THE UNIVERSITY OF CHICAGO

EXTREME VALUES OF LOG-CORRELATED GAUSSIAN FIELDS

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ABSTRACT

In this thesis we discuss extremes of log-correlated Gaussian processes on integer lattices. The first four chapters show that the centered maximum of a sequence of log-correlated Gaussian fields, with mild assumptions in any fixed dimension, converges in distribution. The final chapter is on the behavior of a typical vertex of a branching random walk(BRW) when placed against a hard wall.

Chapter 1 introduces log-correlated Gaussian processes on the integer lattice and talks about previous related works. We make a few assumptions about the correlation structure, firstly about the form which is sufficient to prove tightness. Next we make further assumptions about convergence of covariances in a suitable sense, for convergence in distribution and discuss examples which show that for logarithmically correlated fields these additional structural assumptions, of the type we make, are needed for convergence of the centered maximum.

The second chapter deals with expectation of the maxima and its tightness after recentering. This is achieved by approximating the process in the sense of covariance comparison by other known Gaussian processes whose similar properties have been proved previously. We also provide an upper bound on the left tail as a complimentary result.

Chapter 3 covers the topic of robustness of log-correlated Gaussian fields. We observe no change in distribution of the maxima, except for shifting of mean, on perturbation at microscopic and macroscopic levels by Gaussian variables. We also study the locations of near-peaks of the field.

Chapter 4 is mainly based on the proof the convergence in distribution of the recentered maxima of the log-correlated Gaussian field. We identify the limit as a randomly shifted Gumbel distribution, and characterize the random shift as the limit in distribution of a sequence of random variables, reminiscent of the derivative martingale in the theory of BRW and Gaussian chaos. We also discuss applications of the main convergence theorem.

Chapter 5 talks about the behavior of the BRW on a d-ary tree when pressed against

a hard wall. To this end, the field is approximated by a new Gaussian field switching sign BRW, and left tail estimates on this field gives our desired result.

CHAPTER 1

INTRODUCTION

1.1 Motivation

The extremes of various log-correlated Gaussian fields (including branching Brownian motion, branching random walk, two-dimensional discrete Gaussian free field, etc.) have been topics of intensive research(see [11], [26], [16], [2], [4], [13], [5], [6]). The Gaussian free field is an analog of Brownian motion. Many constructions in quantum field theory are based on the Gaussian free field. Particularly, the 2D GFF is important in the theory of random surfaces. The extreme values of this process are by themselves important statistics.

The question of extremes of 2D GFF also arises while dealing with the entropic repulsion of the discrete Gaussian free field as is talked about in [9], which deals with the behavior of the field when pressed against a hard wall. The discrete Gaussian free field in two dimension with zero boundary condition on V_N, a 2-dimensional box of side length N with leftmost corner at the origin, is a mean zero Gaussian field with covariance structure given by the random walk Green function which is killed on hitting the boundary of V_N. Estimates from random walks illustrate the logarithmic structure of its covariance, in the interior of the box. The lattice, on being divided into four sub-blocks, motivates a *tree structure*(see Figure 1.1) coming from self-similarity. This establishes a connection between the *Gaussian free field* and *branching random walk*. This fact has been utilized in the analysis of the extremes of the Gaussian free field in [14], [19], [10] and [12].

The Gaussian membrane model in dimension 4 is another object of wide importance in statistical mechanics. This Gaussian process also admits a covariance structure in the logarithmic form. Log-correlated Gaussian fields have been used in [24] to mathematically model Gaussian multiplicative chaos(see further [20], [21]). Maxima of log-correlated Gaussian fields, whose covariance admits a kernel representation, has been worked upon in [30]. In the continuous setup the tightness of the recentered maxima for log-correlated Gaussian

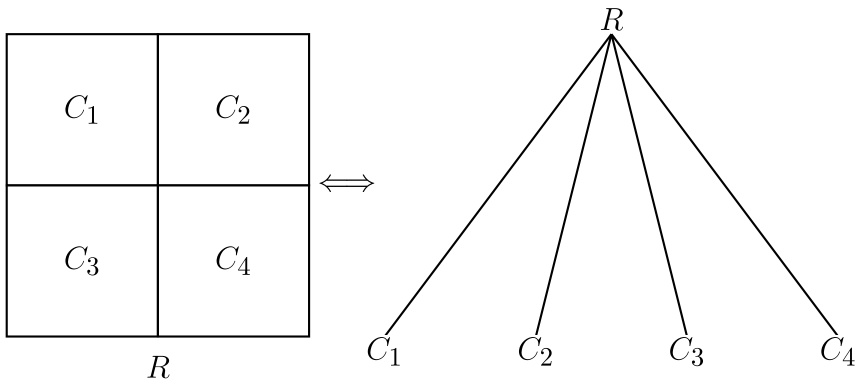

Figure 1.1: Tree structure for the lattice

fields has been shown in [1]. Similar and further questions for generalized log-correlated Gaussian fields in the discrete setup is therefore of great importance. Major focus of my research during my graduate studies has been on the question of convergence in distribution of recentered maxima of general log-correlated Gaussian fields, with minimal assumptions on the correlation structure. With this we start with the definition of the log-correlated Gaussian field and the minimal assumptions we make for proving different properties of the field.

1.2 Definition

Fix $d \in \mathbb{N}$ and let $V_N = \mathbb{Z}_N^d$ be the d-dimensional box of side length N with the left bottom corner located at the origin. Let us also define $V_N^0 = ((0, N-1) \cap \mathbb{Z})^d$. We call the outer boundary to be $\partial V_N = \{y \in V_N : y \notin V_N^0 \text{ and } \exists z \in V_N^0 \text{ with } |y - z| = 1\}$. For convenience, we consider a suitably normalized version of Gaussian fields $\{\varphi_{N,v} : v \in V_N\}$, of mean zero, satisfying the following.

(A.0) **(Logarithmically bounded fields)** There exists a constant $\alpha_0 > 0$ such that for all $u, v \in V_N$,

$$\operatorname{Var} \varphi_{N,v} \leq \log N + \alpha_0$$

and

$$\mathbb{E}(\varphi_{N,v} - \varphi_{N,u})^2 \leq 2 \log_+ |u - v| - |\operatorname{Var} \varphi_{N,v} - \operatorname{Var} \varphi_{N,u}| + 4\alpha_0,$$

where $|\cdot|$ denotes the Euclidean norm and $\log_+ x = \log x \vee 0$.

We introduce the sets $V_N^\delta = \{z \in V_N : d(z, \partial V_N) \geq \delta N\}$ and $V^\delta = [\delta, 1-\delta]^d$, where $d(z, \partial V_N) = \min\{\|z - y\|_\infty : y \notin V_N\}$. Then, introduce the following assumption.

(A.1) **(Logarithmically correlated fields)** For any $\delta > 0$ there exists a constant $\alpha^{(\delta)} > 0$ such that for all $u, v \in V_N^\delta$, $|\operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}) - (\log N - \log_+ |u - v|)| \leq \alpha^{(\delta)}$.

First, we assume convergence of the covariance in finite scale around the diagonal.

(A.2) **(Near diagonal behavior)** There exist a continuous function $f : (0,1)^d \mapsto \mathbb{R}$ and a function $g : \mathbb{Z}^d \times \mathbb{Z}^d \mapsto \mathbb{R}$ such that the following holds. For all $L, \epsilon, \delta > 0$, there exists $N_0 = N_0(\epsilon, \delta, L)$ such that for all $x \in V^\delta$, $u, v \in [0, L]^d$ and $N \geq N_0$ we have

$$|\operatorname{Cov}(\varphi_{N,xN+v}, \varphi_{N,xN+u}) - \log N - f(x) - g(u,v)| < \epsilon \,.$$

Next, we introduce an assumption concerning convergence of the covariance in case of the off-diagonal terms (in a macroscopic scale). Let $\mathcal{D}^d = \{(x,y) : x, y \in (0,1)^d, x \neq y\}$.

(A.3) **(Off diagonal behavior)** There exists a continuous function $h : \mathcal{D}^d \mapsto \mathbb{R}$ such that the following holds. For all $L, \epsilon, \delta > 0$, there exists $N_1 = N_1(\epsilon, \delta, L) > 0$ such that for all $x, y \in V^\delta$ with $|x - y| \geq \frac{1}{L}$ and $N \geq N_1$ we have

$$|\operatorname{Cov}(\varphi_{N,xN}, \varphi_{N,yN}) - h(x,y)| < \epsilon \,.$$

As we move to the Chapter 2, Chapter 3 and Chapter 4, we go deeper into the relevance of the assumptions about the covariance structure of the Gaussian field, but for some basic ideas we discuss two examples in Section 1.4.

1.3 Previous Results

The study of log-correlated Gaussian fields is motivated by the study of discrete Gaussian free free field in two dimensions, which is one of the most popular examples of log-correlated fields in the recent times. The analysis is also deeply connected to the analysis of 2D GFF, which we give a brief account of.

As discussed in the Section 1.1, the study of Gaussian free field starts with the work of [9], which proves a law of large number result for the maxima of GFF. This makes use of the tree structure in proving so. The next important work along the direction of convergence of recentered maxima of GFF is that of [10], which proves the tightness of the GFF along a deterministic subsequence. It also showed that tightness for the recentered maxima itself can be proved by computing the expected the maxima up to the order of a constant. This was achieved in [15]. This work involved the introduction of modified branching random walk and comparison of it with the GFF.

Few other works which were useful in proving the convergence in distribution of the recentered maxima were on the geometry of set of large values which are within a multiplicative constant from the maxima([16]), and the work in [19] which studied the relative distances between peaks and gave an estimate on the order of the right tail of the maxima of the GFF.

The convergence in distribution of the recentered maxima of the GFF was proved in [12] by splitting the field into two independent fields, the course field and the fine field. Then a modified second moment method was used to obtain a refined estimate on the right tail of the fine field. This, along with the behavior of the course field showed the convergence in distribution.

These results appear as a build up to our problem and the methods we use in our computations.

1.4 On the assumptions

The basis of Assumption (A.1).

Set $M_N = \max_{v \in V_N} \varphi_{N,v}$ and

$$m_N = \sqrt{2d} \log N - \frac{3}{2\sqrt{2d}} \log\log N \,. \tag{1.1}$$

Proposition 1.4.1. *Under Assumption (A.0), there exists a constant $C = C(\alpha_0) > 0$ such that for all $N \in \mathbb{N}$ and $z \geq 1$,*

$$\mathbb{P}(M_N \geq m_N + z) \leq Cze^{-\sqrt{2d}z} e^{-C^{-1}z^2/n} \,. \tag{1.2}$$

Furthermore, for all $z \geq 1, y \geq 0$ and $A \subseteq V_N$ we have

$$\mathbb{P}(\max_{v \in A} \varphi_{N,v} \geq m_N + z - y) \leq C \left(\frac{|A|}{|V_N|}\right)^{1/2} ze^{-\sqrt{2d}(z-y)} \,. \tag{1.3}$$

Here we denote by $|A|$ the cardinality of the set A.

The proof of Proposition 1.4.1 is provided in Section 2.2.

By Proposition 1.4.1, if one has a complementary lower bound showing that for a large enough constant C, $\max_{v \in V_N} \varphi_{N,v} > m_N - C$ with high probability, it follows that the maximizer of the Gaussian field is away from the boundary with high probability. Therefore, in the study of convergence of the centered maximum, it suffices to consider the Gaussian field away from the boundary (more precisely, with distance δN away from the boundary where $\delta \to 0$ after $N \to \infty$).

The basis of Assumptions (A.2) and (A.3). We next construct two examples that demonstrate that one cannot totally dispense of Assumptions (A.2) and (A.3). Since the examples are only ancillary to our main result, we will only give a brief sketch for the verification of the claims made concerning these examples.

Example 1.4.2. *Fix $d = 2$ and let $\{\varphi_{N,v} : v \in V_N\}$ be the DGFF on V_N (normalized so that it satisfies Assumptions (A.0), (A.1), (A.2) and (A.3)), with $Z_N = \max_{v\in V_N} \varphi_{N,v}$. Let $V_{N,1}$ and $V_{N,2}$ be the left and right halves of the box V_N. Let $\{\epsilon_{N,v} : v \in V_N\}$ and X be i.i.d. standard Gaussian variables. Let $\sigma'_N > 0$ be selected later. Define*

$$\tilde{\varphi}_{N,v} = \begin{cases} \varphi_{N,v} + \sigma X + \epsilon_{N,v}, & v \in V_{N,1} \\ \varphi_{N,v}, & v \in V_{N,2} \end{cases}, \quad \hat{\varphi}_{N,v} = \begin{cases} \varphi_{N,v} + \sigma X, & v \in V_{N,1} \\ \varphi_{N,v} + \sigma'_N \epsilon_{N,v}, & v \in V_{N,2} \end{cases}.$$

Set $\tilde{M}_N = \max_{v\in V_N} \tilde{\varphi}_{N,v}$ and $\hat{M}_N = \max_{v\in V_N} \hat{\varphi}_{N,v}$. We first claim that there exist σ'_N depending on (N, σ) but bounded from above by an absolute constant such that $\mathbb{E}\tilde{M}_N = \mathbb{E}\hat{M}_N$. In order to see that, note that, by Theorem 2.0.4,

$$\mathbb{E}\tilde{M}_N \leq \mathbb{E} \max_{v\in V_{N/2}} \varphi_{N,v} + \sigma \mathbb{E} \max(0, X) + O(1),$$

where $O(1)$ is an error term independent of all parameters, while

$$\mathbb{E}\tilde{M}_N \geq \mathbb{E} \max_{v\in V_{N/2}} \varphi_{N,v} + \sigma \mathbb{E} \max(0, X).$$

In addition, by considering a $N/2$-box in the left side and dividing the right half box into two copies of $N/2$-boxes, one gets that

$$\begin{aligned} \mathbb{E}\hat{M}_N &\geq \mathbb{E} \max(Z_{N/2} + \sigma X, Z'_{N/2} + \sigma'_N \epsilon', Z''_{N/2} + \sigma'_N \epsilon'') \\ &\geq \mathbb{E} Z_{N/2} + \frac{1}{2}\sigma'_N \mathbb{E} \max(\epsilon', \epsilon'') + \sigma \mathbb{E} X \mathbf{1}_{X\geq 0}. \end{aligned}$$

where $Z_{N/2}, Z'_{N/2}, Z''_{N/2}$ are three independent copies with law $\max_{v\in V_{N/2}} \varphi_{N,v}$ and $\epsilon' = \epsilon_{N,v^1_*}, \epsilon'' = \epsilon''_{N,v^*_2}$ (here v^*_1 and v^*_2 are the maximizers of the DGFF in the two $N/2$-boxes on the right half of V_N, respectively). The claim follows from combining the last two displays.

Now, choose σ to be a large fixed constant so that for $0 < \lambda < \log\log N$,

$$\begin{aligned}
\mathbb{P}(\tilde{M}_N \geq \mathbb{E}Z_N + \lambda) &\geq \mathbb{P}(\max_{v \in V_{N,1}} \{\varphi_{N,v} + \sigma X + \epsilon_{N,v}\} \geq \mathbb{E}Z_N + \lambda) \\
&\geq \mathbb{P}((1 + 1/4\log N) \max_{v \in V_{N,1}} \{\varphi_{N,v} + \sigma X\} \geq \mathbb{E}Z_N + \lambda) \\
&\geq \mathbb{P}(\max_{v \in V_{N,1}} \varphi_{N,v} + \sigma X \geq \mathbb{E}Z_N + \lambda - 1/10)\,. \qquad (1.4)
\end{aligned}$$

(Here, the second inequality is due to Slepian's comparison lemma (Lemma 2.2.1) and the fact that σ is large, while the last inequality uses that $2/(1+1/(4\log N)) \leq 2 - (\log N)/10$ for N large.) Further,

$$\begin{aligned}
\mathbb{P}(\hat{M}_N \geq \mathbb{E}Z_N + \lambda) &\leq \mathbb{P}(\max_{v \in V_{N,1}} \varphi_{N,v} + \sigma X \geq \mathbb{E}Z_N + \lambda) \\
&+ \mathbb{P}(\max_{v \in V_{N,2}} \varphi_{N,v} + \epsilon'_{N,v} \geq \mathbb{E}Z_N + \lambda) \\
&\leq \mathbb{P}(\max_{v \in V_{N,1}} \varphi_{N,v} + \sigma X \geq \mathbb{E}Z_N + \lambda) + O(1)\lambda e^{-2\lambda}\,, \qquad (1.5)
\end{aligned}$$

where the last inequality follows from Proposition 1.4.1. Combining (1.4) *and* (1.5) *and using the form of the limiting right tail of the two-dimensional DGFF as in [12, Proposition 4.1], one obtains that for λ, σ sufficiently large but independent of N,*

$$\limsup_{N\to\infty} \mathbb{P}(\tilde{M}_N \geq \mathbb{E}Z_N + \lambda) \geq (1+c) \limsup_{N\to\infty} \mathbb{P}(\hat{M}_N \geq \mathbb{E}Z_N + \lambda) \geq c(\sigma)\lambda e^{-2\lambda}\,,$$

where $c > 0$ is an absolute constant and $c(\sigma)$ satisfies $c(\sigma) \to_{\sigma\to\infty} \infty$. This implies that the laws of $\tilde{M}_N - EM_N$ and $\hat{M}_N - E\hat{M}_N$ do not coincide in the limit $N \to \infty$.

Finally, set $\bar{\varphi}_{N,v} = \tilde{\varphi}_{N,v}$ for all $v \in V_N$ and odd N, and $\bar{\varphi}_{N,v} = \hat{\varphi}_{N,v}$ for all $v \in V_N$ and even N. One then sees that the sequence of Gaussian fields $\{\bar{\varphi}_{N,v} : v \in V_N\}$ satisfies Assumptions (A.0), (A.1) and (A.3) (while not satisfying (A.2)), but the law of the centered maximum does not converge.

Example 1.4.3. *Let $\{\varphi_{N,v} : v \in V_N\}$ be a sequence of Gaussian fields satisfying (A.0),* *(A.1) and (A.2), such that the law of the centered maximum converges. Consider the fields* $\{\tilde{\varphi}_{N,v} : v \in V_N\}$ *where* $\tilde{\varphi}_{N,v} = \varphi_{N,v} + \mathbf{1}_{N \text{ is even}} X_N$ *with* X_N *a sequence of i.i.d. standard Gaussian variables. Then, the field* $\{\tilde{\varphi}_{N,v} : v \in V_N\}$ *satisfies (A.0), (A.1) and (A.2) (possibly increasing the values of* $\alpha^{(\delta)}$ *by 1 for all* $0 \leq \delta \leq 1$*). However, the centered law of the maximum of* $\{\tilde{\varphi}_{N,v} : v \in V_N\}$ *cannot converge.*

CHAPTER 2

EXPECTATION AND TIGHTNESS OF MAXIMUM

This chapter covers the expectation and tightness of the maximum of log-correlated Gaussian field.

The main result of this chapter is showing the tightness of the sequence $\{M_N - m_N\}_N$. Assumptions (A.0) and (A.1) are enough to ensure the tightness of the sequence $\{M_N - m_N\}_N$.

Theorem 2.0.4. *Under Assumptions (A.0) and (A.1), we have that* $\mathbb{E}M_N = m_N + O(1)$ *where the* $O(1)$ *term depends on* α_0 *and* $\alpha^{(1/10)}$*. In addition, the sequence* $M_N - \mathbb{E}M_N$ *is tight.*

(The constant 1/10 in Theorem 2.0.4 could be replaced by any positive number that is less than 1/3.)

A similar result (in the slightly different setup of fields indexed by a continuous parameter) appears in [1].

The rest is devoted to the proofs of Proposition 1.4.1 and Theorem 2.0.4, and to an auxiliary lower bound on the right tail of the distribution of the maximum(see Lemma 2.0.6). The proof of the proposition is very similar to the proof in the case of the DGFF in dimension two, using a comparison with an appropriate BRW; Essentially, the proposition gives the correct right tail behavior of the distribution of the maximum. In contrast, given the proposition, in order to prove Theorem 2.0.4, one needs an upper bound on the *left* tail of that distribution. In the generality of this work, one cannot hope for a universal sharp estimate on the left tail, as witnessed by the drastically different left tails exhibited in the cases of the modified branching random walk and the two-dimensional DGFF, see [17]. We will however provide the following universal *upper* bound for the decay of the left tail.

Lemma 2.0.5. *Under Assumption (A.1) there exist constants* $C, c > 0$ *(depending only on*

$\alpha_{1/10}, d$) *so that for all* $n \in \mathbb{N}$ *and* $0 \leqslant \lambda \leqslant (\log n)^{2/3}$,

$$\mathbb{P}(\max_{v \in V_N} \varphi_{N,v} \leqslant m_N - \lambda) \leqslant Ce^{-c\lambda}.$$

Theorem 2.0.4 follows at once from Proposition 1.4.1 and Lemma 2.0.5.

Later, we will need the following complimentary lower bound on the right tail.

Lemma 2.0.6. *Under Assumption (A.1), there exists a constant* $C > 0$ *depending only on* $(\alpha_0, \alpha^{(1/10)}, d)$ *such that for all* $\lambda \in [1, \sqrt{\log N}]$,

$$\mathbb{P}(M_N > m_N + \lambda) \geq C^{-1} \lambda e^{-\sqrt{2d}\lambda}.$$

2.1 Branching random walk and modified branching random walk

The study of extrema for log-correlated Gaussian fields is possible because they exhibit an approximate tree structure and can be efficiently compared with branching random walk and the modified branching random walk introduced in [15]. In this subsection, we briefly review the definitions of BRW and MBRW in $\mathbb{Z}^d$. We remark that the MBRW can be seen as an discrete analogue of the $*$-scale invariant log-correlated fields studied in [30]; we further remark that the natural continuous construction of MBRW is not exactly a $*$-scale invariant field since the corresponding kernel function (in the language of [30]) is not continuous.

Suppose $N = 2^n$ for some $n \in \mathbb{N}$. For $j = 0, 1, \ldots, n$, define $\mathcal{B}_j$ to be the set of d-dimensional cubes of side length 2^j with corners in $\mathbb{Z}^d$. Define $\mathcal{BD}_j$ to be those elements of $\mathcal{B}_j$ which are of the form $\left([0, 2^j - 1] \cap \mathbb{Z}\right)^d + (i_1 2^j, i_2 2^j, \ldots, i_d 2^j)$, where $i_1, i_2, \ldots, i_d$ are integers. For $x \in V_N$, define $\mathcal{B}_j(x)$ to be those elements of $\mathcal{B}_j$ which contains x. Define $\mathcal{BD}_j(x)$ similarly.

