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THE WONDERFUL COMPACTIFICATION AND THE UNIVERSAL CENTRALIZER

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ABSTRACT

This thesis describes work of the author involving the geometry of the wonderful compactification of a complex semisimple algebraic group G of adjoint type. We consider the centralizer G^e in G of a regular nilpotent element $e \in \mathrm{Lie}(G)$, and we prove that its closure in the wonderful compactification is isomorphic to the Peterson variety. We generalize this result to show that the closure in the wonderful compactification of the centralizer G^x of any regular element $x \in \mathrm{Lie}(G)$ is isomorphic to the closure of a general G^x-orbit in the flag variety.

We then consider the family $\overline{\mathcal{X}}$ of all closures of such centralizers, parametrized by conjugacy classes of regular elements in $\mathrm{Lie}(G)$. This is a relative compactification of the universal centralizer $\mathcal{X}$, which has a natural symplectic structure arising from a Hamiltonian reduction of the cotangent bundle of G. We prove that the symplectic structure on $\mathcal{X}$ extends to a log-symplectic Poisson structure on $\overline{\mathcal{X}}$, by realizing $\overline{\mathcal{X}}$ as a Hamiltonian reduction of the logarithmic cotangent bundle of the wonderful compactification.

CHAPTER 1

INTRODUCTION

The work described in this thesis falls broadly under the scope of geometric representation theory, but its motivation and methods are closely linked to the theory of equivariant embeddings, spherical varieties, and Poisson geometry. It primarily discusses problems of an algebraic and geometric nature that involve the wonderful compactification of a semisimple algebraic group, which we approach using an interplay of techniques from representation theory, algebraic geometry, and symplectic geometry.

The *wonderful compactification* $\overline{G}$ of a complex semisimple algebraic group G with trivial center is a smooth projective variety that contains a copy of G as an open dense subset, and on which G acts in a way that extends the left- and right-multiplication within the group itself. It is well-established in the world of Lie theory and algebraic geometry, where it holds a distinguished place among the equivariant compactifications of G, but in recent years it has also come to play a pivotal role in many areas of study within representation theory, notably in the work of Bezrukavnikov and Kazhdan [7], of Bezrukavnikov, Finkelberg, and Ostrik [5], of Drinfeld and Gaitsgory [15], and of Sakellaridis and Venkatesh [35].

The wonderful compactification was first introduced by deConcini and Procesi [13] in the 1980s in the more general context of symmetric spaces, and has since received much attention in the theory of equivariant embeddings. (See [28], [8], and [23].) When the symmetric space is a semisimple algebraic group G, the wonderful compactification $\overline{G}$ of G is an analogue of the compactification of a torus into a projective toric variety.

The boundary of $\overline{G}$ is a smooth divisor with normal crossings that consists of hypersurfaces corresponding to simple roots. The $G \times G$-orbits on the boundary are precisely the intersections of these hypersurfaces, and they are indexed combinatorially by collections of simple roots. Every such orbit can be described as a fibration over a product of partial flag varieties, and the unique closed orbit, which is the orbit of minimal dimension, is isomorphic to a product $\mathcal{B} \times \mathcal{B}$ of two copies of the flag variety of G [16]. The boundary of $\overline{G}$

encodes the asymptotic behavior of G “at infinity,” and through this perspective $\overline{G}$ plays a prominent role in many areas of modern representation theory, for example in the work of Bezrukavnikov and Kazhdan [7] and of Sakellaridis and Venkatesh [35].

In the simplest case of rank 1, when $G \cong PGL_2(\mathbb{C})$, the wonderful compactification is isomorphic to the projective space $\mathbb{P}(M_{2\times 2}) \cong \mathbb{CP}^3$ of 2×2 matrices modulo scalars. The group G sits inside this space as the set of invertible matrices, and the boundary divisor is the single $PGL_2 \times PGL_2$-orbit of 2×2 projective matrices with determinant 0. This divisor is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, a product of two copies of the flag variety $\mathbb{CP}^1$ of PGL_2.

It is natural to characterize the closures of subgroups of G inside $\overline{G}$. Of particular interest are the centralizers of regular elements in the Lie algebra $\mathfrak{g}$ of G—an element $x \in \mathfrak{g}$ is called regular if its centralizer has minimal dimension, equal to the rank of $\mathfrak{g}$. For example, if x is regular and semisimple, its centralizer is a maximal torus T, and the closure of T in $\overline{G}$ is the very interesting projective toric variety corresponding to the fan of Weyl chambers. This toric variety also arises in the work of Dabrowski [12] and of Klyachko [22] as the closure of a general orbit of T in the flag variety $\mathcal{B}$ of G.

If one instead considers a regular nilpotent element $e \in \mathfrak{g}$, its centralizer is a unipotent abelian subgroup G^e of the same dimension as T. In the case $\mathfrak{g} = \mathfrak{sl}_n$, e is the element consisting of a single nilpotent Jordan block, and its centralizer $G^e \subset PGL_n$ is the subgroup of strictly upper-triangular matrices with constant entries along each superdiagonal.

It is known from the general structure theory of semisimple Lie algebras that the element e is contained in a unique Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ [24]. Viewing the flag variety $\mathcal{B}$ as the variety of Borel subalgebras of $\mathfrak{g}$, let $\mathfrak{b}^- \in \mathcal{B}$ be the opposite Borel. The closure of the G^e-orbit of $\mathfrak{b}^-$ in $\mathcal{B}$ is called the *Peterson variety* and is denoted $\mathbf{P}_e$. This variety was introduced by Dale Peterson in the 1990s and has proved essential to the study of the quantum cohomology of flag varieties, in work of Kostant [27], Rietsch [33, 34], and Tymoczko [38].

The work described in Chapter 2 uses the behavior of $\mathfrak{g}^e = \mathrm{Lie}(G^e)$ in the representation theory of $\mathfrak{g}$, which was developed by Kostant in [24, 25] and by Ginzburg in [18], to show the following result, conjectured by Ginzburg and Kazhdan:

Theorem 1.1. *[2] There is an isomorphism of G^e-varieties between the closure of G^e in the wonderful compactification $\overline{G}$ and the Peterson variety $\mathbf{P}_e$.*

We then generalize this result to the case of an arbitrary regular element $x \in \mathfrak{g}$ by defining the notion of a *general* orbit of G^x in $\mathcal{B}$. Intuitively, this is an orbit on which sufficiently many Plücker coordinates do not vanish, and our definition in [2] agrees with Dabrowski's definition of a general torus orbit in [12]. This leads to the following theorem:

Theorem 1.2. *[2] There is an isomorphism of G^x-varieties between the closure of G^x in the wonderful compactification $\overline{G}$ and the closure of a general G^x-orbit in the flag variety $\mathcal{B}$.*

Both of these theorems rely on finding appropriate projective embeddings of the two varieties in question, and then constructing an isomorphism between the resulting homogeneous coordinate rings. To obtain these projective embeddings and their homogeneous coordinate rings, we use the total coordinate ring of $\overline{G}$, an object that arises from the construction of $\overline{G}$ as a GIT quotient of the Vinberg semigroup. (See [39], [29].) To find the relations that cut out these varieties in the homogeneous coordinate rings of $\overline{G}$ and $\mathcal{B}$, we analyze the action of the universal enveloping algebra $\mathcal{U}\mathfrak{g}^e$ on the representations of $\mathfrak{g}$.

In Charpter 3 we consider a regular nilpotent element $e \in \mathfrak{g}$ and a corresponding $\mathfrak{sl}_2$-triple $\{e, h, f\}$. By the the work of Kostant [24] on the structure of semisimple Lie algebras, f is unique up to conjugation by an element of G^e, and the affine space $f + \mathfrak{g}^e$ is a transverse slice to the regular conjugacy classes of $\mathfrak{g}$—that is, it intersects each regular conjugacy class transversely and exactly once.

The variety

$$\mathcal{X} = \{(g, \xi) \in G \times \mathfrak{g} \mid \xi \in (f + \mathfrak{g}^e), g \in G^{\xi}\}$$

is known as the *universal centralizer*, and it parametrizes the family of centralizers of regular elements of $\mathfrak{g}$, up to conjugation. It plays an important role in modern representation theory and in the geometric Langlands program, for instance in the work of Ngo [30, 31], Bezrukavnikov and Finkelberg [6], Bezrukavnikov, Finkelberg, and Mirkovic [4], and Riche [32].

It can be obtained by a Hamiltonian reduction from the cotangent bundle T^*G under the two-sided action of the unipotent radical $N \times N \subset G \times G$. Letting $\mathfrak{n} = \mathrm{Lie}(N)^* \cong \mathfrak{g}/\mathfrak{b}$ via the Killing form, this action gives rise to a moment map

$$\mu : T^*G \longrightarrow \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}.$$

The equivalence class of the principal nilpotent pair (f, f) in $\mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}$ is fixed by $N \times N$, and the action of $N \times N$ on the fiber $\mu^{-1}(f, f)$ is free. There is an isomorphism of algebraic varieties

$$N\backslash\mu^{-1}(f, f)/N \cong \mathcal{X},$$

and thus $\mathcal{X}$ acquires a natural symplectic structure, pulled back from the canonical symplectic structure on the two-sided quotient $N\backslash\mu^{-1}(f, f)/N$ [37].

The work described in Chapter 3 describes the closure of the universal centralizer in $\overline{G} \times \mathfrak{g}$,

$$\overline{\mathcal{X}} = \{(x, \xi) \in \overline{G} \times \mathfrak{g} \mid \xi \in (f + \mathfrak{g}^e), x \in \overline{G^\xi}\}.$$

We show that the symplectic structure on $\mathcal{X}$ extends to a log-symplectic structure—with poles of order at most one—on the boundary of $\overline{\mathcal{X}}$. This is achieved by performing a similar Hamiltonian reduction, this time from the logarithmic cotangent bundle $T^*\overline{G}(-\log D)$—the vector bundle whose sections are differential forms with at most first-order poles along the boundary $D = \overline{G}\backslash G$. The bundle $T^*\overline{G}(-\log D)$ restricts to the ordinary cotangent bundle along the open subset $G \subset \overline{G}$.

In order to prove the results in [3], one must first observe that the logarithmic cotangent bundle of $\overline{G}$ has a natural Poisson structure that is log-symplectic—which is to say, the Poisson bivector gives an isomorphism between the logarithmic cotangent bundle and the logarithmic tangent bundle [19]. This is known for all smooth varieties with smooth normal crossing divisors, and the construction closely mirrors the construction of the canonical symplectic structure on the ordinary cotangent bundle.

The action of $N \times N$ on $T^*\overline{G}(-\log D)$ is Hamiltonian, and gives rise to a moment map

$$\overline{\mu} : T^*\overline{G}(-\log D) \longrightarrow \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}.$$

As before, the equivalence class of (f, f) in $\mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}$ is a regular value of $\overline{\mu}$, and the action of $N \times N$ on the fiber $\overline{\mu}^{-1}(f, f)$ is free. In [3] I show that the quotient of this fiber by the two-sided $N \times N$-action is isomorphic to $\overline{\mathcal{X}}$, and I obtain the following result:

Theorem 1.3. *[3] There is an isomorphism of algebraic varieties*

$$\overline{\mathcal{X}} \cong N\backslash\overline{\mu}^{-1}(f, f)/N$$

between the closure of the universal centralizer and the Hamiltonian reduction of $T^*\overline{G}(-\log D)$ *with respect to* $N\times N$ *at the point* $(f, f) \in \mathfrak{g}/\mathfrak{b}\times\mathfrak{g}/\mathfrak{b}$. *Therefore* $\overline{\mathcal{X}}$ *has a natural log-symplectic structure extending the symplectic structure on* $\mathcal{X} \subset \overline{\mathcal{X}}$.

CHAPTER 2

THE PETERSON VARIETY AND THE WONDERFUL COMPACTIFICATION

2.1 Introduction

The wonderful compactification of a semisimple complex algebraic group G of adjoint type is a special case of the compactification of symmetric spaces introduced by DeConcini and Procesi in [13]. Its boundary is a divisor with normal crossings with a unique closed $G \times G$-orbit, and in some sense it encodes the behavior of the group "at infinity". A survey of its structure can be found in [16].

