THE UNIVERSITY OF CHICAGO

POTENTIAL MODULARITY OF K3 SURFACES WITH LARGE PICARD RANK

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

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CHICAGO, ILLINOIS
AUGUST 2023

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ACKNOWLEDGMENTS

First and foremost, I would like to express my most sincere gratitude to my advisor Professor Frank Calegari for suggesting this problem to me and giving me an in-depth introduction to the related topics and methods with patience.

I would like to thank my family and all my friends for their lasting support and encouragement. This journey as a graduate student would never be as enjoyable without any one of them.

I would like to thank the University of Chicago and the Department of Mathematics for offering me this chance to study as a graduate student, and I want to thank all professors in the Department of Mathematics for various helpful discussions, especially Professor Matthew Emerton, who also agreed to be my thesis reader and member of my thesis committee.

I would also like to thank the staff in the department for all the administrative support in completing my graduate requirements.

ABSTRACT

The first part of this thesis studied GSp_4-type abelian varieties and the corresponding compatible systems of GSp_4 representations. Techniques in [BCGP21] are applied to show that one can prove the potential modularity of these abelian varieties and compatible systems under some conditions that guarantee a sufficient amount of good primes. Then, in the second part, we use the potential modularity theorems to prove that K3 surfaces over totally real field F with Picard rank ≥ 17 are potentially modular.

CHAPTER 1

BACKGROUNDS

1.1 Introduction and Notations

Let F be a totally real field and X/F be an algebraic $K3$ surface of geometric Picard rank ρ. Then the singular cohomology $H = H^2(X, \mathbb{Q})$ with its Hodge structure splits as a direct sum

$$H = T \oplus NS(X)$$

consisting of the Néron-Severi group with rational coefficients and the transcendental part T of rank $22 - \rho$ ([Mor84]), and associated to it is a corresponding compatible system of Galois representations

$$\rho_p : G_F \to \mathrm{GO}_{22-\rho}(\mathbb{Q}_p)$$

Let $\zeta_X(s)$ denote the Hasse-Weil zeta function of X, then there is an equality

$$\frac{1}{\zeta_X(s)} = \zeta(s)\zeta(s-2)L(T, s-1)L(NS, s-1)$$

The Hasse-Weil conjecture states that $\zeta_X(s)$ should extend to a meromorphic function for all $s \in \mathbb{C}$ and satisfy a functional equation. It is known that this holds for the L-function of the Artin motive $NS(X)$ due to Artin-Braeur theorem [Bra47]. So the conjecture would follow from the conjecture that the motive T (or the associated compatible system $\{\rho_p\}$) is potentially automorphic. That is, there is a finite extension F'/F such that the restrictions of the motive become automorphic. The main theorem of this thesis proves that this is true when the geometric Picard rank is larger than or equal to 17.

Theorem 1.1. Suppose that the geometric Picard rank of X is $\rho \geq 17$. Then T is potentially modular, and the Hasse-Weil conjecture holds for X.

Example 1.2. Suppose that $A/\mathbb{Q}$ is an abelian variety. Then associated to A is a $K3$ surface X known as the Kummer surface of A; it is a resolution of $A/[-1]$, and it has geometric Picard rank ≥ 17 (more precisely, $16 + \dim NS(A)$). The main theorem in this case then follows from the main theorem of [BCGP21].

One key point is that are many other examples of X with Picard rank ≥ 17 which are *not* of this form. Such X *do* however turn out to be associated with either abelian surfaces A or *fake* abelian surfaces A. Moreover, the varieties A in question need not be defined over F, but rather are defined over some finite Galois extensions E/F and such that A is isogenous to A^σ for any $\sigma \in \mathrm{Gal}(E/F)$. These are analogues of the so-called $\mathbb{Q}$-curves for GL_2 considered by Ribet [Rib04]. The next step is to prove that all such varieties (with motivic descent data to F) are potentially modular.

In the rest of this chapter, we will introduce the formal definitions of compatible systems of Galois representations, especially the GSp_4 representations which will be our main interest of study in this paper.

In chapter 2 we introduce the possible abelian varieties that give compatible systems of GSp_4 representations. We call these abelian varieties of GSp_4-type, and we say that a compatible system is *geometric* if it comes from such an abelian variety.

In chapter 3 we discuss the large image property of geometric compatible systems of GSp_4 representations and proved that under the assumption of the large image property there are an ample source of *good* primes l such that the abelian variety satisfies some ordinariness condition for at least one prime λ above l.

In chapter 4 we apply the Serre-Tate theory to construct a desired lift of $\bar{A}/\mathbb{F}_l$ for some good prime l, and in chapter 6 we first use the theorem of Moret-Bailly to construct an abelian variety $\mathcal{A}$ together with two primes $\mathfrak{p}$ and $\mathfrak{q}$ such that $\mathcal{A}[\mathfrak{p}]$ coincides with that of the original abelian variety A, and $\mathcal{A}[\mathfrak{q}]$ is isomorphic to an induced representation. This process requires smoothness and connectedness of the moduli space of the GSp_4 type abelian

varieties, which is resolved in chapter 5 through a construction of Shimura varieties. Finally, we apply the modularity lifting theorems to show the potential modularity of the constructed abelian variety $\mathcal{A}$, and hence the original abelian variety A and the compatible system $\mathcal{R}_A$.

In chapter 7, we apply the motivic lifting results from [Pat19] to construct a compatible system $\mathcal{R}$ of Galois representations from the transcendental motive T, and we show that potential modularity theorem proved in chapter 6 applies to $\mathcal{R}$, hence proving the main result.

Note that our arguments do not apply to general compatible families of dimension 5 with Hodge-Tate weights $[-1, 0, 0, 0, 1]$, even those assumed to arise from geometry. Not only do we need purity, but we also use the fact that the Tate conjecture is known for $K3$ surfaces [And96]. More generally, we exploit the existence of the Kuga-Satake construction in order to find motivic lifts of the representations ρ_p to $\mathrm{GSp}_4(\bar{\mathbb{Q}}_p)$. The required ingredients we use can all be found in [Pat19], though we also prove that the representations ρ_p are absolutely irreducible in general which requires more work but also strongly relies on the assumption of large Picard rank.

To prove potential modularity (§6), we naturally use the main modularity lifting theorem of [BCGP21, §7] as well as the methods of [BCGP21, §9]. One notable difference with [BCGP21] is that for general GSp_4 type abelian varieties we need to assume the base field is $\mathbb{Q}$ rather than a totally real field, or at least the abelian variety should have *descent data* to $\mathbb{Q}$. In particular, we do not prove potential modularity for all abelian varieties A of GSp_4-type, that is, abelian varieties A such that $\mathrm{End}(A)\otimes\mathbb{Q}$ contains a totally real field K with $\dim(A) = 2[K : \mathbb{Q}]$ (§2). The reason that we are not able to do this is that we are unable to prove that such abelian varieties admit enough primes p splitting completely in F such that A is "distinguished ordinary", that is, primes p for which A is ordinary at all $v \mid p$ and the unit crystalline Frobenius eigenvalues are distinct modulo p (§3). As a result, more assumptions, including the ordinariness assumption (Definition 1.8), and the large image

assumption (Definition 3.2), need to be added to apply the potential modularity theorems.

This paper will mainly use the notations as in [BCGP21]: For a perfect field K, we let $\mathcal{O}_K$ be the ring of integers, $\bar{K}$ denote an algebraic closure of K and G_K the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. For each prime p not equal to the characteristic of K, we let ϵ_p denote the p-adic cyclotomic character and $\bar{\epsilon}_p$ its reduction modulo p, and will commonly omit the p and write as ϵ and $\bar{\epsilon}$ for simplicity.

Let $K/\mathbb{Q}$ be a finite extension. If v is a finite place of K we write K_v for the corresponding local field and $K_{(v)}$ for its residue field, and $\mathrm{Frob}_v := \mathrm{Frob}_{K_v}$ be the geometric Frobenius element in G_{K_v}. If v is a real place of K, then we let $[c_v]$ denote the conjugacy class in G_K consisting of complex conjugations associated to v.

1.2 Compatible System of GSp_4 representations

Definition 1.3. Let F be a number field. By a rank n weakly compatible system of l-adic representations $\mathcal{R}$ of G_F defined over M we mean a 5-tuple

$$\big(M, S, Q_v(X), \{\rho_\lambda\}, \{H_\tau\}\big)$$

where

(1) M is a number field;

(2) S is a finite set of primes of F;

(3) For each prime $v \notin S$ of F, $Q_v(X)$ is a monic degree n polynomial in $M[X]$;

(4) For each prime $\lambda \mid l$ of M, we have

$$\rho_\lambda : G_F \to \mathrm{GL}_n(\bar{M}_\lambda)$$

a continuous semi-simple representation such that

- if $v \notin S$ and $v \nmid l$ is a prime of F then ρ_λ is unramified at v and $\rho_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$;
- if $v \mid l$, then $\rho_\lambda|_{G_{F_v}}$ is de Rham, and in the case $v \notin S$ crystalline;

(5) For $\tau : F \hookrightarrow \bar{M}$, H_τ is a multiset of n integers such that for any $\bar{M} \hookrightarrow \bar{M}_\lambda$ over M we have $\mathrm{HT}_\tau(\rho_\lambda) = H_\tau$.

We will call $\mathcal{R}$ strictly compatible if for each finite place v of F there is a Weil–Deligne representation $\mathrm{WD}_v(\mathcal{R})$ of the Weil group W_{F_v} over $\bar{M}$ such that for each place λ of M and every M-linear embedding $\iota : \bar{M} \hookrightarrow \bar{M}_\lambda$ we have

$$\iota \mathrm{WD}_v(\mathcal{R}) \cong \mathrm{WD}(\rho_\lambda|_{G_{F_v}})^{F-ss}.$$

We say that the compatible system $\mathcal{R}$ is pure of weight $w \in \mathbb{Z}$ if for a density one set of primes v of F with residue character p, each root of α of $Q_v(X)$ in $\bar{M}$ satisfies $|\iota(\alpha)|^2 = p^w$ for all embeddings $\iota : \bar{M} \hookrightarrow \mathbb{C}$.

In particular, we say $\mathcal{R}$ is a (weakly) compatible system of GSp_4 representation if $n = 4$ and $\rho_\lambda : G_F \to \mathrm{GSp}_4(\bar{K}_\lambda)$, here

$$\mathrm{GSp}_4(K) = \{g \in \mathrm{GL}_4(K) : gJg^t = \nu(g)J\},$$

where $\nu(g) : \mathrm{GSp}_4 \to \mathbb{G}_m$ is the similitude character, and

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Compatible systems of representations naturally arise from abelian varieties, as we have the following result:

Proposition 1.4. (Theorem 2.8.1, [BCGP21]) If A is an abelian variety over a number field F, then for each $0 \leq i \leq 2\,\mathrm{dim}X$, the l-adic cohomology groups $H^i(A_{\bar{F}}, \bar{\mathbb{Q}}_l)$ form a strictly compatible system which is pure of weight i and which is defined over $\mathbb{Q}$.

Definition 1.5. If A/F is an abelian variety, then we may write the Galois representation $\rho_{A,l}$ as $H^1(A_{\bar{F}}, \bar{\mathbb{Q}}_l)$ (which is also the dual of the Tate module $T_l A(\bar{F})$), and $\mathcal{R}_A$ to be the compatible system $\{\rho_{A,l}\}$.

The next lemma shows that if one representation ρ comes from $H^1(A, \bar{\mathbb{Q}}_p)$ for one p, then we can extend it to a compatible system of representations that all comes from the same abelian variety.

Lemma 1.6. Let ρ denote an absolutely irreducible representation of G_F. Suppose that ρ occurs inside $H^1(A, \bar{\mathbb{Q}}_p)$ for some abelian variety A/F, and assuming that the Tate conjecture holds for A. Then ρ extends to a weakly compatible system of irreducible Galois representations $(M, S, \{Q_v(X)\}, \{\rho_\lambda\}, H)$ for some number field M such that ρ_λ occurs inside $H^1(A, \bar{\mathbb{Q}}_l)$ for all $\lambda \mid l$.

Proof. We may assume that A/F is simple. Let $D = \mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, and let M denote the center of D. For any prime l, the group $H^1(A, \mathbb{Q}_l)$ is a module over $M \otimes \mathbb{Q}_l = \prod_{\lambda|l} M_\lambda$, and we may correspondingly write $H^1(A, \mathbb{Q}_l) = \prod_{\lambda|l} W_\lambda$. The Galois representations ρ_λ already give rise to a weakly compatible family of representations of dimension $2\,\mathrm{dim}(A)/[M : \mathbb{Q}]$ with coefficients in M (see [Rib04, Thm 2.1.2] and [Shi67, §11.10]).

On the other hand, by the Tate conjecture, we know that

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_l = \mathrm{End}_{G_F} H^1(A, \mathbb{Q}_l),$$

and hence $D \otimes_M M_\lambda = \mathrm{End}_{G_F}(W_\lambda)$. It follows that, over the algebraic closure $\bar{M}_\lambda$, we may write the extension of scalars of W_λ as V_λ^n for some (absolutely) irreducible representation V_λ of dimension $2\dim(A)/(n[M:\mathbb{Q}])$ with $n^2 = [D:M]$. Here notice that an extension of scalars is only necessary for the finitely many primes in M which D/M is ramified. Since the representations V_λ^n form the compatible system associated to W_λ, it follows that the V_λ themselves form a compatible system also with coefficients in some fixed finite extension of M. □

There is another way to get compatible systems of GSp_4 representations, which is to consider the induction of compatible systems of GL_2 representations. Let F/E be a quadratic extension and $r : \rho_F \to \mathrm{GL}(V) = \mathrm{GL}_2(K)$ is a 2-dimensional representation. Choose σ to be an element in $G_E \setminus G_F$, and suppose it satisfies that $\det r = \det r^\sigma$. Then $\det r$ extends to two possible characters χ_1 and χ_2 on G_E. Let

$$\rho = \mathrm{Ind}_{G_F}^{G_E} r : G_E \to \mathrm{GL}(V \oplus \sigma V) = \mathrm{GL}_4(K),$$

and notice that $\wedge^2\rho$ is the sum of χ_1, χ_2 and some 4-dimensional representation. Therefore under some suitable basis ρ gives simplectic representations $\rho_i : G_E \to \mathrm{GSp}_4(K)$ with similitude character χ_i for $i = 1$ or 2.

With our choice of J, the image will land in the subgroup

$$\begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \cap \mathrm{GSp}_4(K).$$

In this paper we are only interested in the case where $\det r = \det r^\sigma = \epsilon^{-1}$ is the inverse cyclotomic character of G_F, and we take $\mathrm{Ind}_{G_F}^{G_E} r$ to be the symplectic representation that

has similitude character the inverse cyclotomic character of G_E. For example, if C/F is an elliptic curve and r be the representation that corresponds to the dual of the Tate module of C, then $\mathrm{Ind}_{G_F}^{G_E} r$ would correspond to the dual of the Tate module of the abelian variety $\mathrm{Res}_E^F C$, with the symplectic structure coming from the Weil pairing.

