

THE UNIVERSITY OF CHICAGO

AN ALGEBRAIC CHARACTERIZATION OF THE POINT-PUSHING SUBGROUP

A DISSERTATION SUBMITTED TO  
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES  
IN CANDIDACY FOR THE DEGREE OF  
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY  
VICTORIA SUZANNE AKIN

CHICAGO, ILLINOIS

JUNE 2017

To my father who reminds me to work hard.

You'd surf across the kitchen sink upon a stick of gum.  
You couldn't hug your mama, you'd just have to hug her thumb.  
You'd run from people's feet in fright,  
To move a pen would take all night,  
(This poem took fourteen years to write—  
'Cause I'm just one inch tall).

—Shel Silverstein, *Where the Sidewalk Ends*

# TABLE OF CONTENTS

LIST OF TABLES . . . . .	v
LIST OF FIGURES . . . . .	vi
ACKNOWLEDGEMENTS . . . . .	vii
ABSTRACT . . . . .	viii
1 INTRODUCTION . . . . .	1
1.1 Structure of the proof . . . . .	4
2 LINEAR DETECTORS . . . . .	7
2.1 Action on homology: $N \subset \mathcal{J}(\Sigma)$ . . . . .	7
2.2 The second term of the Johnson filtration. . . . .	10
2.2.1 Johnson filtration of $\text{Mod}(\Sigma_{g,1})$ and lower central series of $P$ . . . . .	10
2.2.2 Abelization of $\mathcal{J}(\Sigma)$ . . . . .	15
2.2.3 Intersection of $N$ with $\mathcal{J}_2(\Sigma)$ . . . . .	20
2.2.4 $\text{Sp}_{2g}(\mathbb{Q})$ representation . . . . .	21
2.3 Commutator containment: $[N, N] \subset [P, P]$ . . . . .	25
3 CHARACTERIZING $P$ . . . . .	29
3.1 Proof of proposition 3.0.2 by bounding pair maps . . . . .	29
3.2 Proof of Proposition 3.0.2 using amalgamated product normal form . . . . .	30
3.2.1 Cyclic reduction of normal form in amalgamated products . . . . .	31
3.2.2 Dividing the surface with a separating curve . . . . .	39
3.3 Conclusion: $N = P$ . . . . .	42
3.4 A new proof that $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1}))$ is trivial . . . . .	44
REFERENCES . . . . .	47

## LIST OF TABLES

1.1	Key Surface Group Obstacles . . . . .	2
-----	---------------------------------------	---

## LIST OF FIGURES

3.1	The Dehn twist $T_c$ . . . . .	31
3.2	Curve $b'_1$ . . . . .	41

## ACKNOWLEDGEMENTS

I would like to extend a deep thank you to my advisor, Benson Farb, for suggesting this project and offering continued guidance. He surprised me by brilliantly and consistently saying exactly what I needed to hear. Having him in my corner was crucial to my success. I don't know what I would have achieved without the opportunity to borrow his confidence. I would also like to thank my second advisor, Jesse Wolfson, for his insight, advice, and comments not only regarding mathematical material but also with respect to teaching and broader life decisions.

I have many thanks to offer for discussions that lifted me out of deep ruts. I would like to thank Nick Salter for important corrections and comments. I am grateful to Zev Chonoles, Simion Filip, Victor Ginzburg, Sebastian Hensel, Dan Margalit, Chen Lei, Andy Putman, Emily Smith, and Wouter van Limbeek for helpful comments and conversations.

I am lucky enough to share my home with MurphyKate Montee, who has offered me her ear, her encouragement, and, occasionally, her cat. I would like to thank her for helpful mathematical conversations and for reminding me of my progress when I was focused on my failures. For his perspective, both level and atilt, both broad and narrow, I thank Tyler Johnson.

Thanks are due to The University of Chicago for financial support over the last five years as well as affording me the opportunity to teach curious students.

Finally, I thank my parents Danny and Chris, my Aunt Lynn, and my sister Joanna for their love.

## ABSTRACT

The *point-pushing subgroup*  $P(\Sigma_g)$  of the mapping class group  $\text{Mod}(\Sigma_{g,1})$  of a surface with marked point is an embedding of  $\pi_1(\Sigma_g)$  given by pushing the marked point around loops. We prove that for  $g \geq 3$ , the subgroup  $P(\Sigma_g)$  is the unique normal, genus  $g$  surface subgroup of  $\text{Mod}(\Sigma_{g,1})$ . As a corollary to this uniqueness result, we give a new proof that  $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1})) = 1$ , where  $\text{Out}$  denotes the outer automorphism group; a proof which does not use automorphisms of complexes of curves. Ingredients in our proof of this characterization theorem include combinatorial group theory, representation theory, the Johnson theory of the Torelli group, surface topology, and the theory of Lie algebras.



# CHAPTER 1

## INTRODUCTION

Let  $\Sigma_g$  (respectively,  $\Sigma_{g,1}$ ) be a compact, connected surface of genus  $g$  (respectively, with one marked point). Let  $\Sigma$  be either  $\Sigma_g$  or  $\Sigma_{g,1}$ . The *mapping class group*  $\text{Mod}(\Sigma)$  is the group of orientation-preserving homeomorphisms of  $\Sigma$  modulo isotopy. The map  $\Sigma_{g,1} \rightarrow \Sigma_g$  given by “forgetting” the marked point induces an injection

$$F : \text{Mod}(\Sigma_{g,1}) \hookrightarrow \text{Mod}(\Sigma_g).$$

For  $g \geq 2$ , the *point-pushing subgroup* is defined by  $P(\Sigma_g) := \ker(F)$ .

Informally,  $P(\Sigma_g)$  is the subgroup of  $\text{Mod}(\Sigma_{g,1})$  consisting of elements that “push” the marked point along closed curves in the surface. Birman in [3] (see also [4]) proved that  $P(\Sigma_g) \cong \pi_1(\Sigma_g)$ . A *genus  $h$  surface group* is any group isomorphic to  $\pi_1(\Sigma_h)$ . In particular,  $P(\Sigma_g)$  is an example of a normal, genus  $g$  surface subgroup of  $\text{Mod}(\Sigma_{g,1})$ .

**Theorem 1.0.1** (Uniqueness of  $P(\Sigma_g)$ ). *Let  $g \geq 3$ . The point-pushing subgroup  $P(\Sigma_g)$  is the unique normal, genus  $g$  surface subgroup inside  $\text{Mod}(\Sigma_{g,1})$ .*

*Remarks* (on Theorem 1.0.1).

1. Theorem 1.0.1 has a beautiful free group analogue, proven by Formanek in 1990. Our proof follows in outline the proof given by Formanek in [11]. Even so, in establishing the main result we will have to overcome several obstacles to reconcile the differences between free groups and surface groups. Let  $\{a_1, b_1, \dots, a_g, b_g\}$  be a standard set of generators for  $\pi_1(\Sigma_g)$ . In general, the surface relation  $(\prod_i^g [a_i, b_i] = 1)$  pervades the objects associated to  $\pi_1(\Sigma_g)$  and muddies the analogy between free and surface groups. Some key differences between  $F_n \triangleleft \text{Aut}(F_n)$  and  $\pi_1(\Sigma_g) \triangleleft \text{Mod}(\Sigma_{g,1})$  are summarized in Table1.1.

Table 1.1: Key Surface Group Obstacles

$F_n \triangleleft \text{Aut}(F_n)$	$\pi_1(\Sigma_g) \triangleleft \text{Mod}(\Sigma_{g,1})$
<p>The representation theory of <math>\text{GL}_n(\mathbb{Q})</math> reveals properties of <math>\text{Aut}(F_n)</math> from the canonical surjection</p> $\text{Aut}(F_n) \twoheadrightarrow \text{Aut}(F_n/\gamma_2(F_n)) \cong \text{GL}_n(\mathbb{Z}).$	<p>The representation theory of <math>\text{Sp}_{2g}(\mathbb{Q})</math> reveals properties of <math>\text{Mod}(\Sigma_{g,1})</math> from the standard symplectic representation</p> $\text{Mod}(\Sigma_{g,1}) \twoheadrightarrow \text{Sp}_{2g}(\mathbb{Z}).$
<p>Let <math>\mathcal{J}(F_n)</math> be the Torelli subgroup of <math>\text{Aut}(F_n)</math>, see Definition 1.1.2. <math>\mathcal{J}(F_n)</math> has torsion-free abelianization. Specifically, <math>H_1(\mathcal{J}(F_n); \mathbb{Z}) \cong \Lambda^3 \mathbb{Z}^n</math>.</p>	<p>Let <math>\mathcal{J}(\Sigma)</math> be the Torelli subgroup of <math>\text{Mod}(\Sigma_{g,1})</math>, see Definition 1.1.2. The abelianization of <math>\mathcal{J}(\Sigma)</math> contains 2-torsion. That is, <math>H_1(\mathcal{J}(\Sigma); \mathbb{Z}) \cong \Lambda^3 \mathbb{Z}^{2g} \oplus \mathcal{B}/\langle \alpha \rangle</math> where <math>\mathcal{B}/\langle \alpha \rangle</math> is 2-torsion, see Proposition 2.2.5. The existence of this 2-torsion comes from the Rochlin invariant in 3-manifold theory.</p>
<p>Let <math>\mathcal{J}_2(F_n)</math> be the second term in the Andreadakis-Johnson filtration, see Definition 1.1.2. Then, <math>[\mathcal{J}(F_n), \mathcal{J}(F_n)] = \mathcal{J}_2(F_n)</math>.</p>	<p>Let <math>\mathcal{J}_2(\Sigma)</math> be the second term in the Andreadakis-Johnson filtration, see Definition 1.1.2. Then, <math>[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \neq \mathcal{J}_2(\Sigma)</math>.</p>
<p>The graded Lie algebra associated to the lower central series of <math>F_n</math> is free. See Section 2.2.1.</p>	<p>The graded Lie algebra associated to the lower central series of <math>\pi_1(\Sigma_g)</math> is a nontrivial quotient of the free Lie algebra. See Section 2.2.1</p>

2. That  $P(\Sigma_g)$  is normal in  $\text{Mod}(\Sigma_{g,1})$  is necessary for the uniqueness result stated in Theorem 1.0.1. Clay-Leininger-Mangahas in [8, Cor.1.3] construct infinitely many nonconjugate genus  $g$  surface subgroups in  $\text{Mod}(\Sigma_g)$ . See also the work of Leininger-Reid [26, Cor.5.6]. Specifically for a surface with one marked point, we can find

surface subgroups  $\pi_1(\Sigma_h) < \text{Mod}(\Sigma_{g,1})$  for infinitely many  $h$  using the Thurston norm (see Example 3.3.1 below).

3. Theorem 1.0.1 does not hold for  $g = 1$ . Because  $\text{Mod}(\Sigma_{1,1}) \cong SL_2\mathbb{Z}$  has a finite index, free subgroup,  $\text{Mod}(\Sigma_{1,1})$  has no surface subgroups. It is not known whether or not  $P(\Sigma_2)$  is the only normal, genus 2 surface subgroup in  $\text{Mod}(\Sigma_{2,1})$ .

The *extended mapping class group*  $\text{Mod}^\pm(\Sigma)$  is the group of all homeomorphisms (orientation preserving and reversing) of  $\Sigma$ , modulo isotopy. The Dehn-Nielsen-Baer theorem establishes an isomorphism

$$\Phi : \text{Mod}^\pm(\Sigma_{g,1}) \xrightarrow{\cong} \text{Aut}(\pi_1(\Sigma_g)).$$

Let  $\text{Inn}(\pi_1(\Sigma_g))$  be the group of inner automorphisms of  $\pi_1(\Sigma_g)$ . As a consequence of the Dehn-Nielsen-Baer theorem,

$$\Phi(P(\Sigma_g)) = \text{Inn}(\pi_1(\Sigma_g)) \triangleleft \text{Aut}(\pi_1(\Sigma_g)).$$

Burnside in [7, pp. 261] proved that for a centerless group  $G$  (which gives  $G \cong \text{Inn}(G) \triangleleft \text{Aut}(G)$ ), if every  $\phi \in \text{Aut}(\text{Aut}(G))$  satisfies  $\phi(G) = G$ , then

$$\text{Aut}(\text{Aut}(G)) = \text{Inn}(\text{Aut}(G)) \cong \text{Aut}(G).$$

See Section 3.4 below for a short proof of this fact of Burnside. Since  $\pi_1(\Sigma_g)$  is centerless for  $g > 1$ , Burnside's result together with Theorem 1.0.1 implies:

**Corollary 1.0.2** (Ivanov-McCarthy's Theorem). *Let  $g \geq 3$ . Then  $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1}))$  is trivial.*

*Remarks* (on Corollary 1.0.2).

1. For  $g \geq 3$ , Ivanov-McCarthy proved that  $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1}))=1$ , from which they deduced that  $\text{Out}(\text{Mod}(\Sigma_g)) \cong \mathbb{Z}/2\mathbb{Z}$ , see [16], [15, Th.5] and [29, Th.1]. In fact, Ivanov-McCarthy proved a much stronger result for injective homeomorphisms of finite index subgroups of  $\text{Mod}(\Sigma_g)$ . Their work uses the deep theorem of Ivanov that

the automorphism group of the complex of curves is the extended mapping class group. Our proof does not use this theorem.

2. The result of Ivanov-McCarthy that  $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1}))=1$  implies that  $P(\Sigma_g)$  is characteristic in  $\text{Mod}(\Sigma_{g,1})$ , since all automorphisms of  $\text{Mod}^\pm(\Sigma_{g,1})$  are inner. In contrast, our characterization theorem (Theorem 1.0.1) implies that  $P(\Sigma_g)$  is characteristic, from which we deduce (with Burnside) that  $\text{Mod}(\Sigma_{g,1})$  has no outer automorphisms.
  
3. McCarthy in [29] proved that  $\text{Out}(\text{Mod})^\pm(\Sigma_{2,1})$  is nontrivial, which implies (with Burnside) that  $P(\Sigma_2)$  is not characteristic in  $\text{Mod}^\pm(\Sigma_{2,1})$ . Thus,  $P(\Sigma_2)$  is not the only normal, genus 2 surface subgroup in  $\text{Mod}^\pm(\Sigma_{2,1})$ . However, it is unknown whether or not these additional normal, genus 2 surface subgroups are contained in  $\text{Mod}(\Sigma_{2,1})$ .

## 1.1 Structure of the proof

Ingredients in our proof of Theorem 1.0.1 include combinatorial group theory, representation theory, the Johnson theory of the Torelli group, the theory of Lie algebras, and surface topology. These tools allow us to characterize  $P(\Sigma_g)$  in terms of two filtrations: the lower central series of  $P(\Sigma_g)$  and the Andreadakis-Johnson filtration of  $\text{Mod}(\Sigma_{g,1})$ . By showing any arbitrary normal, genus  $g$  surface subgroup must also have those same characterizing properties, we demonstrate that  $P(\Sigma_g)$  is unique.

To condense notation, let  $P := P(\Sigma_g)$ . Let  $N \triangleleft \text{Mod}(\Sigma_{g,1})$  be a normal subgroup abstractly isomorphic to  $\pi_1(\Sigma_g)$ . We must prove that  $N = P$ .

**Definition 1.1.1.** The *lower central series* of a group  $G$ , denoted as

$$G = \gamma_1(G) \supset \gamma_2(G) \supset \dots$$

is defined inductively as  $\gamma_{i+1}(G) = [\gamma_i(G), G]$ .

Let  $Z(G)$  denote the center of a group  $G$ . The lower central series is *central*, i.e.  $\gamma_k(G)/\gamma_{k+1}(G) \subset Z(G/\gamma_{k+1}(G))$  for each  $k$ . Further, each  $\gamma_k(\pi_1(\Sigma_g))$  is *characteristic* in  $\pi_1(\Sigma_g)$ , i.e. invariant under automorphisms of  $\pi_1(\Sigma_g)$ . As such, there is a family of well-defined maps

$$\Psi_k : \text{Mod}(\Sigma_{g,1}) \rightarrow \text{Aut}(\pi_1(\Sigma_g)/\gamma_{k+1}(\pi_1(\Sigma_g))).$$

**Definition 1.1.2.** The *Johnson filtration* of  $\text{Mod}(\Sigma_{g,1})$ , denoted as

$$\mathcal{J}(\Sigma) = \mathcal{J}_1(\Sigma) \supset \mathcal{J}_2(\Sigma) \supset \dots$$

is defined as

$$\mathcal{J}_k(\Sigma) := \ker(\Psi_k).$$

The first term  $\mathcal{J}(\Sigma)$  is referred to as the *Torelli group* of  $\text{Mod}(\Sigma_{g,1})$ .