We will consider regular elements in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and their centralizers in G, and describe the closure of these centralizers in the wonderful compactification $\overline{G}$. In particular, we will be interested in the unique conjugacy class of regular nilpotent elements, also called principal nilpotents. All the relevant structure theory of semisimple Lie algebras and of their regular orbits was developed by Kostant in [24] and [27].

A principal nilpotent element sits inside a unique Borel subgroup B, and its centralizer is a unipotent abelian subgroup of B. In the full flag variety determined by the opposite Borel the closure of a general orbit of this centralizer is called the Peterson variety. This variety has been well-studied, and is known to be singular and non-normal except in very small rank [27]. It was introduced by Dale Peterson in the 1990s and it has proved essential in the study of the quantum cohomology of flag varieties, for example in [27], [34], and [38].

We will show that the closure of the centralizer of the principal nilpotent in $\overline{G}$ is isomorphic to the Peterson variety. This will lead to the main result of this paper, which states that the closure in $\overline{G}$ of the centralizer of any regular element of $\mathfrak{g}$ is isomorphic to the closure of a sufficiently general orbit of this centralizer in the flag variety. Both of these results are shown by choosing appropriate projective embeddings given by very ample line bundles, and then establishing an isomorphism between the resulting homogeneous coordinate rings.

This extends the case of a maximal torus T, which is the centralizer of a regular semisimple element—the closure of T in the wonderful compactification is the toric variety whose fan is the fan of Weyl chambers (see [16], Remark 4.5), and it is isomorphic to the closure of a general T-orbit in the flag variety [12].

In Section 2.2 we recall some basics about the flag variety $\mathcal{B}$, the Peterson variety, and the very ample G-equivariant line bundles on $\mathcal{B}$. In Section 2.3 we present some analogous facts about the wonderful compactification by describing its construction via the Vinberg semigroup. In Section 2.4 we construct an isomorphism between the homogeneous coordinate rings of the closure of the regular nilpotent centralizer in $\overline{G}$ and the Peterson variety. In Section 2.5 we extend the results of Section 2.4 to the case of the centralizer of an arbitrary regular element. In Section 2.6 we give an explicit description of the orbits of the regular nilpotent centralizer on the Peterson variety, from which it becomes clear, in particular, that except in very few cases they are infinite in number.

2.2 The Peterson Variety

Let G be, as above, a complex semisimple algebraic group of adjoint type and rank l, T a maximal torus, and B a Borel subgroup containing T. Let N be the unipotent radical of B and $\alpha_1, \ldots, \alpha_l$ the set of positive simple roots. If $e_1, \ldots, e_l$ are corresponding simple root vectors in the Lie algebra $\mathfrak{g}$, then

$$e = e_1 + \ldots + e_l$$

is a principal nilpotent sitting inside $\mathfrak{b}$, and we denote by G^e its centralizer in G.

The centralizer G^e is a unipotent abelian subgroup of N of dimension equal to l. In type A, e is the single nilpotent Jordan block, and G^e is the group of unipotent matrices with constant entries along each superdiagonal.

Let B^- be the opposite Borel subgroup and $\mathfrak{b}^-$ its Lie algebra. Viewing Borel subalgebras as points in the flag variety, $\mathfrak{b}^-$ is the basepoint of the flag variety $\mathcal{B} = G/B^-$, and the Peterson variety is the closure

$$\mathbf{P}_e := \overline{G^e \cdot \mathfrak{b}^-} \subset \mathcal{B}.$$

Recall that the G-equivariant line bundles on $\mathcal{B}$ are indexed by integral weights of $\mathfrak{g}$, and for a dominant weight λ the space of global sections of the line bundle $\mathcal{L}_\lambda$ is identified with V_λ, the irreducible representation of highest weight λ, via

$$\begin{aligned} V_\lambda &\xrightarrow{\sim} \Gamma(\mathcal{B}, \mathcal{L}_\lambda) \\ v &\longmapsto \left[gB^- \mapsto (gB^-, v_\lambda^*(g^{-1} \cdot v))\right], \end{aligned}$$

where v_λ^* is the lowest weight vector of V_λ^*. (Note that the space of global sections is V_λ and not its dual, because we are taking the flag variety relative to the opposite Borel B^-.)

Let $\omega_1, \ldots, \omega_l$ be the fundamental weights of $\mathfrak{g}$, $V_1, \ldots, V_l$ the fundamental representations, and for each i, let V_i^* be the dual representation with lowest weight vector v_i^*. The Plücker embedding realizes the flag variety as a multi-projective variety

$$\begin{aligned} \mathcal{B} &\hookrightarrow \prod_{i=1}^{l} \mathbb{P}(V_i^*) \\ gB^- &\longmapsto (g \cdot [v_1^*], \ldots, g \cdot [v_l^*]) \end{aligned}$$

and its total coordinate ring is the multi-graded algebra given by summing the spaces of global sections of all G-equivariant line bundles:

$$R[\mathcal{B}] := \bigoplus_{\lambda \text{ dom}} \Gamma(\mathcal{B}, \mathcal{L}_\lambda) = \bigoplus_{\lambda \text{ dom}} V_\lambda.$$

(See [1] for a detailed introduction to total coordinate rings, also called Cox rings.) Multiplication is given by projection onto the highest weight component:

$$V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}.$$

The very ample line bundles on $\mathcal{B}$ correspond to regular dominant weights, and such a weight λ produces a $\mathbb{Z}$-graded homogeneous coordinate ring, denoted $R_\lambda[\mathcal{B}]$, that is a quotient of the total coordinate ring given by taking a generic line in the semigroup of dominant weights:

$$R_\lambda[\mathcal{B}] := \bigoplus_{n\geq 0} \Gamma(\mathcal{B}, \mathcal{L}_{n\lambda}) = \bigoplus_{n\geq 0} V_{n\lambda}.$$

The homogeneous coordinate ring of $\mathbf{P}_e$ is then

$$R_\lambda[\mathbf{P}_e] = R_\lambda[\mathcal{B}]/\mathcal{I}_{\mathbf{P}_e},$$

where $\mathcal{I}_{\mathbf{P}_e}$ is the ideal of global sections that vanish on the Peterson variety.

2.3 The Wonderful Compactification

Let $\widetilde{G}$ be the simply-connected cover of G, $\widetilde{T}$ the corresponding maximal torus, and $\widetilde{Z}$ its center. Identifying $\mathrm{End}(V_i)$ with $V_i \otimes V_i^*$ gives representation maps

$$\rho_i : \widetilde{G} \longrightarrow V_i \otimes V_i^*.$$

We recall briefly the construction of the wonderful compactification via the Vinberg semigroup [39]. Consider $\widetilde{G} \times_{\widetilde{Z}} \widetilde{T}$, where $\widetilde{Z} \hookrightarrow \widetilde{G} \times \widetilde{T}$ is the anti-diagonal embedding.

Define the embedding

$$\chi: \widetilde{G} \times_{\widetilde{Z}} \widetilde{T} \longhookrightarrow \mathbb{C}^l \times \prod_{i=1}^{l} V_i \otimes V_i^*$$

$$(g,t) \longmapsto (\alpha_1(t), \ldots, \alpha_l(t), \omega_1(t)\rho_1(g), \ldots, \omega_l(t)\rho_l(g))$$

The closure of the image of χ is the Vinberg semigroup V_G, and the first projection is a flat family of semigroups over $\mathbb{C}^l$ (see [39], Section 4.) The closure V_G^0 of the image of χ in the space

$$\mathbb{C}^l \times \prod_{i=1}^{l} (V_i \otimes V_i^* - \{0\})$$

is a smooth open subset, and both V_G and V_G^0 are equipped with a natural action of $\widetilde{G} \times_{\widetilde{Z}} \widetilde{T}$. In particular, the torus $\{1\} \times \widetilde{T}$ acts freely on V_G^0 via coordinate-wise multiplication by

$$(\alpha_1(t), \ldots, \alpha_l(t), \omega_1(t), \ldots, \omega_l(t)),$$

and the wonderful compactification of G is defined to be the quotient of V_G^0 by this action:

$$\overline{G} := V_G^0 / \widetilde{T}$$

(see [29], 5.3.) It contains $G \cong \widetilde{G}/\widetilde{Z}$ as a dense open subset, and it has a natural $\widetilde{G} \times \widetilde{G}$-action on $\overline{G}$ that extends the two-sided action of $\widetilde{G}$ on G itself.

The $\widetilde{G} \times \widetilde{G}$-equivariant line bundles on $\overline{G}$ correspond to integral weights of the group $\widetilde{G}$. For such a weight λ, the global sections of the line bundle $\mathcal{M}_\lambda$ are given by

$$\Gamma(\overline{G}, \mathcal{M}_\lambda) \cong \bigoplus_{\mu \leq \lambda} V_\mu^* \otimes V_\mu$$

as a $\widetilde{G} \times \widetilde{G}$-module, where the sum is over all dominant weights μ less than λ—i.e. dominant weights μ such that $\lambda - \mu$ is a sum of simple roots with non-negative integral coefficients (see

[9], 3.2.3.) The line bundle $\mathcal{M}_\lambda$ is very ample exactly when λ is a regular dominant weight.

The total coordinate ring of $\overline{G}$—that is, the affine coordinate ring of the Vinberg semigroup—is the multi-graded algebra

$$R[\overline{G}] = \bigoplus_{\lambda} \Gamma(\overline{G}, \mathcal{M}_\lambda) = \bigoplus_{\lambda} \left(\bigoplus_{\mu \leq \lambda} V_\mu^* \otimes V_\mu \right) t^\lambda,$$

with multiplication on the right hand side given by viewing the algebra as a subalgebra of $\mathbb{C}[\widetilde{G} \times \widetilde{T}]$. In particular, the multiplication map has the property

$$m : (V_\mu^* \otimes V_\mu) \otimes (V_\nu^* \otimes V_\nu) \cong (V_\mu \otimes V_\nu)^* \otimes (V_\mu \otimes V_\nu) \longrightarrow \bigoplus_{\xi \leq \mu+\nu} V_\xi^* \otimes V_\xi$$

by decomposing $(V_\mu \otimes V_\nu)^*$ and $V_\mu \otimes V_\nu$ separately into irreducible representations and then projecting onto the components of the form $V_\xi^* \otimes V_\xi$.

From now on fix a regular dominant weight λ in the root lattice. Then for any $\mu \leq \lambda$ the $\widetilde{G}$-representations V_λ and V_μ descend to representations of the adjoint group G. The $\mathbb{Z}$-graded homogeneous coordinate ring of $\overline{G}$ produced by the very ample line bundle $\mathcal{M}_\lambda$ is a quotient algebra of $R[\overline{G}]$ corresponding to the generic line given by λ in the cone of dominant weights:

$$R_\lambda[\overline{G}] := \bigoplus_{n \geq 0} \Gamma(\overline{G}, \mathcal{M}_{n\lambda}) = \bigoplus_{n \geq 0} \left(\bigoplus_{\mu \leq n\lambda} V_\mu^* \otimes V_\mu \right) t^{n\lambda}.$$

Let $\overline{\mathrm{G^e}}$ be the closure of the centralizer of the principal nilpotent in the wonderful compactification of G. Its homogeneous coordinate ring is then

$$R_\lambda[\overline{G^e}] = R_\lambda[\overline{G}]/\mathcal{I}_{\overline{\mathrm{G^e}}},$$

where $\mathcal{I}_{\overline{\mathrm{G^e}}}$ is the homogeneous ideal of global sections vanishing on $\overline{\mathrm{G^e}}$.

Remark 2.1. Because of the choice of λ above, from now on whenever a representation V_μ with highest weight μ appears, the weight μ will be an element of the root lattice, and V_μ will descend to a representation of G.