In order to derive potential automorphy results we need the following ordinary properties of the compatible system.

Definition 1.7. For a set of primes S of $\mathbb{Q}$, we say S is of strongly positive density if for any finite extension $F/\mathbb{Q}$, the set of primes over S in F has positive density.

Definition 1.8. For a prime p in $\mathbb{Q}$ that totally splits in F and M, for a prime $\lambda \mid p$ in M, we say that $\mathcal{R}$ is λ-ordinary if the representations $\rho_\lambda|_{G_{F_v}}$ are ordinary mod λ for all $v \mid p$ in F, and p-ordinary if it is λ-ordinary for all $\lambda \mid p$ in M.

We say $\mathcal{R}$ is strongly positive ordinary if it is p-ordinary for a strongly positive set of primes in $\mathbb{Q}$.

Given λ-ordinariness, for a prime $\lambda \mid p$ in M, $\mathcal{R}$ is called λ-distinguished ordinary if for any $v \mid p$ in F, $Q_v(X)$ has distinct unit roots mod λ.

CHAPTER 2

ABELIAN VARIETIES OF GSp_4-TYPE

In this chapter we introduce a list of abelian varieties that give compatible systems of GSp_4 representations. Let F be a totally real Galois extension of $\mathbb{Q}$ and let A be a simple principally polarized abelian variety over F of dimension g with no CM. We write $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$.

2.1 Real GSp_4 Type Abelian Varieties

Definition 2.1. We say that the abelian variety A is of real GSp_4 type if $K = \mathrm{End}^0(A)$ is a totally real field of degree d, with $g = 2d$.

Let l be a prime in $\mathbb{Q}$ that totally splits in F and suppose $l = \lambda_1 \cdots \lambda_d$ also totally splits in K. Then $A[l] \cong \bigoplus_{i=1}^{d} A[\lambda_i]$ as $\mathcal{O}_K$-modules and thus we have the Tate modules

$$T_l(A) = \varprojlim A[l^n] = \bigoplus_{i=1}^{d} T_{\lambda_i}(A)$$

also decomposes into d components. Also, the l-divisible groups $A[l^\infty]$, defined as the inductive system $(A[l^n], \iota_n)_n$, where the exact sequence holds:

$$0 \longrightarrow A[l^n] \xhookrightarrow{\iota_n} A[l^{n+1}] \xrightarrow{[l^n]} A[l^{n+1}] \ ,$$

also decomposes into the "λ_i-divible" groups defined in the same way. That is,

$$A[l^\infty] \cong \bigoplus_{i=1}^{d} A[\lambda_i^\infty].$$

For such primes l, let $\lambda \mid l$ be a prime over l in K, then we can write

$$\rho_{A,\lambda}^{\vee} : G_F \to \mathrm{GL}(T_\lambda A)$$

to be the Galois representation coming from the Tate module, and

$$\bar{\rho}_{A,\lambda}^{\vee} : G_F \to \mathrm{GL}(A[\lambda])$$

be the residual representation that comes from $A[\lambda]$. Note that $\rho_{A,\lambda}^{\vee}$ and $\bar{\rho}_{A,\lambda}^{\vee}$ are the dual of $\rho_{A,\lambda}$ and $\bar{\rho}_{A,\lambda}$ respectively, the latter being the Galois representation and the residual representation coming from $V_{A,\lambda} = H^1(A, K_\lambda)$.

The following propositions are a series of well-known results of l-adic Galois representations that come from geometry:

Proposition 2.2. When A is of real GSp_4 type, we have

$$\rho_{A,\lambda} : G_F \to \mathrm{GSp}_4(K_\lambda)$$

and

$$\bar{\rho}_{A,\lambda} : G_F \to \mathrm{GSp}_4(K_{(\lambda)}) \cong \mathrm{GSp}_4(\mathbb{F}_l).$$

Let p be a prime that totally splits in F with $v \mid p$ a prime above p in F, we have the following well-known property for representations $\rho_{A,\lambda}|_{G_{F_v}}$.

Proposition 2.3. If the abelian variety A is of GSp_4-type, then the characteristic polynomial of Frobenius $\rho_{A,\lambda}(\mathrm{Frob}_v)$ is of form

$$Q_v(X) = X^4 - a_v X^3 + b_v X^2 - p\chi(\mathrm{Frob}_v) a_v X + p^2 \chi^2(\mathrm{Frob}_v)$$

is independent from the choice of l and λ and the coefficients therefore are in $\mathcal{O}_K$. Here χ

is a totally even finite order character hence $\zeta := \chi(\mathrm{Frob}_v)$ is a root of unity with uniformly bounded order. The roots of the polynomials are of form

$$\alpha, p\zeta\alpha^{-1}, \beta, p\zeta\beta^{-1},$$

and are all Weil numbers, which are numbers π such that for any embedding $\psi : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\psi(\pi)| = \sqrt{p}$.

By an argument in [BCGP21] Theorem 8.5.2, since χ is finite order and totally even we can find a totally real extension F'/F such that $\chi|_{G_{F'}} \cong \psi^2$ and we can *untwist* χ by considering the compatible system $\rho_{A,\lambda}|_{G_{F'}} \otimes (\psi^{-1})$. Therefore, from now on (in the following two cases) we may assume that χ is trivial and the compatible system of GSp_4 representations will have similitude ϵ^{-1}.

Proposition 2.4. Abelian variety A of real GSp_4-type gives a weakly compatible system of GSp_4 representations of G_F pure of weight 1, still denoted by $\mathcal{R}_A$, over K.

2.2 Fake GSp_4 Type Abelian Varieties

Now we introduce a type of abelian varieties that also give compatible systems of GSp_4 representations of G_F.

Definition 2.5. We say that the abelian variety A is of fake GSp_4 type if $D = \mathrm{End}^0(A)$ is a quarternion algebra over a totally real field K of degree d, with $g = 4d$.

Proposition 2.6. Abelian variety A of fake GSp_4 type gives a weakly compatible system $\mathcal{R}_A$ of GSp_4 representations of G_F defined over K and is pure of weight 1.

Proof. For a prime l in $\mathbb{Q}$ that totally splits in K and F and $\lambda \mid l$ in K that is unramified in D, we have $D \otimes K_\lambda \cong M_2(K_\lambda)$ and the characteristic polynomial of Frob_v for $v \mid p$ in F

would thus have form

$$Q_v(X) = Q_{A,v}(X)^2,$$

where

$$Q_{A,v}(X) = X^4 - a_v X^3 + b_v X^2 - p a_v X + p^2$$

with coefficients in K. Also, if we let $\rho_\lambda : G_{F_v} \to \mathrm{GL}_8(K_\lambda)$ to be the the Galois representation that comes from $W_{A,\lambda} = H^1(A, K_\lambda)$ (which is also the dual of λ-adic Tate module $T_\lambda A$), then the image of ρ_λ lies in $\big(\mathrm{GSp}_4(K_\lambda)\big)^2$. In other words, we can write $W_{A,\lambda} = V_{A,\lambda} \oplus V_{A,\lambda}$ and $\rho_\lambda = \rho_{A,\lambda} \oplus \rho_{A,\lambda}$ for some $\rho_{A,\lambda} : G_F \to \mathrm{GSp}_4(K_v)$, such that the characteristic polynomial of Frob_v with respect to $\rho_{A,\lambda}$ is $Q_{A,v}(X)$.

For primes that are ramified in D, then after a quadratic extension K'/K such that the quarternion algebra D splits, or $D \otimes K' \cong M_2(K')$, we can still have $\rho_\lambda = \rho_{A,\lambda} \oplus \rho_{A,\lambda}$ for some $\rho_{A,\lambda} : G_F \to \mathrm{GSp}_4(K'_\lambda)$. Thus the characteristic polynomial of Frobenius would also have form

$$Q_v(X) = Q_{A,v}(X)^2,$$

where

$$Q_{A,v}(X) = X^4 - a_v X^3 + b_v X^2 - p a_v X + p^2,$$

and with coefficients of $Q_{A,v}$ in the integer ring of an at most some quadratic extension of K (since the coefficients of $Q_v(X)$ are in K). Thus the characteristic polynomial of Frob_v with respect to $\rho_{A,\lambda}$ is $Q_{A,v}(X)$.

Now that we have definition of $\rho_{A,\lambda}$ on all v's we are now able to show that $\rho_{A,\lambda}$'s form a compatible system of GSp_4 representations. This comes from the fact that ρ_λ's form a rank 8 compatible system of representations and $\rho_\lambda \cong \rho_{A,\lambda} \oplus \rho_{A,\lambda}$. □

2.3 Abelian Varieties with Descent Data

In [Rib04], Ribet introduced the $\mathbb{Q}$-curves, which are elliptic curves A/F such that A^σ is isogenous to A for all $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. He then proved that all $\mathbb{Q}$-curves are modular. Here we introduce an analogous definition for GSp_4 type abelian varieties.

Definition 2.7. Let $F/E/\mathbb{Q}$ be two Galois extensions of $\mathbb{Q}$. A (real or fake) GSp_4 type abelian variety A over F is said to be with descent data to E if A^σ is F-isogenous to A for all $\sigma \in \mathrm{Gal}(F/E)$.

Proposition 2.8. If A is with descent data to E, then A gives a compatible system of twisted GSp_4 representations of G_E. That is, there exists a compatible system $\mathcal{R} = (K, S, \{\rho_\lambda\}, \{Q_v\}, \{H_\tau\})$ of GSp_4 representations $\rho_\lambda : G_E \to \mathrm{GSp}_4(\bar{K}_\lambda)$ such that

$$\rho_\lambda|_{G_F} = \chi \otimes \rho_{A,\lambda},$$

where $\chi : G_E \to \bar{K}_\lambda^\times$ a totally even finite order character and $\rho_{A,\lambda}$ is the l-adic representation of that comes from $V_{A,\lambda} \subseteq H^1(A, K_\lambda)$. The coefficients will not necessarily be in K, but will be in some fixed finite extension of K.

Proof. Since A^σ is isogenous to A, we have isomorphisms denoted by $\psi_\sigma : V_{A^\sigma,\lambda} \to V_{A,\lambda}$ for $\sigma \in \mathrm{Gal}(F/E)$ (and thus well defined for each $\sigma \in G_E$) and we may first define the map $\tilde{\rho}_\lambda : G_E \to \mathrm{GSp}(V_{A,\lambda})$ as

$$\tilde{\rho}_\lambda(\sigma) = \psi_\sigma \circ \sigma.$$

Note that the map isn't necessarily a homomorphism from G_E to $\mathrm{GSp}(V_{A,\lambda})$, but the composition

$$\tilde{\rho}_\lambda(\sigma)\tilde{\rho}_\lambda(\tau)\tilde{\rho}_\lambda^{-1}(\sigma\tau)$$

acts on $V_{A,\lambda}$ as $\psi_\sigma \psi_\tau^\sigma \psi_{\sigma\tau}^{-1}$, which can also be viewed as some $\bar{K}^\times$-valued function $c(\rho, \tau)$ on $G_E \times G_E$. It is easy to check that c is a locally constant 2-cocycle on G_E with value in $\bar{K}^\times$.

This means that there exists a well defined representation of

$$\mathrm{P}\bar{\rho}_\lambda : G_E \to \mathrm{PGSp}_4(K_\lambda).$$

Now by a theorem of Tate [Pat19], we have $H^2(G_E, \bar{K}^\times)$ being trivial and this means that we can find a lift $\rho_\lambda : G_E \to \mathrm{GSp}_4(\bar{K}_\lambda)$ such that its restriction to G_F is a twist of the original representation $\rho_{A,\lambda}$ by a finite order character. In other words, we can write $\rho_\lambda|_{G_F} = \chi \otimes \rho_{A,\lambda}$ such that χ is a finite order character.

In addition, for a prime v in E that totally splits in F, since the Frobenii of primes above v are G_E-conjugates of the other, we can deduce that after the twist χ, the characteristic polynomial of Frob_{v_i} should be the same, thus the coefficients a_{v_i} and b_{v_i} are only dependent on v and we may write the characteristic polynomials as

$$Q_v(X) = X^4 - a_v X^3 + b_v X^2 - p a_v X + p^2$$

(here p is the residue characteristic) and will have coefficients in some fixed finite extension of K.

□

Now we may define the compatible system of GSp_4 representations of G_F over M to be *geometric*, if there exists an abelian variety A/H over some Galois extension H/F with descent data to F such that either

- A is of real GSp_4 type with $\mathrm{End}^0(A) = M$ and $H^1(A, M_\lambda) = V_{A,\lambda}$ gives the compatible system, or
- A is of fake GSp_4 type with $\mathrm{End}^0(A) = D$ for some quaternion algebra D/M and $H^1(A, M_\lambda) = V_{A,\lambda} \oplus V_{A,\lambda}$ for sufficiently large prime l and $\lambda \mid l$, and such that $V_{A,\lambda}$ gives the compatible system.

Now that we have formulated the correspondence between geometric compatible systems of GSp_4 representations and GSp_4-type abelian varieties, we can define distinguished ordinariness for abelian varieties as the following:

Definition 2.9. Let A be a GSp_4-type abelian variety. For a prime p in $\mathbb{Q}$ that totally splits in F and M, let $\mathfrak{p}$ be a prime above p in M, we say that A is $\mathfrak{p}$-ordinary or p-ordinary if the corresponding GSp_4 representations $\rho_{A,\mathfrak{p}}$ are $\mathfrak{p}$-ordinary or p-ordinary.

We say A is strongly positive ordinary if it is p-ordinary for a strongly positive set of primes in $\mathbb{Q}$.

Given $\mathfrak{p}$-ordinariness, for a prime $\mathfrak{p} \mid p$ in M, we say that A is $\mathfrak{p}$-distinguished ordinary if $Q_v(X)$ has distinct unit roots mod $\mathfrak{p}$ for all $v \mid p$.

An immediate proposition of the above definition of distinguished ordinariness is the following:

Proposition 2.10. If A is p-ordinary then the corresponding compatible system $\mathcal{R}_A$ is p-ordinary, and if the characteristic polynomial of Frob_v on $\rho_{A,\lambda}$ is denoted by

$$Q_v(x) = x^4 - a_v x^3 + b_v x^2 - p a_v x + p^2,$$

then $\mathfrak{p}$-ordinariness is equivalent to $\mathfrak{p} \nmid b_v$ for all primes $v \mid p$ in F.

If A is $\mathfrak{p}$-distinguished ordinary, then further we have $\mathfrak{p} \nmid a_v^2 - 4b_v$ for all $v \mid p$.

CHAPTER 3

LARGE IMAGE PROPERTY

In this chapter we derive certain large image properties of the compatible system of Galois representations associated to abelian varieties of GSp_4-type, and show that we can find enough *good* primes under certain large image hypotheses.

3.1 The Large Image Hypothesis

We first derive an analogous result to [BGK03, Theorem 3.5]. Notice that the arguments there don't apply directly in our case of real or fake GSp_4 type abelian varieties since $\dim A/[K:\mathbb{Q}] = 2$ or 4 are even numbers.

