THE UNIVERSITY OF CHICAGO

PROPERTIES OF SCHRAMM LOEWNER EVOLUTION AND SUPERCRITICAL LIOUVILLE QUANTUM GRAVITY METRIC EXPONENTS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
STEPHEN GLENN YEARWOOD-DAVIDSON

CHICAGO, ILLINOIS
AUGUST 2023

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to thank my advisor, Gregory Lawler, for all his patience and attention over the years, and for working so closely with me on some very interesting problems. I would also like to thank my second advisor, Ewain Gwynne, for his constant help, advice, and generosity with his time.

I have been very fortunate to have highly supportive colleagues and friends in the department, including Bill Cooperman, Nixia Chen, Sehyun Ji, Nikiforos Mimikos-Stamatopoulos, Thomas Hameister, Jinwoo Sung, Chloe Avery, Rosemary Elliott Smith, Joshua Mundinger, Sam Quinn, and many others. I am extremely grateful for their support, and the many conversations about Mathematics (and otherwise) that we have had over the years.

My deepest gratitude goes out to my family, and in particular my mother, Glenis. Thank you for your undying love and support throughout this journey.

ABSTRACT

Schramm–Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a plethora of natural probabilistic processes in two dimensions, such as critical percolation interfaces, loop erased random walks, and (in conjecture) self-avoiding walks. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory. The first part of this thesis will focus mainly on two key properties of SLE; namely, reversibility and topological invariance.

For $\gamma \in (0, 2)$, $U \subset \mathbb{C}$, and an instance h of the Gaussian free field (GFF) on U, the γ-Liouville quantum gravity (LQG) surface associated with (U, h) is formally described by the Riemannian metric tensor $e^{\gamma h}(dx^2 + dy^2)$ on U. It is known that one can define a canonical metric (distance function) D_h on U associated with a γ-LQG surface. We show that this metric is conformally covariant in the sense that it respects the coordinate change formula for γ-LQG surfaces. We consider a discrete analog of this metric, and show, in the final chapter of the present work, that it has the same distance exponent as in the continuum case.

CHAPTER 1

INTRODUCTION

1.1 Initial Overview

One of the main objectives of statistical physics and probability theory is to examine macroscopic systems that comprise a large number of small, random microscopic components, which become more significant as the number of components increases towards infinity. There are two possible outcomes: in the limit, the behavior of the macroscopic system can either become deterministic (known as 'law of large number' outcomes), with large deviations that can be used to some extent within this framework, or random. Brownian motion is a prototype for continuous random objects that manifest as the scaling limit of finite systems. Notably, it is the scaling limit of a wide range of random walks, which makes it more universal than the discrete model (simple random walk) since it doesn't require specifying a lattice or jump-distribution. Rather, it captures the general properties of walks, such as stationary increments and mean zero. The central objects studied in this thesis, *Schramm Loewner evolution* and *Liuoville quantum gravity*, also exhibit this universality in that these objects arise as the scaling limits of several discrete models.

The Schramm Loewner evolution (SLE_κ) describes a one parameter family of probability measures on curves in the plane that stand as the only reasonable conformally invariant scaling limits of several discrete lattice models, given certain conditions are met. SLE_κ lies at the intersection of probability and complex analysis, as these curves arise as one-parameter families of solutions to the Loewner differential equation driven by a Brownian motion.

SLE_κ describes the random growth of a set K_t, as seen through a conformal map $g_t(z)$ on the complement of this set. This map is the solution of the Loewner differential equation driven by a Brownian motion, whose "speed" is determined by a single parameter κ. Rohde and Schramm in [40] showed that for $\kappa \neq 8$, a.s. there is a (unique) continuous path

$\eta : [0, \infty) \to \overline{\mathbb{H}}$ such that for each $t > 0$ the set K_t is the union of $\eta[0, t]$ and the bounded connected components of $\mathbb{H} \setminus \eta[0, t]$. This was later extended to $\kappa = 8$ [30]. We call the path η the SLE trace or SLE curve. We will need the following facts about the curve[40]:

- If $\kappa \leq 4$, then η is simple with $\eta(0, \infty) \subset \mathbb{H}$.
- If $4 < \kappa < 8$, then $\eta(0, \infty)$ has double points and intersects $\mathbb{R}$.
- If $\kappa \geq 8$, the curve is space-filling.

There are three variants of SLE : chordal SLE, which connects two boundary points (prime ends) in a given domain; radial SLE, which connects a boundary point to an interior point; and whole-plane SLE, which connects two points on the Riemann sphere. We will provide preliminary details on the chordal and radial cases in the next chapter; see [28, 3, 45] for some expository work on SLE which go into further details.

The study of canonical probability measures on the space of two dimensional Riemannian manifolds is often called "two-dimensional quantum gravity". The final chapter of the thesis focuses on the study *Liouville quantum gravity*, which realizes one natural way to produce a "random geometry" from the Gaussian free field. Recall that the Riemann uniformization theorem states that every smooth simply connected Riemannian manifold can be conformally mapped to either the unit disc, the complex plane, or the complex sphere. In other words, $\mathcal{M}$ can be parametrized by points $z = ix + y$ in one of these spaces such that the metric takes the form $e^{\lambda(z)}(dx^2 + dy^2)$ for some real-valued function λ. LQG shows a way to extend this parametrization to a setting where λ is a generalized function (or a random distribution).

In Liouville quantum gravity, one takes λ to be a multiple of the GFF and seeks to define a measure $\mu^h = e^{\gamma h(z)} dz$ where h is an instance of the Gaussian free field on a simply connected domain $D \subset \mathbb{C}$ and $\gamma \in (0, 2]$. Since h is a distribution, not a function, we require a regularization procedure to make this precise. One natural approach is to consider averages of the GFF over a given region, and then take a limit. For example, we can set $h_\epsilon(z)$ be the average value of h on the circle of radius ϵ centered at z (or an analogous average defined using a bump function supported inside that circle) and then write $\mu^h = \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(z)} d(z)$. We interpret the pair (D, μ) as describing a"random surface" $\mathcal{W}$ conformally parametrized by D with area measure μ^h.

We may also parameterize the same surface with a different domain, under the LQG *coordinate change* rule. If $\varphi : \tilde{D} \to D$ is a conformal map, one can write

$$\tilde{h} = h \circ \varphi + Q \log |\varphi'|, \tag{1.1}$$

where $Q = \dfrac{2}{\gamma} + \dfrac{\gamma}{2}$. The measure $\mu^{\tilde{h}}$ on $\tilde{D}$ is then a.s. equivalent to the pullback via φ^{-1} of the measure μ^h on D. It is known that the coordinate change rule a.s. holds simultaneously for all possible φ. Two domain/field pairs $(D, h), (\tilde{D}, \tilde{h})$ are said to be equivalent as LQG surfaces if they are related as in (1.1). An LQG surface is an equivalence class of domain/field pairs with respect to this equivalence relation. We think of two equivalent pairs as being two embeddings of the same surface. We remark that the set of pairs (D, μ^h) obtained from the set of pairs (D, h) in an equivalence class is itself an equivalence class with respect to the usual measure pullback relation

It is known that LQG surfaces admit a canonical metric, i.e., a distance function D_h. This metric is realized as a continuum limit of a family of random metrics known as the ϵ-*Liouville first passage percolation* (LFPP). This metric is characterized by a specific set of axioms, and obeys a coordinate change rule similar to that of the measure, at least for specific choices of the parameter γ. See [19], for example, for introductory material on the study of LQG.

1.2 Discrete models

While we will focus primarily on SLE, which is the continuum model, familiarity with some of the discrete models aids greatly in understanding it. Thus, we will present some of these discrete models. Through the application of conformal invariance assumptions, we will derive certain properties that we expect the continuum measure to exhibit. For example, we know that the chordal SLE_6 is the scaling limit of the lattice interface of the site percolation on the triangular lattice where one imposes monochromatic boundary conditions on $\mathbb{R}_+$ and $\mathbb{R}_-$ [44]; chordal SLE_8 is the scaling limit of uniform spanning tree Peano curve [30]; chordal SLE_4 is the scaling limit of the contour line of the two-dimensional discrete Gaussian free field with appropriate boundary values [41]; chordal SLE_2 is the scaling limit of loop erased random walk with two marked points, started at one and conditioned to leave the domain near the other [27]. There is also strong mathematical evidence in support of $\mathrm{SLE}_{8/3}$ realizing the scaling limit of the self avoiding walk.

1.2.1 Self-Avoiding walk

A self-avoiding walk (SAW) of length n on the integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ is a sequence of lattice points $\omega = [\omega_0, \ldots, \omega_n]$ where $|\omega_j - \omega_{j-1}| = 1$ for $j = 1, \ldots, n$, and $\omega_j \neq \omega_k$ for $j < k$. We denote by J_n the number of SAWs of length n with $\omega_0 = 0$. It is known that as n approaches infinity, J_n grows exponentially with n and $\log J_n \asymp \beta n$. We call e^β the *connective constant.* We define the exponent ν by saying that the typical diameter (with respect to the uniform probability measure on SAWs of length n with $\omega_0 = 0$) of a SAW is of order n^ν. We remark that we would expect the fractal dimension of the paths in the continuum limit to be $d = \frac{1}{\nu}$.

To take a continuum limit, we scale the lattice by letting $\delta > 0$ and defining $\omega^\delta(j\delta^d) = \delta\omega(j)$, where ω^δ is a SAW on the lattice $\delta\mathbb{Z}^2$ parametrized so that it goes a distance of order one in time of order one. Linear interpolation allows us to make $\omega^\delta(t)$ a continuous curve.

Fix $D \subset \mathbb{Z}^2$. We can then consider a finite measure on continuous curves $\gamma : (0, t_\gamma) \to D$ with $\gamma(0+) = z$, $\gamma(t_\gamma) = w$ for each integer N. This measure is obtained by giving measure $e^{-\beta n}$ to each SAW ω of length n in $\mathbb{Z}^2$ with $\omega_0 = -N$, $\omega_n = N$ and $\omega_1, \ldots, \omega_{n-1} \in ND$, where we identify ω with $\omega^{1/N}$. This gives a measure on curves in D from z to w, with total mass

$$Z(D; z, w) := \sum_{\omega: Nz \to Nw;\, \omega \subset ND} e^{-\beta|\omega|}.$$

It is conjectured that as $N \to \infty$, $Z_N(D; z, w)$ is asymptotic to $C(D; z, w)N^{-2b}$, where b is a constant. Re-scaling by N^{2b} and taking a limit yields a limiting measure $\mu_D(z, w)$ of total mass $C(D; z, w)$ supported on simple (non self-intersecting) curves from z to w in D. The dimension of these curves will be $d = 1/\nu$.

It is believed that this scaling limit obeys conformal invariance, the domain Markov property, and a restriction property. Specifically, if $D_1 \subset D$, then the measure $\mu_{D_1}(z, w)$ is obtained by restricting $\mu_D(z, w)$ to paths that lie in D_1. If this limit exists, then it is known that it must be $\mathrm{SLE}_{8/3}$.

1.2.2 Loop erased random walk

For any $\omega = [\omega_0, \ldots, \omega_m]$, we define the loop-erasure $L(\omega)$ of ω inductively as follows: $L_0 = \omega_0$, and for all $j \geq 0$, we define inductively $n_j = \max\{n \leq m : x_n = L_j\}$ and $L_{j+1} = (\omega_{1+n_j}, \ldots, \omega_m)$ until $j = \sigma$ where $L_\sigma := \omega_m$. In other words, we have erased the loops of ω in chronological order. The number of steps σ of L is not fixed.

Suppose that $(X_n, n \geq 0)$ is a recurrent Markov chain on a discrete state-space S started from $X_0 = x$. Suppose that $A \subset S$ is non-empty, and let τ_A denote the hitting time of A by X. Let $p(x, y)$ denote the transition probabilities for the Markov chain X. We define the loop-erasure $L = L(X[0, \tau_A]) = L^A$ of X up to its hitting time of A. We call σ the number of steps of L^A. For $y \in A$ such that with positive probability $L_A(\sigma) = X(\tau_A) = y$, we call $\mathcal{L}(x, y; A)$ the law of L^A conditioned on the event $\{L^A(\sigma) = y\}$. In other words, it is the law of the loop-erasure of the Markov chain X conditioned to hit A at y.

We can prove a Markovian property for Loop erased random walks, as a means of significantly restricting the candidate pool for its possible scaling limits. More precisely, consider $y_0, \ldots, y_j \in S$ so that with positive probability for $\mathcal{L}(x, y_0; A)$,

$$\{L_\sigma = y_0, L_{\sigma-1} = y_1, \ldots, L_{\sigma-j} = y_j\}.$$

Then, the conditional law of $L[0, \sigma - j]$ given this event is $\mathcal{L}(x, y_j; A \cup y_1, \ldots, y_j)$. This property shows that it is in fact fairly natural to index the loop-erased path backwards (define $\gamma_j = L^A_{\sigma-j}$, so that γ starts on A and goes back to $\gamma_\sigma = x$). Then, the time-reversal of loop-erased (conditioned and stopped) Markov chains have themselves a Markovian-type property.

Let us now come back to our two-dimensional setting: Suppose that ω is a simple random walk on the grid $\delta\mathbb{Z}^2$ (we will then let the mesh δ of the lattice go to 0) that is started from 0. Let $D \subsetneq \mathbb{C}$ be a simply connected domain, and let $D_\delta = \delta\mathbb{Z}^2 \cap D$, $A = A_\delta = \delta\mathbb{Z}^2 \setminus D$.

We are interested in the law of γ^δ as $\delta \to 0$, which is defined as the time-reversed loop-erasure of $\omega[0, \tau_A]$. Note that the law of ω_{τ_A} converges to the harmonic measure on ∂D from 0, so that it is possible to study the behavior of γ^δ conditional on the value of $\{\gamma^\delta = y_0^\delta\}$ where $y_0^\delta \to y \in \partial D$ as $\delta \to 0$. It is natural to keep in mind that simple random walk converges to planar Brownian motion which is conformally invariant, and that on the other hand the chronological loop-erasing procedure is purely geometrical, and so it it reasonable to guess that when $\delta \to 0$, the law of γ^δ should converge to a conformal invariant curve that should be the loop-erasure of planar Brownian motion.

This doesn't quite work as the geometry of planar Brownian motion becomes far too complicated. Indeed, there is no simple (even random) algorithm to loop-erase a Brownian path in chronological order. Yet, the previous heuristic strongly suggests the law of γ_δ should converge, and that the limiting law is invariant under conformal transformations: The scaling limit of LERW in D should be (modulo timechange) identical to the conformal image of the scaling limit of LERW in D'.

Again, we are looking for a continuum limiting measure on paths $\mu_D(z,w)$ with paths of dimension d (not the same d as for SAW). The limit should satisfy:

- Conformal covariance
- Domain Markov property

However, we would not expect the limit to satisfy the restriction property. The reason is that the measure given to each self-avoiding walk ω by this procedure is determined by the number of ordinary random walks which produce ω after loop erasure. If we make the domain smaller, then we lose some random walks that would produce ω and hence the measure would be smaller. In terms of Radon-Nikodym derivatives, we would expect:

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)} < 1.$$

1.2.3 Percolation

Suppose that every point in the triangular lattice in the upper half plane is colored black or white independently with each color having equal probability.

We introduce a boundary condition on the bottom row such that it is entirely black on one side of the origin and entirely white on the other side. For any color realization, there exists a unique path called the percolation exploration process that starts at the bottom row and has all white vertices on one side and all black vertices on the other side. Similarly, we can start with a domain D and two boundary points z, w, where one arc has a black boundary condition and the other has a white boundary condition. We then place a fine triangular lattice inside D, color vertices independently in black or white with probability $1/2$, and consider the path connecting z and w. We hope for a continuous interface in the limit. Unlike the previous examples, the total mass of the lattice measures is 1, meaning $b = 0$. We assume that the curve is conformally invariant and satisfies the domain Markov property. Additionally, the scaling limit of percolation satisfies the *locality property*, which is stronger than the restriction property satisfied by SAW. If D_1 is a subset of D and z, $w \in \partial D \cap \partial D_1$, and only an initial segment of γ is visible, then to determine the measure of the initial segment, we only need to observe the percolation cluster's value at vertices adjoining γ. As a result, the measure of the path is the same whether it is considered a curve in D_1 or a curve in D.

CHAPTER 2

PRELIMINARIES

In this chapter, we discuss a number of known results about SLE that are needed in proving results about the SLE trace.

2.1 Loewner Chains

Let $\mathbb{H}$ denote the upper half plane and let $\gamma\colon (0,\infty) \to \mathbb{H}$ be a continuous curve starting at the origin such that $|\gamma_t| \to \infty$ as $t \to \infty$. We set $K_t := \gamma(0,t]$, and let H_t be the unbounded component of $\mathbb{H} \setminus K_t$. As H_t is a simply connected domain, thus the Riemann Mapping theorem says there is a conformal transformation

$$g_t(z) : H_t \to \mathbb{H},$$

satisfying the **hydrodynamic normalization** at infinity, i.e, as $z \to \infty$,

$$g_t(z) = z + \frac{a(t)}{z} + \mathrm{O}(z^{-2}). \tag{2.1}$$

Notice this uniquely determines g_t among conformal transformations from H_t to the upper half plane. The quantity $a(t)$ is known as the *half plane capacity* of γ_t, written as $\mathrm{hcap}(\gamma_t)$, and functions as a natural parametrization for the curve. Moreover, the map $t \mapsto \frac{\mathrm{hcap}(\gamma_t)}{2}$ is a non-decreasing homeomorphism on $[0,T)$ and so we may reparametrize so that $\mathrm{hcap}(\gamma_t) = 2t$.

The chordal Loewner equation establishes a one-to-one correspondence between continuous valued paths $(U_t)_{t>0}$ and increasing families $(K_t)_{t>0}$ of compact $\mathbb{H}$-hulls having a certain local growth property.

Definition 2.1.1. Let $(K_t)_{t>0}$ be a family of increasing $\mathbb{H}$-hulls. For $K_{t+} = \bigcap_{s>t} K_s$ and for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that $(K_t)_{t>0}$ has the local growth property if

$$\mathrm{rad}(K_{t,t+h}) \to 0 \text{ as } h \to 0 \quad \text{uniformly on compact sets in } t,$$

where

$$\mathrm{rad}(K) := \inf\{r \geq 0 : K \subset r\mathbb{D} + x \text{ for some } x \in \mathbb{R}\}.$$

The first connection between the family of growing compact $\mathbb{H}$-hulls and the real-valued path $(U_t)_{t>0}$ is done in the following proposition.

Proposition 2.1.1. *Let $(K_t)_{t>0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then, $K_{t+} = K_t$ for all t, and the mapping $t \mapsto \mathrm{hcap}(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Moreover, for all $t > 0$, there is a unique $U_t \in \mathbb{R}$ such that $U_t \in \overline{K_{t,t+h}}$ for all $h > 0$, and the process $(U_t)_{t>0}$ is continuous.*

The process $(U_t)_{t>0}$ is called the *driving function* of $(K_t)_{t>0}$. We note that the map $t \mapsto \frac{\mathrm{hcap}(K_t)}{2}$ is a non-decreasing homeomorphism on $[0, T)$ and so we may reparametrize so that $\mathrm{hcap}(K_t) = 2t$.

Theorem 2.1.2. *Let $(K_t)_{t>0}$ be a family of increasing compact hulls in $\mathbb{H}$ satisfying the local growth property with hcap$(K_t) = 2t$. Let $(U_t)_{t>0}$ be its driving function. Set $g_t = g_{K_t}$ and $T(z) = \inf\{t > 0 : z \in K_t\}$. Then, for all $z \in \mathbb{H}$, the function $(g_t(z) : t \in [0, T(z)))$ is differentiable with respect to t and satisfies the Loewner differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z. \tag{2.2}$$

Here, $\dot{g}_t(z)$ denotes the derivative of $g_t(z)$ with respect to t. Moreover, if $T(z) < \infty$, then $|g_t(z) - U_t| \to 0$ as $t \to T(z)$.

We note that the converse of the above statement is also true, i.e., we may recover the family of hulls $(K_t)_{t>0}$ from the driving function.

Theorem 2.1.3. *Let U_t be a continuous, real-valued function. For all $z \in \mathbb{C} \setminus \{U_0\}$, there exists a unique time $T_z \in (0, \infty)$ and a unique continuous map $(g_t(z) : t \in [0, T_z))$ in $\mathbb{H} \setminus K_t = H_t$ such that, for all $t \in [0, T_z)$, we have $g_t(z) \neq U_t$ and*

$$g_t(z) = z + \int_0^t \frac{2\, ds}{g_s(z) - U_s},$$

and such that $|g_t(z) - U_t| \to 0$ as $t \to T(z)$ whenever $T_z < \infty$. Set $T_{U_0} = 0$, and define

$$H_t = \{z \in \mathbb{H} : T_z > t\}.$$

Then, for all $t > 0$, H_t is open and $g_t : H_t \to \mathbb{H}$ is conformal onto $\mathbb{H}$. Moreover, the family of sets $K_t = \{z \in \mathbb{H} : T(z) \leq t\}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property with $\operatorname{hcap}(K_t) = 2t$, *and $g_{K_t} = g_t$, for all t.*

We can make sense of Theorem 2.1.3 as follows. Suppose $t \mapsto U_t$ is a continuous, real-valued function. For each $z \in \mathbb{H}$, if we define $g_t(z)$ as the solution to (2.4), then one can show that the solution exists up to some time $T_z \in (0, \infty]$. If $H_t = \{z \in \mathbb{H} : T_z > t\}$ as defined previously, then it can be shown that g_t is a conformal transformation of H_t onto $\mathbb{H}$ with $g_t(z) - z = o(1)$ as $z \to \infty$. We would like to define a curve γ by the limit

$$\gamma(t) = g_t^{-1}(U_t) = \lim_{y \to 0^+} g_t^{-1}(U_t + iy). \tag{2.3}$$

The quantity $g_t^{-1}(U_t + iy)$ always makes sense, but it is not true that the limit can be taken for every continuous U_t. The "problem" functions U_t have the property that they move faster along the real line than the hull is growing. From the simple example above, we see that if the driving function remains constant, then in time $O(t)$ the hull grows at rate $O(\sqrt{t})$. If $U_t = o(\sqrt{t})$ for small t, then we are fine. In fact, the following holds.

Theorem 2.1.4. *[28]*

- *There exists* $c_0 > 0$ *such that if* U_t *satisfies* $|U_{t+s} - U_t| \leq c_0\sqrt{s}$ *for all* s *sufficiently small, then the curve* γ *exists and is a simple curve.*

- *There exists* $c_1 < \infty$ *and a function* U_t *satisfying* $|U_{t+s} - U_t| \leq c_1\sqrt{s}$ *for all* t, s *for which the limit* (2.3) *does not exist for some* t.

Definition 2.1.2. Suppose $t \mapsto U_t$ is a driving function. We say that U_t generates the curve $\gamma : [0, \infty) \to \mathbb{H}$ if for each t, D_t is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$.

2.2 Schramm Loewner Evolution

We now work our way to the definition of *SLE*, which we view as a measure on curves in $\mathbb{H}$; see [28, 40, 3] for a more thorough treatment of basic properties of SLE. To make things precise, we highlight the assumptions needed on these probability measures on curves γ_t, namely *scale invariance* and the *domain Markov property*. That is, for a probability measure $\mathbb{P}$ on curves $\gamma : [0, \infty) \to \mathbb{H}$ we require the following the following:

- (*Scale Invariance*) If $r > 0$ and $\mathbb{P}_r$ denotes the probability measure obtained by scaling the curve γ, i.e. considering the curve $r\gamma$, then $\mathbb{P}_r = \mathbb{P}$.

- (*Conformal Markov Property*) Suppose the segment $\gamma[0, t]$ is known, and we let $g : H_t \to \mathbb{H}$ be a conformal transformation defined in the previous section with $g(\gamma(t)) = 0$ and $g(\infty) = \infty$. Then the conditional distribution of $g(\gamma(t, \infty))$ given $\gamma[0, t]$ is $\mathbb{P}$.

