THE UNIVERSITY OF CHICAGO

NETWORK AND PRICE DYNAMICS IN A MODEL OF CONTAGION

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ABSTRACT

In the first chapter of the thesis I study network dynamics. Given an initial financial network and liquidity needs of the agents forming it, I am interested in how that network evolves over time. Who lends to whom? I show that when agents are allowed to lend in the interbank market, they have a bigger incentive to lend to agents to whom they already have a big exposure to, which I relate to agents Bonacich centrality in the financial network. I also show that this is the unique equilibrium when the exposure of the lender is relatively big.

In the second chapter I study an environment of optimal inattention to the stock market where the investor has Epstein-Zin preferences. As in [1], the solution strategy of an optimal inattention to the stock market can be done in four easy steps. Although the set-up is in continuous time, the fact that the consumer only decides to evaluate his wealth at discrete times allows us to focus on discrete time Epstein-Zin preferences which simplifies the problem considerably. The first important difference with respect to [1] is that the optimal portfolio will depend on the risk aversion coefficient. Only through the optimal portfolio will the optimal choice of consumption and stopping time be affected. Otherwise, the problem is identical to the case with expected utility.

CHAPTER 1

NETWORK AND PRICE DYNAMICS IN A MODEL OF CONTAGION

1.1 Introduction

In view of the recent financial crisis, a large body of work has focused on contagion though interlinkages as an important mechanism in the propagation of liquidity shocks. Most of the literature until now has modeled interlinkages through network models, and it has mainly focused on how a few illiquid financial institutions have an impact on, and potentially cause the breakdown of the whole financial system, which the literature calls cascading effects. Part of that literature analyzes the properties of different financial networks taking interlinkages as given. Given that financial institutions are signing contracts with each other voluntarily, more recent work has acknowledged the exogeneity assumption as a weakness, and we now have several frameworks where interlinkages are determined endogenously.

Although they open the door for important policy questions, their applicability is undermined by the fact that these models are essentially static (see for example [15]). Questions regarding how the network evolves after a liquidity shock, or perhaps most importantly, how fast, is of first order importance, especially in times of crises. In fact, [8] and [21] document empirically that period of crises exhibit major changes in the network structure. But, with a few salient exceptions, most models of network formation assume exogenous prices and surpluses. In addition, most of the analysis is performed ex-ante. All these features limit their interpretation in a dynamic setting.

My goal in this paper is precisely to study how financial networks evolve over time. More specifically, given an initial financial network that I take as given, and liquidity shocks that partition financial institutions into liquid and illiquid, the question I ask is "who lends to whom and under what terms?". I do that by building on a model of contagion as in [2]. Absent any external financing, illiquid agents face costly default and can even trigger the default of other liquid agents. Recent literature has focused on the extent of these cascading effects. I depart from the current literature by allowing agents to borrow from one another and I show that the presence of the interagent market limits the impact of the liquidity shocks and that it has strong predictions regarding the new contractual relationships that arise in equilibrium.

More precisely, I show that when all agents are solvent, then the impact of the idiosyncratic liquidity shocks is completely muted - the interagent market functions smoothly, and liquid agents arise endogenously as lenders to illiquid agents, and no agents default in equilibrum. Then I show that liquid agents have a bigger incentive to lend to illiquid agents to whom they have already a big exposure, making interlinkages persistent over time. Most strikingly, this is the only equilibrium that survives when aggregate exposure of lenders to borrowers is relatively big. I start by proving the results for a general four player network in section (1.6), and then generalize them on a N-player network in section (1.7).

This framework helps answer one of the natural questions raised by the contagion literature: what is the impact that \$1 liquidity injection has on number of defaults? Or put differently, from the point of view of a planner that wants to prevent inefficient default, how can he do that with the minimum liquidity transfer? Counterintuitively, in the model presented here, the central planner can minimize the needed liquidity injection by making transfers to the surviving agents, instead of the agents that are directly affected by the liquidity shocks. But such an intervention comes at a cost: liquid agents don't necessarily lend to the most productive or solvent agents, but to the ones they are most exposed to. In equilibrium, if there simply isn't enough liquidity to save all agents, then I show that it is not the most solvent ones that survive, but the most interconnected or central in terms their Bonacich centrality.

There are many definitions of interconnectedness or centrality as documented by [11] degree, closeness, and betweenness centrality, which in turn can be classified by walk type, walk property, walk positions, and summary type. The number of possible combinations is immense and each measure makes underlying assumption regarding the propagation mechanism of shocks. In an effort to summarize complex financial networks, the empirical literature on contagion (for example [8], [14], [21], [17], [12] just to name a few) has used some of these off-the-shelf centrality measures to describe and predict past and future contagions. In the spirit of [10], I show that the most commonly used centrality measures are not appropriate for studying contagion and this paper offers guidance in that regard. In fact, my definition of centrality or exposure can be interpreted as a generalized measure of Bonacich/ Katz centrality ([9], [18]).

The paper most similar to mine in the financial networks literature is [19]. There, each agent has to choose between investing into a project or not, out of an endowment that can be less or more than the initial cost of the investment. Moreover, in order to capture the network effects, the investment succeeds only if all agents that are connected to each other invest. Heterogeneity in agent's initial allocation suggests that there are gains from trade between the members of a given network, but [19] specifically assumes that intra-network lending is not an option. Although his emphasis is on ex-ante network formation, he finds that ex-post, groups of liquid agents find in their best interest to collectively bail-out other illiquid agents through unilateral transfers, mainly due to fear of contagion. In my setup, the same mechanism holds. Because of contagion effects, liquid agents lend to illiquid agents they are already most exposed to. And, similarly to a bailout, liquid agents are willing to lend at a discount. [19] focuses on unilateral transfers, that is, by assumption agents cannot borrow. By contrast to [19], I allow agents to borrow against their fundamental value.

Papers that analyze strategic internetwork lending in a network generally ignore the network dimension. For example [22] looks at how binding and non-binding credit arrangements impact contagion. But his analysis is restricted to a liquidity shock that affects one single agent at a time, ignoring the strategic effects altogether. [2] explicitly models internetwork lending and interest rates, but again the lending takes place ex-ante, when there are no network connections, and debt covenants make interest rates contingent on the borrower's behavior.

My paper also relates to the broader literature on liquidity risk sharing.

1.2 The European Debt Crisis

Although my model is general enough that it can capture a general dynamics between borrowers and lenders when contagion is a risk, I provide as an example the European Debt Crises of 2009, where the dynamics presented here are at play.

During 2009 the situation in Greece deteriorated up to the point where Greece was at risk of defaulting on its sovereign debt. European banks at the time owned a significant amount in sovereign debt, making banking system and sovereign solvency reinforcing. By April 2010 it was apparent that the country was becoming unable to borrow on the international market; on 23 April 2010, the Greek government requested an initial loan of 45 billion from the EU and International Monetary Fund (IMF), to cover its financial needs for the remaining part of 2010.

The stated reason behind the bailout at the time was the domino effect and the repercussion to other European countries facing debt crises of their won. In a leak to a Spanish newspaper, the May 2010 minutes of the IMF meeting regarding the bailout plan and a subsequent interview by the former Central Bank head Karl Otto Pöhl, reveal the true considerations at the time. In his interview Karl Otto Pöhl discusses the fact that the ultimate consideration was the saving of the French banks, and ultimately the German ones.

Although the German banks had a small exposure to Greece, German bank would have been severely effected by the default or distress of their French counter-parties, as we see from data on the cross-holdings are for the end of December 2011 from the BIS (Bank for International Settlements) Quarterly Review. The data used for this exercise are the consolidated foreign claims of banks from one country on debt obligations of another country. The data looks at the immediate borrower rather than the final borrower when a bank from a country different from the final borrower serves as an intermediary.

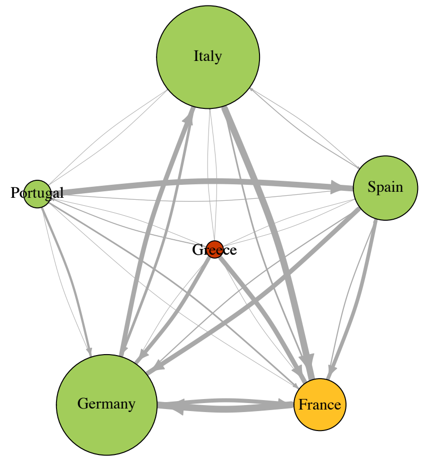

Figure 1.1: European Interbank exposure as of end of December 2011 according to the International Bank of Settlements.

1.3 The Setup

I study an economy with perfect information in the spirit of [2]. The economy is composed of n risk neutral agents indexed by $i \in N = \{1, 2, \ldots, n\}$. Agents live for T periods, and they only care about their final assets or account balance, a_{iT}.

Agents can borrow from and lend to each other using one period debt contracts. Contracts are standardized and are restricted to have the same notional, which I normalize to 1. Agents can engage in multiple contracts with the same counterparty. Each contract x_{ij} specifies the lender, j, the borrower, i, and the gross interest payments at $t+1$ given by R_{ijt+1}.

The terms of the contracts are negotiated in one round as follows:

1. All agents $i \in N$ simultaneously decide which contracts to post as borrowers. Posting a contract has a small cost of κ;

2. Given all agents borrowing strategies, each agent i decides simultaneously which contracts to accept among those that specify them as lenders;

3. Given agents lending strategy, payments to time $t-1$ creditors are made; end of period account balance receives the competitive gross interest rate $R > 1$.

Let X_t denote the set of all contracts signed at time t. I call the matrix $\mathbf{R}_t \circ \mathbf{b}_{t-1}$ the time-t financial network where the entry ij is defined as the total amount that i owes to j at time t according to the contracts signed at time $t-1$. Let $\mathcal{W}_i$ denote the strategy space of borrower $i \in N$ and $\mathcal{W} = \cup_i \mathcal{W}_i$ the space where all contracts live. For any set of contracts $W_t \subseteq \mathcal{W}$, let $W_{\rightarrow it} \subseteq W_t$ denote the set of contracts that involve agent i as a borrower, and $W_{i\rightarrow t} \subseteq W_t$ denote the set of contracts that involve agent i as a lender:

$$W_{\rightarrow it} = \{x_{ij} : x_{ij} \in W_t\}$$
$$W_{i\rightarrow t} = \{x_{ji} : x_{ji} \in W_t\}$$

At time t, in addition to a cash inflow $\tilde{y}_{it} \geq 0$, agent i is entitled to payments from its debtors worth

$$\sum_{j:x_{ji}\in W_{i\rightarrow t-1}} R_{jit} b_{jit-1}$$

and it also has obligations worth

$$\sum_{j:x_{ij}\in W_{\rightarrow it-1}} R_{ijt} b_{ijt-1}.$$

Therefore, agent i's book value liquidity position is given by:

$$a_{it-1} + y_{it} + \sum_{j:x_{ji}\in W_{i\rightarrow t-1}} R_{jit} b_{jit-1} - \sum_{j:x_{ij}\in W_{\rightarrow it-1}} R_{ijt} b_{ijt-1}. \tag{1.1}$$

A positive book value assets indicates that, on paper, agent i is liquid at time t. That is, if all of agent i's debtors pay their obligations in full, given the cash that i has available,

i will also be able to meet all its obligations in full.[1] If book value liquidity is negative, then agent i is below target with respect to their account balance, and I call them illiquid. I call agent i liquid if book value liquidity position is positive. Absent any financing, illiquid agents default on their obligations. Let d_{it} denote the amount that i defaults at time t.

In case of default at time t_0, agents exit the market and they lose access to all their future cash flows. Agents can reenter the market and regain the right to operate by paying a penalty equal to the amount they defaulted d_{it-1}, which I define and characterize in the next section, plus interest $(R-1)d_{it-1}$. Agents have to pay all default penalties first before using the cash flows for borrowing or lending purposes. Moreover, I assume that, although agents are required to pay all their default penalties in full, there is a social loss and lenders that have been defaulted upon in the past only receive a fraction of what they are entitled to. For computational simplicity I assume that that loss is 100% and that past lenders are never paid back once they are defaulted upon.

Let D_{it} denote i's remaining balance of default payments, $\tilde{y}_{it} > 0$ the cash flow at time t, $d_{it-1} > 0$ is the total amount defaulted at $t-1$, and $|X_{\to it}|$ represents the number of contracts offered at time t. Then the law of motion of the remaining balance on the default penalties is given by:

$$D_{it} = d_{it} + [-(a_{it-1} + \tilde{y}_{it} - RD_{it-1})]^{+}$$

Note that $D_{it-1} > 0$ implies that $a_{it-1} = 0$, so we can write

$$D_{it} = d_{it} + [-(\tilde{y}_{it} - RD_{it-1})]^{+}.$$

1. But a positive book value assets does not guarantee that agent i will pay its obligations in full. As a matter of fact, it represents an upper bound on agent i's realized position: if agent i's debtors default, i receives less cash than entitled to, making its realized liquidity position smaller that its book value liquidity position. It might even become negative, which triggers i's own default.

Let y_{it} denote the effective cash flows left for payment of time $t-1$ obligations:

$$y_{it} = [\tilde{y}_{it} - RD_{it-1} - \kappa|X_{\to it}|]^{+}.$$

Notice that a penalty below the competitive rate R implies that default is not costly, and illiquid agents have an incentive to lend whatever funds they have to the outside financiers at a rate R and always incur the default penalty. Similarly, if the default penalty exceeds R, illiquid agents would never have an incentive to default as borrowing from the outside financiers is always profitable. In order to observe some default on the equilibrium path it has to be that the default penalty is exactly R. As we will see later on, illiquid agents will use the threat of a potential default to extract better terms of trade from its lenders.

In order to avoid default, each agent can borrow from other agents using one period debt contracts. In addition, I assume that transfers from new issuances happen after that period cash flow is received, but before old debt obligations are paid. Agent cannot transfer more than their initial assets. Let b_{ijt} is the total transfer from j to i at time t.

Definition 1 (Feasibility)**.** *A set of contracts W_{it} is feasible if*

$$a_{it-1} + y_{it} \geq \sum_{j:\ x_{ji} \in W_{i\to t}} b_{jit} \geq 0, \forall i.$$

Note that feasibility imposes no restriction on which agents can borrow in terms of their liquidity position. Illiquid agents can also be lenders as long as their assets $a_{it-1} + y_{it}$ are positive. Similarly, a liquid agent can borrow from another liquid agent. Later on I show that in this model liquid agents arise endogenously as lenders and illiquid agents as borrowers.

Relevant for the off the equilibrium path, it is possible that the proceeds from the issuance of new debt could potentially not be enough to cover illiquid agents' liquidity deficit, and they still default. In that case I assume that newly issued debt is junior to outstanding debt. In practice this means that new creditors cannot receive any payments if old creditors are not paid in full.[2]

Assumption 2 (Priority of Debt)**.** *Newly issued debt is junior to outstanding debt.*

1.4 Payment Equilibrium and the Default Outcome

If an illiquid agent fails to raise enough liquidity through new debt issuance, it is not able to pay its creditors in full. Exactly how much each agent ends up defaulting on its $t-1$ creditors, which I call a default output, depends on the lending decision made by all liquid agents, more precisely on the set of contracts signed at time t. In this section I show that for any set of contracts signed at $t-1$, the default outcome exists, it is generically unique and I characterize it.

I use $\bar{d}_{it-1}$ to denote the total obligations that agent i has incurred at time $t-1$, payable at time t:

$$\bar{d}_{it-1} = \bar{d}(X_{it-1}) = \sum_j R_{ijt} b_{ijt-1} :$$

Let the the relative liabilities matrix be given by:

$$\boldsymbol{\pi}_{ij}(X_{t-1}) = \begin{cases} \frac{R_{ijt} b_{ijt-1}}{\bar{d}_{it-1}} & \text{if } \bar{d}_{it-1} > 0 \\ 0 & \text{ow.} \end{cases} \tag{1.2}$$

2. Here I assume that lender j commits to transfer the money to borrower i independently of the default outcome of borrower i. Other papers, [3], assume that agents can write contracts contingent on the default outcome, i.e. j transfers funds to i only when i doesn't default. State contingent contracts eliminate bad equilibria due to coordination problems.