Lemma 3.1. (cf. [BGK03] Lemma 3.2) Suppose $\mathcal{R}$ is a geometric compatible system of GSp_4 representations of G_F over M. Then for sufficiently large prime l that completely splits in F and M and $\lambda \mid l$ the λ-adic monodromy group has Lie algebra $\mathfrak{sp}_4$ or $\mathfrak{so}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Proof. Throughout the following proof l should be chosen as completely split in M and F. Recall that the λ-adic monodromy group G_l^{alg} is the algebraic closure of $\rho_{A,l} \otimes \mathbb{Q}_l$ in the group $\mathrm{GL}_4/\mathbb{Q}_l$, and we write $(G_l^{\mathrm{alg}})'$ to be the derived subgroup.

If we write $\mathfrak{g}^{ss} = \mathrm{Lie}(G_l^{\mathrm{alg}})'$, and

$$V_{A,l} = H^1(A, \mathbb{Q}_l) = \bigoplus_{\lambda|l} V_{A,\lambda}$$

where $V_{A,\lambda} = V_{A,l} \otimes K_\lambda$, and let $\bar{V}_{A,\lambda} = V_{A,\lambda} \otimes \bar{\mathbb{Q}}_l$. By the relation

$$G_l^{\mathrm{alg}} \subset \prod_{\lambda|l} \mathrm{GSp}(V_{A,\lambda}) \cong \prod_{\lambda|l} \mathrm{GSp}_4/\mathbb{Q}_l$$

we have

$$\mathfrak{g}^{ss} \subset \bigoplus_{\lambda|l} \mathfrak{sp}_4(V_{A,\lambda})$$

Projecting onto the λ component we see that the image of $\mathfrak{g}^{ss} \otimes \bar{\mathbb{Q}}_l$ in $\mathfrak{sp}_4(\bar{V}_{A,\lambda})$ is semisimple, and we can write it as a decomposition

$$\bar{V}_{A,\lambda} = E(\omega_1) \otimes_{\bar{\mathbb{Q}}_l} \cdots \otimes_{\bar{\mathbb{Q}}_l} E(\omega_r),$$

where $E(\omega_i)$ for all $1 \leq i \leq r$ are the irreducible (orthogonal or symplectic) Lie algebra modules of the highest weight ω_i corresponding to simple Lie algebras $\mathfrak{g}_i$ which are summands of the image

$$\mathrm{Im}\big(\mathfrak{g}^{ss} \otimes \bar{\mathbb{Q}}_l \to \mathfrak{sp}_4(\bar{V}_{A,\lambda})\big) = \bigoplus_{i=1}^{r} \mathfrak{g}_i.$$

By [[Pin98], Corollary 5.11] all simple factors of $\mathfrak{g}^{ss} \otimes \bar{\mathbb{Q}}_l$ are of classical type A, B, C or D and the weights ω_i are minimal. Since $\dim_{\bar{\mathbb{Q}}_l} \bar{V}_{A,\lambda} = 4$, there are only two possible choices of decomposition: Either the product $E(\omega_1) \otimes_{\bar{\mathbb{Q}}_l} \cdots \otimes_{\bar{\mathbb{Q}}_l} E(\omega_r)$ consists of a single space $E(\omega_1)$ and the $\mathfrak{g}_1$-action on $E(\omega_1)$ is of type C symplectic representation, thus $\bigoplus_{i=1}^{r} \mathfrak{g}_i = \mathfrak{sp}_4$, or it consists of two spaces $E(\omega_1) \otimes_{\bar{\mathbb{Q}}_l} E(\omega_2)$ each of dimension 2, with $\mathfrak{g}_i$ acts as $\mathfrak{sl}_2$ on $E(\omega_i)$, thus $\bigoplus_{i=1}^{r} \mathfrak{g}_i = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \cong \mathfrak{so}_4$.

If A is of fake GSp_4 type, then for sufficiently large prime l, since the representation $W_{A,l} = H^1(A, \mathbb{Q}_l)$ can be decomposed into two identical components $W_{A,l} = V_{A,l} \oplus V_{A,l}$ and thus the original Galois representation ρ_l can be decomposed as $\rho_l \cong \rho_{A,l} \oplus \rho_{A,l}$ for some GL_{2g}-representation $\rho_{A,l}$, now the monodromy group G_l^{alg} is the algebraic closure of $\rho_{A,l} \otimes \mathbb{Q}_l$ and and $(G_l^{\mathrm{alg}})'$ be the derived subgroup. Also we have

$$V_{A,\lambda} = V_{A,l} \otimes_K K_\lambda$$

thus

$$V_{A,l} = \bigoplus_{\lambda|l} V_{A,\lambda}$$

By the exact same argument above on the semisimple part of the Lie algebra in the projection to $\mathfrak{sp}_4(V_{A,\lambda})$ we have the λ-adic monodromy group has Lie algebra $\mathfrak{sp}_4$, or $\mathfrak{so}_4$.

□

Definition 3.2. We say that a geometric compatible system $\mathcal{R}$ of GSp_4 representations (or the corresponding real or fake GSp_4 type abelian variety) has large image property if for sufficiently large l that completely splits in F and M and any $\lambda \mid l$ in M we have the Lie algebra of the λ-adic monodromy group being $\mathfrak{sp}_4$.

Proposition 3.3. Given large image property, for sufficiently large l, the image of the residual representation $\bar{\rho}_{A,\lambda}$ at least contains $\mathrm{Sp}_4(\mathbb{F}_l)$.

Proof. Note that Lemma 3.1 proved that the $h = 2$ case in Lemma 3.2 in [BGK03] holds when $\{\rho_{A,\lambda}\}$ has large image property, and the same argument from Lemma 3.2-Lemma 3.5 implies that the $h = 2$ case of Lemma 3.5 also holds for real or fake abelian varieties of GSp_4 type that has large image property.

So if we consider the residual representation

$$\bar{\rho}_{A,l} = \prod_{\lambda|l} \bar{\rho}_{A,\lambda},$$

then for sufficiently large l we have

$$\left[\bar{\rho}_{A,l}(G_F), \bar{\rho}_{A,l}(G_F)\right] = \prod_{\lambda|l} \mathrm{Sp}_4(\mathbb{F}_l),$$

or

$$[\bar{\rho}_{A,\lambda}(G_F), \bar{\rho}_{A,\lambda}(G_F)] = \mathrm{Sp}_4(\mathbb{F}_l).$$

This means that for sufficiently large l and $\lambda \mid l$, The residual representations

$$\bar{\rho}_{A,\lambda} : G_F \to \mathrm{GSp}_4(\mathbb{F}_l)$$

will have image at least containing $\mathrm{Sp}_4(\mathbb{F}_l)$. □

We now define a residual representation to be *vast and tidy* in the sense of [BCGP21] Definition 7.5.6 and Definition 7.5.11. We also have the following lemma that shows the implication from large image property to vast and tidiness for GSp_4 residual representations:

Lemma 3.4. (cf. [BCGP21] Lemma 7.5.15) For $p > 3$, if $\bar{\rho} : G_F \to \mathrm{GSp}_4(\mathbb{F}_p)$ that has image at least containing Sp_4 and with similitude character $\bar{\epsilon}^{-1}$, then $\bar{\rho}$ is vast and tidy.

Proof. For vastness notice that we still have the image of $\bar{\rho}|_{G_{F(\zeta_p^N)}}$ equal to $\mathrm{Sp}_4(\mathbb{F}_p)$ for all $N \in \mathbb{Z}^+$ (here ζ_p is the p-th root of unity). The rest follows from the arguments in [BCGP21] Lemma 7.5.15.

For $p > 3$, tidiness follows from the fact that the center of the image of $\bar{\rho}$ (since the similitude character is $\bar{\epsilon}^{-1}$) will have order $p - 1$ and [BCGP21] Lemma 7.5.12. □

Apart from this lemma we will also need the large image property for certain induced representation, which we will use in chapter 7. Namely, we have:

Lemma 3.5. ([BCGP21] Lemma 7.5.22) Suppose that $p > 3$, F_1/F is a quadratic extension such that F_1 is unramified at p, and $\bar{r} : G_{F_1} \to \mathrm{GL}_2(\mathbb{F}_p)$ restricted to $G_{F_1(\zeta_p)}$ has image $\mathrm{SL}_2(\mathbb{F}_p)$. Choose $\sigma \in G_F \setminus G_{F_1}$, and assume that the determinants $\det \bar{r}^\sigma = \det \bar{r}$ are equal to $\bar{\epsilon}^{-1}$ but the projective images $\mathrm{Proj}\, \bar{r}^\sigma \ncong \mathrm{Proj}\, \bar{r}$ are distinct. Let $\bar{\rho} := \mathrm{Ind}_{G_{F_1}}^{G_F} \bar{r} : G_F \to \mathrm{GSp}_4(\mathbb{F}_p)$, then $\bar{\rho}$ is vast and tidy.

3.2 Distinguished Ordinary Primes

Lemma 3.6. (cf. Lemma 4.2 [CGH19]) For a GSp_4-type abelian variety A/F that satisfies the large image property, given the characteristic polynomial of Frob_v for a prime v with residue characteristic p,

$$Q_v(X) = X^4 - a_v X^3 + b_v X^2 - p\zeta a_v X + p^2\zeta^2$$

with coefficients in K and ζ a root of unity with uniformly bounded order, then there is no linear relation between p, a_v^2, b_v that can hold for a set of primes in F with positive density.

Proof. Note that p, a_v^2 and b_v are respectively the trace of Frob_v on $K_\lambda(1)$, $V_{A,\lambda} \otimes V_{A,\lambda}$, and $\wedge^2 V_{A,\lambda}$. If there were such a relation, we would have a representation W built out of copies of the above three spaces such that Frob_v has zero trace for a set of primes with positive density.

From Lemma 3.1, we know that for sufficiently large l and $\lambda \mid l$, the image of the residual representation $\bar{\rho}_{A,\lambda}$ contains at least $\mathrm{Sp}_4(K_{(\lambda)})$, so the image of G_F on $\mathrm{GL}(V_{A,\lambda})$ is at least a subset of $\mathrm{GSp}_4(\mathcal{O}_{K_\lambda})$ with positive measure. Thus a corresponding relation must hold for elements in GSp_4 on a set of positive measure. Namely, if we write the eigenvalue of an element in $\mathrm{GSp}_4(\mathcal{O}_{K_\lambda})$ as α, β, $p\zeta\beta^{-1}$, $p\zeta\alpha^{-1}$, then there would exist a linear relation between

$$p,\ (\alpha + \beta + p\zeta\alpha^{-1} + p\zeta\beta^{-1})^2, \text{ and}$$

$$2p\zeta + \alpha\beta + p\zeta\alpha^{-1}\beta + p\zeta\alpha\beta^{-1} + p^2\zeta^2\alpha^{-1}\beta^{-1}$$

that holds on a set with positive measure, hence for all α, β, p, which is not possible.

□

Definition 3.7. We define p a good prime in $\mathbb{Q}$ if it satisfies the following properties:

(1) p totally splits in K and F, and unramified in D if applicable.

(2) There exists a prime $\mathfrak{p} \mid p$ of K such that A is $\mathfrak{p}$-distinguished ordinary.

(3) For all $v \mid p$ in F, The residual representations $\bar{\rho}_{A,\mathfrak{p}} : G_{F_v} \to \mathrm{GSp}_4(\mathbb{F}_p)$ at least contain $\mathrm{Sp}_4(\mathbb{F}_p)$.

Lemma 3.8. Let A/F be an abelian surface with $\mathrm{End}(A) = \mathbb{Z}$, or a fake abelian surface with $\mathrm{End}(A)$ an order in a quaternion algebra D over $\mathbb{Q}$. If A satisfies the large image property, then the good primes have relative density one in the set of primes that splits totally in F.

Proof. This is essentially Lemma 9.2.5 of [BCGP21].

For primes p that splits totally in F and $v \mid p$ in F, since each root π of the characteristic polynomial satisfies that for any embedding $\psi : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\psi(\pi)| = \sqrt{p}$, we have $|\psi(a_v)| \leq 4\sqrt{p}$ and $|\psi(b_v) - 2p| \leq 4p$ for every ψ, thus $|\psi(a_v^2 - 4b_v)| \leq 40p$ for every ψ.

Now Lemma 3.6 shows that for each integer c such that $|c| \leq 6$, the relation $b_v = cp$ can only hold for a set of primes with zero density. This means that the primes such that $p \nmid b_v$, or the ordinary primes, will have relative density one.

Similarly, for each integer $|c| \leq 40$, the relation $a_v^2 - 4b_v = cp$ can only hold for a set of primes with zero density. This means that the distinguished primes will also have relative density one and this proves the lemma. □

The previous lemma suggests that there are a sufficient amount of good primes for abelian varieties of GSp_4 type when the coefficient field $K = \mathbb{Q}$. For general K, we have the following result:

Lemma 3.9. Let A/F be a strongly positive ordinary abelian variety of GSp_4 type with descent data to $\mathbb{Q}$ and has large image property, then the good primes have positive relative density in the set of primes that splits totally in K and F.

Proof. Lemma 3.1 shows that primes p that satisfy (3) have relative density one, and thus the primes that satisfy both (3) and p-ordinariness have positive relative density. From now on we assume that p splits completely in both F and the coefficient field K. Again we use the fact from the previous lemma that $|\psi(a_v^2 - 4b_v)| \leq 40p$ for every ψ.

Recall that $\mathfrak{p}$-distinguished ordinariness is equivalent to that $\mathfrak{p} \nmid a_v^2 - 4b_v$ for all $v \mid p$. But since A is with descent data to $\mathbb{Q}$, for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have $a_v^\sigma = \chi(\sigma)a_v$ and $b_v^\sigma = \chi(\sigma)^2 b_v$. Thus for sufficiently large p, $\mathfrak{p} \mid a_v^2 - 4b_v$ is equivalent to $\mathfrak{p} \mid (a_v^\sigma)^2 - 4b_v^\sigma$.

If A is not $\mathfrak{p}$-distinguished ordinary for any $\mathfrak{p}$, then for each $\mathfrak{p}$, $\mathfrak{p} \mid a_v^2 - 4b_v$ for at least one $v \mid p$ thus for all $v \mid p$ due to the above deduction. Therefore for all $v \mid p$, $a_v^2 - 4b_v$ must be divisible by every prime above p, which means that $p \mid a_v^2 - 4b_v$. There are only finite number of choices for $\frac{1}{p}(a_v^2 - 4b_v)$ since they are algebraic integers in some fixed finite extension of K. And by Lemma 3.6, for each choice such v has zero density. This means that primes satisfying (2) has positive relative density in the set of primes that totally splits in K. □

Suppose A/F is a strongly positive ordinary abelian variety of GSp_4 type with descent data to $\mathbb{Q}$, then we can choose p and q be two good primes and $\mathfrak{p}$, $\mathfrak{q}$ be two primes in K above p and q respectively such that A is both $\mathfrak{p}$- and $\mathfrak{q}$- distinguished ordinary. Furthermore, we can choose $\mathfrak{q}$ to be such that it does not divide the discriminant of the characteristic polynomial of Frob_v for all $v \mid p$ in F.