2.2.1 Chordal SLE

Consider a scale invariant measure $\mathbb{P}$ satisfying the domain Markov property, and supported on curves γ_t that are parametrized by hcap. For our choice of g_t defined in the previous section, Theorem 2.1.2 implies that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z. \tag{2.4}$$

Conformal invariance and the domain Markov property imply that U_t must be a continuous, real-valued function with stationary, independent increments. This implies that U_t must be a standard one-dimensional Brownian motion, with drift m and variance κ. Scale invariance forces $m = 0$, and thus we are left with a one-parameter collection of maps $\{g_t\}$. This gives us a precise definition of SLE; a one-parameter family of solutions to the Loewner equation driven by a Brownian motion. For ease of notation we will reparametrize the Loewner equation under the time change $t \mapsto \frac{t}{\kappa}$ so that (2.4) becomes

$$\partial_t g_t(z) = \frac{a}{g_t(z) - \widetilde{U}_t} \qquad g_0(z) = z, \tag{2.5}$$

where the parameter $a = \frac{2}{\kappa}$ and $U_t := U_{t/\kappa}$, which is a standard Brownian motion. This allows us to formally define SLE as follows.

Definition 2.2.1. Suppose $a = \frac{2}{\kappa} > 0$ and $U_t := -B_t$ is a standard Brownian motion. Let g_t solve

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \qquad g_0(z) = z \tag{2.6}$$

Then g_t is called the *Schramm Loewner Evolution* with single parameter κ from 0 to ∞ in $\mathbb{H}$. The parametrized family of maps $\{g_t\}_{t \geq 0}$ is called chordal SLE_κ.

Remark 2.2.1. The above definition exploits a particular parametrization, but often times it is more convenient, for the sake of analyzing the curves, to reparametrize by other natural quantities. For example, when illustrating the various phases of the SLE curves (which, for now, we assume exists), it is convenient to parametrize by conformal radius, so that a two dimensional problem can be reduced to a simpler, one dimensional radial Bessel equation.

Let

$$f_t(z) := g_t^{-1}(z), \qquad \tilde{f}_t(z) := f_t(z + U_t).$$

Then we may define the *SLE* **curve** (which is often referred to as the SLE *trace*) as the (formal) limit $\lim_{z \downarrow 0} \tilde{f}_t(z)$, where z tends to 0 in $\mathbb{H}$.

Theorem 2.2.2. *Chordal SLE_κ is generated by a continuous curve.*

This was proved in [40] for $\kappa \neq 8$. The $\kappa = 8$ case is more delicate, and was proved by showing that the measure is obtained as a limit of measures on discrete curves [30].

Proposition 2.2.3.

(i) SLE_κ is scale invariant in the following sense. For $r > 0$, the process $(t, z) \mapsto r^{-1} g_{r^2 t}(rz)$ is distributed as the process $(t, z) \mapsto g_t(z)$

(ii) Let $t_0 > 0$. The map $(t, z) \mapsto \tilde{g}_t(z) := g_{t+t_0} \circ g_{t_0}^{-1}(z + U_{t_0})$ is distributed as $(t, z) \mapsto g_t(z)$.

2.2.2 Radial SLE

Another form of *SLE* is radial *SLE* which, unlike the previous case, is concerned about connecting boundary points to a distinguished point on the interior of a given domain. There are many ways we can construct this, but we start with the most natural way, which deals with a process connecting a boundary point on the unit disc to the origin. The half-plane capacity parametrization is convenient for curves going from one boundary point to another (∞ is a boundary point of $\mathbb{H}$). When considering paths going from a boundary point to an interior point, it is convenient to consider the radial parametrization which is another kind of capacity parametrization. We expect this to be true for all simply connected Domains by conformal invariance, so it suffices to consider paths from the boundary of the unit disc to the origin.

Definition 2.2.2. If $D \subset \mathbb{C}$ is a simply connected domain and $z \in \mathbb{D}$, then the conformal radius of z in D is defined to be $|f'(0)|$ where $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = z$. We let $\Upsilon_D(z)$ denote one-half times the conformal radius.

By definition, $\Upsilon_{\mathbb{D}}(0) = 1/2$ and a straightforward calculation shows that $\Upsilon_{\mathbb{H}}(z) = \mathrm{Im}(z)$. If γ is a simple curve, let $D_t = D \setminus \gamma(0, t]$.

Definition 2.2.3. The curve γ has a radial parametrization (with respect to z) if

$$\log \Upsilon_{D_t}(z) = -at + r,$$

for some $a, r \in \mathbb{R}$.

Suppose $\gamma : (0, \infty) \to \mathbb{D} \setminus \{0\}$ is a simple curve with $\gamma(0^+) = w \in \partial\mathbb{D}$, and $\gamma(\infty) = 0$. For each t, let g_t be the unique conformal transformation of D_t onto $\mathbb{D}$ with $g_t(0) = 0$, $g_t'(0) > 0$. We assume that the curve has the radial parametrization with $\log[2\Upsilon(D_t(0))] = -2at$. In other words, $g_t'(0) = e^{2at}$. With this setup, we may state the following theorem.

Theorem 2.2.4. *Suppose γ is a simple curve as above. Then for $z \in \mathbb{D}$, $g_t(z)$ satisfies the differential equation*

$$\frac{\partial g_t(z)}{\partial t} = \frac{a}{2} g_t(z) \frac{g_t(z) + e^{2iU_t}}{g_t(z) - e^{2iU_t}}, \quad g_0(z) = z,$$

where $e^{2iU_t} = g_t(\gamma(t)) = \lim_{w' \to \gamma(t)} g_t(w')$. Moreover, the function $t \mapsto U_t$ is continuous. If $z \notin \gamma(0, \infty)$, then the equation is valid for all t. If $z = \gamma(s)$, then the equation is valid for $t < s$.

Definition 2.2.4. For $\kappa > 0$, radial *SLE* is defined to be the solution to the differential equation

$$\partial_t g_t(z) = 2a g_t(z) \frac{e^{2iU_t} + g_t(z)}{e^{2iU_t} - g_t(z)}, \tag{2.7}$$

with $U_t = -B_t$ a standard Brownian motion.

Originally, radial SLE was defined with the parametrization that gives conformal radius e^{2iU_t}, so that the conformal maps g_t are just defined as solutions to

$$\partial_t g_t(z) = g_t(z)\frac{U_t + g_t(z)}{U_t - g_t(z)}, \tag{2.8}$$

for $z \in \mathbb{D}$. The sets K_t and H_t are defined as in the chordal case.

2.2.3 Phases of SLE

The next theorem shows the three "phases" of SLE from the perspective of a point $z \in \mathbb{H}$. Recall that one can scale a standard Brownian motion, either in time or space, to obtain a Brownian motion of any diffusivity. In a sense, all Brownian motions "look the same", with different rates of growth. SLE, on the other hand, exhibits markedly different behaviour as the parameter κ is varied. In particular, SLE runs through three phases which we summarize below; see [40] for a proof of this result.

Theorem 2.2.5. *Suppose γ is a chordal SLE$_\kappa$ in $\mathbb{H}$.*

- *If $\kappa \leq 4$, then γ is a simple curve with $\gamma(0,\infty) \subset \mathbb{H}$.*
- *If $4 < \kappa < 8$, then γ has double points and $\gamma(0,\infty) \cap \mathbb{R} \neq \emptyset$. The curve is not plane-filling, that is to say, $\mathbb{H} \setminus \gamma(0,\infty) \neq \emptyset$.*
- *If $\kappa \geq 8$, then the curve is space-filling, that is, $\gamma[0,\infty) = \mathbb{H}$.*

2.3 Gaussian free field

There is a strong connection between SLE and the Gaussian free field (GFF). In particular, SLE curves are realized as *flow lines* of the GFF, as seen in [31, 32, 33, 34]. Here we discuss the basic construction of the field, from the viewpoint of indexing a family of Gaussian random variables by continuous, compactly supported functions. We follow the construction given in [42]. Let $D \subseteq \mathbb{C}$ be a simply connected domain. Consider the real L^2 space with the inner product

$$(f,g) := \int_D f(z)g(z)\,d\mu(z), \quad f,g \in L^2(D),$$

where $\mu(z)$ is the Lebesgue measure on $\mathbb{C}$; $d\mu(z) = \sqrt{-1}\,dz\,d\bar{z}/2$. Let Δ be the Dirichlet Laplacian acting on $L^2(D)$. In Then $-\Delta$ has positive discrete eigenvalues so that $-\Delta e_n = \lambda_n e_n$, $e_n \in L^2(D), n \in \mathbb{N}$. We assume that the eigenvalues are labeled in non-decreasing order; $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. The system of eigenfunctions $\{e_n\}_{n\in\mathbb{N}}$ forms a countable orthonormal basis of $L^2(D)$. The asymptotic behavior of eigenvalues obeys *Weyl's formula*:

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = O(1).$$

For $f, g \in C_c^\infty(D)$, the Dirichlet inner product is defined by

$$(f,g)_\Delta := \frac{1}{2\pi}\int_D \nabla f(z)\nabla g(z)\,d\mu(z). \tag{2.9}$$

The Hilbert space completion of $C_c^\infty(D)$ with respect to $(\cdot,\cdot)_\Delta$ will be denoted by $\mathcal{W}(D)$. We write $\|f\|_\Delta = \sqrt{(f,f)_\Delta}$. If we set $u_n = \sqrt{\dfrac{2\pi}{\lambda_n}} e_n$, $n \in \mathbb{N}$, then integration by parts tells us that $\{u_n\}_{n\in\mathbb{N}}$ forms a countable orthonormal basis of $\mathcal{W}(D)$.

Let $\hat{\mathcal{H}}(D)$ be the space of formal infinite series in $\{u_n\}_{n\in\mathbb{N}}$, which is obviously isomorphic to $\mathbb{R}^{\mathbb{N}}$ by setting $\hat{\mathcal{H}}(D) \ni \sum_{n\in\mathbb{N}} f_n u_n \mapsto \{f_n\}_{n\in\mathbb{N}}$. As a subspace of $\hat{\mathcal{H}}(D)$, $\mathcal{W}(D)$ is isomorphic to $\ell^2(\mathbb{N}) \subset \mathbb{R}^{\mathbb{N}}$. For two formal series $f = \sum_{n\in\mathbb{N}} f_n u_n, \quad g = \sum_{n\in\mathbb{N}} g_n u_n$, such that $\sum_{n\in\mathbb{N}} |f_n g_n| < \infty$, we define their pairing $(f,g) := \sum_{n\in\mathbb{N}} f_n g_n$. In the case when $f, g \in \mathcal{W}(D)$, their pairing, of course, coincides with the Dirichlet inner product (2.9).

Notice that, for any $a \in \mathbb{R}$, the operator $(-\Delta)^a$ acts on $\hat{\mathcal{H}}(D)$ as $(-\Delta)^a \sum_{n\in\mathbb{N}} f_n u_n := \sum_{n\in\mathbb{N}} \lambda_n^a f_n, \quad \{f_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. Using this fact, we define $\mathcal{H}_a(D) := (-\Delta)^a \mathcal{W}(D), \quad a \in \mathbb{R}$, each of which is a Hilbert space with the inner product $(f,g)_a := ((-\Delta)^{-a} f, (-\Delta)^{-a} g)_\nabla, \quad f, g \in$

$\mathcal{H}_a(D)$. We can prove that $\mathcal{H}_a(D) \subset \mathcal{H}_b(D)$ for $a < b$ using Weyl's formula for $\{\lambda_n\}_{n\in\mathbb{N}}$, and that the dual space of $\mathcal{H}_a(D)$ is given by $\mathcal{H}_{-a}(D)$.

Define $\mathcal{E}(D) := \bigcup_{a>\frac{1}{2}} \mathcal{H}_a(D)$. Then, its dual Hilbert space is identified with $\mathcal{E}(D)^* := \bigcap_{a<-\frac{1}{2}} \mathcal{H}_a(D))$ and so $\mathcal{E}(D)^* \subset \mathcal{W}(D) \subset \mathcal{E}(D)$ is established. Here $(\mathcal{E}(D), \mathcal{E}(D)^*, \mathcal{W}(D))$ is called a Gel'fand triple. We set $\Sigma_{\mathcal{E}(D)} := \sigma(\{(\cdot, f)_\nabla : f \in \mathcal{E}(D)^*\})$. On such a setting, the following is proved.

Theorem 2.3.1 (Bochner–Minlos theorem)**.** *Let ϕ be a positive, continuous function on $\mathcal{W}(D)$, such that $\phi(0) = 1$. Then there exists a unique probability measure $\mathbb{P}$ on $(\mathcal{E}(D), \Sigma_{\mathcal{E}(D)})$ such that*

$$\phi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1}(h,f)_\nabla}\, \mathbb{P}(dh), \qquad f \in \mathcal{E}(D)^* \tag{2.10}$$

With this, we may state the following definition.

Definition 2.3.1 (Dirichlet boundary GFF)**.** Let $D \subsetneq \mathbb{C}$ be a simply connected domain. A Dirichlet boundary GFF in D is an isomatry $h : \mathcal{W}(D) \to (\Omega_D^{\mathrm{GFF}}, \mathcal{F}_D^{\mathrm{GFF}}, \mathbb{P}_D^{\mathrm{GFF}})$, where $(\Omega_D^{\mathrm{GFF}}, \mathcal{F}_D^{\mathrm{GFF}}, \mathbb{P}_D^{\mathrm{GFF}})$ is a probability space for which each $h(f), f \in \mathcal{W}(D)$ is a mean-zero Gaussian random variable [31].

One can construct such an isometry relying on the Bochner-Minlos theorem that is an analogue of Bochner's theorem applicable to the case when the source Hilbert space is infinite dimensional. It is also known that, in this construction, the sigma field $\mathcal{F}_D^{\mathrm{GFF}}$ is generated by the image of $\mathcal{W}(D)$ under h, i.e., h is *full*.

Thus, we may view the zero-boundary GFF on D as a random sum of the form $h = \sum_{i=1}^{\infty} \zeta_i u_i$, where ζ_i are i.i.d. standard normal random variables and $\{u_i\}_{i\geq 0}$ an orthonormal basis for $\mathcal{W}(D)$. This sum almost surely diverges within $\mathcal{W}(D)$; however, it does converge almost surely in the space of distributions — that is, as $n \to \infty$, the limit of $\sum_{i=1}^{n} \zeta_i (u_i, \rho)_\nabla$ exists almost surely for all $\rho \in C_c^\infty(D)$, and we may define $(h, \rho) := \sum_{i=1}^{\infty} \zeta_i (u_i, \rho)_\nabla$. The limiting value as a function of ρ is almost surely a continuous functional on $C_c^\infty(D)$. In general, for any harmonic function h_0 on D, we define the GFF with boundary data h_0 by $h := \tilde{h} + h_0$ where $\tilde{h}$ is the zero-boundary GFF on D. For a thorough treatment on this construction, see [42].

We will also present a more intuitive construction of the Dirichlet GFF on a given domain. Let $D \subset \mathbb{C}$ be a domain on which the Green's function $G_D(z, w)$ is well defined and finite. Intuitively, we view the field as a collection of centered Gaussian random variables $\{h(z) : z \in D\}$ with covariance $\mathbb{E}[h(z)h(w)] = G_D(z, w)$. Note that $G_D(z, z)$ is infinite and hence $h(z)$ is a "Gaussian random variable with infinite variance". We can still make sense of this viewpoint as follows. If ρ is a smooth function with compact support on D, we write formally

$$h(\rho) = \int_D h(z)\rho(z)d\mu(z).$$

More precisely, $h(\rho)$ is a centered Gaussian random variable with variance

$$G_D(\rho) := \int_D \int_D G_D(z, w)\rho(z)\rho(w)d\mu(z)d\mu(w),$$

where

$$G_D(\rho)(z) = \int_D G_D(z, w)\rho(w)d\mu(w) = \mathbb{E}_z\left[\int_0^{\tau_D} \rho(B_t)dt\right].$$

Here B_t is a complex Brownian motion and τ_D is the exit time from the domain D. Recall that

$$\frac{1}{2}\Delta G_D(\rho)(z) = -\rho(z).$$

Definition 2.3.2. The Gaussian free field on D is a centered Gaussian process $\{h(\rho)\}$ indexed by smooth functions with compact support on D satisfying linearity,

$$h(a_1\rho_1 + a_2\rho_2) = a_1 h(\rho_1) + a_2 h(\rho_2), \quad a_1, a_2 \in \mathbb{R}, \tag{2.11}$$

with covariance

$$\mathbb{E}[h(\rho)h(\psi)] = G_D(\rho, \psi) := \int_D \int_D G_D(z, w)\rho(dz)\psi(dw).$$

Remark 2.3.2. If h satisfies (2.11), then to show that h is a Gaussian free field in D, it suffices to show that for each ρ, $h(\rho)$ is a centered Gaussian random variable with variance $G_D(\rho)$. Indeed, since

$$\sum_{j=1}^{n} c_j h(\rho_j) = h\left(\sum_{j=1}^{n} c_j \rho_j\right),$$

this implies that every finite linear combination has a normal distribution and hence $\{G_D(\rho)\}$ has a joint Gaussian distribution. The covariance formula follows from

$$G_D(\rho + \psi) = G_D(\rho) + G_D(\psi) + 2G_D(\rho, \psi).$$

Remark 2.3.3. As in the previous construction, we can extend to the Hilbert space completion of $C_c^\infty(D)$ under the inner product $G_D(\cdot, \cdot)$. Indeed, we can consider (signed) measures μ with the property that $G_D(|\mu|) < \infty$. If Q is a closed subspace of $\mathcal{W}(D)$, $\varphi \in C_c^\infty(D)$, and $\hat{\varphi}$ is the projection of φ onto Q, then $\hat{\varphi}$ and $\varphi - \hat{\varphi}$ are orthogonal, and hence $h(\hat{\varphi})$ and $h(\varphi - \hat{\varphi})$ are independent random variables.

CHAPTER 3

A NEW PROOF OF THE REVERSIBILITY OF SLE_κ FOR $\kappa \le 4$

In thie chapter, we give a new proof of the reversibility of the Schramm Loewner evolution for $\kappa \le 4$. The main ideas used in the proof are similar to those used in the original proof of this result, given by Zhan [47].

3.1 Introduction

While SLE is a model for curves in equilibrium, the definition uses conditional probabilities given the path up to a certain time and hence adds an artificial dynamic. One disadvantage is that some properties that are expected of the limit curve, in particular reversibility, do not follow immediately. Zhan showed this to be true [47] for $\kappa \le 4$, while Miller and Sheffield were able to extend these results to $\kappa \in (0, 8)$ [31, 32, 33] by realizing SLE_κ curves as flow lines of the Gaussian free field.

The purpose of this chapter is to give a new proof of reversibility for $\kappa \le 4$; we hope in future work to extend this to $4 < \kappa < 8$ to give a proof that does not make use of the tools of the Gaussian free field. While we say that it is a new proof, the basic idea of the proof is the same as that given by Zhan. Our hope is that our argument simplifies some of the details. We write SLE for SLE_κ.

- We compare SLE from 0 to x in $\mathbb{H}$ to SLE from x to 0. These are probability measures on bounded curves γ with $\operatorname{hcap}(\gamma) < \infty$. While $\operatorname{hcap}(\gamma)$ is a random quantity, it is almost immediate from the definition that the distribution of $\operatorname{hcap}(\gamma)$ is the same for SLE in both directions.

- We view SLE connecting two points in $\mathbb{R}$ as a probability measure on the final mapping-out functions g_γ.

- We then focus on SLE from 0 to x and x to 0 conditioned to have a specific half-plane capacity. We show that these two probability measures agree on the conformal maps g_γ for each value of $\mathrm{hcap}[\gamma]$. By scaling it suffices to prove this for all x assuming $\mathrm{hcap}[\gamma] = a$.

- For each $r \in [0,1]$ we consider the probability measure μ_r which corresponds to the following:

  - Take SLE from 0 to x conditioned to have $\mathrm{hcap} = a$ stopped at time r, that is, when $\mathrm{hcap} = ra$ giving γ^1.
  - Given γ^1, let γ^2 be SLE from x to $\gamma^1(r)$ in $\mathbb{H} \setminus \gamma^1$ conditioned so that $\mathrm{hcap}(\gamma^1 \cup \gamma^2) = a$.
  - Output g_γ where $\gamma = \gamma^1 \oplus \tilde{\gamma}$ where $\tilde{\gamma}$ is the reversal of γ^2.

  This gives a probability measure on transformations g_γ with $\mathrm{hcap}[\gamma] = a$ which we denote by μ_r.

- We consider this as a measure on continuous functions on a fixed closed ball $K = K_h \subset \mathbb{H}$ where h is large enough so that $\mathrm{Im}[g_\gamma(z)] \geq a$ for all $z \in K$ and $\mathrm{hcap}[\gamma] = a$. We show that the Prokhorov distance between μ_r and μ_s is less than $c|s-r|^{1+\delta}$ for some $\delta > 0$. We conclude that μ_s is a constant function of s. In particular, $\mu_0 = \mu_1$ which is the main result.

- The main local commutation relation which is similar to the relations in [47] and [12] is expressed in terms of Radon-Nikodym derivatives of independent SLE paths tilted by a Brownian loop term. This relation is nicest for $\kappa \leq 4$, but we discuss the $\kappa < 8$ case here in order to prepare for future work.

The chapter is organized as follows. In Section 3.2, we review SLE connecting two points on the boundary, together with some other basic notation, and then we state the main theorem of this chapter. In Section 3.3, we describe the commutation relation, and show explicitly that the measures under consideration have the same Radon-Nikodym derivative with respect to a particular measure. In Section 5.2.1 we prove the main theorem in a sequence of steps, relying on a few Loewner chain estimates. Finally, in Section 3.5, we give the (delayed) proof of a basic Bessel process fact.

Throughout this chapter we fix $\kappa = 2/a \in (0, 8)$ and allow constants, both implicit and explicit, to depend on κ. We write just SLE for SLE_κ. For a number of the results, we need $\kappa \leq 4$ and we say that. Let

$$b = \frac{6-\kappa}{2\kappa} = \frac{3a-1}{2}$$

be the boundary scaling exponent.

3.2 SLE in $\mathbb{H}$ from x_1 to x_2

There are several equivalent characterizations of SLE connecting two real points; here we will take the perspective of SLE from 0 to $x \in \mathbb{R} \setminus \{0\}$ as SLE from 0 to ∞ in $\mathbb{H}$ tilted by the partition function Ψ. For simply connected domains and locally analytic boundary points z, w, the partition function for SLE is $\Psi_D(z, w) = H_{\partial D}(z, w)^b$. Here $H_{\partial D}(z, w)$ is the boundary Poisson kernel normalized so that $H_{\mathbb{H}}(0, x) = x^{-2}$. We also define $\Psi_{\mathbb{H}}(x, \infty) = 1$ for all $x \in \mathbb{H}$. The partition function satisfies the scaling rule: if $f : D \to f(D)$ is a conformal transformation, then

$$\Psi_D(z, w) = |f'(z)|^b \, |f'(w)|^b \, \Psi_{f(D)}(f(z), f(w)).$$

Although this definition of $\Psi_D(z, w)$ requires that z, w be locally analytic boundary points, ratios of partition functions can often be defined using the scaling rule as we will see below.

Suppose that g_t satisfies (4.1) where $U_t = -B_t$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\gamma(t)$ denote the corresponding SLE_κ curve and we write

$\gamma_t = \gamma(0,t]$. Under the measure $\mathbb{P}$, γ has the distribution of an SLE_κ path from 0 to ∞. We will tilt the measure $\mathbb{P}$ using an appropriate local martingale to get SLE from 0 to x. Suppose $x \in \mathbb{R} \setminus \{0\}$, and let $X_t = g_t(x) - U_t$ and $T = T_x = \inf\{t > 0 : X_t = 0\}$. For $t < T$, let D_t be the unbounded component of $\mathbb{H} \setminus \gamma_t$, and define the local martingale M_t formally by

$$M_t = x^{1-3a} \frac{\Psi_{H_t}(\gamma(t), x)}{\Psi_{D_t}(\gamma(t), \infty)}, \qquad t < T.$$

The partition functions on the right-hand side are not well defined but the ratio is well defined using the scaling rule,

$$\frac{\Psi_{D_t}(\gamma(t), x)}{\Psi_{D_t}(\gamma(t), \infty)} = \frac{\left|g'(\gamma_t)\right|^b \, g'(x)^b \, \Psi_{D_t}(U_t, g_t(x))}{\left|g'(\gamma_t)\right|^b \, g'(\infty) \, \Psi_{\mathbb{H}}(U_t, \infty)} = g_t'(x)^b X_t^{1-3a}.$$

While this is formal, this shows that we can define

$$M_t = \left(\frac{X_t}{X_0}\right)^{1-3a} g_t'(x)^b, \quad t < T$$

and one can use Itô's formula and the Loewner equation to see that M_t is a local martingale satisfying $M_0 = 1$ and

$$dM_t = \frac{1-3a}{X_t} \, M_t \, dB_t, \quad 0 \le t < T. \tag{3.1}$$

Let $\mathbb{P}^*$ be the measure obtained by tilting by M_t. More precisely, if $\tau < T$ is a stopping time such that $M_{t\wedge\tau}$ is a martingale and V is an event measurable with respect to $\mathcal{F}_{t\wedge\tau}$, then $\mathbb{P}^*(V) = \mathbb{E}[M_{t\wedge\tau} \, 1_V]$. The Girsanov theorem states that

$$dB_t = \frac{1-3a}{X_t} \, dt + dW_t, \quad 0 \le t < T,$$

where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$ and hence

$$dX_t = \frac{1-2a}{X_t}\,dt + dW_t, \quad 0 \leq t < T.$$

The following is well known.