By assumption,  $N \cong \pi_1(\Sigma_g)$ . As such, for some surface  $\hat{\Sigma}_{g,1}$ , we can define an injection  $N \hookrightarrow \text{Mod}(\hat{\Sigma}_{g,1})$  so that the image of  $N$  is the point-pushing subgroup in  $\text{Mod}(\hat{\Sigma}_{g,1})$ . In this paper, we will consider both the Johnson filtration for  $\text{Mod}(\hat{\Sigma}_{g,1})$  and for  $\text{Mod}(\Sigma_{g,1})$ . To distinguish these two filtrations we will use the notation  $\mathcal{J}_k(N)$  for  $\text{Mod}(\hat{\Sigma}_{g,1})$  and  $\mathcal{J}_k(\Sigma)$  for  $\text{Mod}(\Sigma_{g,1})$ . We will gradually “push”  $P$  and  $N$  through the terms of these Johnson filtrations in order to capture salient properties. Eventually, we will establish the following chain of containments:

$$\gamma_2(N) \subseteq \gamma_2(P) \subseteq \mathcal{J}_2(\Sigma) \subseteq \mathcal{J}(\Sigma) \subseteq \mathcal{J}(N).$$

Furthermore, we will give the following useful characterization of  $P(\Sigma_g)$  in terms of the linear central filtrations defined above.

**Proposition 3.0.2 (Characterization of  $P$ ).** *Let  $g \geq 3$ . Then*

$$P(\Sigma_g) = \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P(\Sigma_g))\}.$$

By proving Proposition 3.0.2 we will also characterize  $N$  as

$$N = \{x \in \mathcal{J}(N) \mid [x, \mathcal{J}(N)] \subset \gamma_2(N)\}.$$

Notice that Proposition 3.0.2 together with the two inclusions  $\mathcal{J}(\Sigma) \subset \mathcal{J}(N)$  and  $\gamma_2(N) \subset \gamma_2(P)$  implies the following chain of containments:

$$\begin{aligned}
N &= \{x \in \mathcal{J}(N) \mid [x, \mathcal{J}(N)] \subset \gamma_2(N)\} \\
&\subseteq \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(N)\} \\
&\subseteq \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\} \\
&= P.
\end{aligned}$$

That is,  $N \subseteq P$ . Applying the index formula  $[N : P] \cdot \chi(\Sigma) = \chi(\hat{\Sigma})$ , we can conclude that  $N = P$ .

In summary, we divide our proof into the following two main parts:

- **Chapter 2:** Demonstrate the chain of containments

$$\gamma_2(N) \subset \gamma_2(P) \subset \mathcal{J}_2(\Sigma) \subset \mathcal{J}(\Sigma) \subset \mathcal{J}(N).$$

- **Chapter 3:** Characterize the point-pushing subgroup as

$$P = \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}.$$

From these two steps, it follows that  $N=P$ .

## CHAPTER 2

### LINEAR DETECTORS

As above, let  $P$  be the point-pushing subgroup of  $\text{Mod}(\Sigma_{g,1})$ . Let  $N \triangleleft \text{Mod}(\Sigma_{g,1})$  be abstractly isomorphic to  $\pi_1(\Sigma_g)$ . In chapter two, we will use the lower central series of  $\pi_1(\Sigma_g)$ , the Johnson filtration of the Torelli group, and  $\text{Sp}_{2g}(\mathbb{Q})$  representations to detect salient features of  $P$ . In particular, we will show that  $N$  must have the same action on homology, and the image under the Johnson homomorphism as the subgroup  $P$ .

#### 2.1 Action on homology: $N \subset \mathcal{J}(\Sigma)$ .

While  $N$  need not act as the point-pushing subgroup on  $\Sigma_g$ , we can choose  $N$  to be the point-pushing subgroup of  $\text{Mod}(\hat{\Sigma}_{g,1})$  for some surface  $\hat{\Sigma}_{g,1}$ . We will use  $\mathcal{J}_k(\Sigma)$  to denote the Johnson filtration for  $\text{Mod}(\Sigma_{g,1})$ , and we will use  $\mathcal{J}_k(N)$  to denote the Johnson filtration for  $\text{Mod}(\hat{\Sigma}_{g,1})$ . For the remainder of the paper, let  $g \geq 3$ .

In this section, we will work toward establishing the chain of containments

$$\gamma_2(N) \subset \gamma_2(P) \subset \mathcal{J}_2(\Sigma) \subset \mathcal{J}(\Sigma) \subset \mathcal{J}(N)$$

by proving that  $\mathcal{J}(\Sigma) \subset \mathcal{J}(N)$  and  $N \subset \mathcal{J}(\Sigma)$ .

Let  $\beta$  be a loop in  $\Sigma_{g,1}$  based at the marked point,  $x_0$ . This loop defines an isotopy from the marked point to itself which can be extended to all of  $\Sigma_{g,1}$ . (For a more precise explanation see e.g. [10, Setc.4.2].) Denote this homeomorphism by  $\phi_\beta$ . The point-pushing subgroup  $P \triangleleft \text{Mod}(\Sigma_{g,1})$  is exactly the subgroup of isotopy classes of homeomorphisms of the form  $\phi_\beta$  for any based loop  $\beta$ . Let  $[\beta] \in \pi_1(\Sigma_g)$  be the homotopy class of loops containing  $\beta$ . There is a well-defined map

$$\text{Push} : \pi_1(\Sigma_g) \rightarrow P$$

given by

$$\text{Push}([\beta]) = [\phi_\beta].$$

Birman in [3] proved that the map  $Push$  is an isomorphism. Because  $P$  is normal,  $\text{Mod}(\Sigma_{g,1})$  acts on  $P$  via conjugation. Alternately, the action of  $\text{Mod}(\Sigma_{g,1})$  on  $\Sigma_{g,1}$  induces an action on the fundamental group  $\pi_1(\Sigma_g)$ . The map  $Push$  respects the action of  $\text{Mod}(\Sigma_{g,1})$ . That is, for  $\psi \in \text{Mod}(\Sigma_{g,1})$  and  $[\beta] \in \pi_1(\Sigma_g)$

$$Push(\psi_*([\beta])) = \psi Push([\beta]) \psi^{-1}. \quad (\dagger)$$

For convenience, we will sometimes equate  $P$  with  $\pi_1(\Sigma_g)$ . For full details regarding the isomorphism between  $P$  and  $\pi_1(\Sigma_g)$  see Section 4.2 of [10].

The point-pushing subgroup  $P$  acts by free homotopies on the unmarked surface  $\Sigma_g$ . As such,  $P$  acts trivially on  $H_1(\pi_1(\Sigma_g); \mathbb{Z}) \cong \pi_1(\Sigma_g)/\gamma_2(\pi_1(\Sigma_g))$ . That is,  $P \subset \mathcal{J}(\Sigma)$ . We want to show that  $N$  also has the property  $N \subset \mathcal{J}(\Sigma)$ .

Given  $\phi \in \text{Mod}(\Sigma_{g,1})$  and  $n \in N$ , define the map

$$\alpha : \text{Mod}(\Sigma_{g,1}) \rightarrow \text{Aut}^\pm(N) \cong \text{Mod}^\pm(\hat{\Sigma}_{g,1})$$

by

$$\alpha(\phi)(n) = \phi n \phi^{-1}.$$

We have the following exact sequences for  $P$  and  $N$ :

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{J}(\Sigma) & \longrightarrow & \text{Mod}(\Sigma_{g,1}) & \xrightarrow{\Psi_\Sigma} & \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1 \\ & & & & \downarrow \alpha & & \downarrow \bar{\alpha} \\ 1 & \longrightarrow & \mathcal{J}(N) & \longrightarrow & \text{Aut}^\pm(N) & \xrightarrow{\Psi_N} & \text{Sp}_{2g}^\pm(\mathbb{Z}) \longrightarrow 1 \end{array} \quad (\ddagger)$$

where  $\text{Sp}_{2g}^\pm(\mathbb{Z})$  is the subgroup of  $\text{GL}_{2g}(\mathbb{Z})$  generated by  $\text{Sp}_{2g}(\mathbb{Z})$  and the image of any orientation-reversing homeomorphism. Because all orientation reversing homeomorphisms are nontrivial on  $H_1(\Sigma_{g,1}; \mathbb{Z})$ , we have the equality

$$\ker(\Psi_N) = \mathcal{J}(N) = \ker(\Psi_N|_{\text{Aut}^+} : \text{Mod}(\hat{\Sigma}_{g,1}) \rightarrow \text{Sp}_{2g}(\mathbb{Z})).$$



Section 2.1 is divided into the following steps:

- 1.A. The map  $\alpha$  is injective.
- 1.B. The Torelli group  $\mathcal{J}(\Sigma)$  is contained in the Torelli group  $\mathcal{J}(N)$ .
- 1.C. The map  $\bar{\alpha}$  is an isomorphism onto its image.

The containment  $N \subset \mathcal{J}(\Sigma)$  will follow easily from part 1.C.

**1.A. Injectivity of  $\alpha$**  The map  $\alpha$  is defined by the conjugation action of  $\text{Mod}(\Sigma_{g,1})$  on  $N$ . Thus,  $\ker(\alpha)$  centralizes  $N$ . Since  $N \cong \pi_1(\Sigma_g)$  has trivial center, it follows that  $\ker(\alpha) \cap N = 1$ .

Let  $\phi \in N \triangleleft \text{Mod}(\Sigma_{g,1})$  be nontrivial. There is some  $x \in \pi_1(\Sigma_g)$  such that  $\phi(x) \neq x$ . That is, by Equation (†), the element  $\phi x \phi^{-1} x^{-1}$  of  $\text{Mod}(\Sigma_{g,1})$  is nontrivial. However, because both  $N$  and  $P$  are normal in  $\text{Mod}(\Sigma_{g,1})$  it follows that  $\phi x \phi^{-1} x^{-1} \in N \cap P$ . Therefore, the intersection  $N \cap P \neq \emptyset$ .

Likewise, if  $\ker(\alpha) \neq 1$ , then there is a nontrivial element of  $\ker \alpha \cap P$ .

The two subgroups  $\ker(\alpha) \cap P$  and  $P \cap N$  are commuting subgroups of  $P$ . Because  $\ker(\alpha) \cap N = 1$ , the intersection  $(\ker(\alpha) \cap P) \cap (P \cap N)$  is trivial. However,  $x_1, x_2 \in \pi_1(\Sigma_g)$  commute if and only if  $x_1 = \omega^k$  and  $x_2 = \omega^m$  for some  $\omega \in \pi_1(\Sigma_g)$ , see e.g. [10, Sect.1.1.3]. For  $\ker(\alpha) \cap P$  and  $P \cap N$  to intersect trivially and also commute, it must be that  $\ker(\alpha) = 1$ . Therefore  $\alpha$  is injective.

**1.B. Containment of Torelli groups** Let  $\Psi_\Sigma, \Psi_N$ , and  $\alpha$  be defined as in (‡). The following theorem of Korkmaz relates the two homomorphisms  $\Psi_\Sigma$  and  $\Psi_N \circ \alpha$ .

**Theorem 2.1.1** (Korkmaz [23] Thm.1). *For  $g \geq 3$ , any group homomorphism  $\phi : \text{Mod}(\Sigma_{g,1}) \rightarrow \text{Gl}_{2g}(\mathbb{C})$  is either trivial or else conjugate to the standard representation  $\Psi_\Sigma : \text{Mod}(\Sigma_{g,1}) \rightarrow \text{Sp}_{2g}(\mathbb{Z})$ .*

Two homomorphisms  $\phi, \psi : G \rightarrow H$  are *conjugate* if there exists an element  $h \in H$  such that  $h\phi h^{-1} = \psi(g)$  for all  $g \in G$ . Note that conjugate homomorphisms have the same kernel.

By Theorem 2.1.1, the composition

$$\Psi_N \circ \alpha : \text{Mod}(\Sigma_{g,1}) \rightarrow \text{Sp}_{2g}^{\pm}(\mathbb{Z}) \subset \text{GL}_{2g}(\mathbb{C})$$

is either trivial or conjugate to  $\Psi_{\Sigma}$ . Thus, the kernel of  $\Psi_N \circ \alpha$  is either all of  $\text{Mod}(\Sigma_{g,1})$  or exactly  $\mathcal{J}(\Sigma)$ . In either case,  $\alpha(\mathcal{J}(\Sigma)) \subset \ker(\Psi_N) = \mathcal{J}(N)$ . Using the injectivity of  $\alpha$  to simplify notation,  $\mathcal{J}(\Sigma) \subset \mathcal{J}(N)$ .

**1.C. The map  $\bar{\alpha}$  is an isomorphism** Using the fact that  $\ker(\Psi_{\Sigma}) \subset \ker(\Psi_N)$ , there is a well-defined homomorphism  $\bar{\alpha} : \text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}^{\pm}(\mathbb{Z})$  which makes the diagram  $(\ddagger)$  commute. Note that  $\text{Mod}(\Sigma_{g,1})$  contains torsion elements but  $\mathcal{J}(N)$  is torsion-free (see [13, Sect.2 pp.101]). Therefore,  $\alpha(\text{Mod}(\Sigma_{g,1})) \not\subset \mathcal{J}(N)$ , and  $\Psi_N \circ \alpha \neq 1$ . The commutativity of  $(\ddagger)$  implies  $\bar{\alpha} \circ \Psi_{\Sigma} \neq 1$ . Again applying Theorem 2.1.1, the image of  $\bar{\alpha}$  must be conjugate to  $\text{Sp}_{2g}(\mathbb{Z})$ . Therefore  $\bar{\alpha}$  is an isomorphism onto its image.

Because  $N \subset \ker(\Psi_N)$  and because the diagram  $(\ddagger)$  commutes, it follows that  $N \subset \ker(\bar{\alpha} \circ \Psi_{\Sigma})$ . However,  $\ker(\bar{\alpha}) = 1$  implies  $N \subset \ker(\Psi_{\Sigma})$ . That is

$$N \subset \mathcal{J}(\Sigma) \subset \mathcal{J}(N).$$

## 2.2 The second term of the Johnson filtration.

In this section we will “push”  $P$  and  $N$  deeper into the second term of Johnson filtration. We will show that  $N \subset P \cdot \mathcal{J}_2(\Sigma)$ . To that end, we will prove  $N \cdot \mathcal{J}(\Sigma)/\mathcal{J}(\Sigma) = P \cdot \mathcal{J}(\Sigma)/\mathcal{J}(\Sigma)$  by using the Johnson homomorphism and the representation theory of  $\text{Sp}_{2g}(\mathbb{Q})$ .

### 2.2.1 Johnson filtration of $\text{Mod}(\Sigma_{g,1})$ and lower central series of $P$

In this subsection, we will consider the quotient  $P \cdot \mathcal{J}(\Sigma)/\mathcal{J}(\Sigma) \cong P/(P \cap \mathcal{J}(\Sigma))$ . We will prove that  $P \cap \mathcal{J}(\Sigma) = [P, P]$ . Moreover, we will establish the following general fact.

**Proposition 2.2.1.**  $P \cap \mathcal{J}(\Sigma)_k = \gamma_k(P)$  for all  $k \geq 1$ .

To condense notation, let  $\gamma_k := \gamma_k(\pi_1(\Sigma_g))$  be the  $k$ th term of the lower central series. Notice that

$$\pi_1(\Sigma_g) \cap \mathcal{J}_k(\Sigma) = \{x \in \pi_1(\Sigma_g) \mid xyx^{-1}y^{-1} \in \gamma_{k+1} \text{ for all } y \in \pi_1(\Sigma_g)\}. \quad (2.2.1)$$

That is,  $x \in \pi_1(\Sigma_g) \cap \mathcal{J}_k(\Sigma)$  if and only if the left coset  $x\gamma_{k+1}$  is contained in the center  $Z(\gamma_1/\gamma_{k+1})$ . Thus, Proposition 2.2.1 is equivalent to showing that:

$$Z(\gamma_1/\gamma_{k+1}) = \gamma_k/\gamma_{k+1} \text{ for all } k \geq 1. \quad (2.2.2)$$

This equality was first established by Asada and Kaneko in [1, Prop. A]. For the reader, we provide an alternate proof below. We will demonstrate Equality 2.2.2 by establishing two containments. The containment  $Z(\gamma_1/\gamma_{k+1}) \supset \gamma_k/\gamma_{k+1}$  follows from the definition of the lower central series. The opposite containment relies on an analysis of the center of the Lie algebra associated to the lower central series of  $\pi_1(\Sigma_g)$ , described below.