Thus Frob_v will have distinct unit roots, denoted by α, β, and are also distinct mod $\mathfrak{q}$. Thus the representation $\rho_{A,\mathfrak{q}}|_{G_{F_v}}$ and the corresponding residual representation will be of form

$$\rho_{A,\mathfrak{q}}|_{G_{F_v}} = \begin{pmatrix} \lambda_\alpha & 0 & 0 & 0 \\ 0 & \lambda_\beta & 0 & 0 \\ 0 & 0 & \epsilon^{-1}\lambda_\beta^{-1} & 0 \\ 0 & 0 & 0 & \epsilon^{-1}\lambda_\alpha^{-1} \end{pmatrix},$$

$$\bar{\rho}_{A,\mathfrak{q}}|_{G_{F_v}} = \begin{pmatrix} \lambda_{\bar{\alpha}} & 0 & 0 & 0 \\ 0 & \lambda_{\bar{\beta}} & 0 & 0 \\ 0 & 0 & \bar{\epsilon}^{-1}\lambda_{\bar{\beta}}^{-1} & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1}\lambda_{\bar{\alpha}}^{-1} \end{pmatrix},$$

where λ_α (λ_β, etc.) is the character that sends Frob_v to α (β, etc.).

CHAPTER 4

THE SERRE-TATE CANONICAL LIFT

Let p be a prime that totally splits in K and $\mathfrak{p} \mid p$ be a prime above p in K. In this chapter's first part, we derive the analogous results of Serre-Tate canonical lifts for $\mathfrak{p}$-distinguished ordinary principally polarized abelian varieties A over a finite field k of characteristic p.

We first recall the original Serre-Tate theorem.

Theorem 4.1. (Serre-Tate) Let k be a finite field of characteristic $p > 0$, and R be a ring with a nilpotent ideal $I \subset R$ satisfying $R/I \cong k$. We define $\mathscr{A}(R)$ be the category of abelian schemes over $\mathrm{Spec}(R)$, and $\mathscr{D}(R)$ be the category of triples $(A_k, G, \mathcal{E})$, where

- A_k is an abelian scheme over $\mathrm{Spec}(k)$;
- G is a p-divisible group over $\mathrm{Spec}(R)$;
- $\mathcal{E}$ is an isomorphism $A_k[p^\infty] \xrightarrow{\sim} G \times_R k$.

Then the functor $\Phi : \mathscr{A} \to \mathscr{D}$, where

$$A \mapsto (A \times_R k, A[p^\infty], \text{natural } \mathcal{E}),$$

is an equivalence of categories.

Now assume that the abelian variety $\bar{A}/k$ has good ordinary reduction at $\mathfrak{p}$. Let R be an artinian local ring and $\mathfrak{m}_R$ be the maximal ideal and $R/\mathfrak{m}_R = k$. Then there exists an integer n such that $\mathfrak{m}_R^n = (0)$. By the original Serre-Tate theorem, the set of liftings of A to R is equivalent to the liftings of the p-divisible groups

$$A[p^\infty] = \bigoplus_{v|p} A[v^\infty] = A[\mathfrak{p}^\infty] \oplus A[\bar{\mathfrak{p}}^\infty],$$

where

$$A[\bar{\mathfrak{p}}^\infty] = \bigoplus_{v|p,\, v\neq\mathfrak{p}} A[v^\infty].$$

We now want to consider the lift of $A[\mathfrak{p}^\infty]$ to $\mathcal{A}[\mathfrak{p}^\infty]$ in R. Since A is $\mathfrak{p}$-ordinary, we have a canonical product structure of $\mathfrak{p}$-divisible group

$$A[\mathfrak{p}^\infty] = \hat{A} \times T_{\mathfrak{p}} A \otimes (K_{\mathfrak{p}}/\mathcal{O}_{K_{\mathfrak{p}}}),$$

where $T_{\mathfrak{p}} A \otimes (K_{\mathfrak{p}}/\mathcal{O}_{K_{\mathfrak{p}}})$ is the étale quotient, and the toroidal formal group $\hat{A}$ satisfies

$$\hat{A} \cong \mathrm{Hom}_{\mathcal{O}_{K_{\mathfrak{p}}}}(T_{\mathfrak{p}} A^\vee, \hat{\mathbb{G}}_m). \tag{4.1}$$

The identification of (4.1) is given by the following steps:

(i) We have the Weil pairings

$$e_{\mathfrak{p}^n} : A[\mathfrak{p}^n] \times A^\vee[\mathfrak{p}^n] \to \mu_{p^n}.$$

(ii) Restricting the pairing to $\hat{A}$ gives the pairing

$$e_{\mathfrak{p}^n} : \hat{A}[\mathfrak{p}^n] \times A^\vee[\mathfrak{p}^n] \to \mu_{p^n}.$$

This gives an isomorphism

$$\hat{A}[\mathfrak{p}^n] \cong \mathrm{Hom}_{\mathcal{O}_K}(A^\vee[\mathfrak{p}^n], \mu_{p^n}).$$

(iii) Taking inverse limit gives the isomorphism

$$\hat{A} \cong \mathrm{Hom}_{\mathcal{O}_{K_{\mathfrak{p}}}}(T_{\mathfrak{p}} A^\vee, \hat{\mathbb{G}}_m).$$

For a lift $\mathcal{A}/R$ of A/k, the $\mathfrak{p}$-divisible part of $\mathcal{A}$ also has a canonical structure of extension

$$0 \longrightarrow \hat{\mathcal{A}} \longrightarrow \mathcal{A}[\mathfrak{p}^\infty] \longrightarrow T_\mathfrak{p} A \otimes (K_\mathfrak{p}/\mathcal{O}_{K_\mathfrak{p}}) \longrightarrow 0$$

The isomorphism of k-groups (4.1) extends to isomorphism of R-groups

$$\hat{\mathcal{A}} \cong \mathrm{Hom}_{\mathcal{O}_{K_\mathfrak{p}}}(T_\mathfrak{p} A^\vee, \hat{\mathbb{G}}_m)$$

via the pairing

$$E_\mathcal{A} : \hat{\mathcal{A}} \times T_\mathfrak{p} A^\vee \to \hat{\mathbb{G}}_m$$

that canonically extends from the Weil pairing.

Lemma 4.2. Let p be a prime that totally splits in the number fields K and F. If an abelian variety A/F is $\mathfrak{p}$-distinguished ordinary with an $\mathcal{O}_K$ or $\mathcal{O}_D$ (which is an order of a quaternion algebra over K) action, then for any $v \mid p$ in F there exists a lift $\tilde{A}/F_v$ which is also $\mathfrak{p}$-distinguished ordinary and with $\mathcal{O}_K$ or $\mathcal{O}_D$ action, and satisfies that

$$\bar{\rho}_{\tilde{A},\mathfrak{p}}|_{G_{F_v}} = \begin{pmatrix} \bar{\chi}_1 & 0 & 0 & 0 \\ 0 & \bar{\chi}_2 & 0 & 0 \\ 0 & 0 & \bar{\epsilon}^{-1}\bar{\chi}_2^{-1} & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1}\bar{\chi}_1^{-1} \end{pmatrix} = (\bar{\rho}_{A,\mathfrak{p}}|_{G_{F_v}})^{ss},$$

a diagonal matrix.

Proof. Consider the reduction modulo v of A, denoted by the $\mathfrak{p}$-distinguished ordinary abelian variety $\bar{A}$ over the residue field $F_{(v)}$ of F_v (which is isomorphic to $\mathbb{F}_p$). Let $W(F_{(v)})$ be the ring of Witt vectors over $F_{(v)}$, and take a series of Artinian local rings $R_n = W(F_{(v)})/(v^n)$, and let $\mathcal{A}_n$ be the lift of $\bar{A}$ on R_n such that:

(i) $\mathcal{A}_n[\bar{\mathfrak{p}}^\infty] = A|_{W(F_{(v)})/(v^n)}[\bar{\mathfrak{p}}^\infty]$; That is, $\mathcal{A}$ fixes everything else but the $\mathfrak{p}$-divisible part in $A[p^\infty]$.

(ii) The lift $\mathcal{A}_n[\mathfrak{p}^\infty]$ is the unique split lift of $\bar{A}[\mathfrak{p}^\infty]$, that is, the extension

$$0 \longrightarrow \hat{\mathcal{A}}_n \longrightarrow \mathcal{A}_n[\mathfrak{p}^\infty] \longrightarrow T_{\mathfrak{p}}A \otimes (K_{\mathfrak{p}}/\mathcal{O}_{K_{\mathfrak{p}}}) \longrightarrow 0$$

is split.

If we take the limit of the R_n's we obtain a lift

$$\mathcal{A} = \varinjlim \mathcal{A}_n$$

on $W(F_{(v)})$ that also has the canonical product structure

$$\mathcal{A} = \hat{\mathcal{A}} \times T_{\mathfrak{p}}A \otimes (K_{\mathfrak{p}}/\mathcal{O}_{K_{\mathfrak{p}}}).$$

Since $W(F_{(v)})$ is isomorphic to $\mathcal{O}_{F_v}$, we may as well think of $\mathcal{A}$ as defined over F_v.

Now consider any endomorphism $f : A \to A$ that gives an endomorphism $\bar{f}$ on $\bar{A}$ hence an endomorphism on the p-divisible group $\bar{A}[p^\infty]$.

For each n, since $\mathcal{A}_n$ does not change the $\bar{\mathfrak{p}}$-divisible part, the action lifts to an endomorphism on $\mathcal{A}_n[\bar{\mathfrak{p}}^\infty]$, and the split extension structure also canonically gives an endomorphism on the $\mathfrak{p}$-divisible part $\mathcal{A}_n[\mathfrak{p}^\infty]$ that lifts $\bar{f}|_{\bar{A}[\mathfrak{p}^\infty]}$. This means that we can canonically lift $\bar{f}$ to an endomorphism on the whole p-divisible group, thus defining a morphism

$$f_n : (\bar{A}, G, \mathcal{E}) \to (\bar{A}, G, \mathcal{E})$$

on $\mathscr{D}(R_n)$. By Serre-Tate Theorem (considering the inverse functor of Φ), this gives an endomorphism $f_n \in \mathrm{End}(\mathcal{A}_n)$. Passing to limits we know that any endomorphism of A gets

deformed to an endomorphism on $\mathcal{A}$ thus we still have an $\mathcal{O}_K$ or $\mathcal{O}_D$ action on $\mathcal{A}$.

In particular, we have the Frobenius morphism Frob_v on $\bar{A}$ lifts and since the action of it on the dual of $T_{\mathfrak{p}}\bar{A}$ is of form $\begin{pmatrix} \bar{\chi}_1 & 0 \\ 0 & \bar{\chi}_2 \end{pmatrix}$ due to $\mathfrak{p}$-distinguished ordinariness, the Galois representation associated to the lift $\mathcal{A}$ would be of form

$$\bar{\rho}_{\mathcal{A},\mathfrak{p}}\big|_{G_{F_v}} \cong \begin{pmatrix} \bar{\chi}_1 & 0 & 0 & 0 \\ 0 & \bar{\chi}_2 & 0 & 0 \\ 0 & 0 & \bar{\epsilon}^{-1}\bar{\chi}_2^{-1} & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1}\bar{\chi}_1^{-1} \end{pmatrix},$$

a diagonal matrix due to the product structure on $\mathcal{A}[\mathfrak{p}^\infty]$.

□

We call the $\tilde{A}$ we choose in Lemma 4.2 the $\mathfrak{p}$-*canonical lift* of $\bar{A}$ to F_v.

Corollary 4.3. For a strongly positive ordinary abelian variety A/F of GSp_4 type with descent data to $\mathbb{Q}$ and satisfies the large image property, there exist good primes p and q of A with $\mathfrak{p} \mid p$ and $\mathfrak{q} \mid q$ primes in K such that A is $\mathfrak{p}$- and $\mathfrak{q}$-distinguished ordinary, and for each $w \mid q$ in F we can find a lift $\tilde{A}_w/F_w$ of $\bar{A}/F_{(w)}$ such that both $\bar{\rho}_{\tilde{A}_w,\mathfrak{q}}|_{G_{F_v}}$ (for all $v \mid p$ in F) and $\bar{\rho}_{\tilde{A}_w,\mathfrak{q}}|_{G_{F_w}}$ are diagonal matrices.

Proof. We choose the good primes p and q in $\mathbb{Q}$ with $\mathfrak{p} \mid p$ and $\mathfrak{q} \mid q$ in K according to the arguments in the end of §3.2, and we know that $\bar{\rho}_{A,\mathfrak{q}}|_{G_{F_v}}$ is of form

$$\begin{pmatrix} \lambda_{\bar{\alpha}} & 0 & 0 & 0 \\ 0 & \lambda_{\bar{\beta}} & 0 & 0 \\ 0 & 0 & \bar{\epsilon}^{-1}\lambda_{\bar{\beta}}^{-1} & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1}\lambda_{\bar{\alpha}}^{-1} \end{pmatrix},$$

a diagonal matrix.

Now if we take $\tilde{A}_w/F_w$ to be the $\mathfrak{q}$-canonical lift of $\bar{A}/F_{(w)}$, then it does not change the Galois representation on G_{F_v}. (in other words, it does not change the action of Frob_v acting on the q-divisible group $\tilde{A}_w[q^\infty]$.) Thus $\bar{\rho}_{\tilde{A}_w,\mathfrak{q}}|_{G_{F_v}} = \bar{\rho}_{A,q}|_{G_{F_v}}$ would also be diagonal.

On the other hand, since $\bar{A}/F_{(w)}$ is $\mathfrak{q}$-distinguished ordinary with an $\mathcal{O}_K$ or $\mathcal{O}_D$ action, by Lemma 4.2 the $\mathfrak{q}$-canonical lift $\tilde{A}_w$ satisfies that

$$\bar{\rho}_{\tilde{A}_w,\mathfrak{q}}|_{G_{F_w}} = \left(\bar{\rho}_{A,\mathfrak{q}}|_{G_{F_w}}\right)^{ss} = \begin{pmatrix} \bar{\chi}1 & 0 & 0 & 0 \\ 0 & \bar{\chi}2 & 0 & 0 \\ 0 & 0 & \bar{\epsilon}^{-1}\bar{\chi}_2^{-1} & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1}\bar{\chi}_1^{-1} \end{pmatrix},$$

is also a diagonal matrix. □

CHAPTER 5

MODULI SPACES OF GSp_4 TYPE ABELIAN VARIETIES

The goal of this chapter is to prove the following statement:

Theorem 5.1. Given a type (real or fake) and a totally real number field F of degree d, and a quarternion algebra B/F if the type is fake, suppose $\mathfrak{p}, \mathfrak{q}$ are two primes of F above p and q respectively that split completely in F, then if we consider the moduli space Y/E of all triples $(A, \iota_{\mathfrak{p}}, \iota_{\mathfrak{q}})$ consisting of

- An abelian variety A of dimension g over E of GSp_4 type with $\mathrm{End}^0(A) = F$ (if real, so $g = 2d$) or B (if fake, so $g = 4d$);
- A full level-$\mathfrak{pq}$ structure, in the sense of
  - An isomorphism of group schemes $\iota_{\mathfrak{p}} : A[\mathfrak{p}] \to (\mathbb{Z}/p\mathbb{Z})^g \times \mu_p^g$;
  - An isomorphism of group schemes $\iota_{\mathfrak{q}} : A[\mathfrak{q}] \to (\mathbb{Z}/q\mathbb{Z})^g \times \mu_q^g$;

  then Y is smooth and geometrically connected.