Proposition 3.2.1. *Suppose* $0 < x$ *and* g_t *is the solution to the Loewner equation* (4.1) *where* $U_t = g_t(x) - X_t$ *and* X_t *satisfies*

$$dX_t = \frac{1-2a}{X_t}\,dt + dW_t, \quad X_0 = x, \quad 0 \leq t < T, \tag{3.2}$$

where W_t *is a standard Brownian motion and* $T = \inf\{t : X_t = 0\}$. *Then* $\gamma(t), 0 \leq t \leq T$ *has the distribution of* SLE_κ *from* 0 *to* x *parametrized by half plane capacity from infinity stopped at the time that* γ_T *disconnects* x *from infinity. In particular,* $\text{hcap}[\gamma_T] = aT$.

Indeed to verify this, one needs only check that the conformal image of SLE from 0 to ∞ by a conformal transformation $F : \mathbb{H} \to \mathbb{H}$ with $F(0) = 0, F(\infty) = x$ gives the same distribution on the driving function as (3.2).

Similarly, if X_t satisfies (3.2) and we define $\tilde{U}_t = g_t(0) - X_t$ with corresponding curve $\tilde{\gamma}$, then $\tilde{\gamma}(t), 0 \leq t \leq T$ has the distribution of SLE_κ from x to 0 parametrized by half plane capacity from infinity stopped at the time that γ_T disconnects 0 from infinity.

While we may use the same X_t for SLE in both directions, the distribution of the driving functions $U_t, \tilde{U}_t$ are different. Indeed, $U_0 = x, \tilde{U}_0 = 0$. For this reason, we cannot conclude the reversibility immediately from this fact. One thing that does follow is that the distribution of the stopping time T is the same for SLE from 0 to x as for SLE from x to 0. It is the same as the time to reach the origin for the Bessel process (3.2). Since $a > 1/4$ the process reaches the origin in finite time.

There is a significant difference between $\kappa \in (0, 4]$ and $\kappa \in (4, 8)$. Let us consider SLE from 0 to x with $x > 0$ stopped at time T. The following statements are with probability one with respect to the tilted measure $\mathbb{P}^*$.

- If $0 < \kappa \leq 4$, then $\gamma(t), 0 \leq t \leq T$ is a simple curve with $\gamma(0) = 0, \gamma(T) = x, \gamma(0, T) \subset \mathbb{H}$.

- If $4 < \kappa < 8$, then $\gamma(T) \in (x, \infty)$. Although the SLE curve continues after time T, D_∞, the unbounded connected component of $\mathbb{H} \setminus \gamma$ is the same as the unbounded connected component of $\mathbb{H} \setminus \gamma_T$.

In particular, for $\kappa \leq 4$, the domain D_∞ determines the entire curve while for $4 < \kappa < 8$, the domain D_∞ gives only the "curve as viewed from infinity", that is $\gamma \cap \overline{D}_\infty$. We will prove reversibility for the domain D_∞. In this chapter, we do the $\kappa \leq 4$ case reproving Zhan's result.

Theorem 3.2.2. *If $\kappa \leq 4$, the distribution of D_∞, is the same for SLE from x_1 to x_2 and for SLE from x_2 to x_1. Equivalently, the distribution of the conformal transformation g_∞ is the same.*

Our proof is in the same spirit as Zhan's proof. One novel aspect is that we choose a realization of the Bessel process (3.2) in a two step process: we first choose a value $T = t_0$ and then given T we run the Bessel process conditioned so that $T = t_0$.

If X_t satisfies (3.2) where W_t is a $\mathbb{P}^*$-Brownian motion, then the transition probability of the process killed at the origin is

$$q_s(x, y) = \frac{y}{x^{4a+1}\, s^{2a+\frac{1}{2}}} \exp\left\{-\frac{x^2 + y^2}{2s}\right\} h\left(\frac{xy}{s}\right),$$

where $h = h_a$ is an entire function with $h(0) > 0$. The density of T in the measure $\mathbb{P}^*$ is a constant times

$$\phi(x, t) := x^{4a-1}\, t^{-\frac{1}{2}-2a} \exp\left\{-\frac{x^2}{2t}\right\}. \tag{3.3}$$

The Bessel process conditioned so that $T = t_0$ is this process tilted by the $\mathbb{P}^*$-martingale

$$N_t := \phi(X_t, t_0 - t), \qquad 0 \le t < t_0 \tag{3.4}$$

which satisfies

$$dN_t = N_t \left[\frac{4a-1}{X_t} - \frac{X_t}{t_0 - t}\right] dW_t. \tag{3.5}$$

Formally one can write $\phi(X_t, t_0 - t) = \mathbb{E}^*[\mathbb{1}_{T=t_0}|X_t]$ which can be thought of as a Doob martingale in the measure $\mathbb{P}^*$. Otherwise, the unconvinced reader may engage in a brief Itô calculus exercise to derive (3.5). Since M_t is a $\mathbb{P}$ local martingale and N_t is a $\mathbb{P}^*$ local martingale, we can see that $\tilde{M}_t := M_t\, N_t$ is a $\mathbb{P}$ local martingale. Again, one can check this again using Itô calculus. If we let

$$\tilde{M}_t = \tilde{M}_{t,t_0} = M_t\, N_t = x^{1-3a}\, X_t^{3a-1}\, g_t'(x)^{(3a-1)/2}\, \phi(X_t, t_0 - t), \quad 0 \le t < t_0,$$

then using (3.1) and (3.4) we see that $\tilde{M}_t, 0 \le t < t_0$ is a $\mathbb{P}$-martingale satisfying

$$d\tilde{M}_t = \left[\frac{a}{X_t} - \frac{X_t}{t_0 - t}\right] \tilde{M}_t\, dB_t, \qquad 0 \le t < t_0.$$

If we tilt in the Girsanov sense as above by $\tilde{M}_t$ giving the new measure $\hat{\mathbb{P}}$ we have

$$dB_t = \left[\frac{a}{X_t} - \frac{X_t}{t_0 - t}\right] dt + d\tilde{W}_t,$$

$$dX_t = d[g_t(x) + B_t] = \left[\frac{2a}{X_t} - \frac{X_t}{t_0 - t}\right] dt + d\tilde{W}_t,$$

where $\tilde{W}_t$ is a $\tilde{\mathbb{P}}$-Brownian motion.

Definition 3.2.1. Suppose x_1, x_2 are distinct real numbers, $0 < \kappa < 8$, and $0 < t_0 < \infty$. Then SLE_κ from x_1 to x_2 in $\mathbb{H}$ of time duration t_0 is defined to be the solution of (4.1) where the driving function $U_t = g_t(x_2) - X_t$, and X_t satisfies

$$dX_t = \left[\frac{2a}{X_t} - \frac{X_t}{t_0 - t}\right] dt + dW_t, \qquad X_0 = x_2 - x_1, \tag{3.6}$$

$$U_t = g_t(x_2) - X_t = x_2 + \int_0^t \frac{a\, ds}{X_s} - X_t,$$

where W_t is a standard Brownian motion.

If $q_t(x, y)$ denotes the transition probability for a Bessel process satisfying (3.2), killed upon reaching the origin, then the density for a process satisfying (3.6) is

$$\psi_t(x, y; t_0) = q_t(x, y)\, \frac{\phi(y, t_0 - t)}{\phi(x, t_0)}.$$

We will need one very believable fact about this process. The proof uses standard techniques but we delay the proof to Section 3.5. This estimate is not optimal but will be more than sufficient for our purposes.

Proposition 3.2.3. *For every* $0 < \kappa < 8$, *there exists* $c < \infty, u > 0$ *such that if* X_t *satisfies* (3.6), *then for all* $r > 0$,

$$\mathbb{P}\left\{\max_{0 \le t \le t_0} |U_t - x_1| \ge \sqrt{t_0}\,(|x_2 - x_1| + r^2)\right\} \le c\, e^{-ur}.$$

We denote the corresponding probability measure on paths (modulo reparametrization) by $\mu^{\#}(x_1, x_2; t_0)$. Assuming $x_1 < x_2$, we have the following:

- $\gamma(0) = x_1$, $T = t_0$;
- $\mathrm{hcap}[\gamma_t] = at,\ 0 \le t \le T$;
- If $\kappa \le 4$, then γ_T is a simple curve with $\gamma(0, t_0) \subset \mathbb{H}$ and $\gamma(t_0) = x_2$. Moreover, $\partial D_\infty \cap \mathbb{H} = \gamma(0, t_0)$;

- If $4 < \kappa < 8$, then

$$\gamma(T) = x_+ := \max\{y \in \mathbb{R} : y \in \gamma_{t_0}\} > x_2,$$

$$x_- := \min\{y \in \mathbb{R} : y \in \gamma_{t_0}\} < x_1.$$

  Indeed, $\partial D_\infty \cap \mathbb{H}$ is a curve connecting x_- to x_+.

To prove Theorem 3.2.2 it suffices to prove the following.

Theorem 3.2.4. *If $\kappa \leq 4$ then for every $t_0 > 0$ and $x_1 < x_2$, the measure $\mu^{\#}(x_1, x_2; t_0)$ is the same as $\mu^{\#}(x_2, x_1; t_0)$ if considered as probability measures on the conformal transformation $g = g_{t_0}$.*

By scaling and translation invariance it suffices to prove this with $x_1 = 0, x_2 = x > 0$ and $t_0 = 1$.

Fix $x > 0$ and consider the measure $\mu_r = \mu_{r,x}$, $0 \leq r \leq 1$ obtained as follows:

- Grow the curve γ under the measure $\mu^{\#}(0, x; 1)$ until time r giving curve γ_r and corresponding map g_r. Let $z_1 = g_r(\gamma(r)), w_1 = g_r(1)$.

- Given γ_r, let $\tilde{\gamma}$ be SLE from x to $\gamma(r)$ in $\mathbb{H} \setminus \gamma_r$ conditioned so that $\text{hcap}[\gamma_r \cup \tilde{\gamma}] = a$. Equivalently, let η be chosen from $\mu^{\#}(w_1, z_1; 1 - r)$ and let $\tilde{\gamma} = g_r^{-1} \circ \eta$. Let $h = g_\eta$ and $g = h \circ g_r$.

Note that $\mu_0 = \mu^{\#}(0, x; 1), \mu_1 = \mu^{\#}(x, 0; 1)$. We will prove the following stronger result,

Proposition 3.2.5. *If $\kappa \leq 4$ and $x > 0$, then for all $0 \leq r \leq 1$, $\mu_r = \mu_0$.*

Since $\text{hcap}[\gamma_1] = a$, we know that $\gamma_1 \subset \{z : \text{Im}(z)^2 \leq 2a\}$ and hence with probability one for each μ_r, $D_\infty \supset \{z : \text{Im}(z)^2 > 2a\}$. Let $\mathcal{I} = \{z : |z - (\sqrt{8a} + 1)i| \leq 1\}$. Let S denote the set of continuous functions from $\mathcal{I}$ to $\mathbb{C}$ endowed with the supremum norm $\|\cdot\|$. We also write ρ for the corresponding Prokhorov metric on probability measures on S. Since the conformal map g is determined by its values on $\mathcal{I}$, it suffices to prove that for every $\epsilon > 0$ and $0 \leq r < s \leq 1$, $\rho(\mu_r, \mu_s) < \epsilon$. We will show the following.

Proposition 3.2.6. *For every $K < \infty$, there exists c, δ such that if $0 < x \leq K$ and $0 \leq r \leq s \leq 1$, we can couple $(g, \tilde{g})$ on the same probability space such that g has distribution μ_r, $\tilde{g}$ has distribution μ_s and*

$$\mathbb{P}\{\|g - \tilde{g}\| \geq c\,(s-r)^{1+\delta}\} \leq c\,(s-r)^{\delta},$$

$$\mathbb{P}\{\|g - \tilde{g}\| \geq c\,(s-r)\} \leq c\,(s-r)^{1+\delta}.$$

We state it this way in preparation for later work in the $4 < \kappa < 8$ case. For $0 < \kappa \leq 4$, we do significantly better by giving a coupling that satisfies $\|g - \tilde{g}\| \leq c\,(s-r)$ for all $(g, \tilde{g})$ and such that $\mathbb{P}\{\|g - \tilde{g}\| \geq (s-r)^{5/4}\}$ decays faster than every power of $s - r$.

Note that Proposition 3.2.6 implies that there exist c, δ

$$\rho(\mu_r, \mu_s) \leq c\,(s-r)^{1+\delta}.$$

This shows that μ_r is Hölder continuous of order $1 + \delta$ in r and a standard argument shows that this means that μ_r is a constant function of r and hence Proposition 3.2.5 holds.

3.3 Local commutation relation

In this section we will state the basic "commutation" relation that we will use. In order to state the relation precisely we will set up some notation. Although we only use it for $\kappa \leq 4$ in this chapter, we will also give a result that holds for all $4 < \kappa < 8$. We fix $x_1 \neq x_2$ and $t_0 > 0$. Suppose $\gamma : [0, t_0] \to \mathbb{H}$ is a non-crossing curve parametrized by capacity from x_1 to x_2 in $\mathbb{H}$. Let $\hat{\gamma}^R$ denote the reversed curve from x_2 to x_1 defined by $\hat{\gamma}^R(t) = \gamma(t_0 - t),\ 0 \leq t \leq t_0$.

Although $\hat{\gamma}^R$ is not parametrized by capacity, we can reparametrize it $\gamma^R(t) = \hat{\gamma}^R(\sigma(t))$ so that for each t, $\mathrm{hcap}[\gamma^R_t] = at$. The total time duration of γ^R is the same as that of γ, t_0.

If $0 < s_1 < s_2 < t_0$, we can write

$$\gamma = \gamma_{s_1} \oplus \gamma[s_1, s_2] \oplus \gamma[s_2, t_0],$$

Let us write γ^1 for γ_{s_1} and γ^2 for the reversal of $\gamma[s_2, t_0]$, so that we have

$$\gamma = \gamma^1 \oplus \eta \oplus (\gamma^2)^R. \tag{3.7}$$

Let us view this at the moment as a decomposition modulo reparametrization but still remember that $\mathrm{hcap}[\gamma] = at_0$ and we assume that

$$\mathrm{hcap}[\gamma^1 \cup (\gamma^2)^R] = \mathrm{hcap}[\gamma^1 \cup \gamma^2] < at_0.$$

We will also assume that

$$\gamma^1 \cap \gamma^2 = \emptyset.$$

If $\kappa \leq 4$, this will happen with probability one since SLE_κ is supported on simple curves, but for $\kappa > 4$ this is a nontrivial constraint.

Suppose $r_1 + r_2 \leq t_0$, V_1, V_2 fixed subsets of $\mathbb{C}$, and τ_1, τ_2 are stopping times for γ^1, γ^2 of the form

$$\tau_j = \min\{s : \mathrm{hcap}[\gamma^j_s] = a\, r_j \text{ or } \gamma^j_s \not\in V_j\}.$$

We view probability measures on curves from x_1 to x_2 of half-plane capacity at_0 as probability measures on ordered pairs

$$\boldsymbol{\gamma} = (\gamma^1, \gamma^2) := (\gamma^1_{\tau_1}, \gamma^2_{\tau_2}).$$

Here γ^1, γ^2 are parametrized by capacity, that is, $\mathrm{hcap}[\gamma^j_s] = as$. Note that if γ^1, γ^2 are

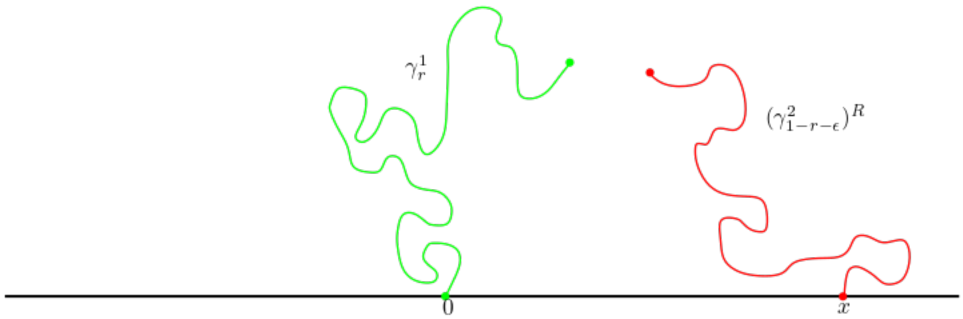

Figure 3.1: We grow SLE from 0 to x until we reach hcap ar (note that, by our choice of parametrization, this corresponds to time r). We then start SLE from x to $\gamma^1(r)$ stopped before its hcap reaches $a(1-r)$. The difference in the construction of the measures $\mathbb{P}_1^*$ and $\mathbb{P}_2^2$ comes in the middle piece, which we may construct in two ways.

nontrivial, then

$$\text{hcap}[\gamma^1 \cup \gamma^2] < \text{hcap}[\gamma^1] + \text{hcap}[\gamma^2] \leq a(r_1 + r_2) = t_0,$$

and hence the η in (3.7) is nontrivial. We will also assume that the stopping time is such that with probability one, $\gamma^1 \cap \gamma^2 = \emptyset$. If $\kappa \leq 4$, t since η is not trivial. For $\kappa > 4$, we will guarantee it by choosing stopping times such that $\gamma^1 \subset V_1, \gamma^2 \subset V_2$ for some deterministic V_1, V_2 with $V_1 \cap V_2 = \emptyset$. We now let $\mathbb{P}_j^*$ be the probability measure on $\boldsymbol{\gamma}$ given by

- Choose γ^j from SLE_κ from x_j to x_{3-j}, conditioned to have total capacity at_0, stopped at time τ_j. Let $z_j = \gamma^j(\tau_j)$.
- Given γ^j, choose γ^{3-j} from SLE_κ from x_{3-j} to z_j in $\mathbb{H} \setminus \gamma^j$, conditioned to that the total capacity of the union of the curve and γ^j is at_0, stopped at time τ_{3-j}.

The commutation result is that $\mathbb{P}_1^* = \mathbb{P}_2^*$. We sketch the proof by giving the Radon-Nikodym derivative of each of the measures with respect to $\mathbb{P}$, the measure obtained from independent SLE_κ paths. To state this we give some notation. Let $D^j = \mathbb{H} \setminus \gamma^j, D = \mathbb{H} \setminus \boldsymbol{\gamma}$. Let g^j, g be the corresponding conformal maps; let $z_j = \gamma^j(\tau_j), U^j = g^j(z_j)$ and define h_2, h_1 by $h_2 \circ g^1 = g = h_1 \circ g_2$.

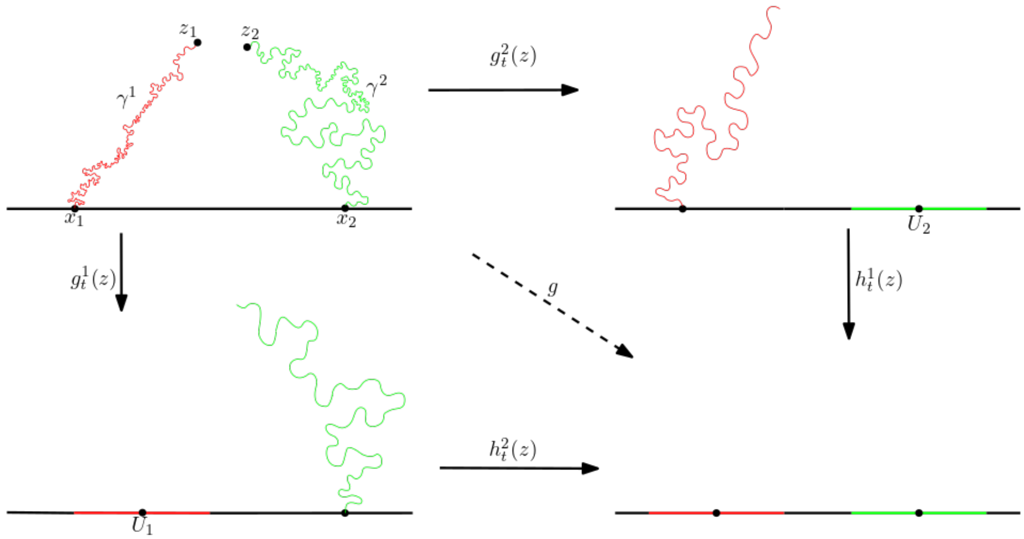

Figure 3.2: The maps g^1, g^2, h^1 and h^2 exhibit a commutative relation.

Proposition 3.3.1. *The Radon-Nikodym derivative of $\mathbb{P}_j^*$ with respect to $\mathbb{P}$, the measure obtained from independent SLE paths from 0 to infinity stopped at times τ_1, τ_2, is given by*

$$\frac{d\mathbb{P}_j^*}{d\mathbb{P}}(\boldsymbol{\gamma}) = h_1'(U^2)^b \, h_2'(U^1)^b \, \exp\left\{\frac{\mathbf{c}}{2}\, m_{\mathbb{H}}(\gamma^1, \gamma^2)\right\}$$
$$\times \frac{|g(z_2) - g(z_1)|^{2b}}{|x_2 - x_1|^{2b}} \, \frac{\phi(|U^2 - U^1|, t_0 - \tau_1 + \tau_2)}{\phi(|x_2 - x_1|, 1)}.$$

Here $b = (6-\kappa)/2\kappa$ is the boundary scaling exponent, $\mathbf{c} = (6-\kappa)(3\kappa - 8)/2\kappa$ is the central charge, and $m_{\mathbb{H}}(\gamma^1, \gamma^2)$ denotes the Brownian loop measure of loops in $\mathbb{H}$ that intersect both γ^1 and γ^2 and ϕ is as in (3.3)*. In particular, $\mathbb{P}_1^* = \mathbb{P}_2^*$.*

Proof. Without loss of generality, we assume $t_0 = 1$. We will prove the result for $j = 1$.

- We start by choosing γ^1 using SLE_κ from x_1 to x_2 stopped at time τ_1. Here we are not conditioning on the total time duration of the path. The Radon-Nikodym derivative of this with respect to SLE from 0 to infinity, restricted to the event that the total time duration is greater than τ_1 is

$$g_1'(x_2)^b \, \frac{|g_1(x_2) - U_t^1|^{2b}}{|x_2 - x_1|^{2b}}.$$

Let $\eta_2 = g_1 \circ \gamma^2$.

- Given γ^1, we will choose γ^2 using SLE from x_2 to z_1 in the domain D_1. We will do this in two steps.

- We first choose γ^2 using SLE from x_2 to infinity in D_1. Using the basic martingale of the restriction property this gives Radon-Nikodym derivative

$$\exp\left\{\frac{\mathbf{c}}{2}\, m_D(\gamma^1,\gamma^2)\right\}\, \frac{h_1'(U^2)^b}{g_1'(x_2)^b}.$$

Note that $\eta_2 := g_1 \circ \gamma_2$ is an SLE from $g_1(x_2)$ to infinity.

