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TOPICS IN SEN THEORY AND APPLICATIONS

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GAL PORAT

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Abstract. This thesis consists of three articles related to Sen theory and its applications.

In the first article, we let K be a finite extension of $\mathbf{Q}_p$ and let Γ be the Galois group of the cyclotomic extension of K. Fontaine's theory gives a classification of p-adic representations of $\mathrm{Gal}(\overline{K}/K)$ in terms of (φ, Γ)-modules. A useful aspect of this classification is Berger's dictionary which expresses invariants coming from p-adic Hodge theory in terms of these (φ, Γ)-modules. We use the theory of locally analytic vectors to generalize this dictionary to the setting where Γ is the Galois group of a Lubin-Tate extension of K. As an application, we show that if F is a totally real number field and v is a place of F lying above p, then the p-adic representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ associated to a finite slope overconvergent Hilbert eigenform which is F_v-analytic up to a twist is Lubin-Tate trianguline. Furthermore, we determine a triangulation in terms of a Hecke eigenvalue at v. This generalizes results in the case $F = \mathbf{Q}$ obtained previously by Chenevier, Colmez and Kisin.

In the second article, we develop a version of Sen theory for equivariant vector bundles on the Fargues-Fontaine curve. We show that every equivariant vector bundle canonically descends to a locally analytic vector bundle. A comparison with the theory of (φ, Γ)-modules in the cyclotomic case then recovers the Cherbonnier-Colmez decompletion theorem. Next, we focus on the subcategory of de Rham locally analytic vector bundles. Using the p-adic monodromy theorem, we show that each locally analytic vector bundle $\mathcal{E}$ has a canonical differential equation for which the space of solutions has full rank. As a consequence, $\mathcal{E}$ and its sheaf of solutions $\mathrm{Sol}(\mathcal{E})$ are in a natural correspondence, which gives a geometric interpretation of a result of Berger on (φ, Γ)-modules. In particular, if V is a de Rham Galois representation, its associated filtered (φ, N, G_K)-module is realized as the space of global solutions to the differential equation. A key to our approach is a vanishing result for the higher locally analytic vectors of representations satisfying the Tate-Sen formalism, which is also of independent interest.

Finally, in the third article, we prove a conjecture of Emerton, Gee and Hellmann concerning the overconvergence of étale (φ, Γ)-modules in families parametrized by topologically finite type $\mathbf{Z}_p$-algebras. As a consequence, we deduce the existence of a natural map from the rigid fiber of the Emerton-Gee stack to the rigid analytic stack of (φ, Γ)-modules.

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Part I
Introduction

This novel ideas in this thesis revolve around the topic of Sen theory. My encounter with the subject originated when I first met with my PhD advisor, where he recommended I read the paper of Sen ([Se81]). I found the paper a pleasure to read - it was completely elementary and absolutely fascinating. Although more than 40 years have passed since Sen published this paper, and although many people have since given thought to these ideas, it seems that

1. There are still basic and natural questions in the subject which have yet to be answered,

2. The setup is relatively clean - you do not need to keep to many things in mind when thinking about questions,

3. One still generates ideas from reading the classical papers in the topic, and

4. There are deep connections to other classical areas of mathematics that are not yet completely understood (especially Hodge theory and the Simpson correspondence).

For these reasons I have found it enjoyable (and at times developed an obsession) to try to understand further what is at the root of the phenomenons of Sen theory.

On top of this, in recent years there has been somewhat a revolution in some aspects of this field, as well as some fantastic applications to some of the deepest questions in modern number theory, such as the Fontaine-Mazur and Calegari-Emerton conjectures, as well as the p-adic Langlands correspondence. It is exciting to see how the interest in Sen theory is spreading out to the rest of the number theory community as the applications are unfolding.

1 The origins of Sen theory

The subject has its roots in the seminal paper of Tate ([Ta67]), and in particular in its proof of the following fundamental result. Let p be a prime and let K be finite extension of $\mathbf{Q}_p$ with absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$. Let $\mathbf{C}_p$ be the completion of the algebraic closure of $\mathbf{Q}_p$, and let $\mathbf{C}_p(i)$ be the 1-dimensional semilinear representation of G_K twisted by the i-power of the cyclotomic character. Then for $i \in \mathbf{Z}_{\geq 0}$ and $j \in \mathbf{Z}$, the

Galois cohomology of $\mathbf{C}_p(j)$ is given by

$$\mathrm{H}^i(G_K, \mathbf{C}_p(j)) \cong \begin{cases} K & (i,j) \in \{(0,0),(1,0)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Tate proved this by considering the intermediate cyclotomic extension $K_{\mathrm{cyc}} := K(\mu_{p^\infty})$. Writing G_K as an extension of $H_K = \mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})$ by $\Gamma_K = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$, Tate computed the cohomology in two steps:

Step 1 - pro-étale descent. Tate showed that $\mathrm{H}^i(H_K, \mathbf{C}_p(j)) \cong 0$ and hence

$$\mathrm{H}^i(G_K, \mathbf{C}_p(j)) \cong \mathrm{H}^i(\Gamma_K, \widehat{K}_{\mathrm{cyc}}(j)).$$

Step 2 - decompletion. Tate showed that

$$\mathrm{H}^i(\Gamma_K, \widehat{K}_{\mathrm{cyc}}(j)) \cong \mathrm{H}^i(\Gamma_K, K_{\mathrm{cyc}}(j)).$$

It is this latter cohomology group $\mathrm{H}^i(\Gamma_K, K_{\mathrm{cyc}}(j))$ which is straightforward to compute.

Ultimately, both of these steps inspired generations of mathematicians.

The first step - pro-étale descent - was studied further and generalized by many people, perhaps most notably by Faltings and Scholze. It is now understood to be a special case of the **almost purity theorem**, which holds in general for coverings of perfectoid spaces. In a sense, this step does not depend on the finer arithmetic properties of the extension K_{cyc} - it only matters that $\widehat{K}_{\mathrm{cyc}}$ is perfectoid, and many other intermediate fields would also have this property. In fact, since the work of Scholze, we can carefully say that most of the basic aspects of this step are understood.

The second step - decompletion - also saw many innovations and generalizations by many people. It is this step which is at the core of what is currently studied in Sen theory, so in what follows we shall give an (extremely partial, not neccesarily chronological) elaboration on its history.

The first person to study this idea further was Sen. He showed how to use the ideas of Tate to decomplete the cohomology of general semilinear representations. More precisely, Sen showed how that if W is a finite dimensional $\widehat{K}_{\mathrm{cyc}}$-semilinear representation of Γ_K, then there exists a canonical decompletion of W, a K_{cyc}-subspace denoted $\mathbf{D}_{\mathrm{Sen}}(W)$, and a canonical isomorphism

$$\mathbf{D}_{\mathrm{Sen}}(W) \otimes_{K_{\mathrm{cyc}}} \widehat{K}_{\mathrm{cyc}} \cong W.$$

Another important discovery due to Sen is that of the Sen operator. Sen showed that $\mathbf{D}_{\mathrm{Sen}}(W)$ is equipped with a canonical linear operator Θ_{Sen}, which encodes in a simple way deep arithmetic properties of the representation W. For example, if W originated from the étale cohomology of a p-adic variety X, then the (negatives of the) eigenvalues of Θ_{Sen}, the so called Hodge-Tate weights, are given by certain Hodge numbers of X.

2 Decompletion of (φ, Γ)-modules

Another important idea came about when Colmez realized the theory connection with the problem of decompleting (φ, Γ)-modules. Motivated by a conjecture of Fontaine, he was able to prove together with Cherbonnier the overconvergence theorem ([CC98]) for (φ, Γ)-modules mirroring the results of Sen. Namely, given a (φ, Γ)-module D over the ring $\mathbf{A}_K$, there exists a canonical sub-$\mathbf{A}_K^\dagger$ module $D^\dagger$ and a canonical isomorphism

$$D^\dagger \otimes_{\mathbf{A}_K^\dagger} \mathbf{A}_K \cong D.$$

Following this, Berger and Colmez ([BC08]) provided the Tate-Sen formalism. This is a single framework for decompletion which gave a uniform proof for both the original results of Sen and the Cherbonnier-Colmez theorem.

3 Sen theory in families

In another direction, for the sake of applications, many people thought about decompletion in various contexts of families. Among others, it is worth mentioning the following works:

- The work of Sen ([Se93]) on the original theory in families
- The work of Brinon ([Br03]) on Sen theory in the relative setting
- The work of Andreatta-Brinon ([AB08]) on overconvergence of (φ, Γ)-modules in the relative setting
- The work of Berger-Colmez ([BC08]) on decompletion in families
- The work of Shimizu ([Sh18]) on geometric families of p-adic Galois representations
- The work of Bellovin ([Bel20]) on families of overconvergent (φ, Γ)-modules for pseudoaffinoind families

4 The connection with the theory of locally analytic representations

An important and more recent discovery of Berger-Colmez [BC16] was the link of Sen theory with the locally analytic representation theory of Schneider and Teitelbaum. Berger-Colmez showed that the theory of Sen can be understood and generalized through by using locally analytic vectors. Namely, if K_∞ is any infinitely ramified extension such that $\mathrm{Gal}(K_\infty/K)$ is a p-adic Lie group, and W is finite dimensional $\widehat{K}_\infty$-semilinear representation of $\Gamma_K := \mathrm{Gal}(K_\infty/K)$, then

$$\mathbf{D}_{\mathrm{Sen}}(W) \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \widehat{K}_\infty \cong W.$$

Furthermore, they showed two related and fascinating phenomenons:

1. If $d = \dim \Gamma_K$, then $\widehat{K}_\infty^{\mathrm{la}}$ is isomorphic to an increasing union of $d-1$-dimensional power series rings over the smooth vectors K_∞.

2. There exists a canonical Sen operator $\Theta_{\mathrm{Sen}}(\widehat{K}_\infty) \in \mathbb{C}_p \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma_K$ which annihilates $\widehat{K}_\infty^{\mathrm{la}}$.

This led to several fantastic developments:

1. In a subsequent paper, Berger ([Be13]) showed that at least in the rational, K-analytic setting (and in particular in the cyclotomic setting), the theory of locally analytic vectors plays a similar role in the theory of (φ, Γ)-modules. This led to applications in Iwasawa theory. It is worth mentioning a yet unsettled conjecture of Kedlaya ([Ke13]), according to which the natural generalization of these results should hold in general, for all K_∞.

2. Remarkable recent work of Pan ([Pa21],[Pa22]) used the ideas of Berger-Colmez [BC16] in families, to give a new proof of the Fontaine-Mazur conjecture, by studying the locally analytic vectors of the completed cohomology of the modular curve. The results of [Pa21] were later generalized by Rodríguez Camargo ([RC22]), where as an application he resolved the rational Calegari-Emerton conjecture.

5 A brief description of the results of this thesis

This thesis consists of three projects, in which I have explored various aspects of Sen theory using the locally analytic theory, and gave applications to the p-adic Langlands program. In addition to abstracts appearing in the body of the text, I give some description of the ideas appearing in them, in light of this introduction.

I. Lubin-Tate theory and overconvergent Hilbert modular forms of low weight. In this paper, I develop some ideas and results in Sen theory in the K-analytic setting. These are natural extensions of some ideas of Berger appearing in [Be02] and [Be08]. The theory turns out to behave very elegantly. As a consequence, we prove some results in the Langlands program, namely, a new Lubin-Tate triangulinity result for the representations coming from Hilbert eigenforms of weight $[k, 1, ..., 1]$. This is an extension of a classical result of Kisin ([Ki03]) in the elliptic setting, and we hope that in the future this will imply more cases of the Fontaine-Mazur conjecture, as in the work of loc. cit. Some results of this paper have been used by Léo Poyeton, see ([Po22]).

II. Locally analytic vector bundles on the Fargues-Fontaine curve. In this paper, I was inspired by the work of Pan [Pa21] and more classical work of Berger ([Be08]) to develop a theory of locally analytic vector bundles on the Fargues-Fontaine curve. The idea was to understand more geometrically the decompletion theorem of Cherbonnier-Colmez, using the theory of locally analytic vectors. What turns out is that the theory behaves very nicely, and in particular the results of Berger can be interperted in terms of a Riemann-Hilbert correspondence on the Fargues-Fontaine curve. Along the way, we prove a fundamental theorem about the vanishing of higher locally analytic vectors in the Tate-Sen setting. This theorem explains an additional link between decompletion methods and locally analytic vectors. This result has been used by Poyeton (unpublished) and Gao-Min-Wang ([GMW22]). Further, Rodríguez Camargo proved a similar result in ([RC22]), a key result of that paper.

III. Overconvergence of étale (φ, Γ)-modules in families. In this paper, I resolve a conjecture of Emerton, Gee and Hellmann ([EGH22]) regarding overconvergence of (φ, Γ)-modules in families. I do this by generalizing results from my master thesis, and by developing a new variant of the Tate-Sen method of Berger-Colmez ([BC08]). This result has direct applications to the p-adic Langlands program, because it links between two stacks of Langlands parameters, as explained in ([EGH22]). In particular with some more work it should be possible to derive some properties of the "analytic" stack of Langlands parameters. Furthermore, it implies the results of Berger-Colmez ([BC08]) and Bellovin [Bel20].

6 Work in progress, questions and speculations

Finally, we discuss the future of the ideas discussed in this introduction

1. Sen theory beyond the cyclotomic setting. The work of Berger-Colmez ([BC16]) shows the existence of a Sen operator $\Theta_{\mathrm{Sen}}(\widehat{K}_\infty)$ for any infinitely ramified, Galois and p-adic Lie extension K_∞ over K. How much of Sen theory can be done in this more general setting? In some unpublished work, we explore these questions further. It turns out it is possible to carry out much of the cyclotomic theory, and interesting phenomenon happens, especially in the context of $\mathbf{B}_{\mathrm{dR}}^+$. In particular, $\mathbf{D}_{\mathrm{Sen}}$ and $\mathbf{D}_{\mathrm{dif}}^+$ make sense and have nice properties with respect to K_∞ (generalizing the cyclotomic case) for a large class of extensions. One surprising discovery we have found is that $\mathbf{B}_{\mathrm{dR}}^{+,\mathrm{pa}}$ is naturally a power series ring in one variable over its residue field $\widehat{K}_{\infty}^{\mathrm{la}}$ (unlike what happens for $\mathbf{B}_{\mathrm{dR}}^{+}$ and $\mathbf{C}_p$). This result uses the fundamental theorem of II mentioned above.

2. (φ, Γ)-modules and locally analytic vector bundles beyond the cyclotomic setting. A related and more difficult problem is to understand the locally analytic theory of (φ, Γ)-modules for general K_∞. This was essentially achieved by Berger ([Be16]) in both the cyclotomic and Lubin-Tate setting, but beyond that, not much is known and much is mysterious. In particular, the conjecture of Kedlaya ([Ke13]) mentioned above is the analogue of Cherbonnier-Colmez for general extensions. In work in progress with Hui Gao and Léo Poyeton, we investigate this question further. We believe the conjecture of Kedlaya is false, and the anticyclotomic extension is a counterexample! In fact in a certain sense the conjecture should fail generically - there should be an obstruction coming from locally analytic vectors. However, it is possible and likely that the decompletion for locally analytic vector bundles holds always, because the obstruction should vanish locally. This will give further justification to our study of these objects. Understanding of these questions should ultimately have application both to the p-adic Langlands program and to Iwasawa theory, as is already seen in the cyclotomic and Lubin-Tate settings.

3. Absolute Sen theory and the p-adic Simpson correspondence.

One question that came up in our investigations, and which we have not been able to answer, is the extent to which there exists a sort of universal theory which explains all the different instances of Sen theory. Among others, this theory should at least consist of:

i. A single framework giving rise to both the p-adic Simpson correspondence (see e.g. ([MW22, Wa21])) and the theory of Sen.

ii. An interpertation of the Sen operator as an arithmetic Higgs field, and a universal construction thereof.

iii. A dimensions theory for locally analytic vectors, having the feature that for an ind-syntomic, generically étale covering R_∞/R with Galois group Γ, we have some sort of equality

$$\dim_R \widehat{R}_{\infty}^{\mathrm{la}} + \text{number of Higgs fields} = \dim \Gamma$$

(generalizing the equality $d - 1 + 1 = d$ in the setting K_∞/K mentioned previously). Here I believe a letter of Fargues to Illusie ([Fa]) seems to have an indication where these Higgs fields might be coming from.

iv. Some sort of "locally analytic site" and a notion of a "locally analytic covering", which would imply vanishing of locally analytic cohomology, as in the work of Pan ([Pa21]). This idea was also observed by Rodríguez Camargo ([RC22]).

v. Compatiblity with a mixed characteristic notion (and not just a rational) version of “locally analytic vectors”. Note that a characteristic p analogue of the locally analytic condition was recently discovered by Berger and Rozensztajn ([BR22a, BR22b]), but the precise relationship with the characteristic p theory is still mysterious.

References

[AB08] Andreatta F., Brinon O., *Surconvergence des représentations p-adiques: le cas relatif*, Astérisque 319 (Représentations p-adiques de groupes p-adiques I : représentations galoisiennes et (φ, Γ)-modules), pp. 39-116 (2008).

[Bel20] Bellovin R., *Galois representations over pseudorigid spaces*, arXiv:2002.06687, 2020.

[Be02] Berger L., *Représentations p-adiques et équations différentielles*, Invent. Math. 148 (2002), no. 2, 219--284.

[Ber08] Berger L., *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Astérisque No. 319 (2008), 13--38.

[Be16] Berger L., *Multivariable (φ, Γ)-modules and locally analytic vectors*, Duke Math. J. 165 (2016), no. 18, 3567--3595.

[BC08] Berger L., Colmez P., *Familles de représentations de de Rham et monodromie p-adique*, Astérisque No. 319 (2008), 303--337.

[BC16] Berger L., Colmez P., *Théorie de Sen et vecteurs localement analytiques*, Ann. Sci. École Norm. Sup. 49 (2016), no. 4, 947--970.

[BR22a] Berger L., Rozensztajn S., *Super-Holder vectors and the field of norms.* arXiv:2209.02572 (2022).

[BR22b] Berger L., Rozensztajn S., *Decompletion of cyclotomic perfectoid fields in positive characteristic.* arXiv:2201.09688 (2022).

[Br03] Brinon O., *Une généralisation de la théorie de Sen,* Mathematische Annalen 327, pp. 793-813 (2003).

[CC98] Cherbonnier F., Colmez P., *Représentations p-adiques surconvergentes*, Invent. Math. 133 (1998), no. 3, p. 581–611.

[dSP19] de Shalit E., Porat G., *Induction and restriction of (φ, Γ)-modules*, Muenster J. Math., vol. 12 (2019), 215-237.

[EGH22] Emerton M., Gee T., Hellmann E.,*"An introduction to the categorical p-adic Langlands program"*, notes for the IHES summer school on the Langlands program (2022).

[Fa] Fargues L., *B_{dR} via de Rham dérivé,* letter to Illusie L.

[FF18] Fargues L, Fontaine J. M., *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Société mathématique de France (2018).

[GMW22] Gao H, Min Y, Wang Y. *de Rham crystals on the prismatic site of $\mathcal{O}_K$*. arXiv:2206.10276. 2022 Jun 21.

[Ke13] Kedlaya K. *Some slope theory for multivariate Robba rings*. arXiv preprint arXiv:1311.7468. 2013 Nov.

[Ki03] Kisin M., *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. 153 (2003), no. 2, p. 373–454.

[MW22] Min Y, Wang Y. *p-adic Simpson correpondence via prismatic crystals*. arXiv:2201.08030. 2022 Jan 20.

[Pa21] Pan L., *On locally analytic vectors of the completed cohomology of modular curves*, arXiv:2008.07099 (2021).

[Pa22] Pan L., *On locally analytic vectors of the completed cohomology of modular curves* II, arXiv:2209.06366 (2022).

[Po22] Poyeton L., *Trianguline periods,* arXiv:2202.08075 (2022).

[RC22] Rodríguez Camargo J. E., *Locally analytic completed cohomology of Shimura varieties and overconvergent BGG maps,* arXiv preprint arXiv:2205.02016 (2022).

[Se81] Sen S., *Continuous cohomology and p-adic Galois representations*, Invent. Math. 62 no. 1, p. 89–116 (1980/81).

[Se93] Sen, S., *An infinite dimensional Hodge-Tate theory*, Bulletin de la Société Mathématique de France 121.1 (1993): 13-34.

[Sh18] Shimizu, K. *Constancy of generalized Hodge–Tate weights of a local system*. Compositio Mathematica, 154(12), 2606-2642 (2018).

[Ta67] Tate, J.T., *p-Divisible groups*. In Proceedings of a conference on Local Fields (pp. 158-183). Springer, Berlin, Heidelberg (1967).

[Wa21] Wang Y. *A p-adic Simpson correspondence for rigid analytic varieties*. arXiv:2101.00263. 2021 Jan 1.

Part II

Lubin-Tate theory and overconvergent Hilbert modular forms of low weight

Abstract

Let K be a finite extension of $\mathbf{Q}_p$ and let Γ be the Galois group of the cyclotomic extension of K. Fontaine's theory gives a classification of p-adic representations of $\mathrm{Gal}\left(\overline{K}/K\right)$ in terms of (φ, Γ)-modules. A useful aspect of this classification is Berger's dictionary which expresses invariants coming from p-adic Hodge theory in terms of these (φ, Γ)-modules.

In this article, we use the theory of locally analytic vectors to generalize this dictionary to the setting where Γ is the Galois group of a Lubin-Tate extension of K. As an application, we show that if F is a totally real number field and v is a place of F lying above p, then the p-adic representation of $\mathrm{Gal}\left(\overline{F}_v/F_v\right)$ associated to a finite slope overconvergent Hilbert eigenform which is F_v-analytic up to a twist is Lubin-Tate trianguline. Furthermore, we determine a triangulation in terms of a Hecke eigenvalue at v. This generalizes results in the case $F = \mathbf{Q}$ obtained previously by Chenevier, Colmez and Kisin.

7 Introduction

Let p be a prime number. Kisin showed in [Ki03] that the p-adic representation ρ_f of $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ attached to a finite slope p-adic eigenform f has a very special property: its restriction to $\mathrm{Gal}\left(\overline{\mathbf{Q}}_p/\mathbf{Q}_p\right)$ always has a crystalline period. Even better, this period is an eigenvector for crystalline Frobenius, with eigenvalue coinciding with that arising from the Hecke action of U_p on f. Consequently, Kisin was able to verify the Fontaine-Mazur conjecture for these p-adic representations. In a subsequent work [Co08], Colmez gave a reinterpertation of Kisin's result to the effect that the (φ, Γ)-module attached to $\rho_f|_{\mathrm{Gal}\left(\overline{\mathbf{Q}}_p/\mathbf{Q}_p\right)}$ is an extension of (φ, Γ)-modules of rank 1, where $\Gamma \cong \mathbf{Z}_p^\times$ is the Galois group of the cyclotomic extension of $\mathbf{Q}_p$. Colmez coined the term "trianguline" for the p-adic representations satisfying this property, and studied them in detail in dimension 2. Then in [Co10] he attached to any 2-dimensional trianguline representation of $\mathrm{Gal}\left(\overline{\mathbf{Q}}_p/\mathbf{Q}_p\right)$ a unitary Banach representation of $\mathrm{GL}_2\left(\mathbf{Q}_p\right)$. By a suitable continuity argument he was able to extend this procedure to any 2-dimensional p-adic representation of Gal $\left(\overline{\mathbf{Q}}_p/\mathbf{Q}_p\right)$, thereby constructing the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. This circle of ideas came to a satisfying conclusion when Emerton used this correspondence in [Em11] to show that the trianguline property at Gal $\left(\overline{\mathbf{Q}}_p/\mathbf{Q}_p\right)$ characterizes these 2-dimensional representations of Gal $\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ which are attached to finite slope p-adic eigenforms.

In this article, we are concerned with performing the reinterpertation step of Colmez in an analogous story when $\mathbf{Q}$ is replaced with a totally real number field F. Namely, the p-adic representation ρ_f of $\mathrm{Gal}(\overline{F}/F)$ will be attached to a finite slope p-adic Hilbert eigenform f, and we would like to show ρ_f is trianguline at a place $v \mid p$. However, when $F_v \neq \mathbf{Q}_p$, there is more than one meaning one can attach to the phrase "ρ_f is trianguline at a place $v \mid p$". On the one hand, there are *cyclotomic trianguline* $\mathrm{Gal}(\overline{F}_v/F_v)$-representations. These are the trianguline representations in the sense of Nakamura in [Na09]; in that setting, Γ is the Galois group of the cyclotomic extension of F_v and is isomorphic to an open subgroup of $\mathbf{Z}_p^\times$. On the other hand, there are *Lubin-Tate trianguline* $\mathrm{Gal}(\overline{F}_v/F_v)$-representations in the sense of Fourquaux and Xie in [FX13], where $\Gamma = \Gamma_{F_v}$ is the Galois group of a Lubin-Tate extension and is isomorphic to $\mathcal{O}_{F_v}^\times$. The representation $\rho_f|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ has been known to be cyclotomic trianguline for a while by the global triangulation theory of Kedlaya-Pottharst-Xiao in [KPX14] and Liu in [Li12]. Although these results have found many applications, it seems like this notion of cyclotomic triangulinity may not be the most suitable for applications to a hypothetical p-adic Langlands correspondence for $\mathrm{GL}_2(F_v)$. Rather, the replacement of $\mathbf{Z}_p^\times$ by $\mathcal{O}_{F_v}^\times$ seems more natural, which leads us in this work to focus on the notion of Lubin-Tate triangulinity of Fourquaux and Xie.

Let us explain what the difficulties are in proving such a result when $F_v \neq \mathbf{Q}_p$. The methods of [KPX14] and [Li12] can still be used to show the existence of a crystalline period which is a Frobenius eigenvector. However, Colmez's reformulation of this condition in terms of (φ, Γ)-module for $F_v = \mathbf{Q}_p$ relies on Berger's dictionary, which expresses invariants coming from p-adic Hodge theory in terms of these (φ, Γ)-modules. This dictionary is only available in the cyclotomic setting. Indeed, the proof of this dictionary ultimately relies on Sen theory and the Cherbonnier-Colmez theorem. Unfortunately, a direct attempt to use these methods breaks down whenever $F_v \neq \mathbf{Q}_p$, because of the failure of the Tate-Sen axioms for Lubin-Tate extensions.

Now let K be a finite extension of $\mathbf{Q}_p$. Recall that a representation V of $\mathrm{Gal}(\overline{K}/K)$ with coefficients in K is called K-analytic if $\mathbf{C}_p \otimes_K^\tau V$ is trivial for each nontrivial embedding $\tau : K \to \overline{K}$. In [BC16], Berger and Colmez were able to find a certain generalization of Sen theory for Lubin-Tate extensions of K using ideas coming from the theory of K-locally analytic vectors. Berger then used this theory in [Be16] to prove that K-analytic representations are overconvergent, so that we can associate to V a Lubin-Tate (φ_q, Γ_K)-module $\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)$ over the (Lubin-Tate) Robba ring $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ (see §5). By adapting the original techniques of [Be02] to the setting of K-locally analytic vectors, we are able to extend Berger's dictionary to Lubin-Tate extensions of K. Our first main result is the following (see Theorem 5.5).

Theorem A. *Let V be a K-analytic representation of G_K. For*

$$* \in \{\mathrm{Sen}, \mathrm{dif}, \mathrm{dR}, \mathrm{cris}, \mathrm{st}\}$$

there is a natural isomorphism

$$\mathbf{D}_{*,K}(V) \cong \mathbf{D}_{*,K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)\right).$$

For the definition of the functors $\mathbf{D}_{*,K}$ we refer to §3. When $K = \mathbf{Q}_p$ they coincide with the usual definitions and the theorem is already known, but note that in contrast, it is not a-priori clear how to even define $\mathbf{D}_{\mathrm{Sen},K}$ and $\mathbf{D}_{\mathrm{dif},K}$ in the general case.

When V is 2-dimensional, one can deduce from Theorem A that the Lubin-Tate triangulinity of V can be detected from the existence of a crystalline period which is a Frobenius eigenvector. On the other hand, techniques going back to the original paper of Colmez show that the cyclotomic triangulinity of V can also be detected in a similar way. From this we deduce that the two notions of triangulinity actually coincide for K-analytic representations of dimension 2 (see Theorem 6.8 for a more precise version).

Theorem B. *Let V be a 2-dimensional K-analytic representation of G_K. Then V is Lubin-Tate trianguline if and only if V is cyclotomic trianguline.*

Although we do not pursue this here, our methods show that under some genericity assumptions the theorem is true for V of arbitrary dimension. On the other hand, after the completion of this paper, we have been informed that a previously unpublished result of Léo Poyeton [Po20] establishes an equivalence of categories between K-analytic Lubin-Tate (φ_q, Γ_K)-modules and K-analytic $\mathbf{B}$-pairs. As an immediate consequence, this gives an independent proof of Theorem B which extends to V of arbitrary dimension. Indeed, the triangulinity can be checked in terms of $\mathbf{B}$-pairs, and the rank 1 $\mathbf{B}$-pairs that appear in the triangulation are attached to both rank 1 cyclotomic (φ, Γ)-modules and to rank 1 Lubin-Tate (φ_q, Γ_K)-modules.

As mentioned above, it is known that the local Galois representations associated to finite slope overconvergent Hilbert eigenforms are cyclotomic trianguline. It is then natural to use Theorem B to translate this into a Lubin-Tate triangulinity result, provided that this local representation is analytic. Furthermore, it is possible to explicitly determine this triangulation, generalizing previous work of Chenevier and Colmez (see Theorem 7.4 for a more precise statement). To state the result, let ρ_f be the p-adic representation of $\mathrm{Gal}(\overline{F}/F)$ associated to a finite slope overconvergent Hilbert eigenform of weights $(k, 1, ..., 1)$ at v and determinant $\eta\chi_{\mathrm{cyc}}^{w-1}$ for some potentially unramified character η. For the sake of simplifying the introduction, assume here that the restriction of ρ_f to a decomposition group $G_{F_v} = \mathrm{Gal}(\overline{F}_v/F_v)$ has coefficients in F_v and that $k, w \in \mathbf{Z}$. Removing these assumptions only requires introducing more notation. To state the result, choose a uniformizer π_v of F_v, write χ_{π_v} for the corresponding Lubin-Tate character and let $a_v \in F_v^\times$ be such that $U_v f = a_v f$. If $y \in F_v^\times$, write $\mu_y : F_v^\times \to F_v^\times$ for the character defined by $\mu_y(z) = y^{\mathrm{val}_{\pi_v}(z)}$. Let $x : F_v^\times \to F_v^\times$ be the character $x(z) = z$ and $x_0 : F_v^\times \to F_v^\times$ be the character $x_0(z) = x/\mu_{\pi_v}$. We say that f is F_v-analytic up to a twist if the same holds for $\rho_f|_{G_{F_v}}$.

Theorem C. *Suppose that f is F_v-analytic up to a twist. Then it is Lubin-Tate trianguline. If $\mathbf{D}^\dagger_{\mathrm{rig},F_v}\left(\rho_f|^\vee_{G_{F_v}}\right)$ is the $(\varphi_q, \Gamma_{F_v})$-module over $\mathbf{B}^\dagger_{\mathrm{rig},F_v}$ associated to $\rho_f|^\vee_{G_{F_v}}$, a triangulation is given by the short exact sequence*

$$0 \to \mathbf{B}^\dagger_{\mathrm{rig},F_v}(\delta_1) \to \mathbf{D}^\dagger_{\mathrm{rig},K}\left(\rho_f|^\vee_{G_{F_v}}\right) \to \mathbf{B}^\dagger_{\mathrm{rig},F_v}(\delta_2) \to 0,$$

where $\delta_2 = \delta_1^{-1}\det(V)$ and $\delta_1 : F_v^\times \to F_v^\times$ is a character. Here and δ_1 and $\rho_f|_{G_{F_v}}$ satisfy the following.

1. *If $k \notin \mathbf{Z}_{\geq 1}$ then $\delta_1 = \mu_{a_v} x_0^{\frac{k-1}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ x_0\right)^{\frac{1-w}{2}}$ and $\rho_f|_{G_{F_v}}$ is irreducible and not Hodge-Tate.*

2. *If $k \in \mathbf{Z}_{\geq 1}$ and $\mathrm{val}_{\pi_v}(a_v) < \frac{k-1}{2} + \frac{w-1}{2}[F_v : \mathbf{Q}_p]$, then*

$$\delta_1 = \mu_{a_v} x_0^{\frac{k-1}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ x_0\right)^{\frac{1-w}{2}}$$

   and $\rho_f|_{G_{F_v}}$ is irreducible and potentially semistable.

3. *If $k \in \mathbf{Z}_{>1}$ and $\mathrm{val}_{\pi_v}(a_v) = \frac{k-1}{2} + \frac{w-1}{2}[F_v : \mathbf{Q}_p]$,* then either

   (a) $\delta_1 = \mu_{a_v} x_0^{\frac{k-1}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ x_0\right)^{\frac{1-w}{2}}$ and *$\rho_f|_{G_{F_v}}$ is reducible, nonsplit and potentially ordinary, or*

   (b) $\delta_1 = x^{1-k}\mu_{a_v} x_0^{\frac{k-1}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ x_0\right)^{\frac{1-w}{2}}$ and *$\rho_f|_{G_{F_v}}$ is a sum of two characters and potentially crystalline.*

4. *If* $k \in \mathbf{Z}_{\geq 1}$ *and* $\mathrm{val}_{\pi_v}(a_v) > \frac{k-1}{2} + \frac{w-1}{2}[F_v : \mathbf{Q}_p]$*, then*

$$\delta_1 = x^{1-k}\mu_{a_v}x_0^{\frac{k-1}{2}}\left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ x_0\right)^{\frac{1-w}{2}}$$

and $\rho_f|_{G_{F_v}}$ *is irreducible, Hodge-Tate and not potentially semistable.*

The condition on the weights at v to be of the form $(k, 1, ..., 1)$ is necessary but not sufficient for f to be F_v-analytic up to a character twist. In fact, the computations of [Na09, Proposition 2.10] and [FX13, Theorem 0.3] suggest that this stronger condition of F_v-analyticity cuts out a locus of codimension $[F_v : \mathbf{Q}_p] - 1$ inside the locus of weights $(k, 1, ..., 1)$ at v of the Hilbert eigenvariety. However, under suitable local-global compatibility conjectures, it contains all classical points of weights $(k, 1, ..., 1)$. We believe it might be possible to obtain a version of Theorem C for arbitrary f if one works with $(\varphi_q, \Gamma_{F_v})$-modules over multivariable Robba rings as in [Be13].

Finally, we make some further speculations. For simplicity, assume that p is inert in F. The small slope condition $\mathrm{val}_p(a_p) < \frac{k-1}{2} + \frac{w-1}{2}[F_p : \mathbf{Q}_p]$ in Theorem C agrees with the optimal bound in partial weight 1 conjectured in an unpublished note of Breuil (see Proposition 4.3 of [Br10]). This suggests that F_p-analytic finite slope p-adic Hilbert eigenforms of weights $(k, 1, ..., 1)$ and $\mathrm{val}_p(a_p) < \frac{k-1}{2} + \frac{w-1}{2}[F_p : \mathbf{Q}_p]$ should be classical. If such a classicality criterion were known, an argument as in Theorem 6.6 of [Ki03] using our Theorem 7.4 would verify the Fontaine-Mazur conjecture for representations which arise from F_v-analytic finite slope p-adic Hilbert eigenforms. Conversely, if the Fontaine-Mazur conjecture were known in our context then Theorem 7.4 would imply such a classicaility criterion. See §7.2 for a more precise discussion.

In another direction, suppose again that p is inert in F and that V is a p-adic representation of $\mathrm{Gal}(\overline{F}/F)$ which is irreducible, totally odd, unramified at almost all primes and Lubin-Tate trianguline at p. Recall again the theorem of Emerton (Theorem 1.2.4 of [Em11]) which asserts that if $F = \mathbf{Q}$ and $\overline{V}$ satisfies certain technical conditions then V is the character twist of the Galois representation attached to an (elliptic) overconvergent p-adic eigenform of finite slope. In light of Theorem 7.4, we ask the following.

Question 7.1. *Is* V *necessarily the character twist of a representation attached to an* F_p*-analytic overconvergent* p*-adic Hilbert eigenform of finite slope?*

7.1 Structure of the article

§2 contains preliminaries regarding locally K-analytic vectors. In §3 we define the functors $\mathbf{D}_{*,K}$ for

$$* \in \{\mathrm{Sen}, \mathrm{dif}, \mathrm{dR}, \mathrm{cris}, \mathrm{st}\}$$

and prove their basic properties. In §4 we study some big period rings and their K-locally analytic vectors. Theorem A is proved in §5. This proof involves reinterperting several constructions in p-adic Hodge theory in terms of K-locally analytic vectors, as well as some computations with rather large rings of periods, and so involves most of the work done in §§2-5. In §6 we relate this to Lubin-Tate triangulinity and prove Theorem B. Finally, in §7 we prove Theorem C, concluding with an example.

7.2 Notations and conventions

The field K denotes a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field k. The field E is a finite extension of $\mathbf{Q}_p$ which contains K. It will serve as a field of coefficients for the objects we consider. The field $K_0 = \mathrm{W}(k)[1/p]$ is the maximal unramified subextension of K. Let $q = p^f$ be the cardinality of the residue field and e the absolute ramification index of K, so that $[K : \mathbf{Q}_p] = ef$. We let Σ_K denote the set of embeddings of K into $\overline{\mathbf{Q}}_p$.

Denote by G_K the absolute Galois group of K. If $\mathcal{F}$ is the formal Lubin-Tate group associated to π, then $K_n = K(\mathcal{F}[\pi^n])$ and $K_\infty = \cup_{n\geq 1} K_n$ are abelian extensions of K which depend only on π. The Lubin-Tate character $\chi_\pi : G_K \to \mathcal{O}_K^\times$ is the character given by the action of G_K on $\mathcal{F}[\pi^\infty]$. It induces an isomorphism of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ with $\mathcal{O}_K^\times$. Its kernel is $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$, and $G_K/H_K = \Gamma_K$. The cyclotomic character χ_{cyc} of G_K satisfes the relation $\mathrm{N}_{K/\mathbf{Q}_p} \circ \chi_\pi = \chi_{\mathrm{cyc}}\eta$ for an unramified character η.

An E-linear representation V of $\mathrm{Gal}(\overline{K}/K)$ is called K-analytic if $\mathbf{C}_p \otimes_K^\tau V$ is trivial for each $\tau \in \Sigma_K \setminus \{\mathrm{Id}\}$.

Finally, all characters and representations appearing in this article are assumed to be continuous. We normalize the p-adic valuation and p-adic logarithm so that $\mathrm{val}_p(p) = 1$ and $\log(p) = 0$. The Hodge-Tate weight of χ_{cyc} is 1.

7.3 Acknowledgments

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8 Locally K-analytic and pro K-analytic vectors

In this section we give reminders on locally analytic and pro analytic vectors and gather a few results that will be used in §3, §4 and §5.

8.1 Locally analytic and pro analytic vectors

We briefly recall the treatment given in §2 of [Be16] and in §2 of [BC16].

Let W be a Banach $\mathbf{Q}_p$-linear representation of Γ_K. Given an open subgroup H of Γ_K with coordinates $c_1, ..., c_d : H \xrightarrow{\sim} \mathbf{Z}_p^d$, we have the subspace $W^{H-\mathrm{an}}$ of H-analytic vectors in W. These are the elements $w \in W$ for which there exists a sequence of vectors $\{w_k\}_{k\in\mathbf{N}^d}$ with $w_k \to 0$ and $g(w) = \sum_{k\in\mathbf{N}^d} c_k(g)^k w_k$ for all $g \in H$. We write $W^{\mathrm{la}} = \cup_H W^{H-\mathrm{an}}$ for the subspace of locally analytic vectors of W. If W is a Fréchet space whose topology is defined by a countable sequence of seminorms, let W_i be the Hausdorff completion of W for the i'th norm, so that $W = \varprojlim W_i$ is a projective limit of Banach spaces. We write $W^{\mathrm{pa}} = \varprojlim W_i^{\mathrm{la}}$ for the subspace of pro analytic vectors.

For $n \gg 0$, we have an isomorphism $l : \Gamma_{K_n} \to \pi^n \mathcal{O}_K$, given by $g \mapsto \log(\chi_\pi(g))$. We have the subspace $W^{\Gamma_{K_n}-\mathrm{an}, K-\mathrm{la}}$ of vectors which are K-analytic on Γ_{K_n}, i.e. such that there exists a sequence $\{w_k\}_{k\in\mathbb{N}}$ with $\pi^{nk} w_k \to 0$ and $g(w) = \sum_{k\in\mathbb{N}^d} l(g)^k w_k$ for all $g \in \Gamma_{K_n}$. We write

$$W^{K-\mathrm{la}} = \cup_{n\gg 0} W^{\Gamma_{K_n}-\mathrm{an}, K-\mathrm{la}}$$

for the subspace of K-locally analytic vectors of W. If $W = \varprojlim W_i$ is a Fréchet space as above, we write $W^{K-\mathrm{pa}} = \varprojlim W_i^{K-\mathrm{la}}$ for the subspace of pro K-analytic vectors. We extend the definitions of locally K-analytic vectors and pro K-analytic vectors to LB and LF spaces (i.e. filtered colimits of Banach spaces and Fréchet spaces) in the obvious way.

For each $\tau \in \Sigma_K$, there is a differential operator $\nabla_\tau \in K^{\mathrm{Gal}} \otimes_{\mathbf{Q}_p} \mathrm{Lie}(\Gamma_K)$ (see §2 of [Be16]) where K^{Gal} is the Galois closure of K. It is defined in such a way that if W is K^{Gal}-linear, then for $w \in W^{\mathrm{la}}$ and $g \in \Gamma_{K_n}$ with

$n \gg 0$ we have $g(w) = \sum_{k \in \mathbf{N}^{\Sigma_K}} l(g)^k \frac{\nabla^k(w)}{k!}$, where $l(g)^k = \prod_{\tau \in \Sigma_K} \tau(l(g))^{k_\tau}$ and $\nabla^k(w) = \prod_{\tau \in \Sigma_K} \nabla_\tau(w)^{k_\tau}$. In other words, we can think of $\tau \circ l$ as giving coordinates for Γ_K and $\nabla^k(w)$ as being an iterated directional derivative of w. In particular, $W^{K-\mathrm{la}}$ is the subspace of W^{la} where $\nabla_\tau = 0$ for each $\tau \in \Sigma_K \setminus \{\mathrm{Id}\}$. On $W^{K-\mathrm{la}}$ (or on $W^{K-\mathrm{pa}}$ if W is Fréchet) we write $\nabla = \nabla_{\mathrm{Id}}$ when there is no danger of confusion; it is given by the forumla

$$\nabla(w) = \lim_{g \to 1} \frac{g(w) - w}{\log(\chi_\pi(g))},$$

and we have $\nabla(w) = \frac{\log(g)(w)}{\log(\chi_\pi(g))}$ when g is sufficiently close to 1.
The next lemma is proved in the same way as Proposition 2.2 and Proposition 2.4 of [Be16].

Lemma 8.1. *Let B be a Banach (resp. Fréchet) Γ_K-ring and let W be a free B-module of finite rank, equipped with a compatible action of Γ_K. If the B module has a basis $w_1, ..., w_d$ in which the function $\Gamma_K \to \mathrm{GL}_d(B) \subset M_d(B)$, $g \mapsto \mathrm{Mat}(g)$ is locally K-analytic (resp. pro K-analytic), then $W^{K-\mathrm{la}} = \oplus_{j=1}^d B^{K-\mathrm{la}} w_i$ (resp. $W^{K-\mathrm{pa}} = \oplus_{j=1}^d B^{K-\mathrm{pa}} w_i$).*

8.2 Locally analytic vectors in $\widehat{K}_\infty$-semilinear representations

Let L be a finite extension of K, and write $\Gamma_L = \mathrm{Gal}(L_\infty/L)$ where $L_\infty = LK_\infty$. Recall that following result (Proposition 2.10 of [Be16]):

Proposition 8.2. *$\widehat{L}_\infty^{K-\mathrm{la}} = L_\infty$.*

The purpose of this subsection is to prove a similar descent result for representations.

Theorem 8.3. *Let W be a finite dimensional $\widehat{L}_\infty$-semilinear representation of Γ_L. Then the natural map $\widehat{L}_\infty \otimes_{L_\infty} W^{K-\mathrm{la}} \to W$ is an isomorphism.*

This is proved in §4 of [BC16] under the assumption that K is Galois over $\mathbf{Q}_p$. Here we shall adapt the methods of ibid. to get rid of this assumption. First, we reduce to the case where L is Galois over $\mathbf{Q}_p$.

Lemma 8.4. *Suppose that Theorem 2.3 holds for $M = L^{\mathrm{Gal}}$, the Galois closure of L over $\mathbf{Q}_p$. Then Theorem 2.3 holds for L.*

Proof. Let L be any finite extension of K and let W be a finite dimensional $\widehat{L_\infty}$-semilinear representation of Γ_K. Write $W_M = \widehat{M}_\infty \otimes_{\widehat{L}_\infty} W$, so that W_M is a finite dimensional $\widehat{M}_\infty$-semilinear representation of Γ_M. Note that W_M

is actually endowed with a semilinear $\mathrm{Gal}(M_\infty/L)$-action, which restricts to a Γ_M action. By the assumption, we have $\widehat{M}_\infty \otimes_{M_\infty} W_M^{K-\mathrm{la}} \cong W_M$. On the other hand, the extension $\mathrm{Gal}(M_\infty/L_\infty)$ is finite, so we are in the setting for completed Galois descent (see §2.2 of [BC09]). We have

$$W^{K-\mathrm{la}} = W_M^{K-\mathrm{la}} \cap W = \left(W_M^{K-\mathrm{la}}\right)^{\mathrm{Gal}(M_\infty/L_\infty)}$$

which implies that $\widehat{M}_\infty \otimes_{M_\infty} W_M^{K-\mathrm{la}} \cong \widehat{M}_\infty \otimes_{L_\infty} W^{K-\mathrm{la}}$. We then have the following chain of natural isomorphisms

$$\begin{aligned}
\widehat{L}_\infty \otimes_{L_\infty} W^{K-\mathrm{la}} &\cong \left(\widehat{M}_\infty \otimes_{L_\infty} W^{K-\mathrm{la}}\right)^{\mathrm{Gal}(M_\infty/L_\infty)} \\
&\cong \left(\widehat{M}_\infty \otimes_{M_\infty} W_M^{K-\mathrm{la}}\right)^{\mathrm{Gal}(M_\infty/L_\infty)} \\
&\cong (W_M)^{\mathrm{Gal}(M_\infty/L_\infty)} \\
&\cong W
\end{aligned}$$

whose composition is the natural map $\widehat{L}_\infty \otimes_{L_\infty} W^{K-\mathrm{la}} \to W$, which proves the claim. □

Proposition 8.5. *If $\tau \in \Sigma_K \setminus \{\mathrm{Id}\}$ and $K^{\mathrm{Gal}} \subset L$, there exists an element $x_\tau \in \widehat{L}_\infty$ such that $g(x_\tau) = x_\tau + \tau(l(g))$ for $g \in G_{K^{\mathrm{Gal}}}$. In particular, $\nabla_\tau(x_\tau) = 1$ and $\nabla_\sigma(x_\tau) = 0$ for $\sigma \neq \tau$.*

Proof. By §3.2 of [Fo09], there exists an element $\xi_\tau \in \mathbf{C}_p^\times$ such that $\xi_\tau/g(\xi_\tau) = \tau(\chi_\pi(g))$ for $g \in G_{K^{\mathrm{Gal}}}$. This equation makes it clear that ξ_τ lies in the completion of $K^{\mathrm{Gal}}K_\infty$, which is contained in $\widehat{L}_\infty$. Now take $x_\tau = -\log \xi_\tau$. □

For each $n \geq 1$ and for each $\tau \neq \mathrm{Id}$, choose x_τ as in Proposition 2.5 and let $x_{n,\tau} \in L_\infty$ be such that $||x_\tau - x_{n,\tau}|| \leq p^{-n}$. For $k \in \mathbf{N}^{\Sigma_K \setminus \{\mathrm{Id}\}}$ we write $(x - x_n)^k = \prod_{\tau \in \Sigma_K \setminus \{\mathrm{Id}\}} (x_\tau - x_{n,\tau})^{k_\tau}$.

Proof of Theorem 2.3. By Lemma 2.4, we may assume L is Galois over $\mathbf{Q}_p$. Recall that by Theorem 1.7 of [BC16], the natural map $\widehat{L}_\infty \otimes_{\widehat{L}_\infty^{\mathrm{la}}} W^{\mathrm{la}} \to W$ is an isomorphism. Therefore, it is enough to prove that the natural map $\widehat{L}_\infty^{\mathrm{la}} \otimes_{L_\infty} W^{K-\mathrm{la}} \to W^{\mathrm{la}}$ is an isomorphism. To prove injectivity, take $\sum_{i=1}^n \alpha_i \otimes x_i \in \widehat{L}_\infty^{\mathrm{la}} \otimes_{L_\infty} W^{K-\mathrm{la}}$ of minimal length such that $\sum_{i=1}^n \alpha_i x_i = 0$. We may assume that $\alpha_1 = 1$. For each $\tau \neq \mathrm{Id}$, we have $\nabla_\tau \left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=2}^n \nabla_\tau(\alpha_i) x_i$, so by minimality $\nabla_\tau(\alpha_i) = 0$ for all i. This means that each $\alpha_i \in L_\infty$, so $\sum_{i=1}^n \alpha_i \otimes x_i = 0$.

To prove surjectivity, we give a sketch, omitting all details of convergence; these can be provided in exactly the same way as in §4 of [BC16]. For each $z \in W^{\mathrm{la}}$, and for each $i \in \mathbf{N}^{\Sigma_K \setminus \{\mathrm{Id}\}}$, let

$$y_i = \sum_{k \in \mathbf{N}^{\Sigma_K \setminus \{\mathrm{Id}\}}} (-1)^{|k|} (x - x_n)^k \frac{\nabla^{k+i}(z)}{(k+i)!} \binom{k+i}{k}.$$

One can show that $y_i \in W^{\mathrm{la}}$. By Proposition 2.5, for each $\tau \in \Sigma_K \setminus \{\mathrm{Id}\}$ we have $\nabla_\tau\left((x-x_n)^k\right) = k_\tau (x-x_n)^{k-1_\tau}$, where 1_τ is the tuple $(k_\sigma) \in \mathbf{N}^{\Sigma_K}$ with $k_\tau = 1$ and $k_\sigma = 0$ for $\sigma \neq \tau$. By a direct calculation this implies that $\nabla_\tau(y_i) = 0$, so that $y_i \in W^{K-\mathrm{la}}$. Finally, the identity

$$z = \sum_{i \in \mathbf{N}^{\Sigma_K \setminus \{\mathrm{Id}\}}} y_i (x-x_n)^i$$

shows that $z \in \widehat{L}_\infty \otimes_{\widehat{L}_\infty^{\mathrm{la}}} W^{\mathrm{la}}$. □

8.3 Pro analytic vectors in $\mathbf{B}_{\mathrm{dR}}$

The ring $\mathbf{B}_{\mathrm{dR}}^+$ contains an element t_K for which each $g \in G_K$ acts by $g(t_K) = \chi_\pi(g) t_K$. It differs from the usual t by a unit, but it has the advantage that it carries an action of Γ_K, which is moreover K-analytic. As $\mathbf{B}_{\mathrm{dR}}^+/t_K \cong \mathbf{C}_p$, the quotients $\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)^{H_K}$ for $l \geq 1$ are Banach Γ_K-rings. The ring and $\left(\mathbf{B}_{\mathrm{dR}}^+\right)^{H_K}$ is a Fréchet Γ_K-ring and $\left(\mathbf{B}_{\mathrm{dR}}\right)^{H_K}$ is an LF Γ_K-ring.

Proposition 8.6. *1.* $\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)^{H_K, K-\mathrm{la}} = K_\infty[t_K]/t_K^l$.
2. $\left(\mathbf{B}_{\mathrm{dR}}^+\right)^{H_K, K-\mathrm{pa}} = K_\infty[[t_K]]$.
3. $\left(\mathbf{B}_{\mathrm{dR}}\right)^{H_K, K-\mathrm{pa}} = K_\infty\left((t_K)\right)$.

Proof. (3) follows from (2) and (2) follows from (1). To prove (1), we argue by induction. For $l = 1$, this is Proposition 2.2. For $l \geq 2$, we have a short exact sequence

$$0 \to \mathbf{C}_p(l-1) \to \mathbf{B}_{\mathrm{dR}}^+/t_K^l \to \mathbf{B}_{\mathrm{dR}}^+/t_K^{l-1} \to 0.$$

Taking H_K invariants and K-locally analytic vectors is left exact, so we have

$$0 \to K_\infty(l-1) \to \left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)^{H_K, K-\mathrm{la}} \to \left(\mathbf{B}_{\mathrm{dR}}^+/t_K^{l-1}\right)^{H_K, K-\mathrm{la}} = K_\infty[t_K]/t_K^{l-1}.$$

This shows that $\dim_{K_\infty} \left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)^{H_K, K-\mathrm{la}} \leq l$, so the containment

$$K_\infty[t_K]/t_K^l \subset \left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)^{H_K, K-\mathrm{la}}$$

has to be an equality, concluding the proof. □

9 Lubin-Tate p-adic Hodge theory

To goal of this section is to provide constructions and properties of several of Fontaine's functors on p-adic representations where $\mathbf{Q}_p$-coefficients are systematically replaced by K-coefficients. Recall from §1.2 that E is a finite extension of K. Throughout, we fix an E-linear G_K-representation V of dimension d over K.

9.1 The modules $\mathbf{D}_{\mathrm{Sen},K}$ and $\mathbf{D}_{\mathrm{dif},K}$

When $K = \mathbf{Q}_p$, the modules $\mathbf{D}_{\mathrm{Sen},K}$ and $\mathbf{D}_{\mathrm{dif},K}$ can be defined using the method of Sen (see §4 of [BC08]). It is unavailable for $K \neq \mathbf{Q}_p$, so we make use of locally analytic and pro analytic vectors instead.

We set $W_{+,l} = \left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l \otimes_K V\right)^{H_K}$ for $l \geq 1$, $W_+ = \left(\mathbf{B}_{\mathrm{dR}}^+ \otimes_K V\right)^{H_K}$ and $W = (\mathbf{B}_{\mathrm{dR}} \otimes_K V)^{H_K}$. By Proposition 2.6, we have $K_\infty[t_K]/t_K^l$-submodules $\mathbf{D}_{\mathrm{dif},K}^{+,l}(V) = W_{+,l}^{K-\mathrm{la}}$ for $l \geq 1$, a $K_\infty[[t_K]]$-submodule $\mathbf{D}_{\mathrm{dif},K}^{+}(V) = W_{+}^{K-\mathrm{pa}}$ and a $K_\infty((t_K))$-vector space $\mathbf{D}_{\mathrm{dif},K}(V) = W^{K-\mathrm{pa}}$. The subspace $\mathbf{D}_{\mathrm{dif},K}^{+,1}(V)$ is also called $\mathbf{D}_{\mathrm{Sen},K}(V)$, and was already constructed in [BC16].

Lemma 9.1. *The natural map $\mathbf{B}_{\mathrm{dR}}^+/t_K^l \otimes_{K_\infty[t_K]/t_K^l} W_{+,l} \to \mathbf{B}_{\mathrm{dR}}^+/t_K^l \otimes_K V$ is an isomorphism.*

Proof. It suffices to prove that $\mathrm{H}^1\left(H_K, \mathrm{GL}_d\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)\right) = 1$. When $l = 1$ this is true by almost étale descent. For $l \geq 2$, we have a short exact sequence

$$1 \to I + t_K^{l-1}\mathrm{M}_d\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right) \to \mathrm{GL}_d\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right) \to \mathrm{GL}_d\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^{l-1}\right) \to 1.$$

As $I + t_K^{l-1}\mathrm{M}_d\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right) \cong \mathrm{M}_d\left(\mathbf{C}_p(l-1)\right)$, the group

$$\mathrm{H}^1\left(H_K, I + t_K^{l-1}\mathrm{M}_d\left(\mathbf{B}_{\mathrm{dR}}^+/t_K^l\right)\right)$$

is trivial, and we conclude by induction. □

We will also need the following.

Lemma 9.2. *Let $w \in W_{+,l}$ and suppose that $t_K \cdot w \in \mathbf{D}_{\mathrm{dif},K}^{+,l}(V)$. Then $w \in \mathbf{D}_{\mathrm{dif},K}^{+,l}(V)$.*

Proof. Since $\nabla(t_K w) = t_K(w + \nabla(w))$, we have that $\frac{\nabla^k(t_K w)}{k!}$ is divisible by t_K for $k \geq 1$. If $t_K \cdot w \in \mathbf{D}_{\mathrm{dif},K}^{+,l}(V)$, there exists an $n \gg 0$ such that for $g \in \Gamma_n$, we have $g(t_K w) = \sum_{k\geq 0} l(g)^k t_K w_k$, where $w_k = t_K^{-1}\frac{\nabla^k(t_K w)}{k!}$. Therefore, $g(w) = \chi_\pi(g^{-1})\sum_{k\geq 0} l(g)^k w_k$, so w is locally K-analytic. □

Proposition 9.3. *1. The natural map $\mathbf{B}_{\mathrm{dR}}^+/t_K^l \otimes_{K_\infty[t_K]/t_K^l} \mathbf{D}_{\mathrm{dif},K}^{+,l}(V) \to \mathbf{B}_{\mathrm{dR}}^+/t_K^l \otimes_K V$ is an isomorphism, and $\mathbf{D}_{\mathrm{dif},K}^{+,l}(V)$ is a free $K_\infty[t_K]/t_K^l$-module of rank d.*

2. The natural map $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{K_\infty[[t_K]]} \mathbf{D}_{\mathrm{dif},K}^{+}(V) \to \mathbf{B}_{\mathrm{dR}}^+ \otimes_K V$ is an isomorphism, and $\mathbf{D}_{\mathrm{dif},K}^{+}(V)$ is a free $K_\infty[[t_K]]$-module of rank d.

3. The natural map $\mathbf{B}_{\mathrm{dR}} \otimes_{K_\infty((t_K))} \mathbf{D}_{\mathrm{dif},K}(V) \to \mathbf{B}_{\mathrm{dR}} \otimes_K V$ is an isomorphism, and $\mathbf{D}_{\mathrm{dif},K}(V)$ is a $K_\infty((t_K))$-vector space of dimension d.

Proof. Recall that $\mathbf{D}^{+,l}_{\mathrm{dif},K}(V) = W^{K-\mathrm{la}}_{+,l}$. By Lemma 3.1, proving (1) reduces to showing that the natural map $\widehat{K}_\infty \otimes_{K_\infty} W^{K-\mathrm{la}}_{+,l} \to W_{+,l}$ is an isomorphism and that $W^{K-\mathrm{la}}_{+,l}$ is a free $K_\infty[t_K]/t_K^l$-module of rank d. By Theorem 2.3, this is true if $l = 1$. For $l \geq 2$, we have a short exact sequence

$$0 \to W^{K-\mathrm{la}}_{+,1}(l-1) \to W^{K-\mathrm{la}}_{+,l} \to W^{K-\mathrm{la}}_{+,l-1}.$$

By the case $l = 1$, we know that $W^{K-\mathrm{la}}_{+,1}(l-1)$ contains linearly independent elements $e_1, ..., e_d$ which are all divisible by t_K^{l-1}. Writing $f_i = t_K^{1-l} e_i$ for $1 \leq i \leq d$, the elements $f_1, ..., f_d$ span a free submodule $W'_{+,l}$ of $W_{+,l}$ which surjects onto $W_{+,l-1}$ and which contains $W_{+,1}$; so $W'_{+,l} = W_{+,l}$. It now suffices to show that the f_i are locally K-analytic, and this follows from Lemma 3.2. This concludes the proof of (1).

As each $W^{K-\mathrm{la}}_{+,l}$ is a free $K_\infty[t_K]/t_K^l$-module of rank d, we have that $W^{K-\mathrm{pa}}_{+} = \varprojlim W^{K-\mathrm{la}}_{+,l}$ is a free $K_\infty[[t_K]]$-module of rank d, and the chain of isomorphisms

$$\begin{aligned}
\mathbf{B}^+_{\mathrm{dR}} \otimes_{K_\infty[[t_K]]} \mathbf{D}^+_{\mathrm{dif},K}(V) &\cong \mathbf{B}^+_{\mathrm{dR}} \otimes_{K_\infty[[t_K]]} W^{K-\mathrm{pa}} \\
&\cong \varprojlim \left(\mathbf{B}^+_{\mathrm{dR}}/t_K^l \otimes_{K_\infty[t_K]/t_K^l} W_l^{K-\mathrm{la}} \right) \\
&\cong \varprojlim \left(\mathbf{B}^+_{\mathrm{dR}}/t_K^l \otimes_K V \right) \\
&\cong \mathbf{B}^+_{\mathrm{dR}} \otimes_K V,
\end{aligned}$$

whose composition is the natural map $\mathbf{B}^+_{\mathrm{dR}} \otimes_{K_\infty[[t_K]]} \mathbf{D}^+_{\mathrm{dif},K}(V) \to \mathbf{B}^+_{\mathrm{dR}} \otimes_K V$. This proves (2), and (3) follows immediately since

$$\mathbf{D}_{\mathrm{dif},K}(V) = \mathrm{colim}_i \mathbf{D}^+_{\mathrm{dif},K}(V(i)).$$

□

Recall from §2 that the modules $\mathbf{D}_{\mathrm{Sen},K}$ and $\mathbf{D}_{\mathrm{dif},K}$ are both endowed with a canonical differential operator. We write $\Theta_{\mathrm{Sen},K}$,$\nabla_{\mathrm{dif},K}$ respectively for the operators acting on $\mathbf{D}_{\mathrm{Sen},K}$,$\mathbf{D}_{\mathrm{dif},K}$. The operator $\Theta_{\mathrm{Sen},K}$ is K_∞-linear, while $\nabla_{\mathrm{dif},K}$ is a derivation over $\nabla_{K_\infty((t_K))} = t_K \frac{\partial}{\partial t_K}$.

The following result serves to complete the analogy with the usual $\mathbf{D}_{\mathrm{Sen}}$.

