$$

Letting $R \to \infty$ implies that $\beta(t_2) = \beta(t_1)$ as desired. This completed Step 1.

Step 2. By Step 1, we have that $\beta(t) = \beta(0) =: \beta$ for all $t \in \mathbb{R}$. In this step, we show that $\beta = 0$ which concludes the proof of Lemma 2.6.15. By Lemma 2.6.14 and Step 1, for all $R \geq R_1$ and for all $t \in \mathbb{R}$ we have

$$\beta = R\int_R^{\infty} \partial_t u_e(t,r)rdr + O(R^{-1}),$$

where the $O(R^{-1})$ term is uniform in time. Integrating the previous expression from 0 to T, dividing by T, and using (2.6.44) yield for all $T > 0$ and $R \geq R_1$

$$\begin{aligned}
\beta &= \frac{R}{T}\int_R^{\infty}\int_0^T \partial_t u_e(t,r)dtrdr + O(R^{-1}) \\
&= \frac{R}{T}\int_R^{\infty}[u_e(T,r) - u_e(0,r)]rdr + O(R^{-1}) \\
&= O\left(\frac{R^{1/6}}{T}\right) + O(R^{-1}).
\end{aligned}$$

If we now choose $R = T$ and let $T \to \infty$, we conclude that $\beta = 0$ as desired. This concludes

Step 2 and the proof of Lemma 2.6.15. ☐

Lemma 2.6.16. *There exists* $\alpha \in \mathbb{R}$ *such that*

$$|\lambda(r) - \alpha| \lesssim r^{-1}.$$

Proof. The proof of Lemma 2.6.16 is very similar to the proof for Lemma 2.6.14 and we only sketch it. Fix $r_0 \geq R_1$. By Lemma 2.6.15, the difference estimate 2.6.33, and the growth estimate 2.6.37 we have

$$\begin{aligned}|\lambda(2^{n+1}r_0) - v_0(2^n r_0)| &\lesssim (2^n r_0)^{-5/3}(2^n r_0)^{1/6} + (2^n r_0)^{-10/3}(2^n r_0)^{1/3} + (2^n r_0)^{-3}(2^n r_0)^{1/2} \\ &\quad + (2^n r_0)^{-2/3}(2^n r_0)^{-1} + (2^n r_0)^{-4/3}(2^n r_0)^{-2} + (2^n r_0)^{-3} \\ &\lesssim (2^n r_0)^{-3/2}.\end{aligned}$$

Thus,

$$\sum_{n \geq 0} |\lambda(2^{n+1}r_0) - \lambda(2^n r_0)| < \infty,$$

so that there exists $\alpha \in \mathbb{R}$ such that $\lim_n \lambda(2^n r_0) = \alpha$. As in the proof of Lemma 2.6.14, we then use the fact that the sequence $\{\lambda(2^n r_0)\}_n$ is bounded and the difference estimate 2.6.37 to conclude that for $r \geq r_0$

$$|\lambda(r) - \alpha| \lesssim r^{-1}$$

as desired. ☐

We have shown that there exists $\alpha \in \mathbb{R}$ such that

$$r^3 u_e(0,r) = \alpha + O(r^{-1}),$$
$$r \int_r^\infty \partial_t u_e(0,\rho)\rho d\rho = O(r^{-1}),$$

as $r \to \infty$. In the case $\alpha = 0$, we conclude that $\vec{u}(0) = (0,0)$ on $r \geq \eta$.

Lemma 2.6.17. *Let α be as in Lemma 2.6.16. If $\alpha = 0$, then $\vec{u}(0,r) = (0,0)$ for $r \geq \eta$.*

Proof. The proof of Lemma 2.6.17 is split into two steps.

Claim 2.6.18. *Let α be as in Lemma 2.6.16. If $\alpha = 0$, then $\vec{u}(0,r)$ is compactly supported in r.*

Proof of Claim 2.6.18. Since $\alpha = 0$,

$$\lambda(r) = O(r^{-1}), \tag{2.6.45}$$
$$\mu(r) = O(r^{-1}). \tag{2.6.46}$$

Then, for $r_0 \geq R_1$, we have

$$|\lambda(2^n r_0)| + |\mu(2^n r_0)| \lesssim (2^n r_0)^{-1}. \tag{2.6.47}$$

By the difference estimate (2.6.33) and the growth estimates (2.6.45), (2.6.46), we conclude that

$$|\lambda(2^{n+1} r_0)| \geq (1 - C_1\delta_1)|\lambda(2^n r_0)| - C_1 (2^n r_0)^{-2/3} |\mu(2^n r_0)|,$$
$$|\mu(2^{n+1} r_0)| \geq (1 - C_1\delta_1)|\mu(2^n r_0)| - C_1 (2^n r_0)^{-8/3} |\lambda(2^n r_0)|.$$

The constant C_1 is independent of δ_1 and r_0. Thus

$$|\lambda(2^{n+1}r_0)| + |\mu(2^{n+1}r_0)| \geq \left(1 - C_1\delta_1 - C_1 r_0^{-2/3}\right)[|\lambda(2^n r_0)| + |\mu(2^n r_0)|].$$

Take r_0 large and δ_1 small enough so that $C_1(\delta_1 + r_0^{-2/3}) < 1/4$. Then

$$|\lambda(2^{n+1}r_0)| + |\mu(2^{n+1}r_0)| \geq \frac{3}{4}[|\lambda(2^n r_0)| + |\mu(2^n r_0)|].$$

Proceeding inductively we obtain

$$|\lambda(2^{n+1}r_0)| + |\mu(2^{n+1}r_0)| \geq \left(\frac{3}{4}\right)^n [|\lambda(r_0)| + |\mu(r_0)|].$$

By (2.6.47) we conclude that

$$\left(\frac{3}{4}\right)^n [|\lambda(r_0)| + |\mu(r_0)|] \lesssim (2^n r_0)^{-1}$$

which implies

$$\left(\frac{3}{2}\right)^n [|\lambda(r_0)| + |\mu(r_0)|] \lesssim 1,$$

where the implied constant is uniform in n. Hence $(\lambda(r_0), \mu(r_0)) = (0,0)$. By (2.6.30) $\|\pi_{r_0}\vec{u}_e(0)\|_{\mathcal{H}(r\geq r_0)} = 0$. By Lemma 2.6.9 $\|\pi_{r_0}^{\perp}\vec{u}_e(0)\|_{\mathcal{H}(r\geq r_0)} = 0$. Hence $\|\vec{u}_e(0)\|_{\mathcal{H}(r\geq r_0)} = 0$. Since $\lim_{r\to\infty} u_{e,0}(r) = 0$, we conclude that $(u_{e,0}(r), u_{e,1}(r)) = (0,0)$ for $r \geq r_0$. Since $u(t,r) = r^2\langle r\rangle^{-2}u_e(t,r)$, we conclude that $\vec{u}(0,r) = (0,0)$ on $r \geq r_0$ as well. This concludes the proof of the claim. □

Claim 2.6.19. *If $\vec{u}(0,r)$ is compactly supported in (η,∞), then $\vec{u}(t,r) = (0,0)$ on (η,∞).*

Proof of Claim 2.6.19. Suppose not, i.e. $\vec{u}(0,r)$ is not identically 0 on (η,∞).

Then $(u_{e,0}, u_{e,1})$ is not identically 0 on (η, ∞). Define

$$\rho_0 = \inf \left\{ \rho : \|\vec{u}_e(0)\|_{\mathcal{H}(r \geq \rho)} = 0 \right\}.$$

By our assumptions we have that $\eta < \rho < \infty$. Let $\rho_1 = \rho_1(\delta_1)$ be so close to ρ_0 so that $\eta < \rho_1 < \rho_0$ and

$$0 < \|\vec{u}_e(0)\|^2_{\mathcal{H}(r \geq \rho_1)} \leq \delta_1^2,$$

where δ_1 is as in Lemma 2.6.11.

By (2.6.29) and (2.6.30) and our choice of ρ_1, we have that

$$\begin{aligned} &\int_{\rho_1}^{\infty} \left(\frac{1}{r} \partial_r \lambda(r) \right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r \mu(r))^2 dr \\ &\quad + 3\rho_1^{-3} \lambda^2(\rho_1) + \rho_1^{-1} \mu^2(\rho_1) = \|\pi_{\rho_1}^{\perp} \vec{u}_e(0)\|^2_{\mathcal{H}(r \geq \rho_1)} + \|\pi_{\rho_1} \vec{u}_e(0)\|^2_{\mathcal{H}(r \geq \rho_1)} < \delta_1^2. \end{aligned} \tag{2.6.48}$$

If we define $(u_{0,\rho_1}, u_{1,\rho_1})$ as in (2.6.24) and (2.6.25), we have for ρ_1 close to ρ_0,

$$\|(u_{0,\rho_1}, u_{1,\rho_1})\|_{\mathcal{H}(r \geq \eta)} \leq C(\rho_0) \|\vec{u}_e(0)\|_{\mathcal{H}(r \geq \rho_1)} \leq \delta_1.$$

Thus, by Lemma 2.6.9 we obtain

$$\begin{aligned} \int_{\rho_1}^{\infty} \left(\frac{1}{r} \partial_r \lambda(r) \right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r \mu(r))^2 dr &\lesssim \rho_1^{-19/3} \lambda^2(\rho_1) + \rho_1^{-29/3} \lambda^4(\rho_1) + R^{-\rho_1} \lambda^6(\rho_1) \\ &\quad + \rho_1^{-13/3} \mu^2(\rho_1) + \rho_1^{-17/3} \mu^4(\rho_1) + \rho_1^{-3} \mu^6(\rho_1) \\ &\leq C(\rho_0) \left[|\lambda(\rho_1)|^2 + |\mu(\rho_1)|^2 \right], \end{aligned} \tag{2.6.49}$$

as long as ρ_1 is sufficiently close to ρ_0. Using the previous estimate and the fact that

$\lambda(\rho_0) = 0$, we argue as in the proof of Lemma 2.6.11 to obtain

$$\begin{aligned} |\lambda(\rho_1)|^2 &= |\lambda(\rho_1) - \lambda(\rho_0)|^2 \\ &\leq (\rho_0 - \rho_1)^3 \left(\int_{\rho_1}^{\rho_0} \left(\frac{1}{r} \partial_r \lambda(r) \right)^2 dr \right) \\ &\leq C(\rho_0)(\rho_0 - \rho_1)^3 \left[|\lambda(\rho_1)|^2 + |\mu(\rho_1)|^2 \right]. \end{aligned}$$

Similarly,

$$|\mu(\rho_1)|^2 \leq C(\rho_0)(\rho_0 - \rho_1) \left[|\lambda(\rho_1)|^2 + |\mu(\rho_1)|^2 \right].$$

We conclude that for all ρ_1 close to ρ_0,

$$|\lambda(\rho_1)|^2 + |\mu(\rho_1)|^2 \leq 2C(\rho_0)(\rho_0 - \rho_1) \left[|\lambda(\rho_1)|^2 + |\mu(\rho_1)|^2 \right]$$

Thus, $(\lambda(\rho_1), \mu(\rho_1)) = (0, 0)$ for $\rho_1 < \rho_0$ close to ρ_0. By (2.6.48) and (2.6.49) we conclude that $\|\vec{u}_e(0)\|_{\mathcal{H}(r \geq \rho_1)} = 0$. This contradicts our definition of ρ_0 and the fact that $\rho_1 < \rho_0$. Thus, $\rho_0 = \eta$ and $\|\vec{u}_e(0)\|_{\mathcal{H}(r \geq \eta)} = 0$ as desired. □

Lemma 2.6.17 now follows immediately from Claim 2.6.18 and Claim 2.6.19. □

Using the previous arguments we can, in fact, conclude more in the case $\alpha = 0$.

Lemma 2.6.20. *Let α be as in Lemma 2.6.16. If $\alpha = 0$, then*

$$\vec{u}(t, r) = (0, 0)$$

for all $t \in \mathbb{R}$ and $r > 0$.

Proof. By Lemma 2.6.17 we know that if $\alpha = 0$ then $\vec{u}(0, r) = (0, 0)$ on $\{r \geq \eta\}$. By finite speed of propagation, we conclude that

$$\vec{u}(t,r) = (0,0) \quad \text{on } \{r \geq |t| + \eta\}. \tag{2.6.50}$$

Let $t_0 \in \mathbb{R}$ be arbitrary and define $u_{t_0}(t,r) = u(t+t_0, r)$. Then $\vec{u}_{t_0}$ inherits the following compactness property from $\vec{u}$:

$$\forall R \geq 0, \quad \lim_{|t|\to\infty} \|\vec{u}_{t_0}(t)\|_{\mathcal{H}(r\geq R+|t|;(r^2+1)^2dr)} = 0,$$
$$\lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}_{t_0}(t)\|_{\mathcal{H}(r\geq R+|t|;(r^2+1)^2dr)}\right] = 0,$$

and by (2.6.50) $\vec{u}_{t_0}(0,r)$ is supported in $\{0 < r \leq \eta + |t_0|\}$. By Claim 2.6.19 applied to $\vec{u}_{t_0}$ we conclude that $\vec{u}_{t_0}(0,r) = (0,0)$ on $r \geq \eta$. Since t_0 was arbitrary, we conclude that

$$\vec{u}(t_0,r) = (0,0) \quad \text{on } \{r \geq \eta\},$$

for any $t_0 \in \mathbb{R}$. Since $\eta > 0$ was arbitrarily fixed in the beginning of this subsection, we conclude that

$$\vec{u}(t,r) = (0,0)$$

for all $t \in \mathbb{R}$ and $r > 0$ as desired. □

Step 3: Conclusion of the proof of Proposition 2.6.4

We now conclude the proof of Proposition 2.6.4 by proving the following.

Lemma 2.6.21. *Let α be as in Lemma 2.6.16. As before, we denote the unique finite energy* *harmonic map of degree n by Q and recall that there exists a unique $\alpha_n > 0$ such that*

$$Q(r) = n\pi - \alpha_n r^{-2} + O(r^{-4}).$$

Let $Q_{\alpha-\alpha_n}$ denote the unique solution to (2.2.1) *with the property that*

$$Q_{\alpha-\alpha_n}(r) = n\pi + (\alpha - \alpha_n) r^{-2} + O(r^{-4}) \tag{2.6.51}$$

as $r \to \infty$. Note that $Q_{\alpha-\alpha_n}$ exists and is unique by Proposition 2.2.4. Define a static solution U_+ to (2.5.1) *via*

$$U_+(r) = \langle r \rangle^{-1} \big(Q_{\alpha-\alpha_n}(r) - Q(r) \big).$$

Then

$$\vec{u}(t, r) = (U_+(r), 0)$$

for all $t \in \mathbb{R}$ and $r > 0$.

Proof. Lemma 2.6.21 follows from the proof of the $\alpha = 0$ case and a change of variables. Let $Q_{\alpha-\alpha_n}$ be as in the statement of the lemma. We define

$$\begin{aligned} u_\alpha(t, r) &:= u(t, r) - \langle r \rangle^{-1} \left(Q_{\alpha-\alpha_n}(r) - Q(r) \right) \\ &= u(t, r) - U_+(r) \end{aligned} \tag{2.6.52}$$

and observe that u_α solves

$$\partial_t^2 u_\alpha - \partial_r^2 u_\alpha - \frac{4r}{r^2 + 1} \partial_r u_\alpha + V_\alpha(r) u_\alpha = N_\alpha(r, u_\alpha),$$

where the potential V_α is given by

$$V_\alpha(r) = \langle r\rangle^{-4} + 2\langle r\rangle^{-2}(\cos 2Q_{\alpha-\alpha_n} - 1), \tag{2.6.53}$$

and $N_\alpha(r,u) := F_\alpha(r,u) + G_\alpha(r,u)$ with

$$\begin{aligned} F_\alpha(r,u) &:= 2\langle r\rangle^{-3}\sin^2(\langle r\rangle u)\sin 2Q_{\alpha-\alpha_n}, \\ G_\alpha(r,u) &:= \langle r\rangle^{-3}\left[2\langle r\rangle u - \sin(2\langle r\rangle u)\right]\cos 2Q_{\alpha-\alpha_n}. \end{aligned} \tag{2.6.54}$$

By (2.6.51), the potential V_α is smooth and satisfies

$$V_\alpha(r) = \langle r\rangle^{-4} + O(\langle r\rangle^{-6}),$$

as $r \to \infty$ and the nonlinearities F_α and G_α satisfy

$$\begin{aligned} |F_\alpha(r,u)| &\lesssim \langle r\rangle^{-3}|u|^2, \\ |G_\alpha(r,u)| &\lesssim |u|^3, \end{aligned}$$

for $r \geq 0$. Moreover, by (2.6.52) we see that $\vec{u}_\alpha$ inherits the compactness property from $\vec{u}$:

$$\begin{aligned} \forall R \geq 0, \qquad \lim_{|t|\to\infty} \|\vec{u}_\alpha(t)\|_{\mathcal{H}(r\geq R+|t|;(1+r^2)^2dr)} &= 0, \\ \lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}_\alpha(t)\|_{\mathcal{H}(r\geq R+|t|;(1+r^2)^2dr)}\right] &= 0. \end{aligned} \tag{2.6.55}$$

Let $\eta > 0$. We now define for $r \geq \eta$,

$$u_{\alpha,e}(t,r) := \frac{r^2+1}{r^2}u_\alpha(t,r) \tag{2.6.56}$$

and note that $u_{\alpha,e}$ satisfies an equation analogous to u_e:

$$\partial_t^2 u_{\alpha,e} - \partial_r^2 u_{\alpha,e} - \frac{4}{r}\partial_r u_{\alpha,e} + V_{\alpha,e}(r)u_{\alpha,e} = N_{\alpha,e}(r, u_{\alpha,e}), \quad t \in \mathbb{R}, r \geq \eta, \tag{2.6.57}$$

where

$$V_{\alpha,e}(r) = V_\alpha(r) - \frac{2}{r^2(r^2+1)},$$

and $N_{\alpha,e}(r, u_e) = F_{\alpha,e}(r, u_e) + G_{\alpha,e}(r, u_e)$ where

$$\begin{aligned} F_{\alpha,e}(r, u_{\alpha,e}) &= \frac{r^2+1}{r^2} F_\alpha\left(r, \frac{r^2}{r^2+1} u_{\alpha,e}\right), \\ G_{\alpha,e}(r, u_{\alpha,e}) &= \frac{r^2+1}{r^2} G_\alpha\left(r, \frac{r^2}{r^2+1} u_{\alpha,e}\right). \end{aligned}$$

In particular, we have the analogues of (2.6.8), (2.6.9), and (2.6.10): for all $r \geq \eta$,

$$|V_{\alpha,e}(r)| \lesssim r^{-4}, \tag{2.6.58}$$

$$|F_{\alpha,e}(r, u)| \lesssim r^{-3}|u|^2, \tag{2.6.59}$$

$$|G_{\alpha,e}(r, u)| \lesssim |u|^3. \tag{2.6.60}$$

Moreover, $u_{\alpha,e}$ inherits the following compactness properties from u_α:

$$\begin{aligned} \forall R \geq \eta, \quad \lim_{|t|\to\infty} \|\vec{u}_{\alpha,e}(t)\|_{\mathcal{H}(r\geq R+|t|;r^4dr)} &= 0, \\ \lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}_{\alpha,e}(t)\|_{\mathcal{H}(r\geq R+|t|;r^4dr)}\right] &= 0. \end{aligned} \tag{2.6.61}$$

Finally, by construction we see that

$$r^3 u_{\alpha,e,0}(r) = O(r^{-1}), \tag{2.6.62}$$
$$r \int_r^\infty u_{\alpha,e,1}(\rho)\rho d\rho = O(r^{-1}). \tag{2.6.63}$$

Using (2.6.57)–(2.6.63), we may repeat the previous arguments with $u_{e,\alpha}$ in place of u_e to conclude the following analog of Lemma 2.6.17:

Lemma 2.6.22. $\vec{u}_\alpha(0, r) = (0, 0)$ *for* $r \geq \eta$.

Finally, we obtain the following analog of Lemma 2.6.20:

Lemma 2.6.23. *We have*

$$\vec{u}_\alpha(t, r) = (0, 0)$$

for all $t \in \mathbb{R}$ *and* $r > 0$.

Equivalently, Lemma 2.6.23 states that

$$\vec{u}(t, r) = (U_+(r), 0)$$

for all $t \in \mathbb{R}$ and $r > 0$. This concludes the proof of Lemma 2.6.21 and Proposition 2.6.4.

□

2.6.2 Proof of Proposition 2.6.3

Using Proposition 2.6.4 and its analog for $r < 0$, we quickly conclude the proof of Proposition 2.6.3. Indeed, we know that there exists static solutions $U_\pm$ to (2.5.1) such that

$$\vec{u}(t, r) = (U_\pm(r), 0) \tag{2.6.64}$$

for all $\pm r > 0$ and $t \in \mathbb{R}$. In particular, $\partial_t u(t,r) = 0$, $\partial_r u(t,r) = \partial_r u(0,r)$ and $u(t,r) = u(0,r)$ for all t and almost every r. Let $\psi \in C_0^\infty(\mathbb{R})$ with $\int \psi dt = 1$ and let $\varphi \in C_0^\infty(\mathbb{R})$. Then since u solves (2.5.1) in the weak sense, we conclude that

$$\begin{aligned} 0 &= \int\int \big[\psi'(t)\varphi(r)\partial_t u(t,r) + \psi(t)\varphi'(r)\partial_r u(t,r) + V(r)\psi(t)\varphi(r)u(t,r) \\ &\qquad - \psi(t)\varphi(r)N(r,u(t,r))\big](r^2+1)^2 drdt \\ &= \int\int \psi(t)\big[\varphi'(r)\partial_r u(0,r) + V(r)\varphi(r)u(0,r) - \varphi(r)N(r,u(0,r))\big](r^2+1)^2 drdt \\ &= \int \big[\varphi'(r)\partial_r u(0,r) + V(r)\varphi(r)u(0,r) - \varphi(r)N(r,u(0,r))\big](r^2+1)^2 dr. \end{aligned}$$

Since φ was arbitrary, we see that $u(0,r)$ is a weak solution in $H^1(\mathbb{R})$ to the static equation $-\partial_r^2 u - \frac{4r}{r^2+1}\partial_r u + V(r)u = N(r,u)$ on $\mathbb{R}$. By standard arguments we conclude that $u(0,r)$ is a classical solution. Thus, $\vec{u}(t,r) = (U(r),0) := (u(0,r),0)$ for all $t, r \in \mathbb{R}$ as desired.

□

2.6.3 Proofs of Proposition 2.6.1 and Theorem 2.4.1

We briefly summarize the proofs of Proposition 2.6.1 and Theorem 2.4.1. From Proposition 2.6.3, we obtain Proposition 2.6.1.

Proof of Proposition 2.6.1. By Proposition 2.6.3, we have that $\vec{u} = (U,0)$ for some finite energy static solution U to (2.5.1). Thus, $\psi = Q + \langle r\rangle U$ is a finite energy static solution to (2.4.1), i.e. a harmonic map. By Proposition 2.2.1, the harmonic map Q is the unique finite energy static solution to (2.4.1) so that $Q = \psi = Q + \langle r\rangle U$ whence $\vec{u} = (0,0)$ as desired. □

Using Proposition 2.5.3 and Proposition 2.6.1, we conclude the proof of our main result Theorem 2.4.1 (equivalently Theorem 2.1.1).