- We now tilt again so that $\eta_2 := g_1 \circ \gamma_2$ is an SLE from $g_1(x_2)$ to U^1. This gives a Radon-Nikodym derivative

$$h_2'(U^1)^b\, \frac{|h_2(\eta_2(\tau_2)) - h_2(U_1)|^{2b}}{|g_1(x_2) - U^1|^{2b}} = h_2'(U^1)^b\, \frac{|g(z_2) - g(z_1)|^{2b}}{|g_1(x_2) - U^1|^{2b}}.$$

- Multiplying the last two gives

$$\exp\left\{\frac{\mathbf{c}}{2}\, m_D(\gamma^1,\gamma^2)\right\}\, \frac{h_1'(U^2)^b\, h_2'(U^1)^b}{g_1'(x_2)^b}\, \frac{|g(z_2) - g(z_1)|^{2b}}{|g_1(x_2) - U^1|^{2b}}.$$

- We thus have that the Radon-Nikodym derivative restricted to the event that the total time duration is greater than $\tau_1 + \tau_2$ is given by:

$$h_1'(U^2)^b\, h_2'(U^1)^b\, \exp\left\{\frac{\mathbf{c}}{2}\, m_D(\gamma^1,\gamma^2)\right\}\, \frac{|g(z_2) - g(z_1)|^{2b}}{|x_2 - x_1|^{2b}}.$$

- If we now condition so that the total time duration is one we get

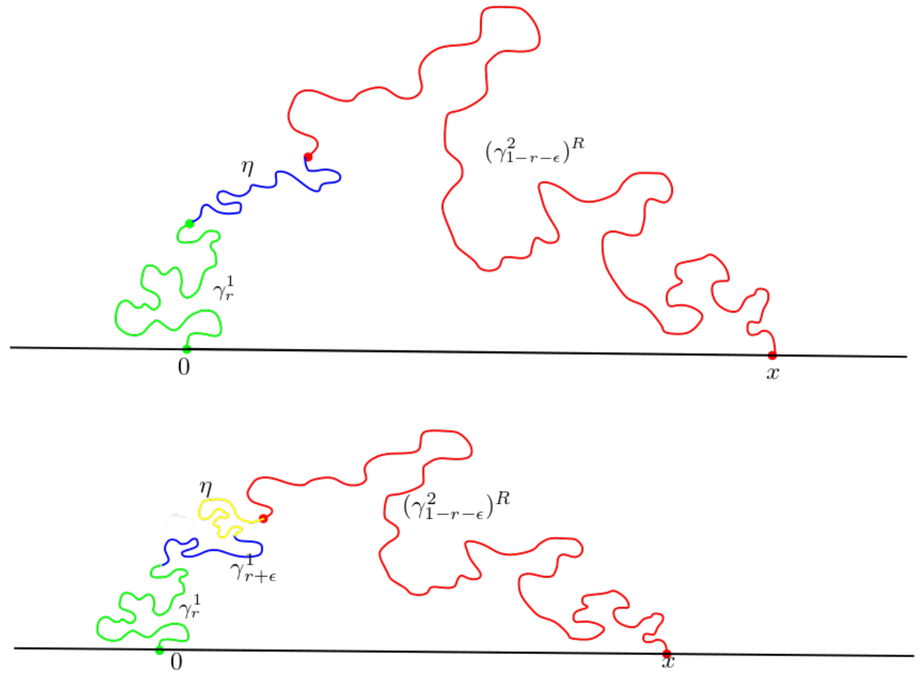

Figure 3.3: The difference in the construction comes in the curves η and $\tilde{\eta}$. Given this, we may sample from μ_r and $\mu_{r+\epsilon}$ respectively, allowing us to conclude using basic facts about the Loewner equation.

$$h_1'(U^2)^b\, h_2'(U^1)^b \,\exp\left\{\frac{\mathbf{c}}{2}\, m_D(\gamma^1,\gamma^2)\right\}\, \frac{|g(z_2)-g(z_1)|^{2b}}{|x_2-x_1|^{2b}}\, \frac{\phi(|U^2-U^1|, 1-(\tau_1+\tau_2))}{\phi(|x_2-x_1|,1)}.$$

$\square$

3.4 Proof of main Theorem

We will use some basic facts about the Loewner equation.

Proposition 3.4.1. *[28, Proposition 3.46] There exists $c<\infty$ such that if $D=\mathbb{H}\setminus K$ is a simply connected domain with $r=\sup\{|z| : z\in K\}$ and $h=\mathrm{hcap}(K)$, then the corresponding* *conformal map $g : D \to \mathbb{H}$ satisfies for $|z| \geq 2r$,*

$$\left| g_D(z) - z - \frac{h}{z} \right| \leq \frac{c\, r h}{|z|^2}.$$

In particular, if $K, \tilde{K}$ are two such hulls with $h = \tilde{h}$, then for $|z| \geq 2(r \wedge \tilde{r})$,

$$|g(z) - \tilde{g}(z)| \leq \frac{c\,(r + \tilde{r})\, h}{|z|^2}.$$

Proposition 3.4.2. *[28, Proposition 4.13] There exists $c < \infty$, such that if U_t is a driving function with $U_0 = 0$ and γ_t is the corresponding curve, then*

$$\operatorname{diam}[\gamma_t] \leq c \left[\sqrt{t} + \max_{0 \leq s \leq t} |U_s| \right].$$

We also need some easy estimates about our Bessel process conditioned to reach the origin at a given time.

Lemma 3.4.3. *If $K < \infty$, there exists $\epsilon_0 > 0$ such that if X_t satisfies (3.6) with $t_0 = 1$ and $|x_2 - x_1| \leq K$, then as $\epsilon \to 0$,*

$$\mathbb{P}\left\{ |X_{1-\epsilon}| \geq \sqrt{\epsilon}\, \log(1/\epsilon) \right\}$$

decays faster than every power of ϵ.

Proof. By a coupling argument, the probability on the left restricted to $|x_2 - x_1| \leq K$ is maximized when $x_2 - x_1 = K$. In this case, we can look at the transition probability. □

We write $\epsilon = s - r$. We decompose a simple path γ from 0 to x with $\operatorname{hcap}[\gamma] = a$ as

$$\gamma = \gamma^1 \oplus \eta \oplus (\eta')^R \oplus (\gamma^2)^R,$$

where the decomposition is defined by

$$\operatorname{hcap}[\gamma^1] = ra, \quad \operatorname{hcap}[\gamma^1 \cup \eta] = sa, \qquad \operatorname{hcap}[\gamma^1 \cup \gamma^2] = (t_0 - \epsilon)a.$$

Using the definition and the conformal Markov property, we can see that when we sampling from μ_s we choose the paths in order $\gamma^1, \eta, \gamma^2, \eta'$. When we sample from μ_r we use the order $\gamma^1, \gamma^2, \eta', \eta^R$. In each case the distribution is SLE to the endpoint of the other curve in the domain slit by the curves at that point, conditioned to have the appropriate total half-plane capacity and stopped as specified above.

We now use Proposition 3.3.1 to say that another way to sample from μ_s is to choose the paths in order $\gamma^1, \gamma^2, \eta, \eta'$. Hence we can write the sampling as follows. Steps 1 and 2 are the same for both sampling methods. Step 3a is used for μ_s and Step 3b is used for μ_r.

- **Step 1:** Choose γ^1 from SLE from 0 to x_0 conditioned to have total half-plane capacity a stopped at time r, that is, stopped when $\operatorname{hcap}[\gamma^1] = ar$. Let $z_1 = \gamma(r)$, let $\hat{g} : \mathbb{H} \setminus \gamma^1 \to \mathbb{H}$ be the corresponding transformation, and let $y_1 = \hat{g}(z_1), x_1 = \hat{g}(x_0)$.

- **Step 2:** Choose η from SLE from x_1 to y_1 conditioned to have total half-plane capacity $a(1-r)$ stopped at time $1-s$, that is, stopped when $\operatorname{hcap}[\eta] = a(1-s)$. Let $h : \mathbb{H} \setminus \eta \to \mathbb{H}$ be the corresponding transformation, and let $y_2 = h(y_1), x_2 = h(\eta(1-s))$. Let $\gamma^2 = \hat{g}^{-1} \circ \eta$ and $w_1 = g^{-1}(\eta(1-s))$. Let $\hat{h} = h \circ g$ and note that $\hat{h} : \mathbb{H} \setminus (\gamma^1 \cup \gamma^2) \to \mathbb{H}$ is the corresponding conformal transformation which satisfies $\hat{h}(z_1) = y_2, \hat{h}(w_1) = x_2$.

- **Step 3a:** Choose ω^1 from SLE from y_2 to x_2 conditioned to have total half-plane capacity $a\epsilon$ stopped at the first time that

$$\operatorname{hcap}[h^{-1} \circ \omega^1] = a\epsilon.$$

This is the same as the first time that

$$\text{hcap}[\gamma^1 \cup \hat{h}^{-1} \circ \omega^1] = as.$$

Let this time be u and let $\phi : \mathbb{H} \setminus \omega^1 \to \mathbb{H}$ be the corresponding transformation with $y_3 = \phi(\omega_u^1), x_3 = \phi(x_2)$. Let $\tilde{\omega}^2$ be chosen from SLE from x_3 to y_3 conditioned to have half-plane capacity $a(\epsilon - u)$ giving conformal map $\hat{\phi}$ and let $\omega^2 = \hat{\phi}^{-1} \circ \tilde{\omega}^2$ and

$$\omega = \omega^1 \oplus [\omega^2]^R.$$

Let $\psi : \mathbb{H} \setminus \omega \to \mathbb{H}$ be the corresponding conformal transformation.

- **Step 3b** Choose ω^* from SLE from x_2 to y_2 conditioned to have total half-plane capacity $a\epsilon$ and set

  $$\hat{\omega} = [\omega^*]^R.$$

  Let $\hat{\psi} : \mathbb{H} \setminus \hat{\omega} \to \mathbb{H}$ be the corresponding conformal transformation.

In our coupling we use the complete coupling for steps 1 and 2. Hence we write

$$g = \psi \circ h, \qquad \tilde{g} = \tilde{\psi} \circ h,$$

where h is the same in both cases. If $z \in \mathcal{I}$, then $\text{Im}(h(z)) \geq \sqrt{4a}$. Except for an event of probability that decays faster than every power of ϵ, we have $x_2 - y_2 \leq \epsilon^{1/2} \log(1/\epsilon)$. Using this, we see that in step 3a and in step 3b we get a curve with the same initial and terminal points, of half plane capacity $a\epsilon$ and such that, except for an event of probability that decays faster than every power of ϵ, has diameter bounded by $\epsilon^{1/2} \log^2 \epsilon$. Let $\psi, \tilde{\psi}$ be the conformal transformations. Then if $\text{Im}(z) \geq \sqrt{a}$ we have

$$|\psi(z) - \tilde{\psi}(z)| \leq c\,\epsilon,$$

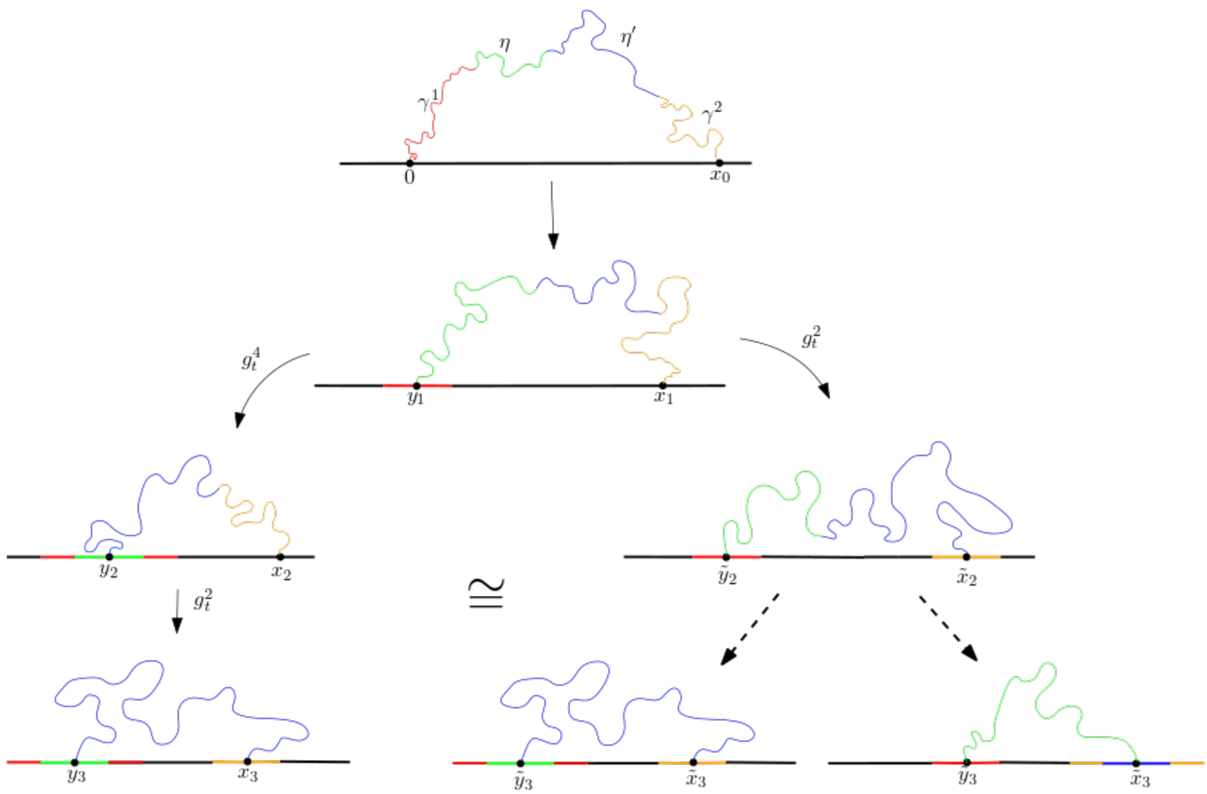

Figure 3.4: A schematic showing the full picture, though not drawn to scale (in particular, the yellow and blue segments ought not to have comparable lengths). The dotted arrows on the right correspond to the commutation relation, and together with some Loewner estimates, we may conclude that the laws of the measures obtained, regardless of the path one chooses in the schematic, are the same.

and, except for an event of probability that decays faster than every power of ϵ,

$$|\psi(z) - \tilde{\psi}(z)| \leq \epsilon^{5/4}.$$

Therefore, in this coupling, with probability one $\|g - \tilde{g}\| \leq c\,\epsilon$ and

$$\mathbb{P}\{\|g - \tilde{g}\| \geq \epsilon^{5/4}\} \leq c\,\epsilon^3.$$

3.5 Proof of Lemma 3.2.3

We fix $a > 1/4$ and allow constants to depend on a. We assume that X_t satisfies (3.6). For ease we will assume $x > 0$ but the proof with $x < 0$ is essentially the same.

The proof follows from the easy estimate

$$-X_t \leq U_t \leq \int_0^t \frac{a}{X_s}\,ds$$

and the following two lemmas that handle the two sides of the inequality. For the lower bound, we get a somewhat sharper estimate.

Lemma 3.5.1. *There exists $c < \infty$ such that if X_t satisfies* (3.6) *with $X_0 = x_0\,\sqrt{t_0} > 0$, then for all $r > 0$,*

$$\mathbb{P}\left\{\max_{0 \leq t \leq t_0} (X_t/\sqrt{t_0}) \geq x_0 + r\right\} \leq c\,\exp\left\{-\frac{r^2}{4}\right\}.$$

Proof. We may assume that $r^2 \geq 1 + 4a$ and by scaling we may assume $t_0 = 1$. Let $y = x_0 + r$ and let $\sigma = \inf\{t : X_t = y\}$. The equation (3.6) can be obtained by starting with X_t satisfying (3.2) where W_t is a $\mathbb{P}^*$ Brownian motion and then tilting by the martingale N_t as in (3.5) to get the measure $\mathbb{P}$. Hence,

$$\mathbb{P}\{\sigma < 1\} \leq M_0^{-1}\, \mathbb{E}^*\left[M_{\sigma_\epsilon}; \sigma_\epsilon < \infty\right].$$

Note that

$$M_0 = x_0^{4a-1}\, \exp\left\{-\frac{x_0^2}{2}\right\},$$

and if $\sigma < 1$,

$$M_\sigma \leq \max_{0\leq t\leq 1} y^{4a-1}\, (1-t)^{-\frac{1}{2}-2a}\, \exp\left\{-\frac{y^2}{2(1-t)}\right\} = y^{4a-1}\, e^{-y^2/2}.$$

The equality uses $r^2 \geq 1+4a$. Therefore,

$$\frac{M_\sigma}{M_0} \leq \left[1 + \frac{\log(1/\epsilon)}{x_0}\right]^{4a-1} \exp\left\{-x_0\, r - \frac{r^2}{2}\right\} \leq c\, \exp\left\{-\frac{r^2}{4}\right\}.$$

$\square$

Lemma 3.5.2. *If $a > 1/4$, there exist $u > 0$ and $c < \infty$ such that for any $x > 0$ and $t_0 > 0$ if X_t satisfies* (3.6), *then for all $r > 0$,*

$$\mathbb{P}^x \left\{\int_0^{t_0} \frac{ds}{X_s}\, dy \geq r\sqrt{t_0}\right\} \leq c\, e^{-ur}.$$

Proof. Let

$$I_n = \int_0^1 \frac{ds}{X_s}\, 1\{2^{-n} \leq X_s < 2^{-n+1}\}\, ds.$$

Our first goal is to show that there exists $c_* < \infty$ such that for all x, t_0, n,

$$\mathbb{E}^x[I_n] \leq c_*\, 2^{-n}, \qquad \mathbb{P}^x\{I_n \geq c_*\, 2^{-n+1}\} \leq \frac{1}{2}. \tag{3.8}$$

The second follows from the first by the Markov property; by scaling, It suffices to show the first inequality for $n = 0$. By the strong Markov property, we may assume that $1 \leq x \leq 2$; otherwise, we first run the process until it reaches $[1, 2]$. Also, note that

$$\mathbb{E}^x[I_0] \leq \int_1^2 \int_0^{t_0} \phi_t(x, y; t_0)\, dt\, dy.$$

Using the immediate estimate

$$\int_1^2 \left[\int_{t_0 - 1}^{t_0} \phi_t(x, y; t_0)\, dt + \int_0^1 \phi_t(x, y; t_0)\, dt \right] dy \leq 2,$$

we see that it suffices to show that there exists c such that for all $1 \leq x, y \leq 2$ and $t_0 \geq 1$,

$$\int_1^{t_0 - 1} \phi_t(x, y; t_0)\, dt \leq c.$$

This can be done in a straightforward way by looking at the transition probability. Indeed, if $1 \leq s \leq t_0 - 1$ and $1 \leq x, y \leq 2$,

$$\phi_t(x, y; t_0) \leq c \left[\frac{t_0}{t_0 - t} \right]^{2a + \frac{1}{2}} \frac{1}{t^{2a + \frac{1}{2}}}.$$

For $t < t_0/2$ we estimate this by $c\, t^{-(2a + \frac{1}{2})}$ and for $t \geq t_0/2$, we estimate this by $c\,(t_0 - t)^{-(2a + \frac{1}{2})}$. Provided that $a > 1/4$ we see that this integral is uniformly bounded in t_0. This gives (3.8).

By scaling it suffices to prove our main result for $t_0 = 1$. Note that

$$\int_0^1 \frac{ds}{X_s}\, ds \leq 1 + \sum_{n=1}^{\infty} I_n,$$

where

$$I_n = \int_0^1 \frac{ds}{X_s}\, 1\{2^{-n} \leq X_s < 2^{-n+1}\}\, ds.$$

By iterating (3.8) using the strong Markov property, we see that for all positive integers k, $\mathbb{P}\{I_n \geq 2k\, c_*\, 2^{-n}\} \leq 2^{-k}$ and hence for all $r > 0$,

$$\mathbb{P}\{I_n \geq r\, 2^{-n}\} \leq c'\, e^{-ur},$$

where $u = (\log 2)/(2c_*), c' = e^u$. In particular,

$$\mathbb{P}\left\{\sum_{n=1}^{\infty} I_n \geq 2r\right\} \leq \sum_{n=1}^{\infty} \mathbb{P}\left\{I_n \geq r(2/3)^n\right\} \leq c' \sum_{n=1}^{\infty} \exp\left\{-ur(4/3)^n\right\} \leq c\, e^{-2u(2r)/3}.$$

$\square$

CHAPTER 4

RANDOMNESS OF THE TOPOLOGY SLE$_\kappa$ FOR $\kappa > 4$

In this chapter, we study the topology of SLE curves for $\kappa > 4$. More precisely, we show that, a.s., there is no homeomorphism $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$, taking the range of one independent SLE curve to another for $\kappa \in (4, 8)$. Furthermore, we extend the result to $\kappa \geq 8$ by showing that there is no homeomorphism taking one SLE curve to another, when viewed as curves modulo parametrization.

4.1 Introduction

4.1.1 Initial Overview

Most works on SLE have focused on its geometric and probabilistic properties, e.g., Hausdorff dimensions of various subsets of the curves, formulas for the probabilities of various events, and connections to other random objects. In this work, we will address a very basic question about the topology of SLE: namely, is the topology of the curve deterministic? Said differently, if we have two independent chordal SLE$_\kappa$ curves η^1 and η^2 (viewed as curves modulo time parametrization), does there a.s. exist a homeomorphism $\overline{\mathbb{H}} \to \overline{\mathbb{H}}$ taking η^1 to η^2?

Since SLE$_\kappa$ is a simple curve for $\kappa \leq 4$, the answer to the above question is clearly affirmative in this case. For $\kappa > 4$, however, the answer is less obvious. On the one hand, many events for SLE$_\kappa$ occur with probability strictly between 0 and 1 (see Section 2 of [36]) so there are many opportunities for one of η^1 or η^2 to do something that the other does not. On the other hand, it is common for seemingly very different fractal sets to be homeomorphic. For example, if K_1 and K_2 are compact, non-empty, totally disconnected subsets of $\mathbb{C}$ without isolated points (e.g., Cantor-type sets), then there is a homeomorphism from $\mathbb{C}$ to $\mathbb{C}$ which takes K_1 to K_2 [37]. The main results of this chapter show that the topology of SLE$_\kappa$ is random for $\kappa > 4$. Indeed, for $\kappa \in (4, 8)$, we show that the topology of the range is random. The results of this chapter are in a similar vein to those of [35], which shows that an SLE_κ curve for $\kappa \in (4,8)$ is not determined by its range. Both this chapter and [35] answer an easily posed question about SLE whose answer is much less obvious than one might initially expect.

4.1.2 Summary of results

The following theorem assets that the topology of SLE_κ is not deterministic for $\kappa \in (4,8)$.

Theorem 4.1.1. *Suppose $\kappa \in (4,8)$, and let η^1 and η^2 be two independent SLE_κ curves in $\mathbb{H}$. Then a.s. there is no homeomorphism on $\overline{\mathbb{H}}$ taking the range of η^1 to the range of η^2.*

We consider the left and right boundaries of an SLE curve η (which are boundary-touching $\mathrm{SLE}_{16/\kappa}(\bar{\rho})$ curves, to be defined later). These curves form 'bubbles' in $\mathbb{H}$ (which we characterize explicitly in a later section) which we use as the primary observable to prove Theorem 5.3.1.

The result also holds for $\kappa \geq 8$, except the curves are viewed modulo time parametrization. The proof is similar, though a bit more work is needed in the setup.

Theorem 4.1.2. *Consider two independent SLE_κ curves, η^1 and η^2 in $\mathbb{H}$. Then a.s. there is no reparametrization of η^2, and no homeomorphism $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ such that $\Phi(\eta^1) = \eta^2$.*

Remark 4.1.3. Notice that in Theorem 5.4.1, we care about parametrized curves, because preservation of ranges in this setting makes less sense. Recall SLE in this instance is plane filling.

As a natural extension, one can think about the behavior of these curves for varying κ.

Conjecture 4.1.4. *Let $\kappa_1, \kappa_2 > 4$ be distinct. Let (η^1, η^2) be any coupling of a chordal SLE_{κ_1} and a chordal SLE_{κ_2}. Almost surely, there is no homeomorphism $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ such that $\Phi(\eta^1) = \eta^2$ viewed as curves modulo time parametrization. If one of κ_1 or κ_2 is in $(4,8)$, a.s., there is no such homeomorphism which takes the range of η^1 to the range of η^2.*

The conjecture says, roughly speaking, that the topologies of SLE_{κ_1} and SLE_{κ_2} are mutually singular. We expect that this conjecture can be proved using similar ideas to the ones in this chapter, but one would have to explicitly compute some of the quantities involved to show that they are κ-dependent.