Proposition 9.4. *The following are equivalent.*

1. $\mathbf{C}_p \otimes_K V \cong \oplus_{i=1}^d \mathbf{C}_p(\chi_\pi^{n_i})$*, where the* $n_i \in \mathbf{Z}$*.*

2. $\mathbf{D}_{\mathrm{Sen},K}(V) \cong \oplus_{i=1}^d K_\infty(\chi_\pi^{n_i})$*, where the* $n_i \in \mathbf{Z}$*.*

3. $\Theta_{\mathrm{Sen},K}$ *is semisimple with integer eigenvalues* $\{n_i\}_{i=1}^d$*.*

Proof. Suppose $v_1, ..., v_d$ is a basis of $\mathbf{C}_p \otimes_K V$ for which $g(v_i) = \chi_\pi^{n_i}(g) v_i$ for $g \in G_K$. The action of G_K on each v_i factors through Γ_K and is locally K-analytic, so $v_i \in \mathbf{D}_{\mathrm{Sen},K}(V)$, which shows (1) implies (2). Next, (2) implies

(3) because the action of $\Theta_{\mathrm{Sen},K}$ on $K_\infty(\chi_\pi^{n_i})$ is given by multiplication with n_i. Finally, suppose that (3) holds, and let $v_1, ..., v_d$ be a basis of $\mathbf{D}_{\mathrm{Sen},K}(V)$ for which $\Theta_{\mathrm{Sen},K}(v_i) = n_i v_i$. By integrating the action of Γ_K, we see that $g \in \Gamma_K$ acts by $g(v_i) = \eta_i(g)\chi_\pi^{n_i}(g)v_i$, where η_i is a finite order character of Γ_K. Then $\mathbf{C}_p(\eta_i\chi_\pi^{n_i}) \cong \mathbf{C}_p(\chi_\pi^{n_i})$, so

$$\begin{aligned}\mathbf{C}_p \otimes_K V &\cong \mathbf{C}_p \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen},K}(V)\\ &\cong \oplus_{i=1}^d \mathbf{C}_p(\eta_i\chi_\pi^{n_i})\\ &\cong \oplus_{i=1}^d \mathbf{C}_p(\chi_\pi^{n_i}).\end{aligned}$$

□

If the conditions of Proposition 3.4 hold for some $n_i \in \mathbf{Z}$, the n_i are called the K-Hodge-Tate weights of V.

9.2 The modules $\mathbf{D}_{\mathrm{HT},K}$ and $\mathbf{D}_{\mathrm{dR},K}$

Let $\mathbf{B}_{\mathrm{HT},K}, \mathbf{B}_{\mathrm{dR},K}, \mathbf{B}_{\mathrm{dR},K}$ respectively be the rings $\mathbf{C}_p[t_K, t_K^{-1}], \mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{dR}}$. We set

$$\mathbf{D}_{\mathrm{HT},K}(V) = (\mathbf{B}_{\mathrm{HT},K} \otimes_K V)^{G_K},$$

$$\mathbf{D}_{\mathrm{dR},K}^+(V) = \left(\mathbf{B}_{\mathrm{dR},K}^+ \otimes_K V\right)^{G_K},$$

and

$$\mathbf{D}_{\mathrm{dR},K}(V) = (\mathbf{B}_{\mathrm{dR},K} \otimes_K V)^{G_K}.$$

We say that V is K-Hodge-Tate (resp. positive K-de Rham, resp. K-de Rham) if

$$\dim_K \mathbf{D}_{\mathrm{HT},K}(V) = d$$

(resp.

$$\dim_K \mathbf{D}_{\mathrm{dR},K}^+(V) = d,$$

resp.

$$\dim_K \mathbf{D}_{\mathrm{dR},K}(V) = d).$$

Lemma 9.5. $(\mathbf{C}_p \otimes_K V)^{G_K} = (\mathbf{D}_{\mathrm{Sen},K}(V))^{\Gamma_K}$.

Proof. By Proposition 3.3, we have

$$(\mathbf{C}_p \otimes_K V)^{G_K} = (\mathbf{C}_p \otimes \mathbf{D}_{\mathrm{Sen},K}(V))^{G_K} = \left(\hat{K}_\infty \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen},K}(V)\right)^{\Gamma_K}.$$

As $\left(\hat{K}_\infty \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen},K}(V)\right)^{\Gamma_K}$ is fixed by the action of Γ_K, it is also locally K-analytic on Γ_K, so it is contained in $\left(\hat{K}_\infty \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen},K}(V)\right)^{K-\mathrm{la}}$. But according to Lemma 2.1,

$$\left(\hat{K}_\infty \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen},K}(V)\right)^{K-\mathrm{la}} = \mathbf{D}_{\mathrm{Sen},K}(V).$$

□

Proposition 9.6. *1.* $\mathbf{D}_{\mathrm{HT},K}(V) = \oplus_{l\in\mathbf{Z}} \left(\mathbf{D}_{\mathrm{Sen},K}(V)t_K^l\right)^{\Gamma_K}$.

2. The natural map $\mathbf{B}_{\mathrm{HT},K} \otimes_K \mathbf{D}_{\mathrm{HT},K}(V) \to \mathbf{B}_{\mathrm{HT},K} \otimes_K V$ *is an isomorphism if* V *is* K*-Hodge-Tate.*

3. V *is* $\mathbf{C}_p$*-admissible if and only if* $\Theta_{\mathrm{Sen},K} = 0$ *on* $\mathbf{D}_{\mathrm{Sen},K}(V)$.

Proof. We have $\mathbf{D}_{\mathrm{HT},K}(V) = \oplus_{l\in\mathbf{Z}} \left(\mathbf{C}_p t_K^l \otimes_K V\right)^{G_K}$, so (upon twisting V with an appropriate power of χ_π) the equality

$$\mathbf{D}_{\mathrm{HT},K}(V) = \oplus_{l\in\mathbf{Z}} \left(\mathbf{D}_{\mathrm{Sen},K}(V)t_K^l\right)^{\Gamma_K}$$

reduces to the verification that $(\mathbf{C}_p \otimes_K V)^{G_K} = (\mathbf{D}_{\mathrm{Sen},K}(V))^{\Gamma_K}$, which was done in the previous lemma. This proves (1). To prove (2), suppose V is K-Hodge-Tate. Then by (1) and Proposition 3.4 we have $\mathbf{D}_{\mathrm{Sen},K}(V) \cong \oplus_{i=1}^d K_\infty(\chi_\pi^{n_i})$, which gives the second isomorphism in

$$\begin{aligned}\mathbf{B}_{\mathrm{HT},K} \otimes_K \mathbf{D}_{\mathrm{HT},K}(V) &\cong \mathbf{B}_{\mathrm{HT},K} \otimes_K \left(\oplus_{l\in\mathbf{Z}} \left(\mathbf{D}_{\mathrm{Sen},K}(V)t_K^l\right)^{\Gamma_K}\right)\\ &\cong \mathbf{B}_{\mathrm{HT},K} \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen},K}(V)\\ &\cong \mathbf{B}_{\mathrm{HT},K} \otimes_K V.\end{aligned}$$

Finally, (3) follows from Proposition 3.4. □

By the same logic one obtains similar results for $\mathbf{D}_{\mathrm{dR},K}$.

Proposition 9.7. *1.* $\mathbf{D}_{\mathrm{dR},K}(V) = \mathbf{D}_{\mathrm{dif},K}(V)^{\Gamma_K}$.

2. The natural map $\mathbf{B}_{\mathrm{dR},K} \otimes_K \mathbf{D}_{\mathrm{dR},K}(V) \to \mathbf{B}_{\mathrm{dR},K} \otimes_K V$ *is an isomorphism if* V *is* K*-de Rham.*

3. V *is* K*-de Rham if and only if* $\nabla_{\mathrm{dif},K}$ *has a full set of sections on* $\mathbf{D}_{\mathrm{dif},K}(V)$.

9.3 The modules $\mathbf{D}_{\mathrm{cris},K}$ and $\mathbf{D}_{\mathrm{st},K}$

Recall that $\mathbf{B}^+_{\max}$ is a period ring similar to Fontaine's $\mathbf{B}^+_{\mathrm{cris}}$. The element[1] t_K introduced in §2.3 actually lies in $\mathbf{B}^+_{\max} \otimes_{K_0} K$. Denote by $\mathbf{B}^+_{\max,K}$, $\mathbf{B}^+_{\mathrm{st},K}$, $\mathbf{B}_{\max,K}$, $\mathbf{B}_{\mathrm{st},K}$ respectively the rings $\mathbf{B}^+_{\max} \otimes_{K_0} K$, $\mathbf{B}^+_{\mathrm{st}} \otimes_{K_0} K$, $\mathbf{B}^+_{\max,K}\left[\frac{1}{t_K}\right]$ and $\mathbf{B}^+_{\mathrm{st},K}\left[\frac{1}{t_K}\right]$. These rings carry a $\varphi_q = \varphi^f$-action, and the usual monodromy operator N of $\mathbf{B}^+_{\mathrm{st}}$ extends to $\mathbf{B}_{\mathrm{st},K}$ with $\mathbf{B}^{N=0}_{\mathrm{st},K} = \mathbf{B}_{\max,K}$. If L is a finite extension of K, we set $\mathbf{D}^+_{\mathrm{cris},K}(V)$, $\mathbf{D}^+_{\mathrm{st},K}(V)$, $\mathbf{D}_{\mathrm{cris},K}(V)$, $\mathbf{D}_{\mathrm{st},K}(V)$ respectively to be $\mathbf{D}^+_{\mathrm{cris},K}(V) = \left(\mathbf{B}^+_{\max,K} \otimes_K V\right)^{G_L}$, $\mathbf{D}^+_{\mathrm{st},K}(V) = \left(\mathbf{B}^+_{\mathrm{st},K} \otimes_K V\right)^{G_L}$, $\mathbf{D}_{\mathrm{cris},K}(V) = (\mathbf{B}_{\max,K} \otimes_K V)^{G_L}$ and $\mathbf{D}_{\mathrm{st},K}(V) = (\mathbf{B}_{\mathrm{st},K} \otimes_K V)^{G_L}$.

[1] It is $\log_{\mathcal{F}} u$ for the element u introduced in §4.1 below.

Lemma 9.8. *If V is a K-analytic representation, then*

$$\mathbf{D}_{\mathrm{cris},K}(V) = (\mathbf{B}_{\max} \otimes_{K_0} V)^{G_L}.$$

Proof. By Lemma 8.17 of [Co02], the element t_K divides the usual t in $\mathbf{B}^+_{\max,K}$. Therefore, $\mathbf{B}_{\max,K} = \mathbf{B}^+_{\max,K}\left[\frac{1}{t_K}\right]$ is contained in $\mathbf{B}_{\max} \otimes_{K_0} K = \mathbf{B}^+_{\max,K}\left[\frac{1}{t}\right]$, and it follows that $\mathbf{D}_{\mathrm{cris},K}(V) \subset (\mathbf{B}_{\max} \otimes_{K_0} V)^{G_L}$ for any representation V. It suffices to show when V is K-analytic, this inclusion is an equality. If all the integer-valued Hodge-Tate weights (in the usual sense) of V are nonnegative, this is clear, because

$$\mathbf{D}_{\mathrm{cris},K}(V) = \mathbf{D}^+_{\mathrm{cris},K}(V) = \left(\mathbf{B}^+_{\max} \otimes_{K_0} V\right)^{G_L} = (\mathbf{B}_{\max} \otimes_{K_0} V)^{G_L}.$$

In general, it suffices to prove $\mathbf{D}_{\mathrm{cris},K}(V) \subset (\mathbf{B}_{\max} \otimes_{K_0} V)^{G_L}$ after twisting V by a power of χ_π. Given that V is K-analytic, there exists an $n \gg 0$ so that all the integer-valued Hodge-Tate weights of $V(\chi_\pi^n)$ are nonnegative, and this case was already dealt with. □

To a filtered (φ_q, N)-module $\mathbf{D}$ over $L_0 \otimes_{K_0} E$ one can associate two polygons. The Hodge polygon $P_H(\mathbf{D})$, whose slopes have lengths according to the jumps in the filtration; and the Newton polygon $P_N(\mathbf{D})$, whose slopes match the slopes of φ_q with respect to the valuation val_π. We say $\mathbf{D}$ is admissible if the endpoints of $P_H(\mathbf{D})$ and $P_N(D)$ are the same and if $P_H(\mathbf{D}_0)$ lies below $P_N(\mathbf{D}_0)$ for every subobject $\mathbf{D}_0$ of $\mathbf{D}$.
If $\mathbf{D}$ is a filtered (φ_q, N)-module over $L_0 \otimes_{K_0} E$, then

$$\mathrm{I}^K_{\mathbf{Q}_p}(\mathbf{D}) := \left(L_0 \otimes_{\mathbf{Q}_p} E\right)[\varphi] \otimes_{\left(L_0 \otimes_{K_0} E\right)[\varphi_q]} \mathbf{D}$$

is a filtered (φ, N)-module over $L_0 \otimes_{\mathbf{Q}_p} E$. The following is proved in §3 of [KR09] under the assumption that $N = 0$, but the proof in the general case is the same. Note that in loc. cit. this statement is actually proved for $(\mathbf{B}_{\max} \otimes_{K_0} V)^{G_L}$ instead of $\mathbf{D}_{\mathrm{cris},K}(V)$, but these coincide for K-analytic representations by Lemma 3.8.

Proposition 9.9. *Suppose that $V \in \mathrm{Rep}_E(G_L)$ is K-analytic. Then*

$$\mathrm{I}^K_{\mathbf{Q}_p}(\mathbf{D}_{\mathrm{st},K}(V)) = \mathbf{D}_{\mathrm{st},\mathbf{Q}_p}(V).$$

Furthermore, $\mathbf{D}_{\mathrm{st},K}(V)$ is admissible if and only if $\mathbf{D}_{\mathrm{st},\mathbf{Q}_p}(V)$ is admissible.

We say that V is K-potentially semistable if for some finite extension L of K we have $\mathrm{rank}_{L_0 \otimes_{K_0} E} \mathbf{D}_{\mathrm{st},K}(V) = \dim_E V$.

Corollary 9.10. *Suppose V is a K-analytic representation. Then the following are equivalent.*

1. V is de Rham.

2. V is K-de Rham.

3. V is potentially semistable.

4. V is K-potentially semistable.

Proof. The equivalence between (1) and (3) is the p-adic monodromy theorem, which is Theorem 0.7 of [Be02]. The equivalence between (3) and (4) follows from Proposition 3.9. It remains to prove that (1) and (2) are equivalent. Indeed, we have $\mathbf{D}_{\mathrm{dR}}(V) \cong \oplus_{\tau \in \Sigma_K} \mathbf{D}_{\mathrm{dR},K}(V^\tau)$, with V^τ being the τ-twist of V. Since V is assumed K-analytic, the representation V^τ is $\mathbf{C}_p$-admissible for $\tau \neq \mathrm{Id}$, so it also K-de Rham. This implies that $\dim_K \mathbf{D}_{\mathrm{dR},K}(V^\tau) = d$, so $\dim_K \mathbf{D}_{\mathrm{dR}}(V) = \dim_{\mathbf{Q}_p} V$ if and only if $\dim_K \mathbf{D}_{\mathrm{dR},K}(V) = \dim_K V$, as required. □

We conclude with a lemma that will be used in the proof of Theorem 6.8.

Lemma 9.11. *Suppose that $V \in \mathrm{Rep}_E(G_K)$ is K-analytic and that $L = K$, and let $\alpha \in E^\times$. Then $\mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)^{\varphi_q=\alpha} = \left(K_0 \otimes_{\mathbf{Q}_p} E\right)[\varphi] \otimes_{E[\varphi_q]} \mathbf{D}_{\mathrm{cris},K}(V)^{\varphi_q=\alpha}$.*

Proof. In general, if $\mathbf{D}$ is a filtered φ_q-module over E, then $\mathrm{I}_{\mathbf{Q}_p}^K(\mathbf{D})^{\varphi_q=\alpha} = \left(K_0 \otimes_{\mathbf{Q}_p} E\right)[\varphi] \otimes_{E[\varphi_q]} \mathbf{D}^{\varphi_q=\alpha}$ because the φ_q action is $\left(K_0 \otimes_{\mathbf{Q}_p} E\right)[\varphi]$-linear. Explicitly, there is a decomposition of $E[\varphi_q]$-modules

$$\mathrm{I}_{\mathbf{Q}_p}^K(\mathbf{D}) = \left(K_0 \otimes_{\mathbf{Q}_p} E\right)[\varphi] \otimes_{E[\varphi_q]} \mathbf{D} = \oplus_{i=0}^{f-1} \left(K_0 \otimes_{\mathbf{Q}_p} E\right) \varphi^i \otimes \mathbf{D},$$

and taking $\varphi_q = \alpha$ of both sides yields the desired equality. The lemma now follows from Proposition 3.9. □

10 Big period rings

In this section, we recall the big period rings which will be used in the proofs of §5. The most important result of this section is the determination of pro-K-analytic vectors in Theorem 4.6.

If $\mathcal{F}$ is the formal $\mathcal{O}_K$-module associated to π as in §1.2, we choose a coordinate T for $\mathcal{F}$ so that for $a \in \mathcal{O}_K$ we have a power series $[a] = [a](T)$ corresponding to the action of a on $\mathcal{F}$. For $n \geq 0$ we choose elements $u_n \in \mathcal{O}_{\mathbf{C}_p}$ such that $u_0 = 0$, $u_1 \neq 0$ and $[\pi](u_n) = u_{n-1}$.

10.1 The rings $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ and $\widetilde{\mathbf{B}}^{\dagger}_{\log}$

This subsection provides ramified counterparts for the constructions given in §2 of [Be02] in the case $K = K_0$. Some of the content of this subsection can also be found in §3 of [Be16], though our notation is slightly different in parts. Recall the notations from §1.2 and set

$$\mathcal{O}_{\mathbf{C}_p^\flat} = \lim \left(\mathcal{O}_{\mathbf{C}_p}/\pi \xleftarrow{x \mapsto x^q} \mathcal{O}_{\mathbf{C}_p}/\pi \xleftarrow{x \mapsto x^q} \ldots \right).$$

We equip $\mathcal{O}_{\mathbf{C}_p^\flat}$ with the valuation $\left|(\overline{x}_n)_{n\geq 0}\right| = \lim_{n\to\infty} |x_n|^{q^n}$ where $x_n \in \mathcal{O}_{\mathbf{C}_p}$ is a lift of $\overline{x}_n$. Denote by $\widetilde{\mathbf{A}}_0^+$, $\widetilde{\mathbf{A}}^+$ respectively the rings $W\left(\mathcal{O}_{\mathbf{C}_p^\flat}\right)$ and $\widetilde{\mathbf{A}}_0^+ \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K$. Then $\overline{u} = (\overline{u}_n)_{n\geq 0}$ lies in $\in \mathcal{O}_{\mathbf{C}_p^\flat}$, and by §8 of [Co02] there exists an element $u \in \widetilde{\mathbf{A}}^+$ which lifts $\overline{u}$ and which satisfies $\varphi_q(u) = [\pi](u)$ and $g(u) = [\chi_\pi(g)](u)$ for $g \in \Gamma_K$.
Let $\varpi \in \mathcal{O}_{\mathbf{C}_p^\flat}$ be any element with $|\varpi| = p^{-p/p-1}$. Given $r, s \in \mathbf{Z}_{\geq 0}[1/p]$ with $r \leq s$, and given $\mathrm{A} \in \left\{\widetilde{\mathbf{A}}_0, \widetilde{\mathbf{A}}\right\}$, we set

$$\mathrm{A}^{[r,s]} = \mathrm{A}^+ \left\langle \frac{p}{[\varpi]^r}, \frac{[\varpi]^s}{p} \right\rangle,$$

the completion of $\mathrm{A}^+\left[\frac{p}{[\varpi]^r}, \frac{[\varpi]^s}{p}\right]$ with respect to the $(p, [\varpi])$-adic topology.[2]
We write $\widetilde{\mathbf{B}}_0^{[r,s]} = \widetilde{\mathbf{A}}_0^{[r,s]}[1/p]$ and $\widetilde{\mathbf{B}}^{[r,s]} = \widetilde{\mathbf{A}}^{[r,s]}[1/\pi]$.

Lemma 10.1. *1.* $\widetilde{\mathbf{A}}_0^{[r,s]} \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K = \widetilde{\mathbf{A}}^{[r,s]}$.
2. $\widetilde{\mathbf{B}}_0^{[r,s]} \otimes_{K_0} K = \widetilde{\mathbf{B}}^{[r,s]}$.

Proof. (2) follows from (1). To prove (1), we write

$$\widetilde{\mathbf{A}}^+ \left[\frac{p}{[\varpi]^r}, \frac{[\varpi]^s}{p}\right] = \oplus_{i=0}^{e-1} \pi^i \widetilde{\mathbf{A}}_0^+ \left[\frac{p}{[\varpi]^r}, \frac{[\varpi]^s}{p}\right]$$

as $\widetilde{\mathbf{A}}_0^+$-modules. Now take the $(p, [\varpi])$-adic completion of both sides. □

We denote by $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},0}$,$\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$, $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},0}$ and $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ respectively the rings

$$\cap_{r\leq s} \widetilde{\mathbf{B}}_0^{[r,s]}, \cap_{r\leq s} \widetilde{\mathbf{B}}^{[r,s]}, \cup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},0}$$

and $\cup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. The φ and G_K actions on $\widetilde{\mathbf{A}}_0^+$ (resp. the φ_q and G_K actions on $\widetilde{\mathbf{A}}^+$) extend to $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},0}$ (resp. to $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$). The following is proved in Proposition 2.23 of [Be02] in the case $K = \mathbf{Q}_p$, but the same proof works in the general case.

[2] In some references the completion is taken with respect to the p-adic topology, but this makes no difference because p divides a power of $[\varpi]$.

Proposition 10.2. *There exists a unique map* $\log : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[X]$ *satisfying* $\log(\pi) = 0$, $\log[\overline{u}] = X$, $\log[x] = 0$ *for* $x \in \overline{\mathbb{F}}_q$ *and* $\log(xy) = \log(x) + \log(y)$, *such that if* $[x] - 1$ *is sufficiently close to* 1, *we have*

$$\log[x] = \sum_{n \geq 1} (-1)^{n-1} \frac{([x]-1)^n}{n}.$$

Moreover, if $x \in \mathcal{O}^\times_{\mathbf{C}_p^\flat}$ *then* $\log[x] \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$.

Write $p^\flat$ and $\pi^\flat$ for the elements $\left(\overline{p}, \overline{p^{1/q}}, ...\right)$ and $\left(\overline{\pi}, \overline{\pi^{1/q}}, ...\right)$ of $\mathcal{O}_{\mathbf{C}_p^\flat}$. We set $\widetilde{\mathbf{B}}^\dagger_{\log,0} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},0}\left[\log\left[p^\flat\right]\right]$ and $\widetilde{\mathbf{B}}^\dagger_{\log} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\left[\log\left[\pi^\flat\right]\right]$. Since $u/[\overline{u}] \equiv 1 \mod \pi$, we have $\log(u/[\overline{u}]) \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$; on the other hand, $\overline{u}^{q-1}/\left(\pi^\flat\right)^q$ is a unit of $\mathcal{O}^\times_{\mathbf{C}_p^\flat}$, so $\log[\overline{u}]^{q-1}/\left[\pi^\flat\right]^q \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ as well. Combining these two observations, we see that in $\widetilde{\mathbf{B}}^\dagger_{\log}$ we have

$$\begin{aligned} q\log\left[\pi^\flat\right] &= (q-1)\log u - (q-1)\log(u/[\overline{u}]) - \log[\overline{u}]^{q-1}/\left[\pi^\flat\right]^q \\ &\equiv (q-1)\log u \mod \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}, \end{aligned}$$

so we also have $\widetilde{\mathbf{B}}^\dagger_{\log} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\log u]$.

The φ (resp. φ_q) action on $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},0}$ (resp. on $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$) extends to $\widetilde{\mathbf{B}}^\dagger_{\log,0}$ (resp. to $\widetilde{\mathbf{B}}^\dagger_{\log}$) by setting $\varphi(\log\left[p^\flat\right]) = p\log\left[p^\flat\right]$ and $g(\log\left[p^\flat\right]) = \log\left[g\left(p^\flat\right)\right]$ (resp. $\varphi_q(\log\left[\pi^\flat\right]) = q\log\left[\pi^\flat\right]$ and $g(\log\left[\pi^\flat\right]) = \log\left[g\left(\pi^\flat\right)\right]$). We have a monodromy operator N which acts on $\widetilde{\mathbf{B}}^\dagger_{\log,0}$ (resp. on $\widetilde{\mathbf{B}}^\dagger_{\log}$) by $-\frac{d}{d\log\left[p^\flat\right]}$ (resp. by $-\frac{1}{e}\frac{d}{d\log\left[\pi^\flat\right]}$), and $N\varphi = p\varphi N$ (resp. $N\varphi_q = q\varphi_q N$).

Proposition 10.3. *1.* $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},0} \otimes_{K_0} K = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$.
2. $\widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{K_0} K = \widetilde{\mathbf{B}}^\dagger_{\log}$.

Proof. For $r \leq s$ we have by Lemma 4.1 that $\widetilde{\mathbf{B}}^{[r,s]}_0 \otimes_{K_0} K = \widetilde{\mathbf{B}}^{[r,s]}$. As K is finite free over K_0, this implies

$$\begin{aligned} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},0} \otimes_{K_0} K &= \left(\cap_{r\leq s}\widetilde{\mathbf{B}}^{[r,s]}_0\right) \otimes_{K_0} K \\ &= \cap_{r\leq s}\left(\widetilde{\mathbf{B}}^{[r,s]}_0 \otimes_{K_0} K\right) \\ &= \cap_{r\leq s}\widetilde{\mathbf{B}}^{[r,s]} \\ &= \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}. \end{aligned}$$

For (2), we write $\pi^e = pv$ with $v \in \mathcal{O}_K^\times$. We can find $v^\flat = \left(\overline{v}, \overline{v^{1/q}}, ...\right) \in \mathcal{O}_{\mathbf{C}_p^\flat}$ such that $\left[\pi^\flat\right]^e = \left[p^\flat\right]\left[v^\flat\right]$, so $e \log\left[\pi^\flat\right] \equiv \log\left[p^\flat\right] \mod \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$, and

$$\begin{aligned}\widetilde{\mathbf{B}}^\dagger_{\log} &= \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\left[\log\left[p^\flat\right]\right]\\ &= \left(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},0} \otimes_{K_0} K\right)\left[\log\left[p^\flat\right]\right]\\ &= \widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{K_0} K.\end{aligned}$$

. □

10.2 Pro K-analytic vectors

Let $\mathbf{B}^\dagger_{\mathrm{rig},K}$ be the Robba ring, i.e. the ring of power series $f(T) = \sum_{k \in \mathbf{Z}} a_k T^k$ with $a_k \in K$ and such that $f(T)$ converges on some nonempty annulus $r < |T| < 1$. The ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$ can be viewed as a subring of $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} = \left(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\right)^{H_K}$ by identifying T with the element u of §4.1. It has induced φ_q and Γ_K actions.

Recall the following result (Theorem B of [Be16]), which determines the ring of pro K-analytic vectors in $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}$.

Theorem 10.4. $\left(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}\right)^{K-\mathrm{pa}} = \cup_{n \geq 0} \varphi_q^{-n}\left(\mathbf{B}^\dagger_{\mathrm{rig},K}\right).$

On the other hand, we can also write $\widetilde{\mathbf{B}}^\dagger_{\log,K} = \left(\widetilde{\mathbf{B}}^\dagger_{\log}\right)^{H_K}$. The goal of this subsection is to obtain an analogous result for $\widetilde{\mathbf{B}}^\dagger_{\log,K}$.

Proposition 10.5. *We have* $\log u \in \left(\widetilde{\mathbf{B}}^\dagger_{\log,K}\right)^{K-\mathrm{pa}}$.

Before we give a proof of Proposition 4.5, we record the following consequence. Let $\mathbf{B}^\dagger_{\log,K} = \mathbf{B}^\dagger_{\mathrm{rig},K}[\log T]$, thought of as a subring of $\widetilde{\mathbf{B}}^\dagger_{\log,K}$. The φ_q action on $\log T$ is given by $\varphi_q(\log T) = q \log T + \log\left([\pi](T)/T^q\right)$, where $\log\left([\pi](T)/T^q\right) \in \mathbf{B}^\dagger_{\mathrm{rig},K}$.

Theorem 10.6. $\left(\widetilde{\mathbf{B}}^\dagger_{\log,K}\right)^{K-\mathrm{pa}} = \cup_{n \geq 0} \varphi_q^{-n}\left(\mathbf{B}^\dagger_{\log,K}\right).$

Proof. Fix $d \geq 0$. As $g(\log u) = \log u + \log \frac{g(u)}{u}$ for $g \in \Gamma_K$, the submodule $\oplus_{i=0}^d \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \cdot (\log u)^i$ is closed under the Γ_K-action. By Proposition 4.5, the elements $1, \log u, ..., (\log u)^i$ from a $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}$-basis of this submodule for which the action is pro K-analytic. Combining Lemma 2.1 and Theorem 4.4, we obtain

$$\left(\oplus_{i=0}^{d}\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}\cdot(\log u)^i\right)^{K-\mathrm{pa}} = \oplus_{i=0}^{d}\left(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}\right)^{K-\mathrm{pa}}\cdot(\log u)^i$$
$$= \oplus_{i=0}^{d}\left(\cup_{n\geq 0}\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K}\right)\right)(\log u)^i.$$

Taking the colimit as $d\to\infty$ shows that

$$\left(\widetilde{\mathbf{B}}^{\dagger}_{\log,K}\right)^{K-\mathrm{pa}} = \left(\cup_{n\geq 0}\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K}\right)\right)[\log u].$$

It remains to show that $\cup_{n\geq 0}\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\log,K}\right)$ is contained in

$$\left(\cup_{n\geq 0}\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K}\right)\right)[\log u],$$

as the inclusion in the other direction is obvious. Assume the opposite and let $f=\sum_{i=0}^{d}a_i(\log u)^i$ be an element of $\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\log,K}\right)$ with $a_i\in\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}$ and d minimal such that f is not contained in $\left(\cup_{n\geq 0}\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K}\right)\right)[\log u]$. As $\varphi_q^n(f)\in\mathbf{B}^{\dagger}_{\log,K}$ and $\varphi_q(\log u)\equiv q\log u \mod \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, examining the coefficient of $(\log u)^d$ reveals that $\varphi_q^n(a_d)\in\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, providing a contradiction. □

We now proceed to prove Proposition 4.5. We do so in several steps, following the method appearing in §4 of [Be16]. If $t\geq 1$, we denote by $\mathrm{LA}_t(\mathcal{O}_K)$ the space functions of $\mathcal{O}_K$ which are analytic on closed discs of radius $|\pi|^t$. For $a\in\mathcal{O}_K$, write $[a](T)=\sum_{n\geq 1}c_n(a)T^n$. Each $c_n(a)$ is a polynomial of degree at most n in a, and $c_n(\mathcal{O}_K)\subset\mathcal{O}_K$.

Lemma 10.7. $||c_n||_{\mathrm{LA}_t}\leq|\pi|^{-\frac{n}{q^t(q-1)}}$.

Proof. Recall that de Shalit has constructed in [dS16] a Mahler basis

$$\{g_n(T)\}_{n\geq 0}^{\infty}$$

such that $g_n(T)$ is a polynomial in $K[T]$ of degree n and and such that $||\pi^{w_{n,t}}g_n||_{\mathrm{LA}_t}=1$, where $w_{n,t}$ is an integer satisfying $w_{n,t}\leq\frac{n}{q^t(q-1)}$. As c_n has degree at most n, we can write $c_n=\sum_{i=0}^{n}b_{n,i}g_i$ for some $b_{n,i}\in\mathcal{O}_K$, and so $||c_n||_{\mathrm{LA}_t}\leq\sup_{1\leq i\leq n}||g_i||\leq|\pi|^{-\frac{n}{q^t(q-1)}}$. □

Recall that $g(\log u)=\log u+\log\left(\frac{[a](u)}{u}\right)$, where $a=\chi_\pi(g)$. We write

$$\log\left(\frac{[a](u)}{u}\right)=\sum_{n=1}^{\infty}d_n(a)u^n.$$

Lemma 10.8. $||d_n||_{\mathrm{LA}_t} \le |\pi|^{-\frac{2n}{q^t(q-1)}+o(n)}$.

Proof. Write $\frac{[a](u)}{u} - 1 = \sum_{n\ge 0} e_n(a)u^n$, where $e_0(a) = a-1$ and $e_n(a) = c_{n+1}(a)$ for $n \ge 1$. Then d_n is a sum of functions of the form

$$\frac{(-1)^{m-1}}{m} \sum_{\substack{(k_1,...,k_m)\in \mathbf{Z}_{\ge 0}^m \\ k_1+...+k_m=n}} \prod_{i=1}^m e_{k_i},$$

and it suffices to bound each such function by $|\pi|^{-\frac{2n}{q^t(q-1)}+o(n)}$, where $o(n)$ does not depend on m.

Fix $(k_1,...,k_m) \in \mathbf{Z}_{\ge 0}^m$ with $k_1 + ... + k_m = n$. Let h be the number of $1 \le i \le m$ such that $k_i \ge 1$. Then by Lemma 4.7 we have

$$\left|\left|\prod_{i:1\le k_i} e_{k_i}\right|\right|_{\mathrm{LA}_t} \le |\pi|^{-(n+h)/q^t(q-1)} \le |\pi|^{-2n/q^t(q-1)}.$$

On the other hand, $||e_0||_{\mathrm{LA}_t} \le |\pi|^t$, so

$$\left|\left|\frac{1}{m}\prod_{i:k_i=0} e_{k_i}\right|\right|_{\mathrm{LA}_t} \le \left|\frac{1}{m}\right| |\pi|^{t(m-h)} \le p^{[v_p(m)-t/e\max\{0,m-n\}]} = |\pi|^{o(n)}.$$

Combining the two inequalities we obtain the claim. □

Proof of Proposition 4.5. Write $r_n = p^{nf-1}(p-1)$ and let $s \le t$. It is enough to show that $\log u$ is K-analytic on $\Gamma_{K_{t+1}}$ as a vector of $\widetilde{\mathbf{B}}_K^{[r_s,r_t]} = \left(\widetilde{\mathbf{B}}^{[r_l,r_t]}\right)^{H_K}$. Since $g(\log u) = \log u + \log\left(\frac{[a](u)}{u}\right)$ for $a = \chi_\pi(g)$, we need to verify that $||d_n||_{\mathrm{LA}_{t+1}} ||u^n||_{[r_s,r_t]} \to 0$ as $n \to 0$. By the maximum principle, we have $||u||_{[r_s,r_t]} = ||u||_{r_t} = |\pi|^{1/q^{t-1}(q-1)}$, and so by Lemma 4.7

$$||d_n||_{\mathrm{LA}_{t+1}} ||u^n||_{[r_s,r_t]} \le |\pi|^{n\left[\frac{1}{q^{t-1}(q-1)} - \frac{2}{q^{t+1}(q-1)}\right]+o(1)},$$

which approaches 0, as required. □

11 Lubin-Tate (φ_q, Γ_K)-modules

In this section we recall how to attach a (φ_q, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ to a K-analytic p-adic representation of G_K, and we express the invariants of §3 in terms of these (φ_q, Γ_K)-modules. Recall from §1.2 that the field E is a finite extension of K which serves as a field of coefficients.

11.1 K-analytic (φ_q, Γ_K)-modules

Let $\mathbf{A}_K$ be the ring of power series $f(T) = \sum_{k \in \mathbf{Z}} a_k T^k$ with $a_k \in \mathcal{O}_K$ such that $\mathrm{val}_p(a_k) \to 0$ as $k \to -\infty$, and let $\mathbf{B}_K = \mathbf{A}_K[1/\pi]$. The rings $\mathbf{A}_K$ and $\mathbf{B}_K$ have a φ_q-action given by $\varphi_q(T) = [\pi](T)$ and a Γ_K-action given by $g(T) = [\chi_\pi(g)](T)$ for $g \in \Gamma_K$. These actions are K-linear, and we extend them E-linearly to $\mathbf{A}_K \otimes_K E$ and $\mathbf{B}_K \otimes_K E$. A (φ_q, Γ_K)-module over $\mathbf{B}_K \otimes_K E$ is a finite free $\mathbf{B}_K \otimes_K E$ module $\mathbf{D}_K$ which has commuting semilinear φ_q and Γ_K actions. We say it is étale if there exists a basis of $\mathbf{D}_K$ for which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathbf{A}_K \otimes_K E)$.

Kisin and Ren have shown in §1 of [KR09] how to associate to any $V \in \mathrm{Rep}_E(G_K)$ an étale (φ_q, Γ_K)-module over $\mathbf{B}_K \otimes_K E$ which we denote $\mathbf{D}_K(V)$. Furthermore, one has the following result.

Theorem 11.1. *The functor $V \mapsto \mathbf{D}_K(V)$ induces an equivalence of categories*

$$\{E\text{-representations of } G_K\} \longleftrightarrow \{\text{étale } (\varphi_q, \Gamma_K)\text{-modules over } \mathbf{B}_K \otimes_K E\}.$$

Now let $\mathbf{B}_K^\dagger$ be the subring of $\mathbf{B}_K$ which consists of power series which converge on some nonempty annulus $r \leq |T| < 1$. It is preserved by the (φ_q, Γ_K)-structure, and we say that a (φ_q, Γ_K)-module over $\mathbf{B}_K \otimes_K E$ is *overconvergent* if $\mathbf{D}_K = \mathbf{D}_K^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_K$ where $\mathbf{D}_K^\dagger$ is a (φ_q, Γ_K)-module over $\mathbf{B}_K^\dagger \otimes_K E$. A representation $V \in \mathrm{Rep}_E(G_K)$ is said to be overconvergent if $\mathbf{D}_K(V)$ is. As $\mathbf{B}_K^\dagger$ is a subring of $\mathbf{B}_{\mathrm{rig},K}^\dagger$, for such a (φ_q, Γ_K)-module we can form $\mathbf{D}_{\mathrm{rig},K}^\dagger = \mathbf{D}_K^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_{\mathrm{rig},K}^\dagger$ and $\mathbf{D}_{\log,K}^\dagger = \mathbf{D}_K^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_{\log,K}^\dagger$.

In the case $K = \mathbf{Q}_p$, Cherbonnier and Colmez have proven in [CC98] that $\mathbf{D}_K(V)$ is always overconvergent. Unfortunately, this is no longer true whenever $K \neq \mathbf{Q}_p$ (see Theorem 0.6 of [FX13]). However, the analogue of the Cherbonnier-Colmez theorem does hold for K-analytic representations, and, even better, we can characterize the (φ_q, Γ_K)-modules which arise in this way. More precisely, a (φ_q, Γ_K)-module $\mathbf{D}_{\mathrm{rig},K}^\dagger$ over $\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_K E$ is called K-analytic if $\mathbf{D}_{\mathrm{rig},K}^\dagger = \left(\mathbf{D}_{\mathrm{rig},K}^\dagger\right)^{K-\mathrm{pa}}$. Then one has the following result (Theorems C and D of [Be16]).

Theorem 11.2. *1. If $V \in \mathrm{Rep}_E(G_K)$ is K-analytic, then $\mathbf{D}_K(V)$ is overconvergent.*

2. The functor $V \mapsto \mathbf{D}_{\mathrm{rig},K}^\dagger(V)$ gives an equivalence of categories

$$\{K\text{-analytic } E\text{-linear representations of } G_K\}$$

$$\longleftrightarrow \left\{\text{étale } K\text{-analytic } (\varphi_q, \Gamma_K)\text{-modules over } \mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_K E\right\}$$

3. If $V \in \mathrm{Rep}_E(G_K)$ is K-analytic, then there exists a natural G_K-equivariant isomorphism $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_K V \cong \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)$.

All characters are overconvergent, so split 2-dimensional representation are always overconvergent. For nonsplit representations, Theorem 5.2 implies the following.

Corollary 11.3. *Let $V \in \mathrm{Rep}_E(G_K)$ be a nonsplit 2-dimensional representation. The following are equivalent.*

1. V is overconvergent.

2. Either V is K-analytic up to a character twist or V is an extension of the trivial representation by itself.

Proof. If $V(\delta)$ is K-analytic then it is overconvergent by Theorem 5.2. In addition, Theorem 0.3 of [FX13] shows that every extension of the trivial representation by itself is overconvergent, so (2) implies (1). In the converse direction, let V be an overconvergent representation. It is either absolutely irreducible or reducible nonsplit after possibly extending scalars.

Case 1: V is absolutely irreducible. Then Corollary 4.3 of [Be13] implies that $V(\delta)$ is K-analytic for some character δ. Corollary 4.3 of [Be13] is proved there in the setting where K is an unramified extension of $\mathbf{Q}_p$; it is a consequence of Theorem 4.2 of ibid. This assumption can be removed, because Theorem 4.2 of ibid is reproven in [Be16] without assuming K is unramified.

Case 2: V is reducible nonsplit after extending scalars. By the following lemma, extending scalars does not matter for the question of oveconvergence. Thus, we may assume V is reducible and nonsplit, and after performing a character twist we may further assume it is an extension of 1 by $E(\delta)$ with $\det(V) = \delta$. If $\delta = 1$, we are done. Otherwise, by Theorem 0.4 of [FX13] a nontrivial overconvergent extension of 1 by $E(\delta)$ can only occur if δ is K-analytic, and since $\delta \neq 1$ this implies that V itself is K-analytic by Theorem 0.3 of [FX13]. □

The following lemma was used in the proof of Corollary 5.3.

Lemma 11.4. *Let $V \in \mathrm{Rep}_E(G_K)$ be a representation, and let E' be a finite extension of E. Then V is overconvergent if and only if $V \otimes_E E'$ is overconvergent.*

Proof. Clearly, if V is overconvergent, so is $V \otimes_E E'$. This being said, when proving the converse direction we are free to enlarge E', and in particular we may assume E'/E is Galois. Write $\mathbf{D}_{E'} = \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V \otimes_E E')$, which is an étale (φ_q, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E'$. By Galois descent, $\mathbf{D} = \mathbf{D}_{E'}^{\mathrm{Gal}(E'/E)}$

is an étale (φ_q, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E$. An explicit description of the inverse functor to $\mathbf{D}^{\dagger}_{\mathrm{rig},K}$ (provided for example in Proposition 1.5 of [FX13]) reveals that $\mathbf{D}$ corresponds to V under the equivalence of Theorem 5.2, because the $\mathrm{Gal}(E'/E)$-action commutes with all the structure involved. This shows that V is overconvergent. □

11.2 The modules $\mathbf{D}_{*,K}$ and the extended dictionary

Recall that for $n \geq 0$ we set $r_n = p^{nf-1}(p-1)$. For $r > 0$ we let $n(r)$ be the minimal n such that $r_n \geq r$. If I is a closed interval and $r_0 = \frac{p-1}{p} \in I$, then for $\widetilde{\mathbf{B}}^I$ as in §4 the usual completion map $\widetilde{\mathbf{A}}^+ \to \mathbf{B}^+_{\mathrm{dR}}$ extends to a map $\iota_0 : \widetilde{\mathbf{B}}^I \to \mathbf{B}^+_{\mathrm{dR}}$. More generally if $r_n \in I$ then one has the map $\iota_n = \iota_0 \circ \varphi_q^{-n} : \widetilde{\mathbf{B}}^I \to \mathbf{B}^+_{\mathrm{dR}}$. Now let $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K} = \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K} \cap \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, then for $n \geq n(r)$ the map above restricts to give $\iota_n : \mathbf{B}^{\dagger,r}_{\mathrm{rig},K} \to K_n[[t_K]] \subset \mathbf{B}_{\mathrm{dR}}$. By Théorème I.3.3 of [Be08], if $r \gg 0$, there exists a unique $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-submodule $\mathbf{D}^{\dagger,r}_{\mathrm{rig},K}$ of $\mathbf{D}^{\dagger}_{\mathrm{rig},K}$ such that $\mathbf{D}^{\dagger}_{\mathrm{rig},K} = \mathbf{D}^{\dagger,r}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and such that the $\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}$-module $\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{D}^{\dagger,r}_{\mathrm{rig},K}$ has a basis contained in $\varphi_q\left(\mathbf{D}^{\dagger,r}_{\mathrm{rig},K}\right)$. Finally, let $t_K = \log_{\mathcal{F}}(T) \in \mathbf{B}^{\dagger}_{\mathrm{rig},K}$; it belongs to $\mathbf{B}^{\dagger,r_0}_{\mathrm{rig},K}$ and $\iota_0(t_K)$ coincides with the usual t_K of $\mathbf{B}^+_{\mathrm{dR}}$ as in §2 and §3. We set

$$\mathrm{D}_{\mathrm{Sen},K}\left(\mathrm{D}^{\dagger}_{\mathrm{rig},K}\right) = \left(D^{\dagger,r}_{\mathrm{rig},K} \otimes_{\theta\circ\varphi_q^{-n}} K_n\right) \otimes_{K_n} K_\infty,$$
$$\mathbf{D}_{\mathrm{dif},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right) = \left(\mathbf{D}^{\dagger,r}_{\mathrm{rig},K} \otimes_{\iota_n} K_n((t_K))\right) \otimes_{K_n} K_\infty((t_K)),$$
$$\mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right) = \mathbf{D}_{\mathrm{dif},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right)^{\Gamma_K},$$
$$\mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right) = \left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}[1/t_K]\right)^{\Gamma_K},$$
$$\mathbf{D}_{\mathrm{st},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right) = \left(\mathbf{D}^{\dagger}_{\log,K}[1/t_K]\right)^{\Gamma_K}.$$

One verifies that $\mathbf{D}_{\mathrm{Sen},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right)$ and $\mathbf{D}_{\mathrm{dif},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right)$ are independent of the choice of n. The main theorem of this section is the following.

Theorem 11.5. *Let V be K-analytic representation of G_K. For*

$$* \in \{\mathrm{Sen}, \mathrm{dif}, \mathrm{dR}, \mathrm{cris}, \mathrm{st}\},$$

we have a natural isomorphism

$$\mathbf{D}_{*,K}(V) \cong \mathbf{D}_{*,K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)\right).$$

Proof. Set $\mathbf{D}^{\dagger}_{\mathrm{rig},K} = \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)$. For $r, n \gg 0$, we have a natural map $\mathbf{D}^{\dagger,r}_{\mathrm{rig},K} \xrightarrow{\theta\circ\varphi_q^{-n}} W$, where $W = (\mathbf{C}_p \otimes_K V)^{H_K}$. The image of $\theta \circ \varphi_q^{-n}$ is by definition $\mathbf{D}_{\mathrm{Sen},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right)$, which is a K_∞-submodule of rank $d = \dim_K V$. As $\theta \circ \varphi_q^{-n}$ is Γ_K equivariant, it maps pro K-analytic vectors to locally K-analytic vectors, so the image lands in $W^{K-\mathrm{la}} = \mathbf{D}_{\mathrm{Sen},K}(V)$. Comparing ranks we get the desired isomorphism for $* = \mathrm{Sen}$. Replacing $\theta \circ \varphi_q^{-n}$ by $\iota_0 \circ \varphi_q^{-n}$ we similarly get a map $\mathbf{D}^{+}_{\mathrm{dif},K}(V) \to \mathbf{D}^{+}_{\mathrm{dif},K}\left(\mathbf{D}^{\dagger,r}_{\mathrm{rig},K}\right)$ of two $K_\infty[[t_K]]$-modules of rank d, whose reduction mod t_K is the isomorphism $\mathbf{D}_{\mathrm{Sen},K}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{Sen},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}\right)$. Thus by Nakayama's lemma we have $\mathbf{D}^{+}_{\mathrm{dif},K}(V) \cong \mathbf{D}^{+}_{\mathrm{dif},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)\right)$ and we deduce

$$\mathbf{D}_{\mathrm{dif},K}(V) \cong \mathbf{D}_{\mathrm{dif},K}\left(\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)\right)$$

by passing to colimits.

As $\mathbf{D}_{\mathrm{cris},K} = \mathbf{D}^{N=0}_{\mathrm{st},K}$, it remains to prove the comparison for $* = \mathrm{st}$. Twisting V by an appropriate power of χ_π, we further reduce to proving that $\mathbf{D}^{+}_{\mathrm{st},K}(V) = \left(\mathbf{D}^{\dagger}_{\log,K}\right)^{\Gamma_K}$. By Lemma 5.6 below, we have

$$\begin{aligned}
\mathbf{D}^{+}_{\mathrm{st},K}(V) &= \left(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_K V\right)^{G_K} \\
&= \left(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V)\right)^{G_K} \\
&= \left(\widetilde{\mathbf{B}}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V)\right)^{\Gamma_K}.
\end{aligned}$$

On the one hand, this implies that $\mathbf{D}^{+}_{\mathrm{st},K}(V) \subset \left(\mathbf{D}^{\dagger}_{\log,K}\right)^{\Gamma_K}$. On the other hand, vectors which are fixed by Γ_K are also pro K-analytic on Γ_K, so

$$\begin{aligned}
\mathbf{D}^{+}_{\mathrm{st},K}(V) &= \left(\widetilde{\mathbf{B}}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V)\right)^{\Gamma_K, K-\mathrm{pa}} \\
&= \left(\left(\widetilde{\mathbf{B}}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V)\right)^{K-\mathrm{pa}}\right)^{\Gamma_K}.
\end{aligned}$$

Since V is K-analytic, Theorem 5.2 implies that $\mathbf{D}^{\dagger}_{\log,K}(V)$ is pro K-analytic, and so by Lemma 2.1 we have

$$\left(\widetilde{\mathbf{B}}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V)\right)^{K-\mathrm{pa}} = \left(\widetilde{\mathbf{B}}^{\dagger}_{\log,K}\right)^{K-\mathrm{pa}} \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V).$$

Applying Theorem 4.6, we deduce

$$\mathbf{D}^{+}_{\mathrm{st},K}(V) \subset \left(\cup_{n\geq 0}\varphi_q^{-n}\left(\mathbf{B}^{\dagger}_{\log,K}\right) \otimes_{\mathbf{B}^{\dagger}_{\log,K}} \mathbf{D}^{\dagger}_{\log,K}(V)\right)^{\Gamma_K}.$$

Thus $\varphi_q^n\left(\mathbf{D}^+_{\mathrm{st},K}(V)\right) \subset \left(\mathbf{D}^\dagger_{\log,K}\right)^{\Gamma_K}$ for some $n \gg 0$. If $e_1, ..., e_l$ is a basis of $\mathbf{D}^+_{\mathrm{st},K}(V)$ then $\varphi_q^n(e_1), ..., \varphi_q^n(e_l)$ gives another basis of $\mathbf{D}^+_{\mathrm{st},K}(V)$ which lies in $\left(\mathbf{D}^\dagger_{\log,K}\right)^{\Gamma_K}$. This concludes the proof. □

The following lemma was used in the proof of Theorem 5.5.

Lemma 11.6. *Let* $V \in \mathrm{Rep}_E(G_K)$. *Then*

$$\mathbf{D}^+_{\mathrm{st},K}(V) = \left(\widetilde{\mathbf{B}}^\dagger_{\log} \otimes_K V\right)^{G_K}.$$

Proof. Given an automorphism $\sigma : K_0 \to K_0$, let V^σ be the σ-twist of V. Then we have G_K-compatible identifications $\mathbf{B}^+_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V \cong \oplus_\sigma \mathbf{B}^+_{\mathrm{st}} \otimes_{K_0} V^\sigma$ and $\widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{\mathbf{Q}_p} V \cong \oplus_\sigma \widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{K_0} V^\sigma$. Now by Proposition 3.4 of [Be02] we have $\left(\mathbf{B}^+_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V\right)^{G_K} = \left(\widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{\mathbf{Q}_p} V\right)^{G_K}$ and hence by projecting to the $\sigma = \mathrm{Id}$ component

$$\mathbf{D}^+_{\mathrm{st},K}(V) = \left(\mathbf{B}^+_{\mathrm{st}} \otimes_{K_0} V\right)^{G_K} = \left(\widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{K_0} V\right)^{G_K}.$$

Finally, by Proposition 4.3 we have $\widetilde{\mathbf{B}}^\dagger_{\log} = \widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{K_0} K$ so $\left(\widetilde{\mathbf{B}}^\dagger_{\log,0} \otimes_{K_0} V\right)^{G_K} = \left(\widetilde{\mathbf{B}}^\dagger_{\log} \otimes_K V\right)^{G_K}$. □

Remark 11.7. The definitions given in this section for

$$\mathbf{D}_{\mathrm{Sen},K}, \mathbf{D}_{\mathrm{dif},K}, \mathbf{D}_{\mathrm{dR},K}, \mathbf{D}_{\mathrm{cris},K}, \mathbf{D}_{\mathrm{st},K}$$

make sense for non étale K-analytic (φ_q, Γ_K)-modules. The properties of these modules which were proved in §3 carry over with no difficulty to this more general case.

Remark 11.8. For each $\mathbf{D}^\dagger_{\mathrm{rig},K}$, we define filtrations on $\mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$ and $\mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$. For $i \in \mathbf{Z}$, we set

$$\mathrm{Fil}^i\left(\mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right) = \left(K_\infty \otimes_{K_n} t_K^i K_n[[t_K]] \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{D}^{\dagger,r}_{\mathrm{rig},K}\right)^{\Gamma_K}$$

for $n \geq n(r)$. Recall there are injections

$$\iota_n : \mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right) \hookrightarrow \mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$$

for $n \gg 0$, and set

$$\varphi_q^n\left(\mathrm{Fil}^i\mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right)=\iota_n^{-1}\left(\mathrm{Fil}^i\left(\mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right)\right).$$

The filtrations $\mathrm{Fil}^i\left(\mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right)$ and $\mathrm{Fil}^i\left(\mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right)$ are independent of all choices. If $\mathbf{D}=\mathbf{D}^\dagger_{\mathrm{rig},K}(V)$, one checks that the usual filtration of $\mathbf{D}_{\mathrm{dR},K}(V)$ (resp. $\mathbf{D}_{\mathrm{cris},K}(V)$) induced from that of $\mathbf{B}_{\mathrm{dR}}$ (resp. $\mathbf{B}_{\mathrm{max},K}$) coincides with the above defined filtration via the identification of Theorem 5.5.

12 Lubin-Tate trianguline representations of dimension 2

We keep the convention that E is a finite extension of $\mathbf{Q}_p$ which contains K.

Definition 12.1. 1. A (φ_q,Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes_K E$ is called Lubin-Tate trianguline if it can be written as a successive extension of (φ_q,Γ_K)-modules of rank 1.
2. An E-linear K-analytic representation V is called Lubin-Tate trianguline if $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)$ is Lubin-Tate trianguline.

In the case $K=\mathbf{Q}_p$, trianguline (φ_q,Γ_K)-modules of dimension 2 were first studied by Colmez in [Co08]. We shall be concerned with Lubin-Tate trianguline (φ_q,Γ_K)-modules of dimension 2 which were studied by Fourquaux and Xie in [FX13].

12.1 Characters of the Weil group

Recall that if W_K is the Weil group of K, local class field theory gives a natural isomorphism $W_K^{\mathrm{ab}}\cong K^\times$. This allows us to identify characters $\delta: K^\times\to E^\times$ with characters $W_K^{\mathrm{ab}}\to E^\times$, the identification given by

$$\delta(\mathrm{Frob}_\pi^{-n}g)=\delta(\pi)^n\delta(\chi_\pi(g))$$

for $g\in\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{un}})$ and $n\in\mathbf{Z}$. To such characters δ we associate the (φ_q,Γ_K)-module $\left(\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes_K E\right)(\delta)$. It is a (φ_q,Γ_K)-module of rank 1 with a basis e_δ, where $\varphi_q(e_\delta)=\delta(\pi)e_\delta$ and $g(e_\delta)=\delta(\chi_\pi(g))$ for $g\in\Gamma_K$. Note that this (φ_q,Γ_K)-module is étale exactly when δ is unitary; in this case, if δ is locally K-analytic, the module $\left(\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes_K E\right)(\delta)$ corresponds under the equivalence of categories in §5.1 to the extension of δ to $\mathrm{Gal}(\overline{K}/K)$. Proposition 1.9 of [FX13] shows that all K-analytic (φ_q,Γ_K)-modules of rank 1 over $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E$ are obtained in this way. We write $\mathscr{I}_{\mathrm{an}} = \mathscr{I}_{\mathrm{an}}(E)$ for the set of locally K-analytic Weil characters. There are two characters in $\mathscr{I}_{\mathrm{an}}$ of particular interest: the inclusion character $x : K^{\times} \to E^{\times}$ and the character $\mu_{\lambda}(z) = \lambda^{\mathrm{val}_{\pi}(z)}$. To $\delta \in \mathscr{I}_{\mathrm{an}}$ we associate to the weight $w(\delta) = \frac{\log_p \delta(u)}{\log_p u}$ where $u \in \mathcal{O}_K^{\times}$ is any element with $\log_p u \neq 0$, and then $w(\delta)$ does not depend on u. If δ is unitary and $w(\delta) \in \mathbf{Z}$ then $w(\delta)$ is the K-Hodge-Tate weight of the associated character of $\mathrm{Gal}(\overline{K}/K)$ (see §3).

12.2 Extensions

Given $\delta_1, \delta_2 \in \mathscr{I}_{\mathrm{an}}$ we consider the set of extensions

$$0 \to \left(\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E\right)(\delta_1) \to \mathbf{D}^{\dagger}_{\mathrm{rig},K} \to \left(\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E\right)(\delta_2) \to 0$$

in the category of K-analytic (φ_q, Γ_K)-modules. These extensions are classified by a finite-dimensional E-vector space $\mathrm{H}^1_{\mathrm{an}}\left(\delta_1\delta_2^{-1}\right)$, whose dimension is determined in Theorem 0.3 of [FX13] as follows.

Theorem 12.2. $\dim_E \mathrm{H}^1_{\mathrm{an}}\left(\delta_1\delta_2^{-1}\right) = 2$ *if* $\delta_1\delta_2^{-1} = x^{-i}$ *for* $i \in \mathbf{Z}_{\geq 1}$ *or if* $\delta_1\delta_2^{-1} = \mu_{q^{-1}}x^i$ *for* $i \in \mathbf{Z}_{\geq 0}$*. Otherwise,* $\dim_E \mathrm{H}^1_{\mathrm{an}}\left(\delta_1\delta_2^{-1}\right) = 1$.

12.3 Spaces of Lubin-Tate trianguline (φ_q, Γ_K)-modules of dimension 2

There is an action of $\mathbb{G}_m(E) = E^{\times}$ on $\mathrm{H}^1_{\mathrm{an}}\left(\delta_1\delta_2^{-1}\right)$, and extensions which lie in the same orbit of this action give rise to isomorphic (φ_q, Γ_K)-modules. Following §6 of [FX13] we write

$$\mathscr{S}^{\mathrm{an}} = \mathscr{S}^{\mathrm{an}}(E) =$$

$$\left\{s = (\delta_1, \delta_2, \mathcal{L}) : \delta_1, \delta_2 \in \mathscr{I}_{\mathrm{an}}(E), \mathcal{L} \in \mathrm{H}^1_{\mathrm{an}}(\delta_1\delta_2^{-1}) \setminus \{0\}\right\} / \mathbb{G}_m(E).$$

By Theorem 6.2, each pair of characters $\delta_1, \delta_2 \in \mathscr{I}_{\mathrm{an}}$ give rise either to a unique point $(\delta_1, \delta_2, \infty)$ of $\mathscr{S}^{\mathrm{an}}$ in the generic case or a $\mathbb{P}^1(E)$-family of points of $\mathscr{S}^{\mathrm{an}}$ in the non generic case. To each such s we associate the correspnding (φ_q, Γ_K)-module $\mathbf{D}^{\dagger}_{\mathrm{rig},K}(s)$ which is an extension of $\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E\right)(\delta_1)$ by $\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E\right)(\delta_2)$.

Inside $\mathscr{S}^{\mathrm{an}}$, we consider the subset $\mathscr{S}^{\mathrm{an}}_{+}$ of real interest given by these $s \in \mathscr{S}^{\mathrm{an}}$ such that

$$\mathrm{val}_{\pi}(\delta_1\delta_2) = 0 \text{ and } \mathrm{val}_{\pi}(\delta_1(\pi)) \geq 0.$$

For example, all étale K-analytic Lubin-Tate trianguline (φ_q, Γ_K)-modules appear in $\mathscr{S}^{\mathrm{an}}_{+}$.

For an element $s \in \mathscr{S}_+^{\mathrm{an}}$ we associate two invariants: the slope $u(s) = \mathrm{val}_\pi(\delta_1(\pi))$ and the weight $w(s) = w(\delta_1\delta_2^{-1}) = w(\delta_1) - w(\delta_2)$. We then have the following partition

$$\mathscr{S}_+^{\mathrm{an}} = \mathscr{S}_+^{\mathrm{ng}} \amalg \mathscr{S}_+^{\mathrm{cris}} \amalg \mathscr{S}_+^{\mathrm{st}} \amalg \mathscr{S}_+^{\mathrm{ord}} \amalg \mathscr{S}_+^{\mathrm{ncl}},$$

where

$$\begin{aligned}
\mathscr{S}_+^{\mathrm{ng}} &= \left\{ s \in \mathscr{S}_+^{\mathrm{an}} : w(s) \notin \mathbf{Z}_{\geq 1} \right\}, \\
\mathscr{S}_+^{\mathrm{cris}} &= \left\{ s \in \mathscr{S}_+^{\mathrm{an}} : w(s) \in \mathbf{Z}_{\geq 1}, u(s) < w(s), \mathcal{L} = \infty \right\}, \\
\mathscr{S}_+^{\mathrm{st}} &= \left\{ s \in \mathscr{S}_+^{\mathrm{an}} : w(s) \in \mathbf{Z}_{\geq 1}, u(s) < w(s), \mathcal{L} \neq \infty \right\}, \\
\mathscr{S}_+^{\mathrm{ord}} &= \left\{ s \in \mathscr{S}_+^{\mathrm{an}} : w(s) \in \mathbf{Z}_{\geq 1}, u(s) = w(s) \right\}, \\
\mathscr{S}_+^{\mathrm{ncl}} &= \left\{ s \in \mathscr{S}_+^{\mathrm{an}} : w(s) \in \mathbf{Z}_{\geq 1}, u(s) > w(s) \right\}.
\end{aligned}$$

(Here ng and ncl are abbreviations for "non-geometric" and "non-classical"). We also write $\mathscr{S}_0^{\mathrm{an}} = \left\{ s \in \mathscr{S}_+^{\mathrm{an}} : u(s) = 0 \right\}$ and $\mathscr{S}_0^* = \mathscr{S}_+^* \cap \mathscr{S}_0^{\mathrm{an}}$. Each subset $\mathscr{S}_+^*$ above is named according to the behaviour that the $s \in \mathscr{S}_+^*$ exhibit. For example, $s \in \mathscr{S}_+^{\mathrm{an}}$ is étale if and only if $s \notin \mathscr{S}_+^{\mathrm{ncl}}$, and in that case if $s \in \mathscr{S}_+^{\mathrm{cris}}$ (resp. $s \in \mathscr{S}_+^{\mathrm{st}}$, $s \in \mathscr{S}_+^{\mathrm{ord}}$) then $\mathbf{D}_{\mathrm{rig},K}^\dagger(s)$ comes from a potentially crystalline (resp. semistable but non-crystalline, potentially ordinary) E-representation up to a twist. See §6 of [FX13] for more details.