This choice is not necessary, and the same argument goes through essentially unchanged with an arbitrary choice of regular dominant λ—however, this will allow us to apply G directly to the spaces V_μ without having to repeatedly refer to the simply-connected cover $\widetilde{G}$.

The following lemmas and propositions use only the fact that G is semisimple, that λ, μ, ν are weights of G, and that G^e is an abelian unipotent subgroup of G that centralizes the principal nilpotent e. Therefore they will apply also to the setting of Section 2.5, where the group under consideration will not necessarily be of adjoint type.

Before we begin to prove our results, we introduce some notation: For any dominant weight μ in the root lattice, and any $u \in V_\mu$ and $v^* \in V_\mu^*$, denote by $f^\mu_{v^*,u}$ the function of G corresponding to the matrix entry $v^* \otimes u \in V_\mu^* \otimes V_\mu$—that is,

$$f^\mu_{v^*,u}(g) = v^*(g \cdot u).$$

Let v_μ^* denote the lowest weight vector of V_μ^*, and make this choice such that, under the multiplication map

$$V_\mu^* \otimes V_\nu^* \longrightarrow V_{\mu+\nu}^*,$$

$v_{\mu+\nu}^*$ is the image of $v_\mu^* \otimes v_\nu^*$, for any dominant weights μ and ν. (This can be done inductively, beginning from the fundamental representations.) Then, since

$$v_\mu^* \otimes v_\nu^* \in V_\mu^* \otimes V_\nu^*$$

always belongs to the irreducible component of the tensor isomorphic to $V^*_{\mu+\nu}$, we have

$$m(v^*_\mu \otimes u_1, v^*_\nu \otimes u_2) = v^*_{\mu+\nu} \otimes u \in V^*_{\mu+\nu} \otimes V_{\mu+\nu}$$

where u is the projection of the tensor $u_1 \otimes u_2 \in V_\mu \otimes V_\nu$ onto the irreducible component $V_{\mu+\nu}$. In other words,

$$f^{\mu}_{v^*_\mu, u_1} \cdot f^{\nu}_{v^*_\nu, u_2} = f^{\mu+\nu}_{v^*_{\mu+\nu}, u}. \tag{2.1}$$

2.4 The Principal Nilpotent Case

In this section we will show that the varieties $\mathbf{P}_e$ and $\overline{\mathrm{G}^{\mathrm{e}}}$ are isomorphic, by establishing an isomorphism between the homogeneous coordinate rings $R_\lambda[\mathbf{P}_e]$ and $R_\lambda[\overline{G^e}]$. Define, component-wise, a map

$$\begin{aligned} \Phi' :\ R_\lambda[\mathcal{B}] &\longrightarrow R_\lambda[\overline{G}] \\ u &\longmapsto (v^*_{n\lambda} \otimes u)t^{n\lambda} \end{aligned} \tag{2.2}$$

for $u \in V_{n\lambda}$. We will show

Theorem 2.2. *The map Φ' descends to an isomorphism of graded algebras*

$$\Phi : R_\lambda[\mathbf{P}_e] \longrightarrow R_\lambda[\overline{\mathrm{G}^{\mathrm{e}}}].$$

Remark 2.3. The argument that follows can be applied directly to the multi-graded total coordinate rings as well, but this approach is significantly more technical. Choosing a suitable $\mathbb{Z}$-graded homogeneous coordinate ring for each variety circumvents these technicalities.

Lemma 2.4 ([18], Corollary 1.6)**.** *For any vector* $v^* \in V_\mu^*$*, one has*

$$\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v_\mu^*) \subseteq \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*).$$

Remark 2.5. This lemma follows from Ginzburg's results on the cohomology of the loop Grassmannian. We were unable to find a direct algebraic proof in the literature.

Proposition 2.6. *Let* $v^* \otimes u \in V_\mu^* \otimes V_\mu$*. Then there exists an element* $w \in V_\mu$ *such that for any* $g \in G^e$*,*

$$v^*(g \cdot u) = v_\mu^*(g \cdot w).$$

Proof. We will first show this for linear functions on the universal enveloping algebra $\mathcal{U}\mathfrak{g}^e$ of the nilpotent abelian subalgebra $\mathfrak{g}^e = \mathrm{Lie}(G^e)$ of $\mathfrak{g}$.

Let $v_1, \ldots, v_r$ be a basis of weight vectors for V_μ, and let $v_1^*, \ldots, v_r^*$ be the dual basis for V_μ^*. One can make this choice so that $v_1^* = v_\mu^*$. Then the representation map is

$$\begin{aligned} \varphi : \mathcal{U}\mathfrak{g}^e &\longrightarrow V_\mu \otimes V_\mu^* \\ x &\longmapsto \sum v_i^*(x \cdot v_j) v_i \otimes v_j^*. \end{aligned}$$

Let $\{e, h, f\}$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$—this triple is unique up to conjugation by G^e, and the element h is regular and semisimple, with $[h, e] = 2e$ (see [24].) Then $\mathfrak{g}$, the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, and the vector space V_μ all have natural $\mathbb{Z}$-gradings by the eigenvalues of h, and $\mathfrak{g}^e$ sits in strictly positive degrees.

If $x \in \mathcal{U}\mathfrak{g}$ has degree m, and $v \in V_\mu$ has degree k, then $x \cdot v \in V_\mu$ has degree $m + k$. Therefore, for any m greater than the maximum eigenvalue M of h on $\mathrm{End}(V_\mu) = V_\mu \otimes V_\mu^*$, we have

$$\varphi|_{\mathcal{U}^m\mathfrak{g}^e} = 0,$$

where $\mathcal{U}^m\mathfrak{g}^e$ denotes the component of $\mathcal{U}\mathfrak{g}^e$ of degree m. So without loss of generality we can restrict to considering

$$\varphi : \mathcal{U}^{\leq M}\mathfrak{g}^e \longrightarrow V_\mu \otimes V_\mu^*.$$

Since all of these spaces are finite-dimensional, we will be able to dualize without issue.

The dual map $\varphi^* : V_\mu^* \otimes V_\mu \longrightarrow (\mathcal{U}^{\leq M}\mathfrak{g}^e)^*$ realizes the elements of $V_\mu^* \otimes V_\mu$ as functions on the universal enveloping algebra, via

$$\varphi^*(v_i^* \otimes v_j)(x) = v_i^*(x \cdot v_j) \quad \text{for any } x \in \mathcal{U}^{\leq M}\mathfrak{g}^e.$$

Consider the commutative diagram

$$\begin{array}{ccc} V_\mu^* \otimes V_\mu & \xrightarrow{\varphi^*} & (\mathcal{U}^{\leq M}\mathfrak{g}^e)^* \\ \uparrow & \nearrow_{\psi^*} & \\ v_\mu^* \otimes V_\mu & & \end{array} \tag{2.3}$$

where the vertical map is the inclusion induced by

$$\mathbb{C}v_\mu^* \hookrightarrow V_\mu^*,$$

and ψ^* is the restriction of φ^* to the subspace $v_\mu^* \otimes V_\mu$.

We would like to first show that every function in $V_\mu^* \otimes V_\mu$ on $\mathcal{U}^{\leq M}\mathfrak{g}^e$ comes from a function in $v_\mu^* \otimes V_\mu$—that is, that the image of φ^* is equal to the image of ψ^*, or in other words that

$$\operatorname{coker}(\varphi^*) = \operatorname{coker}(\psi^*).$$

Dualizing diagram (2.3), we obtain

$$\begin{array}{ccc} V_\mu \otimes V_\mu^* & \xleftarrow[\varphi]{} & \mathcal{U}^{\leq M}\mathfrak{g}^e \\ \downarrow & \swarrow_{\psi} & \\ v_\mu \otimes V_\mu^* & & \end{array} \tag{2.4}$$

and it is now equivalent to show that

$$\ker(\varphi) = \ker(\psi).$$

Since diagram (2.4) is commutative, $\ker(\varphi) \subseteq \ker(\psi)$. Conversely, if $x \in \ker(\psi)$, then

$$\psi(x) = \sum_i v_\mu^*(x \cdot v_i) v_\mu \otimes v_i^* = 0$$

and so $v_\mu^*(x \cdot v_i) = 0$ for each v_i. But then $x \cdot v_\mu^* = 0$ and so by Lemma 2.4 the element x annihilates every $v^* \in V_\mu^*$, so $x \in \ker(\varphi)$.

Thus $\ker(\varphi) = \ker(\psi)$, and in diagram (2.3)

$$\operatorname{coker}(\varphi^*) = \operatorname{coker}(\psi^*).$$

In other words, for any $v^* \otimes u \in V_\mu^* \otimes V_\mu$, there is a $w \in V_\mu$ such that for any $x \in \mathcal{U}\mathfrak{g}^e$,

$$v^*(x \cdot u) = v_\mu^*(x \cdot w).$$

From this we can obtain the same result for functions on the group G. Because G^e is a unipotent group and V_μ is a finite-dimensional representation, it is a general fact that $\varphi(G^e) \subset \varphi(\mathcal{U}\mathfrak{g}^e)$. So for any $g \in G^e$ we have

$$v^*(g \cdot u) = v_\mu^*(g \cdot w). \qquad \square$$

Remark 2.7. Proposition 2.6 tells us that the ideal $\mathcal{I}_{\overline{\mathrm{G^e}}}$ contains, in each graded component

$$\left(\bigoplus_{\mu \leq n\lambda} V_\mu^* \otimes V_\mu \right) t^{n\lambda},$$

all elements of the form

$$(f^{\mu}_{v^*,u} - f^{\mu}_{v^*_\mu,w})t^{n\lambda}$$

for v^*, u, and w as above.

We prove two more results that partially reverse the correspondence in Proposition 2.6 and that will be useful in Section 2.5.

Lemma 2.8. *Let $v^* \in V^*_\mu$ be such that $v^*(v_\mu) \neq 0$. Then*

$$\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu) = \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*).$$

Proof. From Lemma 2.4 there is an inclusion,

$$\iota : \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu) \hookrightarrow \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*).$$

As in the proof of Proposition 2.6, the space V^*_μ is $\mathbb{Z}$-graded and the algebras $\mathcal{U}\mathfrak{g}^e$ and $\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu)$ are $\mathbb{N}$-graded by the eigenvalues of h. For $m \in \mathbb{N}$, the collection

$$\mathcal{U}^{\geq m}\mathfrak{g}^e := \bigoplus_{i \geq m} \mathcal{U}^i\mathfrak{g}^e.$$

is a decreasing filtration that induces decreasing filtrations on both $\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu)$ and $\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*)$. It is sufficient to show that the induced map

$$\mathrm{gr}\iota : \mathrm{gr}(\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu)) = \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu) \hookrightarrow \mathrm{gr}(\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*))$$

on associated graded algebras is surjective.