For notational simplicity I use

$$\boldsymbol{\pi}_{t-1} = \boldsymbol{\pi}(X_{t-1}).$$

Assumption 3. *All in-network time t creditors have the same seniority and in case of default they are paid pro-rata according to the relative liabilities matrix.*

I define next the default outcome which is a mirror image of the clearing payment as defined in [13]. The details for the equivalence can be found in appendix (A.1). Let $\boldsymbol{\gamma}_t \in \mathbb{R}^n$ denote the vector of time t cash flows, assuming that all agents pay their $t-1$ debt in full:

$$\gamma_{it} \coloneqq \gamma(X_{it-1}, X_{it}) = y_{it} + \sum_j R_{jit} - \sum_j R_{ijt} + \sum_j b_{ijt} - \sum_j b_{jit},$$

and d_{it} the default outcome, that is, the dollar amount that i defaults on its $t-1$ creditors. By definition, $d_{it} \in [0, \bar{d}_{it-1}]$. y_{it} is the effective cash flow one default payments and the cost of posting contracts are paid.

Definition 4 (Default Outcome)**.** *For any financial arrangement (X_t, X_{t-1}), the default outcome $\mathbf{d}_t \in \mathbb{R}^n_+$ is a fixed point of the following function*

$$\Phi(x; X_{t-1}, X_t) = \left[\boldsymbol{\pi}^T_{t-1} x - \mathbf{a}_{t-1} - \boldsymbol{\gamma}_t\right]^+ \tag{1.3}$$

$$d(X_{t-1}, X_t) = FIX(\Phi(x; X_{t-1}, X_t)) \tag{DO}$$

$$LGD(X_{t-1}, X_t) = \boldsymbol{\pi}^T_{t-1} \cdot d(X_{t-1}, X_t) \tag{LGD}$$

where $[\cdot]^+$ denotes $\max\{0, \cdot\}$; when applied to a vector it denotes an element-wise operation. Similarly for $[\cdot]^- = \min\{0, \cdot\}$.

To simplify notation, sometimes I use:

$$d_{it} \coloneqq d_i(X_{t-1}, X_t)$$

$$LGD_{it} \coloneqq LGD_i(X_{t-1}, X_t).$$

The book value liquidity position of agent i, is an upper bound on its ex-transfers liquidity position. In each period, there is a gap between both concepts for two reasons. First, agent i can potentially agree on a set of contracts where it promises to transfer to other agents in the financial system a net amount of $\sum_j b_{jit} - \sum_j b_{ijt}$. If this quantity is negative so i is a net lender, transfers reduce i's ex-transfers liquidity position by $\sum_j b_{jit} - \sum_j b_{ijt}$.

There is a second reason why the ex-transfers liquidity position of agent i can drop even further: the book value liquidity position includes all agent i's accounts receivable, $\sum_j R_{jit} b_{jit-1}$. But if some illiquid agents fail to raise the liquidity they need to survive, through cascading effects, it might be that some of i's debtors might be affected and default on i a total amount equal to the loss given default given by equation (LGD).[3] Agent i will therefore have to write off part of its debtors obligations and its available resources are reduced further by LGD_{it}. It is now apparent that, although liquid at the beginning of the period, an agent can actually prove to be ex-post illiquid.

How much will i itself default on its creditors? It depends on i's ex-transfers liquidity position, which in turn depends on its book value liquidity position, on how much debt it has to write-off, LGD_{it}, and on the net transfer to the rest of the financial system $\sum_j b_{jit} - \sum_j b_{ijt}$. More specifically, if the ex-transfers liquidity position is a surplus, by definition agent i has enough cash on hand to meet all its obligations and the default outcome is $d_{it} = 0$. If, on the other hand, agent i has a ex-transfers liquidity deficit, then the total amount that i defaults on its creditors depends on the size of the deficit. When the ex-

3. The literature generally expresses the LGD as a proportion of total debt outstanding. Here I express it in terms of total dollar amount. Moreover, here LGD_{it} is the total loss from i's debt portfolio, not from a unique debtor.

transfers liquidity deficit exceeds its obligations, then agent i defaults on all its debt and $d_{it} = \bar{d}_{it}$.

On the contrary, if the ex-transfers liquidity deficits is smaller that agent i's total obligations, then agent i has at least some cash to meet its in-network obligations, and, by definition, i defaults a total amount equal to its ex-transfers liquidity deficit.

Lastly, note that feasibility of offered contracts implies that lenders always cover their debt at least partially:

$$\left[\boldsymbol{\pi}_{t-1}^{T} x - \mathbf{a}_{t-1} - \boldsymbol{\gamma}_t\right]^{+} \leq \bar{\mathbf{d}}_{t-1}, \forall x \in [0, \bar{\mathbf{d}}_{t-1}].$$

Now we can define the law of motion of assets

$$a_{it} \coloneqq a_i(X_t, X_{t-1}) = a_{it-1} + \gamma_{it} - LGD_{it} + d_{it}$$

where the adjustments to $\boldsymbol{\gamma}$ are meant to capture how much debt agent i actually collects and pays. If there is no default, $d_{it} = 0$, and $a_{it-1} + \gamma_{it} - LGD_{it} \geq 0$. Assets are therefore positive. Assets are 0 if $0 < d_{it} < \bar{d}_{it-1}$.

Notice that a_{it} can never be negative. Instances where the ex-transfer liquidity position $\gamma_{it} - LGD_{it}$ exceed total obligations $\bar{d}_{it-1}$ are ruled out through feasibility. This happens because agents are not allowed to transfer more than they can afford ex-ante, their initial liquidity surplus.

The definition of default outcome mirrors the one of a clearing payment as in [13] and [2]. The difference here is that cash on hand is an endogenous object so the default outcome is a function of the ex-transfers liquidity surplus $\mathbf{a}_{t-1} + \boldsymbol{\gamma}_t$, which in turn depends on the contracts signed by agents. It is apparent now the dependence of the default outcome on the contracts signed, X_t, but also on the financial structure inherited from $t-1$, X_{t-1}. To simplify notation, for the most part I drop the dependence on the primitives of the model, (X_{t-1}, X_t).

As in the related literature, [2] and [4], the default outcome captures the possibility of cascading effects, where the number of defaulting agents can potentially be larger that the number of illiquid agents.

The next proposition comes from [2] and shows that the default outcome exists and it is generically unique.

Proposition 5. *For any financial arrangement* (X_t, X_{t-1}) *a default outcome exists and it is unique.*

The following lemma (6) is a restatement of lemma (5) in [13] and uses the fact that the fixed point of a function inherits some of its properties to show that the default outcome is concave in the book value assets:

Lemma 6. *The function* $\mathbf{a}_{t-1} + \boldsymbol{\gamma}_t \to FIX(\Phi(x; X_{t-1}, X_t))$ *is concave.*

A direct corollary is that dropping individual contracts independently has a smaller impact on the loss given default that dropping all contracts at the same time. The main idea is that borrowers involved in the dropped contracts can end up will a liquidity position insufficient to cover all their time $t-1$ contractual obligations. The default of some agents can therefore trigger the default of other borrowers who are liquid ex-ante. The concavity arrises from the fact that one such ex-ante liquid agent can withstand the default of two different agents one at a time, but not their joint default.

Corollary 7. *For any* X_{t-1}*, the loss given default of removing a set of contracts at time t exceeds the loss given default of removing one contract at a time:*

$$\boldsymbol{LGD}(X_{t-1}, X_t \setminus Y_t \setminus W_t) \geq \boldsymbol{LGD}(X_{t-1}, X_t \setminus Y_t) + \boldsymbol{LGD}(X_{t-1}, X_t \setminus W_t)\,.$$

for any X_{t-1}*,* X_t *and* $Y_t, W_t \subseteq X_t$*.*

[13] characterize default outcomes and most importantly, provide the algorithm for finding the clearing payments. We can therefore use that algorithm to find the set of defaulting agents for any financial arrangement (X_{t-1}, X_t). In addition, the algorithm will prove useful to representing the default outcome as a measure of the Bonacich centrality measure which is a measure of how important an agent (or node) is in channeling losses to the rest of the network. I present here a simplified version modified to solve for the default outcome rather than the clearing payment.

For any financial arrangement, X_{t-1}, X_t, and proposed default outcome x, let the set of defaulting agents, denoted by $D(x)$, be given by the set of nodes i, such that $\Phi(x; X_{t-1}, X_t) > 0$. Let $\Lambda(x)$ represent the $n \times n$ diagonal matrix defined as follows:

$$\mathbf{\Lambda}(x; X_{t-1}, X_t) = \begin{cases} 1 & \text{if } i = j \text{ and } i \in D(x; X_{t-1}, X_t) \\ 0 & \text{ow} \end{cases}$$

$\mathbf{\Lambda}(x)$ is a diagonal matrix whose values equal 1 along the diagonal in those rows representing nodes not in default under x, and equal to 0 otherwise. Thus, when multiplied by other matrices or vectors, the $\mathbf{\Lambda}(x)$ matrix converts the entries corresponding to the non defaulting node to 0. The complementary matrix $\mathbb{I} - \mathbf{\Lambda}(x)$ converts entries corresponding to defaulting nodes to 0.

For fixed x', define the map $x \to FF_{x'}(x)$ as follows:

$$FF_{x'}(x) = \mathbf{\Lambda}(x') \left[\boldsymbol{\pi}_{t-1}^T \left(\mathbf{\Lambda}(x')x \right) - \boldsymbol{\gamma}_t - \mathbf{a}_{t-1} \right]$$

This map, $FF_{x'}(x)$, simply returns, for all nodes not defaulting under x', 0, and, for all other nodes, returns the node's liquidity deficit assuming that non defaulting nodes under x' pay in full and defaulting nodes under x' default x. We know from lemma (1) in [13] that under $\sum_i y_{it} > 0$, it is not possible that all agents to default. Thus, $\mathbf{\Lambda}(x')$ has a row sum that is less than 1, and no row sum exceeds 1; this, in turn, implies that $FF_{x'}(x)$ has a unique fixed point by standard input-output matrix results (Karlin 1959, Theorem 8.3.2) given by:

$$f(x') = \left[\mathbb{I} - \mathbf{\Lambda}(x')\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(x')\right]^{-1}\left[-\mathbf{\Lambda}(x')(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})\right]$$

[13] show that the fixed point can be obtained inductively using the following sequence of payment vectors:

$$x^0 = \vec{0} \qquad x^j = f(x^{j-1})$$

I summarize the findings in the next lemma and relate the default outcome to the Bonacich centrality defined in detail in Appendix (A.3).

Proposition 8. *The default outcome is a function of each agent Bonacich centrality with respect to the defaulting set of agents $D(\mathbf{d}_t)$:*

$$\mathbf{d}_t = \mathbf{c}\left[\mathbf{\Lambda}(\mathbf{d}_t)\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(\mathbf{d}_t), -\mathbf{\Lambda}(\mathbf{d}_t)(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1}), 1\right]. \tag{1.4}$$

Note that $\mathbf{\Lambda}(\mathbf{d}_t)\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(\mathbf{d}_t)$ is a subnetwork of the initial network $\boldsymbol{\pi}_{t-1}^T$ which selects only the defaulting agents as nodes. And $-\mathbf{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})$ select those exact defaulting nodes liquidity position once new debt is issued and transfers are realized.

Read literally, equation (1.4) therefore tells us that the default outcome of a particular node is its Bonacich centrality with respect to the full set of defaulting agents, which measures all the ways in which i channels the defaulting losses to the rest of the system. In particular, it counts the total strength of all paths in the defaulting network, $\mathbf{\Lambda}(\mathbf{d}_t)\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(\mathbf{d}_t)$, that start at any other defaulting node j and end at i. Paths that start at defaulting node j are weighted by that defaulting node's *ex-transfers* liquidity *deficit*, $-(\gamma_{jt}+a_{jt-1})$. A defaulting agent affects therefore the default outcome in two distinct way. Firstly, it can affect the number and strength of the paths connecting the defaulting set through $\mathbf{\Lambda}(\mathbf{d}_t)\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(\mathbf{d}_t)$. Secondly, it will add to the default outcome through its own weight given by its ex-transfer liquidity deficit.

More precisely, nodes that are ex-transfers illiquid agents, $\gamma_{jt} + a_{jt-1} < 0$, add to i's Bonacich centrality and therefore to its default outcome, whereas ex-transfers liquid nodes $\gamma_{jt} + a_{jt-1} > 0$, subtract from i's default outcome as they have a liquidity buffer that absorbs partially the losses given default that that agent experiences. Worth mentioning is that nodes that are just balanced, $\gamma_{jt} + a_{jt-1} = 0$, have a weight of zero, but still affect the default outcome $\mathbf{d}_t$, by potentially affecting the number of paths connecting the default set, $\mathbf{\Lambda}(\mathbf{d}_t)\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(\mathbf{d}_t)$.

Unlike the standard definition of the Bonacich centrality, here the strength of the paths, $\mathbf{\Lambda}(\mathbf{d}_t)\boldsymbol{\pi}_{t-1}^T\mathbf{\Lambda}(\mathbf{d}_t)$, depends in turn on the vector of the Bonacich centrality which can be computed as the fixed point of a recursive algorithm described in detail in this section. Like the standard definition of the Bonacich centrality, the default outcome is therefore linear in the ex-transfers liquidity deficit of defaulting agents, once we keep the default set constant.

Through equation (LGD), the loss given default inherits all the properties of the default outcome.

Lemma 9. *Total losses to non-defaulting agents is given by defaulting agents total liquidity* *deficit and I call this property* loss conservation property.

$$\sum_{i \in \mathbf{N} \setminus \mathbf{D}(\mathbf{d}_t)} LGD_i(X_{t-1}, X_t) = - \sum_{i \in \mathbf{D}(\mathbf{d}_t)} \gamma_i(X_{t-1}, X_t)$$

Definition 10. *i is* exposed *to j under (X_{t-1}, X_t) if*

1. *j defaults under (X_{t-1}, X_t); and*

2. *$b_j(X_{t-1}, X_t)\pi_{ji} > 0$.*

1.5 Full Equilibrium Definition

There is no intermediate consumption, and agents maximize their final account balance which is paid off to equity holders. Final payoff depends on the full history of signed contracts.

$$V_i(X^T) = a_{iT} \geq 0.$$

The non-negativity of final payoffs is due to the fact that agents are not allowed to be in debt when the game ends.

Note that the payoffs to agent i depends not only on the specifics of the contracts that i is involved in, but also on the contracts signed by everybody else through the loss given default. Essentially, signing a contract has an externality on the rest of the financial system. I use example (11) to illustrate this dependence.

Example 11. *In figure* (1.2) *agent 1 and agent 2 have a liquidity deficit whereas 3 and 4 have a liquidity surplus worth σ and the arrows indicate the inherited network structure at time t. This implies that, when the green agents do not lend, 1 and 2 are the only ones defaulting. What is the payoff of agent 3 if it decides to lend to 2? The answer necessarily* *depends on whether 4 is already lending to 1 or not. If 4 already lends to 1, then 3 collects the interest rate on the loan with 2. If 4 doesn't lend to 1 to start with, in addition to collecting the interest on the loan, 3 also has to write off σ.*

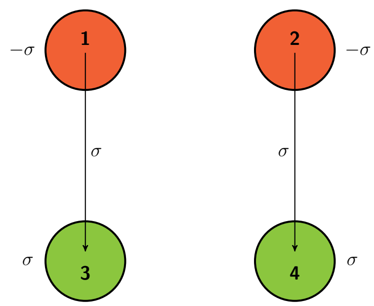

Figure 1.2: A 4 player network.

I am studying a finite game of perfect information and the equilibrium concept is subgame perfection.

Definition 12. *A subgame perfect equilibrium is a set of feasible posted contracts $W_t^* = \cup_i W_{it}^*$, $W_i^* \subseteq \mathcal{W}_i$ and lending strategies $X_t^* = \cup_i X_{it}^*$, $X_{it}^* \subseteq W_i^*$ such that at each point in time*

1. *borrowers maximize final payoff by choosing their borrowing strategy, W_{it}, taking as given the inherited network structure $\boldsymbol{\pi}_{t-1}$, and other borrowers lending strategies W_{-it}*

$$W_{it}^* = \arg\max_{W_{it} \in \mathcal{W}} V_i \left(X^*(W_{1t}^*, \cdots, W_{it}, \cdots, W_{nt}^*) \right);$$

2. *lenders maximize final payoff by choosing which contracts to accept among the ones posted $X_{it} \subseteq W_{i\to t}$, taking as given the inherited network structure $\boldsymbol{\pi}_{t-1}$, borrowers posting strategies W_t, and other lenders lending strategies X_{-it}*

$$X_{it}^{*} = \arg \max_{X_{it} \subseteq W_{i \to t}^{*}} V_i \left(X_{1t}^{*}, \cdots, X_{it}, \cdots, X_{nt}^{*} \right);$$

3. *signed contracts are feasible.*

The equilibrium definitions puts no restrictions on who the borrower and the lenders are in terms of their initial asset levels other than lenders cannot lend more that their cash available at the beginning of the period.