Lemma 12.3. *Let $\mathbf{D}_{\mathrm{rig},K}^\dagger$ be a K-analytic (φ_q, Γ_K)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_K E$. Then*

$$\nabla\left(t_K \mathbf{D}_{\mathrm{dif},K}^+\left(\mathbf{D}_{\mathrm{rig},K}^\dagger\right)\right) \subset t_K \mathbf{D}_{\mathrm{dif},K}^+\left(\mathbf{D}_{\mathrm{rig},K}^\dagger\right).$$

.

Proof. Use the identity $\nabla(t_K x) = t_K x + t_K \nabla(x) = t_K(x + \nabla(x))$. □

The following shows that if $s \in \mathscr{S}_+^{\mathrm{an}} \backslash \left(\mathscr{S}_+^{\mathrm{ng}} \cup \mathscr{S}_+^{\mathrm{ncl}}\right)$ then $\mathbf{D}_{\mathrm{rig},K}^\dagger(s)$ is comes from a de Rham E-representation up to a twist (see Corollary 3.10).

Corollary 12.4. *Let $s = (\delta_1, \delta_2, \mathcal{L}) \in \mathscr{S}_+^{\mathrm{an}}$ and suppose that $w(\delta_1), w(\delta_2) \in \mathbf{Z}$ with $w(\delta_1) > w(\delta_2)$. Then $\mathbf{D}_{\mathrm{rig},K}^\dagger(s)$ is K-de Rham.*

Proof. We may assume that $\delta_1 = 1$. Write $\delta = \delta_2$ so that and $w(\delta) < 0$. Then $\mathbf{D}_{\mathrm{dif},K}^+ = \mathbf{D}_{\mathrm{dif},K}^+\left(\mathbf{D}_{\mathrm{rig},K}^\dagger\right)$ is an extension of the form

$$0 \to K_\infty[[t_K]] \otimes_K E \to \mathbf{D}_{\mathrm{dif},K}^+ \to \mathbf{D}_{\mathrm{dif},K}^+\left(\left(\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_K E\right)(\delta)\right) \to 0.$$

Take $e_\delta \in \mathbf{D}_{\mathrm{dR},K}^+\left(\left(\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_K E\right)(\delta)\right) = \mathbf{D}_{\mathrm{dif},K}^+\left(\left(\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_K E\right)(\delta)\right)^{\Gamma_K=1}$ and lift it to $\mathbf{D}_{\mathrm{dif},K}^+$. If we take $e = 1 \in K_\infty[[t_K]] \otimes_K E$ then e, e_δ is a basis of $\mathbf{D}_{\mathrm{dif},K}^+$, and the action of $\nabla_{\mathrm{dif},K}$ on $\mathbf{D}_{\mathrm{dif},K}^+$ in the basis e, e_δ is given by

$$\mathrm{Mat}(\nabla_{\mathrm{dif},K}) = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix},$$

where $f \in K_\infty[[t_K]] \otimes_K E$. As $w(\delta) < 0$, we have that e_δ is divisible by t_K so by Lemma 6.3 we have that f is divisible by t_K. As $\nabla_{\mathrm{dif},K} = t_K \frac{\partial}{\partial t_K}$ on $K_\infty[[t_K]]$ we may find an $h \in K_\infty[[t_K]] \otimes_K E$ with $\nabla_{\mathrm{dif},K}(h) = f$. Then e and $e_\delta - he$ give a full set of sections for $\nabla_{\mathrm{dif},K}$ on $\mathbf{D}^+_{\mathrm{dif},K}$. By Proposition 3.7 (which applies according to Remark 5.6) we are done. □

Remark 12.5. The argument of Corollary 6.4 can be generalized to show that if $\mathbf{D}^\dagger_{\mathrm{rig},K}$ is a K-analytic Lubin-Tate trianguline (φ_q, Γ_K)-module which is a successive extension of $\left(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E\right)(\delta_i)$ for $i = 1, ..., d$ with $w(\delta_i)$ integers satisfying $w(\delta_1) > ... > w(\delta_d)$, then $\mathbf{D}^\dagger_{\mathrm{rig},K}$ is K-de Rham.

12.4 Lubin-Tate triangulation of a p-adic representation of dimension 2

The following generalizes Lemma 2.4.2 of [BC09]. Using Theorem 5.5, we may follow the same proof of loc. cit.

Proposition 12.6. *Let $\mathbf{D}^\dagger_{\mathrm{rig},K}$ be a K-analytic (φ_q, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E$, $\alpha \in E^\times$ and $i \in \mathbf{Z}$. If $x \in \mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)^{\varphi_q=\alpha} \subset \mathbf{D}^\dagger_{\mathrm{rig},K}[1/t_K]$, then $x \in \mathrm{Fil}^i\left(\mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right)$ if and only if $x \in t_K^i \mathbf{D}^\dagger_{\mathrm{rig},K}$.*

Proof. We may reduce to the case $i = 0$ by twisting. By Theorem 5.5,

$$\mathrm{Fil}^0\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right) = \mathbf{D}^+_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right) = \left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)^{\Gamma_K}$$

and

$$\mathbf{D}^+_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right) = \mathbf{D}^+_{\mathrm{dif},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)^{\Gamma_K}.$$

Recall that for $n \gg 0$ we have an injection

$$\iota_n : \mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right) \hookrightarrow \mathbf{D}_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$$

and by the definition of filtration in Remark 5.8, we have

$$\varphi_q^n\left(\mathbf{D}^+_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right) = \iota_n^{-1}\left(\mathbf{D}^+_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right).$$

If $x \in \mathbf{D}^\dagger_{\mathrm{rig},K} \cap \mathbf{D}_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)^{\varphi_q=\alpha}$ we have

$$\iota_n(x) \in \iota_n\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right) \subset \mathbf{D}^+_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$$

for $n \gg 0$, so $x \in \varphi_q^n\left(\mathbf{D}^+_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)\right)$ and the relation $x = \alpha^n\varphi_q^{-n}(x)$ implies $x \in \mathbf{D}^+_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$. Conversely, if $x \in \mathbf{D}^+_{\mathrm{cris},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$ satisfies $\varphi_q(x) = x\alpha$, then $x = \alpha^{-n}\varphi_q^n(x)$, so $\iota_n(x)$ lies in $\mathbf{D}^+_{\mathrm{dR},K}\left(\mathbf{D}^\dagger_{\mathrm{rig},K}\right)$ for $n \gg 0$. Choose a $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-basis $e_1, ..., e_d$ of $\mathbf{D}^{\dagger,r}_{\mathrm{rig},K}$ for $r \gg 0$, and write $x = \sum f_j e_j$ for $f_j \in \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}[1/t_K]$. For each j, we have that $\iota_n(f_j)$ lies in $K_n[[t_K]]$ for $n \gg 0$. This implies by Lemma 4.6 of [Be02] that each $f_j \in \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$, whence $x \in \mathbf{D}^\dagger_{\mathrm{rig},K}$. □

Following §3 of [Ch08], we compute the triangulation of a representation of dimension 2 in terms of a crystalline period.

Proposition 12.7. *Let V be a 2-dimensional E-linear K-analytic representation of G_K. Then V is Lubin-Tate trianguline if and only if there exists a K-analytic character $\eta : G_K \to \mathcal{O}_E^\times$ and $\alpha \in E^\times$ such that*

$$\mathbf{D}_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha} \neq 0.$$

Moreover, if i is the largest integer such that

$$\mathrm{Fil}^i\mathbf{D}_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha} \not\subseteq \mathrm{Fil}^{i+1}\mathbf{D}_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha},$$

then $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)$ is an extension of $\left(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E\right)(\delta_1)$ by $\left(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E\right)(\delta_2)$ where $\delta_1 = \eta^{-1}\mu_\alpha x^{-i}$ and $\delta_2 = \eta\mu_{\alpha^{-1}}x^i\det(V)$.

Proof. If V is Lubin-Tate trianguline, then $\mathbf{D}^\dagger_{\mathrm{rig,K}}(V)$ contains a submodule of rank 1 isomorphic to $\left(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E\right)(\delta)$ for some $\delta \in \mathscr{I}_{\mathrm{an}}$. Taking $\eta : G_K \to \mathcal{O}_E^\times$ defined by $\eta(g) = \delta^{-1}(\chi_\pi(g))$ we have $\mathbf{D}^+_{\mathrm{cris},L}(V(\eta))^{\varphi_q=\delta(\pi)} = \mathbf{D}^\dagger_{\mathrm{rig}}(V(\eta))^{\Gamma_L=1,\varphi_q=\delta(\pi)} \neq 0$. Conversely, suppose that such an α and η exist. We shall shows V is Lubin-Tate trianguline with the described triangulation. Twisting by a power of χ_π, we may assume that $i = 0$ and that $\mathbf{D}^+_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha}$ contains an element $f \notin \mathrm{Fil}^1\mathbf{D}_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha}$. By what we have proven in §5, we have

$$\mathbf{D}^+_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha} = \mathbf{D}^\dagger_{\mathrm{rig},K}(V(\eta))^{\Gamma_K=1,\varphi_q=\alpha},$$

so $f \in \mathbf{D}^\dagger_{\mathrm{rig},K}(V(\eta))^{\Gamma_K=1,\varphi_q=\alpha}$. By taking its span and twisting by η^{-1} we get a rank 1 sub (φ_q, Γ_K)-module of $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)$. The ideal I generated by the coefficients of f in a basis of in $\mathbf{D}^\dagger_{\mathrm{rig},K}(V(\eta))$ is stable under the actions of φ_q and Γ_K. As $\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E$ is a Bézout domain and I is finitely generated, it is principal, and we conclude from Lemma 1.1 of [FX13] that $I = (t_K^n)$ for $n \in \mathbf{Z}_{\geq 0}$. Proposition 6.6 shows that $n = 0$, and this means that

$$\left(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E\right) \cdot f(\eta^{-1}) \cong \left(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_K E\right)(\eta^{-1}\mu_\alpha)$$

is a rank 1 saturated submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)$. We then have

$$\mathbf{D}^{\dagger}_{\mathrm{rig},K}(V)/\left(\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E\right)\left(\eta^{-1}\mu_\alpha\right) \cong \left(\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_K E\right)\left(\eta\mu_{\alpha^{-1}}\det(V)\right)$$

by the classification of (φ_q, Γ_K)-modules of rank 1. □

Finally, we conclude with the proof of Theorem B from the introduction. To do so, we first recall what are cyclotomic trianguline representations. Let $K_\infty^{\mathrm{cyc}} = K(\mu_{p^\infty})$ be the cyclotomic extension of K and let K_0' be the maximal unramified extension of K_0 in K_∞^{cyc}. The ring $\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K}$ is the ring of power series $\sum_{n\in\mathbf{Z}} a_n T^n$ with $a_n \in K_0'$ and such that $f(T)$ converges on some nonempty annulus $r < |T| < 1$. The ring is endowed with a Frob_p-semilinear φ action and a semilinear Γ_K^{cyc}-action. If $K = K_0$ then $\varphi(T) = (1+T)^p - 1$ and $\gamma(T) = (1+T)^{\chi_{\mathrm{cyc}}(\gamma)} - 1$, but in general the action has to do with the theory of lifting the field of norms and is more complicated.

We can then define a notion of a $(\varphi, \Gamma_K^{\mathrm{cyc}})$-module over $\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E$ analogous to the notion of a (φ, Γ_K)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E$. If V is an E-linear representation of G_K, one can associate to V a $(\varphi, \Gamma_K^{\mathrm{cyc}})$-module $\mathbf{D}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K}(V)$ over $\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E$. Now let $\delta : K^\times \to E^\times$ a continuous character; we can define a $(\varphi, \Gamma_K^{\mathrm{cyc}})$-module $\left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta)$ in the following way. If δ is unitary, then it corresponds to a character $\delta : G_K \to E^\times$, and we set

$$\left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta) = \mathbf{D}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K}(V)(E(\delta)).$$

If $\delta|_{\mathcal{O}_K^\times} = 1$, set

$$\left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta) = \left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)[\varphi] \otimes_{\left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)[\varphi_q]} E e_\delta,$$

where $\varphi_q(e_\delta) = \delta(\pi)e_\delta$. For general δ, write $\delta = \delta_1\delta_2$ where δ_1 is unitary and $\delta_2|_{\mathcal{O}_K^\times}$ and set

$$\left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta) = \left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta_1) \otimes \left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta_2).$$

An an E-linear representation V of G_K is said to be cyclotomic trianguline if $\mathbf{D}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K}(V)$ is a successive extension $(\varphi, \Gamma_K^{\mathrm{cyc}})$-modules of the form $\left(\mathbf{B}^{\dagger,\mathrm{cyc}}_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E\right)(\delta)$. This is the same notion of triangulinity which appears in [Na09, KPX14, Li12], but we give it a different name here to distinguish it from Lubin-Tate triangulinity.

Theorem 12.8. *Let V be a 2-dimensional E-linear K-analytic representation of G_K. The following are equivalent.*

1. *V is cyclotomic trianguline.*
2. *There exists a K-analytic character $\eta : \mathcal{O}_K^\times \to E^\times$ and $\alpha \in E^\times$ such that $\mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V(\eta))^{\varphi_q=\alpha}$ is nonzero.*
3. *There exists a K-analytic character $\eta : \mathcal{O}_K^\times \to E^\times$ and $\alpha \in E^\times$ such that $\mathbf{D}_{\mathrm{cris},K}(V(\eta))^{\varphi_q=\alpha}$ is nonzero.*
4. *V is Lubin-Tate trianguline.*

Proof. The equivalence between 3 and 4 was proven in Proposition 6.7, while the equivalence between 2 and 3 follows from Lemma 3.11. It remains to prove the equivalence of 1 and 2. This equivalence seems to be well known but due to a lack of suitable reference when $K \neq \mathbf{Q}_p$ we give a proof here.
If V is cyclotomic trianguline, then $\mathbf{D}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}}(V)$ can be written as an extension

$$0 \to \left(\mathbf{B}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}} \otimes_{\mathbf{Q}_p} E\right)(\delta_1) \to \mathbf{D}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}}(V) \to \left(\mathbf{B}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}} \otimes_{\mathbf{Q}_p} E\right)(\delta_2) \to 0.$$

Since V is K-analytic, δ_1 is also K-analytic. Twisting by $\delta_1|_{\mathcal{O}_K^\times}^{-1}$, we may assume $\delta_1|_{\mathcal{O}_K^\times} = 1$. It then follows from [KPX14, Example 6.2.6] that

$$\mathbf{D}_{\mathrm{cris}}\left(\left(\mathbf{B}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}} \otimes_{\mathbf{Q}_p} E\right)(\delta_1)\right) = \mathrm{I}_{\mathbf{Q}_p}^{K}\left(Ee_{\delta_1}\right),$$

where $\varphi_q(e_{\delta_1}) = \delta_1(\pi)e_{\delta_1}$. It follows that $\mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)^{\varphi_q=\delta_1(\pi)} \neq 0$.
Conversely, suppose that 2 holds. By replacing V with a K-analytic twist, we may assume that $\mathbf{D}^+_{\mathrm{cris},\mathbf{Q}_p}(V)^{\varphi_q=\alpha} = \mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)^{\varphi_q=\alpha} \neq 0$. It follows from Berger's dictionary that

$$\mathbf{D}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}}(V)^{\Gamma_K^{\mathrm{cyc}},\varphi_q=\alpha} = \mathbf{D}^+_{\mathrm{cris},\mathbf{Q}_p}(V)^{\varphi_q\neq 0},$$

so that $\mathbf{D}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}}(V)^{\Gamma_K^{\mathrm{cyc}},\varphi_q=\alpha}$ contains a $(\varphi_q, \Gamma_K^{\mathrm{cyc}})$ invariant E-line, and hence $\left(\mathbf{B}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}} \otimes_{\mathbf{Q}_p} E\right)(\delta)$ where $\delta|_{\mathcal{O}_K^\times} = 1$ and $\delta(\pi_K) = \alpha$. This sub $\mathbf{B}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}} \otimes_{\mathbf{Q}_p} E$-module may not be saturated, but it follows from [KPX14, Corollary 6.2.9] that $\mathbf{D}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}}(V)$ contains a saturated module of the form $\left(\mathbf{B}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}} \otimes_{\mathbf{Q}_p} E\right)(\delta')$. In particular, $\mathbf{D}_{\mathrm{rig},K}^{\dagger,\mathrm{cyc}}(V)$ is an extension of two rank 1 $(\varphi_q, \Gamma_K^{\mathrm{cyc}})$-modules, so V is cyclotomic trianguline. □

13 Overconvergent Hilbert modular forms

13.1 Overconvergent Hilbert eigenforms

We briefly recall what we need about the cuspidal Hilbert eigenvariety of Andreatta, Iovita and Pilloni (see [AIP16]).

Let F be a totally real number field, Σ the set of embeddings of F in $\overline{\mathbf{Q}}$ and $N \in \mathbf{Z}_{\geq 4}$. A choice of an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ determines a decomposition $\Sigma = \amalg_{v:v|p} \Sigma_{F_v}$ where each v is a place of F lying over p. Let L be a finite extension of $\mathbf{Q}_p$ which contains F^{Gal}. The weight space for the algebraic group $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\mathrm{GL}_2$ is $\mathcal{W} = \mathrm{Spf}\left(\mathcal{O}_L\left[\left[(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)^\times \times \mathbf{Z}_p^\times\right]\right]\right)^{\mathrm{rig}}$. If f is a classical Hilbert eigenform on F of tame level N, its weight is a tuple $\mathrm{wt}(f) = \left(\{k_\tau\}_{\tau\in\Sigma}, w\right) \in \mathbf{Z}_{\geq 1}^{\Sigma} \times \mathbf{Z}$ satisfying $k_\tau \equiv w \mod 2$ for each $\tau \in \Sigma$. It is then identified with the point in $\mathcal{W}$ corresponding to the character $(z_1, z_2) \mapsto \left(\prod_{\tau\in\Sigma} \tau(z_1)^{k_\tau}\right) z_2^w$. The cuspidal Hilbert eigenvariety of tame level N is a certain rigid analytic space $\mathcal{E}$ which gives a p-adic interpolation of classical Hilbert eigenforms. More precisely, it is a rigid analytic space together with a weight map $\mathrm{wt} : \mathcal{E} \to \mathcal{W}$ whose points parametrize overconvergent Hilbert modular forms of finite slope together with a choice of Hecke eigenvalues at places $v|p$. We summarize its properties below (see §5 of [AIP16]).

Theorem 13.1. *1. The map* $\mathrm{wt} : \mathcal{E} \to \mathcal{W}$ *is, locally on* $\mathcal{E}$ *and* $\mathcal{W}$*, finite and surjective.*

2. For each $\kappa \in \mathcal{W}(\mathbf{C}_p)$*, the fiber* $\mathrm{wt}^{-1}(\kappa)$ *is in bijection with finite slope Hecke eigenvalues appearing in the space of overconvergent cusp forms of weight* κ*, level* N *and coefficients in* $\mathbf{C}_p$*.*

3. There exists a universal Hecke character $\lambda : \mathcal{H}^{Np} \otimes \mathcal{U}_p \to \mathcal{O}_{\mathcal{E}}$*. Here,* $\mathcal{H}^{Np}$ *is the abstract Hecke algebra away from* Np*, and* $\mathcal{U}_p$ *is the* $\mathbf{Q}_p$*-algebra generated by the* U_v*-operators for* $v|p$*.*

4. There is a universal pseudo-character $T : \mathrm{Gal}(\overline{F}/F) \to \mathcal{O}_{\mathcal{E}}$ *which is unramified for* $\mathfrak{l} \nmid Np$ *such that* $T(\mathrm{Frob}_{\mathfrak{l}}) = \lambda(T_{\mathfrak{l}})$ *for the arithmetic Frobenius* $\mathrm{Frob}_{\mathfrak{l}}$*.*

5. For each $x \in \mathcal{E}$ *there exists a semisimple Galois representation* $\rho_x : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(k(x))$ *which is unramified for* $\mathfrak{l} \nmid Np$ *and which is characterized by* $\mathrm{Tr}(\rho_x) = T_x$ *and* $\det(\rho_x) = \mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{l})\,\lambda_x(S_{\mathfrak{l}})$*.*

6. The generalized Hodge-Tate weights of $\rho_x|_{G_{F_v}}$ *are* $\left\{\frac{w-k_\tau}{2}, \frac{w+k_\tau-2}{2}\right\}_{\tau\in\Sigma_{F_v}}$*.*

We fix a place $v|p$ in F and place ourselves in the setting of §1.2 with $K = F_v$, $\pi = \pi_v$ a uniformizer of F_v, etc. We extend scalars if necessary so that $\rho_x|_{G_{F_v}}$ is F_v^{Gal}-linear.

Proposition 13.2. *For* $x \in \mathcal{E}$*, we have*

$$\mathbf{D}^+_{\mathrm{cris},F_v}\left(\rho_x^\vee|_{G_{F_v}}\left(\prod_{\tau\in\Sigma_{F_v}} (\tau\circ\chi_{\pi_v})^{\frac{w-k_\tau}{2}}\right)\right)^{\varphi_q = \prod_{\tau\in\Sigma_{F_v}} \tau(\pi_v)^{\frac{k_\tau - w}{2}} U_v} \neq 0.$$

Proof. For classical Hilbert modular forms of cohomological weights this is known by Saito's local-global compatibility results in [Sa09]. The regular classical points are Zariski dense in $\mathcal{E}$ by the classicality criterion in [Bi16], so the claim follows from the global triangulation results in Theorem 6.3.13 of [KPX14] or in Theorem 4.4.2 of [Li12].

For example, in the notation of [Li12], Theorem 7.1, Saito's results and the classicality criterion in [Bi16] imply that $\rho_x^\vee|_{G_{F_v}}$ is a family of refined p-adic representations of G_{F_v} of dimension 2, where $\kappa_1 = \left\{\frac{k_\tau - w}{2}\right\}_\tau$, $\kappa_2 = \left\{\frac{-k_\tau - w}{2} + 1\right\}_\tau$, $F_1 = \prod_\tau \tau(\pi_v)^{\frac{k_\tau - w}{2}} U_v$, $\eta_1 = \prod_{\tau \in \Sigma_{F_v}} (\tau \circ \chi_{\pi_v})^{\frac{k_\tau - w}{2}}$ and Z is the set of crystalline points of $\mathcal{E}$. Then Theorem 4.4.2 of [Li12] (or rather, its proof) establishes the desired result after applying Berger's dictionary to pass from $\mathbf{D}^\dagger_{\mathrm{rig},F_v}$ to $\mathbf{D}^+_{\mathrm{cris},F_v}$. □

13.2 Lubin-Tate triangulation

Let $x \in \mathcal{E}$ and and consider $\rho_x|_{G_{F_v}}$ as an E-linear representation for some finite extension $\mathbf{Q}_p \subset E$ which contains F_v^{Gal} and $\overline{k}(x)$. In this section we shall describe exactly when $\rho_x|_{G_{F_v}}$ is either overconvergent or Lubin-Tate trianguline and explicitly describe the triangulations. We shall assume $\rho_x|_{G_{F_v}}$ is nonsplit. The split case is less interesting because in that case it is obvious that $\rho_x|_{G_{F_v}}$ is overconvergent and that Lubin-Tate triangulations exist in a degenerate sense.

Corollary 5.3 gives the following.

Proposition 13.3. *$\rho_x|_{G_{F_v}}$ is overconvergent if and only it is F_v-analytic up to a twist.*

Let us assume then that $\rho_x|_{G_{F_v}}$ is s F_v-analytic up to a twist, so that that the weights at Σ_{F_v} are $(k, 1, ..., 1)$ where $k = k_{\mathrm{Id}}$. Let a_v be eigenvalue of U_v for the corresponding Hecke operator of v. Then $\alpha_v = \pi_v^{\frac{k-1}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p}(\pi_v)\right)^{\frac{1-w}{2}} a_v$ interpolates to a function on $\mathcal{E}$ (see Remark 4.7 of [AIP16]). Upon writing

$$V = \rho_x^\vee|_{G_{F_v}} \left(\chi_{\pi_v}^{\frac{1-k}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ \chi_{\pi_v}\right)^{\frac{w-1}{2}}\right),$$

Proposition 7.2 becomes the statement $\mathbf{D}^+_{\mathrm{cris},F_v}(V)^{\varphi_q = \alpha_v} \neq 0$. The representations $\rho_x|_{G_{F_v}}$ and V differ by a dual and a character twist, so according to Corollary 5.3 their overconvergence and Lubin-Tate triangulinity are equivalent. However, V is F_v-analytic with F_v-Hodge-Tate weights 0 and $k-1$ which makes it nicer to work with.

The following is a generalization of Proposition 5.2 of [Ch08].

Theorem 13.4. *The representation V is Lubin-Tate trianguline. We have $\mathbf{D}^\dagger_{\mathrm{rig},F_v}(V) = \mathbf{D}^\dagger_{\mathrm{rig},F_v}(s)$ for $s = \left(\delta_1, \delta_1^{-1}\det(V), \mathcal{L}\right) \in \mathscr{S}_+^{\mathrm{an}}$, where*

1. If $k \notin \mathbf{Z}_{\geq 1}$ then $\delta_1 = \mu_{\alpha_v}$, $\mathcal{L} = \infty$ and $s \in \mathscr{S}_+^{\mathrm{ng}}$.

2. If $k \in \mathbf{Z}_{\geq 1}$ and $\mathrm{val}_{\pi_v}(\alpha_v) < k-1$ then $\delta_1 = \mu_{\alpha_v}$ and either

   (a) $\mathcal{L} = \infty$, in which case $s \in \mathscr{S}_+^{\mathrm{cris}}$.

   (b) $\mathcal{L} \neq \infty$, in which case $s \in \mathscr{S}_+^{\mathrm{st}}$. This is only possible if $2\mathrm{val}_{\pi_v}(\alpha_v) + [F_v : \mathbf{Q}_p] = k-1$.

3. If $k \in \mathbf{Z}_{>1}$ and $\mathrm{val}_{\pi_v}(\alpha_v) = k-1$, then $\delta_1 = \mu_{\alpha_v}$, $\mathcal{L} = \infty$ and $s \in \mathscr{S}_+^{\mathrm{ord}}$.

4. If $k \in \mathbf{Z}_{\geq 1}$ and $\mathrm{val}_{\pi_v}(\alpha_v) > k-1$ then $\delta_1 = x^{1-k}\mu_{\alpha_v}$, $\mathcal{L} = \infty$ and $s \in \mathscr{S}_+^{\mathrm{ng}}$.

Proof. By Proposition 6.7, we know V is Lubin-Tate trianguline and a triangulation is determined the by largest $i \in \mathbf{Z}$ with $\mathrm{Fil}^i \mathbf{D}_{\mathrm{cris},F_v}(V)^{\varphi_q = \alpha_v} \not\subseteq \mathrm{Fil}^{i+1}\mathbf{D}_{\mathrm{cris},F_v}(V)^{\varphi_q=\alpha_v}$. It remains to determine i in each case; it is a non-negative F_v-Hodge-Tate weight of V. If $k \notin \mathbf{Z}_{>1}$ then $i = 0$, so (1) is settled and we may assume $k \in \mathbf{Z}_{>1}$.

Assume that $\mathrm{val}_{\pi_v}(\alpha_v) < k-1$ and suppose by contradiction that $i = k-1$. Then $\mathbf{D}^{\dagger}_{\mathrm{rig},F_v}(V)$ has $\left(\mathbf{B}^{\dagger}_{\mathrm{rig},F_v} \otimes_{F_v} E\right)(x^{1-k}\mu_{\alpha_v})$ as a subobject, and the latter has slope $\mathrm{val}_{\pi_v}(\alpha_v) - (k-1) < 0$ which contradicts Kedlaya's slope filtration theorem (theorem 6.10 of [Ke04]). Thus $i = 0$. For the equality in part (b) of (2), observe that $\mathcal{L} \neq \infty$ can only occur if $\dim_E \mathrm{H}^1_{\mathrm{an}}\left(\delta_1\delta_2^{-1}\right) > 1$, which by Theorem 6.2 implies $\delta_1\delta_2^{-1} = \mu_{q^{-1}}x^{k-1}$. This proves (2).

For (3), suppose by contradiction that $i = k-1$. Then $\delta_1 = x^{1-k}\mu_{\alpha_v}$ and $s \in \mathscr{S}_0^{\mathrm{cris}} \amalg \mathscr{S}_0^{\mathrm{st}}$, so by Corollary 6.4 we have that V is de Rham. A similar argument to Lemma 6.7 of [Ki03] shows that V must be split, contradicting our assumption that $\rho_x|_{G_{F_v}}$ is nonsplit.

Finally, suppose that $\mathrm{val}_{\pi_v}(\alpha_v) > k-1$ and suppose by contradiction that $i = 0$. Then $\mathbf{D}^{\dagger}_{\mathrm{rig},F_v}(V)$ is an extension of

$$\left(\mathbf{B}^{\dagger}_{\mathrm{rig},F_v} \otimes_{F_v} E\right)(\delta_1)$$

by

$$\left(\mathbf{B}^{\dagger}_{\mathrm{rig},F_v} \otimes_{F_v} E\right)(\delta_2)$$

with $w(\delta_1) = 0$ and $w(\delta_2) = 1-k$. This implies by Corollary 6.4 that V is F_v-de Rham, and hence also F_v-potentially semistable by Corollary 3.10. But this contradicts admissiblity because $\mathrm{val}_{\pi_v}(\alpha_v) > k-1$. □

Remark 13.5.

1. If $k, w \in \mathbf{Z}$ then $\mathrm{val}_{\pi_v}(\alpha_v) = \frac{k-1}{2} + \frac{w-1}{2}[F_v : \mathbf{Q}_p] + \mathrm{val}_{\pi_v}(a_v)$. The small slope condition $0 \leq \mathrm{val}_{\pi_v}(\alpha_v) \leq k-1$ can then be rewritten as

$$\frac{1-k}{2} + \frac{w-1}{2}[F_v : \mathbf{Q}_p] \leq \mathrm{val}_{\pi_v}(a_v) \leq \frac{k-1}{2} + \frac{w-1}{2}[F_v : \mathbf{Q}_p].$$

2. The parameter $\mathcal{L} \neq 0$ appearing in the case 2(b) is described in the work of Ding (Corollary 2.3 of [Di17]) in the following way. Upon considering these points of $\mathcal{E}$ with weights $(\kappa, 1, ..., 1)$ in a small affinoid neighborhood of x, one has

$$\mathcal{L}(x) = -2\frac{d\log\alpha_v}{d\kappa}|_{\kappa=k}.$$

3. When $F = \mathbf{Q}$ and $k \geq 2$, Coleman's classicality theorem ([Co97]) says that f is classical if and only if $\mathrm{val}_p(a_p) < k-1$ or $\mathrm{val}_p(a_p) = k-1$ and f is not in the image of Θ^{k-1}, where Θ is the operator which acts on q-expansions by $q\frac{d}{dq}$. Analogously, we can give a prediction in general when p is an inert prime in F. We expect, based on the conjectures appearing in §4 of [Br10], that an F_p-analytic form f is classical if and only if $\mathrm{val}_p(\alpha_p) < k-1$ or $\mathrm{val}_p(\alpha_p) = k-1$ and f is not in the image of $\Theta_{\mathrm{Id}}^{k-1}$. Here Θ_{Id} is the Theta operator in the direction of the identity embedding, as constructed in §15 of [AG05]. If such a classicality statement were known, one could argue as in §6 of [Ki03] and deduce the Fontaine-Mazur conjecture for the representations attached to F_p-analytic finite slope Hilbert eigenforms.

4. If we allow $\rho_f|_{G_{F_v}}$ to be split, it is also possible that $k \in \mathbf{Z}_{\geq 1}$, $\mathrm{val}_{\pi_v}(\alpha_v) = k-1$ and $\mathrm{Fil}^{k-1}\mathbf{D}_{\mathrm{cris},F_v}(V)^{\varphi_q=\alpha_v} \neq 0$. Our expectation is that if f itself is not classical then $f = \Theta_{\mathrm{Id}}^{k-1}g$ for some eigenform g, so that this is the only case where $\rho_f|_{G_{F_v}}$ can be de Rham without f itself being classical. In the case of $F = \mathbf{Q}$, this is known by §6 of [Ki03].

13.3 Example: the eigenform of Moy and Specter

In this section we shall test our results for a classical Hilbert eigenform. It is not too easy to find *explicit* classical Hilbert eigenforms for which Theorem 7.4 gives any new information beyond that which already exists in the literature. The case where v splits in F is well understood, and for CM Hilbert eigenforms of F the local representation at F_v splits so Theorem 7.4 is rather trivial. That's why we shall consider in this subsection the non-CM Hilbert eigenform of partial weight 1 found by Moy and Specter in [MS15]. To the best of the author's knowledge, it is the only example in the literature of a non-CM classical Hilbert eigenform of partial weight 1.

Recall that if f is a classical Hilbert eigenform of level $\Gamma_1(N)$ and nebentypus ε, the Hecke polynomial $P_v(X)$ at a place v with $a_v \neq 0$ is given by

$$P_v(X) = \begin{cases} X - a_v & \text{if } v \mid N \\ X^2 - c(v,f)X + \varepsilon(v)\mathrm{N}_{F/\mathbf{Q}}(v)^{w-1} & \text{if } v \nmid N \end{cases},$$

where $c(v, f)$ is the T_v-eigenvalue. When $v \nmid N$, raising the level of f gives two eigenforms f_1, f_2 whose attached p-adic representations ρ_f coincide and such that $\{a_v(f_1), a_v(f_2)\}$ are the two roots of $P_v(X)$. Using Theorem 7.4, this gives rise to two different triangulations of $V = \rho_x^\vee|_{G_{F_v}} \left(\chi_{\pi_v}^{\frac{1-k}{2}} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ \chi_{\pi_v}\right)^{\frac{w-1}{2}}\right)$. Whenever local-global compatibility holds, the Hecke polynomial is equal to the characteristic polynomial of the action of φ_q on $\mathbf{D}_{\mathrm{cris}}^+\left(\rho_f|_{G_{F_v}}^\vee\right)$. Thus the valuation of $c(v, f)$ determines the valuations of the eigenvalues of φ_q by the method of the Newton polygon. This observation is used in the computations below.

Next recall that the main theorem of [MS15] finds for $F = \mathbf{Q}(\sqrt{5})$ a non CM cuspidal Hilbert eigenform f of weights $(k_1, k_2, w) = (5, 1, 5)$, level $\Gamma_1(14)$, nebentypus ε with conductor $7\,(\infty_1)\,(\infty_2)$. For the following examples, we let p be a prime in the range $[2, 11]$, v a place of F lying over that prime and ρ_f the associated p-adic Galois representation of f. We set

$$V = \rho_x^\vee|_{G_{F_v}} \left(\chi_{\pi_v}^{-2} \left(\mathrm{N}_{F_v/\mathbf{Q}_p} \circ \chi_{\pi_v}\right)^2\right),$$

which differs from $\rho_f|_{G_{F_v}}$ only by a dual and a crystalline twist. We shall examine the behaviour of V for different v. When $v \neq (2)$ local-global compatibility holds by Remark 1.5 of [Ne15], while in $v = (2)$ we shall assume it holds, though it seems to be still conjectural in this case. Local-global compatibility implies that $\rho_f|_{G_{F_v}}$ is de-Rham, and since its Hodge-Tate weights at each nontrivial embedding of F_v are $\{0, 0\}$, it is also F_v-analytic. Given an eigenvalue a_v of U_v, Theorem 7.4 produces a point $s \in \mathscr{S}_+^{\mathrm{an}}$. The table in §3 of [MS15] computes the values of a_v for such v. It has to lie in the range given by Remark 7.5(1).

Examples.

1. The place $v = (2)$ lies over the inert prime $p = 2$ and the valuation bound is $\mathrm{val}_v(a_v) \in [2, 6]$. Since the character has conductor prime to 2 and the level at 2 is $\Gamma_0(2)$, the local component $\pi_2(f)$ is Steinberg (up to an unramified quadratic twist). A suitable local-global compatibility theorem predicts that V is semistable noncrystalline and $s \in \mathscr{S}_+^{\mathrm{st}}$. In particular, the condition of case 2(b) of Theorem 7.4 predicts that $\mathrm{val}_2(a_2) = 3$, which is confirmed by §3 of [MS15].

2. The place $v = (3)$ lies over the inert prime $p = 3$ and the valuation bound is $\mathrm{val}_v(a_v) \in [2, 6]$. The place v is coprime to the level, so the local component $\pi_3(f)$ is unramified principal series. By local-global compatibility, V is crystalline. By §3 of [MS15], we have $\mathrm{val}_3\left(c(3, f)\right) = 2$, so that the two U_v-eigenvalues have valuations 2 and 4. Then V has two triangulations, giving rise to $s_1 \in \mathscr{S}_0^{\mathrm{cris}}$ and $s_2 \in \mathscr{S}_+^{\mathrm{ord}}$.

3. The place $v = \left(\sqrt{5}\right)$ lies over the ramified prime $p = 5$ and the valuation

bound is $\mathrm{val}_v(a_v) \in [2,6]$. By §3 of [MS15], we have $\mathrm{val}_v\left(c(v,f)\right) = 2$, and the triangulations in this case behave similar to the case of $v = (3)$.

4. The place $v = (7)$ lies over the inert prime $p = 7$ and the valuation bound is $\mathrm{val}_v(a_v) \in [2,6]$. The character has conductor divisible by 7 and the level at 7 is $\Gamma_0(7)$, so the local component $\pi_7(f)$ is ramified prinicpal series. After an abelian extension it becomes unramified principal series, so by local-global compatibility V is crystabelline. By §3 of [MS15], we have $\mathrm{val}_7\left(a_7\right) = 3$, so V gives rise to $s \in \mathscr{S}_+^{\mathrm{cris}} \backslash \mathscr{S}_0^{\mathrm{cris}}$.

5. The place $v = \left(\frac{7+\sqrt{5}}{2}\right)$ lies over the split prime $p = 11$ and the valuation bound is $\mathrm{val}_v(a_v) \in [0,4]$. By §3 of [MS15], we have $\mathrm{val}_v\left(c(v,f)\right) = 0$, and the triangulations in this case behave similar to the case of $v = (3)$ and $v = \left(\sqrt{5}\right)$.

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References

[AG05] Andreatta F., Goren E., *Hilbert modular forms: mod p and p-adic aspects*, Memoirs of Amer. Math. Soc. 819, 2005.

[AIP16] Andreatta F., Iovita A., Pilloni V., *On overconvergent hilbert modular cusp forms*, Astérisque 382 (2016): 163-193.

[BC09] Bellaïche J., Chenevier G., F*amilies of Galois Representations and Selmer Groups,* Astérisque 324, Soc. Math. France, 314 p. (2009).

[Be02] Berger L., *Représentations p-adiques et équations différentielles*, Invent. Math. 148 (2002), no. 2, 219--284.

[Be08] Berger L., *Équations différentielles p-adiques et (φ, N)-modules filtré*s, Astérisque No. 319 (2008), 13--38.

[Be13] Berger L., *Multivariable Lubin-Tate (φ, Γ)-modules and filtered φ-modules*, Math. Res. Lett. 20 (2013), no. 3, 409--428.

[Be16] Berger L., *Multivariable (φ, Γ)-modules and locally analytic vectors*, Duke Math. J. 165 (2016), no. 18, 3567--3595.

[BC08] Berger L., Colmez P., *Familles de représentations de de Rham et monodromie p-adique*, Astérisque No. 319 (2008), 303--337.

[BC16] Berger L., Colmez P., *Théorie de Sen et vecteurs localement analytiques*, Ann. Sci. École Norm. Sup. 49 (2016), no. 4, 947--970.

[Bi16] Bijakowski S., *Arithmetique p-adique des formes de Hilbert*, Asterisque 382 (2016), 49-71.

[Br10] Breuil C., *Conjectures de classicité sur les formes de Hilbert surconvergentes de pente finie*, unpublished note.

[Ch08] Chenevier G., *Quelques courbes de Hecke se plongent dans l'espace de Colmez*, J. Number Theory 128 (2008), no. 8, p. 2430–2449.

[CC98] Cherbonnier F., Colmez P., *Représentations p-adiques surconvergentes*, Invent. Math. 133 (1998), no. 3, p. 581–611.

[Co97] Coleman, R. Classical and overconvergent modular forms of higher level. J. Théor. Nombres Bordx. 9 (1997), 395–403.

[Co02] Colmez P., *Espaces de Banach de dimension finie*, J. Inst. Math. Jussieu 1 (2002), no. 3, p. 331–439.

[Co08] Colmez P., *Représentations triangulines de dimension 2*, Astérisque (2008), no. 319, p. 213–258.

[Co10] Colmez P., *La série principale unitaire de* $\mathrm{GL}_2(\mathbf{Q}_p)$, Astérisque (2010), no. 330, p. 213– 262.

[Di17] Ding, Yiwen. *$\mathcal{L}$-invariants, partially de Rham families, and local-global compatibility.* Annales de l'Institut Fourier, Volume 67 (2017) no. 4, pp. 1457-1519.

[dS16] de Shalit E., *Mahler bases and elementary p-adic analysis*, J. de Theorie de Nombres Bordeaux, vol. 28 (2016), 597-620.

[Em11] Emerton M., *Local-global compatibility in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbf{Q}}$, preprint 2011.

[Fo09] Fourquaux L., *Applications $\mathbf{Q}_p$-linéaires, continues et Galois-équivariantes de $\mathbf{C}_p$ dans lui-même*, J. Number Theory 129 (2009), no. 6, p. 1246–1255.

[FX13] Fourquaux L., Xie B., *Triangulable $\mathcal{O}_F$ -analytic (φ_q, Γ)-modules of rank* 2, Algebra Number Theory 7 (2013), no. 10, p. 2545–2592.

[Ke04] Kedlaya K., *A p-adic local monodromy theorem*, Ann. of Math. (2) 160 (2004), no. 1, p. 93–184.

[Ki03] Kisin M., *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. 153 (2003), no. 2, p. 373–454.

[KR09] Kisin M., Ren W., *Galois Representations and Lubin-Tate Groups*. Documenta Mathematica 14 (2009) 441-461.

[KPX14] Kedlaya K., Pottharst J., Xiao L., *Cohomology of arithmetic families of (φ, Γ)-modules*, Journal of the American Mathematical Society Vol. 27, No. 4 (2014), pp. 1043-1115.

[Li12] Liu R., *Triangulation of refined families*. Commentarii Mathematici Helvetici 90 (2012): 831-904.

[MS15] Moy R., Specter J., *There Exist Non-CM Hilbert Modular Forms of Partial Weight 1*. International Mathematics Research Notices, vol. 2015, no. 24, pp. 13047-13061

[Na09] Nakamura K., *Classification of two-dimensional split trianguline representations of p-adic field*. Compos. Math. 145, 2009, no. 4, 865-914.

[Ne15] Newton J., *Towards local-global compatibility for Hilbert modular forms of low weight*. Algebra Number Theory 9 (2015), no. 4, 957--980.

[Po20] Poyeton L., *A note on F-analytic B-pairs,* arXiv:2011.04900, 2020.

[Sa09] Saito T., *Hilbert modular forms and p-adic Hodge theory*. Compositio Mathematica 145, (2009) 1081-1113.

Part III

Locally analytic vector bundles on the Fargues-Fontaine curve

Abstract

In this article, we develop a version of Sen theory for equivariant vector bundles on the Fargues-Fontaine curve. We show that every equivariant vector bundle canonically descends to a locally analytic vector bundle. A comparison with the theory of (φ, Γ)-modules in the cyclotomic case then recovers the Cherbonnier-Colmez decompletion theorem. Next, we focus on the subcategory of de Rham locally analytic vector bundles. Using the p-adic monodromy theorem, we show that each locally analytic vector bundle $\mathcal{E}$ has a canonical differential equation for which the space of solutions has full rank. As a consequence, $\mathcal{E}$ and its sheaf of solutions $\mathrm{Sol}(\mathcal{E})$ are in a natural correspondence, which gives a geometric interpretation of a result of Berger on (φ, Γ)-modules. In particular, if V is a de Rham Galois representation, its associated filtered (φ, N, G_K)-module is realized as the space of global solutions to the differential equation. A key to our approach is a vanishing result for the higher locally analytic vectors of representations satisfying the Tate-Sen formalism, which is also of independent interest.

Contents

14 Introduction

The study of p-adic Galois representations has been conditioned to an extent by two dogmas. One is the *analytic* dogma; its main idea is to associate to every such representation a (φ, Γ)-module over the Robba ring and to study these objects using p-adic analysis. The other dogma is *geometric:* to every p-adic Galois representation one associates an equivariant vector bundle over the Fargues-Fontaine curve. The aim of this article is, roughly speaking, to find a framework where both analysis and geometry can be carried out. In recent years, much of the theory of p-adic Galois representations has been understood in terms of the geometry of the Fargues-Fontaine curve. A notable exception has been the p-adic Langlands program, where the analyic approach has been dominant. Thus we are motivated to reduce this discrepancy by introducing corresponding objects on the Fargues-Fontaine curve which are also amenable to analytic methods. These are the locally analytic vector bundles, the main new objects introduced in this article.

We shall now explain this in more detail. Let K be a finite extension of $\mathbf{Q}_p$ with absolute Galois group G_K. Let K_{cyc} be the cyclotomic extension of K and write $\Gamma = \text{Gal}(K_{\text{cyc}}/K)$. For the sake of simplifying the introduction, we shall focus now on the cyclotomic setting, though as we shall explain later, the content of this paper will apply to a wider class of Galois extensions K_∞/K. The Robba ring $\mathcal{R}$ is the ring of power series over a certain finite extension of $\mathbf{Q}_p$ in a variable T which converge in some annuli $r \leq |T| < 1$. The Fargues-Fontaine curve $\mathcal{X} = \mathcal{X}(\widehat{K}_{\text{cyc}})$ associated to the perfectoid field $\widehat{K}_{\text{cyc}}$ (see §3) has an action of Γ. Combining theorems of Cherbonnier-Colmez, Fargues-Fontaine and Kedlaya, it is known that there is an equivalence

$$\{(\varphi, \Gamma)\text{-modules over } \mathcal{R}\} \cong \{\Gamma\text{-equivariant vector bundles on } \mathcal{X}\}$$

with the category $\text{Rep}_{\mathbf{Q}_p}(G_K)$ of finite dimensional $\mathbf{Q}_p$-representations of G_K embedding fully faithfully into each of these categories.

If D is a (φ, Γ)-module over $\mathcal{R}$, a fundamental fact is that the Γ-action on D can be differentiated, namely, there is a well defined action of $\text{Lie}\Gamma$ on D. Since $\text{Lie}\Gamma$ is 1-dimensional, this data is the same as that of a connection ∇ which acts on functions of T by a multiple of d/dT. It is this structure which allows the introduction of p-adic analysis into the picture. Note that in the construction of the p-adic Langlands correspondence for $\text{GL}_2(\mathbf{Q}_p)$ given in [Co10] the use of this analytic structure is ubiquitous, and so we find it desirable to have it available for equivariant vector bundles as well.

Unfortunately, the action of Γ on an equivariant vector bundle on $\mathcal{X}$ *cannot* be differentiated. This is already true for the structure sheaf $\mathcal{O}_{\mathcal{X}}$. Here is a simplified model of the situation which illustrates why there is no action of $\text{Lie}\Gamma$ on $\mathcal{O}_{\mathcal{X}}$. The functions on an open subset of $\mathcal{X}$ can roughly be thought of as power series in T^{1/p^∞} satisfying certain convergence conditions. When we try to apply the operator d/dT to such a power series, the result will often not converge since the derivative

$$d(T^{1/p^n})/dT = (1/p^n)T^{k/p^n-1}$$

grows exponentially larger p-adically as n goes to infinity. Nevertheless, there is a way to single out these sections for which the action of $\text{Lie}\Gamma$ does not explode. This is achieved by considering only these sections on which the action of Γ is regular enough. In this toy model picture, this will amount to considering only these power series where the coefficient of the exponent of T^{k/p^n} will decay proportionally to p^n.

More canonically and more generally, these elements for which differentation is possible are precisely the locally analytic elements. Given an equivariant vector bundle $\widetilde{\mathcal{E}}$ on $\mathcal{X}$, there is a subsheaf of locally analytic sections $\widetilde{\mathcal{E}}^{\mathrm{la}} \subset \widetilde{\mathcal{E}}$. This sheaf is a module over $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ which is preserved under the Γ-action, and, crucially, $\mathrm{Lie}\Gamma$ acts on $\widetilde{\mathcal{E}}^{\mathrm{la}}$. We are thus naturally lead to the definition of a *locally analytic vector bundle on* $\mathcal{X}$: by this we shall mean a locally free $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$-module together with a Γ-action.

Our first main result is saying that there is no loss of information in this process: each equivariant vector bundle canonically descends to a locally analytic vector bundle.

Theorem A. *The functor* $\widetilde{\mathcal{E}} \mapsto \widetilde{\mathcal{E}}^{\mathrm{la}}$ *gives rise to an equivalence of categories*

$$\{\text{equivariant vector bundles on } \mathcal{X}\} \cong \{\text{locally analytic vector bundles on } \mathcal{X}\}.$$

Its inverse is given by the functor $\mathcal{E} \mapsto \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$.

This theorem fits naturally into the framework of Sen theory, as we shall now explain. Let $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Then according to Sen's theory, proven by Sen in [Se81], there is a canonical isomorphism

$$(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})} \cong \widehat{K}_{\mathrm{cyc}} \otimes_{K_{\mathrm{cyc}}} \mathbf{D}_{\mathrm{Sen}}(V)$$

where $\mathbf{D}_{\mathrm{Sen}}(V)$ is the K_{cyc}-subspace of elements with finite Γ-orbit in $V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$. Later, Fontaine (see §3−4 of [Fo04]) proved an analogue of this theorem for $\mathbf{B}_{\mathrm{dR}}^+$: he showed there is an isomorphism

$$(V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})} \cong (\mathbf{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})} \otimes_{K_{\mathrm{cyc}}[[t]]} \mathbf{D}_{\mathrm{dif}}^+(V)$$

where $\mathbf{D}_{\mathrm{dif}}^+(V)$ is a canonical $K_{\mathrm{cyc}}[[t]]$-submodule of $V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$.

Both of these results are obtained from Theorem A by specializing at the "point at infinity" $x_\infty \in \mathcal{X}$. Indeed, when $\widetilde{\mathcal{E}}$ is the equivariant vector bundle associated to $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ and $\mathcal{E} = \widetilde{\mathcal{E}}^{\mathrm{la}}$, specializing the isomorphism $\widetilde{\mathcal{E}} \cong \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$ at the fiber of x_∞ gives rise to an isomorphism

$$\widetilde{\mathcal{E}}_{k(x_\infty)} \cong \mathcal{O}_{\mathcal{X},k(x_\infty)} \otimes_{\mathcal{O}_{\mathcal{X},k(x_\infty)}^{\mathrm{la}}} \mathcal{E}_{k(x_\infty)}$$

which is none other than Sen's theorem. Similarly, there is an isomorphism of the completed stalks at x_∞

$$\widetilde{\mathcal{E}}_{x_\infty}^{\wedge,+} \cong \mathcal{O}_{\mathcal{X},x_\infty}^{\wedge,+} \otimes_{\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{la},\wedge,+}} \mathcal{E}_{x_\infty}^{\wedge,+}$$

which recovers Fontaine's theorem. In this way, Theorem A is a sheaf theoretic version of Sen theory on $\mathcal{X}$ which specializes at x_∞ to classical Sen theory.

In the interest of applications, we give a proof of this equivalence not just for the cyclotomic extension, but more generally for any p-adic Lie group

$\Gamma = \mathrm{Gal}(K_\infty/K)$ where K_∞ is an infinitely ramified Galois extension of K which contains an unramified twist of the cyclotomic extension. Notably, this condition holds when K_∞ is the extension generated by the torsion points of a formal group.

As we shall explain in the article, these ideas are closely related to the decompletion of (φ, Γ)-modules, especially in the case $K_\infty = K_{\mathrm{cyc}}$. This is not too surprising, because such (φ, Γ)-modules are also obtained by a Sen theory type of idea through the theorem of Cherbonnier and Colmez in [CC98], and further, these objects relate to $\mathbf{D}_{\mathrm{Sen}}$ and $\mathbf{D}^+_{\mathrm{dif}}$ in a similar way. In fact, Theorem A is equivalent to the Cherbonnier-Colmez theorem on decompletion of (φ, Γ)-modules (after inverting p). Our proof is not independent from the ideas of Cherbonnier-Colmez, since we still use their trace maps in our arguments. However, it is logically different - more on this below.

First, let us discuss an application of Theorem A. We give a geometric reinterpertation of Berger's work on p-adic differential equations and filtered (φ, N)-modules [Be08B]. In that article, Berger relates between (φ, Γ)-modules with locally trivial connection and invariants of p-adic Hodge theory. It turns out that the constructions appearing there can be made to work on a sheaf theoretic level on $\mathcal{X}$ in a way which is reminiscient of the Riemann-Hilbert correspondence. It can be described as follows.

To each de Rham locally analytic vector bundle $\mathcal{E}$ we associated a sheaf $\mathrm{Sol}(\mathcal{E})$ on $\mathcal{X}$. It is the sheaf of solutions to a differential equation $\nabla = 0$ on a modification $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$ of $\mathcal{E}$ obtained from the action of $\mathrm{Lie}\Gamma$. We write $\mathcal{X}_{\log,\overline{K}}$ for a certain surface which surjects onto $\mathcal{X}$. Essentially $\mathcal{X}_{\log,\overline{K}}$ is obtained by adjoining scalars and a logarithm to the functions on $\mathcal{X}$, which appear when we try to solve the equation $\nabla = 0$. We shall also consider a variant $\mathrm{Sol}^{\varphi}(\mathcal{E})$, the solutions on the pullback of $\mathcal{E}$ along the usual covering $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$. We then have the following (see §8 for more precise statements).

Theorem B. *Let $\mathcal{E}$ be a de Rham locally analytic vector bundle.*

(i) The sheaf of solutions $\mathrm{Sol}(\mathcal{E})$ is locally free over the subsheaf of potentially log smooth sections $\mathcal{O}^{\mathrm{plsm}}_{\mathcal{X}} \subset \mathcal{O}^{\mathrm{la}}_{\mathcal{X}}$ and its rank is equal to the rank of $\mathcal{E}$. There is a canonical isomorphism

$$\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{\log,\overline{K}}} \otimes_{\mathcal{O}^{\mathrm{plsm}}_{\mathcal{X}}} \mathrm{Sol}(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{\log,\overline{K}}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

(ii) The space of global solutions $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E}))$ is naturally a filtered (φ, N, G_K)-module and the functor $\mathcal{E} \mapsto \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E}))$ induces an equivalence of categories

$$\{\text{de Rham locally analytic vector bundles}\} \cong \{\text{filtered } (\varphi, N, G_K)\text{-modules}\}.$$

(iii) The stalk of $\mathrm{Sol}(\mathcal{E})$ at x_∞ is canonically isomorphic to $\overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$.

Remark 14.1. 1. In particular, if V is a de Rham representation of G_K with associated locally analytic vector bundle $\mathcal{E}$, then $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) = \mathbf{D}_{\mathrm{pst}}(V)$ and the stalk $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ is identified with $\overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$. The localization map corresponds to the natural map $\mathbf{D}_{\mathrm{pst}}(V) \to \overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$.
2. If $\mathcal{E}$ becomes crystalline after extending K to a finite extension $L \subset K_\infty$, the sheaf $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})^{\nabla=0} \subset \mathrm{Sol}(\mathcal{E})$ is locally free over the subsheaf of smooth sections $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ of rank equal to the rank of $\mathcal{E}$, and there is a canonical isomorphism

$$\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E})^{\nabla=0} \xrightarrow{\sim} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

3. The sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$ is much smaller than $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. Though we have not been quite able to show this, $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$ seems to be "almost" a locally constant sheaf except that the base field becomes slightly larger when localizing; for that reason, we think of $\mathrm{Sol}(\mathcal{E})$ as morally being close to a local system on $\mathcal{X}$. In this sense the (φ, N, G_K)-structure is related to the monodromy of the p-adic differential equation $\nabla = 0$.

Finally, let us discuss the proof of Theorem A. The essential point is to show that if $\widetilde{\mathcal{E}}$ is an equivariant vector bundle on $\mathcal{X}$, the natural map $\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \widetilde{\mathcal{E}}^{\mathrm{la}} \to \widetilde{\mathcal{E}}$ is an isomorphism. Fargues and Fontaine observe that the only point of $\mathcal{X}$ with finite Γ-orbit is x_∞. The idea is then to use a very simple geometric argument: once one knows that $\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \widetilde{\mathcal{E}}^{\mathrm{la}} \to \widetilde{\mathcal{E}}$ is injective, everything can be understood by arguing locally at x_∞. Indeed, if this map is an isomorphism after localizing and completing along $\mathcal{O}_{\mathcal{X}} \to \widehat{\mathcal{O}}^{+}_{\mathcal{X},x_\infty}$, then the cokernel has to be supported at finitely many points outside x_∞. But these points also form a finite Γ-orbit, so the cokernel cannot be supported anywhere.

It therefore remains to understand the properties of our spaces of locally analytic vectors under certain localizations and completions. To do this, we are naturally led to consider higher locally analytic vectors and their vanishing, and we prove a representation-theoretic result which is of independent interest. To state the result, let G be a p-adic Lie group and let $\widetilde{\Lambda}$ be a Banach ring with a continuous action of G. Assume the topology on $\widetilde{\Lambda}$ is p-adic.

Theorem C. *Suppose G and $\widetilde{\Lambda}$ satisfy the Tate-Sen axioms (TS1)-(TS3) of [BC08] as well as an additional axiom (TS4). Then for any finite free $\widetilde{\Lambda}$-semilinear representation M of G, the higher locally analytic vectors $\mathrm{R}^i_{G\text{-}la}(M)$ are zero for $i \geq 1$.*

Here are two special cases of the theorem where we conclude that $\mathrm{R}^i_{G\text{-la}}(M) = 0$ for $i \geq 1$.

1. If M is a finite dimensional $\widehat{K}_\infty$-module with a semilinear action of Γ, for K_∞ containing an unramified twist of K_{cyc}. In fact, the vanishing of $\mathrm{R}^i_{G\text{-la}}(M)$ can be established for arbitrary K_∞, see §5.

2. If M a finite free $\widetilde{\mathbf{B}}_I(\widehat{K}_\infty)$-module with a semilinear action of Γ, under the same assumptions on K_∞.

Note that the vanishing of higher locally analytic vectors is automatic for admissible representations, but the examples above are not admissible. Theorem C illustrates how the Tate-Sen axioms can serve as a substitute for admissibility.

Theorem C is especially useful for making cohomological computations. Here is an example application, which follows directly from the main results of [RJRC21] (see §5): if M satisfies assumptions of the theorem, then for $i \geq 0$ we have natural isomorphisms

$$\mathrm{H}^i(G, M) \cong \mathrm{H}^i(G, M^{\mathrm{la}}) \cong \mathrm{H}^i(\mathrm{Lie}G, M^{\mathrm{la}})^G.$$

Finally, let us mention that in the recent work [RC22], Juan Esteban Rodríguez Camargo proves similar results to our Theorem C. He then applies them in the setting of rigid adic spaces with fantastic applications to the Calegari-Emerton conjecture, among others.

14.1 Structure of the article

§2 contains reminders on locally analytic vectors and their derived functors. In §3 we give reminders on the Fargues-Fontaine curve and equivariant vector bundles. In §4 we introduce locally analytic bundles and we discuss their basic properties. §5 is the longest and most technical section of the paper, in which we prove Theorem C. Theorem A is proved in §6. In §7 we compare our results to the theory of (φ, Γ)-modules. Finally, in §8 we discuss p-adic differential equations on the Fargues-Fontaine curve and explain Theorem B.

At several points in the article we have taken the liberty to raise speculations and ask questions to which we do not yet know the answer.

14.2 Notations and conventions

The field K denotes a finite extension of $\mathbf{Q}_p$. We write $K_{\mathrm{cyc}} = K(\mu_{p^\infty})$ for the cyclotomic extension. Its Galois group $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$ is an open subgroup of $\mathbf{Z}_p^\times$. We denote by K_∞ an infinitely ramified Galois extension of K with $\Gamma = \mathrm{Gal}(K_\infty/K)$ a p-adic Lie group. If $\overline{K}$ denotes the algebraic closure of K, we let $G_K = \mathrm{Gal}(\overline{K}/K)$ and $H = \mathrm{Gal}(\overline{K}/K_\infty)$ so that $G_K/H = \Gamma$.

The p-adic completion $\widehat{K}_\infty$ of K_∞ is a perfectoid field. Write ϖ for a pseudouniformizer of $\widehat{K}_\infty$ with valuation $\mathrm{val}\,(\varpi) = p$ that admits a sequence of

p'th power roots ϖ^{1/p^n} (such a choice is always possible, and the constructions in this paper never depend on this choice). Let $\varpi^\flat = (\varpi, \varpi^{1/p}, ...)$ be the corresponding pseudouniformizer of the tilt $\widehat{K}^\flat_\infty$.

Denote by LieΓ for the Lie algebra of Γ. It is a finite dimensional $\mathbf{Q}_p$-vector space, and if $v \in \text{Lie}\Gamma$ is sufficiently small, we have a corresponding element $\exp(v) \in \Gamma$.

All representations and group actions appearing in this article are assumed to be continuous. Galois cohomology groups are always taken in the continuous sense.

If W is a Banach space over $\mathbf{Q}_p$ we write W^+ for its unit ball.

All completed tensor products appearing in this article are projective. In other words, if V^+ and W^+ are unit balls of two Banach spaces V and W over $\mathbf{Q}_p$, then

$$V^+ \widehat{\otimes}_{\mathbf{Z}_p} W^+ = \varprojlim_n (V^+ \otimes_{\mathbf{Z}_p} W^+)/p^n$$

and $V \widehat{\otimes}_{\mathbf{Q}_p} W = (V^+ \widehat{\otimes}_{\mathbf{Z}_p} W^+)[1/p]$.

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15 Locally analytic and pro analytic vectors

In this section we give reminders on locally analytic and pro analytic vectors and quote results that will be used in §4, §5 and §6.

15.1 Locally analytic and pro analytic vectors

We shall say a compact p-adic Lie group G is *small* if there exists a saturated integral valued p-valuation on G which defines its topology and if for some $N \in \mathbf{Z}_{\geq 1}$ there exists an embedding of G into $1 + p^2 M_N(\mathbf{Z}_p)$, the group of N by N matrices congruent to 1 mod p^2. See §23 and §26 of [Sch11] for the first condition. If G is small, there exists an ordered basis $g_1, ..., g_d$ such that $(x_1, ..., x_d) \mapsto g_1^{x_1} \cdot ... \cdot g_1^{x_d}$ gives a homeomorphism of $\mathbf{Z}_p^d$ with G. We then have coordinates on G

$$c = (c_1, ..., c_d) : G \xrightarrow{\sim} \mathbf{Z}_p^d$$

defined by the inverse map where $c_i(g_1^{x_1} \cdot ... \cdot g_1^{x_d}) = x_i$.