Let k be the degree of v^*_μ under the grading—this is the minimal eigenvalue of h on V^*_μ. Then we have

$$v^* = cv^*_\mu + w^*$$

for a nonzero constant c and for w^* sitting in degrees strictly higher than k. Let

$$x \in \mathrm{gr}(\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*))_n \subset \mathcal{U}^{\geq m}\mathfrak{g}^e / \mathcal{U}^{\geq m+1}\mathfrak{g}^e$$

be nontrivial, with representative $x' \in \mathcal{U}^{\geq m}\mathfrak{g}^e$. We write

$$x' = x^{(m)} + x'',$$

where $x^{(m)}$ is a nonzero element in degree m and x'' sits in degree strictly higher than m. Then

$$0 = x' \cdot v^* = cx' \cdot v^*_\mu + x' \cdot w^* = cx^{(m)} \cdot v^*_\mu + cx'' \cdot v^*_\mu + x^{(m)} \cdot w^* + x'' \cdot w^*.$$

The first term has degree $m + k$, and all the other terms sit in strictly higher degrees, so we must have

$$x^{(m)} \cdot v^*_\mu = 0.$$

Therefore, $x^{(m)} \in \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu)$ is such that

$$\mathrm{gr}\iota \left(x^{(m)} \right) = x. \qquad \square$$

Proposition 2.9. *Let $w \in V_\mu$, and let v^* be as in Lemma 2.8. Then there exists an element $u \in V_\mu$ such that for any $g \in G^e$,*

$$v^*_\mu(g \cdot w) = v^*(g \cdot u).$$

Proof. Let φ be the representation map from the proof of Proposition 2.6, and consider the restriction φ^*_{res} of φ^* to $v^* \otimes V_\mu$.

$$\begin{array}{ccc} v^* \otimes V_\mu & \xrightarrow{\varphi^*_{res}} & (\mathcal{U}^{\leq M}\mathfrak{g}^e)^* \\ & \overset{\psi^*}{\nearrow} & \\ v^*_\mu \otimes V_\mu & & \end{array}$$

We would like to show that

$$\mathrm{Im}(\varphi^*_{res}) = \mathrm{Im}(\psi^*).$$

The first inclusion already follows from 2.6, and to show the second it is sufficient to show that

$$\ker(\varphi_{res}) \subset \ker(\psi)$$

in the following diagram, where $v \in V_\mu$ is the dual vector to v^* under the choice of weight vector basis in the proof of Proposition 2.6:

$$\begin{array}{ccc} v \otimes V^*_\mu & \xleftarrow[\varphi_{res}]{} & \mathcal{U}^{\leq M}\mathfrak{g}^e \\ & \underset{\psi}{\swarrow} & \\ v_\mu \otimes V^*_\mu & & \end{array}$$

If $x \in \ker(\varphi_{res})$, then

$$\varphi_{res}(x) = \sum_i v^*(x \cdot v_i) v \otimes v^*_i = 0$$

and so $v^*(x \cdot v_i) = 0$ for each v_i. Then $x \cdot v^* = 0$ and by Lemma 2.8 the element x annihilates v^*_μ, so $x \in \ker(\psi)$.

As in the proof of Proposition 2.6, since $\mathrm{Im}(\varphi^*_{res}) = \mathrm{Im}(\psi^*)$, it follows that there is an element $u \in V_\mu$ such that

$$v^*_\mu(g \cdot w) = v^*(g \cdot u) \qquad \text{for any } g \in G^e. \qquad \square$$

Next we will show that if $\mu \leq \lambda$, then every function $f^\mu_{v^*_\mu, w} \in v^*_\mu \otimes V_\mu$ on G^e is equivalent to a function $f^{\lambda}_{v^*_\lambda, z}$. For this we will need a result similar to Corollary 2.4, and it will follow from the same theorem of Ginzburg:

Theorem 2.10 ([18], Theorem 1.5)**.** *Let $\mathbb{O}_\lambda$ be the orbit of the affine Grassmannian of the Langlands dual $\check{G}$ of G corresponding to the dominant weight λ of G. Then there is a natural isomorphism of graded algebras*

$$H^\bullet(\overline{\mathbb{O}}_\lambda, \mathbb{C}) \simeq \mathcal{U}\mathfrak{g}^e / \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\lambda). \tag{2.5}$$

Lemma 2.11. *Let $\mu \leq \lambda$ be dominant weights. Then*

$$\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\lambda) \subseteq \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu).$$

Proof. The orbits of $\check{G}(\mathbb{C}[[t]])$ on the affine Grassmannian $\check{G}(\mathbb{C}((t)))/\check{G}(\mathbb{C}[[t]])$ are indexed by the dominant weights of G, and since $\mu \leq \lambda$, we have

$$\mathbb{O}_\mu \subset \overline{\mathbb{O}}_\lambda.$$

(See Theorem 2.17 in [36].)

The induced restriction map on cohomology

$$H^\bullet(\overline{\mathbb{O}}_\lambda, \mathbb{C}) \longrightarrow H^\bullet(\overline{\mathbb{O}}_\mu, \mathbb{C})$$

is surjective since $\overline{\mathbb{O}_\lambda}$ and $\overline{\mathbb{O}_\mu}$ have compatible decompositions into affine strata. In view of Theorem 2.5, this gives a surjection

$$\mathcal{U}\mathfrak{g}^e / \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\lambda) \longrightarrow \mathcal{U}\mathfrak{g}^e / \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu),$$

and implies that

$$\mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\lambda) \subseteq \mathrm{Ann}_{\mathcal{U}\mathfrak{g}^e}(v^*_\mu). \qquad \square$$

Proposition 2.12. *Let $\mu \leq \lambda$ and $v_\mu^* \otimes w \in V_\mu^* \otimes V_\mu$. Then there is an element $z \in V_\lambda$ such that for any $g \in G^e$,*

$$v_\mu^*(g \cdot w) = v_\lambda^*(g \cdot z).$$

Proof. As in the proof of Proposition 2.6, we will show this first for linear functions on the universal enveloping algebra. Consider the following representation maps:

$$\varphi_\mu : \mathcal{U}\mathfrak{g}^e \longrightarrow v_\mu \otimes V_\mu^*$$
$$\varphi_\lambda : \mathcal{U}\mathfrak{g}^e \longrightarrow v_\lambda \otimes V_\lambda^*$$

As in the previous proof, we can restrict to considering

$$\varphi_\mu : \mathcal{U}^{\leq M}\mathfrak{g}^e \longrightarrow v_\mu \otimes V_\mu^*$$
$$\varphi_\lambda : \mathcal{U}^{\leq M}\mathfrak{g}^e \longrightarrow v_\lambda \otimes V_\lambda^*$$

for some sufficiently large integer M. The dual maps realize the elements of $v_\mu^* \otimes V_\mu$ and of $v_\lambda^* \otimes V_\lambda$ as functions on the universal enveloping algebra

$$\begin{array}{ccc} v_\mu^* \otimes V_\mu & \xrightarrow{\varphi_\mu^*} & (\mathcal{U}^{\leq M}\mathfrak{g}^e)^* \\ & & \uparrow{\scriptstyle \varphi_\lambda^*} \\ & & v_\lambda^* \otimes V_\lambda, \end{array} \tag{2.6}$$

and we would like to show that every function in $v_\mu^* \otimes V_\mu$, when restricted to $\mathcal{U}\mathfrak{g}^e$, is equivalent to a function in $v_\lambda^* \otimes V_\lambda$.

Therefore, we will prove that the image of φ_μ^* is contained in the image of φ_λ^*. Dualizing diagram (2.6),

$$\begin{array}{ccc} v_\mu \otimes V_\mu^* & \xleftarrow[\varphi_\mu]{} & \mathcal{U}^{\leq M}\mathfrak{g}^e \\ & & \downarrow{\scriptstyle \varphi_\lambda} \\ & & v_\lambda \otimes V_\lambda^* \end{array} \tag{2.7}$$

it is equivalent to show that

$$\ker(\varphi_\lambda) \subseteq \ker(\varphi_\mu).$$

Suppose $x \in \ker(\varphi_\lambda)$. Then

$$\varphi_\lambda(x) = \sum_i v_\lambda^*(x \cdot v_i) v_\lambda \otimes v_i^* = 0$$

and so $v_\lambda^*(x \cdot v_i) = 0$ for each v_i. But then $x \cdot v_\lambda^* = 0$, and so by Lemma 2.11 we also have $x \cdot v_\mu^* = 0$, and therefore $x \in \ker(\varphi_\mu)$.

So $\ker(\varphi_\lambda) \subseteq \ker(\varphi_\mu)$, and therefore $\mathrm{Im}(\varphi_\lambda^*) \supseteq \mathrm{Im}(\varphi_\mu^*)$. For any $v_\mu^* \otimes w \in v_\mu^* \otimes V_\mu$, there is an element $z \in V_\lambda$ such that for any $x \in \mathcal{U}^{\leq M}\mathfrak{g}^{\mathfrak{e}}$,

$$v_\mu^*(x \cdot w) = v_\lambda^*(x \cdot z).$$

As in the proof of Proposition 2.6, it follows that

$$v_\mu^*(g \cdot w) = v_\lambda^*(g \cdot z) \qquad \text{for any } g \in G^e.$$

□

Remark 2.13. Proposition 2.12 implies that the ideal $\mathcal{I}_{\overline{\mathrm{G^e}}}$ also contains all elements of the form

$$(f^{n\lambda}_{v_\lambda^*, z} - f^{\mu}_{v_\mu^*, w}) t^{n\lambda} \in \left(\bigoplus_{\mu \leq n\lambda} V_\mu^* \otimes V_\mu \right) t^{n\lambda}$$

for w and z as above. We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. First note that by (2.1), the function Φ' defined in (2.2) is a homomorphism of graded algebras. We have $R_\lambda[\mathbf{P}_e] = R_\lambda[\mathcal{B}]/\mathcal{I}_{\mathbf{P}_e}$, where

$$\begin{aligned} \mathcal{I}_{\mathbf{P}_e} &= \bigoplus_{n \geq 0} \left\{ u \in V_{n\lambda} \mid v_{n\lambda}^*(g \cdot u) = 0, \, \forall g B^- \in \mathbf{P}_e \right\} \\ &= \bigoplus_{n \geq 0} \left\{ u \in V_{n\lambda} \mid v_{n\lambda}^*(g \cdot u) = 0, \, \forall g \in G^e \right\}, \end{aligned}$$

since the image of G^e is dense in $\mathbf{P}_e$. Similarly $R_\lambda[\overline{\mathrm{G^e}}] = R_\lambda[\overline{G}]/\mathcal{I}_{\overline{\mathrm{G^e}}}$, where

$$\mathcal{I}_{\overline{\mathrm{G^e}}} = \bigoplus_{n\geq 0}\left\{\sum f^{\mu}_{v^*,u}t^{n\lambda} \in \bigoplus_{\mu\leq n\lambda} V_\mu^* \otimes V_\mu \mid \sum f^{\mu}_{v^*,u}(g)\lambda(t)^n = 0,\, \forall (g,t) \in G^e \times T\right\}$$
$$= \bigoplus_{n\geq 0}\left\{\sum f^{\mu}_{v^*,u}t^{n\lambda} \in \bigoplus_{\mu\leq n\lambda} V_\mu^* \otimes V_\mu \mid \sum f^{\mu}_{v^*,u}(g) = 0,\, \forall g \in G^e\right\},$$

since the function $t^{n\lambda} = \lambda(t)^n$ is always nonzero.

We will check everything on graded components. First, Φ' does indeed descend to a homomorphism of algebras

$$\Phi : R_\lambda[\mathbf{P}_e] \longrightarrow R_\lambda[\overline{\mathrm{G^e}}],$$

since for any $u \in \mathcal{I}_{\mathbf{P}_e} \cap V_{n\lambda}$ and $(g,t) \in G^e \times T$

$$\Phi'(u)(g,t) = f^{n\lambda}_{v^*_{n\lambda},u}(g)\lambda(t)^n = v^*_{n\lambda}(g\cdot u)\lambda(t)^n = 0,$$

and so $\Phi'(u) \in \mathcal{I}_{\overline{\mathrm{G^e}}}$.

Second, the homomorphism Φ is injective: if $\Phi'(u) \in \mathcal{I}_{\overline{G^e}}$ for some $u \in V_{n\lambda}$, then

$$v^*_{n\lambda}(g\cdot u)\lambda(t)^n = 0$$

for all $(g,t) \in G^e \times T$, so $u \in \mathcal{I}_{\mathbf{P}_e}$. Thus, $\ker(\Phi') = \mathcal{I}_{\mathbf{P}_e}$, and $\ker(\Phi) = 0$.