Let $\{a_{j,B}\}_{j \geq 0, B \in \mathcal{BD}_j}$ be a family of i.i.d. Gaussian variables of variance $\log 2$. Define the *branching random walk* (BRW) $\{\mathcal{R}_{N,z}\}_{z\in V_N}$ by

$$\mathcal{R}_{N,z} = \sum_{j=0}^{n} \sum_{B\in \mathcal{BD}_j(z)} a_{j,B}\,, \quad z \in V_N\,.$$

Let $\mathcal{B}_j^N$ be the subset of $\mathcal{B}_j$ consisting of elements of the latter with lower left corner in V_N. Let $\{b_{j,B} : j \geq 0, B \in \mathcal{B}_j^N\}$ be a family of independent Gaussian variables such that $\operatorname{Var} b_{j,B} = \log 2 \cdot 2^{-dj}$ for all $B \in \mathcal{B}_j^N$. Write $B \sim_N B'$ if $B = B' + (i_1 N, \ldots, i_d N)$ for some integers $i_1, \ldots, i_d \in \mathbb{Z}$. Let

$$b_{j,B}^N = \begin{cases} b_{j,B} & B \in \mathcal{B}_j^N\,, \\ b_{j,B'} & B \sim_N B' \in \mathcal{B}_j^N\,. \end{cases}$$

Define the *modified branching random walk* (MBRW) $\{\mathcal{S}_{N,z}\}_{z\in V_N}$ by

$$\mathcal{S}_{N,z} = \sum_{j=0}^{n} \sum_{B\in \mathcal{B}_j(z)} b_{j,B}^N\,, \quad z \in V_N\,. \tag{2.1}$$

The proof of the following lemma is an straightforward adaption of [15, Lemma 2.2] for dimension d, which we omit.

Lemma 2.1.1. *There exists a constant C depending only on d such that for $N = 2^n$ and $x, y \in V_N$*

$$|\operatorname{Cov}(\mathcal{S}_{N,x}, \mathcal{S}_{N,y}) - (\log N - \log(|x-y|_N \vee 1))| \leq C\,,$$

where $|x-y|_N = \min_{y'\sim_N y} |x-y'|$.

In the rest of the calculations, we assume that the constants $\alpha_0, \alpha^{(\delta)}$ in Assumptions (A.0) and (A.1) are taken large enough so that the MBRW satisfies the assumptions.

2.2 Comparison of right tails

The following Slepian's comparison lemma for Gaussian processes [32] will be useful.

Lemma 2.2.1. *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a : a \in \mathcal{A}\}$ and $\{Y_a : a \in \mathcal{A}\}$ be two centered Gaussian processes such that: $\mathbb{E}(X_a - X_b)^2 \geq \mathbb{E}(Y_a - Y_b)^2$, for all $a, b \in \mathcal{A}$ and $\mathrm{Var}(X_a) = \mathrm{Var}(Y_a)$ for all $a \in \mathcal{A}$. Then $\mathbb{P}(\max_{a \in \mathcal{A}} X_a \geq \lambda) \geq \mathbb{P}(\max_{a \in \mathcal{A}} Y_a \geq \lambda)$ for all $\lambda \in \mathbb{R}$.*

The next lemma compares the right tail for the maximum of $\{\varphi_{N,v} : v \in V_N\}$ to that of a BRW.

Lemma 2.2.2. *Under Assumption (A.0), there exists an integer $\kappa = \kappa(\alpha_0) > 0$ such that for all N and $\lambda \in \mathbb{R}$ and any subset $A \subseteq V_N$*

$$\mathbb{P}(\max_{v \in A} \varphi_{N,v} \geq \lambda) \leq 2\mathbb{P}(\max_{v \in 2^\kappa A} \mathcal{R}_{2^\kappa N, v} \geq \lambda)\,. \tag{2.2}$$

Proof. For $\kappa \in \mathbb{N}$, consider the map

$$\psi_N = \psi_N^{(\kappa)} : V \mapsto 2^\kappa V \text{ such that } \psi_N(v) = 2^\kappa v\,. \tag{2.3}$$

By Assumption (A.0), we can choose a sufficiently large κ depending on α_0 such that $\mathrm{Var}(\varphi_{N,v}) \leq \mathrm{Var}(\mathcal{R}_{2^\kappa N, \psi_N(v)})$ for all $v \in V_N$. So, we can choose a collection of positive numbers

$$a_v^2 = \mathrm{Var}\,\mathcal{R}_{2^\kappa N, \psi_N(v)} - \mathrm{Var}\,\varphi_{N,v}\,,$$

such that $\mathrm{Var}(\varphi_{N,v} + a_v X) = \mathrm{Var}(\mathcal{R}_{2^\kappa N, \psi_N(v)})$ for all $v \in V_N$, where X is a standard Gaussian random variable, independent of everything else. Since the BRW has constant variance over all vertices, we get that

$$\begin{aligned}\mathbb{E}(\varphi_{N,u} + a_u X - \varphi_{N,v} - a_v X)^2 &\le \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2 + (a_v - a_u)^2 \\ &\le \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2 + |\operatorname{Var}\varphi_{N,v} - \operatorname{Var}\varphi_{N,u}|.\end{aligned}$$

Combined with Assumption (A.0), it yields that

$$\mathbb{E}(\varphi_{N,u} + a_u X - \varphi_{N,v} - a_v X)^2 \le 2\log_+ |u - v| + 4\alpha_0\,.$$

Since $\mathbb{E}(\mathcal{R}_{2^\kappa N, \psi_N(u)} - \mathcal{R}_{2^\kappa N, \psi_N(v)})^2 - 2\log_+ |u-v| \ge \log 2\kappa - C_0$ (where C_0 is an absolute constant), we can choose sufficiently large κ depending only on α_0 such that

$$\mathbb{E}(\varphi_{N,u} + a_u X - \varphi_{N,v} - a_v X)^2 \le \mathbb{E}(\mathcal{R}_{2^\kappa N, \psi_N(u)} - \mathcal{R}_{2^\kappa N, \psi_N(v)})^2\,, \text{ for all } u, v \in V_N\,.$$

Combined with Lemma 2.2.1, it gives that for all $\lambda \in \mathbb{R}$ and $A \subseteq V_N$

$$\mathbb{P}(\max_{v\in A} \varphi_{N,v} + a_v X \ge \lambda) \le \mathbb{P}(\max_{v \in A} \mathcal{R}_{2^\kappa N, \psi_N(v)} \ge \lambda)\,.$$

In addition, by independence and symmetry of X we have

$$\mathbb{P}(\max_{v\in A} \varphi_{N,v} + a_v X \ge \lambda) \ge \mathbb{P}(\max_{v\in A} \varphi_{N,v} \ge \lambda, X \ge 0) = \tfrac{1}{2}\mathbb{P}(\max_{v\in A} \varphi_{N,v} \ge \lambda)\,.$$

This completes the proof of the desired bound.

□

Proof of Proposition 1.4.1. An analogous statement was proved in [12, Lemma 3.8] for the case of 2D DGFF. In the proof of [12, Lemma 3.8], the desired inequality was first proved for BRW on the 2D lattice and then deduced for 2D DGFF applying [19, Lemma 2.6], which is the analogue of Lemma 2.2.2 above. The argument for BRW in [12, Lemma 3.8] carries out (essentially with no change) from dimension two to dimension d. Given that, an application of Lemma 2.2.2 completes the proof of the proposition. $\square$

A complimentary lower bound on the right tail is also available.

Lemma 2.2.3. *Under Assumption (A.1), there exists an integer $\kappa = \kappa(\alpha^{(1/10)}) > 0$ such that for all N and $\lambda \in \mathbb{R}$*

$$\mathbb{P}(\max_{v \in V_N} \varphi_v^N \geq \lambda) \geq \tfrac{1}{2}\mathbb{P}(\max_{v \in V_{2^{-\kappa}N}} \mathcal{S}_{2^{-\kappa}N,v} \geq \lambda)\,. \tag{2.4}$$

Proof. It suffices to consider $M_N^{(1/10)} = \max_{v \in V_N^{1/10}} \varphi_{N,v}$. By Assumption (A.1) and an argument analogous to that used in the proof of Lemma 2.2.2 (which can be raced back to the proof of [19, Lemma 2.6]), one deduces that for $\kappa = \kappa(\alpha^{(1/10)})$,

$$\mathbb{P}(M_N^{(1/10)} \geq \lambda) \geq \tfrac{1}{2}\mathbb{P}(\max_{v \in V_{2^{-\kappa}N}} \mathcal{S}_{2^{-\kappa}N,v} \geq \lambda) \text{ for all } \lambda \in \mathbb{R}\,.$$

This completes the proof of the lemma. $\square$

We also need the following estimate on the right tail for MBRW in d-dimension. The proof is a routine adaption of the proof of [19, Lemma 3.7] to arbitrary dimension, and is omitted.

Lemma 2.2.4. *There exists an absolute constant $C > 0$ such that for all $\lambda \in [1, \sqrt{\log n}]$, we have*

$$C^{-1}\lambda e^{-\sqrt{2d}\lambda} \leq \mathbb{P}(\max_{v \in V_N} \mathcal{S}_{N,v} > m_N + \lambda) \leq C\lambda e^{-\sqrt{2d}\lambda}\,.$$

Proof of Lemma 2.0.6. Combine Lemma 2.2.3 and Lemma 2.2.4. $\square$

2.3 An upper bound on the left tail

This section is devoted to the proof of Lemma 2.0.5. The proof consists of two steps: (1) a derivation of an exponential upper bound on the left tail for the MBRW; (2) a comparison of the left tail for general log-correlated Gaussian field to that of the MBRW.

Lemma 2.3.1. *There exist constants $C, c > 0$ so that for all $n \in \mathbb{N}$ and $0 \leqslant \lambda \leqslant (\log n)^{2/3}$,*

$$\mathbb{P}(\max_{v \in V_N} \mathcal{S}_{N,v} \leqslant m_N - \lambda) \leqslant Ce^{-c\lambda}\,.$$

Proof. A trivial extension of the arguments in [15] (for the MBRW in dimension two) yields the tightness for the maximum of the MBRW in dimension d arounds its expectation, with the latter given by (1.1). Therefore, there exist constants $\kappa, \beta > 0$ such that for all $N \geq 4$,

$$\mathbb{P}(\max_{v \in V_N} \mathcal{S}_{N,v} \geqslant m_N - \beta) \geqslant 1/2\,. \tag{2.5}$$

In addition, a simple calculation gives that for all $N \geq N' \geq 4$ (adjusting the value of κ if necessary),

$$\sqrt{2d}\log(N/N') - \frac{3}{4d}\log(\log N/\log N') - \kappa \leqslant m_N - m_{N'} \leqslant \sqrt{2d}\log(N/N') + \kappa\,. \tag{2.6}$$

Let $\lambda' = \lambda/2$ and $N' = N\exp(-\frac{1}{\sqrt{2d}}(\lambda' - \beta - \kappa - 4))$. By (2.6), one has $m_N - m_{N'} \leqslant \lambda' - \beta$. Divide V_N into disjoint boxes of side length N', and consider a maximal collection $\mathcal{B}$ of N'-boxes such that all the pairwise distances are at least $2N'$, implying that $|\mathcal{B}| \geq \exp(\frac{\sqrt{d}}{\sqrt{2}}(\lambda' - \beta - \kappa - 8 - 4\sqrt{d}))$. Now consider the modified MBRW

$$\tilde{\mathcal{S}}_{N,v} = g_{N',v} + \phi \quad \forall v \in B \in \mathcal{B}\,,$$

where ϕ is an zero mean Gaussian variable with variance $\log(N/N')$ and $\{g_{N',v} : v \in B\}_B$ are the MBRWs defined on the boxes B, independently of each other and of ϕ. It is straightforward to check that

$$\operatorname{Var}\mathcal{S}_{N,v} = \operatorname{Var}\tilde{\mathcal{S}}_{N,v} \text{ and } \mathbb{E}\mathcal{S}_{N,v}\mathcal{S}_{N,u} \leq \mathbb{E}\tilde{\mathcal{S}}_{N,v}\tilde{\mathcal{S}}_{N,u} \text{ for all } u, v \in \cup_{B \in \mathcal{B}} B\,.$$

Combined with Lemma 2.2.1, it gives that

$$\mathbb{P}(\max_{v\in V_N} \mathcal{S}_{N,v} \leq t) \leq \mathbb{P}(\max_{v\in\cup_{B\in\mathcal{B}}B} \mathcal{S}_{N,v} \leq t) \leq \mathbb{P}(\max_{v\in\cup_{B\in\mathcal{B}}B} \tilde{\mathcal{S}}_{N,v} \leq t) \text{ for all } t\in\mathbb{R}\,. \tag{2.7}$$

By (2.5), one has that for each $B \in \mathcal{B}$,

$$\begin{aligned}\mathbb{P}(\sup_{v\in B} g_{N',v} \geq m_N - \lambda') &= \mathbb{P}(\sup_{v\in B} g_{N',v} \geq m_{N'} + m_N - m_{N'} - \lambda') \\ &\geq \mathbb{P}(\sup_{v\in B} g_{N',v} \geq m_{N'} - \beta) \geqslant \frac{1}{2},\end{aligned}$$

and therefore

$$\mathbb{P}(\sup_{v\in\cup_{B\in\mathcal{B}}B} g_{N',v} < m_N - \lambda') \leq (\tfrac{1}{2})^{|\mathcal{B}|}\,.$$

Thus,

$$\mathbb{P}(\max_{v\in\cup_{B\in\mathcal{B}}B} \tilde{\mathcal{S}}_{N,v} \leq m_N - \lambda) \leq \mathbb{P}(\sup_{v\in\cup_{B\in\mathcal{B}}B} g_{N',v} < m_N - \lambda') + \mathbb{P}(\phi \leq -\lambda') \leq C\mathrm{e}^{-c\lambda'}\,,$$

for some constants $C, c > 0$. Combined with (2.7), this completes the proof of the lemma. □

Proof of Lemma 2.0.5. In order to prove Lemma 2.0.5, we will compare the maximum of a sparsified version of the log-correlated field to that of a modified version of MBRW. By Assumption (A.1) and Lemma 2.1.1 , there exists a $\kappa_0 = \kappa_0(\alpha^{(1/10)})$ such that for all $\kappa \geq \kappa_0$,

$$\mathrm{Var}(\varphi_{2^\kappa N, 2^\kappa v}) \leqslant \mathrm{Var}(\mathcal{S}_{2^{2\kappa}N,v}) \text{ for all } v \in V_N^{1/10}\,.$$

Therefore, one can choose a collection of positive numbers $\{a_v : v \in V_N^{1/10}\}$ such that

$$\mathrm{Var}(\varphi_{2^\kappa N, 2^\kappa v} + a_v X) = \mathrm{Var}(\mathcal{S}_{2^{2\kappa}N,v})\,,$$

where X is a standard Gaussian variable. Since the MBRW has constant variance, we have that $|a_v - a_u| \leqslant C_1$ for some constant $C_1 = C_1(\alpha^{(1/10)}) > 0$. By Lemma 2.1.1 again, one has

$$\mathbb{E}(\mathcal{S}_{2^{2\kappa}N,v} - \mathcal{S}_{2^{2\kappa}N,u})^2 \le 2\log_+ |u-v| + O(1)\,,$$

where the $O(1)$ term is bounded by a absolute constant. On the other hand, for all $u, v \in V_N^{1/10}$,

$$\mathbb{E}(\varphi_{2^\kappa N, 2^\kappa v} + a_v X - \varphi_{2^\kappa N, 2^\kappa u} - a_u X)^2 \ge \log 2 \cdot \kappa + 2\log_+ |u-v| - O_{\alpha^{(1/10)}}(1)\,,$$

where $O_{\alpha^{(1/10)}}(1)$ is a term that is bounded by a constant depending only on $\alpha^{(1/10)}$. Therefore, there exists a $\kappa = \kappa(\alpha^{(1/10)})$ such that for all $u, v \in V_N^{1/10}$,

$$\mathbb{E}(\varphi_{2^\kappa N, 2^\kappa v} + a_v X - \varphi_{2^\kappa N, 2^\kappa u} - a_u X)^2 \ge \mathbb{E}(\mathcal{S}_{2^{2\kappa}N,v} - \mathcal{S}_{2^{2\kappa}N,u})^2\,.$$

Combined with Lemma 2.2.1, this implies that for a suitable C_κ depending on κ,

$$\begin{aligned}\mathbb{P}(\max_{v \in V_N} \varphi_{2^\kappa N, 2^\kappa v} \leqslant m_N - \lambda) &\le \mathbb{P}(\max_{v \in V_N^{1/10}} (\varphi_{2^\kappa N, 2^\kappa v} + a_v X) \leqslant m_N - \lambda/2) \qquad (2.8)\\ &\quad + \mathbb{P}(X \le -\lambda/C_\kappa)\\ &\le \mathbb{P}(\max_{v \in V_N^{1/10}} \mathcal{S}_{2^{2\kappa}N,v} \leqslant m_N - \lambda/2) + \mathbb{P}(X \le -\lambda/C_\kappa)\,. \qquad (2.9)\end{aligned}$$

There are number of ways to bound $\mathbb{P}(\max_{v \in V_N^{1/10}} \mathcal{S}_{2^{2\kappa}N,v} \leqslant m_N - \lambda/2)$, and we choose not to optimize the bound, but instead simply apply the FKG inequality [31]. More precisely, we note that there exists a collection of boxes $\mathcal{V}$ with $|\mathcal{V}| \le 2^{4d\kappa}$ where each box is a translated copy of $V_N^{1/10}$ such that $V_{2^{2\kappa}N} \subseteq \cup_{V \in \mathcal{V}} V$. Since $\{\max_{v \in V_{2^{2\kappa}N}} \mathcal{S}_{2^{2\kappa}N,v} \leqslant m_N - \lambda/2\} = \cap_{V \in \mathcal{V}} \{\max_{v \in V} \mathcal{S}_{2^{2\kappa}N,v} \leqslant m_N - \lambda/2\}$, the FKG inequality gives that

$$\mathbb{P}(\max_{v \in V_{2^{2\kappa}N}} \mathcal{S}_{2^{2\kappa}N,v} \leqslant m_N - \lambda/2) \ge (\mathbb{P}(\max_{v \in V_N^{1/10}} \mathcal{S}_{2^{2\kappa}N,v} \leqslant m_N - \lambda/2))^{2^{4d\kappa}}\,,$$

Combined with (2.8) and Lemma 2.3.1, this completes the proof of the lemma. $\square$

2.4 Tightness for Gaussian Membrane model

The Gaussian membrane model in dimension 4 is an example we study now. The maxima and entropic repulsion for the Gaussian membrane model was discussed in [25]. Let us denote the field by $\{\psi_v^N : v \in V_N\}$, where V_N is the cube as defined in the introduction but restricted to dimension 4. Let us call the covariance function $\Gamma_N(x, y)$, for $x, y \in V_N$, changing the notation used in [25] due standard notations for Green's function in literature. The law of this field is the Gibbs measure on $\mathbb{R}^{V_N}$ with 0 boundary conditions outside V_N and Hamilton as $\frac{1}{2}\sum_{x\in\mathbb{Z}^d}(\Delta\psi_x^N)^2$. In [25, Section 2] we see that the covariace function $\Gamma_N(.,.)$ is approximated by another function $\bar{G}_N(.,.)$ inside V_N^δ. In order to define this function we first define another function(Green's function of random walk)

$$G_N(x, y) = \mathbb{E}^x(\sum_{k=0}^{\tau_{V_N}} 1_{X_k=y}).$$

Here $\{X_k\}$ is a random walk starting from from x and τ_{V_N} is the time the random walk exits V_N. Further we define $G(x, y) =: \lim_{N\to\infty} G_N(x, y)$, which exists for all $x, y \in \mathbb{Z}^d$, and as $|x - y| \to \infty$,

$$G(x, y) = a_4\frac{1}{|x - y|^2} + O(|x - y|^{-3}),$$

with $a_4 = \frac{1}{\omega_4}$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$. From estimates in [27], we further have for ball B_N of radius N around 0,

$$G_{B_N}(o, x) = a_4\left(\frac{1}{|x|^2} - \frac{1}{N^2}\right) + O(|x|^{-3}).$$

Finally we have,

$$\bar{G}_N(x, y) = \sum_{z\in V_N} G_N(x, z)G_N(z, y)$$

. Using this estimates, we can compute the order of the covariances to show that the Gaussian membrane model is log-correlated. In continuation to [25, Lemma 2.2], which gives logarithmic order of the variance term, we have the following result.

Lemma 2.4.1. *If $\delta \in (0, 1/2)$ then there exists constants c_3, c_4 independent of N such that for $x, y \in V_N^\delta$, we have*

$$\frac{8}{\pi^2}(\log N - \log|x-y|) + c_3 \le \overline{G}_N(x,y) \le \frac{8}{\pi^2}(\log N - \log|x-y|) + c_4$$

Proof. Let $B_r(x)$ denote the ball of radius r around $x \in V_N$. We look at $B_{2N}(x)$ where $x \in V_N$. We apply a transformation to polar co-ordinates after using random walk estimates. The rest of the calculations follow easily from that. Since $G_N(x,y) \le G(x,y)$, we can begin as follows.

$$\begin{aligned}
\overline{G}_N(x,y) &\le \sum_{z \in B_{2N}} G(x,z)G(z,y) \\
&\le a_4^2 \sum_{z \in B_{2N}, z \ne x,y} \frac{1}{|z-x|^2|z-y|^2} + O(1) \\
&= a_4^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{2N} \frac{r^3 \sin^2\theta_1 \sin\theta_2}{r^2(r^2+d^2-2dr\cos\theta_1)} dr d\theta_1 d\theta_2 d\theta_3 + O(1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } d = |x-y| \\
&= a_4^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin^2\theta_1 \sin\theta_2 \int_0^{2N} \frac{r - d\cos\theta_1 + d\cos\theta_1}{(r-d\cos\theta_1)^2 + d^2\sin^2\theta_1} dr d\theta_1 d\theta_2 d\theta_3 \\
&+ O(1) \\
&\le a_4^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin^2\theta_1 \sin\theta_2 \{\log|2N+d| - \log d\} d\theta_1 d\theta_2 d\theta_3 \\
&+ a_4^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\theta_1 \sin\theta_2 \cos\theta_1 \{\tan^{-1}\left(\frac{2N - d\cos\theta_1}{d\sin\theta_1}\right) \\
&- \tan^{-1}(-d\cot\theta_1)\} d\theta_1 d\theta_2 d\theta_3 + O(1) \\
&\le 2a_4^2\pi^2\{\log N - \log d\} + c_4 \\
&\le \frac{8}{\pi^2}\{\log N - \log d\} + c_4
\end{aligned}$$

The lower bound will similarly follow by taking $B_{\delta N}(x)$, instead of $B_{2N}(x)$. We transform the integral into polar co-ordinates, following by routine calculations.

$$
\begin{aligned}
\overline{G}_N(x,y) &\geq \sum_{z\in B_{\delta N}} G_{\delta N}(x,z)G_{\delta N}(z,y) \\
&\geq a_4^2 \sum_{z\in B_{\delta N}, z\neq x,y} \frac{1}{|z-x|^2|z-y|^2} + O(1) \\
&\geq 4a_4^2\omega_4 \int_0^{\delta N} \frac{r^3}{r^2(r+d)^2} dr + O(1) \qquad \text{where } d = |x-y| \\
&\geq 4a_4^2\omega_4 \times \frac{1}{2} \times 2\{\log \delta N - \log d\} + O(1) \\
&\geq \frac{8}{\pi^2}\{\log N - \log d\} + c_3
\end{aligned}
$$

□

CHAPTER 3

ROBUSTNESS OF THE MAXIMUM UNDER PERTURBATIONS

The main goal of this chapter is to establish that the law of the maximum for a log-correlated Gaussian field is robust under certain perturbations. These invariance properties will be crucial in Section 4.1 when constructing a new field that approximates our target field.