1.5.1 Subgame Perfection vs. Group Stability

Here each contract imposes an externality on the agents not involved in the contact through repercussions on the default outcome. In a novel and recent paper, similarly to this paper, [23] agents payoffs depend on everybody else's actions through the network effects. They prove that under certain conditions, substitutability and irrelevance of rejected alternatives, group stability can be extended to a setup with externalities and use a modification of the deferred acceptance algorithm to prove existence of a stable outcome and show that the equilibrium, which is generically unique, still conserves some of the classical properties.

Unfortunately here agents payoffs do not satisfy substitutability, a sufficient requirement for the convergence of the deferred acceptance algorithm. The reason is that, as the set of defaulting banks shrinks, previously accepted contracts might become unacceptable. Because banks are willing to accept lower interest payments from banks that cause a bigger loss given default, as the set of defaulting banks shrinks and the loss given default decreases with it, lenders ask for a higher interest rate thereby dropping previously signed contracts. But in that case, the set of defaulting banks expands, and the algorithm may fail to converge.

I study the technical detail of the substitutability requirement in appendix (A.2).

1.6 Equilibrium Characterization for Four Agents and Two Periods

In this section I solve fully for a general network of 4 agents with an example showed in figure (1.3). I show that agents have a bigger incentive to lend to agents that are already most exposed to, and that is the unique equilibrium when the aggregate exposure of liquid agents is relatively high.

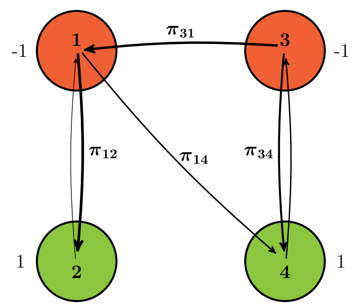

Figure 1.3: A general 4 player network.

I consider here a two period game where the inherited network structure is given by X_{-1}, the set of obligations contracted before the game started and can be summarized by the matrix of relative liabilities $\boldsymbol{\pi}$ (shorthand for $\boldsymbol{\pi}_{-1}$). I assume that agent 1 and 3 are illiquid at time 0 and have a liquidity need of 1. Agents 3 and 4 on the other hand can lend up to one unit. This assumption implies that each liquid agent can engage in at most one contract which simplifies the game. At time 0 the financing game is played and at time 1 agents collect their last cash flow.

Given that this is a game of perfect information, I solve it by backward induction. I characterize the equilibrium in four steps:

Step 1 There is no default at $t = 1$.

Step 2 There is no default at $t = 0$ so liquid agents lend to illiquid ones.

Step 3 Liquid agents lending to the illiquid ones they are most exposed to is always an equilibrium.

Step 4 Liquid agents lending to the illiquid ones they are most exposed to is the unique equilibrium when the aggregate exposure of the lenders is relatively big.

Lemma 13. *All agents surviving to $t = 1$ are liquid.*

At time 1, all debt outstanding is debt issued at $t = 0$ to avoid default. Given that there is no uncertainty at $t = 0$, all agents surviving to that point are liquid, as their $t = 0$ lenders will not/ cannot ask for interest payments in excess of what the borrower can pay.

As a consequence, there is no new debt being issued and no default at that time. The question is weather agents default at $t = 0$ to start with and, if that is the case, what are the borrower-lender pairs that are being formed.

Let X_{-1} denote the inherited contracts from the previous period and X_0 the contracts signed at $t = 0$. In equilibrium it has to be that the borrower accepts the posted contract. If not, the lender is better off withdrawing the contract given that posting such a contract is costly.

I assume that it is always profitable for agents to reenter the market and pay the default penalty in case of default. Later on I analyze the conditions on the parameters of the model under which this assumption is correct. If agents always pay the default penalty, then our problem simplifies and the payoffs for the case of bankruptcy and no bankruptcy are symmetric and we do not need to disentangle the different cases. To see this clearly, it is useful to go back to equations (1.3) and (DO) and contrast it to what happens in the situation of no default.

In case j does not default at time 0, its payoff at time 1 is simply:

$$V_j(X_{-1}, X_0) = [a_{j0} + \gamma_{j0} - LGD_j(X_{-1}, X_0)]R + y_{j1} + \sum_i R_{ij} - \sum_i R_{ji}$$

where feasibility implies that at R_{ij} is zero for at least one $j \in \{1,3\}$. In case of no default, $a_{j0} + \gamma_{j0} - LGD_j(X_{-1}, X_0) > 0$ represent the cash left in the account at the end of $t = 0$ which delivers the competitive (gross) interest rate R.

In case j defaults (and reenters the market at $t = 1$ by repaying the default penalty), j has to pay the default penalty, obligations contracted at $t = 0$, but it is also entitled to receiving interest payments. Its payoff is therefore:

$$V_j(X_{-1}, X_0) = y_{j1} - d_{j0}R + + \sum_i R_{ij} - \sum_i R_{ji}.$$

From equations (1.3) and (DO) we get that

$$d_{j0} = a_{j0} + \gamma_{j0} - LGD_j(X_{-1}, X_0),$$

therefore final payoff has the same functional form both for the default and the no default case. Remembering that all contracts have size 1, and any balance left after payments are made receive a interest payment of R if positive or pay a default payment of R if negative, j would accept the contract x_{ij} as long as:

$$\begin{aligned} V_j(X_{-1}, X_0) &\geq V_j(X_{-1}, X_0 \setminus \{x_{ij}\}) \\ [a_{j0} + \gamma_{j0} - 1 - LGD_j(X_{-1}, X_0)]R + y_{j1} + R_{ij} &\geq [a_{j0} + \gamma_{j0} - LGD_j(X_{-1}, X_0 \setminus \{x_{ij}\})]R + y_{j1} \\ R - R_{ij} &\leq [LGD_j(X_{-1}, X_0 \setminus \{x_{ij}\}) - LGD_j(X_{-1}, X_0)]R \end{aligned}$$

Note that if i is an illiquid agent, then i may default under $X_0 \setminus \{x_{ij}\}$ and, if j is affected in any form by i's default, j is willing to offer a more competitive interest rate that what it would get on the market.

In our example, if 2 is affected more by 1's default, then 2 is willing to lend at a lower interest rate to 1 than to 3 which would get the competitive interest rate. Which brings us to the next lemma.

I show next that the borrowers are the illiquid agents and the lenders are the liquid ones. Notice that it is allowed that a liquid agent lends to another liquid agent, but given that the interest rate changed on the loan is at most the competitive rate R, illiquid agents always prefer borrowing to defaulting, so it is always a profitable deviation to offer more advantageous terms that the liquid agent. Given that liquid agents never default if the illiquid ones don't, then the reverse is not true.

Lemma 14. *Liquid agents (2 and 4) are the lenders, and illiquid agents (1 and 3) the borrowers and* $\mathbf{LGD}_0 = 0$.

We know that illiquid agents (weakly) prefer borrowing to defaulting and paying the default cost. Left to prove is whether liquid agents would rather lend to other liquid agents. See for example figure (1.4). In that case 2 is the only agent directly exposed to both illiquid agents 1 and 3, whereas 4 is only exposed to them indirectly through 2. In that case, 4 might prefer lending to 2 directly as the defaulting cascade would stop at 2. I show that this cannot be the case in equilibrium because illiquid agents can borrow from 2 instead and 2 would always be willing to accept such a contract. The main idea is that once one of the illiquid agents borrows from 2, 2's loss given default is also reduced.

Assume that is not the case and that at least one liquid agent is a borrower. Given feasibility, only one contract is signed between one borrower and one lender - simply exchanging cash between liquid agents is equivalent to not lending at all- and $X_0 = \{x_{ij}\}$. j is happy to sign such a contract if:

$$R_{ij} - R \geq [LGD_j(X_{-1}, \{x_{ij}\}) - LGD_j(X_{-1}, \emptyset)]R$$

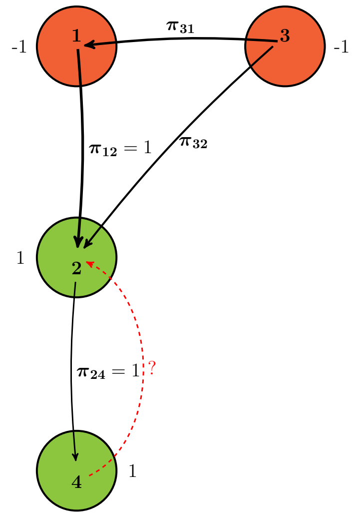

Figure 1.4: A stipulated equilibrium where 4 lends to 2 and the default set is comprised by 1 and 3.

i is willing to offer such a contract if

$$V_i(X_{-1}, \{x_{ij}\}) \geq V_i(X_{-1}, \emptyset)$$
$$-R_{ij} + [1 - LGD_i(X_{-1}, \{x_{ij}\})]R \geq -LGD_i(X_{-1}, \emptyset)R$$
$$R_{ij} - R \leq [LGD_i(X_{-1}, \emptyset) - LGD_i(X_{-1}, \{x_{ij}\})]R$$

There are benefits to trade if

$$LGD_j(X_{-1}, \{x_{ij}\}) - LGD_j(X_{-1}, \emptyset) < LGD_i(X_{-1}, \emptyset) - LGD_i(X_{-1}, \{x_{ij}\}),$$

which can be rewritten as

$$LGD_j(X_{-1}, \{x_{ij}\}) + LGD_i(X_{-1}, \{x_{ij}\}) < LGD_i(X_{-1}, \emptyset) + LGD_j(X_{-1}, \emptyset). \qquad (1.5)$$

1 can deviate and offer a contract x_{1i} to i. i accepts the new contract if the interest rate

is attractive enough:

$$\begin{aligned} V_i(X_{-1}, \{x_{1i}, x_{ij}\}) &\geq V_i(X_{-1}, \{x_{ij}\}) \\ R_{1i} - LGD_i(X_{-1}, \{x_{1i}, x_{ij}\})R &\geq [1 - LGD_i(X_{-1}, \{x_{ij}\})]R \\ R_{1i} - R &\geq [LGD_i(X_{-1}, \{x_{1i}, x_{ij}\}) - LGD_i(X_{-1}, \{x_{ij}\})]R \end{aligned}$$

In addition x_{1i} is a profitable deviation if:

$$\begin{aligned} V_1(X_{-1}, \{x_{1i}, x_{ij}\}) &> V_1(X_{-1}, \{x_{ij}\}) \\ -R_{1i} + [1 - LGD_1(X_{-1}, \{x_{1i}, x_{ij}\})]R &> -LGD_1(X_{-1}, \{x_{ij}\})R \\ R_{1i} - R &< [LGD_1(X_{-1}, \{x_{ij}\}) - LGD_1(X_{-1}, \{x_{1i}, x_{ij}\})]R. \end{aligned}$$

Note that $\mathbf{LGD}(X_{-1}, \{x_{1i}, x_{ij}\}) = \mathbf{LGD}(X_{-1}, \{x_{1j}\})$. There's always a profitable deviation as long as:

$$LGD_i(X_{-1}, \{x_{1j}\}) - LGD_i(X_{-1}, \{x_{ij}\}) < LGD_1(X_{-1}, \{x_{ij}\}) - LGD_1(X_{-1}, \{x_{1j}\}) \tag{1.6}$$

In addition, the default set under $\{x_{1i}, x_{ij}\}$ is smaller that the default set under x_{ij}: 1 and 3 default for sure under $X_0 = \{x_{ij}\}$, whereas j for sure doesn't default. 1 and 3 default because they are illiquid and do not borrow. j doesn't default because its liquidity position is 2 and its loss given default is at most 2 (sum of the liquidity position of the illiquid agents). Under $X_0 = \{x_{1i}, x_{ij}\}$, 3 defaults for sure, and i still doesn't default as it has a liquidity position of 1 and its losses given default are at most 1 (when $\pi_{3i} = 1$). Therefore $\mathbf{LGD}(X_{-1}, \{x_{1j}\}) \leq \mathbf{LGD}(X_{-1}, \{x_{ij}\})$, and

$$LGD_i(X_{-1}, \{x_{1j}\}) - LGD_i(X_{-1}, \{x_{ij}\}) \leq LGD_1(X_{-1}, \{x_{ij}\}) - LGD_1(X_{-1}, \{x_{1j}\})$$

given that LHS is negative and RHS positive. Equality happens when neither 1 nor i are exposed directly to 3, and indirectly through j. But in that case $LGD_i(X_{-1}, \{x_{ij}\}) = LGD_i(X_{-1}, \emptyset)$, and $LGD_j(X_{-1}, \{x_{ij}\}) = LGD_j(X_{-1}, \emptyset)$, therefore there are no benefits for i and j to trade given their losses given default are not reduced when they sign x_{ij}.

Proposition 15. *Let*

$$[LGD_2(X_{-1}, \{x_{32}\}) + LGD_4(X_{-1}, \{x_{14}\})] - [LGD_2(X_{-1}, \{x_{12}\}) + LGD_4(X_{-1}, \{x_{34}\})] \geq 0.$$

If illiquid agents prefer borrowing to defaulting, then $X_0^* = \{x_{12}, x_{34}\}$ is an equilibrium.

$X_0 = \{x_{12}, x_{34}\}$ is an equilibrium if neither 1 nor 3 have an incentive to deviate unilaterally by offering x_{14} or x_{32} instead. In order for 4 to accept x_{14} it has to be that the interest offered on the new contract x_{14} is attractive enough:

$$\begin{aligned}
V_4(X_{-1}, X_0 \setminus \{x_{12}, x_{34}\} \cup \{x_{14}\}) &\geq V_4(X_{-1}, X_0 \setminus \{x_{12}\}) \\
V_4(X_{-1}, \{x_{14}\}) &\geq V_4(X_{-1}, \{x_{34}\}) \\
R_{14} - LGD_4(X_{-1}, \{x_{14}\})R &\geq R_{34} - LGD_4(X_{-1}, \{x_{34}\})R \\
R_{14} &\geq R_{34} + [LGD_4(X_{-1}, \{x_{14}\}) - LGD_4(X_{-1}, \{x_{34}\})]\, R
\end{aligned}$$

But offering such a contract has to be profitable for 1, and the new interest rate offered has to be low enough:

$$\begin{aligned}
V_1(X_{-1}, X_0 \setminus \{x_{12}, x_{34}\} \cup \{x_{14}\}) &> V_1(X_{-1}, X_0) \\
V_1(X_{-1}, \{x_{14}\}) &> V_1(X_{-1}, X_0) \\
-R_{14} - LGD_1(X_{-1}, \{x_{14}\})R &> -R_{12} - LGD_1(X_{-1}, X_0)R \\
R_{14} &< R_{12} - [LGD_1(X_{-1}, \{x_{14}\}) - LGD_1(X_{-1}, X_0)]\, R
\end{aligned}$$

Offering x_{14} is not a profitable deviation as long as:

$$R_{12} - [LGD_1(X_{-1}, \{x_{14}\}) - LGD_1(X_{-1}, X_0)]\, R \leq$$
$$R_{34} + [LGD_4(X_{-1}, \{x_{14}\}) - LGD_4(X_{-1}, \{x_{34}\})]\, R$$

Similarly, x_{32} is not a profitable deviation if:

$$R_{34} - [LGD_3(X_{-1}, \{x_{32}\}) - LGD_3(X_{-1}, X_0)]\, R \leq$$
$$R_{12} + [LGD_2(X_{-1}, \{x_{32}\}) - LGD_2(X_{-1}, \{x_{12}\})]\, R$$

If neither x_{12} nor x_{34} are a profitable deviation, then it has to be that:

$$[LGD_1(X_{-1}, X_0) - LGD_1(X_{-1}, \{x_{14}\})] + [LGD_3(X_{-1}, X_0) - LGD_3(X_{-1}, \{x_{32}\})] \leq$$
$$[LGD_2(X_{-1}, \{x_{32}\}) - LGD_2(X_{-1}, \{x_{12}\})] + [LGD_4(X_{-1}, \{x_{14}\}) - LGD_4(X_{-1}, \{x_{34}\})]$$

Notice that both $LGD_1(X_{-1}, X_0) - LGD_1(X_{-1}, \{x_{14}\})$ and $LGD_3(X_{-1}, X_0) - LGD_3(X_{-1}, \{x_{33}\})$ are negative given that in equilibrium $\mathbf{LGD}(X_{-1}, X_0) = 0$ from lemma (14), therefore the left hand side is negative. If the right hand side is positive, then there are no profitable deviations and $X_0 = \{x_{12}, x_{34}\}$ is an equilibrium.