Associated to the lower central series of any group is a graded Lie algebra. (For a discussion of this graded Lie algebra, see e.g. work of Lazard in [25], or Labute in [24]. Mal'cev is credited with first using this nilpotent filtration to study groups in [28].) Specifically, for  $G = \pi_1(\Sigma_g)$  define

$$\Lambda_i := \gamma_i(\pi_1(\Sigma_g))/\gamma_{i+1}(\pi_1(\Sigma_g)) \text{ for } i \geq 1.$$

Each  $\Lambda_i$  is a  $\mathbb{Z}$ -module. The sum

$$\Lambda := \bigoplus_i \Lambda_i$$

can be given the structure of a graded Lie algebra over  $\mathbb{Z}$  as follows. Let  $(, )$  be the commutator in  $\pi_1(\Sigma_g)$ . The Lie bracket  $[, ]$  is induced on  $\Lambda$  by the commutator. That is, for  $x \in \Lambda_k$ ,  $y \in \Lambda_j$ , and  $\bar{x}, \bar{y}$  lifts of  $x, y$  respectively to  $\pi_1(\Sigma_g)$ , we define

$$[x, y] := (\bar{x}, \bar{y})\gamma_{k+j+1} \in \Lambda_{k+j}.$$

Despite the fact that  $\pi_1(\Sigma_g)$  is written as a multiplicative group, we will write  $\Lambda_i$  additively as a  $\mathbb{Z}$ -vector space. In particular, the left coset  $1\gamma_{k+1}$  is 0 as an element of  $\Lambda_k$ .

To prove Proposition 2.2.1 it remains to show  $Z(\gamma_1/\gamma_{k+1}) \subset \gamma_k/\gamma_{k+1}$ . We will divide the proof into two steps as follows:

2.2.1.A.  $\Lambda$  has trivial center  $\implies Z(\pi_1/\gamma_{k+1}(\pi_1)) \subset \gamma_k(\pi_1)/\gamma_{k+1}(\pi_1)$ .

2.2.1.B. The universal enveloping algebra  $U(\Lambda) \cong \mathcal{A}_{2g}/\mathfrak{A}$  where  $\mathcal{A}_{2g}$  is the free associative algebra on  $2g$  indeterminates, and  $\mathfrak{A}$  is the ideal generated by  $\sum_{i=1}^g (a_i \otimes b_i - b_i \otimes a_i)$ .

To conclude, we will check that  $U(\Lambda)$  has trivial center. Because all relations in  $\Lambda$  must hold in its universal enveloping algebra, if  $U(\Lambda)$  has trivial center, then so does  $\Lambda$ . Proposition 2.2.1 follows.

*Proof of Proposition 2.2.1.A.* Let  $x \in Z(\gamma_1/\gamma_{k+1})$ . Suppose for the sake of contradiction  $x \notin \Lambda_k$ . We will show that  $x \in Z(\Lambda)$ .

There is some smallest  $i \geq 1$  such that  $x \in \gamma_{k-i}/\gamma_{k+1}$ . Let  $y \in \Lambda_j$ . In order to show that  $x$  is central, because the Lie bracket is bilinear, it suffices to check that  $[x, y] = 0$  for any  $y$  and any  $j$ . That is, the commutator  $(\bar{x}, \bar{y}) \in \gamma_{k-i+j+1}$ .

First, let  $1 \leq j \leq i$ . Notice, for any  $\bar{y} \in \pi_1(\Sigma_g)$ , the commutator  $(\bar{x}, \bar{y}) \in \gamma_{k+1}$  because  $x \in Z(\gamma_1/\gamma_{k+1})$ . Since  $j \leq i$ , it follows that  $\gamma_{k+1} \subset \gamma_{k-i+j+1}$ . Thus,

$$(\bar{x}, \bar{y}) \in \gamma_{k+1} \subset \gamma_{k-i+j+1}.$$

Therefore, if  $y \in \Lambda_j$  for  $j \leq i$  then

$$[x, y] = 1\gamma_{k-i+j+1} = 0 \in \Lambda_{k-i+j}.$$

Otherwise, let  $1 \leq i < j$ . We will prove  $(\bar{x}, \bar{y}) \in \gamma_{k-i+j+1}$  by induction on  $j$ . Suppose first  $j = 2$  (forcing  $i = 1$ ). Without loss of generality we may assume  $\bar{y} = (\bar{a}, \bar{b})$ . The Jacobi identity provides

$$[x, y] = [x, [a, b]] = -[b, [x, a]] - [a, [b, x]].$$

However,  $(\bar{x}, \bar{a}), (\bar{b}, \bar{x}) \in \gamma_{k+1}$  because  $x \in Z(\gamma_1/\gamma_{k+1})$ . This implies

$$-(b, (x, a)) - (a, (b, x)) \in \gamma_{k+2} = \gamma_{k-1+2+1} = \gamma_{k-i+j+1}.$$

Therefore,  $[x, y] = 0$ .

To complete the induction, let  $M < k$ . Assume if  $i \leq j \leq M$ , then  $(\bar{x}, \bar{y}) \in \gamma_{k-i+j+1}$  for all  $y \in \Lambda_j$ . Suppose  $y \in \Lambda_{M+1}$ . Without loss of generality, we may assume that  $y$  is an  $(M+1)$ -fold commutator, i.e.  $y = (\bar{a}, \bar{b})\gamma_{M+2}$  for some  $a \in \Lambda_1$  and  $b \in \Lambda_M$ . By assumption,  $(\bar{x}, \bar{a}) \in \gamma_{k+1}$ , which implies

$$(\bar{b}, (\bar{x}, \bar{a})) \in \gamma_{k+1+M+1} \subset \gamma_{k-i+M+1}.$$

By the inductive hypothesis,  $(\bar{b}, \bar{x}) \in \gamma_{k-i+M+1}$ . Therefore,

$$(\bar{x}, (\bar{a}, \bar{b})) = -(\bar{b}, (\bar{x}, \bar{a})) - (\bar{a}, (\bar{b}, \bar{x})) \in \gamma_{k-1+M+1} \subset \gamma_{k-i+M+1}$$

implying that  $[x, y] = 0$  for all  $y \in \Lambda$ .

Therefore,  $x \neq 0$  is central in  $\Lambda$ . This proves the implication

$$Z(\Lambda) = 0 \Rightarrow Z(\gamma_1/\gamma_{k+1}) \subset \gamma_k/\gamma_{k+1}. \quad (2.2.3)$$

□

*Proof of Proposition 2.2.1.B.* The following theorem of Labute shows that the graded Lie algebra  $\Lambda$  is a quotient of the free Lie algebra on  $2g$  generators by a principal ideal.

**Theorem 2.2.2.** (Labute [24]). *Let  $\mathcal{L}_{2g}$  be the free Lie algebra on  $2g$  generators (denoted  $a_1, b_1, \dots, a_g, b_g$ ). Let  $\mathcal{R}$  be the ideal generated by  $\sum_i [a_i, b_i]$ . Then  $\Lambda \cong \mathcal{L}_{2g}/\mathcal{R}$ .*

Let  $T(\Lambda)$  be the tensor algebra on the vector space underlying  $\Lambda$ . Let  $U(\Lambda)$  be the universal enveloping algebra. Define  $U(\Lambda)$  as

$$U(\Lambda) := T(\Lambda)/\langle a \otimes b - b \otimes a - [a, b] \rangle.$$

Let  $\mathcal{A}_{2g}$  be the free associative algebra on  $2g$  indeterminates. Let  $\mathfrak{R}$  be the ideal in  $\mathcal{A}_{2g}$  generated by  $\sum_{i=1}^g (a_i \otimes b_i - b_i \otimes a_i)$ . We will prove below that the universal enveloping algebra  $U(\Lambda)$  is isomorphic to  $\mathcal{A}/\mathfrak{R}$ .

The analogous fact for free groups,  $U(\mathcal{L}_n) \cong \mathcal{A}_n$ , was established by Magnus-Karrass-Solitar, see [27, pp.347 ex.5]. Let  $U : \mathcal{L} \rightarrow \mathcal{A}$  be the functor from the category of Lie algebras to the category of associative algebras that takes a Lie algebra to its universal enveloping algebra. Let  $G : \mathcal{A} \rightarrow \mathcal{L}$  be the functor from the category of associative algebras to the category of Lie algebras, that induces the Lie bracket by the commutator in the associative algebra.  $U$  is left-adjoint to  $G$ .

Define an injection

$$\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_{2g}$$

via

$$\phi(1) := \sum_i [a_i, b_i].$$

Notice that

$$U(\mathcal{L}_1 \xrightarrow{\phi} \mathcal{L}_{2g}) = \mathcal{A}_1 \xrightarrow{U(\phi)} \mathcal{A}_{2g}.$$

The map  $U(\phi)$  is the injective map defined by  $U(\phi)(1) := \sum (a_i \otimes b_i - b_i \otimes a_i)$ . Note,  $\text{coker}(\phi) \cong \Lambda$  and  $\text{coker}(U(\phi)) \cong \mathcal{A}_{2g}/\mathfrak{R}$ . Since  $U$  is left-adjoint to  $G$  it preserves cokernels, meaning  $U(\text{coker}(\phi)) \cong \text{coker}(U(\phi))$ . Therefore  $U(\Lambda) \cong \mathcal{A}_{2g}/\mathfrak{R}$ .  $\square$

In order to show that  $\Lambda$  is centerless, it suffices to show that the universal enveloping algebra,  $U(\Lambda)$  is centerless. A computation of Crawley-Boevey-Etingof-Ginzburg in [9, Thm.8.4.1(ii)] shows that the Hochschild cohomology  $HH^0(\mathcal{A}_{2g}/\mathfrak{R}) \cong \mathbb{Z}$ . For an associative algebra  $\mathcal{A}$ , the center  $Z(\mathcal{A}) = HH^0(\mathcal{A})$  (see e.g. [35, Sect.9.1.1]). That is, only  $\mathbb{Z}$  is central in the associative  $\mathbb{Z}$ -algebra  $\mathcal{A}/\mathfrak{R} = U(\Lambda)$ . All relations in  $\Lambda$  must be preserved in  $U(\Lambda)$ . Thus,  $Z(\Lambda) = 0$ .

Because  $Z(\Lambda) = 0$ , it follows from Equation (2.2.3) that  $Z(\gamma_1/\gamma_k) \subset \gamma_k/\gamma_{k+1}$ . Thus,  $Z(\gamma_1/\gamma_k) = \gamma_k/\gamma_{k+1}$ . Then, by Equation (2.2.1), it follows that  $P \cap \mathcal{J}_k = \gamma_k(P)$ .

In particular, we have shown that  $P \cap \mathcal{J}_2(\Sigma) = \gamma_2(P)$ , and equivalently  $N \cap \mathcal{J}_2(N) = \gamma_2(N)$ .

## 2.2.2 Abelianization of $\mathcal{J}(\Sigma)$

We have already established that:

$$\begin{aligned} N \cap [N, N] &\subseteq N \cap [\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \\ &\subseteq N \cap [\mathcal{J}(N), \mathcal{J}(N)] \\ &\subseteq N \cap \mathcal{J}_2(N) \\ &= [N, N]. \end{aligned}$$

The first containment follows from  $N \subset \mathcal{J}(\Sigma)$  (Section 2). The second containment follows from  $\mathcal{J}(\Sigma) \subset \mathcal{J}(N)$  (Section 2). Johnson's work showing that  $\mathcal{J}(N)/\mathcal{J}_2(N)$  is abelian implies the third containment (see [21, 18] or e.g. [10, Sect.6.6.3]). The final equality is a consequence of Proposition 2.2.1. To conclude that  $N \cap \mathcal{J}_2(\Sigma) = [N, N]$  it suffices to check that  $\mathcal{J}_2(\Sigma) \subset \mathcal{J}_2(N)$ . To establish this containment, we need to study the Johnson homomorphism and Johnson filtration.

Let  $\Sigma_g^1$  be a compact surface of genus  $g$  with one boundary component. Let  $\text{Mod}(\Sigma_g^1)$  be the group of isotopy classes of orientation preserving homeomorphisms of  $\Sigma_g^1$  fixing the boundary pointwise. Define the Torelli group for a surface with boundary as

$$\mathcal{J}_g^1 := \ker(\Psi : \text{Mod}(\Sigma_g^1) \rightarrow \text{Sp}_{2g}(\mathbb{Z}))$$

where  $\Psi$  is the standard symplectic representation. For emphasis, we will sometimes distinguish as  $\mathcal{J}_{g,1}$  the Torelli group for a once marked surface. Unless otherwise specified  $\mathcal{J} = \mathcal{J}_{g,1}$ . Let  $x \in H_1(\Sigma_g^1; \mathbb{Z})$ , let  $\bar{x} \in \pi_1(\Sigma_g^1)/\gamma_3(\pi_1(\Sigma_g^1))$  be a representative of  $x$ . Let  $\phi \in \mathcal{J}_g^1$ . The Johnson homomorphism for  $\mathcal{J}_g^1$  is

$$\tau_g^1 : \mathcal{J}_g^1 \rightarrow \text{Hom}(H_1(\Sigma_g^1; \mathbb{Z}), \gamma_2(\pi_1(\Sigma_g^1))/\gamma_3(\pi_1(\Sigma_g^1)))$$

given by

$$\tau_g^1(\phi)(x) = \phi(\bar{x})\bar{x}^{-1}.$$

Many properties of  $\mathcal{J}_{g,1}$  follow directly from the properties of  $\mathcal{J}_g^1$ . In a series of papers, Johnson established several important results summarized in the following theorem.

**Theorem 2.2.3** (Johnson). *Let the notation be as above. For  $g \geq 3$ , the following hold:*

A.  $\text{Im}(\tau_g^1) \cong \Lambda^3 H_1(\Sigma_g^1; \mathbb{Z})$  [21].

B.  $H_1(\mathcal{J}_g^1; \mathbb{Q}) \cong \Lambda^3 H_1(\Sigma_g^1; \mathbb{Q})$  [20, 22, 21].

C.  $\ker(\tau_g^1) = (\mathcal{J}_g^1)_2$  see, e.g. [10, Th.6.18].

D.  $\tau_g^1 : \mathcal{J}_g^1 / (\mathcal{J}_g^1)_2 \rightarrow \Lambda^3 H_1(\Sigma_g^1; \mathbb{Z})$  is an  $Sp_{2g}(\mathbb{Z})$ -equivariant isomorphism, see, e.g. [10, Eq.6.1].

E. The quotient  $\mathcal{J}_g^1 / (\mathcal{J}_g^1)_2$  is the universal torsion-free abelian quotient of  $\mathcal{J}_g^1$  see [21, 18] or e.g. [10, Sect.6.6.3]).

To condense notation, let

$$H_{\mathbb{Z}} := H_1(\Sigma_g^1; \mathbb{Z}),$$

$$H_{\mathbb{Q}} := H_1(\Sigma_g^1; \mathbb{Q}),$$

$$\pi_1 := \pi_1(\Sigma_g).$$

We can define the Johnson homomorphism for  $\mathcal{J}_{g,1}$  as follows. Let  $x \in H_{\mathbb{Z}}$ , let  $\bar{x}$  a representative of  $x$  in  $\pi_1$ , and  $\phi \in \mathcal{J}$ . Define

$$\tau : \mathcal{J}_{g,1} \rightarrow \text{Hom}(H_{\mathbb{Z}}, \gamma_2(\pi_1) / \gamma_3(\pi_1))$$

by

$$\tau(\phi)(x) = \phi(\bar{x})\bar{x}^{-1}.$$

The map  $\tau$  is well-defined by Proposition 2.2.1. Let  $T_{\partial}$  be the Dehn twist about the boundary curve of  $\Sigma_g^1$ . The fact that  $T_{\partial} \in \ker(\tau_g^1)$  implies that  $\tau_g^1 : \mathcal{J}_g^1 \rightarrow \Lambda^3 H_{\mathbb{Z}}$  factors through  $\mathcal{J}_{g,1}$  (see [21]). As such, Theorem 2.2.3 A-E holds for  $\mathcal{J}_{g,1}$  and  $\tau$ .