4.2 Preliminaries

Here we recall a few SLE basics as well as how one defines the more general $\mathrm{SLE}_\kappa(\bar{\rho})$ processes. We write $\mathbb{H} := \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$. If K is a bounded closed subset of $\mathbb{H}$ such that $\mathbb{H} \setminus K$ is simply connected, then we call K a hull in $\mathbb{H}$ w.r.t. ∞. For such K, there is a unique g_K that maps $\mathbb{H} \setminus K$ conformally onto $\mathbb{H}$ such that $g_K(z) = z + \frac{a}{z} + O\left(\frac{1}{z^2}\right)$ as $z \to \infty$, for some real a. The quantity a is known as the ***half plane capacity*** of K, and is denoted by $\mathrm{hcap}K$. It can be shown that $a \geq 0$. The map g_K is said to satisfy the hydrodynamic normalization at infinity. For a real interval I, let $\mathcal{C}(I)$ denote the real-valued continuous functions on I. Suppose $U \in \mathcal{C}([0,T])$ for some $T \in (0,\infty]$. For each $z \in \overline{\mathbb{H}} \setminus \{0\}$, let $g_t(z)$ be the solution of the ordinary differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z. \tag{4.1}$$

Note that for $z \in \mathbb{C} \setminus 0$, the solution to (4.1) holds $\forall\, t < T_z$ where

$$T_z = \sup_t \{\min\{|g_s(z) - U_s| : 0 \leq s \leq t\} > 0\}.$$

Set

$$K_t := \{z \in \overline{\mathbb{H}} : T_z \leq t\}.$$

The sets K_t are the chordal Loewner hulls, and the collection of maps $\{g_t : t \geq 0\}$ are called the chordal Loewner maps driven by U_t. Suppose that for every $t \in [0, T)$,

$$\eta_t := \lim_{z \in \mathbb{H}, z \to U_t} g_t^{-1}(z) \in \mathbb{H} \cup \mathbb{R}$$

exists, and $\eta[0, T)$ is a continuous curve. Then for every $t \in [0, T)$, K_t is the complement of the unbounded component of $\mathbb{H} \setminus \eta((0, t])$. We call η the chordal Loewner trace driven by U_t. In general, however, such a curve may not exist depending on the choice of driving function.

An SLE_κ in $\mathbb{H}$ from 0 to ∞ is defined by the random family of conformal maps g_t obtained by solving the Loewner ODE driven by Brownian motion. In particular, we let $U_t = \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion. An SLE_κ connecting boundary points x and y of an arbitrary simply connected Jordan domain can be constructed as the image of an SLE_κ on $\mathbb{H}$ under a conformal transformation $\Psi \colon \mathbb{H} \to D$ sending 0 to x and ∞ to y. SLE curves are characterized by scale invariance and the domain Markov property, and are viewed modulo reparametrization. It is shown in [40, 30] that the SLE_κ processes are generated by curves.

$\mathrm{SLE}(\kappa; \bar{\rho})$, which is often written as $\mathrm{SLE}_\kappa(\bar{\rho}_L; \bar{\rho}_R)$, is the stochastic process one obtains by solving (4.1) with a modification on the driving process U_t, which we now discuss. It is a natural generalization of SLE_κ in which one keeps track of additional marked points which are called force points. In this chapter, we need only the two force point regime, but the following definitions are easily extended to the multiple force point setting. Fix $x_1 < 0 < x_2$. We associate with each x_i for $i \in \{1, 2\}$ a weight $\rho_i \in \mathbb{R}$. An $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ process with force points $(x_1; x_2)$ is the measure on continuously growing compact hulls K_t generated by the Loewner chain with U_t given by the solution to the system of SDEs given by

$$dU_t = \frac{\rho_1}{U_t - V_t^1} dt + \frac{\rho_2}{U_t - V_t^2} dt + \sqrt{\kappa}\, dB_t, \tag{4.2}$$

$$dV_t^i = \frac{2}{V_t^i - U_t} dt; \quad V_0^i = x_i, \quad i \in \{1, 2\}. \tag{4.3}$$

The existence and uniqueness of solutions to this SDE is discussed in [41], and follows from results in [39]. These results are extended to the more general setting of multiple force points in [31].

For $\kappa > 4$, there is also significant interest in the hulls that are generated by the SLE_κ curves. Duplantier conjectured in [15, 16] the duality between SLE_κ and $\mathrm{SLE}_{16/\kappa}$, which says that the boundary of an SLE_κ hull behaves like an $\mathrm{SLE}_{16/\kappa}$ curve, for $\kappa > 4$. Many versions of this duality have been shown in [46, 48, 12, 31, 34].

Lemma 4.9 in [31] asserts that, for $\kappa > 4$, the outer boundary η' of an SLE_κ curve is an $\mathrm{SLE}_\kappa(\bar{\rho})$ process. This is done in the setting of imaginary geometry, in which the SLE curves (for $\kappa \in (0,4)$), are realized as ***flow lines*** of the Gaussian free field (i.e $\mathrm{SLE}_\kappa(\bar{\rho})$ curves coupled with the Gaussian free field in $\mathbb{H}$), with the outer boundaries, themselves SLE curves (for $\kappa \in [4,\infty)$) described as ***counterflow lines*** (in which the coupling is done with the negation of the Gaussian field). Though we do not need this machinery as presented in [31] and [36], it serves as an excellent framework for proving some general properties of $\mathrm{SLE}_\kappa(\bar{\rho})$, some of which we rely on to prove the main results. We state one such fact as follows:

Lemma 4.2.1. *Fix $\kappa > 0$. Suppose that η is an $SLE_\kappa(\bar{\rho}_L;\bar{\rho}_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points located at $(\bar{x}_L;\bar{x}_R)$ with $x_{1,L} = 0^-$ and $x_{1,R} = 0^+$ (possibly by taking $\rho_{1,q} = 0$ for $q \in \{L,R\}$). Assume that $\rho_{1,L}$, $\rho_{1,R} > -2$. Fix $k \in \mathbb{N}$ such that $\rho = \sum_{j=1}^{k} \rho_{j,R} \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$ and $\epsilon > 0$. There exists $p_1 > 0$ depending only on κ, $\max_{i,q}|\rho_{i,q}|$, ρ, and ϵ such that if $|x_{2,q}| \geq \epsilon$ for $q \in \{L,R\}, x_{k+1,R} - x_{k,R} \geq \epsilon$, and $x_{k,R} \leq \epsilon^{-1}$ then the following is true. Suppose that $\gamma \colon [0,T] \to \mathbb{H}$ is a simple curve starting from 0, terminating in $[x_{k,R}, x_{k+1,R}]$, and otherwise does not hit $\partial\mathbb{H}$, for some $T \in [0,\infty)$. Let $A(\epsilon)$ be the ϵ-neighborhood of $\gamma([0,T])$ and let*

$$\sigma_1 = \inf\{t \geq 0 : \eta(t) \in (x_{k,R}, x_{k+1,R})\} \quad \text{and} \quad \sigma_2 = \inf\{t \geq 0 : \eta(t) \notin A(\epsilon)\}.$$

Then $\mathbb{P}[\sigma_1 < \sigma_2] \geq p_1$.

Intuitively, Lemma 4.2.1 tells us that an $\mathrm{SLE}_\kappa(\bar{\rho}_L;\bar{\rho}_R)$ process has a positive chance to stay close to any fixed deterministic curve for a positive amount of time.

Proof. This is Lemma 2.5 in [36]. $\square$

4.3 Proof of Theorem 4.1.1

Consider the left and right boundaries of the SLE curve η, which are boundary-touching $\mathrm{SLE}_{\frac{16}{\kappa}}(\rho)$ curves, with force points starting at 0. In fact, the left boundary of SLE_κ turns out to be $\mathrm{SLE}_{16/\kappa}(\frac{16}{\kappa}-4;\frac{8}{\kappa}-2)$ and by symmetry, the right boundary is $\mathrm{SLE}_{16/\kappa}(\frac{8}{\kappa}-2;\frac{16}{\kappa}-4)$. This can be deduced from Theorem 5.3 in [46]. These curves are shown in Figure 4.1. The open region between the left and right boundaries has countably many connected components, which are separated by the intersection points of the left and right boundaries, i.e., the cut points of η. These connected components have a total ordering, and come in four types:

- Type 0: Neither the left nor the right boundary of the component intersects the real line.
- Type 1: Only the right boundary intersects the real line.
- Type 2: Only the left boundary intersects the real line.
- Type 3: The left and right boundaries both intersect the real line.

Note that η is a continuous curve that travels between the positive and negative real axes between any two consecutive components of type 3. This shows that the components of type 3 form a discrete set, to which we may assign a labeling by the integers - written as

$$(\dots U_{-1}, U_0, U_1, U_2, \dots)$$

uniquely, modulo index shift. For concreteness, we choose the indexing for the sequence so

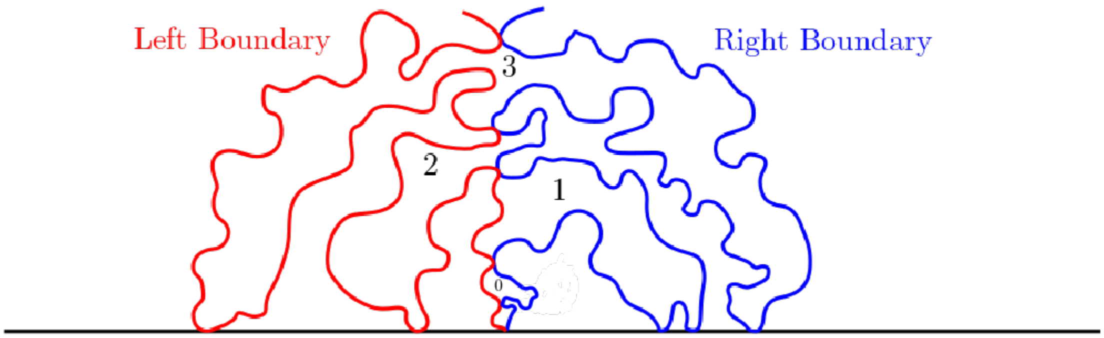

Figure 4.1: We view the complement of the SLE curve as the union of two boundary-touching $\mathrm{SLE}_\kappa(\bar{\rho})$ processes. We observe 'bubbles' of four types, which we use in constructing the observable invariant.

that U_0 is the first type 3 bubble which has Euclidean diameter at least 1. We remark here that our construction relies on a few tail triviality arguments, and so we require the following:

Lemma 4.3.1. *Suppose $t > 0$ and let a_t (resp. b_t) be the last time before t at which η hits the left (resp. right) boundary. Then $\eta|_{[0,t]}$ determines the set of bubbles (i.e. connected components of the region between the left and right boundaries) which are formed before time $\min\{a_t, b_t\}$ as well as their types.*

Proof. This follows from the fact that η cannot cross itself and $\eta([\min\{a_t, b_t\}, t])$ disconnects all of the bubbles formed before time $\min\{a_t, b_t\}$ from $\eta(t)$. □

Between pairs of consecutive type 3 bubbles, U_i and U_{i+1}, we might observe type 1 or type 2 bubbles. Let E_i be the event that there is a type 1 or type 2 bubble between U_i and U_{i+1}, and define

$$X = (\dots \mathbb{1}_{E_{-1}}, \mathbb{1}_{E_0}, \mathbb{1}_{E_1}, \mathbb{1}_{E_2}, \dots)$$

the bi-infinite sequence of 0's and 1's consisting of the indicators of the E_i's.

Lemma 4.3.2. *For any fixed deterministic bi-infinite sequence of 0's and 1's x, we have $\mathbb{P}[X = x] = 0$.*

In order to prove Lemma 4.3.2 we must first introduce some notation, and prove a few preliminary results. The proof requires a few key observations which we discuss below.

Consider a left-infinite sequence $y = (\dots y_{-2}, y_{-1}, y_0)$. For $k \in \mathbb{N}$, let A_k be the event that $\{\dots X_{-k-1}, X_{-k} = y\}$. We wish to show that $\mathbb{P}[A_0] = 0$. We will argue this by contradiction, but we first require a bit of setup. For $r \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$, let $K_r^{(n)}$ be the n^{th} smallest k such that the Euclidean diameter of U_k is at least r. Now, we claim that $\mathbb{P}[A_{K_1^{(n)}}] = 0$ for all n. We argue to the contrary, and so we assume that there exists some n such that $\mathbb{P}\left[A_{K_1^{(n)}}\right] > 0$. Note that by scale invariance, $\mathbb{P}\left[A_{K_r^{(n)}}\right]$ is independent of r, and so depends only on n. Consider the event

$$\mathcal{A}_n := \bigcap_{i=0}^{\infty} \bigcup_{m \geq i} A_{K_{\frac{1}{m}}^{(n)}}.$$

We claim that, for every n, $\mathcal{A}_n$ is a tail event for the Brownian motion that drives the SLE. Indeed, Lemma 4.3.1 implies that for each t, $\mathcal{F}_t$ determines $A_{K_r^{(n)}}$ for each r which is small enough so that the bubble $U_{K_r^{(n)}}$ is formed before time $\min\{a_t, b_t\}$. With this, and by continuity from above, we note that

$$\mathbb{P}\left[\mathcal{A}_n\right] \geq \mathbb{P}\left[A_{K_1^{(n)}}\right] > 0,$$

and so the Blumenthal $0-1$ law implies that, a.s., there exists a sequence $\{r_j\} \to 0$ such that the events $A_{K_{r_j}^{(n)}}$ occur for all j. This implies that there exist infinitely many k such that A_k occurs. Thus, it follows that a.s., $\exists$ infinitely many k such that

$$(\dots X_{-k-1}, X_{-k}) = y$$

forcing the sequence y to be periodic. We claim that this implies that the sequence $\{X_k\}$ is periodic. Indeed, let m be the period of y. Since there are arbitrarily large k for which $(\dots X_{-k-1}, X_{-k}) = y$ and y is periodic, it follows that with probability tending to 1 as $r \to 0$, the sequence

$(\dots X_{-K_r^{(n)}-1}, X_{-K_r^{(n)}})$ is equal to $(\dots y_{-j-1}, y_{-j})$ for some $j = 1, \dots, m$. By scale invariance, the probability that this is the case for all values of r is equal to 1. Thus, as $r \to \infty$, we see that the entire sequence $\{X_k\}$ is equal to y, shifted by some $j = 1, ..., m$. This means that if we observe $(...X_{-k-1}, X_{-k})$ for some k, we can determine the rest of the sequence $\{X_k\}$, forcing this sequence to be itself periodic.

For $t > 0$, we have that by Lemma 4.3.1 $\mathcal{F}_t$ determines the sequence $(\dots X_{-l-1}, X_{-l})$ for some l, which by periodicity is enough to determine the sequence $\{X_k\}$. Thus, by Lemma 4.3.1, $\mathcal{F}_t$ determines $\{X_k\}$ modulo an index shift for each $t > 0$, and hence the sequence $\{X_k\}$ is deterministic modulo an index shift. The goal now is to recursively apply Lemma 4.2.1 to arrive at a contradiction.

Proposition 4.3.3. *Let $\mathcal{Z}$ be a finite sequence of 0's and 1's which does not appear in y, with $|\mathcal{Z}| = m$. Then it must hold that*

$$\mathbb{P}\left[\{X_1, X_2, \dots, X_m\} = \{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_m\}\right] > 0.$$

Note that the existence of such a $\mathcal{Z}$ follows from the periodicity of y. With this result, we can conclude that the sequence $\{X_k\}$ can contain any finite sequence of 0's and 1's with positive probability, and hence cannot be periodic and deterministic modulo index shift. We delay the proof of the proposition to state the following key lemma, which uses the fact that the outer boundaries of the curve are $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ processes, and more specifically the right boundary, η^R, conditioned on the left boundary, η^L, has distribution of $\mathrm{SLE}_{\frac{16}{\kappa}}(-\frac{8}{\kappa}; \frac{16}{\kappa} - 4)$ (see Lemma 7.1 in [31]):

Lemma 4.3.4. *Let τ be a stopping time for η^R given η^L, at which η^R forms a type 3 bubble denoted U_{k_τ}. Let E_{k_τ} be the event that there is a type 1 or type 2 bubble between U_{k_τ} and $U_{k_\tau+1}$, as defined previously. Then,*

$$0 < \mathbb{P}\left[E_{k_\tau} \,\middle|\, \eta^L, \eta^R_{|[0,\tau]}\right] < 1.$$

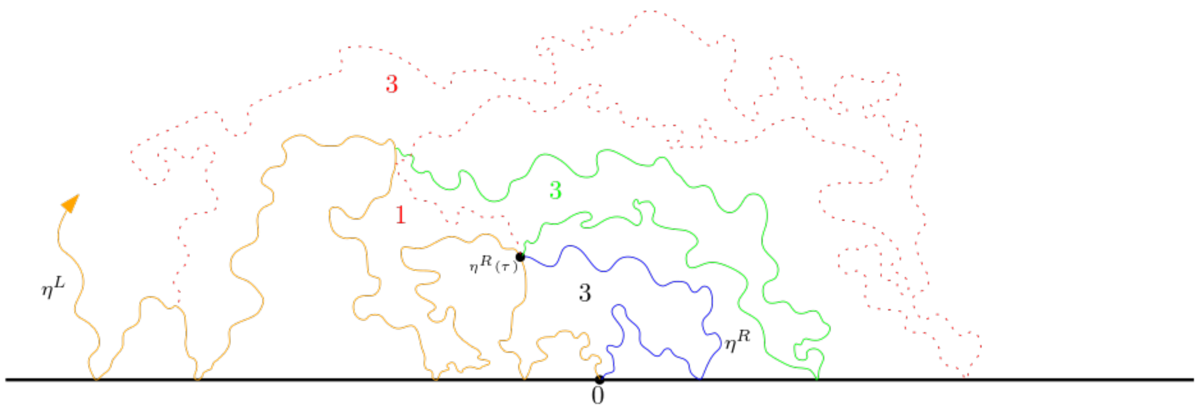

Figure 4.2: We condition on the left boundary (pictured as the orange curve) and run the right boundary until we first form a type 3 bubble of diameter at least 1 (blue). At this time (denoted $\eta^R(\tau)$), we have two options: either the right boundary hits $[0, \infty)$ before hitting the left boundary again (green), thus forming a type 3 bubble, or it hits the left boundary first (red), forming a type 1 bubble before forming the next type 3 bubble. These events each occur with positive probability.

Proof. With some setup, this is a straightforward application of Lemma 4.2.1. Indeed, let $z_\tau := \eta^R(\tau)$ and define C_{z_τ} to be the connected component of $\eta^L \setminus \mathbb{R}$ containing z_τ. Set

$$s^1 := \inf\{t > \tau : \eta^R \cap [0, \infty) \neq \emptyset\}, \qquad s^2 := \inf\{t > \tau : \eta^R \cap \eta^L \setminus (C_{z_\tau} \cup (-\infty, 0]) \neq \emptyset.\}$$

By Lemma 4.2.1, we have that

$$\mathbb{P}\left[s^2 > s^1 \middle| \eta^L, \eta^R_{|_{[0,\tau]}}\right] > 0; \qquad \mathbb{P}\left[s^2 \leq s^1 \middle| \eta^L, \eta^R_{|_{[0,\tau]}}\right] > 0$$

where the second inequality follows from symmetry considerations. Indeed, we can simply apply Lemma 4.2.1 to the curve η^R, under the conditional law given η^L. In this case, an interval on the left boundary corresponds to a segment of η^L. Note that these probabilities are strictly less than 1 as they are both positive and complementary. With this, and appealing to the setting of Fig. 4.2, we have that $\eta^R[\tau, \infty)$, conditioned on $\eta^L, \eta^R_{|_{[0,\tau]}}$, will either first intersect the left boundary and form a type 1 bubble before forming another type 3 bubble, or it will intersect $[0, \infty)$ before hitting the left boundary again, forming another type 3 bubble.

In particular, the event that a type 1 bubble is formed after U_{k_τ} occurs with probability strictly between 0 and 1 as desired. $\square$

Proof of Proposition 4.3.3. We define a sequence of stopping times as follows: For a given bubble U_i, let τ_i be the corresponding time at which U_i is formed. By our choice of indexing of the type 3 bubbles, we have that

$$\tau_0 := 1^{\text{st}} \text{ time we form a type 3 bubble of Euclidean diameter at least 1}$$

$$\tau_1 := 1^{\text{st}} \text{ time after } \tau_0 \text{ we form a type 3 bubble}$$

$$\vdots$$

$$\tau_m := 1^{\text{st}} \text{ time after } \tau_{m-1} \text{ we form a type 3 bubble.}$$

Note that $E_{k_{\tau_i}}$ is measurable with respect to η^L and $\eta^R|_{[0,\tau_{i+1}]}$, and for each $i \in \{1, 2, \ldots, m\}$, we have that by Lemma 4.3.4,

$$0 < \mathbb{P}\left[E_{k_{\tau_i}} \,\middle|\, \eta^L, \eta^R_{|_{[0,\tau_i]}}\right] < 1.$$

Thus, it follows that

$$\mathbb{P}[X_i = \mathcal{Z}_i | X_1 = \mathcal{Z}_1, ..., X_{i-1} = \mathcal{Z}_{i-1}] > 0.$$

To finish the proof, we note that since $\{X_j = Z_j\}$ is determined by η^L and $\eta^R|_{[0,\tau_i]}$ for $i < j$, so

$$\mathbb{P}[X_1 = \mathcal{Z}_1, ..., X_i = \mathcal{Z}_i] = \mathbb{E}\left[\mathbb{P}[X_i = \mathcal{Z}_i \,|\, \eta^L, \eta^R|_{[0,\tau_i]}]\mathbb{1}_{X_1 = \mathcal{Z}_1, \ldots, X_{i-1} = \mathcal{Z}_{i-1}}\right].$$

The probability within the expectation on the right hand side is always positive, and so inducting on i (and setting $i = m$ as a final step) yields the desired result. $\square$

Proof of Lemma 4.3.2. By Proposition 4.3.3, we see that $\{X_k\}$ can contain any finite sequence of 0's and 1's not contained in y, implying that $\{X_k\}$ cannot be deterministic modulo index shift. This is a contradiction. Thus, $\mathbb{P}[A_{K_1^{(n)}}] = 0$ for every n.

Thus, by scale invariance we see that $\mathbb{P}[A_{K_r^{(n)}}] = 0$ for every r and n. Note that every k is equal to $K_r^{(n)}$ for some rational r and some n. Indeed, every k^{th} bubble has some positive diameter, and there are at most finitely many bubbles before it of larger diameter. Thus, we can set n to be the number of bubbles before the k^{th} bubble with diameter exceeding that of the k^{th} bubble, and simply let r be any rational number slightly smaller than this diameter. From this, it follows that

$$\mathbb{P}[\exists\, k \text{ such that } A_k \text{ occurs }] \le \mathbb{P}\left[\bigcup_{n\in\mathbb{N}} \bigcup_{r\in\mathbb{Q}_{>0}} A^{(n)}_{K_r}\right] = 0.$$

In particular, we have that $\mathbb{P}[A_0] = 0$.

$\square$

Proof of Theorem 4.1.1. Now let η^1 and η^2 be two independent SLE's. In order for $\eta^1 \cup \mathbb{R}$ and $\eta^2 \cup \mathbb{R}$ to be homeomorphic via a homeomorphism that takes $\mathbb{R}$ to $\mathbb{R}$, it must be the case that the corresponding bi-infinite sequences X^1 and X^2 differ by at most an index shift. Indeed, any homeomorphism has to preserve the bi-infinite sequence of connected components lying between the left and right boundaries of the curve, as well as the types of these components. Thus, by the above argument, the probability that X^1 is equal to any of the countably many possible index shifted versions of X^2 is zero. Hence the probability that $\eta^1 \cup \mathbb{R}$ and $\eta^2 \cup \mathbb{R}$ are homeomorphic, via a homeomorphism that takes $\mathbb{R}$ to $\mathbb{R}$, is 0. $\square$

4.4 Proof of Theorem 4.1.2

Here, we require a more subtle argument that relies on a less obvious observable. In this section, we fix $\kappa \ge 8$. Let η be an instance of SLE_κ in $\overline{\mathbb{H}}$. We are interested in the successive crossing times (about the origin) of the curve η, i.e., the times at which η hits the real line again, just after having hit it on the opposite side of the origin. Consider one such crossing time, i.e., a single left right crossing about the origin. The SLE goes back and forth between the left and right boundaries of this crossing at some times, and the set of times when it does so has to be a discrete set since the SLE is continuous. As pictured below in Fig. 4.3, these left and right crossings (within the curve) define a sequence of marked points $\{X_k\}$ along the boundary, which accumulate only at the tip of the curve. Via the corresponding Loewner map g_t^η, we may conformally map this configuration as shown in Fig. 4.3, so that the tip goes to 0, and we obtain a sequence of marked points along the left boundary. Notice these marked points are determined by the past, so we can condition on (all of) their locations, and the future will still be an SLE by the Markov property.