Last, Φ is surjective: suppose $f^{\mu}_{v^*,u}t^{n\lambda} \in (V_\mu^* \otimes V_\mu)t^{n\lambda}$. By Proposition 2.6, there is a $w \in V_\mu$ such that

$$f^{\mu}_{v^*,u}t^{n\lambda} \equiv f^{\mu}_{v^*_\mu,w}t^{n\lambda} \pmod{\mathcal{I}_{\overline{\mathrm{G^e}}}},$$

as noted in Remark 2.7. By Proposition 2.12 there is a $z \in V_{n\lambda}$ such that

$$f^{\mu}_{v^*_\mu,w}t^{n\lambda} \equiv f^{n\lambda}_{v^*_\lambda,z}t^{n\lambda} \pmod{\mathcal{I}_{\overline{\mathrm{G^e}}}},$$

as in Remark 2.13. Then

$$\Phi(z) \equiv f^{\mu}_{v^*,u} t^{n\lambda} \pmod{\mathcal{I}_{\overline{\mathrm{G}^{\mathrm{e}}}}}. \qquad \square$$

2.5 The General Case

Now let $x \in \mathfrak{g}$ be a regular element, not necessarily nilpotent, and let $G^x \subset G$ be its centralizer. By the Jordan decomposition and by conjugating appropriately,

$$x = s + n$$

for some semisimple $s \in \mathfrak{t}$ and a nilpotent $n \in \mathfrak{n}$ such that

$$n = \sum_{i \in I} e_i$$

is a sum of the simple root vectors indexed by the set $I \subset \{1, \ldots, l\}$.

The centralizer of s in the group G is the centralizer of the one-parameter subgroup $\{\exp(ts) \mid t \in \mathbb{C}^*\}$ and is therefore a Levi subgroup $L \subset G$ (see [14], Proposition 1.22). The centralizer $G^x = L^n$, being abelian, decomposes as

$$G^x = C \times A,$$

where C is the center of L and $A = [L, L]^n \cap N$ is the unipotent part of the centralizer of n in the derived subgroup $[L, L]$. The element n is a principal nilpotent of $[L, L]$, and A is a unipotent subgroup that centralizes it, so all the results from Section 2.4 apply to A as a subgroup of the semisimple group $[L, L]$. (See Remark 2.1.)

For any dominant weight λ of G, the irreducible representation V_λ decomposes into irreducible representations of L

$$V_\lambda \simeq \bigoplus_{(\alpha,\rho) \in [\lambda]} W^{\alpha}_{\rho},$$

where W_ρ^α is the irreducible representation of $[L,L]$ of highest weight ρ with an action of C by the character α, and $[\lambda]$ denotes the set of pairs (α,ρ) that appear in the decomposition of V_λ. Let w_ρ^α denote the highest weight vector of W_ρ^α.

As before, fix a regular dominant weight λ in the root lattice of G, and let V_λ^* be the dual of the corresponding representation. There is a decomposition

$$V_\lambda^* \simeq \bigoplus_{(\alpha,\rho)\in[\lambda]} W_\rho^{\alpha*},$$

and we denote the lowest weight vector of $W_\rho^{\alpha*}$ by $w_\rho^{\alpha*}$.

The dominant weight λ gives rise to the line bundle $\mathcal{L}_\lambda$ on $\mathcal{B}$, with space of global sections

$$\Gamma(\mathcal{B},\mathcal{L}_\lambda) = V_\lambda$$

as in Section 2.2.

Definition 2.14. An element $\mathfrak{b}\in\mathcal{B}$ is *general* if for all $(\alpha,\rho)\in[\lambda]$,

$$w_\rho^\alpha(\mathfrak{b}) \neq 0,$$

where $w_\rho^\alpha \in V_\lambda$ is viewed as a global section of $\mathcal{L}_\lambda$. The G^x-orbit of such an element is a *general orbit* of G^x.

Generality is independent of the basepoint of a G^x-orbit, and it is an open and nonempty condition. Let $h\in G$ be such that the h-translate $h\cdot\mathfrak{b}^-$ is general. This is the case if and only if

$$(h\cdot v_\lambda^*)(w_\rho^\alpha) \neq 0$$

for all $(\alpha,\rho)\in[\lambda]$, and then $h\cdot v_\lambda^*$ satisfies the condition of Lemma 2.8 and Proposition 2.9.

Let $\mathbf{P}_x$ be the closure of the (general) G^x-orbit of $h\cdot\mathfrak{b}^-$ in $\mathcal{B}$, and let $\overline{G^x}$ be the closure of G^x in the wonderful compactification $\overline{G}$. We will use the methods of Section 2.4 to show that the varieties $\mathbf{P}_x$ and $\overline{G^x}$ are isomorphic.

Consider the homogeneous coordinate rings of the flag variety and of the wonderful compactification given by the projective embeddings corresponding to λ. As before, we have

$$R_\lambda[\mathbf{P}_x] = R_\lambda[\mathcal{B}]/\mathcal{I}_{\mathbf{P}_x}$$
$$R_\lambda[\overline{G^x}] = R_\lambda[\overline{G}]/\mathcal{I}_{\overline{G^x}}$$

where $\mathcal{I}_{\mathbf{P}_x}$ and $\mathcal{I}_{\overline{G^x}}$ are the ideals of global sections that vanish on $\mathbf{P}_x$ and $\overline{G^x}$ respectively.

Define, component-wise, a map

$$\begin{aligned} \Psi' :\ R_\lambda[\mathcal{B}] &\longrightarrow R_\lambda[\overline{G}] \\ u &\longmapsto (h \cdot v^*_{n\lambda} \otimes u)t^{n\lambda} \end{aligned} \tag{2.8}$$

for $u \in V_{n\lambda}$. We will show

Theorem 2.15. *The map Ψ' descends to an isomorphism of graded algebras*

$$\Psi : R_\lambda[\mathbf{P}_x] \longrightarrow R_\lambda[\overline{G^x}].$$

Proposition 2.16. *Let $w^* \otimes u \in W^{\alpha *}_\rho \otimes W^\alpha_\rho$. There exists an element $v \in W^\alpha_\rho$ such that for all $g \in G^x$*

$$w^*(g \cdot u) = w^{\alpha *}_\rho(g \cdot v).$$

Proof of the Proposition. We decompose the centralizer G^x as

$$G^x \simeq C \times A.$$

When we restrict W^α_ρ to $[L, L] \subset L$ the representation remains irreducible, and by Proposition 2.6 there is a $v \in W_\rho^\alpha$ such that for any $a \in A$

$$w^*(a \cdot u) = w_\rho^{\alpha *}(a \cdot v).$$

We can write any $g \in G^x$ as $g = ca$ with $c \in C$ and $a \in A$, and since C acts on W_ρ^α by α we have

$$\begin{aligned} w^*(g \cdot u) &= w^*(ca \cdot u) \\ &= \alpha(c) w^*(a \cdot u) \\ &= \alpha(c) w_\rho^{\alpha *}(a \cdot v) \\ &= w_\rho^{\alpha *}(ca \cdot v) = w_\rho^{\alpha *}(g \cdot v). \end{aligned}$$ □

Proposition 2.16 is an analogue to Proposition 2.6. Proposition 2.18 will give an analogous result to Proposition 2.12, and the following lemma will allow us to apply it to the proof of Theorem 2.15.

We introduce an new item of notation: If two integral weights θ and ξ of T differ by a linear combination of simple roots of $[L, L]$ with positive integral coefficients, we will write $\theta \leq_L \xi$ to indicate that θ is less than ξ in the partial ordering on the weight lattice of $[L, L]$.

Lemma 2.17. *Suppose μ and λ are dominant weights of G such that $\mu \leq \lambda$. Then for any $(\alpha, \sigma) \in [\mu]$ there exists a dominant weight ρ of $[L, L]$ such that $\sigma \leq_L \rho$ and $(\alpha, \rho) \in [\lambda]$.*

Proof. Let $\mathrm{Spec}(\mu)$ and $\mathrm{Spec}(\lambda)$ denote the set of all weights of G that appear in the irreducible representations V_μ and V_λ respectively. Since $\mu \leq \lambda$, $\mathrm{Spec}(\mu) \subset \mathrm{Spec}(\lambda)$. (See [17], Section 14.1.)

If $(\alpha, \sigma) \in [\mu]$, then $\alpha + \sigma \in \mathrm{Spec}(\mu) \subset \mathrm{Spec}(\lambda)$, so there is some $(\beta, \rho) \in [\lambda]$ such that $\alpha + \sigma$ appears as a weight in W_ρ^β.

Since the center C acts by the same character on all of W_ρ^β, we must have $\beta = \alpha$. When we restrict the representation W_ρ^α to the derived subgroup $[L, L]$, it is the irreducible representation of $[L, L]$ of highest weight ρ. Since σ appears as a weight in this representation, $\sigma \leq_L \rho$. $\square$

Proposition 2.18. *Let σ and ρ be dominant weights of $[L, L]$ such that $\sigma \leq_L \rho$, and let α be a character of C. Let $v \in W_\sigma^\alpha$. Then there exists an element $z \in W_\rho^\alpha$ such that for all $g \in G^x$,*

$$w_\sigma^{\alpha*}(g \cdot v) = w_\rho^{\alpha*}(g \cdot z).$$

Proof. Since $\sigma \leq_L \rho$, by Proposition 2.12 there exists an element $z \in W_\rho^{\alpha*}$ such that for any $a \in A$,

$$w_\sigma^{\alpha*}(a \cdot v) = w_\rho^{\alpha*}(a \cdot z).$$

Then we can write any $g \in G^x$ as $g = ca$ with $c \in C$ and $a \in A$, and since C acts by the character α on both $W_\sigma^{\alpha*}$ and $W_\rho^{\alpha*}$ we have

$$\begin{aligned} w_\sigma^{\alpha*}(g \cdot v) &= w_\sigma^{\alpha*}(ca \cdot v) \\ &= \alpha(c) w_\sigma^{\alpha*}(a \cdot v) \\ &= \alpha(c) w_\rho^{\alpha*}(a \cdot z) \\ &= w_\rho^{\alpha*}(ca \cdot z) = w_\rho^{\alpha*}(g \cdot z). \end{aligned}$$ $\square$

As in Section 2.3, we will use the notation $f_{w^*,v}^{\alpha,\sigma}$ to denote a global section arising from an element $w^* \otimes v \in W_\sigma^{\alpha*} \otimes W_\sigma^\alpha$.

Proof of Theorem 2.15. As before, by (2.1) the function Ψ' is a homomorphism of graded algebras. We have $R_\lambda[\mathbf{P}_x] = R_\lambda[\mathcal{B}]/\mathcal{I}_{\mathbf{P}_x}$, where

$$\begin{aligned} \mathcal{I}_{\mathbf{P}_x} &= \bigoplus_{n \geq 0} \left\{ u \in V_{n\lambda} \mid v_{n\lambda}^*(h^{-1} g \cdot u) = 0, \forall g^{-1} h B^- \in \mathbf{P}_x \right\} \\ &= \bigoplus_{n \geq 0} \left(\bigoplus_{(\alpha,\rho) \in [n\lambda]} \left\{ v \in W_\rho^\alpha \mid (h \cdot v_{n\lambda}^*)(g \cdot v) = 0, \forall g \in G^x \right\} \right). \end{aligned}$$

Moreover, $R_\lambda[\overline{G^x}] = R_\lambda[\overline{G}]/\mathcal{I}_{\overline{G^x}}$, where

$$\mathcal{I}_{\overline{G^x}} = \bigoplus_{n\geq 0}\left(\bigoplus_{\mu\leq n\lambda}\left\{\sum f^{\mu}_{v^*,u} \in V^*_\mu \otimes V_\mu \mid \sum f^{\mu}_{v^*,u}(g)\lambda(t)^n = 0, \forall (g,t) \in G^x \times T\right\}\right)$$
$$= \bigoplus_{n\geq 0}\left(\bigoplus_{\mu\leq n\lambda}\bigoplus_{(\alpha,\sigma)\in[\mu]}\left\{\sum f^{\alpha,\sigma}_{w^*,v} \in W^{\alpha *}_\sigma \otimes W^\alpha_\sigma \mid \sum f^{\alpha,\sigma}_{w^*,v}(g)\lambda(t)^n = 0, \forall (g,t) \in G^x \times T\right\}\right).$$

We will check everything on graded components. The homomorphism Ψ' does indeed descend to a homomorphism of algebras

$$\Psi : R_\lambda[\mathbf{P}_x] \longrightarrow R_\lambda[\overline{G^x}],$$

since for any $u \in \mathcal{I}_{\mathbf{P}_x}$ and $(g,t) \in G^x \times T$,

$$\Psi'(u)(g,t) = f^{n\lambda}_{h\cdot v^*_{n\lambda},u} t^{n\lambda}(g,t) = v^*_{n\lambda}(h^{-1}g \cdot u)\lambda(t)^n = 0.$$

Moreover, Ψ is injective: if $\Psi'(u) = 0$ for some $u \in V_{n\lambda}$, then

$$v^*_{n\lambda}(h^{-1}g \cdot u)\lambda(t)^n = 0$$

for all $(g,t) \in G^x \times T$, so $u \in \mathcal{I}_{\mathbf{P}_x}$. So $\ker(\Psi') = \mathcal{I}_{\mathbf{P}_x}$, and $\ker(\Psi) = 0$.