Lemma 16. *Let*

$$[LGD_2(X_{-1}, \{x_{32}\}) + LGD_4(X_{-1}, \{x_{14}\})] - [LGD_2(X_{-1}, \{x_{12}\}) + LGD_4(X_{-1}, \{x_{34}\})] \geq$$
$$[LGD_1(X_{-1}, \{x_{12}\}) + LGD_3(X_{-1}, \{x_{34}\})] \geq 0$$

Then $X_0 = \{x_{12}, x_{34}\}$ *is the only equilibrium.*

Similarly to previous computations $X_0 = \{x_{14}, x_{12}\}$ is not an equilibrium if any of the illiquid agents have an incentive to deviate unilaterally and offer x_{12} or x_{34} instead. Neither x_{12} nor x_{34} are a profitable deviation if:

$$[LGD_1(X_{-1}, X_0) - LGD_1(X_{-1}, \{x_{12}\})] + [LGD_3(X_{-1}, X_0) - LGD_3(X_{-1}, \{x_{34}\})] \leq$$
$$[LGD_2(X_{-1}, \{x_{12}\}) - LGD_2(X_{-1}, \{x_{32}\})] + [LGD_4(X_{-1}, \{x_{34}\}) - LGD_4(X_{-1}, \{x_{14}\})]$$

Note that $LGD_1(X_{-1}, X_0) = LGD_3(X_{-1}, X_0) = 0$. We can rewrite the equation as follows:

$$[LGD_2(X_{-1}, \{x_{32}\}) + LGD_4(X_{-1}, \{x_{14}\})] - [LGD_2(X_{-1}, \{x_{12}\}) + LGD_4(X_{-1}, \{x_{34}\})] \geq$$
$$[LGD_1(X_{-1}, \{x_{12}\}) + LGD_3(X_{-1}, \{x_{34}\})]$$

Therefore the unique equilibrium that emerges is the one where loss given default are maximized, that is, liquid agents lend to the illiquid agents to which they already have a big exposure to start with, in the aggregate.

And lastly, notice that all liquid agents prefer paying the default penalty rather than exiting the market forever when they default. If they pay the default penalty and reenter the market then they receive the following cash flow at $t = 1$:

$$[a_{j0} + \gamma_{j0} - LGD_{i0}]R + y_{j1} + \sum_i b_{ij0} R_{ij1} \geq$$

$$[a_{j0} + y_{j0} + \sum_i b_{ij-1} R_{ij0} - \sum_i b_{ji-1} R_{ji0} - LGD_i(X_{-1}, X_0)]R + y_{j1} + \sum_i b_{ij0}[R_{ij1} - R] \geq$$

$$[a_{j0} + y_{j0} - \sum_i b_{ji-1} R_{ji0}]R + y_{j1} - \sum_i b_{ij0}[LGD_j(X_{-1}, X_0 \setminus \{x_{ij}\}) - LGD_j(X_{-1}, X_0)] \geq$$

$$[a_{j0} + y_{j0} - \sum_i b_{ji-1} R_{ji0}]R + y_{j1} - \sum_i b_{ij0} \geq$$

$$[a_{j0} + y_{j0}](R-1) - \left[\sum_i b_{ji-1} R_{ji0}\right] R + y_{j1}$$

where I used the fact that $\sum_i b_{ij} R_{ij} \geq LGD_i(X_{-1}, X_0)$ in the second line, $LGD_j(X_{-1}, X_0 \setminus \{x_{ij}\}) - LGD_j(X_{-1}, X_0) \leq 1$ in the third, and in the last line $\sum_i b_{ij0} < a_{j0} + y_{j0}$.

If $[a_{j0}+y_{j0}](R-1) - \left[\sum_i b_{ji-1} R_{ji0}\right] R + y_{j1} > 0$, then liquid agents always prefer paying the penalty to exiting the market forever. This is an assumption that I carry throughout.

[show that the interest rate of all contracts has to be the same]

1.7 Equilibrium Characterization for N Agents

I characterize next the general case where all agents have perfect foresight, loans are standardized to a size of 1, all liquid agents can lend up to one unit and illiquid agents have a liquidity need of one. That is, each liquid agent can lend to at most one agent. Moreover, I assume that the number of liquid and illiquid agents is exactly the same. That is aggregate liquidity is positive.

I analyze the equilibrium through backward induction and characterize first the set of contracts offered at time $T-1$. First, note that, given that the game ends at time T, the only feasible contracts at this time are transfers between lenders to borrowers at time T.

Lemma 17. *$W_T^* = \emptyset$ and $\mathbf{LGD}_T = 0$.*

The only debt that agents have have at time T is debt incurred in the previous period. Here, the only incentive for the lender to lend is to reduce its loss given default through contagion, given that payments at a future date are impossible. But, given that there are no liquidity shocks at time T, no lender would ask for time T payments in excess of what the borrowers can afford and

$$LGD_{iT} = 0 \quad \forall i.$$

Now I turn to characterizing how the in-network lending works. First note that all posted contracts are accepted in equilibrium. Posting contracts costs κ, so in equilibrium, all posted contracts are accepted, otherwise the borrowers could simply withdraw the contracts posted and save on the posting cost.

Next I study the contracts that are signed at time $T-1$, taking as given contracts signed at $T-2$, and I start by putting limits on the interest rates that can arise in equilibrium. First, note that if lender j does not default under X^*_{T-1}, it will not default under $X^*_{T-1} \setminus \{x_{ij}\}$ either. The reason is that losses increase by at most b_{ijT-1}, the principal specified by contract x_{ij}, but j's cash buffer increases by b_{ijT-1}. There are therefore three different cases:

1. The lender does not default neither when it signs x_{ij}, nor when it doesn't:

$$d_j(X_{T-2}, X^*_{T-1}) = d_j(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\}) = 0;$$

2. the lender defaults in both cases:

$$d_j(X_{T-2}, X^*_{T-1}), d_j(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\}) > 0;$$

3. lastly, the lender defaults under X^*_{T-1}, but not under $X^*_{T-1} \setminus \{x_{ij}\}$.

Given that the default penalty β is the competitive interest rate R, payoffs are symmetric for all three cases as showed in section (1.6). The lender decides to accept a given contract as long as:

$$V_{jT}(X_{T-2}, X^*_{T-1}) \geq V_{jT}(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\})$$
$$b_{ijT-1}R_{ijT} + (\gamma_{jT-1} - LGD_j(X_{T-2}, X^*_{T-1}))R \geq (\gamma_{jT-1} + b_{ijT-1} - LGD_j(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\}))R,$$

or,

$$b_{ijT-1}R_{ijT} \geq [b_{ijT-1} - (LGD_j(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\}) - LGD_j(X_{T-2}, X^*_{T-1}))]R,$$

where $\gamma_{jT-1} = \gamma_j(X_{T-2}, X^*_{T-1})$. Moreover, $R_{ijT} \leq R$, otherwise borrowers are better off defaulting and paying the default penalty next period. Therefore the bound on interest rate is given by:

$$(R - R_{ijT})/Rb_{ijT-1} \in \left[0, LGD_j(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\}) - LGD_j(X_{T-2}, X^*_{T-1})\right]. \quad (1.7)$$

Note that if i defaults as a result of not being able to sign x_{ij}, then j might be potentially affected by i's default through the loss given default. If $LGD_j(X_{T-2}, X^*_{T-1} \setminus \{x_{ij}\}) > LGD_j(X_{T-2}, X^*_{T-1})$, then j is willing to lend to i at a discount. In that case, an illiquid agent i would rather borrow from other agents at a discount rather than paying the default cost R.

Lemma 18. *Liquid agents are the lenders and illiquid agents are the borrowers.*

Assume that there exist two agents i' and j' such that $x_{i'j'} \in X_{T-1}$. Then, by equation (1.5), there are benefits from trade between i and j as long as their combined exposure under X_{T-1} is less than under $X_{T-1} \setminus \{x_{i'j'}\}$:

$$
\begin{aligned}
&LGD_{i'}(X_{T-2}, X_{T-1}) + LGD_{j'}(X_{T-2}, X_{T-1}) < \\
&LGD_{i'}(X_{T-2}, X_{T-1} \setminus \{x_{i'j'}\}) + LGD_{j'}(X_{T-2}, X_{T-1} \setminus \{x_{i'j'}\})
\end{aligned}
$$

If there are agents defaulting under X_{T-1}, there there is at least one illiquid agent i that has a liquidity deficit and one liquid j agents that has a liquidity surplus. There are gains from trade from j lending to i if

$$
LGD_i(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) + LGD_j(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) < LGD_i(X_{T-2}, X_{T-1}) + LGD_j(X_{T-2}, X_{T-1}) \tag{1.8}
$$

That is, losses given default to the parties involved have to decrease as a consequence of signing the contract therefore at least one of the parties involved in x_{ij} is exposed to the default set. If i is a sink node then i is indifferent between paying the default cost and borrowing.

Moreover, going back to the formula for the loss given default,

$$
\mathbf{LGD}_{T-1} = \boldsymbol{\pi}_{T-2}^T \left[\mathbb{I} - \boldsymbol{\Lambda}(\mathbf{d}_{T-1}) \boldsymbol{\pi}_{T-2}^T \boldsymbol{\Lambda}(\mathbf{d}_{T-1}) \right]^{-1} \left[-\boldsymbol{\Lambda}(\mathbf{d}_{T-1})(\boldsymbol{\gamma}_{T-1} + \mathbf{a}_{T-2}) \right], \tag{1.9}
$$

LGD can decline because of (1) default set shrinks; and/ or (2) liquidity deficit of defaulting agents decreases. Therefore, if j does not default under $(X_{T-2}, X_{T-1} \cup \{x_{ij}\})$, then $\mathbf{LGD}(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) < \mathbf{LGD}(X_{T-2}, X_{T-1})$ as (1) either i switches from defaulting to no defaulting and **LGD** drops; or (2) i still defaults under $(X_{T-2}, X_{T-1} \cup \{x_{ij}\})$, but its liquidity deficit is reduced and **LGD** drops. Either way, it is a profitable deviation for i to offer x_{ij}.

The problematic case is when j does not default under X_{T-1}, but it defaults under

$(X_{T-2}, X_{T-1} \cup \{x_{ij}\})$. In that case, the default set does not shrink and the effect on **LGD** is ambiguous. This can only happen if there is at least one more illiquid agent that default under (X_{T-2}, X_{T-1}), which I call i_2. If only i defaults in (X_{T-2}, X_{T-1}), then $LGD_j(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) = LGD_i(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) = 0$, so i would not default, making the deviation profitable.

Moreover, if i is not exposed to j to start with then $LGD_i(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) = LGD_i(X_{T-2}, X_{T-1})$, and there are gains from trade.

If j does not default under (X_{T-2}, X_{T-1}), then

$$\gamma_j(X_{T-2}, X_{T-1}) - LGD_j(X_{T-2}, X_{T-1}) \geq 0.$$

But if j defaults under $(X_{T-2}, X_{T-1} \cup \{x_{ij}\})$, then

$$\gamma_j(X_{T-2}, X_{T-1}) - 1 - LGD_j(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) < 0.$$

Putting everything together it has to be that

$$LGD_j(X_{T-2}, X_{T-1}) - LGD_j(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) \leq 1.$$

Therefore we need $LGD_j(X_{T-2}, X_{T-1}) > 0$, and

$$LGD_j(X_{T-2}, X_{T-1}) - LGD_j(X_{T-2}, X_{T-1} \cup \{x_{ij}\}) \leq 1,$$

that is i owes money to other agents in the network.

Proposition 19. *Let $f : N \to N$ denote the function that assigns an agent to its lending* *partner under X_{T-1}. If*

$$\sum_{j\in L}\sum_{k\in L,k\neq j} LGD_j(X_{T-2}, X_{T-1}\setminus\{x_{f(j)k}\}) \geq \sum_{j\in L}\sum_{k\in L,k\neq j} LGD_j(X_{T-2}, X_{T-1}\setminus\{x_{f(k)k}\})$$

then X_{T-1} is a no-default equilibrium.

Proof. Consider the following deviation by any borrower i: $\tilde{W}_{\to iT-1} = W_{\to iT-1}\setminus\{x_{if(i)}\}\cup\{x_{ij}\}$, with $x_{ij} = (1, R_{ij})$ for any $j \in N$. j has to drop one of the contracts signed under X_{T-1}, which I call $x_{f(j)j}$. Lender j will accept such a deviation if

$$V_j(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}, x_{f(j)j}\}\cup\{x_{ij}\}) \geq V_j(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}\})$$

$$R_{ij} - LGD_j(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}, x_{f(j)j}\}\cup\{x_{ij}\})R \geq R_{f(j)j} - LGD_j(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}\})R$$

Solving for R_{ij} and

$$R_{ij} \geq R_{f(j)j} + [LGD_j(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}, x_{f(j)j}\}\cup\{x_{ij}\}) - LGD_j(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}\})]R$$

At the same time, the deviation is profitable to i if:

$$V_i(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}, x_{f(j)j}\}\cup\{x_{ij}\}) > V_i(X_{T-2}, X_{T-1}\})$$

$$-R_{ij} - LGD_i(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}, x_{f(j)j}\}\cup\{x_{ij}\})R > -R_{if(i)} - LGD_i(X_{T-2}, X_{T-1})R$$

Solving for R_{ij}:

$$R_{ij} < R_{if(i)} + [LGD_i(X_{T-2}, X_{T-1}) - LGD_i(X_{T-2}, X_{T-1}\setminus\{x_{if(i)}, x_{f(j)j}\}\cup\{x_{ij}\})]R$$

Such a deviation is not profitable if the interest rate required for j to accept the proposed contract exceeds what i is already paying:

$$R_{if(i)}+[LGD_i(X_{T-2},X_{T-1})-LGD_i(X_{T-2},X_{T-1}\setminus\{x_{if(i)},x_{f(j)j}\}\cup\{x_{ij}\})]R\leq$$
$$R_{f(j)j}+[LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)},x_{f(j)j}\}\cup\{x_{ij}\})-LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)}\})]R\forall i,j.$$

Given that all contracts have same size 1, note that $X_{T-1}\setminus\{x_{f(j)j},x_{if(i)}\}\cup\{x_{ij}\}$ is default equivalent to $\tilde{X}_{T-1}\setminus\{x_{f(j)f(i)}\}$ where X_{T-1} and $\tilde{X}_{T-1}$ have the same end of period liquidity vector.[4] Therefore I use X_{T-1} and $\tilde{X}_{T-1}$ interchangeably. Moreover $LGD_i(X_{T-2},X_{T-1})=0$ given that we know that no agents default in equilibrium. We can then rewrite the previous expression as follows:

$$R_{if(i)}\leq R_{f(j)j}+[LGD_i(X_{T-2},X_{T-1}\setminus\{x_{f(j)f(i)}\})+LGD_j(X_{T-2},X_{T-1}\setminus\{x_{f(j)f(i)}\})-$$
$$-LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)}\})]R.$$

X_{T-1} is an equilibrium if there are no profitable deviations and the previous equation holds for all i and j. Summing across all borrowers i and lenders j, X_{T-1} is an equilibrium at $T-1$ if:

$$\sum_{i\in IL}\sum_{j\in L}LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)}\})\leq$$
$$\sum_{i\in IL}\sum_{j\in L}\left[LGD_i(X_{T-2},X_{T-1}\setminus\{x_{f(j)f(i)}\})+LGD_j(X_{T-2},X_{T-1}\setminus\{x_{f(j)f(i)}\})\right]$$

Which can be rewritten as:

4. The allocation of borrowers and lenders is different, but under both X_{T-1} and $\tilde{X}_{T-1}$, all liquid agents lend to illiquid agents therefore the end of period liquidity vector is identical.

$$\sum_{i\in IL}\sum_{j\in L}[LGD_{f(j)}(X_{T-2},X_{T-1}\setminus\{x_{ij}\})+LGD_{f(i)}(X_{T-2},X_{T-1}\setminus\{x_{ij}\})]\geq$$
$$\sum_{i\in IL}\sum_{j\in L}LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)}\}).$$

True if

$$\sum_{i\in IL}\sum_{j\in L}[LGD_{f(i)}(X_{T-2},X_{T-1}\setminus\{x_{ij}\})]\geq\sum_{i\in IL}\sum_{j\in L}LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)}\}).$$

which can be rewritten as

$$\sum_{j\in L}\sum_{k\in L,k\neq j}LGD_j(X_{T-2},X_{T-1}\setminus\{x_{f(j)k}\})]\geq\sum_{j\in L}\sum_{i\in ILi\neq f(j)}LGD_j(X_{T-2},X_{T-1}\setminus\{x_{if(i)}\})$$
$$\sum_{j\in L}\sum_{k\in L,k\neq j}[LGD_j(X_{T-2},X_{T-1}\setminus\{x_{f(j)k}\})-LGD_j(X_{T-2},X_{T-1}\setminus\{x_{f(k)k}\})]\geq 0$$

□

In the next lemma I show that as long as a posted contract x_{ij} satisfies (1.7) assuming all contracts are accepted in equilibrium, then lender j will accept the contract independently of other lenders strategies. If other lenders deviate, and they do not accept the posted contracts, then the borrowers will default. This makes the loss given default to lender j increase. But by equation (1.7), we know that lender j is willing to lend to i at an even lower interest rate than the one posted. Therefore, if j was willing to lend in the situation where all other lenders accept the posted contracts, then it will also lend if any or all of the lenders deviate.