By showing that  $\mathcal{J}_2(\Sigma) \subset \mathcal{J}_2(N)$ , we will conclude that

$$[N, N] = N \cap [\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] = N \cap \mathcal{J}_2(\Sigma) = N \cap \mathcal{J}_2(N) = [N, N].$$



*Remark 2.2.4.* The quotient  $\mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma)$  differs from the universal abelian quotient of  $\mathcal{J}(\Sigma)$  only in torsion. That is,

$$\begin{aligned} \mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma) \otimes \mathbb{Q} &\cong H_1(\mathcal{J}(\Sigma); \mathbb{Q}) \\ &\cong \mathcal{J}(\Sigma)/[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \otimes \mathbb{Q}. \end{aligned}$$

Therefore,  $[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \subset \mathcal{J}_2(\Sigma)$  and the quotient  $\mathcal{J}_2(\Sigma)/[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)]$  is isomorphic to the torsion subgroup of  $\mathcal{J}(\Sigma)/[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)]$ .

To see that  $\mathcal{J}_2(\Sigma) \subset \mathcal{J}_2(N)$ , we will consider the difference between  $H_1(\mathcal{J}; \mathbb{Z})$  and  $H_1(\mathcal{J}, \mathbb{Q})$ . The abelianization,  $H_1(\mathcal{J}_{g,1}; \mathbb{Z})$ , can be computed using techniques employed by Johnson in [22] to compute  $H_1(\mathcal{J}_g^1; \mathbb{Z})$ . We could not find this exact computation in the literature, so we give it below.

**Proposition 2.2.5.**  $H_1(\mathcal{J}_{g,1}; \mathbb{Z}) \cong \Lambda^3 H_{\mathbb{Z}} \oplus \mathcal{B}_2/\langle a \rangle$  where  $\mathcal{B}_2/\langle a \rangle$  is 2-torsion (defined explicitly below).

A *boolean polynomial* is a polynomial with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ . Define  $\mathcal{B}_i$  to be the group of boolean polynomials  $p$  on  $2g$  indeterminates with  $\deg(p) \leq i$ . Building on the work of Birman-Craggs in [5], Johnson constructed in [19, Th.6] (see also e.g. [10, Th.6.19]) a surjective homomorphism

$$\sigma : H_1(\mathcal{J}_g^1; \mathbb{Z}) \rightarrow \mathcal{B}_3$$

such that the torsion of  $H_1(\mathcal{J}_g^1; \mathbb{Z})$  is captured by  $\mathcal{B}_2$ . In addition, Johnson constructed the surjective  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -equivariant homomorphism

$$q : \mathcal{B}_3 \rightarrow \Lambda^3 H_{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z};$$

for details see [22][Prop.4]. Explicitly, Johnson computed  $H_1(\mathcal{J}_g^1; \mathbb{Z}) \cong \Lambda^3 H \oplus \mathcal{B}_2$  using these two homomorphisms and pullback diagrams of groups. A *pullback diagram* for the group

homomorphisms  $\psi_1 : A \rightarrow C$  and  $\psi_2 : B \rightarrow C$  is

$$\begin{array}{ccc}
 & D & \\
 \phi_2 \swarrow & & \searrow \phi_1 \\
 A & & B \\
 \psi_1 \searrow & & \swarrow \psi_2 \\
 & C &
 \end{array} \tag{2.2.4}$$

a commutative square (2.2.4) that is terminal among all such squares. That is, the pullback  $(D, \phi_1, \phi_2)$  is universal with respect to Diagram (2.2.4). For a diagram of groups, the pullback is

$$D \cong \{(a, b) \in A \times B \mid \psi_1(a) = \psi_2(b)\}.$$

$D$  is unique up to canonical isomorphism.

Diagram (2.2.5) (below) is a pullback diagram, from which Johnson in [22] concludes that  $H_1(\mathcal{J}_g^1; \mathbb{Z}) \cong \Lambda^3 H \oplus \mathcal{B}_2$ .

$$\begin{array}{ccc}
 & H_1(\mathcal{J}_g^1, \mathbb{Z}) & \\
 \sigma \swarrow & & \searrow \tau \\
 \mathcal{B}_3 & & \Lambda^3 H_{\mathbb{Z}} \\
 q \searrow & & \swarrow \otimes \mathbb{Z}/2\mathbb{Z} \\
 & \Lambda^3 H_{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z} &
 \end{array} \tag{2.2.5}$$

Let  $T_{\partial}$  be the Dehn-twist about the boundary component in  $\mathcal{J}_g^1$ . In order to compute  $H_1(\mathcal{J}_{g,1}; \mathbb{Z})$  note that  $\mathcal{J}_{g,1} \cong \mathcal{J}_g^1 / \langle T_{\partial} \rangle$ . Define  $a \in \mathcal{B}_2$  as

$$a := \sum_i a_i b_i.$$

In [22], Johnson computes  $\sigma(T_{\partial}) = a$ .

*Proof of Proposition 2.2.5.* We will use two additional pullback diagrams to compute  $H_1(\mathcal{J}_{g,1}; \mathbb{Z}) \cong \Lambda^3 H_{\mathbb{Z}} \oplus \mathcal{B}_2 / \langle a \rangle$ . Define the quotient map

$$f : \mathcal{J}_g^1 \rightarrow \mathcal{J}_g^1 / \langle T_{\partial} \rangle \cong \mathcal{J}_{g,1}.$$

The inverse image of the commutator subgroup of  $\mathcal{J}_{g,1}$  is

$$f^{-1}([\mathcal{J}_{g,1}, \mathcal{J}_{g,1}]) = f^{-1}([\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle / \langle T_{\partial} \rangle) = [\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle.$$

Define the quotient map

$$g : \mathcal{J}_g^1 \rightarrow \frac{(\mathcal{J}_g^1 / \langle T_{\partial} \rangle)}{([\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle) / \langle T_{\partial} \rangle} \cong H_1(\mathcal{J}_{g,1}; \mathbb{Z}).$$

The kernel of  $g$  is exactly  $[\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle$ . Thus, there is an isomorphism

$$g : \mathcal{J}_g^1 / ([\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle) \rightarrow H_1(\mathcal{J}_{g,1}; \mathbb{Z}).$$

Notice that:

$$H_1(\mathcal{J}_g^1; \mathbb{Z}) \rightarrow \frac{(\mathcal{J}_g^1 / [\mathcal{J}_g^1, \mathcal{J}_g^1])}{([\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle) / [\mathcal{J}_g^1, \mathcal{J}_g^1]} \cong \frac{\mathcal{J}_g^1}{([\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle)}.$$

Therefore we have a map  $H_1(\mathcal{J}_g^1; \mathbb{Z}) \rightarrow H_1(\mathcal{J}_{g,1}; \mathbb{Z})$  with kernel  $[\mathcal{J}_g^1, \mathcal{J}_g^1] \cdot \langle T_{\partial} \rangle / [\mathcal{J}_g^1, \mathcal{J}_g^1]$ . We construct the following two pullback diagrams:

$$\begin{array}{ccc}
 & (\langle T_{\partial} \rangle \cdot [\mathcal{J}_g^1, \mathcal{J}_g^1]) / [\mathcal{J}_g^1, \mathcal{J}_g^1] & \\
 \sigma \swarrow & & \searrow \tau \\
 \langle a \rangle & & 1 \\
 & \searrow & \swarrow \\
 & 1 & 
 \end{array} \tag{2.2.6}$$

Taking a quotient of Diagram (2.2.5) by Diagram (2.2.6) results in the following pullback diagram:

$$\begin{array}{ccc}
& H_1(\mathcal{J}_{g,1}, \mathbb{Z}) & \\
\sigma \swarrow & & \searrow \tau \\
B_3/\langle a \rangle & & \Lambda^3 H_{\mathbb{Z}} \\
q \searrow & & \swarrow \otimes \mathbb{Z}/2\mathbb{Z} \\
& \Lambda^3 H_{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z} &
\end{array} \tag{2.2.7}$$

Johnson showed that Diagram (2.2.5) is a pullback diagram in [22]. Diagram (2.2.6) is a pullback diagram because  $\langle a \rangle \cong (\langle T_{\partial} \rangle[\mathcal{J}_g^1, \mathcal{J}_g^1]) / [\mathcal{J}_g^1, \mathcal{J}_g^1] \cong \mathbb{Z}/2\mathbb{Z}$ . Since D3 is a quotient of two pullback diagrams and one terminal homomorphism of Diagram (2.2.5) is surjective, it follows that D3 is also a pullback diagram. Therefore,  $H_1(\mathcal{J}_{g,1}; \mathbb{Z}) \cong \Lambda^3 H \oplus \mathcal{B}_2/\langle a \rangle$ .  $\square$

### 2.2.3 Intersection of $N$ with $\mathcal{J}_2(\Sigma)$

The homomorphism  $\alpha$ , as defined in Section 2, gives the injection  $\text{Mod}(\Sigma_{g,1}) \hookrightarrow \text{Aut}^{\pm}(N)$ . From Section 2, the containment  $\mathcal{J}(\Sigma) \subset \mathcal{J}(N)$  implies  $[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \subset [\mathcal{J}(N), \mathcal{J}(N)]$ . Define  $\bar{\tau}_P$  (respectively,  $\bar{\tau}_N$ ) as the quotient map

$$\bar{\tau}_P : \mathcal{J}(\Sigma) \rightarrow \mathcal{J}(\Sigma) / [\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \cong \Lambda^3 H_{\mathbb{Z}} \oplus \mathcal{B}_2/\langle a \rangle.$$

Since  $[\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \subseteq [\mathcal{J}(N), \mathcal{J}(N)]$  it follows that  $\ker(\bar{\tau}_P) \subset \ker(\bar{\tau}_N)$ . Thus, we can define a homomorphism  $\tilde{\alpha}$  so that the right hand square of (2.2.8) commutes.

$$\begin{array}{ccccccc}
1 & \longrightarrow & [\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] & \longrightarrow & \mathcal{J}(\Sigma) & \xrightarrow{\bar{\tau}_P} & \Lambda^3 H_{\mathbb{Z}} \oplus \mathcal{B}_2/\langle a \rangle \longrightarrow 1 \\
& & \downarrow \alpha & & \downarrow \alpha & & \downarrow \tilde{\alpha} \\
1 & \longrightarrow & [\mathcal{J}(N), \mathcal{J}(N)] & \longrightarrow & \mathcal{J}(N) & \xrightarrow{\bar{\tau}_N} & \Lambda^3 H_{\mathbb{Z}} \oplus \mathcal{B}_2/\langle a \rangle \longrightarrow 1
\end{array} \tag{2.2.8}$$

The fact that  $\tilde{\alpha}$  must map torsion to torsion implies that  $\tilde{\alpha}(\mathcal{B}_2/\langle a \rangle) \subset \mathcal{B}_2/\langle a \rangle$ . Thus,

$$\bar{\tau}_N(\alpha(\mathcal{J}_2(\Sigma))) = \tilde{\alpha}(\bar{\tau}_p(\mathcal{J}_2(\Sigma))) \subset \mathcal{B}_2/\langle a \rangle.$$

This containment implies

$$\alpha(\mathcal{J}_2(\Sigma)) \subset \bar{\tau}_N^{-1}(\mathcal{B}_2/\langle a \rangle) = \mathcal{J}_2(N).$$

Therefore  $\mathcal{J}_2(\Sigma) \subset \mathcal{J}_2(N)$ .

The containment  $\mathcal{J}_2(\Sigma) \subset \mathcal{J}_2(N)$  allows us to deduce the following:

$$[N, N] \subseteq N \cap [\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \subseteq N \cap \mathcal{J}_2(\Sigma) \subseteq N \cap \mathcal{J}_2(N) = [N, N].$$

Therefore,  $N \cap \mathcal{J}_2(\Sigma) = \gamma_2(N)$ .

## 2.2.4 $\mathrm{Sp}_{2g}(\mathbb{Q})$ representation

In this subsection, we will use  $\mathrm{Sp}_{2g}(\mathbb{Q})$  representations to show that  $N \cdot \mathcal{J}_2(\Sigma) = P \cdot \mathcal{J}_2(\Sigma)$ .

$\mathrm{Mod}(\Sigma_{g,1})$  acts on  $\mathcal{J}(\Sigma)$  via conjugation. The kernel of  $\tau$  is exactly the set of elements that act trivially on  $\pi_1/\gamma_3(\pi_1)$ , i.e.  $\ker(\tau) = \mathcal{J}_2(\Sigma)$ . The quotient  $\mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma)$  is the universal torsion-free abelian quotient of  $\mathcal{J}(\Sigma)$ . Thus, the conjugation action of  $\mathcal{J}(\Sigma)$  on  $\mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma)$  is trivial. Therefore, we have a well-defined action of  $\mathrm{Mod}(\Sigma_{g,1})/\mathcal{J} \cong \mathrm{Sp}_{2g}(\mathbb{Z})$  on  $\mathcal{J}/\mathcal{J}_2$ . Similarly,  $\mathrm{Sp}_{2g}(\mathbb{Z})$  has a canonical action on  $\Lambda^3 H_{\mathbb{Z}}$ . The isomorphism

$$\tau : \mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma) \rightarrow \Lambda^3 H_{\mathbb{Z}}$$

is  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -equivariant.

To prove that  $N \cdot \mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma) = P \cdot \mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  we will establish the following bijective correspondence:

$$\left\{ \begin{array}{l} \mathrm{Sp}_{2g}(\mathbb{Q})\text{-irreps in } \Lambda^3 H_{\mathbb{Q}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathrm{Sp}_{2g}(\mathbb{Z})\text{-invariant } \mathbb{Z}\text{-module} \\ \text{direct summands in } \mathcal{J}/\mathcal{J}_2 \end{array} \right\}.$$

We will check that there is exactly one  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant, dimension  $2g$  subspace of  $\Lambda^3 H_{\mathbb{Q}}$ . To conclude, we will show that both  $N\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  and  $P\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  are rank  $2g$  direct summands of  $\mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma)$  invariant under the action of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ .

**Lemma 2.2.6** ((Bijective correspondence)). *There is a bijective correspondence between  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant dimension  $m$   $\mathbb{Q}$ -vector subspaces of  $\Lambda^3 H_{\mathbb{Q}}$  and  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -invariant rank  $m$   $\mathbb{Z}$ -module direct summands of  $\Lambda^3 H_{\mathbb{Z}}$ .*

*Proof of Lemma 2.2.6.* Define the map

$$f : \{\mathrm{Sp}_{2g}(\mathbb{Z})\text{-invariant direct summands of } \Lambda^3 H_{\mathbb{Z}}\} \rightarrow \{\text{subspaces of } \Lambda^3 H_{\mathbb{Q}}\}$$

via

$$f(V) = V \otimes \mathbb{Q}.$$

To establish the bijective correspondence, we need to check that the image of  $f$  lies in  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant subspaces of  $\Lambda^3 H_{\mathbb{Q}}$ .