5.1 Construction of the Shimura Variety

We will first construct the Shimura variety corresponding to GSp_4 type abelian varieties following the arguments of [KR00]. Now in either cases (real or fake) we let B be an indefinite quaternion algebra over a totally real field F with canonical involution ι and a maximal order $\mathcal{O}_B$ such that $\mathcal{O}_B^{\iota} = \mathcal{O}_B$, and let $C = M_2(B)$ with the involution $x' = (x^{\iota})^t$. Let $\{\sigma_1, \cdots, \sigma_d\}$ be the set of real embeddings $F \hookrightarrow \mathbb{R}$.

Let $V = \{x \in C : x' = x, \mathrm{tr}(x) = 0\}$ be a five dimensional vector space over F with a quadratic form q defined by $x^2 = q(x) \cdot \mathbf{1}_2$ of signature (3,2). Then the natural

homomorphism $C(V) \to C$ extending the inclusion $V \hookrightarrow C$ to its Clifford algebra gives an isomorphism $C^+(V) \cong C$. We may let

$$G = \mathrm{Res}_{F/\mathbb{Q}}\big(\mathrm{GSpin}(V)\big)$$

be the restriction of scalars from F to $\mathbb{Q}$ of the general Spin group

$$\mathrm{GSpin}(V) = \{g \in C^\times : gg' = \nu(g)\}$$

of V, so

$$G(\mathbb{Q}) \cong \mathrm{GSpin}(3, 2, F),$$

and $G(\mathbb{R}) = G(\mathbb{Q}) \otimes \mathbb{R}$ acts on $V^d(\mathbb{R})$, which is d copies of $V(\mathbb{R})$, via the real embeddings σ_i on the i-th component. We define $G(\mathbb{Q})^+$ to be the connected component of elements in $G(\mathbb{Q})$ such that $\nu(g)$ is a totally positive element in $F^\times$.

Let $\mathcal{D}$ be the set of $z = (z_1, \cdots z_d)$ such that $z_i = (z_{i1}, z_{i2})$ represents a negative 2-plane in the i-th copy of $V(\mathbb{R})$ in $V^d(\mathbb{R})$ with z_{i1} and z_{i2} be a properly oriented basis such that the restriction of the quadratic form q from $V(\mathbb{R})$ to z_i has matrix $-\mathbf{1}_2$ for the basis z_{i1}, z_{i2}. Then

$$J_z := (z_{i1} z_{i2})_{i=1}^d \in C^d(\mathbb{R})$$

lies in $G(\mathbb{R})$ and defines a morphism on $\mathbb{R}$:

$$h_z : \mathbb{S} \to G, \quad h_z(i) = J_z$$

since $J_z' = -J_z$ and $J_z J_z' = \mathbf{1}$. Thus $\mathcal{D}$ can be viewed as the space of conjugacy classes of such maps under the action of the group $G(\mathbb{R})$ by

$$J_{gz} = g J_z g^{-1}.$$

Choose an element $\tau \in B^\times$ such that $\tau^\iota = -\tau$, $\tau^2 = -D(B)$ (where $D(B)$ is the discriminant of B) and $\tau\mathcal{O}_B\tau^{-1} = \mathcal{O}_B$, then the map $x \mapsto x^* = \tau x^\iota \tau^{-1}$ gives a positive involution on $\mathcal{O}_B$. If we write $\mathcal{O}_C = M_2(\mathcal{O}_B)$ and $W = \mathcal{O}_C^d$, viewed as left and right C (thus, $G(\mathbb{Q})$)-module, and let $\alpha = \mathrm{diag}(\tau, \tau) \in \mathcal{O}_C$, then $\alpha' = -\alpha$ and the map

$$x \mapsto x^* := \alpha x' \alpha^{-1}$$

is a positive involution on $\mathcal{O}_C$.

We may define the alternating form

$$\langle \cdot, \cdot \rangle : W \times W \to \mathbb{Q}$$

by

$$\langle x, y \rangle = \sum_{i=1}^{d} \mathrm{tr}(y_i' \alpha^{-1} x_i).$$

Then for $c \in C$, we have

$$\langle cx, y \rangle = \sum_{i=1}^{d} \mathrm{tr}(y_i' \alpha^{-1}(cx_i)) = \sum_{i=1}^{d} \mathrm{tr}(y_i' \alpha^{-1} c \alpha \alpha^{-1} x_i) = \langle x, c^* y \rangle$$

and

$$\langle xc, y \rangle = \sum_{i=1}^{d} \mathrm{tr}(y_i' \alpha^{-1} x_i c) = \sum_{i=1}^{d} \mathrm{tr}(c y_i' \alpha^{-1} x_i) = \langle x, yc' \rangle.$$

In particular,

$$\langle xg, yg \rangle = \nu(g) \langle x, y \rangle,$$

for $g \in G$ and especially,

$$\langle xJ_z, yJ_z \rangle = \langle x, y \rangle.$$

This means that $(W(R), \langle \cdot, \cdot \rangle)$ is a skew-Hermitian $(\mathcal{O}_C(R), *)$-module with G being

the algebraic group of symplectic F-linear automorphisms up to a similitude factor $\nu \in \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \otimes_{\mathbb{Q}} R$.

Now for a compact open subgroup $K \subset G(\mathbb{A}^\infty)$, consider the functor M_K that associates to a locally noetherian scheme S over $\mathbb{Q}$ the set of quadruples $(A, \iota, \lambda, \bar{\eta})$, such that

- A is a abelian scheme over S up to isogeny;
- $\iota : C \to \mathrm{End}^0(A)$ is a homomorphism that satisfies a determinant condition

$$\det\big(\iota(c) \mid \mathrm{Lie}(A)\big) = N^2(c),$$

  where N is the reduced norm on C. It also induces a homomorphism from C to the endomorphism of the dual of A denoted by $\iota^\vee : C \to \mathrm{End}^0(A^\vee)$.

- λ is a $\mathbb{Q}$-class of polarizations on A that is compatible to the action by C, that is,

$$\lambda \circ \iota(c) = \iota^\vee(c) \circ \lambda.$$

- $\bar{\eta}$ is a K-class of $C \otimes \mathbb{A}^\infty$-linear isomorphisms

$$\eta : V_{\mathbb{A}^\infty}(A) \to W(\mathbb{A}^\infty) = W(\hat{\mathbb{Z}}) \otimes \mathbb{Q}.$$

  that respect the symplectic forms on both sides up to a scalar multiple in $\mathbb{A}^\infty$. Here

$$V_{\mathbb{A}^\infty}(A) = \prod_l T_l(A) \otimes \mathbb{Q} \cong H_1(A, \mathbb{A}^\infty)$$

  is the adelic Tate module of A.

Note that this abelian scheme A will have dimension $8d$ over S.

Proposition 5.2. (cf. [KR00, Prop. 1.1]) For K neat (in the sense of [Lan13]) this moduli problem is representable by a smooth quasi-projective scheme M_K over $\mathbb{Q}$ and

$$M_K(\mathbb{C}) \cong Sh_K(G, \mathcal{D})(\mathbb{C}),$$

where $Sh_K(G, \mathcal{D})$ is the Shimura variety defined by the PEL datum $(G, \mathcal{D})$.

Proof. The representability is proved in §5 of [Kot92] (our moduli problem is in case C so no further discussion is required), and we now show the correspondence of complex points following [KR00].

For each $i \in \{1, 2, \cdots d\}$ and thus an inclusion $\sigma_i : F \hookrightarrow \mathbb{R}$, We can find $\tau_i = D(B)^{-1/2}\tau \in B^\times(\mathbb{R})$ so $\tau^2 = -1$, and let $\alpha_i = D(B)^{-1/2}\alpha$. Choose an element $\beta_i \in B^\times(\mathbb{R})$ such that $\beta_i\tau = -\tau\beta_i$, $\beta^\iota = -\beta$ and normalized such that $\beta^2 = 1$.

Then $\begin{pmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \tau_i\beta_i \\ -\tau_i\beta_i & 0 \end{pmatrix}$ forms a standard basis of a negative 2-plane on the i-th component of $V^d(\mathbb{R})$, denoted by z_i, and

$$\begin{pmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau_i\beta_i \\ -\tau_i\beta_i & 0 \end{pmatrix} = \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix} = \alpha_i$$

.

So we write $z_0 = (z_i)_{i=1}^d$ and $J_{z_0} = (\alpha_i)_{i=1}^d$. Then if $h = h_{z_0}$ be the map such that $h(i) = J_{z_0}$ then

$$\langle xJ_{z_0}, y\rangle = -\langle x, yJ_{z_0}\rangle = \langle yJ_{z_0}, x\rangle,$$

and

$$\langle xJ_{z_0}, x\rangle = \sum_{i=1}^d \mathrm{tr}\big(x_i'\alpha^{-1}x_i\alpha_i\big) = D(B)^{-1/2}\sum_{i=1}^d \mathrm{tr}(x_i^*x_i) > 0.$$

Moreover, for $g \in G$,

$$\langle xJ_{gz_0}, x\rangle = \langle xgJ_{z_0}g^{-1}, x\rangle = \nu(g)^{-1}\langle xgJ_{z_0}, xg\rangle.$$

This means that if we write $\mathcal{D}^+$ be the connected component of $\mathcal{D}$ containing z_0 then for $z \in \mathcal{D}^+$, $z = gz_0$ for some $g \in G^+$ and the pairing $\langle xJ_z, y\rangle$ is symmetric and positive definite.

Now for each $z \in \mathcal{D}^+$ we obtain a principally polarized abelian variety A_z be the $8d$-dimensional complex structure $(W(\mathbb{R}), J_z)$ modulo the lattice $W(\mathbb{Z})$, and with principle polarization λ induced by the pairing $\langle\cdot,\cdot\rangle$. It carries an action of $\mathcal{O}_C$ by left multiplication (thus the homomorphism $\iota : C \to \mathrm{End}^0(A_z)$) and such that it is compatible with the polarization λ. Also, we have the isomorphism of the finite adelic Tate module

$$V_{\mathbb{A}^\infty}(A_z) \cong W(\hat{\mathbb{Z}}) \otimes \mathbb{Q} \cong W(\mathbb{A}^\infty).$$

We can correspond the pair $(z, gK) \in \mathcal{D}^+ \times G(\mathbb{A}^\infty)/K$ to the quadruple $(A_z, \iota, \lambda, \bar{\eta})$ where ι and λ are defined as above and $\bar{\eta}$ be the K-class of isomorphism induced by right multiplication by g on $W(\mathbb{A}^\infty)$, that is, the composition

$$V_{\mathbb{A}^\infty}(A_z) \cong W(\mathbb{A}^\infty) \xrightarrow{\cdot g} W(\mathbb{A}^\infty).$$

For $\gamma \in G(\mathbb{Q})^+$, we can establish an isomorphism between quadruples corresponding to pairs (z, gK) and $(\gamma z, \gamma gK)$ via the element in $\mathrm{Hom}(A_z, A_{\gamma z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined upon $W(\mathbb{R})$ by right multiplication by γ^{-1}, and thus we have the correspondence

$$M_K(\mathbb{C}) \cong G(\mathbb{Q})^+\backslash\mathcal{D}^+ \times G(\mathbb{A}^\infty)/K \cong G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}^\infty)/K.$$

□

5.2 Proof of Theorem 5.1

Now we are able to describe the moduli problem in Theorem 5.1. If B is of fake type (an indefinite quaternion algebra that is division), then choose an idempotent element $e \in \mathcal{O}_C$ and eA gives an abelian variety of dimension $4d$ and has action by $\mathcal{O}_B$. Conversely for a fake GSp_4 type abelian variety A, $A \times A$ gives an abelian variety that has $\mathcal{O}_C = M_2(\mathcal{O}_B)$ action.

On the other hand, if B is of real type (so $B \cong M_2(F)$), then we may choose idempotent elements e in $\mathcal{O}_C$ and e' in $\mathcal{O}_B$ and $e'eA$ gives an abelian variety of dimension $2d$ and has action by $\mathcal{O}_F$. Conversely for a real GSp_4 type abelian variety A, $A \times A$ gives an ablian variety that has $\mathcal{O}_B = M_2(\mathcal{O}_F)$ action, and the product of four copies of A gives an abelian variety that has $\mathcal{O}_C$ action.

Therefore the moduli problems in Theorem 5.1 can be identified with the moduli problem described in §5.1 over E with a certain full level-$\mathfrak{pq}$ structure. Smoothness easily follows from the construction of the Shimura variety, and in order to prove Theorem 5.1 we need to check connectedness.

Notice that the algebraic group $G \cong \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_4$ is connected, we can deduce that $(G, \mathcal{D})$ is a PEL Shimura datum of type C ([Mil05]), and connectedness now follows from the arguments in §8 of [Kot92], and the Hasse principle of G.

CHAPTER 6

POTENTIAL MODULARITY OF GSp_4 TYPE ABELIAN VARIETIES

6.1 The Theorem of Moret-Bailly

In this chapter, we first introduce the well-known theorem of Moret-Bailly,

Theorem 6.1. (Proposition 3.1.1, [BLGGT14]) Let $K^0/K/E$ be number fields with K^0/K and K/E Galois. Suppose S be a finite set of places of E and S^K be the set of places of K above S. For $v \in S^K$, let L'_v/K_v be a finite Galois extension with $L'_{\sigma v} = \sigma L'_v$ for $\sigma \in G_E$. Suppose also that T/K is a smooth geometrically connected variety and that for each $v \in S^K$ we are given a non-empty, $\mathrm{Gal}(L'_v/K_v)$-invariant, open subset $\Omega_v \subset T(L'_v)$.

Then there exists a finite Galois extension L/K and a point $P \in T(L)$ such that

- L/E is Galois;
- L/K is linearly disjoint from K^0/K;
- if $v \in S^K$ and w a prime of L above v then L_w/K_v is isomorphic to L'_v/K_v and $P \in \Omega_v \subset T(L'_v) \cong T(L_w)$.

The following proposition is a consequence of the theorem of Moret-Bailly, which is originally proved in [Cal12] and generalized in [BCGP21].

Lemma 6.2. Let G be a finite group, $E/\mathbb{Q}$ a finite Galois extension, and S a set of primes of E. Let E'/E, E^0/E be finite extensions, linearly disjoint from each other.