Lemma 20. *If X_{T-1} is an equilibrium in the second stage of the game, then it is the only equilibrium.*

Proof. I show that it is always a dominant strategy to lend provided that the postulated equilibrium satisfies equation (1.7).

I prove the claim by contradiction. Assume there are two possible equilibria X_{T-1} and $X'_{T-1} \subseteq X_{T-1}$. Let S be the set of contracts defined as $X_{T-1} = X'_{T-1} \cup S$. X_{T-1} is an equilibrium if for all $x_{ij} \in X_{T-1}$, the lender is better off signing the contract.

$$\begin{aligned}
V_j(X_{T-2}, X_{T-1}) \geq V_j(X_{T-2}, X_{T-1} \setminus \{x_{ij}\}) \Leftrightarrow \\
R_{ijT} \geq (1 - LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{ij}\}))R \Leftrightarrow \\
(R_{ijT} - R) \geq \left[LGD_j(X_{T-2}, X_{T-1}) - LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{ij}\})\right] R.
\end{aligned}$$

Moreover, X'_{T-1} is an equilibrium if no lender is better off signing additional contracts.

$$V_j(X_{T-2}, X'_{T-1} \cup \{x_{ij}\}) < V_j(X_{T-2}, X'_{T-1}), \forall j$$

Given that payoff are symmetric for the lender in case of default, the same analysis flows through for different default outcomes for the lender.

$$\begin{aligned}
V_j(X_{T-2}, X'_{T-1} \cup \{x_{ij}\}) < V_j(X_{T-2}, X'_{T-1}) \\
R_{ijT} - LGD_j(X_{T-2}, X'_{T-1} \cup \{x_{ij}\})R < \left[1 - LGD_j(X_{T-2}, X'_{T-1})\right] R \Leftrightarrow \\
R_{ijT} - R < [LGD_j(X_{T-2}, X'_{T-1} \cup \{x_{ij}\}) - LGD_j(X_{T-2}, X'_{T-1})]R
\end{aligned}$$

Therefore, for contracts x_{ij} in X_{T-1} but not in X'_{T-1} we have:

$$(R_{ijT} - R)/R \in [LGD_j(X_{T-2}, X_{T-1}) - LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{ij}\}),$$
$$LGD_j(X_{T-2}, X'_{T-1} \cup \{x_{ij}\}) - LGD_j(X_{T-2}, X'_{T-1})).$$

But the interval does not exist given that

$$LGD_j(X_{T-2}, X_{T-1}) - LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{ij}\}) \geq$$
$$LGD_j(X_{T-2}, X'_{T-1} \cup \{x_{ij}\}) - LGD_j(X_{T-2}, X'_{T-1})$$

where the inequality follows from lemma (7). Therefore, if the interest rates on x_{ij} is high enough such that j accepts it given that everybody else signs $X_{T-1} \setminus \{x_{ij}\}$, then it is also high enough to accept if everybody else signs X''_{T-1}.

□

The previous lemma therefore allows borrowers to deviate in the first stage without worrying whether their contract will be accepted or not in the second stage. If borrowers offer a high enough interest rate to start with, then it is a dominant strategy for lenders to lend.

Proposition 21. *If*

$$\sum_{j \in L} \sum_{k \in L, k \neq j} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{f(j)k}\}) \geq$$
$$\sum_{j \in L} \sum_{k \in L, k \neq j} [LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{g(j)k}\}) + LGD_{g(k)}(X_{T-2}, X_{T-1} \setminus \{x_{g(j)k}\})]$$

for all $g(\cdot)$, then $f(\cdot)$ is the unique equilibrium.

Proof. $f : N \to N$ is an equilibrium if

$$\sum_{j \in L} \sum_{k \in L, k \neq j} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{f(j)k}\}) \geq \sum_{j \in L} \sum_{k \in L, k \neq j} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{f(k)k}\}).$$

Let $g : N \to N$ denote any other allocation of lenders and borrowers. Then g is not an equilibrium allocation if:

$$\sum_{j \in L} \sum_{k \in L, k \neq j} [LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{g(j)k}\}) - LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{g(k)k}\})] \leq$$
$$-\sum_{j \in L} \sum_{k \in L, k \neq j} LGD_{g(k)}(X_{T-2}, X_{T-1} \setminus \{x_{g(j)k}\})$$

Note that

$$\sum_{j \in L} \sum_{k \in L, k \neq j} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{f(k)k}\}) = \sum_{j \in L} \sum_{k \in L, k \neq j} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{g(k)k}\})$$

Then $f : N \to N$ is the unique allocation of borrowers to lenders if

$$\sum_{j \in L} \sum_{k \in L, k \neq j} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{f(j)k}\}) \geq$$
$$\sum_{j \in L} \sum_{k \in L, k \neq j} [LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{g(j)k}\}) + LGD_{g(k)}(X_{T-2}, X_{T-1} \setminus \{x_{g(j)k}\})]$$

for any allocation of borrowers to lenders $g(\cdot)$.

□

I show next that in the case of a bipartite graph the equilibrium is unique.

1.7.1 A special case: bipartite graph

In the case of a bipartite graph, no-default is the unique equilibrium as equation (1.8) is satisfied given that for all illiquid agents i

$$LGD_i(X_{T-2}, X_{T-1}) = 0, \forall X_{T-1},$$

and

$$LGD_j(X_{T-2}, X_{T-1}) > LGD_j(X_{T-2}, X_{T-1} \cup \{x_{ij}\}).$$

Moreover,

$$LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{ik}\}) = LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{il}\}),$$

for any liquid agents l and k.

The condition for uniqueness becomes:

$$\sum_{j \in L} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{f(j)j}\}) \geq \sum_{j \in L} LGD_j(X_{T-2}, X_{T-1} \setminus \{x_{g(j)j}\}).$$

In aggregate, liquid agents lend to the illiquid agents they are most exposed to.

1.8 Conclusion

I study a model of contagion where one agent's default causes other agents to incur losses and potentially their own default, and I allow agents to borrow from and lend to other agents in the network. In the case four agents, two liquid and two illiquid, I show that in equilibrium there is no default and liquid agents lend to illiquid ones. Moreover, liquid agents lend to the illiquid agents whom they are most exposed to, which is the unique equilibrium if aggregate losses to lenders are relatively big. In the second part of the paper I generalize the results for a general N player netwok.

CHAPTER 2

OPTIMAL INATTENTION WITH EPSTEIN ZIN PREFERENCES

2.1 Introduction

I introduce Epstein-Zin preferences (EZ from now on) in a optimal inattention problem as in [1].

2.2 The Consumer's Optimization Problem

The consumer has EZ stochastic differential utility. For reasons that are going to become obvious later, I am going to use EZ preferences as limits of the discrete time counterpart. The consumer will maximize recursive utility as in [5] and [7]:

$$V(t) = \left[(1 - e^{-\rho\epsilon})\frac{c(t)^{1-\alpha}}{1-\alpha} + e^{-\rho\epsilon}\mathcal{R}_t V(t+\epsilon)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}$$

where $\mathcal{R}_t V(t+\epsilon) \equiv E_t[V(t+\epsilon)^{1-\gamma}]^{\frac{1}{1-\gamma}}$.

The continuous time version will be given by:

$$\rho V(t)^{1-\alpha} = \rho\frac{c(t)^{1-\alpha}}{1-\alpha} + \frac{1-\alpha}{1-\gamma}V(t)^{\gamma-\alpha}\lim_{\epsilon\to 0}\frac{E_t V(t+\epsilon)^{1-\gamma} - V(t)^{1-\gamma}}{\epsilon} \tag{2.1}$$

where ρ is the discount rate and $1/\alpha$ the intertemporal elasticity of substitution (IES).

The investment portfolio holds a riskless bond with rate of return $r > 0$ and a nondividend-paying stock with price P_t that follows a geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu dt + \sigma dz \tag{2.2}$$

$X_{t_j}(\tau)$ is the amount of riskless liquid asset used to finance consumption from time t_j to $t_{j+1} = t_j + \tau$, so:

$$X_{t_j}(\tau) = \int_0^{\tau} c_{t_j+s} e^{-r^L s} ds \qquad (2.3)$$

At time $t_j + \tau$, the amount held in the riskless asset will just have reached zero since $r^L < r$, and the value of wealth, after paying the cost of observing the value of the investment portfolio is:

$$W_{t_j}(\tau) = (1-\theta)(W_{t_j} - X_{t_j})R(t_j, t_j + \tau) \qquad (2.4)$$

The investment portfolio is managed such that the fraction invested in the risky asset is constant between adjustments:

$$\frac{dR(t_j, t_j + s)}{R(t_j, t_j + s)} = [r + \phi(\mu - r)]ds + \phi\sigma dz \qquad (2.5)$$

As in [1], the solution strategy of an optimal inattention to the stock market can be done in four steps. Although the set-up is in continuous time, the fact that the consumer only decides to evaluate his wealth at discrete times allows us to focus on discrete time EZ preferences which simplifies the problem considerably. The first important difference with respect to [1] is that the optimal portfolio will depend on the risk aversion coefficient. Only through the optimal portfolio will the optimal choice of consumption and stopping time be affected. Otherwise, the problem is identical to the case with expected utility.

This problem can be solved in four steps.

A. Step 1: Given τ and X_t, Choose c_{t+s}, $0 < s \leq \tau$

Between 0 and τ there is no uncertainty so $\lim_{\epsilon \to 0} \frac{E_t V(t+\epsilon)^{1-\gamma} - V(t)^{1-\gamma}}{\epsilon} = dV(t)^{1-\gamma}$. Thus (2.1) becomes:

$$
\begin{aligned}
\rho V(t)^{1-\alpha} &= \rho \frac{c(t)^{1-\alpha}}{1-\alpha} + \frac{1-\alpha}{1-\gamma} V(t)^{\gamma-\alpha} d\left(V(t)^{1-\gamma}\right) \\
&= \rho \frac{c(t)^{1-\alpha}}{1-\alpha} + (1-\alpha) V(t)^{\gamma-\alpha} V(t)^{-\gamma} dV(t) \\
&= \rho \frac{c(t)^{1-\alpha}}{1-\alpha} + d\left(V(t)^{1-\alpha}\right)
\end{aligned}
$$

obtaining expected utility.

We can express consumer's problem in the following form:

$$
\begin{aligned}
& U_{t_j} = \max_{\{c_{t_j+s}\}_{s=0}^{\tau}} \int_0^{\tau} \rho e^{-\rho s} \frac{c(t_j+s)^{1-\alpha}}{1-\alpha} ds \\
& \text{s.t. } X_{t_j}(\tau) = \int_0^{\tau} c_{t_j+s} e^{-r^L s} ds
\end{aligned}
$$

With first order conditions:

$$
c(t_j+s) = c(t_j) e^{\frac{1}{\alpha}(r^L-\rho)s}
$$

then

$$
\frac{c(t_j+s)}{c(t_j)} = e^{\frac{1}{\alpha}(r^L-\rho)s} \tag{2.6}
$$

Rewrite X_{t_j} and U_{t_j} as functions of c_{t_j}:

$$
X_{t_j} = c(t_j) h(\tau) \tag{2.7}
$$

where

$$h(\tau) \equiv \frac{\alpha}{(1-\alpha)r^L - \rho}\left(e^{\frac{(1-\alpha)r^L-\rho}{\alpha}\tau} - 1\right) = \frac{1-e^{-\omega\tau}}{\omega} \tag{2.8}$$

assuming that

$$\omega \equiv \frac{\rho - (1-\alpha)r^L}{\alpha} > 0 \tag{2.9}$$

We can also write utility between adjustments as a function of aggregate consumption between adjustments, X_{t_j}:

$$U_{t_j} = \rho\frac{1}{1-\alpha}X_{t_j}^{1-\alpha}h(\tau)^{\alpha} \tag{2.10}$$

B. Step 2: Given τ, Choose X_{t_j} and ϕ

Given τ, the problem becomes a standard portfolio problem with discrete-time EZ preferences. At the time when the consumer chooses his portfolio, the value function is given by:

$$V(W_{t_j})^{1-\alpha} = \max_{X_{t_j},\phi}\left[(1-e^{-\rho\tau})U_{t_j}(\tau) + e^{-\rho\tau}\left(\mathcal{R}_{t_j}V_{t_j+\tau}\right)^{1-\alpha}\right]$$
$$\text{s.t. } W_{t_j+\tau} = (1-\theta)\left(W_{t_j} - X_{t_j}\right)R(t_j, t_j+\tau)$$

where $\mathcal{R}_{t_j}V_{t_j+\tau} \equiv E_{t_j}\left[V_{t_j+\tau}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$

First conjecture a solution for $V_{t_j}(W_{t_j})^{1-\alpha} = \rho(1-e^{-\rho\tau})\Gamma_{t_j}\frac{W_{t_j}^{1-\alpha}}{1-\alpha}$, i.e. the value function is homogenous of degree one in wealth with a value to wealth ratio that is allowed to be time dependent (deterministically) simplifies to

$$\Gamma_{t_j} W_{t_j}^{1-\alpha} = \max_{X_{t_j},\phi} \left[X_{t_j}^{1-\alpha} h(\tau)^\alpha + e^{-\rho\tau}(1-\theta)^{1-\alpha} \Gamma_{t_j+\tau} \left(W_{t_j} - X_{t_j} \right)^{1-\alpha} \left\{ E_{t_j} \left[R_{t_j,t_j+\tau}^{1-\gamma} \right] \right\}^{\frac{1-\alpha}{1-\gamma}} \right]$$

If we normalize by wealth:

$$\Gamma_{t_j} = \max_{X_{t_j},\phi} \left[\left(\frac{X_{t_j}}{W_{t_j}} \right)^{1-\alpha} h(\tau)^\alpha + e^{-\rho\tau}(1-\theta)^{1-\alpha} \Gamma_{t_j+\tau} \left(\frac{W_{t_j} - X_{t_j}}{W_{t_j}} \right)^{1-\alpha} \left\{ E_{t_j} \left[R_{t_j,t_j+\tau}^{1-\gamma} \right] \right\}^{\frac{1-\alpha}{1-\gamma}} \right]$$

Or simply:

$$\Gamma_{t_j} = \max_{x_{t_j}} \left[x_{t_j}^{1-\alpha} h(\tau)^\alpha + e^{-\rho\tau}(1-\theta)^{1-\alpha} \Gamma_{t_j+\tau} \left(1 - x_{t_j} \right)^{1-\alpha} y_{t_j}^* \right] \tag{2.11}$$

where I define:

$$x_{t_j} \equiv \frac{X_{t_j}}{W_{t_j}}$$
$$y_{t_j}^* \equiv \max_{\phi} \left\{ E_{t_j} R_{t_j,t_j+\tau}^{1-\gamma} \right\}^{\frac{1-\alpha}{1-\gamma}}$$