Fix a basis of  $\mathbb{Q}^{2g}$  so that  $\mathrm{Sp}_{2g}(\mathbb{Q}) < \mathrm{GL}_{2g}(\mathbb{Z})$  is the subgroup that fixes the symplectic form  $\left( \begin{array}{c|c} 0 & I_{g \times g} \\ \hline -I_{g \times g} & 0 \end{array} \right)$ . The group  $\mathrm{Sp}_{2g}(\mathbb{Q})$  is generated by matrices of the following forms, where  $\lambda$  varies in  $\mathbb{Q}$ , and  $e_{ij}$  is the  $g \times g$  matrix with 1 in the  $i, j$  entry and 0 elsewhere (see e.g. [30, Sect.2.2]):

$$\begin{aligned} & \left( \begin{array}{c|c} I_{g \times g} & \lambda e_{ii} \\ \hline & I_{g \times g} \end{array} \right), \quad \left( \begin{array}{c|c} I_{g \times g} & \\ \hline \lambda e_{ii} & I_{g \times g} \end{array} \right), \quad \left( \begin{array}{c|c} I_{g \times g} & \\ \hline \lambda(e_{ij} + e_{ji}) & I_{g \times g} \end{array} \right), \\ & \left( \begin{array}{c|c} I_{g \times g} & \lambda(e_{ij} + e_{ji}) \\ \hline & I_{g \times g} \end{array} \right), \quad \left( \begin{array}{c|c} I_{g \times g} + \lambda e_{ij} & \\ \hline & I_{g \times g} - \lambda e_{ji} \end{array} \right). \end{aligned} \quad (2.2.9)$$

Let  $V$  be an  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -invariant direct summand of  $\Lambda^3 H_{\mathbb{Z}}$ , and let  $v \in V$ . Let  $A$  be any of the generators of  $\mathrm{Sp}_{2g}(\mathbb{Q})$  given in (2.2.9) and let  $A_{\mathbb{Z}}$  be the matrix  $A$  with  $\lambda = 1$ . Notice that  $A_{\mathbb{Z}} \in \mathrm{Sp}_{2g}(\mathbb{Z})$  and  $A = \lambda A_{\mathbb{Z}} - (\lambda - 1)I_{2g \times 2g}$ . Therefore, for any  $q \in \mathbb{Q}$

$$Aqv = qAv = q((\lambda)(A_{\mathbb{Z}}v) - \lambda v + v).$$

Since  $V$  is an  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -invariant direct summand,  $q((\lambda)(A_{\mathbb{Z}}v) - \lambda v + v) \in V \otimes \mathbb{Q}$ . Therefore  $V \otimes \mathbb{Q}$  is an  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant subspace.

Let  $W$  be an  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant subspace of  $\Lambda^3 H_{\mathbb{Q}}$ . Let  $W_{\mathbb{Z}}$  be the  $\mathbb{Z}$ -module consisting of all integral points of  $W$ . Define the map

$$g: \{\mathrm{Sp}_{2g}(\mathbb{Q})\text{-invariant subspaces of } \Lambda^3 H_{\mathbb{Q}}\} \rightarrow \{\mathrm{Sp}_{2g}(\mathbb{Z})\text{-invariant direct summands of } \Lambda^3 H_{\mathbb{Z}}\}$$

via

$$g(W) = W_{\mathbb{Z}}.$$

The composition  $f \circ g$  is the identity because  $W_{\mathbb{Z}} \otimes \mathbb{Q} = W$ .

On the other hand, consider  $v \in g \circ f(V) = (V \otimes \mathbb{Q})_{\mathbb{Z}}$ . Decompose  $\Lambda^3 H_{\mathbb{Z}} = V \oplus V^{\perp}$ . If  $v \notin V$  then the projection of  $v$  onto  $V^{\perp} \neq 0$ . Let  $p_{\perp}(v)$  be the projection onto  $V^{\perp}$ . Because  $v \in V \otimes \mathbb{Q}$ , it follows that  $nv \in V$  for some large enough  $n \in \mathbb{Z}$ . However, that implies  $p_{\perp}(nv) = 0$ , or equivalently  $n(p_{\perp}(v)) = 0$ , a contradiction.

Therefore,  $g$  is a bijection and the correspondence is established. □

The representation  $\Lambda^3 H_{\mathbb{Q}}$  decomposes as an  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -representation in the following way (see, e.g. [6, Sect.3]):

$$\Lambda^3 H_{\mathbb{Q}} \cong H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}}/H_{\mathbb{Q}}.$$

Note that  $\dim_{\mathbb{Q}}(H_{\mathbb{Q}}) = 2g$  and  $\dim_{\mathbb{Q}}(\Lambda^3 H_{\mathbb{Q}}/H_{\mathbb{Q}}) = \binom{2g}{3} - 2g$ . Thus, for genus  $g \geq 3$ , there is exactly one  $\mathrm{Sp}_{2g}(\mathbb{Q})$ -invariant, dimension  $2g$  subspace of  $\Lambda^3 H_{\mathbb{Q}}$ .

From Section 2.2.3 we have  $N \cap \mathcal{J}_2(\Sigma) = [N, N]$ . Thus,

$$N\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma) \cong N/(N \cap \mathcal{J}_2(\Sigma)) \cong N/[N, N] \cong \mathbb{Z}^{2g}.$$

Therefore,  $N\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  is a  $\mathbb{Z}$ -module of rank  $2g$ . Likewise,  $P\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  is a  $\mathbb{Z}$ -module of rank  $2g$ .

To see that the submodule  $P\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  is a direct summand of  $\mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma)$ , it suffices to check that the generators of  $P$  surject onto a partial basis of  $\Lambda^3 H_{\mathbb{Z}}$  under the Johnson homomorphism. A *partial basis* is any set of linearly independent vectors that can be completed to a  $\mathbb{Z}$ -basis.

Consider a fixed generating set for  $P$  and a corresponding basis for  $H_{\mathbb{Z}}$ , given by  $\{a_1, b_1, \dots, a_g, b_g\}$ . Then,  $\tau(a_i) = \theta \wedge a_i$  where  $\theta = \sum_i a_i \wedge b_i$ . For details of this computation, see Johnson's work in [18]. The image of the standard generators of  $P$  gives a partial basis of  $\Lambda^3 H_{\mathbb{Z}}$ . Therefore the image of  $P$  is a direct summand in  $\Lambda^3 H_{\mathbb{Z}}$ .

It remains to be seen that  $N \cdot \mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma)$  is a direct summand. Because  $[N, N] \subset [\mathcal{J}(\Sigma), \mathcal{J}(\Sigma)] \subset [\mathcal{J}(N), \mathcal{J}(N)]$ , the following diagram given by restrictions of quotient maps commutes:

$$\begin{array}{ccc}
 N/[N, N] & \xrightarrow{k} & \mathcal{J}(\Sigma)/\mathcal{J}_2(\Sigma) \\
 & \searrow j & \downarrow \\
 & & \mathcal{J}(N)/\mathcal{J}_2(N)
 \end{array} \tag{2.2.10}$$

The image  $j(N/[N, N]) = N\mathcal{J}_2(N)/\mathcal{J}_2(N) \cong \mathbb{Z}^{2g}$  is a direct summand in  $\mathcal{J}(N)/\mathcal{J}_2(N)$ . Further,  $k(N/[N, N]) = N\mathcal{J}_2(\Sigma)/\mathcal{J}_2(\Sigma) \cong \mathbb{Z}^{2g}$ .

**Lemma 2.2.7.** *Suppose that the diagram below commutes*

$$\begin{array}{ccc}
 \mathbb{Z}^{2g} & \xrightarrow{L_1} & \mathbb{Z}^{2g} \oplus \mathbb{Z}^{n-2g} \\
 & \searrow L_2 & \downarrow L_3 \\
 & & \mathbb{Z}^{2g} \oplus \mathbb{Z}^{n-2g}
 \end{array}$$

and the maps  $L_i$  are linear. If  $L_2(\mathbb{Z}^{2g}) \cong \mathbb{Z}^{2g}$  is a direct summand in  $\mathbb{Z}^{2g} \oplus \mathbb{Z}^{n-2g}$ , then so is  $L_1(\mathbb{Z}^{2g})$ .

*Proof of Lemma 2.2.7.* Because  $L_2(\mathbb{Z}^{2g})$  is a direct summand in  $\mathbb{Z}^n$ , there exists a retract  $R : \mathbb{Z}^n \rightarrow \mathbb{Z}^{2g}$  of  $L_2$  with  $R \circ L_2 = Id_{\mathbb{Z}^{2g}}$ . Further, since  $L_2 = L_3 \circ L_1$ , the homomorphism  $R \circ L_3 : \mathbb{Z}^n \rightarrow \mathbb{Z}^{2g}$  is a retract of  $L_1$ . That is  $R \circ L_3 \circ L_1 = Id_{\mathbb{Z}^{2g}}$ . Consider  $L_1 \circ R \circ L_3 : \mathbb{Z}^n \rightarrow \mathbb{Z}^{2g}$ . Note that

$$(L_1 \circ R \circ L_3)^2 = L_1 \circ (R \circ L_3 \circ L_1) \circ R \circ L_3 = L_1 \circ (Id_{\mathbb{Z}^{2g}}) \circ R \circ L_3 = L_1 \circ R \circ L_3.$$



It follows that  $L_1 \circ R \circ L_3$  is a projection with image  $L_1(\mathbb{Z}^{2g})$ . Thus,  $L_1(\mathbb{Z}^{2g})$  is a direct summand.  $\square$

Applying Lemma 2.2.7 to commutative diagram (2.2.10), it follows that  $k(N/[N, N]) = NJ_2(\Sigma)/J_2(\Sigma) \cong \mathbb{Z}^{2g}$  is a direct summand in  $J(\Sigma)/J_2(\Sigma)$ .

Because  $N, P$ , and  $J_2(\Sigma)$  are normal in  $\text{Mod}(\Sigma_{g,1})$ , both of the above  $\mathbb{Z}$ -module direct summands are invariant under the action of  $\text{Sp}_{2g}(\mathbb{Z})$ . There is exactly one rank  $2g$  direct summand  $\mathbb{Z}$ -submodule of  $J(\Sigma)/J_2(\Sigma)$ . Thus,  $NJ_2(\Sigma)/J_2(\Sigma) = PJ_2(\Sigma)/J_2(\Sigma)$ . Equivalently,  $NJ_2(\Sigma) = PJ_2(\Sigma)$ .

### 2.3 Commutator containment: $[N, N] \subset [P, P]$ .

From Section 3, we have the containment  $N \subset PJ_2(\Sigma)$ . Furthermore, since  $[N, N] \subset J_2(\Sigma)$ , it is also true that  $[N, N] \subset PJ_2(\Sigma)$ . In this section, we will use an inductive argument to confirm that  $[N, N] \subset PJ_k(\Sigma)$  for all  $k$ . Grossman's Property A Lemma (see Lemma 2.3.1) implies that for any surface group  $\pi_1(\Sigma_g)$ , if  $q \in \text{Aut}(\pi_1(\Sigma_g))$  preserves conjugacy classes in  $\pi_1(\Sigma_g)$ , then  $q \in \pi_1(\Sigma_g)$ . Using Grossman together with the conjugacy  $p$ -separability of surface groups, we will show that  $\cap_k PJ_k = P$ . This will prove  $[N, N] \subset P \cap J_2(\Sigma) = [P, P]$ .

We have already established the following facts:

- i. The Johnson filtration is a central series [2].
- ii.  $N \subset J(\Sigma)$  (Sect. 2).
- iii.  $N \subset PJ_2(\Sigma)$  (from Sect. 3).
- iv.  $N \cap J_k(N) = \gamma_k(N)$  (Prop 2.2.1).
- v.  $J_2(\Sigma) \subset J_2(N)$  (from Sect. 3).
- vi.  $[J_2(\Sigma), N] \subset N$  (because  $N$  is normal in  $\text{Mod}(\Sigma_{g,1})$ ).
- vii.  $[J_2(N), N] \subset J_3(N)$  (because  $N \subset J$  and the Johnson filtration is a central series).

We will establish an eighth fact:

viii.  $[PG, N] \subset P[G, N]$  for any  $G \triangleleft \text{Mod}(\Sigma_{g,1})$  (below).

To prove (viii), let  $G \triangleleft \text{Mod}(\Sigma_{g,1})$ . Let  $g \in G$ ,  $p \in P$ , and  $n \in N$  be given. Then

$$[pg, n] = pgn g^{-1} p^{-1} n^{-1} = pgn p^{-1} n^{-1} p g^{-1} p^{-1} (p g p^{-1} n (p g p^{-1})^{-1} n^{-1}).$$

However,  $P$  normal in  $\text{Mod}(\Sigma_{g,1})$  implies that

$$p g n p^{-1} n^{-1} p g^{-1} p^{-1} \in P.$$

Furthermore, because  $G$  is normal in  $\text{Mod}(\Sigma_{g,1})$  it follows that

$$((p g p^{-1}) n (p g p^{-1})^{-1} n^{-1}) \in [G, N].$$

Therefore  $[PG, N] \subset P[G, N]$  for any  $G \triangleleft \text{Mod}(\Sigma_{g,1})$ . In particular,  $[PJ_2(\Sigma), N] \subset P[J_2(\Sigma), N]$ .

With reference to the above list of facts,

$$[N, N] \stackrel{(iii.)}{\subset} [PJ_2(\Sigma), N] \stackrel{(viii.)}{\subset} P[J_2(\Sigma), N] \stackrel{(v.)}{\subset} P[J_2(N), N] \stackrel{(i.), (ii.)}{\subset} P[J_3(N) \cap N] \stackrel{(iv.)}{\subset} P\gamma_3(N).$$

Therefore,  $[N, N] = \gamma_2(N) \subset P\gamma_3(N)$ .

We will induct on  $m$  to check that  $\gamma_2(N) \subset P\gamma_m(N)$  for all  $m > 0$ . Let  $M \in \mathbb{N}$  with  $M \geq 3$ . Suppose for all  $m \leq M$  we have  $\gamma_2(N) \subset P\gamma_m(N)$ . It follows that

$$[N, N] = \gamma_2(N) \subset P\gamma_3(N) = P[\gamma_2(N), N] \subset P[P\gamma_M(N), N] \subset P^2[\gamma_M(N), N] \subset P\gamma_{M+1}(N).$$

Therefore,  $[N, N] \subset P\gamma_m(N)$  for all  $m \geq 1$ .

We will use a second inductive argument to show that  $\gamma_k(N) \subset J_k(\Sigma)$  for all  $k \geq 2$ . For the base case, note that

$$[N, N] \subset [J(\Sigma), J(\Sigma)] \subset J_2(\Sigma).$$

Assume as inductive hypothesis that  $\gamma_k(N) \subset \mathcal{J}(\Sigma)_k$  for all  $k < K$ . Then

$$\gamma_K(N) = [\gamma_{K-1}(N), N] \subseteq [\mathcal{J}_{K-1}(\Sigma), \mathcal{J}(\Sigma)] \subseteq \mathcal{J}_K(\Sigma).$$

The above containment implies that

$$[N, N] = \gamma_2(N) \subseteq \cap_k P\gamma_k(N) \subseteq \cap_k P\mathcal{J}(\Sigma)_k.$$

In order to confirm that  $[N, N] \subset P$ , it remains to be shown that  $\cap_k P\mathcal{J}(\Sigma)_k = P$ . We will use the following Lemma due to Grossman:

**Lemma 2.3.1** ((Grossman's Property A [12])). *Let  $P$  be a surface group of genus  $g \geq 1$ . Let  $q \in \text{Aut}(P)$ . If  $q$  preserves conjugacy classes in  $P$ , then  $q \in P$ .*

To apply Lemma 2.3.1, choose  $q \in \cap_k P\mathcal{J}_k$  and  $x \in P$ . Since  $q \in P\mathcal{J}_k$  for all  $k \geq 1$ , we can find  $u_k \in P$  and  $i_k \in \mathcal{J}_k(\Sigma)$  such that  $q = u_k i_k$ . However, because  $i_k \in \mathcal{J}_k$  it follows that  $i_k x i_k^{-1} x^{-1} \in \gamma_{k+1}(P)$ . This can be rewritten in terms of left cosets as

$$i_k x i_k^{-1} \gamma_{k+1}(P) = x \gamma_{k+1}(P).$$

Conjugating by  $u_k$  gives

$$u_k i_k x i_k^{-1} u_k^{-1} \gamma_{k+1}(P) = u_k x u_k^{-1} \gamma_{k+1}(P).$$

That is,  $q x q^{-1}$  is conjugate to  $x$  in  $P/\gamma_{k+1}(P)$  for all  $k \geq 1$ .

Finite  $p$ -groups are nilpotent. Furthermore, any homomorphism  $\phi : P \rightarrow H$  where  $H$  is  $i$ -step nilpotent factors through  $P/\gamma_{i+1}(P)$ . Thus, any homomorphism  $\phi : P \rightarrow H$  where  $H$  is a  $p$ -group factors through  $P/\gamma_k(P)$  for some  $k$ .