For a positive integer r, let $\mathcal{B}_r$ be a collection of sub-boxes of side length r which forms a partition of $V_{\lfloor N/r \rfloor r}$. Write $\mathcal{B} = \cup_{r \in [N]} \mathcal{B}_r$. Let $\{g_B : B \in \mathcal{B}\}$ be a collection of i.i.d. standard Gaussian variables. For $v \in V_N$, denote by $B_{v,r} \in \mathcal{B}_r$ the box that contains v. For $\sigma = (\sigma_1, \sigma_2)$ with $\|\sigma\|_2^2 = \sigma_1^2 + \sigma_2^2$ and r_1, r_2, define,

$$\tilde{\varphi}_{N,r_1,r_2,\sigma,v} = \varphi_{N,v} + \sigma_1 g_{B_{v,r_1}} + \sigma_2 g_{B_{v,N/r_2}}\,, \tag{3.1}$$

and set $\tilde{M}_{N,r_1,r_2,\sigma} = \max_{v \in V_N} \tilde{\varphi}_{N,r_1,r_2,\sigma,v}$.

For probability measures ν_1, ν_2 on $\mathbb{R}$, let $d(\nu_1, \nu_2)$ denote the Lévy distance between ν_1, ν_2, i.e.

$$d(\nu_1, \nu_2) = \inf\{\delta > 0 : \nu_1(B) \le \nu_2(B^\delta) + \delta \quad \text{for all open sets } B\},$$

where $B^\delta = \{y : |x - y| < \delta \text{ for some } x \in B\}$. In addition, define

$$\tilde{d}(\nu_1, \nu_2) = \inf\{\delta > 0 : \nu_1((x, \infty)) \le \nu_2((x - \delta, \infty)) + \delta \quad \text{for all } x \in \mathbb{R}\}\,.$$

If $\tilde{d}(\nu_1, \nu_2) = 0$, then ν_1 is stochastically dominated by ν_2. Thus, $\tilde{d}(\nu_1, \nu_2)$ measures approximate stochastic domination of ν_1 by ν_2; in particular, unlike $d(\cdot, \cdot)$, the function $\tilde{d}(\cdot, \cdot)$ is not symmetric.

With a slight abuse of notation, if X, Y are random variables with laws μ_X, μ_Y respectively, we also write $d(X,Y)$ for $d(\mu_X,\mu_Y)$ and $\tilde{d}(X,Y)$ for $\tilde{d}(\mu_X,\mu_Y)$.

A notation convention: By Proposition 1.4.1, one has that

$$\limsup_{\delta\to 0}\limsup_{N} d(\max_{v\in V_N^\delta}\varphi_{N,v},\max_{v\in V_N}\varphi_{N,v})=0\,.$$

Therefore, in order to prove the convergence in distribution of recentered maxima, it suffices to show that for each fixed $\delta>0$, the law of $\max_{v\in V_N^\delta}\varphi_{N,v}-m_N$ converges. To this end, one only needs to consider the Gaussian field restricted to V_N^δ. For convenience of notation, we will treat V_N^δ as the whole box that is under consideration. Equivalently, throughout the rest of the chapters when assuming (A.1), (A.2) or (A.3) holds, we assume these assumptions hold with $\delta=0$, and we set $\alpha:=\max(\alpha_0,\alpha^{(0)})$.

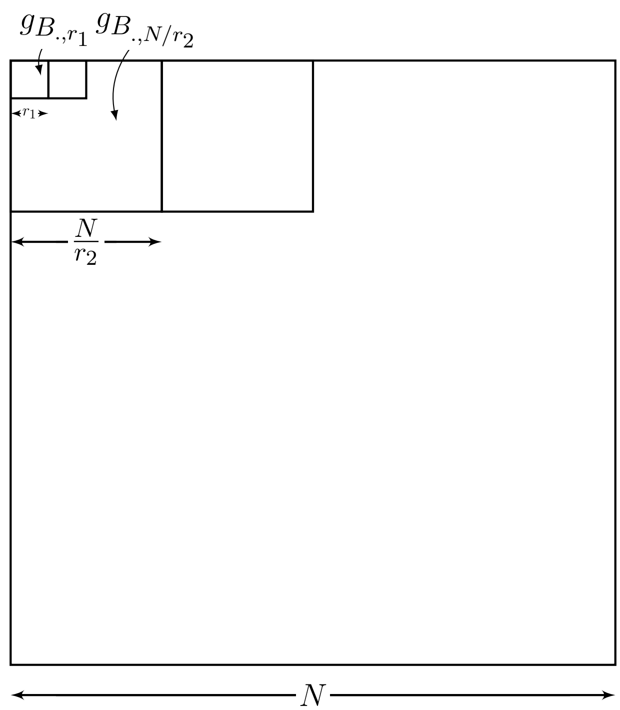

Figure 3.1: Perturbation levels of the Gaussian field

The following lemma, which is one of the main results of this section, relates the laws of M_N and $\tilde{M}_{N,r_1,r_2,\sigma}$.

Lemma 3.0.2. *The following holds uniformly for all Gaussian fields* $\{\varphi_{N,v} : v \in V_N\}$ *satisfying Assumption (A.1):*

$$\limsup_{r_1,r_2\to\infty} \limsup_{N\to\infty} d(M_N - m_N, \tilde{M}_{N,r_1,r_2,\sigma} - m_N - \|\sigma\|_2^2\sqrt{d/2}) = 0\,. \tag{3.2}$$

The next lemma states that under Assumption (A.1), the law of the maximum is robust under small perturbations (in the sense of ℓ_∞ norm) of the covariance matrix.

Lemma 3.0.3. *Let* $\{\varphi_{N,v} : v \in V_N\}$ *be a sequence of Gaussian fields satisfying Assumption (A.1), and let* σ *be fixed. Let* $\{\bar{\varphi}_{N,v} : v \in V_N\}$ *be Gaussian fields such that for all* $u, v \in V_N$

$$|\operatorname{Var}\varphi_{N,v} - \operatorname{Var}\bar{\varphi}_{N,v}| \le \epsilon, \text{ and } \mathbb{E}\bar{\varphi}_{N,v}\bar{\varphi}_{N,u} \le \mathbb{E}\varphi_{N,v}\varphi_{N,u} + \epsilon\,.$$

Then, there exists $\iota = \iota(\epsilon)$ *with* $\iota \to_{\epsilon\to 0} 0$ *such that*

$$\limsup_{N\to\infty} \tilde{d}(M_N - m_N, \max_{v\in V_N} \bar{\varphi}_{N,v} - m_N) \le \iota\,.$$

A key step in the proof of Lemma 3.0.2 is the following characterization of the geometry of vertices achieving large values in the fields, an extension of [19, Theorem 1.1]; it states that near maxima are either at microscopic or macroscopic distance from each other. This may be of independent interest.

Lemma 3.0.4. *There exists a constant* $c > 0$ *such that, uniformly for all Gaussian fields satisfying Assumption (A.1), we have*

$$\lim_{r\to\infty} \lim_{N\to\infty} \mathbb{P}(\exists u, v : |u - v| \in (r, \tfrac{N}{r}), \varphi_{N,v}, \varphi_{N,u} \ge m_N - c\log\log r) = 0\,.$$

3.1 Maximal sum over restricted pairs

As in the case of 2D DGFF discussed in [19], in order to prove Lemma 3.0.4, we will study the maximum of the sum over restricted pairs. For any Gaussian field $\{\eta_{N,v} : v \in V_N\}$ and $r > 1$, define

$$\eta^{\diamond}_{N,r} = \max\{\eta_{N,u} + \eta_{N,v} : u, v \in V_N, r \leq |u - v| \leq N/r\}\,.$$

Lemma 3.1.1. *There exist constants c_1, c_2 depending only on d and $C > 0$ depending only on (α, d) such that for all r, n with $N = 2^n$ and all Gaussian fields satisfying Assumption (A.1), we have*

$$2m_N - c_2 \log\log r - C \leq \mathbb{E}\varphi^{\diamond}_{N,r} \leq 2m_N - c_1 \log\log r + C\,. \tag{3.3}$$

Proof. In order to prove Lemma 3.1.1, we will show that

$$\mathbb{E}\mathcal{S}^{\diamond}_{2^{-\kappa}N,r} \leq \mathbb{E}\varphi^{\diamond}_{N,r} \leq \mathbb{E}\mathcal{S}^{\diamond}_{2^{\kappa}N,r}\,. \tag{3.4}$$

To this end, we recall the following Sudakov-Fernique inequality [22] which compares the first moments for maxima of two Gaussian processes.

Lemma 3.1.2. *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a : a \in \mathcal{A}\}$ and $\{Y_a : a \in \mathcal{A}\}$ be two centered Gaussian processes such that:*

$$\mathbb{E}(X_a - X_b)^2 \geq \mathbb{E}(Y_a - Y_b)^2, \qquad \textit{for all } a, b \in \mathcal{A}\,.$$

Then $\mathbb{E}(\max_{a\in\mathcal{A}} X_a) \leq \mathbb{E}(\max_{a\in\mathcal{A}} Y_a)$.

We will give a proof for the upper bound in (3.3). The proof of the lower bound follows using similar arguments. For $\kappa \in \mathbb{N}$, recall the definition of the restriction map ψ_N as in

(2.3). By Lemma 2.1.1, there exists a $\kappa > 0$ (depending only on (α, d)) such that for all $u, v, u', v' \in V_N$,

$$\mathbb{E}(\varphi_{N,u} + \varphi_{N,v} - \varphi_{N,u'} - \varphi_{N,v'})^2 \le \mathbb{E}(\mathcal{S}^{2^\kappa N}_{\psi_N(u)} + \mathcal{S}^{2^\kappa N}_{\psi_N(v)} - \mathcal{S}^{2^\kappa N}_{\psi_N(u')} - \mathcal{S}^{2^\kappa N}_{\psi_N(v')})^2 .$$

(To see this, note that the variance of $\mathcal{S}^{2^\kappa N}_{\psi_N(u)}$ increases with κ but the covariance between $\mathcal{S}^{2^\kappa N}_{\psi_N(u)}$ and $\mathcal{S}^{2^\kappa N}_{\psi_N(v)}$ does not.) In addition, note that for $r \le |u - v| \le N/r$ one has $r \le |\psi_N(u) - \psi_N(v)| \le 2^\kappa N/r$. Combined with Lemma 3.1.2, this yields $\mathbb{E}\varphi^\diamond_{N,r} \le \mathbb{E}\mathcal{S}^\diamond_{2^\kappa N,r}$, completing the proof of the upper bound in (3.4).

To complete the proof of Lemma 3.1.2, note that [19, Lemma 3.1] readily extends to MBRW in d-dimension, and thus

$$2m_N - c_2 \log\log r - C \le \mathbb{E}\mathcal{S}^\diamond_{N,r} \le 2m_N - c_1 \log\log r + C ,$$

where c_1, c_2 are constants depending only on d and C is a constant depending on (α, d). Combined with (3.4), this completes the proof of the lemma. □

We will also need the following tightness result.

Lemma 3.1.3. *Under Assumption (A.1), the sequence* $\{\frac{(\varphi^\diamond_{N,r} - \mathbb{E}\varphi^\diamond_{N,r})}{\log\log r}\}_{N\in\mathbb{N}, r\ge 100}$ *is tight. Further, there exists a constant $C > 0$ depending only on d such that for all $r \ge 100$ and $N \in \mathbb{N}$,*

$$|(\varphi^\diamond_{N,r} - \mathbb{E}\varphi^\diamond_{N,r})| \le C \log\log r .$$

Proof. Take $N' = 2N$ and partition $V_{N'}$ into 2^d copies of V_N, denoted by $V_N^{(1)}, \ldots, V_N^{(2^d)}$. For each $i \in [2^d]$, let $\{\varphi^{(i)}_{N,v} : v \in V_N^{(i)}\}$ be an independent copy of $\{\varphi_{N,v} : v \in V_n\}$ where we identify V_N and $V_N^{(i)}$ by the suitable translation such that the two boxes coincide. Denote by

$$\hat\varphi_{N',v} = \varphi^{(i)}_{N,v} \text{ for } v \in V_N^{(i)} \text{ and } i \in [2^d] . \tag{3.5}$$

Clearly, $\{\varphi_{N',v}\}$ is a Gaussian field that satisfies Assumption (A.1) (with α increased by an absolute constant). Therefore, by Lemma 3.1.1, we have

$$2m_N - c_2 \log\log r - C \le \mathbb{E}\hat{\varphi}^{\diamond}_{N,r} \le 2m_N - c_1 \log\log r + C\,, \tag{3.6}$$

where $c_1, c_2, C > 0$ are constants depending only on (d, α). In addition, we have

$$\mathbb{E}(\hat{\varphi}^{\diamond}_{N',r}) \ge \mathbb{E}\max\{\varphi^{(1),\diamond}_{N,r}, \varphi^{(2),\diamond}_{N,r}\}\,.$$

Combined with Lemma 3.1.1 and (3.6), and the simple algebraic fact that $|a-b| = 2(a \vee b) - a - b$, it yields that

$$\mathbb{E}|\varphi^{(1),\diamond}_{N,r} - \varphi^{(2),\diamond}_{N,r}| \le 2(\mathbb{E}\hat{\varphi}^{\diamond}_{N',r} - \mathbb{E}\varphi^{\diamond}_{N,r}) \le C' \log\log r\,,\ \text{ for all } r \ge 100\,,$$

where $C' > 0$ is a constant depending only on d. This completes the proof of the lemma. □

3.2 Location of near maxima

In this section we will prove Lemma 3.0.4, by contradiction. Suppose otherwise that Lemma 3.0.4 does not hold. Then for any constant $c > 0$, there exists $\epsilon > 0$ and a subsequence $\{r_k\}$ such that for all $k \in \mathbb{N}$

$$\lim_{N\to\infty} \mathbb{P}\big(\exists u, v : |u - v| \in \left(r_k, \frac{N}{r_k}\right), \varphi_{N,v}, \varphi_{N,u} \ge m_N - c\log\log r_k\big) > \epsilon\,. \tag{3.7}$$

Now fix $\delta > 0$ and consider $N' = 2^\kappa N$ where κ is an integer to be selected. Partition $V_{N'}$ into $2^{\kappa d}$ disjoint boxes of side length N, denoted by $V_N^{(1)}, \ldots, V_N^{(2^{\kappa d})}$. Define $\{\hat{\varphi}_{N',v} : v \in V_{N'}\}$ in the same manner as in (3.5) except that now we take $2^{\kappa d}$ copies of $\{\varphi_{N,v} : v \in V_N\}$ (one for each $V_N^{(i)}$ with $i \in [2^{\kappa d}]$). Clearly, $\{\hat{\varphi}_{N',v} : v \in V_{N'}\}$ is a Gaussian field satisfies Assumption (A.1) with α replaced by a constant α' depending only on (α, d, κ). Therefore, by Lemma 3.1.1,

$$2m_N - c_2 \log\log r - C \le \mathbb{E}\hat{\varphi}^{\diamond}_{N',r} \le 2m_N - c_1 \log\log r + C\,, \tag{3.8}$$

where $c_1, c_2 > 0$ are two constants depending only on d and $C > 0$ is a constant depending only on (α, d, κ).

Next we derive a contradiction to (3.8). Set $z_{N,r} = 2m_N - c\log\log r$, $Z_{N,r} = (\hat{\varphi}^{\diamond}_{N',r} - z_{N,r})_-$ and $Y^{(i)}_{N,r} = (\varphi^{(i),\diamond}_{N,r_k} - z_{N,r})_-$. Then (3.7) implies that

$$\lim_{N\to\infty} \mathbb{P}(Y^{(1)}_{N,r_k} > 0) \le 1 - \epsilon \text{ for all } k \in \mathbb{N}\,. \tag{3.9}$$

In addition, by Lemmas 3.1.1 and 3.1.3, there exists a constant $C' > 0$ depending only on d such that for all $r \ge 100$ and $N \in \mathbb{N}$, we have

$$\mathbb{E}Y^{(1)}_{N,r} \le C' \log\log r\,. \tag{3.10}$$

Clearly, $Z_{N,r} \le \min_{i\in[2^{\kappa d}]} Y^{(i)}_{N,r}$. Combined with the fact that $Y^{(i)}_{N,r}$ are i.i.d. random variables, one obtains

$$\begin{aligned}
\mathbb{E}Z_{N,r_k} &\le \int_0^\infty (\mathbb{P}(Y^{(1)}_{N,r_k} > y))^{2^{\kappa d}} dy \\
&\le (1-\epsilon)^{2^{\kappa d}-1} \int_0^\infty (\mathbb{P}(Y^{(1)}_{N,r_k} > y)) dy \\
&\le (1-\epsilon)^{2^{\kappa d}-1} \mathbb{E}Y^{(1)}_{N,r_k},
\end{aligned}$$

where (3.9) was used in the second inequality. Combined with (3.10), one concludes that for all $r \ge 100$ and N

$$\mathbb{E}Z_{N,r_k} \le (1-\epsilon)^{2^{\kappa d}-1} C' \log\log r_k\,.$$

Now set $c = c_1/4$ and choose κ depending on (ϵ, d, C', c_1) such that $(1-\epsilon)^{2^{\kappa d}-1} C' \le c_1/4$.

Then,

$$\mathbb{E}\hat{\varphi}^{\diamond}_{N',r_k} \geq 2m_N - c_1 \log\log r_k/2\,,$$

for all $k \in \mathbb{N}$ and sufficiently large $N \geq N_k$ where N_k is a number depending only on k. Sending $N \to \infty$ first and then $k \to \infty$ contradicts (3.8), thereby completing the proof of the lemma. □

3.3 Behavior of maxima on perturbation

The next lemma, which extends [12, Lemma 3.9] to the current setup, will be useful for the proof of Lemma 3.0.2 and later as well.

Lemma 3.3.1. *Let Assumptions (A.0) and (A.1) holds. Let $\{\phi_u^N : u \in V_N\}$ be a collection of random variables independent of $\{\varphi_{N,u} : u \in V_N\}$ such that*

$$\mathbb{P}(\phi_u^N \geq 1 + y) \leq e^{-y^2} \text{ for all } u \in V_N\,. \tag{3.11}$$

Then, there exists $C = C(\alpha, d) > 0$ such that, for any $\epsilon > 0, N \in \mathbb{N}$ and $x \geq -\epsilon^{-1/2}$,

$$\mathbb{P}(\max_{u\in V_N}(\varphi_{N,u}+\epsilon\phi_u^N) \geq m_N+x) \leq \mathbb{P}(\max_{u\in V_N}\varphi_{N,u} \geq m_N+x-\sqrt{\epsilon})(1+C(e^{-C^{-1}\epsilon^{-1}}))\,. \tag{3.12}$$

Proof. We first give the proof for $\epsilon \leq 1$. Define $\Gamma_y = \{u \in V_N : y/2 \leq \epsilon\phi_u^N \leq y\}$. Then,

$$\begin{aligned}\mathbb{P}(\max_{u\in V_N}(\varphi_{N,u} + \epsilon\phi_u^N) \geq m_N + x) \leq & \mathbb{P}(M_N \geq m_N + x - \sqrt{\epsilon}) \\ & + \sum_{i=0}^{\infty} \mathbb{E}(\mathbb{P}(\max_{u\in\Gamma_{2^i\sqrt{\epsilon}}} \varphi_{N,u} \geq m_N + x - 2^i\sqrt{\epsilon}|\Gamma_{2^i\sqrt{\epsilon}}))\,.\end{aligned}$$

By Proposition 1.4.1, one can bound the second term on the right hand side above by

$$\sum_{i=0}^{\infty} \mathbb{E}(\mathbb{P}(\max_{u\in V_N} \varphi_{N,u} \geq m_N + x - 2^i\sqrt{\epsilon}|\Gamma_{2^i\sqrt{\epsilon}})) \lesssim \frac{x \vee 1}{e^{\sqrt{2d}x}} \sum_{i=0}^{\infty} \mathbb{E}(|\Gamma_{2^i\sqrt{\epsilon}}|/N^d)^{1/2} e^{\sqrt{2d}2^i\sqrt{\epsilon}}\,.$$

By (3.11), one has $\mathbb{E}(|\Gamma_{2^i\sqrt{\epsilon}}|/N^d)^{1/2} \le e^{-4^i(C\epsilon)^{-1}}$. Altogether, one gets

$$\sum_{i=0}^{\infty} \mathbb{E}(\mathbb{P}(\max_{u\in V_N} \varphi_{N,u} \ge m_N + x - 2^i\sqrt{\epsilon}|\Gamma_{2^i\sqrt{\epsilon}})) \lesssim \frac{x\vee 1}{e^{\sqrt{2d}x}} e^{-(C\epsilon)^{-1}} ,$$

completing the proof of the lemma when $\epsilon \le 1$. The case $\epsilon > 1$ is simpler and follows by repeating the same argument with $\Gamma_{2^i\epsilon}$ replacing $\Gamma_{2^i\sqrt{\epsilon}}$. We omit further details. □

We next consider a combination of two independent copies of $\{\varphi_{N,v}\}$. For $\sigma > 0$, define

$$\varphi^*_{N,\sigma,v} = \varphi_{N,v} + \sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,v} \text{ for } v \in V_N\,, \text{ and } M^*_{N,\sigma} = \max_{v\in V_N} \varphi^*_{N,\sigma,v}\,. \tag{3.13}$$

where $\{\varphi'_{N,v} : v \in V_N\}$ is an independent copy of $\{\varphi_{N,v} : v \in V_N\}$. Note that the field $\{\varphi^*_{N,\sigma,v}\}$ is distributed like the field $\{a_N\varphi_{N,v}\}$ where $a_N = \sqrt{1 + \|\sigma\|_2^2/\log N}$.