Let S' be the set of places in E' over S. For each finite prime $v \in S'$, let H'_v/E'_v be a finite Galois extension with a fixed inclusion $\phi_v : \mathrm{Gal}(H'_v/E'_v) \to G$ with image D_v. For each real prime $v \in S'$, let $c_v \in G$ be an element of order dividing 2.

Then there exists a number field K/E and a finite Galois extension L/K such that if $K' = KE'$ and $L' = LE'$, then

(1) There is an isomorphism $\mathrm{Gal}(L/K) \to G$;

(2) L/E is linearly disjoint from E^0E'/E;

(3) All places of S' splits completely in K';

(4) For all local places w above $v \in S'$, the local extension L'_w/K'_w is equal to H'_v/E'_v, with a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(L'_w/K'_w) & \longrightarrow & D_w \\ \| & & \| \\ \mathrm{Gal}(H'_v/E'_v) & \xrightarrow{\phi_v} & D_v \end{array}$$

(5) For all real places w of K' above $v \in S'$, complex conjugation $c_w \in G$ is conjugate to c_v.

Proof. This is Proposition 9.1.12 of [BCGP21].

□

We are going to use this lemma to show that in our construction in Lemma 4.3, the diagonal representations $\bar{\rho}_{\tilde{A}_w,\mathfrak{q}}$ restricted to G_{F_w} and all G_{F_v}'s are potentially induced representations.

Lemma 6.3. (cf. Lemma 9.2.7 [BCGP21]) Let A/F be a strongly positive ordinary GSp_4-type abelian variety with descent data to $\mathbb{Q}$ and satisfying the large image property, and let p, q be good primes chosen as in Lemma 4.3. Fix a real quadratic extension F'/F such that p and q split completely and that it is linearly disjoint from the kernel of the action G_F on $A[\mathfrak{p}]$ and $A[\mathfrak{q}]$. Then we can find a totally real Galois extension F_1/F of F and a quadratic extension $F_2/F_1 := F_1F'/F_1$, and a representation $\bar{r} : G_{F_2} \to \mathrm{GL}_2(\mathbb{F}_q)$, such that

(1) p, q splits totally in F_1;

(2) F_1/F is linearly disjoint from F'/F and the kernel of the action $G_{\mathbb{Q}}$ on $A[\mathfrak{p}]$ and $A[\mathfrak{q}]$;

(3) $\det \bar{r} = \bar{\epsilon}^{-1}$.

(4) $\bar{r}(G_{F_2}) = \mathrm{GL}_2(\mathbb{F}_q)$, and the projective image of $\bar{r}$ is not equal to its conjugate under $\mathrm{Gal}(F_2/F_1)$.

(5) Let $\bar{\rho} = \mathrm{Ind}_{G_{F_2}}^{G_{F_1}} \bar{r}$ with similitude character $\bar{\epsilon}^{-1}$, then it is vast and tidy, and satisfies

(i) For all primes $w \mid q$ of F and $w_1 \mid w$ of F_1,

$$\bar{\rho}|_{G_{F_{1,w_1}}} \cong \bar{\rho}_{\tilde{A}_w,\mathfrak{q}}|_{G_{F_w}} = \left(\bar{\rho}_{A,\mathfrak{q}}|_{G_{F_w}}\right)^{ss},$$

(ii) For all primes $v \mid p$ of F and $v_1 \mid v$ of F_1,

$$\bar{\rho}|_{G_{F_{1,v_1}}} \cong \bar{\rho}_{\tilde{A}_w,\mathfrak{q}}|_{G_{F_v}} = \bar{\rho}_{A,\mathfrak{q}}|_{G_{F_v}}.$$

Proof. Fix a prime $\ell \nmid pq$ in F. We are going to apply Lemma 6.2 with $G = \mathrm{GL}_2(\mathbb{F}_q)$, and with $E = F$, E^0/F be the extension containing the kernels of $\bar{\rho}_{A,v}$ and $\bar{\rho}_{A,w}$ in G_F. Let E'/F be any quadratic extension linearly disjoint from E^0/F. Let $S = \{\ell, p, q, \infty\}$. For infinite places v, we let c_v to have eigenvalues 1 and -1.

For prime $v \mid p$ in F, if we write $v = v_1 v_2$ in E', then we can let ϕ_{v_1} to be the representation corresponding to $\begin{pmatrix} \lambda_{\bar{\alpha}} & 0 \\ 0 & \bar{\epsilon}^{-1}\lambda_{\bar{\alpha}}^{-1} \end{pmatrix}$ and ϕ_{v_2} corresponding to $\begin{pmatrix} \lambda_{\bar{\beta}} & 0 \\ 0 & \bar{\epsilon}^{-1}\lambda_{\bar{\beta}}^{-1} \end{pmatrix}$. Similarly, we can write $w = w_1 w_2$ in E' for $w \mid q$ and let ϕ_{w_1} correspond to $\begin{pmatrix} \bar{\chi}_1 & 0 \\ 0 & \bar{\epsilon}^{-1}\bar{\chi}_1^{-1} \end{pmatrix}$ and ϕ_{w_2} correspond to $\begin{pmatrix} \bar{\chi}_2 & 0 \\ 0 & \bar{\epsilon}^{-1}\bar{\chi}_2^{-1} \end{pmatrix}$.

Finally for $\ell = l_1 l_2$ in E', we let ϕ_{l_1} and ϕ_{l_2} to be such that they both have determinant $\bar{\epsilon}^{-1}$, ϕ_{l_1} be unramified while $\mathrm{Proj}\,\phi_{l_2}$ be ramified.

Given the above setup, we can find a totally real extension F_1 (corresponding to the field K as in Lemma 6.2), and $F_2 = F'F_1$ be the quadratic extension that satisfies (1) and (2), together with a representation $\bar{r} : G_{F_1} \to \mathrm{GL}_2(\mathbb{F}_q)$ that is vast and tidy due to Lemma 3.5. It satisfies (5) due to the construction in Lemma 4.3. (4) is satisfied because of the behavior of $\bar{r}$ on primes above ℓ. By further replacing F by a totally real quadratic extension (in which p, q, ℓ totally split) we can assume that the character $\bar{\epsilon}\det(\bar{r})$ has a square root in F_1. Replacing $\bar{r}$ by its twist by the square root of $\bar{\epsilon}\det(\bar{r})$ gives a representation that satisfies (3).

□

Theorem 6.4. Given a non-CM abelian variety A/F of GSp_4-type with descent data to $\mathbb{Q}$ that is strongly positive ordinary and has large image property. Let the totally real field K be $\mathrm{End}^0(A)$ if A is of real type and the base field of the quarternion algebra $D = \mathrm{End}^0(A)$ if A is of fake type. Then we can find primes p, q totally split in F, K (and unramified in D if A is of fake type) and an abelian variety $\mathcal{A}/F'$, where F' is a totally real extension of F in which p, q totally split, such that there exists $\mathfrak{p} \mid p$, $\mathfrak{q} \mid q$ in K such that $\mathcal{A}[\mathfrak{p}] \cong A[\mathfrak{p}]$, and $\mathcal{A}[\mathfrak{q}]$ is induced as representations.

Proof. We choose the good primes p and q, together with $\mathfrak{p} \mid p$ and $\mathfrak{q} \mid q$ in K. For $w \mid q$ in F, we can find a $\mathfrak{q}$-canonical lifting $\tilde{A}_w$ as in Lemma 4.3. Then from Lemma 6.3 there exists a totally real extension F_1/F such that $\tilde{A}_w[\mathfrak{q}]$ gives the dual of the induced representation $\bar{\rho} = \mathrm{Ind}_{G_{F_2}}^{G_{F_1}} \bar{r}$.

Now let Y/F_1 to be the moduli space of all triples $(B, \iota_{\mathfrak{p}}, \iota_{\mathfrak{q}})$ consisting of GSp_4-type abelian varieties B with endomorphism ring either $\mathcal{O}_K$ or $\mathcal{O}_D$, together with symplectic isomorphisms

- $\iota_{\mathfrak{p}} : B[\mathfrak{p}] \to A[\mathfrak{p}]|_{G_{F_1}}$;
- $\iota_{\mathfrak{q}} : B[\mathfrak{q}] \to \bar{\rho}^{\vee}$ if B is of real type, or $\iota_{\mathfrak{q}} : B[\mathfrak{q}] \to (\bar{\rho}^{\vee})^2$ if B is of fake type,

as in §5. For finite primes λ, let $\Omega_\lambda = Y^{\mathrm{ord}}(F_{1,\lambda}) \subset Y(F_{1,\lambda})$ be the subspace of those that has good ordinary reduction. Then for every prime $\lambda \mid pq$, $\Omega_\lambda \neq \emptyset$, since we can choose A itself for $\lambda \mid p$, and $\tilde{A}_w$ for $\lambda \mid w \mid q$ due to the construction in Lemma 4.3. For primes $\lambda \mid \infty, Y(F_{1,\lambda})$ is also nonempty because of the assumption $\det \bar{r} = \bar{\epsilon}^{-1}$.

By the theorem of Moret-Bailly, we apply Theorem 6.1 wth $E = F, K = F_1$, K^0 be the compositum of F_2 and the kernels of G_F on $A[\mathfrak{p}]$, $\tilde{A}_w[\mathfrak{q}]$, the extensions L'_v/K_v be trivial, and $T/K = Y/F_1$ be the smooth geometrically connected variety, together with the open subsets Ω_v's.

Thus we can find a totally real field F'/F, linearly disjoint from F_2/F_1 and the kernels of G_F on $A[\mathfrak{p}]$, $\tilde{A}_w[\mathfrak{q}]$, such that p, q splits completely, and $Y(F') \cap \bigcap_{\lambda|pq} \Omega_\lambda$ is non-empty. This means that there exists an abelian variety $\mathcal{A}/F'$ such that $\mathcal{A}$ has good ordinary reduction mod every prime above p, q, and such that $\mathcal{A}[\mathfrak{p}] \cong A[\mathfrak{p}]$, and $\mathcal{A}[\mathfrak{q}]$ is induced as Galois representation.

□

Remark 6.5. Note that when K is $\mathbb{Q}$, which is the case where A is an abelian surface or fake abelian surface, then we no longer need the descent data and the ordinariness assumption to find the good primes p and q due to Theorem 3.8.

In these cases, we have $\mathfrak{p} = p$ and $\mathfrak{q} = q$, so the $\mathfrak{q}$-canonical lifts $\tilde{A}_w$ become the usual canonical lift of $\bar{A}/F_{(w)}$ to F_w, and the above Lemma 6.3 becomes Lemma 9.2.7 in [BCGP21]. Theorem 6.4 holds due to the smoothess and connectedness on moduli spaces of abelian surfaces or fake abelian surfaces with full level-pq structure.

6.2 Potential Modularity Theorems

First we recall the Galois representations associated to automorphic GSp_4 representations.

Theorem 6.6. ([BCGP21], Theorem 2.7.1-2) Suppose that F is a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ of parallel weight 2 and has central character $|\cdot|^2$.

Fix a prime p and a prime $\lambda \mid p$ in K, then there is a continuous semisimple representation $\rho_{\pi,\lambda} : G_F \to \mathrm{GSp}_4(\bar{K}_\lambda)$ satisfying the following properties:

(1) The similitude character is $\nu(\rho_{\pi,\lambda}) = \epsilon^{-1}$.

(2) For each prime v of F, we have

$$\mathrm{WD}(\rho_{\pi,\lambda}|_{G_{F_v}})^{ss} \cong \mathrm{rec}_\lambda(\pi_v \otimes |\nu|^{-3/2})^{ss},$$

where the rec_λ map that sends admissible irreducible complex representations of $\mathrm{GSp}_4(F_v)$ to a Frobenius-semisimple complex Weil–Deligne representations of the Weil group W_{F_v} given by the local Langlands correspondence [GT11]. (Here we are fixing an embedding of coefficients $\bar{K}_\lambda \to \mathbb{C}$.)

(3) If $v \mid p$, then $\rho_{\pi,\lambda}|_{G_{F_v}}$ is de Rham with Hodge-Tate weights $[0, 0, 1, 1]$. Furthermore, if π is λ-ordinary, then

$$\rho_{\pi,\lambda}|_{G_{F_v}} = \begin{pmatrix} \lambda_{\alpha_v} & * & * & * \\ 0 & \lambda_{\beta_v} & * & * \\ 0 & 0 & \epsilon^{-1}\lambda_{\beta_v}^{-1} & * \\ 0 & 0 & 0 & \epsilon^{-1}\lambda_{\alpha_v}^{-1} \end{pmatrix}.$$

(4) If $\rho_{\pi,\lambda}$ is irreducible, then for each finite prime v of F, $\rho_{\pi,\lambda}|_{G_{F_v}}$ is pure.

Definition 6.7. Let F be a totally real number field. We say that a representation $\rho : G_F \to \mathrm{GSp}_4(\bar{K}_\lambda)$ is modular if there is a cuspidal automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_F)$ of parallel weight 2 and central character $|\cdot|^2$ that satisfies $\rho \cong \rho_{\pi,\lambda}$. We say that ρ is potentially modular if there is a finite Galois extension F'/F of totally real fields such that $\rho|_{G_{F'}}$ is modular.

For a compatible system of GSp_4 representations $\mathcal{R}$, we say that $\mathcal{R}$ is modular (potentially modular) if ρ_λ is modular for some (equivalently, for any) λ.

If A/F is an abelian variety of GSp_4 type, then we say A is modular (potentially modular) if $\mathcal{R}_A$ is modular, in other words, $\rho_{A,\lambda}$ is modular (potentially modular) for some (equivalently, for any) λ.

Theorem 6.8. ([BCGP21], Theorem 8.4.1) Suppose that $p \geq 3$ splits completely in K and the totally real field $F/\mathbb{Q}$. Fix $\mathfrak{p} \mid p$ a prime in K. Suppose that $\rho : G_F \to \mathrm{GSp}_4(\bar{K}_\mathfrak{p})$ satisfies:

(1) The similitude character of ρ is ϵ^{-1};

(2) The representation $\bar{\rho}$ is vast and tidy;

(3) For all $v \mid p$, $\bar{\rho}|_{G_{F_v}}$ is conjugate to a representation of form

$$\begin{pmatrix} \bar{\lambda}_{\alpha_v} & 0 & * & * \\ 0 & \bar{\lambda}_{\beta_v} & * & * \\ 0 & 0 & \bar{\epsilon}^{-1}\bar{\lambda}_{\beta_v}^{-1} & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1}\bar{\lambda}_{\alpha_v}^{-1} \end{pmatrix}$$

where $\alpha_v \neq \beta_v$;

(4) There exists π of parallel weight 2 and central character $|\cdot|^2$, which is ordinary at all $v \mid p$, such that $\bar{\rho}_{\pi,\mathfrak{p}} \cong \bar{\rho}$.

(5) For all finite places $\mathfrak{p}$ of F, $\rho|_{G_{F_v}}$ and $\rho_{\pi,\mathfrak{p}}|_{G_{F_v}}$ are pure.