The portfolio choice is separated from the consumption decision. The first order condition with respect to x_{t_j} is given by:

$$(x_{t_j}^*)^{-\alpha} h(\tau)^\alpha = e^{-\rho\tau}(1-\theta)^{1-\alpha} \Gamma_{t_j+\tau} \left(1 - x_{t_j}^* \right)^{-\alpha} y_{t_j}^* \tag{2.12}$$

which we can use together with (2.11) to find an expression for $x_{t_j}^*$:

$$\Gamma_{t_j} = \left(x_{t_j}^* \right)^{1-\alpha} h(\tau)^\alpha + \left(1 - x_{t_j}^* \right) \left(x_{t_j}^* \right)^{-\alpha} h(\tau)^\alpha$$
$$= \left(x_{t_j}^* \right)^{-\alpha} h(\tau)^\alpha \Rightarrow \tag{2.13}$$
$$x_{t_j}^* = \left(\Gamma_{t_j}^* \right)^{-\frac{1}{\alpha}} h(\tau) \tag{2.14}$$

Now we can derive the optimal portfolio:

$$\max_{\phi}\left\{E_{t_j}R_{t_j,t_j+\tau}^{1-\gamma}\right\}^{\frac{1-\alpha}{1-\gamma}}$$

so we can consider the following maximization problem instead:

$$\max_{\phi} E_{t_j}R_{t_j,t_j+\tau}^{1-\gamma}$$

Using Ito's lemma:

$$E_{t_j}R_{t_j,t_j+\tau}^{1-\gamma} = e^{(1-\gamma)[r+\phi(\mu-r)-\frac{1}{2}\gamma\phi^2\sigma^2]\tau}$$

which implies an optimal composition of the portfolio of $\phi^* = \frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$ which is a similar expression to the one in the original paper, where γ now is risk aversion. Substituting back into y_{t_j} we get the following:

$$y_{t_j}^* = e^{(1-\alpha)\left[r+\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]\tau}$$

Now, going back to (2.11) we need the following restriction in the parameters to ensure that the present value is finite:

$$\rho > (1-\alpha)\left[r+\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]$$

Going back to the value function:

$$\Gamma_{t_j} = \max_{x_{t_j}} \left[x_{t_j}^{1-\alpha} h(\tau)^{\alpha} + e^{-\rho\tau} \Gamma_{t_j+\tau} (1-\theta)^{1-\alpha} \left(1 - x_{t_j}\right)^{1-\alpha} \left\{ e^{(1-\gamma)\left[r + \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right]\tau} \right\}^{\frac{1-\alpha}{1-\gamma}} \right]$$

$$\Gamma_{t_j} = \max_{x_{t_j}} \left[x_{t_j}^{1-\alpha} h(\tau)^{\alpha} + e^{-\rho\tau} \Gamma_{t_j+\tau} (1-\theta)^{1-\alpha} \left(1 - x_{t_j}\right)^{1-\alpha} e^{(1-\alpha)\left[r + \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right]\tau} \right]$$

where $r + \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2 > r^L$ for an interior solution and $\alpha\lambda = \rho - (1-\alpha)\left[r + \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right] > 0$ to ensure that the above expression converges. In the original's paper notation $\lambda = \frac{\rho - (1-\alpha)\Omega(\gamma)}{\alpha}$, with $\Omega(\gamma) = r + \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2$ being a function of the risk aversion.

$$\Gamma_{t_j} = \max_{x_{t_j}} \left[x_{t_j}^{1-\alpha} h(\tau)^{\alpha} + \Gamma_{t_j+\tau} (1-\theta)^{1-\alpha} \left(1 - x_{t_j}\right)^{1-\alpha} e^{-\alpha\lambda\tau} \right]$$

the FOC for X_{t_j}:

$$x_{t_j} = \Gamma_{t_j+\tau}^{-\frac{1}{\alpha}} (1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} h(\tau) \left(1 - x_{t_j}\right)$$

Now define:

$$A_{t_j+\tau} \equiv \Gamma_{t_j+\tau}^{-\frac{1}{\alpha}} (1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} h(\tau) \tag{2.15}$$

where A could potentially depend on time in addition to τ. Thus

$$x_{t_j} = \frac{A_{t_j+\tau}}{1 + A_{t_j+\tau}} \tag{2.16}$$

C. Step 3: Given τ, Compute the Value Function

Substituting for $X^*_{t_j}$ into (2.13):

$$\Gamma_{t_j} = h(\tau)^{\alpha} \left(\frac{A_{t_j+\tau}}{1 + A_{t_j+\tau}} \right)^{-\alpha} \tag{2.17}$$

and

$$A_{t_j+\tau} = \Gamma_{t_j+\tau}^{-\frac{1}{\alpha}} (1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} h(\tau)$$

or, equivalently:

$$A_{t_j} = \frac{A_{t_j+\tau}}{1 + A_{t_j+\tau}} (1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau}$$

This equation in differences either goes to zero or diverges to $+\infty$, unless A_{t_j} starts in its steady state. Thus A can only be a constant given by:

$$A = \frac{A}{1 + A} (1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau}$$

Thus

$$A(\tau) = (1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} - 1 \tag{2.18}$$

Note that positivity on A implies $(1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} > 1$.

From (2.17), we know that:

$$
\begin{aligned}
\Gamma &= (1-\theta)^{-(1-\alpha)} e^{\alpha\lambda\tau} h(\tau)^{\alpha} A(\tau)^{-\alpha} \\
&= (1-\theta)^{-(1-\alpha)} e^{\alpha\lambda\tau} h(\tau)^{\alpha} \left((1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} - 1 \right)^{-\alpha} \\
&= h(\tau)^{\alpha} \left[(1-\theta)^{\frac{1-\alpha}{\alpha}} e^{-\lambda\tau} \left((1-\theta)^{-\frac{1-\alpha}{\alpha}} e^{\lambda\tau} - 1 \right) \right]^{-\alpha} \\
&= h(\tau)^{\alpha} \left[1 - (1-\theta)^{\frac{1-\alpha}{\alpha}} e^{-\lambda\tau} \right]^{-\alpha} \\
&= \left(\frac{1-e^{-\omega\tau}}{\omega} \right)^{\alpha} \left[1 - (1-\theta)^{\frac{1-\alpha}{\alpha}} e^{-\lambda\tau} \right]^{-\alpha} \\
&= \omega^{-\alpha} \left[\frac{1 - (1-\theta)^{\frac{1-\alpha}{\alpha}} e^{-\lambda\tau}}{1-e^{-\omega\tau}} \right]^{-\alpha} \\
&= \omega^{-\alpha} \left[\frac{1-e^{-\omega\tau}}{1-\chi e^{-\lambda\tau}} \right]^{\alpha}
\end{aligned}
$$

where

$$
\begin{aligned}
\chi &\equiv (1-\theta)^{\frac{1-\alpha}{\alpha}} \\
\alpha\lambda &\equiv \rho - (1-\alpha) \left[r + \frac{1}{2\gamma} \left(\frac{\mu - r}{\sigma} \right)^2 \right] \\
\alpha\omega &\equiv \rho - (1-\alpha) r^L
\end{aligned}
$$

Also note that if the consumer decides not to invest in the risky asset, the value of Γ would be $\omega^{-\alpha}$.

D. Step 4: Choose τ to Maximize the Value Function

Remember that $V(W_{t_j})^{1-\alpha} = \frac{\rho(1-e^{-\rho\tau})}{1-\alpha} \Gamma W_{t_j}^{1-\alpha}$. Thus choose τ such that the value function is maximized is equivalent to maximizing:

$$
F(\tau) = (1 - e^{-\rho\tau}) \Gamma(\tau)
$$

Firstly I would like to note that, with respect to the time separable utility, the value function is scaled by $1 - e^{-\rho\tau}$. For this reason, I will consider the redefined function:

$$\max_{\tau} \mathcal{F}(\tau) = \max_{\tau} \log \frac{F(\tau)}{1 - e^{-\rho\tau}} = \max_{\tau} \left\{ \log\left(1 - e^{-\omega\tau}\right) - \log\left(1 - \chi e^{-\lambda\tau}\right) \right\}$$

$$\begin{aligned}
\mathcal{F}'(\tau) &= \frac{\omega e^{-\omega\tau}}{1 - e^{-\omega\tau}} - \frac{\chi\lambda e^{-\lambda\tau}}{1 - \chi e^{-\lambda\tau}} \\
&= \frac{\omega}{e^{\omega\tau} - 1} - \frac{\chi\lambda}{e^{\lambda\tau} - \chi} \\
\mathcal{F}''(\tau) &= \frac{\omega e^{-\omega\tau}}{1 - e^{-\omega\tau}} - \frac{\chi\lambda e^{-\lambda\tau}}{1 - \chi e^{-\lambda\tau}} \\
&= \frac{\omega}{e^{\omega\tau} - 1} - \frac{\chi\lambda}{e^{-\lambda\tau} - \chi}
\end{aligned}$$

$\mathcal{F}(\tau)$ is a concave function (- for a proof see the original paper) thus we are guaranteed that a maximum exists. Moreover $\mathcal{F}'(0) > 0$ so $\tau^* > 0$. τ^* will be given by $\mathcal{F}'(\tau) = 0$.

As in the original paper, the optimal τ is given by the intersection of χ^{-1} and $\mathcal{M}(\tau)$, where:

$$\begin{aligned}
\mathcal{M}(\tau) &= (\omega - \lambda + \lambda e^{\omega\tau})\frac{1}{\omega}e^{-\lambda\tau} \\
\mathcal{M}'(\tau) &= (\omega - \lambda)(e^{\omega\tau} - 1)\frac{1}{\omega}e^{-\lambda\tau} \\
\mathcal{M}''(\tau) &= \left[e^{\omega\tau} - (e^{\omega\tau} - 1)\frac{\lambda}{\omega}\right](\omega - \lambda)\lambda e^{-\lambda\tau}
\end{aligned}$$

Note that $\mathcal{M}' > 0$ if $\alpha < 1$ given that $\omega - \lambda > 0$ (from $r > r^L$) and $\omega\tau > 0$ (since the optimal stopping time is strictly positive and we are imposing $\lambda > 0$ in order for the value function to be finite).

The only difference with the original paper is in the parameters.

Parameter	EZ Preferences	Time Separable Preferences
χ	$(1-\theta)^{\frac{1-\alpha}{\alpha}}$	$(1-\theta)^{\frac{1-\alpha}{\alpha}}$
$\alpha\lambda$	$\rho-(1-\alpha)\left[r+\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]$	$\rho-(1-\alpha)\left[r+\frac{1}{2\alpha}\left(\frac{\mu-r}{\sigma}\right)^2\right]$
$\alpha\omega$	$\rho-(1-\alpha)r^L$	$\rho-(1-\alpha)r^L$
Ω	$r+\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2$	$r+\frac{1}{2\alpha}\left(\frac{\mu-r}{\sigma}\right)^2$

Proposition 22. *$d\tau^*/d\theta > 0$. Moreover, with EZ preferences, for sufficiently high α ($\alpha > 1$), the response of the optimal stopping time to a greater adjustment cost is increasing in risk aversion.*

Proof. Starting from $\mathcal{M}(\tau^*) = \chi^{-1}$, I differentiate with respect to χ and obtain $d\tau^*/d\chi = -\mathcal{M}(\tau^*)/[\chi\mathcal{M}'(\tau^*)] < 0$. Moreover, $d\chi/d\theta = -(1-\alpha)\chi[\alpha(1-\theta)]^{-1}$. After applying the chain rule, $d\tau^*/d\theta = \omega e^{\omega\tau^*}/[\lambda\chi(1-\theta)(\Omega(\gamma)-r^L)(e^{\omega\tau^*}-1)] > 0$. Moreover, $\Omega'(\gamma) < 0$ and $d(\lambda(\Omega(\gamma)-r^L))/d\Omega > 0$ if $\alpha > 1$.

□

An implication of this proposition is that in the case of early resolution of uncertainty, $\gamma > \alpha > 1$, the response is greater with respect to the time separable case, for same IES.

Proposition 23. *$d\tau^*/dr^L > 0$. Moreover, the response increases with risk aversion γ.*

Proof. After using the chain rule in $\mathcal{M}(\tau^*)\chi = 1$, I obtain $d\tau^*/dr^L = -\mathcal{M}_\omega/\mathcal{M}'(\tau^*)d\omega/dr^L$. But $\mathcal{M}_\omega = \lambda\frac{e^{-\lambda\tau}}{\omega^2}\left[1-(1-\omega\tau)e^{\omega\tau}\right]$ which implies

$$d\tau^*/dr^L = \left[1-(1-\omega\tau)e^{\omega\tau}\right]\left(\Omega(\gamma)-r^L\right)^{-1}(e^{\omega\tau}-1)^{-1}\omega^{-1} > 0.$$

□

Again, a direct implication of this proposition is that in the case of early resolution of uncertainty, $\gamma > \alpha$, the response is greater with respect to the time separable case, for same IES.

Lemma 24. *If $\alpha > 1$, then $d\tau^*/d\Omega(\gamma) < 0$.*

With a higher expected return on their portfolio, agents asses their wealth more frequently.

Proof. Starting from $\mathcal{M}(\tau^*)\chi = 1$, by the implicit function theorem,

$$d\tau^*/d\Omega = -\mathcal{M}_\lambda \mathcal{M}'^{-1} (d\lambda/d\Omega)^{-1}.$$

If $\alpha > 1$, then $\mathcal{M}' < 0$ and $d\lambda/d\Omega(\gamma) > 0$. Moreover,

$$\begin{aligned}\mathcal{M}_\lambda &= \frac{1}{\omega\lambda} e^{-\lambda\tau^*} \left(\omega - \lambda + \lambda e^{\omega\tau^*}\right) \left[-\tau^*\lambda + \frac{-\lambda + \lambda e^{\omega\tau}}{\omega - \lambda + \lambda e^{\omega\tau}}\right] \\ &= \frac{1}{\omega\lambda} e^{-2\lambda\tau^*} \left(\omega + \lambda(e^{\omega\tau^*} - 1)\right) \left[(1 - \tau^*\lambda)e^{\lambda\tau^*} + \mathcal{M}^{-1}(\tau^*)\right].\end{aligned}$$

We also know that the function $(1 - \lambda\tau^*)e^{\lambda\tau^*} < 1 \forall \lambda\tau^* > 0$ and that, in equilibrium, $\mathcal{M}^{-1}(\tau^*) = \chi$. Moreover, if $\alpha > 1$, $\chi > 1$. This implies $\mathcal{M}_\lambda < 0$.Thus $d\tau^*/d\omega(\gamma) < 0$. □

Conjecture 25. *For $\alpha > 1$, $\exists \underline{\chi}$ defined as $\underline{\chi} = \frac{e^{\lambda * \tau^*(\underline{\chi})}}{\lambda\tau^*(\underline{\chi}) - 1}$ s.t. $\forall \chi > \underline{\chi}$ then $d\tau^*/d\Omega(\gamma)$ declines in risk aversion.*

Or, put differently, under sufficiently high transaction costs, for very risk averse agents, the frequency of wealth assessment is unaffected by the expected return on the portfolio, i.e. it will approach zero from below.

The sketch of the proof goes as follows: $\frac{d^2\tau}{d\Omega d\gamma} = \left(-\frac{\alpha-1}{\alpha}\right) \frac{\mathcal{M}_\tau \mathcal{M}_{\lambda\gamma} - \mathcal{M}_\lambda \mathcal{M}_{\tau\gamma}}{\mathcal{M}_\tau^2}$. This means that $sign\left(\frac{d^2\tau}{d\Omega d\gamma}\right) = sign(\mathcal{M}_\lambda \mathcal{M}_{\tau\gamma} - \mathcal{M}_\tau \mathcal{M}_{\lambda\gamma})$. Moreover, $\mathcal{M}_\lambda < 0$, $\mathcal{M}_{\tau\gamma} < 0$, $\mathcal{M}_\tau > 0$, and $\mathcal{M}_{\lambda\gamma} > 0$ for $\chi > \underline{\chi}$.