Remark 3.3.2. *The idea of writing a Gaussian field as a sum of two independent Gaussian fields has been extensively employed in the study of Gaussian processes. In the context of the study of extrema of the 2D DGFF, this idea was first used in [7], where (combined with an invariance result from [29] as well as the geometry of the maxima of DGFF [19], see Lemma 3.1.1) it led to a complete description of the extremal process of 2D DGFF. The definition* (3.13) *is inspired by [7].*

The following is the key to the proof of Lemma 3.0.2.

Proposition 3.3.3. *Let Assumption (A.1) hold. Let $\{\tilde\varphi_{N,r,\sigma,v} : v \in V_N\}$ and $\{\varphi^*_{N,\sigma,v} : v \in V_N\}$ be defined as in* (3.1) *and* (3.13) *respectively. Then for any fixed σ,*

$$\lim_{r_1,r_2\to\infty} \limsup_{N\to\infty} d(\tilde M_{N,r_1,r_2,\sigma} - m_N, M^*_{N,\sigma} - m_N) = 0\,. \tag{3.14}$$

Proof. Partition V_N into boxes of side length N/r_2 and denote by $\mathcal{B}$ the collection of boxes. Fix an arbitrary small $\delta > 0$, and let B_δ denote the box in the center of B with side length $(1-$

$\delta)N/r_2$ for each $B \in \mathcal{B}$. Write $V_{N,\delta} = \cup_{B\in\mathcal{B}} B_\delta$. Set $\tilde{M}_{N,r_1,r_2,\sigma,\delta} = \max_{v\in V_{N,\delta}} \tilde{\varphi}_{N,r_1,r_2,\sigma,v}$ and $M^*_{N,\sigma,\delta} = \max_{v\in V_{N,\delta}} \varphi^*_{N,\sigma,v}$. By (1.3), one has

$$\lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}(\tilde{M}_{N,r_1,r_2,\sigma,\delta} \neq \tilde{M}_{N,r_1,r_2,\sigma}) = \lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}(M^*_{N,\sigma,\delta} \neq M^*_{N,\sigma}) = 0\,.$$

Therefore, it suffices to prove (3.14) with $\tilde{M}_{N,r_1,r_2,\sigma,\delta}$ and $M^*_{N,\sigma,\delta}$ replacing $\tilde{M}_{N,r_1,r_2,\sigma}$ and $M^*_{N,\sigma}$ respectively. To this end, let z_B be such that

$$\max_{v\in B_\delta} \varphi_{N,v} = \varphi_{N,z_B} \text{ for every } B \in \mathcal{B}\,.$$

We will show below that

$$\begin{aligned}
&\lim_{r_1,r_2\to\infty}\limsup_{N\to\infty} \mathbb{P}(|\tilde{M}_{N,r_1,r_2,\sigma,\delta} - \max_{B\in\mathcal{B}} \tilde{\varphi}_{N,r_1,r_2,\sigma,z_B}| \geq 1/\log\log N)\\
=&\limsup_{N\to\infty} \mathbb{P}(|M^*_{N,\sigma,\delta} - \max_{B\in\mathcal{B}} \varphi^*_{N,\sigma,z_B}| \geq 1/\log\log N) = 0\,. \qquad (3.15)
\end{aligned}$$

Note that the field $\{\varphi_{N,v} : v \in V_N\}$ and $\{\sqrt{\|\sigma\|_2^2/\log N}\varphi'_{N,v} : v \in V_N\}$ are independent of each other. Thus, conditioning on the field $\{\varphi_{N,v} : v \in V_N\}$, the field $\{\sqrt{\|\sigma\|_2^2/\log N}\varphi'_{N,z_B} : B \in \mathcal{B}\}$ is a centered Gaussian field with pairwise correlation bounded by $O(1/\log N)$. Therefore the conditional covariance matrix of $\{\sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,z_B} : B \in \mathcal{B}\}$ and that of $\{\sigma_1 g_{B_{z_B,r_1}} + \sigma_2 g_{B_{z_B,N/r_2}} : B \in \mathcal{B}\}$ are within additive $O(1/\log N)$ of each other entrywise. In addition, $|\mathcal{B}| \leq (2r_2)^d$. Therefore, it is clear that there exists a coupling between the two fields such that

$$\begin{aligned}
\lim_{N\to\infty} \quad &\mathbb{P}(\max_{B\in\mathcal{B}} |\sqrt{\|\sigma\|_2^2/\log N}\varphi'_{N,z_B} - (\sigma_1 g_{B_{z_B,r_1}} + \sigma_2 g_{B_{z_B,N/r_2}})|\\
&\geq 1/\log\log N \mid \{\varphi_{N,v} : v \in V_N\}) = 0
\end{aligned}$$

(here the term $1/\log\log N$ is somewhat arbitrary, any negative power larger than 1/2 of $(\log N)$ would work). Note that the preceding equality holds for almost all realizations of

$\{\varphi_{N,v} : v \in V_N\}$. Combined with (3.15), it then yields the proposition.

It remains to prove (3.15). Write $r = r_1 \wedge r_2$ and let C be a constant which we will send to infinity after sending first $N \to \infty$ and then $r \to \infty$, and let c be the constant from Lemma 3.0.4. Suppose that either of the events that are considered in (3.15) occurs. In this case, one of the following events has to occur:

- The event $E_1 = \{\tilde{M}_{N,r_1,r_2,\sigma,\delta} \notin (m_N - C, m_N + C)\} \cup \{M^*_{N,\sigma,\delta} \notin (m_N - C, m_N + C)\}$.
- The event E_2 that there exists $u, v \in (r, N/r)$ such that $\varphi_{N,u} \wedge \varphi_{N,v} > m_N - c \log\log r$.
- The event $E_3 = \tilde{E}_3 \cup E_3^*$ where $\tilde{E}_3$ (E_3^*) is the event that $\tilde{M}_{N,r_1,r_2,\sigma}$ ($M^*_{N,\sigma,\delta}$) is achieved at a vertex v such that $\varphi_{N,v} \leq m_N - c \log\log r$.
- The event E_4 that there exists $v \in B \in \mathcal{B}$ with $\varphi_{N,v} \geq m_N - c \log\log r$ and

$$\sqrt{\tfrac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,v} - \sqrt{\tfrac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,z_B} \geq \tfrac{1}{\log\log N}\,.$$

By Theorem 2.0.4, $\lim_{C\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_1) = 0$. By Lemma 3.0.4,

$$\lim_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_2) = 0.$$

In addition, writting $\Gamma_x = \{v \in V_N : \tilde{\varphi}_{N,r_1,r_2,\sigma,v} - \varphi_{N,v} \in (x, x+1)\}$, one has

$$\begin{aligned}
\mathbb{P}(E_1^c \cap \tilde{E}_3) &\leq \mathbb{P}(\max_{x \geq c\log\log r - C} \max_{v\in\Gamma_x} \tilde{\varphi}_{N,r_1,r_2,\sigma,v} \geq m_N - C) \\
&\leq \sum_{x \geq c\log\log r - C} \mathbb{P}(\max_{v\in\Gamma_x} \tilde{\varphi}_{N,r_1,r_2,\sigma,v} \geq m_N - C) \\
&\leq \sum_{x \geq c\log\log r - C} \mathbb{E}(\mathbb{P}(\max_{v\in\Gamma_x} \varphi_{N,v} \geq m_N - x - C|\Gamma_x)) \\
&\lesssim_C \sum_{x \geq c\log\log r - C} \mathbb{E}(|\Gamma_x|/N^d)^{1/2} x e^{\sqrt{2d}x}\,,
\end{aligned}$$

where the last inequality follows from (1.3). From simple estimates using the Gaussian distribution one has $\mathbb{E}(|\Gamma_x|/N^d)^{1/2} \leq e^{-c'x^2}/c'$ where $c' = c'(\sigma) > 0$. Therefore, one concludes that

$$\limsup_{C\to\infty}\limsup_{r\to\infty}\limsup_{N\to\infty}\mathbb{P}(E_1^c \cap \tilde{E}_3) = 0\,.$$

A similar argument leads to the same estimate with E_3^* replacing E_3. Thus,

$$\limsup_{C\to\infty}\limsup_{r\to\infty}\limsup_{N\to\infty}\mathbb{P}(E_1^c \cap E_3) = 0\,.$$

Finally, let $\Gamma'_r = \{v : \varphi_{N,v} \geq m_N - c\log\log r\}$. On the event E_2^c, one has $|\Gamma'_r| \leq r^4$. Further, for each $v \in B \cap \Gamma'_r$, on E_2^c one has $|v - z_B| \leq r$ and thus (by the independence between $\{\varphi_{N,v}\}$ and $\{\varphi'_{N,v}\}$),

$$\mathbb{P}\Big(\sqrt{\tfrac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,v} - \sqrt{\tfrac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,z_B} \geq 1/\log\log N\Big) = o_N(1)\,.$$

Therefore, a union bound gives that

$$\limsup_{r\to\infty}\limsup_{N\to\infty}\mathbb{P}(E_4 \cap E_2^c) \leq \limsup_{r\to\infty}\limsup_{N\to\infty} r^4 o_N(1) = 0\,.$$

Altogether, this completes the proof of (3.15) and hence of the proposition. □

Proof of Lemma 3.0.2. Define

$$\bar{\varphi}_{N,\sigma,v} = \left(1 + \frac{\|\sigma\|_2^2}{2\log N}\right)\varphi_{N,v} \text{ for } v \in V_N\,, \text{ and } \bar{M}_{N,\sigma} = \max_{v\in V_N}\bar{\varphi}_{N,v}\,.$$

Clearly we have $\bar{M}_{N,\sigma} = (1 + \frac{\|\sigma\|_2^2}{2\log N})M_N$. Combined with (1.1), it gives that $\mathbb{E}\bar{M}_{N,\sigma} = \mathbb{E}M_N + \sigma^2\sqrt{d/2} + o(1)$ and that $d(M_N - \mathbb{E}M_N, \bar{M}_{N,\sigma} - \mathbb{E}\bar{M}_{N,\sigma}) \to 0$ as $N \to \infty$. Further define $\{\varphi^*_{N,\sigma,v} : v \in V_N\}$ as in (3.13). By the fact that the field $\{\bar{\varphi}_{N,\sigma,v}\}$ can be seen as a sum of $\{\varphi^*_{N,\sigma,v}\}$ and an independent field whose variances are $O((1/\log N)^3)$ across the field, we see that $\mathbb{E}\bar{M}_{N,\sigma} = \mathbb{E}M^*_{N,\sigma} + o(1)$ and that

$$d(\bar{M}_{N,\sigma} - \mathbb{E}\bar{M}_N, M^*_{N,\sigma} - \mathbb{E}M^*_N) \to 0\,. \tag{3.16}$$

Combined with Proposition 3.3.3, this completes the proof of the lemma. □

Proof of Lemma 3.0.3. Let ϕ and $\phi_{N,v}$ be i.i.d. standard Gaussian variables, and for $\epsilon^* > 0$ let

$$\varphi_{\mathrm{lw},N,\epsilon^*,v} = (1 - \epsilon^*/\log N)\varphi_{N,v} + \epsilon'_{N,v}\phi \text{ and } \bar{\varphi}_{\mathrm{up},N,\epsilon^*,v} = (1 - \epsilon^*/\log N)\bar{\varphi}_{N,v} + \epsilon''_{N,v}\phi_{N,v}\,,$$

where $\epsilon'_{N,v}, \epsilon''_{N,v}$ are chosen so that $\mathrm{Var}\,\varphi_{\mathrm{lw},N,\epsilon^*,v} = \mathrm{Var}\,\bar{\varphi}_{\mathrm{up},N,\epsilon^*,v} = \mathrm{Var}\,\varphi_{N,v} + \epsilon$. We can choose $\epsilon^* = \epsilon^*(\epsilon)$ with $\epsilon^* \to_{\epsilon \to 0} 0$ so that $\mathbb{E}\varphi_{\mathrm{lw},N,\epsilon^*,v}\varphi_{\mathrm{lw},N,\epsilon^*,u} \geq \mathbb{E}\bar{\varphi}_{\mathrm{up},N,\epsilon^*,v}\bar{\varphi}_{\mathrm{up},N,\epsilon^*,u}$ for all $u, v \in V_N$. By Lemma 2.2.1, one has

$$\tilde{d}(\max_{v \in V_N} \varphi_{\mathrm{lw},N,\epsilon^*,v} - m_N, \max_{v \in V_N} \bar{\varphi}_{\mathrm{up},N,\epsilon^*,v} - m_N) = 0\,.$$

Combined with Lemma 3.3.1, this completes the proof of the lemma. □

CHAPTER 4

CONVERGENCE OF RECENTERED MAXIMA

Our main result is the following theorem.

Theorem 4.0.4. *Under Assumptions (A.0), (A.1), (A.2) and (A.3), the sequence* $\{M_N - \mathbb{E}M_N\}_N$ *converges in distribution.*

As a byproduct of our proof, we also characterize the limiting law of $(M_N - m_N)$ as a Gumbel distribution with random shift, given by a positive random variable $\mathcal{Z}$ which is the weak limit of a sequence $\mathcal{Z}_N$, defined as

$$\mathcal{Z}_N = \sum_{v \in V_N} (\sqrt{2d} \log N - \varphi_{N,v}) e^{-\sqrt{2d}(\sqrt{2d} \log N - \varphi_{N,v})} . \tag{4.1}$$

In the case of BBM, the corresponding sequence $\mathcal{Z}_N$ is precisely the derivative martingale, introduced in [26]. It also occurs in the case of BRW, see [3], and plays a similar role in the study of critical Gaussian multiplicative chaos [20]. Even though in our case the sequence $\mathcal{Z}_N$ is not necessarily a martingale, in analogy with these previous situations we keep refering to it as *the derivative martingale.* The definition naturally extends to a derivative martingale *measure* on V_N by setting, for $A \subset V_N$,

$$\mathcal{Z}_{N,A} = \sum_{v \in A} (\sqrt{2d} \log N - \varphi_{N,v}) e^{-\sqrt{2d}(\sqrt{2d} \log N - \varphi_{N,v})}.$$

Theorem 4.0.5. *Suppose that Assumptions (A.0), (A.1), (A.2) and (A.3) hold. Then the derivative martingale* $\mathcal{Z}_N$ *converges in law to a positive random variable* $\mathcal{Z}$*. In addition, the limiting law* μ_∞ *of* $M_N - m_N$ *can be expressed by*

$$\mu_\infty((-\infty, x]) = \mathbb{E} e^{-\beta^* \mathcal{Z} e^{-\sqrt{2d} x}} , \text{ for all } x \in \mathbb{R} ,$$

where β^* *is a positive constant.*

Theorems 4.0.4 and 4.0.5 shall be seen as a generalization of [12, Theorems 1.1 and 2.5] and [30, Theorem 1.1] (see Remark 4.0.6 below). In fact, Theorem 4.0.4 and 4.0.5 also overlap with [8], which *effectively* studied the *conformal symmetry* (in language of [8]) for the law of the maximum of GFF in general domains— the main results in [8] were presented in terms of the intensity measure for the extremal process, but this corresponds to the law of the maximum. In terms of proof strategies, the works of [12, 8] relied heavily on the Markov field property for DGFF, and the work [30] relied crucially on the integral representation for the covariances of $*$-scale invariant field. Comparing to [12, 30, 8], our current work *aims* to study the universality aspects for the law of the maximum of log-correlated Gaussian fields under *minimal* assumptions (which is the main novelty), and notably our result manifests that Markov field property plays no role in the limiting law for the maximum.

Remark 4.0.6. *Despite the fact that our result is directly on discrete log-correlated fields, it should imply [30, Theorem 1.1] on the convergence in law for the centered supremum of $*$-scale invariant log-correlated fields (which is constructed in the continuous setting). Precisely, one could apply our result to the $*$-scale invariant field over a discretized index set and then use the smoothness of the $*$-scale invariant field.*

Remark 4.0.7. *Our proof will show that the random variable $\mathcal{Z}$ appearing in Theorem 4.0.5 depends only on the functions $f(x), h(x,y)$ appearing in Assumptions (A.2) and (A.3), while the constant β^* depends on other parameters as well. In particular, two sequences of fields that differ only at the microscopic level will have the same limit law for their centered maxima, up to a (deterministic) shift. We provide more details at the end of Section 4.*

Remark 4.0.8. *In the same spirit as the preceeding remark, if the field $\{\varphi_{N,v}\}_{v\in V_N}$ is stationary, then Assumption (A.2) can be removed, at the cost of replacing in Theorems 4.0.4 and 4.0.5 m_N by an appropriate sequence $\tilde{m}_N$ with $|m_N - \tilde{m}_N| = O(1)$. This is proved by a diagonalization procedure similar to that used for Remark 4.0.7. We omit further details.*

In [7, 8], the authors used the convergence of the centered maximum, a-priori information on the geometric properties of the clusters of near-maxima of the DGFF and a beautiful invariance argument and derived the convergence in law of the process of near extrema of the two-dimensional DGFF, and its properties. A natural extension of our work would be to study the extremal process in the class of processes studied here, and tie it to properties of the derivative martingale measure.

A word on proof strategy. This work is closely related to [12], which dealt with 2D GFF. The proof in [12] consists of three main steps:

1. Decompose the DGFF to a sum of a coarse field and a fine field (which itself is a DGFF), and further approximate the fine field as a sum of modified branching random walk (see Section 2.1 for definition) and a local DGFF. It is crucial for the proof that the different components are independent of each other, and that the approximation error is small enough so that the value of the maximum is not altered significantly. These approximations were constructed using heavily the Markov field property of DGFF, and detailed estimates for the Green function of random walk.

2. Use a modified second moment method in order to compute the asymptotics of the right tail for the distribution of the maximum of the fine field, as well as derive a limiting distribution for the location of the maximizer in the fine field.

3. Combine the limiting right tail estimates for the maximum of the fine field and the behavior of the coarse field to deduce the convergence in law.

In the general setup of logarithmically correlated fields, it is not a priori clear how can one decompose the field by an (independent) sum of a coarse field, an MBRW and a local field, as the Markov field property is no longer available. A natural approach under our assumptions is to employ the self-similarity of the fields, and to approximate the coarse and local fields by an instance of $\{\varphi_{K,v} : v \in V_K\}$ for some $K \ll N$. One difficulty in this attempt is to control the error of the approximation and its influence on the law of the maximum. In order to address this issue, we partition the box V_N to sub-boxes congruent to V_L, and borrow a key idea from [7] to show that the law of the maximum of a log-correlated fields has the following invariance property: if one adds i.i.d. Gaussian variables with variance $O(1)$ to each sub-box of the field (here the same variable will be added to each vertex in the same sub-box), where the size L of the sub-box is either K or N/K (assuming K grows to infinity arbitrarily slow in N), then the law of the maximum for the perturbed field is simply a shift of the original law where the shift can be explicitly determined (see Lemma 3.0.2). In light of this, in Section 4.1 we approximate the field $\{\varphi_{N,v}\}$ by the sum of coarse field (which is given by $\{\varphi_{KL,v} : v \in V_{KL}\}$), an MBRW, and a local field (which is given by independent copies of $\{\varphi_{K'L',v} : v \in V_{K'L'}\}$) (here the parameters satisfy $N \gg K' \gg L' \gg K \gg L$). In this construction, we make sure that the error in the covariance between two vertices is $o(1)$ if their distance is not in between L and N/L', and the error is $O(1)$ otherwise. Then we apply Lemma 3.0.2 (and Lemma 3.0.3) to argue that our approximation indeed recovers the law of the maximum for the original field. In Subsection 4.2, we present the proof for the convergence in law for the centered maximum of the approximated field we constructed and, as in [12], it readily also yields the convergence in distribution for the derivative martingale constructed from the original field.

As in the case of the DGFF in two dimensions, a number of properties for the log-correlated fields are needed, and are proved by adapting or modifying the arguments used in that case. Those properties are:

1. The tightness of $M_N - m_N$, and the bounds on the right and left tails of $M_N - m_N$ as well as certain geometric properties of maxima for the log-correlated fields under consideration, follow from modifying arguments in [15, 19, 17]. This has been shown in Chapter 2.

2. Precise asymptotics for the right tail of the distribution of the maximum of the fine field follow from arguments similar to [12] with a number of simplifications, as our fine field has a nicer structure than its analogue in [12], whereas the coarse field employed in this paper is constant over each box; in particular, there is no need to consider the distribution for the location of the maximizer in the fine field as done in [12]. The

adaption is explained in the end of the chapter.

In this chapter we assume (A.0)–(A.3) and prove Theorem 4.0.4. Toward this end, in Section 4.1 we will approximate the field $\{\varphi_{N,v} : v \in V_N\}$ by a field which is simpler to analyze, in such a way that the results of Chapter 3 apply and yield the asymptotic equivalence of their respective laws of the centered maximum. In Section 4.2 we prove the convergence in law for the centered maximum of the new field. Our method of proof yields Theorem 4.0.5 as a byproduct.

4.1 An approximation of the log-correlated Gaussian field

In this section, we approximate the log-correlated Gaussian field. Let $R_N(u, v) = E(\varphi_{N,u}\varphi_{N,v})$. We consider three scales for the approximation of the field $\{\varphi_{N,v}\}$:

1. The top (macroscopic) scale, dealing with $R_N(u, v)$ for $|u - v| \asymp N$.

2. The bottom (microscopic) scale, dealing with $R_N(u, v)$ for $|u - v| \asymp 1$.

3. The middle (mesoscopic) scale, dealing with $R_N(u, v)$ for $1 \ll |u - v| \ll N$.

By Assumptions (A.2) and (A.3), $R_N(u, v)$, properly centered, converges in the top and bottom scale. So in those scales, we approximate $\{\varphi_{N,u}\}$ by the corresponding "limiting" fields. In the middle scale, we simply approximate $\{\varphi_{N,u}\}$ by the MBRW. One then expects that this approximation gives an additive $o(1)$ error for $R_N(u, v)$ in the top and bottom scale, and an additive $O(1)$ error in the middle scale. It turns out that this guarantees that the limiting laws of the centered maxima coincide.

In what follows, for any integer t we refer to a box of side length t as an t-box. Take two large integers $L = 2^\ell$ and $K = 2^k$. Consider first $\{\varphi_{KL,u} : u \in V_{KL}\}$ in a KL-box whose left-bottom corner is identified as the origin, and let Σ denote its covariance matrix.

Recall that by Proposition 1.4.1, with probability tending to 1 as $N \to \infty$, the maximum of $\varphi_{N,v}$ over V_N occurs in a sub-box of V_N with side length $\lfloor N/KL \rfloor \cdot KL$. Therefore, one may neglect the maximization over the indices in $V_N \setminus V_{\lfloor N/KL \rfloor \cdot KL}$. For notational convenience, we will assume throughout that KL divides N in what follows.