A bit more care is needed in defining these quantities. Let $\tau(t)$ be the last time before t such that $\eta(t) \in \mathbb{R}$. Define the sets

$$T_- := \{t : \eta(\tau(t)) < 0\} \qquad T_+ := \{t : \eta(\tau(t)) > 0\}$$

and set $\mathcal{S} = \bar{T}_- \cap \bar{T}_+$. Notice that $\mathcal{S}$ is a discrete set since η is continuous, and so it cannot cross back and forth between $(-\infty, 0)$ and $(0, \infty)$ infinitely many times during any compact time interval contained in $(0, \infty)$. Thus, we may index the elements of $\mathcal{S}$ as a countable sequence of well defined crossing times $\{\tau_j\}$.

Notice that these are not stopping times (which poses a problem in applying the strong Markov property), but this can be addressed by adopting some notation from the previous section as follows. Let $\eta_j := \eta|_{[\tau_{j-1}, \tau_j]}$, which is the j^{th} left-right crossing around 0 that we observe, i.e., the crossing of index $j \in \mathbb{Z}$. For $r > 0$, let $J_r^{(n)}$ be the n^{th} smallest j for which the Euclidean diameter of η_j is at least r. It is not difficult to see that the set of times $\{\tau_{J_r^{(n)}}\}$ is indeed a set of stopping times. To see this, let $t > 0$. If one sees $\eta|_{[0,t]}$, then one can determine the set $\{\tau_j : \tau_j \le t\}$. This follows from the definition of the times $\{\tau_j\}$ as the intersection points of T_- and T_+, as shown previously. Hence $\eta|_{[0,t]}$ determines the set of excursions $\{\eta_j : \tau_j \leq t\}$. We have $\tau_{J_r^{(n)}} \leq t$ if and only if this set of excursions includes at least n elements which have Euclidean diameter at least r. Hence $\{\tau_{J_r^{(n)}} \leq t\}$ is determined by $\eta|_{[0,t]}$, which holds for any choice of t.

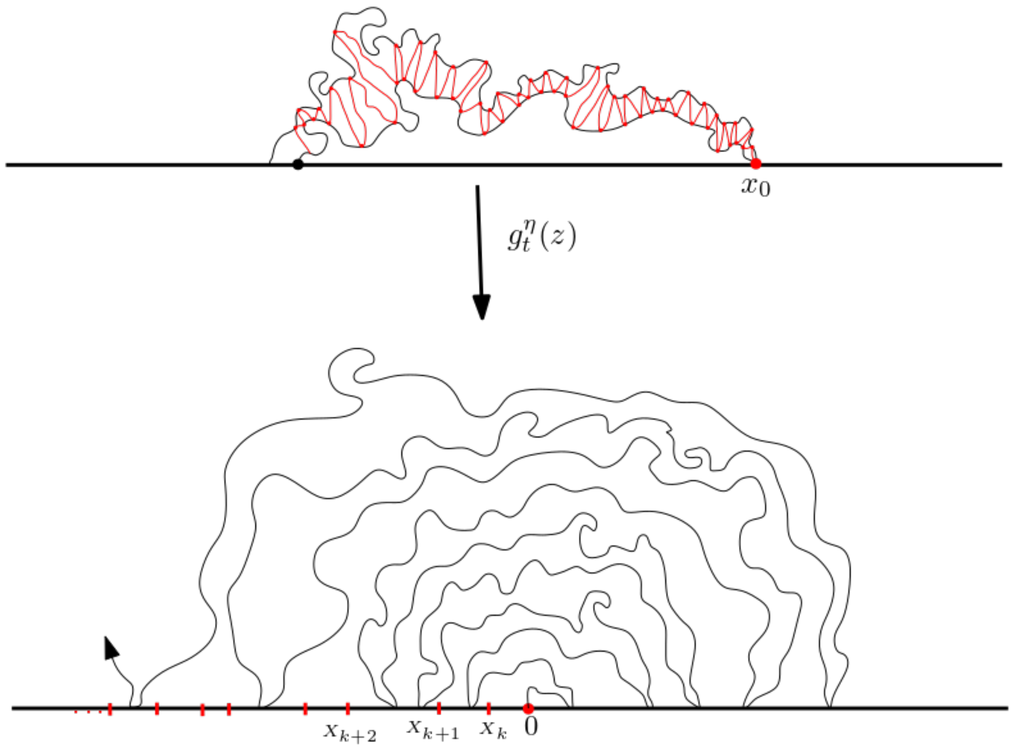

Figure 4.3: The top picture illustrates a single left-right crossing around 0, with $x_0 = \eta(\tau_J)$ and the corresponding triangulation in red, determined by the (past) piece of the curve making boundary crossings. The marked points X_k define the locations of the tips of the triangles in the triangulation, after conformally mapping to the real line via g_t^η. We thus consider intervals $[X_{k+1}, X_k]$ in which the tips of future triangles, obtained by left right crossings about 0, may lie. Some intervals may have multiple, while some may have none.

We fix some r and some n, and set $J := J_r^{(n)}$. Between the outer boundaries of the crossing η_J, we can keep track of the times at which $\{\eta_t : t < \tau_J\}$ sequentially hits these boundaries. More precisely, we let L_J be the outer boundary of $\eta[0, \tau_J]$. We define our sequence of crossing times inductively as follows:

$$\sigma_{J,1} := \min\{t > \tau_{J-1} : \eta_t \cap L_J \neq \emptyset\}$$

$$\tilde{\sigma}_{J,1} := \min\{t > \sigma_{J,1} : \eta_t \cap L_{J-1} \neq \emptyset\}$$

$$\vdots$$

$$\tilde{\sigma}_{J,k} := \min\{t > \sigma_{J,k} : \eta_t \cap L_{J-1} \neq \emptyset\}$$

$$\sigma_{J,k+1} := \min\{t > \tilde{\sigma}_{J,k} : \eta_t \cap L_J \neq \emptyset\}$$

and so on. The sequences $\{\sigma_{J,k}\}_{k\geq 1}$ and $\{\tilde{\sigma}_{J,k}\}_{k\geq 1}$ define two discrete sets of times that our curve successively hits the outer boundaries L_J and L_{J-1} respectively. We assume without loss of generality that the J^{th} excursion goes from left to right. By considering only the outer boundary L_J (as a priori τ_J is a well-defined stopping time), we can construct a sequence of marked points $\{X_{J,k}\}_{k\geq 1}$ along the negative real axis, via the (shifted) Loewner map which sends $\eta(\tau_J)$ to 0. That is to say, $X_{J,k} := g_{\tau_J}(\eta(\sigma_{J,k})) - U_{\tau_J}$. As we are considering a fixed J, we may write $X_{J,k} := X_k$ for ease.

Consider the points where the future of the SLE process, $\eta|_{[\tau_J,\infty)}$, hits the negative real axis after having hit the real line to the right of 0, which we call crossing endpoints. More precisely, we define these crossing endpoint times as follows:

$$\sigma_1^* := \min\{t > \tau_J : \eta_t \cap \mathbb{R}_{>0} \neq \emptyset\}$$

$$\tilde{\sigma}_1^* := \min\{t > \sigma_1^* : \eta_t \cap \mathbb{R}_{<0} \neq \emptyset\}$$

$$\vdots$$

$$\tilde{\sigma}_i^* := \min\{t > \sigma_i^* : \eta_t \cap \mathbb{R}_{<0} \neq \emptyset\}$$

$$\sigma_{i+1}^* := \min\{t > \tilde{\sigma}_i^* : \eta_t \cap \mathbb{R}_{>0} \neq \emptyset\}$$

and so on. We let

$$N_k = \#\left\{i : \tilde{\sigma}_i^* \in [X_{k+1}, X_k]\right\}.$$

In other words, we are looking at $\eta' := g_{\tau_J}(\eta|_{[\tau_J,\infty)}) - U_{\tau_J}$ as it successively makes left-right crossings about 0, conditioned on the past, and for each interval we are keeping track of how many crossing endpoints it contains. We wish to show that for every deterministic sequence of integers $\{n_k\}_{k\in\mathbb{N}}$, we have that

$$\mathbb{P}[N_k = n_k;\, \forall\, k] = 0. \tag{4.4}$$

It suffices to show that there are arbitrarily large k such that $\mathbb{P}[N_k = n_k]$ is bounded away from 1. Indeed, the event $\{N_k = n_k$ for all sufficiently large $k\}$ is a tail event for the Brownian motion driving the SLE, and the Blumenthal $0-1$ law implies that this has probability 0 or 1. Thus, being bounded away from 1 guarantees that we have (5.2.1). We do this in cases as follows:

Case 1: Assume there exist arbitrarily large k such that $n_k \neq 0$. We claim that there exists $q > 0$ such that

$$\mathbb{P}[N_k = n] \leq 1 - q \quad \forall\, n \geq 1.$$

To see this, we consider the segment of the curve η' , just after the $(n-1)^{th}$ crossing about 0 is completed. Let $\mathcal{T}_n$ denote the n^{th} time we have a crossing in the interval $[X_{k+1}, X_k]$. Thus $\mathcal{T}_n$ is a stopping time, and conditioned on what we have seen up until this time, the future of the curve is still SLE. The goal is to have an upper bound on the probability that there are exactly n crossings, and we do so by comparing the harmonic measure (from ∞)

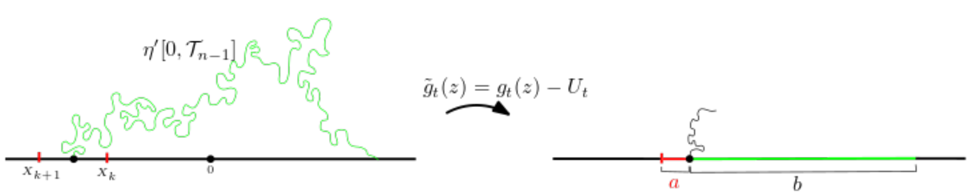

Figure 4.4: We stop the SLE after it has made its $(n-1)^{\text{th}}$ crossing in the interval shown. Under the map $\tilde{g}$, we send the tip of the curve to the origin, and analyze the likelihood of either observing two more crossings in the red interval of length a, or no more crossings, in which case the interval is swallowed.

of the interval $[X_{k+1}, \eta'(\mathcal{T}_{n-1})]$, to that of the outer boundary of the curve $\eta'[0, \mathcal{T}_{n-1}]$(and more precisely, this is the harmonic measure from ∞ in $\mathbb{H} \setminus \eta'[0, \mathcal{T}_{n-1}])$. These quantities are denoted a and b respectively, as shown in Fig. 4.4.

The proof relies on the following intuitive argument which we formalize later: If a is larger than b, then with positive probability we observe 2 further crossings, hence $n+1$ total crossings. If a is smaller than b, then, with positive probability, we expect the interval $[X_{k+1}, \eta'(\mathcal{T}_{n-1})]$ to be covered before we observe the next crossing. In other words, there is always a positive chance that we observe either $n-1$ crossings or $n+1$ crossings, and so

$$\mathbb{P}[N_k \neq n] > 0.$$

Proposition 4.4.1. *Let η be an SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$ with $\kappa > 4$. For marked points $a < 0 < c$ along the real line, let $E_{a,c}$ be the event that the chordal SLE$_\kappa$ trace visits $[c, \infty)$ before $(-\infty, a]$. Then*

$$\mathbb{P}\left[E_{a,c}\right] = F\left(\frac{-a}{c-a}\right) \qquad \text{where } F(x) = \frac{1}{Z_\kappa}\int_0^x \frac{du}{u^{\frac{4}{\kappa}}(1-u)^{\frac{4}{\kappa}}}$$

and Z_κ is chosen so that $F(1) = 1$.

Proof. This is Theorem 10 in [2], which is a generalized restatement of Theorem 3.2 in [29]. □

Remark 4.4.2. It is possible to get an estimate which is weaker than Theorem 3 above, but which is still sufficient for our purposes, via the following elementary argument. For $x \in \mathbb{R}$, let $t_x := \inf[t \geq 0 : \eta(t) = x]$. If we let $P(n) = \mathbb{P}[t_n < t_{-1}]$, a bit of thought shows that

$$P(n) \geq P(n-1)[1 - P(n)]$$

which thus implies that

$$P(n) \geq \frac{P(n-1)}{1+P(n-1)}.$$

The equality case can be realized as $P(n) = \dfrac{1}{n+1}$, the details of which we omit. By considering $f(x) = \dfrac{x}{x+1}$, which is increasing on $\mathbb{R}_{\geq 0}$, we find that

$$P(n) \geq f(P(n-1)) \geq f^{(2)}(P(n-2) \cdots \geq f^{(n)}\left(\frac{1}{2}\right) = \frac{1}{n+1}$$

which gives a rough (yet easy to compute) estimate. Note, for our purposes, we only require a positive probability.

We return to the notation introduced in Fig. 4.4, and we consider the the behavior of the SLE curve given the relative quantities a and b. In particular, we require the following two key lemmas to prove the original claim:

Lemma 4.4.3. *If $a \leq b$, it holds with conditional probability at least $\frac{1}{2}$, given $\eta'|_{[0,\mathcal{T}_{n-1}]}$, that $\eta'|_{[\mathcal{T}_{n-1},\infty)}$ hits X_{k+1} before $[b,\infty)$.*

Proof. Notice that by symmetry, there is a positive chance that we disconnect $[X_{k+1}, \eta'(\mathcal{T}_{n-1})]$ before hitting b. Indeed, this follows from the fact that $\mathbb{P}[t_{-1} < t_1] = \frac{1}{2}$. □

Lemma 4.4.4. *There exists a deterministic κ-dependent constant $c > 0$ such that if $a > b$, it holds with conditional probability at least c given $\eta'|_{[0,\mathcal{T}_{n-1}]}$ that $\eta'|_{[\mathcal{T}_{n-1},\infty)}$ crosses between $(-\infty, 0)$ and $(0,\infty)$ at least twice before hitting X_{k+1}.*

Proof. If $a > b$, then we can apply the estimate given in Proposition 4.4.1 via a two step process. We retain the notation from Remark 4.4.2, and define t_{-a} and t_b as discussed, after having mapped $\eta'([0,\mathcal{T}_{n-1}])$ to the real line via the map $\tilde{g}$. Note that Proposition 4.4.1 implies that $\exists\, p > 0$ such that $\mathbb{P}[t_b < t_{-a/2}] \geq p$. In fact, we have assumed $a > b$, so p in this instance can be thought of as a universal bound. We condition on this event occurring, and we look at the harmonic measure of the outer boundary curve $\tilde{\eta}$ of this most recent crossing. Note that $\mathrm{hm}_{\mathbb{H}\setminus\tilde{\eta}}(\infty, \tilde{\eta})$ is bounded above by the harmonic measure of the outer boundary at the time we hit $-\frac{a}{2}$. This follows from the fact that the harmonic measure can only increase, as we observe more of the curve. Moreover, the law of this harmonic measure, divided by a, is independent of a by scale invariance, and is almost surely finite. This implies that $\exists\, C = C(p) > 0$ such that

$$\mathbb{P}\left[\mathrm{hm}_{\mathbb{H}\setminus\tilde{\eta}}(\infty, \tilde{\eta}) \leq Ca\right] \geq 1 - \frac{p}{2}$$

from which it follows that

$$\mathbb{P}\left[\mathrm{hm}_{\mathbb{H}\setminus\tilde{\eta}}(\infty, \tilde{\eta}) \leq Ca,\, t_b < t_{-a/2}\right] \geq \frac{p}{2}.$$

This bound guarantees a positive probability that, after we have observed the first crossing, the harmonic measure of the outer boundary is not too large. Now we condition on this event, and we apply Proposition 4.1 to the quantities Ca and $\frac{a}{2}$. In particular, This yields a positive κ-dependent constant lower bound for the probability that $\eta'|_{[\mathcal{T}_{n-1},\infty)}$ has at least two crossings before hitting X_{k+1}. $\square$

Case 2: $n_k = 0$ for all but finitely many k.

This condition implies that the SLE travels a positive distance of time without any left-right crossings, which happens with probability 0. This shows that for any fixed deterministic sequence $\{n_k\}_{k\in\mathbb{N}}$ with only finitely many non-zero elements, we have that

$$\mathbb{P}[\{N_k\} = \{n_k\}] = 0.$$

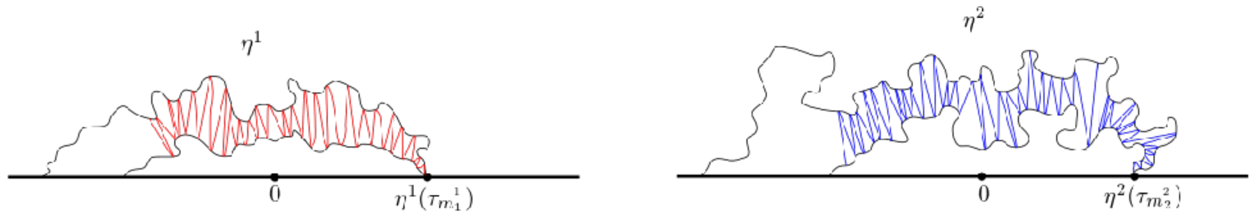

Figure 4.5: We observe two instances of SLE, η^1 and η^2, stopped after the $m_1{}^{\text{th}}$ and $m_2{}^{\text{th}}$ crossings respectively. Any homeomorphism between the two should send one tip to the other, and retain the structure of the future crossings (i.e., preserve the corresponding sequences $\{N_k^j\}$).

21‘

Proof of Theorem 4.1.2. Consider two instances of SLE_κ in $\mathbb{H}$, η^1 and η^2, with corresponding sequences of points $\{X^1_{m_1,k}\}_{k\in\mathbb{N}}$ and $\{X^2_{m_2,k}\}_{k\in\mathbb{N}}$ respectively, for fixed indicies $m_1, m_2 \in \mathcal{S}$, corresponding to the $m_1{}^{\text{th}}$ crossing of η^1 and $m_2{}^{\text{th}}$ crossing of η^2 respectively. Here, we indicate objects associated with η^j for $j \in \{1, 2\}$ by a superscript j. Note that by construction, $m_1 = J^{(n_1),1}_{r_1}$ and $m_2 = J^{(n_2),2}_{r_2}$ for some n_1, n_2 and (rational) r_1, r_2. Each sequence of points $\{X^j_k\}_{k\in\mathbb{N}}$ generates a sequence $\{N^j_k\}_{k\in\mathbb{N}}$ for $j \in \{1, 2\}$ and so by the independence of η^1 and η^2, as well as (5.2.1), we have that for any choice of m_1, m_2 and number l

$$\mathbb{P}\left[N^1_k = N^2_{k+l};\ \forall\, k\right] = \mathbb{P}\left[N^1_k = N^2_{k+l};\ \forall\, k \,|\, \eta^2\right] = 0.$$

This implies that

$$\mathbb{P}\left[\exists\, l \text{ s.t } N^1_k = N^2_{k+l};\, \forall\, k\right] = 0 \tag{4.5}$$

as there are countably many possible choices of l, meaning we can apply this very argument for each fixed choice of l, and apply the union bound.

Observe that a homeomorphism from $\overline{\mathbb{H}}$ to itself taking η^1 to η^2, modulo time parametrization, must preserve the number of left right crossings of the 'future' curves, which correspond to the sequences $\{N^j_k\}$, and it must take $\eta^1(\tau^1_{m_1})$ to $\eta^2(\tau^2_{m_2})$ for some m_2. In particular, as in the setting of Figure 4, for any fixed m_1 and m_2 there is no homeomorphism which takes

η^1 to η^2 and $\eta^1(\tau^1_{m_1})$ to $\eta^2(\tau^2_{m_2})$ by (4.5). As the set $\mathcal{S}$ of crossing times is discrete, this holds for any choice of indices $m_1$and m_2 , where there are only countably many choices. Thus, it must hold that,

$$\mathbb{P}\left[\exists \text{ a homeomorphism } \Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}} \text{ taking } \eta^1 \text{ to } \eta^2\right] = 0.$$

$\square$

CHAPTER 5

STRICT MONOTONICITY OF THE SUPERCRITICAL LIOUVILLE QUANTUM GRAVITY METRIC

Recent works have shown that for each $\xi > 0$, one can define a metric (distance function) on the plane obtained by weighting lengths of paths by $e^{\xi h}$, where h is the planar Gaussian free field. This metric is related to Liouville quantum gravity with matter central charge $c_{\mathrm{M}} = 25 - 6Q(\xi)^2 < 25$, where $Q = Q(\xi)$ is a non-explicit function of ξ. We show that $Q(\xi)$ is a strictly decreasing function of ξ, and hence that there is a one-to-one correspondence between ξ and c_{M} for the whole parameter range (previously, this was known only for $\xi < 0.07$). To do this, we relate ξ to the dyadic subdivision model for supercritical LQG introduced by Gwynne, Holden, Pfeffer, and Remy (2018).

5.1 Introduction

Liouville quantum gravity (LQG) is a family of random surfaces that realize a coupling of two-dimensional quantum gravity with conformal matter fields, depending on a single parameter. It was first introduced in physics by Polyakov [38] to define a "sum over Riemannian metrics" in two dimensions within the context of bosonic string theory. To define LQG, we select a *central charge* $c_{\mathrm{M}} \in (-\infty, 25)$ which comes from the conformal field theory (CFT) associated with the matter fields. For a compact surface D with Riemannian metric g, and the corresponding Laplace-Beltrami operator Δ_g, we may define (at least informally) an LQG surface with central charge c_{M} as a random surface that is sampled from the measure on Riemannian metric tensors g on D, whose probability density with respect to the "Lebesgue measure on the space of metrics g on D" is proportional to $(\det \Delta_g)^{-c_{\mathrm{M}}/2}$. Of course, this is far from rigorous as this space of metric tensors is infinite-dimensional, and so any notion of uniform measure on this space is far from obvious. The determinant $(\det \Delta_g)^{-c_{\mathrm{M}}/2}$ can be thought of as the partition function of a statistical mechanics model, which is described by a CFT with central charge c_{M} in the scaling limit.

The notions of area and distance are natural quantities that come into question when considering these 'random Riemannian surfaces'. We restrict to the case $c_{\mathrm{M}} \leq 1$, and let h be an instance of the Gaussian Free Field (GFF) on D. Let $\gamma \in (0, 2]$ be the unique solution of the equation

$$c_{\mathrm{M}} = 25 - 6Q^2 \qquad \text{where } Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{5.1}$$

We may think of the LQG surface associated with (D, h), with parameter γ, as the random Riemannian manifold parametrized by D, endowed with the 'random metric' $e^{\gamma h}(dx^2 + dy^2)$, where $dx^2 + dy^2$ is the usual Euclidean metric tensor. Of course, this definition is also far from rigorous as h itself is a distribution rather than a function, and so formalizing the notion of the exponential of h is highly nontrivial. Note that (5.1) gives us a bijection between $c_{\mathrm{M}} \in (-\infty, 1]$ and $\gamma \in (0, 2]$, and so we may parametrize LQG surfaces by γ, yielding the more canonical characterization γ-LQG surfaces. We describe the *phases* of LQG surfaces, however, via their central charges as follows.

Definition 5.1.1. LQG with $c_M \in (-\infty, 1)$, $c_M = 1$, and $c_M \in (1, 25)$, is referred to as *subcritical, critical* and *supercritical* respectively.

One major motivation for studying LQG surfaces is that they arise, in many cases in conjecture, as the scaling limits of various planar map models. Some of these convergence results have been shown in the subcritical and supercritical cases, and can be better understood through expository works done in [4, 21, 19]. The supercritical case has proven to be a bit more mysterious, at least in the more geometric sense. Nevertheless, it is still expected that supercritical LQG, in some sense, corresponds to some random geometry related to the GFF. See [20] for an overview of various motivations and conjectures coming from the physics literature of supercritical LQG.

	c_M	Q	γ
c_M		$25-6Q^2$	$25-6\left(\frac{2}{\gamma}-\frac{\gamma}{2}\right)^2$
Q	$\sqrt{\frac{25-c_M}{6}}$		$\frac{2}{\gamma}+\frac{\gamma}{2}$
γ	$\sqrt{\frac{13-c_M-\sqrt{(25-c_M)(1-c_M)}}{3}}$	$Q-\sqrt{Q^2-4}$	

Figure 5.1: Table of relationships between the values of the matter central charge c_{M}, the background charge Q, and the constant γ. When $c_{\mathrm{M}} \leq 1$, all three parameters are real. When $c_{\mathrm{M}} \in (1, 25)$, γ is complex, but Q is real and nonzero.