Proof of Theorem 2.4.1. Suppose that Theorem 2.4.1 fails. Then by Proposition 2.5.3, there exists a nonzero solution u_* to (2.4.4) such that the trajectory

$$K := \{\vec{u}_*(t) : t \in \mathbb{R}\},$$

is precompact in $\mathcal{H}$. However, by Proposition 2.6.1, we must have that $\vec{u}_* = (0,0)$, which contradicts the fact that u_* is nonzero. Thus, Theorem 2.4.1 holds. □

CHAPTER 3

SOLITON RESOLUTION FOR HIGHER EQUIVARIANT WAVE MAPS ON A WORMHOLE

3.1 Main Result and Outline of the Chapter

In this chapter, we establish soliton resolution for all finite energy ℓ–equivariant wave maps on a wormhole for all $\ell > 2$ (Chapter 2 established this for $\ell = 1$). We recall that an ℓ–equivariant wave map $U : \mathbb{R} \times \mathcal{M} \to \mathbb{S}^3$ is determined completely by its associated azimuth angle $\psi = \psi(t, r)$ on $\mathbb{S}^3$ (with respect to spherical coordinates) which satisfies the single scalar semilinear wave equation

$$\begin{aligned}
&\partial_t^2\psi - \partial_r^2\psi - \frac{2r}{r^2+1}\partial_r\psi + \frac{\ell(\ell+1)}{2(r^2+1)}\sin 2\psi = 0, \quad (t,r) \in \mathbb{R}\times\mathbb{R}, \\
&\vec{\psi}(0) = (\psi_0, \psi_1), \\
&\psi(t, -\infty) = 0, \quad \psi(t,\infty) = n\pi, \quad \forall t.
\end{aligned} \tag{3.1.1}$$

Here $n \in \mathbb{N} \cup \{0\}$ is the topological degree of the solution, and the fact that such a n exists and is an integer follows from us requiring that $\vec{\psi}$ have finite energy

$$\begin{aligned}
\mathcal{E}_\ell(\vec{\psi}(t)) &:= \frac{1}{2}\int_{\mathbb{R}} \left[|\partial_t\psi(t,r)|^2 + |\partial_r\psi(t,r)|^2 + \frac{\ell(\ell+1)}{r^2+1}\sin^2\psi(t,r) \right](r^2+1)dr \\
&= \mathcal{E}_\ell(\vec{\psi}(0)).
\end{aligned}$$

For each $\ell \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}$, we denote the set of finite energy pairs of degree n by

$$\mathcal{E}_{\ell,n} := \left\{ (\psi_0, \psi_1) : \mathcal{E}_\ell(\psi_0,\psi_1) < \infty, \quad \psi_0(-\infty) = 0, \quad \psi_0(\infty) = n\pi \right\}.$$

As in the corotational case treated in Chapter 2, it can be shown that for every $\ell \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$ there exists a unique solution $Q_{\ell,n} \in \mathcal{E}_{\ell,n}$ to

$$\begin{aligned} &Q'' + \frac{2r}{r^2+1} Q' - \frac{\ell(\ell+1)}{2(r^2+1)} \sin 2Q = 0, \quad r \in \mathbb{R}, \\ &Q(-\infty) = 0, \quad Q(\infty) = n\pi. \end{aligned} \tag{3.1.2}$$

See Section 3.2 for more details.

We now state our main result. Let $\ell \in \mathbb{N}$ be a fixed equivariance class, and let $n \in \mathbb{N} \cup \{0\}$ be a fixed topological degree. In the $n = 0$ case, the natural space to place the solution $\vec{\psi}(t)$ to (3.1.1) in is the *energy space* $\mathcal{H}_0 := \mathcal{H}((-\infty, \infty); (r^2+1)dr)$. Indeed, it is easy to show that $\|\vec{\psi}\|_{\mathcal{E}_{\ell,0}} \simeq \|\vec{\psi}\|_{\mathcal{H}_0}$. For $n \geq 1$, we measure distance relative to $(Q_{\ell,n}, 0)$ and define $\mathcal{H}_{\ell,n} := \mathcal{E}_{\ell,n} - (Q_{\ell,n}, 0)$ with 'norm'

$$\|\vec{\psi}\|_{\mathcal{H}_{\ell,n}} := \|\vec{\psi} - (Q_{\ell,n}, 0)\|_{\mathcal{H}_0}.$$

Note that $\psi(r) - Q_{\ell,n}(r) \to 0$ as $r \to \pm\infty$. The main result of this chapter is the following.

Theorem 3.1.1. *For all $(\psi_0, \psi_1) \in \mathcal{E}_{\ell,n}$, there exists a unique global solution $\vec{\psi}(t) \in C(\mathbb{R}; \mathcal{H}_{\ell,n})$ to (3.1.1) which scatters forwards and backwards in time to the harmonic map $(Q_{\ell,n}, 0)$, i.e. there exist solutions $\varphi_L^{\pm}$ to the underlying linear equation*

$$\partial_t^2 \varphi - \partial_r^2 \varphi - \frac{2r}{r^2+1} \partial_r \varphi + \frac{\ell(\ell+1)}{r^2+1} \varphi = 0,$$

such that

$$\vec{\psi}(t) = (Q_{\ell,n}, 0) + \vec{\varphi}_L^{\pm}(t) + o_{\mathcal{H}_0}(1),$$

as $t \to \pm\infty$.

We now give an outline of the proof and the chapter. The proof is a generalization of that for the corotational case $\ell = 1$ in Chapter 2 and draws from the work [18]. For this model, the set up is as follows. We first note that the existence and uniqueness of the harmonic map $Q_{\ell,n}$ follows nearly verbatim from the arguments in Chapter 2 for the special case $\ell = 1$ which are classical ODE type arguments. This is discussed more in Section 3.2. In the remainder of Section 3.2 we give an equivalent reformulation of Theorem 3.1.1 which is simpler to work with. Instead of studying the azimuth angle ψ, we study the function u defined by the relation $\psi = Q_{\ell,n} + \langle r \rangle^{\ell} u$. A simple computation shows that u satisfies a radial semilinear wave equation on a higher dimensional wormhole $(\mathcal{M}^d, g)$

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + V(r)u = N(r,u), \quad (t,r) \in \mathbb{R} \times \mathbb{R}, \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H} := \dot{H}^1 \times L^2(\mathcal{M}^d). \end{aligned} \tag{3.1.3}$$

Here, $d = 2\ell + 3$ and the potential V and nonlinearity N are explicit with V arising from linearizing (3.1.1) about the harmonic map $Q_{\ell,n}$. Our main result, Theorem 3.1.1, is shown to be equivalent to the statement that every solution to (3.1.3) is global and scatters to free waves on $\mathcal{M}^d$ as $t \to \infty$ (see Theorem 3.2.3 for the precise statement). The remainder of the work is then devoted to proving Theorem 3.2.3 (the 'u–formulation' of our main result). As in Chapter 2, we use the concentration–compactness/rigidity method introduced by Kenig and Merle in there work on the energy–critical Schrödinger and wave equations [14] [15]. The method has three main steps and is by contradiction. In the first step, we show that solutions to (3.1.3) starting from small initial data scatter to free waves as $t \to \pm\infty$. In the second step, we then show that if our main result fails, then there exists a nonzero solution u_* to (3.1.3) which doesn't scatter in either direction and is, in a certain sense, minimal. This minimality imposes the following compactness property on u_*: the set

$$K = \{\vec{u}_*(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H}$. These two steps are carried out in Section 3.3. We remark here that in the work [18] the authors established these steps by using delicate estimates and arguments developed in [4] and [23] for the energy–critical wave equation on flat space in high dimensions. This is done by using a Strauss estimate to reduce the nonlinearity to an energy–critical power on $\mathbb{R}^{1+(2\ell+3)}$. However, the arguments we give in this work are much simpler and bypass all of this technical machinery by using only basic Strichartz and Strauss estimates (in fact, our argument also applies to the analogous step in the exterior wave map problem). In the final and most difficult step, we establish the following rigidity result: if u solves (3.1.3) and

$$K = \{\vec{u}(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H}$ then $\vec{u} = (0, 0)$. This step contradicts the second step and we conclude that our main result Theorem 3.1.1 holds. This is proved in Section 3.4. In particular, we show that such a solution u must be a static solution to (3.1.3) which implies $\psi = Q_{\ell,n} + \langle r \rangle^{\ell} u$ is a harmonic map. By the uniqueness of $Q_{\ell,n}$, it follows that $\vec{u} = (0, 0)$ as desired. The proof that u must be a static solution to (3.1.3) uses channels of energy arguments rooted in [7] which were then generalized and used in the works [16] [18] on exterior wave maps. These arguments focus only on the behavior of solutions in regions exterior to light cones, and this is what allows us to adapt them to our asymptotically Euclidean setting. These arguments, however, become increasingly complex as the equivariance integer ℓ grows and represent a major technical obstacle. For the argument in the specific case of $\ell = 1$ which is much simpler, see Section 2.6.

3.2 Harmonic Maps and a Reduction to Higher Dimensions

For the remainder of the paper we fix an equivariance class $\ell \in \mathbb{N}$, topological degree $n \in \mathbb{N} \cup \{0\}$ and study solutions to the wave map on a wormhole equation

$$\begin{aligned} &\partial_t^2 \psi - \partial_r^2 \psi - \frac{2r}{r^2+1}\partial_r \psi + \frac{\ell(\ell+1)}{2(r^2+1)} \sin 2\psi = 0, \quad (t,r) \in \mathbb{R} \times \mathbb{R} \\ &\psi(t,-\infty) = 0, \quad \psi(t,\infty) = n\pi, \quad \forall t, \\ &\vec{\psi}(0) = (\psi_0, \psi_1). \end{aligned} \tag{3.2.1}$$

We recall that the energy

$$\mathcal{E}_\ell(\psi) = \frac{1}{2} \int \left[|\partial_t \psi|^2 + |\partial_r \psi|^2 + \frac{\ell(\ell+1)}{r^2+1} sin^2 \psi \right] (r^2+1) dr \tag{3.2.2}$$

is conserved along and the flow, and so, we take initial data (ψ_0, ψ_1) in the metric space

$$\mathcal{E}_{\ell,n} = \left\{ (\psi_0, \psi_1) : \mathcal{E}_\ell(\psi_0, \psi_1) < \infty, \quad \psi_0(-\infty) = 0, \quad \psi_0(\infty) = n\pi \right\}.$$

In this section we review the theory of static solutions to (3.2.1) (i.e. harmonic maps) and reduce the study of ℓ–equivariant wave maps on a wormhole to the study of a semilinear wave equation on a higher dimensional wormhole.

3.2.1 Harmonic Maps

In this subsection, we briefly review the theory of harmonic maps for (3.2.1). The main result is the following.

Proposition 3.2.1. *There exists a unique solution $Q_{\ell,n} \in \mathcal{E}_{\ell,n}$ to the equation*

$$Q'' + \frac{2r}{r^2+1} Q' - \frac{\ell(\ell+1)}{2(r^2+1)} \sin 2Q = 0. \tag{3.2.3}$$

In the case $n = 0$, $Q_{\ell,0} = 0$. *If* $n \in \mathbb{N}$, *then* $Q_{\ell,n}$ *is increasing on* $\mathbb{R}$, *satisfies* $Q_{\ell,n}(r) + Q_{\ell,n}(-r) = n\pi$ *and there exists* $\alpha_{\ell,n} \in \mathbb{R}$ *such that*

$$Q_{\ell,n}(r) = n\pi - \alpha_{\ell,n} r^{-\ell-1} + O(r^{-\ell-3}), \quad \textit{as } r \to \infty,$$
$$Q_{\ell,n}(r) = \alpha_{\ell,n} |r|^{-\ell-1} + O(r^{-\ell-3}), \quad \textit{as } r \to -\infty.$$

The $O(\cdot)$ *terms satisfy the natural derivative bounds.*

The proof of Proposition 3.2.1 is nearly identical to the proof of the corresponding statement in Chapter 2 for the corotational case which was inspired by arguments in [21]. We briefly sketch the argument.

Sketch of Proof. We first can use simple ODE arguments to show that every solution Q to (3.2.3) is defined on $\mathbb{R}$ and has limits $Q(\pm\infty)$ in $\mathbb{Z}\pi$ or $(\mathbb{Z} + \frac{1}{2})\pi$. Moreover, if $Q(\pm\infty) \in \mathbb{Z}\pi$, then Q is monotonic and there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} Q(r) &= Q(\infty) + \alpha r^{-\ell-1} + O(r^{-\ell-3}), \quad \text{as } r \to \infty, \\ Q(r) &= Q(-\infty) + \beta r^{-\ell-1} + O(r^{-\ell-3}), \quad \text{as } r \to -\infty. \end{aligned} \tag{3.2.4}$$

For existence, we use a classical shooting argument. For $b > 0$, we consider the solution Q_b to

$$Q_b'' + \frac{2r}{r^2+1} Q_b' - \frac{\ell(\ell+1)}{2(r^2+1)} \sin 2Q_b = 0,$$
$$Q_b(0) = \frac{n\pi}{2}, \quad Q_b'(0) = b,$$

and show the existence of a special value b_* for the shooting parameter b such that $Q_{b_*}(\infty) = n\pi$. Indeed, using the properties of general solutions to (3.2.3) already outlined and simple

ODE arguments, we can show that the sets

$$B_< = \{b > 0 : Q_b(\infty) < n\pi\},$$
$$B_> = \{b > 0 : Q_b(\infty) > n\pi\},$$

are both nonempty, open, proper subsets of $(0, \infty)$. By connectedness, there exists b_* such that $Q_{b_*}(\infty) = n\pi$. From the initial condition $Q_{b_*} = 0$ and the symmetry $Q(r) \mapsto n\pi - Q(-r)$ of (3.2.3), we conclude that $Q_{b_*}(r) = n\pi - Q_{b_*}(-r)$ as well as $Q_{b_*}(-\infty) = 0$. We then set $Q_{\ell,n} = Q_{b_*}$.

For the uniqueness of $Q_{\ell,n}$, suppose that there are two solutions Q_1, Q_2 to (3.2.3). By the previous discussion each solution is monotonic increasing on $\mathbb{R}$ and satisfies (3.2.4). We change variables to $x = \operatorname{arcsinh} r$ so that (3.2.3) becomes

$$Q'' + \tanh x Q' - \frac{\ell(\ell+1)}{2} \sin 2Q = 0. \tag{3.2.5}$$

Based on (3.2.4) (in the x–variable) and (3.2.5), we can then show that if we assume, without loss of generality, that $\frac{dQ_2}{dx} > \frac{dQ_1}{dx}$ for x large and positive, then

$$\frac{dQ_2}{dx} > \frac{dQ_1}{dx}, \quad \forall x \in \mathbb{R}.$$

However, this can easily be shown to be incompatible with (3.2.4) as $x \to -\infty$. Thus, $Q_1 = Q_2$, and the solution $Q_{\ell,n}$ constructed is unique. For the full details of the argument in the $\ell = 1$ case, see Chapter 2. □

A fact that will be essential in the final section of this work is that we may always find a unique solution to (3.2.3) with prescribed asymptotics as either $r \to \infty$ or $r \to -\infty$ (but not necessarily both).

Proposition 3.2.2. *Let $\alpha \in \mathbb{R}$. Then there exists a unique solution Q_α^+ to* (3.2.3) *such that*

$$Q_\alpha^+ = n\pi + \alpha r^{-\ell-1} + O(r^{-\ell-3}), \quad \text{as } r \to \infty.$$

Similarly, given $\beta \in \mathbb{R}$, there exists a unique solution Q_β^- to (3.2.3) *such that*

$$Q_\beta^- = \beta r^{-\ell-1} + O(r^{-\ell-3}), \quad \text{as } r \to -\infty.$$

Proof. The proof is nearly identical to the proof of Proposition 2.2.4 in Chapter 2 and we omit the details. □

3.2.2 Reduction to a Wave Equation on a Higher Dimensional Wormhole

In this subsection we reduce the study of the large data solutions to (3.2.1) to the study of large data solutions to a semilinear wave equation on a higher dimensional wormhole geometry. This process is a generalization of the analogous step in the corotational case in Chapter 2.

By Proposition 3.2.1, there exists a unique static solution $Q_{\ell,n}(r) \in \mathcal{E}_{\ell,n}$ to (3.2.1). For a solution ψ to (3.2.1), we define φ by

$$\psi(t, r) = Q_{\ell,n}(r) + \varphi(t, r).$$

Then (3.2.1) implies that φ satisfies

$$\begin{aligned}
&\partial_t^2 \varphi - \partial_r^2 \varphi - \frac{2r}{r^2+1}\partial_r \varphi + \ell(\ell+1)\frac{\cos 2Q_{\ell,n}}{r^2+1}\varphi = Z(r, \varphi), \\
&\varphi(t, -\infty) = \varphi(t, \infty) = 0, \quad \forall t, \qquad (3.2.6)\\
&\vec{\varphi}(0) = (\psi_0 - Q_{\ell,n}, \psi_1),
\end{aligned}$$

where

$$Z(r,\phi) = \frac{\ell(\ell+1)}{2(r^2+1)}\left[2\varphi - \sin 2\varphi\right]\cos 2Q_{\ell,n} + (1-\cos 2\varphi)\sin 2Q_{\ell,n}.$$

The left–hand side of (3.2.6) has more dispersion than a free wave on $\mathcal{M}^3$ due to the repulsive potential

$$\ell(\ell+1)\frac{\cos 2Q}{r^2+1} = \frac{\ell(\ell+1)}{r^2+1} + O(\langle r\rangle^{-2\ell-4})$$

as $r \to \pm\infty$. Here and throughout this work, we use the Japanese bracket notation $\langle r\rangle = (r^2+1)^{1/2}$. The $O(\cdot)$ term is a consequence of the asymptotics from Proposition 3.2.1. We now make a standard reduction that incorporates this extra dispersion. We define u and d via the relations

$$\varphi = \langle r\rangle^{\ell} u,$$
$$d = 2\ell + 3.$$

We define the d–dimensional wormhole $\mathcal{M}^d = \mathbb{R}\times\mathbb{S}^{d-1}$ with metric

$$ds^2 = dr^2 + (r^2+1)d\Omega^2_{d-1},$$

where $d\Omega^2_{d-1}$ is the standard round metric on $\mathbb{S}^{d-1}$. Since we will only be dealing with functions depending solely on r, we will abuse notation slightly and denote the radial part of the Laplacian on $\mathcal{M}^d$ by $-\Delta_g$, i.e.

$$-\Delta_g u = -\partial_r^2 u - \frac{(d-1)r}{r^2+1}\partial_r u.$$

By (3.2.6), u satisfies the radial semilinear wave equation

$$\begin{aligned}
&\partial_t^2 u - \Delta_g u + V(r)u = N(r,u), \\
&u(t,-\infty) = u(t,\infty) = 0, \quad \forall t, \\
&\vec{u}(0) = (u_0, u_1),
\end{aligned} \tag{3.2.7}$$

where the potential term is given by

$$V(r) = \frac{\ell^2}{\langle r\rangle^4} + \ell(\ell+1)\frac{\cos 2Q - 1}{\langle r\rangle^2}, \tag{3.2.8}$$

and the nonlinearity $N(r,u) = F(r,u) + G(r,u)$ is given by

$$\begin{aligned}
F(r,u) &= \frac{\ell(\ell+1)}{\langle r\rangle^{\ell+2}} \sin^2(\langle r\rangle^\ell u) \sin 2Q_{\ell,n}, \\
G(r,u) &= \frac{\ell(\ell+1)}{2\langle r\rangle^{\ell+2}} \left[2\langle r\rangle^\ell u - \sin(2\langle r\rangle^\ell u)\right] \cos 2Q_{\ell,n}.
\end{aligned} \tag{3.2.9}$$

By Proposition 3.2.1, the potential V is smooth and satisfies

$$V(r) = \frac{\ell^2}{\langle r\rangle^4} + O(\langle r\rangle^{-2\ell-4}). \tag{3.2.10}$$

Also, by Proposition 3.2.1 $Q_{\ell,n}(-r) + Q_{\ell,n}(r) = n\pi$ which implies that $V(r)$ is an even function. The nonlinearities F and G satisfy

$$F(r,u) = \left(\ell(\ell+1)\langle r\rangle^{\ell-2} \sin 2Q_{\ell,n}\right) u^2 + F_0(r,u), \tag{3.2.11}$$

where

$$|F_0(r,u)| \lesssim \langle r\rangle^{2\ell-3}|u|^4, \tag{3.2.12}$$

and

$$|G(r,u)| \lesssim \langle r\rangle^{2\ell-2}|u|^3, \tag{3.2.13}$$

where the implied constants depend only on ℓ. Since the original azimuth angle $\psi = Q_{\ell,n} + \langle r\rangle^{\ell} u \in \mathcal{E}_{\ell,n}$, we take initial data $(u_0, u_1) \in \mathcal{H}(\mathbb{R}; \langle r\rangle^{d-1}dr)$ for (3.2.7). For the remainder of this section and the next we denote

$$\mathcal{H}_0 := \mathcal{H}(\mathbb{R}; \langle r\rangle^2 dr), \quad \mathcal{H} := \mathcal{H}(\mathbb{R}; \langle r\rangle^{d-1}dr),$$

and note that $\mathcal{H}_0$ is simply the space of radial functions in $\dot{H}^1 \times L^2(\mathcal{M}^3)$ while $\mathcal{H}$ is the space of radial functions in $\dot{H}^1 \times L^2(\mathcal{M}^d)$.

In the remainder of the paper, we work in the 'u–formulation' rather than with the original azimuth angle ψ. We first show that a solution $\vec{\psi}(t) \in C(\mathbb{R}; \mathcal{H}_n)$ to (3.2.1) with initial data $(\psi_0, \psi_1) \in \mathcal{E}_{\ell,n}$ yields a solution $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H})$ with initial data $(u_0, u_1) = \langle r\rangle^{-\ell}(\psi_0 - Q_{\ell,n}, \psi_1) \in \mathcal{H}$ and vice versa. The only fact that needs to be checked is that

$$\|\vec{u}\|_{\mathcal{H}} \simeq \left\|\vec{\psi} - (Q_{\ell,n}, 0)\right\|_{\mathcal{H}_0}. \tag{3.2.14}$$

We define $\varphi = \psi - Q_{\ell,n} = \langle r\rangle^{\ell} u$ and compute

$$\partial_r\varphi = \langle r\rangle^{\ell}\partial_r u + \ell r\langle r\rangle^{\ell-2}u. \tag{3.2.15}$$

We first note that by the fundamental theorem of calculus, we have the Strauss estimates

$$\begin{aligned} |\varphi(r)| &\lesssim \langle r\rangle^{-1/2}\left(\int |\partial_r\varphi|^2\langle r\rangle^2 dr\right)^{1/2}, \\ |u(r)| &\lesssim \langle r\rangle^{(2-d)/2}\left(\int |\partial_r u|^2\langle r\rangle^{d-1}dr\right)^{1/2}. \end{aligned} \tag{3.2.16}$$

Using the Strauss estimates and integration by parts, we have the following Hardy's inequalities,

$$\int |\varphi|^2 dr \lesssim \int |\partial_r \varphi|^2 \langle r \rangle^2 dr,$$
$$\int |u|^2 \langle r \rangle^{d-3} dr \lesssim \int |\partial_r u|^2 \langle r \rangle^{d-1} dr. \tag{3.2.17}$$

Recalling that d and ℓ are related by $d = 2\ell + 3$, we see that the relation (3.2.15) and the two Hardy's inequalities immediately imply (3.2.14). Hence, the two Cauchy problems (3.2.1) and (3.2.7) are equivalent.

The equivalent u–formulation of our main result, Theorem 3.1.1, is the following.