Now let G is an be any compact p-adic Lie group. By Theorem 27.1 of [Sch11] and Ado's theorem (see 2.1.3 of [Pa21]), the collection of small open subgroups of G forms a fundamental system of open neighborhoods of the identity element. Let W be a Banach $\mathbf{Q}_p$-linear representation of G (or G-Banach space for short). If H is a small open subgroup of G, choose coordinates c on H and write $c(h)^{\mathbf{k}} = \prod_{i=1}^{d} c_i(h)^{k_i}$ if $\mathbf{k} = (k_1, ..., k_d)$ for $h \in H$. We have the subspace $W^{H\text{-an}}$ of H-analytic vectors in W; it is the subspace of elements $w \in W$ for which there exists a sequence of vectors $\{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{N}^d}$ with $w_{\mathbf{k}} \to 0$ and

$$h(w) = \sum_{\mathbf{k} \in \mathbf{N}^d} c(h)^{\mathbf{k}} w_{\mathbf{k}}$$

for all $h \in H$. The norm $||w||_{H\text{-an}} = \sup_{\mathbf{k}} ||w_{\mathbf{k}}||$ makes $W^{H\text{-an}}$ into a Banach space. Note that $W^{H\text{-an}}$ does not depend on the choice of coordinates. We write $W^{\text{la}} = \bigcup_H W^{H\text{-an}}$ for the subspace of locally analytic vectors of W. If W is a Fréchet space whose topology is defined by a countable sequence of seminorms, let W_i be the Hausdorff completion of W for the i'th seminorm, so that $W = \varprojlim W_i$ is a projective limit of Banach spaces. We write $W^{\text{pa}} = \varprojlim_i W_i^{\text{la}}$ for the subspace of pro analytic vectors. Finally, we extend the definitions of locally analytic vectors and pro analytic vectors to LB and LF spaces (i.e. filtered colimits of Banach spaces and Fréchet spaces) in the obvious way.

The Lie algebra $\text{Lie}(G)$ acts on each $W^{H\text{-an}}$ (and hence also on W^{la} and W^{pa}) through derivations. This action is given as follows. If $v \in \text{Lie}(G)$ then $\exp(p^k v) \in H$ for $k \gg 0$, and we define

$$\nabla_v(w) = \lim_{k \to \infty} \frac{\exp(p^k v)(w) - w}{p^k}.$$

The operator $\nabla_v : W^{H\text{-an}} \to W^{H\text{-an}}$ is bounded, see [BC16, Lemma 2.6].

Locally analytic and pro analytic vectors behave well when we have a basis of such vectors ([BC16, Proposition 2.3] and [Be13, Proposition 2.4]):

Proposition 15.1. *Let B be a Banach or Fréchet G-ring and let W be a free B-module of finite rank, equipped with a compatible action of G. If the B module has a basis $w_1, ..., w_d$ in which the function $G \to \text{GL}_d(B) \subset \text{M}_d(B)$,*

$g \mapsto \mathrm{Mat}(g)$ is H-analytic (resp. locally analytic, resp. pro analytic), then $W^{H\text{-}an} = \oplus_{j=1}^{d} B^{H\text{-}an} \cdot w_i$ (resp. $W^{\mathrm{la}} = \oplus_{j=1}^{d} B^{\mathrm{la}} \cdot w_i$, resp. $W^{\mathrm{pa}} = \oplus_{j=1}^{d} B^{\mathrm{pa}} \cdot w_i$).

It will often be useful for us to choose a specific fundamental system of open neighborhoods of G as follows. Fix a small compact open $G_0 \subset G$ which with coordinates c. For $n \geq 0$ we set

$$G_n = G^{p^n} = \left\{ g^{p^n} : g \in G_0 \right\}.$$

These are subgroups ([Sch11, 26.9]) which have induced coordinates $c|_{G_n} : G_n \xrightarrow{\sim} (p^n \mathbf{Z}_p)^d$. The normalization is such that for $w \in W^{G_n\text{-an}}$ we can write

$$g(w) = \sum_{\mathbf{k} \in \mathbb{N}^d} c(g)^{\mathbf{k}} w_{\mathbf{k}}$$

for $g \in G_n$ and $\{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{N}^d}$ with $p^{n|\mathbf{k}|} w_{\mathbf{k}} \to 0$, and the Banach norm is given by

$$||w||_{G_n\text{-an}} = \sup_{\mathbf{k}} \left|\left| p^{n|\mathbf{k}|} w_{\mathbf{k}} \right|\right|.$$

It is easy to check if $w \in W^{G_n\text{-an}}$ then $||w||_{G_m\text{-an}} \leq ||w||_{G_{m+1}\text{-an}}$ for $m \geq n$ and $||w||_{G_m\text{-an}} = ||w||$ for $m \gg n$.

15.2 Rings of analytic functions

Suppose first that G is small. Let $\mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$ be the space of analytic functions on G. These are those functions that after pullback by the coordinates $c : G \xrightarrow{\sim} \mathbf{Z}_p^d$ are of the form

$$\mathbf{x} = (x_1, \cdots, x_d) \mapsto \sum_{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{N}^d} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}.$$

where $b_{\mathbf{k}} \to 0$ as $|\mathbf{k}| \to \infty$. The norm $||f||_G = \sup_{\mathbf{k} \in \mathbb{N}^d} |||b_{\mathbf{k}}||$ makes $\mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$ into a Banach space. We shall regard $\mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$ as a representation of G through the left G-action.

If now G is any compact p-adic Lie group with a system of small neighborhoods $\{G_n\}_{n \geq 0}$ as in §2.1, we have for each $n \geq 0$ the space of analytic functions $\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ on G_n. Using the coordinates $c : G_n \xrightarrow{\sim} (p^n \mathbf{Z}_p)^d$ as in §2.1, we shall regard $\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ as the ring of functions that under the bijection are identified with functions of the form

$$\mathbf{x} = (x_1, \cdots, x_d) \mapsto \sum_{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{N}^d} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}.$$

where $p^{n|\mathbf{k}|}b_{\mathbf{k}} \to 0$ as $|\mathbf{k}| \to \infty$. Under this normalization

$$||f||_{G_n} = \sup_{\mathbf{k}\in\mathbb{N}^d} \left|\left|p^{n|\mathbf{k}|}b_{\mathbf{k}}\right|\right|$$

for $f \in \mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$.

The following lemma will be used in §5.

Lemma 15.2. *For $k \geq 1$ the subgroup G_{n+k} acts trivially on $\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)^+/p^k$.*

Proof. This is an easy exercise using the coordinates. See Lemma 2.1.2 of [Pa21] for the case $k = 1$. □

The following is shown in Proposition 2.1.3 of [Pa21] and in its proof (originally in the proof of Théoréme 6.1 of [BC16]).

Proposition 15.3. *Suppose that G is small. There is a dense subspace $\varinjlim_{l\in\mathbb{N}} V_l \subset \mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$, where each V_l is a finite-dimensional subrepresentation of $\mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$ with coefficients in $\mathbf{Q}_p$ such that for any $k, l \in \mathbb{N}$, we have $V_k \cdot V_l \subset V_{k+l}$.*

Furthermore, if we fix G and consider small open subgroups $G' \subset G$, we may choose $V_l(G') \subset \mathcal{C}^{\mathrm{an}}(G', \mathbf{Q}_p)$ at once for all G' in such a way that the natural map $\mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p) \to \mathcal{C}^{\mathrm{an}}(G', \mathbf{Q}_p)$ restricts to $V_l(G) \to V_l(G')$.

15.3 Higher locally analytic vectors

Suppose first that G is small and let W be a G-Banach space. There is an isometry $W\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p) \cong \mathcal{C}^{\mathrm{an}}(G, W)$, where $\mathcal{C}^{\mathrm{an}}(G, W)$ is the space of W-valued analytic functions on G. We then have $(\mathcal{C}^{\mathrm{an}}(G, W))^G = W^{G\text{-an}}$, the identification given by $f \mapsto f(1)$. This gives an alternative description of G-analytic vectors that we will use in what follows.

The functor $W \mapsto W^{G\text{-an}}$ is left exact. Following §2.2 of [Pa21] and [RJRC21], define right derived functors for $i \geq 0$:

$$\mathrm{R}^i_{G\text{-an}}(W) = \mathrm{H}^i\left(G, W\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G)\right).$$

If G is a compact p-adic Lie group with subgroups $\{G_n\}_{n\geq 1}$ as in §2.1 − §2.2, taking the colimit over n, there are right derived functors for $W \mapsto W^{G\text{-la}}$ given by

$$\mathrm{R}^i_{G\text{-la}}(W) = \varinjlim_n \mathrm{R}^i_{G_n\text{-an}}(W) = \varinjlim_n \mathrm{H}^i\left(G_n, W\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n)\right).$$

We shall call these groups the higher locally analytic vectors of W. If G is understood from the context we shall just write $\mathrm{R}^i_{\mathrm{la}}$ instead of $\mathrm{R}^i_{G\text{-la}}$.

If

$$0 \to V \to W \to X \to 0$$

is a short exact sequence of G-Banach spaces, then we have a long exact sequence

$$0 \to V^{\mathrm{la}} \to W^{\mathrm{la}} \to X^{\mathrm{la}} \to \mathrm{R}^1_{\mathrm{la}}(V) \to \mathrm{R}^1_{\mathrm{la}}(W) \to \mathrm{R}^1_{\mathrm{la}}(X) \to \ldots$$

Lemma 15.4. *Let H be an open subgroup of G and let $H_n = G_n \cap H$. Then for $n \gg 0$ and each $i \geq 0$ there are natural isomorphisms $\mathrm{R}^i_{H_n\text{-}an} \cong \mathrm{R}^i_{G_n\text{-}an}$. In particular, $\mathrm{R}^i_{H\text{-}la} \cong \mathrm{R}^i_{G\text{-}la}$.*

Proof. We have $H_n = G_n$ for $n \gg 0$. □

Suppose that G be a small compact p-adic Lie group, and let H be a small closed normal subgroup. Let W be a G-Banach space. Using the method of Hochshild-Serre we obtain the following spectral sequences.

Proposition 15.5. *(i) There is a spectral sequence*

$$E_2^{ij} = \mathrm{H}^i\left(G/H, \mathrm{H}^j\left(H, W \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)\right)\right) \Rightarrow \mathrm{R}^{i+j}_{G\text{-}an}(W).$$

(ii) There is a spectral sequence

$$E_2^{ij} = \mathrm{R}^i_{G/H\text{-}an}\left(\mathrm{H}^j(H, W)\right) \Rightarrow \mathrm{H}^{i+j}\left(G, W \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(G/H, \mathbf{Q}_p)\right).$$

(iii) Suppose additionally that there is a splitting $G \cong H \times G/H$. Then there is a spectral sequence

$$E_2^{ij} = \mathrm{H}^i\left(G/H, \mathrm{R}^j_{H\text{-}an}(W)\right) \Rightarrow \mathrm{H}^{i+j}\left(G, W \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(H, \mathbf{Q}_p)\right).$$

Proof. Apply the Hochshild-Serre spectral sequence to $W \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$, $W \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(G/H, \mathbf{Q}_p)$ and $W \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(H, \mathbf{Q}_p)$ respectively. The condition in (*iii*) is needed to make sense of the action of G on $\mathcal{C}^{\mathrm{an}}(H, \mathbf{Q}_p)$. □

16 Equivariant vector bundles

In this section we give reminders on the Fargues-Fontaine curve and equivariant vector bundles. For more details, see Chapter 9 of [FF18] and Lectures 12-13 of [SW20].

16.1 The spaces $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$

Let F be a perfectoid field, with tilt $F^\flat$. We have Fontaine's ring $\mathbf{A}_{\text{inf}} = \mathbf{A}_{\text{inf}}(F)$, defined as the Witt vectors of the ring of integers $\mathcal{O}_{F^\flat}$. Write $\text{Spa}(\mathbf{A}_{\text{inf}})$ for short for the adic space associated to the Huber pair $(\mathbf{A}_{\text{inf}}, \mathbf{A}_{\text{inf}})$. For any pseudouniformizer ϖ of F, we define

$$\mathcal{Y} = \mathcal{Y}(F) = \text{Spa}\mathbf{A}_{\text{inf}} - \{(p, [\varpi]) = 0\}$$

$$\mathcal{Y}_{(0,\infty)} = \mathcal{Y}_{(0,\infty)}(F) = \text{Spa}\mathbf{A}_{\text{inf}} - \{p[\varpi] = 0\}.$$

The spaces $\mathcal{Y}$ and $\mathcal{Y}_{(0,\infty)}$ have a Frobenius automorphism φ induced from the Witt vectors strucure of $\mathbf{A}_{\text{inf}}$.

The space $\mathcal{Y}_{(0,\infty)}$ is a preperfectoid space. The (adic) Fargues-Fontaine curve associated to F is defined as the quotient

$$\mathcal{X} = \mathcal{X}(F) = \mathcal{Y}_{(0,\infty)}(F)/\varphi^{\mathbf{Z}},$$

which makes sense because the Frobenius action is proper and discontinuous. The natural projection $\pi : \mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ is a local isomorphism, so $\mathcal{X}$ is a preperfectoid space, by virtue of $\mathcal{Y}_{(0,\infty)}$ being so. The space $\mathcal{Y}_{(0,\infty)}$ has a canonical point called x_∞, the "point at infinity". It corresponds to the kernel of Fontaine's map

$$\theta : \mathbf{A}_{\text{inf}} \to \mathcal{O}_F,$$
$$\sum_{n\geq 0}[x_n]p^n \mapsto \sum_{n\geq 0} x_n^\sharp p^n.$$

Identify x_∞ with its image $\pi(x_\infty) \in \mathcal{X}$.

If $F = \widehat{K}_\infty$, there is an induced action of the group $\Gamma = \text{Gal}(K_\infty/K)$ on each of the spaces mentioned above, and the map $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ is Γ-equivariant. The point $x_\infty \in \mathcal{X}$ is the unique Γ-fixed point; in fact, it is the unique point with finite Γ-orbit ([FF18, Proposition 10.1.1]).

16.2 The spaces $\mathcal{Y}_I$ and $\mathcal{X}_I$

It be fruitful to consider certain open subsets of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$. By Lecture 12 of [SW20] there is a surjective continuous map $\kappa : \mathcal{Y} \to [0,\infty]$ given by[3]

$$\kappa(x) = \frac{\log|p(\widetilde{x})|}{\log|[\varpi^\flat](\widetilde{x})|},$$

where $\widetilde{x}$ is the maximal generization of x. For each interval $I \subset (0,\infty)$, let $\mathcal{Y}_I$ be the interior of the preimage of $\mathcal{Y}$ under κ. These spaces are Γ-stable

[3] Our normalization of κ is the inverse of loc. cit.

if such a Γ action is present. Furthermore, the map φ induces isomorphisms $\varphi : \mathcal{Y}_{pI} \xrightarrow{\sim} \mathcal{Y}_I$. Write $\log(I) = \{\log x : x \in I\}$. Whenever I is sufficiently small so that the inequality $|\log(I)| < \log(p)$ holds, we have $\overline{I} \cap p\overline{I} = 0$ and π maps $\mathcal{Y}_I$ isomorphically onto its image $\pi(\mathcal{Y}_I) = \mathcal{X}_I \subset \mathcal{X}$. Note that $x_\infty \in \mathcal{X}_I$ if and only if I contains an element of $(p-1)p^{\mathbf{Z}}$.
For $I \subset (0,\infty)$, we have the coordinate rings

$$\widetilde{\mathrm{B}}_I = \widetilde{\mathrm{B}}_I(\widehat{K}_\infty) = \mathrm{H}^0(\mathcal{Y}_I, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}).$$

If I is compact, the geometry of $\mathcal{Y}_I$ is simple.

Proposition 16.1. *Suppose $I \subset (0,\infty)$ is a compact interval.*
(i) $\mathcal{Y}_I = \mathrm{Spa}(\widetilde{\mathrm{B}}_I, \widetilde{\mathbf{A}}_I)$, where $\widetilde{\mathbf{A}}_I$ is the ring of power bounded elements of $\widetilde{\mathrm{B}}_I$. In particular, $\mathcal{Y}_I$ is affinoid.
(ii) $\widetilde{\mathrm{B}}_I$ is a principal ideal domain.
(iii) The global sections functor induces an equivalence of categories between vector bundles on $\mathcal{Y}_I$ and finite free $\widetilde{\mathrm{B}}_I$-modules.

Proof. Parts (i) and (ii) follow from 3.5.1.2 of [FF18]. Part (iii) follows from Theorem 5.2.8 of [SW20] (originally Theorem 2.7.7 of [KL13]), since finite projective $\widetilde{\mathrm{B}}_I$-modules are finite free. □

16.3 Equivariant vector bundles

The action of Γ on $\mathcal{X}$ gives an automorphism $\gamma : \mathcal{X} \xrightarrow{\sim} \mathcal{X}$ for each $\gamma \in \Gamma$.

Definition 16.2. A Γ-equivariant vector bundle (or simply Γ-vector bundle) on $\mathcal{X}$ is a vector bundle $\widetilde{\mathcal{E}}$ on $\mathcal{X}$ equipped with an isomorphism $c_\gamma : \gamma^*\widetilde{\mathcal{E}} \xrightarrow{\sim} \widetilde{\mathcal{E}}$ for each $\gamma \in \Gamma$ such that the cocycle condition $c_{\gamma_2} \circ \gamma_2^* c_{\gamma_1} = c_{\gamma_1\gamma_2}$ holds for every $\gamma_1, \gamma_2, \in \Gamma$.

Similarly, we have a notion of a (φ,Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$. This consists of a Γ-vector bundle $\widetilde{\mathcal{M}}$ on $\mathcal{Y}_{(0,\infty)}$ together with an additional isomorphism $c_\varphi : \varphi^*\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{M}}$ such that $c_\varphi \circ \varphi^* c_\gamma = c_\gamma \circ \gamma^* c_\varphi$ for every $\gamma \in \Gamma$.
Descent along φ gives the following.

Proposition 16.3. *There is an equivalence of categories*

$$\{\Gamma\text{-vector bundles on } \mathcal{X}\} \cong \{(\varphi,\Gamma)\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\}.$$

The equivalence is given by the following functors: if $\widetilde{\mathcal{E}}$ is an equivariant vector bundle, we map it to $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}} \otimes_{\mathcal{O}_\mathcal{X}} \widetilde{\mathcal{E}}$. Conversely, if $\widetilde{\mathcal{M}}$ is a (φ,Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$, we map it to $\pi_*(\widetilde{\mathcal{M}})^{\varphi=1}$.

If $\widetilde{\mathcal{E}}$ is a Γ-vector bundle on $\mathcal{X}$ and $U \subset \mathcal{X}$ is an open subset stable under Γ, there is an induced action of Γ on $\mathrm{H}^0(U, \widetilde{\mathcal{E}})$. In particular, there is a natural action of Γ on $\mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}})$ when $|\log(I)| < \log(p)$. For a general open subset U, one only has a map

$$c_\gamma : \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}) \otimes_{\mathrm{H}^0(\gamma^{-1}(U), \mathcal{O}_{\mathcal{X}})} \mathrm{H}^0(\gamma^{-1}(U), \widetilde{\mathcal{E}}) \to \mathrm{H}^0(U, \widetilde{\mathcal{E}}).$$

Similar remarks apply for (φ, Γ)-equivariant vector bundles on $\mathcal{Y}_{(0,\infty)}$.

Example 16.4. Let V be a finite dimensional $\mathbf{Q}_p$-representation of G_K. Then

$$\widetilde{\mathcal{E}}(V) := (V \otimes_{\mathbf{Q}_p} \mathcal{O}_{\mathcal{X}(\mathbf{C}_p)})^H$$

is a Γ-vector bundle on $\mathcal{X}$. More generally, recall that the category of finite dimensional G_K-representations embeds fully faithfully to the category of (φ, Γ)-modules, with essential image the subcategory of étale (φ, Γ)-modules. We can extend the domain of the functor $V \mapsto \widetilde{\mathcal{E}}(V)$ from G_K-representations to (φ, Γ)-modules. Conversely, any Γ-vector bundle on $\mathcal{X}$ gives rise to a (φ, Γ)-module, and this correspondence results in a equivalence of categories. This will be discussed in detail in §7.

17 Locally analytic vector bundles

In this section, we introduce the category a locally analytic vector bundles and discuss their basic properties.

17.1 Locally analytic functions of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$

Let $U \subset \mathcal{X}$ be an open affinoid. Then U is quasicompact and hence stable under the action of a finite index subgroup $\Gamma' \leq \Gamma$. The space of functions $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})$ is a Banach Γ'-ring, and so it makes sense to speak of its subring of Γ'-locally analytic functions. This does not depend on the choice of Γ', and so we shall write $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$ for the Γ'-locally analytic functions in $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})$ for any Γ'. These can be glued and we obtain a sheaf of rings $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ on $\mathcal{X}$ that satisfies

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$$

for every open affinoid $U \subset \mathcal{X}$. More generally, suppose U is an open subset of $\mathcal{X}$ which is not necessarily affinoid, but for which there is a cover $U = \bigcup_i U_i$ with each U_i affinoid and a single finite index subgroup $\Gamma' \leq \Gamma$ stabilizing all of the U_i simultaneously. This condition will be satisfied in any situation we shall consider. Then the sections of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ on U are the pro analytic functions

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \varprojlim_i \mathrm{H}^0(U_i, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}} = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{pa}}.$$

The sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ is stable for the action of Γ on $\mathcal{O}_{\mathcal{X}}$, in the sense that the inclusion $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \subset \mathcal{O}_{\mathcal{X}}$ induces isomorphisms

$$c_\gamma : \gamma^* \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \otimes_{\gamma^{-1}\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}.$$

The preceding discussion then applies equally well to $\mathcal{Y}_{(0,\infty)}$, so we have a sheaf $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}$ of locally analytic functions on $\mathcal{Y}_{(0,\infty)}$ endowed with isomorphisms c_γ. Since the φ-action on $\mathcal{Y}_{(0,\infty)}$ commutes with the Γ-action, it preserves the Γ-locally analytic functions, and this gives an isomorphism

$$c_\varphi : \varphi^* \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}} = \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}} \otimes_{\varphi^{-1}\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}} \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}$$

which commutes with the Γ-action as usual.

17.2 A flatness result

For our application at §6 it would be useful to know the inclusion $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \subset \mathcal{O}_{\mathcal{X}}$ is flat. We are only able to establish this in the cyclotomic case where $K_\infty = K_{\mathrm{cyc}}$, and only for certain open subsets. Nevertheless, this will suffice for our needs.

So in this subsection suppose $K_\infty = K_{\mathrm{cyc}}$ and let I be a closed interval of the form $I = [r, s]$ with $r \geq (p-1)/p$. We write $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ for $\widetilde{\mathbf{B}}_I(\widehat{K}_{\mathrm{cyc}})$ of §3.2. Let K_0' be the maximal unramified extension of $\mathbf{Q}_p$ contained in K_{cyc}. Then we write $\mathbf{B}_{I,\mathrm{cyc},K}$ for the ring of power series $f(T) = \sum_{k \in \mathbf{Z}} a_k T^k$ with $a_k \in K_0'$, such that $f(T)$ converges on some nonempty annulus $|T| \in I$. By a classical result, $\mathbf{B}_{I,\mathrm{cyc},K}$ is a principal ideal domain. There is an embedding $\mathbf{B}_{I,\mathrm{cyc},K} \hookrightarrow \widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ for which $\mathbf{B}_{I,\mathrm{cyc},K}$ is Γ_{cyc}-stable. If K is unramified over $\mathbf{Q}_p$, this embedding can be described as follows: the variable T is mapped to $[\varepsilon] - 1$, where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \widehat{K}_{\mathrm{cyc}}^\flat$. Further, one calculates that $\gamma(T) = (1+T)^{\chi_{\mathrm{cyc}}(\gamma)} - 1$, so $\mathbf{B}_{I,\mathrm{cyc},K}$ is indeed stable under the action of Γ_{cyc}.

Proposition 17.1. *Suppose $I = [r, (p-1)p^{k-1}]$ with $k \geq 1$. Then*

(i) $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}^{\mathrm{la}} = \bigcup_{n \geq 0} \varphi^{-n}(\mathbf{B}_{p^n I,\mathrm{cyc},K})$.

(ii) $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$ *is a Prüfer domain.*

(iii) The map $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}^{\mathrm{la}} \to \widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ *is flat.*

Proof. Part (i) is Theorem 4.4 (2) of [Be13]. Note that in loc. cit. this is stated only for I of the form $\left[(p-1)p^{l-1}, (p-1)p^{k-1}\right]$, but the argument given there (see also §13 of [Be21]) is valid for any interval of the form $[r, (p-1)p^{k-1}]$. (ii) follows, because each $\mathbf{B}_{p^n I,\mathrm{cyc}}$ is a principal ideal domain, and an increasing union of such rings is a Prüfer domain. Finally, the ring

$\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ is a domain and hence torsionfree over the subring $\widetilde{\mathbf{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}$. Part (iii) is established by recalling that a torsionfree module over a Prüfer domain is flat. □

Question 17.2. *To what extent do (ii) and (iii) of Proposition 4.1 hold for coordinate rings of general open subsets in $\mathcal{X}$ and general K_∞? We do not expect $\widetilde{\mathbf{B}}^{\mathrm{la}}_I$ to be a Prüfer domain when Γ has dimension larger than 1. Nevertheless, it might still be the case that $\widetilde{\mathbf{B}}^{\mathrm{la}}_I \to \widetilde{\mathbf{B}}_I$ is flat.*

17.3 Locally analytic vector bundles

Definition 17.3. A locally analytic vector bundle on $\mathcal{X}$ is a locally free $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}$-module $\mathcal{E}$ on $\mathcal{X}$ equipped with an isomorphism $c_\gamma : \gamma^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ for each $\gamma \in \Gamma$ such that the cocycle condition $c_{\gamma_2} \circ \gamma_2^* c_{\gamma_1} = c_{\gamma_1\gamma_2}$ holds for every $\gamma_1, \gamma_2, \in \Gamma$. We require the action to be continuous with respect to the locally analytic topology.

Example 17.4. 1. Let $\widetilde{\mathcal{E}}$ be a Γ-vector bundle on $\mathcal{X}$. Define a sheaf $\widetilde{\mathcal{E}}^{\mathrm{la}}$ by generalizing the definition of $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}$. Namely, for every open affinoid $U \subset \mathcal{X}$ choose $\Gamma' \leq \Gamma$ stabilizing U. Then $\mathrm{H}^0(U, \widetilde{\mathcal{E}})$ is a Banach Γ'-ring and it makes sense to speak of $\mathrm{H}^0(U, \widetilde{\mathcal{E}})^{\mathrm{la}}$, which does not depend on the choice of Γ'. Glue these together to form a sheaf $\widetilde{\mathcal{E}}^{\mathrm{la}}$. The sheaf $\widetilde{\mathcal{E}}^{\mathrm{la}}$ is an $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}$ -module with a Γ-action. We shall show in §6 that $\widetilde{\mathcal{E}}^{\mathrm{la}}$ is locally free and therefore an example of a locally analytic vector bundle.
2. Conversely, if $\mathcal{E}$ is a locally analytic vector bundle, we can associate to it a Γ-vector bundle $\widetilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}} \mathcal{E}$. If $U \subset \mathcal{X}$ is an open affinoid such that $\mathcal{E}|_U$ is free, it follows from Proposition 2.1 that

$$\mathrm{H}^0(U, \mathcal{E}) = \mathrm{H}^0(U, \widetilde{\mathcal{E}})^{\mathrm{la}},$$

and so $\mathcal{E} = \widetilde{\mathcal{E}}^{\mathrm{la}}$. This shows that the functor from Γ-vector bundles to locally analytic vector bundles mapping $\widetilde{\mathcal{E}}$ to $\widetilde{\mathcal{E}}^{\mathrm{la}}$ is essentially surjective.

It follows from example 2 above that if $\mathcal{E}$ is a locally analytic vector bundle, we have an action by derivations

$$\mathrm{Lie}\,(\Gamma) \times \mathcal{E} \to \mathcal{E},$$

or, what amounts to the same, a connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathbf{Q}_p} (\mathrm{Lie}\Gamma)^\vee$$

satisfying the identity

$$\nabla(fx) = \nabla(f)x + f\nabla(x)$$

for local sections f of $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}$ and x of $\mathcal{E}$.

Remark 17.5. We emphasize that if $U \subset \mathcal{X}$ is an arbitrary open subset then we have an induced action of $\mathrm{Lie}\,(\Gamma)$ on $\mathrm{H}^0\,(U, \mathcal{E})$. This is unlike the Γ-action, which only maps $\mathrm{H}^0\,(U, \mathcal{E})$ to itself if U is Γ-stable. This is one pleasant aspect of working with locally analytic vector bundles instead of Γ-vector bundles.

Finally, we have the following proposition computing sections of interest. We may define a locally analytic φ-vector bundle on $\mathcal{Y}_{(0,\infty)}$ by imitating Definition 4.3. Then given a (φ, Γ)-vector bundle $\widetilde{\mathcal{M}}$ on $\mathcal{Y}_{(0,\infty)}$, one can define a locally analytic φ-vector bundle $\widetilde{\mathcal{M}}^{\mathrm{la}}$ on $\mathcal{Y}_{(0,\infty)}$ as in Example 4.4.

Proposition 17.6. *Let $\widetilde{\mathcal{E}}$ (resp. $\widetilde{\mathcal{M}}$) be a Γ-vector bundle on $\mathcal{X}$ (resp. a (φ, Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$) and let $\widetilde{\mathcal{E}}^{\mathrm{la}}$ (resp. $\widetilde{\mathcal{M}}^{\mathrm{la}}$) be its associated locally analytic vector bundle (resp. locally analytic φ-vector bundle). There are natural isomorphisms*
(i) $\mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}})^{\mathrm{la}}$ *for I a closed interval.*
(ii) $\mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}})^{\mathrm{pa}}$ *for I an open interval.*
(iii) $\mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}})^{\mathrm{la}}$ *for I a closed interval with* $|\log(I)| < \log(p)$.
(iv) $\mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}})^{\mathrm{pa}}$ *for I an open interval with* $|\log(I)| < \log(p)$.
(v) $\mathrm{H}^0(\mathcal{X}, \widetilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}, \widetilde{\mathcal{E}})^{\mathrm{la}}$.
(vi) $\mathrm{H}^0(\mathcal{X} - x_\infty, \widetilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X} - x_\infty, \widetilde{\mathcal{E}})^{\mathrm{pa}}$.

Proof. Parts (i) and (iii) are immediate from the definition. For (ii) and (iv), use the coverings $\mathcal{Y}_I = \bigcup_{J \subset I} \mathcal{Y}_J$ and $\mathcal{X}_I = \bigcup_{J \subset I} \mathcal{X}_J$ ranging over $J \subset I$ closed. For (v), consider the covering

$$\mathcal{X} = \mathcal{X}_{[1,\sqrt{p}]} \cup \mathcal{X}_{[\sqrt{p},p]}$$

with intersection $\mathcal{X}_{[\sqrt{p},\sqrt{p}]}$. This yields exact sequences

$$0 \to \mathrm{H}^0(\mathcal{X}, \widetilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(\mathcal{X}_{[1,\sqrt{p}]}, \widetilde{\mathcal{E}}^{\mathrm{la}}) \oplus \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}, \widetilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},\sqrt{p}]}, \widetilde{\mathcal{E}}^{\mathrm{la}})$$

and

$$0 \to \mathrm{H}^0(\mathcal{X}, \widetilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(\mathcal{X}_{[1,\sqrt{p}]}, \widetilde{\mathcal{E}})^{\mathrm{la}} \oplus \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}, \widetilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},\sqrt{p}]}, \widetilde{\mathcal{E}})^{\mathrm{la}}.$$

By virtue of (iii) the kernels of these sequences are identified. This proves part (v).
For (vi), use the covering

$$\mathcal{X} - x_\infty = \mathcal{X}_{[1,\sqrt{p}]} \cup (\mathcal{X}_{[\sqrt{p},p]} - x_\infty)$$

with intersection $\mathcal{X}_{[\sqrt{p},\sqrt{p}]}$. Choosing an element ξ generating $\ker\theta$ gives rise to a uniformizer at x_∞. We may write $\mathcal{X}_{[\sqrt{p},p]} - x_\infty$ as a union of Γ-stable rational open subsets

$$\mathcal{X}_{[\sqrt{p},p]} - \infty = \cup_{n\geq 1}\mathcal{X}_{[\sqrt{p},p]}\left\{|\xi| \geq p^{-n}\right\}.$$

Thus

$$\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]} - x_\infty, \widetilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]} - x_\infty, \widetilde{\mathcal{E}})^{\mathrm{pa}}.$$

Repeating the argument which proved part (v), we conclude. □

Example 17.7. We place ourselves in the cyclotomic setting so that $\Gamma = \Gamma_{\mathrm{cyc}}$ and $H = \mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})$, and we write $\mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}}) = (\mathbf{B}^+_{\mathrm{cris}})^H$. For $n \in \mathbf{Z}$ take $\widetilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}}(n)$ to be the Γ-line bundle corresponding to the graded module

$$\bigoplus_{m\geq 0}\mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^{m+n}}$$

(see §10.2 of [FF18]). We shall compute that the global sections of $\widetilde{\mathcal{E}}^{\mathrm{la}} = \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}}$ are given by

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}}) = \begin{cases} 0 & n < 0 \\ \mathbf{Q}_p(n) & n \geq 0 \end{cases}.$$

Note that a similar computation appears in §3.3 of [BC16] in the case $n = 1$.

To show this, notice first that

$$\mathrm{H}^0\left(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)\right) = \mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n} = \begin{cases} 0 & n < 0 \\ \mathbf{Q}_p & n = 0 \\ \mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n} & n > 0 \end{cases}.$$

If $n > 0$ then by [FF18, 6.4.2] there is an exact sequence

$$0 \to \mathbf{Q}_p(n) \to \mathbf{B}^{+,\varphi=p^n}_{\mathrm{cris}} \to \mathbf{B}^+_{\mathrm{dR}}/t^n\mathbf{B}^+_{\mathrm{dR}} \to 0.$$

Take H-invariants and locally analytic vectors. By Théorème 4.11 of [BC16] we know that $(\mathbf{B}^+_{\mathrm{dR}}/t^n\mathbf{B}^+_{\mathrm{dR}})^{H,\mathrm{la}} = K_{\mathrm{cyc}}[[t]]/t^n$, so we are left with an exact sequence

$$0 \to \mathbf{Q}_p(n) \to \mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} \to K_{\mathrm{cyc}}[[t]]/t^n.$$

Claim. $\mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} = \mathbf{Q}_p(n)$.

Given the claim the computation is finished because part (v) of the preceding proposition implies that

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}}) = \mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} = \begin{cases} 0 & n < 0 \\ \mathbf{Q}_p(n) & n \geq 0 \end{cases}.$$

To show the claim, take $x \in \mathbf{B}_{\mathrm{cris}}^{+}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}}$. Its image in $K_{\mathrm{cyc}}[[t]]/t^n$ is killed by the polynomial

$$P_n(\gamma) := \prod_{i=0}^{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-i}\gamma - 1)$$

for γ which generates an open subgroup of Γ. It follows that $P_n(\gamma)(x) \in \mathbf{Q}_p(n)$ for this γ. Since $P_n(\gamma)$ acts on $\mathbf{Q}_p(n)$ by a nonzero element we reduce to showing that $\mathbf{B}_{\mathrm{cris}}^{+}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,P_n(\gamma)=0}$ is 0. In fact, if K' is the subfield of K_{cyc} corresponding to $\gamma^{\mathbf{Z}_p} \subset \Gamma$ with maximal unramified subextension K'_0, we shall compute that

$$\mathbf{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_n(\gamma)=0} = \bigoplus_{i=0}^{n-1} K'_0 t^i,$$

and in particular there are no nonzero elements with $\varphi = p^n$.

To show this latter description of the elements killed by $P_n(\gamma)$, we argue by induction. If $n = 0$ then $P_n(\gamma) = \gamma - 1$ and the equality follows from the usual description of the Galois invariants of $\mathbf{B}_{\mathrm{cris}}$. For $n \geq 1$, we have $P_n(\gamma)/(\gamma - 1) = P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)$ and

$$\mathbf{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)=0} = t\mathbf{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\gamma)=0} = \bigoplus_{i=1}^{n-1} K'_0 t^i.$$

Thus there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K'_0 & \longrightarrow & \bigoplus_{i=0}^{n-1} K'_0 t^i & \longrightarrow & \bigoplus_{i=1}^{n-1} K'_0 t^i & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & \mathbf{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\gamma-1=0} & \longrightarrow & \mathbf{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_n(\gamma)=0} & \longrightarrow & \mathbf{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)=0} & &
\end{array}$$

whose rows are exact and whose outer vertical maps are isomorphisms. We conclude by the applying the five lemma.

Remark 17.8. In particular, if we set $\mathbf{B}_e(\widehat{K}_\infty) = \mathbf{B}_e^H$ for the usual ring $\mathbf{B}_e$, then $\mathbf{B}_e \subset \mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}})$. This inclusion is not an equality: the ring $\mathbf{B}_e$ allows only meromorphic functions at x_∞ while in $\mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}})$ there will be functions with essential singularities. The subring $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} \subset \mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$ is more tractable and we can understand its structure to an extent.

Remark. In particular, let us consider the subring $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} = \mathbf{B}_e \cap \mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$ in the case $\Gamma = \Gamma_{\mathrm{cyc}}$. We shall claim that in fact $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} = \mathbf{Q}_p$.

To see this, take $x \in \mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{pa}}$, and restrict it to $\mathcal{X}_{[\sqrt{p},p]} - x_\infty$. Multiplying by a bounded power of t, which makes sense on this open subset, the function $t^n x$ extends to an element of $\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$, which shows that x itself is actually locally analytic. Therefore, $t^n x \in \mathrm{H}^0(\mathcal{O}_{\mathcal{X}}, \mathbf{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n})^{\mathrm{la}}$ which is equal to $\mathbf{Q}_p t^n$ as was shown in the previous example. This shows that $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} = \mathbf{Q}_p$.

Question 17.9. *1. Is it true that* $\mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathbf{Q}_p$ *if* $\Gamma = \Gamma_{\mathrm{cyc}}$*?*

2. If $\dim \Gamma > 1$ *then one can sometimes produce elements in* $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$ *which do not belong to* $\mathbf{Q}_p$*. For example, in the Lubin-Tate setting, the element* $(t_{-\sqrt{p}}/t_{\sqrt{p}})^2$ *lies in* $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$ *, for* $t_{\pm\sqrt{p}}$ *being the analogue of Fontaine's element attached to the uniformizer* $\pi = \pm\sqrt{p}$ *(see §8.3 of [Co02] for the notation appearing here). Is it true that in some generality* $\mathbf{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$ *will be* $d-1$ *dimensional for* $d = \dim \Gamma$*? See Théoréme 6.1 of [BC16] for a related statement.*

18 Acyclicity of locally analytic vectors for semilinear representations

In this section, we shall prove vanishing the of $\mathrm{R}^i_{\mathrm{la}}$-groups for certain semilinear representations. These results will be used to prove the descent result in §6 but are also of independant interest.

18.1 Statement of the results

To state the main result of this section, we recall the Tate-Sen axioms of [BC08, 3]. Let G be a profinite group and let $\widetilde{\Lambda}$ be a G-Banach ring endowed with a valuation val for which the G action is continuous and unitary. We suppose there is a character $\chi : G \to \mathbf{Z}_p^\times$ with open image and let $H = \ker \chi$. Given an open subgroup $G_0 \subset G$ we let $H_0 = G_0 \cap H$ and $\Gamma_{H_0} = G/H_0$.

The Tate-Sen axioms are the following.

Axiom (TS1). There exists $c_1 > 0$ such that for an open subgroup $H_1 \subset H_2$ of H_0 there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ with $\mathrm{val}(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

Axiom (TS2). There exists $c_2 > 0$ and for each H_0 open in H an integer $n(H_0)$ depending on H_0 such that for $n \geq n(H_0)$, we have the extra data of

- Closed subalgebras $\Lambda_{H_0,n} \subset \widetilde{\Lambda}^{H_0}$, and
- Trace maps $\mathrm{R}_{H_0,n} : \widetilde{\Lambda}^{H_0} \to \Lambda_{H_0,n}$

satisfying:

1. For $H_1 \subset H_2$ we have $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and $\mathrm{R}_{H_1,n}|_{\Lambda_{H_2,n}} = \mathrm{R}_{H_2,n}$.
2. $\mathrm{R}_{H_0,n}$ is $\Lambda_{H_0,n}$-linear and $\mathrm{R}_{H_0,n}(x) = x$ for $x \in \Lambda_{H_0,n}$.
3. $g(\Lambda_{H_0,n}) = \Lambda_{gH_0g^{-1},n}$ and $g(\mathrm{R}_{H_0,n}(x)) = \mathrm{R}_{gH_0g^{-1},n}(gx)$ if $g \in G$.
4. $\lim_{n\to\infty} \mathrm{R}_{H_0,n}(x) = x$ for $x \in \widetilde{\Lambda}^{H_0}$.
5. If $n \geq n(H_0)$ and $x \in \widetilde{\Lambda}^{H_0}$ then $\mathrm{val}(R_{H_0,n}(x)) \geq \mathrm{val}(x) - c_2$.

Axiom (TS3). There exists $c_3 > 0$ and for each open normal subgroup G_0 of G an integer $n(G_0) \geq n(H_0)$ such that if $n \geq n(G_0)$ and $\gamma \in \Gamma_{H_0}$ has $n(\gamma) = \mathrm{val}_p(\chi(\gamma) - 1) \leq n$, then $\gamma - 1$ acts invertibely on $\mathrm{X}_{H_0,n} = (1 - \mathrm{R}_{H_0,n})(\widetilde{\Lambda}^{H_0})$ and $\mathrm{val}((\gamma - 1)^{-1}(x)) \geq \mathrm{val}(x) - c_3$.

We introduce an additional possible axiom which does not appear in [BC08].

Axiom (TS4). For any open $G_0 \subset G$ with $H_0 = G_0 \cap H$ and for any $n \geq n(G')$, there exists a positive real number $t = t(H_0, n) > 0$ such that if $\gamma \in G_0/H_0$ and $x \in \Lambda_{H_0,n}$ then

$$\mathrm{val}((\gamma - 1)(x)) \geq \mathrm{val}(x) + t.$$

We then have the following result.

Theorem 18.1. *Let M be a finite free $\widetilde{\Lambda}$-semilinear representation of G. Suppose there exists an open subgroup $G_0 \subset G$, a G-stable $\widetilde{\Lambda}^+$-lattice $M^+ \subset M$ and an integer $k > c_1 + 2c_2 + 2c_3$ such that in some basis of M^+, we have $\mathrm{Mat}(g) \in 1 + p^k\mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G_0$. Then*
(i). If (TS1)-(TS3) are satisfied then for $i \geq 2$

$$\mathrm{R}^i_{G\text{-}la}(M) = 0.$$

In fact, $\mathrm{R}^i_{G_0\text{-}an}(M) = 0$ for any sufficiently small open subgroup $G_0 \subset G$.
(ii). If in addition (TS4) is satisfied then

$$\mathrm{R}^1_{G\text{-}la}(M) = 0.$$

In fact, for every sufficiently small open subgroup G_0 there is an open subgroup $G_1 \subset G_0$ such that the map $\mathrm{R}^1_{G_0\text{-}an}(M) \to \mathrm{R}^1_{G_1\text{-}an}(M)$ is 0.
(iii). In particular, if (TS1)-(TS4) are satisfied then M has no higher locally analytic vectors. In fact, M is strongly $\mathfrak{LA}$-acyclic in the sense of [Pa21, 2.2].

Remark 18.2. If G and $\widetilde{\Lambda}$ satisfy (TS1)-(TS4) and if in addition the topology on $\widetilde{\Lambda}$ is p-adic, then for any M the higher locally analytic vectors $\mathrm{R}^i_{\mathrm{la}}(M)$

vanish for $i \geq 1$. Indeed, we claim that any finite free $\widetilde{\Lambda}$-semilinear representation of G satisfies the assumptions of the theorem after possibly replacing G by a smaller open subgroup G'. This suffices because, by Lemma 2.4, higher locally analytic vectors do not change when we replace G by G'. To see why the claim is true, suppose M is a finite free representation and choose any $\widetilde{\Lambda}$-basis $e_1, .., e_d$ of M. If we take $M^+ = \bigoplus_{i=1}^d \widetilde{\Lambda}^+ e_i$ then M^+ is stable under the action of an open subgroup G' of G, and since the topology on $\widetilde{\Lambda}$ is p-adic, we can always find another open subgroup $G'_0 \subset G'$ such that $\mathrm{Mat}(g) \in 1 + p^k \mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G'_0$.

Before giving the proof of Theorem 5.1, we record a few applications.

Corollary 18.3. *Suppose G and $\widetilde{\Lambda}$ satisfy (TS1)-(TS4) and let M be as in the statement of the theorem. Then for all $i \geq 0$,*

$$\mathrm{H}^i(G, M) \cong \mathrm{H}^i\left(G, M^{\mathrm{la}}\right) \cong \mathrm{H}^i\left(\mathrm{Lie}G, M^{\mathrm{la}}\right)^G.$$

Proof. Apply Corollary 1.6 and Theorem 1.7 of [RJRC21]. □

Two main cases of interest are the following. Let F be an algebraic extension of K which contains an unramified twist of the cyclotomic extension, i.e. the field extension of K cut out by $\eta\chi_{\mathrm{cyc}}$ for η an unramified character.

1. Take $G = \mathrm{Gal}(F/K)$ and $\widetilde{\Lambda} = \widehat{F}$. Then G and $\widetilde{\Lambda}$ satisfy the axioms (TS1)-(TS3) for arbitrary $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$. See 4.1.1 of [BC08] for the case $F = \overline{K}$, which goes back to Tate. For general F the same proof works.
In addition, we claim that G and $\widetilde{\Lambda}$ satisfy the axiom (TS4). Indeed, if G_0 is an open subgroup of G corresponding to a finite extension L of K, then $\Lambda_{H_0,n} = L(\zeta_{p^n})$ and $G_0/H_0 = \mathrm{Gal}(L_{\mathrm{cyc}}/L)$. For $\gamma \in \mathrm{Gal}(L_{\mathrm{cyc}}/L)$,

$$\mathrm{val}((\gamma-1)(\zeta_{p^n})) \geq \frac{1}{(p-1)p^{n-1}} = \mathrm{val}(\zeta_{p^n}) + \frac{1}{(p-1)p^{n-1}}$$

which shows that (TS4) holds with $t = \frac{1}{(p-1)p^{n-1}}$.

2. Take $G = \mathrm{Gal}(F/K)$ and for a closed interval $I \subset (p/p-1, \infty)$ let $\widetilde{\Lambda} = \widetilde{\mathbf{B}}_I\left(\widehat{F}\right)$. Then again G and $\widetilde{\Lambda}$ satisfy the axioms (TS1)-(TS4) for arbitrary $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$. Here if $G_0 \subset G$ is an open subgroup corresponding a finite extension L of K then one takes $\Lambda_{H_0,n} = \varphi^{-n}(\mathbf{B}_{p^nI,\mathrm{cyc},L})$ with notations as in §4.2. For (TS1)-(TS3), see [Be08A, 1.1.12]. (TS4) follows from Corollary 9.5 of [Co08]. If L is unramified over $\mathbf{Q}_p$, it is easy to verify that axiom (TS4) holds directly. Indeed, using the notation of §4.2, the ring $\mathbf{B}_{p^nI,\mathrm{cyc},L}$ is a power series in a variable T, and we have an inequality

$$\mathrm{val}_I((\gamma-1)(T)) = \mathrm{val}_I\left((1+T)^{\chi_{\mathrm{cyc}}(\gamma)} - (1+T)\right)$$

$$\geq \mathrm{val}_I(T) + p^{\mathrm{val}_p(\chi_{\mathrm{cyc}}(\gamma)-1)}\mathrm{val}_I(T),$$

which shows that (TS4) holds with $t = p^{\mathrm{val}_p(\chi_{\mathrm{cyc}}(\gamma)-1)}\mathrm{val}_I(T)$ for γ being the generator of G_0/H_0.

Corollary 18.4. *(i). If M is a finite free $\widehat{F}$-semilinear representation of* $\mathrm{Gal}\,(F/K)$ *then* $\mathrm{R}^i_{\mathrm{la}}\,(M) = 0$ *for* $i \geq 1$.

(ii). If $I \subset (p/p-1, \infty)$ is a closed interval and M is a finite free $\widetilde{\mathbf{B}}_I(\widehat{F})$-semilinear representation of $\mathrm{Gal}\,(F/K)$ *then* $\mathrm{R}^i_{\mathrm{la}}\,(M) = 0$ *for* $i \geq 1$.

Proof. In both of these cases the topology on $\widetilde{\Lambda}$ is p-adic, so the theorem applies by Remark 5.2. □

Remark 18.5. Suppose F/K is any infinitely ramified p-adic Lie extension of K (not necessarily containing an unramified twist of the cyclotomic extension), and let M be a finite free $\widehat{F}$-semilinear representation of $\mathrm{Gal}\,(F/K)$. Then $\mathrm{R}^i_{\mathrm{la}}\,(M) = 0$ for $i \geq 1$. To prove this, one is always allowed to replace K by a finite extension. Then the extension FK_{cyc}/F can be assumed to be either trivial or infinite. In the first case, the group $\mathrm{R}^i_{\mathrm{la}}\,(M)$ vanishes by the corollary. In the second case, one can argue as in §3.6 of [Pa21]. We omit the details since this result will not be used in the article.

18.2 Vanishing of H-cohomology

If $t \in \mathbb{R}$ we write

$$p^{-t}\widetilde{\Lambda}^+ := \text{elements in } \widetilde{\Lambda} \text{ with val} \geq -t.$$

The first result we shall need for the proof of Theorem 5.1 is the following.

Proposition 18.6. *Suppose that $(G, H, \widetilde{\Lambda})$ satisfies (TS1) for some $c_1 > 0$. If $H_0 \subset H$ is an open subgroup, and $r \geq 1$, we have*

(i) The map $\mathrm{H}^r(H_0, \widetilde{\Lambda}^+) \to \mathrm{H}^r(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ is 0.

(ii) If M^+ is a finite free $\widetilde{\Lambda}^+$-semilinear representation of H_0 which has a fixed H_0-basis, then the map $\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$ is 0.

(iii)$M^+ = \widehat{\bigcup_{k\in\mathbf{N}} M_k^+}$ be the completion of an increasing union of finite free $\widetilde{\Lambda}^+$-semilinear representation of H_0, each having an H_0-fixed basis, then the map $\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$ is 0.

In particular, in each of the cases (i)-(iii) the rational cohomology $\mathrm{H}^r(H_0, M)$ is equal to zero.

Proof. We have $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$. So it is enough to prove (i). This statement is probably well known, but for lack of a suitable reference, we provide a proof here. It is essentially a fiber product of the arguments appearing in Corollary 1 of §3.2 of [Ta67] and Proposition 10.2 of [Co08].

Let $\xi \in Z^r(H_0, \widetilde{\Lambda}^+)$ be an r-cocycle of H_0 valued in $\widetilde{\Lambda}^+$. By a valuation of a cocochain we shall mean the infimum of its valuation on elements. We shall construct a sequence of $r-1$ cocochains $x_n \in C^{r-1}(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ for $n \geq -1$ such that

1. $\mathrm{val}(\xi - \delta x_n) \geq nc_1$ for $\sigma \in H_0$, and
2. $\mathrm{val}(x_n - x_{n-1}) \geq (n-2)c_1$.

This will suffice, since $x_n \to x$ for some $x \in C^{r-1}(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ which shows that $\xi = \delta x$ is 0 in $\mathrm{H}^r(H_0, \widetilde{\Lambda}^+)$.

To do this, choose $x_{-1} = 0$, which clearly satisfies the first condition. Suppose x_n has been constructed, we construct x_{n+1}. Let ξ_n be the r-cocycle

$$\xi_n := \xi - \delta x_n$$

which is valued in $p^{nc_1}\widetilde{\Lambda}^+$. Choose $H_1 \subset H_0$ an open subgroup such that for every $\sigma_1, ..., \sigma_r \in H_0$ and $\sigma \in H_1$ we have

$$\mathrm{val}(\xi_n(\sigma_1, ..., \sigma_r) - \xi_n(\sigma_1, ..., \sigma_r\sigma)) \geq (n+2)c_1.$$

Such a choice is possible by the continuity of ξ_n as well as the compactness of H_0.

Now by the axiom (TS1) there is an element $\alpha \in \widetilde{\Lambda}^{H_1}$ such that $\mathrm{val}(\alpha) > -c_1$ and $\sum_{\tau \in H_0/H_1} \tau(\alpha) = 1$. Let S be a system of representatives for H_0/H_1, and let

$$x_S := (-1)^r \sum_{\tau_1, ..., \tau_r \in S} (\tau_1\tau_2 \cdot ... \cdot \tau_r)(\alpha)\xi_n(\tau_1, ..., \tau_r).$$

Each term in the sum has val $\geq (n-1)c_1 \geq -2c_1$, so

$$\mathrm{val}\,(x_S) \geq (n-1)\,c_1.$$

In particular, $x_S \in Z^r(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$.

A straightforward computation shows that for $\sigma_1, ..., \sigma_r \in H_0$ we have

$$(\xi_n - \delta x_S)(\sigma_1, ..., \sigma_r) =$$

$$\sum_{\tau \in S}(\sigma_1 \cdot ... \cdot \sigma_{r-1}\tau)(\alpha)\xi_n(\sigma_1, ..., \sigma_{r-1}, \tau) - \sum_{\tau \in S}(\sigma_1 \cdot ... \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, ..., \sigma_r\tau).$$

Let $\sigma'_{r,\tau} \in H_1$ be such that $\tau\sigma_{r,\tau} \in \sigma_r S$. Then the term on the right hand side of the previous equation becomes

$$\sum_{\tau \in S}(\sigma_1 \cdot ... \cdot \sigma_{r-1}\tau)(\alpha)[\xi_n(\sigma_1, ..., \sigma_{r-1}, \tau) - \xi_n(\sigma_1, ..., \tau\sigma_{r,\tau})],$$

so by the the choice of H_1 we have

$$\mathrm{val}(\xi_n - \delta x_S) \geq (n+1)c_1.$$

Finally, set $x_{n+1} := x_n + x_S$ where S is arbitrary. The calculations we have done show that $\mathrm{val}(x_{n+1} - x_n) \geq (n-1)c_1$ and $\mathrm{val}(\xi_\sigma - (1-\sigma)x_{n+1}) \geq (n+1)c_1$, as required. This concludes the induction and with it the proof. □

18.3 Descent of semilinear representations

In this subsection we suppose that G and $\widetilde{\Lambda}$ satisfy the axioms (TS1), (TS2) and (TS3).

Given an integer $k > c_1 + 2c_2 + 2c_3$ and an open subgroup $G_0 \subset G$ we write $\mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ for the category of finite free $\widetilde{\Lambda}^+$-semilinear representations M^+ of G such that in some basis of M^+, we have $\mathrm{Mat}(g) \in 1 + p^k\mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G_0$.

The following will allow us to descent coefficients from $\widetilde{\Lambda}^+$ to the much smaller ring $\Lambda^+_{H_0,n} = \widetilde{\Lambda}^+ \cap \Lambda_{H_0,n}$. It is a simple modification of Proposition 3.3.1 of [BC08] and is proved in exactly the same way.

Proposition 18.7. *Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$. Then for $n \geq n(G_0)$ and $H_0 = H \cap G_0$ there exists a unique finite free $\Lambda^+_{H_0,n}$-submodule $\mathbf{D}^+_{H_0,n}(M^+)$ of M^+ such that*

(1) $\mathbf{D}^+_{H_0,n}(M^+)$ is fixed by H_0 and stable by G.

(2) The natural map $\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_0,n}} \mathbf{D}^+_{H_0,n}(M^+) \to M^+$ is an isomorphism. In particular, $\mathbf{D}^+_{H_0,n}(M^+)$ is free of rank $= \mathrm{rank} M^+$.

(3) $\mathbf{D}^+_{H_0,n}(M^+)$ has a basis which is c_3-fixed by G_0/H_0, meaning that for $\gamma \in G_0/H_0$ we have $\mathrm{val}(\mathrm{Mat}\,(\gamma) - 1) > c_3$.

Corollary 18.8. *Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$, $\backslash M = M^+ \otimes_{\widetilde{\Lambda}^+} \widetilde{\Lambda}$ and $r \geq 1$. The map*

$$\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$$

is 0 and $H^r(H_0, M) = 0$.

Proof. This follows from Proposition 5.6 since M^+ has a basis fixed by H_0. □

Lemma 18.9. *Let H_0 be an open subgroup of H, $n \geq n(H_0)$ an integer, $\gamma \in \Gamma_H$ an element such that $n(\gamma) \leq n$ and $B \in \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H_0,+})$ a matrix. Suppose there are $V_1 \in \mathrm{GL}_l(\Lambda_{H_0,n})$ and $V_2 \in \mathrm{GL}_d(\Lambda_{H_0,n})$ such that $\mathrm{val}(V_1 - 1), \mathrm{val}(V_2 - 1) > c_3$ and $\gamma(B) = V_1 B V_2$. Then $B \in \mathrm{M}_{l\times d}(\Lambda_{H_0,n})$.*

Proof. The proof is exactly the same as that of Lemma 3.2.5 of [BC08]. The only difference between that lemma and the statement appearing here is that there one further assumes $l = d$ and $B \in \mathrm{GL}_d(\Lambda_{H_0,n})$, but these assumptions are not used in the proof. $\square$

In fact, the very same argument shows the result holds for matrices of infinite size, as long as we understand that an infinite matrix has coefficients which tend to zero as the indexes tend to ∞. We record this lemma for later use. Namely, if R is a ring with valuation and $l, d \in \mathrm{N} \cup \{\infty\}$, let $\mathrm{M}_{l\times d}(R)$ be the set of matrices $A = (a_{ij})$ of size $l \times d$ and $a_{ij} \in R$ such that $\mathrm{val}(a_{ij}) \to \infty$ as $i + j \to \infty$. With this, we have:

Lemma 18.10. *The previous lemma holds for $d = \infty$.*

Using Lemma 5.9, we have the following description of $\mathbf{D}^+_{H_0,n}(M^+)$. It explains why $\mathbf{D}^+_{H_0,n}(M^+)$ is functorial in M^+.

Proposition 18.11. *Given $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$, the module $\mathbf{D}^+_{H_0,n}(M^+)$ is the union of all finitely generated $\Lambda^+_{H_0,n}$-submodules of M^+ which are G-stable, H_0-fixed by are generated by a c_3-fixed set of generators.*

Proof. Indeed, if we have a submodule generated by c_3-fixed elements $f_1, ..., f_l$ and if $e_1, ..., e_d$ is a c_3-fixed basis, write

$$f_i = Be_i$$

for some matrix $B \in \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H_0,+})$. Then we have

$$\mathrm{Mat}_{f_i}(\gamma)\, B = \gamma(B)\mathrm{Mat}_{e_i}(\gamma)\,.$$

Here by $\mathrm{Mat}_{f_i}(\gamma)$ we mean any matrix which represents the action in terms of the f_i. It is not a priori unique as the submodule may not be free. Neverthelss, we have $\mathrm{val}(\mathrm{Mat}_{f_i}(\gamma) - 1) > c_3$ by the assumption, and this implies that $\mathrm{Mat}_{f_i}(\gamma)$ is invertible by Lemma 3.1.1 of [BC08]. So by Lemma 5.9

$$B \in \mathrm{M}_{l\times d}(\Lambda_{H_0,n}) \cap \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H_0,+}) = \mathrm{M}_{l\times d}(\Lambda^+_{H_0,n}),$$

hence the submodule generated by the f_i is contained in $\mathbf{D}^+_{H_0,n}(M^+)$. $\square$

Corollary 18.12. *Let $M^+, N^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$. Then for $n \geq n(G_0)$,*
(i) There are natural isomorphisms

$$\mathbf{D}^+_{H_0,n}(M^+) \otimes_{\Lambda^+_{H_0,n}} \mathbf{D}^+_{H_0,n}(N^+) \xrightarrow{\sim} \mathbf{D}^+_{H_0,n}(M^+ \otimes_{\widetilde{\Lambda}^+} N^+)$$

and

$$\mathbf{D}^+_{H_0,n}(M^+) \oplus \mathbf{D}^+_{H_0,n}(N^+) \xrightarrow{\sim} \mathbf{D}^+_{H_0,n}(M^+ \oplus N^+).$$

(ii) If $M^+ \subset N^+$ then $\mathbf{D}^+_{H_0,n}(M^+) = \mathbf{D}^+_{H_0,n}(N^+) \cap M^+$.

18.4 Descent of $\mathcal{C}^{\mathrm{an}}(G_0, M)$

We continue to assume G and $\widetilde{\Lambda}$ satisfy the axioms (TS1), (TS2) and (TS3). Assume further that G_0 is small. Then, by Proposition 2.3, we have for $V_l^+ = V_l(G_0) \cap \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$ an equality

$$\widehat{\varinjlim_{l \in \mathbb{N}} V_l^+} = \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+ .$$

For $M \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ we have

$$\varinjlim_{l \in \mathbb{N}} \widehat{M^+ \otimes_{\mathbf{Z}_p} V_l^+} \cong M^+ \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+ .$$

Each $M^+ \otimes_{\mathbf{Z}_p} V_l^+$ is a finite free $\widetilde{\Lambda}^+$-semilinear representation of G_0. The action of G_k on each of the V_l^+ is trivial mod p^k by Lemma 2.2, and hence its action on $M^+ \otimes V_l^+$ is trivial mod p^k. So if $n \geq n(G_k)$, we may define using Proposition 5.7 a $\Lambda^+_{H_k,n}$-submodule of $M^+ \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$ given by

$$\mathbf{D}^+_{H_k,n,\infty}\left(M^+\right) := \varinjlim_{l \in \mathbb{N}} \widehat{\mathbf{D}^+_{H_k,n}\left(M^+ \otimes V_l^+\right)}.$$

The module $\mathbf{D}^+_{H_k,n,\infty}(M^+)$ is then G_0-stable and fixed by H_k. By Proposition 5.7 we have natural isomorphisms

$$\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_k,n}} \mathbf{D}^+_{H_k,n}\left(M^+ \otimes V_l^+\right) \xrightarrow{\sim} M^+ \otimes V_l^+.$$

This shows that $\mathbf{D}^+_{H_k,n,\infty}(M^+)$ is generated by c_3-fixed elements which give it the sup norm, and there is an isometry

$$\widetilde{\Lambda}^+ \widehat{\otimes}_{\Lambda^+_{H_k,n}} \mathbf{D}^+_{H_k,n,\infty}\left(M^+\right) \xrightarrow{\sim} M^+ \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+ .$$

The next proposition is proved in the same way as Proposition 5.11.