We use Σ to approximate the macroscopic scale of $R_N(u,v)$, as follows. Partition V_N into a disjoint union of boxes of side length N/KL, denoted $\mathcal{B}_{N/KL} = \{B_{N/KL,i} : i = 1, \ldots, (KL)^d\}$. Let $v_{N/KL,i}$ be the left bottom corner of box $B_{N/KL,i}$ and write $w_i = \frac{v_{N/KL,i}}{N/KL}$. Let Ξ^c be a matrix of dimension $N^d \times N^d$ such that $\Xi^c_{u,v} = \Sigma_{w_i,w_j}$ for $u \in B_{N/KL,i}$ and $v \in B_{N/KL,j}$. Note that Ξ^c is a positive definite matrix with diagonal terms $\log(KL) + O_{KL}(1)$.

Next, take two other integers $K' = 2^{k'}$ and $L' = 2^{\ell'}$. As above, we assume that $K'L'$ divides N. Consider $\{\varphi_{K'L',u} : u \in V_{K'L'}\}$ in a $K'L'$-box whose left-bottom corner is identified as the origin, and denote by Σ' the covariance matrix for $\{\varphi_{K'L',u} : u \in V_{K'L'}\}$. As above, assume for notational convenience that $K'L'$ divides N. Partition V_N into a disjoint union of boxes of side length $K'L'$, denoted $\mathcal{B}_{K'L'} = \{B_{K'L',i} : i = 1, \ldots, (N/K'L')^d\}$. Let $v_{K'L',i}$ be the left bottom corner of $B_{K'L',i}$. Let Ξ^b be a matrix of dimension $N^d \times N^d$ so that

$$\Xi^b_{u,v} = \begin{cases} \Sigma'_{u-v_{K'L',i},v-v_{K'L',i}}, & u,v \in B_{K'L',i} \\ 0, & u \in B_{K'L',i}, v \in B_{K'L',j}, i \neq j \end{cases}.$$

Note that Ξ^b is a positive definite matrix with diagonal terms $\log(K'L') + O_{K'L'}(1)$.

Let $\{\xi^c_{N,v} : v \in V_N\}$ be a Gaussian field with covariance matrix Ξ^c, which we occasionally refer to as the coarse field, and let $\{\xi^b_{N,v} : v \in V_N\}$ be a Gaussian field with covariance matrix Ξ^b, which we occasionally refer to as the bottom field. Note that the coarse field is constant in each box $B_{N/KL,i}$, and the bottom fields in different boxes $B_{K'L',i}$ are independent of each other.

We will consider the limits when L, K, L', K' are sent to infinity in that order. In what follows, we denote by $(L, K, L', K') \Rightarrow \infty$ sending these parameters to infinity in the order of K', L', K, L (so $K' \gg L' \gg K \gg L$).

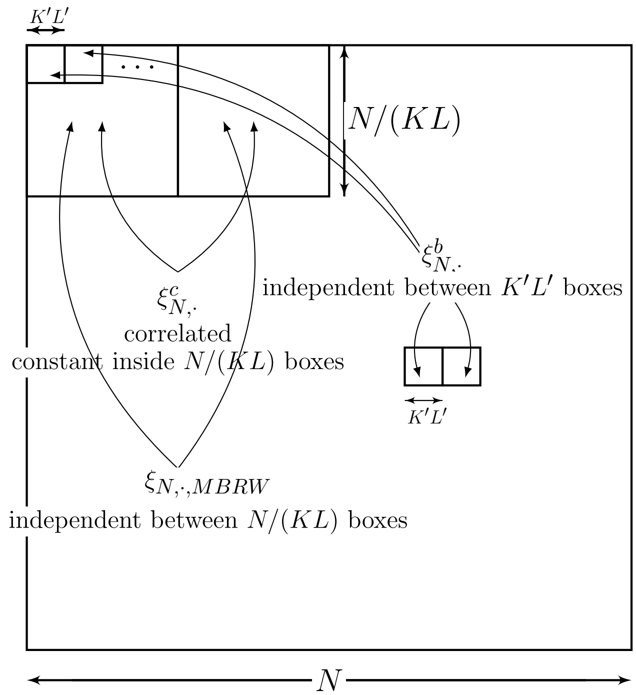

Figure 4.1: Hierarchy of construction of the approximating Gaussian field

Finally, we give the MBRW approximation for the mesoscopic scales. Recall the definitions of $\mathcal{B}_j^N$ and $\mathcal{B}_j(v)$ in Subsection 2.1, and recall that $\{b_{i,k,B} : k \geq 0, 1 \leq i \leq (KL)^d, B \in \mathcal{B}_k^N\}$ is a family of independent Gaussian variables such that $\operatorname{Var} b_{i,j,B} = \log 2 \cdot 2^{-dj}$ for all $B \in \mathcal{B}_j^N$ and $1 \leq i \leq (KL)^d$. For $v \in B_{N/KL,i} \cap B_{K'L',i'}$ (where $i = 1, \ldots, (KL)^d$ and $i' = 1, \ldots, (N/K'L')^d$), define

$$\xi_{N,v,\mathrm{MBRW}} = \sum_{j=\ell'+k'}^{n-k-\ell} \sum_{B \in \mathcal{B}_j(v_{K'L',i'})} b_{i,j,B}^N \,. \tag{4.2}$$

Note that by our construction $\{\xi_{N,v,\mathrm{MBRW}} : v \in B_{N/KL,i}\}$ are independent of each other for $i = 1, \ldots, (KL)^d$, and in addition $\xi_{N,\cdot,\mathrm{MBRW}}$ is constant over each $K'L'$-box. Further, let $\{\xi_{N,v}^c : v \in V_N\}$, $\{\xi_{N,v}^b : v \in V_N\}$ and $\{\xi_{N,v,\mathrm{MBRW}} : v \in V_N\}$ be independent of each other. One has by Assumption (A.1) that

$$|\operatorname{Var}(\xi_{N,v}^c + \xi_{N,v}^b + \xi_{N,v,\mathrm{MBRW}}) - \operatorname{Var} \varphi_{N,v}| \leq 4\alpha \,.$$

Let $a_{N,v}$ be a sequence of numbers such that for all $v \in B_{N/KL,i}$ and all $1 \le i \le (KL)^d$,

$$\operatorname{Var}(\xi^c_{N,v} + \xi^b_{N,v} + \xi_{N,v,\mathrm{MBRW}}) + a^2_{N,v} = \operatorname{Var}\varphi_{N,v} + 4\alpha\,. \tag{4.3}$$

(Here, the sequence $a_{N,v}$ implicitly depends on (KL).) It is clear that

$$\max_{v \in V_N} a_{N,v} \le \sqrt{8\alpha}\,. \tag{4.4}$$

For $v \in B_{N/KL,i}$ and $v \equiv \bar{v} \mod K'L'$, one has

$$\begin{aligned} a^2_{N,v} &= \operatorname{Var}\varphi_{N,v} + 4\alpha - \operatorname{Var}\varphi_{KL,w_i} - \operatorname{Var}\varphi_{K'L',\bar{v}} - \log(\tfrac{N}{KLK'L'}) \\ &= \log N - \log(KL) + \epsilon_{N,KL,K'L'} + 4\alpha - \operatorname{Var}\varphi_{K'L',\bar{v}} - \log(\tfrac{N}{KLK'L'}) \ge 0, \end{aligned}$$

where, by Assumptions (A.2),

$$\limsup_{(L,K,L',K') \Rightarrow \infty} \limsup_{N \to \infty} \epsilon_{N,KL,K'L'} = 0\,. \tag{4.5}$$

Therefore, one can write

$$a^2_{N,v} = a^2_{K',L',\bar{v}} + \epsilon_{N,KL,K'L'}\,, \tag{4.6}$$

where $a_{K'L',\bar{v}}$ depends on $(K'L', \bar{v})$. By Assumption (A.2) and the continuity of f, one has

$$\limsup_{(L,K,L',K') \Rightarrow \infty} \sup_{u,v: \|u-v\|_\infty \le L'} \limsup_{N \to \infty} |\operatorname{Var}\xi^b_{N,v} - \operatorname{Var}\xi^b_{N,u}| = 0.$$

Therefore, we can further require that

$$|a_{K'L',\bar{v}} - a_{K'L',\bar{u}}| \le \epsilon_{N,KL,K'L'} \text{ for all } \|\bar{v} - \bar{u}\|_\infty \le L'\,. \tag{4.7}$$

Let ϕ_j be i.i.d. standard Gaussian variables. For $v \in B_{K'L',j}$ and $v \equiv \bar{v} \mod K'L'$, define

$$\xi_{N,v} = \xi^c_{N,v} + \xi^b_{N,v} + \xi_{N,v,\mathrm{MBRW}} + a_{K'L',\bar{v}}\phi_j \,. \tag{4.8}$$

It follows from (4.3) and (4.6) that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} |\operatorname{Var}\xi_{N,v} - \operatorname{Var}\varphi_{N,v} - 4\alpha| = 0 \,. \tag{4.9}$$

Finally, we partition V_N into a disjoint union of boxes of side length N/L which we denote by $\mathcal{B}_{N/L} = \{B_{N/L,i} : 1 \le i \le L^d\}$, as well as a disjoint union of boxes of side length L which we denote by $\mathcal{B}_L = \{B_{L,i} : 1 \le i \le (N/L)^d\}$. Again, we denote by $v_{N/L,i}$ and $v_{L,i}$ the left bottom corner of the boxes $B_{N/L,i}$ and $B_{L,i}$, respectively.

For $\delta > 0$ and any box B, denote by $B^\delta \subseteq B$ the collection of all vertices in B that are $\delta\ell_B$ away from its boundary ∂B (here ℓ_B is the side length of B). Let

$$V^*_{N,\delta} = (\cup_i B^\delta_{N/L,i}) \cap (\cup_i B^\delta_{N/KL,i}) \cap (\cup_i B^\delta_{L,i}) \cap (\cup_i B_{KL,i}) \,.$$

One has $|V^*_{N,\delta}| \ge (1 - 100d\delta)|V_N|$.

The following lemma suggests that $\{\xi_{N,v} : v \in V_N\}$ is a good approximation of $\{\varphi_{N,v} : v \in V_N\}$.

Lemma 4.1.1. *Let Assumptions (A.1), (A.2) and (A.3) hold. Then there exist* $\epsilon'_{N,K,L,K',L'} > 0$ *with* $\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \epsilon'_{N,K,L,K',L'} = 0$*, such that the following hold for all* $u, v \in V^*_{N,\delta}$ *:*

(a) If $u, v \in B_{L',i}$ *for some* $1 \le i \le (N/L')^d$*, then* $|\mathbb{E}(\xi_{N,u} - \xi_{N,v})^2 - \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2| \le \epsilon'_{N,K,L,K',L'}$*.*

(b) If $u \in B_{N/L,i}$*,* $v \in B_{N/L,j}$ *with* $i \neq j$*, then* $|\mathbb{E}\xi_{N,u}\xi_{N,v} - \mathbb{E}\varphi_{N,v}\varphi_{N,u}| \le \epsilon'_{N,K,L,K',L'}$*.*

(c) Otherwise, $|\mathbb{E}\xi_{N,u}\xi_{N,v} - \mathbb{E}\varphi_{N,v}\varphi_{N,u}| \le 4\log(1/\delta) + 40\alpha$*.*

Proof. **(a):** Let i' be such that $B_{L',i} \subseteq B_{K'L',i'}$. By (4.7) and (4.8), one has

$$|\mathbb{E}(\xi_{N,u} - \xi_{N,v})^2 - \mathbb{E}(\varphi_{KL,u-v_{KL,i'}} - \varphi_{KL,v-v_{KL,i'}})^2| \leq 4\epsilon_{N,KL,K'L'} ,$$

where $\epsilon_{N,KL,K'L'}$ satisfies (4.5) (and was defined therein). By Assumption (A.2), one has

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} |\mathbb{E}(\varphi_{KL,u-v_{KL,i'}} - \varphi_{KL,v-v_{KL,i'}})^2 - \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2| = 0 .$$

Altogether, this completes the proof for (a).

(b): Let i', j' be such that $u \in B_{N/KL,i'}$ and $v \in B_{N/KL,j'}$, and assume w.l.o.g. that $K' \gg L' \gg K \gg L \gg 1/\delta$. The definition of $\{\xi_{N,v}\}$ gives

$$\mathbb{E}\xi_{N,v}\xi_{N,u} = \mathbb{E}\varphi_{KL,w_{i'}}\varphi_{KL,w_{j'}}$$

where $w_{i'} = \frac{v_{N/KL,i'}}{N/KL}$ and $w_{j'} = \frac{v_{N/KL,j'}}{N/KL}$. In this case, we have $|w_{i'} - w_{j'}| \geq \delta K$. Writing $x_u = u/N, x_v = v/N$ and $y_u = w_{i'}/KL, y_v = w_{j'}/KL$, one obtains

$$|y_u - y_v| \geq \delta/L, |x_u - x_v| \geq \delta/L, |x_u - y_u| \leq 1/K, |x_v - y_v| \leq 1/K .$$

Therefore, Assumption (A.3) yields

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} |\mathbb{E}\xi_{N,u}\xi_{N,v} - \mathbb{E}\varphi_{N,v}\varphi_{N,u}| = 0,$$

completing the proof of (b).

(c). In this case, one has

$$\begin{aligned}\mathbb{E}\xi_{N,v}\xi_{N,u} =& \mathbb{E}\xi^c_{N,v}\xi^c_{N,u} + \mathbb{E}\xi^b_{N,v}\xi^b_{N,u} + \mathbb{E}\xi_{N,u,\mathrm{MBRW}}\xi_{N,v,\mathrm{MBRW}} + \mathrm{err}_1 \\ =& \log KL - \log_+\left(\tfrac{|u-v|}{N/KL}\right) \\ &+ \mathbf{1}_{|u-v|\le N/KL}\left(\log \tfrac{N}{(KLK'L')} - \log_+ \tfrac{|u-v|}{K'L'}\right) + \mathrm{err}_2 \\ =& \log N - \log_+ |u-v| + \mathrm{err}_2,\end{aligned}$$

where $|\mathrm{err}_1| \le 8\alpha$ and $|\mathrm{err}_2| \le 2\log 1/\delta + 20\alpha$. Combined with Assumption (A.1), this completes the proof of (c) and hence of the lemma. □

Lemma 4.1.2. *Let Assumptions (A.0), (A.1), (A.2) and (A.3) hold. Then,*

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} d(M_N - m_N, \max_{v\in V_N} \xi_{N,v} - m_N - 2\alpha\sqrt{2d}) = 0\,.$$

Proof. By Proposition 1.4.1, it suffices to show that for all $\delta > 0$

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \limsup_{N\to\infty} d(\max_{v\in V^*_{N,\delta}} \varphi_{N,v} - m_N, \max_{v\in V^*_{N,\delta}} \xi_{N,v} - m_N - 2\alpha\sqrt{2d}) = 0\,.$$

Consider a fixed $\delta > 0$. Let $\sigma_*^2 = 4\log(1/\delta) + 60\alpha$. Let $\sigma_{\mathrm{lw}} = (0, \sqrt{\sigma_*^2 + 4\alpha})$ and $\sigma_{\mathrm{up}} = (\sigma^*, 0)$. Define $\{\tilde\varphi_{N,L',L,\sigma_{\mathrm{lw}},v} : v \in V_N\}$ as in (3.1) with $r_1 = L'$, $r_2 = L$ and $\sigma = \sigma_{\mathrm{lw}}$. Analogously, define $\{\tilde\xi_{N,L',L,\sigma_{\mathrm{up}},v} : v \in V_N\}$. By (4.8) and Lemma 4.1.1, one has for all $u, v \in V^*_{N,\delta}$,

$$\begin{aligned}&|\operatorname{Var}\tilde\varphi_{N,L',L,\sigma_{\mathrm{lw}},v} - \operatorname{Var}\tilde\xi_{N,L',L,\sigma_{\mathrm{up}},v}| \le \bar\epsilon_{N,K,L,K',L'}\,, \\ &\qquad\qquad \mathbb{E}\tilde\xi_{N,L,\sigma_{\mathrm{up}},v}\tilde\xi_{N,L,\sigma_{\mathrm{up}},u} \le \mathbb{E}\tilde\varphi_{N,L,\sigma_{\mathrm{lw}},v}\tilde\varphi_{N,L,\sigma_{\mathrm{lw}},u} + \bar\epsilon_{N,K,L,K',L'}\,,\end{aligned}$$

where $\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \bar\epsilon_{N,K,L,K',L'} = 0$. Since $\{\tilde\varphi_{N,L',L,\sigma_{\mathrm{lw}},v} : v \in V^*_{N,\delta}\}$ satisfies Assumption (A.1) with α being replaced by $10\alpha + \sigma_*^2$, one may apply Lemma 3.0.3

and obtain that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty} \tilde{d}(\max_{v\in V^*_{N,\delta}} \tilde{\varphi}_{N,L',L,\sigma_{\mathrm{lw}},v} - m_N, \max_{v\in V^*_{N,\delta}} \tilde{\xi}_{N,L',L,\sigma_{\mathrm{up}},v} - m_N) = 0\,.$$

By Lemma 3.0.2 (it is clear that the same statement holds for maximum over $V^*_{N,\delta}$), one gets

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \overline{\lim_{N\to\infty}}\, d(\max_{v\in V^*_{N,\delta}} \tilde{\varphi}_{N,L',L,\sigma_{\mathrm{lw}},v} - m_N - (\sigma_*^2+4\alpha)\sqrt{\tfrac{d}{2}}, \max_{v\in V_{N,\delta^*}} \varphi_{N,v} - m_N) = 0,$$

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \overline{\lim_{N\to\infty}}\, d(\max_{v\in V^*_{N,\delta}} \tilde{\xi}_{N,L',L,\sigma_{\mathrm{up}},v} - m_N - (\sigma_*^2)\sqrt{\tfrac{d}{2}}, \max_{v\in V_{N,\delta^*}} \xi_{N,v} - m_N) = 0\,.$$

Altogether, this gives that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty} \tilde{d}(\max_{v\in V^*_{N,\delta}} \varphi_{N,v} - m_N, \max_{v\in V^*_{N,\delta}} \xi_{N,v} - m_N - 2\alpha\sqrt{2d}) = 0\,.$$

The other direction of stochastic domination follows in the same manner. Altogether, this completes the proof of the lemma. □

4.2 Convergence in law for the centered maximum

In light of Lemma 4.1.2, in order to prove Theorem 4.0.4 it remains to show the convergence in law for the centered maximum of $\{\xi_{N,v} : v \in V_N\}$. To this end, we will follow the proof of the convergence in law in the case of the 2D DGFF given in [12]. Let the *fine field* be defined as $\xi^f_{N,v} = \xi_{N,v} - \xi^c_{N,v}$, and note that it implicitly depends on $K'L'$. As in [12], a key step in the proof of convergence of the centered maximum is the following sharp tail estimate on the right tail of the distribution of $\max_{v\in B} \xi^f_{N,v}$ for $B \in \mathcal{B}_{N/KL}$. The proof of this estimate is postponed to Section 4.4.

Proposition 4.2.1. *Let Assumptions (A.1), (A.2) and (A.3) hold. Then there exist con-*

stants $C_\alpha, c_\alpha > 0$ depending only on α and constansts $c_\alpha \le \beta^*_{K',L'} \le C_\alpha$ such that

$$\lim_{z\to\infty} \limsup_{L'\to\infty} \limsup_{K'\to\infty} \limsup_{N\to\infty} |z^{-1} e^{\sqrt{2d}z} \mathbb{P}(\max_{v\in B_{N/KL,i}} \xi^f_{N,v} \ge m_{N/KL} + z) - \beta^*_{K',L'}| = 0. \quad (4.10)$$

Remark 4.2.2. *Proposition 4.2.1 is analogous to [12, Proposition 4.1], but there are two important differences:*

1. *In Proposition 4.2.1 the convergence is to a constant $\beta^*_{K',L'}$ which depends on K', L', while in [12, Proposition 4.1] the convergence is to an absolute constant α^*. This is because the fine field $\xi_{N,v}$ here implicitly depends on K', L', and thus a priori one is not able to eliminate the dependence on (K', L') from the limit. However, in the same spirit as in [12], the dependence on (K', L') is not an issue for deducing a convergence in law — the crucial requirement is the independence of N. Eventually, we will deduce the convergence of $\beta^*_{K',L'}$ as $K', L' \to \infty$ in that order from the convergence in law of the centered maximum.*

2. *In [12, Proposition 4.1], one also controls the limiting distribution of the location of the maximizer while in Proposition 4.2.1 this is not mentioned. This is because in the current situation and unlike the construction in [12], the coarse field $\{\xi^c_{N,v}\}$ is constant over each box $B_{N/KL,i}$, and thus the location of the maximizer of the fine field in each of the boxes $B_{N/KL,i}$ is irrelevant to the value of the maximum for $\{\xi_{N,v}\}$.*

Next, we construct the limiting law of the centered maximum of $\{\xi_{N,v} : v \in V_N\}$. We partition $[0,1]^d$ into $R = (KL)^d$ disjoint boxes of equal sizes. Let $\beta^*_{K',L'}$ be as defined in the statement of Proposition 4.2.1. By that proposition, there exists a function $\gamma : \mathbb{R} \mapsto \mathbb{R}$ that grows to infinity arbitrarily slowly (in particular, we may assume that $\gamma(x) \le \log\log\log x$) such that

$$\lim_{z'\to\infty} \overline{\lim}_{L'} \overline{\lim}_{K'} \overline{\lim}_{N} \sup_{z'\le z\le \gamma(K'L')} |z^{-1} e^{\sqrt{2d}z} \mathbb{P}(\max_{v\in B_{N/KL,i}} \xi^f_{N,v} \ge m_{N/KL} + z) - \beta^*_{K',L'}| = 0,$$

with each of the limsups with respect to the corresponding independent variables tending to infinity.

Let $\{\varrho_{R,i}\}_{i=1}^{R}$ be independent Bernoulli random variables with

$$\mathbb{P}(\varrho_{R,i} = 1) = \beta^*_{K',L'}\gamma(KL)e^{-\sqrt{2d}\gamma(KL)}\,.$$

In addition, consider independent random variables $\{Y_{R,i}\}_{i=1}^{R}$ such that

$$\mathbb{P}(Y_{R,i} \geqslant x) = \tfrac{\gamma(KL)+x}{\gamma(KL)}e^{-\sqrt{2d}x} \qquad x \geqslant 0. \tag{4.11}$$

Let $\{Z_{R,i} : 1 \le i \le R\}$ be an independent Gaussian field with covariance matrix Σ (recall that Σ is of dimension $R \times R$). We then define

$$G^*_{K,L,K',L'} = \max_{1\le i\le R, \varrho_{R,i}=1} G_{R,i} \text{ where } G_{R,i} = \varrho_{R,i}(Y_{R,i} + \gamma(KL)) + Z_{R,i} - \sqrt{2d}\log(KL)$$

(here we use the convention that $\max\emptyset = 0$). Let $\bar{\mu}_{K,L,K',L'}$ be the distribution of $G^*_{K,L,K',L'}$. We note that $\bar{\mu}_{K,L,K',L'}$ does not depend on N.