Theorem 3.2.3. *For any initial data $(u_0, u_1) \in \mathcal{H}$, there exists a unique global solution $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H})$ to (3.2.7) which scatters to free waves on $\mathcal{M}^d$, i.e. there exist solutions $v_L^{\pm}$ to*

$$\partial_t^2 v - \partial_r^2 v - \frac{(d-1)r}{r^2+1} \partial_r v = 0, \quad (t, r) \in \mathbb{R} \times \mathbb{R},$$

such that

$$\lim_{t \to \pm\infty} \|\vec{u}(t) - \vec{v}_L^{\pm}(t)\|_{\mathcal{H}} = 0.$$

The remainder of this work is devoted to proving Theorem 3.2.3.

3.3 Small Data Theory and Concentration–Compactness

In this section we begin the proof of Theorem 3.2.3 and the study of the nonlinear evolution introduced in the previous section:

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + V(r)u = N(r,u), \quad (t,r) \in \mathbb{R}\times\mathbb{R}, \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H}, \end{aligned} \tag{3.3.1}$$

where $\mathcal{H} := \mathcal{H}(\mathbb{R}; \langle r\rangle^{d-1}dr)$, $d = 2\ell + 3$, $-\Delta_g$ is the (radial) Laplace operator on the d–dimensional wormhole $\mathcal{M}^d$, and $V(r)$ and $N(r,u)$ are given in (3.2.8) and (3.2.9).

As summarized in the introduction, the proof of Theorem 3.2.3, or equivalently Theorem 3.1.1, uses the powerful concentration–compactness/rigidity methodology introduced by Kenig and Merle in their study of energy–critical dispersive equations [14] [15]. This methodology was used in the corototational case, $\ell = 1$, $d = 5$, in Chapter 2. The general situation $\ell \in \mathbb{N}$ requires many refinements due to the growing dimension d.

The proof of Theorem 3.2.3 is split up into three main steps and is by contradiction. In the first step, we establish small data global well–posedness and scattering for (3.3.1). In particular, we establish Theorem 3.2.3 if $\|(u_0,u_1)\|_{\mathcal{H}} \ll 1$. In the second step, we use the first step and a concentration–compactness argument to show that the *failure* of Theorem 3.2.3 implies that that there exists a nonzero 'critical element' u_*; a minimal non–scattering global solution to (3.3.1). The minimality of u_* imposes the following compactness property on u_*: the trajectory

$$K = \{\vec{u}_*(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H}$. In the third and final step, we establish the following rigidity theorem: every solution u with $\{\vec{u}(t) : t \in \mathbb{R}\}$ precompact in $\mathcal{H}$ must be identically 0. This contradicts the second step which implies that Theorem 3.2.3 holds.

In this section we complete the first two two steps in the program: small data theory and concentration–compactness. The proofs for these steps are straightforward generalizations of or nearly identical to those in the corototational case in Chapter 2. We will therefore only outline the main steps and refer the reader to the relevant proofs in Chapter 2 for full details.

3.3.1 Small Data Theory

In this subsection, we establish global well–posedness and scattering for small data solutions to (3.3.1). The key tools for establishing this and facts found later in this section are Strichartz estimates for the inhomogeneous wave equation with potential

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + V(r)u = h(t,r), \quad (t,r) \in \mathbb{R}\times\mathbb{R}, \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H}. \end{aligned} \tag{3.3.2}$$

Here, as in the previous section,

$$-\Delta_g u = -\partial_r^2 u - \frac{(d-1)r}{r^2+1}\partial_r u,$$

and the potential V is given by

$$V(r) = \frac{\ell^2}{\langle r\rangle^4} + \ell(\ell+1)\frac{\cos 2Q_{\ell,n} - 1}{\langle r\rangle^2},$$

where $Q_{\ell,n}$ is the unique ℓ–equivariant harmonic map of degree n. The conserved energy for the homogeneous problem, $h \equiv 0$ in (3.3.2), is given by

$$\mathcal{E}_V(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}} \left(|\partial_t u|^2 + |\partial_r u|^2 + V(r)|u|^2\right)\langle r\rangle^{d-1} dr.$$

In exactly the same fashion as in the corotational case, it can be shown that the operator $-\Delta_g + V(r)$ defined (densely) on $L^2(\mathcal{M}^d) = L^2(\mathbb{R}; \langle r\rangle^{d-1}dr)$ is a nonnegative self–adjoint operator and 0 is neither an eigenvalue nor a resonance. Moreover, from this spectral information we conclude $\|\vec{u}\|_{\mathcal{H}}^2 \simeq \mathcal{E}_V(\vec{u})$ along with the following Strichartz estimates (see Section 2.4 and Section 2.5 for full details of the arguments).

We say that a triple (p, q, γ) is *admissible* if

$$p > 2, q \geq 2, \quad \frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \quad \frac{1}{p} \leq \frac{d-1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big).$$

In the sequel, we use the notation for spacetime norms over $I \times \mathcal{M}^d$ via

$$\|u\|_{L^p_t L^q_x(I)} := \left(\int_I \left(\int_{\mathbb{R}} |u(t,r)|^q \langle r\rangle^{d-1} dr\right)^{p/q} dt\right)^{1/p}.$$

Proposition 3.3.1. *Let (p, q, γ) and (r, s, ρ) be admissible. Then any solution u to* (3.3.2) *satisfies*

$$\||\nabla|^{-\gamma}\nabla u\|_{L^p_t L^q_x(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \||\nabla|^{\rho} h\|_{L^{r'}_t L^{s'}_\rho(I)},$$

where r' and s' are the conjugates of r and s.

Proposition 3.3.1 with $V = 0$ was proved in Section 2.3. Using the spectral information for $-\Delta_g + V$ we can then transfer these estimates to the perturbed wave operator $\partial_t^2 - \Delta_g + V$. This is done by first reducing Proposition 3.3.1 to a pair of local energy estimates. These estimates are then established using the spectral information and a distorted Fourier basis for $-\Delta_g + V$ (the fact that V is even also plays a role in the analysis). Again, for full details see Section 2.4.

For $I \subseteq \mathbb{R}$, we denote the following spacetime norms

$$\begin{aligned}
\|u\|_{S(I)} &:= \left\| \langle r \rangle^{(d-5)/3} u \right\|_{L_t^3 L_x^6 (I)} + \|u\|_{L_t^3 L_x^{\frac{3d}{2}}(I)}, \\
\|u\|_{W(I)} &:= \|u\|_{L_t^3 \dot{W}_x^{\frac{1}{2}, \frac{6d}{3d-5}}(I)}, \\
\|h\|_{N(I)} &:= \|F\|_{L_t^1 L_x^2(I) + L_t^{3/2} \dot{W}_x^{\frac{1}{2}, \frac{6d}{3d+5}}(I)}.
\end{aligned}$$

We first use Proposition 3.3.1 to show that for any solution u to (3.3.2), we have the estimate

$$\|u\|_{S(I)} + \|u\|_{W(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{N(I)}. \tag{3.3.3}$$

Indeed, we have directly from Proposition 3.3.1

$$\|u\|_{W(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{N(I)}. \tag{3.3.4}$$

We claim that for all radial $f \in C_0^\infty(\mathcal{M}^d)$, we have

$$\left\| \langle r \rangle^{(d-5)/3} f \right\|_{L_x^6} + \|f\|_{L_x^{\frac{3d}{2}}} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2}, \frac{6d}{3d-5}}} \tag{3.3.5}$$

(we recall that the volume element is $\langle r \rangle^{d-1} dr$). Define $m_0, m_1 > 1$ by the relations

$$\begin{aligned}
\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{m_0} &= \frac{3d-5}{6d}, \\
\frac{1}{2} \cdot \frac{4}{3d} + \frac{1}{2} \cdot \frac{1}{m_1} &= \frac{3d-5}{6d},
\end{aligned} \tag{3.3.6}$$

i.e. $\frac{1}{m_0} = \frac{2d-5}{3d}$ and $\frac{1}{m_1} = \frac{3d-9}{3d}$. By the fundamental theorem of calculus

$$|f(r)| \lesssim \|f\|_{\dot{W}^{1,m_0}} \langle r \rangle^{-\frac{2}{3}(d-4)},$$
$$|f(r)| \lesssim \|f\|_{\dot{W}^{1,m_1}} \langle r \rangle^{-(d-4)}.$$

Thus, we have the embeddings

$$\begin{aligned} \left\| \langle r \rangle^{\frac{2}{3}(d-4)} f \right\|_{L^\infty_x} &\lesssim \|f\|_{\dot{W}^{1,m_0}}, \\ \left\| \langle r \rangle^{d-4} f \right\|_{L^\infty_x} &\lesssim \|f\|_{\dot{W}^{1,m_1}}. \end{aligned} \tag{3.3.7}$$

From the trivial embedding $L^3_x \hookrightarrow L^3_x$, (3.3.7), (3.3.6) and interpolation we conclude that

$$\left\| \langle r \rangle^{(d-4)/3} f \right\|_{L^6_x} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2}, \frac{6d}{3d-5}}}$$

which implies

$$\left\| \langle r \rangle^{(d-5)/3} f \right\|_{L^6_x} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2}, \frac{6d}{3d-5}}}.$$

Similarly, from the trivial embedding $L_x^{\frac{3d}{4}} \hookrightarrow L_x^{\frac{3d}{4}}$, (3.3.7), (3.3.6) and interpolation we conclude that

$$\left\| \langle r \rangle^{(d-4)/2} f \right\|_{L_x^{\frac{3d}{2}}} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2}, \frac{6d}{3d-5}}}$$

which implies

$$\|f\|_{L_x^{\frac{3d}{2}}} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2}, \frac{6d}{3d-5}}}.$$

This proves the claim. In particular, $\|u\|_{S(I)} \lesssim \|u\|_{W(I)}$ which along with (3.3.4) proves

(3.3.3). Although it may seem redundant to also use the $S(I)$ norm along with the $W(I)$ norm, it is essential in later concentration–compactness arguments to use the weaker norm $\|\cdot\|_{S(I)}$ rather than $\|\cdot\|_{W(I)}$ to measure errors.

We now use (3.3.4) to establish an a priori estimate for solutions to (3.3.1). The case $\ell = 1$, $d = 5$ was covered in Chapter 2, so we assume that $d \geq 7$. By the conservation of energy (3.2.2), the Strauss estimate (3.2.16), and Hardy's inequality (3.2.17) it is easy to show by a contraction mapping/time–stepping argument that given $(u_0, u_1) \in \mathcal{H}$, there exists a unique global solution $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H}) \cap L^\infty(\mathbb{R}, \mathcal{H})$ to (3.3.1). By the Strichartz estimate (3.3.3), we have that if u solves (3.3.1), then for any $I \subseteq \mathbb{R}$,

$$\begin{aligned}\|u\|_{S(I)} + \|u\|_{W(I)} &\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|N(\cdot, u)\|_{N(I)} \\ &\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|F(\cdot, u)\|_{N(I)} + \|G(\cdot, u\|_{N(I)},\end{aligned} \tag{3.3.8}$$

where the nonlinearities F, G are given by (3.2.9). By (3.2.13) and the relation $d = 2\ell + 3$, we may estimate

$$\|G(\cdot, u\|_{N(I)} \lesssim \left\| \langle r \rangle^{d-5} u^3 \right\|_{L^1_t L^2_x(I)} \lesssim \|u\|^3_{S(I)}. \tag{3.3.9}$$

By (3.2.11), (3.2.12) and the Strauss estimate (3.2.16) we have that

$$\begin{aligned}\|F(\cdot, u)\|_{N(I)} &\lesssim \left\| \left(\langle r \rangle^{\ell-2} \sin 2Q_{\ell,n} \right) u^2 \right\|_{L^{3/2}_t \dot{W}^{\frac{1}{2}, \frac{6d}{3d+5}}_x (I)} + \|F_0\|_{L^1_t L^2_x(I)} \\ &\lesssim \left\| \left(\langle r \rangle^{\ell-2} \sin 2Q_{\ell,n} \right) u^2 \right\|_{L^{3/2}_t \dot{W}^{\frac{1}{2}, \frac{6d}{3d+5}}_x (I)} + \|\vec{u}\|_{L^\infty_t \mathcal{H}} \|u\|^3_{S(I)}.\end{aligned} \tag{3.3.10}$$

By Proposition 3.2.1 we have

$$\begin{aligned} \langle r\rangle^{\ell-2}\sin 2Q_{\ell,n} &= O(\langle r\rangle^{-3}),\\ \frac{d}{dr}\Big(\langle r\rangle^{\ell-2}\sin 2Q_{\ell,n}\Big) &= O(\langle r\rangle^{-4}), \end{aligned}$$

so that

$$\Big\|\Big(\langle r\rangle^{\ell-2}\sin 2Q_{\ell,n}\Big)\Big\|_{L^d_x\cap \dot W^{\frac{1}{2},d}_x} < \infty \tag{3.3.11}$$

by interpolation. By the Leibniz rule for Sobolev spaces (see [5] for asymptotically conic manifolds) and (3.3.11), we conclude that

$$\begin{aligned} \Big\|\Big(\langle r\rangle^{\ell-2}\sin 2Q_{\ell,n}\Big)u^2\Big\|_{\dot W_x^{\frac{1}{2},\frac{6d}{3d+5}}} &\lesssim \Big\|\Big(\langle r\rangle^{\ell-2}\sin 2Q_{\ell,n}\Big)\Big\|_{\dot W_x^{\frac{1}{2},d}}\|u^2\|_{L_x^{\frac{6d}{3d-1}}}\\ &\quad + \Big\|\Big(\langle r\rangle^{\ell-2}\sin 2Q_{\ell,n}\Big)\Big\|_{L^d_x}\|u\|_{L_x^{\frac{3d}{2}}}\|u\|_{\dot W_x^{\frac{1}{2},\frac{6d}{3d-5}}}\\ &\lesssim \|u\|^2_{L_x^{\frac{12d}{3d-1}}} + \|u\|_{L_x^{\frac{3d}{2}}}\|u\|_{\dot W_x^{\frac{1}{2},\frac{6d}{3d-5}}}. \end{aligned}$$

By Hölder's inequality and the fact that $d\geq 7$,

$$\begin{aligned} \Big(\int_{\mathbb{R}}|u|^{\frac{12d}{3d-1}}\langle r\rangle^{d-1}\Big)^{\frac{3d-1}{12d}} &\leq \Big(\int_{\mathbb{R}}\Big|\langle r\rangle^{\frac{d-5}{3}}u\Big|^6\langle r\rangle^{d-1}dr\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}}\langle r\rangle^{\frac{20d-4d^2}{d-1}}\langle r\rangle^{d-1}dr\Big)^{\frac{d-1}{3d-1}}\\ &\lesssim \Big(\int_{\mathbb{R}}\Big|\langle r\rangle^{\frac{d-5}{3}}u\Big|^6\langle r\rangle^{d-1}dr\Big)^{\frac{1}{6}}. \end{aligned}$$

Thus,

$$\Big\|\Big(\langle r\rangle^{\ell-2}\sin 2Q_{\ell,n}\Big)u^2\Big\|_{\dot W_x^{\frac{1}{2},\frac{6d}{3d+5}}} \lesssim \Big\|\langle r\rangle^{\frac{d-5}{3}}u\Big\|^2_{L^6_x} + \|u\|_{L_x^{\frac{3d}{2}}}\|u\|_{\dot W_x^{\frac{1}{2},\frac{6d}{3d-5}}}$$

so that by Hölder's inequality in time

$$\left\| \left(\langle r \rangle^{\ell-2} \sin 2Q_{\ell,n} \right) u^2 \right\|_{L_t^{3/2} \dot{W}_x^{\frac{1}{2}, \frac{6d}{3d+5}}(I)} \lesssim \|u\|_{S(I)}^2 + \|u\|_{S(I)} \|u\|_{W(I)}. \tag{3.3.12}$$

Combining (3.3.12) with (3.3.10) we obtain

$$\|F(\cdot, u)\|_{N(I)} \lesssim \|u\|_{S(I)}^2 + \|u\|_{S(I)} \|u\|_{W(I)} + \|\vec{u}\|_{L_t^\infty \mathcal{H}} \|u\|_{S(I)}^3. \tag{3.3.13}$$

The estimates (3.3.8), (3.3.9), and (3.3.13) imply the following a priori estimate for u:

$$\|u\|_{S(I)} + \|u\|_{W(I)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|u\|_{S(I)}^2 + \|u\|_{S(I)} \|u\|_{W(I)} + \|\vec{u}\|_{L_t^\infty \mathcal{H}} \|u\|_{S(I)}^3 + \|u\|_{S(I)}^3. \tag{3.3.14}$$

Based on (3.3.14) and continuity arguments we have the following small data theory and long–time perturbation theory for (3.3.1). For full details, see the proofs of Proposition 2.5.1 and Proposition 2.5.2.

Proposition 3.3.2. *For every* $(u_0, u_1) \in \mathcal{H}$*, there exists a unique global solution* u *to* (3.3.1) *such that* $\vec{u}(t) \in C(\mathbb{R}; \mathcal{H}) \cap L^\infty(\mathbb{R}; \mathcal{H})$*. A solution* u *scatters to a free wave on* $\mathcal{M}^d$ *as* $t \to \infty$*, i.e. there exists a solution* v_L *to*

$$\partial_t^2 v - \partial_r^2 v - \frac{(d-1)r}{r^2+1} \partial_r v = 0, \quad (t, r) \in \mathbb{R} \times \mathbb{R},$$

such that

$$\lim_{t \to \infty} \|\vec{u}(t) - \vec{v}_L^{\pm}(t)\|_{\mathcal{H}} = 0,$$

if and only if

$$\|u\|_{S(0,\infty)} < \infty.$$

A similar characterization of u scattering to a free wave on $\mathcal{M}^d$ as $t \to -\infty$ also holds. Moreover, there exists $\delta > 0$ such that if $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, then

$$\|\vec{u}\|_{L_t^\infty \mathcal{H}} + \|u\|_{S(\mathbb{R})} + \|u\|_{W(\mathbb{R})} \lesssim \|\vec{u}(0)\|_{\mathcal{H}}.$$

Proposition 3.3.3 (Long–time perturbation theory)**.** *Let $A > 0$. Then there exists $\epsilon_0 = \epsilon_0(A) > 0$ and $C = C(A) > 0$ such that the following holds. Let $0 < \epsilon < \epsilon_0$, $(u_0, u_1) \in \mathcal{H}$, and $I \subseteq \mathbb{R}$ with $0 \in I$. Assume that $\vec{U}(t) \in C(I; \mathcal{H})$ satisfies on I*

$$\partial_t^2 U - \Delta_g U + VU = N(\cdot, U) + e,$$

such that

$$\begin{aligned} \sup_{t \in I} \|\vec{U}(t)\|_{\mathcal{H}} + \|U\|_{S(I)} &\leq A, \\ \|\vec{U}(0) - (u_0, u_1)\|_{\mathcal{H}} + \|e\|_{N(I)} &\leq \epsilon. \end{aligned} \tag{3.3.15}$$

Then the unique global solution u to (3.3.1) *with initial data $\vec{u}(0) = (u_0, u_1)$ satisfies*

$$\sup_{t \in I} \|\vec{u}(t) - \vec{U}(t)\|_{\mathcal{H}} + \|u - U\|_{S(I)} \leq C(A)\epsilon.$$

3.3.2 Concentration–Compactness

In this subsection we complete the second step of the concentration–compactness/rigidity method outlined in the beginning of this section. A crucial tool used in completing this step is the following linear *profile decomposition* of a bounded sequence in $\mathcal{H}$.

Lemma 3.3.4 (Linear Profile Decomposition). *Let $\{(u_{0,n}, u_{1,n})\}_n$ be a bounded sequence in $\mathcal{H}$. Then after extraction of subsequences and relabeling, there exist a sequence of solutions $\left\{U_L^j\right\}_{j\geq 1}$ to (3.3.2) with $h \equiv 0$ which are bounded in $\mathcal{H}$ and a sequence of times $\{t_{j,n}\}_n$ for $j \geq 1$ that satisfy the orthogonality condition*

$$\forall j \neq k, \quad \lim_{n\to\infty} |t_{j,n} - t_{k,n}| = \infty,$$

such that for all $J \geq 1$,

$$(u_{0,n}, u_{1,n}) = \sum_{j=1}^{J} \vec{U}_L^j(-t_{j,n}) + (w_{0,n}^J, w_{1,n}^J),$$

where the error $w_n^J(t) := S_V(t)(w_{0,n}^J, w_{1,n}^J)$ satisfies

$$\lim_{J\to\infty} \overline{\lim_{n\to\infty}} \|w_n^J\|_{L_t^\infty L_x^r(\mathbb{R})\cap S(\mathbb{R})} = 0, \quad \forall \, \frac{2d}{d-2} < r < \infty. \tag{3.3.16}$$

Moreover, we have the following Pythagorean expansion of the energy

$$\mathcal{E}_V(\vec{u}_n) = \sum_{j=1}^{J} \mathcal{E}_V(\vec{U}_L^j) + \mathcal{E}_V(\vec{w}_n^J) + o(1), \tag{3.3.17}$$

as $n \to \infty$.

The proof is exactly the same as the corototational case which follows from the proof of Lemma 3.2 in [20]. However, we will explain why the error w_n^J satisfies (3.3.16) since the reasoning is subtle. The $d = 5$ case is contained in Chapter 2, so we assume that $d \geq 7$. The proof from Lemma 3.2 in [20] shows that we have

$$\lim_{J\to\infty} \overline{\lim_{n\to\infty}} \|w_n^J\|_{L_t^\infty L_x^r(\mathbb{R})} = 0, \quad \forall \, \frac{2d}{d-2} < r < \infty, \tag{3.3.18}$$

as well as

$$\overline{\lim_{J\to\infty}}\,\overline{\lim_{n\to\infty}}\,\|\vec{w}_n^J\|_{\mathcal{H}} < \infty. \tag{3.3.19}$$

We recall that in proving (3.3.5), we in fact proved the stronger claim that

$$\left\|\langle r\rangle^{(d-4)/2} f\right\|_{L_x^{\frac{3d}{2}}} + \left\|\langle r\rangle^{(d-4)/3} f\right\|_{L_x^6} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2},\frac{6d}{3d-5}}}. \tag{3.3.20}$$

We also observe that the admissible triple $\left(\frac{1}{2}, 3, \frac{6d}{3d-5}\right)$ is not sharp if $d \geq 7$, i.e.

$$\frac{1}{3} < \frac{d-1}{2}\Big(\frac{1}{2} - \frac{3d-5}{6d}\Big). \tag{3.3.21}$$

The two observations (3.3.20) and (3.3.21) and continuity imply the following. Let $r > \frac{2d}{d-2}$ and $0 < \theta < 1$, and define a triple (p, q, γ) and exponent s by

$$\begin{aligned}
\gamma &= \frac{1}{\theta}\frac{1}{2},\\
\frac{1}{p} &= \frac{1}{\theta}\frac{1}{3},\\
\frac{1}{p} + \frac{d}{q} &= \frac{d}{2} - \gamma,\\
\frac{1}{s} &= \theta\frac{1}{q} + (1-\theta)\frac{1}{r}.
\end{aligned} \tag{3.3.22}$$

Then as long as r is sufficiently large and θ is sufficiently close to 1, we have that (p, q, γ) is admissible, $s > 1$, and

$$\|f\|_{L_x^{\frac{3d}{2}}} + \left\|\langle r\rangle^{(d-5)/3} f\right\|_{L_x^6} \lesssim \|f\|_{\dot{W}_x^{\frac{1}{2},s}}, \quad \forall f \in C_0^\infty. \tag{3.3.23}$$

Indeed, the fact that (p, q, γ) defined by (3.3.22) are admissible for r large and θ close to 1 follows from (3.3.21) and continuity in θ. Similarly, by (3.3.20) if r is large and θ is close to

1, then we can find $m_0 = m_0(\theta)$ and $m_1 = m_1(\theta)$, analogous to m_0, m_1 from (3.3.6), so that

$$\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{m_0} = \frac{1}{s},$$
$$\frac{1}{2} \cdot \frac{4}{3d} + \frac{1}{2} \cdot \frac{1}{m_1} = \frac{1}{s},$$

with

$$|f(r)| \lesssim \|f\|_{\dot{W}^{1,m_0}} \langle r \rangle^{-\alpha},$$
$$|f(r)| \lesssim \|f\|_{\dot{W}^{1,m_1}} \langle r \rangle^{-\beta},$$

where $\alpha > \frac{2(d-5)}{3}$ and $\beta > 0$. By interpolation we conclude (3.3.23). We now fix r sufficiently large and θ sufficiently close to 1 so that if (p, q, γ) and s are defined as in (3.3.22), then (p, q, γ) is an admissible triple and (3.3.23) holds. Then by (3.3.23), interpolation, and Strichartz estimates we have that the errors satisfy

$$\begin{aligned} \|w_n^J\|_{S(\mathbb{R})} &\lesssim \|w_n^J\|_{L_t^3 \dot{W}_x^{\frac{1}{2},s}(\mathbb{R})} \\ &\lesssim \|w_n^J\|^{\theta}_{L_t^p \dot{W}_x^{\gamma,q}(\mathbb{R})} \|w_n^J\|^{1-\theta}_{L_t^\infty L_x^r(\mathbb{R})} \\ &\lesssim \|w_n^J\|^{\theta}_{\mathcal{H}} \|w_n^J\|^{1-\theta}_{L_t^\infty L_x^r(\mathbb{R})} \end{aligned}$$

whence by (3.3.18) and (3.3.19)

$$\lim_{J\to\infty} \overline{\lim_{n\to\infty}} \|w_n^J\|_{S(\mathbb{R})} = 0$$

as desired. We remark here that is unclear whether or not the errors satisfy the stronger condition $\lim_J \overline{\lim}_n \|w_n^J\|_{W(\mathbb{R})} = 0$. It is for this reason that we used the weaker $S(I)$ norm in the previous subsection.