Proposition 18.13. *A finitely generated $\Lambda^+_{H_k,n}$-submodule of*

$$M^+ \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$$

which is stable by G_0, fixed by H_k and is generated by a c_3-fixed set of elements is contained in $\mathbf{D}^+_{H_k,n,\infty}(M^+)$.

In particular, we have the function $\log : G_0 \twoheadrightarrow G_0/H_0 \cong \mathbf{Z}_p^\times$ lying in $\mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$. Note that for $g \in G_0$,

$$g(\log) = \log - \log\left(g^{-1}\right) = \log + \log(g).$$

Lemma 18.14. *The elements* 1 *and* log *of*

$$\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$$

lie in $\mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$.

Proof. The $\Lambda^+_{H_k,n}$-submodule generated by 1 and log in $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$ is stable under the G_0 action and fixed by H_k. Furthermore, we claim the elements 1 and log are c_3-fixed by the action of G_k/H_k. This is clear for 1. To show this for log, notice that if $g^{p^k} \in G_k/H_k$ (recalling that $G_k = G_0^{p^k}$) then

$$\mathrm{val}(g^{p^k} - 1)\,(\log) \geq k > c_1 + 2c_2 + 2c_3 > c_3.$$

We conclude by the previous proposition. □

Proposition 18.15. *(i)* $\mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$ *is a subring of* $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$.

(ii) The module structure of $M^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$ *over* $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^+$ *restricts to a module structure of* $\mathbf{D}^+_{H_k,n,\infty}(M^+)$ *over* $\mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$.

Proof. $\mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$ contains 1 by the previous proposition. Now the ring and module structure maps

$$\widetilde{\Lambda}^+ \otimes \widetilde{\Lambda}^+ \to \widetilde{\Lambda}^+, \widetilde{\Lambda}^+ \otimes M^+ \to \widetilde{\Lambda}^+$$

give rise by functoriality and Proposition 5.13 maps

$$\mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \otimes \mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \to \mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$$

and

$$\mathbf{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \otimes \mathbf{D}^+_{H_k,n,\infty}(M^+) \to \mathbf{D}^+_{H_k,n,\infty}(M^+),$$

giving the desired ring and module structures. □

18.5 Computation of higher locally analytic vectors I

Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ and $M = M^+ \otimes_{\widetilde{\Lambda}^+} \widetilde{\Lambda}$. In this subsection we shall do a first simplifcation towards the computation of of the groups $\mathrm{R}^i_{G\text{-la}}(M)$ for $i \geq 1$.

If G_0 is any open subgroup of G, we have $\mathrm{R}^i_{G\text{-la}}(M) = \mathrm{R}^i_{G_0\text{-la}}(M)$ so that if $G_n = G_0^{p^n}$ we have

$$\mathrm{R}^i_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^i(G_n, M \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)).$$

Upon possibly making G_0 smaller, we may assume that G_0 is small and that $\chi : G_0/H_0 \to \mathbf{Z}_p^\times$ has image isomorphic to $\mathbf{Z}_p$. Write $\Gamma_n = G_n/H_n$.

Lemma 18.16. *For $i \geq 1$,*

$$\mathrm{H}^i(G_n, M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)) \cong \mathrm{H}^i(\Gamma_{n+k}, (M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{H_{n+k}})$$

Proof. By the Hochshild-Serre spectral sequence and the vanishing of H_{n+k} cohomologies in (iii) of Proposition 5.6 (taking $M_k^+ = M^+ \otimes V_k^+$), we have

$$\mathrm{H}^i(G_n, M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)) \cong \mathrm{H}^i\left(G_n/H_{n+k}, (M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{H_{n+k}}\right).$$

Now the inclusion $\Gamma_{n+k} \hookrightarrow G_n/H_{n+k}$ induces an isomorphism

$$\mathrm{H}^i(G_n/H_{n+k}, (M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{H_{n+k}}) \cong \mathrm{H}^i\left(G_n/H_{n+k}, (M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{H_{n+k}}\right).$$

This again follows from Hochshild-Serre, once we notice all the higher cohomologies of G_n/G_{n+k} appearing vanish. This is because G_n/G_{n+k} is finite and the coefficients are rational. □

Corollary 18.17. $\mathrm{R}^i_{G_n\text{-}la}(M) = 0$ *for* $i \geq 2$ *and* $n \geq 0$.

Proof. Because $\Gamma_{n+k} \cong \mathbf{Z}_p$. □

This proves the first part of Theorem 5.1. It remains to study the 1st derived group

$$\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^1(\Gamma_{n+k}, (M\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{H_{n+k}}).$$

Now for $m \geq m(G_n) = m(G, n)$ we have natural isomorphisms

$$\widetilde{\Lambda}^+\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}}\mathbf{D}^+_{H_{n+k},m,\infty}\left(M^+\right) \xrightarrow{\sim} M^+\widehat{\otimes}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)^+$$

and thus

$$\widetilde{\Lambda}^{+,H_{n+k}}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}}\mathbf{D}^+_{H_{n+k},m,\infty}\left(M^+\right) \xrightarrow{\sim} (M^+\widehat{\otimes}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)^+)^{H_{n+k}}.$$

On the other hand, recall we have the trace maps

$$\mathrm{R}_{H_{n+k},m} : \widetilde{\Lambda}^{H_{n+k}} \to \Lambda_{H_{n+k},m}$$

which induce for $X_{H_{n+k},m} = \ker R_{H_{n+k},m}$ a decomposition

$$\widetilde{\Lambda}^{H_{n+k}} = \Lambda_{H_{n+k},m} \oplus X_{H_{n+k},m}.$$

Therefore, we can decompose

$$\widetilde{\Lambda}^{H_{n+k}}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}_{H_{n+k},m,\infty}\left(M\right) \cong$$

$$\mathbf{D}_{H_{n+k},m,\infty}\left(M\right) \oplus (X_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}_{H_{n+k},m,\infty}\left(M\right)),$$

and so we get the description

$$\mathrm{R}^1_{G\text{-la}}(M) =$$

$$\varinjlim_n \mathrm{H}^1(\Gamma_{n+k}, \mathbf{D}_{H_{n+k},m,\infty}\left(M\right)) \oplus \mathrm{H}^1(\Gamma_{n+k}, X_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}^+_{H_{n+k},m,\infty}\left(M\right)).$$

where in each object of the direct limit, we take $m \geq m(G_n)$.

18.6 Computation of higher locally analytic vectors II

If $m \geq 0$ is an integer and γ is an element of a group, write γ_m for γ^{p^m}. The following simple lemma will be used to compare the behaviour of $(\gamma-1)^m$ and $\gamma_m - 1$.

Lemma 18.18. *Let $l \geq 0$. The element $X^{p^l} - 1$ of the ring $\mathbf{Z}_p[X]$ is in the ideal generated by the elements $p^i(X-1)^{l+1-i}$ for $0 \leq i \leq l$.*

So far we have only used the axioms (TS1),(TS2) and (TS3). We will now use the final axiom (TS4).

Proposition 18.19. *If (TS4) holds, then*
(i) $\Lambda_{H,n}$ is Γ_t-analytic for an open subgroup of Γ depending on t.
(ii) There exists an element $s = s(t, c_3) = s(n, m, G_0, c_3)$ such that for $\gamma \in G_{n+k}/H_{n+k}$ we have

$$(\gamma - 1)\mathbf{D}^+_{H_{n+k},m,\infty}\left(M^+\right) \subset p^s \mathbf{D}^+_{H_{n+k},m,\infty}\left(M^+\right).$$

(iii) $\mathbf{D}_{H_{n+k},m,\infty}(M)$ is Γ-analytic for some open subgroup Γ of Γ_{n+k} which depends on n, m, G and c_3.

Proof. Once (ii) is established, we claim parts (i) and (iii) follow from Example 2.1.9 of [Pa21]. Let us elaborate a little bit. Take l large enough so that $(l-i)+(i+1)t = l+t+(t-1)i$ is at least 2 for each $0 \leq i \leq l$. Then for such l (which only depends on t) we have by the previous lemma

$$(\gamma_l - 1)(\Lambda^+_{H,n}) \subset p^2 \Lambda^+_{H,n},$$

So that if $b \in \Lambda_{H,n}$, the series

$$\gamma_l^x(b) = \sum_{n \geq 0} \binom{x}{n} (\gamma_l - 1)^n(b)$$

converges. This shows b is analytic for the subgroup generated by γ_l. The argument for (iii) given (ii) is similar.
To show part (ii), recall the identity

$$(\gamma - 1)(ab) = (\gamma - 1)(a)b + \gamma(a)(\gamma - 1)(b).$$

(TS4) implies that if $a \in \Lambda^+_{H,m}$ and $b \in \mathbf{D}^+_{H_{n+k},m,\infty}(M^+)$ is c_3-fixed, then ab is $\min(c_3, t)$-fixed. Since the c_3-fixed elements generate $\mathbf{D}^+_{H_{n+k},m,\infty}(M^+)$, it follows that every element of $\mathbf{D}^+_{H_{n+k},m,\infty}(M^+)$ is $s = \min(c_3, t)$-fixed. □

Using this we can show

Lemma 18.20. *Given n there is m sufficiently large depending only on n (and not on M) such that $\mathrm{H}^1(\Gamma_{n+k}, X_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}_{H_{n+k},m,\infty}(M)) = 0$.*

Proof. (This argument is adapted from Lemma 3.6.6 of [Pa21]) Fix $m_0 \geq n(G_{n+l})$. For $m \geq m_0$ the natural map

$$\Lambda_{H_{n+k},m}\widehat{\otimes}\mathbf{D}_{H_{n+k},m_0,\infty}(M) \to \mathbf{D}_{H_{n+k},m,\infty}(M)$$

is an isomorphism. Let $X^+_{H_{n+k},m} = X_{H_{n+k},m} \cap \widetilde{\Lambda}^+$. We have an induced isomorphism

$$X^+_{H_{n+k},m}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}}\mathbf{D}^+_{H_{n+k},m,\infty}\left(M^+\right) \cong X^+_{H_{n+k},m}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m_0}}\mathbf{D}^+_{H_{n+k},m_0,\infty}(M).$$

Let γ be a generator of Γ_{n+k}. By the previous proposition, there is some s such that

$$(\gamma-1)\mathbf{D}^+_{H_{n+k},m_0,\infty}\left(M^+\right) \subset p^s\mathbf{D}^+_{H_{n+k},m_0,\infty}\left(M^+\right).$$

If l is sufficiently large Lemma 5.19 implies that

$$(\gamma_l-1)\mathbf{D}^+_{H_{n+k},m_0,\infty}(M) \subset p^{2c_3}\mathbf{D}^+_{H_{n+k},m_0,\infty}(M)$$

Choose such an l, and take m large enough so that $n(\gamma_l) \leq m$. Then by (TS3) we have $\mathrm{val}((\gamma_l-1)^{-1}(x)) \geq \mathrm{val}(x) - c_3$ for $x \in X^+_{H_{n+k},m}$.

We will now show any element of $X_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}_{H_{n+k},m,\infty}(M)$ is in the image of $\gamma_l - 1$. This will also imply any element is in the image of $\gamma - 1$, since $\gamma_l - 1$ is divisible by $\gamma - 1$, and hence it will further imply that the cohomology

$$\mathrm{H}^1(\Gamma_{n+k}, X_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}_{H_{n+k},m,\infty}(M)) \cong$$

$$X_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\mathbf{D}_{H_{n+k},m,\infty}(M)/(\gamma-1)$$

is 0.

To do this last step, it suffices to show that each simple tensor

$$a \otimes b \in X^+_{H_{n+k},m}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m_0}}\mathbf{D}^+_{H_{n+k},m_0,\infty}\left(M^+\right) \cong$$

$$X^+_{H_{n+k},m}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}}\mathbf{D}^+_{H_{n+k},m,\infty}\left(M^+\right)$$

is in the image of $\gamma_l - 1$. Choose an integer r so that $p^r a$ is in the image of $(\gamma_l-1)^{-1}$ restricted to $X^+_{H_{n+k},m}$ (choose any $r \geq c_3$). It suffices to show $p^r a\otimes b$ is in the image of $\gamma_l - 1$. So write $p^r a = (\gamma_l-1)^{-1}(c)$ for $c \in X^+_{H_{n+k},m}$, and consider the series

$$y = \sum_{i=0}^{+\infty}(\gamma_l^{-1}-1)^{-i}(c) \otimes (\gamma_l-1)^i(b) = \sum_{i=0}^{+\infty}\gamma_l^i(1-\gamma_l)^{-i}(c) \otimes (\gamma_l-1)^i(b).$$

This series converges, because by our choices $\mathrm{val}((\gamma_l - 1)^{-1}(x)) \geq \mathrm{val}(x) - c_3$ on $X^+_{H_{n+k},m}$ and $(\gamma_l - 1)(x) \geq \mathrm{val}(x) + 2c_3$ on $\mathbf{D}^+_{H_{n+k},m_0,\infty}(M^+)$! A direct computation then gives

$$(\gamma_l - 1)(y) = (\gamma_l - 1)(c) \otimes b = p^r a \otimes b,$$

so $p^r a \otimes b$ is in the image of $\gamma_l - 1$, as required. □

Combing this lemma with the results of the previous subsection, we get the following description of $\mathrm{R}^1_{G\text{-la}}(M)$.

Proposition 18.21. *We have*

$$\mathrm{R}^1_{G\text{-}la}(M) = \varinjlim_{n,m} \mathrm{H}^1(\Gamma_{n+k}, \mathbf{D}_{H_{n+k},m,\infty}(M))$$

where the direct limit is taken over pairs n, m.

18.7 Computation of higher locally analytic vectors III

We are now almost ready to prove our theorem. First we prove a lemma that will be used.

Lemma 18.22. *Let $\Gamma = \gamma^{\mathbf{Z}_p}$ and let B be a Banach representation of Γ. Suppose $B = B^{\Gamma\text{-}an}$, and that*

$$||\gamma - 1|| < p^{-\frac{1}{p-1}}.$$

Then $||b|| = ||b||_{\Gamma\text{-}an}$ for any $b \in B$.

Proof. We have for $x \in \mathbf{Z}_p$ that

$$\gamma^x(b) = \sum \frac{\nabla^k_\gamma(b)}{k!} x^k$$

where $\nabla_\gamma = \log(\gamma)$. By definition

$$||b||_{\Gamma\text{-an}} = \sup_{k\geq 0} \left\{ \left|\left|\nabla^k_\gamma(b)/k!\right|\right| \right\}.$$

Now recall we have

$$\nabla_\gamma = (\gamma - 1) \sum_{m\geq 0} (-1)^m \frac{(\gamma - 1)^m}{m+1},$$

so $||\nabla_\gamma(b)|| \leq ||\gamma - 1|| \, ||b||$, and more generally

$$\left|\left|\nabla^k_\gamma(b)\right|\right| \leq ||\gamma - 1||^k \, ||b|| .$$

It follows that for $k \geq 1$ we have

$$\left|\left|\nabla^k_\gamma(b)/k!\right|\right| \leq p^{-\frac{k}{p-1}} ||\gamma - 1||^k \, ||b|| < ||b|| ,$$

so that $||b||_{\Gamma\text{-an}} = ||b||$. □

Proof of Theorem 5.1. By the Proposition

$$\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_{n,m} \mathrm{H}^1(\Gamma_{n+k}, \mathbf{D}_{H_{n+k},m,\infty}(M)).$$

Fix n and m. Given $b \in \mathbf{D}_{H_{n+k},m,\infty}(M)$ we shall show it becomes zero in some $\mathbf{D}_{H_{l+k},m',\infty}(M)$ for some $l \geq n$, $m' \geq m$ - this will show the direct limit is zero. By Proposition 5.19 we know there is an open subgroup $\Gamma \subset \Gamma_{n+k}$ such that $\mathbf{D}_{H_{n+k},m,\infty}(M)$ is Γ-analytic. Writing γ for a generator of Γ, we may take Γ small enough so that $||\gamma - 1|| < p^{-\frac{1}{p-1}}$, and hence the previous lemma applies. Thus, writing $||\cdot||_n$ for the norm on $\mathbf{D}_{H_{n+k},m,\infty}(M)$ induced from its inclusion into $M\widehat{\otimes}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$, we have $||b||_n = ||b||_{\Gamma\text{-an}}$ for $b \in \mathbf{D}_{H_{n+k},m,\infty}(M)$. We know there is a real number $D > 0$ such that if $b \in \mathbf{D}_{H_{n+k},m,\infty}(M)$ then

$$||\nabla_\gamma(b)||_n = ||\nabla_\gamma(b)||_{\Gamma\text{-an}} \leq D\,||b||_{\Gamma\text{-an}} = D\,||b||_n\,.$$

Now choose $l \geq n$ such that Γ_l has index p^t in Γ, where t is taken large enough so that

$$2p^{\frac{1}{p-1}} D \leq p^t.$$

Let $\gamma_t = \gamma^{p^t}$ be the generator of Γ_l, and let $\log_l \in \mathcal{C}^{\mathrm{an}}(G_l) : G_l \twoheadrightarrow G_l/H_l \to \mathbf{Z}_p$ be the logarithm so that $\log_l(\gamma_t) = 1$. Now let $m' \geq m$ be large enough so that $\mathbf{D}_{H_{l+k},m',\infty}(M)$ is defined. Recall that by Lemma 5.14, $\log_l \in \mathbf{D}_{H_{l+k},m',\infty}\left(\widetilde{\Lambda}^+\right)$. Let $\Gamma' \subset \Gamma_{l+k}$ be an open subgroup so that $\mathbf{D}_{H_{l+k},m',\infty}(M)$ is Γ'-analytic and write p^q for the index of Γ' in Γ_{l+k}. Finally, write γ' for the generator of Γ'. Again by making Γ' smaller we may assume $||\gamma' - 1|| < p^{\frac{-1}{p-1}}$ on $\mathbf{D}_{H_{l+k},m',\infty}(M)$. We have

$$\gamma' = (\gamma_t^{p^k})^{p^q} = \gamma^{p^{t+k+q}}.$$

Let $z_l = \log_l / p^{k+q} \in \mathbf{D}_{H_{l+k},m',\infty}\left(\widetilde{\Lambda}\right)$, the one computes that $\gamma'(z_l) = z_l + 1$. Therefore, $\nabla_{\gamma'}(z_l) = 1$. Now consider the series

$$bz_l - \nabla_{\gamma'}(b)\frac{z_l^2}{2!} + \nabla^2_{\gamma'}(b)\frac{z_l^3}{3!} - \ldots$$

in $\mathbf{D}_{H_{l+k},m',\infty}(M)$. We claim first it converges with respect to the norm $||\cdot||_l$ of $\mathbf{D}_{H_{l+k},m',\infty}(M)$. Indeed, we have

$$||z_l||_l = p^{k+q}$$

and (noting that $\nabla^i_{\gamma'} = p^{i(t+k+q)}\nabla^i_\gamma$)

$$\left|\left|\nabla^i_{\gamma'}(b)\right|\right|_l = p^{-i(t+k+q)} \left|\left|\nabla^i_\gamma(b)\right|\right|_l \leq p^{-i(t+k+q)} \left|\left|\nabla^i_\gamma(b)\right|\right|_n$$

$$\leq p^{-i(t+k+q)} D^i \left|\left|b\right|\right|_n,$$

so the general term of series has size

$$\left|\left|\nabla^i_{\gamma'}(b)/\left(i+1\right)! \cdot z_l^{i+1}\right|\right|_l \ll p^{-i(t+k+q)} D^i p^{i(k+q)} p^{\frac{i}{p-1}}$$

$$= (p^{-t} D p^{\frac{1}{p-1}})^i \leq 2^{-i},$$

so the series converges in the in the $||\cdot||_l$ norm. But then the series must also converge with respect to $||\cdot||_{\Gamma'\text{-an}}$ because of our lemma. So if we write y for the sum of the series, it makes sense to speak of the derivative $\nabla_{\gamma'}(y)$, and one computes that $\nabla_{\gamma'}(y) = b$. So b is in the image of $\nabla_{\gamma'} : \mathbf{D}_{H_{l+k},m',\infty}(M) \to \mathbf{D}_{H_{l+k},m',\infty}(M)$, hence also in the image of $\gamma' - 1$, which divides $\nabla_{\gamma'}$. But $\gamma' = \gamma_{t+k}^{p^q}$ so $\gamma_{t+k} - 1$ divides $\gamma' - 1$. It follows that b is also in the image of $\gamma_{t+k} - 1$. This means that b is 0 in

$$\mathbf{D}_{H_{l+k},m',\infty}(M) / (\gamma_{t+k} - 1) \cong \mathrm{H}^1(\Gamma_{l+k}, \mathbf{D}_{H_{l+k},m',\infty}(M))$$

and we are done! □

Remark 18.23. This argument proves a little bit more. Since the choices of l and m' did not depend on b, each $\mathbf{D}_{H_{n+k},m,\infty}(M)$ maps in its entirety to 0 in some $\mathbf{D}_{H_{l+k},m',\infty}(M)$. This gives the stronger form of (ii) in Theorem 5.1.

19 Descent to locally analytic vectors

Work again in the setting of §3 – §4. We shall assume in this section that K_∞ contains an unramified twist of the cyclotomic extension. The purpose of this section is to prove the following theorem.

Theorem 19.1. *The functor $\mathcal{E} \mapsto \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$ gives rise to an equivalence of categories*

$$\{\textit{locally analytic vector bundles on } \mathcal{X}\} \cong \{\Gamma\textit{-vector bundles on } \mathcal{X}\}.$$

The inverse functor is given by $\widetilde{\mathcal{E}} \mapsto \widetilde{\mathcal{E}}^{\mathrm{la}}$.

In the rest of this section, we shall prove that given a Γ-vector bundle $\widetilde{\mathcal{E}}$ on $\mathcal{X}$, the natural map

$$\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \widetilde{\mathcal{E}}^{\mathrm{la}} \to \widetilde{\mathcal{E}}$$

is an isomorphism. This is enough for proving Theorem 6.1. Indeed, if this isomorphism is granted, then in particular it follows from Proposition 2.2 that $\widetilde{\mathcal{E}}^{\mathrm{la}}$ is locally free over $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$, so that the functor $\widetilde{\mathcal{E}} \mapsto \widetilde{\mathcal{E}}^{\mathrm{la}}$ is valued in the correct category and is fully faithful. On the other hand, it follows from Example 4.2(2) that it is also essentially surjective.

19.1 Computations at the stalk

Let $\widetilde{\mathcal{E}}$ be a Γ-vector bundle. We have the fiber $\widetilde{\mathcal{E}}_{k(x_\infty)}$ at x_∞, a finite dimensional $\widehat{K}_\infty$-semilinear representation of Γ, and the completed stalk $\widetilde{\mathcal{E}}^{\wedge,+}_{x_\infty}$, a finite free $\mathbf{B}^+_{\mathrm{dR}}(\widehat{K}_\infty) = \mathbf{B}^{+,H}_{\mathrm{dR}}$-module. We define

$$\mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}}) = (\widetilde{\mathcal{E}}_{k(x_\infty)})^{\mathrm{la}}$$

and

$$\mathbf{D}^+_{\mathrm{dif}}(\widetilde{\mathcal{E}}) = (\widetilde{\mathcal{E}}^{\wedge,+}_{x_\infty})^{\mathrm{pa}}.$$

If V is a p-adic representation and $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(V)$ as in Example 3.4, and if $\Gamma = \Gamma_{\mathrm{cyc}}$, then we recover the classical invariant $\mathbf{D}_{\mathrm{Sen}}(V)$ according to Théorème 3.2 of [BC16]. The invariant $\mathbf{D}^+_{\mathrm{dif}}(V)$ is also recovered, see Proposition 3.3 of [Po20]. It is therefore natural to extend these definitions to arbitrary $\widetilde{\mathcal{E}}$ and Γ as we have done here.

There is the following decompletion result.

Theorem 19.2. *(i) The natural map $\widehat{K}_\infty \otimes_{\widehat{K}^{\mathrm{la}}_\infty} \mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}}_{k(x_\infty)}$ is an isomorphism.*

(ii) The natural map $\mathbf{B}^+_{\mathrm{dR}}(\widehat{K}_\infty) \otimes_{\mathbf{B}^+_{\mathrm{dR}}(\widehat{K}_\infty)^{\mathrm{pa}}} \mathbf{D}^+_{\mathrm{dif}}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}}^{\wedge,+}_{x_\infty}$ is an isomorphism.

Proof. The fiber $\widetilde{\mathcal{E}}_{k(x_\infty)}$ is a finite dimensional $\widehat{K}_\infty$-semilinear representation of Γ. So (i) follows from Théorème 3.4. of [BC16]. For (ii), write I_θ for the maximal ideal of $\mathbf{B}^+_{\mathrm{dR}}(\widehat{K}_\infty)$. It suffices to prove that for $n \geq 1$ the natural map

$$(*) \quad \mathbf{B}^+_{\mathrm{dR}}(\widehat{K}_\infty)/I^n_\theta \otimes_{(\mathbf{B}^+_{\mathrm{dR}}/I^n_\theta)^{\mathrm{la}}} (\widetilde{\mathcal{E}}_{x_\infty}/I^n_\theta)^{\mathrm{la}} \to \widetilde{\mathcal{E}}_{x_\infty}/I^n_\theta$$

is an isomorphism.

By Theorem 5.1 we have $\mathrm{R}^1_{\mathrm{la}}(I^{n-1}_\theta \widetilde{\mathcal{E}}_{x_\infty}/I^n_\theta) = 0$, so by devissage the map

$$(\widetilde{\mathcal{E}}_{x_\infty}/I^n_\theta)^{\mathrm{la}} \to (\widetilde{\mathcal{E}}_{x_\infty}/I_\theta)^{\mathrm{la}} = \mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}})$$

is surjective. It follows from the case $n = 1$ and Nakayama's lemma that $(*)$ is surjective too.

For injectivity, we argue as follows. Let $\overline{e}_1, ..., \overline{e}_d$ be a basis of $\mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}})$ over the field $\widehat{K}^{\mathrm{la}}_\infty$. By what was just proved, we may choose a lifting $e_1, ..., e_d$ of this basis to $(\widetilde{\mathcal{E}}_{x_\infty}/I^n_\theta)^{\mathrm{la}}$. Then $1 \otimes e_1, ..., 1 \otimes e_d$ generate

$$\mathbf{B}^+_{\mathrm{dR}}(\widehat{K}_\infty)/I^n_\theta \otimes_{(\mathbf{B}^+_{\mathrm{dR}}/I^n_\theta)^{\mathrm{la}}} (\widetilde{\mathcal{E}}_{x_\infty}/I^n_\theta)^{\mathrm{la}}$$

according to Nakayama's lemma.

Now suppose that

$$\sum x_i \otimes e_i \in \mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\mathbf{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\widetilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$$

is in the kernel of $(*)$, so its image is 0 mod I_θ^n. Choose a generator ξ of I_θ. Reducing mod I_θ and using the injectivity of $(*)$ for $n = 1$, we get the relation $\sum \overline{x}_i \otimes \overline{e}_i = 0$. As the $\overline{e}_i$ form a basis, each x_i must be divisible by ξ. Writing $x_i = \xi x_i'$, we have

$$\sum x_i \otimes e_i = \sum \xi x_i' \otimes e_i = \xi \sum x_i' \otimes e_i,$$

so the image of

$$\sum x_i' \otimes y_i \in \mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^{n-1} \otimes_{(\mathbf{B}_{\mathrm{dR}}^+/I_\theta^{n-1})^{\mathrm{la}}} (\widetilde{\mathcal{E}}_{x_\infty}/I_\theta^{n-1})^{\mathrm{la}}$$

in $\widetilde{\mathcal{E}}_{x_\infty}/I_\theta^{n-1}$ is 0. The injectivity now follows from induction. □

Theorem 5.1 of §5 allows us to prove the following proposition. In 6.5 below we shall prove a stronger statement.

Proposition 19.3. *Let I be a closed interval with $|\log(I)| < \log(p)$ and let $\widetilde{M}_I = \mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}})$. There are natural isomorphisms $\mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}}) \cong \widetilde{M}_I^{\mathrm{la}}/(I_\theta \widetilde{M}_I)^{\mathrm{la}}$ and $\mathbf{D}_{\mathrm{dif}}^+(\widetilde{\mathcal{E}}) \cong \varprojlim_n \widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$.*

Proof. As I_θ is principal, $I_\theta \widetilde{M}_I$ is finite free over $\widetilde{\mathbf{B}}_I$. By Theorem 5.1, the cohomology $\mathrm{R}^1_{\mathrm{la}}(I_\theta \widetilde{M}_I)$ vanishes. Applying la to the short exact sequence

$$0 \to I_\theta \widetilde{M}_I \to \widetilde{M}_I \to \widetilde{M}_I I_\theta \widetilde{M}_I \to 0$$

we get $\widetilde{M}_I^{\mathrm{la}}/(I_\theta \widetilde{M}_I)^{\mathrm{la}} \xrightarrow{\sim} (\widetilde{M}_I/I_\theta \widetilde{M}_I)^{\mathrm{la}} = \mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}})$, which gives the first isomorphism. By the same argument $\widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}} \xrightarrow{\sim} (\widetilde{M}_I/I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$ for $n \geq 1$. To get the second isomorphism, take the limit over n. □

19.2 Descent to locally analytic vectors

In this subsection we will give a proof of Theorem 6.1.

We start with the following key proposition, which builds upon all of the work done in §4, §5 and the previous subsections of §6.

Proposition 19.4. *Let $I = [r, (p-1)p^n]$ be an interval with $n \geq 1$ and $|\log(I)| < \log(p)$. Then the natural map*

$$(*) \quad \widetilde{\mathbf{B}}_I \otimes_{\widetilde{\mathbf{B}}_I^{\mathrm{la}}} \widetilde{M}_I^{\mathrm{la}} \to \widetilde{M}_I$$

is an isomorphism.

Proof. First let us explain how to reduce to the cyclotomic case. After an unramified twist, which causes no obstructions to descent, we may assume $K_{\mathrm{cyc}} \subset K_\infty$. We then have $\widetilde{M_I} \cong \widetilde{\mathbf{B}}_I \otimes_{\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}} \widetilde{M}_{I,\mathrm{cyc}}$, and if the conclusion of the proposition holds for the cyclotomic case, we have

$$\widetilde{M_I} \cong \widetilde{\mathbf{B}}_I \otimes_{\widetilde{\mathbf{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}.$$

This shows that $\widetilde{M_I}$ has a basis of locally analytic vectors and by Proposition 2.1 the map $(*)$ is an isomorphism.

It remains to establish the proposition in the cyclotomic case where $\widetilde{\mathbf{B}}_I = \widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$. By Proposition 4.1, $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ is flat as a $\widetilde{\mathbf{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}$-module. Since $\widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}$ is torsionfree as a $\widetilde{\mathbf{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}$-module, it follows from [Sta, 15.22.4] that $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}} \otimes_{\widetilde{\mathbf{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}$ is also torsionfree. By Theorem 6.2, the completion at $I_\theta \subset \widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ of $(*)$ is nothing but the map

$$\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{B}^{+,\mathrm{pa}}_{\mathrm{dR}}} \mathbf{D}^+_{\mathrm{dif}}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}}^{\wedge,+}_{x_\infty},$$

so by Proposition 6.3, the map $(*)$ is an isomorphism at least after taking this completion. As $\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ is a PID, it follows that $(*)$ is injective with cokernel supported at finitely many maximal ideals. These maximal ideals correspond to a finite set of points on $\mathcal{X}$, and this set must form a finite orbit under the action of Γ. But by 10.1.1 of [FF18], the only point with finite orbit under the Γ-action is x_∞! Thus the cokernel of $(*)$ is supported at I_θ. But then it must be 0, as we have just shown the completion at I_θ is an isomorphism. □

Proof of Theorem 6.1. Let U be an open subaffinoid of $\mathcal{X}_I$ for $I = [r, (p-1)p^n]$. Then we claim that the natural map

$$\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}(U)} \mathrm{H}^0(U, \widetilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(U, \widetilde{\mathcal{E}})$$

is an isomorphism. Indeed, we have

$$\mathrm{H}^0(U, \widetilde{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{X}}(U) \otimes_{\widetilde{\mathbf{B}}_{I,\mathrm{cyc}}} \widetilde{M}_{I,\mathrm{cyc}} \cong \mathcal{O}_{\mathcal{X}}(U) \otimes_{\widetilde{\mathbf{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}.$$

Thus $\mathrm{H}^0(U, \widetilde{\mathcal{E}})$ has a basis of locally analytic elements. By Proposition 2.1, we have an isomorphism

$$\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)^{\mathrm{la}}} \mathrm{H}^0(U, \widetilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(U, \widetilde{\mathcal{E}}),$$

from which the claim follows.

Now let $(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}} \widetilde{\mathcal{E}}^{\mathrm{la}})^\circ$ be the presheaf on $\mathcal{X}$ sending

$$U \mapsto \mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}(U)} \widetilde{\mathcal{E}}^{\mathrm{la}}(U).$$

The $\mathcal{X}_I$ for various I's of the form $I = [r, (p-1)p^n]$ with $|\log(I)| < \log(p)$ give a covering of $\mathcal{X}$, so the claim shows that the natural map

$$(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \widetilde{\mathcal{E}}^{\mathrm{la}})^{\circ} \to \widetilde{\mathcal{E}}$$

is an isomorphism on stalks. Theorem 6.1 follows. □

The proof just given essentially shows that $\mathcal{E}$ is quasi-coherent. This leads to a simple interpertation of $\mathbf{D}_{\mathrm{Sen}}$ and $\mathbf{D}^+_{\mathrm{dif}}$ in terms of $\mathcal{E}$ as follows. Given a locally analytic vector bundle define

$$\mathbf{D}_{\mathrm{Sen}}(\mathcal{E}) = \mathcal{E}_{k(x_\infty)},$$

the fiber of $\mathcal{E}$ at x_∞, and

$$\mathbf{D}^+_{\mathrm{dif}}(\mathcal{E}) = \widehat{\mathcal{E}}^+_{x_\infty},$$

the completed stalk of $\mathcal{E}$ at x_∞. These would not *a priori* be the same as $\mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}})$ and $\mathbf{D}^+_{\mathrm{dif}}(\widetilde{\mathcal{E}})$, because quotients in general do not commute with locally analytic vectors, but they do in this case.

Theorem 19.5. *Let $\widetilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$. There are natural isomorphisms $\mathbf{D}_{\mathrm{Sen}}(\widetilde{\mathcal{E}}) \cong \mathbf{D}_{\mathrm{Sen}}(\mathcal{E})$ and $\mathbf{D}^+_{\mathrm{dif}}(\widetilde{\mathcal{E}}) \cong \mathbf{D}^+_{\mathrm{dif}}(\mathcal{E})$.*

Proof. For $I = [r, (p-1)p^n]$ with $|\log(I)| < \log(p)$ write $\widetilde{M_I} = \mathrm{H}^0(\mathcal{X}_I, \widetilde{\mathcal{E}})$. For any sufficiently small U containing x_∞, the proof just given shows that

$$\mathrm{H}^0(U, \mathcal{E}) \cong \mathcal{O}_{\mathcal{X}}(U)^{\mathrm{la}} \otimes_{\widetilde{\mathbf{B}}_I^{\mathrm{la}}} \widetilde{M}_I^{\mathrm{la}}.$$

It follows that the quotient $\mathcal{E}_{x_\infty}/m^n_{x_\infty}\mathcal{E}_{x_\infty}$ of the stalk $\mathcal{E}_{x_\infty}$ by the n'th power of the maximal idea $m_{x_\infty} \subset \mathcal{O}^{\mathrm{la}}_{\mathcal{X},x_\infty}$ is identified with the quotient $\widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$. Now use Proposition 6.3. □

20 The comparison with (φ, Γ)-modules

In this section, we give reminders on (φ, Γ)-modules and compare them to locally analytic vector bundles. We keep the notation from §6 and the assumption that $K^\eta_{\mathrm{cyc}} \subset K_\infty$ for some η.

20.1 Galois representations and (φ, Γ)-modules

Recall the notations from §3 and let

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(\widehat{K}_\infty) = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathcal{O}_{\mathcal{Y}}) = \varinjlim_r \varprojlim_{s \geq r} \mathrm{H}^0(\mathcal{Y}_{[r,s]}, \mathcal{O}_{\mathcal{Y}})$$

be the extended Robba ring. The (φ, Γ)-actions on $\mathcal{Y}$ induce actions on $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$.

Definition 20.1. A (φ,Γ)-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is a finite free $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$-module with commuting semilinear (φ,Γ)-actions such that in some basis $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})$..

We can compare these objects to (φ,Γ)-vector bundles using two functors. On the one hand, if $\widetilde{\mathcal{M}}$ is a (φ,Γ)-vector bundle, then

$$\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}} = \varinjlim_{r} \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \widetilde{\mathcal{M}})$$

is a (φ,Γ)-module. Here, the nontrivial thing one needs to check is that $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}$ is free, and this follows from $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ being Bézout (Theorem 3.20 of [Ke04]).

One the other hand, given a (φ,Γ)-module $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}$ we define a (φ,Γ)-vector bundle $\mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})$ as follows. If $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}$ is a (φ,Γ)-module then for every $r \gg 0$ we have a finite free $\widetilde{\mathbf{B}}_{[r,\infty)}$-semilinear Γ-representation $\widetilde{\mathrm{M}}_{[r,\infty)}$ together with isomorphisms

$$\varphi^* \widetilde{\mathbf{B}}_{[r,\infty)} \otimes_{\widetilde{\mathbf{B}}_{[r/p,\infty)}} \widetilde{\mathrm{M}}_{[r/p,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{[r,\infty)}$$

as well as identifications

$$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\widetilde{\mathbf{B}}_{[r,\infty)}} \widetilde{\mathrm{M}}_{[r,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}.$$

Using the isomorphisms $\varphi : \widetilde{\mathbf{B}}_{[r,\infty)} \xrightarrow{\sim} \widetilde{\mathbf{B}}_{[r/p,\infty)}$ we can then uniquely extend this to all $r > 0$ by inductivey defining $\widetilde{\mathrm{M}}_{[r/p^n,\infty)}$ through the isomorphisms

$$\varphi^* \widetilde{\mathbf{B}}_{[r/p^{n-1},\infty)} \otimes_{\widetilde{\mathbf{B}}_{[r/p^n,\infty)}} \widetilde{\mathrm{M}}_{[r/p^n,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{[r/p^{n-1},\infty)}.$$

Setting for every $r > 0$

$$\mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})) = \widetilde{\mathrm{M}}_{[r,\infty)},$$

we obtain a (φ,Γ)-vector bundle $\mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})$.

Proposition 20.2. *The functors $\widetilde{\mathcal{M}} \mapsto \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \widetilde{\mathcal{M}})$ and* FT *induce an equivalence of categories*

$$\left\{(\varphi,\Gamma)\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\right\} \cong \left\{(\varphi,\Gamma)\text{-modules over } \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\right\}.$$

Proof. This is well known. See for example the discussion after 13.4.3 in [SW20]. The treatment there is given in the situation where there is no Γ-action present, but the same proof works in our setting. $\square$

The following theorem due to Fontaine and Kedlaya gives the relation of these objects with Galois representations. To formulate it, we need to introduce some terminology. Let y be the point of $\mathcal{Y}$ corresponding to $p = 0$. A (φ, Γ)-module over over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is called étale if it has a basis for which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{Y},y})$. We also have the notion of a semistable slope 0 vector bundle on $\mathcal{X}$ - we refer the reader to [FF18].

Theorem 20.3. *The following categories are equivalent.*

1. *Finite dimensional $\mathbf{Q}_p$-representations of G_K.*
2. *Étale (φ, Γ)-modules over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$.*
3. *Γ-vector bundles on $\mathcal{X}$ which are semistable of slope 0.*

Proof. The equivalence of 2 and 3 follows from the previous proposition and Proposition 3.3. The category in 1 is equivalent to (φ, Γ)-modules over $\widetilde{\mathbf{B}} = \widehat{\mathcal{O}}_{\mathcal{Y},y}[1/p]$, where $\widehat{\mathcal{O}}_{\mathcal{Y},y}$ is the p-adic completion of $\mathcal{O}_{\mathcal{Y},y}$, by the theorem of Fontaine [Fo90]. Next, by a relatively elementary argument, this category is equivalent to the category of (φ, Γ)-modules over $\widetilde{\mathbf{B}}^{\dagger}$, see for example Theorem 2.4.5 of [Ke15] or Theorem 4.3 of [dSP19]. Finally, one can replace $\widetilde{\mathbf{B}}^{\dagger}$ by $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ by Proposition 5.11 and Corollary 5.12 of [Ke04]. See also Proposition 11.2.24 of [FF18]. □

20.2 The comparison with locally analytic vector bundles

Let $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$ be the subring of pro-analytic vectors in $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ for the action of Γ. We have a corresponding version of (φ, Γ)-modules.

Definition 20.4. A (φ, Γ)-module $\mathrm{M}^{\dagger}_{\mathrm{rig}}$ over $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$ is a finite free $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$-module with commuting semilinear (φ, Γ)-actions such that in some basis $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}})$. It is étale if $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}} \mathrm{M}^{\dagger}_{\mathrm{rig}}$ is so.

The following theorem explains the relationship between (φ, Γ)-modules and locally analytic vector bundles.

Theorem 20.5. *The following categories are all equivalent.*

1. *(φ, Γ)-modules over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$.*
2. *(φ, Γ)-modules over $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$.*
3. *(φ, Γ)-vector bundles over $\mathcal{Y}_{(0,\infty)}$.*
4. *Locally analytic φ-vector bundles on $\mathcal{Y}_{(0,\infty)}$.*
5. *Γ-vector bundles on $\mathcal{X}$.*
6. *Locally analytic vector bundles on $\mathcal{X}$.*

Proof. The equivalences 1 $\Leftrightarrow$ 3 $\Leftrightarrow$ 5 are Proposition 7.2 and Proposition 3.3. 4 $\Leftrightarrow$ 6 is similar to Proposition 3.3. The proof of 5 $\Leftrightarrow$ 6 was given in Theorem 6.1, and 3 $\Leftrightarrow$ 4 can be proved in a similar way. It remains to give an equivalence between 2 and 4. The Frobenius trick functor of §7.1 induces a functor

$$\mathrm{FT} : \{(\varphi, \Gamma)\text{-modules over } \widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}\} \to$$
$$\{\text{Locally analytic } \varphi\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\}.$$

In the other direction we map a locally analytic φ-vector bundle $\mathcal{M}$ to $\mathcal{M}^{\dagger}_{\mathrm{rig}} = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathcal{M})$. It is easy to check from the definitions these two are inverses to each other once we know that $\mathcal{M} \mapsto \mathcal{M}^{\dagger}_{\mathrm{rig}}$ is valued in the correct category. So it remains to prove the following.

Claim. $\mathcal{M}^{\dagger}_{\mathrm{rig}}$ is (φ, Γ)-module over $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$.

Proof of the claim. We only need to explain why $\mathcal{M}^{\dagger}_{\mathrm{rig}}$ is a free $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$-module. Since we can always descend along unramified extensions, we may assume $K_{\mathrm{cyc}} \subset K_{\infty}$. Then $\mathcal{M}$ and $\mathcal{M}^{\dagger}_{\mathrm{rig}}$ are both base changed from their cyclotomic counterparts $\mathcal{M}^{\mathrm{Gal}(K_{\infty}/K_{\mathrm{cyc}})}$ and $\mathcal{M}^{\dagger,\mathrm{Gal}(K_{\infty}/K_{\mathrm{cyc}})}_{\mathrm{rig}}$, so we reduce to the cyclotomic case.

To deal with this case, recall the rings $\mathbf{B}_{I,\mathrm{cyc}}$ from §4. The (cyclotomic) Robba ring is defined as

$$\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}} = \varinjlim_{r} \varprojlim_{s \geq r} \mathbf{B}_{[r,s],\mathrm{cyc}}.$$

The maps $\mathbf{B}_{[r,s],\mathrm{cyc}} \hookrightarrow \widetilde{\mathbf{B}}_{I,\mathrm{cyc}}$ of §4 induce an embedding $\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}} \hookrightarrow \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig,cyc}} = \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(\widehat{K}_{\mathrm{cyc}})$. By Theorem B of [Be13] we have

$$\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}} = \bigcup_{n \geq 0} \varphi^{-n}(\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}}),$$

and since each $\varphi^{-n}\left(\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}}\right)$ is a Bézout domain, the conclusion follows. □

In particular, we recover decompletion result entirely phrased in terms of (φ, Γ)-modules:

$$\{(\varphi, \Gamma)\text{-modules over } \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}\}.$$

This result recovers the decompletion theorem of Cherbonnier-Colmez [CC98] and Kedlaya [Ke04].

Theorem 20.6. *If $K_{\infty} = K_{\mathrm{cyc}}$, base extension induces an equivalence of categories*

$$\{(\varphi, \Gamma)\text{-modules over } \mathbf{B}^{\dagger}_{\mathrm{rig,cyc}}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig,cyc}}\}.$$

Proof. If M is a (φ, Γ)-module over $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig,cyc}} = \bigcup_n \varphi^{-n}(\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}})$ then there exists $n \gg 0$ such that M is defined over $\varphi^{-n}(\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}})$. If $e_1, ..., e_d$ is a basis of M then $\varphi^n(e_1), ..., \varphi^n(e_d)$ is a basis defined over $\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}}$. Therefore the category of (φ, Γ)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig,cyc}}$ is equivalent to the category of (φ, Γ)-modules over $\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig,cyc}}$. But this latter category is equivalent to (φ, Γ)-modules over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig,cyc}}$ by the previous theorem. □

21 Locally analytic vector bundles and p-adic differential equations

21.1 Modifications of locally analytic vector bundles

We first introduce the following category. It is the locally analytic version of Berger's category of $\mathbf{B}$-pairs, see [Be08A].

Definition 21.1. A locally analytic pair is a pair $\mathcal{W} = (\mathcal{W}_e, W^+_{\mathrm{dR}})$ where $\mathcal{W}_e$ is a locally free $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}\text{-}\{\infty\}} = \mathcal{O}^{\mathrm{la}}_{\mathcal{X}}|_{\mathcal{X}\text{-}\{\infty\}}$-module with a semilinear Γ-action and $W^+_{\mathrm{dR}} \subset \mathbf{B}^{\mathrm{pa}}_{\mathrm{dR}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}\text{-}\{\infty\}}} \mathcal{W}_e$ is a Γ-stable $\mathbf{B}^{+,\mathrm{pa}}_{\mathrm{dR}}$-lattice.

Proposition 21.2. *The functor from locally analytic vector bundles to locally analytic pairs mapping $\mathcal{E}$ to $(\mathcal{E}|_{\mathcal{X}\text{-}\{\infty\}}, \mathbf{D}^+_{\mathrm{dif}}(\mathcal{E}))$ is an equivalence of categories.*

Proof. There is an obvious functor from the category of locally analytic pairs to the category of $\mathbf{B}$-pairs. This leads to a commutative diagram

$$\begin{array}{ccc}
\{\text{locally analytic vector bundles}\} & \longrightarrow & \{\text{locally analytic pairs}\}\,. \\
\downarrow \cong & & \downarrow \\
\{\Gamma\text{-vector bundles}\} & \xrightarrow{\ \cong\ } & \{\mathbf{B}\text{-pairs}\}
\end{array}$$

The left vertical arrow is an equivalence by Theorem 6.1. The lower horizontal arrow is also an equivalence, as explained in §10.1.2 of [FF18]. It follows that the functor from locally analytic pairs to $\mathbf{B}$-pairs is essentially surjective. It is also fully faithful by Proposition 2.1. By commutativity of the diagram, the upper horizontal arrow is an equivalence. □

Definition 21.3. Given two locally analytic vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ we say that $\mathcal{E}_2$ is a modification of $\mathcal{E}_1$ if $\mathcal{E}_1|_{\mathcal{X}\text{-}\{\infty\}} \cong \mathcal{E}_2|_{\mathcal{X}\text{-}\{\infty\}}$.

Any Γ-stable $\mathbf{B}^{+,\mathrm{pa}}_{\mathrm{dR}}$-lattice $N \subset \mathbf{D}_{\mathrm{dif}}(\mathcal{E})$ defines a modification of $\mathcal{E}$ by taking the pair $\big(\mathcal{E}|_{\mathcal{X}\text{-}\{\infty\}}, N\big)$.

Remark 21.4. We could have also defined this notion of modification in terms of usual $\mathbf{B}$-pairs. Our choice of presentation is meant to illustrate that one can speak of modifications without leaving the locally analytic realm.

21.2 de Rham and $\mathbf{C}_p$-admissible locally analytic vector bundles

Let $\mathcal{E}$ be a locally analytic vector bundle. We say that

- $\mathcal{E}$ is $\mathbf{C}_p$-admissible if $\dim_K \mathcal{E}_{x_\infty}^{\Gamma=1} = \operatorname{rank}(\mathcal{E})$.
- $\mathscr{E}$ is de Rham if $\mathbf{D}_{\mathrm{dR}}(\mathcal{E}) := \dim_K \widehat{\mathcal{E}}_{x_\infty}^{\Gamma=1} = \operatorname{rank}(\mathcal{E})$.

If V is a p-adic representation and $\mathcal{E} = \widetilde{\mathcal{E}}(V)^{\mathrm{la}}$ then $\mathcal{E}_{x_\infty}^{\Gamma=1} = (\mathbf{C}_p \otimes V)^{G_K}$ and $\mathbf{D}_{\mathrm{dR}}(\mathcal{E}) = \mathbf{D}_{\mathrm{dR}}(V)$, so this extends the usual definitions.

In what follows, note that $\mathbf{D}_{\mathrm{dR}}(\mathcal{E})$ has a natural filtration induced from the I_θ filtration on $\widehat{\mathcal{E}}_{x_\infty}$.

Definition 21.5. Suppose $\mathcal{E}$ is de Rham.

1. $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$ is the modification of $\mathcal{E}$ given by the lattice $\mathbf{D}_{\mathrm{dR}}(\mathcal{E}) \otimes_K \mathbf{B}_{\mathrm{dR}}^{+,\mathrm{pa}} \subset \mathbf{D}_{\mathrm{dif}}(\mathcal{E})$. It is $\mathbf{C}_p$-admissible.
2. $\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$ is the locally analytic φ-vector bundle corresponding to $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$.

21.3 The surfaces $\mathcal{Y}_{\log,L}$ and $\mathcal{X}_{\log,L}$

In §10.3 of [FF18], Fargues and Fontaine define a scheme $X_{\log}$. It is a line bundle over the schematic Fargues-Fontaine curve $X_{\mathrm{FF}} = X_{\mathrm{FF}}(\mathbf{C}_p)$ with a natural projection $\pi : X_{\log} \to X$; further, it has a G_K-action and π is G_K-equivariant.

We let $\mathcal{X}_{\log}$ be the analytification of $X_{\log}$. If L is a finite extension of K, we set

$$\mathcal{X}_{\log,L} := \mathcal{X}_{\log}/\mathrm{Gal}(\overline{K}/L_\infty).$$

Similarly, write $\mathcal{Y}_{\log} = \mathcal{Y}_{(0,\infty)} \times_{\mathcal{X}} \mathcal{X}_{\log}$ and $\mathcal{Y}_{\log,L} = \mathcal{Y}_{\log}/\mathrm{Gal}(\overline{K}/L_\infty)$; then $\mathcal{Y}_{\log,L}/\varphi = \mathcal{X}_{\log,L}$. These spaces have an action of $\mathrm{Gal}(L_\infty/L)$, an open subgroup of Γ.

Write p_L (resp. $p_{\log,L}$) for the projection maps $\mathcal{Y}_L \to \mathcal{Y}$ or $\mathcal{X}_L \to \mathcal{X}$ (resp. $\mathcal{Y}_{\log,L} \to \mathcal{Y}$ or $\mathcal{X}_{\log,L} \to \mathcal{X}$). If $I \subset (0,\infty)$ is closed interval, let $\mathcal{Y}_{\log,L,I} = p_{\log,L}^{-1}(\mathcal{Y}_I)$ and similarly $\mathcal{X}_{\log,L,I} = p_{\log,L}^{-1}(\mathcal{X}_I)$ for $\mathcal{X}$ if I is sufficiently small.

Define

$$\widetilde{\mathbf{B}}_{\log,L,I} = \mathrm{H}^0(\mathcal{Y}_{\log,L,I}, \mathcal{O}_{\mathcal{Y}_{\log,L}}).$$

As explained in loc. cit., there is a natural G_K-equivariant morphism of sheaves

$$d : \mathcal{O}_{X_{\log}} \to \Omega^1_{X_{\log}/X} \cong \pi^*\mathcal{O}_X(-1)$$

which induces an $\mathcal{O}_{\mathcal{X}}$-linear morphism

$$N : \mathcal{Y}_{\log} \to \mathcal{Y}_{\log} \otimes \Omega^1_{\mathcal{Y}_{\log}/\mathcal{Y}_{(0,\infty)}}.$$

This then further induces an $\widetilde{\mathbf{B}}_{L,I}$-linear differential operator $N : \widetilde{\mathbf{B}}_{\log,L,I} \to \widetilde{\mathbf{B}}_{\log,L,I}$. If $T \in \widetilde{\mathbf{B}}_{\log,L,I}$ is such that $N(T) = 1$ then $\widetilde{\mathbf{B}}_{\log,L,I} = \widetilde{\mathbf{B}}_{L,I}[T]$ and $N = d/dT$. Such a T exists: if ϖ is any nonunit $\varpi \in \widehat{L}_\infty^\times$ and $\varpi^\flat = (\varpi, \varpi^{1/p}, ...)$, take $T = \log[\varpi^\flat]$.

Lemma 21.6. *There exists $T \in \widetilde{\mathbf{B}}^{\mathrm{la}}_{\log,L,I}$ with $N(T) = 1$. Consequently, $\widetilde{\mathbf{B}}^{\mathrm{la}}_{\log,L,I} = \widetilde{\mathbf{B}}^{\mathrm{la}}_{L,I}[T]$.*

Proof. The second claim follows from the first using Proposition 2.1. To find such an element T, consider the exact sequence

$$0 \to \widetilde{\mathbf{B}}_{L,I} \to \widetilde{\mathbf{B}}^{N^2=0}_{\log,L,I} \xrightarrow{N} \widetilde{\mathbf{B}}_{L,I} \to 0.$$

After taking locally analytic vectors the sequence stays exact by Theorem 5.1. Thus the sequence

$$0 \to \widetilde{\mathbf{B}}^{\mathrm{la}}_{L,I} \to \widetilde{\mathbf{B}}^{\mathrm{la},N^2=0}_{\log,L,I} \xrightarrow{N} \widetilde{\mathbf{B}}^{\mathrm{la}}_{L,I} \to 0$$

is exact. This means we can lift 1 to an element $T \in$ with $N(T) = 1$, as required. □

Proposition 21.7. *Suppose $\varphi^{\mathbf{Z}}(x_\infty) \cap \mathcal{Y}_I \neq \emptyset$.*
(i) If M is a finite extension of L contained in L_∞, then $\widetilde{\mathbf{B}}^{\mathrm{Gal}(L_\infty/M)}_{\log,L,I} = M_0$ where M_0 is the maximal unramified extension of $\mathbf{Q}_p$ contained in M.
(ii) $\widetilde{\mathbf{B}}^{\mathrm{la},\mathrm{Lie}\Gamma=0}_{\log,L,I} = L'_0$, the maximal unramified extension of $\mathbf{Q}_p$ contained in L_∞.

Proof. (i) follows from [FF18, 10.3.15] and (ii) follows from (i). □

One way to construct de Rham locally analytic vector bundles is as follows. Write $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K)$ for the category of finite dimensional vector spaces D over $\mathbf{Q}_p^{\mathrm{un}}$ together with a semilinear action of φ, a monodromy operator N with $\varphi N = pN\varphi$, a filtration on $D \otimes_{\mathbf{Q}_p^{\mathrm{un}}} K^{\mathrm{un}}$ and a discrete action of G_K on D which respects the filtration. For example, if V is a potentially semistable representation then $\mathbf{D}_{\mathrm{pst}}(V)$ is an object of $\mathrm{Mod}^{\mathrm{Fil},\varphi}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K)$.
There is a functor

$$\mathcal{E} : \mathrm{Mod}^{\mathrm{Fil},\varphi}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K) \to \{\text{de Rham locally analytic vector bundles}\}$$

defined as follows: given $D \in \mathrm{Mod}^{\mathrm{Fil},\varphi}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K)$, choose L such that D is defined over L, i.e. $D = \mathbf{Q}_p^{\mathrm{un}} \otimes_{L_0} D_0$. Such an L exists because the action of G_K is discrete. Then $\mathcal{E}(D)$ is defined to be the locally analytic vector bundle corresponding to the pair

$$((\mathcal{O}^{\mathrm{la}}_{\mathcal{Y}_{\log,L} - p^{-1}_{\log,L}(\infty)} \otimes_{L_0} D)^{\varphi=1,N=0,\mathrm{Gal}(L_\infty/K_\infty)}, \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}^{H_L,\mathrm{pa}} \otimes_{L_0} D_0)^{\mathrm{Gal}(L_\infty/K_\infty)}).$$

It is de Rham because

$$D \subset \mathbf{B}_{\mathrm{dR}}^{H_K,\mathrm{pa}} \otimes \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}^{H_L,\mathrm{pa}} \otimes_{L_0} D_0)^{\mathrm{Gal}(L_\infty/K_\infty)}$$

is fixed by an open subgroup of Γ. If we choose any larger L we get the same pair, so the construction $D \mapsto \mathcal{E}(D)$ is independent of the choice of L.

21.4 Sheaves of smooth functions

In this subsection we introduce certain sheaves of functions on $\mathcal{X}$. All of these can be defined equally well for $\mathcal{Y}_{(0,\infty)}$.

Definition 21.8. We define the following sheaves of functions on $\mathcal{X}$.
(i) Smooth functions: $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{la},\mathrm{Lie}\Gamma=0}$.
(ii) For $[L:K] < \infty$, L-smooth functions:

$$\mathcal{O}_{\mathcal{X}}^{L\text{-sm}} = p_{L,*}\left(p_L^*\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}\right)^{\mathrm{Lie}\Gamma=0}.$$

(iii) For $[L:K] < \infty$, L log-smooth functions:

$$\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}} = p_{\log,L,*}\left(p_{\log,L}^*\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}\right)^{\mathrm{Lie}\Gamma=0}.$$

(iv) Potentially smooth functions:

$$\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}} = \varinjlim_{[L:K]<\infty} \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}.$$

(v) Potentially log-smooth functions:

$$\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}} = \varinjlim_{[L:K]<\infty} \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}.$$

The following proposition has been essentially explained to us by Kedlaya.

Proposition 21.9. *Let U be a connected open affinoid subset of $\mathcal{X}$.*
(i) The sections of each of $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}$, $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$ and $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}$ at U is a field which injects (noncanonically) into $\mathbf{C}_p$.
(ii) If $x_\infty \in U$ then there are canonical injections $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}) \hookrightarrow K_\infty$, $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}) \hookrightarrow L_\infty$ and $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}) \hookrightarrow \overline{K}$.
(iii) If $x_\infty \in U$ and $U = \mathcal{X}_I$, we have $\mathrm{H}^0(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}) = K_0'$, $\mathrm{H}^0(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}) = L_0'$ and $\mathrm{H}^0(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}) = K_0^{\mathrm{un}}$.
(iv) We have $\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{sm}} = K_\infty$, $\mathcal{O}_{\mathcal{X},x_\infty}^{L\text{-sm}} = L_\infty$ and $\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{psm}} = \overline{K}$.

Proof. Each of the assertions (i)-(iv) for $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}$ follows from the corresponding assertion for $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$. We shall give below arguments proving (i)-(iv) for $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}$; the proofs for $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$ are the same once K is replaced by L.

After passing to an open subgroup of Γ, we may assume Γ stabilizes U. By Theorem 8.8 of [Ke16], the ring $\mathcal{O}_{\mathcal{X}}(U)$ is a Dedekind domain. Each rank 1 point x of U defines a maximal ideal of $\mathcal{O}_{\mathcal{X}}(U)$, so $f \in \mathcal{O}_{\mathcal{X}}(U)$ can belong to only finitely many of these points. If $f \in \mathcal{O}_{\mathcal{X}}(U)$ is killed by $\mathrm{Lie}\Gamma$ then f is fixed by a finite subgroup of Γ, so these finitely many maximal ideals must form a finite orbit under the Γ-action. But the only rank 1 point with finite orbit is the point x_∞, again by 10.1.1 of [FF18]. So every $f \in \mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U)$ either vanishes only at x_∞ or is invertible.

If $x_\infty \notin U$, this proves that $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U)$ is a field. In particular, it injects into the residue field of each rank 1 point, and there is a dense subset of $\mathcal{X}$ with residue field a subfield of $\mathbf{C}_p$. This proves (i) in this case. On the other hand, if $x_\infty \in U$ then there is a Γ-equivariant embedding of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(U)$ into $\mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)^{\mathrm{la}}$ which gives an embedding of $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U)$ into $\mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)^{\mathrm{la},\mathrm{Lie}\Gamma=0} = K_\infty$. This simultaenously proves (i) and (ii) for $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}$.

Next, (iii) follows immediately from Proposition 8.7. For (iv), we have already shown that $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U) \subset K_\infty$ for each U which contains x_∞, so $\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{sm}} \subset K_\infty$. To show the converse inclusion, use the henselian property of local rings of adic spaces [Mo19, III.6.3.7] to show first that $K_\infty \subset \mathcal{O}_{\mathcal{X},x_\infty}$. It then follows that $K_\infty \subset \mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{sm}}$, which concludes the proof. □

We raise a few questions to which we expect a positive answer but have not answered in this article.

Question 21.10. *1. We can show that $\overline{K} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}$ if x is any rank 1 point. Indeed, any untilt of $\mathbf{C}_p^\flat$ is algebraically closed, and one can use this to show that the completed local rings $\mathbf{B}_{\mathrm{dR},x}^+$ contain $\overline{K}$. This implies by the same argument that $\overline{K} \subset \mathcal{O}_{\mathcal{X},x}$. But every element of $\overline{K}$ has finite degree over K_0, which is fixed by G_K. This implies that every $x \in \overline{K}$ is fixed by an open subgroup G_K so $\overline{K} \subset \mathcal{O}_{\mathcal{X},x}^{\mathrm{psm}}$.*

Is it true that $\overline{K} = \mathcal{O}_{\mathcal{X},x}^{\mathrm{psm}}$ for any rank 1 point x?

2. Is it true that for every connected open affinoid $U \subset \mathcal{X}$, the field $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}(U)$ is a finite extension of K_0^{un}? In particular, this would imply a positive answer to question 1.