Suppose  $\phi : P \rightarrow H$  gives a homomorphism to some  $p$ -group  $H$ . Because  $q x q^{-1}$  is conjugate to  $x$  in  $P/\gamma_k(P)$  for all  $k \geq 1$ , it must be that  $\phi(q x q^{-1})$  is conjugate to  $\phi(x)$  in  $H$ . Because  $P$  is conjugacy  $p$ -separable (see [31]),  $q x q^{-1}$  is conjugate to  $x$  in  $P$ . Applying Lemma 2.3.1, it follows that  $q \in P$ . Therefore,  $\cap_k P\mathcal{J}_k(\Sigma) = P$ .

We have established for all  $k \geq 1$ ,

$$\gamma_2(N) \subset P\gamma_k(N) \subset P\mathcal{J}_k(\Sigma).$$

That is,  $\gamma_2(N) \subset \bigcap_k P\mathcal{J}_k(\Sigma) = P$ . From Section 3, we have the containment  $\gamma_2(N) \subset \mathcal{J}_2(\Sigma)$ . Thus,  $\gamma_2(N) \subset P \cap \mathcal{J}_2(\Sigma) = \gamma_2(P)$ . This concludes the first main goal in the proof of Theorem 1.0.1:

$$\gamma_2(N) \subseteq \gamma_2(P) \subseteq \mathcal{J}_2(\Sigma) \subseteq \mathcal{J}(\Sigma) \subseteq \mathcal{J}(N).$$

## CHAPTER 3

### CHARACTERIZING $P$

In this chapter we will characterize  $P$  in terms of  $\mathcal{J}(\Sigma)$  and  $\gamma_2(P)$ . We will provide two distinct proofs. In each case, we will show that any  $\phi \in \mathcal{J}(\Sigma)$  satisfying certain conditions must fix a filling set of curves up to conjugation. Then, we will apply the Alexander method to show that  $\phi$  must be isotopic to the identity in  $\text{Mod}(\Sigma_g)$ .

**Proposition 3.0.2** ((Characterization of  $P$ )). *For  $g \geq 3$ ,*

$$P(\Sigma_g) = \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P(\Sigma_g))\}.$$

### 3.1 Proof of proposition 3.0.2 by bounding pair maps

The proof of Proposition 3.0.2 was greatly simplified by Chen Lei.

*Proof.* Because  $P \triangleleft \text{Mod}(\Sigma_{g,1})$  and  $P \subset \mathcal{J}(\Sigma)$  it follows that for any  $p \in P$

$$[p, \mathcal{J}(\Sigma)] \subset (\mathcal{J}_2(\Sigma) \cap P) = \gamma_2(P).$$

Therefore,

$$P \subseteq \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}.$$

For the opposite containment, let  $\phi \in \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}$ .

Our goal is to apply the Alexander method by demonstrating that  $\phi x_i \phi^{-1}$  is isotopic to each  $x_i$  in some filling set  $\{x_1, \dots, x_k\}$  of simple closed curves. This would force  $\phi$  to be isotopic to the identity in  $\text{Mod}(\Sigma_g)$ . That is,  $\phi \in P$ .

Take any bounding pair map,  $T_a T_b^{-1}$  where  $a$  and  $b$  are disjoint, homologous, non-isotopic simple closed curves. Because  $a$  and  $b$  are homologous, it follows that  $T_a T_b^{-1}$  acts trivially on  $H_1(\Sigma_{g,1})$ . That is,  $T_a T_b^{-1} \in \mathcal{J}(\Sigma)$ . By assumption,  $\phi T_a T_b^{-1} \phi^{-1} (T_a T_b^{-1})^{-1} \in P$ . Mapping into  $\text{Mod}(\Sigma_g)$  via the forgetful map, we obtain,  $F(\phi T_a T_b^{-1} \phi^{-1} (T_a T_b^{-1})^{-1}) = 1$ . That is

$$\phi T_a T_b^{-1} \phi^{-1} (T_a T_b^{-1})^{-1} = 1 \text{ in } \text{Mod}(\Sigma_g).$$

Therefore

$$\begin{aligned}
\phi T_a T_b^{-1} \phi^{-1} (T_a T_b^{-1})^{-1} &= 1 \\
\phi T_{(a)} \phi^{-1} \phi T_{(b)}^{-1} \phi^{-1} (T_a T_b^{-1})^{-1} &= 1 \\
T_{\phi(a)} T_{\phi(b)}^{-1} (T_a T_b^{-1})^{-1} &= 1 \\
T_{\phi(a)} T_{\phi(b)}^{-1} &= T_a T_b^{-1}.
\end{aligned}$$

Bounding pair maps commute if and only if they have the same canonical reduction system. Thus,  $T_{\phi(a)} T_{\phi(b)}^{-1}$  and  $T_a T_b^{-1}$  have the same canonical reduction system, namely  $\{a, b\}$ . As such, the curves  $\phi(a)$  and  $\phi(b)$  are isotopic to  $a$  and  $b$ , respectively in  $\text{Mod}(\Sigma_g)$ .

For any non-separating simple closed curve  $c$  there is a bounding pair map  $T_c T_{c'}$  where  $c$  and  $c'$  are homologous, disjoint, and non-isotopic. It follows that  $\phi c$  is isotopic in  $\text{Mod}(\Sigma_g)$  to  $c$  for any non-separating simple closed curve  $c$ .

In particular, for a filling set of simple closed curves,  $\{x_1, \dots, x_k\}$ , we have  $\phi x_i \phi^{-1}$  is isotopic to  $x_i$  for each  $i$ . By the Alexander method, the map  $\phi$  must be trivial in  $\text{Out}(\pi_1(\Sigma_g))$ . That is,  $\phi \in P$ . Proposition 3.0.2 follows.  $\square$

### 3.2 Proof of Proposition 3.0.2 using amalgamated product normal form

As above, we wish to demonstrate the containment

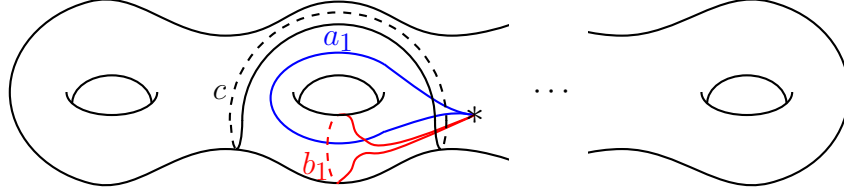
$$P \supseteq \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}.$$

Suppose that

$$\beta \in \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}.$$

Let  $\{a_1, b_1, \dots, a_g, b_g\}$  be a standard set of generators for  $\pi_1(\Sigma_g)$ . If we can show that  $\beta a_1 \beta^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to  $a_1$ , then it follows that  $\beta \phi(a_1) \beta^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to  $\phi(a_1)$  for any  $\phi \in \text{Mod}(\Sigma_{g,1})$ . By the Alexander method, if  $\beta x_i \beta^{-1}$  is conjugate to  $x_i$  for

Figure 3.1: The Dehn twist  $T_c$



The simple closed curve  $c$  separates  $\Sigma_g$  into a surface of genus 1 and a surface of genus  $g - 1$ . The Dehn twist  $T_c$  acts by conjugation on  $a_1$  and  $b_1$ . We can always choose  $c$  to separate the marked point from the curves  $a_1$  and  $b_1$ .

a filling set  $\{x_i\}$  of simple closed curves, then  $\beta$  is isotopic in  $\text{Mod}(\Sigma_g)$  to the identity. That is,  $\beta \in P$ . Thus, it is sufficient to demonstrate that  $\beta a_1 \beta^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to  $a_1$ .

Let  $c$  be the separating simple closed curve given by a representative of the homotopy class  $[a_1, b_1]$  as shown in Figure 3.1. Let  $T_c$  be the Dehn twist about the curve  $c$ .

We break the proof of Proposition 3.0.2 into the following two steps:

3.0.2.A. Let  $x$  be a simple closed curve in  $\Sigma_g$ . If  $T_c(x)$  is conjugate to  $x$  in  $\pi_1(\Sigma_g)$ , then  $x$  is freely homotopic to a curve representing an element of the subgroup generated by  $\{a_1, b_1\}$  or to a curve representing an element of the subgroup generated by  $\{a_2, b_2, \dots, a_g, b_g\}$ .

3.0.2.B.  $\beta x \beta^{-1}$  is conjugate to  $x$  in  $\pi_1(\Sigma_g)$  for every simple closed curve  $x$ .

### 3.2.1 Cyclic reduction of normal form in amalgamated products

Let  $\delta = [a_1, b_1] = \left(\prod_{i>1} [a_i, b_i]\right)^{-1}$ . The homeomorphism  $T_c$  acts on  $\pi_1(\Sigma_g)$  as follows (see e.g. [10, Sect 6.6.2]):

$$\begin{aligned} T_c(a_1) &= \delta a_1 \delta^{-1} \\ T_c(b_1) &= \delta b_1 \delta^{-1} \\ T_c(x) &= x \text{ for any } x \in \langle a_2, b_2, \dots, a_g, b_g \rangle. \end{aligned}$$

Let  $A = \langle a_1, b_1 \rangle$ ,  $B = \langle a_2, b_2 \dots, a_g, b_g \rangle$ , and  $C = \langle [a_1, b_1] \rangle = \langle \delta \rangle$ . We can write  $\pi_1(\Sigma_g)$  as an amalgamated product:

$$\pi_1(\Sigma_g) \cong A *_C B.$$

There is a unique normal form for elements of an amalgamated product (see [33]) defined as follows. For each left coset in  $A/C$  (respectively, in  $B/C$ ) choose a representative  $x_i$  (respectively,  $y_i$ ) subject to the constraint that the identity coset be represented by 1. Then, any element of  $\pi_1(\Sigma_g)$  can be written uniquely as a reduced word in the form  $x_{i_1} y_{i_1} \dots x_{i_k} y_{i_k} \delta^\mu$  where  $x_{i_1}$  and  $y_{i_k}$  may be the identity coset, but all other coset representatives are nonidentity elements. Intuitively, elements of  $Z$  can be “pushed to the right.”

**Definition 3.2.1.** Let  $w \in \pi_1(\Sigma_g) \cong A *_C B$  be given in normal form as  $w = x_1 y_1 \dots x_k y_k \delta^\mu$ , where  $x_i \in A$  with  $x_i \neq 1$  for  $i \geq 2$  and  $y_i \in B$  with  $y_i \neq 1$  for  $i \leq k - 1$ . The *normal form length*  $l(w)$  is the number of  $A$ - $B$  pairs of coset representatives in the reduced word. That is,  $l(w) = l(x_1 y_1 \dots x_k y_k \delta^\mu) = k$ .

We will establish the following lemma concerning normal form length of concatenated words:

**Lemma 3.2.2.** Let  $\pi_1(\Sigma_g) \cong A *_C B$  as above. Let  $q, q' \in \pi_1(\Sigma_g)$  be given in normal form as  $q = x_1 y_1 \dots x_k y_k \delta^\mu$  and  $q' = z_1 w_1 \dots z_l w_l \delta^\nu$  where  $x_i, z_i \in A$  and  $y_i, w_i \in B$ . Let  $a \in A$  and  $b \in B$ .

i.  $l(w) = l(\delta w \delta^{-1})$  for all  $w \in \pi_1(\Sigma_g)$ .

ii.  $l(aq) = l(q)$ .

iii.  $l(qb) = l(q)$ .

iv.  $l(q) - 1 \leq l(qa) \leq l(q) + 1$ .

v.  $l(q) - 1 \leq l(bq) \leq l(q) + 1$ .

vi. If  $z_1, y_k \neq 1$  then

$$l(qq') = l(q) + l(q').$$

vii. If  $z_1 = y_k = 1$  then

$$l(qq') = l(q) + l(q') - 1.$$



*Proof of Lemma 3.2.2.*

i. Consider  $q \in \pi_1(\Sigma_g)$  as above. Take the conjugate by  $\delta$ :

$$\delta q \delta^{-1} = \delta x_1 y_1 \dots x_k y_k \delta^\mu \delta^{-1}.$$

This conjugate can be put in normal form by moving  $\delta$  to the right and relabeling cosets. That is

$$l(\delta p \delta^{-1}) = l(x'_1 y'_1 \dots x'_k y'_k \delta^{\mu'})$$

for relabeled coests  $x'_i, y'_i$  and some power  $\mu'$ . We need to show  $x'_i = 1$  (respectively,  $y'_i = 1$ ) if and only if  $x_i = 1$  (respectively,  $y_i = 1$ ) for all  $i$ .

If  $x_i = 1$  then  $\delta^r x_i = \delta^r \in \langle \delta \rangle$  which implies  $x'_i = 1$ .

Conversely, if  $x'_i = 1$  then  $\delta^r x_i = \delta^s$  for some integers  $r$  and  $s$ . This implies that  $x_i = \delta^{r-s}$ . Because the identity coset must be represented by 1, it follows that  $x_i = 1$ . An identical argument holds to show  $y_i = 1$  if and only if  $y'_i = 1$ . Therefore,  $l(q) = l(\delta q \delta^{-1})$ .

ii. & iii. Take the product:

$$aq = ax_1 y_1 \dots x_k y_k \delta^\mu.$$

In this product, either  $ax_1 \notin C$  or  $ax_1 \in C$ . If  $ax_1 \notin C$  then  $l(aq) = l(q)$ . Otherwise, if  $ax_1 \in C$ , it still follows that  $l(q) = l(aq)$  because  $y_1 \neq 1$ .

A similar argument shows  $l(qb) = l(q)$ .

iv. & v. Consider the product:

$$qa = x_1 y_1 \dots x_k y_k \delta^\mu a.$$

We can restrict attention to the last 3 coests:

$$x_k y_k \delta^\mu a = x_k y_k a' \delta^{\mu'}$$

where  $a'$  is the relabeled coset representative. Assume  $a' \neq 1$  (otherwise length is unchanged). If  $y_k \neq 1$ , then because  $y_k \in B$  and  $a' \in A$  it follows that:

$$l(x_k y_k a' \delta^{\mu'}) = l(x_k y_k) + 1.$$

In this case,  $l(qa) = l(q) + 1$ .

On the other hand, if  $y_k = 1$  then

$$l(x_k y_k a' \delta^{\mu'}) = l(x_k a' \delta^{\mu'}).$$

Either  $x_k a' \in C$  or  $x_k a' \notin C$ . If  $x_k a' \in C$ , then  $l(qa) = l(q) - 1$ . Otherwise, if  $x_k a' \notin C$ , then  $l(qa) = l(q)$ .

A similar argument shows  $len(q) - 1 \leq len(yq) \leq len(q) + 1$ .

vi. Consider the product  $qq'$  where  $y_k \neq 1$  and  $z_1 \neq 1$ . We have

$$qq' = x_1 y_1 \dots x_k y_k \delta^\mu z_1 w_1 \dots z_l w_l \delta^\nu.$$

This is not necessarily in normal form. The normal form length of  $qq'$  depends on whether or not there is cancellation between adjacent cosets. Because  $y_k \in B$  and  $z_1 \in A$ , if  $y_k \neq 1$  and  $z_1 \neq 1$ , then there is no cancellation between cosets and  $l(qq') = l(q) + l(q')$ .

vii. Let  $y_k = z_1 = 1$ . Then

$$qq' = x_1 y_1 \dots x_k \delta^\mu w_1 z_2 \dots z_l w_l \delta^\nu.$$

Because  $x_k \in A$  and  $w_1 \in B$ , there is no cancellation between adjacent cosets in  $A$  and  $B$ . The number of  $A$ - $B$  coset pairs is  $k + l - 1$ . That is  $\text{len}(qq') = \text{len}(q) + \text{len}(q') - 1$ .

□

**Definition 3.2.3.** A word  $w \in \pi_1(\Sigma_g) = A *_C B$  is *cyclically reduced* if  $l(w) \leq l(pwp^{-1})$  for all  $p \in \pi_1(\Sigma_g)$ .

As in a free group, any word  $w \in \pi_1(\Sigma_g)$  is conjugate to a cyclically reduced word. The process of cyclic reduction terminates in one of the following states, see [32]:

1.  $w' = x_1 y_1 \dots x_k y_k \delta^\mu$  where  $x_1, y_k \neq 1$ .
2.  $w' = \delta^\nu$  for some  $\nu$ .
3.  $w' = x\delta^\nu$  or  $w' = y\delta^\nu$  for some  $x \in A$ , respectively,  $y \in B$  and some power  $\nu$ .