Using Lemma 3.3.4 and Proposition 3.3.3, we establish that if our main result, Theorem

3.2.3, fails, then there exists a nonzero 'critical element.' In particular, we establish the following.

Proposition 3.3.5. *Suppose that Theorem 3.2.3 fails. Then there exists a nonzero global solution u_* to* (3.3.1) *such that the set*

$$K = \{\vec{u}_*(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H}$.

The proof of Proposition 3.3.5 is the same as in the corototational case; see the proof of Proposition 2.5.3 for full details. We remark that proving Proposition 3.3.5 uses the nonlinear perturbation theory, Proposition 3.3.3, applied to the linear profile decompositions provided by Lemma 3.3.4. What makes this possible is that the perturbation theory is established with certain errors measured in the weaker norm $\| \cdot \|_{S(\mathbb{R})}$ (see (3.3.15)) and the errors w_n^J in the linear profile decomposition satisfy $\lim_J \overline{\lim}_n \|w_n^J\|_{S(\mathbb{R})} = 0$ (but possibly not $\lim_J \overline{\lim}_n \|w_n^J\|_{W(\mathbb{R})} = 0$).

3.4 Rigidity Theorem

In this section we prove that the critical element from Proposition 3.3.5 does not exist and conclude the proof of our main result Theorem 3.2.3 (equivalently Theorem 3.1.1). The main result of this section is the following.

Proposition 3.4.1. *Let u be a global solution of* (3.3.1) *such that the trajectory*

$$K = \{\vec{u}(t) : t \in \mathbb{R}\}$$

is precompact in $\mathcal{H} := \mathcal{H}(\mathbb{R}; \langle r \rangle^{d-1} dr)$. Then $\vec{u} = (0, 0)$.

We first note that for a solution u as in Proposition 3.4.1, we have the following uniform control of the energy in exterior regions.

Lemma 3.4.2. *Let u be as in Proposition 3.4.1. Then we have*

$$\begin{aligned} \forall R \geq 0, \quad \lim_{|t|\to\infty} \|\vec{u}(t)\|_{\mathcal{H}(|r|\geq R+|t|;\langle r\rangle^{d-1}dr)} &= 0, \\ \lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}(t)\|_{\mathcal{H}(|r|\geq R+|t|;\langle r\rangle^{d-1}dr)}\right] &= 0. \end{aligned} \tag{3.4.1}$$

To prove that $\vec{u} = (0,0)$, we proceed as in the corotational case and show that u is a finite energy static solution to (3.3.1).

Proposition 3.4.3. *Let u be as in Proposition 3.4.1. Then there exists a static solution U to* (3.3.1) *such that $\vec{u} = (U, 0)$.*

We will first show that $\vec{u}$ is equal to static solutions $(U_\pm, 0)$ on $\pm r > 0$ separately. The proof for $r < 0$ is identical to the proof for $r > 0$ so we will only consider the case $r > 0$. The major part of this section is devoted to proving the following.

Proposition 3.4.4. *Let u be as in Proposition 3.4.1. Then there exists a static solution $(U_+, 0)$ such that $\vec{u}(t,r) = (U_+(r), 0)$ for all $t \in \mathbb{R}$ and $r > 0$.*

3.4.1 Proof of Proposition 3.4.4

Let $\eta > 0$ be arbitrary, and let u be as in Proposition 3.4.1. As in Chapter 2, we will show that $\vec{u}(t,r)$ is equal to a static solution $(U_+(r), 0)$ to (3.3.1) on $\{t \in \mathbb{R}, r \in (\eta, \infty)\}$. We now introduce a function related to u that will be integral in the proof. Define

$$u_e(t,r) := \frac{\langle r\rangle^{(d-1)/2}}{r^{(d-1)/2}} u(t,r), \quad (t,r) \in \mathbb{R} \times (0,\infty).$$

If u solves (3.3.1) then u_e solves the following radial semilinear wave equation on $\mathbb{R}^{1+d}$

$$\partial_t^2 u_e - \partial_r^2 u_e - \frac{d-1}{r}\partial_r u_e + V_e(r) u_e = N_e(r, u_e), \quad (t, r) \in \mathbb{R} \times (0, \infty), \tag{3.4.2}$$

where

$$V_e(r) = V(r) - \frac{(d-1)(d-4)}{2} r^{-2} \langle r \rangle^{-2} + \frac{(d-1)(d-5)}{4} r^{-2} \langle r \rangle^{-4}, \tag{3.4.3}$$

and $N_e(r, u_e) = F_e(r, u_e) + G_e(r, u_e)$ with

$$F_e(r, u_e) = \frac{\langle r \rangle^{(d-1)/2}}{r^{(d-1)/2}} F\left(r, \frac{r^{(d-1)/2}}{\langle r \rangle^{(d-1)/2}} u_e\right), \tag{3.4.4}$$

$$G_e(r, u_e) = \frac{\langle r \rangle^{(d-1)/2}}{r^{(d-1)/2}} G\left(r, \frac{r^{(d-1)/2}}{\langle r \rangle^{(d-1)/2}} u_e\right), \tag{3.4.5}$$

where F and G are given in (3.2.9). Note that for all $R > 0$, we have

$$\|\vec{u}_e(t)\|_{\mathcal{H}(r \geq R; r^{d-1} dr)} \leq C(R) \|\vec{u}\|_{\mathcal{H}(r \geq R; \langle r \rangle^{d-1} dr)}, \tag{3.4.6}$$

so that by Lemma 3.4.2, u_e inherits the compactness properties

$$\begin{aligned} \forall R > 0, \quad & \lim_{|t| \to \infty} \|\vec{u}_e(t)\|_{\mathcal{H}(r \geq R + |t|; r^{d-1} dr)} = 0, \\ & \lim_{R \to \infty} \left[\sup_{t \in \mathbb{R}} \|\vec{u}_e(t)\|_{\mathcal{H}(r \geq R + |t|; r^{d-1} dr)} \right] = 0. \end{aligned} \tag{3.4.7}$$

We also note that due to (3.2.10)–(3.2.13) and the definition of V_e, F_e, and G_e, we have for all $r > 0$,

$$|V_e(r)| \lesssim r^{-4}, \tag{3.4.8}$$

$$|F_e(u_e, r)| \lesssim r^{-3}|u_e|^2, \tag{3.4.9}$$

$$|G_e(u_e, r)| \lesssim r^{d-5}|u_e|^3, \tag{3.4.10}$$

where the implied constants depend on $Q_{\ell,n}$ and d.

To prove Proposition 3.4.4, we use channels of energy arguments that originate in the seminal work [7] on the $3d$ energy–critical wave equation. These arguments have since been used in the study of equivariant exterior wave maps [16] [18] and in the proof of the corotational case of Theorem 3.1.1 in Chapter 2. The arguments of this section are derived from those in [18]. The proof is split into three main steps. In the first two steps, we determine the precise asymptotics of

$$(u_{e,0}(r), u_{e,1}(r)) := (u_e(0, r), \partial_t u_e(0, r)) \quad \text{as } r \to \infty.$$

In particular, we show that there exists $\alpha \in \mathbb{R}$ such that

$$r^{d-2} u_{e,0}(r) = \alpha + O(r^{-2}), \tag{3.4.11}$$

$$\int_r^\infty u_{e,1}(\rho)\rho^{2j-1} d\rho = O(r^{2j-d-1}), \quad j = 1, \dots, \left\lfloor \frac{d}{4} \right\rfloor, \tag{3.4.12}$$

as $r \to \infty$. In the final step, we use this information and channels of energy arguments to conclude the proof of Proposition 3.4.4. In the remainder of this subsection we denote $\mathcal{H}(r \geq R) := \mathcal{H}(r \geq R; r^{d-1}dr)$.

As in the study of corotational wave maps on a wormhole, the key tool used in establishing (3.4.11) and (3.4.12) is the following exterior energy estimate for radial free waves on

Minkowski space $\mathbb{R}^{1+d}$ with d odd. The case $d = 5$ used for corotational wave maps on a wormhole and exterior wave maps was proved in [16], and the general case of $d \geq 3$ and odd was proven in [17].

Proposition 3.4.5 (Theorem 2, [17])**.** *Let $d \geq 3$ be odd. Let v be a radial solution to the free wave equation in $\mathbb{R}^{1+d}$*

$$\begin{aligned} &\partial_t^2 v - \Delta v = 0, \quad (t,x) \in \mathbb{R}^{1+d}, \\ &\vec{v}(0) = (f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^d). \end{aligned}$$

Then for every $R > 0$,

$$\max_{\pm} \inf_{\pm t \geq 0} \int_{r \geq R+|t|} |\nabla_{t,r} v(t,r)|^2 r^{d-1} dr \geq \frac{1}{2} \|\pi_R^{\perp}(f,g)\|_{\mathcal{H}(r \geq R)}, \tag{3.4.13}$$

where $\pi_R = I - \pi_R^{\perp}$ is the orthogonal projection onto the plane

$$P(R) = span\Big\{(r^{2i-d}, 0), (0, r^{2j-d}) : i = 1, \dots, \Big\lfloor \frac{d+2}{4} \Big\rfloor, j = 1, \dots, \Big\lfloor \frac{d}{4} \Big\rfloor \Big\}$$

in $\mathcal{H}(r \geq R)$. The left–hand side of (3.4.13) *is identically 0 for data satisfying $(f,g)|_{r \geq R} \in P(R)$*

We remark here that Proposition 3.4.5 states, quantitatively, that generic solutions to the free wave equation on $\mathbb{R}^{1+d}$ with d odd emit a fixed amount of energy into regions exterior to light cones. However, this property fails in the case $R = 0$ for general data (f,g) in even dimensions (see [6]).

In the remainder of this subsection, we denote

$$\tilde{k} := \Big\lfloor \frac{d+2}{4} \Big\rfloor, \quad k := \Big\lfloor \frac{d}{4} \Big\rfloor.$$

For $R \geq 1$, we define the projection coefficients $\lambda_i(t,R), \mu_j(t,R)$ for $i = 1, \ldots, \tilde{k}$, $j = 1, \ldots, k$, via

$$\pi_R \vec{u}_e(t,r) = \left(u_e(t,r) - \sum_{i=1}^{\tilde{k}} \lambda_i(t,R) r^{2i-d}, \partial_t u_e(t,r) - \sum_{j=1}^{k} \mu_j(t,R) r^{2j-d} \right). \qquad (3.4.14)$$

We now give identities relating u_e to the coefficients $\lambda_j(t,r), \mu_i(t,r)$ and an equivalent way of expressing the relative size of $\|\pi_R \vec{u}_e(t)\|_{\mathcal{H}(r\geq R)}$ and $\|\pi_R^{\perp} \vec{u}_e(t)\|_{\mathcal{H}(r\geq R)}$ using projection coefficients.

Lemma 3.4.6 (Lemma 4.5, Lemma 5.10, [18])**.** *For each fixed $R > 0$, and $(t,r) \in \{r \geq R + |t|\}$, we have the following identities:*

$$\begin{aligned}
u_e(t,r) &= \sum_{j=1}^{\tilde{k}} \lambda_j(t,r) r^{2j-d}, \\
\int_r^\infty \partial_t u_e(t,\rho)\rho^{2i-1} d\rho &= \sum_{j=1}^{k} \mu_j(t,r) \frac{r^{2i+2j-d}}{d-2i-2j}, \quad \forall 1 \leq i \leq k, \\
\mu_j(t,r) &= \sum_{i=1}^{k} r^{d-2i-2j} \frac{c_i c_j}{d-2i-2j} \int_r^\infty \partial_t u_e(t,\rho)\rho^{2i-1} d\rho, \quad \forall 1 \leq i \leq k, \\
\lambda_j(t,r) &= \frac{d_j}{d-2j} \Big(u_e(t,r) r^{d-2j} \\
&\qquad + \sum_{i=1}^{\tilde{k}-1} \frac{(2i) d_{i+1} r^{d-2i-2j}}{d-2i-2j} \int_r^\infty u_e(t,\rho)\rho^{2i-1} d\rho \Big),
\end{aligned}$$

where the last identity holds for all $j \leq \tilde{k}$ and

$$\begin{aligned}
c_j &:= \frac{\Pi_{1\leq l\leq k}(d-2j-2l)}{\Pi_{1\leq l\leq k, l\neq j}(2l-2j)}, \quad 1 \leq j \leq k, \\
d_j &:= \frac{\Pi_{1\leq l\leq \tilde{k}}(d+2-2j-2l)}{\Pi_{1\leq l\leq \tilde{k}, l\neq j}(2l-2j)}, \quad 1 \leq j \leq \tilde{k}.
\end{aligned}$$

Also, the following estimates hold

$$\|\pi_R\vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} \simeq \sum_{i=1}^{\tilde{k}}\Big(\lambda_i(t,R)R^{2i-\frac{d+2}{2}}\Big)^2 + \sum_{j=1}^{k}\Big(\mu_j(t,R)R^{2j-\frac{d}{2}}\Big)^2,$$
$$\|\pi_R^{\perp}\vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} \simeq \int_R^\infty \sum_{i=1}^{\tilde{k}}\Big(\partial_r\lambda_i(t,r)r^{2i-\frac{d+1}{2}}\Big)^2 + \sum_{j=1}^{k}\Big(\partial_r\mu_j(t,r)r^{2j-\frac{d-1}{2}}\Big)^2 dr,$$

where the implied constants depend only on d.

We now proceed to the first step in proving Proposition 3.4.4.

Step 1: Decay rate for $\pi_R^{\perp}\vec{u}_e(t)$ in $\mathcal{H}(r \geq R)$

In this step we establish the following decay estimate for $\pi_R^{\perp}\vec{u}_e(t)$.

Lemma 3.4.7. *There exists $R_0 > 1$ such that for all $R \geq R_0$ and for all $t \in \mathbb{R}$ we have*

$$\begin{aligned}\|\pi_R^{\perp}\vec{u}_e(t)\|_{\mathcal{H}(r\geq R)} \lesssim R^{-2}\|\pi_R\vec{u}_e(t)\|_{\mathcal{H}(r\geq R)} + R^{-d/2}\|\pi_R\vec{u}_e(t)\|^2_{\mathcal{H}(r\geq R)} \\ + R^{-1}\|\pi_R\vec{u}_e(t)\|^3_{\mathcal{H}(r\geq R)}.\end{aligned} \tag{3.4.15}$$

Since we are only interested in the behavior of $\vec{u}_e(t,r)$ in exterior regions $\{r \geq R + |t|\}$, we first consider a modified Cauchy problem. In particular, we can, by finite speed of propagation, alter V_e, F_e, and G_e appearing in (3.4.2) in the interior region $\{r \leq R + |t|\}$ without affecting the behavior of $\vec{u}_e$ on the exterior region $\{r \geq R + |t|\}$.

Definition 3.4.8. *Let $R \geq \eta$. For a function $f = f(r,u) : [\eta,\infty)\times\mathbb{R} \to \mathbb{R}$, we define*

$$f_R(t,r,u) := \begin{cases} f(R+|t|,u) & \text{if } \eta \leq r \leq R+|t| \\ f(r,u) & \text{if } r \geq R+|t| \end{cases}, \quad (t,r,u) \in \mathbb{R}\times[\eta,\infty)\times\mathbb{R}.$$

We now consider solutions to a modified version of (3.4.2):

$$\begin{aligned}
&\partial_t^2 h - \partial_r^2 h - \frac{d-1}{r}\partial_r h = N_R(t,r,h), \quad (t,r) \in \mathbb{R} \times (\mathbb{R}\backslash B(0,\eta)), \\
&\vec{h}(0) = (h_0, h_1) \in \mathcal{H}_0(r \geq \eta),
\end{aligned} \tag{3.4.16}$$

where $\mathcal{H}_0(r \geq \eta) = \{(h_0, h_1) \in \mathcal{H}(r \geq \eta) : h_0(\eta) = 0\}$ and

$$N_R(t,r,h) = -V_{e,R}(t,r)h + F_{e,R}(t,r,h) + G_{e,R}(t,r,h).$$

We note that from Definition 3.4.8 and (3.4.8), (3.4.9), and (3.4.10), we have

$$|V_{e,R}(t,r)| \lesssim \begin{cases} (R+|t|)^{-4} & \text{if } \eta \leq r \leq R+|t|, \\ r^{-4} & \text{if } r \geq R+|t|, \end{cases} \tag{3.4.17}$$

$$|F_{e,R}(t,r,h)| \lesssim \begin{cases} (R+|t|)^{-3}|h|^2 & \text{if } \eta \leq r \leq R+|t|, \\ r^{-3}|h|^3 & \text{if } r \geq R+|t|, \end{cases} \tag{3.4.18}$$

$$|G_{e,R}(t,r,h)| \lesssim \begin{cases} (R+|t|)^{d-5}|h|^3 & \text{if } \eta \leq r \leq R+|t|, \\ r^{d-5}|h|^3 & \text{if } r \geq R+|t|. \end{cases} \tag{3.4.19}$$

Lemma 3.4.9. *There exist $R_0 > 0$ large and $\delta_0 > 0$ small such that for all $R \geq R_0$ and all $(h_0, h_1) \in \mathcal{H}_0(r \geq \eta)$ with*

$$\|(h_0, h_1)\|_{\mathcal{H}(r \geq \eta)} \leq \delta_0,$$

there exists a unique globally defined solution h to (3.4.16) *such that*

$$\left\| r^{(d-4)/3} h \right\|_{L_t^3 L_x^6(\mathbb{R} \times (\mathbb{R}^d \backslash B(0,\eta)))} \lesssim \|\vec{h}(0)\|_{\mathcal{H}(r \geq \eta)}. \tag{3.4.20}$$

Moreover, if we define h_L to be the solution to the free equation $\partial_t^2 h_L - \Delta h_L = 0$, $(t,x) \in \mathbb{R} \times (\mathbb{R}^d \backslash B(0,\eta))$, $\vec{h}_L(0) = (h_0, h_1)$, then

$$\sup_{t\in\mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}(r\geq\eta)} \lesssim R^{-2}\|\vec{h}(0)\|_{\mathcal{H}(r\geq\eta)} + R^{-d/2}\|\vec{h}(0)\|^2_{\mathcal{H}(r\geq\eta)} + R^{-1}\|\vec{h}(0)\|^3_{\mathcal{H}(r\geq\eta)}. \tag{3.4.21}$$

Proof. For the proof, we use the shorthand notation $\mathbb{R}^d_* = \mathbb{R}^d \backslash B(0,\eta)$. The small data global well–posedness and spacetime estimate (3.4.20) follow from standard contraction mapping and continuity arguments using the following Strichartz estimate: if h is a radial solution to $\partial_t^2 h - \Delta h = F$ on $\mathbb{R} \times \mathbb{R}^d_*$ with $h(t,\eta) = 0$, $\forall t$, then

$$\left\| r^{(d-4)/3} h \right\|_{L^3_t L^6_x(\mathbb{R}\times\mathbb{R}^d_*)} \lesssim \|\vec{h}(0)\|_{\mathcal{H}(r\geq\eta)} + \|F\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)}.$$

This estimate follows from [12] and an argument similar to the one used to establish (3.3.3). Rather than give the details for proving (3.4.20), we prove (3.4.21) since the argument is similar. By the Duhamel formula and Strichartz estimates we have

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}(r\geq\eta)} &\lesssim \|N_R(\cdot,\cdot,h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)} \\ &\lesssim \|V_{e,R} h\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)} + \|F_{e,R}(\cdot,\cdot,h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)} \\ &\quad + \|G_{e,R}(\cdot,\cdot,h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)}.\end{aligned}$$

The third term is readily estimated by (3.4.19) and (3.4.20)

$$\|G_{e,R}(\cdot,\cdot,h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)} \lesssim R^{-1}\|r^{d-4}h^3\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d_*)} \lesssim R^{-1}\|\vec{h}(0)\|^3_{\mathcal{H}(r\geq\eta)}.$$

For the first term we have

$$\|V_{e,R}h\|_{L^1_tL^2_x(\mathbb{R}\times\mathbb{R}^d_*)} \leq \left\| r^{-(d-4)/3}V_{e,R}\right\|_{L^{3/2}_tL^3_x(\mathbb{R}\times\mathbb{R}^d_*)} \left\| r^{(d-4)/3}h\right\|_{L^3_tL^6_x(\mathbb{R}\times\mathbb{R}^d_*)}.$$

By (3.4.17)

$$\left\| r^{-(d-4)/3}V_{e,R}\right\|_{L^{3/2}_tL^3_x(\mathbb{R}\times\mathbb{R}^d_*)} \lesssim R^{-2}.$$

Thus, by (3.4.20)

$$\left\| r^{-(d-4)/3}V_{e,R}\right\|_{L^{3/2}_tL^3_x(\mathbb{R}\times\mathbb{R}^d_*)} \left\| r^{(d-4)/3}h\right\|_{L^3_tL^6_x(\mathbb{R}\times\mathbb{R}^d_*)} \lesssim R^{-2}\|\vec{h}(0)\|_{\mathcal{H}(r\geq\eta)}.$$

Similarly, using (3.4.18), (3.4.20) and the Strauss estimate valid for all radial $f \in C^\infty_0(\mathbb{R}^d_*)$

$$|f(r)| \lesssim r^{\frac{2-d}{2}}\|\nabla f\|_{L^2(\mathbb{R}^d_*)},$$

we conclude that $\|F_{e,R}(\cdot,\cdot,h)\|_{L^1_tL^2_x(\mathbb{R}\times\mathbb{R}^d_*)} \lesssim R^{-d/2}\|h(0)\|^2_{\mathcal{H}(r\geq\eta)}$ which proves (3.4.21). □