Proposition 26. *If $\alpha > 1$, then $d\tau^*/d\mu < 0$, $d\tau^*/d\sigma^2 > 0$, and $d\tau^*/d\gamma > 0$.*

Proof. Straightforward application of the chain rule. $\Omega(\gamma)$ is increasing in μ and decreasing in σ^2 and γ. □

Proposition 27. *If $\alpha > 1$, then $d\tau^*/dr < 0$ as $\phi^* < 1$ and vice versa.*

Proof. As in the original paper. □

2.3 Illustrative Calculations

As in the original paper, I consider the baseline case with $\theta = 0.0001$, $\alpha = 4$, $\rho = 0.01$, $r^L = 0.01$, $r = 0.02$, $\mu = 0.06$, and $\sigma^2 = (0.16)^2$, where ρ, r^L, r, μ, and σ are rates per year. I compare the time separable case, $\gamma = \alpha$, with $\gamma = 2\alpha$.

	$\gamma = \alpha$	$\gamma = 8$
Baseline	0.696	0.843
$\theta = 0.001$	2.223	2.690
$\rho = 0.02$	0.662	0.795
$\alpha = 2$	0.587	0.924
$r^L = 0$	0.557	0.643
$r = 0.03$	0.541	0.584
$\mu = 0.07$	0.584	0.753
$\sigma = 0.2$	0.796	0.913

Table 2.1: Optimal decision intervals τ^*, in years.

2.4 Conclusion

As in [1], the solution strategy of an optimal inattention to the stock market can be done in four easy steps. Although the set-up is in continuous time, the fact that the consumer only decides to evaluate his wealth at discrete times allows us to focus on discrete time Epstein-Zin

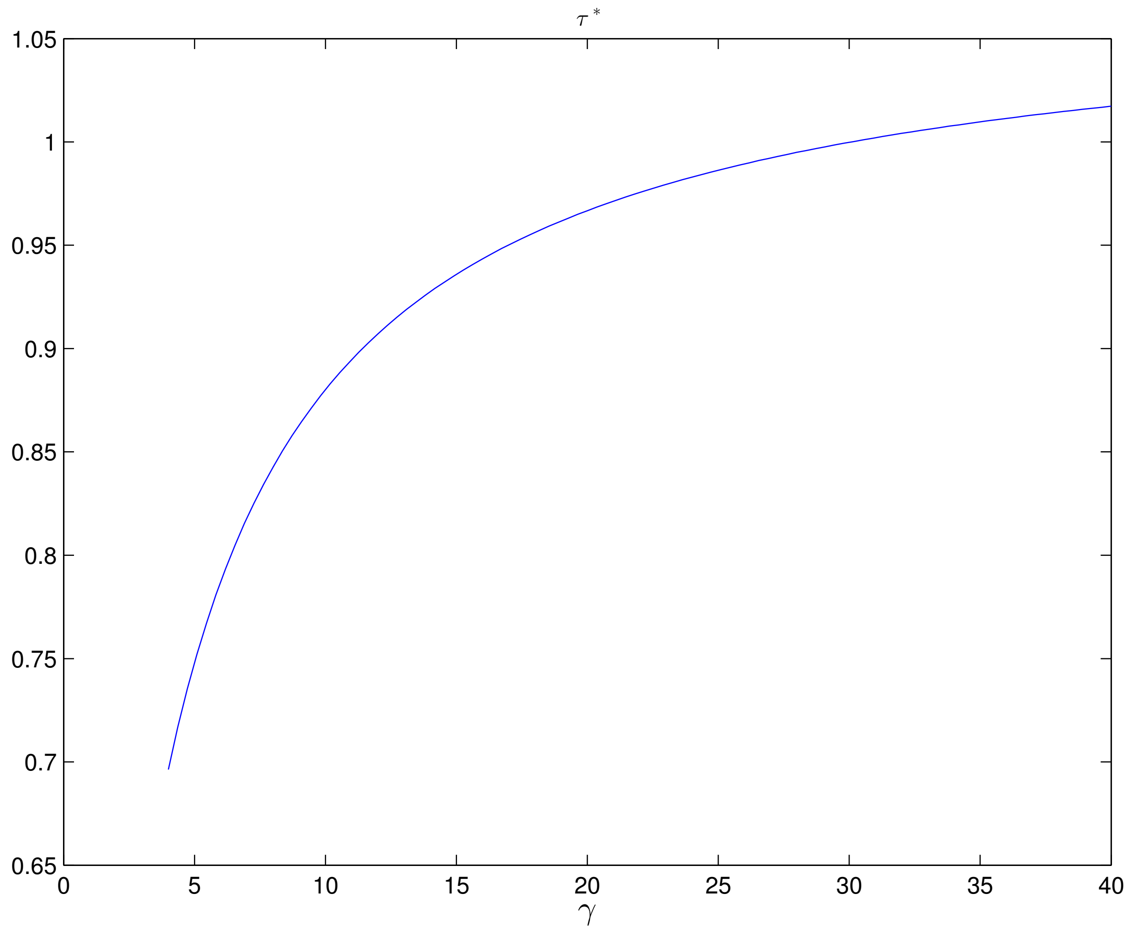

Figure 2.1: Optimal stopping time as a function of γ for the baseline case.

preferences which simplifies the problem considerably. The first important difference with respect to [1] is that the optimal portfolio will depend on the risk aversion coefficient. Only through the optimal portfolio will the optimal choice of consumption and stopping time be affected. Otherwise, the problem is identical to the case with expected utility.

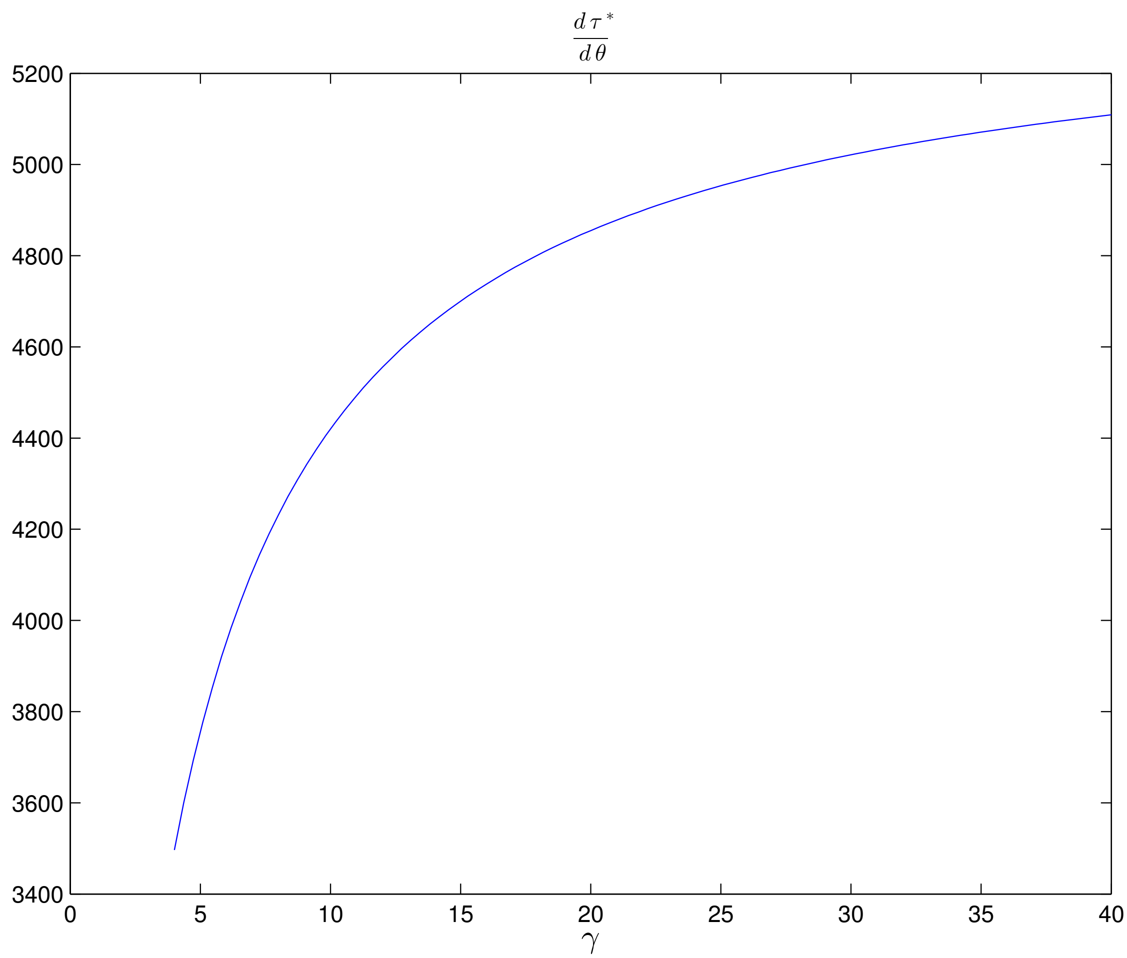

Figure 2.2: Response of the optimal stopping time to observational cost as a function of γ for the baseline case.

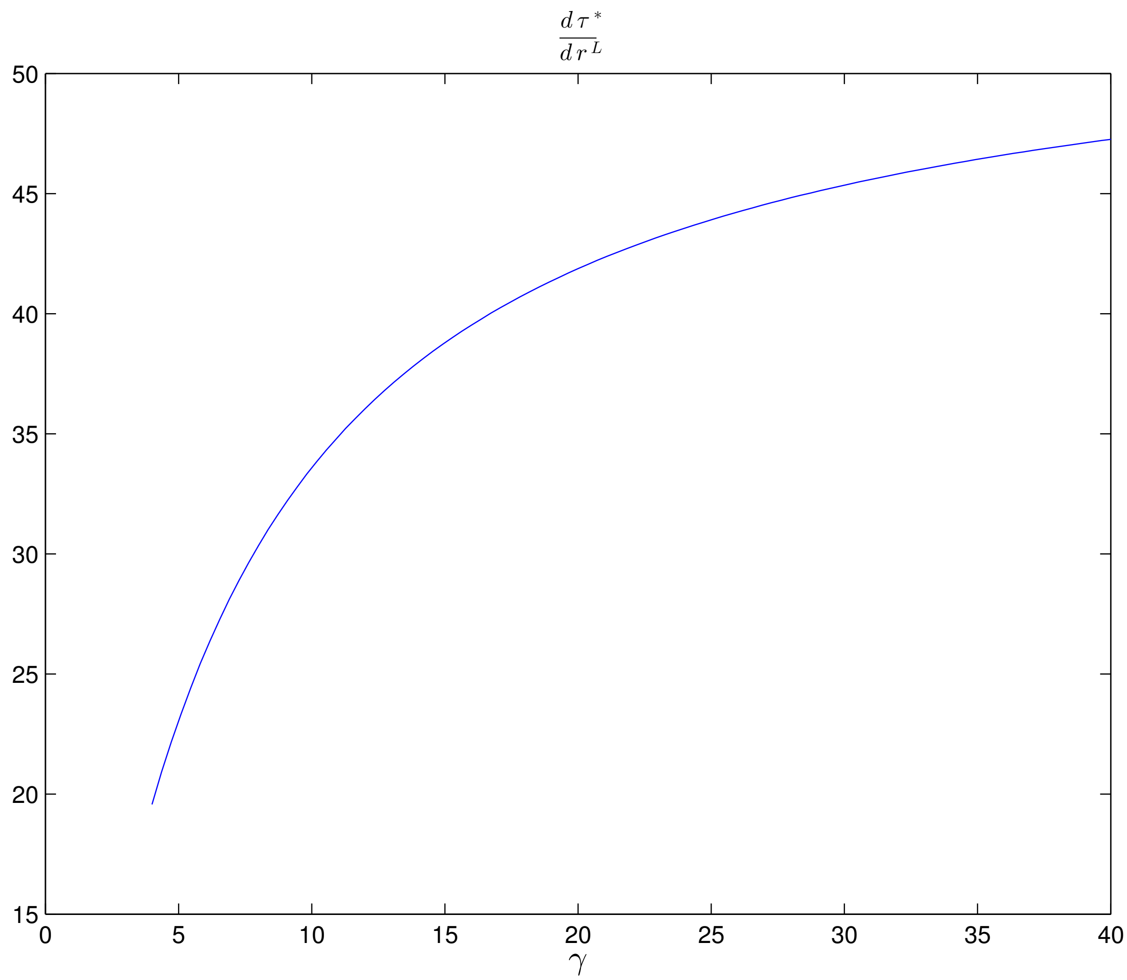

Figure 2.3: Response of the optimal stopping time to interest rate paid on the liquid asset as a function of γ for the baseline case.

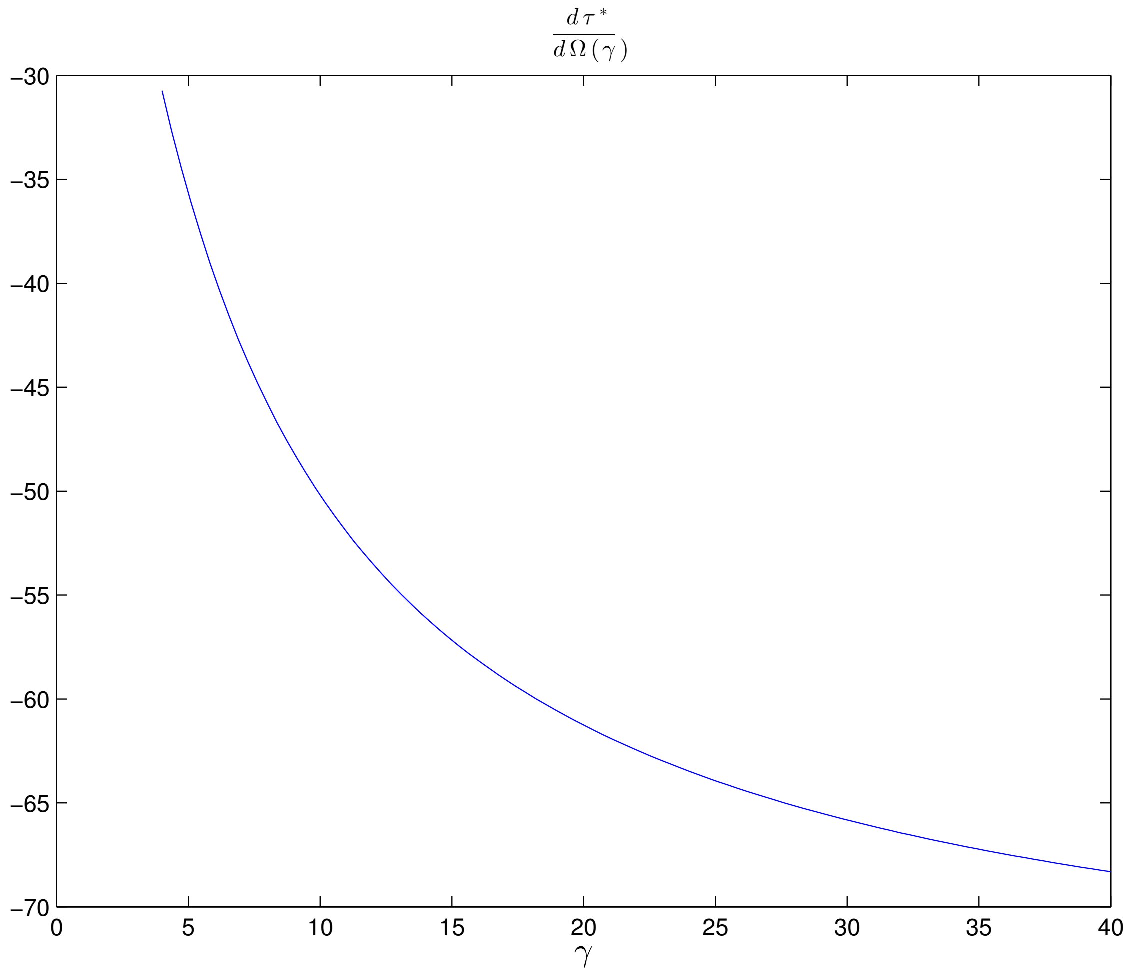

Figure 2.4: Response of the optimal stopping time to portfolio return as a function of γ for the baseline case.

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APPENDIX A

NETWORK AND PRICE DYNAMICS IN A MODEL OF CONTAGION

A.1 From Clearing Payment to Default Outcome

In this appendix I show the equivalence between clearing payments as defined in [13] and the default outcome.