Then ρ is modular. More precisely, there exists an ordinary automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A}_F)$ of parallel weight 2 and central character $|\cdot|^2$ that satisfies $\rho_{\pi',\mathfrak{p}} \cong \rho$.

For twisted Galois representations, we also have the following theorem.

Theorem 6.9. ([BCGP21], Theorem 8.5.2) Suppose that $p \geq 3$ splits completely in K and the totally real field $F/\mathbb{Q}$. Fix $\mathfrak{p} \mid p$ a prime in K. Suppose that $\rho : G_F \to \mathrm{GSp}_4(\bar{K}_{\mathfrak{p}})$ satisfies:

(1) The similitude character of ρ is $\epsilon^{-1}\chi$, where χ is a totally even finite order character, and is unramified at all places above p;

(2) The representation $\bar{\rho}$ is vast and tidy;

(3) For all $v \mid p$, $\bar{\rho}|_{G_{F_v}}$ is conjugate to a representation of form

$$\begin{pmatrix} \bar{\lambda}_{\alpha_v} & 0 & * & * \\ 0 & \bar{\lambda}_{\beta_v} & * & * \\ 0 & 0 & \bar{\chi}\bar{\epsilon}^{-1}\bar{\lambda}_{\beta_v}^{-1} & 0 \\ 0 & 0 & 0 & \bar{\chi}\bar{\epsilon}^{-1}\bar{\lambda}_{\alpha_v}^{-1} \end{pmatrix}$$

where $\alpha_v \neq \beta_v$;

(4) There exists π of parallel weight 2 and central character $|\cdot|^2$, which is a twist of an ordinary character at all $v \mid p$, such that $\bar{\rho}_{\pi,\mathfrak{p}} \cong \bar{\rho}$.

(5) For all finite places v of F, $\rho|_{G_{F_v}}$ and $\rho_{\pi,\mathfrak{p}}|_{G_{F_v}}$ are pure.

Then ρ is modular. More precisely, there exists a twisted ordinary automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A}_F)$ of parallel weight 2 and central character $|\cdot|^2$ that satisfies $\rho_{\pi',\mathfrak{p}} \cong \rho$.

In §6, we showed that we can find an abelian variety $\mathcal{A}$ such that $\bar{\rho}_{\mathcal{A},\mathfrak{q}}$ is induced from GL_2 representations. The next theorem shows that such GL_2 representations are potentially modular.

Theorem 6.10. ([BCGP21] Theorem 9.1.11) Let F_1/F be a finite extension of totally real fields and let $p, q > 2$ be distinct primes that splits completely in K and F_1. Fix a prime $\mathfrak{q} \mid q$ in K. Let $\bar{r} : G_{F_1} \to \mathrm{GL}_2(\bar{K}_{(\mathfrak{q})}) \cong \mathrm{GL}_2(\bar{\mathbb{F}}_q)$ be a representation with determinant $\bar{\epsilon}^{-1}$.

Suppose that for each place $w \mid q$ of F_1, $\bar{r}|_{F_{1,w}}$ is of form $\begin{pmatrix} \bar{\lambda}_{\alpha_w} & 0 \\ 0 & \bar{\epsilon}^{-1}\bar{\lambda}_{\alpha_w}^{-1} \end{pmatrix}$, and suppose that $\bar{r}$ is unramified at all places above p.

Then r is potentially modular as GL_2 representation. More precisely, let F^0/F be a finite extension, then there is a finite Galois extension F'/F of totally real fields in which p and q split completely and which is linearly disjoint from F_1F^0/F, and a $\mathfrak{q}$-ordinary cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_{F_1F'})$ of weight 0 and trivial central character which is unramified at all places dividing pq and satisfies $\bar{\rho}_{\Pi,\mathfrak{q}} \cong \bar{r}|_{G_{F_1F'}}$.

Theorem 6.11. Let $\mathcal{R}$ be a geometric compatible system of GSp_4 representation satisfying the large image property and falls in one of the two cases:

- $\mathcal{R}$ is defined over $G_{\mathbb{Q}}$ and is strongly positive ordinary; In other words, $\mathcal{R}$ comes from a non-CM strongly positive ordinary abelian variety A/F of GSp_4-type that has descent data to $\mathbb{Q}$ and satisfies the large image property;

- $\mathcal{R}$ has coefficients in $\mathbb{Q}$ for a set of primes of density one, which includes the case where $\mathcal{R}$ comes from an abelian surface A with endomorphism ring $\mathbb{Z}$ or a fake abelian surface with endomorphism ring $\mathcal{O}_D$ for a quaternion algebra $D/\mathbb{Q}$.

Then $\mathcal{R}$ (hence A) is potentially modular.

Proof. We first apply Lemma 3.9 in the first case and Lemma 3.8 in the second case to get the good primes p and q. Following the notations of Theorem 6.4, we know that there exists

an abelian variety $\mathcal{A}/F'$ over a totally real extension field F' in which p, q splits completely, such that $\mathcal{A}[\mathfrak{p}] \cong A[\mathfrak{p}]$, and $\mathcal{A}[\mathfrak{q}] \cong \bar{\rho}^\vee$ or $(\bar{\rho}^\vee)^2$, where $\bar{\rho} = \mathrm{Ind}_{G_{F'}}^{G_F} \bar{r}$ is induced with $\bar{r}$ satisfying properties in Lemma 6.3.

By Theorem 6.10, after replacing F' with a further totally real extension, we can maintain all assumptions in Theorem 6.4 and there exists a $\mathfrak{q}$-ordinary automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_{F_1F'})$ of weight 0 and trivial central character and is unramified at all places dividing pq that satisfies $\bar{\rho}_{\Pi,\mathfrak{q}} \cong \bar{r}|_{G_{F_1F'}}$.

It follows from [Rob01, Thm. 8.6] that there is an automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_{F'})$ of parallel weight 2 and trivial central character whose transfer to $\mathrm{GL}_4(\mathbb{A}_{F'})$ is the automorphic induction of $\Pi \otimes |\cdot|$. So $\rho_{\pi,\mathfrak{q}} \cong \mathrm{Ind}_{G_{F_1F'}}^{G_{F'}} \rho_{\Pi,\mathfrak{q}}$, and $\bar{\rho}_{\pi,\mathfrak{q}} \cong \bar{\rho}|_{G_{F'}}$. In addition, π is ordinary, and $\rho_{\pi,\lambda}$ is pure at all finite places since $\rho_{\Pi,\lambda}$ is.

Now we first apply Theorem 6.8 to $\rho_{\mathcal{A},\mathfrak{q}}$ and conclude that it is modular. Thus $\rho_{\mathcal{A},\mathfrak{p}}$ is modular, and we may apply Theorem 6.8 again to deduce that $\rho_{A,\mathfrak{p}}|_{G_{F'}}$ is modular, thus A is potentially modular. $\square$

CHAPTER 7

$K3$ SURFACES WITH LARGE PICARD RANK

7.1 Compatible System of Representations Associated to $K3$ Surfaces

Let F be a totally real field and X/F be a $K3$ surface with geometric Picard number 17. We are interested in considering the transcendental motive T which is complement of the image of the cycle classes. Let $\mathcal{S} := (\mathbb{Q}, S, \{P_v(X)\}, \{\rho_p\}, H_\tau)$ denote the compatible system of Galois representations $V_p := T_p(-1)$ which are the p-adic realizations of $T(-1)$ inside $H^2(X, \mathbb{Q}_p)$. There is a perfect orthogonal pairing

$$H^2(X, \mathbb{Q}_p(1)) \times H^2(X, \mathbb{Q}_p(1)) \to \mathbb{Q}_p$$

which gives rise to such a pairing on T_p. It follows that $\det(\rho_p) = \epsilon_p^5 \eta_p$ where ϵ_p is the p-adic cyclotomic character and $\eta_p = \det(\rho_p(-1))$ is also self-dual, and thus $\eta_p = \eta_p^\vee = \eta_p^{-1}$ is at most quadratic. The determinant is also a compatible system and thus is independent of p, and hence we write it simply as η.

We have the following well-known properties for the compatible system $\mathcal{S}$:

(1) If $v \notin S$ and $v \nmid p$ then ρ_p is unramified at v and $\rho_p(\mathrm{Frob}_v)$ has characteristic polynomial $P_v(X) \in \mathbb{Z}[x]$.

(2) For all p, the representation ρ_p is de Rham with Hodge–Tate weights $H = [0, 1, 1, 1, 2]$. If in addition $v \notin S$, then ρ_p is crystalline, and the characteristic polynomial of crystalline Frobenius Frob_v is $P_v(X)$.

(3) The representation ρ_p is pure of weight 2.

(4) If $v \in S$ and $v \nmid p$, then the semi-simplification of the representation to $\rho_p|_{I_{F_v}}$ has uniformly bounded order. That is, there exists a fixed finite extension of K/F_v such that $\rho_p|_{G_K}$ is unipotent on inertia.

Our main goal would be to construct from $\mathcal{S}$ a compatible family $\mathcal{R}$ of 4-dimensional representations to $\mathrm{GSp}_4(\bar{\mathbb{Q}}_p)$ that lifts the representations $\rho_p \otimes \eta$ via the degree 2 isogeny $\mathrm{GSp}_4 \to \mathrm{GO}_5$.

Theorem 7.1. Suppose that there exists a prime p for which T_p has monodromy group with Lie algebra $\mathfrak{so}_5$. Then:

(1) There exists a simple abelian variety A/F with $\mathrm{End}^0(A) = D$ having center M and a weakly compatible system

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{\tilde{\rho}_\lambda\}, [0, 0, 1, 1])$$

of Galois representations such that, for all l that totally splits in M and all $\lambda \mid l$ in M, the representation $\tilde{\rho}_\lambda$ is irreducible, occurs inside $H^1(A, \bar{\mathbb{Q}}_l)$, and the composition

$$\tilde{\rho}_\lambda : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_l) \to \mathrm{GO}_5(\bar{\mathbb{Q}}_l)$$

is isomorphic to $\rho_l \otimes \eta$.

(2) There exists a finite Galois extension E/F with $\mathrm{Gal}(E/F) = (\mathbb{Z}/2\mathbb{Z})^m$ such that the restriction of this compatible system comes from either an abelian surface A/E or a fake abelian surface A/E with descent data to F.

Proof. The irreducible representation $\rho_p \otimes \eta$ lifts to a representation $\tilde{\rho} : G_F \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_p)$ with Hodge-Tate weights $[0, 0, 1, 1]$ by [Pat19] §3.2. Furthermore, according to Proposition 4.2.31 of [Pat19], $\tilde{\rho}$ lies in some abelian variety A after some finite extension. Thus by Lemma

1.6, it can be extended to some compatible system $\mathcal{R} = (M, S, \{Q_v(X)\}, \{\tilde{\rho}_\lambda\}, [0, 0, 1, 1])$ that comes from the abelian variety A.

Then we have $\mathrm{Gal}(M/\mathbb{Q})$ acting on $\mathcal{R}$ via coefficients. If $\sigma \in \mathrm{Gal}(M/\mathbb{Q})$, we see that $\mathcal{R}^\sigma$ is also a compatible system, and the corresponding compatible system of GO_5 representations given by $\wedge^2\mathcal{R}^\sigma$ is equal to the compatible system associated to $\rho_p \otimes \eta$. We deduce that $\mathcal{R}^\sigma$ is isomorphic to $\mathcal{R}$ up to at most a quadratic twist by irreducibility. Hence there exists an extension E/F (with Galois group $(\mathbb{Z}/2\mathbb{Z})^m$ for some m) where all these twists become trivial. But then $\mathcal{R}$ restricted to G_E is invariant under $\mathrm{Gal}(M/\mathbb{Q})$, and hence forms a compatible system with coefficients in $\mathbb{Q}$. But now the proof of [BCGP21] Lemma 10.3.2 implies that $\mathcal{R}$ over E/F is associated to either an abelian surface or fake abelian surface A over E, and the fact that $\mathcal{R}$ can be extended to G_F and gives the same Galois representations up to twist proves the descent data.

□

We now show the irreducibility of one (hence for all) T_p together with the large image property:

Lemma 7.2. The representation T_p can not have any one-dimensional factors.

Proof. Any one dimensional factor must correspond to a representation of G_F with Hodge–Tate weight -1, 0, or 1. If it has Hodge-Tate weight 0, then the character has finite order. By the Tate conjecture for $K3$ surfaces over a number field it must come from a ($\bar{\mathbb{Q}}_p$-linear combination of) cycle classes, which contradicts the assumption that T is the transcendental motive. If the Hodge–Tate weight is -1 or 1, it must come from $\chi\epsilon^{-1}$ or $\chi\epsilon$ where χ is some finite order character and ϵ is the cyclotomic character. But this contradicts purity. □

Lemma 7.3. If T_p is irreducible for one p, then it is absolutely irreducible for all p and the Lie algebra of the monodromy group contains $\mathfrak{so}_5$.

Proof. Suppose that T_p is irreducible for one p. We first claim that T_p must be absolutely irreducible. Note that $\dim T_p = 5$ is prime. Hence if it is reducible but not absolutely irreducible, then $T_p \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$ has to decompose into the product of 1-dimensional representations which are conjugate by the action of $G_{\mathbb{Q}_p}$. These one dimensional representations must be Hodge-Tate, and thus of the form $\chi\,\epsilon^m$ for some m where χ is a finite order character and ϵ is the cyclotomic character. But $G_{\mathbb{Q}_p}$ fixes ϵ, which means that all the Hodge-Tate weights of T_p are equal, a contradiction.

Hence we may assume that T_p is absolutely irreducible. Suppose that T_p becomes reducible over a finite extension. Since G_F acts transitively on the factors they must all have the same dimension, which must therefore be equal 1. Moreover the dimension of each isotypic component is independent of the character. The characters cannot all be the same since T_p has distinct Hodge-Tate weights. Hence all the characters are distinct, and thus T_p is induced from a character of a degree 5 extension K/F which is de Rham and hence an algebraic character by [CG18, Lemma 4.3]. But such a K cannot be CM and hence the only algebraic characters are finite order characters times characters of G_F, and the induction of such a character has all Hodge-Tate weights equal, which is again a contradiction.

Thus the monodromy representation of T_p is connected. The only proper Lie subalgebra of $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$ with irreducible representations of dimension 5 is $\mathfrak{sl}_2$ acting via the 4th symmetric power representation. But if T_p is the 4th symmetric power of some representation over some finite extension of G_F, then the Hodge-Tate weights at any embedding will necessarily be in arithmetic progression which they are not.

Hence T_p must have monodromy group $\mathfrak{so}_5$. But now by Theorem 7.1 there exists a compatible system $\mathcal{R}$ of absolutely irreducible GSp_4 representations $\tilde{\rho}_\lambda$ and it follows by constancy of ranks that the monodromy at each prime has semisimple part $\mathfrak{so}_5 = \mathfrak{sp}_4$ or $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$. In the latter case, the representation T_l decomposes into a direct sum of a one-dimensional representation and a 4 dimensional representation, which is not possible

from the previous lemma. □

It follows that if T_p is not absolutely irreducible then it must decompose as $A_p \oplus B_p$ where $\dim(A_p) = 2$ and $\dim(B_p) = 3$ are both irreducible.