Theorem 4.2.3. *Let Assumptions (A.0), (A.1), (A.2) and (A.3) hold. Then,*

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty} d(\mu_N, \bar{\mu}_{K,L,K',L'}) = 0, \tag{4.12}$$

where μ_N *is the law of* $\max_{v\in V_N} \xi_{N,v} - m_N$.

(Note that μ_N does depend on $KL, K'L'$.)

Proof of Theorem 4.0.4. Theorem 4.0.4 follows from Lemma 4.1.2 and Theorem 4.2.3. □

Next, we give the proof of Theorem 4.2.3. Our proof is conceptually simpler than that of its analogue [12, Theorem 2.4], since our coarse field is constant over a box of size N/KL (and thus no consideration of the location for the maximizer in the fine field is needed).

Proof of Theorem 4.2.3. Denote by $\tau = \arg\max_{v\in V_N} \xi_{N,v}$. Applying Theorem 2.0.4 to the Gaussian fields $\{\xi_{N,v} : v \in V_N\}$ and $\{\xi^c_{N,v} : v \in V_N\}$ (where the maximum of $\{\xi^c_{N,v} : v \in V_N\}$ is equivalent to the maximum of a log-correlated Gaussian field in a KL-box), we deduce that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\xi^f_{N,\tau} \geq m_{N/KL} + \gamma(KL) + 1) = 1\,. \tag{4.13}$$

Therefore, in what follows, we assume w.l.o.g. the occurrence of the event

$$\{\xi^f_{N,\tau} \geq \sqrt{2d}\log\tfrac{N}{KL} - \tfrac{3}{2\sqrt{2d}}\log\log\tfrac{N}{KL} + \gamma(KL) + 1\}\,.$$

Let $\mathcal{E} = \cup_{1\leq i\leq R}\{\max_{v\in B_{N/KL,i}} \xi^f_{N,v} \geq m_{N/KL} + KL + 1\}$. A simple union bound over i gives that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\mathcal{E}) = 0\,. \tag{4.14}$$

Thus in what follows we assume without loss that $\mathcal{E}$ does not occur. Analogously, we let $\mathcal{E}' = \cup_{1\leq i\leq R}\{Y_{R,i} \geq KL + 1 - \gamma(KL)\}$. We see from the union bound that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\mathcal{E}') = 0\,. \tag{4.15}$$

In what follows, we assume without loss that $\mathcal{E}'$ does not occur.

For convenience of notation, we denote by

$$M^f_{N,i} = \max_{v\in B_{N/KL,i}} \xi^f_{N,v} - (m_{N/KL} + \gamma(KL))\,.$$

By Proposition 4.2.1, there exists $\epsilon^* = \epsilon^*(N, K, L, K', L')$ with

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \epsilon^*(N, K, L, K', L') = 0\,,$$

such that for some $|\epsilon^\diamond| \le \epsilon^*/4$

$$\mathbb{P}(\epsilon^\diamond \le M^f_{N,i} \le KL - \gamma(KL) + 1) = \mathbb{P}(\varrho_{R,i} = 1, Y_{R,i} \le KL - \gamma(KL) + 1)\,,$$

and that for all $-1 \le t \le KL - \gamma(KL) + 1$

$$\mathbb{P}(\varrho_{R,i=1}, Y_{R,i} \le t - \epsilon^*/2) \le \mathbb{P}(\epsilon^\diamond \le M^f_{N,i} \le t) \le \mathbb{P}(\varrho_{R,i=1}, Y_{R,i} \le t + \epsilon^*/2)\,.$$

Therefore, there exists a coupling between $\{M^f_{N,i} : 1 \le i \le R\}$ and $\{\varrho_i, Y_{R,i} : 1 \le i \le R\}$ such that on the event $(\mathcal{E} \cup \mathcal{E}')^c$,

$$\varrho_{R,i} = 1, |Y_{R,i} - M^f_{N,i}| \le \epsilon^* \text{ if } M^f_{N,i} \ge \epsilon^*\,, \text{ and } |Y_{R,i} - M^f_{N,i}| \le \epsilon^* \text{ if } \varrho_{R,i} = 1\,. \qquad (4.16)$$

In addition, it is trivial to couple such that $\xi^c_{N,v} = Z_{R,i}$ for all $v \in B_{N/KL,i}$ and $1 \le i \le R$. Also, notice the following simple fact

$$\limsup_{L\to\infty} \limsup_{K\to\infty} \limsup_{N\to\infty} (m_N - m_{N/KL} - \sqrt{2d}\log(KL)) = 0\,.$$

Altogether, we conclude that there exists a coupling such that outside an event of probability tending to 0 as $N \to \infty$ and then $(L, K, L', K') \Rightarrow \infty$ (c.f. (4.13), (4.14), (4.15)) we have

$$\max_{v \in V_N} (\xi_{N,v} - m_N) - G^*_{K,L,K',L'} \le 2\epsilon^*\,.$$

Now, let $\tau' = \arg\max_{1 \le i \le R} G_{R,i}$. Applying Theorem 2.0.4 to the Gaussian field $\{Z_{R,i}\}$ and using the preceding inequality, we see that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\varrho_{R,\tau'} = 1) = 1\,. \qquad (4.17)$$

Combined with (4.16), this yields that there exists a coupling such that except with probability tending to 0 as $N \to \infty$ and then $(L, K, L', K') \Rightarrow \infty$ we have

$$|\max_{v \in V_N} (\xi_{N,v} - m_N) - G^*_{K,L,K',L'}| \leq 2\epsilon^* .$$

thereby completing the proof of Theorem 4.2.3. $\square$

Proof of Theorem 4.0.5. Recall that $G^*_{K,L,K',L'}$ is a random variable with law $\bar{\mu}_{K,L,K',L'}$. We will construct random variables $\mathcal{Z}_{K,L}$, measurable with respect to $\mathcal{F}^c := \sigma(\{Z_{R,i}\})$, so that

$$\limsup_{(L,K,L',K') \Rightarrow \infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}(\mathrm{e}^{-\beta^*_{K',L'} \mathcal{Z}_{K,L} \mathrm{e}^{-\sqrt{2d}x}})} = \liminf_{(L,K,L',K') \Rightarrow \infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}(\mathrm{e}^{-\beta^*_{K',L'} \mathcal{Z}_{K,L} \mathrm{e}^{-\sqrt{2d}x}})} = 1 . \quad (4.18)$$

for all x. To demonstrate (4.18), due to (4.17), we may and will assume without loss that $\varrho_{R,\tau'} = 1$. Define $S_{R,i} := \sqrt{2d} \log(KL) - Z_{R,i}$. Then, for any real x,

$$\mathbb{P}(G^*_{K,L,K',L'} \leq x) = \mathbb{E}\left(\prod_{i=1}^{R} \left(1 - \mathbb{P}(\varrho_{R,i} Y_{R,i} > S_{R,i} + x - \gamma(KL) \,|\, \mathcal{F}^c)\right) \right) . \quad (4.19)$$

In addition, the union bound gives that

$$\limsup_{KL \to \infty} \mathbb{P}(\mathcal{D}) = 1 \text{ where } \mathcal{D} = \{\min_{1 \leq i \leq R} S_{R,i} \geq 2\gamma(KL)\} .$$

So in the sequel we assume that $\mathcal{D}$ occurs. By the definition of $\varrho_{R,i}$ and $Y_{R,i}$, we get that

$$\mathbb{P}(\varrho_{R,i} Y_{R,i} > S_{R,i} + x - \gamma(KL) \,|\, \mathcal{F}^c) = \beta^*_{K',L'}(S_{R,i} + x) e^{-\sqrt{2d}(S_{R,i}+x)} \to 0 \text{ as } KL \to \infty .$$

Therefore,

$$\begin{aligned} \exp(-(1+\epsilon_{K,L}) \beta^*_{K',L'} S_{R,i} \mathrm{e}^{-\sqrt{2d}(x+S_{R,i})}) \;&\leq\; \mathbb{P}(\varrho_{R,i} Y_{R,i} \leq S_{R,i} + x - \gamma(KL) \,|\, \mathcal{F}^c) \\ &\leq \exp(-(1-\epsilon_{K,L}) \beta^*_{K',L'} S_{R,i} \mathrm{e}^{-\sqrt{2d}(x+S_{R,i})}) , \end{aligned} \quad (4.20)$$

for $\epsilon_{K,L} > 0$ with

$$\limsup_{KL\to\infty} \epsilon_{K,L} = 0.$$

Define $\mathcal{Z}_{K,L} = \sum_{i=1}^{R} S_{R,i} e^{-\sqrt{2d} S_{R,i}}$ (this is the analogue of a derivative martingale, see (4.1)). Substituting (4.20) into (4.19) completes the proof of (4.18). Now, combining (4.18) and Theorem 4.2.3, we see that we necessarily have

$$\limsup_{K'\to\infty} \limsup_{L'\to\infty} |\beta^*_{K',L'} - \beta^*| = 0$$

for a number β^* that does not depend on (K', L'). Plugging the preceding inequality into (4.18), we deduce that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}(\mathrm{e}^{-\beta^* \mathcal{Z}_{K,L} \mathrm{e}^{-\sqrt{2d}x}})} = \liminf_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}(\mathrm{e}^{-\beta^* \mathcal{Z}_{K,L} \mathrm{e}^{-\sqrt{2d}x}})} = 1\,. \qquad (4.21)$$

Combining (4.21) with Theorem 4.2.3 again, we see that $\mathcal{Z}_{K,L}$ converges weakly to a random variable $\mathcal{Z}$ as $K \to \infty$ and then $L \to \infty$. Also note that $\mathcal{Z}_{K,L}$ depends only on the product KL. Therefore, this implies that $\mathcal{Z}_N$ converges weakly to a random variable $\mathcal{Z}$. From the tightness of the laws $\bar{\mu}_{K,L,K',L'}$, it follows that $\mathcal{Z} > 0$ a.s. This completes the proof of Theorem 4.0.5. □

Proof of Remark 4.0.7. Consider two sequences $\{\varphi_{N,v}\}$ and $\{\tilde{\varphi}_{N,v}\}$ that satisfy assumptions (A.0)–(A.3) with the same functions $h(x, y)$ and $f(x)$ but possibly different functions $g(u, v), \tilde{g}(u, v)$ and different constants $\alpha^{(\delta)}, \alpha^{(\delta),'}$ and α_0, α_0'. Introduce the corresponding fields

$$\xi_{N,KL,K'L'} = \xi^c_{N,KL,K'L'} + \xi^f_{N,KL,K'L'}\,, \quad \tilde{\xi}_{N,KL,K'L'} = \tilde{\xi}^c_{N,KL,K'L'} + \tilde{\xi}^f_{N,KL,K'L'}\,,$$

see Section 4.1. Set also

$$\hat{\xi}_{N,KL,K'L'} = \tilde{\xi}^c_{N,KL,K'L'} + \xi^f_{N,KL,K'L'}\,.$$

Let $\nu_N, \tilde{\nu}_N$ denote the laws of the centered maxima $\max_{v\in V_N}\varphi_{N,v} - m_N$, $\max_{v\in V_N}\tilde{\varphi}_{N,v} - \tilde{m}_N$, and let $\mu_N, \tilde{\mu}_N, \hat{\mu}_N$ denote the laws of the centered maxima of the $\xi_N, \tilde{\xi}_N, \hat{\xi}_N$ fields. (Recall that the latter depend also on $KL, K'L'$ but we drop that fact from the notation.) By Lemma 4.1.2, we have

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\left(d(\mu_N,\nu_N) + d(\tilde{\mu}_N,\tilde{\nu}_N)\right) = 0\,. \tag{4.22}$$

For $s \in \mathbb{R}$, let $\theta_s\mu$ denote the shift of a probability measure μ on $\mathbb{R}$, that is $\theta_s\mu(A) = \mu(A+s)$ for any measurable set A. Recall the construction of $\bar{\mu}_{K,L,K',L'}$, see Theorem 4.2.3, and construct similarly $\tilde{\mu}_{K,L,K',L'}$ and $\hat{\mu}_{K,L,K',L'}$. Note that, by construction, there exists $s = s(KL)$, bounded uniformly in KL, so that $\theta_s\hat{\mu}_{K,L,K',L'} = \tilde{\mu}_{K,L,K',L'}$. In particular, from Theorem 4.2.3 we get that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\Big(d(\mu_N,\bar{\mu}_{K,L,K',L'}) + d(\tilde{\mu}_N,\theta_s\hat{\mu}_{K,L,K',L'})\Big) = 0\,. \tag{4.23}$$

From (4.22) and (4.23), one can find a sequence $L(N), K(N), K'(N), L'(N)$ along which the convergence still holds (as $N\to\infty$). Let $\{\eta_{v,N}\}$ and $\{\hat{\eta}_{v,N}\}$ denote the fields $\{\xi_{v,N}\}$ and $\{\hat{\xi}_{v,N}\}$ with this choice of parameters, and let $\bar{\mu}_N$ and $\hat{\mu}_N$ denote the corresponding laws of the maximum. Let μ_∞, $\tilde{\mu}_\infty$ denote the limits of μ_N and $\tilde{\mu}_N$, which exist by theorem 4.0.4. From the above considerations we have that $\bar{\mu}_N \to \mu_\infty$ and $\theta_{s(N)}\hat{\mu}_N \to \tilde{\mu}_\infty$. On the other hand, the fields $\eta_{N,\cdot}$ and $\hat{\eta}_{N,\cdot}$ both satisfy assumptions (A.0)-(A.3) with the same functions f, g, h and thus, interleaving between then one deduces that the laws of their centered maxima converge to the same limit, denoted Θ_∞. It follows that necessarily, $s(N)$ converges and $\mu_\infty = \theta_s\tilde{\mu}_\infty = \Theta_\infty$. Using the characterization in Theorem 4.0.5, this yields the claim in the remark. □

4.3 An example: the circular logarithmic REM

In the important paper [23], the authors introduce a one dimensional logarithmically correlated Gaussian field, which they call the *circular logarithmic REM* (CLREM). Fyodorov and Bouchaud consider the CLREM as a prototype for Gaussian fields exhibiting Carpentier-LeDoussal freezing. (We do not discuss here the notion of freezing, referring instead to [23] and to [33].) Explicitly, fix an integer N, set $\theta_k = 2\pi k/N$, and introduce the matrix

$$R_{k,\ell} = -\frac{1}{2}\log\left(4\sin^2\left(\frac{\theta_k-\theta_\ell}{2}\right)\right)\mathbf{1}_{k\neq\ell} + (\log N + W)\mathbf{1}_{k=\ell}\,,$$

where W is a constant independent on N. It is not hard to verify (and this is done explicitly in [23]) that one can choose W so that the matrix R is positive definite for all N; the resulting Gaussian field $\varphi_{N,v}$ with correlation matrix R is the CLERM. One may think of the CLREM as indexed by V_N in dimension $d = 1$, or (as the name indicates) by an equally spaced collection of N points on the unit circle in the complex plane.

Let $M_N = \max_{v\in V_N}\varphi_{N,v}$. The following is a corollary of Theorems 2.0.4 and 4.0.5.

Corollary 4.3.1. $\mathbb{E}M_N = \sqrt{2}\log N - (3/2\sqrt{2})\log\log N + O(1)$ *and there exist a constant* β^* *and a random variable* $\mathcal{Z}$ *so that*

$$\lim_{N\to\infty}\mathbb{P}(M_N - \mathbb{E}M_N \le x) = \mathbb{E}(e^{-\beta^*\mathcal{Z}e^{-\sqrt{2}x}})\,. \tag{4.24}$$

Proof. Assumptions (A.0) and (A.1) are immediate to check. An explicit computation reveals that Assumption (A.2) holds with $f(x) = 0$ and

$$g(u,v) = \begin{cases} -W, & u = v \\ \log(4\pi) + |u-v|, & u \neq v \end{cases}\,.$$

Finally, it is clear that Assumption (A.3) holds with $h(x,y) = \log(4\sin^2(2\pi|x-y|))$. Thus, Theorems 4.0.4 and 4.0.5 apply and yields (4.24). $\square$

Remark 4.3.2. *Remarkably, in [23] the authors compute explicitly, albeit non-rigorously, the law of the maximum of the CLREM, up to a deterministic shift that they do not compute. It was observed in [33] that the law computed in [23] is in fact the law of a convolutions of two Gumbel random variables. In the notation of Corollary 4.3.1, this means that one expects that $2^{-1/2}\log(\beta^* \mathcal{Z})$ is Gumbel distributed. We do not have a rigorous proof for this claim.*

4.4 Precise estimate of right tail

Our proof of Proposition 4.2.1 is highly similar to the proof in [12, Proposition 4.1], but simpler in a number of places. We will sketch the outline of the arguments, and refer to [12] extensively (it is helpful to recall Remark 4.2.2). To start, we note that by Lemmas 2.0.6 and 2.2.1, there exists $c_\alpha > 0$ depending only on α such that

$$\mathbb{P}(\max_{v \in B_{N/KL,i}} \xi^f_{N,v} \geq m_{N/KL} + z) \geq c_\alpha z e^{-\sqrt{2d}z} \text{ for all } 1 \leq z \leq \sqrt{\log N/KL}, 1 \leq i \leq (KL)^d . \tag{4.25}$$

In addition, adapting the proof of (1.2), we deduce that there exists $C_\alpha > 0$ depending only on α such that

$$\mathbb{P}(\max_{v \in B_{N/KL,i}} \xi^f_{N,v} \geq m_{N/KL} + z) \leq C_\alpha z e^{-\sqrt{2d}z} \text{ for all } z \geq 1, 1 \leq i \leq (KL)^d . \tag{4.26}$$

Recall the definition of $\{\xi_{N,v}\}$ as in (4.8). In what follows we consider a fixed i and a box $B_{N/KL,i}$. We note that the law of the fine field $\{\xi^f_{N,v} : v \in B_{N/KL,i}\}$ does not depend on K, L, i, and hence $\beta^*_{K',L'}$ does not depend on K, L, i. Write $\bar{N} = N/KL = 2^{\bar{n}}$ and $\bar{L} = K'L' = 2^{\bar{\ell}}$. For convenience of notation, we will refer to the box $B_{N/KL,i}$ as $V_{\bar{N}}$ and let $\Xi_{\bar{N}}$ be the collection of all left bottom corners of $\bar{L}$-boxes of form $B_{\bar{L},j}$ in $B_{N/KL,i}$. In addition, write $n^* = \frac{\mathrm{Var}\, X_{v,N}}{\log 2} = \bar{n} - \bar{\ell}$, where we denote $X_{v,N} = \xi_{N,v,\mathrm{MBRW}}$.

For convenience, we now view each $X_{v,N}$ as the value at time n^* of a Brownian motion with variance rate $\log 2$. More precisely, we assign to each Gaussian variable $b^N_{j,B}$ in (4.2) an independent Brownian motion, with variance rate $\log 2$, that runs for 2^{-2j} time units and ends at the value $b^N_{j,B}$. We now define a Brownian motion $\{X_{v,N}(t) : 0 \leq t \leq n^*\}$ by concatenating each of the previous Brownian motions associated with $v \in \Xi_{\bar{N}}$, with earlier times corresponding to larger boxes. From our construction, we see that $X_{v,N}(n^*) = X_{v,N}$. We partition $V_{\bar{N}}$ into disjoint $\bar{L}$-boxes, for which we denote $\mathcal{B}_{\bar{L}}$. Further, denote by B_v the $\bar{L}$-box in $\mathcal{B}_{\bar{L}}$ that contains v. Define

$$\begin{aligned}
E_{v,N}(z) &= \{X_{v,N}(t) \leq z + \frac{m_{\bar{N}}}{\bar{n}} t \text{ for all } 0 \leq t \leq n^*, \text{ and } \max_{u \in B_v} \xi^f_{u,N} \geq m_{\bar{N}} + z\}\,, \\
F_{v,N}(z) &= \{X_{v,N}(t) \leq z + \frac{m_{\bar{N}}}{\bar{n}} t + 10(\log(t \wedge (n^* - t)))_+ + z^{1/20} \\
&\qquad\qquad \text{for all } 0 \leq t \leq n^*, \text{ and } \max_{u \in B_v} \xi^f_{u,N} \geq m_{\bar{N}} + z\}\,, \\
G_N(z) &= \bigcup_{v \in \Xi_{\bar{N}}} \bigcup_{0 \leq t \leq n^*} \{X_{v,N}(t) > z + \frac{m_{\bar{N}}}{\bar{n}} t + 10(\log(t \wedge (n^* - t)))_+ + z^{1/20}\}\,.
\end{aligned} \tag{4.27}$$

Also define

$$\Lambda_{\bar{N},z} = \sum_{v \in \Xi_{\bar{N}}} \mathbf{1}_{E_{v,N}(z)}\,, \text{ and } \Gamma_{\bar{N},z} = \sum_{v \in \Xi_{\bar{N}}} \mathbf{1}_{F_{v,N}(z)}\,.$$

In words, the random variable $\Lambda_{N,z}$ counts the number of boxes in $\mathcal{B}_{\bar{L}}$ whose "backbone" path $X_{v,N}(\cdot)$ stays below a linear path connecting z to roughly $m_{\bar{N}} + z$, so that one of its "neighbors" achieves a terminal value that is at least $m_{\bar{N}} + z$; the random variable $\Gamma_{N,z}$ similarly counts boxes in $\mathcal{B}_{\bar{L}}$ whose backbone is constrained to stay below a slightly "upward bent" curve. Clearly, $E_{v,N}(z) \subseteq F_{v,N}(z)$ always holds, as does $\Lambda_{\bar{N},z} \leq \Gamma_{\bar{N},z}$.