3. Is it true that $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}} = \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$ (and hence $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$)? If $x_\infty \in U$ then $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}(U) = \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}(U)$. This can be seen by using the embedding into $\mathbf{B}_{\mathrm{dR}}^+$ as in the proof of Proposition 8.7.

21.5 The solution functor

In this subsection, we assume $\mathcal{E}$ is a de Rham locally analytic vector bundle. Given L finite over K, we define the sheaves of solutions on $\mathcal{X}$:

1. $\mathrm{Sol}_L(\mathcal{E}) := p_{L,*}(p_L^* \mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}\Gamma=0}$, a module over $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$,
2. $\mathrm{Sol}_{\log,L}(\mathcal{E}) := p_{\log,L,*}(p_{\log,L}^* \mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}\Gamma=0}$, a module over $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$,
3. $\mathrm{Sol}(\mathcal{E}) := \varinjlim_{[L:K]<\infty} \mathrm{Sol}_{\log,L}(\mathcal{E})$, a module over $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$.

We have similar versions of these sheaves on $\mathcal{Y}_{(0,\infty)}$, denoted by $\mathrm{Sol}_*^{\varphi}(\mathcal{E})$ for $* \in \{L, \{\log, L\}, \emptyset\}$. Since the φ action on $\mathcal{Y}_{\log,L}$ is Γ-equivariant, there are natural identifications $\mathrm{Sol}_*(\mathcal{E}) = (\mathrm{Sol}_*^{\varphi}(\mathcal{E}))^{\varphi=1}$ and $\mathrm{Sol}_*^{\varphi}(\mathcal{E}) \cong \mathcal{O}^{\bullet}_{\mathcal{Y}_{(0,\infty)}} \otimes_{\mathcal{O}^{\bullet}_{\mathcal{X}}} \mathrm{Sol}_*(\mathcal{E})$ where $(*, \bullet) = \{(L, L\text{-sm}), (\{\log, L\}, L\text{-lsm}), (\emptyset, \mathrm{plsm})\}$.

To make the link with $\mathcal{E}$ clear, we shall need the following form of the p-adic monodromy theorem due to André [An02], Kedlaya [Ke04] and Mebkhout [Me02].

Proposition 21.11. *There exists a finite extension L over K such that if U is an open subset of $\mathcal{Y}_{[r,\infty)}$ for some $r \gg 0$ then the natural map*

$$\mathcal{O}^{\mathrm{la}}_{\mathcal{Y}_{\log,L}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}^{L\text{-lsm}}_{\mathcal{Y}_{(0,\infty)}}(U)} \mathrm{Sol}^{\varphi}_{\log,L}(\mathcal{E})(U) \to$$

$$\mathcal{O}^{\mathrm{la}}_{\mathcal{Y}_{\log,L}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{Y}_{(0,\infty)}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U).$$

is an isomorphism. Consequently, if $U \subset \mathcal{X}_I$ for some I then

$$\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{\log,L}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}^{L\text{-lsm}}_{\mathcal{X}}(U)} \mathrm{Sol}_{\log,L}(\mathcal{E})(U) \xrightarrow{\sim} \mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{\log,L}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}(U)} \mathcal{N}_{\mathrm{dR}}(\mathcal{E})(U).$$

Proof. Let $\widetilde{\mathbf{D}}^{\dagger}_{\mathrm{rig}}$ be the (φ, Γ)-module corresponding to $\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$. By the p-adic monodromy theorem, we know there is an isomorphism

$$\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\log,L} \otimes_{L'_0} (\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\log,L} \otimes_{\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig},K}} \widetilde{\mathbf{D}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}})^{\mathrm{Lie}\Gamma=0} \xrightarrow{\sim} \widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\log,L} \otimes_{\widetilde{\mathbf{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig},K}} \widetilde{\mathbf{D}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$$

in the cyclotomic setting (see [Be08B, III.2.1]). More generally, we may descend along unramified extensions to give it in the twisted cyclotomic case, and by base changing we get it in our setting as well as per the usual argument.

It follows that for $r \gg 0$ we also have an isomorphism

$$\widetilde{\mathbf{B}}^{\mathrm{pa}}_{\log,[r,\infty),L} \otimes_{L'_0} (\widetilde{\mathbf{B}}^{\mathrm{pa}}_{\log,[r,\infty),L} \otimes_{\widetilde{\mathbf{B}}^{\mathrm{pa}}_{[r,\infty),K}} \widetilde{\mathbf{D}}^{\mathrm{pa}}_{[r,\infty)})^{\mathrm{Lie}\Gamma=0} \xrightarrow{\sim} \widetilde{\mathbf{B}}^{\mathrm{pa}}_{\log,[r,\infty),L} \otimes_{\widetilde{\mathbf{B}}^{\mathrm{pa}}_{[r,\infty),K}} \widetilde{\mathbf{D}}^{\mathrm{pa}}_{[r,\infty)}.$$

Pulling back along Frobenius, we obtain this isomorphism for any r. Then by finding $r \gg 0$ so that $U \subset \mathcal{Y}_{[r,\infty)}$, we can base change the isomorphism along the map $\widetilde{\mathbf{B}}^{\mathrm{pa}}_{\log,[r,\infty),L} \to \mathcal{O}^{\mathrm{la}}_{\mathcal{Y}_{\log,L}}(p_{\log,L}^{-1}U)$ to conclude. □

Note that whether we need to adjoin log and/or perform a finite extension L of K depends exactly on whether $\mathcal{E}$ becomes crystalline or semistable after restricting G_K to G_L. Applying this observation and taking $\mathrm{Lie}\Gamma = 0$ of both sides of the proposition, we obtain the following.

Theorem 21.12. *The sheaf* $\mathrm{Sol}\,(\mathcal{E})$ *is a locally free* $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$*-module of rank equal to* $\mathrm{rank}(\mathcal{E})$*. More precisely:*

i. If $\mathcal{E}$ *becomes crystalline after restricting* G_K *to* $G_{L'}$ *for some* $L \subset L' \subset L_\infty$ *then* $\mathrm{Sol}_L(\mathcal{E})$ *is a locally free* $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$*-module of rank equal to* $\mathrm{rank}(\mathcal{E})$*, and there is a natural isomorphism*

$$\mathcal{O}_{\mathcal{X}_L}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}} \mathrm{Sol}_L(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_L}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}\,(\mathcal{E})\,.$$

ii. If $\mathcal{E}$ *becomes semistable after restricting* G_K *to* $G_{L'}$ *for some* $L \subset L' \subset L_\infty$ *then* $\mathrm{Sol}_{\log,L}(\mathcal{E})$ *is a locally free* $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$*-module of rank equal to* $\mathrm{rank}(\mathcal{E})$*, and there is a natural isomorphism*

$$\mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}} \mathrm{Sol}_{\log,L}(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}\,(\mathcal{E})\,.$$

Lemma 21.13. *For each sufficiently small open connected affinoid* U *of* $\mathcal{Y}_{(0,\infty)}$ *which contains an element of* $\varphi^{\mathbf{Z}}(x_\infty)$*, and for* L *large enough so that* G_L *stablizes* U*, there is a natural* G_L*-embedding* $\mathrm{H}^0(U, \mathrm{Sol}_{\log,L}^{\varphi}(\mathcal{E})) \hookrightarrow L_\infty \otimes_K \mathbf{D}_{\mathrm{dR}}\,(\mathcal{E})\,.$

Proof. Taking the completed stalk at a φ-translate of x_∞, we obtain an injection

$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U) \hookrightarrow \widehat{L}_\infty^{\mathrm{la}} \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \mathbf{D}_{\mathrm{dif}}\,(\mathcal{E})\,.$$

On the other hand, Proposition 8.7 gives an isomorphism

$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{L\text{-lsm}}(U)} \mathrm{Sol}_{\log,L}^{\varphi}(\mathcal{E})(U) \xrightarrow{\sim}$$
$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U).$$

Applying $\mathrm{Lie}\Gamma = 0$ to the composition of these maps gives the desired embedding. □

We can now give an interpertation of the stalk at x_∞:

Proposition 21.14. *There following are each naturally isomorphic to each other.*

1. *The stalk* $\mathrm{Sol}(\mathcal{E})_{x_\infty}$.
2. *The stalk* $\mathrm{Sol}(\mathcal{E})_y^{\varphi}$ *for any* $y \in \varphi^{\mathbf{Z}}(x_\infty)$.
3. $\overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}\,(\mathcal{E})$.

In particular, $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ *is naturally a filtered* $\overline{K}$*-representation of* G_K *of dimension* $\mathrm{rank}(\mathcal{E})$ *and* G_K*-fixed points* $\mathbf{D}_{\mathrm{dR}}\,(\mathcal{E})$*.*

Proof. It is clear 1 and 2 are isomorphic. By the previous lemma, we have a natural embedding of $\mathrm{Sol}(\mathcal{E})_y$, and hence of $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ into $\overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}(\mathcal{E})$. By Theorem 8.12, $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ is a finite free module of rank equal to $\dim_K \mathbf{D}_{\mathrm{dR}}(\mathcal{E})$ over $\mathcal{O}^{\mathrm{plsm}}_{\mathcal{X},x_\infty}$. But by Proposition 8.7 $\mathcal{O}^{\mathrm{plsm}}_{\mathcal{X},x_\infty} = \overline{K}$ so this embedding must be an isomorphism. □

Finally, we consider the global solutions to the differential equation, namely

$$D(\mathcal{E}) = \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) = \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathcal{O}^{\mathrm{plsm}}_{\mathcal{Y}_{(0,\infty)}} \otimes_{\mathcal{O}^{\mathrm{plsm}}_{\mathcal{X}}} \mathrm{Sol}(\mathcal{E})).$$

Proposition 21.15. *$D(\mathcal{E})$ is naturally an object of $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K)$ and $\dim_{\mathbf{Q}_p^{\mathrm{un}}} D(\mathcal{E}) = \mathrm{rank}\,(\mathcal{E})$.*

Proof. We know each $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}_{\log,L}(\mathcal{E}))$ is an L_0' vector space for U sufficiently small (independentely of L), so $D(\mathcal{E})$ is a $\mathbf{Q}_p^{\mathrm{un}}$-vector space. The filtration is induced from the embedding $\mathrm{H}^0\left(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})\right) \hookrightarrow \mathrm{Sol}(\mathcal{E})_{x_\infty} \cong \overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}(\mathcal{E})$. The φ-action is induced from the map $\varphi : \mathcal{Y}_{(0,\infty)} \to \mathcal{Y}_{(0,\infty)}$. The monodromy operator N is induced from the equivariant connection $p^*_{\log,L}\mathcal{M}_{\mathrm{dR}}(\mathcal{E}) \to p^*_{\log,L}\mathcal{N}_{\mathrm{dR}}(\mathcal{E}) \otimes \Omega^1_{\mathcal{Y}_{\log}/\mathcal{Y}_{(0,\infty)}}$. Finally, G_K acts on the smooth elements in $p^*_{\log,L}\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$, and this action is discrete because every element is killed by $\mathrm{Lie}\Gamma$, hence by an open subgroup of $\mathrm{Gal}(L_\infty/L)$. To compute the dimension use Theorem 8.12. □

Using this language, Berger's theorem ([Be08B, Théoréme III.2.4]) admits the following interpertation.

Theorem 21.16. *The functors $D \mapsto \mathcal{E}(D)$ and $\mathcal{E} \mapsto D(\mathcal{E})$ are mutual inverses and induce an equivalence of categories*

$$\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K) \cong \{\textit{de Rham locally analytic vector bundles}\}.$$

Remark 21.17. If $\mathcal{E}$ is the locally analytic vector bundle associated to a p-adic representation V, we see that the global-to-local map

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) \hookrightarrow \mathrm{Sol}(\mathcal{E})_{x_\infty}$$

is nothing but the more familiar map

$$\mathbf{D}_{\mathrm{pst}}(V) \hookrightarrow \overline{K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V).$$

Question 21.18. *Theorem 8.17 allows us to consider objects of*

$$\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbf{Q}_p^{\mathrm{un}}}(G_K)$$

as global solutions to p-adic differential equations. The filtration is coming from the behavious of orders of vanishing at $x_\infty = 0$, while the (φ, N, G_K)-structure comes from some sort of monodromy of the map $\varprojlim_L \mathcal{Y}_{\log,L} \to \mathcal{X}$.

In our description the space $\varprojlim_L \mathcal{Y}_{\log,L}$ behaves as a substitute for a universal cover of $\mathcal{X}$. It would be interesting if it can be replaced by a more literal cover of $\mathcal{X}$ for which the (φ, N, G_K)-actions can be interperted as monodromy actions. One could even speculate that in an appropriate sense, the analytic fundamental group of $\mathcal{X}(\mathbf{C}_p)_{\overline{K}}$ should be a tame Weil group with its two dimensions reflecting the φ and N operators.

We conclude with an example.

Example 21.19. Take $\alpha \in \mathbf{Z}_p^\times$, and given $g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ let $\xi_\alpha(g) \in \mathbf{Z}_p$ be the element such that $\zeta_{p^n}^{\xi_\alpha(g)} = g(\alpha^{1/p^n})/\alpha^{1/p^n}$ for each $n \geq 1$. The Kummer extension

$$0 \to \mathbf{Q}_p(\chi_{\mathrm{cyc}}) \to V = V_\alpha \to \mathbf{Q}_p \to 0$$

is given by mapping in a basis e, f the element g to the matrix

$$\begin{pmatrix} \chi_{\mathrm{cyc}}(g) & \xi_\alpha(g) \\ 0 & 1 \end{pmatrix}.$$

The associated locally analytic vector bundle $\mathcal{E}$ sits in an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(\chi_{\mathrm{cyc}}) \to \mathcal{E} \to \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \to 0.$$

We have

$$\mathcal{N}_{\mathrm{dR}}(\mathcal{E}) = \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} x \oplus \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} y \cong \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(1) \oplus \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$$

where at a neighborhood of x_∞ we have $x = t^{-1}e$ and $y = -\log[\alpha^\flat]t^{-1}e + f$. Thus

$$\begin{aligned} \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}_{\mathbf{Q}_p}(\mathcal{E})) &= \mathrm{H}^0(\mathcal{O}^{\mathrm{sm}}_{\mathcal{Y}_{(0,\infty)}} x \oplus \mathcal{O}^{\mathrm{sm}}_{\mathcal{Y}_{(0,\infty)}} y) \\ &= \mathbf{Q}_p x \oplus \mathbf{Q}_p y. \end{aligned}$$

The action of φ is given by $\varphi(x) = p^{-1}x$ and $\varphi(y) = y$. This gives the underlying φ-module of $\mathbf{D}_{\mathrm{cris}}(V)$.

To get the filtration, we consider the stalk of $\mathrm{Sol}_{\mathbf{Q}_p}(\mathcal{E})$ at x_∞. Observe that Fil^0 consists exactly of these smooth sections which do not have a pole at x_∞. As $\log[\alpha^\flat] \equiv \log_p \alpha \bmod t$, we have $\mathrm{Fil}^0\mathrm{Sol}_{\mathbf{Q}_p}(\mathcal{E})_{x_\infty} = \mathbf{Q}_{p,\mathrm{cyc}}(x \log_p \alpha + y)$ and so the filtration on $\mathbf{D}_{\mathrm{cris}}(V)$ is given by

$$\mathrm{Fil}^{-1} = \mathbf{D}_{\mathrm{cris}}(V) \supset \mathrm{Fil}^0 = \mathbf{Q}_p(x \log_p \alpha + y) \supset \mathrm{Fil}^1 = 0.$$

References

[An02] André Y., *Filtrations de Hasse-Arf et monodromie p-adique*. Invent. Math. 148 (2002), no. 2, 285–317.

[Be08A] Berger L., *Construction de (φ,Γ)-modules: représentations p-adiques et B-paires*, Algebra Number Theory 2 (2008), no. 1, 91--120.

[Be08B] Berger L., *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Astérisque No. 319 (2008), 13--38.

[Be13] Berger L., *Multivariable Lubin-Tate (φ,Γ)-modules and filtered φ-modules*, Math. Res. Lett. 20 (2013), no. 3, 409--428.

[Be21] Berger L., *Errata for my articles.* perso.ens-lyon.fr/laurent.berger/articles.php. (2021).

[BC08] Berger L., Colmez P., *Familles de représentations de de Rham et monodromie p-adique*, Astérisque No. 319 (2008), 303--337.

[BC16] Berger L., Colmez P., *Théorie de Sen et vecteurs localement analytiques*, Ann. Sci. École Norm. Sup. 49 (2016), no. 4, 947--970.

[CC98] Cherbonnier F., Colmez P., *Représentations p-adiques surconvergentes*, Invent. Math. 133 (1998), no. 3, p. 581–611.

[Co02] Colmez P., *Espaces de Banach de dimension finie*, J. Inst. Math. Jussieu 1 (2002), no. 3, p. 331–439.

[Co08] Colmez P., *Espaces Vectoriels de dimension finie et représentations de de Rham*, Astérisque 319 (2008), 117–186.

[Co10] Colmez P., *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (φ,Γ)*-modules*, Astérisque 330 (2010), 281–509.

[dSP19] de Shalit E., Porat G., *Induction and restriction of (φ,Γ)-modules*, Muenster J. Math., vol. 12 (2019), 215-237.

[FF18] Fargues L, Fontaine J. M., *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Société mathématique de France (2018).

[Fo90] Fontaine J.M., *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, 249-309, Progr. Math., 87, Birkhäuser Boston, Boston, MA (1990)

[Fo04] Fontaine J.M., *Arithmétique des représentations galoisiennes p-adiques* Astérisque, 295 (2004), pp.1-115.

[Ke04] Kedlaya K., *A p-adic local monodromy theorem*, Ann. of Math. (2) 160 (2004), no. 1, p. 93–184.

[Ke15] Kedlaya K., *New methods for* (φ, Γ)*-modules.* Research in the Mathematical Sciences, 1-31 (2015).

[Ke16] Kedlaya KS. *Noetherian properties of Fargues-Fontaine curves.* International Mathematics Research Notices, 2544-2567.

[KL13] Kedlaya K., Liu R. *Relative p-adic Hodge theory: foundations,* arXiv preprint arXiv:1301.0792. (2013).

[Me02] Mebkhout Z., *Analogue p-adique du théoréme de Turrittin et le théoréme de la monodromie p-adique.* Invent. math. 148 (2002), 319–351

[Mo19] Morel S. *Adic spaces.* Lecture Notes. https://web. math. princeton. edu/~ smorel/adic_notes. pdf.

[Pa21] Pan L., *On locally analytic vectors of the completed cohomology of modular curves*, arXiv preprint arXiv:2008.07099 (2021).

[Po20] Porat G., *Lubin-Tate theory and overconvergent Hilbert modular forms of low weight,* arXiv preprint arXiv:2010.14574. (2020).

[Po19] Poyeton L. *Extensions de Lie p-adiques et* (φ, Γ)*-modules.* Doctoral dissertation, Université de Lyon (2019).

[RJRC21] Rodrigues Jacinto J., Rodríguez Camargo J. E., *Solid locally analytic representations of p-adic Lie groups,* arXiv preprint arXiv:2110.11916 (2021).

[RC22] Rodríguez Camargo J. E., *Locally analytic completed cohomology of Shimura varieties and overconvergent BGG maps,* arXiv preprint arXiv:2205.02016 (2022).

[Sch11] Schneider P., *p-adic Lie groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 344, Springer, Heidelberg, 2011.

[Se81] Sen S., *Continuous cohomology and p-adic Galois representations*, Invent. Math. 62 no. 1, p. 89–116 (1980/81).

[Sta] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu.

[SW20] Scholze P., Weinstein J., *Berkeley lectures on p-adic geometry,* Princeton University Press (2020).

[Ta67] Tate, J.T., *p-Divisible groups.* In Proceedings of a conference on Local Fields (pp. 158-183). Springer, Berlin, Heidelberg (1967).

Part IV

Overconvergence of étale (φ, Γ)-modules in families

Abstract

We prove a conjecture of Emerton, Gee and Hellmann concerning the overconvergence of étale (φ, Γ)-modules in families parametrized by topologically finite type $\mathbf{Z}_p$-algebras. As a consequence, we deduce the existence of a natural map from the rigid fiber of the Emerton-Gee stack to the rigid analytic stack of (φ, Γ)-modules.

Contents

22 Introduction

In recent years, there has been growing interest in realizing the collection of Langlands parameters in various settings as a moduli space with a geometric structure. In particular, in the p-adic Langlands program for $\mathrm{Gal}(\overline{K}/K)$, where K is a finite extension of $\mathbf{Q}_p$, this space should come in two different forms corresponding to the two different flavours of the p-adic Langlands correspondence. In the so called "Banach" case, this space has been constructed and studied by Emerton and Gee in their manuscript [EG19]. For $d \geq 1$, it is the moduli stack $\mathcal{X}_d$ whose value on a $\mathbf{Z}/p^a$-finite type scheme $\mathrm{Spec} A$ is the groupoid of d-dimensional projective étale (φ, Γ)-modules over the ring $\mathbf{A}_{K,A}$. In the "analytic" case, this space has been defined by Emerton, Gee and Hellmann in [EGH22]. For $d \geq 1$, it is the rigid analytic stack $\mathfrak{X}_d$ whose value on an affinoid space $\mathrm{Sp} A$ is the groupoid of d-dimensional projective (φ, Γ)-modules over the Robba ring $\mathcal{R}_{K,A}$.

These two cases are believed to be related. In [EGH22], the authors express the following expectation: there should exist a map

$$\pi_d : \mathcal{X}_d^{\mathrm{rig}} \to \mathfrak{X}_d$$

obtained by forgetting the étale lattice. Unfortunately, the existence of π_d is not immediate from the definitions. Indeed, if A is a p-adically complete, topologically of finite type $\mathbf{Z}_p$-algebra, there is no natural map from $\mathbf{A}_{K,A}$ to $\mathcal{R}_{K,A}$. Rather, there is a ring $\mathbf{A}^{\dagger}_{K,A}$ of overconvergent periods which is naturally contained in both of $\mathbf{A}_{K,A}$ and $\mathcal{R}_{K,A}$, and it is through this subring that we obtain the link between the two types of (φ, Γ)-modules. In light of this, Emerton, Gee and Hellmann make the following conjecture [EGH22].

Conjecture. *Every étale (φ, Γ)-module over $\mathbf{A}_{K,A}$ canonically descends to an étale (φ, Γ)-module over $\mathbf{A}^{\dagger}_{K,A}$. Consequently, the map π_d exists.*

The main result of this article confirms this expectation.

Theorem. *The conjecture is true.*
More precisely, the functor $M^{\dagger} \mapsto M := \mathbf{A}_{K,A} \otimes_{\mathbf{A}^{\dagger}_{K,A}} M^{\dagger}$ induces an equivalence of categories from the category of projective étale (φ, Γ)-modules over $\mathbf{A}^{\dagger}_{K,A}$ to the category of projective étale (φ, Γ)-modules over $\mathbf{A}_{K,A}$.

22.1 Previous results

The main result of this article is already known in some cases. The first result of overconvergence, proved by Cherbonnier-Colmez in [CC98], can be thought of as the case $A = \mathbf{Z}_p$ of the conjecture. Later, Berger and Colmez in [BC08] extended these ideas. Though the current setting is a little different, loc. cit. makes it clear that overconvergence of free étale (φ, Γ)-modules would hold whenever the family comes from a family of Galois representations, at least in the case A is p-torsionfree. Other two works worth mentioning are those of Gao ([Ga19]), which establishes overconvergence in the case $A = \mathbf{Z}_p$ without appealing to Galois representations; and Bellovin's article ([Bel20]), which proves overconvergence of families of étale (φ, Γ)-modules coming from Galois representations in the pseudorigid setting. One novel feature of the latter two works is that they prove overconvergence in settings where A could have p-torsion.

22.2 The ideas of the proof

The difficult part of the theorem is the essential surjectivity of the functor. The scheme of the proof is to introduce two additional perfect rings of periods $\widetilde{\mathbf{A}}_{K,A}, \widetilde{\mathbf{A}}^{\dagger}_{K,A}$, with inclusions

$$\mathbf{A}_{K,A} \subset \widetilde{\mathbf{A}}_{K,A} \supset \widetilde{\mathbf{A}}^{\dagger}_{K,A} \supset \mathbf{A}^{\dagger}_{K,A}.$$

Then, starting with a projective étale (φ, Γ)-module over $\mathbf{A}_{K,A}$, we extend it to $\widetilde{\mathbf{A}}_{K,A}$, and then descend in two steps, first from $\widetilde{\mathbf{A}}_{K,A}$ to $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ and then from $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ to $\mathbf{A}^{\dagger}_{K,A}$.
Let us emphasize that when a family of étale (φ, Γ)-modules comes from a family of Galois representations, the first descent step from $\widetilde{\mathbf{A}}_{K,A}$ to $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ can be completely avoided. Therefore in previous work, the entire focus was on the second step. However, outside of the case where A is a finite type $\mathbf{Z}_p$-algebra, families of étale (φ, Γ)-modules over $\mathbf{A}_{K,A}$ do not in general come from Galois representations ([Ch09, KL11]), so we want to avoid using them.

Thus, a method for descending from $\widetilde{\mathbf{A}}_{K,A}$ to $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ is required. Fortunately, this was worked out in the case $A = \mathbf{Z}_p$ in the article of de Shalit and the author in [dSP19], based on the author's master thesis, by using the contracting properties of Frobenius. The first key idea of this article is to generalize these results to the setting of families; since the method of [dSP19] is relatively elementary, this does not cause too many technical difficulties. Next, for the descent step from $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ to $\mathbf{A}^{\dagger}_{K,A}$, we use ideas based on the Tate-Sen method of [BC08]. However, to apply the descent results of loc. cit., one needs the existence of an open subgroup for which the matrices of the group action are congruent to 1 mod some power of p, which is arranged there by restricting attention to étale (φ, Γ)-modules coming from Galois representations, and using the lattice of such a representation. This causes two technical problems for us: first, we want to avoid using Galois representations; and second, we work in a setting where A could have p-torsion, so in any case one cannot expect such a congruence to occur in general. The second key idea of this article is to develop a variant of the Tate-Sen method for Tate rings which replaces the role of p by a pseudouniformizer f. Here, we were inspired by the article [Bel20] which showed how the role of p can be replaced by that of a psuedounifomizer (though still in the context of Galois representations). This idea ends up solving both of the aforementioned problems at the same time. In fact, in terms of applications, our method turns out to be flexible enough to also reprove results of both [BC08] and [Bel20].

22.3 Structure of the article

In §2, we give the definitions and establish the basic properties of the coefficient rings involved. Since the rings involved are a little bit subtle, this is required for many of the arguments appearing later to make sense, and ends up being the longest section of this article. On a first reading, the reader may decide to skim it and come back to it as needed. In §3 we prove the descent from $\widetilde{\mathbf{A}}_{K,A}$ to $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$. In §4, we develop the variant of the Tate-Sen method to be used in §5. This might be of independent interest for future applications. In §5, we prove the descent from $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ to $\mathbf{A}^{\dagger}_{K,A}$. Finally, §6 is a short section putting everything together for the proof of the main theorem.

22.4 Notations and conventions

The field K denotes a finite extension of $\mathbf{Q}_p$. The group $G_K = \mathrm{Gal}(\overline{K}/K)$ denotes the absolute Galois group of K. We write $K_\infty = K(\mu_{p^\infty})$ for the cyclotomic extension. Its absolute Galois group is $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$, and the cyclotomic character identifies the quotient $\Gamma_K = G_K/H_K \cong \mathrm{Gal}(K_\infty/K)$ with an open subgroup of $\mathbf{Z}_p^\times$.

We write $\mathbf{C}$ for the p-adic completion of an algebraic closure of $\mathbf{Q}_p$, and $\widehat{K}_\infty$ for the p-adic completion of K_∞. Both of these are perfectoid fields. We choose a compatible system of roots of unity $\zeta_p, \zeta_{p^2}, \ldots$ and let $\varepsilon = (\zeta_p, \zeta_{p^2}, \ldots)$. We let $\varpi = \varepsilon - 1$; it is a pseudouniformizer of both $\widehat{K}_\infty^\flat$ and $\mathbf{C}^\flat$, and has valuation $p/p-1$.

By a valuation on a ring R, we mean a map $\mathrm{val}_R : R \to (-\infty, \infty]$ satisfying the following properties for $x, y \in R$:

1. $\mathrm{val}_R(x) = \infty$ if and only if $x = 0$ (i.e. R is separated);
2. $\mathrm{val}_R(xy) \geq \mathrm{val}_R(x) + \mathrm{val}_R(y)$;
3. $\mathrm{val}_R(x+y) \geq \min(\mathrm{val}_R(x), \mathrm{val}_R(y))$.

Occasionally, we also allow $\mathrm{val}_R(x) = -\infty$; we shall point out when this is the case. Given a matrix M with coefficients in R, we let $\mathrm{val}_R(M)$ be the minimum of the valuation of its entries.

Whenever we introduce a module or a ring as an inverse limit (resp. direct limit or localization) of topological modules or rings, we always endow it with the inverse limit topology (resp. direct limit topology[4]) unless otherwise stated.

If R is a topological ring endowed with continuous, commuting (φ, Γ_K)-actions, then:

- an étale φ-module M over R is a finitely generated R-module together with a φ-semilinear continuous map $\varphi : M \to M$, such the linearized morphism $\varphi^* M \to M$ is an isomorphism.
- an étale (φ, Γ_K)-module M over R is an étale φ-module together with a continuous semilinear action of Γ_K such that the actions of φ and Γ_K commute.

The ring A is a p-adically complete $\mathbf{Z}_p$-algebra which is topologically of finite type. In §2, we allow it to merely be a $\mathbf{Z}_p$-module.

22.5 Acknowledgments

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[4]Recall that the direct limit topology on a direct limit $X = \varinjlim X_i$ of topological spaces is the finest topology on X for which each $X_i \to X$ is continuous.

Gee for their helpful comments on the first draft of this article. Next, I want to thank Lie Qian for informing me of the paper [Bel20] during the Arizona winter school of 2022. Special thanks goes to Ehud de Shalit, who advised me when I was a master student, for teaching me tricks you can do with the Frobenius map. The conjecture regarding the existence of a natural map from $\mathcal{X}_d^{\mathrm{rig}}$ to $\mathfrak{X}_d$ was presented in a course given by Matthew Emerton, Toby Gee and Eugen Hellmann in the IHES summer school on the Langlands program, in which the author was present. It is during the school that the key ideas of the proof were understood. I would like to thank the organizers of the school for the pleasant event, and the institution for its hospitality.

23 Coefficient rings

In this section we introduce the various modules and rings which will play a role in this article and establish some of their properties.

23.1 The modules and rings

We introduce the following objects.

- Let A be a p-adically complete $\mathbf{Z}_p$-module which is topologically of finite type.
- Set $\widetilde{\mathbf{A}}^+ = \mathbf{A}_{\mathrm{inf}} := W(\mathcal{O}_{\mathbf{C}}^\flat)$ and $\widetilde{\mathbf{A}} := W(\mathbf{C}^\flat)$. We endow $\widetilde{\mathbf{A}}^+$ with its $(p, [\varpi])$-topology and $\widetilde{\mathbf{A}}$ with the topology making $W(\mathcal{O}_{\mathbf{C}}^\flat)$ into an open subring.
- For $1/r \in \mathbf{Z}[1/p]_{>0}$ we write

  $$\widetilde{\mathbf{A}}^{(0,r],\circ} := \left\{ \sum_{k\geq 0} p^k[x_k] \in \widetilde{\mathbf{A}} : 0 \leq \mathrm{val}_{\mathbf{C}^\flat}(x_k) + (pr/p-1)k \to \infty \right\}.$$

  According to [CC98, Rem. II.1.3], this ring may be equivalently defined as $\widetilde{\mathbf{A}}^+ \left\langle p/[\varpi]^{1/r} \right\rangle$, the p-adic completion of $\widetilde{\mathbf{A}}^+[p/[\varpi]^{1/r}]$. We endow the ring $\widetilde{\mathbf{A}}^{(0,r],\circ}$ with its $[\varpi]$-adic topology, or what is the same, the topology defined by the valuation

  $$\mathrm{val}^{(0,r]}(x) := (p/p-1)\mathrm{sup}\left\{ t \in \mathbf{Z}[1/p] : x \in [\varpi]^t \widetilde{\mathbf{A}}^{(0,r],\circ} \right\}.$$

  One checks that

  $$\mathrm{val}^{(0,r]}(\sum_{k\geq 0} p^k[x_k]) = \mathrm{inf}_k[\mathrm{val}_{\mathbf{C}^\flat}(x_k) + (pr/p-1)k].$$

- For $1/r \in \mathbf{Z}[1/p]_{>0}$ we set

$$\widetilde{\mathbf{A}}^{(0,r]} := \left\{ x = \sum_{k \gg -\infty} p^k [x_k] \in \widetilde{\mathbf{A}} : \mathrm{val}_{\mathbf{C}^\flat}(x_k) + (pr/p-1)k \to \infty \right\}.$$

  Equivalently, $\widetilde{\mathbf{A}}^{(0,r]} = \widetilde{\mathbf{A}}^{(0,r],\circ}[1/[\varpi]]$. We endow $\widetilde{\mathbf{A}}^{(0,r]}$ with the topology which makes $\widetilde{\mathbf{A}}^{(0,r],\circ}$ into an open subring. This topology is the same as that defined by the valuation

$$\mathrm{val}^{(0,r]}(x) := (p/p-1)\sup\left\{ t \in \mathbf{Z}[1/p] : x \in [\varpi]^t \widetilde{\mathbf{A}}^{(0,r],\circ} \right\}.$$

- For $r = \infty$ we set $\widetilde{\mathbf{A}}^{(0,\infty),\circ} = \widetilde{\mathbf{A}}^+$ with its $(p,[\varpi])$-topology and $\widetilde{\mathbf{A}}^{(0,\infty)} = \widetilde{\mathbf{A}}^+[1/[\varpi]]$ with its p-adic topology.

For each of the rings introduced above, there are versions with coefficients in A, which we introduce next. In general, these are just topological $\mathbf{Z}_p$-modules, but if A is a $\mathbf{Z}_p$-algebra, they all become topological $\mathbf{Z}_p$-algebras. We have:

- For $1/r \in \mathbf{Z}[1/p]_{>0}$,

$$\widetilde{\mathbf{A}}_A^{(0,r],\circ} = \widetilde{\mathbf{A}}^{(0,r],\circ} \widehat{\otimes}_{\mathbf{Z}_p} A := \varprojlim_i (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A)/[\varpi^i]$$

  and $\widetilde{\mathbf{A}}_A^{(0,r]} = \widetilde{\mathbf{A}}_A^{(0,r],\circ}[1/[\varpi]]$

- $\widetilde{\mathbf{A}}_A^{(0,r],+} :=$ the image of $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$ in $\widetilde{\mathbf{A}}_A^{(0,r]}$, endowed with its subspace topology. (The map $\widetilde{\mathbf{A}}_A^{(0,r],\circ} \to \widetilde{\mathbf{A}}_A^{(0,r]}$ may not be injective if A has p-torsion).

- For $a \geq 1$ the a-typical Witt vectors $W_a(\mathcal{O}_{\mathbf{C}}^\flat) = W(\mathcal{O}_{\mathbf{C}}^\flat)/p^a$, with the $[\varpi]$-adic topology.

- For $a \geq 1$,

$$W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A = W_a(\mathcal{O}_{\mathbf{C}}^\flat) \widehat{\otimes}_{\mathbf{Z}_p} A := \varprojlim_i (W_a(\mathcal{O}_{\mathbf{C}}^\flat) \otimes_{\mathbf{Z}_p} A)/[\varpi^i].$$

  .

- For $r = \infty$, $\widetilde{\mathbf{A}}_A^+ = \widetilde{\mathbf{A}}_A^{(0,\infty),\circ} := W(\mathcal{O}_{\mathbf{C}}^\flat)_A = \varprojlim_a W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A$ and $\widetilde{\mathbf{A}}_A^{(0,\infty)} = W(\mathcal{O}_{\mathbf{C}}^\flat)_A[1/[\varpi]]$.

- $\widetilde{\mathbf{A}}_A = W(\mathbf{C}^\flat)_A := \varprojlim_a W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A[1/[\varpi]]$.

- $\widetilde{\mathbf{A}}_A^\dagger := \varinjlim_r \widetilde{\mathbf{A}}_A^{(0,r]}$.

For each $R_A \in \{\widetilde{\mathbf{A}}_A^+, \widetilde{\mathbf{A}}_A, \widetilde{\mathbf{A}}_A^{(0,r]}, \widetilde{\mathbf{A}}_A^{(0,\infty]}, \widetilde{\mathbf{A}}_A^\dagger\}$ we introduce versions relative to K, by setting $R_{K,A} := (R_A)^{H_K}$. The following proposition shows that each of these can be defined in terms of $\widehat{K}_\infty$. Define $W(\mathcal{O}^\flat_{\widehat{K}_\infty})_A, W(\widehat{K}^\flat_\infty)_A$ in a similar way to the definition of $W(\mathcal{O}^\flat_{\mathbf{C}})_A$, $W(\mathbf{C}^\flat)_A$ respectively.

Proposition 23.1. *We have natural isomorphisms*
i. $\widetilde{\mathbf{A}}^+_{K,A} \cong W(\mathcal{O}^\flat_{\widehat{K}_\infty})_A$.
ii. $\widetilde{\mathbf{A}}_{K,A} \cong W(\widehat{K}^\flat_\infty)_A$.
iii. $\widetilde{\mathbf{A}}^{(0,r]}_{K,A} \cong (\widetilde{\mathbf{A}}^+_K \langle p/[\varpi]^{1/r} \rangle \widehat{\otimes} A)[1/[\varpi]]$.
iv. $\widetilde{\mathbf{A}}^{(0,\infty)}_{K,A} \cong W(\mathcal{O}^\flat_{\widehat{K}_\infty})_A [1/[\varpi]]$.
v. $\widetilde{\mathbf{A}}^\dagger_{K,A} = \varinjlim_r (\widetilde{\mathbf{A}}^+_K \langle p/[\varpi]^{1/r} \rangle \widehat{\otimes} A)[1/[\varpi]]$.
Here, in iii and v, the tensor products are completed with respect to the $[\varpi]$*-adic topology.*

Proof. It is clear that $W(\mathcal{O}^\flat_{\widehat{K}_\infty}) = W(\mathcal{O}^\flat_{\mathbf{C}})^{H_K}$ and that $[\varpi]$ is fixed by H_K. Taking fixed points commutes with inverse limits so this proves parts i and ii. In addition, i implies iv and iii implies v. It remains to prove iii. For ease of notation, we write H for H_K. We prove this by showing two claims.
Claim 1. The natural map $\widetilde{\mathbf{A}}^{(0,r],\circ,H} \widehat{\otimes} A := (\widetilde{\mathbf{A}}^{(0,r],\circ,H} \otimes A)^\wedge_{[\varpi]} \to (\widetilde{\mathbf{A}}^{(0,r],\circ}_A)^H$ is injective.
To prove this, we argue in steps.
Step 1. The natural maps $\widetilde{\mathbf{A}}^{(0,r],\circ,H}/p \to \widetilde{\mathbf{A}}^{(0,r],\circ}/p$ and $\mathcal{O}^{\flat,H}_{\mathbf{C}}[X]/X\varpi^{1/r} \to \mathcal{O}^\flat_{\mathbf{C}}[X]/X\varpi^{1/r}$ are injective.
Indeed, the cokernel of $\widetilde{\mathbf{A}}^{(0,r],\circ,H} \to \widetilde{\mathbf{A}}^{(0,r],\circ}$ (resp of $\mathcal{O}^{\flat,H}_{\mathbf{C}}[X] \to \mathcal{O}^\flat_{\mathbf{C}}[X]$) is p-torsionfree (resp is $X\varpi^{1/r}$-torsionfree).
Step 2. The natural isomorphism $\widetilde{\mathbf{A}}^{(0,r],\circ}/p \xrightarrow{\sim} \mathcal{O}^\flat_{\mathbf{C}}[X]/X\varpi^{1/r}$ induces a natural isomorphism $\widetilde{\mathbf{A}}^{(0,r],\circ,H}/p \xrightarrow{\sim} \mathcal{O}^{\flat,H}_{\mathbf{C}}[X]/X\varpi^{1/r}$.
For this, consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbf{A}}^{(0,r],\circ,H}/p & \longrightarrow & \mathcal{O}^{\flat,H}_{\mathbf{C}}[X]/X\varpi^{1/r} \\ \downarrow & & \downarrow \\ \widetilde{\mathbf{A}}^{(0,r],\circ}/p & \longrightarrow & \mathcal{O}^\flat_{\mathbf{C}}[X]/X\varpi^{1/r} \end{array}.$$

The bottom horizontal map is an isomorphism, hence injective. The two vertical maps are also injective, by step 1. Hence $\widetilde{\mathbf{A}}^{(0,r],\circ,H}/p \to \mathcal{O}^{\flat,H}_{\mathbf{C}}[X]/X\varpi^{1/r}$ is injective. For surjectivity, it suffices to show that $\mathbf{A}^H_{\mathrm{inf}} \to \mathcal{O}^{\flat,H}_{\mathbf{C}}$ is surjective, which is clear, for example because the Teichmuller map gives a section.
Step 3. Set $S = \widetilde{\mathbf{A}}^{(0,r],\circ}/\widetilde{\mathbf{A}}^{(0,r],\circ,H}$. Then S is p-torsionfree.

Step 4. If A is killed by p, we claim that the ϖ^∞-torsion in $K \otimes A$ is $\varpi^{1/r}$-torsion.

Indeed, choose an isomorphism $A \cong \bigoplus_{i\in I} \mathbf{F}_p$. By step 3, we have an exact sequence

$$0 \to \widetilde{\mathbf{A}}^{(0,r],\circ,H} \otimes A \to \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A \to S \otimes A \to 0,$$

which by step 2 is isomorphic to

$$0 \to \bigoplus_{i\in I} \mathcal{O}_{\mathbf{C}}^{\flat,H}[X]/X\varpi^{1/r} \to \bigoplus_{i\in I} \mathcal{O}_{\mathbf{C}}^{\flat}[X]/X\varpi^{1/r} \to$$

$$\bigoplus_{i\in I} \mathcal{O}_{\mathbf{C}}^{\flat}[X]/(\mathcal{O}_{\mathbf{C}}^{\flat,H}[X], X\varpi^{1/r}) \to 0,$$

so this is clear.

Step 5. If A is killed by p^N, we claim that the $[\varpi]^\infty$-torsion in $K \otimes A$ is killed by $[\varpi]^{N/r}$.

To see this, step 3 to obtain an exact sequence

$$0 \to S \otimes pA \to S \otimes A \to S \otimes A/p \to 0$$

which implies the statement by an obvious devissage, using step 4.

Step 6. If A is p-torsionfree, we claim that $S \otimes A$ is $[\varpi]$-torsionfree.

This follows from flat base change, because

$$\mathrm{Tor}_1^{\mathbf{A}_{\mathrm{inf}}}(\mathbf{A}_{\mathrm{inf}}/[\varpi], K) \cong \mathrm{Tor}_1^{\mathbf{A}_{\mathrm{inf}}\otimes A}((\mathbf{A}_{\mathrm{inf}} \otimes A)/[\varpi], S \otimes A).$$

As $[\varpi]$ is a nonzero divisor in $\mathbf{A}_{\mathrm{inf}} \otimes A$ and S is $[\varpi]$-torsionfree, the claim follows.

Step 7. Since A is noetherian, we have $A[p^\infty] = A[p^N]$ for $N \gg 0$. Since $A/A[p^N]$ is p-torsionfree, we have an exact sequence

$$0 \to S \otimes A[p^N] \to S \otimes A \to S \otimes A/A[p^N] \to 0,$$

which, combining steps 5 and 6, shows that $S \otimes A$ is bounded $[\varpi]$-torsion.

Step 8. Finally, we have an exact sequence

$$0 \to \widetilde{\mathbf{A}}^{(0,r],\circ,H} \otimes A \to \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A \to S \otimes A \to 0.$$

Since $K \otimes A$ has bounded $[\varpi]$-torsion, it follows from Lemma 2.2 below that the sequence remains exact after $[\varpi]$-adic completion. Hence the map $\widetilde{\mathbf{A}}^{(0,r],\circ,H}\widehat{\otimes} A \to \widetilde{\mathbf{A}}_A^{(0,r],\circ}$ is injective, which concludes the proof of claim 1.

Claim 2. The map $\widetilde{\mathbf{A}}^{(0,r],\circ,H}\widehat{\otimes} A \to (\widetilde{\mathbf{A}}_A^{(0,r],\circ})^H$ is almost surjective.

Since both sides are $[\varpi]$-adically complete, is enough to prove almost surjectivity mod $[\varpi]^{1/r}$. We have a natural isomorphism

$$\widetilde{\mathbf{A}}_A^{(0,r],\circ}/[\varpi]^{1/r} \cong \mathcal{O}_{\mathbf{C}}^\flat/\varpi^{1/r}[X] \otimes_{\mathbf{F}_p} A/p.$$

Since A/p is an $\mathbf{F}_p$-vector space, it is free over $\mathbf{F}_p$, and so

$$(\widetilde{\mathbf{A}}_A^{(0,r],\circ}/[\varpi]^{1/r})^H = (\mathcal{O}_{\mathbf{C}}^\flat/\varpi^{1/r}[X])^H \otimes_{\mathbf{F}_p} A/p.$$

As $\widetilde{\mathbf{A}}^{(0,r],\circ,H}\widehat{\otimes}A$ surjects onto $\mathcal{O}_{\mathbf{C}}^{\flat,H}/\varpi^{1/r}[X] \otimes_{\mathbf{F}_p} A/p$ (as follows from step 2 in the proof of the previous claim), it suffices to prove that $\mathcal{O}_{\mathbf{C}}^{\flat,H}/\varpi^{1/r}$ almost surjects onto $(\mathcal{O}_{\mathbf{C}}^\flat/\varpi^{1/r})^H$. This follows from $\mathrm{H}^1(H, \varpi\mathcal{O}_{\mathbf{C}}^\flat)$ being almost zero, which is [Co98, Lem. IV.2.3].

Combining both of the claims, we deduce that there is a natural isomorphism $(\widetilde{\mathbf{A}}_A^{(0,r]})^H \cong \widetilde{\mathbf{A}}^{(0,r],H}\widehat{\otimes}A$. This is simply different notation for $\widetilde{\mathbf{A}}_{K,A}^{(0,r]} \cong (\widetilde{\mathbf{A}}_K^+ \langle p/[\varpi]^{1/r}\rangle \widehat{\otimes}A)[1/[\varpi]]$, and so proves part *iii* of the proposition. ☐

Finally, we have imperfect versions of the rings relative to K as above. They are defined as follows.

- We have standard rings $\mathbf{A}_K^+$ and $\mathbf{A}_K$, which are defined in §2.1 of [EG19] (where they are denoted by $(\mathbf{A}'_K)^+$ and $\mathbf{A}'_K$). We have a certain element $T \in \mathbf{A}_K^+$ lifting ϖ, denoted[5] by $T_{\mathbf{Q}_p}$ in loc. cit. Note that by the theory of the field of norms, we have canonical embeddings $\mathbf{A}_K^+ \hookrightarrow \widetilde{\mathbf{A}}$ and $\mathbf{A}_K \hookrightarrow \widetilde{\mathbf{A}}$ which map T to $[\varepsilon] - 1$.
- We set[6]

$$\mathbf{A}_{K,A}^+ = \mathbf{A}_K^+\widehat{\otimes}_{\mathbf{Z}_p}A := \varprojlim_m(\varprojlim_n \mathbf{A}_K^+/(p^m, T^n) \otimes_{\mathbf{Z}_p} A),$$

  endowed with the inverse limit topology, and

$$\mathbf{A}_{K,A} = \mathbf{A}_K\widehat{\otimes}_{\mathbf{Z}_p}A := \varprojlim_m(\varprojlim_n(\mathbf{A}_K^+/(p^m, T^n) \otimes_{\mathbf{Z}_p} A)[1/T]).$$

- We let $\mathbf{A}_K^{(0,r],\circ} = \mathbf{A}_K \cap \widetilde{\mathbf{A}}^{(0,r],\circ}$ and

$$\mathbf{A}_{K,A}^{(0,r],\circ} = \varprojlim_i(\mathbf{A}_K^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A)/T^i.$$

- Let $\mathbf{A}_{K,A}^{(0,r]} := \mathbf{A}_{K,A}^{(0,r],\circ}[1/T]$ and write $\mathbf{A}_{K,A}^{(0,r],+}$ for the image of $\mathbf{A}_{K,A}^{(0,r],\circ}$ in $\mathbf{A}_{K,A}^{(0,r]}$.
- Finally, let $\mathbf{A}_{K,A}^\dagger := \varinjlim_r \mathbf{A}_{K,A}^{(0,r]}$.

[5]More generally there is also an element T_K defined in loc. cit. when K is an unramified extension of $\mathbf{Q}_p$. We shall not need it for our arguments, however.

[6]Compare with §2.2 of loc. cit. Note that there, T can be used in the corresponding definitions instead of T_K.

23.2 Torsion and completions

The main goal of this subsection is to construct a natural continuous map $\widetilde{\mathbf{A}}_A^{(0,r]} \to \widetilde{\mathbf{A}}_A$ and to prove it is injective. This will allow us to prove that the direct limits defining $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ and $\mathbf{A}^{\dagger}_{K,A}$ have injective transition maps.

To establish basic properties of the modules and rings introduced above, it will be necessary to prove that certain completion operations are well behaved. Our rings will usually be nonnoetherian, so such results are not automatic in our setting. However, it will turn out that in the situations we consider here the torsion appearing is bounded. Fortunately, this weaker finiteness condition will suffice for controlling the completions appearing in this article by virtue of the following simple lemma.

Lemma 23.2. *Let R be a ring, x a nonzerodivisor of R, and $0 \to M \to N \to P \to 0$ a short exact sequence of R-modules.*

If P has bounded x-torsion then the x-adic completion $0 \to M^{\wedge}_x \to N^{\wedge}_x \to P^{\wedge}_x \to 0$ is exact.

Proof. The exactness on the right is automatic by Nakayama's lemma.[7]

We now turn to explain the exactness elsewhere. Firstly, we note that since x is a nonzerodivisor, x^n is also a nonzerodivisor. Replacing x by its own power, we may and do assume $P[x] = P[x^\infty]$. Next, we observe that $\mathrm{Tor}_1(R/x^n, P) = P[x^n]$, and the map $\mathrm{Tor}_1(R/x^n, P) \to \mathrm{Tor}_1(R/x^{n-1}, P)$ induced from $R/x^n \to R/x^{n-1}$ corresponds to the map $P[x^n] \xrightarrow{x} P[x^{n-1}]$.

With this given, we have exact sequences

$$P[x^n] \to M/x^n \to N/x^n \to P/x^n \to 0.$$

Let K_n be the image of $P[x^n]$ in M/x^n. Then we have short exact sequences

$$0 \to (M/x^n)/K_n \to N/x^n \to P/x^n \to 0,$$

and taking the inverse limit we obtain an exact sequence

$$0 \to \varprojlim_n (M/x^n)/K_n \to N^{\wedge}_x \to P^{\wedge}_x.$$

It remains to show that $\varprojlim_n (M/x^n) \to \varprojlim_n (M/x^n)/K_n$ is an isomorphism. To show this, it suffices to show that $\varprojlim_n K_n$ and $\mathrm{R}^1 \varprojlim_n K_n$ both vanish. Since $P[x^\infty] = P[x]$, the transition maps $f_{n+1} : K_{n+1} \to K_n$, which are induced from the multiplication by x from $P[x^{n+1}]$ to $P[x^n]$, are all 0. This implies that the complex

$$\prod_{n\geq 1} K_n \to \prod_{n \geq 1} K_n$$

[7] This does not require the modules to be assumed finitely generated, see [Sta, Tag 0315 (2)]

$$(x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$

is exact. This complex computes $\mathrm{R}^i \varprojlim_n K_n$, so we are done. □

The next proposition allows us to control torsion.

Proposition 23.3. *Let v be an element of the maximal ideal of $\mathcal{O}_{\mathbf{C}}^\flat$.*
i. A has bounded p-torsion.
ii. $\mathbf{A}_{\mathrm{inf}}/[v]$ is p-torsionfree.
iii. $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A$ is $[v]$-torsionfree.
iv. $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ has bounded $[v]$-torsion.
v. $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$ has bounded $[v]$-torsion.

Proof. i. The module $\mathbf{Z}_p$-A is noetherian, since it is topologically of finite type as a $\mathbf{Z}_p$-module. It therefore has bounded p-torsion.
ii. Let $x \in \mathbf{A}_{\mathrm{inf}}$ and suppose that $px \in [v]\mathbf{A}_{\mathrm{inf}}$. If we write $x = \sum_{i\geq 0}[x_i]p^i$ for the Teichmuller expansion then $px = \sum_{i\geq 0}[x_i]p^{i+1} \in [v]\mathbf{A}_{\mathrm{inf}}$, which implies by uniqueness of the expansion that $x_i \in v\mathcal{O}_{\mathbf{C}}^\flat$, hence x itself is divisible by $[v]$.
iii. The ring $\mathbf{A}_{\mathrm{inf}}$ is $[v]$-torsionfree and $\mathbf{Z}_p$-flat. Tensoring the exact sequence

$$0 \to \mathbf{A}_{\mathrm{inf}} \xrightarrow{[v]} \mathbf{A}_{\mathrm{inf}} \to \mathbf{A}_{\mathrm{inf}}/[v] \to 0$$

with A, we see that the $[v]$-torsion in $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A$ is isomorphic to

$$\mathrm{Tor}_1^{\mathbf{Z}_p}(A, \mathbf{A}_{\mathrm{inf}}/[v]).$$

This vanishes by *ii*.
iv. The ring $\widetilde{\mathbf{A}}^{(0,r],\circ}$ ihas no p or $[v]$-torsion. This is because it is a subring of $\widetilde{\mathbf{A}} = W(\mathbf{C}^\flat)$, which has these properties. Tensoring the exact sequence

$$0 \to \widetilde{\mathbf{A}}^{(0,r],\circ} \xrightarrow{[v]} \widetilde{\mathbf{A}}^{(0,r],\circ} \to \widetilde{\mathbf{A}}^{(0,r],\circ}/[v] \to 0$$

with A shows that there is a natural isomorphism between the $[v]$-torsion in $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ and $\mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[v], A)$. Then for $N \gg 0$ we have

$$\mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[v], A) \cong \mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[v], A[p^\infty]) = \mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[v], A[p^N]),$$

so we deduce that the $[v]$-torsion in $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ is isomorphic to the $[v]$-torsion in $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A[p^N]$.
Replacing A by $A[p^N]$, we may assume $p^N A = 0$. We now prove the $[v]$-torsion is bounded by induction on N. We may assume $v = \varpi$. When

$N = 1$, we have $pA = 0$, so A is an $\mathbf{F}_p$-vector space. Upon choosing a basis and using the isomorphism

$$\widetilde{\mathbf{A}}^{(0,r],\circ}/p \cong \mathcal{O}_{\mathbf{C}}^{\flat}[X]/(X\varpi^{1/r}),$$

we see that the ϖ^{∞}-torsion in $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ is killed by $\varpi^{1/r}$.

For general N, we see that the ϖ^{∞}-torsion in $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ is killed by $\varpi^{N/r}$ by devissage through use of the exact sequence

$$0 \to (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A[p]) \to (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A) \xrightarrow{p} (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} pA) \to 0.$$

v. As in iv there is $N \gg 0$ such that $A[p^N] = A[p^{\infty}]$, and we may assume that $v = \varpi$. Consider $M \geq N/r$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[\varpi]^{N/r}, A) & \longrightarrow & \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A & \xrightarrow{[\varpi]^{N/r}} & [\varpi]^{N/r}\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle [\varpi]^{M-N/r}} & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[\varpi]^{M}, A) & \longrightarrow & \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A & \xrightarrow{[\varpi]^{M}} & [\varpi]^{M}\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A & \longrightarrow & 0 \end{array}$$

We claim that the leftmost map is an equality. To see this, notice that by the argument proving *iv*, we have natural isomorphisms

$$\mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[\varpi]^{N/r}, A) \cong \mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[\varpi]^{N/r}, A[p^N])$$

and similarly

$$\mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[\varpi]^{M}, A) \cong \mathrm{Tor}_1^{\mathbf{Z}_p}(\widetilde{\mathbf{A}}^{(0,r],\circ}/[\varpi]^{M}, A[p^N])$$

and we know the two respective right hand sides are equal by what was proven in *iv*.

It follows from the snake lemma that the commutative diagram gives an isomorphism between the two rows. Now again by iv, the $[\varpi]^{\infty}$-torsion in $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ is bounded. So by Lemma 2.2, taking the $[\varpi]$-completion of both rows is exact. In addition, by [Sta, 05GG], the $[\varpi]$-adic completion of $[\varpi]^k\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes_{\mathbf{Z}_p} A$ for any k is equal to $[\varpi]^k\widetilde{\mathbf{A}}_A^{(0,r],\circ}$. We deduce that the $[\varpi]^{N/r}$-torsion in $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$ is equal to the $[\varpi]^M$ -torsion in $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$ for $M \geq N/r$. This proves the $[\varpi]^{\infty}$-torsion is bounded in $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$. □

We record the following result that will be used later in §5.

Corollary 23.4. *For $1/r \in \mathbf{Z}[1/p]_{>0}$ the topology on $\widetilde{\mathbf{A}}_A^{(0,r]}$ is defined by the valuation given by*

$$\mathrm{val}^{(0,r]}(x) = (p/p-1)\sup\{t \in \mathbf{Z}[1/p] : x \in [\varpi]^t\widetilde{\mathbf{A}}_A^{(0,r],+}\}.$$

Proof. It follows from the definitions that $\widetilde{\mathbf{A}}_A^{(0,r]}$ has $\widetilde{\mathbf{A}}_A^{(0,r],+}$ as an open subring, for which the topology is $[\varpi]$-adic. The only thing left to check is that $\widetilde{\mathbf{A}}_A^{(0,r],+}$ is $[\varpi]$-adically separated, so that $\mathrm{val}^{(0,r]}$ defines a valuation. But for $N \gg 0$ we have by Proposition 2.3.v that $\widetilde{\mathbf{A}}_A^{(0,r],\circ}[[\varpi]^\infty] = \widetilde{\mathbf{A}}_A^{(0,r],\circ}[[\varpi]^N]$, so that

$$\widetilde{\mathbf{A}}_A^{(0,r],+} = \widetilde{\mathbf{A}}_A^{(0,r],\circ}/\widetilde{\mathbf{A}}_A^{(0,r],\circ}[[\varpi]^N],$$

with $\widetilde{\mathbf{A}}_A^{(0,r],\circ}[[\varpi]^N]$ closed. Since $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$ is $[\varpi]$-adically separated, the corollary follows. $\square$

Next, we define two p-adically completed $\mathbf{Z}_p$-modules

$$\mathbf{A}_{\mathrm{inf},A} := \varprojlim_a (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)/p^a,$$

$$\mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle := \varprojlim_a (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)\left[p/[\varpi]^{1/r}\right]/p^a,$$

which are rings if A is. Clearly, there is a map $\mathbf{A}_{\mathrm{inf},A} \to \mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle$ which is continuous with respect to the p-adic topology.

These two will play an auxiliary role in what follows. We shall need these as it will be easier to construct maps out of them, and then later extend these to maps to the objects we are concerned with. More precisely, note that $\mathbf{A}_{\mathrm{inf},A}$ is quite close to $\widetilde{\mathbf{A}}_A^{(0,\infty],\circ} = W(\mathcal{O}_F^\flat)_A$ while $\mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle$ is close to being equal to $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$, with these latter rings being those of true importance. The subtle differences in the two pairs occur because of the distinction between the $[\varpi]$-adic, p-adic and $(p,[\varpi])$-adic completions.

Lemma 23.5. *$[\varpi]$-adic completion induces an isomorphism*

$$\mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle^\wedge_{[\varpi]} \cong \widetilde{\mathbf{A}}_A^{(0,r],\circ}.$$

Proof. Recall that $\widetilde{\mathbf{A}}_A^{(0,r],\circ}$ is defined as the $[\varpi]$-adic completion of

$$\mathbf{A}_{\mathrm{inf}}\left\langle p/[\varpi]^{1/r}\right\rangle \otimes_{\mathbf{Z}_p} A.$$

We have

$$\begin{aligned}\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle^\wedge_{[\varpi]} &= \left[(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)\left[\frac{p}{[\varpi]^{1/r}}\right]^\wedge_p\right]^\wedge_{[\varpi]} \\ &\cong (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)\left[\frac{p}{[\varpi]^{1/r}}\right]^\wedge_{[\varpi]}.\end{aligned}$$

Further,

$$
\begin{aligned}
(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^a &\cong (\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)[X]/(X[\varpi]^{1/r}-p)/[\varpi]^a\\
&\cong (\mathbf{A}_{\inf}[X]/(X[\varpi]^{1/r}-p)\otimes_{\mathbf{Z}_p}A)/[\varpi]^a\\
&\cong (\mathbf{A}_{\inf}\left[\frac{p}{[\varpi]^{1/r}}\right]\otimes_{\mathbf{Z}_p}A)/[\varpi]^a.
\end{aligned}
$$

Taking the limit over a we obtain the desired isomorphism. □

Lemma 23.6. *The natural map $\mathbf{A}_{\inf,A}\to\mathbf{A}_{\inf,A}\left\langle p/[\varpi]^{1/r}\right\rangle$ is injective.*

Proof. We start by showing that $\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A\to(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)[p/[\varpi]^{1/r}]$ is injective. It suffices to show that

$$(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)\cap(X[\varpi]^{1/r}-p)(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)[X]=0,$$

where the intersection is taken in $(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)[X]$. Indeed, suppose $f=f(X)$ is in the intersection, then we may write

$$f(X)=(X[\varpi]^{1/r}-p)g(X)$$

with $g(X)=a_0+\dots+a_dX^d\in(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)[X]$ and $d\geq 0$, with $a_d\neq 0$ unless $g(X)=0$. Since $f\in\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A$, the coefficient of X^{d+1} in $f(X)$ is 0, which gives $a_d[\varpi]^{1/r}=0$ in $\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A$. By 2.3.*iii*, the ring $\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A$ is $[\varpi]^{1/r}$-torsionfree so we must have $a_d=0$ which means $g(X)=0$, and hence $f(X)=0$.
Now if $p^aA=0$ the proposition holds because what we have just shown, since in this case $\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A=\mathbf{A}_{\inf,A}$ and $(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)[p/[\varpi]^{1/r}]=\mathbf{A}_{\inf,A}\left\langle p/[\varpi]^{1/r}\right\rangle$. In general, the map $\mathbf{A}_{\inf,A}\to\mathbf{A}_{\inf,A}\left\langle p/[\varpi]^{1/r}\right\rangle$ is obtained by taking the inverse limit over a of the injective maps $\mathbf{A}_{\inf,A/p^a}\to\mathbf{A}_{\inf,A/p^a}\left\langle p/[\varpi]^{1/r}\right\rangle$, so it is injective. This concludes the proof. □

We may now construct a natural map $\widetilde{\mathbf{A}}_A^{(0,r]}\to\widetilde{\mathbf{A}}_A$ as follows. First, we have a natural map $\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A\to W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A\left[\frac{1}{[\varpi]}\right]$, defined as the composition

$$\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A=W(\mathcal{O}_{\mathbf{C}}^\flat)\otimes_{\mathbf{Z}_p}A\to W_a(\mathcal{O}_{\mathbf{C}}^\flat)\otimes_{\mathbf{Z}_p}A\to W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A\to W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A\left[\frac{1}{[\varpi]}\right].$$

This induces a map

$$(\mathbf{A}_{\inf}\otimes_{\mathbf{Z}_p}A)\left[\frac{p}{[\varpi]^{1/r}}\right]/p^a\to W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A\left[\frac{1}{[\varpi]}\right],$$

and taking limits as $a\to\infty$, we get a map $\mathbf{A}_{\inf,A}\left\langle p/[\varpi]^{1/r}\right\rangle\to\widetilde{\mathbf{A}}_A$, which is by construction continuous for the p-adic topology on $\mathbf{A}_{\inf,A}\left\langle p/[\varpi]^{1/r}\right\rangle$ and the natural topology on $\widetilde{\mathbf{A}}_A$. Recall this latter topology is the inverse limit topology induced from $\widetilde{\mathbf{A}}_A = \varprojlim_a W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A\,[1/[\varpi]]$, for which a basis of open neighborhoods of 0 in $\widetilde{\mathbf{A}}_A$ is given by $\{[\varpi]^{k/r}W(\mathcal{O}_{\mathbf{C}}^\flat)_A + p^k W(\mathbf{C}^\flat)_A\}_{k\geq 1}$. The construction of the map $\widetilde{\mathbf{A}}_A^{(0,r]} \to \widetilde{\mathbf{A}}_A$ is concluded by the Lemma 2.5 and the following lemma.