Proof of Lemma 3.4.7. We first prove Lemma 3.4.7 for $t = 0$. For $R > \eta$, define the truncated initial data $\vec{u}_R(0) = (u_{0,R}, u_{1,R}) \in \mathcal{H}_0(r \geq \eta)$ via

$$u_{0,R}(r) = \begin{cases} u_e(0,r) & \text{if } r \geq R, \\ \frac{r-\eta}{R-\eta}u_e(0,R) & \text{if } r < R, \end{cases} \tag{3.4.22}$$

$$u_{1,R}(r) = \begin{cases} \partial_t u_e(0,r) & \text{if } r \geq R, \\ 0 & \text{if } r < R. \end{cases} \tag{3.4.23}$$

Note that for R large,

$$\|\vec{u}_R(0)\|_{\mathcal{H}(r\geq\eta)} \lesssim \|\vec{u}_e(0)\|_{\mathcal{H}(r\geq R)}. \tag{3.4.24}$$

In particular, by (3.4.7) there exists $R_0 \geq 1$ such that for all $R \geq R_0$, $\|\vec{u}_R(0)\|_{\mathcal{H}(r\geq\eta)} \leq \delta_0$ where δ_0 is from Lemma 3.4.9. Let $u_R(t)$ be the solution to (3.4.16) with initial data $(u_{0,R}, u_{1,R})$, and let $\vec{u}_{R,L}(t) \in \mathcal{H}_0(r \geq \eta)$ be the solution to the free wave equation $\partial_t^2 u_{R,L} - \Delta u_{R,L} = 0$, $(t,x) \in \mathbb{R} \times \mathbb{R}^d_*$, $\vec{u}_{R,L}(0) = (u_{0,R}, u_{1,R})$. By finite speed of propagation

$$r \geq R + |t| \implies \vec{u}_R(t,r) = \vec{u}_e(t,r).$$

By Proposition 3.4.5, for all $t \geq 0$ or for all $t \leq 0$,

$$\|\pi_R^\perp \vec{u}_{R,L}(0)\|_{\mathcal{H}(r\geq R)} \lesssim \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)}.$$

Suppose, without loss of generality, that the above bound holds for all $t \geq 0$. By (3.4.21) we conclude that for all $t \geq 0$

$$\begin{aligned}\|\vec{u}_e(t)\|_{\mathcal{H}(r\geq R+|t|)} &\geq \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)} - \|\vec{u}_R(t) - \vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq\eta)} \\ &\geq c\|\pi_R^\perp \vec{u}_{R,L}(0)\|_{\mathcal{H}(r\geq R)} \\ &\quad - C\Big[R^{-2}\|u_R(0)\|_{\mathcal{H}(r\geq\eta)} + R^{-d/2}\|\vec{u}_R(0)\|^2_{\mathcal{H}(r\geq\eta)} \\ &\qquad + R^{-1}\|\vec{u}_R(0)\|^3_{\mathcal{H}(r\geq\eta)}\Big].\end{aligned}$$

Letting $t \to \infty$ and using the decay property (3.4.7) and the definition of $(u_{0,R}, u_{1,R})$, we conclude that

$$\|\pi_R^\perp \vec{u}_e(0)\|_{\mathcal{H}(r\geq R)} \lesssim R^{-2}\|u_e(0)\|_{\mathcal{H}(r\geq R)} + R^{-d/2}\|\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} + R^{-1}\|\vec{u}_e(0)\|^3_{\mathcal{H}(r\geq R)}.$$

Note that $\|\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} = \|\pi_R^\perp \vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R \vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)}$. Thus, if we take R_0 large enough to absorb terms involving $\|\pi_R^\perp \vec{u}_e(0)\|_{\mathcal{H}(r\geq R)}$ into the left hand side in the previous estimate, we obtain for all $R \geq R_0$

$$\begin{aligned}\|\pi_R^\perp \vec{u}_e(0)\|_{\mathcal{H}(r\geq R)} \lesssim R^{-2}\|\pi_R u_e(0)\|_{\mathcal{H}(r\geq R)} + R^{-d/2}\|\pi_R \vec{u}_e(0)\|^2_{\mathcal{H}(r\geq R)} \\ + R^{-1}\|\pi_R \vec{u}_e(0)\|^3_{\mathcal{H}(r\geq R)},\end{aligned}$$

as desired. This proves Lemma 3.4.7 for $t = 0$.

For general $t = t_0$ in (3.4.15), we first set

$$u_{0,R,t_0} = \begin{cases} u_e(t_0, r) & \text{if } r \geq R, \\ \frac{r-\eta}{R-\eta} u_e(t_0, R) & \text{if } r < R, \end{cases}$$

$$u_{1,R,t_0} = \begin{cases} \partial_t u_e(t_0, r) & \text{if } r \geq R, \\ 0 & \text{if } r < R. \end{cases}$$

By (3.4.7) we can find $R_0 = R_0(\delta_0)$ independent of t_0 such that for all $R \geq R_0$

$$\|(u_{0,R,t_0}, u_{1,R,t_0})\|_{\mathcal{H}(r\geq\eta)} \lesssim \|\vec{u}_e(t_0)\|_{\mathcal{H}(r\geq R)} \lesssim \delta_0.$$

The previous argument for $t_0 = 0$ repeated with obvious modifications yield (3.4.15) for $t = t_0$. □

Before proceeding to the next step, we reformulate the conclusion of Lemma 3.4.7 using the projection coefficients $\lambda_i(t, R), \mu_j(t, R)$ for $\vec{u}_e(t)$. The following is an immediate consequence of Lemma 3.4.6 and Lemma 3.4.7.

Lemma 3.4.10. *Let $\lambda_i(t, R), \mu_j(t, R)$, $1 \leq i \leq \tilde{k}$, $1 \leq j \leq k$ be the projection coefficients as* *in* (3.4.14). *Then there exists* $R_0 \geq 1$ *such that uniformly in* $R > R_0$ *and* $t \in \mathbb{R}$

$$\begin{aligned}\int_R^\infty \sum_{i=1}^{\tilde{k}} \Big(\partial_r \lambda_i(t,r) r^{2i-\frac{d+1}{2}}\Big)^2 + \sum_{j=1}^{k} \Big(\partial_r \mu_j(t,r) r^{2i-\frac{d-1}{2}}\Big)^2 dr \\ \lesssim \sum_{i=1}^{\tilde{k}} R^{4i-d-6} |\lambda_i(t,R)|^2 + R^{8i-3d-4} |\lambda_i(t,R)|^4 + R^{12i-3d-8} |\lambda_i(t,R)|^6 \\ + \sum_{i=1}^{k} R^{4i-d-4} |\mu_i(t,R)|^2 + R^{8i-3d} |\mu_i(t,R)|^4 + R^{12i-3d-2} |\mu_i(t,R)|^6.\end{aligned}$$

Step 2: Asymptotics for $\vec{u}_e(0)$

In this step, we prove that $\vec{u}_e(0)$ has the asymptotic expansions (3.4.11), (3.4.12) which we now formulate as a proposition.

Proposition 3.4.11. *Let* u_e *be a solution to* (3.4.2) *which satisfies* (3.4.7). *Let* $\vec{u}_e(0) = (u_{e,0}, u_{e,1})$. *Then there exists* $\alpha \in \mathbb{R}$ *such that*

$$\begin{aligned}r^{d-2} u_{e,0}(r) = \alpha + O(r^{-2}), \\ \int_r^\infty u_{e,1}(\rho) \rho^{2j-1} d\rho = O(r^{2j-d-1}), \quad j = 1, \dots, k,\end{aligned}$$

as $r \to \infty$.

The proof of Proposition 3.4.11 is split up into a several lemmas. First, we use Lemma 3.4.10 to prove the following difference estimate for the projection coefficients.

Lemma 3.4.12. *Let* $\delta_1 \leq \delta_0$ *where* δ_0 *is from Lemma 3.4.9. Let* $R_1 \geq R_0 > 1$ *be large enough so that for all* $R \geq R_1$ *and for all* $t \in \mathbb{R}$

$$\begin{aligned}\|\vec{u}_e(t)\|_{\mathcal{H}(r \geq R)} \leq \delta_1, \\ R^{-2} \leq \delta_1.\end{aligned}$$

Then for all r, r' with $R_1 \leq r \leq r' \leq 2r$ and uniformly in t

$$\begin{aligned}|\lambda_j(t,r) - \lambda_j(t,r')| \lesssim r^{-2j+1} \sum_{i=1}^{\tilde{k}} r^{2i-3} |\lambda_i(t,r)| + r^{4i-d-2} |\lambda_i(t,r)|^2 + r^{6i-d-4} |\lambda_i(t,r)|^3 \\ + r^{-2j+1} \sum_{i=1}^{k} r^{2i-2} |\mu_i(t,r)|^2 + r^{4i-d} |\mu_i(t,r)|^2 + r^{6i-d-1} |\mu_i(t,r)|^3,\end{aligned} \tag{3.4.25}$$

and

$$\begin{aligned}|\mu_j(t,r) - \mu_j(t,r')| \lesssim r^{-2j} \sum_{i=1}^{\tilde{k}} r^{2i-3} |\lambda_i(t,r)| + r^{4i-d-2} |\lambda_i(t,r)|^2 + r^{6i-d-4} |\lambda_i(t,r)|^3 \\ + r^{-2j} \sum_{i=1}^{k} r^{2i-2} |\mu_i(t,r)|^2 + r^{4i-d} |\mu_i(t,r)|^2 + r^{6i-d-1} |\mu_i(t,r)|^3.\end{aligned} \tag{3.4.26}$$

Proof. By the fundamental theorem of calculus and Lemma 3.4.10 we have, for all r, r' such that $R_1 \leq r \leq r' \leq 2r$,

$$\begin{aligned}|\lambda_j(t,r) - \lambda_j(t,r')|^2 &= \left(\int_r^{r'} \partial_\rho \lambda_j(t,\rho) d\rho \right)^2 \\ &\leq \left(\int_r^{r'} \rho^{-4j+d+1} d\rho \right) \left(\int_r^{r'} \left(\rho^{2j-\frac{d+1}{2}} \partial_\rho \lambda(t,\rho) \right)^2 d\rho \right) \\ &\lesssim r^{-4j+d+2} \sum_{i=1}^{\tilde{k}} r^{4i-d-6} |\lambda_i(t,r)|^2 + r^{8i-3d-4} |\lambda_i(t,r)|^4 + r^{12i-3d-8} |\lambda_i(t,r)|^6 \\ &+ r^{-4j+d+2} \sum_{i=1}^{k} r^{4i-d-4} |\mu_i(t,r)|^2 + r^{8i-3d} |\mu_i(t,r)|^4 + r^{12i-3d-2} |\mu_i(t,r)|^6\end{aligned}$$

which proves (3.4.25).

Similarly, we have

$$\begin{aligned}|\mu_j(t,r)-\mu_j(t,r')|^2 &\leq r^{-4j+d}\left(\int_r^{r'}(\rho^{2j-\frac{d-1}{2}}\partial_\rho\mu_j(t,\rho))^2 d\rho\right)\\ &\lesssim r^{-4j+d}\sum_{i=1}^{\tilde{k}} r^{4i-d-6}|\lambda_i(t,r)|^2+r^{8i-3d-4}|\lambda_i(t,r)|^4+r^{12i-3d-8}|\lambda_i(t,r)|^6\\ &\quad+r^{-4j+d}\sum_{i=1}^{k} r^{4i-d-4}|\mu_i(t,r)|^2+r^{8i-3d}|\mu_i(t,r)|^4+r^{12i-3d-2}|\mu_i(t,r)|^6\end{aligned}$$

which proves (3.4.26). □

We note that with δ_1 and R_1 fixed as in Lemma 3.4.12, we have by Lemma 3.4.6 for all $r\geq R_1$ and uniformly in time

$$\begin{aligned}|\lambda_i(t,r)| &\lesssim \delta_1 r^{\frac{d+2}{2}-2i},\quad \forall 1\leq i\leq \tilde{k},\\ |\mu_j(t,r)| &\lesssim \delta_1 r^{\frac{d}{2}-2j},\quad \forall 1\leq j\leq k.\end{aligned}\tag{3.4.27}$$

By using this observation, a simple consequence of Lemma 3.4.12 is the following.

Corollary 3.4.13. *Let δ_1 and R_1 be as in Lemma 3.4.12. Then for all r, r' with $R_1\leq r\leq r'\leq 2r$ and for all $t\in\mathbb{R}$*

$$|\lambda_j(t,r)-\lambda_j(t,r')|\lesssim \delta_1\left(\sum_{i=1}^{\tilde{k}} r^{2i-2j}|\lambda_i(t,r)|+\sum_{i=1}^{k} r^{2i-2j+1}|\mu_i(t,r)|\right),\tag{3.4.28}$$

$$|\mu_j(t,r)-\mu_j(t,r')|\lesssim r^{-1}\delta_1\left(\sum_{i=1}^{\tilde{k}} r^{2i-2j}|\lambda_i(t,r)|+\sum_{i=1}^{k} r^{2i-2j+1}|\mu_i(t,r)|\right).\tag{3.4.29}$$

Before proceeding further, we state point wise and averaged difference (in time) estimates for the projection coefficients that will be used in the sequel.

Lemma 3.4.14 (Lemma 5.10, Lemma 5.12 [18]). *For each $R>0$, $r\geq R$, and $t_1\neq t_2$ with*

$(t_j, r) \in \{r \geq R + |t|\}$ *we have for any* $1 \leq j \leq j' \leq \tilde{k}$

$$\begin{aligned} &|\lambda_j(t_1, r) - \lambda_j(t_2, r)| \\ &\qquad \lesssim r^{2j'-2j}|\lambda_{j'}(t_1, r) - \lambda_{j'}(t_2, r)| + \sum_{m=1}^{k} \left| r^{2m-2j} \int_{t_2}^{t_1} \mu_m(t, r) dt \right|, \end{aligned} \tag{3.4.30}$$

as well as for any $1 \leq j \leq k$

$$\frac{1}{R} \int_R^{2R} \mu_j(t_1, r) - \mu_j(t_2, r) dr = \sum_{i=1}^{\tilde{k}} \frac{c_i c_j}{d - 2 - 2j} \int_{t_1}^{t_2} I(i, j) + II(i, j) dt, \tag{3.4.31}$$

with

$$\begin{aligned} I(i, j) &= -\frac{1}{R}(u_e(t, r) r^{d-2j-1})\Big|_{r=R}^{r=2R} + (2i - 2j - 1)\frac{1}{R} \int_R^{2R} u_e(t, r) r^{d-2j-2} dr \\ &\quad - \frac{(2\ell - 2i + 3)(2i - 2)}{R} \int_R^{2R} r^{d-2i-2j} \int_r^{\infty} u_e(t, \rho) \rho^{2i-3} d\rho dr, \\ II(i, j) &= \frac{1}{R} \int_R^{2R} r^{d-2i-2j} \int_r^{\infty} \left[-V_e(\rho) u_e(t, \rho) + N_e(\rho, u_e(t, \rho)) \right] \rho^{2i-1} d\rho dr. \end{aligned} \tag{3.4.32}$$

Subtleties arise depending on when $d = 7, 11, 15, \ldots$ (ℓ is even) and when $d = 5, 9, 13, \ldots$ (ℓ is odd) which are due to the relationship between $\tilde{k}$ and k. We first prove Proposition 3.4.11 in the case that ℓ **is even**. When ℓ is even, we have the relations

$$d = 4\tilde{k} - 1, \quad \tilde{k} = k + 1.$$

We now establish a growth estimate which improves (3.4.27).

Lemma 3.4.15. *Let* $\epsilon > 0$ *be fixed and sufficiently small. Then as long as* δ_1 *as in Lemma*

3.4.12 is sufficiently small, we have uniformly in t,

$$\begin{aligned}
|\lambda_{\tilde{k}}(t,r)| &\lesssim r^{\epsilon}, \\
|\mu_k(t,r)| &\lesssim r^{\epsilon}, \\
|\lambda_i(t,r)| &\lesssim r^{2\tilde{k}-2j-2+3\epsilon}, \quad \forall 1 \le i < \tilde{k}, \\
|\mu_i(t,r)| &\lesssim r^{2\tilde{k}-2j-3+3\epsilon}, \quad \forall 1 \le i < k.
\end{aligned} \tag{3.4.33}$$

Proof. If $r > R_1$, by Corollary 3.4.13 we have,

$$|\lambda_j(t,2r)| \le (1+C\delta_1)|\lambda_j(t,r)| + C\delta_1 \left(\sum_{i=1}^{\tilde{k}} r^{2i-2j}|\lambda_i(t,r)| + \sum_{i=1}^{k} r^{2i-2j+1}|\mu_i(t,r)| \right), \tag{3.4.34}$$

$$|\mu_j(t,2r)| \le (1+C\delta_1)|\mu_j(t,r)| + r^{-1}C\delta_1 \left(\sum_{i=1}^{\tilde{k}} r^{2i-2j}|\lambda_i(t,r)| + \sum_{i=1}^{k} r^{2i-2j+1}|\mu_i(t,r)| \right). \tag{3.4.35}$$

Fix $r_0 > R_1$ and define

$$a_n := \sum_{i=1}^{\tilde{k}} (2^n r_0)^{2i-2\tilde{k}} |\lambda_i(t, 2^n r_0)| + \sum_{i=1}^{k} (2^n r_0)^{2i-2\tilde{k}+1} |\mu_i(t, 2^n r_0)|$$

then (3.4.34) and (3.4.35) imply

$$a_{n+1} \le (1 + C(k+\tilde{k})\delta_1) a_n.$$

By induction

$$a_n \le (1 + C(k+\tilde{k})\delta_1)^n a_0$$

Choose δ_1 so small so that $1 + C(k + \tilde{k})\delta_1 < 2^\epsilon$. We conclude (using the compactness of $\vec{u}_e$) that

$$a_n \leq 2^{n\epsilon} a_0 \lesssim 2^{n\epsilon},$$

whence by our definition of a_n

$$|\lambda_i(t, 2^n r_0)| \lesssim (2^n r_0)^{2\tilde{k}-2i+\epsilon}, \quad |\mu_i(t, 2^n r_0)| \lesssim (2^n r_0)^{2\tilde{k}-2i-1+\epsilon}, \tag{3.4.36}$$

which is an improvement of (3.4.27).

We now insert (3.4.36) back into our difference estimates (3.4.25) and (3.4.26). We first note that by (3.4.36) and the relation $d = 4\tilde{k} - 1 \geq 7$, we have the estimates

$$\begin{aligned}
(2^n r_0)^{2i-3} |\lambda_i(t, 2^n r_0)| &\lesssim (2^n r_0)^{2\tilde{k}-3+\epsilon}, \\
(2^n r_0)^{4i-d-2} |\lambda_i(t, 2^n r_0)|^2 &\lesssim (2^n r_0)^{-1+2\epsilon} \lesssim (2^n r_0)^{2\tilde{k}-3+3\epsilon}, \\
(2^n r_0)^{6i-d-4} |\lambda_i(t, 2^n r_0)|^3 &\lesssim (2^n r_0)^{2\tilde{k}-3+3\epsilon},
\end{aligned} \tag{3.4.37}$$

as well as

$$\begin{aligned}
(2^n r_0)^{2i-2} |\mu_i(t, 2^n r_0)| &\lesssim (2^n r_0)^{2\tilde{k}-3+\epsilon}, \\
(2^n r_0)^{4i-d} |\mu_i(t, 2^n r_0)|^2 &\lesssim (2^n r_0)^{-1+2\epsilon} \lesssim (2^n r_0)^{2\tilde{k}-3+3\epsilon}, \\
(2^n r_0)^{6i-d} |\mu_i(t, 2^n r_0)|^3 &\lesssim (2^n r_0)^{2\tilde{k}-3+3\epsilon}.
\end{aligned} \tag{3.4.38}$$

Thus, by (3.4.25) and (3.4.26), we deduce that

$$|\lambda_j(t, 2^{n+1} r_0) - \lambda_j(t, 2^n r_0)| \leq C\delta_1 |\lambda_j(2^n r_0)| + C(2^n r_0)^{2\tilde{k}-2j-2+3\epsilon}, \tag{3.4.39}$$

$$|\mu_j(t, 2^{n+1} r_0) - \mu_j(t, 2^n r_0)| \leq C\delta_1 |\mu_j(2^n r_0)| + C(2^n r_0)^{2\tilde{k}-2j-3+3\epsilon}. \tag{3.4.40}$$

From this we obtain

$$|\lambda_j(t, 2^{n+1}r_0)| \leq (1 + C\delta_1)|\lambda_j(2^n r_0)| + C(2^n r_0)^{2\tilde{k}-2j-2+3\epsilon}.$$

Using that we have chosen δ_1 so that $(1 + C\delta_1) < 2^\epsilon$ and iterating we obtain

$$|\lambda_j(t, 2^n r_0)| \leq (2^\epsilon)^n |\lambda_j(t, r_0)| + C \sum_{m=1}^{n} (2^m r_0)^{2\tilde{k}-2j-2+3\epsilon}(2^\epsilon)^{n-m}.$$

In the case $j = \tilde{k}$, the previous estimate is easily seen to be $O\left(2^{\epsilon n}\right)$ since the first term dominates, while if $j < \tilde{k}$, we have the previous estimate is $O\left((2^n r_0)^{2\tilde{k}-2j-2+3\epsilon}\right)$ since then the second term dominates. A similar argument applies to the μ_j's, and we conclude that

$$\begin{aligned}
|\lambda_{\tilde{k}}(t, 2^n r_0)| &\lesssim (2^n r_0)^\epsilon, \\
|\mu_k(t, 2^n r_0)| &\lesssim (2^n r_0)^\epsilon, \\
|\lambda_i(t, 2^n r_0)| &\lesssim (2^n r_0)^{2\tilde{k}-2i-2+3\epsilon}, \quad \forall 1 \leq i < \tilde{k}, \\
|\mu_i(t, 2^n r_0)| &\lesssim (2^n r_0)^{2\tilde{k}-2i-3+3\epsilon}, \quad \forall 1 \leq i < k.
\end{aligned} \tag{3.4.41}$$

The estimate (3.4.41) is uniform in time and an improvement of (3.4.36). Let $r \geq r_0$ with $2^n r_0 \leq r \leq 2^{n+1}r_0$. We plug (3.4.41) into the difference estimate (3.4.25) and obtain

$$|\lambda_{\tilde{k}}(t, r)| \leq (1 + C\delta_1)|\lambda_{\tilde{k}}(t, 2^n r_0)| + C(2^n r_0)^{2\tilde{k}-2j-2+3\epsilon} \lesssim (2^n r_0)^\epsilon \lesssim r^\epsilon.$$

The other estimates in (3.4.33) are obtained by similar reasoning. This concludes the proof.

□

The following corollary is a consequence of the proof of Lemma 3.4.15.

Corollary 3.4.16. *Let ϵ and δ_1 be as in Lemma 3.4.15, let $r_0 > R_1$ be fixed, and let*

$j \in \{1, \ldots, \tilde{k}\}$. *If there exists* $a \geq \epsilon$ *such that for all* $n \in \mathbb{N}$,

$$|\lambda_j(t, 2^{n+1} r_0)| \leq (1 + C\delta_1)|\lambda_j(t, 2^n r_0)| + (2^n r_0)^a,$$

then for all $r \geq r_0$

$$|\lambda_j(t, r)| \lesssim r^a,$$

uniformly in time. A similar statement holds for the μ_j*'s as well.*

We now use the previous lemma as the base case for an induction argument. The main goal is to prove the following decay estimates for the projection coefficients.