The assets available for payments at time t increase with incoming transfers and decrease with outgoing transfers.

$$e_{it} := a_{it-1} + y_{it} + \sum_{j \in N_0} \hat{b}_{ijt} - \sum_{j \in N_0} \hat{b}_{jit}.$$

Note that feasibility and non-negativity of assets guarantee that $e_{it} \geq 0$ for all i. I follow [13] and define the clearing payment as the fixed point of the following map:

$$x \to \min\{\boldsymbol{\pi}_{t-1}^T x + \mathbf{e}_t, \bar{\mathbf{d}}_t\}. \tag{A.1}$$

where $[\cdot]^-$ denotes $\min\{0, \cdot\}$; when applied to a vector it denotes an element-wise operation. Similarly for $[\cdot]^+ = \max\{0, \cdot\}$.

Let the default outcome be denoted by w, where w satisfies $w := \bar{\mathbf{d}}_t - x$. Substituting in the definition of the clearing payment, equation (A.1), we find:

$$\bar{\mathbf{d}}_t - w = \min\{\boldsymbol{\pi}_{t-1}^T(\bar{\mathbf{d}}_t - w) + \mathbf{e}_t, \bar{\mathbf{d}}_t\}$$

$$w = \bar{\mathbf{d}}_t - \min\{\boldsymbol{\pi}_{t-1}^T\bar{\mathbf{d}}_t + \mathbf{e}_t - \boldsymbol{\pi}_{t-1}^T w, \bar{\mathbf{d}}_t\}$$

$$w = \max\{\bar{\mathbf{d}}_t - \boldsymbol{\pi}_{t-1}^T\bar{\mathbf{d}}_t - \mathbf{e}_t + \boldsymbol{\pi}_{t-1}^T w, 0\}$$

$$w = [-\mathbf{a}_{t-1} - \boldsymbol{\gamma}_t + \boldsymbol{\pi}_{t-1}^T w]^+$$

A.2 Substitutability and Relationship with Matching Literature

This economy has two main features. Firstly, each agent is allowed to engage in potentially numerous contracts with different counterparties and the identity of the lenders and borrowers is determined endogenously, without any preset restrictions. Lastly and most importantly, agents payoffs depend on everybody's actions through the network effects: each contract imposes an externality on the agents not involved in the contact through repercussions on the default outcome. I could incorporate these two features using an alternative equilibrium definition group stability as in [16] adjusted to include externalities as in [23], which I discuss briefly here.

Let C_i be bank i choice correspondence, where $C_i(Z \mid X_{-i})$ is the set of contracts that i chooses from Z given that X_{-i} is the set of contracts signed by the other agents:

$$C_i(Z \mid X_{-i}) \coloneqq \arg\max_{W \subseteq Z} V_i(W \cup X_{-i})$$

Definition 28 (Blocking Contracts)**.** *Given a set of contracts X that specifies ex-transfers liquidity surplus $\boldsymbol{\sigma'}$, $\tilde{X}$ is a blocking set of contracts if*

1. $\exists i, j \in N$ *such that*

$$\tilde{X} = X \setminus Z \cup Y, \; Z = Z_i \cup Z_j, \; Z \subseteq X, \; Y = Y_i = Y_j \tag{A.2}$$

2. *for* $k \in \{i, j\}$, *for all* $W \in C_k(\tilde{X} \mid X_{-k})$, *we have* $\tilde{X}_k \subseteq W$.

Definition 29 (Equilibrium)**.** *Given a financial system* $(\boldsymbol{\sigma}, \boldsymbol{v})$, *the set of contracts* X *is an equilibrium if:*

1. *it is feasible;*

2. *it is individually rational:* $X_i \in C_i (X \mid X_{-i})$ *for all* i*;*

3. *it is unblocked: there is no blocking contracts* $\tilde{X}$.

Feasibility requires that no agent promises to pay tomorrow more that its available resources. Individual rationality requires that no agent can become strictly better off by dropping some of the contracts that he is involved in. This is a standard requirement in the matching literature. The third condition states that there are no two agents i and j that can present a new set of contracts $\tilde{X}$ obtained from X through unilateral or bilateral deviations, such that both agents choose the new contracts instead.

In a novel and recent paper [23] prove that under certain conditions, substitutability and irrelevance of rejected alternatives, group stability can be extended to a setup with externalities as the one presented in this paper. Moreover, they use a modification of the deferred acceptance algorithm to prove existence of a stable outcome and show that the equilibrium still conserves some of the classical properties. Unfortunately here bank's payoffs do not satisfy substitutability, a sufficient requirement for the convergence of the deferred acceptance algorithm. The reason is that, as the set of defaulting agents shrinks, previously accepted contracts might become unacceptable. Because banks are willing to accept lower interest payments from banks that cause a bigger loss given default, as the set of defaulting banks shrinks and the loss given default decreases with it, lenders ask for a higher interest rate thereby dropping previously signed contracts. But in that case, the set of defaulting agenst expands, which can potentially lead to an infinite loop.

Lemma 30. *The agents' payoffs do not satisfy substitutability.*

Let $\mathcal{X} \coloneqq N \times N \times \mathbb{R}^2$ be the space where contracts live. Substitutability is defined in [23] as:

Definition 31. *A choice function C_i satisfies substitutability for illiquid agents if for any $X \subseteq X' \subseteq \mathcal{X}$, and $\mu, \mu' \subseteq \mathcal{X}$ such that $\forall i \in N$, $e_i(C_i(X' \mid \mu') \mid \mu') \geq e_i(C_i(X \mid \mu) \mid \mu)$ and $R_i(X \mid \mu) \coloneqq X_i \setminus C_i(X \mid \mu)$,*

$$R_i(X \mid \mu) \subseteq R_i(X' \mid \mu')$$

For that I provide a counter-example. Consider the network in figure (A.1). $\sigma_1 = \sigma_2 = -\$1$ and $\sigma_3 = \sigma_4 = \$1$. If 1 and 2 fail to raise \$1 liquidity, they default on all the intraagent debt they have.

Let $\mu = \emptyset$ and $\mu' = \{x_{24}\}$ where under contract x_{24} 4 transfers \$1 today in exchange of an interest payment of \$0. Clearly $\forall i \in \{1, 2, 3, 4\}$ and all $X \subseteq X' \subseteq \mathcal{X}$, $e_i(c_i(X' \mid \mu') \mid \mu') \geq e_i(C_i(X \mid \mu) \mid \mu)$: 4 always chooses the $\emptyset$ under both μ and μ' and it is weakly better off under μ' than under μ; the rest of the players are strictly better off under μ'.

I show that contract x_{13} under which 3 transfers \$1 today to 1 in exchange of an interest payment of \$0 is rejected under μ, but not under μ', contradicting therefore substitutability. 3 doesn't default if 4 transfers resources, or if it doesn't make a loan, but it will default if 3 is the only one lending, therefore x_{13} is rejected under μ, but not under μ'.

A.3 The Bonacich Centrality Measure

In this section I follow [6] in defining a centrality measure based on [9] that is useful in the analysis that follows. In this setting, the n-squared matrix $\mathbf{v}$ keeps track of the direct

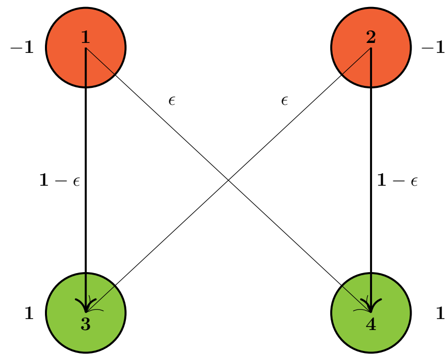

Figure A.1: A 4 player network.

connection in the financial network. If $v_{ij} > 0$, then i is a direct debtor of j and v_{ij} measures its strength. Let $\mathbf{v}^k$ denote the k^{th} power of the matrix $\mathbf{v}$, with coefficients v_{ij}^k, where k is some positive integer. The matrix $\mathbf{v}^k$ keeps track of the indirect connections in the financial network: if $v_{ij}^k > 0$, then i is an indirect debtor of j of order k with a total strength of v_{ij}^k. In graph theoretic terms, if $v_{ij}^k > 0$, there is a path of length k from i to j.[1] In particular $\mathbf{v}^0 = \mathbb{I}$.

Given a scalar $f \geq 0$ and a financial network $\mathbf{v}$, I define the matrix:

$$\mathbf{m}(\mathbf{v}, f) \coloneqq [\mathbb{I} - f\mathbf{v}]^{-1} = \sum_{k=0}^{\infty} f^k \mathbf{v}^k$$

These expressions are well defined if $f < 1/\rho(\mathbf{v})$, where $\rho(\mathbf{v})$ is the spectral value of $\mathbf{v}$.

1. A path of length k is a sequence of banks $< i_0, \cdots, i_k >$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and $v_{i_p i_{p+1}} > 0$, that is players i_p and i_{p+1} are directly linked in $\mathbf{v}$. When the network is unweighted, that is $\mathbf{v}$ is a (0,1) matrix, v_{ij}^k is simply the number of paths of length k from i to j.

The parameter f is a decay factor that scales down the relative strength of longer paths. If $\mathbf{m}(\mathbf{v}, f)$ is a nonnegative matrix, its coefficients $m_{ij}(\mathbf{v}, f) = \sum_{k=0}^{\infty} f^k v_{ij}^k$ count the total strength of paths in $\mathbf{v}$ that start at i and end at j, where paths of length k are weighted by f^k.

Definition 32. *Consider a network* $\mathbf{v}$ *and a scalar* f *such that* $\boldsymbol{m}(\mathbf{v}, f) = [\mathbb{I} - f\mathbf{v}]^{-1}$ *is well defined and nonnegative. The vector of Bonacich centralities of parameter* f *weighted by vector* $\boldsymbol{y}$ *is* $\boldsymbol{c}(\mathbf{v}, \boldsymbol{y}, f) = [\mathbb{I} - f\mathbf{v}]^{-1}\, \boldsymbol{y}$.

The Bonacich centrality of bank i is $c_i(\mathbf{v}, \mathbf{y}, f) = \sum_{j=1}^{n} m_{ij}(\mathbf{v}, f) y_j$ and it counts the total strength of all paths in $\mathbf{v}$ that start at i, where paths that end at node j are weighted by y_j. It is the sum of all weighted loops $m_{ii}(\mathbf{v}, f) y_i$, from i to i itself and all the weighted outer paths $\sum_{j \neq i} m_{ij}(\mathbf{v}, f) y_j$, that is,

$$c_i(\mathbf{v}, \mathbf{y}, f) = m_{ii}(\mathbf{v}, f) y_i + \sum_{j \neq i} m_{ij}(\mathbf{v}, f) y_j$$

By definition $m_{ii}(\mathbf{v}, f) \geq 1$.

A.4 Main Proofs

Proof of Corollary (7). Let γ denote the ex-post liquidity position defined under contracts $X \setminus Y \setminus W$. Similarly, let γ_1 and γ_2 denote the ex-post liquidity positions specifies under contracts Y and W respectively.

In that case, we have that the claim is true as long as the following holds:

$$\begin{aligned} \mathbf{LGD}(X \setminus Y \setminus W) &\geq \mathbf{LGD}(X \setminus Y) + \mathbf{LGD}(X \setminus W) \\ \mathbf{LGD}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2) &\geq \mathbf{LGD}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_1) + \mathbf{LGD}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_2) \geq \\ &\geq \mathbf{LGD}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_1) + \mathbf{LGD}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_2) - \mathbf{LGD}(\boldsymbol{\gamma}) \end{aligned}$$

where the last inequality follows from the fact that the loss given default is non-negative.

After reordering and the following change of variable $\boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}} + \tilde{\gamma}_1 + \tilde{\gamma}_2$ be obtain the the initial claim is correct as long as:

$$\mathbf{LGD}(\tilde{\gamma} + \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2) - \mathbf{LGD}(\tilde{\gamma} + \boldsymbol{\gamma}_1) \geq \mathbf{LGD}(\tilde{\gamma} + \boldsymbol{\gamma}_2) - \mathbf{LGD}(\tilde{\gamma})$$

which holds true given that **LGD** is convex.[2] □

Proof of Proposition (8).

$$\mathbf{d} = -\boldsymbol{\Lambda}\left[\boldsymbol{\gamma} - \boldsymbol{\pi}\boldsymbol{\Lambda}\mathbf{d}\right] \Rightarrow (\mathbb{I} - \boldsymbol{\Lambda}\boldsymbol{\pi}\boldsymbol{\Lambda})\mathbf{d} = -\boldsymbol{\Lambda}\boldsymbol{\gamma}$$

where $\boldsymbol{\Lambda}\boldsymbol{\pi}$ thas a row sum that is less than 1, and no row sum exceeds 1. To see that there is at least one bank that doesn't default assume the opposite $\boldsymbol{\Lambda} = \mathbb{I}$. Then we have the following:

$$\begin{aligned}
\mathbf{d} &= -\left[\boldsymbol{\gamma} + \mathbf{a} - \boldsymbol{\pi}\mathbf{d}\right] \Rightarrow \\
\mathbb{1}^T\mathbf{d} &= -\left[\mathbb{1}^T(\boldsymbol{\gamma} + \mathbf{a}) - \mathbb{1}^T\boldsymbol{\pi}\mathbf{d}\right] \Rightarrow \\
0 &= -\mathbb{1}^T(\boldsymbol{\gamma} + \mathbf{a})
\end{aligned}$$

a contradiction as long as $\sum_i (y_{i0} + a_i) \neq 0$.

As a consequence, $(\mathbb{I} - \boldsymbol{\Lambda}\boldsymbol{\pi}\boldsymbol{\Lambda})^{-1}$ exists and converges. [3] □

Proof lemma (9). Let $D(d)$ denote the default set under contracts (X_{t-1}, X_t). Using the definition of the default outcome we can write

2. Convexity of the loss given default follows from lemma 5 in [13] and the fact that the loss given default is the negative of the clearing payment.

3. See for example corollary to Theorem 4.C.11 in [24] and pg. 696 [20] for a reference.

$$\mathbf{d} = \boldsymbol{\Lambda}(\mathbf{d})\boldsymbol{\pi}_{t-1}^T\boldsymbol{\Lambda}(\mathbf{d})\mathbf{d} - \boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})$$

but $\boldsymbol{\Lambda}(\mathbf{d})\mathbf{d} = \mathbf{d}$ and $\boldsymbol{\pi}_{t-1}^T\mathbf{d} = \mathbf{LGD}$, therefore:

$$\mathbf{d} = \boldsymbol{\Lambda}(\mathbf{d})\mathbf{LGD} - \boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})$$

Premultiplying by $\mathbb{1}^T$:

$$\mathbb{1}^T\mathbf{d} = \mathbb{1}^T\boldsymbol{\Lambda}(\mathbf{d})\mathbf{LGD} - \mathbb{1}^T\boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1}) \Leftrightarrow$$
$$\mathbb{1}^T\mathbf{d} = \mathbb{1}^T[\boldsymbol{\Lambda}(\mathbf{d})\mathbf{LGD}] - \mathbb{1}^T[\boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})]$$

Use the fact that $\mathbb{1}^T\mathbf{LGD} = \mathbb{1}^T\boldsymbol{\pi}_{t-1}^T\mathbf{d} = (\boldsymbol{\pi}_{t-1}\mathbb{1})^T\mathbf{d} = \mathbb{1}^T\mathbf{d}$ to obtain:

$$\mathbb{1}^T\mathbf{LGD} = \mathbb{1}^T[\boldsymbol{\Lambda}(\mathbf{d})\mathbf{LGD}] - \mathbb{1}^T[\boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})] \Leftrightarrow$$
$$\mathbb{1}^T[\mathbb{I} - \boldsymbol{\Lambda}(\mathbf{d})]\mathbf{LGD} = -\mathbb{1}^T[\boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})] \Leftrightarrow$$

Therefore

$$\sum_{i \in N \setminus D} LGD_i(X_{t-1}, X_t) = -\sum_{i \in D}[\gamma_{it} + a_{it-1}]$$

And after some manipulation, the loss given default of the default set is:

$$\mathbb{1}^T(\boldsymbol{\Lambda}(\mathbf{d})\mathbf{LGD}) = \mathbb{1}^T\left[-\mathbb{I} + [\mathbb{I} - \boldsymbol{\Lambda}(\mathbf{d})\boldsymbol{\pi}^T\boldsymbol{\Lambda}(\mathbf{d})]^{-1}\right][-\boldsymbol{\Lambda}(\mathbf{d})(\boldsymbol{\gamma}_t + \mathbf{a}_{t-1})]$$

□