Lemma 23.7. *The map $\mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle \to \widetilde{\mathbf{A}}_A$ is continuous for the natural topology on $\widetilde{\mathbf{A}}_A$ and the $[\varpi]$-adic topology on $\mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle$.*

Proof. A basis of open neighborhoods of 0 in $\widetilde{\mathbf{A}}_A$ is given by $\{[\varpi]^{k/r}W(\mathcal{O}_{\mathbf{C}}^\flat)_A + p^k W(\mathbf{C}^\flat)_A\}_{k\geq 1}$. It suffices to show that

$$[\varpi]^{2k/r}\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle \subset [\varpi]^{k/r}\mathbf{A}_{\mathrm{inf},A} + p^k\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle$$

inside $\mathbf{A}_{\mathrm{inf},A}\left\langle p/[\varpi]^{1/r}\right\rangle$, because the right hand side maps to $[\varpi]^{k/r}W(\mathcal{O}_{\mathbf{C}}^\flat)_A + p^k W(\mathbf{C}^\flat)_A$. (We are implicitly invoking Lemma 2.6 to make sense of this inclusion).

To show this inclusion, start by observing

$$[\varpi]^{1/r}\mathbf{A}_{\mathrm{inf},A}\left[\frac{p}{[\varpi]^{1/r}}\right] \subset [\varpi]^{1/r}\mathbf{A}_{\mathrm{inf},A} + p\mathbf{A}_{\mathrm{inf},A}\left[\frac{p}{[\varpi]^{1/r}}\right]$$

$$\subset [\varpi]^{1/r}\mathbf{A}_{\mathrm{inf},A} + p\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle.$$

Since the right hand side is p-adically complete, we deduce that

$$[\varpi]^{1/r}\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle \subset [\varpi]^{1/r}\mathbf{A}_{\mathrm{inf},A} + p\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle.$$

Arguing inductively, we have

$$[\varpi]^{k/r}\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle \subset$$

$$[\varpi]^{k/r}\mathbf{A}_{\mathrm{inf},A} + p[\varpi]^{k-1/r}\mathbf{A}_{\mathrm{inf},A} + \ldots + p^{k-1}[\varpi]^{1/r}\mathbf{A}_{\mathrm{inf},A} + p^k\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle.$$

Hence,

$$[\varpi]^{2k/r}\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle \subset [\varpi]^{k/r}\mathbf{A}_{\mathrm{inf},A} + p^k\mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}}\right\rangle,$$

as required. ∎

Thus we have a natural continuous map $\widetilde{\mathbf{A}}_A^{(0,r]} \to \widetilde{\mathbf{A}}_A$.

Proposition 23.8. *i. Let $a \in \mathbf{Z}_{\geq 1}$. If $p^a A = 0$, then the kernel of $\widetilde{\mathbf{A}}_A^{(0,r],\circ} \to \widetilde{\mathbf{A}}_A$ is killed by $[\varpi]^{a/r}$.*

ii. If A is p-torsionfree, the map $\widetilde{\mathbf{A}}_A^{(0,r],\circ} \to \widetilde{\mathbf{A}}_A$ is injective.

Proof. i. Start with the case that $a = 1$, so that A is an $\mathbf{F}_p$-vector space. We may choose an isomorphism $A \cong \bigoplus_{i\in I} \mathbf{F}_p$. The map whose kernel we are considering is given by

$$(\mathbf{A}_{\mathrm{inf}} \widehat{\otimes}_{\mathbf{Z}_p} (\oplus_{i\in I} \mathbf{F}_p))[X]/(X[\varpi]^{1/r} - p) \to [\oplus_{i\in I} \mathcal{O}_{\mathbf{C}}^\flat] \left[\frac{1}{\varpi}\right],$$

or, more simply,

$$[\widehat{\oplus}_{i\in I} \mathcal{O}_{\mathbf{C}}^\flat][X]/(X\varpi^{1/r}) \to [\widehat{\oplus}_{i\in I} \mathcal{O}_{\mathbf{C}}^\flat] \left[\frac{1}{\varpi}\right],$$

which maps X to $\frac{p}{\varpi^{1/r}} = 0$. Hence the kernel is given by $X[\widehat{\oplus}_{i\in I} \mathcal{O}_{\mathbf{C}}^\flat][X]/(X\varpi^{1/r})$, which is $\varpi^{1/r}$-torsion.

In general, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & \mathbf{A}_{\mathrm{inf},pA}\left\langle \frac{p}{[\varpi]^{1/r}} \right\rangle^\wedge_{[\varpi]} & \longrightarrow & \mathbf{A}_{\mathrm{inf},A}\left\langle \frac{p}{[\varpi]^{1/r}} \right\rangle^\wedge_{[\varpi]} & \longrightarrow & \mathbf{A}_{\mathrm{inf},A/p}\left\langle \frac{p}{[\varpi]^{1/r}} \right\rangle^\wedge_{[\varpi]} \longrightarrow 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W_{a-1}(\mathcal{O}_{\mathbf{C}}^\flat)_{pA}\left[\frac{1}{[\varpi]}\right] & \longrightarrow & W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A\left[\frac{1}{[\varpi]}\right] & \longrightarrow & (\mathcal{O}_{\mathbf{C}}^\flat \otimes A/p)^\wedge_{[\varpi]}\left[\frac{1}{[\varpi]}\right]
\end{array}$$

Here, the top row is exact because it is given by first tensoring the exact sequence $0 \to pA \to A \to A/p \to 0$ with $\mathbf{A}_{\mathrm{inf}}[X]/(X[\varpi]^{1/r} - p)$ and then $[\varpi]$-completing. According to Proposition 2.3, the $[\varpi]$-torsion in $(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A/p)\left[p/[\varpi]^{1/r}\right]$ is bounded, so by Lemma 2.2 this latter operation preserves exactness.

The bottom row is exact because it is given by first tensoring the same exact sequence with the flat $\mathbf{Z}_p$-module $W(\mathcal{O}_{\mathbf{C}}^\flat)$, then completing $[\varpi]$-adically, and then inverting $[\varpi]$. The second step is exact: this again follows from Lemma 2.2, since $\mathcal{O}_{\mathbf{C}}^\flat \otimes A/p$ is $[\varpi]$-torsionfree.

With the exactness properties of the diagram established, we may use the snake lemma, from which *i* follows by induction on a.

ii. If $A = \mathbf{Z}_p$, this map is known to be injective. Indeed, $\widetilde{\mathbf{A}}^{(0,r],\circ}$ can be defined as a subring of $\widetilde{\mathbf{A}}$.

We shall now reduce to the case. Since A is p-torsionfree and p-adically complete, we may write $A \cong [\bigoplus_{i\in I} \mathbf{Z}_p]^\wedge_p$ as a $\mathbf{Z}_p$-module. We have:

$$
\begin{aligned}
\widetilde{\mathbf{A}}_A^{(0,r],\circ} &= \varprojlim_b (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^b \\
&\cong \varprojlim_b (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} [\bigoplus_{i\in I}\mathbf{Z}_p]_p^{\wedge})\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^b \\
&= \varprojlim_b (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} \bigoplus_{i\in I}\mathbf{Z}_p)\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^b \\
&= \varprojlim_b \bigoplus_{i\in I}\mathbf{A}_{\mathrm{inf}}\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^b \\
&\hookrightarrow \varprojlim_b \prod_{i\in I}\mathbf{A}_{\mathrm{inf}}\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^b \\
&= \prod_{i\in I}\varprojlim_b \mathbf{A}_{\mathrm{inf}}\left[\frac{p}{[\varpi]^{1/r}}\right]/[\varpi]^b = \prod_{i\in I}\widetilde{\mathbf{A}}^{(0,r],\circ}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\widetilde{\mathbf{A}}_A &= \varprojlim_a W_a(\mathcal{O}_{\mathbf{C}}^{\flat})_A[\frac{1}{[\varpi]}] \\
&= \varprojlim_a [\varprojlim_b (W_a(\mathcal{O}_{\mathbf{C}}^{\flat}) \otimes_{\mathbf{Z}_p} A)/[\varpi]^b][\frac{1}{[\varpi]}] \\
&\cong \varprojlim_a [\varprojlim_b (W_a(\mathcal{O}_{\mathbf{C}}^{\flat}) \otimes_{\mathbf{Z}_p} [\bigoplus_{i\in I}\mathbf{Z}_p]_p^{\wedge})/[\varpi]^b][\frac{1}{[\varpi]}] \\
&= \varprojlim_a [\varprojlim_b (W_a(\mathcal{O}_{\mathbf{C}}^{\flat}) \otimes_{\mathbf{Z}_p} \bigoplus_{i\in I}\mathbf{Z}_p)/[\varpi]^b][\frac{1}{[\varpi]}] \\
&= \varprojlim_a [\varprojlim_b \bigoplus_{i\in I} W_a(\mathcal{O}_{\mathbf{C}}^{\flat})/[\varpi]^b][\frac{1}{[\varpi]}] \\
&\hookrightarrow \varprojlim_a [\varprojlim_b \prod_{i\in I} W_a(\mathcal{O}_{\mathbf{C}}^{\flat})/[\varpi]^b][\frac{1}{[\varpi]}] \\
&= \varprojlim_a [\prod_{i\in I}\varprojlim_b W_a(\mathcal{O}_{\mathbf{C}}^{\flat})/[\varpi]^b][\frac{1}{[\varpi]}] \\
&= \varprojlim_a [\prod_{i\in I} W_a(\mathcal{O}_{\mathbf{C}}^{\flat})][\frac{1}{[\varpi]}] \hookrightarrow \varprojlim_a \prod_{i\in I} W_a(\mathcal{O}_{\mathbf{C}}^{\flat})[\frac{1}{[\varpi]}] \\
&= \prod_{i\in I}\varprojlim_a W_a(\mathcal{O}_{\mathbf{C}}^{\flat})[\frac{1}{[\varpi]}] = \prod_{i\in I}\widetilde{\mathbf{A}}.
\end{aligned}
$$

We therefore have a commuative diagram

$$\begin{array}{ccc} \widetilde{\mathbf{A}}_A^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_A \\ \downarrow & & \downarrow \\ \prod_{i\in I}\widetilde{\mathbf{A}}^{(0,r],\circ} & \longrightarrow & \prod_{i\in I}\widetilde{\mathbf{A}} \end{array}$$

where all the maps are have been shown to be injective, except possibly the top horizontal map. It follows that it is injective also, concluding the proof. □

Finally, we have the following result.

Theorem 23.9. *There exists a natural, continuous map* $\widetilde{\mathbf{A}}_A^{(0,r]} \to \widetilde{\mathbf{A}}_A$. *It is injective.*

Proof. It remains to show this map is injective. Recall, by Proposition 2.3.i that $A[p^N] = A[p^\infty]$ for some $N \gg 0$. We have a commutative diagram

$$\begin{array}{ccccccc} & & \widetilde{\mathbf{A}}_{A[p^N]}^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_A^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_{A/A[p^N]}^{(0,r],\circ} & \longrightarrow & 0\ . \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathbf{A}}_{A[p^N]} & \longrightarrow & \widetilde{\mathbf{A}}_A & \longrightarrow & \widetilde{\mathbf{A}}_{A/A[p^N]} & & \end{array}$$

The top row is exact, because it is obtained by tensoring the sequence $0 \to A[p^N] \to A \to A/A[p^N] \to 0$ with $\mathbf{A}_{\inf}[p/[\varpi]^{1/r}]$ and then taking $[\varpi]$-adic completion. This last step is exact because of Lemma 2.2 , since $\mathbf{A}_{\inf}[p/[\varpi]^{1/r}] \otimes A/A[p^N]$ has bounded $[\varpi]$-torsion according to Proposition 2.3. The bottom row is also exact. To see this, start from the exact sequence $0 \to A[p^N] \to A \to A/A[p^N] \to 0$, tensor it with $W_a(\mathcal{O}_{\mathbf{C}}^\flat)$, take $[\varpi]$-adic completion, invert $[\varpi]$, and take inverse limits over a. Here the first step is exact because $A/A[p^N]$ is p-torsionfree, and the second step is exact because $W_a(\mathcal{O}_F) \otimes_{\mathbf{Z}_p} A/A[p^N]$ is $[\varpi]$-torsionfree by Proposition 2.3.

With the exactness established, the snake lemma applies. Using the previous lemma, we learn that the kernel of $\widetilde{\mathbf{A}}_A^{(0,r],\circ} \to \widetilde{\mathbf{A}}_A$ is $[\varpi]$-torsion (even bounded). Since the map $\widetilde{\mathbf{A}}_A^{(0,r]} \to \widetilde{\mathbf{A}}_A$ is induced from inverting $[\varpi]$, the proof is finished. □

Corollary 23.10. *If* $s > r$, *the natural map* $\widetilde{\mathbf{A}}_A^{(0,s]} \to \widetilde{\mathbf{A}}_A^{(0,r]}$ *is injective.*

This proves that in the definition of $\widetilde{\mathbf{A}}_A^\dagger$, the colimit is in fact a union, so that $\widetilde{\mathbf{A}}_A^\dagger = \bigcup_{r>0} \widetilde{\mathbf{A}}_A^{(0,r]}$, and $U \subset \widetilde{\mathbf{A}}_A^\dagger$ is open if and only if $U \cap \widetilde{\mathbf{A}}_A^{(0,r]}$ is open for every r. By the theorem, there is a natural continuous and injective map $\widetilde{\mathbf{A}}_A^\dagger \to \widetilde{\mathbf{A}}_A$. By taking H_K-invariants, we immediately deduce that we have a similar statement $\widetilde{\mathbf{A}}_{K,A}^\dagger = \bigcup_{r>0} \widetilde{\mathbf{A}}_{K,A}^{(0,r]}$ relative to K, with a natural continuous and injective map $\widetilde{\mathbf{A}}_{K,A}^\dagger \to \widetilde{\mathbf{A}}_{K,A}$.

Remark 23.11. The analogue of Theorem 2.9 for $\widetilde{\mathbf{A}}_A^{(0,\infty)}$ is also true. Let us explain briefly how this works. The map $\widetilde{\mathbf{A}}_A^{(0,\infty),\circ} \to \widetilde{\mathbf{A}}_A$ is injective according to [EG19, Rem. 2.2.13]. It therefore suffices to explain why $\widetilde{\mathbf{A}}_A^{(0,\infty),\circ}$ is $[\varpi]$-torsionfree. For each $N \geq 1$ we have exact sequences

$$0 \to p^N W(\mathcal{O}_{\mathbf{C}}^\flat)_A \to p^{N+1} W(\mathcal{O}_{\mathbf{C}}^\flat)_A \to (\mathcal{O}_{\mathbf{C}}^\flat)_{p^N A/p^{N+1}A} \to 0.$$

Assume for a moment that $(\mathcal{O}_{\mathbf{C}}^\flat)_{p^N A/p^{N+1}A}$ is ϖ-torsionfree. Then it follows by devissage that $\widetilde{\mathbf{A}}_A^{(0,\infty),\circ}[[\varpi]] \subset \bigcap_{n\geq 1} p^n \widetilde{\mathbf{A}}_A^{(0,\infty),\circ} = 0$.
Renaming $p^N A/p^{N+1}A$ as A, we reduce to proving that $(\mathcal{O}_{\mathbf{C}}^\flat)_A$ is ϖ-torsionfree. Clearly $\mathcal{O}_{\mathbf{C}}^\flat$ itself is ϖ-torsionfree, so we have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{C}}^\flat \xrightarrow{\varpi} \mathcal{O}_{\mathbf{C}}^\flat \to \mathcal{O}_{\mathbf{C}}^\flat/\varpi \to 0.$$

Applying $\otimes_{\mathbf{Z}_p} A$ to $\mathcal{O}_{\mathbf{C}}^\flat$ is the same as applying $\otimes_{\mathbf{F}_p} A/p$ to it. Since A/p is free, tensoring with it gives an exact sequence

$$0 \to \mathcal{O}_{\mathbf{C}}^\flat \otimes A \xrightarrow{\varpi} \mathcal{O}_{\mathbf{C}}^\flat \otimes A \to \mathcal{O}_{\mathbf{C}}^\flat/\varpi \otimes A \to 0.$$

Now $\mathcal{O}_{\mathbf{C}}^\flat/\varpi \otimes A$ is killed by ϖ, and in particular, its ϖ-torsion is bounded. Hence, by Lemma 2.2, the sequence stays exact after ϖ-adic completion:

$$0 \to (\mathcal{O}_{\mathbf{C}}^\flat)_A \xrightarrow{\varpi} (\mathcal{O}_{\mathbf{C}}^\flat)_A \to (\mathcal{O}_{\mathbf{C}}^\flat)_A/\varpi \to 0.$$

It follows that $(\mathcal{O}_{\mathbf{C}}^\flat)_A$ is ϖ-torsionfree, as required.

We shall now deduce similar properties for the imperfect rings relative to K.

Proposition 23.12. *The natural map $\mathbf{A}_{K,A} \to \widetilde{\mathbf{A}}_A$ is injective.*

Proof. This follows from [EG21, Rem. 2.2.13]. □

Proposition 23.13. *The natural map $\mathbf{A}_{K,A}^{(0,r],\circ} \to \widetilde{\mathbf{A}}_A^{(0,r],\circ}$ is injective.*

Proof. We do this in several steps.
Step 1. The statement is true if A is p-torsionfree.
Since $\mathbf{A}_K^{(0,r],\circ} = \widetilde{\mathbf{A}}^{(0,r],\circ} \cap \mathbf{A}_K$, the intersection taken in $\widetilde{\mathbf{A}}$, we get an exact sequence

$$0 \to \mathbf{A}_K^{(0,r],\circ} \to \widetilde{\mathbf{A}}^{(0,r],\circ} \oplus \mathbf{A}_K \xrightarrow{(x,y)\mapsto x-y} \widetilde{\mathbf{A}}.$$

Tensoring with A we get

$$0 \to \mathbf{A}_K^{(0,r],\circ} \otimes A \to (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A) \oplus (\mathbf{A}_K \otimes A) \xrightarrow{(x,y)\mapsto x-y} \widetilde{\mathbf{A}} \otimes A,$$

so that $\mathbf{A}_K^{(0,r],\circ} \otimes A = (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A) \cap (\mathbf{A}_K \otimes A)$, the intersection taken in $\widetilde{\mathbf{A}} \otimes A$.

Next, we notice that $(\widetilde{\mathbf{A}} \otimes A)/(\mathbf{A}_K \otimes A)$ is T-torsionfree. Indeed, T is invertible in $\mathbf{A}_K$ and $\widetilde{\mathbf{A}}$. Hence,

$$(\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A)/(\mathbf{A}_K^{(0,r],\circ} \otimes A) = (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A)/(\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A) \cap (\mathbf{A}_K \otimes A),$$

which injects into $(\widetilde{\mathbf{A}} \otimes A)/(\mathbf{A}_K \otimes A)$, is also T-torsionfree.

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{A}_K^{(0,r],\circ} \otimes A & \longrightarrow & \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A & \longrightarrow & (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A)/(\mathbf{A}_K^{(0,r],\circ} \otimes A) & \longrightarrow & 0 \\ & & \downarrow T^a & & \downarrow T^a & & \downarrow T^a & & \\ 0 & \longrightarrow & \mathbf{A}_K^{(0,r],\circ} \otimes A & \longrightarrow & \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A & \longrightarrow & (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A)/(\mathbf{A}_K^{(0,r],\circ} \otimes A) & \longrightarrow & 0 \end{array}.$$

By the snake lemma, the maps $(\mathbf{A}_K^{(0,r],\circ} \otimes A)/T^a \to (\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A)/T^a$ are injective. Taking the inverse limit $a \to \infty$, this implies $\mathbf{A}_{K,A}^{(0,r],\circ} \to \widetilde{\mathbf{A}}_A^{(0,r],\circ}$ is injective.

Step 2. The statement is true if $pA = 0$.

First we claim that the quotient $\widetilde{\mathbf{A}}/\mathbf{A}_K$ is p-torsionfree. Indeed, this follows from the fact that $\widetilde{\mathbf{A}}$ is p-torsionfree and the injectivity of $\mathbf{A}_K \otimes \mathbf{F}_p \to \widetilde{\mathbf{A}} \otimes \mathbf{F}_p$. Now since $\widetilde{\mathbf{A}}^{(0,r],\circ}/\mathbf{A}_K^{(0,r],\circ} = \widetilde{\mathbf{A}}^{(0,r],\circ}/\widetilde{\mathbf{A}}^{(0,r],\circ} \cap \mathbf{A}_K$ injects into $\widetilde{\mathbf{A}}/\mathbf{A}_K$, it is also p-torsionfree. In particular, the natural map from $\mathbf{A}_K^{(0,r],\circ} \otimes \mathbf{F}_p$ into $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes \mathbf{F}_p$ is injective.

Next, upon choosing an isomorphism $A \cong \bigoplus_{i\in I} \mathbf{F}_p$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in I}(\mathbf{A}_K^{(0,r],\circ} \otimes \mathbf{F}_p)/T^a & \longrightarrow & \bigoplus_{i\in I}(\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes \mathbf{F}_p)/T^a \\ \downarrow & & \downarrow \\ \prod_{i\in I}(\mathbf{A}_K^{(0,r],\circ} \otimes \mathbf{F}_p)/T^a & \longrightarrow & \prod_{i\in I}(\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes \mathbf{F}_p)/T^a \end{array}.$$

Here, the vertical maps are injective. Taking the limit over a, we obtain

$$\begin{array}{ccc} \mathbf{A}_{K,A}^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_A^{(0,r],\circ} \\ \downarrow & & \downarrow \\ \prod_{i\in I} \mathbf{A}_K^{(0,r],\circ} \otimes \mathbf{F}_p & \longrightarrow & \prod_{i\in I} \widetilde{\mathbf{A}}^{(0,r],\circ} \otimes \mathbf{F}_p \end{array},$$

where the vertical maps are still injective and the lower horizontal map is injective by we have just explained. It follows that the top horizontal map is injective.

Step 3. If $p^N A = 0$ for some N, the statement is true for A.

To case $N = 1$ was already discussed. Now consider the commutative diagram

$$\begin{array}{ccccccccc} & & \mathbf{A}_{K,pA}^{(0,r],\circ} & \longrightarrow & \mathbf{A}_{K,A}^{(0,r],\circ} & \longrightarrow & \mathbf{A}_{K,A/p}^{(0,r],\circ} & \longrightarrow & 0\ . \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathbf{A}}_{pA}^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_{A}^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_{A/p}^{(0,r],\circ} & \longrightarrow & 0 \end{array}$$

The top row is exact because it is obtained from tensoring $0 \to pA \to A \to A/p \to 0$ with $\mathbf{A}_K^{(0,r],\circ}$, which is p-torsionfree, and then completing with respect to T, which is right exact by Nakayama's lemma. The bottom row is exact because of Lemma 2.2, since $\widetilde{\mathbf{A}}^{(0,r],\circ} \otimes A/p$ has bounded $[\varpi]$-torsion (Proposition 2.3.*iv*).

The result now easily follows by devissage using the snake lemma.

Step 4. The statement is true for general A.

This is similar to the proof of step 3. Namely, take N large enough so that $A[p^N] = A[p^\infty]$. Then one has a commutative diagram

$$\begin{array}{ccccccccc} & & \mathbf{A}_{K,A[p^N]}^{(0,r],\circ} & \longrightarrow & \mathbf{A}_{K,A}^{(0,r],\circ} & \longrightarrow & \mathbf{A}_{K,A/A[p^N]}^{(0,r],\circ} & \longrightarrow & 0\ , \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathbf{A}}_{A[p^N]}^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_{A}^{(0,r],\circ} & \longrightarrow & \widetilde{\mathbf{A}}_{A/A[p^N]}^{(0,r],\circ} & \longrightarrow & 0 \end{array}$$

which is exact by the same arguments. We then conclude by using what was prove in steps 1 and 3. ☐

Corollary 23.14. *The natural map $\mathbf{A}_{K,A}^{(0,r]} \to \mathbf{A}_{K,A}$ is injective.*

Proof. From the proposition it follows that $\mathbf{A}_{K,A}^{(0,r]} \to \widetilde{\mathbf{A}}_A^{(0,r]}$ is injective. Use the diagram

$$\begin{array}{ccc} \mathbf{A}_{K,A}^{(0,r]} & \longrightarrow & \mathbf{A}_{K,A}\ , \\ \downarrow & & \downarrow \\ \widetilde{\mathbf{A}}_{A}^{(0,r]} & \longrightarrow & \widetilde{\mathbf{A}}_{A} \end{array}$$

in which we already know by Theorem 2.9, Proposition 2.12 and Proposition 2.13 that every map other than $\mathbf{A}_{K,A}^{(0,r]} \to \mathbf{A}_{K,A}$ is injective. ☐

Corollary 23.15. *If $s > r$, the natural map $\mathbf{A}_{K,A}^{(0,s]} \to \mathbf{A}_{K,A}^{(0,r]}$ is injective.*

As in the perfect case, we now know that $\mathbf{A}_{K,A}^{\dagger} = \bigcup_{r>0} \mathbf{A}_{K,A}^{(0,r]}$.

23.3 Compatibility with reduction

In this subsection we establish a result that will be used in §3.

For $a \geq 1$ we have natural maps $\widetilde{\mathbf{A}}_{K,A/p^{a+1}} \to \widetilde{\mathbf{A}}_{K,A/p^a}$, obtain from the surjection $A/p^{a+1} \to A/p^a$ by tensoring with $W(\mathcal{O}^\flat_{\widehat{K}_\infty})$, completing $[\varpi]$-adically, and inverting $[\varpi]$.

Lemma 23.16. *Suppose A is p-torsionfree. Then for $N \in \mathbf{Z}_{\geq 1}$, the natural map*

$$p^N \widetilde{\mathbf{A}}_{K,A} \to \varprojlim_a p^N \widetilde{\mathbf{A}}_{K,A/p^a}$$

is an isomorphism.

Proof. As A is p-torsionfree, for $a \geq N$ we have exact sequences

$$0 \to p^{a-N}A/p^aA \to A/p^a \xrightarrow{p^N} A/p^a \to A/p^N \to 0.$$

Hence, tensoring with $W(\mathcal{O}^\flat_{\widehat{K}_\infty})$, for $b \geq a$ we have

$$0 \to W_b(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes p^{a-N}A/p^aA \to W_b(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes A/p^a$$

$$\xrightarrow{p^N} W_b(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes A/p^a \to W_b(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes A/p^N \to 0.$$

Each term appearing in the exact sequence has no $[\varpi]$-torsion, by Proposition 2.3. Hence, by Lemma 2.2, completing $[\varpi]$-adically is exact. Inverting $[\varpi]$ also, we get an exact sequence

$$0 \to \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^a} \xrightarrow{p^N} p^N \widetilde{\mathbf{A}}_{K,A/p^a} \to 0.$$

Taking limits, we have a long exact sequence

$$0 \to \varprojlim_a \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^a} \to \widetilde{\mathbf{A}}_{K,A} = \varprojlim_a \widetilde{\mathbf{A}}_{K,A/p^a}$$

$$\xrightarrow{p^N} \varprojlim_a p^N \widetilde{\mathbf{A}}_{K,A/p^a} \to \mathrm{R}^1 \varprojlim_a \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^a},$$

but $\varprojlim_a \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^a}$ and $\mathrm{R}^1 \varprojlim_a \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^a}$ both vanish because the composition of N successive transition maps $\widetilde{\mathbf{A}}_{K,p^{a+1-N}A/p^{a+1}} \to \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^a}$ is zero. This proves the lemma. □

Proposition 23.17. *For $N \in \mathbf{Z}_{\geq 1}$ and $1/r \in \mathbf{Z}[1/p]_{>0}$ we have natural isomorphisms*

$$\widetilde{\mathbf{A}}_{K,A}/p^N \widetilde{\mathbf{A}}_{K,A} \cong \widetilde{\mathbf{A}}_{K,A/p^N} \cong \widetilde{\mathbf{A}}^{(0,r]}_{K,A/p^N} \cong \widetilde{\mathbf{A}}^{(0,\infty)}_{K,A/p^N}.$$

Proof. There are a priori natural inclusions $\widetilde{\mathbf{A}}^{(0,\infty)}_{K,A/p^N} \subset \widetilde{\mathbf{A}}^{(0,r]}_{K,A/p^N} \subset \widetilde{\mathbf{A}}_{K,A/p^N}$ according to Theorem 2.9, Corollary 2.10 and Remark 2.11. The inclusion of the first term in the last term is an isomorphism by Proposition 2.1.*ii*. Hence the middle term is also naturally isomorphic to them.

It remains to construct $\widetilde{\mathbf{A}}_{K,A}/p^N\widetilde{\mathbf{A}}_{K,A} \cong \widetilde{\mathbf{A}}_{K,A/p^N}$. Starting from

$$0 \to p^N(A/p^a) \to A/p^a \to A/p^N \to 0,$$

for $a \geq N$, tensor with $W(\mathcal{O}^\flat_{\widehat{K}_\infty})$, complete $[\varpi]$-adically, and invert $[\varpi]$ to get

$$0 \to \widetilde{\mathbf{A}}_{K,p^N(A/p^a)} \to \widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^N} \to 0.$$

Taking limits, we obtain a surjective map

$$\widetilde{\mathbf{A}}_{K,A} \cong \varprojlim_a \widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^N}$$

which factors through $\widetilde{\mathbf{A}}_{K,A}/p^N\widetilde{\mathbf{A}}_{K,A}$.

Step 1. The map $\widetilde{\mathbf{A}}_{K,A} \to \widetilde{\mathbf{A}}_{K,A/p^N}$ is an isomorphism if A is killed by a power of p.

In this case, consider the exact sequence

$$0 \to A[p^N] \to A \xrightarrow{p^N} A \to A/p^N \to 0.$$

Now tensor with $W(\mathcal{O}^\flat_{\widehat{K}_\infty})$, complete $[\varpi]$-adically (which is exact by Lemma 2.2 and Proposition 2.3), and invert $[\varpi]$ to get

$$0 \to \widetilde{\mathbf{A}}_{K,A[p^N]} \to \widetilde{\mathbf{A}}_{K,A} \xrightarrow{p^N} \widetilde{\mathbf{A}}_{K,A} \to \widetilde{\mathbf{A}}_{K,A/p^N} \to 0$$

(there is no need to take an inverse limit since all the terms will be the same for $a \gg 0$). In particular, $\widetilde{\mathbf{A}}_{K,A}/p^N\widetilde{\mathbf{A}}_{K,A} \cong \widetilde{\mathbf{A}}_{K,A/p^N}$.

Step 2. The map $\widetilde{\mathbf{A}}_{K,A}/p^N \to \widetilde{\mathbf{A}}_{K,A/p^N}$ is an isomorphism if A is p-torsionfree.

In this case, consider the exact sequence

$$0 \to p^{a-N}A/p^aA \to A/p^a \xrightarrow{p^N} A/p^a \to A/p^N \to 0.$$

Now tensor with $W(\mathcal{O}^\flat_{\widehat{K}_\infty})$, complete $[\varpi]$-adically (which is exact by Lemma 2.2 and Proposition 2.3), and invert $[\varpi]$ to get

$$0 \to \widetilde{\mathbf{A}}_{K,p^{a-N}A/p^aA} \to \widetilde{\mathbf{A}}_{K,A/p^a} \xrightarrow{p^N} \widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^N} \to 0.$$

In particular, we have a short exact sequence

$$0 \to p^N\widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^N} \to 0.$$

Taking limits over a, we have

$$0 \to \varprojlim_a p^N \widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A} = \varprojlim \widetilde{\mathbf{A}}_{K,A/p^a} \to \widetilde{\mathbf{A}}_{K,A/p^N}.$$

Hence, the kernel of $\widetilde{\mathbf{A}}_{K,A} \to \widetilde{\mathbf{A}}_{K,A/p^N}$ is equal to $\varprojlim_a p^N \widetilde{\mathbf{A}}_{K,A/p^a} = p^N \widetilde{\mathbf{A}}_{K,A}$, by the previous lemma.

Step 3. The map is an isomorphism in general.

Indeed, let $A[p^\infty] = A[p^M]$ for $M \gg 0$ so that $A/A[p^M]$ is p-torsionfree and $A[p^M]$ is killed by p^M. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathbf{A}}_{K,A[p^M]}/p^N & \longrightarrow & \widetilde{\mathbf{A}}_{K,A}/p^N & \longrightarrow & \widetilde{\mathbf{A}}_{K,A/A[p^M]}/p^N & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathbf{A}}_{K,A[p^M]/p^N} & \longrightarrow & \widetilde{\mathbf{A}}_{K,A/p^N} & \longrightarrow & \widetilde{\mathbf{A}}_{K,A/(A[p^M]+p^N A)} & \longrightarrow & 0 \end{array}$$

The top row is exact because it is obtained by starting from

$$0 \to A[p^M] \to A/p^a A \to A/(A[p^M] + p^a A) \to 0$$

for $a \geq N$, tensoring with $W(\mathcal{O}^\flat_{\widehat{K}_\infty})$, completing $[\varpi]$-adically, inverting $[\varpi]$, taking inverse limits over a, and tensoring with $\mathbf{Z}/p^N$. All of these operations are exact, including the last two ones: inverse limits over a because the transition maps $\widetilde{\mathbf{A}}_{K,A[p^N]/p^{a+1}} \to \widetilde{\mathbf{A}}_{K,A[p^N]/p^a}$ are just the identity for $a \geq N$, and tensoring over $\mathbf{F}_p$ because $\widetilde{\mathbf{A}}_{K,A/A[p^M]}$ is p-torsionfree (as can be seen from the proof of Proposition 2.8.*ii*). The bottom row is exact, because it is obtained by starting from

$$0 \to A[p^M] \to A \to A/A[p^M] \to 0,$$

tensoring with $W_N(\mathcal{O}^\flat_{\widehat{K}_\infty})$, completing $[\varpi]$-adically and inverting $[\varpi]$-adically. The second step is exact because $A/A[p^M]$ is p-torsionfree.

With this given, one now concludes the proof from the snake lemma, using steps 1 and 2. □

23.4 The (φ, G_K)-actions

There are continuous (φ, G_K)-actions on $\widetilde{\mathbf{A}}_A$ by [EG19, Lem. 2.2.18]. We now establish the continuity on all of the rings defined in §2.1.

Given $r \in \mathbf{R}_{>0} \cup \{\infty\}$ we have a continuous map

$$\varphi : \widetilde{\mathbf{A}}_A^{(0,r]} \to \widetilde{\mathbf{A}}_A^{(0,r/p]}$$

induced by extending the action of φ on $\mathbf{A}_{\mathrm{inf}}$. It is continuous since $\varphi([\varpi]) = [\varpi]^p$ and the topology is $[\varpi]$-adic on both the source and the target. Similarly, we have a continuous inverse

$$\varphi^{-1} : \widetilde{\mathbf{A}}_A^{(0,r/p]} \to \widetilde{\mathbf{A}}_A^{(0,r]}.$$

This immediately extends to give continuous φ and φ^{-1} actions on $\widetilde{\mathbf{A}}_A^{\dagger}$.

Lemma 23.18. *Let G be a topological group.*

i. If G is profinite and acts continuously on topological spaces $X_1 \to X_2 \to \dots$, and $X = \varinjlim_i X_i$ is endowed with the direct limit topology, then the natural action of G on X is continuous.

ii. If G acts continuously on topological spaces $X_1 \leftarrow X_2 \leftarrow X_3, \dots$ then the natural action of G on the limit $\varprojlim_i X_i$ is continuous.

Proof. i. Let $U \subset X$ and let $(g, x) \in G \times X$ be such that $\mathrm{act} : G \times X \to X$ maps (g, x) into U. Suppose $x \in X_n$, and let $i_n : X_n \to X$ denote the canonical map. Then since $\mathrm{act}|_{G\times X_n}$ is continuous, there exists $N \subset G$ open compact such that

$$\mathrm{act}|_{G\times X_n}(Ng \times \{x\}) \subset i_n^{-1}(U),$$

which implies

$$\mathrm{act}|_{G\times X}(Ng \times \{x\}) \subset U.$$

By the tube lemma, there exists an open subset $V \subset X$ containing x such that $\mathrm{act}(Ng \times V) \subset U$. This proves that the action is continuous.

ii. It is enough to prove the continuity of the action on the product $\prod X_i$, which is obvious. □

Now, G_K acts continuously on $\mathbf{A}_{\mathrm{inf}}$, hence on $(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)[p/[\varpi]^{1/r}]/[\varpi]^a$. As $\widetilde{\mathbf{A}}_A^{(0,r]}$ is built out of $(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} A)[p/[\varpi]^{1/r}]/[\varpi]^a$ by taking direct limits and projective limits, the lemma implies that the action of G_K on $\widetilde{\mathbf{A}}_A^{(0,r]}$ is continuous. Finally, $\widetilde{\mathbf{A}}_A^{\dagger}$ is built out of $\widetilde{\mathbf{A}}_A^{(0,r]}$ by taking direct limits, so again by the lemma the action of G_K on it is continuous. Via the topological embeddings $\mathbf{A}_{K,A}^{\dagger} \hookrightarrow \widetilde{\mathbf{A}}_{K,A}^{\dagger} \hookrightarrow \widetilde{\mathbf{A}}_A^{\dagger}$ and $\mathbf{A}_{K,A} \hookrightarrow \widetilde{\mathbf{A}}_{K,A} \hookrightarrow \widetilde{\mathbf{A}}_A$, we conclude this subsection with the following result.

Proposition 23.19. *The $(\varphi^{\pm 1}, G_K)$-actions (resp. $(\varphi^{\pm 1}, \Gamma_K)$-actions, resp. (φ, Γ_K)-actions) on $\widetilde{\mathbf{A}}_A^{\dagger}$ and $\widetilde{\mathbf{A}}_A$ (resp. $\widetilde{\mathbf{A}}_{K,A}^{\dagger}$ and $\widetilde{\mathbf{A}}_{K,A}$, resp. $\mathbf{A}_{K,A}^{\dagger}$ and $\mathbf{A}_{K,A}$) are continuous.*

24 Overconvergence for perfect coefficients

The purpose of this section is to prove the following result.

Theorem 24.1. *The functor $\widetilde{M}^{\dagger} \mapsto \widetilde{M} := \widetilde{M}^{\dagger} \otimes_{\widetilde{\mathbf{A}}^{\dagger}_{K,A}} \widetilde{\mathbf{A}}_{K,A}$ induces an equivalence of categories from the category of projective étale (φ, Γ_K)-modules over $\widetilde{\mathbf{A}}^{\dagger}_{K,A}$ to the category of projective étale (φ, Γ_K)-modules over $\widetilde{\mathbf{A}}_{K,A}$.*

Our proof follows §4 of [dSP19] with appropriate modifications. The idea will be to first establish the equivalence for étale φ-modules and then deduce it for étale (φ, Γ_K)-modules.

24.1 Descent of φ-modules

By Corollary 2.4, the topology on $\widetilde{\mathbf{A}}^{(0,r]}_{K,A}$ is defined by a valuation $\mathrm{val}^{(0,r]}$

$$\mathrm{val}^{(0,r]}_A(x) := (p/p-1)\sup\{t \in \mathbf{Z}[1/p] : x \in [\varpi]^t \widetilde{\mathbf{A}}^{(0,r],+}_{K,A}\}.$$

We also define for $x \in \widetilde{\mathbf{A}}^{(0,\infty)}_{K,A}$

$$\mathrm{val}^{(0,\infty)}_A(x) := (p/p-1)\sup\{t \in \mathbf{Z}[1/p] : x \in [\varpi]^t \widetilde{\mathbf{A}}^{(0,\infty),+}_{K,A}\}.$$

We extend $\mathrm{val}^{(0,r]}_A$ and $\mathrm{val}^{(0,\infty)}_A$ to all of $\widetilde{\mathbf{A}}_{K,A}$, by allowing the value $-\infty$, so that $\widetilde{\mathbf{A}}^{(0,r]}_{K,A}$ (resp. $\widetilde{\mathbf{A}}^{(0,\infty)}_{K,A}$) is the subset of elements $x \in \widetilde{\mathbf{A}}_{K,A}$ with $\mathrm{val}^{(0,r]}_A(x) > -\infty$ (resp. $\mathrm{val}^{(0,\infty)}_A(x) > -\infty$).
The following lemma serves as a generalization of the usual Teichmuller digits. Recall from Proposition 2.17 that

$$\widetilde{\mathbf{A}}_{K,A}/p^N \cong \widetilde{\mathbf{A}}_{K,A/p^N} \cong \widetilde{\mathbf{A}}^{(0,\infty)}_{K,A/p^N} \cong \widetilde{\mathbf{A}}^{(0,r]}_{K,A/p^N}.$$

Lemma 24.2. *There exists a (noncanonical) map $[\bullet] : \widetilde{\mathbf{A}}_{K,A}/p \to \widetilde{\mathbf{A}}_{K,A}$ such that*

1. For every $x \in \widetilde{\mathbf{A}}_{K,A}/p$ we have $[x] \equiv x \bmod p$.
2. The map $[\bullet]$ commutes with φ.
3. For every every $x \in \widetilde{\mathbf{A}}_{K,A}/p$, we have $\mathrm{val}^{(0,\infty)}_A([x]) \geq \mathrm{val}^{(0,\infty)}_{A/p}(x)$.
4. For every $n \in \mathbf{Z}_{\geq 0}$ and every $x \in \widetilde{\mathbf{A}}_{K,A}/p$, we have $\mathrm{val}^{(0,r]}_A(p^n[x]) \geq \mathrm{val}^{(0,r]}_{A/p^{n+1}A}(p^n[x] \mod p^{n+1})$.

Proof. Multiplication by p induces surjections

$$A/p \twoheadrightarrow pA/p^2 \twoheadrightarrow p^2A/p^3 \twoheadrightarrow \dots$$

of $\mathbf{F}_p$-vector spaces. Let W_n be the kernel of $\pi_n : A/p \twoheadrightarrow p^nA/p^{n+1}$, so that the W_n are increasing with n. Choose a complement U to the union of the W_n so that

$$A/p = U \oplus \bigcup_{n\geq 0} W_n.$$

For U choose an $\mathbf{F}_p$-basis $\{e_i\}_{i\in I_U}$ and choose, as we may, compatible bases $\{e_i\}_{i\in I_n}$ for W_n so that $\{e_i\}_{i\in I_n} \subset \{e_i\}_{i\in I_{n+1}}$, and set $I = \bigcup_{n\geq 0} I_n$. With this being given, we set $J_n = I_U \cup (I\backslash I_n)$ for $n \geq 0$. We claim that $\{\pi_n(e_i)\}_{i\in J_n}$ gives an $\mathbf{F}_p$-basis of p^nA/p^{n+1}. Indeed, by construction, the map $\pi_n : A/p \twoheadrightarrow p^nA/p^{n+1}$ gives a decomposition

$$A/p = \mathrm{span}\{e_i\}_{i\in I_U\cup I} = W_n \oplus \mathrm{span}\{e_i\}_{i\in J_n},$$

so that applying π_n to the e_i for $i \in J_n$ gives an $\mathbf{F}_p$-basis of p^nA/p^{n+1}. With this choice of $\{e_i\}_{i\in I\cup I_U}$, choose an arbitrary lifting of them to A, which we denote by $\widetilde{e}_i$.

Now each $x \in \mathcal{O}^\flat_{\widehat{K}_\infty} \otimes_{\mathbf{F}_p} A/p$ can be written uniquely as a finite sum of the form $x = \sum x_i \otimes e_i$ with $x_i \in \mathcal{O}^\flat_{\widehat{K}_\infty}$. Let

$$[x] := \sum [x_i] \otimes \widetilde{e}_i \in W(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes_{\mathbf{Z}_p} A,$$

where $[x_i]$ is the usual Teichmuller lift. This defines a map

$$[\bullet] : \mathcal{O}^\flat_{\widehat{K}_\infty} \otimes_{\mathbf{F}_p} A/p \to W(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes_{\mathbf{Z}_p} A.$$

It follows from the definition that

$$[\bullet](\varpi^t \mathcal{O}^\flat_{\widehat{K}_\infty} \otimes_{\mathbf{F}_p} A/p) \subset [\varpi]^t W(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes_{\mathbf{Z}_p} A,$$

hence it is continuous for the $[\varpi]$-topology on the source and the $(p, [\varpi])$-topology on the target. It therefore extends to a map

$$[\bullet] : (\mathcal{O}^\flat_{\widehat{K}_\infty})_{A/p} \to W(\mathcal{O}^\flat_{\widehat{K}_\infty})_A,$$

which we can extend further to a continuous map

$$[\bullet] : \widetilde{\mathbf{A}}_{K,A/p} = (\mathcal{O}^\flat_{\widehat{K}_\infty})_{A/p}[1/\varpi] \to W(\widehat{K}^\flat_\infty)_A = \widetilde{\mathbf{A}}_{K,A}.$$

Clearly properties 1 and 2 are satisfied.

To show property 3 holds, we need to check that $\varpi^{-t}x \in (\mathcal{O}^\flat_{\widehat{K}_\infty})_{A/p}$ implies $[\varpi]^{-t}[x] \in W(\mathcal{O}^\flat_{\widehat{K}_\infty})_A$. Replacing x with $\varpi^{-t}x$, we reduce to the case $t = 0$. Using the continuity of $[\bullet]$, we may assume $x = \sum x_i \otimes e_i \in \mathcal{O}^\flat_{\widehat{K}_\infty} \otimes_{\mathbf{F}_p} A/p$. But then it is clear that $[x] = \sum [x_i] \otimes e_i$ lies $W(\mathcal{O}^\flat_{\widehat{K}_\infty}) \otimes_{\mathbf{Z}_p} A \subset W(\mathcal{O}^\flat_{\widehat{K}_\infty})_A$.

To show property 4 holds, we may again twist, so it suffices to show that $p^n[x] \bmod p^{n+1} \in \widetilde{\mathbf{A}}^{(0,r],+}_{K,A/p^n}$ implies $p^n[x] \in \widetilde{\mathbf{A}}^{(0,r],+}_{K,A}$. Again using the continuity of $[\bullet]$, we may assume $x \in \widehat{K}^\flat_\infty \otimes_{\mathbf{Z}_p} A$. Hence $[x]$, and consequently $p^n[x]$, belongs to the image of $\widetilde{\mathbf{A}}^{(0,r]}_K \otimes_{\mathbf{Z}_p} A$, in $\widetilde{\mathbf{A}}^{(0,r]}_{K,A}$. Explicitly, writing $x = \sum x_i \otimes e_i$ with $x_i \in \widehat{K}^\flat_\infty$, the sum being finite, we have that $p^n[x]$ is the image under the map $\widetilde{\mathbf{A}}^{(0,r]}_K \otimes_{\mathbf{Z}_p} A \to \widetilde{\mathbf{A}}^{(0,r]}_{K,A}$ of $\sum [x_i] \otimes p^n \widetilde{e}_i$. Taking this modulo p^{n+1}, we obtain that $p^n[x] \bmod p^{n+1}$ is the image of $\sum [x_i] \otimes \pi_n(p^n \widetilde{e}_i)$ under the map $\widetilde{\mathbf{A}}^{(0,r]}_K \otimes_{\mathbf{Z}_p} p^n A/p^{n+1} \to \widetilde{\mathbf{A}}^{(0,r]}_{K,A/p^{n+1}}$. On the other hand by assumption $p^n[x] \bmod p^{n+1} \in \widetilde{\mathbf{A}}^{(0,r],+}_{K,A/p^n}$. Since the $\pi_n(p^n \widetilde{e}_i)$ are linearly independent, the assumption $p^n[x] \bmod p^{n+1} \in \widetilde{\mathbf{A}}^{(0,r],+}_{K,A/p^n}$ implies that each $[x_i]$ lies in the image of $\widetilde{\mathbf{A}}^{(0,r],\circ}_K$. Hence $p^n[x]$, which is the image of $\sum [x_i] \otimes p^n \widetilde{e}_i$ under $\widetilde{\mathbf{A}}^{(0,r]}_K \otimes_{\mathbf{Z}_p} A \to \widetilde{\mathbf{A}}^{(0,r]}_{K,A}$, has to lie in $\widetilde{\mathbf{A}}^{(0,r],+}_{K,A}$. This concludes the proof. □

The next key lemma is essentially the same as [dSP19, Lem. 1.3], but we include the proof for completeness. Let R be a commutative ring endowed with a nonarchimedean valuation, where we allow val $= -\infty$. Suppose there exists $p \in \mathbf{Z}_{\geq 2}$ and an invertible morphism $\varphi : R \to R$ such that $p\mathrm{val}(x) = \mathrm{val}(\varphi(x))$.

Lemma 24.3. *Let $X \in \mathrm{GL}_d(R)$. Then for any $c < (1/p-1)[\mathrm{val}(X) + \mathrm{val}(X^{-1})]$ and for any $Y \in \mathrm{M}_d(R)$ there exist $U, V \in \mathrm{M}_d(R)$ such that $\mathrm{val}(V) \geq c$ and*

$$X^{-1}\varphi(U)X - U = Y - V.$$

Proof. Let

$$U = \sum_{i=1}^{N} \varphi^{-1}(X)\varphi^{-2}(X)\cdots\varphi^{-i}(X)\cdot\varphi^{-i}(Y)\cdot\varphi^{-i}(X)^{-1}\cdots\varphi^{-2}(X)^{-1}\varphi^{-1}(X)^{-1}$$

and

$$V = \varphi^{-1}(X)\varphi^{-2}(X)\cdots\varphi^{-N}(X)\cdot\varphi^{-N}(Y)\cdot\varphi^{-N}(X)^{-1}\cdots\varphi^{-2}(X)^{-1}\varphi^{-1}(X)^{-1}.$$

Then

$$X^{-1}\varphi(U)X - U = Y - V.$$

If $Y = 0$ then choose $V = 0$. Otherwise, by selecting N large enough, we can make $\mathrm{val}(\varphi^{-N}(Y))$ as close as we want to 0. In addition,

$$\mathrm{val}(\varphi^{-1}(X)\varphi^{-2}(X)\cdots\varphi^{-N}(X)) \geq (p^{-1} + ... + p^{-N})\mathrm{val}(X)$$

and

$$\mathrm{val}(\varphi^{-N}(X)^{-1}\cdots\varphi^{-2}(X)^{-1}\varphi^{-1}(X)^{-1}) \geq (p^{-1} + ... + p^{-N})\mathrm{val}(X^{-1}),$$

so by taking N sufficiently large we have $\mathrm{val}(V) \geq c$. □

Proposition 24.4. *Let $\widetilde{M}$ be a free étale φ-module over $\widetilde{\mathbf{A}}_{K,A}$. Then there exists a free étale φ-module $\widetilde{M}^\dagger$ over $\widetilde{\mathbf{A}}^\dagger_{K,A}$, contained in $\widetilde{M}$, such that the natural map*

$$\widetilde{\mathbf{A}}_{K,A} \otimes_{\widetilde{\mathbf{A}}^\dagger_{K,A}} \widetilde{M}^\dagger \to \widetilde{M}$$

is an isomorphism.

Proof. Choose a basis of $\widetilde{M}$ and let $X = \mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A})$ be the matrix of φ in this basis. We need to show there exists a matrix $U \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A})$ such that $C = U^{-1}X\varphi(U) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^\dagger_{K,A})$. In fact we shall find U so that $C \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^{(0,\infty)}_{K,A})$.
To do this write $X = \sum_{n\geq 0} p^n[X_n]$ with

$$X_n \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}/p) = \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A/p}) = \mathrm{M}_d(\widetilde{\mathbf{A}}^{(0,\infty)}_{K,A/p}),$$

and

$$\mathrm{val}^{(0,\infty)}_A(p^k[X_n]) = \mathrm{val}^{(0,\infty)}_{p^kA/p^{k+1}}(p^k[X_n] \bmod p^{k+1})$$

as we may according to Lemma 3.2. We shall construct $U = \sum_{n\geq 0} p^n[U_n]$ with $U_n \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}/p)$ and $\mathrm{val}^{(0,\infty)}_A([U_n]) = \mathrm{val}^{(0,\infty)}_{A/p}(U_n)$ by constructing U_n inductively.
For $n = 0$ take $U_0 = \mathrm{Id}$. Now suppose $U_0, ..., U_{n-1}$ have been defined. Let $U' = \sum_{i=0}^{n-1} p^i[U_i]$. We shall also suppose we have chosen matrices $C_0 = X_0, C_1, ..., C_{n-1} \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}/p)$ inductively such that

$$U'^{-1}X\varphi(U') \equiv \sum_{i=0}^{n-1} p^i[C_i] \bmod p^n\mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}).$$

Write

$$U'^{-1}X\varphi(U') - \sum_{i=0}^{n-1} p^i[C_i] = p^nY$$

with $Y \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A})$. Now look for $U_n \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}/p)$ such that

$$(U' + p^n[U_n])^{-1}X\varphi(U' + p^n[U_n]) \equiv \sum_{i=0}^{n} p^i[C_i] \bmod p^{n+1}$$

with $\mathrm{val}^{(0,\infty)}_{A/p}(C_n)$ bounded below. Noting that $(U' + p^n[U_n])^{-1} = U'^{-1} - p^n[U_n] \bmod p^{n+1}$, and letting $\overline{Y}$ denote the reduction of Y mod $p\mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A})$, it suffices to solve the mod p equation

$$U_n - X_0\varphi(U_n)X_0^{-1} = \overline{Y}X_0^{-1} - C_nX_0^{-1}$$

in $\widetilde{\mathbf{A}}_{K,A/p}$.
By Lemma 3.3, this equation can be solved for both U_n and C_n, with C_n, hence also $p^n[C_n]$, with $\mathrm{val}_A^{(0,\infty)}(p^n[C_n])$ bounded independently of n. More precisely, we can solve for U_n and C_n with

$$\mathrm{val}_{A/p}^{(0,\infty)}(C_n X_0^{-1}) \geq c$$

for any $c < \frac{1}{p-1}[\mathrm{val}(X_0) + \mathrm{val}(X_0^{-1})]$. Then we may and do choose the C_n's in a way such that for $n \geq 1$ we have

$$\mathrm{val}_A^{(0,\infty)}(p^n[C_n]) \geq \mathrm{val}_A^{(0,\infty)}([C_n]) \geq \mathrm{val}_{A/p}^{(0,\infty)}(C_n) \geq \mathrm{val}_{A/p}^{(0,\infty)}(X_0) + c.$$

In particular, this bound is independent of n, the bound in fact depending only on X_0. With these choices of the C_n, letting $C = \sum_{n\geq 0}[C_n]p^n$ and $U = \sum_{n\geq 0}[U_n]p^n$ we therefore have $C = U^{-1}X\varphi(U)$,where $C \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A}^{(0,\infty)})$ because $\mathrm{val}_A^{(0,\infty)}(C) \geq c > -\infty$. □

24.2 Descending φ-module morphisms

We will need to regularize the action of φ.

Proposition 24.5. *Let $1/r \in \mathbf{Z}[1/p]_{>0} \cup \{\infty\}$, and let $X \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}^{(0,r]}), Y \in \mathrm{M}_e(\widetilde{\mathbf{A}}_{K,A}^{(0,r]})$ and $U \in \mathrm{M}_{d\times e}(\widetilde{\mathbf{A}}_{K,A})$ satisfy*

$$\varphi(U) = XUY.$$

Then $U \in \mathrm{M}_{d\times e}(\widetilde{\mathbf{A}}_{K,A}^{(0,r]})$.
In particular, if $X \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}^{\dagger}), Y \in \mathrm{M}_e(\widetilde{\mathbf{A}}_{K,A}^{\dagger})$ and $U \in \mathrm{M}_{d\times e}(\widetilde{\mathbf{A}}_{K,A})$, then $U \in \mathrm{M}_{d\times e}(\widetilde{\mathbf{A}}_{K,A}^{\dagger})$.

Proof. Write $X = [\varpi]^t X'$ with $\mathrm{val}_A^{(0,r]}(X') \geq 0$, then we see

$$\mathrm{val}_A^{(0,r]}(\varphi^{-1}(X)\varphi^{-2}(X)\cdots\varphi^{-N}(X)) \geq t(p^{-1} + ... + p^{-N})$$

is bounded independently of N. (This also happens with another bound if $r = \infty$). A similar analysis applies to $\varphi^{-N}(Y)\cdots\varphi^{-2}(Y)\varphi^{-1}(Y)$.
From the equation $U = \varphi^{-1}(X)\varphi^{-1}(U)\varphi^{-1}(Y)$ we get by iteration

$$U = \varphi^{-1}(X)\varphi^{-2}(X)\cdots\varphi^{-N}(X)\cdot\varphi^{-N}(U)\cdot\varphi^{-N}(Y)\cdots\varphi^{-2}(Y)\varphi^{-1}(Y)$$

which we write as $U = X_N\varphi^{-N}(U)Y_N$, with $\mathrm{val}_A^{(0,r]}(X_N)$ and $\mathrm{val}_A^{(0,r]}(Y_N)$ bounded independently of N.

Now write $U = \sum_{n\geq 0}[U_n]p^n$, where $[\bullet]$ denotes the generalized Teichmuller digit of Lemma 3.2, applied successively to the entries of the reduction of U mod p^n. Let $U^{(k)}$ be the k'th-truncation of U, i.e. $U^{(k)} = \sum_{n=0}^{k-1}[U_n]p^n$. We then have

$$U^{(k)} = X_N\varphi^{-N}(U^{(k)})Y_N \mod p^k.$$

Fixing k and choosing N large we can make $\mathrm{val}_A^{(0,r]}(X_N\varphi^{-N}(U^{(k)})Y_N) \geq c$ where c is a constant depending only on X and Y, because $\mathrm{val}_A^{(0,r]}(\varphi^{-N}(U^{(k)})) = p^{-N}\mathrm{val}_A^{(0,r]}(U^{(k)})$.

We then get that

$$\mathrm{val}_{A/p^k}^{(0,r]}(U^{(k)} \mod p^k) \geq \mathrm{val}_A^{(0,r]}(U^{(k)})$$

is bounded below by c.

We shall now prove by induction that $\mathrm{val}_A^{(0,r]}([U_{k-1}]p^{k-1}) \geq c$. By possibly making c smaller, we may assume this is true for $k = 1$. Arguing by induction on k, using $U^{(k)} = [U_{k-1}]p^{k-1} + U^{(k-1)}$, we deduce that

$$\mathrm{val}_{A/p^k}^{(0,r]}([U_{k-1}]p^{k-1} \mod p^k) \geq c$$

is bounded below by c. But by Lemma 3.2, this implies that

$$\mathrm{val}_A^{(0,r]}([U_{k-1}]p^{k-1}) \geq c.$$

Hence for each k, we have $\mathrm{val}_A^{(0,r]}(U^{(k)}) \geq c$, and consequently $\mathrm{val}_A^{(0,r]}(U) \geq c$. This proves that $U \in \mathrm{M}_{d\times e}(\widetilde{\mathbf{A}}_A^{(0,r]})$, as required. □

Corollary 24.6. *Let $\widetilde{M}$ be a free étale φ-module over $\widetilde{\mathbf{A}}_{K,A}$. Then there exists a unique free étale φ-module $\widetilde{M}^\dagger$ over $\widetilde{\mathbf{A}}_{K,A}^\dagger$, contained in $\widetilde{M}$, such that the natural map*

$$\widetilde{M}^\dagger \otimes_{\widetilde{\mathbf{A}}_{K,A}^\dagger} \widetilde{\mathbf{A}}_{K,A} \to \widetilde{M}$$

is an isomorphism.

Proof. The existence was established in Proposition 3.4, and the uniqueness follows from Proposition 3.5. □

Corollary 24.7. *Let $\widetilde{M}$ be a free étale φ-module over $\widetilde{\mathbf{A}}_A$, and let $\widetilde{M}^\dagger$ be the unique free étale φ-module over $\widetilde{\mathbf{A}}_A^\dagger$ with $\widetilde{M}^\dagger \otimes_{\widetilde{\mathbf{A}}_A^\dagger} \widetilde{\mathbf{A}}_A \xrightarrow{\sim} \widetilde{M}$ as in Corollary 3.6. Then*

$$(\widetilde{M}^\dagger)^{\varphi=1} = \widetilde{M}^{\varphi=1}.$$

Proof. Let $e_1, ..., e_d$ be a basis of $\widetilde{M}^\dagger$ and let $X = \mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A}^\dagger)$ be the matrix of φ with respect to this basis. If $m \in \widetilde{M}^{\varphi=1}$ and $m = \sum a_ie_i$ with $a_i \in \widetilde{\mathbf{A}}_{K,A}$, the vector $a = (a_i)$ satisfies the equation

$$\varphi(a) = X^{-1}a,$$

so by Proposition 3.5 we conclude that the a_i belong to $\widetilde{\mathbf{A}}_{K,A}^\dagger$. □

Remark 24.8. Using Proposition 3.5, one can show that $\widetilde{M}^\dagger$ admits the following characterization: it is the subset of all elements m of $\widetilde{M}$ such that $\{\varphi^i(m)\}_{i\geq 0}$ spans a finitely generated $\widetilde{\mathbf{A}}^\dagger_{K,A}$-submodule.