It was shown in a series of papers [6, 24, 22, 25, 13, 14] that subcritical LQG surfaces admit a canonical distance function, i.e., there is a (unique) metric associated to these surfaces. This was later proven in the critical and supercritical cases in [11, 9]. One of the main ideas presented in these constructions is the *Liouville first passage percolation* (LFPP) with parameter $\xi > 0$, which describes a one-parameter family of random metrics. Indeed, the DDK ansatz suggests constructing a random metric associated to LQG as a limit of regularized versions of the heuristic metric

$$(z, w) \to \inf_{P: z \to w} \int_0^1 e^{\xi h(P(t))} |P'(t)| \, dt,$$

for a constant $\xi := \xi(c_{\mathrm{M}})$ which depends on the central charge. More precisely, for $s > 0$ and $z \in \mathbb{C}$, let $p_s(z) := \dfrac{1}{2\pi s} \exp\left\{\dfrac{-|z|^2}{2s}\right\}$ be the usual heat kernel. For $\epsilon > 0$, we consider the modified version of the GFF

$$h_\epsilon(z) := (h * p_{\epsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\epsilon^2/2}(z - w) \, dw,$$

as realized as a mollification of the field with the heat kernel. We consider a parameter ξ which will be later chosen to depend on the central charge.

Definition 5.1.2. Let $\xi > 0$. Liouville first passage percolation with parameter ξ is the family of random metrics $\{d_h^\epsilon\}_{\epsilon>0}$ given by

$$d_h^\epsilon(z,w) := \inf_{P:z\to w} \int_0^1 e^{\xi h_\epsilon(P(t))}|P'(t)|\,dt,$$

where the infimum is over all piecewise continuously differentiable paths from z to w.

We define the re-scaled LFPP as the family of random metrics $\{\mathfrak{a}_\epsilon^{-1} d_h^\epsilon\}_{\epsilon>0}$. The normalizing factor is given by

$$\mathfrak{a}_\epsilon := \text{median of} \inf \left\{ \int_0^1 e^{\xi h_\epsilon(P(t))}|P'(t)|\,dt : P(t) \text{ is a left-right crossing of } [0,1]^2 \right\},$$

where left-right crossings refer to (differentiable) paths connecting the left and right boundaries of the unit square. This choice is somewhat arbitrary, but it is through this rescaling that one extracts a meaningful limit. The value of $\mathfrak{a}_\epsilon$ is not known explicitly, but one can see [10, 1, 26, 5] for estimates of this quantity. It is shown that these re-scaled metrics converge in probability to a random metric D_h on $\mathbb{C}$, with respect to the topology of lower semicontinuous functions on $\mathbb{C}\times\mathbb{C}$. This limiting candidate D_h is thus shown to satisfy a collection of axioms that characterize the LQG metric.

It was shown in [5] and [9] that, for every $\xi > 0$, there exists a $Q = Q(\xi) > 0$ with

$$\mathfrak{a}_\epsilon = \epsilon^{1-\xi Q - o_\epsilon(1)} \qquad \text{as } \epsilon \to 0.$$

The relationship between the LFPP parameter ξ and the LQG central charge c_{M} is discussed in [8, 11]. It is known that the function $\xi \to Q(\xi)$ is continuous, with $Q(\xi) \to \infty$ as $\xi \to 0$ and $Q(\xi) \to 0$ as $\xi \to \infty$, strictly decreasing on $(0, 0.7)$, and thus injective on this interval

[26]. Furthermore, the quantity Q appears in the metric analog of the LQG coordinate change formula from [18], but restricted to translation and scaling [11]. More precisely, Let $U, \tilde{U} \subset \mathbb{C}$ be open and let $\phi \colon U \to \tilde{U}$ be a complex affine map. Then, a.s.,

$$D_h(\phi(z), \phi(w); \tilde{U}) = D_{h\circ\phi + Q\log|\phi'|}(z, w; U), \qquad \forall\, z, w \in \mathbb{C}.$$

We note that with ξ_{crit} chosen so that $Q(\xi_{\text{crit}}) = 2$, we call LFPP for $\xi \in (0, \xi_{\text{crit}})$ the subcritical phase and LFPP for $\xi \in (\xi_{\text{crit}}, \infty)$ the supercritical phase. Equivalently, if we associate LFPP with parameter ξ to the value of matter central charge associated with background charge $Q(\xi)$, then the subcritical phase of LFPP corresponds to $c_{\text{M}} < 1$, and the supercritical phase of LFPP corresponds to $c_{\text{M}} \in (1, 25)$, where c_{M} satisfies

$$c_{\text{M}} := c_{\text{M}}(\xi) = 25 - Q(\xi)^2. \tag{5.2}$$

In the critical case $\xi = \xi_{\text{crit}}$, the limiting metric D_h for the re-scaled LFPP induces the same topology as the Euclidean metric [7], and can be thought of as the Riemannian distance function associated with critical ($\gamma = 2$) LQG. In the supercritical case $\xi > \xi_{\text{crit}}$, the limiting metric does not induce the Euclidean topology on $\mathbb{C}$. Indeed, a.s. there exists an uncountable, Euclidean-dense set of *singular points* $z \in \mathbb{C}$ such that

$$D_h(z, w) = \infty, \forall w \in \mathbb{C} \setminus \{z\}.$$

It is known, however, that for each fixed $z \in \mathbb{C}$, a.s. z is not a singular point, so the set of singular points has zero Lebesgue measure. Moreover, any two non-singular points lie at finite D_h-distance from each other [9]. One can think of singular points as infinite "spikes" which D_h paths must avoid.

The main idea of this chapter is to prove the injectivity of the map $\xi \to Q(\xi)$ on $(0, \infty)$, through analysis of a (modified) discrete model of LQG in the supercritical phase. Note that injectivity implies that there is a unique value of ξ (hence a unique metric) for each value of c_{M}. The model, which was introduced in [20], takes the form of a one-parameter family of random planar maps, indexed by $c_{\mathrm{M}} \in (-\infty, 25)$, which are defined as the adjacency graphs of a family of dyadic tilings of the plane constructed from the Gaussian free field. It is expected that, in this supercritical regime, the tiling should converge in some sense to supercritical continuum LQG model. In particular, the graph distance on the adjacency graph of squares should converge to the LQG metric.

5.2 Preliminaries

In this section, we introduce the basic notation used throughout the chapter, as well as a few basic definitions. In particular, we provide an axiomatic characterization of weak LQG metrics.

5.2.1 *Basic notation and definitions*

If $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to (0,\infty)$, we say that $f(\epsilon) = O_\epsilon(g(\epsilon)))$ (resp. $f(\epsilon) = o_\epsilon(g(\epsilon))))$ as $\epsilon \to 0$ if $f(\epsilon)/g(\epsilon)$ remains bounded (resp. tends to zero) as $\epsilon \to 0$. We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity.

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the open Euclidean ball of radius r centered at z. More generally, for $X \subseteq \mathbb{C}$, we write $B_r(X) = \bigcup_{z \in X} B_r(z)$. Topological concepts such as "open", "closed", "boundary", etc., are always defined with respect to the Euclidean topology unless otherwise stated. For $X \subseteq \mathbb{C}$, we write $\overline{X}$ for its Euclidean closure, $\mathring{X}$ for its interior, and ∂X for its Euclidean boundary.

An annular region is a bounded open set $\mathbb{A} \subset \mathbb{C}$ such that $\mathbb{A}$ is homeomorphic to an open, closed, or half-open Euclidean annulus. If $\mathbb{A}$ is an annular region, then $\partial\mathbb{A}$ has two connected components, one of which disconnects the other from ∞. We call these components the outer and inner boundaries of $\mathbb{A}$, respectively.

Definition 5.2.1. Let (X, d) be a metric space, with d allowed to take on infinite values.

1. A curve in (X, d) is a continuous function $P : [a, b] \to X$ for some interval $[a, b]$.

2. For a curve $P : [a, b] \to X$, the d-length of P is defined by

$$\operatorname{len}(P; d) := \sup_T \sum_{i=1}^{\#T} d(P(t_i), P(t_{i-1}))$$

   where the supremum is over all partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a, b]$ which, by convention, may be infinite. In particular, the d-length of P is infinite if there are times $s, t \in [a, b]$ such that $d(P(s), P(t)) = \infty$.

3. We say that (X, d) is a *length space* if for each $x, y \in X$ and each $\varepsilon > 0$, there exists a curve of d-length at most $d(x, y) + \varepsilon$ from x to y. If $d(x, y) < \infty$, a curve from x to y of d-length exactly $d(x, y)$ is called a *geodesic*.

4. For $Y \subseteq X$, the *internal* metric of d on Y is defined by

$$d(x, y; Y) := \inf_{P \subseteq Y} \operatorname{len}(P; d), \quad \forall x, y \in Y$$

   where the infimum is over all curves P in Y from x to y. Note that $d(\cdot, \cdot; Y)$ is a metric on Y, except that it is allowed to take infinite values.

5. If $X \subseteq \mathbb{C}$, we say that d is a lower semicontinuous metric if the function $(x, y) \mapsto d(x, y)$ is lower semicontinuous with respect to the Euclidean topology. We equip the set of lower semicontinuous metrics on X with the topology on lower semicontinuous functions on $X \times X$, and the associated Borel σ-algebra.

6. If $\mathbb{A}$ is an annular region, we define the d-distance *across* $\mathbb{A}$ as the distance between

the inner and outer boundaries of $\mathbb{A}$, and the d-distance *around* $\mathbb{A}$ as the infimum of the d-distances of closed paths that separate the inner and outer boundaries of $\mathbb{A}$.

Definition 5.2.2 (Weak LQG metric)**.** Let $\mathcal{M}'(\mathbb{C})$ be the space of distributions (generalized functions) on $\mathbb{C}$, equipped with the usual weak topology. For each $\xi > 0$, we define a weak LQG metric with parameter ξ as a measurable function $h \mapsto D_h$ from $\mathcal{M}'(\mathbb{C})$ to the space of lower semicontinuous metrics on $\mathbb{C}$ such that the following is true whenever h is a whole-plane GFF plus a continuous function.

i. **Length space**. Almost surely, $(\mathbb{C}, D_h)$ is a length space.

ii. **Locality**. For each deterministic open set $U \subset \mathbb{C}$, the D_h-internal metric $D_h(\cdot, \cdot; U)$ is determined almost surely by $h|_U$.

iii. **Weyl scaling**. If $f : \mathbb{C} \to \mathbb{R}$ is a continuous function, then almost surely $D_{h+f} = e^{\xi f} D_h$, where

$$(e^{\xi f} \cdot D_h)(z, w) = \inf_{P: z \to w} \int_0^{\mathrm{len}(P; D_h)} e^{\xi f(P(t))} dt, \quad \forall z, w \in \mathbb{C}, \tag{5.3}$$

where the infimum is taken over all paths from z to w parametrized by D_h-length.

iv. **Scale and translation covariance.** Let Q be the LQG background charge. For each fixed deterministic $r > 0$ and $z \in \mathbb{C}$, a.s.

$$D_h(ru + z, rv + z) = D_h(r \cdot u + z) + Q \log r \qquad (u, v) \in \mathbb{C}^2$$

v. **Finiteness.** Let $U \subset \mathbb{C}$ be a deterministic, open, connected set, and let $K_1, K_2 \subset U$ be disjoint, deterministic, compact, connected sets which are not singletons. Almost surely,

$$D_h(K_1, K_2; U) < \infty.$$

5.2.2 Description of the model

We refer to [41, 42, 43] for introductory details on the GFF. Fix $Q > 0$ and let $c_{\mathrm{M}} \in (-\infty, 25)$ be the corresponding matter central charge. Let h be a whole plane GFF. We will define a dyadic tiling associated with h as a realization of a graph approximation to an LQG surface with central charge c_{M}.

For a square $S \subset \mathbb{C}$, we write $|S|$ to be the side length of the square, and v_S to be its center. By convention, we say that a square S is *dyadic* if, for some $n \in \mathbb{N}$, $|S| = 2^{-n}$, and its corners lie on $2^{-n}\mathbb{Z}^2$. For a square $S \subset \mathbb{C}$, we define

$$M_h(S) := e^{h_{|S|/2}(v_S)}|S|^Q, \tag{5.4}$$

where $h_r(z)$ is the circle average of the GFF over $\partial B_r(z)$. Again, we omit details regarding circle averages, but the interested reader can see [17], for example. For $\epsilon > 0$, let

$$\begin{aligned} \mathcal{S}_h^\epsilon(\mathbb{C}) := \{&\text{Dyadic squares } S \subset \mathbb{C} \text{ with } M_h(S) \leq \epsilon \text{ and } M_h(S') > \epsilon \\ &\forall \text{ dyadic ancestors } \mathbb{C} \supset S' \supset S.\} \end{aligned} \tag{5.5}$$

We write $\mathcal{S}_h^\epsilon := \mathcal{S}_h^\epsilon(\mathbb{C})$. For $z, w \in \mathbb{C}$, we let $D_h^\epsilon(z, w)$ be the minimal $\mathcal{S}_h^\epsilon$ graph distance from a square containing z to a square containing w. By convention, we set this infimum equal to ∞ if either z or w is not contained in a square belonging to $\mathcal{S}_h^\epsilon$. We remark that $\mathcal{S}_h^\epsilon$ is locally finite in the regime $Q \geq 2$, but not in the $Q \in (0, 2)$ regime. For basic properties of this discrete LQG distance, see Section 1.2 of [20].

We say that $z \in \mathbb{C}$ is a *singularity* of $\mathcal{S}_h^\epsilon$ if z is not contained in any square of $\mathcal{S}_h^\epsilon$. We observe that if $z \in \mathbb{C}$ is fixed, then a.s. z is not a singularity of $\mathcal{S}_h^\epsilon$. Indeed, if h is a whole-plane GFF normalized so that $h_1(0) = 0$, then since each $h_{|S|/2}(v_S)$ is Gaussian with variance $\log(2/|S|) + O(1)$, the desired statement is easily seen from the Gaussian tail bound and a union bound over the dyadic squares (contained in some bounded domain) which contain

z. The corresponding statement for other variants of the GFF follows by local absolute continuity. From this, it is easily seen that a.s. every singularity is an accumulation point of arbitrarily small squares of $\mathcal{S}_h^\epsilon$. Given this, we will consider a modified version of this metric that establishes a minimal allowable side length for $\mathcal{S}_h^\epsilon$ squares in a given path. To that end, let

$$N^\epsilon := \left(\log \frac{1}{\epsilon}\right)^{7/5} \tag{5.6}$$

and define

$$\tilde{D}_h^\epsilon(z,w) := \inf_{P: z \to w} \#P \tag{5.7}$$

where the infimum is taken over all paths $P = (S_0, S_1, \ldots, S_{\#P})$ with $S_i \in \mathcal{S}_h^\epsilon$, $|S_i| \geq 2^{-N^\epsilon}$ for all i, S_i is adjacent to S_{i-1} for all i (i.e., they intersect along a non-trivial connected line segment), $z \in S_0$, and $w \in S_{\#P}$. The main focus of this chapter will be to establish upper and lower bounds for this modified metric, in terms of the limiting LQG metric D_h.

5.2.3 Statement of main result

Throughout this subsection, we assume that h is a whole-plane GFF normalized so that its circle average over $\partial\mathbb{D}$ is zero.

Theorem 5.2.1. *Suppose $\xi > 0$ is such that $Q(\xi) = Q$ is the parameter for the square tiling model, as defined previously. Fix a bounded open set $\mathcal{D} \subset \mathbb{C}$ with connected boundary and a compact, connected set K, with non-empty interior, contained in $\mathcal{D}$. Then, with probability tending to 1 as $\epsilon \to 0$,*

$$\tilde{D}_h^\epsilon(K, \partial\mathcal{D}) = \epsilon^{-\xi + o_\epsilon(1)}, \tag{5.8}$$

i.e., the distance exponent for the square tiling model is ξ. In particular, the function $\xi \to Q(\xi)$ is strictly decreasing on $(0,\infty)$.

We remark that the ϵ-square tiling model, a priori, depends on the parameter Q, while the equality seen in (5.8) recovers ξ as the distance exponent. This implies that we have constructed an inverse to the continuous mapping $\xi \to Q(\xi)$, from which strict monotonicity of this mapping follows. We prove the result in two main steps. The main idea is to give both an upper and a lower bound for the number of squares in the $\mathcal{S}_h^\epsilon$ tiling, by appealing to a few results already shown in the literature. The main result is similar to the relationship between LFPP and Liouville graph distance proven in [5], but applies in the supercritical case. Moreover, the relationship was conjectured in Section 2.3 of [20]. Indeed, see the paragraph in that section about LFPP.

5.2.4 Initial estimates

We begin by showing that the squares of $\mathcal{S}_h^\epsilon$ are not macroscopic. The following lemma asserts that the maximum size of the squares of $\mathcal{S}_h^\epsilon$ that intersect a fixed bounded set will converge to zero as $\epsilon \to 0$. We state this result more precisely, and we note that the proof follows from two key lemmas in [20].

Lemma 5.2.2. *Let U be a fixed bounded open set in $\mathbb{C}$, and let $Q(\xi) = Q$ be the parameter for the square tiling model. Then, for each $\zeta \in (0,1)$, it holds with probability 1 as $\epsilon \to 0$ that*

$$\max\{|S| : S \in \mathcal{S}_h^\epsilon \text{ and } S \cap U \neq \emptyset\} \leq \epsilon^{\frac{1}{2+Q}-\zeta}.$$

Proof. Lemmas 4.3 and 4.4 in [20] imply that, with probability tending to 1 as $\epsilon \to 0$,

$$\max\{|S| : S \in \mathcal{S}_h^\epsilon \text{ and } S \cap [0,1]^2\} \leq \epsilon^{\frac{1}{2+Q}-\zeta}$$

and so the desired result follows from the fact that U can be covered by finitely many (translated) copies of $[0,1]^2$. $\square$

Next, we require the following concentration bounds for LQG distances around and across annili. Recall the last item of Definition 5.2.1.

Lemma 5.2.3. *Let $\xi > 0$, let h be the whole-plane GFF, and let D_h be a weak LQG metric. Let $\mathbb{A}$ be a bounded subset of the plane with the topology of a Euclidean annulus, and with the property that both the inner and outer boundaries of $\mathbb{A}$ are non-singleton. There are constants $c_0, c_1 > 0$ depending on $\mathbb{A}$ such that the following is true. For each $r > 0$ and each $R > 3$,*

$$\mathbb{P}\left\{D_h(\text{across } r\mathbb{A}) < R^{-1} e^{\xi h_r(0)} r^{\xi Q}\right\} \leq c_0 e^{-c_1 (\log R)^2}, \tag{5.9}$$

and

$$\mathbb{P}\left\{D_h(\text{around } r\mathbb{A}) > R e^{\xi h_r(0)} r^{\xi Q}\right\} \leq c_0 e^{\frac{-c_1 (\log R)^2}{\log\log R}}. \tag{5.10}$$

Proof. This is an immediate consequence of Lemma 2.1 in [7]. □

5.3 Upper bound for D_h in terms of $\tilde{D}_h^\epsilon$

In this section, we fix a bounded open set $\mathcal{D} \subset \mathbb{C}$ with connected boundary and a compact, connected set K with non-empty interior, contained in $\mathcal{D}$.

Theorem 5.3.1. *Suppose $\xi = \xi(c_M)$ is such that $Q = Q(\xi)$ is the parameter for the square tiling model. Then, with probability tending to 1 as $\epsilon \to 0$,*

$$D_h(K, \partial\mathcal{D}) \leq \epsilon^{\xi + o_\epsilon(1)} \tilde{D}_h^\epsilon(K, \partial\mathcal{D}).$$

In particular, $\tilde{D}_h^\epsilon(K, \partial\mathcal{D}) \geq \epsilon^{-\xi + o_\epsilon(1)}$ with probability tending to 1 as $\epsilon \to 0$.

Given a $\tilde{D}_h^\epsilon$-geodesic in the $\mathcal{S}_h^\epsilon$ tiling, we may use (5.10) to construct a new path whose D_h length is upper bounded. We view the geodesic as a minimum length path of squares in the $\mathcal{S}_h^\epsilon$ tiling. For this construction, we require that ϵ be sufficiently small so that the paths around the squares have small enough Euclidean diameter, which forces them to intersect the compact sets.

Lemma 5.3.2. *Almost surely, every small enough dyadic square S that intersects $\mathcal{D}$ can be enclosed by a loop, which is contained in $B_{|S|}(S) \setminus S$, and whose D_h-length is at most $(M_h(S))^{\xi}\, e^{(\log \frac{1}{|S|})^{2/3}}$.*

Proof. Let $d = \operatorname{diam}(\mathcal{D})$. Fix $n \in \mathbb{N}$ and consider dyadic squares of side length 2^{-n}. Let $R_n = e^{n^{2/3}}$, so that $\sum_{n \geq 1} 2^{2n}\, e^{\frac{-(\log R_n)^2}{\log \log R_n}} < \infty$. Using the estimate (5.10) and a simple union bound, we see that

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{\substack{S \cap \mathcal{D} \neq \emptyset \\ |S| = 2^{-n}}} \left\{D_h(\text{around } B_{|S|}(S) \setminus S) > R_n (M_h(S))^{\xi}\right\}\right] \\
&\leq \sum_{\substack{S \in \mathcal{D} \\ |S| = 2^{-n}}} \mathbb{P}\left[D_h(\text{around} B_{|S|}(S) \setminus S) > R_n (M_h(S))^{\xi}\right] \\
&\qquad\qquad < d^2\, 2^{2n} e^{\frac{-n^{4/3}}{2/3 \log n}},
\end{aligned}$$

which follows from the fact that any tiling can admit at most $(2^n d)^2$ squares of side length 2^{-n}. Observe that the final term obtained in the above string of inequalities is summable over all values of n, and so the proof is concluded via Borel-Cantelli. □

Let P^{ϵ} be an $\mathcal{S}_h^{\epsilon}$-geodesic connecting $\partial \mathcal{D}$ and K. Our goal is to construct a new path that is realized as a concatenation of loops around annuli containing squares that are hit by P^{ϵ}. This new path will have bounded D_h length by Lemma 5.3.2, and this, in turn, would give us a lower bound for the number of squares in the square-tiling geodesic P^{ϵ}. To make this construction, we prove the following topological fact.

Lemma 5.3.3. *Let $J = J(K) := \operatorname{diam}(K)$. For $\zeta \in (0, 1)$, and for $\epsilon \leq J^{\frac{2+Q}{1-\zeta(2+Q)}}$, consider the path P^{ϵ}, viewed as a minimal length path of $\mathcal{S}_h^{\epsilon}$ squares between K and $\partial \mathcal{D}$. For each square S in this path, let $\tilde{P}_S$ be a bounded D_h-path around $B_{|S|}(S) \setminus S$, as realized in Lemma 5.3.2. Then, the union of the paths $\tilde{P}_S$, $S \in P^{\epsilon}$ contains a connected path from K to $\partial \mathcal{D}$.*

Proof. We follow the proof of Lemma 4.3 in [23]. Observe that the union of the sets bounded by the paths $\tilde{P}_S$, $S \in P^\epsilon$ realizes a cover for P^ϵ. We denote these sets by $A_{\tilde{P}_S}$ for ease. We then select a sub-collection of squares $\mathcal{S}$ which admit a minimal cover, in the sense that $P^\epsilon \subset \bigcup_{S \in \mathcal{S}} A_{\tilde{P}_S}$, and P^ϵ is not covered by any proper subset of the sets containing the squares in $\mathcal{S}$. Since P^ϵ is connected, it follows that $\bigcup_{S\in\mathcal{S}} A_{\tilde{P}_S}$ is connected. Indeed, if this set had two proper disjoint open subsets, then each would have to intersect P^ϵ (by minimality) which would contradict the connectedness of P^ϵ. Moreover, by minimality, no path surrounding any square in $\mathcal{S}$ is properly contained in another path surrounding a different square in $\mathcal{S}$.