Assume  $T_c(w) = pwp^{-1}$  for some  $p$ . Then for any  $q \in \pi_1(\Sigma_g)$  we have the following:

$$T_c(q^{-1}wq) = (T_c(q^{-1})pq) q^{-1}wy_1 (q^{-1}p^{-1}T_c(q) = p'(q^{-1}wq)p'^{-1}$$

where  $p' = T_c(q^{-1})pq$ . That is, if  $T_c(w)$  is conjugate to  $w$  then  $T_c(q^{-1}wq)$  is conjugate to  $q^{-1}wq$ . Thus,  $w$  may be considered cyclically reduced. If  $w$  cyclically reduces to either state 2 or state 3, then  $w$  is conjugate to either an element of  $A$  or an element of  $B$ , as desired.

It remains to show that if  $T_c(w) = pwp^{-1}$  for some  $p$  in  $\pi_1(\Sigma_g)$  then  $w$  does not cyclically reduce to state 1. We will argue by contradiction, first showing that length preserving conjugates of  $w$  have a particular form (cyclic permutations up to conjugation by  $\delta$ ). Then, we will demonstrate that  $T_c(w)$  does not take such a form.

Let  $w \in \pi_1(\Sigma_g) \cong A *_C B$  be written in normal form as  $w = x_1 y_1 \dots x_k y_k \delta^\mu$ . Make the

following definition:

$$\begin{aligned}
\sigma_0(w) &:= w = x_1y_1 \dots x_ky_k\delta^\mu. \\
\sigma_l(w) &:= x_{l+1}y_{l+1} \dots x_ky_k\delta^\mu x_1y_1 \dots x_ly_l \quad \text{for } l < k. \\
\sigma_k(w) &:= x_1y_1 \dots x_ky_k\delta^\mu = \sigma_0(w). \\
\sigma_{nk+i}(w) &:= \sigma_i(w) \quad \text{for } n \geq 0 \text{ and } i < k.
\end{aligned}$$

**Lemma 3.2.4.** *Let  $w \in \pi_1(\Sigma_g) \cong A *_C B$  be cyclically reduced to state 1. Let  $p \in \pi_1(\Sigma_g)$ . If  $l(pwp^{-1}) = l(w)$ , then  $pwp^{-1} = \delta^\nu \sigma_i(w) \delta^{-\nu}$  for some  $i$  and some power  $\nu$ .*

*Proof of Lemma 3.2.4.* We will induct on the length of  $p$ .

Base case: Let  $p \in \pi_1(\Sigma_g)$ , and assume  $l(p) = 1$ . Either

$$p \in A \text{ (analogously } p \in B)$$

$$\text{or } p = xy\delta^r \text{ for some } x \in A, y \in B, \text{ and } x, y \neq 1$$

If  $p \in A$  and  $p \notin C$ , then Lemma 3.2.2. ii. and iv. implies

$$l(pwp^{-1}) = l(wp^{-1}) = l(w) + 1.$$

Therefore if  $l(pwp^{-1}) = l(w)$ , then  $p \in C$ . Thus,  $pwp^{-1} = \delta^r \sigma_0(w) \delta^{-r}$  for some power  $r$ .

The same holds for  $p \in B$ .

Otherwise, if  $p = xy\delta^r$  for some  $x \in A, y \in B$ , and  $x, y \neq 1$ , then

$$l(pwp^{-1}) = l((xy\delta^r)x_1y_1 \dots x_ky_k\delta^\mu(\delta^{-r}y^{-1}x^{-1})).$$

Consider first  $pw = xy\delta^r x_1y_1 \dots x_ky_k\delta^\mu$ . Since  $y \in B$  but  $x_1 \in A$ , by Lemma 3.2.2. vii. it follows that:

$$l(pw) = l(w) + 1.$$

Restricting attention to  $pw\delta^{-r}y^{-1}$ , notice that you can combine the rightmost two cosets, because  $y_k \in B$  and  $y \in B$ . Lemma 3.2.2. iii. implies

$$l(pwy^{-1}) = l(pw).$$

It is possible that  $x_k y_k \delta^{\mu-r} y^{-1} x^{-1} \in C$  as in Lemma 3.2.2. iv. and it follows that:

$$\begin{aligned}
l(pwp^{-1}) &= l(pw\delta^{-r}y^{-1}x^{-1}) \\
&= l(w\delta^{-r}y^{-1}x^{-1}) + 1 \\
&\geq l(w) - 1 + 1 \\
&= l(w).
\end{aligned}$$

Equality holds exactly when  $x_k y_k \delta^{\mu-r} y^{-1} x^{-1} \in C$ . That is,

$$x_k y_k \delta^{\mu} \delta^{-r} y^{-1} x^{-1} = \delta^{\nu}$$

for some power  $\nu$ . Therefore, equality implies

$$p = xy\delta^r = \delta^{-\nu} x_k y_k \delta^{\mu}.$$

Thus,

$$\begin{aligned}
pwp^{-1} &= \delta^{-\nu} x_k y_k \delta^{\mu} x_1 y_1 \dots x_{k-1} y_{k-1} \delta^{\nu} \\
&= \delta^{-\nu} \sigma_{k-1}(w) \delta^{\nu}.
\end{aligned}$$

Our base case is established.

Let  $p \in \pi_1(\Sigma_g)$  with  $l(p) = n > 1$ . Assume as inductive hypothesis that if  $l(p) < n$  and  $l(pwp^{-1}) = l(w)$  then  $pwp^{-1} = \delta^{\nu} \sigma_i(w) \delta^{-\nu}$  for some  $i$  and some power  $\nu$ . Let  $p$  be given in normal form as

$$p = z_1 w_1 \dots z_n w_n \delta^r.$$

We can divide  $p$  into two pieces:

$$p_1 := z_1 w_1 p_2 := z_2 w_2 \dots z_n w_n \delta^r. \tag{3.2.1}$$

In order to apply the inductive hypothesis, we need to check that  $l(p_2 w p_2^{-1}) = l(w)$ . Suppose for the sake of contradiction that  $l(p_2 w p_2^{-1}) > l(w)$ . Applying Lemma 3.2.2. vi. it follows

that:

$$\begin{aligned}
l(p_1(p_2wp_2^{-1})) &= l(p_1) + l(p_2wp_2^{-1}) \\
&> l(p_1) + l(w) \\
&= 1 + l(w).
\end{aligned}$$

Then, by Lemma 3.2.2 vii. it follows that:

$$\begin{aligned}
l(p_1(p_2wp_2^{-1})p_1^{-1}) &\geq l(p_1(p_2wp_2^{-1})) - 1 \\
&> l(w) + 1 - 1 \\
&= l(w).
\end{aligned}$$

Therefore, if  $l(pwp^{-1}) = l(w)$  then  $l(p_2wp_2^{-1}) = l(w)$ .

By inductive hypothesis,  $p_2wp_2^{-1} = \delta^\nu \sigma_i \delta^{-\nu}$  for some  $i$  and some power  $\nu$ .

Thus, we can rewrite

$$\begin{aligned}
pwp^{-1} &= p_1 \delta^\nu \sigma_i(w) \delta^{-\nu} p_1^{-1} \\
&= (z_1 w_1) \delta^\nu x_{i+1} y_{i+1} \dots x_k y_k \delta^\mu x_1 y_1 \dots x_i y_i \delta^{-\nu} (w_1^{-1} z_1^{-1}).
\end{aligned}$$

If  $z_1 = 1$ , the base case implies that  $p_1 \in C$ . That is, if  $z = 1$  then  $w_1 = 1$ , but this contradicts the definition of normal form.

On the other hand, suppose  $z_1 \neq 1$ . Write  $\sigma_i(w)$  in normal form as

$$\begin{aligned}
\sigma_i(w) &= x_{i+1} y_{i+1} \dots x_k y_k \delta^\mu x_1 y_1 \dots x_i y_i \\
&= x_{i+1} y_{i+1} \dots x_k y_k x'_1 y'_1 \dots x'_i y'_i \delta^{\mu_i}.
\end{aligned}$$

By the base case,  $p_1 \delta^\nu = \delta^s x'_j y'_j \delta^{\mu_i}$  for some power  $s$ . Thus,

$$p_1 \delta^\nu \sigma_i(w) \delta^{-\nu} p_1^{-1} = \delta^s x'_j y'_j \delta^{\mu_i} x_{j+1} y_{j+1} \dots x_k y_k x'_1 y'_1 \dots x'_{j-1} y'_{j-1} \delta^{-s}.$$

However by substitution,

$$\begin{aligned} p_1 \delta^\nu \sigma_i(w) \delta^{-\nu} p_1^{-1} &= \delta^{s+\mu_{j-1}} x_j y_j \dots x_k y_k \delta^\mu x_1 y_1 \dots x_{j-1} y_{j-1} \delta^{-s-\mu_{j-1}} \\ &= \delta^{s'} \sigma_{j-1}(w) \delta^{-s'} \end{aligned}$$

as desired.  $\square$

To prove Proposition 3.0.2.A. we need to show that  $l(T_c(w)) = l(w)$  and that  $T_c(w) \neq \delta^\nu \sigma_i(w) \delta^{-\nu}$  for all  $i$  and all  $\nu$ .

### 3.2.2 Dividing the surface with a separating curve

As above, let  $c$  be the separating curve in Figure 3.1 that divides the generators  $a_1, b_1$  from the rest of the surface.

*Proof of Proposition 3.0.2.A.* By definition of  $T_c$  and by Lemma 3.2.2. i. it follows that:

$$l(T_c(w)) = l(\delta x_1 \delta^{-1} y_1 \dots \delta x_k \delta^{-1} y_k \delta^\mu) = l(w).$$

Assume for the sake of contradiction that  $T_c(w) = pwp^{-1}$  for some  $p \in \pi_1(\Sigma_g)$ . Then  $T_c(w) = pwp^{-1} = \delta^\nu \sigma_i(w) \delta^{-\nu}$  for some  $\nu$  and some  $i$ . Notice that:

$$T_c(pwp^{-1}) = T_c(p)T_c(w)T_c(p^{-1}) = p'w(p')^{-1}.$$

Thus  $(T_c)^s(w) = \delta^{\nu s} \sigma_{i_s}(w) \delta^{-\nu s}$  for any  $s \in \mathbb{Z}$ .

Because  $w$  has finite length, there exist  $0 < s < t$  such that  $T_c^s(w) = \delta^m \sigma_i(w) \delta^{-m}$  and  $T_c^t = \delta^n \sigma_i(w) \delta^{-n}$ . That is for some power  $\nu$ ,

$$T_c^s(w) = \delta^\nu T_c^t(w) \delta^{-\nu}.$$

Therefore,

$$T_c^s(w) = \delta^s x_1 \delta^{-s} y_1 \delta^s x_2 \dots y_k \delta^\mu = \delta^{\nu+t} x_1 \delta^{-t} y_1 \delta^t x_2 \dots \delta^{\mu-\nu}.$$

Equivalently,

$$x_1 \delta^{-s} y_1 \delta^s x_2 \dots \delta^\mu = \delta^{\nu+t-s} x_1 \delta^{-t} y_1 \delta^t x_2 \dots \delta^{\mu-\nu}.$$

By uniqueness of normal form,  $\delta^{\nu+t-s}x_1 = x_1\delta^\rho$  for some  $\rho$ . That is

$$\begin{aligned}\delta^{\nu+t-s}x_1 &= x_1\delta^\rho \\ x_1^{-1}\delta^{\nu+t-s}x_1 &= \delta^\rho \\ x_1^{-1}\delta^{\nu+t-s}x_1\delta^{s-t-\nu} &= \delta^{\rho+(s-t-\nu)}.\end{aligned}$$

Because  $\delta \in \gamma_2(A)$ , it follows that  $x_1^{-1}\delta^{\nu+t-s}x_1\delta^{s-t-\nu} \in \gamma_3(A)$ . That is,  $\delta^{\rho+(s-t-\nu)} \in \gamma_3(A)$ . However since  $\gamma_2(A)/\gamma_3(A)$  is torsion-free, (see e.g. [11] pp.426), it follows that  $\delta^{\rho+(s-t-\nu)} = 1$ . Therefore,

$$x_1^{-1}\delta^{\nu+t-s}x_1\delta^{s-t-\nu} = 1.$$

That is,  $x_1$  commutes with  $\delta^{\nu+t-s}$ . However if  $a, b \in \pi_1(\Sigma_g)$  commute, then  $a$  and  $b$  are powers of some common element. Because  $A/\gamma_2(A)$  is torsion-free, if  $x \notin \langle \delta \rangle$  then  $x^\zeta \neq \delta$  for any power  $\zeta$ . Therefore, either  $x_1 = \delta^\xi$  for some  $\xi \in \mathbb{Z}$ , or  $\nu + t - s = 0$ . Since  $x_1$  does not represent the identity coset, this forces  $\nu + t - s = 0$ . Thus,

$$y_1\delta^s x_2 \dots \delta^\mu = \delta^{-t+s} y_1 \delta^t x_2 \dots \delta^{\mu-\nu}.$$

By the same argument, it follows that  $s - t = 0$ . But this contradicts the assumption that  $s \neq t$ . Therefore, if  $T_c(w)$  is conjugate to  $w$ , then  $w$  does not cyclically reduce to state 1.

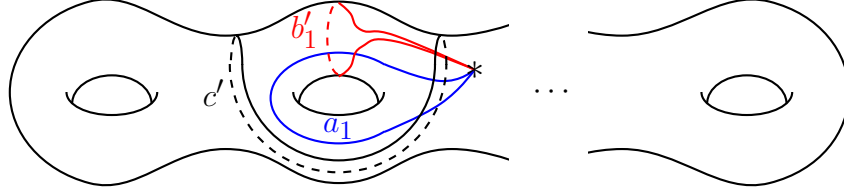
Therefore, if  $T_c(w)$  is conjugate to  $w$  in  $\pi_1(\Sigma_g)$  then  $w$  is conjugate in  $\pi_1(\Sigma_g)$  to an element of  $A$  or an element of  $B$ .  $\square$

In order to prove Proposition 3.0.2.B. we want to isolate the generating curve  $a_1$ . In light of this, the argument above can be repeated for a slightly different set of generators  $\{a_1, b'_1, a_2, b_2, \dots, a_g, b_g\}$ , where  $b'_1$  is homologous to  $b_1$  but is not freely homotopic, as shown in Figure 3.2.

The curve  $c'$  separates  $a_1, b'_1$  from the rest of the surface. By the argument above, if  $w$  is conjugate to  $T_{c'}(w)$ , then either  $w \in \langle a_1, b'_1 \rangle$  or  $w \in \langle a_2, b_2, \dots, a_g, b_g \rangle$ . Let  $A' := \langle a_1, b'_1 \rangle$ . If a simple closed  $x$  curve is freely homotopic to both a curve representing an element of  $A$  and a curve representing an element of  $A'$ , then  $x$  is freely homotopic to a curve representing an element of  $\langle a_1 \rangle$ . That is, if  $w$  is conjugate in  $\pi_1(\Sigma_g)$  to both an element of  $A$  and an



Figure 3.2: Curve  $b'_1$



The curve  $b'_1$  is homologous to  $b_1$  but the two curves are not homotopic.

element of  $A'$ , then  $w$  is conjugate to an element of  $\langle a_1 \rangle$ .

*Proof of Proposition 3.0.2.B.* Let  $\beta \in \pi_1(\Sigma_g)$  be given. Assume  $[\beta, \mathcal{J}(\Sigma)] \subset \gamma_2(\pi_1(\Sigma_g))$ . Let  $z = [\beta, T_c]$ . By assumption,  $z \in \gamma_2(\pi_1(\Sigma_g))$ . Let  $w = T_c \beta \delta \beta^{-1} T_c^{-1} \in \pi_1(\Sigma_g)$ . Then

$$wz\beta a_1 \beta^{-1} z^{-1} w^{-1} = T_c \beta a_1 \beta^{-1} T_c^{-1}.$$

Notice that because  $wz \in \pi_1(\Sigma_g)$ , it follows that  $T_c(\beta a_1 \beta^{-1})$  is conjugate to  $\beta a_1 \beta^{-1}$  in  $\pi_1(\Sigma_g)$ . In the quotient  $\pi_1(\Sigma_g)/\gamma_2(\pi_1(\Sigma_g))$ , we have the following equality of left cosets:

$$\beta a_1 \beta^{-1} \gamma_2(\pi_1(\Sigma_g)) = a_1 \gamma_2(\pi_1(\Sigma_g)).$$

Proposition 3.0.2.A. implies that  $\beta a_1 \beta^{-1}$  must be conjugate to an element generated by  $\{a_1, b_1\}$ .