Lastly, Ψ is surjective. We will prove this first in degree 1—since the homogeneous coordinate ring is generated in degree 1, surjectivity will then follow for all degrees.

Suppose $\mu \leq \lambda$, $(\alpha,\sigma) \in [\mu]$, and $f^{\alpha,\sigma}_{w^*,v} t^\lambda \in (W^{\alpha *}_\sigma \otimes W^\alpha_\sigma)t^\lambda$. By Proposition 2.16, there is a $u \in W^\alpha_\sigma$ such that

$$f^{\alpha,\sigma}_{w^*,v} t^\lambda \equiv f^{\alpha,\sigma}_{w^{\alpha *}_\sigma,u} t^\lambda \pmod{\mathcal{I}_{\overline{G^x}}}.$$

By Lemma 2.17, there is some dominant weight ρ of $[L,L]$ such that $\sigma \leq_L \rho$ and $(\alpha,\rho) \in [\lambda]$, and by Proposition 2.18 there is an element $y \in W_\rho^\alpha$ such that

$$f^{\alpha,\sigma}_{w_\sigma^{\alpha *},u} t^\lambda \equiv f^{\alpha,\rho}_{w_\rho^{\alpha *},y} t^\lambda \pmod{\mathcal{I}_{\overline{G^x}}}.$$

Let $\pi_\rho^\alpha : V_\lambda^* \to W_\rho^{\alpha *}$ denote the projection of V_λ^* onto $W_\rho^{\alpha *}$. Then $\pi_\rho^\alpha(h \cdot v_\lambda^*)(w_\rho^\alpha) \neq 0$, so by Proposition 2.9 there is an element $z \in W_\rho^\alpha$ such that

$$f^{\alpha,\rho}_{w_\rho^{\alpha *},y} t^\lambda \equiv f^{\alpha,\rho}_{\pi_\rho^\alpha(h \cdot v_\lambda^*),z} t^\lambda \pmod{\mathcal{I}_{\overline{G^x}}}.$$

Then

$$\Psi'(z) = \left(h \cdot v_\lambda^* \otimes z\right) t^\lambda = \left(\pi_\rho^\alpha(h \cdot v_\lambda^*) \otimes z\right) t^\lambda,$$

so in fact

$$\Psi(z) \equiv f^{\alpha,\sigma}_{w^*,v} t^\lambda \pmod{\mathcal{I}_{\overline{G^x}}}. \qquad \square$$

2.6 Orbits on the Peterson Variety

We will give a description of the orbits of G^e on $\mathbf{P}_e$, and in particular we will show that in most cases there are infinitely many. For this we will consider the Peterson variety as a subvariety of the flag variety G/B with basepoint $\mathfrak{b}$, coming from the embedding

$$\begin{aligned} G^e &\longhookrightarrow G/B \\ g &\longmapsto g w_0 \cdot \mathfrak{b} \end{aligned}$$

where w_0 is the longest word of the Weyl group W.

Let $f_1, \ldots, f_l$ be the negative simple root vectors in $\mathfrak{g}$. In this setting, the Peterson variety has the following description [34]:

$$\mathbf{P}_e = \left\{ gB \in G/B \mid Ad(g^{-1}) \cdot e \in \mathfrak{b} \oplus \left(\sum_{i=1}^{l} \mathbb{C} f_i \right) \right\}. \tag{2.9}$$

We introduce some notation. For any $I \subseteq \{1, \ldots, l\}$ indexing a subset of the simple roots $\{\alpha_i \mid i \in I\}$, let P_I be the corresponding parabolic subgroup, L_I its Levi subgroup, U_I its unipotent radical, and $\mathfrak{l}_I = \mathrm{Lie}(L_I)$ and $\mathfrak{u}_I = \mathrm{Lie}(U_I)$ their Lie algebras.

Let $N_I \subset L_I$ be the maximal unipotent subgroup of the Levi, and $\mathfrak{n}_I$ its Lie algebra. Let W_I be the subgroup of W generated by the reflections corresponding to the simple roots $\{\alpha_i \mid i \in I\}$, and let w_I be the longest element of W_I. Let

$$e_I = \sum_{i \in I} e_i$$

be a nilpotent element of $\mathfrak{g}$, and note that it is regular in $[\mathfrak{l}_I, \mathfrak{l}_I]$.

The centralizer of e_I in L_I decomposes as a product

$$L_I^{e_I} = C_I \times A_I$$

where $C_I = Z(L_I)$ is the center of L_I and A_I is a unipotent subgroup of L_I, as in Section 2.5.

To find the G^e-orbits on $\mathbf{P}_e$, we will use the Bruhat decomposition. We have

$$\mathbf{P}_e = \bigcup_{w \in W} (\mathbf{P}_e \cap NwB/B)$$

and each intersection $\mathbf{P}_e \cap NwB/B$ is a G^e-stable subset.

Lemma 2.19 ([20], Proposition 5.8)**.** *The intersection of* $\mathbf{P}_e$ *with the Schubert cell* NwB/B *non-empty if and only if* w *is the longest word* w_I *of some parabolic Weyl group* W_I*.*

Proof. Suppose $nwB \in NwB/B$ is in the Peterson variety. Then by (2.9)

$$w^{-1}n^{-1} \cdot e \in \mathfrak{b} \oplus \sum_{i=1}^{l} \mathbb{C} f_i.$$

We can write

$$n^{-1} = \exp(x)$$

for some nilpotent $x \in \mathfrak{n}$, and then

$$\begin{aligned} w^{-1}n^{-1} \cdot e &= w^{-1}\left(e + x \cdot e + \frac{x^2 \cdot e}{2} + \dots\right) \\ &= w^{-1} \cdot e + w^{-1}\left(x \cdot e + \frac{x^2 \cdot e}{2} + \dots\right). \end{aligned} \tag{2.10}$$

Since the two summands in (2.10) belong to disjoint sums of roots spaces, this means in particular that

$$w^{-1} \cdot e \in \mathfrak{b} \oplus \sum_{i=1}^{l} \mathbb{C} f_i.$$

So for any simple root α_i, $w^{-1} \cdot \alpha_i$ is either a simple negative root or a positive root. But this precisely characterizes the longest words of parabolic Weyl groups (see [33], Lemma 2.2), and so $w = w_I$ for some $I \subseteq \{1, \dots, l\}$. □

Remark 2.20. The following result is proved by Insko and Yong [21] in type A, and is known to experts in the general case.

Proposition 2.21. *In the notation above,*

$$\mathbf{P}_e \cap N w_I B/B = A_I w_I B/B.$$

Proof. Suppose first that $h \in A_I$, so that h centralizes e_I, and write $e = e_I + e'_I$, where

$$e'_I = \sum_{i \notin I} e_i \in \mathfrak{u}_I.$$

We will show that $hw_I B \in \mathbf{P}_e$. Using (2.9), we obtain

$$\begin{aligned} w_I h^{-1} \cdot e &= w_I h^{-1} \cdot e_I + w_I h^{-1} \cdot e'_I \\ &= w_I \cdot e_I + w_I h \cdot e'_I \end{aligned}$$

Since w_I negates all the positive roots $\{\alpha_i \mid i \in I\}$, the first term is in $\sum_{i \in I} \mathbb{C} f_i$. Since $\mathfrak{u}_I$ is normalized by P_I and stable under the action of any representative of w_I, the second term is in $\mathfrak{u}_I$. So,

$$w_I h^{-1} \cdot e \in \mathfrak{b} \oplus \sum_{i=1}^{l} \mathbb{C} f_i$$

and $hw_I B \in \mathbf{P}_e$.

Conversely, let $n \in N$ so that $nw_I B \in \mathbf{P}_e$. Then

$$w_I n^{-1} \cdot e \in \mathfrak{b} \oplus \sum_{i=1}^{l} \mathbb{C} f_i.$$

Decomposing e as above,

$$w_I n^{-1} \cdot e = w_I n^{-1} \cdot e_I + w_I n^{-1} \cdot e'_I,$$

and as before the second term is in $\mathfrak{u}_I$, so in fact

$$w_I n^{-1} \cdot e_I \in \mathfrak{b} \oplus \sum_{i=1}^{l} \mathbb{C} f_i.$$

By the Levi decomposition, $n^{-1} = vu$ with $v \in L_I$ and $u \in U_I$. Since $n \in N$, we have $v \in N_I$, so we can write $v = \exp(x)$ and $u = \exp(y)$ for $x \in \mathfrak{n}_I$ and $y \in \mathfrak{u}_I$. Then

$$\begin{aligned} w_I n^{-1} \cdot e_I &= w_I \exp(x) \exp(y) \cdot e_I \\ &= w_I (e_I + x \cdot e_I + \text{terms in } \mathfrak{u}_I). \end{aligned}$$

In particular,

$$w_I x \cdot e_I \in \mathfrak{b} \oplus \sum_{i=1}^{l} \mathbb{C} f_i. \tag{2.11}$$

But x is a sum of positive root vectors of strictly positive height in the levi $\mathfrak{l}_I$, so $x \cdot e_I$ is a sum of root vectors with root height at least 2. That is,

$$x \cdot e_I \in \mathfrak{l}_I \cap \left(\bigoplus_{\mathrm{ht}(\alpha) \geq 2} \mathfrak{g}_\alpha \right),$$

and since w_I flips every root in $\mathfrak{l}_I$,

$$w_I x \cdot e_I \in \mathfrak{l}_I \cap \left(\bigoplus_{\mathrm{ht}(\alpha) \leq -2} \mathfrak{g}_\alpha \right),$$

and to satisfy (2.11) we must have $x \cdot e_I = 0$. Then $h = \exp(x) \in A_I$ and

$$n w_I B = v u w_I B = v w_I B \in A_I w_I B$$

since $u \in U_I$ and U_I is w_I-stable. □

In particular, $A_I w_I B/B$ is G^e-stable, being the intersection of two G^e-stable subvarieties of G/B. The following Proposition describes the G^e-orbits on $A_I w_I B/B$. Define

$$\pi_I : P_I \longrightarrow L_I$$

to be the projection of the parabolic P_I onto its Levi subgroup. The image of G^e under this projection centralizes e_I, because e_I is itself the image of e under the differential $d\pi_I : \mathfrak{p}_I \longrightarrow \mathfrak{l}_I$. Therefore,

$$\pi_I(G^e) \subset A_I.$$

Proposition 2.22. *The G^e-orbits on $A_I w_I B/B = \mathbf{P}_e \cap N w_I B/B$ are in bijection with the cosets of $A_I/\pi_I(G^e)$.*

Proof. Let $h, k \in A_I$ and suppose first that $hw_I \in gkw_I B$ for some $g \in G^e$. Then

$$k^{-1}g^{-1}h \in w_I B w_I$$

and in fact

$$k^{-1}g^{-1}h \in w_I B w_I \cap B = U_I.$$

The Levi decomposition gives $g^{-1} = xu$ for $x = \pi_I(g^{-1}) \in A_I$ and $u \in U_I$, and we can write

$$k^{-1}g^{-1}h = k^{-1}xuh = k^{-1}xhh^{-1}uh = k^{-1}xhu'$$

where $u' \in U_I$ since P_I normalizes U_I. Since this expression is in U_I, we have

$$k^{-1}xh \in U_I$$

and since $k, x, h \in L_I$ we conclude

$$k^{-1}xh = 1.$$

Thus $k = xh$ and the elements h and k of A_I are $\pi_I(G^e)$-translates.