Proposition 3.4.17. *Suppose* $d = 7, 11, 15, \ldots$ *and* ϵ, δ_1,r_0 *are as in Lemma 3.4.15. Then uniformly in time, the following estimates hold:*

$$\begin{aligned} &|\lambda_j(t, r)| \lesssim r^{-2j+3\epsilon}, \quad \forall 1 < j \leq \tilde{k}, \\ &|\lambda_1(t, r)| \lesssim r^{\epsilon}, \\ &|\mu_j(t, r)| \lesssim r^{-2j-1+3\epsilon}, \quad \forall 1 \leq j \leq k. \end{aligned} \tag{3.4.42}$$

Proposition 3.4.17 is a consequence of the following proposition with $P = k$.

Proposition 3.4.18. *With the same hypotheses as in Proposition 3.4.17, for* $P = 0, 1, \ldots, k$ *the following estimates hold uniformly in time:*

$$\begin{aligned} |\lambda_j(t, r)| &\lesssim r^{2(\tilde{k}-P-j)-2+3\epsilon}, \quad \forall 1 \leq j \leq \tilde{k} \text{ with } j \neq \tilde{k} - P, \\ |\lambda_{\tilde{k}-P}(t, r)| &\lesssim r^{\epsilon}, \\ |\mu_j(t, r)| &\lesssim r^{2(k-P-j)-1+3\epsilon}, \quad \forall 1 \leq j \leq k \text{ with } j \neq k - P, \\ |\mu_{k-P}(t, r) &\lesssim r^{\epsilon}. \end{aligned} \tag{3.4.43}$$

Proof of Proposition 3.4.18. As was mentioned before, we prove Proposition 3.4.18 by induction. The base case $P = 0$ is contained in Lemma 3.4.15. We now assume that the estimates (3.4.43) hold for P with $1 \leq P \leq k-1$ and wish to show that the estimates (3.4.43) also hold for $P+1$. The proof is divided into several lemmas. The bulk of the argument is devoted to proving that the coefficients $\lambda_{\tilde{k}-P}$ and μ_{k-P} satisfy certain decay estimates. We first show that they have spatial limits.

Lemma 3.4.19. *There exist bounded functions $\alpha_{\tilde{k}-P}(t)$ and $\beta_{k-P}(t)$ such that*

$$|\lambda_{\tilde{k}-P}(t,r) - \alpha_{\tilde{k}-P}(t)| = O(r^{-2}), \tag{3.4.44}$$

$$|\mu_{k-P}(t,r) - \beta_{k-P}(t)| = O(r^{-1}), \tag{3.4.45}$$

where the $O(\cdot)$ terms are uniform in time.

Proof. Fix $r_0 > R_1$. We insert the estimates (3.4.43) furnished by our induction hypothesis into the difference estimate (3.4.25). We first note that based on (3.4.43), we can estimate the sum excluding the coefficients $\lambda_{\tilde{k}-P}$ and μ_{k-P}:

$$\begin{aligned}
&\sum_{i \neq \tilde{k}-P}^{\tilde{k}} (2^n r_0)^{2i-3} |\lambda_i(t, 2^n r_0)| + (2^n r_0)^{4i-d-2} |\lambda_i(t, 2^n r_0)|^2 + (2^n r_0)^{6i-d-4} |\lambda_i(t, 2^n r_0))|^3 \\
&\quad + \sum_{i \neq k-P}^{k} (2^n r_0)^{2i-2} |\mu_i(t, 2^n r_0)|^2 + (2^n r_0)^{4i-d} |\mu_i(t, 2^n r_0)|^2 + (2^n r_0)^{6i-d-1} |\mu_i(t, 2^n r_0)|^3 \\
&\lesssim (2^n r_0)^{2(\tilde{k}-P-1)-3+3\epsilon} + (2^n r_0)^{-4P-5+6\epsilon} + (2^n r_0)^{2(\tilde{k}-3P-1)-7+9\epsilon}.
\end{aligned}$$

In particular, we have the following estimate which will be used repeatedly,

$$\begin{aligned}&\sum_{i\neq\tilde{k}-P}^{\tilde{k}}(2^nr_0)^{2i-3}|\lambda_i(t,2^nr_0)|+(2^nr_0)^{4i-d-2}|\lambda_i(t,2^nr_0)|^2+r^{6i-d-4}|\lambda_i(t,2^nr_0)|^3\\&+\sum_{i\neq k-P}^{k}(2^nr_0)^{2i-2}|\mu_i(t,2^nr_0)|^2+(2^nr_0)^{4i-d}|\mu_i(t,2^nr_0)|^2+(2^nr_0)^{6i-d-1}|\mu_i(t,2^nr_0)|^3\\&\lesssim(2^nr_0)^{2(\tilde{k}-P-1)-3+3\epsilon}.\end{aligned}\tag{3.4.46}$$

Using (3.4.43) and the relation $d=4\tilde{k}-1$, $k=\tilde{k}-1$, we estimate

$$\begin{aligned}&(2^nr_0)^{2(\tilde{k}-P)-3}|\lambda_{\tilde{k}-P}(t,2^nr_0)|+(2^nr_0)^{4(\tilde{k}-P)-d-2}|\lambda_{\tilde{k}-P}(t,2^nr_0)|^2\\&\quad+(2^nr_0)^{6(\tilde{k}-P)-d-4}|\lambda_{\tilde{k}-P}(t,2^nr_0)|^3+(2^nr_0)^{2(k-P)-2}|\mu_{k-P}(t,2^nr_0)|\\&\quad+(2^nr_0)^{4(k-P)-d}|\mu_{k-P}(t,2^nr_0)|^2+(2^nr_0)^{6(k-P)-d-1}|\mu_{k-P}(t,2^nr_0)|^3\\&\quad\lesssim(2^nr_0)^{2(\tilde{k}-P)-3+3\epsilon}.\end{aligned}\tag{3.4.47}$$

Inserting (3.4.46) and (3.4.47) into our difference estimates (3.4.25) and (3.4.26), we deduce for each $n\in\mathbb{N}$

$$\begin{aligned}|\lambda_{\tilde{k}-P}(t,2^{n+1}r_0)-\lambda_{\tilde{k}-P}(t,2^nr_0)|&\lesssim(2^nr_0)^{-2+3\epsilon},\\|\mu_{k-P}(t,2^{n+1}r_0)-\mu_{k-P}(t,2^nr_0)|&\lesssim(2^nr_0)^{-1+3\epsilon}.\end{aligned}\tag{3.4.48}$$

From (3.4.48), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}|\lambda_{\tilde{k}-P}(t,2^{n+1}r_0)-\lambda_{\tilde{k}-P}(t,2^nr_0)|&\lesssim\sum_{n=0}^{\infty}2^{(-2+3\epsilon)n}\lesssim1,\\\sum_{n=0}^{\infty}|\mu_{k-P}(t,2^{n+1}r_0)-\mu_{k-P}(t,2^nr_0)|&\lesssim\sum_{n=0}^{\infty}2^{(-1+3\epsilon)n}\lesssim1,\end{aligned}$$

uniformly in t. In particular, for all $t \in \mathbb{R}$ there exist $\alpha_{\tilde{k}-P}(t), \beta_{k-P}(t) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \lambda_{\tilde{k}-P}(t, 2^n r_0) = \alpha_{\tilde{k}-P}(t),$$
$$\lim_{n\to\infty} \mu_{k-P}(t, 2^n r_0) = \beta_{\tilde{k}-P}(t),$$

with the estimates

$$\begin{aligned} \left|\alpha_{\tilde{k}-P}(t) - \lambda_{\tilde{k}-P}(t, 2^n r_0)\right| &\lesssim (2^n r_0)^{-2+3\epsilon}, \\ |\beta_{k-P}(t) - \mu_{k-P}(t, 2^n r_0)| &\lesssim (2^n r_0)^{-1+3\epsilon}. \end{aligned} \tag{3.4.49}$$

Since the compactness of $\vec{u}_e(t)$ implies $|\lambda_{\tilde{k}-P}(t, r_0)|$ is uniformly bounded in t, we have via (3.4.49)

$$|\alpha_{\tilde{k}-P}(t)| \leq |\alpha_{\tilde{k}-P}(t) - \lambda_{\tilde{k}-P}(t, r_0)| + |\lambda_{\tilde{k}-P}(t, r_0)| \lesssim 1$$

uniformly in t. Thus,

$$|\lambda_{\tilde{k}-P}(t, 2^n r_0)| \lesssim 1,$$

uniformly in t and n. Similarly, $\beta_{k-P}(t)$ and $|\mu_{k-P}(t, 2^n r_0)|$ are bounded uniformly in t and n. In conclusion, we have

$$|\lambda_{\tilde{k}-P}(t, 2^n r_0)| + |\mu_{k-P}(t, 2^n r_0)| \lesssim 1 \tag{3.4.50}$$

uniformly in t and n.

Let $r \geq r_0$ with $2^n r_0 \leq r \leq 2^{n+1} r_0$. If we insert (3.4.50) back into the difference estimates (3.4.25) and (3.4.26), we deduce that

$$
\begin{aligned}
&|\lambda_{\tilde{k}-P}(t,r) - \lambda_{\tilde{k}-P}(t,2^n r_0)| \lesssim (2^n r_0)^{-2} \lesssim r^{-2},\\
&|\mu_{k-P}(t,r) - \mu_{k-P}(t,2^n r_0)| \lesssim (2^n r_0)^{-1} \lesssim r^{-1},
\end{aligned}
\tag{3.4.51}
$$

which imply the following improvements of (3.4.49)

$$
\begin{aligned}
&\left|\alpha_{\tilde{k}-P}(t) - \lambda_{\tilde{k}-P}(t,2^n r_0)\right| \lesssim (2^n r_0)^{-2},\\
&|\beta_{k-P}(t) - \mu_{k-P}(t,2^n r_0)| \lesssim (2^n r_0)^{-1}.
\end{aligned}
\tag{3.4.52}
$$

Finally, using (3.4.51) and (3.4.52) we conclude that

$$
\begin{aligned}
|\alpha_{\tilde{k}-P}(t) - \lambda_{\tilde{k}-P}(t,r)| &\lesssim |\alpha_{\tilde{k}-P}(t) - \lambda_{\tilde{k}-P}(t,2^n r_0)| + |\lambda_{\tilde{k}-P}(t,r) - \lambda_{\tilde{k}-P}(t,2^n r_0)|\\
&\lesssim (2^n r_0)^{-2} \lesssim r^{-2},
\end{aligned}
$$

and

$$
\begin{aligned}
|\beta_{k-P}(t) - \mu_{k-P}(t,r)| &\lesssim |\beta_{k-P}(t) - \mu_{k-P}(t,2^n r_0)| + |\mu_{k-P}(t,r) - \mu_{k-P}(t,2^n r_0)|\\
&\lesssim (2^n r_0)^{-1} \lesssim r^{-1}.
\end{aligned}
$$

This concludes the proof. □

A corollary of Lemma 3.4.19 is the following preliminary asymptotics for u_e.

Corollary 3.4.20. *We have*

$$
r^{-2(\tilde{k}-P)+d} u_e(t,r) = \alpha_{\tilde{k}-P}(t) + O(r^{-2+3\epsilon}). \tag{3.4.53}
$$

The $O(\cdot)$ term is uniform in time.

Proof. By Lemma 3.4.6, (3.4.44), and our induction hypotheses (3.4.43), we have

$$\begin{aligned} r^{-2(\tilde{k}-P)-d}u_e(t,r) &= \sum_{j=1}^{\tilde{k}} \lambda_j(t,r) r^{2j-2(\tilde{k}-P)} \\ &= \alpha_{\tilde{k}-P}(t) + \sum_{j\neq\tilde{k}-P}^{\tilde{k}} \lambda_j(t,r) r^{2j-2(\tilde{k}-P)} + O(r^{-2}) \\ &= \alpha_{\tilde{k}-P}(t) + O(r^{-2+3\epsilon}) \end{aligned}$$

uniformly in time. □

A corollary of the proof of Lemma 3.4.19 is the following.

Corollary 3.4.21. *Suppose that for all r, r' with $R_1 \leq r \leq r' \leq 2r$, we have*

$$|\lambda_j(t,r') - \lambda_j(t,r)| \lesssim r^{-a},$$

with $a < 0$. Then $\lambda_j(t,r)$ has a limit, $\alpha_j(t)$, as $r \to \infty$. Moreover, $\alpha_j(t)$ is bounded in time and

$$|\lambda_j(t,r) - \alpha_j(t)| \lesssim r^{-a}$$

uniformly in time. A similar statement holds for the μ_j's.

We will now show that

$$\alpha_{\tilde{k}-P}(t) \equiv 0, \quad \beta_{k-P}(t) \equiv 0.$$

We first show that $\alpha_{\tilde{k}-P}(t)$ is constant in time.

Lemma 3.4.22. *The function $\alpha_{\tilde{k}-P}(t)$ is constant in time. From now on, we will write $\alpha_{\tilde{k}-P}$ in place of $\alpha_{\tilde{k}-P}(t)$.*

Proof. Let $t_2 \neq t_1$. By (3.4.30) with $j = \tilde{k} - P$ and $j' = \tilde{k} - P - 1$, (3.4.44), and our induction hypotheses (3.4.43) we have

$$\begin{aligned}
|\alpha_{\tilde{k}-P}(t_2) - \alpha_{\tilde{k}-P}(t_1)| &\lesssim |\lambda_{\tilde{k}-P}(t_2, r) - \lambda_{\tilde{k}-P}(t_1, r)| + O(r^{-2}) \\
&\lesssim r^{-2}|\lambda_{\tilde{k}-P-1}(t_2, r) - \lambda_{\tilde{k}-P-1}(t_1, r)| \\
&\quad + \sum_{m=1}^{k} \int_{t_1}^{t_2} r^{2m-2(\tilde{k}-P)} |\mu_m(t, r)| dt + O(r^{-2}) \\
&\lesssim r^{-2+3\epsilon}(1 + |t_2 - t_1|).
\end{aligned}$$

We let $r \to \infty$ and deduce that $\alpha_{\tilde{k}-P}(t_2) = \alpha_{\tilde{k}-P}(t_1)$ as desired. □

We now show that $\alpha_{\tilde{k}-P} = 0$. As a consequence, we will also obtain the fact that $\beta_{k-P}(t)$ is constant in time.

Lemma 3.4.23. *We have $\alpha_{\tilde{k}-P} = 0$ and $\beta_{k-P}(t)$ is constant in time. From now on, we will write β_{k-P} in place of $\beta_{k-P}(t)$.*

Proof. The key tool for proving both assertions is Lemma 3.4.14. By (3.4.31) and (3.4.45) we have

$$\begin{aligned}
\beta_{k-P}(t_2) - \beta_{k-P}(t_1) &= \frac{1}{R} \int_R^{2R} \beta_{k-P}(t_2) - \beta_{k-P}(t_1) dr \\
&= \frac{1}{R} \int_R^{2R} [\mu_{k-P}(t_2, r) - \mu_{k-P}(t_1, r)] dr + O(R^{-1}) \\
&= \sum_{i=1}^{k} \frac{c_i c_{k-P}}{d - 2i - 2(k-P)} \int_{t_1}^{t_2} I(i, k-P) + II(i, k-P) dt
\end{aligned}$$

where $I(i, k-P)$ and $II(i, k-P)$ are defined as in (3.4.32). The estimates for the potential

V_e and nonlinearity N_e, (3.4.8)–(3.4.10), along with (3.4.53) imply

$$\begin{aligned}\Big|-V_e(r)u_e + N_e(r,u_e)\Big| &\lesssim r^{-2\tilde{k}-2P-3} + r^{-4\tilde{k}-4P-1} + r^{-2\tilde{k}-6P-3} \\ &\lesssim r^{-2\tilde{k}-2P-3}.\end{aligned}$$

Hence, using that $d = 4\tilde{k} - 1$ and $k = \tilde{k} - 1$, we have

$$\begin{aligned}|II(i,k-P)| = \Big|\frac{1}{R}\int_R^{2R} r^{d-2i-2k-2P}\int_r^\infty [-V_e(\rho)u_e(t,\rho) \\ + N_e(\rho,u_e(t,\rho)]\rho^{2i-1}d\rho dr\Big| \lesssim R^{-2}. \qquad (3.4.54)\end{aligned}$$

We now estimate the remaining term,

$$\begin{aligned}I(i,k-P) = &-\frac{1}{R}(u_e(t,r)r^{d-2(k-P)-1})\big|_{r=R}^{r=2R} \\ &+ (2i-2(k-P)-1)\frac{1}{R}\int_R^{2R} u_e(t,r)r^{d-2(k-P)-2}dr \\ &- \frac{(2\ell-2i+3)(2i-2)}{R}\int_R^{2R} r^{d-2i-2(k-P)}\int_r^\infty u_e(t,\rho)\rho^{2i-3}d\rho dr.\end{aligned}$$

By (3.4.53), we have

$$\begin{aligned}r^{d-2(k-P)-2}u_e(t,r) &= \alpha_{\tilde{k}-P} + O(r^{-2+3\epsilon}), \\ r^{d-2(k-P)-2i}\int_r^\infty u_e(t,\rho)\rho^{2i-3}d\rho &= \frac{\alpha_{\tilde{k}-P}}{d-2i-2(k-P)} + O(r^{-2+3\epsilon}),\end{aligned}$$

so that

$$I(i,k-P) = -\frac{2(k-P)(d-2(k-P)-2)}{d-2i-2(k-P)}\alpha_{\tilde{k}-P} + O(R^{-2+3\epsilon}).$$

Thus,

$$\sum_{i=1}^{k} \frac{c_i c_{k-P}}{d-2i-2(k-P)} \int_{t_1}^{t_2} I(i,k-P)dt = C_0(t_2-t_1)\alpha_{\tilde{k}-P} + O(R^{-2+3\epsilon}(t_2-t_1)) \tag{3.4.55}$$

where

$$C_0 := -\sum_{i=1}^{k} \frac{2c_i c_{k-P}(k-P)(d-2(k-P)-2)}{(d-2i-2(k-P))^2}$$

It can be shown using contour integration that $C_0 \neq 0$ (see Remark 5.29 in [18] for the explicit value for C_0). We let $R \to \infty$ in (3.4.55) and deduce that

$$C_0(t_2-t_1)\alpha_{\tilde{k}-P} = \beta_{k-P}(t_2) - \beta_{k-P}(t_1). \tag{3.4.56}$$

Since $|\beta_{k-P}(t)| \lesssim 1$ by Lemma 3.4.19 and $C_0 \neq 0$, we obtain

$$\alpha_{\tilde{k}-P} = \frac{1}{C_0} \lim_{t_2\to\infty} \frac{\beta_{k-P}(t_2)-\beta_{k-P}(t_1)}{t_2-t_1} = 0.$$

Thus, $\alpha_{\tilde{k}-P} = 0$ which by (3.4.56) implies that $\beta_{k-P}(t)$ is constant in time.

□

We now conclude that $\beta_{k-P} = 0$.

Lemma 3.4.24. *We have* $\beta_{k-P} = 0$.

Proof. By Lemma 3.4.19, $\beta_{k-P} = \mu_{k-P}(t,R) + O(R^{-1})$ uniformly in time so that

$$\beta_{k-P} = \frac{1}{T}\int_0^T \mu_{k-P}(t,R)dt + O(R^{-1}).$$

Since $\alpha_{\tilde{k}-P} = 0$, we have by (3.4.53)

$$u_e(t,r) = O(r^{-d+2(\tilde{k}-P)-2+3\epsilon}),$$

uniformly in time. Thus, by Lemma 3.4.6 and the relations $d = 4\tilde{k}-1$, $\tilde{k} = k+1$, we have

$$\begin{aligned}
\Big|\int_0^T \mu_{k-P}(t,R)dt\Big| &\lesssim \sum_{i=1}^{k} R^{d-2i-2(k-P)}\Big|\int_R^\infty \int_0^T \partial_t u_e(t,\rho)dt\rho^{2i-1}d\rho\Big| \\
&\lesssim \sum_{i=1}^{k} R^{d-2i-2(k-P)} \int_R^\infty |u_e(T,\rho) - u_e(0,\rho)|\rho^{2i-1}d\rho \\
&\lesssim R^{3\epsilon}.
\end{aligned}$$

It follows that

$$\beta_{k-P} = O(R^{3\epsilon}/T) + O(R^{-1}).$$

We set $R = T$ and let $T \to \infty$ to conclude that $\beta_{k-P} = 0$ as desired. □

In summary, we have now shown that if (3.4.43) holds, then

$$\begin{aligned}
\lambda_{\tilde{k}-P}(t,r) &= O(r^{-2}), \\
\mu_{k-P}(t,r) &= O(r^{-1}),
\end{aligned} \tag{3.4.57}$$

uniformly in time. We will now insert (3.4.57) back into the difference estimates (3.4.25) and (3.4.26) to obtain (3.4.43) for $P+1$.

Lemma 3.4.25. *Assume* (3.4.43) *is true for* $0 \leq P \leq k-1$. *Then* (3.4.43) *holds for* $P+1$.

Proof. We recall that by (3.4.46), we have for all $r > R_1$

$$\begin{aligned}\sum_{i\neq \tilde{k}-P}^{\tilde{k}} & r^{2i-3}|\lambda_i(t,r)| + r^{4i-d-2}|\lambda_i(t,r)|^2 + r^{6i-d-4}|\lambda_i(t,r)|^3 \\ & + \sum_{i\neq k-P}^{k} r^{2i-2}|\mu_i(t,r)|^2 + r^{4i-d}|\mu_i(t,r)|^2 + r^{6i-d-1}|\mu_i(t,r)|^3, \\ & \lesssim r^{2(\tilde{k}-P-1)-3+3\epsilon}\end{aligned} \tag{3.4.58}$$

with the main contribution coming from the linear terms. By (3.4.57), we have for all $r > R_1$

$$\begin{aligned}& r^{2(\tilde{k}-P)-3}|\lambda_{\tilde{k}-P}(t,r)| + r^{4(\tilde{k}-P)-d-2}|\lambda_{\tilde{k}-P}(t,r)|^2 + r^{6(\tilde{k}-P)-d-4}|\lambda_{\tilde{k}-P}(t,r)|^3 \\ & + r^{2(k-P)-2}|\mu_{k-P}(t,r)| + r^{4(k-P)-d}|\mu_{k-P}(t,r)|^2 + r^{6(k-P)-d-1}|\mu_{k-P}(t,r)|^3 \\ & \lesssim r^{2(\tilde{k}-P-1)-3+3\epsilon}\end{aligned} \tag{3.4.59}$$

with the main contribution coming from the linear terms. Thus, inserting (3.4.58) and (3.4.59) into our difference estimate (3.4.25), we have for all $R_1 \le r \le r' \le 2r$,

$$|\lambda_j(t,r') - \lambda_j(t,r)| \lesssim r^{2(\tilde{k}-(P+1)-j)-2+3\epsilon}. \tag{3.4.60}$$

By our induction hypotheses (3.4.43), if $\tilde{k} - P < j \le k-1$, we have $\lambda_j(t,r) \to 0$. By Corollary 3.4.21 we then deduce that

$$|\lambda_j(t,r)| \lesssim r^{2(\tilde{k}-(P+1)-j)-2+3\epsilon}$$

uniformly in time. If $j = \tilde{k} - (P+1)$, by (3.4.60) and Corollary 3.4.21 we also deduce that

$$|\lambda_{\tilde{k}-(P+1)}(t,r)| \lesssim 1 \lesssim r^{\epsilon}$$

uniformly in time. Finally, if $j > \tilde{k} - (P+1)$, we have $2(\tilde{k} - P - 1 - j) - 2 + 3\epsilon > \epsilon$ so that by (3.4.60) and Corollary 3.4.16

$$|\lambda_j(t,r)| \lesssim r^{2(\tilde{k}-(P+1)-j)-2+3\epsilon}.$$

In conclusion, we have shown that

$$\begin{aligned} |\lambda_j(t,r)| &\lesssim r^{2(\tilde{k}-(P+1)-j)-2+3\epsilon}, \quad \forall 1 \leq j \leq \tilde{k} \text{ with } j \neq \tilde{k}-(P+1), \\ |\lambda_{\tilde{k}-(P+1)}(t,r)| &\lesssim r^{\epsilon}, \end{aligned}$$

uniformly in time. A similar argument establishes

$$\begin{aligned} |\mu_j(t,r)| &\lesssim r^{2(k-(P+1)-j)-1+3\epsilon}, \quad \forall 1 \leq j \leq k \text{ with } j \neq k-(P+1), \\ |\mu_{k-(P+1)}(t,r)| &\lesssim r^{\epsilon}. \end{aligned}$$

This proves Lemma 3.4.25. □

By Lemma 3.4.25 and induction, we have proved Proposition 3.4.18. □

The final step in proving Proposition 3.4.11 is to establish that $\lambda_1(0,r)$ has a limit as $r \to \infty$. In what follows we denote $\lambda_j(r) = \lambda_j(0,r)$ and $\mu_j(r) = \mu_j(0,r)$.