Lemma 7.4. If T_p is reducible, then A_p is the Galois representation associated to a CM modular form of level only divisible by primes in S and bounded independently of p.

Proof. Since T_p carries a generalized orthogonal form, it follows that A_p and B_p are both also orthogonal and in particular self-dual up to twist. Hence either A_p had Hodge-Tate weights $[-1, 1]$ and B_p has Hodge–Tate weights $[0, 0, 0]$, or A_p has Hodge-Tate weights $[0, 0]$ and B_p has Hodge-Tate weights $[-1, 0, 1]$. The Galois representation associated to A_p has image in $\mathrm{GO}_2(\mathbb{Z}_p)$, thus is potentially abelian.

The Fontaine-Mazur conjecture is known for potentially abelian representations. In particular, if A_p has Hodge-Tate weights $[0, 0]$, then it has finite image, which would contradict the Tate conjecture as in the proof of Lemma 7.2. Hence for any such p we may assume that A_p has Hodge-Tate weights $[-1, 1]$ and $A_p(-1)$ is the Galois representation associated to a CM modular form of weight 3 with good reduction outside S. The uniform boundedness of level at primes in S is deduced from property (4) of the compatible system $\mathcal{S}$. □

We now prove the desired irreducibility statement.

Theorem 7.5. The representation T_p is absolutely irreducible for all p with monodromy group containing $\mathfrak{sp}_4$.

Proof. By Lemma 7.3, it suffices to assume that T_p is reducible for all primes p and reach a contradiction. If T_p is reducible, then by the previous lemma the 3-dimensional factor B_p has Hodge-Tate weights $[0, 0, 0]$, and for each p, the 2-dimensional Galois representation $A_p(-1)$ comes from one of only finitely many weight 3 CM modular forms π. All such forms are induced from a finite set of imaginary quadratic fields (those unramified outside S).

Hence we may assume that, for some set of primes Σ of strongly positive density that split completely in each of these quadratic fields, that A_p is associated to a fixed such π, and in particular $A_p = \mathrm{Ind}_{G_E}^{G_F}\chi_p$ for a fixed imaginary quadratic field E/F and Grössencharacter χ_p of E.

Let $P_{A_p,v}(X)$ denote the characteristic polynomial of Frobenius on $A_p(-1)$ at v for $v \notin S \cup \{p\}^F$, and let $P_{A_p,v}(X)$ denote the characteristic polynomial of crystalline Frobenius on $A_p(-1)$ at v assuming that $v \mid p$ and $v \notin S$. Because the Grössencharacters give rise to a weakly compatible system, these polynomials do not depend on p as long as $p \in \Sigma$. Hence we may write them as $P_{A,v}(X)$, and similarly we may write $P_{B,v}(X)$. Since $A_p(-1)$ occurs in $H^2(X, \mathbb{Q}_p)$ we also see that these polynomials have integral coefficients.

We now have a factorization

$$P_v(X) = P_{A,v}(X)P_{B,v}(X).$$

for all v. Recall that the Hodge-Tate weights of ρ_p are $[0, 1, 1, 1, 2]$. On the other hand, since primes $p \in \Sigma$ split in E by construction, the representation A_p is ordinary at p and hence the Newton Polygon of $P_{A,v}(X)$ has slopes 0 and 2. By Newton over Hodge, this implies that the Newton Polygon of B_p has all slopes equal to 1. That means that the roots of $P_{B,v}(X)$ are all divisible by p as algebraic integers. But by purity they also have absolute value p for all complex embeddings, and thus it follows that the roots of $P_{B,v}(pX)$ are roots of unity. Since $P_v(X)$ is a degree 5 polynomial over $\mathbb{Q}$, they are indeed roots of unity of bounded order. Since there are only finitely many such roots of unity, we deduce that for a set Σ of strongly positive density, the polynomial $P_{B,v}(X)$ is given by a fixed polynomial. On the other hand, up to twist, the Galois representation associated to B_p has image inside $\mathrm{SO}_3(\mathbb{Q}_p)$. Hence either the monodromy group of B_p is connected, in which case B_p would have Lie algebra $\mathfrak{sl}_2$ acting via the symmetric square, or B_p is potentially abelian with finite image or induced.

In the latter case we can again apply the Fontaine-Mazur conjecture to get a contradic-

tion, so we may assume that the monodromy group of B_p has Lie algebra $\mathfrak{sl}_2$. But then from the Cebotarev density theorem we see that any subset of primes with a fixed characteristic polynomial has density zero, a contradiction.

□

In order to apply our potential modularity results in chapter 6 we need an extra lemma on ordinariness:

Lemma 7.6. The representation $T_p(-1)$ contains at least one crystalline eigenvalue which is a p-adic unit for a set of primes p of density one. The representations ρ_λ are also ordinary for all $\lambda \mid p$ with distinct unit eigenvalues mod λ for a set of primes p of density one.

Proof. Note that $T_p(-1)$ has Hodge-Tate weights $[0, 1, 1, 1, 2]$. If all crystalline eigenvalues were not p-adic units, then the polynomial $P_v(X) = X^5 + t_v X^4 + \ldots + p^5\eta(p)$ would have $p \mid t_v$. By purity, we have $|t_v| \leq 5p$. Thus we would have to have $t_v/p = a$ for some fixed integer $|a| \leq 5$, and some set of primes v of strongly positive density. Arguing as in the proof of Lemma 3.6, this contradicts the fact that the monodromy group of T_p contains $\mathfrak{so}_5$ for any prime p. This proves the first part of the lemma.

Now suppose $\tilde{\rho}_\lambda : G_F \to \mathrm{GSp}_4(\bar{\mathbb{Q}}_p)$ be the representation lifting $T_p(-1)$. Note that all of the crystalline Frobenius eigenvalues of $\tilde{\rho}_\lambda$ will be algebraic integers. Suppose they are $\alpha_v, \beta_v, p\zeta\alpha_v^{-1}, p\zeta\beta_v^{-1}$, where ζ is a root of unity of uniformly bounded order. Up to symmetry, if $T_p(-1)$ has a crystalline eigenvalue which is an p-adic unit then we may assume that $\alpha_v\beta_v$ is prime to p, and hence that ρ_λ is ordinary for all $\lambda \mid p$ and a set p of density one.

Write

$$\begin{aligned} Q_v(X) &= X^4 - a_v X^3 + b_v X^2 - p\zeta a_v X + p^2\zeta^2 \\ &= (X - \alpha_v)(X - \beta_v)(X - p\zeta\alpha_v^{-1})(X - p\zeta\beta_v^{-1}) \in \mathcal{O}_M[X]. \end{aligned}$$

The ordinary assumption implies that $b_v \not\equiv 0 \bmod \lambda$ for all $\lambda \mid p$. Suppose that the unit

root eigenvalues are the same for some $\lambda \mid p$. Then $a_v^2 - 4b_v \equiv 0 \bmod \lambda$. On the other hand if $\sigma \in \mathrm{Gal}(M/\mathbb{Q})$, then $a_v^\sigma = \chi(\sigma)a_v$ and $b_v^\sigma = \chi(\sigma)^2 b_v$ for some finite order character χ, and so $(a_v^\sigma)^2 - 4b_v^\sigma = \chi(\sigma)^2(a_v^2 - 4b_v)$, hence if this is divisible by one λ then it is divisible by all, and thus (at least for p unramified in M) that $a_v^2 - 4b_v$ is divisible by p. Since ρ_λ is pure of weight one, it follows that $(a_v^2 - 4b_v)/p \in \mathcal{O}_M$ has absolute value bounded uniformly in p, and there are only finitely many such elements in $\mathcal{O}_M$ with this property.

But once more arguing as in the proof of Lemma 3.6, the fact that the image of $\tilde{\rho}_\lambda$ has monodromy group containing $\mathfrak{sp}_4$ for all λ shows that the eigenvalues would be different modulo λ for a set of primes p of relative density one. □

Now that we have constructed the abelian variety A/E with descent data to F that gives the compatible system $\mathcal{R}$ and proved that it is strongly positive ordinary, thus we have:

Corollary 7.7. The compatible system of GSp_4 representation $\mathcal{R}$ constructed in Theorem 7.1 that corresponds to the transcendental motive T is potentially modular. Therefore, Theorem 1.1 is true when X has Picard rank 17.

Proof. The compatible system defined in Theorem 7.1 falls into the second case of Theorem 6.11, where we consider $\mathcal{R}$ to be the compatible system of representations of G_E with coefficients in $\mathbb{Q}$ for a set of primes of density one. In fact, Lemma 7.6 shows that we can find good primes p and q to perform the trick in Theorem 6.4 and thus the desired result just follows from Theorem 6.11. □

7.2 The Geometric Picard Rank ≥ 18

Now suppose X/F has geometric Picard rank 18. Then associated to $T_p(-1)$ is a compatible system $\mathcal{S} = (M, S, \{Q_v(X)\}, \{\rho_p\}, [0, 1, 1, 2])$. We now derive similar results for T_p:

Lemma 7.8. The semisimple part of the Lie algebra of the monodromy group of T_p is $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Proof. Suppose that T_p is not absolutely irreducible. Since Lemma 8.2 also applies here, it follows that $T_p \otimes \bar{\mathbb{Q}}_p$ it is a direct sum of two factors A_p and B_p of dimension 2. We show that this cannot occur even over some finite extension of F. If A_p and B_p are both orthogonal, they are both induced, and then T_p is potentially abelian. Thus without loss of generality A_p is not orthogonal, which implies that the dual of A_p under the orthogonal pairing is a twist of B_p. Since two-dimensional representations are self-dual up to twist, it follows that $T_p \simeq A_p \oplus A_p \otimes \chi$ for some character χ which will be algebraic.

But since T_p is pure of weight 1, it follows that A_p is also pure of weight one, and that χ is pure of weight zero, and thus has finite image. But then the Hodge–Tate weights of T_p must each have multiplicity two, which is a contradiction.

Suppose that T_p is absolutely irreducible but becomes reducible after a finite extension. The factors must each have the same dimension. If they have dimension one then T_p is potentially abelian. If they have dimension 2 then T_p is once more of the form $A_p \oplus B_p$ after restricting to some finite extension which we have already considered. Thus T_p remains irreducible over any finite extension and the only possibility is that the monodromy group is $\mathfrak{so}_4$. □

There is an isogeny $\mathrm{GL}_2(\bar{\mathbb{Q}}_p) \times \mathrm{GL}_2(\bar{\mathbb{Q}}_p) \to \mathrm{GO}_4(\bar{\mathbb{Q}}_p)$ whose image is the connected subgroup of index two. By [LP92], the component group of a compatible family is independent of p, and thus there exists an (at most) degree two extension L/F such that the image of T_p lands inside this image.

Theorem 7.9. Suppose that there exists a prime p for which T_p has monodromy group with Lie algebra $\mathfrak{so}_4$. Then there exists an extension L/F of degree at most two over which the monodromy of T_p is connected such that:

(1) There exists simple abelian varieties E_1/L and E_2/L and weakly compatible families

$$\mathcal{R}_i = (M, S, \{Q_{v,i}(X)\}, \{\tilde{\rho}_{\lambda,i}\}, [0,1])$$

of Galois representations such that, for all l and all $\lambda \mid l$, the representation $\tilde{\rho}_{\lambda,i}$ is irreducible, occurs inside $H^1(E_i, \bar{\mathbb{Q}}_l)$ as a $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$ representation, and the composite

$$\tilde{\rho}_\lambda : G_L \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p) \times \mathrm{GL}_2(\bar{\mathbb{Q}}_p) \to \mathrm{GO}_4(\bar{\mathbb{Q}}_l)$$

is isomorphic to ρ_l restricted to G_L.

(2) There exists a finite Galois extension H/L such that the restriction of these compatible families comes from either elliptic curves E_i/H or fake elliptic curves E_i/H with descent data to L, that is, there exist isogenies between E_i and E_i^σ for any $\sigma \in \mathrm{Gal}(H/L)$.

Proof. By [Pat19, §3.2], the representation $\rho_p : G_E \to \mathrm{GO}_4(\mathbb{Q}_p)$ lifts to a representation $\tilde{\rho} : G_E \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p) \times \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ with Hodge-Tate weights $[0, 0, 1, 1]$ and monodromy group whose Lie algebra has semisimple part $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. By [Pat19, Lemma 4.2.22], over some finite extension E'/E, the representation $\tilde{\rho}$ is part of a compatible family of Galois representations associated to (part of) some abelian variety. In particular, the 2-dimensional constituents also come from abelian varieties E_i/L giving rise to $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, with coefficients jointly in some finite extension F. Now consider the action of $\mathrm{Gal}(M/\mathbb{Q})$ on the coefficients, we have

$$\mathcal{R}_1^\sigma \otimes \mathcal{R}_2^\sigma \simeq \mathcal{R}_1 \otimes \mathcal{R}_2 \simeq \mathcal{S},$$

because the latter has coefficients over $\mathbb{Q}$. If follows that $\mathcal{R}_i^\sigma = \mathcal{R}_j \otimes \chi_\sigma$ for each σ where j may or may not be equal to i, and χ_σ is a finite order character. If there exists a σ such that $\mathcal{R}_1^\sigma$ is a twist of $\mathcal{R}_2$, then their monodromy groups are the same on a finite index, which contradicts the fact that the relevant Lie algebra is $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and not $\mathfrak{sl}_2$. Hence $\mathcal{R}_i^\sigma$ is a twist of $\mathcal{R}_i$ for each σ. Letting H/L denote the fixed field of these characters, we deduce that $\mathcal{R}_i$ as a compatible family over G_L has coefficients in $\mathbb{Q}$. But now we are done using the same idea of the proof of [BCGP21, Theorem 10.3.2]. □

Theorem 7.10. Theorem 1.1 is also true when X has Picard rank ≥ 18.

Proof. Suppose the Picard rank is 18. Then it suffices to show that the compatible families $\mathcal{R}_1$ and $\mathcal{R}_2$ constructed in Theorem 7.9 are potentially modular, and then to use the automorphic of tensor products $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ [Ram00]. But the potential (simultaneous) modularity of $\mathcal{R}_1$ and $\mathcal{R}_2$ follows from [ACC$^+$18, Theorem 7.1.10].

If the Picard rank is 19 or 20, then $\mathcal{S}$ is regular and thus potentially automorphic by [BLGGT14] Theorem A. □

When the Picard rank is 19 or 20, it is easy to deduce the stronger claim that $\mathcal{S}$ is automorphic. In the rank 20 case the representation is orthogonal and thus induced. In the rank 19 case, the compatible system of GO_3 representations lifts as in the proof of Theorem 7.1 to a 2-dimensional family associated with a GL_2-type abelian variety with descent to F, and hence the result follows from [Rib04] and the proof of Serre's conjecture [Kha06, KW09].

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