Conversely, suppose that $k = xh$ for some $x \in \pi_I(G^e)$. Then there is some $u \in U_I$ such that $xu \in G^e$, and we have

$$\begin{aligned} xu \cdot (hw_I B) &= kh^{-1}uhw_I B \\ &= kh^{-1}hvw_I B \quad \text{for some } v \in U_I, \text{ since } h \text{ normalizes } U_I \\ &= kw_I B \quad \text{since } w_I \text{ normalizes } U_I \end{aligned}$$

so the cosets $hw_I B$ and $kw_I B$ are in the same G^e-orbit. □

Proposition 2.22 gives a bijective correspondence between the G^e-orbits on the intersection of the Peterson variety with the Schubert cell Nw_IB and the $\pi_I(G^e)$-cosets in the subgroup A_I of L_I. Since A_I and $\pi_I(G^e)$ are unipotent groups, the coset space $A_I/\pi_I(G^e)$ is in fact a vector space.

Because the dimension of $\pi_I(G^e)$ may be strictly less than the dimension of A_I, there may be infinitely many G^e-orbits in the boundary of the Peterson variety. In type A this is the case for all choices of I for which $[\mathfrak{l}_I, \mathfrak{l}_I]$ is not simple, and such a choice exists in all ranks strictly greater than 2.

CHAPTER 3

THE RELATIVE COMPACTIFICATION OF THE UNIVERSAL CENTRALIZER

3.1 Introduction

The universal centralizer $\mathcal{X}$ of a semisimple algebraic group G of adjoint type is the variety that parametrizes the centralizers of regular elements in G, indexed by the regular conjugacy classes of $\mathfrak{g} = \mathrm{Lie}(G)$ along the affine subspace known as the Kostant slice. It has appeared as an important technical tool in the geometric Langlands program, in the work of Bezrukavnikov and Finkelberg [6] and Ngô [30], [31]. The variety $\mathcal{X}$ can be obtained by a Hamiltonian reduction from the cotangent bundle T^*G of G, and this equips it with a natural symplectic structure inherited from the canonical symplectic structure on T^*G.

We will view the universal centralizer $\mathcal{X}$ as a subvariety of $G \times \mathfrak{g}//G$, where $\mathfrak{g}//G$ is the adjoint quotient of G (see [25]), and we will consider its closure $\overline{\mathcal{X}}$ in $\overline{G} \times \mathfrak{g}//G$. We will show that the symplectic structure on $\mathcal{X}$ extends to a log-symplectic structure on $\overline{\mathcal{X}}$, using a Hamiltonian reduction in the Poisson setting from the logarithmic cotangent bundle of $\overline{G}$, which has a canonical log-symplectic Poisson structure.

In Section 3.2 we recall how the universal centralizer $\mathcal{X}$ is obtained from T^*G by a Hamiltonian reduction. In Section 3.3 we discuss the logarithmic cotangent bundle of $\overline{G}$, the natural $G \times G$-action it inherits from G itself, and the moment map resulting from this action. In Section 3.4 we show that the logarithmic cotangent bundle of $\overline{G}$ has a natural log-symplectic structure, and that the closure $\overline{\mathcal{X}}$ can be obtained from this bundle by a Hamiltonian reduction. This induces a log-symplectic structure on $\overline{\mathcal{X}}$ that extends the ordinary symplectic structure on $\mathcal{X}$.

3.2 The Symplectic Structure on the Universal Centralizer

In this section we perform the Hamiltonian reduction by which the universal centralizer $\mathcal{X}$ is equipped with a natural symplectic structure. The same reduction is outlined in [4] and [37].

Let G be a complex semisimple algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, and $\mathfrak{g}^*$ the dual algebra. The cotangent bundle T^*G is equipped with a natural two-sided $G \times G$-action and it can be trivialized under left translation by G:

$$T^*G \cong G \times \mathfrak{g}^*.$$

Under this identification the $G \times G$-action becomes

$$(h, h') \cdot (g, \xi^*) = (hgh'^{-1}, h' \cdot \xi^*) \qquad \text{for all } h, h' \in G, (g, \xi^*) \in G \times \mathfrak{g}^*.$$

The bundle T^*G has a natural symplectic structure, and the action of $G \times G$ is by symplectomorphisms. Identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form, the $G \times G$-equivariant moment map is given by the projection

$$\begin{aligned} \mu' : T^*G \cong G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \times \mathfrak{g} \\ (g, \xi) &\longmapsto (g \cdot \xi, \xi). \end{aligned} \tag{3.1}$$

Let $B \subset G$ be a Borel subgroup, N its unipotent radical, $\mathfrak{n} = \mathrm{Lie}(N)$, and $\mathfrak{n}^*$ its dual, and consider the action of the maximal unipotent subgroup $N \times N \subset G \times G$ on T^*G. The Killing form identifies $\mathfrak{n}^* \cong \mathfrak{g}/\mathfrak{b}$, and we obtain the moment map

$$\begin{aligned} \mu : T^*G \cong G \times \mathfrak{g} &\longrightarrow \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b} \\ (g, \xi) &\longmapsto (g \cdot \xi + \mathfrak{b}, \xi + \mathfrak{b}), \end{aligned} \tag{3.2}$$

which factors through the moment map μ' in (3.1).

Let $\{e, h, f\}$ be a principal $\mathfrak{sl}_2$-triple such that $e \in B$. (For the general structure theory of semisimple Lie algebras and their $\mathfrak{sl}_2$-triples, see [24].) Since the element $(f + \mathfrak{b}, f + \mathfrak{b}) \in \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}$ is fixed under the action of $N \times N$, the fiber $\mu^{-1}(f + \mathfrak{b}, f + \mathfrak{b})$ is $N \times N$-stable.

Let $\mathfrak{g}^e = \mathrm{Lie}(G^e)$ be the centralizer of e in $\mathfrak{g}$, and consider the affine space $f + \mathfrak{g}^e \subset \mathfrak{g}$. It is known as the *Kostant slice*, and it is transverse to the regular orbits of G on $\mathfrak{g}$, intersecting each exactly once—therefore, it gives a section of the adjoint quotient $\mathfrak{g} \twoheadrightarrow \mathfrak{g}//G$. (See [25].) Moreover, if $\mathfrak{b} = \mathrm{Lie}(B)$, one has an isomorphism of affine spaces

$$N \times (f + \mathfrak{g}^e) \cong f + \mathfrak{b},$$

by [26], Theorem 2.1, and so N acts freely on $(f + \mathfrak{b})$.

The *universal centralzer* of G is the variety

$$\mathcal{X} = \{(g, \xi) \in G \times (f + \mathfrak{g}^e) \mid g \in G^\xi\}.$$

It can be viewed as a family of centralizers in of $G \times \mathfrak{g}//G$, parametrized by the base $\mathfrak{g}//G$, via the injection

$$\begin{aligned} \mathcal{X} &\longrightarrow G \times \mathfrak{g}//G \\ (g, \xi) &\longmapsto (g, \overline{G \cdot \xi}). \end{aligned}$$

The fiber above $(f + \mathfrak{b}, f + \mathfrak{b})$ is

$$\begin{aligned} \mu^{-1}(f + \mathfrak{b}, f + \mathfrak{b}) &= \{(g, \xi) \in G \times \mathfrak{g} \mid \xi \in f + \mathfrak{b}, g \cdot \xi \in f + \mathfrak{b}\} \\ &= \{(g, \xi) \in G \times \mathfrak{g} \mid \xi \in N \cdot (f + \mathfrak{g}^e),\ g \cdot \xi \in N \cdot (f + \mathfrak{g}^e)\}. \end{aligned}$$

Since N acts freely on $f + \mathfrak{b}$, the action of $N \times N$ on this fiber is free, and $(f + \mathfrak{b}, f + \mathfrak{b})$

is a regular value of μ. Moreover, the fiber decomposes naturally as a product $N \times N \times \mathcal{X}$ under the isomorphism

$$N \times N \times \mathcal{X} \longrightarrow \mu^{-1}(f+\mathfrak{b}, f+\mathfrak{b})$$
$$(n_1, n_2, (g,\xi)) \longmapsto (n_1 g n_2^{-1}, n_2 \cdot \xi).$$

Then by the Marsden-Weinstein-Meyer theorem the quotient

$$N\backslash\mu^{-1}(f+\mathfrak{b}, f+\mathfrak{b})/N \cong \mathcal{X}$$

is a symplectic variety with a natural symplectic structure inherited from T^*G.

3.3 The Logarithmic Cotangent Bundle of $\overline{G}$

Let $\overline{G}$ be the wonderful compactification of G, and $D = \overline{G}\backslash G$ its boundary divisor. We will work with the logarithmic cotangent bundle $T^*\overline{G}(-\log D)$ of $\overline{G}$, which is the bundle whose sections are the logarithmic differential forms—differential forms with poles of order at most one along the boundary divisor D.

The bundle $T^*\overline{G}(-\log D)$ restricts to the ordinary cotangent bundle T^*G on $G \subset \overline{G}$, and it can be identified with subbundle $R_{\overline{G}}$ of isotropy Lie subalgebras of the trivial bundle $\overline{G} \times (\mathfrak{g} \times \mathfrak{g})$. (See [9], Proposition 2.1.2.) More precisely, at every $x \in \mathfrak{g}$, the fiber $R_{\overline{G},x} = (\mathfrak{g} \times \mathfrak{g})_{(x)}$ is the kernel of the action of

$$(\mathfrak{g} \times \mathfrak{g})^x = \operatorname{Lie}\left((G \times G)^x\right)$$

on the normal space to the orbit $(G \times G) \cdot x$ at x. When $x \in G$, $(\mathfrak{g} \times \mathfrak{g})_{(x)}$ is the centralizer $(\mathfrak{g} \times \mathfrak{g})^x$ itself.

Under this identification, the action of $G \times G$ on $\overline{G}$ lifts to an action on $T^*\overline{G}(-\log D)$

via

$$(g_1, g_2) \cdot (x, (\xi_1, \xi_2)) = (g_1 x g_2^{-1}, (g_1 \cdot \xi_1, g_2 \cdot \xi_2))$$

for all $g_1, g_2 \in G$, $x \in \overline{G}$, $(\xi_1, \xi_2) \in (\mathfrak{g} \times \mathfrak{g})_{(x)}$.

Remark 3.1. Let $n = \dim(\mathfrak{g})$. There is a realization of $\overline{G}$ inside the Grassmannian $Gr(n, \mathfrak{g} \times \mathfrak{g})$ via the embedding

$$\begin{aligned} \chi : \overline{G} &\longrightarrow Gr(n, \mathfrak{g} \times \mathfrak{g}) \\ x &\longmapsto (\mathfrak{g} \times \mathfrak{g})_{(x)} \end{aligned}$$

(See [16], Section 3.2.) The bundle $R_{\overline{G}}$ is just the restriction of the tautological bundle on $Gr(n, \mathfrak{g} \times \mathfrak{g})$ to the subvariety $\chi(\overline{G})$.

Let $l = \mathrm{rk}(G)$ be the rank of G and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots. The boundary of $\overline{G}$ decomposes as a union

$$\overline{G} \backslash G = D_1 \cup \ldots \cup D_l$$

of l transverse hypersurfaces, and the orbits of $G \times G$ on the boundary $\overline{G} \backslash G$ correspond to subsets $I \subset \{1, \ldots, l\}$ in the sense that for every subset I the closure of the orbit $\mathcal{O}_I$ is

$$\overline{\mathcal{O}_I} = \bigcap_{i \notin I} D_i.$$

(See [16] for details on the orbit structure of the wonderful compactification.)