We claim that $\bigcup_{S\in\mathcal{S}} \partial A_{\tilde{P}_S} = \bigcup_{S\in\mathcal{S}} \tilde{P}_S$ is connected. Assume to the contrary, and partition $\mathcal{S} = \mathcal{S}_1 \bigsqcup \mathcal{S}_2$ so that $\mathcal{S}_1$ and $\mathcal{S}_2$ are non-empty, and the corresponding sets $A_{\tilde{P}_S}$ are such that $\bigcup_{S\in\mathcal{S}_1} \tilde{P}_S$ and $\bigcup_{S\in\mathcal{S}_2} \tilde{P}_S$ are disjoint. Since our chosen cover is minimal, no corresponding set $A_{\tilde{P}_S}$ surrounding a square in $\mathcal{S}_2$ can be contained in $\bigcup_{S\in\mathcal{S}_1} \partial A_{\tilde{P}_S}$. Furthermore, since $\bigcup_{S\in\mathcal{S}_1} \tilde{P}_S$ and $\bigcup_{S\in\mathcal{S}_2} \tilde{P}_S$ are disjoint, it cannot be the case that any set containing a square in $\mathcal{S}_2$ intersects both $\bigcup_{S\in\mathcal{S}_1} A_{\tilde{P}_S}$ and $\mathbb{C} \setminus \bigcup_{S\in\mathcal{S}_1} A_{\tilde{P}_S}$. Therefore, $\bigcup_{S\in\mathcal{S}} A_{\tilde{P}_S}$ and $\bigcup_{S\in\mathcal{S}_2} \tilde{P}_S$ are disjoint. Since no set containing an element of $\mathcal{S}_1$ can be contained in $\bigcup_{S\in\mathcal{S}_2} A_{\tilde{P}_S}$, we get that $\bigcup_{S\in\mathcal{S}_1} A_{\tilde{P}_S}$ and $\bigcup_{S\in\mathcal{S}_2} A_{\tilde{P}_S}$ are disjoint. This contradicts the connectedness of $\bigcup_{S\in\mathcal{S}} A_{\tilde{P}_S}$, and therefore supports our claim.

Since the initial and terminal sets of the form $A_{\tilde{P}_S}$ intersect K and $\partial\mathcal{D}$ respectively given our choice of ϵ, the desired claim follows. $\square$

Proof of Theorem 5.3.1. Define P^ϵ and $\tilde{P}_S$ as in Lemma 5.3.3, and let $\mathcal{S}$ be a sub-collection of P^ϵ squares such that the union of the sets bounded by the paths $\tilde{P}_S$, $S \in \mathcal{S}$, admits a minimal cover of P^ϵ. Lemma 5.3.3 tells us that we can construct a connected path P^* between K and $\partial\mathcal{D}$ realized as a union of segments of (bounded) D_h-paths around $B_{|S|}(S) \setminus S$, for $S \in P^\epsilon$. Thus, by the triangle inequality, we have that

$$
\begin{aligned}
D_h(K, \partial\mathcal{D}) &\le \sum_{\substack{S\in\mathcal{S} \\ P^*\cap B_{|S|}(S)\setminus S\neq\emptyset}} D_h(\text{ segment of } P^* \text{ contained in } B_{|S|}(S)\setminus S) \\
&\le \sum_{S\in\mathcal{S}} D_h(\text{around}\, B_{|S|}(S)\setminus S) \\
&\le \sum_{n=1}^{N^\epsilon} \sum_{\substack{S\in P^\epsilon \\ |S|=2^{-n}}} D_h(\text{around}\, B_{|S|}(S)\setminus S),
\end{aligned}
$$

where the last inequality follows from the fact that, by definition of the model, the squares have minimal side length of 2^{-N^ϵ}, where $N^\epsilon = (\log\frac{1}{\epsilon})^{7/5}$. By Lemma 5.3.2, it holds with probability tending to 1 as $\epsilon \to 0$, that for all $S \in P^\epsilon$, $D_h(\text{around}\, B_{|S|}(S)\setminus S) \le (M_h(S))^\xi e^{(\log\frac{1}{|S|})^{2/3}}$ and by definition, each such square satisfies $M_h(S) \le \epsilon$. With this, and the fact that the multiplicative error, in the worst case, equals $e^{(\log 2^{N^\epsilon})^{2/3}} = \epsilon^{o_\epsilon(1)}$ by construction, we obtain the inequality

$$D_h(K, \partial\mathcal{D}) \le \epsilon^{\xi+o_\epsilon(1)} \tilde{D}_h^\epsilon(K, \partial\mathcal{M}).$$

The D_h distance on the LHS of the above inequality is a finite random variable that does not depend on ϵ, and so it follows that $\tilde{D}_h^\epsilon(K, \partial\mathcal{D}) \ge \epsilon^{-\xi+o_\epsilon(1)}$ with probability tending to 1 as $\epsilon \to 0$. □

5.4 Upper bound for $\tilde{D}_h^\epsilon$ in terms of D_h

In this section, we aim to upper bound the number of $\mathcal{S}_h^\epsilon$-squares that the continuum geodesic can intersect. As in the previous section, we fix a bounded open set $\mathcal{D} \subset \mathbb{C}$ with connected boundary and a compact, connected set K with non-empty interior, contained in $\mathcal{D}$. We state this more precisely as follows.

Theorem 5.4.1. *Suppose $\xi = \xi(c_M)$ is such that $Q = Q(\xi)$ is the parameter for the square* *tiling model. Then, with probability 1 as $\epsilon \to 0$,*

$$D_h(K, \partial\mathcal{D}) \geq \epsilon^{\xi + o_\epsilon(1)} \tilde{D}_h^\epsilon(K, \partial\mathcal{D}).$$

In particular, $\tilde{D}_h^\epsilon(K, \partial\mathcal{D}) \leq \epsilon^{-\xi + o_\epsilon(1)}$ with probability 1 as $\epsilon \to 0$.

We adopt a slightly different strategy, in that we start with a D_h-geodesic, and using the estimate (5.9), we can find an upper bound for the number of squares in the $\mathcal{S}_h^\epsilon$-tiling hit by the D_h-geodesic, and thus giving an upper bound for the $\tilde{D}_h^\epsilon$ distance between two sets in the tiling. To do this we begin by showing that the D_h-geodesic avoids extremely small squares, which relies on estimates similar to Lemma 5.3.2.

Lemma 5.4.2. *Almost surely, every dyadic ancestor S' of every small enough dyadic square S intersecting $\mathcal{D}$ has the property that both the D_h distance around the annulus $B_{100|S'|}(S') \setminus S'$, and the D_h distance across $B_{|S'|}(S') \setminus S'$ are at least $(M_h(S'))^\xi \, e^{-(\log(1/|S'|)^{2/3})}$.*

Proof. In the spirit of Lemma 5.3.2, $n \in \mathbb{N}$, and let $R_n = e^{n^{2/3}}$ so that $\sum_{n \geq 1} 2^{2n} \, e^{-(\log R_n)^2} < \infty$. Observe that, with this choice of R_n, we must have that $R_n^{-1} = e^{-(\log 1/|S'|)^{2/3}}$. The rest of the proof is exactly as in Lemma 5.3.2. □

We will show that the geodesic avoids tiny squares in a sequence of steps. We first give a lower bound for the length of a D_h-path η which disconnects the inner and outer boundaries of $B_{|S|^\alpha}(S) \setminus B_{100|S|}(S)$ for a fixed square S in $S \in \mathcal{S}_h^\epsilon$, and some $\alpha \in (0, 1)$. Then, using a key lemma in [8], we give an upper bound for the D_h-distance around $B_{|S|^\alpha}(S) \setminus S$, and in turn this allows us to give the desired lower bound on the size of the squares the D_h-geodesic actually sees.

Lemma 5.4.3. *For each $\alpha \in (0, 1)$ and $S \in \mathcal{S}_h^\epsilon$ intersecting $\mathcal{D}$ let η be a D_h-path disconnecting the inner and outer boundaries of $B_{|S|^\alpha}(S) \setminus B_{100|S|}(S)$. Then, a.s. for each $\delta > 0$ and ϵ*

small enough, there exists a dyadic ancestor S' of S such that

$$\mathit{len}(\eta) \geq \epsilon^{\xi} |S'|^{-\delta\xi},$$

where ξ is an in Lemma 5.4.2.

Proof. We first claim that there exists a dyadic ancestor S' of S such that either $\eta \subset B_{100|S'|}(S') \setminus S'$, or η crosses from S' to $\partial B_{|S'|}(S')$. Indeed, let T be the smallest ancestor of S with the property that $T \cap \eta \neq \emptyset$, and let $\tilde{S}$ be the immediate dyadic offspring of T containing S. Note that the condition imposed on η guarantees that $\tilde{S} \neq S$. If $\eta \cap \partial B_{25|T|}(T) \neq \emptyset$, then we simply let $S' = T$, as indeed, η must cross from ∂T to $\partial B_{|T|}(T)$ for this to happen. If $\eta \subset B_{25|T|}(T)$, then certainly $\eta \subset B_{100|\tilde{S}|}(\tilde{S})$. In this case, we simply let $S' = \tilde{S}$. Thus, we have the following cases.

- Suppose $\eta \subset B_{100|S'|}(S') \setminus S'$ for some dyadic ancestor $S'$of S. Then, by Lemma 5.4.2, under the formulation for distances around annuli, we have

$$\begin{aligned}
\mathrm{len}(\eta) &\geq D_h(\text{around}\, B_{100|S'|}(S') \setminus S') \\
&\geq D_h(\text{around}\, B_{|S'|}(S') \setminus S') \\
&\geq (M_h(S'))^{\xi} |S'|^{-\delta\xi} \\
&\geq \epsilon^{\xi} |S'|^{-\delta\xi},
\end{aligned}$$

  as each dyadic ancestor has LQG size greater than ϵ.

- Suppose η crosses from S' to $\partial B_{|S'|}(S')$, for some dyadic ancestor $S'$of S. Then, by Lemma 5.4.2 we have

$$\mathrm{len}(\eta) \geq D_h(\text{across}\, B_{|S'|}(S') \setminus S') \geq (M_h(S'))^{\xi} |S'|^{-\delta\xi} \geq \epsilon^{\xi} |S'|^{-\delta\xi},$$

  and this concludes the proof.

□

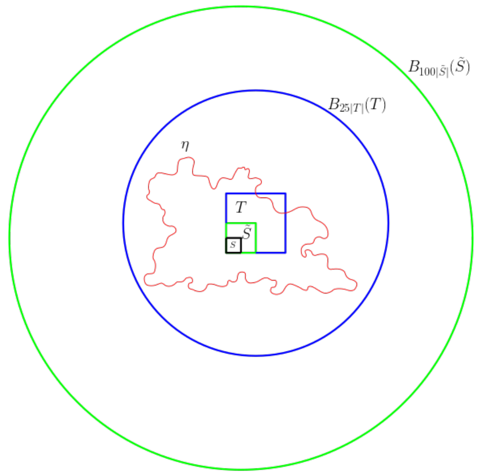

Figure 5.2: A schematic of the geometric argument given in Lemma 5.4.3. The case where η intersects the boundary of the inner (blue) circle gives us the result automatically. In the case that it does not, then in η is contained in the ball of radius $100|\tilde{S}|(\tilde{S})$, and is thus contained in $B_{100|\tilde{S}|}(\tilde{S}) \setminus \tilde{S}$.

Lemma 5.4.4. *For each $\zeta > 0$ and $\alpha \in (0,1)$, there exists $\beta = \beta(\alpha) > 0$ such that for each Euclidean-bounded open set $U \subset \mathbb{C}$, the following holds with probability tending to 1 as $\zeta \to 0$. Suppose $z \in U, x, y \in \mathbb{C} \setminus B_{\zeta^\alpha}(z)$, and $s > 0$ such that there is a D_h-geodesic P from x to y with $P(s) \in B_\zeta(z)$. Then*

$$D_h(\text{around}\, B_{\zeta^\alpha}(z) \setminus B_\zeta(z)) \leq \zeta^\beta s.$$

Roughly speaking, the lemma implies that if a D_h-geodesic does hit a square in $\mathcal{S}_h^\epsilon$, then the D_h distance around an annulus containing that square is bounded above.

Proof. This is Corollary 3.7 in [8]. □

Proposition 5.4.5. *Let $\mathcal{D}'$ be a bounded open set with $\mathcal{D} \subset \overline{\mathcal{D}'}$, and fix a compact, non-singleton set $K' \subset \mathring{K}$. Let η be a D_h geodesic between K' and $\partial\mathcal{D}'$, and let δ, β be defined as in Lemma 5.4.3 and Lemma 5.4.4 respectively. Then, with probability tending to 1 as $\epsilon \to 0$, η does not hit any $\mathcal{S}_h^\epsilon$ squares S intersecting $\mathcal{D} \setminus K$, with $|S| < \epsilon^{\frac{\xi}{\beta+\delta\xi}} (\mathrm{len}(\eta))^{-1}$.*

We remark that the proof of the above proposition relies on Lemma 5.4.4. The lemma specifies that the marked points between which the geodesic traverses remain outside of a ball of given radius, and so the introduction of the sets $\mathcal{D}'$ and K' allow for this.

Proof. Let S be an $\mathcal{S}_h^\epsilon$ square intersecting both η and $\mathcal{D} \setminus K$, with ϵ sufficiently small. For $\alpha \in (0,1)$, Lemma 5.4.3 tells us that, a.s., there exists a dyadic ancestor $S' \supset S$ with the property that

$$D_h(\text{around } B_{|S'|^\alpha}(S') \setminus S') \geq \epsilon^\xi |S'|^{-\delta\xi},$$

for some $\delta > 0$. On the other hand Lemma 5.4.4 implies that, with probability tending to 1 as $\epsilon \to 0$,

$$D_h(\text{around } B_{|S'|^\alpha}(S') \setminus S') \leq |S'|^\beta \operatorname{len}(\eta),$$

for some $\beta = \beta(\alpha)$. Should these two inequalities be satisfied, we must have that $|S'|^\beta \operatorname{len}(\eta) \geq \epsilon^\xi |S'|^{-\delta\xi}$. Rearrangement yields the inequality $|S'| \geq \epsilon^{\frac{\xi}{\beta+\delta\xi}} (\operatorname{len}(\eta))^{-1}$ and the desired claim follows. $\square$

Proof of Theorem 5.4.1. We begin by making the following geometric observations about the interactions of a D_h- geodesic η (as in Proposition 5.4.5), and the $\mathcal{S}_h^\epsilon$ tiling.

- If η hits a square $S \in \mathcal{S}_h^\epsilon$, then certainly η crosses from $\partial B_{|S'|}(S') \setminus S'$ to ∂S, traverses S (and possibly one of the other squares adjacent to it), and crosses this annulus again. We can lower bound the D_h lengths of the segments of η contained in this annulus by twice the distance across it.

- The possible side lengths for the squares in a given $\mathcal{S}_h^\epsilon$ path are 2^{-n} for $n \in (c \log \frac{1}{\epsilon}, N_\epsilon)$, as the side length of the squares is upper bounded by Lemma 5.2.2, and lower bounded by the definition of the model. Thus, if we let $\mathcal{S}_n := \{S \in \mathcal{S}_h^\epsilon : |S| = 2^{-n}, S \cap \eta \neq \emptyset\}$, we see that there must exist a number $k \in \mathbb{N}$ such that

$$\#\mathcal{S}_k \geq \frac{\#\{S \in \mathcal{S}_h^\epsilon : S \cap \eta \neq \emptyset\}}{N^\epsilon},$$

and so it suffices to upper bound $\#\mathcal{S}_k$. Observe that, for every square S such that $|S| = 2^{-n}$, there are at most 8 other squares $\tilde{S}$ with $|\tilde{S}| = 2^{-n}$ such that $\tilde{S} \cap B_{|S|}(S) \neq \emptyset$.

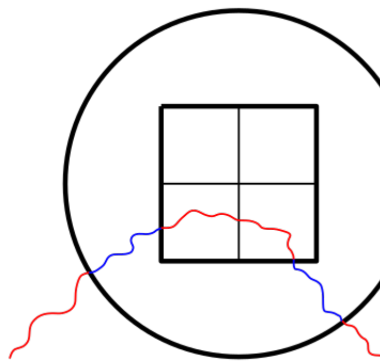

Figure 5.3: A schematic of a D_h-geodesic entering the annulus around S', traversing the smaller cells, and leaving it. The segments in blue are the crossing segments.

Let M^ϵ be the number of $\mathcal{S}_h^\epsilon -$ squares that the D_h geodesic η intersects, and define $\mathcal{S}_k^*$ to be the set of dyadic parents of the squares in $\mathcal{S}_k$. Using the observation above, we deduce that the following inequalities hold with probability going to 1 as $\epsilon \to 0$.

$$\begin{aligned}
D_h(K, \partial\mathcal{D}) &\geq \frac{1}{32} \sum_{S' \in \mathcal{S}_k^*} D_h(\text{segment of } \eta \text{ contained in } B_{|S'|}(S') \setminus S') \\
&\geq \frac{1}{32} \sum_{S' \in \mathcal{S}_k^*} 2D_h(\text{across}\, B_{|S'|}(S') \setminus S') \qquad \text{(by discussion preceding Fig. 5.3)} \\
&\geq \frac{1}{16} \sum_{S' \in \mathcal{S}_k^*} \epsilon^{\xi} e^{-(\log(1/|S'|)^{2/3})} \qquad \text{(by Lemma 5.4.2)} \\
&\geq \frac{1}{16} \sum_{S' \in \mathcal{S}_k^*} \epsilon^{\xi} \epsilon^{(\frac{\xi}{\beta+\delta\xi}) o_\epsilon(1)} \left(D_h(K', \partial\mathcal{D}')\right)^{-1} \qquad \text{(by Proposition 5.4.5)} \\
&\geq \frac{1}{16} \epsilon^{\xi + o_\epsilon(1)} \left(D_h(K', \partial\mathcal{D}')\right)^{-1} \#\mathcal{S}_k \\
&\geq \frac{1}{16} (N^\epsilon)^{-1} \epsilon^{\xi + o_\epsilon(1)} \left(D_h(K', \partial\mathcal{D}')\right)^{-1} M^\epsilon \qquad \text{(by discussion preceding Fig. 5.3)} \\
&= \frac{1}{16} \epsilon^{\xi + o_\epsilon(1)} \left(D_h(K', \partial\mathcal{D}')\right)^{-1} M^\epsilon \qquad (\text{as } N^\epsilon = \epsilon^{o_\epsilon(1)} \text{ by definition}).
\end{aligned}$$

Again, $D_h(K, \partial\mathcal{D})$ and $D_h(K', \partial\mathcal{D}')$ are random variables independent of ϵ, and so we must have that $M^\epsilon \leq \epsilon^{-\xi + o_\epsilon(1)}$. □

REFERENCES

[1] Morris Ang. Comparison of discrete and continuum Liouville first passage percolation. *Electron. Commun. Probab.*, 24:Paper No. 64, 12, 2019.

[2] Vincent Beffara. Schramm-Loewner Evolution and other conformally invariant objects. *Probability and Statistical Physics in Two and More Dimensions*, 15:1–48, 2012.

[3] N. Berestycki and J.R. Norris. Lectures on Schramm-Loewner Evolution. Available at `http://www.statslab.cam.ac.uk/~james/Lectures/`.

[4] Nathanaël Berestycki. Introduction to the Gaussian Free Field and Liouville Quantum Gravity. Available at `https://homepage.univie.ac.at/nathanael.berestycki/articles.html`.

[5] J. Ding and E. Gwynne. The fractal dimension of Liouville quantum gravity: universality, monotonicity, and bounds. *Communications in Mathematical Physics*, 374:1877–1934, 2018.

[6] Jian Ding, Julien Dubédat, Alexander Dunlap, and Hugo Falconet. Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$. *Publ. Math. Inst. Hautes Études Sci.*, 132:353–403, 2020.

[7] Jian Ding and Ewain Gwynne. The critical Liouville quantum gravity metric induces the Euclidean topology. *ArXiv e-prints*, April 2021.

[8] Jian Ding and Ewain Gwynne. Regularity and confluence of geodesics for the supercritical Liouville quantum gravity metric. *ArXiv e-prints*, April 2021.

[9] Jian Ding and Ewain Gwynne. Tightness of supercritical Liouville first passage percolation. *ArXiv e-prints*, 2021.

[10] Jian Ding and Ewain Gwynne. Up-to-constants comparison of Liouville first passage percolation and Liouville quantum gravity. *ArXiv e-prints*, 2021.

[11] Jian Ding and Ewain Gwynne. Uniqueness of the critical and supercritical Liouville quantum gravity metrics. *ArXiv e-prints*, 2022.

[12] Julien Dubédat. Duality of Schramm-Loewner evolutions. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):697–724, 2009.

[13] Julien Dubédat and Hugo Falconet. Liouville metric of star-scale invariant fields: tails and Weyl scaling. *Probab. Theory Related Fields*, 176(1-2):293–352, 2020.

[14] Julien Dubédat, Hugo Falconet, Ewain Gwynne, Joshua Pfeffer, and Xin Sun. Weak LQG metrics and Liouville first passage percolation, 2020.

[15] Bertrand Duplantier. Conformally invariant fractals and potential theory. *Physical Review Letters*, 84(7):1363–1367, Feb 2000.

[16] Bertrand Duplantier. Higher conformal multifractality. *Journal of Statistical Physics*, 110(3–6):691–738, 2003.

[17] Bertrand Duplantier and Scott Sheffield. Duality and the Knizhnik-Polyakov-Zamolodchikov relation in Liouville quantum gravity. *Phys. Rev. Lett.*, 102(15):150603, 4, 2009.

[18] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[19] Ewain Gwynne. Random surfaces and Liouville quantum gravity. *Notices of the American Mathematical Society*, April 2020.

[20] Ewain Gwynne, Nina Holden, Joshua Pfeffer, and Guillaume Remy. Liouville quantum gravity with matter central charge in (1, 25): a probabilistic approach. *Comm. Math. Phys.*, 376(2):1573–1625, 2020.

[21] Ewain Gwynne, Nina Holden, and Xin Sun. Mating of trees for random planar maps and Liouville quantum gravity: a survey. *ArXiv e-prints*, Oct 2019.

[22] Ewain Gwynne and Jason Miller. Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0, 2)$. *Ann. Probab.*, 48(4):1861–1901, 2020.

[23] Ewain Gwynne and Jason Miller. Local metrics of the Gaussian free field. *ArXiv e-prints*, 2020.

[24] Ewain Gwynne and Jason Miller. Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Invent. Math.*, 223(1):213–333, 2021.

[25] Ewain Gwynne and Jason Miller. Random walk on random planar maps: Spectral dimension, resistance and displacement. *Ann. Probab.*, 49(3):1097–1128, 2021.

[26] Ewain Gwynne and Joshua Pfeffer. Bounds for distances and geodesic dimension in Liouville first passage percolation. *ArXiv e-prints*, 2019.

[27] Gregory F. Lawler. A self-avoiding random walk. *Duke Math. J.*, 47(3):655–693, 1980.

[28] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[29] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001.

[30] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[31] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting SLEs. *Probab. Theory Related Fields*, 164(3-4):553–705, 2016.

[32] Jason Miller and Scott Sheffield. Imaginary geometry II: Reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. *Ann. Probab.*, 44(3):1647–1722, 2016.

[33] Jason Miller and Scott Sheffield. Imaginary geometry III: reversibility of SLE_κ for $\kappa \in (4,8)$. *Ann. of Math. (2)*, 184(2):455–486, 2016.

[34] Jason Miller and Scott Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields*, 169(3-4):729–869, 2017.

[35] Jason Miller, Scott Sheffield, and Wendelin Werner. Non-simple SLE curves are not determined by their range. *J. Eur. Math. Soc. (JEMS)*, 22(3):669–716, 2020.

[36] Jason Miller and Hao Wu. Intersections of SLE Paths: the double and cut point dimension of SLE. *Probab. Theory Related Fields*, 167(1-2):45–105, 2017.

[37] Edwin Moise. *Geometric Topology in Dimensions 2 and 3*. Springer, New York, NY, 1977.

[38] A. M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103(3):207–210, 1981.

[39] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[40] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[41] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013.

[42] Scott Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007.

[43] Scott Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5):3474–3545, 2016.

[44] Stanislav Smirnov and Wendelin Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(5-6):729–744, 2001.

[45] Wendelin Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pages 107–195. Springer, Berlin, 2004.

[46] Dapeng Zhan. Duality of chordal SLE. *Invent. Math.*, 174(2):309–353, 2008.

[47] Dapeng Zhan. Reversibility of chordal SLE. *Ann. Probab.*, 36(4):1472–1494, 2008.

[48] Dapeng Zhan. Duality of chordal SLE, II. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(3):740–759, 2010.