Similarly, replacing  $T_c$  with  $T_{c'}$ , the above argument demonstrates that  $T_{c'}(\beta a_1 \beta^{-1})$  is conjugate to  $\beta a_1 \beta^{-1}$  in  $\pi_1(\Sigma_g)$ . Thus,  $\beta a_1 \beta^{-1}$  is conjugate to an element generated by  $\{a_1, b'_1\}$  and an element generated by  $\{a_1, b_1\}$ .

Therefore  $\beta a_1 \beta^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to  $a_1^k$  for some  $k$ . Since  $\beta a_1 \beta^{-1} \equiv a_1 \pmod{\gamma_2(\pi_1(\Sigma_g))}$ , it follows that  $\beta a_1 \beta^{-1}$  is homotopic to  $a_1$ .

For any  $\phi \in \text{Mod}(\Sigma_{g,1})$ , because  $\{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}$  is normal in  $\text{Mod}(\Sigma_{g,1})$ , it follows that:

$$\phi\beta\phi^{-1} \in \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subset \gamma_2(P)\}.$$

Thus,  $\phi\beta\phi^{-1}a_1\phi\beta^{-1}\phi^{-1}$  is conjugate to  $a_1$  in  $\pi_1(\Sigma_g)$ . This means for some  $\rho \in \pi_1(\Sigma_g)$ ,

$$\beta\phi a_1 \phi^{-1} \beta^{-1} = \phi \rho a_1 \rho^{-1} \phi^{-1}.$$

Because  $[\phi, \rho] \in \pi_1(\Sigma_g)$ , it follows that  $\phi \rho a_1 \rho^{-1} \phi^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to

$$[\rho, \phi] \phi \rho a_1 \rho^{-1} \phi^{-1} [\phi, \rho] = \rho \phi a_1 \phi^{-1} \rho^{-1}.$$

That is,  $\beta\phi(a_1)\beta^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to  $\phi(a_1)$ . Therefore,  $\beta x \beta^{-1}$  is conjugate in  $\pi_1(\Sigma_g)$  to  $x$  for any simple closed curve  $x$ .  $\square$

In particular, for a filling set of simple closed curves,  $\{x_1, \dots, x_k\}$ , we have  $\beta x_i \beta^{-1}$  is conjugate to  $x_i$  in  $\pi_1(\Sigma_g)$  for each  $i$ . By the Alexander method, the map  $\beta$  must be trivial in  $\text{Out}(\pi_1(\Sigma_g))$ . That is,  $\beta \in P$ . Proposition 3.0.2 follows.

### 3.3 Conclusion: $N = P$

To conclude the proof of Theorem 1.0.1, write both  $P$  and  $N$  in the form given by Proposition 3.0.2:

$$P = \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subseteq \gamma_2(P)\}.$$

$$N = \{x \in \mathcal{J}(N) \mid [x, \mathcal{J}(N)] \subseteq \gamma_2(N)\}.$$

From section 2.3,  $[N, N] \subset [P, P]$  implies that:

$$N \subset \{x \in \mathcal{J}(N) \mid [x, \mathcal{J}(N)] \subseteq \gamma_2(P)\}.$$

From Section 2,  $\mathcal{J}(\Sigma) \subset \mathcal{J}(N)$  implies that:

$$N \subset \{x \in \mathcal{J}(N) \mid [x, \mathcal{J}(\Sigma)] \subseteq \gamma_2(P)\}.$$

From Section 2,  $N \subset \mathcal{J}(\Sigma)$  implies that:

$$N \subset \{x \in \mathcal{J}(\Sigma) \mid [x, \mathcal{J}(\Sigma)] \subseteq \gamma_2(P)\}.$$

Thus

$$N \subseteq P.$$

Since  $N$  is a subgroup of  $P$  and is not free, the index of  $N$  in  $P$  is finite (see e.g. [17, Th.1]).

We can determine the index of  $N$  via the following formula (see e.g. [14, Sect.2.2 Ex.22]):

$$[N : P] \cdot \chi(\Sigma) = \chi(\Gamma)$$

where  $\chi$  is Euler characteristic. Therefore  $[N : P] = 1$  and  $N = P$ . We have established Theorem 1.0.1.

The following example demonstrates that  $N$  need not equal  $P$  if we remove the condition of normality.

**Example 3.3.1.** Let  $\varphi \in \text{Mod}(\Sigma_g)$ . Construct the mapping torus  $M_\varphi \cong (I \times \Sigma_g)/(1, x) \sim (0, \varphi(x))$ . Note that  $\pi_1(M_\varphi) \cong \pi_1(\Sigma_g) \rtimes \mathbb{Z}$ . Consider the exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \text{Mod}(\Sigma_{g,1}) \xrightarrow{F} \text{Mod}(\Sigma_g) \longrightarrow 1.$$

The preimage  $F^{-1}(\varphi) \cong \pi_1(\Sigma_g) \rtimes \mathbb{Z} < \text{Mod}(\Sigma_{g,1})$ . This induces an injection

$$g : \pi_1(M_\varphi) \hookrightarrow \text{Mod}(\Sigma_{g,1}).$$

$M_\varphi$  fibers over  $S^1$  with fiber  $\Sigma_g$ . As long as  $H_2(M_\varphi; \mathbb{Z}) \geq 2$ , the theory of the Thurston norm [34] implies that  $M_\varphi$  fibers over  $S^1$  with fiber  $\Sigma_h$  for infinitely many  $h$ . (These  $h$  correspond to integer points in the cone over a fibered face of the unit ball in the Thurston

norm.) Fiberings of the form

$$\begin{array}{ccc} \Sigma_h & \longrightarrow & M_\varphi \\ & & \downarrow \\ & & S^1 \end{array}$$

give injections  $i_h : \pi_1(\Sigma_h) \rightarrow \pi_1(M_\varphi)$ . The image of the composition

$$g \circ i_h : \Sigma_h \hookrightarrow \text{Mod}(\Sigma_{g,1})$$

is a surface subgroup of  $\text{Mod}(\Sigma_{g,1})$ . This subgroup is not necessarily normal in  $\text{Mod}(\Sigma_{g,1})$ . Using the fibered faces of the unit ball in the Thurston norm, we can find multiple (non-normal) copies of  $\pi_1(\Sigma_g)$  in  $\text{Mod}(\Sigma_{g,1})$ .

### 3.4 A new proof that $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1}))$ is trivial

**Corollary 1.0.2 (Ivanov-McCarthy's Theorem).** *Let  $g \geq 3$ . Then  $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1}))$  is trivial.*

Theorem 1.0.1 together with the following classical theorem of Burnside implies Corollary 1.0.2. A group  $G$  is *complete* if it is centerless and every automorphism is inner, i.e.  $\text{Aut}(G) \cong \text{Inn}(G) \cong G$ . A subgroup  $H < G$  is *characteristic* if  $H$  is invariant under all automorphisms of  $G$ .

**Theorem 3.4.1** (Burnside [7]). *A centerless group  $G$  is characteristic in its automorphism group if and only if  $\text{Aut}(G)$  is complete.*

*Proof of  $(\Rightarrow)$  for Theorem 3.4.1.* Suppose that  $G$  is centerless and characteristic in  $\text{Aut}(G)$ . Let  $\phi \in \text{Aut}(\text{Aut}(G))$  and let  $g \in G$ . There is a homomorphism

$$i : G \rightarrow \text{Inn}(G)$$

given by

$$i(g)(h) = ghg^{-1}$$

for any  $h \in G$ . The homomorphism  $i$  is an isomorphism because  $G$  is centerless. Additionally, because  $G$  is characteristic,  $\phi$  restricts to an automorphism of  $\text{Inn}(G) \cong G$ . Define

$$\bar{\phi} : G \rightarrow G$$

by

$$i(\bar{\phi}(g)) := \phi(i(g)).$$

To show that  $\text{Aut}(\text{Aut}(G)) = \text{Aut}(G)$ , it suffices to show that

$$\phi(\psi) = \bar{\phi} \circ \psi \circ \bar{\phi}^{-1}$$

for any  $\psi \in \text{Aut}(G)$ . Notice that:

$$\phi(i(\psi(g))) = i(\bar{\phi}(\psi(g))).$$

On the other hand

$$\begin{aligned} \phi(i(\psi(g))) &= \phi(\psi \circ i(g) \circ \psi^{-1}) \\ &= \phi(\psi) \circ i(\bar{\phi}(g)) \circ \phi(\psi)^{-1} \\ &= i(\phi(\psi)(\bar{\phi}(g))). \end{aligned}$$

Because  $i$  is an isomorphism we can equate

$$\phi(\psi)(\bar{\phi}(g)) = \bar{\phi}(\psi(g))$$

for any  $g \in G$ . Therefore,

$$\phi(\psi) \circ \bar{\phi} = \bar{\phi} \circ \psi.$$

As such,

$$\phi(\psi) = \bar{\phi} \circ \psi \circ \bar{\phi}^{-1}.$$

□

*Proof of corollary 1.0.2.* By Theorem 1.0.1,  $P$  is characteristic in  $\text{Mod}(\Sigma_{g,1})$ . By the Dehn-Nielsen-Baer theorem (see e.g. [10] Th. 8.1) it follows that  $\text{Aut}(P) \cong \text{Mod}^\pm(\Sigma_{g,1})$ . To prove the corollary, it suffices to show that  $\text{Mod}(\Sigma_{g,1})$  is characteristic in  $\text{Mod}^\pm(\Sigma_{g,1})$ . Notice that:

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z} &\cong \frac{\text{Mod}^\pm(\Sigma_{g,1})}{[\text{Mod}^\pm(\Sigma_{g,1}), \text{Mod}^\pm(\Sigma_{g,1})]} \cong H_1(\text{Mod}^\pm(\Sigma_{g,1}); \mathbb{Z}) \quad \text{and} \\ \mathbb{Z}/2\mathbb{Z} &\cong \frac{\text{Mod}^\pm(\Sigma_{g,1})}{\text{Mod}(\Sigma_{g,1})}. \end{aligned}$$

For further details on these quotients see [10, Th. 5.2 and Ch. 8]. Because the quotient  $\text{Mod}^\pm(\Sigma_{g,1})/\text{Mod}(\Sigma_{g,1})$  is abelian,  $[\text{Mod}^\pm(\Sigma_{g,1}), \text{Mod}^\pm(\Sigma_{g,1})] \subset \text{Mod}(\Sigma_{g,1})$ . Further, because the quotients are isomorphic and finite, it follows that  $\text{Mod}(\Sigma_{g,1})$  is equal to the commutator subgroup of  $\text{Mod}^\pm(\Sigma_{g,1})$ . Therefore  $\text{Mod}(\Sigma_{g,1})$  is characteristic, and  $\text{Out}(\text{Mod}^\pm(\Sigma_{g,1})) \cong 1$ . □

## REFERENCES

- [1] Mamoru Asada and Masanobu Kaneko, *On the automorphism group of some pro- $l$  fundamental groups*, Advanced studies in Pure Mathematics **12** (1987), 137–159.
- [2] H. Bass and A. Lubotzky, *Linear-central filtrations on groups*, Contemporary Mathematics **169** (1994), 45–45.
- [3] J. Birman, *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. **22** (1969), 213–238. MR 0243519
- [4] ———, *Braids, links, and mapping class groups*, no. 82, Princeton University Press, 1975.
- [5] J. Birman and R. Craggs, *The  $\mu$ -invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold*, Transactions of the American Mathematical Society **237** (1978), 283–309.
- [6] N. Broaddus, B. Farb, and A. Putman, *Irreducible  $Sp$ -representations and subgroup distortion in the mapping class group*, Comment. Math. Helv. **86** (2011), no. 3, 537–556. MR 2803852
- [7] W. Burnside, *Theory of groups of finite order*, Messenger of Mathematics **23** (1909), 112.
- [8] M. Clay, C. Leininger, and J. Mangahas, *The geometry of right-angled Artin subgroups of mapping class groups*, Groups Geom. Dyn. **6** (2012), no. 2, 249–278. MR 2914860
- [9] William Crawley-Boevey, Pavel Etingof, and Victor Ginzburg, *Noncommutative geometry and quiver algebras*, Advances in Mathematics **209** (2007), no. 1, 274 – 336.
- [10] B. Farb and D. Margalit, *A primer on mapping class groups.*, Princeton Mathematical Series, Princeton University Press, 2012.
- [11] E. Formanek, *Characterizing a free group in its automorphism group*, Journal of Algebra **133** (1990), no. 2, 424 – 432.

- [12] E. Grossman, *On the residual finiteness of certain mapping class groups*, Journal of the London Mathematical Society **2** (1974), no. 1, 160–164.
- [13] R. Hain, *Torelli groups and geometry of moduli spaces of curves*, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93) (1995), 97–143.
- [14] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.
- [15] N. Ivanov, *Algebraic properties of the Teichmüller modular group*, Dokl. Akad. Nauk SSSR, vol. 275, 1984, pp. 786–789.
- [16] N. Ivanov and J. McCarthy, *On injective homomorphisms between Teichmüller modular groups. I*, Invent. Math. **135** (1999), no. 2, 425–486. MR 1666775 (99m:57014)
- [17] W. Jaco, *On certain subgroups of the fundamental group of a closed surface*, Proc. Cambridge Philos. Soc, vol. 67, Cambridge Univ Press, 1970, pp. 17–18.
- [18] D. Johnson, *An abelian quotient of the mapping class group  $J_g$* , Mathematische Annalen **249** (1980), no. 3, 225–242.
- [19] ———, *Quadratic forms and the Birman-Craggs homomorphisms*, Transactions of the American Mathematical Society **261** (1980), no. 1, 235–254.
- [20] ———, *The structure of the Torelli group I: a finite set of generators for  $j$* , Annals of Mathematics (1983), 423–442.
- [21] ———, *A survey of the torelli group*, Contemporary Math **20** (1983), 165–179.
- [22] ———, *The structure of the Torelli group III: The abelianization of  $J$* , Topology **24** (1985), no. 2, 127–144.
- [23] M. Korkmaz, *The symplectic representation of the mapping class group is unique*, arXiv preprint arXiv:1108.3241 (2011).
- [24] J. P. Labute, *On the descending central series of groups with a single defining relation*, Journal of Algebra **14** (1970), no. 1, 16 – 23.



- [25] M. Lazard, *Sur les groupes nilpotents et les anneaux de lie*, Annales Scientifiques de l'École Normale Supérieure, vol. 71, Société mathématique de France, 1954, pp. 101–190.
- [26] C. Leininger and A. Reid, *A combination theorem for Veech subgroups of the mapping class group*, Geometric & Functional Analysis GAFA **16** (2006), no. 2, 403–436.
- [27] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Dover books on mathematics, Dover Publications, 2004.
- [28] A. I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation **1951** (1951), no. 39, 33. MR 0039734
- [29] J. McCarthy, *Automorphisms of surface mapping class groups. a recent theorem of N. Ivanov*, Inventiones mathematicae **84** (1986), no. 1, 49–71.
- [30] O. O'Meara, *Symplectic groups*, vol. 16, American Mathematical Soc., 1978.
- [31] L. Paris, *Residual properties of mapping class groups and surface groups*, Transactions of the American Mathematical Society **361** (2009), no. 5, 2487–2507.
- [32] Paul E Schupp, *Small cancellation theory over free products with amalgamation*, Mathematische Annalen **193** (1971), no. 4, 255–264.
- [33] P. Scott and T. Wall, *Topological methods in group theory*, Homological group theory **36** (1979), 137.
- [34] W. Thurston, *A norm for the homology of 3-manifolds*, Memoirs of the American Mathematical Society **59** (1986), no. 339, 99–130.
- [35] C. Weibel, *An introduction to homological algebra*, no. 38, Cambridge University Press, 1995.