Let $L_I \subset G$ be the Levi subgroup generated by the simple roots $\{\alpha_i \mid i \in I\}$, $P_I \subset G$ the corresponding parabolic subgroup, P_I^- the opposite parabolic, and $\mathfrak{l}_I$, $\mathfrak{p}_I$, and $\mathfrak{p}_I^-$ their respective Lie algebras. Let U_I and U_I^- be the unipotent radicals of P_I and P_I^-, with Lie algebras $\mathfrak{u}_I$ and $\mathfrak{u}_I^-$. In each orbit $\mathcal{O}_I$ there is a distinguished basepoint $z_I \in \overline{T}$ such that the centralizer of z_I in $(G \times G)$ is

$$(G \times G)^{z_I} = \{(pa, qb) \in P_I \times P_I^- \mid p \in U_I, q \in U_I^-, a, b \in L_I, ab^{-1} \in Z(L_I)\}.$$

Proposition 3.2. *The fiber* $(\mathfrak{g} \times \mathfrak{g})_{(z_I)}$ *of* $T^*\overline{G}(-\log D)$ *at* $z_I \in \overline{G}$ *is*

$$\mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I^-,$$

the subalgebra of pairs of elements in $\mathfrak{p}_I \times \mathfrak{p}_I^-$ *that have the same component in the Levi subalgebra* $\mathfrak{l}_I$*.*

Proof. The Lie algebra $(\mathfrak{g} \times \mathfrak{g})^{z_I}$ acts on the normal space

$$N_{z_I} = T_{z_I}\overline{G}/T_{z_I}\mathcal{O}_I.$$

Since the orbit $\mathcal{O}_I$ of z_I is open and dense in the intersection of the hypersurfaces $\{D_i \mid i \notin I\}$, the normal space N_{z_I} decomposes naturally as a sum of lines

$$N_{z_I} = \bigoplus_{i \notin I} \ell_i,$$

where each ℓ_i corresponds to the normal direction to the hyperfsurface D_i at z_I. Then

$$(\mathfrak{g} \times \mathfrak{g})^{z_I} = \{(u + x, v + y) \in \mathfrak{p}_I \times \mathfrak{p}_I^- \mid u \in \mathfrak{u}_I, v \in \mathfrak{u}_I^-, x, y \in \mathfrak{l}_I, x - y \in \mathfrak{l}_I\}$$

preserves each ℓ_i, and so it acts on N_{z_I} by a central character. The kernel of this central character is

$$\mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I^- = \{(u + x, v + x) \in \mathfrak{p}_I \times \mathfrak{p}_I^- \mid u \in \mathfrak{u}_I, v \in \mathfrak{u}_I^-, x \in \mathfrak{l}_I\}.$$ □

We now consider the $G \times G$-equivariant projection

$$\overline{\mu'} : T^*\overline{G}(-\log D) \longrightarrow \mathfrak{g} \times \mathfrak{g} \tag{3.3}$$
$$(x, (\xi_1, \xi_2)) \longmapsto (\xi_1, \xi_2).$$

This map is studied in [10] and [23], where it is called the *compactified moment map*, since the restriction of $\overline{\mu}'$ to $T^*G \subset T^*\overline{G}(-\log D)$ coincides with the moment map μ' defined in (3.1).

Proposition 3.3. *The image of $\overline{\mu'}$ is the variety $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ of pairs of elements in $\mathfrak{g} \times \mathfrak{g}$ that lie in the closure of the same G-orbit on $\mathfrak{g}$.*

Proof. By Lemma 3.2,

$$\operatorname{Im}\left(\overline{\mu'}\right) = \bigcup_{I \subseteq \Delta} (G \times G)(\mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I).$$

If $(u + x, v + x) \in \mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I$ for some $I \subseteq \Delta$, then for any invariant polynomial $f \in \mathbb{C}[\mathfrak{g}]^G$,

$$f(u + x) = f(x) = f(v + x),$$

(see [25], Proposition 17) and so $(u + x, v + x) \in \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$.

Conversely, suppose $(x, y) \in \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$. Conjugating x and y by appropriate elements $g, h \in G$, we may write them in Jordan normal form

$$g \cdot x = s + n$$
$$h \cdot y = t + m$$

such that $s, t \in \mathfrak{t}$, $n \in \mathfrak{n}$, and $m \in \mathfrak{n}^-$. Since x and y are in the closure of the same G-orbit, for any $f \in \mathbb{C}[\mathfrak{g}]^G$

$$f(s) = f(g \cdot x) = f(h \cdot y) = f(t),$$

so $s = t$ and $(g \cdot x, h \cdot y) \in \mathfrak{b} \times_{\mathfrak{t}} \mathfrak{b}^- = \mathfrak{p}_\emptyset \times_{\mathfrak{l}_\emptyset} \mathfrak{p}_\emptyset^-$. ☐

Lemma 3.4. *The variety $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is normal.*

Proof. By the Chevalley isomorphism theorem (see [11], let $f_1, \ldots, f_l \in \mathbb{C}[\mathfrak{g}]^G$ be a minimal set of generators for the algebra of G-invariant polynomials on $\mathfrak{g}$. The variety $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ has dimension

$$2 \dim(\mathfrak{g}) - \dim(\mathfrak{g}//G) = 2 \dim(\mathfrak{g}) - l,$$

and is a complete intersection since it is the zero-locus in $\mathfrak{g} \times \mathfrak{g}$ of the l polynomials $g_i(x, y) = f_i(x) - f_i(y)$:

$$\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g} = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid f_i(x) - f_i(y) = 0 \text{ for all } i = 1, \ldots, l\}.$$

Moreover, the subset

$$\mathfrak{g}^{reg} \times_{\mathfrak{g}//G} \mathfrak{g}^{reg} = \{(x, y) \in \mathfrak{g}^{reg} \times \mathfrak{g}^{reg} \mid x \in G \cdot y\} \subset \mathfrak{g}^{reg} \times \mathfrak{g}^{reg}$$

is a smooth open subset of $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$, because the differentials $dg_1, \ldots dg_l$ are linearly independent at every point of $\mathfrak{g}^{reg} \times_{\mathfrak{g}//G} \mathfrak{g}^{reg}$. (See [11], Claim 6.7.10.) Since its complement has codimension two, $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ has no codimension-one singularities, and so it is a normal variety. ☐

3.4 The Closure of the Universal Centralizer

The logarithmic cotangent bundle $T^*\overline{G}(-\log D)$ has a natural Poisson structure extending the symplectic structure on T^*G. In fact, this is true for any smooth variety with a smooth divisor with normal crossings.

Lemma 3.5 ([19]). *Let X be a smooth algebraic variety and $D \subset X$ a divisor with normal crossings. Then there is a canonical log-symplectic Poisson structure $\overline{\omega}$ on $T^*X(-\log D)$.*

In the case of $X = \overline{G}$, the log-symplectic form $\overline{\omega}$ on $T^*\overline{G}(-\log D)$ restricts to the canonical symplectic form on T^*G and it is preserved by the action of $G \times G$. Identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form, the moment map corresponding to this action is precisely the compactified moment map $\overline{\mu'}$ defined in (3.3).

Consider $T^*\overline{G}(-\log D)$ as a variety under the action of the subgroup $N \times N \subset G \times G$. Under the identification $n^* \cong \mathfrak{g}/\mathfrak{b}$, the resulting moment map is simply the projection

$$\begin{aligned} \overline{\mu} : T^*\overline{G}(-\log D) &\longrightarrow \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b} \\ (x, (\xi_1, \xi_2)) &\longmapsto (\xi_1 + \mathfrak{b}, \xi_2 + \mathfrak{b}). \end{aligned} \tag{3.4}$$

Once again we consider the point $(f + \mathfrak{b}, f + \mathfrak{b}) \in \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}$, fixed by the action of $N \times N$. The points in the fiber $\overline{\mu}^{-1}(f + \mathfrak{b}, f + \mathfrak{b})$ are of the form $(x, (\xi_1, \xi_2))$ with $\xi_1, \xi_2 \in f + \mathfrak{b}$.

Since $N \times N$ acts freely on $f + \mathfrak{b}$, it acts freely on the fiber $\overline{\mu}^{-1}(f + \mathfrak{b}, f + \mathfrak{b})$, and so $(f+\mathfrak{b}, f+\mathfrak{b})$ is a regular value of $\overline{\mu}$. This is because the Hamiltonian action of $N \times N$ preserves the symplectic leaves of the Poisson structure on $T^*\overline{G}(-\log D)$, and in the symplectic case it is well-known that a free action on the fiber above a point implies that this point is a regular value of the moment map.

Proposition 3.6. *The fibers of $\overline{\mu}$ are connected.*

Proof. The moment map $\overline{\mu}$ factors through the moment map $\overline{\mu'}$,

$$\begin{array}{ccc} T^*\overline{G}(-\log D) & & \\ \downarrow \overline{\mu'} & \searrow^{\overline{\mu}} & \\ \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b} \end{array} \tag{3.5}$$

and the fibers of the projection π are connected, so it is sufficient to show that the fibers of $\overline{\mu'}$ are connected.

Note that $\overline{\mu'}$ is proper, since it is a composition of proper morphisms:

$$\begin{array}{ccc} T^*\overline{G}(-\log D) & \overset{\iota}{\hookrightarrow} & \overline{G}\times\mathfrak{g}\times\mathfrak{g} \\ & \overset{\overline{\mu'}}{\searrow} & \twoheaddownarrow \\ & & \mathfrak{g}\times\mathfrak{g} \end{array} \tag{3.6}$$

Moreover, its image is normal, and the fiber above a general point is a connected toric variety. (See [10], Example 2.5.) Then by Stein factorization every fiber of $\overline{\mu'}$ is connected. □

Theorem 3.7. *There is an isomorphism of algebraic varieties*

$$N\backslash\overline{\mu}^{-1}(f+\mathfrak{b}, f+\mathfrak{b})/N \cong \overline{\mathcal{X}}$$

giving $\overline{\mathcal{X}}$ a log-symplectic Poisson structure that extends the symplectic structure on $\mathcal{X}$.

Proof. Since this fiber is smooth and connected, it is equal to the closure of its intersection with the ordinary cotangent bundle $T^*G \subset T^*\overline{G}(-\log D)$. But this intersection is precisely

$$\mu^{-1}(f+\mathfrak{b}, f+\mathfrak{b}) = \{(x,(\xi_1,\xi_2)) \in T^*\overline{G}(-\log D) \mid \exists n_1, n_2 \in N, \xi \in f+\mathfrak{g}^e : \\ n_1\cdot\xi_1 = \xi = n_2\cdot\xi_2, g \in n_1 G^{\xi} n_2^{-1}\},$$

and so

$$\overline{\mu}^{-1}(f+\mathfrak{b}, f+\mathfrak{b}) = \{(x,(\xi_1,\xi_2)) \in T^*\overline{G}(-\log D) \mid \exists n_1, n_2 \in N, \xi \in f+\mathfrak{g}^e : \\ n_1\cdot\xi_1 = \xi = n_2\cdot\xi_2, g \in n_1 \overline{G^{\xi}} n_2^{-1}\}.$$

The variety $\overline{\mu}^{-1}(f+\mathfrak{b}, f+\mathfrak{b})$ is then isomorphic to the product

$$N \times N \times \overline{\mathcal{X}}$$

via the isomorphism

$$\begin{aligned} N \times N \times \overline{\mathcal{X}} &\longrightarrow \overline{\mu}^{-1}(f+\mathfrak{b}, f+\mathfrak{b}) \\ (n_1, n_2, (x, \xi)) &\longmapsto (n_1 x n_2^{-1}, (n_1 \cdot \xi_1, n_2 \cdot \xi_2)) \end{aligned}$$

and therefore there is an isomorphism

$$N \backslash \overline{\mu}^{-1}(f+\mathfrak{b}, f+\mathfrak{b})/N \cong \overline{\mathcal{X}}.$$

By the Marsden-Weinstein theorem, $\overline{\mathcal{X}}$ then has a natural Poisson structure. □

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