By (4.8), for each $v \in \Xi_{\bar{N}}$ we can write that

$$\max_{u \in B_v} \xi^f_{N,v} = X_{v,N} + Y_{v,N}\,, \tag{4.28}$$

where $\{Y_{v,N}\}$ are i.i.d. random variables with the same law as $\max_{u \in V_{\bar{L}}} \varphi_{\bar{L},u} + a_{K,L,K',L',u}\phi$

where ϕ is a standard Gaussian variable. Crucially, the law of $Y_{v,N}$ does not depend on N. In addition, by Proposition 1.4.1 and Lemma 2.0.6, there exist C_α depending only on α such that

$$\mathbb{P}(Y_{v,N} \geq m_{\bar{L}} + \lambda) \leq C_\alpha \lambda e^{-\sqrt{2d}\lambda} e^{-C_\alpha^{-1}\lambda^2/\bar{\ell}} \text{ for all } \lambda \geq 1\,. \tag{4.29}$$

When estimating the ratio $\frac{\Lambda_{\bar{N},z}}{\Gamma_{\bar{N},z}}$, it is clear that $\frac{\Lambda_{\bar{N},z}}{\Gamma_{\bar{N},z}} = \frac{\mathbb{P}(E_{v,N}(z))}{\mathbb{P}(F_{v,N}(z))}$ for any fixed $v \in \Xi_{\bar{N}}$, where the latter concerns only the associated Brownian motion to $X_{v,N}$ and the random variable $Y_{v,N}$. As such, the arguments in [12, Lemma 4.10] carry out with merely notation change and give that

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \frac{\Lambda_{\bar{N},z}}{\Gamma_{\bar{N},z}} = 1\,. \tag{4.30}$$

Analogous to the proof of [12, Equation (100)], we can compare the field $\{X_{v,N}\}$ to a BRW and apply [12, Lemma 3.7] to obtain that

$$\mathbb{P}(G_N(z)) \leq C_\alpha e^{-\sqrt{2d}z}\,. \tag{4.31}$$

Note that the dimension does not play a significant role in these estimates, as [12, Lemma 3.7] follows from a union bound calculation. The dimension changes the volume of the box, but the probability

$$\mathbb{P}(X_{v,N}(t) > z + \frac{m_{\bar{N}}}{\bar{n}} t + 10(\log(t \wedge (n^* - t)))_+ + z^{1/20})$$

scales in the dimension (recall that m_N depends on d) which exactly cancels the growth of the volume in d.

The next desired ingredient is the second moment computation for $\Lambda_{\bar{N},z}$. Note that (i) our field $\{X_{v,N} : v \in \Xi_{\bar{N}}\}$ is simply an MBRW (so $\{X_{v,N}\}$ is nicer than its analog in [12], which is a sum of an MBRW and a field with uniformly bounded variance); (ii) our $\{Y_{v,N}\}$ are i.i.d. random variable with desired tail bounds as in (4.28) (so also nicer than its analog in [12], which has weak correlation for two neighboring local boxes). Therefore, the second moment computation in [12, Lemma 4.11] carries out with minimal notation change and gives

$$\lim_{z\to\infty}\limsup_{\bar{L}\to\infty}\limsup_{N\to\infty}\frac{\mathbb{E}(\Lambda_{\bar{N},z})^2}{\mathbb{E}\Lambda_{\bar{N},z}}=1\,. \tag{4.32}$$

Note that in [12, Equation (90)], there is no analog of $\limsup_{\bar{L}\to\infty}$ as in the preceding inequality. That's because we have assumed in [12] that $L\geq 2^{2^{z^4}}$. Our statement as in (4.32) is weaker as it does not give a quantitative dependance on how $\bar{L}$ should grow in z. But this detailed quantitative dependence is not needed for the proof of convergence in law.

Combining (4.25), (4.30), (4.31) and (4.32), we deduce that

$$\lim_{z\to\infty}\limsup_{\bar{L}\to\infty}\limsup_{N\to\infty}\left|\frac{\mathbb{P}(\max_{v\in V_{\bar{N}}}\xi^f_{N,v}\geq m_{\bar{N}}+z)}{\mathbb{E}\Lambda_{\bar{N},z}}-1\right|=0\,. \tag{4.33}$$

Therefore, it remains to estimate $\mathbb{E}\Lambda_{\bar{N},z}$. To this end, we will follow [12, Section 4.3]. We first note that by (4.25) and (4.33), we have

$$\lim_{z\to\infty}\limsup_{\bar{L}\to\infty}\limsup_{N\to\infty}\frac{\mathbb{E}\Lambda_{\bar{N},z}}{ze^{-\sqrt{2d}z}}\geq c_\alpha\,, \tag{4.34}$$

where $c_\alpha>0$ is a constant depending on α.

The main goal is to derive the asympototics for $\mathbb{E}\Lambda_{\bar{N},z}$. For $v\in\Xi_{\bar{N}}$, let $\nu_{v,\bar{N}}(\cdot)$ be the density function (of a sub-probability measure on $\mathbb{R}$) such that, for all $I\subseteq\mathbb{R}$,

$$\int_I\nu_{v,\bar{N}}(y)dy=\mathbb{P}(X_{v,N}(t)\leq z+\frac{m_{\bar{N}}}{\bar{n}}t\text{ for all }0\leq t\leq n^*;X_{v,N}(n^*)-(\bar{n}-\bar{\ell})m_{\bar{N}}/\bar{n}\in I)\,.$$

Clearly, by (4.28),

$$\mathbb{P}(E_{v,N}(z))=\int_{-\infty}^{z}\nu_{v,\bar{N}}(y)\mathbb{P}(Y_{v,N}\geq\bar{\ell}m_{\bar{N}}/\bar{n}+z-y)dy\,.$$

For a given interval J, define

$$\lambda_{v,N,z,J} = \int_J \nu_{v,\bar{N}}(y)\mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)dy\,. \tag{4.35}$$

Set $J_{\bar{\ell}} = [-\bar{\ell}, -\bar{\ell}^{2/5}]$. For convenience of notation, we denote by $A \lesssim B$ that there exists a constant $C_\alpha > 0$ that depends only on α such that $A \leq C_\alpha B$ for two functions/sequences A and B. As in [12, Lemma 4.13], we claim that for any any sequences $x_{v,N}$ such that $|x_{v,N}| \lesssim \bar{\ell}^{1/5}$,

$$\lim_{z\to\infty} \liminf_{\bar{\ell}\to\infty} \liminf_{N\to\infty} \frac{\sum_{v\in\Xi_N} \lambda_{v,N,z,x_{v,N}+J_{\bar{\ell}}}}{\mathbb{E}\Lambda_{N,z}} = 1\,. \tag{4.36}$$

Note that, by containment, the above ratio is always at most 1. We prove (4.36) for the case when $x_{v,N} = 0$; the general case follows in the same manner. Application of the reflection principle (c.f. [12, Equation (28)]) to the Brownian motion with drift, $\bar{X}_{v,N}(\cdot) = X_{v,N}(\cdot) - m_{\bar{N}}t/\bar{n}$, together with the change of measure that removes the drift $m_{\bar{N}}t/\bar{n}$, implies that

$$\nu_{v,\bar{N}}(y) \lesssim e^{-\sqrt{2d}y} 2^{-dn^*} z|y|\,,$$

for $y \leq -\bar{\ell}$, over the given range $z \in (0, \bar{\ell})$ (which implies $z - y \asymp |y|$). Together with (4.29) and independence among $Y_{v,N}$ for $v \in \Xi_{\bar{N}}$, this implies the crude bound

$$\int_{-\infty}^{-\ell} \nu_{v,\bar{N}}(y)\mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)dy \lesssim 2^{-dn^*} \mathrm{e}^{-C_\alpha^{-1}\bar{\ell}}$$

for a constant $C_\alpha > 0$ depending on α. Similarly, for $y \leq z$ (and therefore, for $z - y \geq 0$), application of the reflection principle and (4.29) again implies that

$$\int_{-\bar{\ell}^{2/5}}^{z} \nu_{v,\bar{N}}(y)\mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)dy \lesssim 2^{-dn^*} \bar{\ell}^{-3/10} z \mathrm{e}^{-\sqrt{2d}z}\,.$$

Together with (4.34), this completes the verification of (4.36).

Next, we claim that there exists $\Lambda^*_{K',L',z} > 0$ that does not depend on N such that,

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}}{\Lambda^*_{K',L',z}} = \lim_{z\to\infty} \liminf_{\bar{L}\to\infty} \liminf_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}}{\Lambda^*_{K',L',z}} = 1\,. \tag{4.37}$$

By the reflection principle and change of measure, we get that for all $y \in [-\bar{\ell}, z]$ (see the derivation of [12, Equation (107)])

$$\nu_{v,\bar{N}}(y) = 2^{-dn^*} \mathrm{e}^{-\sqrt{2d}y} \frac{z(z-y)}{\sqrt{2\pi \log 2}} (1 + O(\bar{\ell}^3/\bar{n}))\,. \tag{4.38}$$

Therefore,

$$\begin{aligned}
\sum_{v\in\Xi_{\bar{N}}} \lambda_{v,N,z,J_{\bar{\ell}}} &= (\frac{\bar{N}}{\bar{L}})^d \int_{J_{\bar{\ell}}} \nu_{v_0,\bar{N}}(y + O(\bar{\ell}/\sqrt{\bar{n}}))\mathbb{P}(Y_{v_0,N} \geq \sqrt{2d}\log 2 \cdot \bar{\ell} + z - y)dy \\
&= (1 + O(\bar{\ell}^3/\sqrt{\bar{n}})) \int_{J_{\bar{\ell}}} \frac{z(z-y)}{\sqrt{2\pi \log 2}\mathrm{e}^{\sqrt{2d}y}} \mathbb{P}(Y_{v_0,N} \geq \sqrt{2d}\log 2 \cdot \bar{\ell} + z - y)dy\,,
\end{aligned}$$

where v_0 is any fixed vertex in $\Xi_{\bar{N}}$ and in the last step we have used the fact that $n^* = \bar{n} - \bar{\ell}$. Recall that the law of $Y_{v_0,N}$ is the same as $\max_{u\in V_{\bar{L}}} \varphi_{\bar{L},u} + a_{K',L',u}\phi$, which does not depend on N. Combined with (4.36), this completes the proof of (4.37).

Finally, we analyze how $\mathbb{E}\Lambda_{N,z}$ scales with z. To this end, consider $z_1 < z_2$. For $v \in \Xi_N$ and $j = 1, 2$, recall that

$$\lambda_{v,N,z_j,z_j+J_{\bar{\ell}}} = \int_{J_{\bar{\ell}}+z_j} \nu_{v,\bar{N}}(y)\mathbb{P}(Y_{v,N} \geq \ell m_{\bar{N}}/\bar{n} + z_i - y)dy\,.$$

By (4.38), for any $y \in J_{\bar{\ell}}$ and $z_1, z_2 \ll \log \bar{\ell}$,

$$\begin{aligned}
&\frac{\nu_{v,\bar{N}}(y+z_1)\mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} - y)}{\nu_{v,\bar{N}}(y+z_2)\mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} - y)} \\
&= \frac{\nu_{v,\bar{N}}(y+z_1)}{\nu_{v,\bar{N}}(y+z_2)} = (1+O(\bar{\ell}^3/\bar{n}))\frac{z_1(z_1-y)}{z_2(z_2-y)}\mathrm{e}^{-\sqrt{2\pi}(z_1-z_2)} \\
&= (1+O(\bar{\ell}^3/\bar{n}))\frac{z_1}{z_2}\mathrm{e}^{-\sqrt{2d}(z_1-z_2)}(1+z_2^{-3/5})\,.
\end{aligned}$$

This implies that

$$\frac{\lambda_{v,N,z_1,z_1+J_{\bar{\ell}}}}{\lambda_{v,N,z_2,z_2+J_{\bar{\ell}}}} = (1+O(\bar{\ell}^3/\bar{n}))\frac{z_1}{z_2}\mathrm{e}^{-\sqrt{2\pi}(z_1-z_2)}(1+z_2^{-3/5})\,.$$

Together with (4.36), the above display implies that

$$\lim_{z_1,z_2\to\infty}\limsup_{\bar{L}\to\infty}\limsup_{N\to\infty}\frac{z_2 e^{-\sqrt{2\pi}z_2}\mathbb{E}\Lambda_{N,z_1}}{z_1 e^{-\sqrt{2\pi}z_1}\mathbb{E}\Lambda_{N,z_2}} = \lim_{z_1,z_2\to\infty}\liminf_{\bar{L}\to\infty}\liminf_{N\to\infty}\frac{z_2 e^{-\sqrt{2\pi}z_2}\mathbb{E}\Lambda_{N,z_1}}{z_1 e^{-\sqrt{2\pi}z_1}\mathbb{E}\Lambda_{N,z_2}} = 1\,.$$

Along with (4.37), this completes the proof of (4.10) for some $\beta^*_{K',L'}$. From (4.25) and (4.26), we see that $c_\alpha \leq \beta^*_{K',L'} \leq C_\alpha$ for all K', L'. This completes the proof of the proposition. □

CHAPTER 5

D-ARY TREE PRESSED AGAINST A HARD WALL

Let us consider a d-ary tree of n levels and call it T_n. We define a tree indexed Gaussian process which we call a branching random walk on T_n and denote it by $\{\phi_v^n : v \in T_n\}$. The covariance structure of this Gaussian process is given by the following

$$\begin{aligned} \operatorname{Var} \phi_v^n &= n \quad \text{for all } v \in T_n \\ \operatorname{Cov}(\phi_u^n, \phi_v^n) &= n - d_T(u,v) \quad \text{ for all } u \neq v \in T_n\,. \end{aligned} \tag{5.1}$$

where d_T denotes the tree distance.

We wish to find bounds on the order of the probability of a branching random walk being positive at all vertices. We also want to compute the expected value of a typical vertex under the condition that it is positive everywhere. The behavior that we are considering is that of entropic repulsion for this Gaussian field. This is its behavior of drifting away when pressed against a hard wall so as to have enough room for local fluctuations. This phenomenon has been discussed in [28].

We are interested in $\mathbb{P}(\phi_v^n \geq 0 \ \forall v \in T^n)$ as well as $\mathbb{E}(\phi_u^n \mid \phi_v^n \geq 0 \ \forall v \in T_n)$ and $\operatorname{Var}(\phi_u^n \mid \phi_v^n \geq 0 \ \forall v \in T_n)$. We know from [34] that $\mathbb{E}(\max_{v \in T_n} \phi_v)$ is of the form $c_1 n - c_2 \log n + O(1)$. Let us first define m_n to be equal to $c_1 n - c_2 \log n$. In [9] it has been shown that the conditional expectation under positivity is roughly close to the expected maximum for the discrete GFF in 2 dimensions. Here we show that for a branching random walk the conditional expectation is at least a constant times $\log n$ less than the expected maximum. This is the first such result along the direction of extropic repulsion in case of a Gaussian field. The main result of this chapter is:

Theorem 5.0.1. *There exists positive numbers a, b, $a < b$ such that for all $v \in T_n$,*

$$m_n - b \log n + O(1) \leq \mathbb{E}(\phi_u^n \mid \phi_v^n \geq 0 \ \forall v \in T_n) \leq m_n - a \log n + O(1).$$

The approach that we take for proving this is that we raise the average value of the Gaussian process and then multiply a compensation probability to that. We optimize this average value so as to maximize the probability of positivity. The value at which this probability is maximized should ideally be the required conditional expectation.

In order to prove this in details, we invoke a new model called the switching sign branching random walk, which is similar in structure to the original branching random walk. We begin our calculations with a preliminary upper bound on the left tail of the maxima of the BRW in Section 5.1. Section 5.2 contains the definition of the new model switching sign branching random walk followed by a comparison of positivity for the branching random walk with this model using Slepian's lemma. A left tail computation for the maximum of this model gives us the order of positivity of for the branching random walk which is the concluding result of Section 5.3. Section 5.4 contains the proof of the main theorem of this chapter. The upper bound follows from Section 5.2, while for the lower bound we further have to invoke the Bayes' rule and tail estimates to arrive at our result. Throughout the chapter we will use d_T to denote the tree distance. Let us call the event $\{\phi_v^n \geq 0 \ \forall v \in T^n\}$ as Λ_n^+. First let us consider the sum of all the Gaussian variables at the level n and term it S_n. In mathematical terms $S_n = \sum_{v:v\in T_n} \phi_v^n$, where the sum contains d^n terms.

5.1 Left tail of maximum of BRW

This section is dedicated to proving an exponential upper bound on the left of the maxima of a BRW.

Lemma 5.1.1. *There exists constants $\bar{C}, c^* > 0$ such that for all $n \in \mathbb{N}$ and $0 \leq \lambda \leq (\log n)^{2/3}$,*

$$\mathbb{P}(\max_{v\in T_n} \phi_v^n \leqslant m_n - \lambda) \leq \bar{C} e^{-c^*\lambda} \tag{5.2}$$

Proof. From [34, Section 2.5] we have tightness for $\{\max_{v\in T_n} \phi_v^n - m_n\}_{n\in\mathbb{N}}$. So there exists

$\beta > 0$ such that for all $n \geq 2$,

$$\mathbb{P}(\max_{v \in T_n} \phi_v^n \geqslant m_n - \beta) \geqslant 1/2. \tag{5.3}$$

Further, we also have that for some $\kappa > 0$ and for all $n \geq n' \geq 2$

$$\sqrt{2d}(n - n') - \frac{3}{4d}\log(n/n') - \kappa \leqslant m_n - m_{n'} \leqslant \sqrt{2d}(n - n') + \kappa. \tag{5.4}$$

Now let us fix $\lambda' = \lambda/2$ and $n' = n - \frac{1}{\sqrt{2d}}(\lambda' - \beta - \kappa - 4)$. From (5.4) it follows then that $m_n - m_{n'} \leqslant \lambda' - \beta$. Consider a tree of height n and look at its subtrees at height $n - n'$, which are individually trees of height n'. The total number of subtrees we have is $d^{n-n'}$. Let us call them $\{T_{n'}^{(1)}, T_{n'}^{(2)}, \dots, T_{n'}^{(d^{n-n'})}\}$. Now for all $v \in T_n$, we define

$$\bar{\phi}_v^n = g_v^{n'} + \phi,$$

where g_v^n are the BRWs obtained by adding the Gaussians for the edges only in the subtrees of height n', and ϕ is a Gaussian of mean 0 and variance $n - n'$. Clearly

$$\operatorname{Var} \phi_v^n = \operatorname{Var} \bar{\phi}_v^n \quad \text{and} \quad \mathbb{E}\phi_v^n \phi_u^n \leq \mathbb{E}\bar{\phi}_v^n \bar{\phi}_u^n \ \forall u \neq v \in T_n.$$

So by Lemma 2.2.1, we have

$$\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq t) \leq \mathbb{P}(\max_{v \in T_n} \bar{\phi}_v^n \leq t) \ \forall\, t \in \mathbb{R}. \tag{5.5}$$

Using (5.3) and (5.4), one has for all $i \in \{1, 2, \dots, d^{n-n'}\}$,

$$\begin{aligned}
\mathbb{P}(\sup_{v \in T_{n'}^{(i)}} g_v^{n'} \geq m_n - \lambda') &= \mathbb{P}(\sup_{v \in T_{n'}^{(i)}} g_v^{n'} \geq m_{n'} + m_n - m_{n'} - \lambda') \\
&\geq \mathbb{P}(\sup_{v \in T_{n'}^{(i)}} g_v^{n'} \geq m_{n'} - \beta) \geq 1/2
\end{aligned}$$

and so $\mathbb{P}(\sup_{v\in T_n} g_v^{n'} < m_n - \lambda') \leq (\frac{1}{2})^{d^{n-n'}}$.

Therefore,

$$\mathbb{P}(\sup_{v\in T_n} \bar{\phi}_v^n \leq m_n - \lambda) \leq \mathbb{P}(\sup_{v\in T_n} g_v^{n'} < m_n - \lambda') + \mathbb{P}(\phi \leq -\lambda') \leq \bar{C}e^{-c^*\lambda},$$

for some $\bar{C}, c^* > 0$. Now in conjunction with (5.5), the lemma is proved. □

5.2 Switching Sign Branching Random Walk

At this juncture we start defining a new Gaussian process on the tree, which we call the switching sign branching random walk. This was used to approximate the branching random walk in [18] in case of a 4-ary tree. We have generalized the process for a d-ary tree. The switching sign branching random walk consists of two parts, one that varies across vertices, and the other that is fixed over vertices. The first part of the process, which is not fixed over vertices, is different from the normal branching random walk in the sense that instead of the d-edges coming out of it being associated to independent normal random variables, they are associated to linear combinations of $d-1$ independent Gaussians, such that the covariance between any two of them is the same, and all of them add up to zero. The existence of this is guaranteed by the following Lemma.

Lemma 5.2.1. *There exists $A \in \mathbb{R}^{(d-1)\times(d-1)}$ such that for $X \sim N(0, \sigma^2 I_{(d-1)\times(d-1)})$, the covariance matrix of AX has all its diagonal entries to be σ^2 and all its off-diagonal entries to be equal(say b). Further $\mathrm{Var}(1^T AX) = \sigma^2$ and $\mathrm{Cov}(-1^T AX, (AX)_i) = b$ for all $i \in \{1, 2, \ldots, d-1\}$.*

Proof. We know that the covariance matrix for AX is AA^T. Further from the condition that

$\mathrm{Var}(1^T AX) = \sigma^2$ we get that $b = -\frac{\sigma^2}{d-1}$. So in order for A to exist we must have

$$AA^T = \sigma^2 \begin{bmatrix} 1 & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \\ -\frac{1}{d-1} & 1 & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{d-1} & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & 1 \end{bmatrix}_{(d-1)\times(d-1)}.$$

Since the matrix on the right hand side is a symmetric matrix with non-negative eigenvalues, so by Cholesky decomposition we obtain the existence of such an A. □

A pictorial representation of a node for this process is given in Figure 5.1.

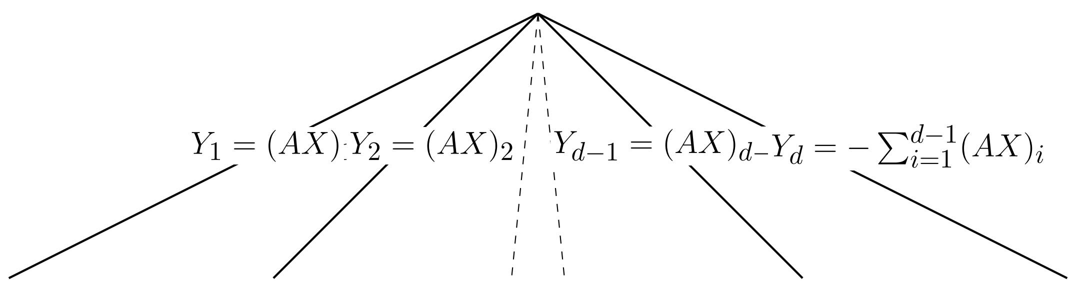

Figure 5.1: Node of the varying part of SSBRW

Now in the actual construction, unlike the BRW, we use a different value for σ^2 for each level l such that $1 \le l \le n$. Here level 1 denotes the edge connecting the root to its children and level n denotes the edges joining the leaf nodes to their parents. Let us denote this switching sign branching random walk on the tree T_n as $\{\xi_v^n : v \in T_n\}$. For $v \in T_n$ let us denote the Gaussian variable that is added on level l, on the path connecting v to the root, by $\phi_v^{n,l}$. Then we assign $\mathrm{Var}(\phi_v^{n,l}) = 1 - d^{-(n-l+1)}$. The switching sign branching random walk will consist of two parts, the first coming from the contribution at different levels in the tree which we call $\tilde{\phi}_v^n \stackrel{def}{=} \sum_{l=1}^n \phi_v^{n,l}$.