Lemma 3.4.26. *There exists $\alpha \in \mathbb{R}$ such that*

$$|\lambda_1(r) - \alpha| = O(r^{-2}). \tag{3.4.61}$$

Moreover, we have the slightly improved decay rates

$$\begin{aligned} |\lambda_j(r)| &\lesssim r^{-2j}, \quad 1 < j \leq \tilde{k}, \\ |\mu_j(r)| &\lesssim r^{-2j-1}, \quad 1 \leq j \leq k. \end{aligned} \tag{3.4.62}$$

Proof. By (3.4.42),

$$\begin{aligned}
&\sum_{i=2}^{\tilde{k}} r^{2i-3}|\lambda_i(t,r)| + r^{4i-d-2}|\lambda_i(t,r)|^2 + r^{6i-d-4}|\lambda_i(t,r)|^3 \\
&\quad + \sum_{i=1}^{k} r^{2i-2}|\mu_i(t,r)|^2 + r^{4i-d}|\mu_i(t,r)|^2 + r^{6i-d-1}|\mu_i(t,r)|^3, \\
&\quad \lesssim r^{-3+3\epsilon}
\end{aligned} \tag{3.4.63}$$

with the main contribution coming from the linear terms, and

$$r^{2-3}|\lambda_1(t,r)| + r^{4-d-2}|\lambda_1(t,r)|^2 + r^{6-d-4}|\lambda_1(t,r)|^3 \lesssim r^{-1+\epsilon}. \tag{3.4.64}$$

We insert (3.4.63) and (3.4.64) into our difference estimate (3.4.25) and conclude that for all r, r' with $R_1 < r \leq r' \leq 2r$

$$|\lambda_1(r') - \lambda_1(r)| \lesssim r^{-2+\epsilon}.$$

By Corollary 3.4.21, we deduce that there exists $\alpha \in \mathbb{R}$ such that

$$|\lambda_1(r) - \alpha| \lesssim r^{-2+\epsilon}.$$

In particular, $|\lambda_1(r)| \lesssim 1$. This improves the estimate (3.4.64) to

$$r^{2-3}|\lambda_1(t,r)| + r^{4-d-2}|\lambda_1(t,r)|^2 + r^{6-d-4}|\lambda_1(t,r)|^3 \lesssim r^{-1}. \tag{3.4.65}$$

We plug (3.4.63) and (3.4.65) back into our difference estimate (3.4.25) and conclude that for all r, r' with $R_1 < r \leq r' \leq 2r$

$$|\lambda_1(r') - \lambda_1(r)| \lesssim r^{-2}.$$

Thus,

$$|\lambda_1(r) - \alpha| \lesssim r^{-2}.$$

By (3.4.63) and (3.4.65) and the difference estimate (3.4.25) we conclude that for the other coefficients, for all r, r' with $R_1 < r \leq r' \leq 2r$

$$|\lambda_j(r') - \lambda_j(r)| \lesssim r^{-2j}, \tag{3.4.66}$$

$$|\mu_j(r') - \mu_j(r)| \lesssim r^{-2j-1}. \tag{3.4.67}$$

By (3.4.42), these coefficients go to 0 as $r \to \infty$ so that by Corollary 3.4.21 we conclude that

$$|\lambda_j(r)| \lesssim r^{-2j},$$

$$|\mu_j(r)| \lesssim r^{-2j-1}.$$

This completes the proof. □

Proof of Proposition 3.4.11. By (3.4.61), (3.4.62) and Lemma 3.4.6

$$\begin{aligned} r^{d-2} u_e(0, r) &= \sum_{j=1}^{\tilde{k}} \lambda_j(r) r^{2j-2} \\ &= \lambda_1(r) + \sum_{j=2}^{\tilde{k}} \lambda_j(r) r^{2j-2} \\ &= \alpha + O(r^{-2}) \end{aligned}$$

as well as

$$\int_r^\infty \partial_t u_e(0,\rho)\rho^{2i-1}d\rho = \sum_{j=1}^{k}\mu_j(r)\frac{r^{2i+2j-d}}{d-2i-2j}$$
$$= O(r^{2i-d-1})$$

as desired. □

We now establish Proposition 3.4.11 in the case that $d = 5, 9, 13, \dots$, i.e. when $d = 2\ell+3$ with ℓ **odd**. The case $\ell = 1$, $d = 5$, is contained in Chapter 2. When ℓ is odd, we have the identities

$$k = \tilde{k} = \frac{\ell+1}{2}, \quad d = 4k+1.$$

The proof of Proposition 3.4.11 for when ℓ is odd is very similar to the case when ℓ is even but contains subtleties because of the above identities. In particular, there is an extra μ coefficient, μ_k, which must be dealt with before we can proceed to showing λ_j, μ_{j-1} tend to 0 by induction.

We first establish an ϵ–growth estimate for the coefficients.

Lemma 3.4.27. *Let $\epsilon > 0$ be fixed and sufficiently small. Then as long as δ_1 as in Lemma 3.4.12 is sufficiently small, we have uniformly in t,*

$$\begin{aligned}
|\lambda_k(t,r)| &\lesssim r^{\epsilon}, \\
|\mu_k(t,r)| &\lesssim r^{\epsilon}, \\
|\lambda_i(t,r)| &\lesssim r^{2k-2j-1+3\epsilon}, \quad \forall 1 \le i < k, \\
|\mu_i(t,r)| &\lesssim r^{2k-2j-2+3\epsilon}, \quad \forall 1 \le i < k.
\end{aligned} \tag{3.4.68}$$

Proof. Let $r > R_1$. By Corollary 3.4.13 we have,

$$|\lambda_j(t,2r)| \leq (1+C\delta_1)|\lambda_j(t,r)| + C\delta_1 \left(\sum_{i=1}^{k} r^{2i-2j}|\lambda_i(t,r)| + \sum_{i=1}^{k} r^{2i-2j+1}|\mu_i(t,r)| \right),$$
$$|\mu_j(t,2r)| \leq (1+C\delta_1)|\mu_j(t,r)| + r^{-1}C\delta_1 \left(\sum_{i=1}^{k} r^{2i-2j}|\lambda_i(t,r)| + \sum_{i=1}^{k} r^{2i-2j+1}|\mu_i(t,r)| \right). \tag{3.4.69}$$

Fix $r_0 > R_1$ and define

$$b_n := \sum_{i=1}^{k} (2^n r_0)^{2i-2k-1}|\lambda_i(t,2^n r_0)| + \sum_{i=1}^{k} (2^n r_0)^{2i-2k}|\mu_i(t,2^n r_0)|.$$

Then by (3.4.69)

$$b_{n+1} \leq (1+2Ck\delta_1)b_n.$$

By iterating we obtain

$$b_n \leq (1+2Ck\delta_1)^n b_0$$

Choose δ_1 so small so that $1+2Ck\delta_1 < 2^{\epsilon}$. By the compactness of $\vec{u}_e(t)$, $b_0 \lesssim 1$ uniformly in t, and we conclude that

$$b_n \leq 2^{n\epsilon} b_0 \lesssim 2^{n\epsilon}.$$

By our definition of b_n it follows that

$$|\lambda_i(t,2^n r_0)| \lesssim (2^n r_0)^{2k-2i+1+\epsilon}, \quad |\mu_i(t,2^n r_0)| \lesssim (2^n r_0)^{2k-2i+\epsilon}, \tag{3.4.70}$$

which is an improvement of (3.4.27).

As in the proof of Lemma 3.4.15, we insert (3.4.70) back into our difference estimates (3.4.25) and (3.4.26) and conclude that

$$|\lambda_j(t, 2^{n+1}r_0) - \lambda_j(t, 2^n r_0)| \leq C\delta_1 |\lambda_j(2^n r_0)| + C(2^n r_0)^{2k-2j-1+3\epsilon}, \tag{3.4.71}$$

$$|\mu_j(t, 2^{n+1}r_0) - \mu_j(t, 2^n r_0)| \leq C\delta_1 |\mu_j(2^n r_0)| + C(2^n r_0)^{2k-2j-2+3\epsilon}, \tag{3.4.72}$$

with the dominant contribution coming from the cubic terms. By Corollary 3.4.16 we conclude that uniformly in t

$$\begin{aligned} |\lambda_k(t, r)| &\lesssim r^{\epsilon}, \\ |\mu_k(t, r)| &\lesssim r^{\epsilon}, \\ |\lambda_i(t, r)| &\lesssim r^{2\tilde{k}-2j-1+3\epsilon}, \quad \forall 1 \leq i < k, \\ |\mu_i(t, r)| &\lesssim r^{2\tilde{k}-2j-2+3\epsilon}, \quad \forall 1 \leq i < k, \end{aligned}$$

as desired. □

We now turn to showing that the extra term μ_k goes to 0 as $r \to \infty$.

Lemma 3.4.28. *There exists a bounded function $\beta_k(t)$ on $\mathbb{R}$ such that*

$$|\mu_k(t, r) - \beta_k(t)| \lesssim r^{-2}, \tag{3.4.73}$$

uniformly in t.

Proof. We insert (3.4.65) into the difference estimate (3.4.26) with $j = k$ and obtain for all $R_1 < r_1 < r' < 2r$

$$|\mu_k(t, r') - \mu_k(t, r)| \lesssim r^{-2+3\epsilon}.$$

The dominant contribution in the difference estimate (3.4.26) comes from the cubic term $|\mu_k|^3$. By Corollary 3.4.21, we conclude that there exists a bounded function $\beta_k(t)$ such that

$$|\mu_k(t,r) - \beta_k(t)| \lesssim r^{-2+3\epsilon}.$$

uniformly in t. In particular, $|\mu_k(t,r)| \lesssim 1$ uniformly in t and r. Using this information, we can improve the difference estimate (3.4.26) with $j = k$ to

$$|\mu_k(t,r') - \mu_k(t,r)| \lesssim r^{-2}$$

and conclude that

$$|\mu_k(t,r) - \beta_k(t)| \lesssim r^{-2}$$

uniformly in t as desired. □

Lemma 3.4.29. *We have $\beta_k(t) \equiv 0$.*

Proof. The proof is in similar spirit to the proofs of Lemmas 3.4.23 and 3.4.24. We first note that by Lemma 3.4.6, (3.4.68), and the relation $d = 4k+1$, we have

$$|u_e(t,r)| = \left|\sum_{j=1}^{k} \lambda_j(t,r) r^{2j-d}\right| \lesssim r^{2k-d+\epsilon} = r^{-2k-1+\epsilon}. \tag{3.4.74}$$

By (3.4.73) and (3.4.31)

$$\begin{aligned}\beta_k(t_2) - \beta_k(t_1) &= \frac{1}{R}\int_R^{2R} [\mu_k(t_2,r) - \mu_k(t_1,r)]dr + O(R^{-2})\\ &= \sum_{i=1}^{k} \frac{c_i c_k}{d-2i-2k}\int_{t_1}^{t_2} I(i,k) + II(i,k)dt\end{aligned}$$

where $I(i,k)$ and $II(i,k)$ are defined as in (3.4.32). The estimates for the potential V_e and nonlinearity N_e, (3.4.8)–(3.4.10), along with (3.4.74) imply

$$\Big|-V_e(r)u_e + N_e(r,u_e)\Big| \lesssim r^{-2k-5+\epsilon}.$$

Using that $d = 4k+1$ we have

$$\begin{aligned}|II(i,k-P)| &= \Big|\frac{1}{R}\int_R^{2R} r^{d-2i-2k-2P}\int_r^\infty [-V_e(\rho)u_e(t,\rho) + N_e(\rho,u_e(t,\rho)]\rho^{2i-1}d\rho dr\Big| \\ &\lesssim R^{-4+\epsilon}.\end{aligned}$$

We now estimate the remaining term,

$$\begin{aligned}I(i,k) = &-\frac{1}{R}(u_e(t,r)r^{d-2k-1})|_{r=R}^{r=2R} + (2i-2k-1)\frac{1}{R}\int_R^{2R} u_e(t,r)r^{d-2k-2}dr \\ &-\frac{(2\ell-2i+3)(2i-2)}{R}\int_R^{2R} r^{d-2i-2k}\int_r^\infty u_e(t,\rho)\rho^{2i-3}d\rho dr.\end{aligned}$$

Using (3.4.74), it is simple to conclude that

$$|I(i,k)| \lesssim R^{-2+\epsilon}.$$

Thus,

$$\beta_k(t_2) - \beta_k(t_1) = O(R^{-2+\epsilon})(1+|t_2-t_1|).$$

We let $R\to\infty$ and conclude that $\beta_k(t_2) = \beta_k(t_1)$ as desired.

We now write β_k in place of $\beta_k(t)$. By the previous paragraph, we have that

$$\beta_k = \mu_k(t,R) + O(R^{-2})$$

where the $O(\cdot)$ term is uniform in time. Integrating the previous expression from 0 to T and dividing by T yields

$$\beta_k = \frac{1}{T}\int_0^T \mu_k(t,R)dt + O(R^{-2}).$$

By Lemma 3.4.6, (3.4.74), and the relations $d = 4k+1$, we have

$$\begin{aligned}\Big|\int_0^T \mu_k(t,R)dt\Big| &\lesssim \sum_{i=1}^{k} R^{d-2i-2k}\Big|\int_R^\infty \int_0^T \partial_t u_e(t,\rho)dt\rho^{2i-1}d\rho\Big| \\ &\lesssim \sum_{i=1}^{k} R^{d-2i-2k}\int_R^\infty |u_e(T,\rho)-u_e(0,\rho)|\rho^{2i-1}d\rho \\ &\lesssim R^{\epsilon}.\end{aligned}$$

It follows that

$$\beta_k = O(R^{3\epsilon}/T) + O(R^{-2}).$$

We set $R = T$ and let $T \to \infty$ to conclude that $\beta_k = 0$ as desired. □

Lemma 3.4.30. *Let $\epsilon > 0$ be fixed and sufficiently small. Then as long as δ_1 as in Lemma 3.4.12 is sufficiently small, we have uniformly in t,*

$$\begin{aligned}|\lambda_k(t,r)| &\lesssim r^{\epsilon}, \\ |\mu_{k-1}(t,r)| &\lesssim r^{\epsilon}, \\ |\lambda_i(t,r)| &\lesssim r^{2k-2j-2+\epsilon}, \quad \forall 1 \le i < k, \\ |\mu_i(t,r)| &\lesssim r^{2k-2j-3+\epsilon}, \quad \forall 1 \le i \le< k, i \neq k-1.\end{aligned} \tag{3.4.75}$$

Proof. We first establish

$$|\mu_k(t,r)| \lesssim r^{-3+\epsilon},$$

uniformly in time. By (3.4.68), we have uniformly in time

$$\begin{aligned}\sum_{i=1}^{k} r^{2i-3}|\lambda_i(t,r)| + r^{4i-d-2}|\lambda_i(t,r)|^2 + r^{6i-d-4}|\lambda_i(t,r)|^3 \\ + \sum_{i=1}^{k-1} r^{2i-2}|\mu_i(t,r)| + r^{4i-d}|\mu_i(t,r)|^2 + r^{6i-d-1}|\mu_i(t,r)|^3 \lesssim r^{2k-3+\epsilon},\end{aligned} \tag{3.4.76}$$

where the dominant contribution comes from the linear term involving λ_k. By (3.4.73) and the fact that $\beta_k = 0$, we have

$$r^{2k-2}|\mu_k(t,r)| + r^{4k-d}|\mu_k(t,r)|^2 + r^{6k-d-1}|\mu_k(t,r)|^3 \lesssim r^{2k-4}, \tag{3.4.77}$$

uniformly in time. Inserting (3.4.76) and (3.4.77) into the difference estimate (3.4.26) with $j = k$ implies that for all r, r' with $R_1 \leq r \leq r' \leq 2r$, we have

$$|\mu_k(t,r') - \mu_k(t,r)| \lesssim r^{-3+\epsilon}$$

uniformly in time. Since $\lim_{r\to\infty} \mu_k(t,r) = 0$, we conclude by Corollary 3.4.21 that

$$|\mu_k(t,r)| \lesssim r^{-3+\epsilon} \tag{3.4.78}$$

uniformly in time.

We now establish the other estimates in (3.4.75). Fix $r_0 > R_1$. By (3.4.78)

$$\begin{aligned}(2^n r_0)^{2k-2}|\mu_k(t,2^n r_0)| + (2^n r_0)^{4k-d}|\mu_k(t,2^n r_0)|^2 + (2^n r_0)^{6k-d-1}|\mu_k(t,2^n r_0)|^3 \\ \lesssim (2^n r_0)^{2k-5+\epsilon}\end{aligned}$$

uniformly in time. This estimate along with (3.4.76) and the difference estimate (3.4.25) imply for all $1 \le j \le k$

$$|\lambda_j(t, 2^{n+1}r_0)| \le (1 + C\delta_1)|\lambda_j(t, 2^n r_0)| + C(2^n r_0)^{2k-2j-2+\epsilon},$$

uniformly in time. By Corollary 3.4.16, we conclude that

$$\begin{aligned} |\lambda_j(t,r)| &\lesssim r^{2k-2j-2+\epsilon}, \quad \forall 1 \le j < k, \\ |\lambda_k(t,r)| &\lesssim r^{\epsilon}, \end{aligned}$$

uniformly in time. A similar argument establishes the remaining bounds in (3.4.75) involving the μ_j's.

□

As in the case that ℓ is even, we use Lemma 3.4.30 as the base case for an induction argument. In particular, we will prove the following.

Proposition 3.4.31. *Suppose $d = 5, 9, 13, \dots$ and ϵ, δ_1,r_0 are as in Lemma 3.4.30. For $P = 0, 1, \dots, k-1$ the following estimates hold uniformly in time:*

$$\begin{aligned} |\lambda_j(t,r)| &\lesssim r^{2(k-P-j)-2+\epsilon}, \quad \forall 1 \le j \le k \text{ with } j \ne k-P, \\ |\lambda_{k-P}(t,r)| &\lesssim r^{\epsilon}, \\ |\mu_j(t,r)| &\lesssim r^{2(k-P-j)-3+\epsilon}, \quad \forall 1 \le j \le k \text{ with } j \ne k-P-1, \\ |\mu_{k-P-1}(t,r)| &\lesssim r^{\epsilon}. \end{aligned} \tag{3.4.79}$$

In we take $P = k-1$ in Proposition 3.4.31, then we obtain the following.

Proposition 3.4.32. *With the same hypotheses as in Proposition 3.4.31, the following es-*

timates hold uniformly in time:

$$\begin{aligned}
&|\lambda_j(t,r)| \lesssim r^{-2j+\epsilon}, \quad \forall 1 < j \leq k, \\
&|\lambda_1(t,r)| \lesssim r^{\epsilon}, \\
&|\mu_j(t,r)| \lesssim r^{-2j-1+\epsilon}, \quad \forall 1 \leq j \leq k.
\end{aligned} \tag{3.4.80}$$

Proof of Proposition 3.4.31. The proof of Proposition 3.4.31 is nearly identical to the proof of Proposition 3.4.18. Therefore, we will only outline the main steps of the proof and refer the reader to the proofs given for the case that ℓ is even for the details. The proof is by induction on P. The case $P = 0$ is covered in Lemma 3.4.30. We now assume that (3.4.79) holds for all P with $0 \leq P < k - 1$.

Step 1 There exist bounded functions $\alpha_{k-P}(t)$ and $\beta_{k-P-1}(t)$ defined on $\mathbb{R}$ such that

$$\begin{aligned}
|\lambda_{k-P}(t,r) - \alpha_{k-P}(t)| &\lesssim r^{-2}, \\
|\mu_{k-P-1}(t,r) - \beta_{k-P-1}(t)| &\lesssim r^{-1},
\end{aligned}$$

uniformly in t. For details, see the proof of Lemma 3.4.19.

Step 2 We have

$$r^{d-2(k-P)}u_e(t,r) = \alpha_{k-P}(t) + O(r^{-2+\epsilon}),$$

where the $O(\cdot)$ term is uniform in time. For details, see the proof of Corollary 3.4.20.

Step 3 The function $\alpha_{k-P}(t)$ is constant in time and from now on we write α_{k-p} in place of $\alpha_{k-P}(t)$. For details, see the proof Lemma 3.4.22.

Step 4 We have $\alpha_{k-P} = 0$ and $\beta_{k-P-1}(t)$ is constant in time. From now on we write β_{k-P-1} in place of $\beta_{k-P-1}(t)$. For details, see the proof of Lemma 3.4.23.

Step 5 We have $\beta_{k-P-1} = 0$. For details, see the proof of Lemma 3.4.24.

From Steps 1–5, we conclude that

$$\begin{aligned} \lambda_{k-P}(t) &= O(r^{-2}), \\ \mu_{k-P-1}(t) &= O(r^{-1}), \end{aligned} \tag{3.4.81}$$

uniformly in time. Inserting (3.4.79) and (3.4.81) into the difference estimates (3.4.25) and (3.4.26), we conclude that the following holds.

Step 6 If (3.4.79) holds for all $0 \leq P < k-1$, then (3.4.79) holds for $P+1$. For details, see the proof of Lemma 3.4.25.

By induction and Step 6, we have proved Proposition 3.4.31.

□

As in the case that ℓ is even, from Proposition 3.4.32 we deduce the following behavior for λ_1.

Lemma 3.4.33. *There exists $\alpha \in \mathbb{R}$ such that*

$$|\lambda_1(r) - \alpha| = O(r^{-2}). \tag{3.4.82}$$

Moreover, we have the slightly improved decay rates

$$\begin{aligned} |\lambda_j(r)| &\lesssim r^{-2j}, \quad \forall 1 < j \leq k, \\ |\mu_j(r)| &\lesssim r^{-2j-1}, \quad \forall 1 \leq j \leq k. \end{aligned} \tag{3.4.83}$$

Proof. The proof is identical to the proof of Lemma 3.4.26. □

The proof of Proposition 3.4.11 for the case that ℓ odd is identical to the case that ℓ is even and we omit it.