Finally we define the switching sign branching random walk as

$$\xi_v^n = \tilde{\phi}_v^n + X \tag{5.6}$$

where X is a Gaussian variable with mean zero and variance $\frac{1-d^{-n}}{d-1}$.

The covariance structure for this new model closely resembles that of the branching random walk. The following lemma deals with this comparison:

Lemma 5.2.2. *The Gaussian fields* $\{\xi_v^n : v \in T_n\}$ *and* $\{\phi_v^n : v \in T_n\}$ *are identically distributed.*

Proof. First we show that the variances are identical for the two processes. To this end,

$$\begin{aligned}\mathrm{Var}(\xi_v^n) &= 1 - d^{-1} + 1 - d^{-2} + \cdots + 1 - d^{-n} + \frac{1-d^{-n}}{d-1} \\ &= n - \frac{1-d^{-n}}{d-1} + \frac{1-d^{-n}}{d-1} = n.\end{aligned}$$

Next in case of the covariances suppose we consider $u, v \in T_n$, such that they are separated until level k i.e $\mathrm{Cov}(\phi_u^n, \phi_v^n) = n - k$. Then we have

$$\mathrm{Cov}(\tilde{\phi}_u^n, \tilde{\phi}_v^n) = -\frac{1-d^{-k}}{d-1} + \sum_{l=k+1}^{n} (1 - d^{-l}) = n - k - \frac{1-d^{-n}}{d-1}.$$

So, the covariance structures for the fields ξ and ϕ match, and hence they are identically distributed. □

A simple corollary of Lemma 5.2.2, is the following, based on the fact that the two processes have identical distributions.

Corollary 5.2.3. *We have the following equality:*

$$\mathbb{P}(\phi_v^n \geq 0 \ \forall v \in T_n) = \mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq X) \tag{5.7}$$

Corollary 5.2.4. *From [34, Theorem 4], we have* $\mathbb{E} \max_{v \in T_n} \phi_v^n = \sqrt{2d}n - \frac{3}{2\sqrt{2d}} \log n + O(1)$. *Therefore,*

$$\mathbb{E} \max_{v \in T_n} \tilde{\phi}_v^n = \sqrt{2d}n - \frac{3}{2\sqrt{2d}} \log n + O(1).$$

Corollary 5.2.5. *There exists constants $\bar{C}', c^* > 0$ such that for all $n \in \mathbb{N}$ and $0 \le \lambda \le (\log n)^{2/3}$,*

$$\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leqslant m_n - \lambda) \le \bar{C}' e^{-c^*\lambda} \tag{5.8}$$

Proof.

$$\frac{1}{2}\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leqslant m_n - \lambda) = \mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leqslant m_n - \lambda, X \le 0) \le \mathbb{P}(\max_{v\in T_n} \phi_v^n \leqslant m_n - \lambda).$$

Now using (5.2), and with $\bar{C}' = 2\bar{C}$ we arrive at (5.8). □

5.3 Estimates on left tail and positivity

From the (5.7) we understand that the probability of positivity for the branching random walk can be computed using bounds on the left tail of the maximum of $\tilde{\phi}_\cdot^n$, a part of the switching sign branching random walk, as the left tail is heavily concentrated around the maximum. This motivates the following computations on the left tail of the maximum.

Lemma 5.3.1. *Let us call $c = 1/c_1$(where $m_n = c_1 n - c_2 \log n$) to be the constant such that $|m_{n-c\lambda} - m_n - \lambda| \to 0$ as $n \to \infty$, where λ is of lower order than n. Then there exists independent constants C', C'', K', K'' such that for sufficiently large n we have*

$$K' \exp(-K'' d^{c\lambda}) \le \mathbb{P}(\max_{v\in T^n} \tilde{\phi}_v \le m_n - \lambda) \le C' \exp(-C'' d^{c\lambda}). \tag{5.9}$$

Proof. We work with $\mathbb{P}(\max_{v\in T^n} \tilde{\phi}_v \le m_{n-c\lambda})$ as due to our definition of c, for sufficiently large n this probability is close to $\mathbb{P}(\max_{v\in T^n} \tilde{\phi}_v \le m_n - \lambda)$. This comes from the fact that, from Chapter 4, $\{\max_{v\in T^n} \tilde{\phi}_v - m_n\}$ converges in distribution, and so by an application of Slutsky's theorem $\mathbb{P}(\max_{v\in T^n} \tilde{\phi}_v \le m_{n-c\lambda})$ and $\mathbb{P}(\max_{v\in T^n} \tilde{\phi}_v \le m_n - \lambda)$ converge to the same value. We know that the SSBRW is a Gaussian field which obtained by the same Gaussian to all vertices of a BRW. This helps us find bounds on lower and upper tails of maxima using results on convergence of maxima of BRW, as proved in [3], [13] etc.

We first consider the tree only up to the level $c\lambda$ and consider the cumulative sum of the Gaussian variables at these vertices till the level $c\lambda$. Let us rename all these Gaussian variables at level $c\lambda$ of this new tree to be $A_1, A_2, \ldots, A_{d^{c\lambda}}$. We know that the definition in Section 5.2 of switching sign branching random walk model guarantees $\sum_{i=1}^{d^{c\lambda}} A_i = 0$. Let us consider the subtrees rooted at the vertex which has values A_i and call its maximum to be M_i. These are trees of height $n - c\lambda$ and hence we have $\mathbb{E}M_i = m_{n-c\lambda} + O(1) \ \ \forall i$ and $M := \max_{v \in T^n} \tilde{\phi}_v = \max_{i=1}^{d^{c\lambda}}(M_i + A_i)$. We want to obtain bounds for the probability $\mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \leq m_{n-c\lambda})$. We condition on the values of $A_1, A_2, \ldots, A_{d^{c\lambda}}$ which in turn breaks down the required probability in a product form since the maxima for the $d^{c\lambda}$ subtrees are independent and have identical distributions. We consider two different cases:

1) When $A_i^- \leq 2\bar{A}$ for at least $d^{c\lambda}/2$ many i, where $\bar{A}$ is a positive constant to be chosen later on.

2) When 1) doesn't happen and so then $\sum_{i=1}^{d^{c\lambda}} A_i^- \geq \bar{A} d^{c\lambda}$.

For the first case we break it down into two parts according to when $\sum_{i=1}^{d^{c\lambda}} A_i^- \geq \bar{A} d^{c\lambda}$ or not. Now we have

$$
\begin{aligned}
& \mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \leq m_{n-c\lambda} \mid A_1, A_2, \ldots, A_{d^{c\lambda}}) \\
= \; & \mathbb{P}(\max_{i=1}^{d^{c\lambda}}(M_i + A_i) \leq m_{n-c\lambda} \mid A_1, A_2, \ldots, A_{d^{c\lambda}}) \\
= \; & \prod_{i=1}^{d^{c\lambda}} \mathbb{P}(M_i + A_i \leq m_{n-c\lambda} \mid A_i) \qquad < \text{from independence} > \\
\leq \; & \prod_{i:A_i>0}^{d^{c\lambda}} \mathbb{P}(M_i \leq m_{n-c\lambda} - A_i \mid A_i) \\
\leq \; & \bar{C'}^{d^{c\lambda}} \exp(-c^* \sum_{i=1}^{d^{c\lambda}} A_i^+) = \exp(d^{c\lambda} \log \bar{C'} - c^* \sum_{i=1}^{d^{c\lambda}} A_i^-)
\end{aligned}
$$

In the final two steps we first make use of (5.8), followed by the fact that $\sum_i A_i = 0$. For the cases where $A_i < 0$ we bound the terms in the product by 1. When 2) holds then clearly this is bounded by $\exp(-(c^* \bar{A} - \log \bar{C'}) d^{c\lambda})$ and now on choosing $\bar{A}$ such that $(c^* \bar{A} - \log \bar{C'}) > 0$

we have $c^{**} > 0$ such that our required term is bounded by $\exp(-c^{**}d^{c\lambda})$. In the other case also

$$\mathbb{P}(M_i \le m_{n-c\lambda} - A_i \mid A_i) \le \mathbb{P}(M_i \le m_{n-c\lambda} + 2\bar{A})$$

for those i for which $A_i^- \le 2\bar{A}$. From lower bound on right tail of maximum, we can find p, independent of n, where $0 < p < 1$ such that $\mathbb{P}(M_i \le m_{n-c\lambda} + 2\bar{A}) < p$ for all sufficiently large n and so the probability is bounded by $\exp(-\bar{c}d^{c\lambda})$. Now from this $\bar{c}$ and c^{**} we select one unified C', C'' so that

$$\mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \le m_{n-c\lambda}) \le C' \exp(-C'' d^{c\lambda}).$$

Again for the lower bound we have

$$\begin{aligned} \mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \le m_{n-c\lambda}) &= \int_{\mathbb{R}^{d^{c\lambda}}} \prod_{i=1}^{2^{c\lambda}} \mathbb{P}(M_i \le m_{n-c\lambda} - A_i) dA_i \\ &\ge (\bar{p})^{d^{c\lambda}} \int_{[-1,1]^{d^{c\lambda}}} \prod_{i=1}^{d^{c\lambda}} dA_i \end{aligned}$$

where $\bar{p}$ is chosen to be a lower bound on $\mathbb{P}(M_i \le m_{n-c\lambda} - 1)$ for all sufficiently large n, which can be obtained from using convergence results on maxima of branching random walk. Now $\{A_1, A_2, \ldots, A_{d^{c\lambda}}\}$ are obtained by linear combinations of $d^{c\lambda} - 1$ independent standard normal random variables, each being obtained from $c\lambda$ many of them, and a way to make all A_i's in the range $[-1, 1]$ is to make absolute value of the contribution at the jth level to be bounded by $\frac{1}{10(c\lambda+1-j)^2}$, for $j = 1, 2, \ldots, c\lambda$. So the independent standard normals at level j are bounded by $\frac{1}{10\sqrt{d}(c\lambda+1-j)^2}$. So this gives, for some constant $K > 0$,

$$\mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \le m_{n-c\lambda}) \ge (\bar{p})^{d^{c\lambda}} \prod_{j=1}^{c\lambda} \left(\frac{1}{10K\sqrt{d}(c\lambda + 1 - j)^2} \right)^{(d-1)d^{j-1}}.$$

Approximation of the sum, as shown below in Lemma 5.3.2 proves (5.9). □

Lemma 5.3.2. $\sum_{j=1}^{n} (\log |n + 1 - j|) d^j$ *is of order* d^n.

Proof. We begin with an upper bound on the sum. We use a trivial bound of $\log|x| \leq |x|$ for $|x| \geq 1$, followed by a few series summations.

$$\begin{aligned}
\sum_{j=1}^{n}(\log|n+1-j|)d^j &\leq \sum_{j=1}^{n}(|n+1-j|)d^j \\
&= (n+1)\sum_{j=1}^{n} d^j - \sum_{j=1}^{n} jd^j \\
&= (n+1)\frac{d^{n+1}-d}{d-1} - \frac{nd^{n+2}-(n+1)d^{n+1}+d}{(d-1)^2} \\
&= \frac{d^{n+2}-(n+1)d^2+nd}{(d-1)^2}
\end{aligned}$$

This gives the upper bound to be of order d^n. The lower bound follows easily. □

We now look back into our question of the branching random walk being positive at all vertices. We know that the maximum of the BRW is heavily concentrated around the expected maximum. Using this fact, in a neighborhood around the maximum, we further try to maximize the probability of the maximum being there. This point where this occurs will also roughly be the typical value of a vertex. This motivates the following proposition which is the main result of this section:

Proposition 5.3.3. *There exists λ' such that $d^{c\lambda'}$ is of order n such that for n sufficiently large we have, for $K_1, K_2, K_3 > 0$ independent of n,*

$$K_1 e^{-\frac{1}{2\sigma^2_{d,n}}(m_n-\lambda')^2 - K_3(m_n-\lambda')} \leq \mathbb{P}(\phi^n_v \geq 0\ \forall v \in T_n) \leq K_2 e^{-\frac{1}{2\sigma^2_{d,n}}(m_n-\lambda')^2 - \frac{m_n-\lambda'}{c\sigma^2_{d,n}\log d}}. \tag{5.10}$$

Proof. Upper bound: From (5.7) we have an upper-bound on the probability of positivity based on the switching signs branching random walk. We optimize this bound by first raising the mean to a level and look at the compensation we have to apply correspondingly. We optimize over these two to obtain our bound. We apply a similar strategy for obtaining the lower bound as well. Let us recall (5.7) at this juncture along with X, and let us call the variance of X to be $\sigma_{d,n}^2 = \frac{1-d^{-n}}{d-1}$. In (5.7), we condition on the value of X to obtain the following:

$$\mathbb{P}(\Lambda_n^+) = \frac{1}{\sigma_{d,n}\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \le x) \exp(-x^2/2\sigma_{d,n}^2) dx$$

Now, since the left tail of the maximum of a log-correlated Gaussian field, is heavily concentrated. So we may as well replace x by $m_n - \lambda$, and then integrate over λ. We split the integral into three parts, first with $\{-\infty < \lambda \le 0\}$, second with $\{\frac{3}{c}\log_d n \le \lambda < \infty\}$ and the rest. From tail estimates of a Gaussian, the first part is bounded by $O(\exp(-\frac{1}{2\sigma_{d,n}^2}(m_n - \lambda')^2))$. From (5.9), we know that the second part is bounded by $C'\exp(-C''n^3)$. The rest part has an upper bound:

$$\frac{C'}{\sqrt{2\pi}} \int_0^{\frac{3}{c}\log_d n} \exp(-C'' d^{c\lambda}) \exp(-(m_n - \lambda)^2/2) d\lambda. \tag{5.11}$$

We maximize the integrand in (5.11), over the range of the integral, to obtain an optimal λ, say λ', which is of order $\log n$. It satisfies the equation

$$m_n - \lambda' = \sigma_{d,n}^2 C'' c d^{c\lambda'} \log d.$$

Plugging in we obtain an upper bound as in (5.10).

Lower bound: Again recalling (5.9) we obtain that

$$\mathbb{P}(\Lambda_n^+) \ge \frac{K'}{\sqrt{2\pi}\sigma_{d,n}} \int_{-n}^{n} e^{-K'' d^{c\lambda}} \exp(-(m_n - \lambda)^2/2\sigma_{d,n}^2) d\lambda.$$

The integrand here is infact a decreasing function of λ in the range $\lambda \in [\lambda', \lambda' + 1]$, where λ' is from the first part of the proof. This gives a lower bound of

$$\frac{K'}{\sqrt{2\pi}\sigma_{d,n}} e^{-K'' d^c d^{c\lambda'}} \exp(-(m_n - \lambda' - 1)^2/2\sigma_{d,n}^2).$$

So, we obtain the required lower bound in (5.10). □

5.4 Expected value of a typical vertex under positivity

Proof. We want to compute $\mathbb{E}\left(\frac{S_n}{d^n} \mid \Lambda_n^+\right)$. Due to Lemma 5.2.2, this is equivalent to computing $\mathbb{E}\left(\frac{\sum_{v=1}^{d^n} \xi_v^n}{d^n} \mid \xi_u^n \geq 0 \;\forall\; v \in T_n\right) = \mathbb{E}\left(X \mid \max_{v\in T_n} \tilde{\phi}_v^n \leq X\right)$.

Upper Bound: We first split the expectation into two parts, one concerning the contribution of the right tail in the integral and the rest. We aim to show that the contribution of the right tail is negligible, thereby implying that the main contribution is from the rest, which gives an upper bound on the expectation. The tail here is motivated by the maximizer in Proposition 5.3.3.

$$\begin{aligned}
\mathbb{E}\left(X \mid \max_{v\in T_n} \tilde{\phi}_v^n \leq X\right) &= \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{m_n - b\log n} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} dx \\
&+ \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{m_n - b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} dx
\end{aligned}$$

Let us call the first term as J_1 and the next one as J_2. We first want to show that the contribution of J_2 in the conditional expectation is negligible. We use a trivial upper bound on the tail probability in the numerator. Then we compute the integral which is the tail expectation of a normal.

$$\begin{aligned}
J_2 &\leq \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{m_n - b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{1}{\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_{d,n}\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)} \int_{m_n - b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} dx \\
&= \frac{\sigma_{d,n} e^{-(m_n - b\log n)^2/2\sigma_{d,n}^2}}{\sqrt{2\pi}\mathbb{P}(\max_{v\in T_n} \tilde{\phi}_v^n \leq X)}
\end{aligned}$$

So we end up showing that contribution from the right tail is negligible. We now move on to the rest part and obtain an upper bound for it. We use a general upper bound on x from the range of the integral, which we can do since the integral exists and is finite by the fact that absolute expectation of a normal exists.

$$\begin{aligned}
J_1 &\leq \frac{m_n - b\log n}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{m_n - b\log n} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)} dx \\
&\leq \frac{m_n - b\log n}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)} dx \\
&= m_n - b\log n
\end{aligned}$$

From (5.10) it is clear that on choosing b such that $b\log n \leq \lambda'$ then the upper bound on the conditional expectation is $m_n - b\log n$.

Lower Bound: We apply a similar technique as in case of the upper bound, the only difference being that we look at the left tail instead, motivated by the left tail of the maximum of the Gaussian process.

$$\begin{aligned}
\mathbb{E}\left(X \mid \max_{v\in T_n}\tilde{\phi}_v^n \leq X\right) &= \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{m_n - \frac{3}{c}\log_d n} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)} dx \\
&+ \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{m_n - \frac{3}{c}\log_d n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)} dx
\end{aligned}$$

Let us call the first term as I_1 and the second as I_2.

When $x \in (-\infty, m_n - \frac{3}{c}\log_d n]$ then $\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x) \leq C' \exp(-C'' n^3)$ following (5.9). Also we have a lower bound on the probability of positivity, which gives the following bounds on I_1 and I_2.

$$\mid I_1 \mid \lesssim e^{\frac{1}{2\sigma_{d,n}^2}(m_n - \lambda')^2 + d^{c\lambda'}(\log\lambda' - \log\bar{p}/K) - C'' n^3} \int_{-\infty}^{\infty} \mid x \mid e^{-x^2/2\sigma_{d,n}^2} dx$$

where we ignore the constants. This shows that this term is negligible. Further,

$$
\begin{aligned}
I_2 &\geq \left(m_n - \frac{3}{c}\log_d n\right)\frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{m_n-\frac{3}{c}\log_d n}^{\infty} e^{-x^2/2\sigma_{d,n}^2}\frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)}dx \\
&= \left(m_n - \frac{3}{c}\log_d n\right)\frac{1}{\sqrt{2\pi}\sigma_{d,n}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma_{d,n}^2}\frac{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v\in T_n}\tilde{\phi}_v^n \leq X)}dx - o(1) \\
&= m_n - \frac{3}{c}\log_d n
\end{aligned}
$$

□

REFERENCES

[1] J. Acosta. Tightness of the recentered maximum of log-correlated Gaussian fields. *Electron. J. Probab.*, 19:no. 90, 25, 2014.

[2] L. Addario-Berry and B. Reed. Minima in branching random walks. *Ann. Probab.*, 37(3):1044–1079, 2009.

[3] E. Aïdékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, 41(3A):1362–1426, 2013.

[4] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching brownian motion. Preprint,available at `http://arxiv.org/abs/1103.2322`.

[5] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of a log-correlated Gaussian field. *Ann. Appl. Probab.*, 24(4):1446–1481, 2014.

[6] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. *Electron. J. Probab.*, 20:no. 59, 19, 2015.

[7] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete gaussian free field. Preprint, available at `http://arxiv.org/abs/1306.2602`, 2013.

[8] M. Biskup and O. Louidor. Conformal symmetries in the extremal process of two-dimensional discrete gaussian free field. Preprint, available at `http://arxiv.org/abs/1410.4676`, 2014.

[9] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):1670–1692, 2001.

[10] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni. Recursions and tightness for the maximum of the discrete, two dimensional gaussian free field. *Elect. Comm. in Probab.*, 16:114–119, 2011.

[11] M. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[12] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete gaussian free field. Preprint, available at `http://arxiv.org/abs/1301.6669`, 2013.

[13] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of nonlattice branching random walk. Preprint, available at `http://arxiv.org/abs/1404.3423`, 2014.

[14] M. Bramson and O. Zeitouni. Tightness for a family of recursion equations. *Ann. Probab.*, 37(2):615–653, 2009.

[15] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65:1–20, 2011.

[16] O. Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3):962–986, 2006.

[17] J. Ding. Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):285–299, 2013.

[18] J. Ding and S. Goswami. First passage percolation on the exponential of two-dimensional branching random walk. Preprint, available at `http://arxiv.org/abs/1511.06932`.

[19] J. Ding and O. Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.*, 42(4):1480–1515, 2014.

[20] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas. Critical Gaussian multiplicative chaos: convergence of the derivative martingale. *Ann. Probab.*, 42(5):1769–1808, 2014.

[21] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas. Critical Gaussian multiplicative chaos: convergence of the derivative martingale. *Ann. Probab.*, 42(5):1769–1808, 2014.

[22] X. Fernique. Regularité des trajectoires des fonctions aléatoires gaussiennes. In *École d'Été de Probabilités de Saint-Flour, IV-1974*, pages 1–96. Lecture Notes in Math., Vol. 480. Springer, Berlin, 1975.

[23] Y. B. Fyodorov and J.-P. Bouchaud. Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A: Math. Theor.*, 41, 2008.

[24] J.-P. Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Quebec*, 9(2):105–150, 1985.

[25] N. Kurt. Maximum and entropic repulsion for a gaussian membrane model in the critical dimension. *Ann. Probab.*, 37(2):687–725, 2009.

[26] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.

[27] G. F. Lawler and V. Limic. *Random walk: A Modern Introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[28] J. Lebowitz and C. Maes. The effect of an external field on an interface, entropic repulsion. *Journal of Statistical Physics*, 46(1-2):39–49, 1987.

[29] T. M. Liggett. Random invariant measures for Markov chains, and independent particle systems. *Z. Wahrsch. Verw. Gebiete*, 45(4):297–313, 1978.

[30] T. Madaule. Maximum of a log-correlated gaussian field. Preprint, available at `http://arxiv.org/abs/1307.1365`, 2013.

[31] L. D. Pitt. Positively correlated normal variables are associated. *Ann. Probab.*, 10(2):496–499, 1982.

[32] D. Slepian. The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.*, 41:463–501, 1962.

[33] E. Subag and O. Zeitouni. Freezing and decorated poisson point processes. Preprint, available at `http://arxiv.org/abs/1404.7346`, to appear, Comm. Math. Phys., 2015.

[34] O. Zeitouni. Branching random walks and gaussian fields. In preparation.