Step 3: Conclusion of the Proof of Proposition 3.4.4

Let α be as in Proposition 3.4.11. We now show that there exists a unique static solution U_+ to (3.4.2) such that $\vec{u}(0) = (U_+, 0)$ on $r \geq \eta$ (where U_+ does not depend on η). We distinguish two cases: $\alpha = 0$ and $\alpha \neq 0$. For the case $\alpha = 0$, we will show that $\vec{u}(0) = (0, 0)$ on $r \geq \eta$. We first show that if $\alpha = 0$, then $\vec{u}(0, r)$ is compactly supported.

Lemma 3.4.34. *Let $\vec{u}_e$ be as in Proposition 3.4.4, and let α be as in Proposition 3.4.11. If $\alpha = 0$, then $\vec{u}(0, r)$ is compactly supported in $r \in (\eta, \infty)$.*

Proof. If $\alpha = 0$, then by Lemma 3.4.26 and Lemma 3.4.33, we have

$$\begin{aligned} |\lambda_j(r)| &\lesssim r^{-2j}, \quad \forall 1 \leq j \leq \tilde{k}, \\ |\mu_j(r)| &\lesssim r^{-2j-1}, \quad \forall 1 \leq j \leq k. \end{aligned} \tag{3.4.84}$$

Thus, there exists C_1 such that for all $r \geq \eta$

$$\sum_{j=1}^{\tilde{k}} r^{2j} |\lambda_j(r)| + \sum_{j=1}^{k} r^{2j+1} |\mu_j(r)| \leq C_1. \tag{3.4.85}$$

Fix $r_0 > R_1$. By Corollary 3.4.13, we have

$$\begin{aligned} |\lambda_j(2^{n+1} r_0)| &\geq (1 - C\delta_1) |\lambda_j(2^n r_0)| \\ &\quad - C\delta_1 (2^n r_0)^{-2j} \left[\sum_{i=1}^{\tilde{k}} (2^n r_0)^{2i} |\lambda_i(2^n r_0)| + \sum_{i=1}^{k} (2^n r_0)^{2i+1} |\mu_i(2^n r_0)|, \right], \end{aligned}$$

and

$$\begin{aligned} |\mu_j(2^{n+1} r_0)| &\geq (1 - C\delta_1) |\mu_j(2^n r_0)| \\ &\quad - C\delta_1 (2^n r_0)^{-2j-1} \left[\sum_{i=1}^{\tilde{k}} (2^n r_0)^{2i} |\lambda_i(2^n r_0)| + \sum_{i=1}^{k} (2^n r_0)^{2i+1} |\mu_i(2^n r_0)|, \right]. \end{aligned}$$

We conclude that

$$\sum_{i=1}^{\tilde{k}}(2^{n+1}r_0)^{2i}|\lambda_i(2^{n+1}r_0)| + \sum_{i=1}^{k}(2^{n+1}r_0)^{2i+1}|\mu_i(2^{n+1}r_0)|$$
$$\geq 4\Big(1 - C\delta_1(\tilde{k}+k+1)2^{2k+1}\Big)\left[\sum_{i=1}^{\tilde{k}}(2^n r_0)^{2i}|\lambda_i(2^n r_0)| + \sum_{i=1}^{k}(2^n r_0)^{2i+1}|\mu_i(2^n r_0)|\right].$$

If we fix δ_1 so small so that $C\delta_1(\tilde{k}+k+1)2^{2k+1} < \frac{1}{2}$, then we conclude that

$$\sum_{i=1}^{\tilde{k}}(2^{n+1}r_0)^{2i}|\lambda_i(2^{n+1}r_0)| + \sum_{i=1}^{k}(2^{n+1}r_0)^{2i+1}|\mu_i(2^{n+1}r_0)|$$
$$\geq 2\left[\sum_{i=1}^{\tilde{k}}(2^n r_0)^{2i}|\lambda_i(2^n r_0)| + \sum_{i=1}^{k}(2^n r_0)^{2i+1}|\mu_i(2^n r_0)|\right].$$

Iterating, we conclude that for all $n \geq 0$,

$$\sum_{i=1}^{\tilde{k}}(2^n r_0)^{2i}|\lambda_i(2^n r_0)| + \sum_{i=1}^{k}(2^n r_0)^{2i+1}|\mu_i(2^n r_0)|$$
$$\geq 2^n\left[\sum_{i=1}^{\tilde{k}}(r_0)^{2i}|\lambda_i(r_0)| + \sum_{i=1}^{k}(r_0)^{2i+1}|\mu_i(r_0)|\right].$$

By (3.4.85), we obtain for all $n \geq 0$,

$$\sum_{j=1}^{\tilde{k}}(r_0)^{2i}|\lambda_i(r_0)| + \sum_{i=1}^{k}(r_0)^{2i+1}|\mu_i(r_0)| \leq 2^{-n}C_1.$$

Letting $n \to \infty$ implies that

$$\lambda_j(r_0) = \mu_i(r_0) = 0, \quad \forall 1 \leq j \leq \tilde{k}, 1 \leq i \leq k.$$

By Lemma 3.4.6 and Lemma 3.4.7, it follows that $\|\vec{u}_e(0)\|_{\mathcal{H}(r\geq r_0)} = 0$. Thus, $(\partial_r u_{e,0}, u_{e,1})$

is compactly supported in (η, ∞). Since

$$\lim_{r\to\infty} u_{e,0}(r) = 0,$$

we conclude that $(u_{e,0}, u_{e,1})$ is compactly supported as well. $\square$

Lemma 3.4.35. *Let $\vec{u}_e$ be as in Proposition 3.4.4, and let α be as in Proposition 3.4.11. If $\alpha = 0$, then $\vec{u}(0, r) = (0, 0)$ on $r \geq \eta$.*

Proof. If $\alpha = 0$, then by Lemma 3.4.34, $\vec{u}_e(0) = (u_{e,0}(r), u_{e,1}(r))$ is compactly supported in (η, ∞). Thus, we may define

$$\rho_0 := \inf \left\{ \rho : \|\vec{u}_e(0)\|_{\mathcal{H}(r\geq\rho)} = 0 \right\} < \infty.$$

We now argue by contradiction and assume that $\rho_0 > \eta$. Let $\epsilon > 0$ to be fixed later, and choose $\rho_1 \in (\eta, \rho_0)$ close to ρ_0 so that

$$0 < \|\vec{u}_e(0)\|_{\mathcal{H}(r\geq\rho_1)} < \epsilon, \tag{3.4.86}$$

where δ_1 is as in Lemma 3.4.12.

By Lemma 3.4.6, we have

$$\begin{aligned}
0 &= \|\vec{u}_e(0)\|^2_{\mathcal{H}(r\geq\rho_0)} \\
&\simeq \sum_{i=1}^{\tilde{k}} \Big(\lambda_i(\rho_0)\rho_0^{2i-\frac{d+2}{2}}\Big)^2 + \sum_{j=1}^{k} \Big(\mu_j(\rho_0)\rho_0^{2i-\frac{d}{2}}\Big)^2, \\
&\quad + \int_{\rho_0}^{\infty} \sum_{i=1}^{\tilde{k}} \Big(\partial_r \lambda_i(r) r^{2i-\frac{d+1}{2}}\Big)^2 + \sum_{j=1}^{k} \Big(\partial_r \mu_j(r) r^{2i-\frac{d-1}{2}}\Big)^2 dr.
\end{aligned}$$

Thus, $\lambda_j(\rho_0) = \mu_i(\rho_0) = 0$ for all $1 \leq j \leq \tilde{k}, 1 \leq i \leq k$.

A simple reworking of the proofs of Lemma 3.4.9 and Lemma 3.4.7 shows that as long as ϵ and $|\rho_0 - \rho_1|$ is sufficiently small, we have for all ρ with $1 < \rho_1 \leq \rho \leq \rho_0$,

$$\|\pi_\rho^\perp \vec{u}_e(t)\|_{\mathcal{H}(r\geq\rho)} \lesssim (\rho_0 - \rho)^{1/3} \|\pi_\rho \vec{u}_e(t)\|_{\mathcal{H}(r\geq\rho)} + \|\pi_\rho \vec{u}_e(t)\|^2_{\mathcal{H}(r\geq\rho)} + \|\pi_\rho \vec{u}_e(t)\|^3_{\mathcal{H}(r\geq\rho)}, \tag{3.4.87}$$

where the implied constant is independent of ρ. In the argument, smallness is achieved by taking ϵ and $|\rho_0 - \rho_1|$ sufficiently small, cutting off the potential term to the exterior region $\{\rho + t \leq r \leq \rho_0 + t\}$, and using the compact support of $\vec{u}_e$ along with finite speed of propagation. By taking ρ_1 even closer to ρ_0 so that $|\rho_0 - \rho_1| < \epsilon^3$, we conclude as in Corollary 3.4.13 that

$$\begin{aligned} |\lambda_j(\rho_0) - \lambda_j(\rho_1)| &\leq C\epsilon \left(\sum_{i=1}^{\tilde{k}} |\lambda_i(\rho_1)| + \sum_{i=1}^{k} |\mu_i(\rho_1)| \right), \\ |\mu_j(\rho_0) - \mu_j(\rho_1)| &\leq C\epsilon \left(\sum_{i=1}^{\tilde{k}} |\lambda_i(\rho_1)| + \sum_{i=1}^{k} |\mu_i(\rho_1)| \right). \end{aligned}$$

Since $\lambda_j(\rho_0) = \mu_j(\rho_0) = 0$ we conclude by summing the previous expressions that

$$\sum_{i=1}^{\tilde{k}} |\lambda_i(\rho_1)| + \sum_{i=1}^{k} |\mu_i(\rho_1)| \leq C(k + \tilde{k})\epsilon \left(\sum_{i=1}^{\tilde{k}} |\lambda_i(\rho_1)| + \sum_{i=1}^{k} |\mu_i(\rho_1)| \right).$$

By fixing ϵ sufficiently small, it follows that

$$\sum_{i=1}^{\tilde{k}} |\lambda_i(\rho_1)| + \sum_{i=1}^{k} |\mu_i(\rho_1)| = 0.$$

Thus, $\lambda_j(\rho_1) = \mu_j(\rho_1) = 0$. By Lemma 3.4.6 and (3.4.87), we conclude that

$$\|\vec{u}_e(0)\|_{r\geq\rho_1)} = 0$$

which contradicts (3.4.86). Thus, we must have $\rho_0 = \eta$ and $\vec{u}_e(0,r) = (0,0)$ for $r \geq \eta$ as desired. $\square$

From the previous argument, we conclude even more for the case $\alpha = 0$.

Lemma 3.4.36. *Let α be as in Lemma 3.4.19. If $\alpha = 0$, then*

$$\vec{u}(t,r) = (0,0), \quad \forall (t,r) \in \mathbb{R} \times (0,\infty).$$

Proof. By Lemma 3.4.35 we know that if $\alpha = 0$ then $\vec{u}(0,r) = (0,0)$ on $\{r \geq \eta\}$. By finite speed of propagation, we conclude that

$$\vec{u}(t,r) = (0,0) \quad \text{ on } \{r \geq |t| + \eta\}. \tag{3.4.88}$$

Let $t_0 \in \mathbb{R}$ be arbitrary and define $u_{t_0}(t,r) = u(t + t_0, r)$. Then $\vec{u}_{t_0}$ inherits the following compactness property from $\vec{u}$:

$$\begin{gathered} \forall R \geq 0, \quad \lim_{|t|\to\infty} \|\vec{u}_{t_0}(t)\|_{\mathcal{H}(r \geq R+|t|; \langle r\rangle^{d-1} dr)} = 0, \\ \lim_{R\to\infty} \left[\sup_{t\in\mathbb{R}} \|\vec{u}_{t_0}(t)\|_{\mathcal{H}(r \geq R+|t|; \langle r\rangle^{d-1} dr)} \right] = 0, \end{gathered}$$

and by (3.4.88) $\vec{u}_{t_0}(0,r)$ is supported in $\{0 < r \leq \eta + |t_0|\}$. By the proof of Lemma 3.4.35 applied to $\vec{u}_{t_0}$ we conclude that $\vec{u}_{t_0}(0,r) = (0,0)$ on $r \geq \eta$. Since t_0 was arbitrary, we conclude that

$$\vec{u}(t_0, r) = (0,0) \quad \text{on } \{r \geq \eta\},$$

for any $t_0 \in \mathbb{R}$. Since $\eta > 0$ was arbitrarily fixed in the beginning of this subsection, we conclude that

$$\vec{u}(t,r) = (0,0), \quad \forall (t,r) \in \mathbb{R} \times (0,\infty).$$

□

We now consider the general case for α.

Lemma 3.4.37. *Let α be as in Lemma 3.4.19. As before, we denote the unique ℓ–equivariant finite energy harmonic map of degree n by Q and recall that there exists a unique $\alpha_{\ell,n} > 0$ such that*

$$Q(r) = n\pi - \alpha_{\ell,n} r^{-\ell-1} + O(r^{-\ell-3}) \quad \text{as } r \to \infty.$$

Let $Q_{\alpha-\alpha_{\ell,n}}$ denote the unique solution to (3.2.3) *with the property that*

$$Q_{\alpha-\alpha_{\ell,n}}(r) = n\pi + (\alpha - \alpha_{\ell,n}) r^{-\ell-1} + O(r^{-\ell-3}) \quad \text{as } r \to \infty. \tag{3.4.89}$$

Note that $Q_{\alpha-\alpha_{\ell,n}}$ exists and is unique by Proposition 3.2.2. Define a static solution U_+ to (3.3.1) *via*

$$U_+(r) = \langle r \rangle^{-\ell} \big(Q_{\alpha-\alpha_{\ell,n}}(r) - Q(r) \big).$$

Then

$$\vec{u}(t,r) = (U_+(r), 0), \quad \forall (t,r) \in \mathbb{R} \times (0,\infty).$$

Proof. Lemma 3.4.37 follows from the proof for the $\alpha = 0$ case and a change of variables.

Let $Q_{\alpha-\alpha_{\ell,n}}$ be as in the statement of the lemma. We define

$$\begin{aligned} u_\alpha(t,r) &:= u(t,r) - \langle r\rangle^{-\ell}\left(Q_{\alpha-\alpha_{\ell,n}}(r) - Q(r)\right) \\ &= u(t,r) - U_+(r) \end{aligned} \tag{3.4.90}$$

and observe that u_α solves

$$\partial_t^2 u_\alpha - \partial_r^2 u_\alpha - \frac{(d-1)r}{r^2+1}\partial_r u_\alpha + V_\alpha(r)u_\alpha = N_\alpha(r,u_\alpha),$$

where the potential V_α is given by

$$V_\alpha(r) = \ell^2\langle r\rangle^{-4} + 2\langle r\rangle^{-2}(\cos 2Q_{\alpha-\alpha_{\ell,n}} - 1), \tag{3.4.91}$$

and $N_\alpha(r,u) = F_\alpha(r,u) + G_\alpha(r,u)$ with

$$\begin{aligned} F_\alpha(r,u) &= \ell(\ell+1)\langle r\rangle^{-\ell-2}\sin^2(\langle r\rangle^\ell u)\sin 2Q_{\alpha-\alpha_{\ell,n}}, \\ G_\alpha(r,u) &= \frac{\ell(\ell+1)}{2}\langle r\rangle^{-\ell-2}\left[2\langle r\rangle^\ell u - \sin(2\langle r\rangle^\ell u)\right]\cos 2Q_{\alpha-\alpha_{\ell,n}}. \end{aligned} \tag{3.4.92}$$

By (3.4.89), the potential V_α is smooth and satisfies

$$V_\alpha(r) = \ell^2\langle r\rangle^{-4} + O(\langle r\rangle^{-2\ell-4}),$$

as $r \to \infty$ and the nonlinearities F_α and G_α satisfy

$$\begin{aligned} |F_\alpha(r,u)| &\lesssim \langle r\rangle^{-3}|u|^2, \\ |G_\alpha(r,u)| &\lesssim \langle r\rangle^{d-5}|u|^3, \end{aligned}$$

for $r \geq 0$. Moreover, by (3.4.90) we see that $\vec{u}_\alpha$ inherits the compactness property from $\vec{u}$:

$$\begin{aligned} \forall R \geq 0, \quad \lim_{|t|\to\infty} \|\vec{u}_\alpha(t)\|_{\mathcal{H}(r\geq R+|t|;\langle r\rangle^{d-1}dr)} &= 0, \\ \lim_{R\to\infty} \left[\sup_{t\in\mathbb{R}} \|\vec{u}_\alpha(t)\|_{\mathcal{H}(r\geq R+|t|;\langle r\rangle^{d-1}dr)} \right] &= 0. \end{aligned} \tag{3.4.93}$$

Let $\eta > 0$. We now define for $r \geq \eta$,

$$u_{\alpha,e}(t,r) := \frac{\langle r\rangle^{(d-1)/2}}{r^{(d-1)/2}} u_\alpha(t,r) \tag{3.4.94}$$

and note that $u_{\alpha,e}$ satisfies an equation analogous to u_e:

$$\partial_t^2 u_{\alpha,e} - \partial_r^2 u_{\alpha,e} - \frac{d-1}{r}\partial_r u_{\alpha,e} + V_{\alpha,e}(r)u_{\alpha,e} = N_{\alpha,e}(r,u_{\alpha,e}), \quad t\in\mathbb{R}, r\geq\eta, \tag{3.4.95}$$

where

$$V_{\alpha,e}(r) = V_\alpha(r) - \frac{(d-1)(d-4)}{2r^2\langle r\rangle^2} + \frac{(d-1)(d-5)}{4r^2\langle r\rangle^4},$$

and $N_{\alpha,e}(r,u_e) = F_{\alpha,e}(r,u_e) + G_{\alpha,e}(r,u_e)$ where

$$\begin{aligned} F_{\alpha,e}(r,u_{\alpha,e}) &= \frac{\langle r\rangle^{(d-1)/2}}{r^{(d-1)/2}} F_\alpha\left(r, \frac{r^{(d-1)/2}}{\langle r\rangle^{(d-1)/2}} u_{\alpha,e}\right), \\ G_{\alpha,e}(r,u_{\alpha,e}) &= \frac{\langle r\rangle^{(d-1)/2}}{r^{(d-1)/2}} G_\alpha\left(r, \frac{r^{(d-1)/2}}{\langle r\rangle^{(d-1)/2}} u_{\alpha,e}\right). \end{aligned}$$

In particular, we have the analogues of (3.4.8), (3.4.9), and (3.4.10): for all $r > 0$,

$$|V_{\alpha,e}(r)| \lesssim r^{-4}, \tag{3.4.96}$$

$$|F_{\alpha,e}(r,u)| \lesssim r^{-3}|u|^2, \tag{3.4.97}$$

$$|G_{\alpha,e}(r,u)| \lesssim r^{d-5}|u|^3. \tag{3.4.98}$$

Moreover, $u_{\alpha,e}$ inherits the following compactness properties from u_α:

$$\begin{aligned} \forall R \geq \eta, \quad \lim_{|t|\to\infty} \|\vec{u}_{\alpha,e}(t)\|_{\mathcal{H}(r\geq R+|t|;r^{d-1}dr)} &= 0, \\ \lim_{R\to\infty}\left[\sup_{t\in\mathbb{R}} \|\vec{u}_{\alpha,e}(t)\|_{\mathcal{H}(r\geq R+|t|;r^{d-1}dr)}\right] &= 0. \end{aligned} \tag{3.4.99}$$

Finally, by construction we see that

$$\begin{aligned} r^{2-d}u_{\alpha,e,0}(r) &= O(r^{-2}), \\ \int_r^\infty u_{\alpha,e,1}(\rho)\rho^{2j-1}d\rho &= O(r^{2j-d-1}), \quad j = 1,\dots,k. \end{aligned} \tag{3.4.100}$$

Using (3.4.95)–(3.4.100), we may repeat the previous arguments with $u_{e,\alpha}$ in place of u_e to conclude the following analog of Lemma 3.4.35:

Lemma 3.4.38. $\vec{u}_\alpha(0,r) = (0,0)$ *for* $r \geq \eta$.

Finally, we obtain the following analog of Lemma 3.4.36:

Lemma 3.4.39. *We have*

$$\vec{u}_\alpha(t,r) = (0,0)$$

for all $t \in \mathbb{R}$ *and* $r > 0$.

Equivalently, Lemma 3.4.39 states that

$$\vec{u}(t,r) = (U_+(r),0)$$

for all $t \in \mathbb{R}$ and $r > 0$. This concludes the proof of Lemma 3.4.37 and Proposition 3.4.4.

□

3.4.2 Proof of Proposition 3.4.3

Using Proposition 3.4.4 and its analog for $r < 0$, we quickly conclude the proof of Proposition 3.4.3. Indeed, we know that there exists static solutions $U_\pm$ to (3.3.1) such that

$$\vec{u}(t,r) = (U_\pm(r), 0) \tag{3.4.101}$$

for all $\pm r > 0$ and $t \in \mathbb{R}$. In particular, $\partial_t u(t,r) = 0$, $\partial_r u(t,r) = \partial_r u(0,r)$ and $u(t,r) = u(0,r)$ for all t and almost every r. Let $\psi \in C_0^\infty(\mathbb{R})$ with $\int \psi dt = 1$ and let $\varphi \in C_0^\infty(\mathbb{R})$. Then since u solves (3.3.1) in the weak sense, we conclude that

$$\begin{aligned} 0 &= \int\int \big[\psi'(t)\varphi(r)\partial_t u(t,r) + \psi(t)\varphi'(r)\partial_r u(t,r) + V(r)\psi(t)\varphi(r)u(t,r) \\ &\qquad - \psi(t)\varphi(r)N(r,u(t,r))\big]\langle r\rangle^{d-1} dr dt \\ &= \int\int \psi(t)\big[\varphi'(r)\partial_r u(0,r) + V(r)\varphi(r)u(0,r) - \varphi(r)N(r,u(0,r))\big]\langle r\rangle^{d-1} dr dt \\ &= \int \big[\varphi'(r)\partial_r u(0,r) + V(r)\varphi(r)u(0,r) - \varphi(r)N(r,u(0,r))\big]\langle r\rangle^{d-1} dr. \end{aligned}$$

Since φ was arbitrary, we see that $u(0,r)$ is a weak solution in $H^1(\mathbb{R})$ to the static equation $-\partial_r^2 u - \frac{(d-1)r}{r^2+1}\partial_r u + V(r)u = N(r,u)$ on $\mathbb{R}$. By simple elliptic arguments we conclude that $u(0,r)$ is a classical solution. Thus, $\vec{u}(t,r) = (U(r), 0) := (u(0,r), 0)$ for all $t, r \in \mathbb{R}$ as desired.

□

3.4.3 Proofs of Proposition 3.4.1 and Theorem 3.2.3

We now briefly summarize the proofs of Proposition 3.4.1 and Theorem 3.2.3.

Proof of Proposition 3.4.1. By Proposition 3.4.3, we have that $\vec{u} = (U, 0)$ for some finite energy static solution to (3.3.1). Thus, $\psi = Q_{\ell,n} + \langle r\rangle^\ell u$ is a finite energy solution to

(3.2.1), i.e. a harmonic map. By the uniqueness part of Proposition 3.2.1, we conclude that $\vec{u} = (0, 0)$ as desired. □

Proof of Theorem 3.2.3. Suppose that Theorem 3.2.3 fails. Then by Proposition 3.3.5, there exists a nonzero solution u_* to (3.3.1) such that the trajectory

$$K := \{\vec{u}_*(t) : t \in \mathbb{R}\},$$

is precompact in $\mathcal{H}(\mathbb{R}; \langle r \rangle^{d-1} dr)$. By Proposition 3.4.1 we conclude that $\vec{u}_* = (0, 0)$, which contradicts the fact that u_* is nonzero. Thus, Theorem 3.2.3 must hold. □

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