24.3 The equivalence of categories

The following lemma will allow us to reduce from the projective case to the free case.

Lemma 24.9. *Let M be a projective étale φ-module over a ring R. Then there exists a free étale φ-module F over R such that M is a direct summand of F.*

Proof. This argument of [EG21, Lem. 5.2.14] carries over. □

Lemma 24.10. *If M, N are projective étale φ-modules over a ring R then $M^\vee \otimes_R N$ is also a projective étale φ-module, and we have a natural identification*

$$\mathrm{Hom}_{R,\varphi}(M,N) \xrightarrow{\sim} (M^\vee \otimes_R N)^{\varphi=1}.$$

Proof. This is [EG19, Lem. 2.5.4]. □

Proposition 24.11. *Let $\widetilde{M}^\dagger, \widetilde{N}^\dagger$ be projective étale φ-modules over $\widetilde{\mathbf{A}}^\dagger_{K,A}$, and let $\widetilde{M} = \widetilde{M}^\dagger \otimes_{\widetilde{\mathbf{A}}^\dagger_{K,A}} \widetilde{\mathbf{A}}_{K,A}$, $\widetilde{N} = \widetilde{N}^\dagger \otimes_{\widetilde{\mathbf{A}}^\dagger_{K,A}} \widetilde{\mathbf{A}}_{K,A}$. Then*

$$\mathrm{Hom}_{\widetilde{\mathbf{A}}^\dagger_{K,A},\varphi}(\widetilde{M}^\dagger, \widetilde{N}^\dagger) \xrightarrow{\sim} \mathrm{Hom}_{\widetilde{\mathbf{A}}_{K,A},\varphi}(\widetilde{M}, \widetilde{N}).$$

Proof. We argue as in [EG19, Prop. 2.6.6]. i.e. let $\widetilde{P}^\dagger = (\widetilde{M}^\dagger)^\vee \otimes \widetilde{N}^\dagger$, then $\widetilde{P}^\dagger$ is a projective étale φ-module over $\widetilde{\mathbf{A}}^\dagger_{K,A}$. By Lemma 3.10, we need to check that $(\widetilde{P}^\dagger)^{\varphi=1} = \widetilde{P}^{\varphi=1}$, where $\widetilde{P} = \widetilde{P}^\dagger \otimes_{\widetilde{\mathbf{A}}^\dagger_A} \widetilde{\mathbf{A}}_A$. Since the formation of φ-invariants is compatible with direct sums we reduce by Lemma 3.9 to the case that $\widetilde{P}^\dagger$ and $\widetilde{P}$ are free. But in this case the equality is known by Corollary 3.7. □

Proposition 24.12. *Let $\widetilde{M}$ be a projective étale φ-module over $\widetilde{\mathbf{A}}_{K,A}$. Then there exists a projective étale φ-module $\widetilde{M}^\dagger$ over $\widetilde{\mathbf{A}}^\dagger_{K,A}$ and an isomorphism $\widetilde{M}^\dagger \otimes_{\widetilde{\mathbf{A}}^\dagger_{K,A}} \widetilde{\mathbf{A}}_{K,A} \xrightarrow{\sim} \widetilde{M}$.*

Proof. By Lemma 3.9, we may write $\widetilde{M}$ as a direct summand of a free étale φ-module $\widetilde{F}$ over $\widetilde{\mathbf{A}}_{K,A}$. By Proposition 3.4, there exists a free étale φ-module $\widetilde{F}^\dagger$ over $\widetilde{\mathbf{A}}^\dagger_{K,A}$ and an isomorphism $\widetilde{F}^\dagger \otimes_{\widetilde{\mathbf{A}}^\dagger_{K,A}} \widetilde{\mathbf{A}}_{K,A} \xrightarrow{\sim} \widetilde{F}$. By Proposition 3.11, the idempotent in $\mathrm{End}(\widetilde{F})$ corresponding to $\widetilde{M}$ comes from an idempotent in $\mathrm{End}(\widetilde{F}^\dagger)$, and we may take $\widetilde{M}^\dagger$ to be the étale φ-module corresponding to this idempotent. □

Combining Propositions 3.11 and 3.12 we get:

Theorem 24.13. *The functor $\widetilde{M}^\dagger \mapsto \widetilde{M}$ induces an equivalence of categories from the category of projective étale φ-modules over $\widetilde{\mathbf{A}}^\dagger_{K,A}$ to the category of projective étale φ-modules over $\widetilde{\mathbf{A}}_{K,A}$.*

Proof of Theorem 3.1. For full faithfulness, take Γ_K-invariants of both sides in the isomorphism of Proposition 3.11.

For essential surjectivity, suppose $\widetilde{M}$ is a projective étale (φ, Γ_K)-module over $\widetilde{\mathbf{A}}_{K,A}$, and let $\widetilde{M}^\dagger$ be the unique étale projective φ-module over $\widetilde{\mathbf{A}}^\dagger_{K,A}$ associated to it in Proposition 3.12. Then by uniquness, $\widetilde{M}^\dagger$ is stable under the action of Γ_K, hence is actually a (φ, Γ_K)-module over $\widetilde{\mathbf{A}}^\dagger_{K,A}$. This concludes the proof. □

25 The Tate-Sen method for Tate rings

In this section, we present a variant of the Tate-Sen method of [BC08] which will later allow us to avoid the use of Galois representations when decompleting (φ, Γ)-modules. Furthermore, it will apply in the case the coefficents have nonzero p-torsion. The idea, inspired by [Bel20], is to replace the role of p in the original Tate-Sen method by that of a pseudouniformizer. Technically speaking, the results presented here are neither a special case nor a generalization of the setting in [BC08], because of the difference in assumptions. However we are not aware of any applications of [BC08] where the method of this section would not apply also.

We work in the following general setting. Let G_0 be a profinite group endowed with a continuous character $\chi : G_0 \to \mathbf{Z}_p^\times$ with open image and let $H_0 = \ker \chi$. If $g \in G_0$, let $n(g) = \mathrm{val}_p(\chi(g) - 1) \in \mathbf{Z}$. For G an open subgroup of G_0, set $H = G \cap H_0$. Let G_H be the normalizer of H in G_0. Note G_H is open in G_0 since $G \subset G_H$. Finally let $\widetilde{\Gamma}_H = G_H/H$ and write C_H for the center of $\widetilde{\Gamma}_H$. By [BC08, Lem. 3.1.1] the group C_H is open in $\widetilde{\Gamma}_H$. Let $n_1(H)$ be the smallest positive integer such that $\chi(C_H)$ contains $1 + p^n\mathbf{Z}_p$.

Let $(\widetilde{\Lambda}, \widetilde{\Lambda}^+)$ be a pair of topological rings with $\widetilde{\Lambda}^+ \subset \widetilde{\Lambda}$, and f an element of $\widetilde{\Lambda}^+$. We shall make the following assumptions.

(i) $\widetilde{\Lambda}$ is a Tate ring, with $\widetilde{\Lambda}^+$ a ring of definition, and f a pseudouniformizer.

(ii) $\widetilde{\Lambda}^+$ is f-adically complete.

(iii) There exists a valuation $\mathrm{val}_\Lambda : \widetilde{\Lambda} \to (-\infty, \infty]$ defining the topology on $\widetilde{\Lambda}$ such that $\mathrm{val}_\Lambda(fx) = \mathrm{val}_\Lambda(f) + \mathrm{val}_\Lambda(x)$ for $x \in \widetilde{\Lambda}$, and such that $\widetilde{\Lambda}^+ = \widetilde{\Lambda}^{\mathrm{val}_\Lambda \geq 0}$.

(iv) The group G_0 acts on $\widetilde{\Lambda}$, and this action is unitary for the valuation val_Λ.

Lemma 25.1. *If* $U \in \mathrm{M}_d(\widetilde{\Lambda})$ *has* $\mathrm{val}_\Lambda(U-1) > 0$ *then* $U \in \mathrm{GL}_d(\widetilde{\Lambda})$ *with inverse* $\sum_{n=0}^{\infty}(1-U)^n$.

25.1 The Tate-Sen axioms

With the previous setting, we define them to be the following.[8]

(TS1) There exists $c_1 > 0$ such that for each pair $H_1 \subset H_2$ of open subgroups of H_0 there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ such that $\mathrm{val}_\Lambda(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

(TS2) There exists $c_2 > 0$ and for each open subgroup H of H_0 an integer $n(H)$, as well as an increasing sequence $(\Lambda_{H,n})_{n \geq n(H)}$ of closed subalgebras of $\widetilde{\Lambda}^H$, **each containing** f**,** and $\Lambda_{H,n}$-linear maps

$$\mathrm{R}_{H,n} : \widetilde{\Lambda}^H \to \Lambda_{H,n}$$

such that

(1) If $H_1 \subset H_2$ then $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and $\mathrm{R}_{H_1,n}|_{\widetilde{\Lambda}^{H_2}} = \mathrm{R}_{H_2,n}$.

(2) $\mathrm{R}_{H,n}(x) = x$ if $x \in \Lambda_{H,n}$.

(3) $g(\Lambda_{H,n}) = \Lambda_{gHg^{-1},n}$ and $g(\mathrm{R}_{H,n}(x)) = \mathrm{R}_{gHg^{-1}}(gx)$ if $g \in G_0$.

(4) If $n \geq n(H)$ and if $x \in \widetilde{\Lambda}^H$ then $\mathrm{val}_\Lambda(\mathrm{R}_{H,n}(x)) \geq \mathrm{val}_\Lambda(x) - c_2$.

(5) If $x \in \widetilde{\Lambda}^H$ then $\lim_{n\to\infty} \mathrm{R}_{H,n}(x) = x$.

(TS3) There exists $c_3 > 0$ and, for each open subgroup G of G_0 an integer $n(G) \geq n_1(H)$ where $H = G \cap H_0$, such that if $n(\gamma) \leq n \leq n(G)$ for $\gamma \in \widetilde{\Gamma}_H$ then $\gamma - 1$ is invertible on $X_{H,n} = (1 - \mathrm{R}_{H,n})(\widetilde{\Lambda}^H)$ and $\mathrm{val}_\Lambda((\gamma-1)^{-1}(x)) \geq \mathrm{val}_\Lambda(x) - c_3$.

For the rest of the section we shall assume (TS1), (TS2) and (TS3) are satisfied.

Remark 25.2. If H_0 is trivial then the conditions simplify, and in particular, (TS1) is automatically satisfied for any $c_1 > 0$. This will be the setting in §5, however, we produce here a more general framework for ease of future applications.

25.2 Descent to H-invariants

Lemma 25.3. *Let* H *be an open subgroup of* H_0, $a > c_1$. *Suppose* $\tau \mapsto U_\tau$ *is a continuous 1-cocycle of* H *valued in* $\mathrm{GL}_d(\widetilde{\Lambda})$ *which verifies and* $\mathrm{val}(U_\tau - 1) \geq a$ *for each* $\tau \in H$. *Then there exists a matrix* $M \in \mathrm{GL}_d(\widetilde{\Lambda})$

[8] The only difference from the usual conditions is in (TS2).

with $\mathrm{val}_\Lambda(M-1) \geq a - c_1$ *such that the cocycle* $\tau \mapsto M^{-1}U_\tau\tau(M)$ *verifies* $\mathrm{val}_\Lambda(M^{-1}U_\tau\tau(M)-1) \geq a+1$.

Proof. This is proven in the same way as [BC08, Lem. 3.2.1]. (The analogue of the condition $M-1 \in p^k\mathrm{M}_d(\widetilde{\Lambda})$, which appears in loc. cit., is $M-1 \in f^k\mathrm{M}_d(\widetilde{\Lambda})$. It is vacuous because with our assumptions f is invertible in $\widetilde{\Lambda}$). □

Corollary 25.4. *Let* H *be an open subgroup of* H_0, $a > c_1$ *and* $\tau \mapsto U_\tau$ *a continuous 1-cocycle of* H *valued in* $\mathrm{GL}_d(\widetilde{\Lambda})$. *Suppose* $\mathrm{val}(U_\tau - 1) \geq a$ *for each* $\tau \in H$. *Then there exists* $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ *such that* $\mathrm{val}_\Lambda(M-1) \geq a - c_1$ *such that the cocycle* $\tau \mapsto M^{-1}U_\tau\tau(M)$ *is trivial.*

Proof. This is proven in the same way as [BC08, Cor. 3.2.2]. □

25.3 Decompletion

The following lemma needs to be slightly modified compared to the treatment of [BC08].

Lemma 25.5. *Let* $\delta > 0$ *and let* $a, b \in \mathbf{R}$ *such that* $a \geq c_2 + c_3 + \delta$ *and* $b \geq \sup(a + c_2, 2c_2 + 2c_3 + \delta)$. *Let* H *be an open subgroup of* H_0, $n \geq n(H)$ *and* $\gamma \in \widetilde{\Gamma}_H$ *such that* $n(\gamma) \leq n$. *Finally, let*

$$U = 1 + f^kU_1 + f^kU_2$$

such that

$$U_1 \in \mathrm{M}_d(\Lambda_{H,n}), \mathrm{val}_\Lambda(U_1) \geq a - \mathrm{val}_\Lambda(f^k)$$

$$U_2 \in \mathrm{M}_d(\widetilde{\Lambda}^H), \mathrm{val}_\Lambda(U_2) \geq b - \mathrm{val}_\Lambda(f^k).$$

Then there exists $M \in \mathrm{M}_d(\widetilde{\Lambda}^H)$ *with* $\mathrm{val}(M-1) \geq b - c_2 - c_3$ *such that* $M^{-1}U\gamma(M) = 1 + f^kV_1 + f^kV_2$ *such that*

$$V_1 \in \mathrm{M}_d(\Lambda_{H,n}), \mathrm{val}_\Lambda(V_1) \geq a - \mathrm{val}_\Lambda(f^k)$$

$$V_2 \in \mathrm{M}_d(\widetilde{\Lambda}^H), \mathrm{val}_\Lambda(V_2) \geq b - \mathrm{val}_\Lambda(f^k) + \delta.$$

Proof. This again just follows the proof of [BC08, Lem. 3.2.3], but there are a few small modifications required, so we give more details. One sets[9]

$$V = f^{-k}(1-\gamma)^{-1}(1 - \mathrm{R}_{H,n})(f^kU_2)$$

[9] Compare this to the proof of loc. cit, where one takes $V = (1-\gamma)^{-1}(1-\mathrm{R}_{H,n})(U_2)$. The change is necessary for us because in general $\gamma - 1$ does not act linearly on f^k)

$$V_1 = U_1 + \mathrm{R}_{H,n}(U_2)$$
$$V_2 = f^{-k}[(1+f^kV)^{-1}U\gamma(1+f^kV) - (1+f^kU_1+f^k\mathrm{R}_{H,n}(U_2))]$$
$$M = 1+f^kV.$$

By explicit calculations, one checks using (TS2), (TS3), the $\Lambda_{H,n}$-linearity of $\mathrm{R}_{H,n}$, and the expansion

$$(1+f^kV)^{-1} = 1 - f^kV + f^{2k}V^2 - ...$$

that all the terms are well defined and that the conditions are satisifed. □

Corollary 25.6. *Let $\delta > 0$ and $b \geq 2c_2 + 2c_3 + \delta$. If $U \in \mathrm{M}_d(\widetilde{\Lambda}^H)$ has $\mathrm{val}_\Lambda(U-1) \geq b$ then there exists $M \in \mathrm{M}_d(\widetilde{\Lambda}^H)$ with $\mathrm{val}_\Lambda(M-1) \geq b - c_2 - c_3$ such that $M^{-1}U\gamma(M) \in M_d(\Lambda_{H,n})$.*

Proof. This is the same proof as that of [BC08, Cor. 3.2.4]. □

Lemma 25.7. *Let H be an open subgroup of H_0 and let $n \geq n(H)$, $\gamma \in \Gamma_H$ such that $n(\gamma) \leq n$ and $B \in \mathrm{M}_{l\times d}(\widetilde{\Lambda}^H)$ be a matrix. Suppose there are $V_1 \in \mathrm{GL}_l(\Lambda_{H,n}), V_2 \in \mathrm{GL}_d(\Lambda_{H,n})$ such that $\mathrm{val}(V_1-1), \mathrm{val}(V_2-1) > c_3$ and $\gamma(B) = V_1BV_2$. Then $B \in \mathrm{M}_{l\times d}(\Lambda_{H,n})$.*

Proof. The proof is exactly the same as that of [BC08, Lem. 3.2.5]. The only difference between that lemma and the statement appearing here is that there one further assumes $l = d$ and $B \in \mathrm{GL}_d(\Lambda_{H,n})$, but these assumptions are not used in the proof. □

25.4 Descent

Proposition 25.8. *Let $\sigma \mapsto U_\sigma$ be a continuous 1-cocycle of G_0 valued in $\mathrm{GL}_d(\widetilde{\Lambda})$. If G is an open subgroup of G_0 such that $\mathrm{val}(U_\sigma - 1) > c_1 + 2c_2 + 2c_3$ when $\sigma \in G$, and if $H = G \cap H_0$, then there exists $M \in \mathrm{M}_d(\widetilde{\Lambda})$ with $\mathrm{val}(M-1) > c_2 + c_3$ such that the 1-cocycle of G_0 given by $\sigma \mapsto V_\sigma = M^{-1}U_\sigma\sigma(M)$ is trivial on H and valued in $\mathrm{GL}_d(\Lambda_{H,n(G)})$.*

Proof. This is the same proof as that of [BC08, Prop. 3.2.6]. □

Let M^+ be a finite free $\widetilde{\Lambda}^+$-semilinear representation of G_0 and for $H \subset H_0$ open let $\Lambda^+_{H,n} = \widetilde{\Lambda}^+ \cap \Lambda_{H,n}$.

Proposition 25.9. *Suppose that G is an open subgroup of G_0 and that M^+ has a basis such that $\mathrm{val}(\mathrm{Mat}(g) - 1) > c_1 + 2c_2 + 2c_3$ for $g \in G$. Let $H = G \cap H_0$.*

Then for $n \geq n(G)$ there exists a unique free $\Lambda^+_{H,n}$-submodule $\mathbf{D}^+_{H,n}(M^+)$ of M^+ such that

(1) $\mathbf{D}^+_{H,n}(M^+)$ is fixed by H and stable by G_0.

(2) The natural map $\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H,n}} \mathbf{D}^+_{H,n}(M^+) \to M^+$ is an isomorphism. In particular, $\mathbf{D}^+_{H,n}(M^+)$ is free of rank $= \mathrm{rank} M^+$.

(3) $\mathbf{D}^+_{H,n}(M^+)$ has a basis which is c_3-fixed by G/H, meaning that for $\gamma \in G/H$ we have $\mathrm{val}(\mathrm{Mat}(\gamma) - 1) > c_3$.

Proof. We follow the proof of [BC08, Prop. 3.3.1], which is closely related. Let $v_1, ..., v_d$ be a basis of M^+ over $\widetilde{\Lambda}^+$. We get a cocycle U in $\mathrm{H}^1(G_0, \mathrm{GL}_d(\widetilde{\Lambda}^+))$. By assumption $\mathrm{val}(U_\sigma - 1) > c_1 + 2c_2 + 2c_3$ if $\sigma \in G$. By Proposition 4.8 there exists $M \in \mathrm{M}_d(\widetilde{\Lambda})$ such that $\mathrm{val}(M - 1) > c_2 + c_3$ and the cocycle $\sigma \mapsto V_\sigma = M^{-1} U_\sigma \sigma(M)$ is trivial on H and is valued in $\mathrm{GL}_d(\Lambda_{H,n(G)})$. Now $\mathrm{val}(M-1) > c_2 + c_3 > 0$ so $M \in \mathrm{GL}_d(\widetilde{\Lambda}^+)$. It follows that V is valued in $\mathrm{GL}_d(\Lambda_{H,n(G)}) \cap \mathrm{GL}_d(\widetilde{\Lambda}^+) = \mathrm{GL}_d(\Lambda^+_{H,n(G)})$.

Now let $M = (m_{ij})$ and $U_\sigma = (u_{ij})$. If we write $e_i = M v_i$ for $i = 1, ..., d$ then

$$\sigma(e_k) = \sum_j \sigma(m_{jk})\sigma(v_j) = \sum_i \sum_j u_{ij}\sigma(m_{ij}) v_i = e_k$$

for $\sigma \in H$. So $e_1, ..., e_d$ is a basis of M^+ fixed by H, and if we write $\mathbf{D}^+_{H,n}(M^+) = \oplus \Lambda^+_{H,n} e_i$ then the natural map $\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H,n}} \mathbf{D}^+_{H,n}(M^+) \to M^+$ is an isomorphism.

Further, if $\gamma \in G/H$ then $W = \mathrm{Mat}(\gamma)$ in the basis $e_1, ..., e_d$ is of the form $M^{-1} U_\sigma \sigma(M)$ where $\sigma \in G$ is a lift of γ. Using the identity

$$W - 1 = M^{-1} U_\sigma \sigma(M) - 1 = (M^{-1} - 1) U_\sigma \sigma(M) + (U_\sigma - 1)\sigma(M) + \sigma(M) - 1,$$

it follows that $\mathrm{val}(W-1) > c_2 + c_3 > c_3$. This implies that the basis $\{e_1, ..., e_d\}$ is c_3-fixed.

Finally, we show that $\mathbf{D}^+_{H,n}(M^+)$ is unique. Choose $\gamma \in C_H$ with $n(\gamma) = n$ and let $e'_1, ..., e'_d$ be another basis. Then $\mathrm{Mat}_{\{e_i\}}(\gamma) = W$ and $\mathrm{Mat}_{\{e'_i\}}(\gamma) = W'$ are both in $\mathrm{GL}_d(\Lambda^+_{H,n(G)})$ with $n \geq n(G)$ and $\mathrm{val}(W-1), \mathrm{val}(W'-1) > c_3$. Let $B \in \mathrm{GL}_d(\widetilde{\Lambda}^+)$ be the matrix expressing e'_i in terms of e_i. Then B is H-invariant and $W' = B^{-1} W \gamma(B)$. According to Lemma 4.7 we have $B \in \mathrm{GL}_d(\Lambda_{H,n})$, so $B \in \mathrm{GL}_d(\widetilde{\Lambda}^+) \cap \mathrm{GL}_d(\Lambda_{H,n}) = \mathrm{GL}_d(\Lambda^+_{H,n})$. It follows that e'_i and e_i generate the same submodule. □

Proposition 25.10. *With the notations of the previous proposition, the module $\mathbf{D}^+_{H,n}(M^+)$ admits the following characterization: it is the union of all finitely generated $\Lambda^+_{H,n}$-submodules of M^+ which are stable by G_0, fixed by H and which are generated by a c_3-fixed set of generators.*

Proof. Indeed, if we have a submodule generated by c_3-fixed elements $f_1, ..., f_l$ and if $e_1, ..., e_d$ is a c_3-fixed basis, let $B \in \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H,+})$ be a matrix expressing the f_i in terms of the e_i. We have

$$\mathrm{Mat}_{\{f_i\}}(\gamma)B = \gamma(B)\mathrm{Mat}_{\{e_i\}}(\gamma).$$

(Here $\mathrm{Mat}_{\{f_i\}}(\gamma)$ is not necessarily a uniquely determined matrix, since the submodule generated by the f_i may not be free, but this does not matter). We have $\mathrm{val}(\mathrm{Mat}_{\{f_i\}}(\gamma) - 1) > c_3$, and this implies that $\mathrm{Mat}_{\{f_i\}}(\gamma)$ is invertible. So by Lemma 4.7,

$$B \in \mathrm{M}_{l\times d}(\Lambda_{H,n}) \cap \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H,+}) = \mathrm{M}_{l\times d}(\Lambda^+_{H,n}),$$

which means the submodule generated by f_i is contained in $\mathbf{D}^+_{H,n}(M^+)$. □

We introduce two additive, $\otimes$-categories:

- $\mathrm{Mod}^{G_0}_{\widetilde{\Lambda}^+}(G)$, the category of finite free $\widetilde{\Lambda}^+$-semilinear representations of G_0 such that for some basis $\mathrm{val}(\mathrm{Mat}(g)-1) > c_1+2c_2+2c_3$ for $g \in G$.
- $\mathrm{Mod}^{G_0}_{\Lambda^+_{H,n}}(G)$, the category of finite free $\Lambda^+_{H,n}$-semilinear representations of G_0 that are fixed by $H = G \cap H_0$ and which have c_3-fixed basis.

For $n \geq n(G)$, Propositions 4.9 and 4.10 give us a functor $M^+ \mapsto \mathbf{D}^+_{H,n}(M^+)$ from $\mathrm{Mod}_{\widetilde{\Lambda}^+}(G)$ to $\mathrm{Mod}_{\Lambda^+_{H,n}}(G)$. We also have a functor $N^+ \mapsto \widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H,n}} N^+$ from $\mathrm{Mod}_{\Lambda^+_{H,n}}(G)$ to $\mathrm{Mod}_{\widetilde{\Lambda}^+}(G)$. We can then express Proposition 4.9 in the following way.

Theorem 25.11. *The functor $M^+ \mapsto \mathbf{D}^+_{H,n}(M^+)$ gives an equivalence of categories between $\mathrm{Mod}^{G_0}_{\widetilde{\Lambda}^+}(G)$ and $\mathrm{Mod}^{G_0}_{\Lambda^+_{H,n}}(G)$ for $n \geq n(G)$.*

26 Deperfection

In this section we explain how to descend (φ, Γ_K)-modules from $\widetilde{\mathbf{A}}^\dagger_{K,A}$ to $\mathbf{A}^\dagger_{K,A}$.

26.1 Verifying the Tate-Sen axioms

We take $G_0 = \Gamma_K$, and $\chi : G_0 \to \mathbf{Z}_p^\times$ is the cyclotomic character, so that $H_0 = 1$.
Let $\widetilde{\Lambda} = \widetilde{\mathbf{A}}^{(0,r]}_{K,A}$ for $1/r \in \mathbf{Z}[1/p]_{>0}$ with $r < 1$ and $\widetilde{\Lambda}^+ = \widetilde{\mathbf{A}}^{(0,r],+}_{K,A}$. Then

(i) $\widetilde{\Lambda}$ is a Tate ring, with ring of definition $\widetilde{\Lambda}^+$, with a pseudouniformizer f taken to be T, the element introduced in §2.1. Note T was introduced as an element of $\mathbf{A}^+_{K,A}$, but it can be thought of as an element of $\mathbf{A}^{(0,r]}_{K,A}$, the latter being a subring of $\widetilde{\Lambda}$ according to Proposition 2.13.
(ii) $\widetilde{\Lambda}^+$ is T-adically complete, by Corollary 2.4.
(iii) By §6 of [Co08], $T/[\varpi]$ is a unit in $\widetilde{\mathbf{A}}^{(0,r],\circ}$ if $r<1$, hence also in $\widetilde{\mathbf{A}}^{(0,r],+}_{K,A}$. We endow $\widetilde{\mathbf{A}}^{(0,r]}_{K,A}$ with the valuation $\mathrm{val}^{(0,r]}$:

$$\mathrm{val}^{(0,r]}(a) = (p/p-1)\sup\{x \in \mathbf{Z}[1/p] : a \in [\varpi]^x \widetilde{\mathbf{A}}^{(0,r],+}_{K,A}\}.$$

Note that $[\varpi]^x\widetilde{\mathbf{A}}^{(0,r],+}_{A} = T^x\widetilde{\mathbf{A}}^{(0,r],+}_{A}$ whenever T^x makes sense. It therefore induces the T-adic topology.
(iv) The group G_0 acts continuously on $\widetilde{\mathbf{A}}^{(0,r]}_{K,A}$, and is unitary for the valuation $\mathrm{val}^{(0,r]}$.

As explained in Remark 4.2, the condition (TS1) is automatic in this setting since $H_0 = 1$.

The axiom (TS2). We shall check this axiom holds in the following setting. Recalling $H_0 = 1$, we set

$$\Lambda_n = \Lambda_{H_0,n} := \varphi^{-n}(\mathbf{A}^{(0,rp^{-n}]}_{K,A}),$$

which is a closed subalgebra of $\widetilde{\mathbf{A}}^{(0,r]}_{K,A}$, and

$$\mathrm{R}_n := \mathrm{R}_{H_0,n} : \widetilde{\mathbf{A}}^{(0,r]}_{K,A} \to \varphi^{-n}(\mathbf{A}^{(0,rp^{-n}]}_{K,A})$$

is the continuous extension of the map $(\widetilde{\mathbf{A}}^{(0,r]}_{K}) \to \varphi^{-n}(\mathbf{A}^{(0,rp^{-n}]}_{K})$ constructed in §8 of [Co08] for the case $A = \mathbf{Z}_p$. This extension exists because this original map is T-adically continuous.

We shall now verify (TS2) holds. The only thing which is not immediate is condition 4, i.e. we need to check that there exists $c_2 > 0$ such that $\mathrm{val}_\Lambda(R_n(x)) \geq \mathrm{val}_\Lambda(x) - c_2$. To do this, choose $c_2 > 0$ which works in the case $A = \mathbf{Z}_p$, which is known to exist by [BC08, Prop. 4.2.1].

Suppose $x \in \widetilde{\mathbf{A}}^{(0,r]}_{K,A}$. If $\mathrm{val}_\Lambda(x) \geq (p/p-1)t$ for $t \in \mathbf{Z}[1/p]$ then $x \in [\varpi]^t\widetilde{\mathbf{A}}^{(0,r],+}_{K,A}$. We may write x as the image of a (possibly infinite) sum

$$\sum x_i \otimes a_i$$

with

$$x_i \in [\varpi]^t\widetilde{\mathbf{A}}^{(0,r],\circ}_{K}, a_i \in A.$$

so that $\mathrm{val}^{(0,r]}(x_i) \geq (p/p-1)t$.

Now $\mathrm{R}_n(x) = \sum \mathrm{R}_n(x_i) \otimes a_i$, and we have from the case $A = \mathbf{Z}_p$ that

$$\mathrm{val}^{(0,r]}(\mathrm{R}_n(x_i)) \geq (p/p-1)t - c_2.$$

Hence

$$\mathrm{R}_n(x_i) \in ([\varpi]^{t-\frac{p-1}{p}c_2} \widetilde{\mathbf{A}}_K^{(0,r],\circ}) \widehat{\otimes} A$$

for each i, which shows that

$$\mathrm{R}_n(x) \in [\varpi]^{t-\frac{p-1}{p}c_2} \widetilde{\mathbf{A}}_{K,A}^{(0,r],+},$$

so that $\mathrm{val}_\Lambda(\mathrm{R}_n(x)) \geq \frac{p}{p-1}t - c_2$. Hence (TS2) holds with the same c_2.

The axiom (TS3). We need to show that there exists $c_3 > 0$ and, for each open subgroup G of G_0 an integer $n(G)$ such that if $n \geq n(G)$ and if $n(\gamma) \leq n$ then $\gamma - 1$ is invertible on $X_n = (1 - \mathrm{R}_n)(\widetilde{\Lambda})$ and $\mathrm{val}_\Lambda((\gamma - 1)^{-1}(x)) \geq \mathrm{val}_\Lambda(x) - c_3$.

We shall show that if $c_2 = c_2(\mathbf{Z}_p)$ and $c_3 = c_3(\mathbf{Z}_p)$ work for the case $A = \mathbf{Z}_p$ as in [BC08, Prop. 4.2.1] then any $c_3' > c_2(\mathbf{Z}_p) + c_3(\mathbf{Z}_p)$ works for general A.

Let $x \in X_n = (1 - \mathrm{R}_n)(\widetilde{\Lambda})$. Let $y \in \widetilde{\Lambda} \cong \widetilde{\mathbf{A}}_{K,A}^{(0,r]}$. We may write y as the image of

$$\sum y_i \otimes a_i$$

in $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$, with $y_i \in \widetilde{\mathbf{A}}_K^{(0,r]}$ (the sum possibly infinite). If $\mathrm{val}^{(0,r]}(y) \geq (p/p-1)t$ for $t \in \mathbf{Z}[1/p]$ then $y \in [\varpi]^t \widetilde{\mathbf{A}}_{K,A}^{(0,r],+}$, which is the image of $[\varpi]^t \widetilde{\mathbf{A}}_K^{(0,r],\circ} \widehat{\otimes} A$, so one can choose $y_i \in [\varpi]^t \widetilde{\mathbf{A}}_K^{(0,r],\circ}$. This shows that we may assume $\mathrm{val}^{(0,r]}(y_i) \geq \mathrm{val}^{(0,r]}(y)$ for every i.

We let $x = (1 - \mathrm{R}_n)(y)$, so that x is the image of $\sum (1 - \mathrm{R}_n)(y_i) \otimes a_i$. Writing $x_i = (1 - \mathrm{R}_n)(y_i)$ we have

$$\mathrm{val}_\Lambda((\gamma - 1)^{-1}(x_i)) \geq \mathrm{val}_\Lambda(x_i) - c_3 \geq \mathrm{val}^{(0,r]}(y_i) - c_3,$$

which approaches zero. The sum $\sum ((\gamma - 1)^{-1}(x_i)) \otimes a_i$ therefore converges in $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$. This shows that $\gamma - 1$ is invertible on X_n.

Finally, suppose $x \in X_n$ with $\mathrm{val}^{(0,r]}(x) \geq (p/p-1)t$, we shall show that

$$\mathrm{val}^{(0,r]}((\gamma - 1)^{-1}(x)) \geq \frac{p}{p-1}t - c_2 - c_3.$$

By assumption, $x \in [\varpi]^t \widetilde{\mathbf{A}}_{K,A}^{(0,r],+}$, so we may write $x = \sum x_i \otimes a_i$ (the sum possibly infinite) with $x_i \in [\varpi]^t \widetilde{\mathbf{A}}_K^{(0,r],\circ}$. Then since $(1 - \mathrm{R}_n)$ is idempotent we have $x = \sum (1 - \mathrm{R}_n)(x_i) \otimes a_i$. Letting $y_i = (1 - \mathrm{R}_n)(x_i)$ we have $y_i \in [\varpi]^{t-\frac{p-1}{p}c_2} \widetilde{\mathbf{A}}_K^{(0,r],\circ}$ and so $(\gamma - 1)^{-1}(x) = \sum (\gamma - 1)^{-1}(y_i) \otimes a_i$. Then

$$(\gamma - 1)^{-1}(y_i) \in [\varpi]^{t-\frac{p-1}{p}c_2-\frac{p-1}{p}c_3} \widetilde{\mathbf{A}}_K^{(0,r],\circ}$$

which shows

$$\mathrm{val}_\Lambda((\gamma-1)^{-1}(x)) \geq \mathrm{val}_\Lambda(x) - c_2 - c_3,$$

which shows we can take $c_3' = c_2 - c_3$, as required.

Remark 26.1. *i*. With a little more work, once can prove that $\widetilde{\mathbf{A}}_A^{(0,r]}$ also satisfies the Tate-Sen axioms. This recovers overconvergence results appearing in §4.2 of [BC08].

ii. The following was pointed out to us by Rebecca Bellovin: if A is the ring of a pseudoaffinoid algebra with pseudouniformizer u, the rings $\widetilde{\mathbf{A}}_{A/u}^{(0,r]}$ are consistent with the reduction mod u of the rings $\widetilde{\Lambda}_{A,(0,r]}$ of [Bel20]. By taking u-adic limits, one should be able to recover the main result of [Bel20] from the results of §4.

iii. More generally one could probably phrase the results of this section as some sort of stablity of our version of the Tate-Sen axioms under base change, but we have not attempted to do so.

26.2 Descent

The following proposition is a variant of [Ber08, Thm. I.3.3].

Proposition 26.2. *If $\widetilde{M}^\dagger$ is a projective étale φ-module over $\widetilde{\mathbf{A}}_{K,A}^\dagger$ then for every $1/r \in \mathbf{Z}[1/p]_{>0}$ there exists a unique projective $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$-submodule $\widetilde{M}^{(0,r]} \subset \widetilde{M}^\dagger$ such that*

i. The natural map $\widetilde{\mathbf{A}}_{K,A}^\dagger \otimes_{\widetilde{\mathbf{A}}_{K,A}^{(0,r]}} \widetilde{M}^{(0,r]} \to \widetilde{M}^\dagger$ is an isomorphism.

ii. φ sends $\widetilde{M}^{(0,r]}$ into $\widetilde{\mathbf{A}}_{K,A}^{(0,r/p]} \otimes_{\widetilde{\mathbf{A}}_{K,A}^{(0,r]}} \widetilde{M}^{(0,r]}$, and the induced map

$$\widetilde{\mathbf{A}}_{K,A}^{(0,r/p]} \otimes^{\varphi}_{\widetilde{\mathbf{A}}_{K,A}^{(0,r]}} \widetilde{M}^{(0,r]} = \varphi^*\widetilde{M}^{(0,r]} \to \widetilde{\mathbf{A}}_{K,A}^{(0,r/p]} \otimes_{\widetilde{\mathbf{A}}_{K,A}^{(0,r]}} \widetilde{M}^{(0,r]}$$

is an isomorphism.

In particular,

1. $\widetilde{\mathbf{A}}_{K,A}^{(0,s]} \otimes_{\widetilde{\mathbf{A}}_{K,A}^{(0,r]}} \widetilde{M}^{(0,r]} = \widetilde{M}^{(0,s]}$ for $s < r$;

2. φ induces an isomorphism $\varphi^*\widetilde{M}^{(0,r]} \xrightarrow{\sim} \widetilde{M}^{(0,r/p]}$;

3. If $\widetilde{M}^\dagger$ is a (φ, Γ_K)-module, then each $\widetilde{M}^{(0,r]}$ is Γ_K-stable.

Furthermore, if $\widetilde{M}^\dagger$ is free, then so is $\widetilde{M}^{(0,r]}$.

Proof. We start by proving the statement in when $\widetilde{M}^\dagger$ is free.

To prove existence, choose any basis $e_1, .., e_d$ of $\widetilde{M}^\dagger$ over $\widetilde{\mathbf{A}}_{K,A}^\dagger$. Then $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A}^\dagger)$, and so for some r_0 and all $r \leq r_0$ we have $\mathrm{Mat}(\varphi), \mathrm{Mat}(\varphi^{-1}) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A}^{(0,r_0]})$. We take $\widetilde{M}^{(0,r]} = \bigoplus \widetilde{\mathbf{A}}_{K,A}^{(0,r]} e_i$, so that i and ii are satisfied.

For $r > r_0$ we may recursively define $\widetilde{M}^{(0,r]} := (\varphi^{-1})^*(\widetilde{M}^{(0,r/p]})$ using the isomorphism $(\varphi^{-1})^*\widetilde{M}^\dagger \xrightarrow{\sim} \widetilde{M}^\dagger$.

For uniqueness, suppose that $\widetilde{M}^{(0,r],(1)}$ and $\widetilde{M}^{(0,r],(2)}$ are two submodules satisfying conditions i and ii. Let $X \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^\dagger_{K,A})$ be the transition matrix between the two bases of $\widetilde{M}^{(0,r],(1)}$ and $\widetilde{M}^{(0,r],(2)}$ and let P_1 and P_2 be the matrices in $\mathrm{GL}_d(\widetilde{\mathbf{A}}^{(0,r/p]}_{K,A})$ of φ in these bases. Then we have the equation

$$X = P_1^{-1}\varphi(X)P_2,$$

which implies by Proposition 3.5 that $X \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^{(0,r/p]}_{K,A})$. Hence

$$\widetilde{\mathbf{A}}^{(0,r/p]}_{K,A} \otimes_{\widetilde{\mathbf{A}}^{(0,r]}_{K,A}} \widetilde{M}^{(0,r],(1)} = \widetilde{\mathbf{A}}^{(0,r/p]}_{K,A} \otimes_{\widetilde{\mathbf{A}}^{(0,r]}_{K,A}} \widetilde{M}^{(0,r],(2)},$$

and it follows from condition ii that $\varphi^*\widetilde{M}^{(0,r],(1)} = \varphi^*\widetilde{M}^{(0,r],(2)}$. Since $\varphi : \widetilde{\mathbf{A}}^{(0,r]}_{K,A} \to \widetilde{\mathbf{A}}^{(0,r/p]}_{K,A}$ is an isomorphism, this gives $\widetilde{M}^{(0,r],(1)} = \widetilde{M}^{(0,r],(2)}$.

We now prove existence and uniqueness when $\widetilde{M}^\dagger$ is only assumed projective. To show existence, given $\widetilde{M}^\dagger$, embded it as a direct summand of a free étale φ-module $\widetilde{F}^\dagger$, as we may according to Lemma 3.9. and set $\widetilde{M}^{(0,r]} = \widetilde{M}^\dagger \cap \widetilde{F}^{(0,r]}$. Let $\pi : \widetilde{F}^\dagger \to \widetilde{M}^\dagger$ be the projection.

Claim. If $x \in \widetilde{F}^{(0,r]}$ then also $\pi(x) \in \widetilde{F}^{(0,r]}$.

To see this, choose a basis $e_1, ..., e_d$ of $\widetilde{F}^{(0,r]}$. Let $\mathrm{Mat}(\varphi)$ be the matrix of φ with respect to this basis. As the proof of the existence in the free case has shown, if r is taken to be sufficiently small, we can actually arrange that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^{(0,r]}_{K,A})$ (and not just $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^{(0,r/p]}_{K,A})$). The relation $\pi \circ \varphi = \varphi \circ \pi$ shows that if the claim is true for sufficiently small r it holds for any r, so we may restrict to the case where $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^{(0,r]}_{K,A})$. Now since $\widetilde{\mathbf{A}}^\dagger_{K,A} \otimes_{\widetilde{\mathbf{A}}^{(0,r]}_{K,A}} \widetilde{F}^{(0,r]} \xrightarrow{\sim} \widetilde{F}^\dagger$, there is a matrix $\mathrm{Mat}(\pi) \in \mathrm{M}_d(\widetilde{\mathbf{A}}^\dagger_{K,A})$ representing π with respect to the basis e_i. We need to show that $\mathrm{Mat}(\pi) \in \mathrm{M}_d(\widetilde{\mathbf{A}}^{(0,r]}_{K,A})$. But again using the relation $\pi \circ \varphi = \varphi \circ \pi$ we deduce

$$\mathrm{Mat}(\varphi)\varphi(\mathrm{Mat}(\pi)) = \mathrm{Mat}(\pi)\mathrm{Mat}(\varphi),$$

which implies once more by Proposition 3.5 that $\mathrm{Mat}(\pi) \in \mathrm{M}_d(\widetilde{\mathbf{A}}^{(0,r]}_{K,A})$, as required.

Now write $\widetilde{P}^\dagger$ for the étale φ-module which is the complement of $\widetilde{M}^\dagger$ in $\widetilde{F}^\dagger$. Letting $\widetilde{P}^{(0,r]} = \widetilde{P}^\dagger \cap \widetilde{F}^{(0,r]}$, the claim implies there is a direct decomposition

$$\widetilde{F}^{(0,r]} = \widetilde{M}^{(0,r]} \oplus \widetilde{P}^{(0,r]},$$

which respects the φ-action. With this given, the fact that conditions i and ii hold for $\widetilde{F}^{(0,r]}$ implies that they also hold for $\widetilde{M}^{(0,r]}$. The existence of this decomposition also shows that $\widetilde{M}^{(0,r]}$ is projective. This finishes the proof of existence of $\widetilde{M}^{(0,r]}$ in general.

To prove uniquness, it suffices to show that if $\widetilde{M}^{(0,r],\prime}$ satisfies conditions i and ii, and if $\widetilde{F}^{(0,r]}$ is constructed as above, and if $\widetilde{M}^{(0,r]} = \widetilde{F}^{(0,r]} \cap \widetilde{M}^{\dagger}$, then $\widetilde{M}^{(0,r],\prime} = \widetilde{M}^{(0,r]}$.

But this follows from the uniqueness proved in the free case, because

$$\widetilde{M}^{(0,r],\prime} \oplus \widetilde{P}^{(0,r]} = \widetilde{F}^{(0,r]} = \widetilde{M}^{(0,r]} \oplus \widetilde{P}^{(0,r]}.$$

Applying the projection $\pi : \widetilde{F}^{\dagger} \to \widetilde{M}^{\dagger}$ we obtain $\widetilde{M}^{(0,r]} = \widetilde{M}^{(0,r],\prime}$. This concludes the proof. □

Proposition 26.3. *Let $\widetilde{M}^{(0,r]}$ be a finite free $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$ semilinear representation of Γ_K.*

There exists an open normal subgroup $\Gamma_L = \mathrm{Gal}(K_\infty/L)$ of Γ_K such that

1. There exists at most one free $\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})$-submodule $M_n^{(0,r]}$ of $\widetilde{M}^{(0,r]}$ such that

i. The natural map $\widetilde{\mathbf{A}}_{K,A}^{(0,r]} \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})} M_n^{(0,r]} \to \widetilde{M}^{(0,r]}$ is an isomorphism;

ii. $M_n^{(0,r]}$ is Γ_K-stable;

iii. $M_n^{(0,r]}$ has a c_3-fixed basis for the Γ_L-action.

2. If $n \geq n(\Gamma_L, \widetilde{M}^{(0,r]})$ then $M_n^{(0,r]}$ exists.

Proof. We start by proving 2. Choose a basis of $\widetilde{M}^{(0,r]}$. Then since $\widetilde{\mathbf{A}}_{K,A}^{(0,r],+}$ is open in $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$, there exists an open subgroup Γ_L of Γ_K such that $\mathrm{Mat}(g) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_{K,A}^{(0,r],+})$ for $g \in \Gamma_L$. By possibly shrinking Γ_L, we may assume it to be normal, and we may further assume for $g \in \Gamma_L$ we have $\mathrm{val}(\mathrm{Mat}(g)-1) > c_3$. Let $\widetilde{M}^{(0,r],+}$ to be the $\widetilde{\mathbf{A}}_{K,A}^{(0,r],+}$-span of this basis. It is a free $\widetilde{\mathbf{A}}_{K,A}^{(0,r],+} = \widetilde{\mathbf{A}}_{L,A}^{(0,r],+}$-semilinear[10] representation of Γ_L, which satisfies the assumptions in Theorem 4.11. Hence there exists a unique finite free $\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n],+})$-submodule $M_n^{(0,r],+} := \mathbf{D}_n^+(\widetilde{M}^{(0,r],+})$ of $M^{(0,r],+}$, satisfying that the natural map

$$\widetilde{\mathbf{A}}_{K,A}^{(0,r],+} \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n],+})} M_n^{(0,r],+} \to \widetilde{M}^{(0,r],+}$$

is an isomorphism, that $M_n^{(0,r],+}$ is Γ_L-stable, and which has a $\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n],+})$-basis which is c_3-fixed for the action of Γ_L.

Set $M_n^{(0,r]} = M_n^{(0,r],+}[1/T]$. In order to finish the proof of existence of $M_n^{(0,r]}$, the only part which is not yet clear is the following.

Claim: by possibly enlargning n, depending on $\widetilde{M}^{(0,r]}$, we can arrange $M_n^{(0,r]}$ to be Γ_K-stable.

[10]The equality $\widetilde{\mathbf{A}}_{K,A}^{(0,r],+} = \widetilde{\mathbf{A}}_{L,A}^{(0,r],+}$ occurs here because $K_\infty = L_\infty$.

Indeed, choose coset representatives $\{g_i\}_{i\in I}$ for Γ_K/Γ_L. For each such $g = g_i$, consider $g(M_n^{(0,r],+})$. If $e_1, ..., e_d$ is the c_3-fixed basis of $M_n^{(0,r],+}$ then $g(e_1), ..., g(e_d)$ is a basis of $g(M_n^{(0,r],+})$. It may not be c_3-fixed, however. By continuity, we may find a nontrivial $\gamma \in \Gamma_L$, with $\mathrm{val}(\mathrm{Mat}_{\{g(e_i)\}}(\gamma) - 1)) > c_3$, and by taking n larger we can arrange that $n \geq n(\gamma)$. Lemma 4.7 then implies that $g(M_n^{(0,r],+}) = \mathbf{D}_n^+(g(\widetilde{M}^{(0,r],+}))$. Choosing n large enough for all the g_i simultaneously, we may arrange that $g(M_n^{(0,r],+}) = \mathbf{D}_n^+(g(\widetilde{M}^{(0,r],+}))$ for every $g \in \Gamma_K$.

With this in mind, let $g \in \Gamma_K$. Then for some $t \in \mathbf{Z}$, we have by continuity

$$g(\widetilde{M}^{(0,r],+}) \subset T^{-t}\widetilde{M}^{(0,r],+},$$

so that

$$g(M_n^{(0,r],+}) = \mathbf{D}_n^+(g(\widetilde{M}^{(0,r],+})) \subset \mathbf{D}_n^+(T^{-t}\widetilde{M}^{(0,r],+}) = T^{-t}M_n^{(0,r],+}.$$

Every element of $M_n^{(0,r]}$ can be written in the form $T^t m$ with $m \in M_n^{(0,r],+}$, and since

$$g(T^t m) = [g(T^t)/T^t]T^t g(m) \in M_n^{(0,r]},$$

we see that $M_n^{(0,r]}$ is Γ_K-stable. This proves the claim.

Finally, we show uniqueness. Suppose $M_n^{(0,r],(1)}$ and $M_n^{(0,r],(2)}$ are two submodules satisfying these properties. Let $M_n^{(0,r],(1),+}$ be the $\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n],+})$-span of a c_3-fixed basis in $M_n^{(0,r],(1)}$. Let $\widetilde{M}^{(0,r],(1),+}$ be the image of

$$\widetilde{\mathbf{A}}_{K,A}^{(0,r],+} \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n],+})} M_n^{(0,r],(1)}$$

in $\widetilde{M}^{(0,r]}$. Define $M_n^{(0,r],(2),+}$ and $\widetilde{M}^{(0,r],(2),+}$ similarly. Then we have for some sufficiently large t the inclusions

$$T^t\widetilde{M}^{(0,r],(2),+} \subset \widetilde{M}^{(0,r],(1),+} \subset T^{-t}\widetilde{M}^{(0,r],(2),+},$$

which implies upon applying $\mathbf{D}_n^+$ that

$$T^t M_n^{(0,r],(2),+} \subset M_n^{(0,r],(1),+} \subset T^{-t}M_n^{(0,r],(2),+}.$$

Hence $M_n^{(0,r],(1)} = M_n^{(0,r],(2)}$. □

26.3 The equivalence of categories

The following is an analogue of Lemma 3.9, with the action of φ replaced by the Γ_K action.

Lemma 26.4. *Let R be a complete topological ring with a continuous action of Γ_K, and let M be a projective R-semilinear representation of Γ_K. Then there exists a finite free R-semilinear representation of Γ_K which contains M as a direct summand.*

Proof. Choose a topological generator γ of Γ_K. Then by the same argument proving Lemma 3.9, we may find a finite free R-module F endowed with an isomorphism $\gamma^*F \cong F$ and which contains M as a direct summand as a semilinear representation of $\gamma^{\mathbf{Z}}$. Namely, to construct F, choose first a free R-module G and a projective R-module P together with an isomorphism $M \oplus P \cong G$. Since G is free, we may declare a basis fixed by γ to give G the structure of a semilinear representation of γ. Then $G \oplus P$ also admits such a structure, by taking the composite

$$\gamma^*(G \oplus P) \cong \gamma^*G \oplus \gamma^*P \cong G \oplus \gamma^*P$$
$$= M \oplus P \oplus \gamma^*P \cong \gamma^*M \oplus P \oplus \gamma^*P \cong \gamma^*G \oplus P \cong G \oplus P.$$

We then take $F := (G \oplus P) \oplus M$.

To extend the action of $\gamma^{\mathbf{Z}}$ on F to all of Γ_K we argue as follows. First, for any $k \in \mathrm{N}$, the subgroups $\gamma^{\mathbf{Z}}$ and $\gamma^{p^k \mathbf{Z}_p}$ generate Γ_K, so it suffices to explain how to extend the action of $\gamma^{p^k \mathbf{Z}}$ to $\gamma^{p^k \mathbf{Z}_p}$ for some k. Take k sufficiently large that $\gamma^{p^k} - 1$ acts topologically nilpotently. This is possible because of the explicit construction of the action of γ on F we have described above. The extension is then given by

$$\gamma^{p^k a}(x) := \sum_{n \geq 0} \binom{a}{n} (\gamma^{p^k} - 1)^n(x),$$

and the proof is concluded. □

Theorem 26.5. *The functor $M^\dagger \mapsto \widetilde{M}^\dagger$ induces an equivalence of categories from the category of projective étale (φ, Γ_K)-modules over $\widetilde{\mathbf{A}}^\dagger_{K,A}$ to the category of projective étale (φ, Γ_K)-modules over $\widetilde{\mathbf{A}}^\dagger_{K,A}$.*

Proof. We start by proving full faithfulness. As usual, we reduce to proving that if $M^\dagger$ is a projective étale (φ, Γ_K)-module over $\widetilde{\mathbf{A}}^\dagger_{K,A}$ then $(M^\dagger)^{\varphi, \Gamma_K} = (\widetilde{M}^\dagger)^{\varphi, \Gamma_K}$.

The injectivity of $(M^\dagger)^{\varphi, \Gamma_K} \to (\widetilde{M}^\dagger)^{\varphi, \Gamma_K}$ is easy, the reason being that we already know that $\mathbf{A}^\dagger_{K,A} \to \widetilde{\mathbf{A}}^\dagger_{K,A}$ is injective, so it remains injective after tensoring with the projective, hence flat, $\mathbf{A}^\dagger_{K,A}$-module $M^\dagger$. It then still remains injective after taking fixed points.

For the surjectivity we argue as follows. Let $x \in (\widetilde{M}^\dagger)^{\varphi, \Gamma_K}$. Then $x \in (\widetilde{M}^{(0,r]})^{\varphi, \Gamma_K}$ for some $r > 0$. Take a finite free $\mathbf{A}^\dagger_{K,A}$-semilinear representation of Γ_K denoted $F^\dagger$ which contains $M^\dagger$ as a direct summand as we may according to Lemma 5.4. Choose a basis $e_1, ..., e_d$ of $F^\dagger$. We can write $\sum a_i e_i = x$ with $a_i \in \widetilde{\mathbf{A}}_{K,A}^{(0,r]}$. Choose a nontorsion $\gamma \in \Gamma_K$. By possibly making r smaller, we can arrange that $\mathrm{Mat}_{\{e_i\}}(\gamma)$ also has coefficients in $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$. Since x is fixed by γ, we obtain the equation of $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$-valued matrices

$$\mathrm{Mat}_{\{e_i\}}(\gamma)\gamma(a) = a.$$

where a is the vector of the a_i. Replacing γ by $\gamma^{p^k}$for $k \gg 0$ we may arrange in addition that $\mathrm{val}(\mathrm{Mat}_{\{e_i\}}(\gamma) - 1) > c_3$. So by Lemma 4.7 we know that for $n \gg 0$, we have $a_i \in \varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^{-n}]})$, which is contained in $\varphi^{-n}(\mathbf{A}_{K,A}^\dagger)$. Hence $x \in \varphi^{-n}(\mathbf{A}_{K,A}^\dagger) \otimes_{\mathbf{A}_{K,A}^\dagger} M^\dagger$, and since x is fixed by φ, we see after n successive applications of φ that $x \in M^\dagger$. This shows that $x \in M^\dagger \cap (\widetilde{M}^\dagger)^{\varphi,\Gamma_K} = (M^\dagger)^{\varphi,\Gamma_K}$, as required.

Next, we prove essential surjectivity. Let $\widetilde{M}^\dagger$ be a projective étale φ-module over $\widetilde{\mathbf{A}}_{K,A}^\dagger$. Let $\widetilde{M}^{(0,r]}$ be as in Proposition 5.2. Then $\widetilde{M}^{(0,r]}$ is a projective $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$-semilinear representation of Γ_K, so by Lemma 5.4, we may find a free $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$-semilinear Γ_K-representation $\widetilde{F}^{(0,r]}$ and a projective $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$-semilinear Γ_K-representation $\widetilde{P}^{(0,r]}$ such that $\widetilde{M}^{(0,r]} \oplus \widetilde{P}^{(0,r]} = \widetilde{F}^{(0,r]}$. By Proposition 5.3, we can find for $n \gg 0$ a free $\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})$-submodule $F_n^{(0,r]} \subset \widetilde{F}^{(0,r]}$ which is Γ_K-stable.

Claim. Let $\pi : \widetilde{F}^{(0,r]} \to \widetilde{M}^{(0,r]}$ denote the projection. If $x \in F_n^{(0,r]}$ then $\pi(x) \in F_n^{(0,r]}$.

To see this, choose a basis $e_1, ..., e_d$ of $F_n^{(0,r]}$. Choose $\gamma \in \Gamma_K$ nontorsion and let $\mathrm{Mat}(\gamma)$ be the matrix of γ with respect to this basis. Since $F_n^{(0,r]}$ spans $\widetilde{F}^{(0,r]}$ as an $\widetilde{\mathbf{A}}_{K,A}^{(0,r]}$-module, there is a matrix $\mathrm{Mat}(\pi) \in \mathrm{M}_d(\widetilde{\mathbf{A}}_{K,A}^{(0,r]})$ representing π with respect to the basis e_i. The relation $\pi \circ \gamma = \gamma \circ \pi$ gives

$$\mathrm{Mat}(\gamma)\gamma(\mathrm{Mat}(\pi)) = \mathrm{Mat}(\pi)\mathrm{Mat}(\gamma),$$

and after replacing γ by γ^{p^k} for $k \gg 0$ we may assume $\mathrm{val}(\mathrm{Mat}(\gamma) - 1) > c_3$. This implies by Lemma 4.7 that $\mathrm{Mat}(\pi) \in \mathrm{M}_d(\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]}))$, as required.

Set $M_n^{(0,r]} = F_n^{(0,r]} \cap \widetilde{M}^{(0,r]}$ and $P_n^{(0,r]} = F_n^{(0,r]} \cap \widetilde{P}^{(0,r]}$. Then the claim shows that $M_n^{(0,r]} \oplus P_n^{(0,r]} = F_n^{(0,r]}$. The isomorphism $\widetilde{\mathbf{A}}_{K,A}^{(0,r]} \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})} F_n^{(0,r]} \xrightarrow{\sim} \widetilde{F}^{(0,r]}$ implies that $\widetilde{\mathbf{A}}_{K,A}^{(0,r]} \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})} M_n^{(0,r]} \xrightarrow{\sim} \widetilde{M}^{(0,r]}$ and hence also

$$\widetilde{\mathbf{A}}_{K,A}^\dagger \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})} M_n^{(0,r]} \xrightarrow{\sim} \widetilde{M}^\dagger.$$

It is also clear that $M_n^{(0,r]}$ is Γ_K-stable. We set

$$M^\dagger := \mathbf{A}_{K,A}^\dagger \otimes_{\mathbf{A}_{K,A}^{(0,r/p^n]}} \varphi^n(M_n^{(0,r]}),$$

then $M^\dagger$ is a Γ_K-stable, projective $\mathbf{A}^\dagger_{K,A}$-submodule of $\widetilde{M}^\dagger$, and the natural map $\widetilde{\mathbf{A}}^\dagger_{K,A} \otimes_{\mathbf{A}^\dagger_{K,A}} M^\dagger \xrightarrow{\sim} \widetilde{M}^\dagger$ is an isomorphism. It remains to show that $M^\dagger$ is φ-stable and an étale φ-module. To do this, simply notice that the uniqueness of $F_n^{(0,r]}$ implies the uniqueness of $M_n^{(0,r]}$, and so if n is sufficiently large so that $M_{n-1}^{(0,r/p]}$ and $M_n^{(0,r]}$ are both defined, we get

$$\varphi(M_n^{(0,r]}) = M_{n-1}^{(0,r/p]} = \varphi^{-(n-1)}(\mathbf{A}_{K,A}^{(0,r/p^{n-1}]}) \otimes_{\varphi^{-n}(\mathbf{A}_{K,A}^{(0,r/p^n]})} M_n^{(0,r]},$$

which implies both that $M^\dagger$ is φ-stable and that the action of φ is invertible. This finishes the proof. □

27 The main theorem

In this section, we conclude with the proof of overconvergence of (φ, Γ_K)-modules over $\mathbf{A}_{K,A}$.

Lemma 27.1. *The functor $M \mapsto \widetilde{M} := \widetilde{\mathbf{A}}_{K,A} \otimes_{\mathbf{A}_{K,A}} M$ from projective étale (φ, Γ_K)-modules over $\mathbf{A}_{K,A}$ to projective étale (φ, Γ_K)-modules over $\widetilde{\mathbf{A}}_{K,A}$ is fully faithful.*

Proof. As usual, using Lemma 3.10 we can reduce to checking that the natural map $M^{\varphi=1} \xrightarrow{\sim} \widetilde{M}^{\varphi=1}$ is an isomorphism. Since $\mathbf{A}_{K,A}$ and $\widetilde{\mathbf{A}}_{K,A}$ are p-adically complete, and since M is free, we have compatible isomorphisms $M \cong \varprojlim_a M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A/p^a}$ and $\widetilde{M} \cong \varprojlim_n \widetilde{M} \otimes_{\widetilde{\mathbf{A}}_{K,A}} \widetilde{\mathbf{A}}_{K,A/p^a}$. Since φ-invariants are compatible with inverse limits, we are reduced to the case where $p^a A = 0$. But in this case the statement is known, by [EG19, Prop. 2.6.6]. □

Finally, we can prove the main theorem.

Theorem 27.2. *The functor $M^\dagger \mapsto M := \mathbf{A}_{K,A} \otimes_{\mathbf{A}^\dagger_{K,A}} M^\dagger$ induces an equivalence of categories from the category projective étale (φ, Γ_K)-modules over $\mathbf{A}^\dagger_{K,A}$ to the category of projective étale (φ, Γ_K)-modules over $\mathbf{A}_{K,A}$.*

Proof. This follows by combining Theorem 3.13, Theorem 5.5 and Lemma 6.1. □

References

[Bel20] Bellovin R., *Galois representations over pseudorigid spaces*, arXiv:2002.06687, 2020.

[Ber08] Berger L., *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Astérisque No. 319 (2008), 13--38.

[BC08] Berger L., Colmez P., *Familles de représentations de de Rham et monodromie p-adique*, Astérisque No. 319 (2008), 303--337.

[Ch09] Chenevier, G., *Une application des variétés de Hecke des groupes unitaires,* preprint available at http://www.math.polytechnique.fr/~chenevier/articles/ (2009).

[CC98] Cherbonnier F., Colmez P., *Représentations p-adiques surconvergentes*, Invent. Math. 133 (1998), no. 3, p. 581–611.

[Co98] Colmez P., *Théorie d'Iwasawa des représentations de de Rham d'un corps local,* Annals of Math. 148 (1998), 485–571.

[Co08] Colmez P., *Espaces Vectoriels de dimension finie et représentations de de Rham*, Astérisque 319 (2008), 117–186.

[dSP19] de Shalit E., Porat G., *Induction and restriction of (φ, Γ)-modules*, Muenster J. Math., vol. 12 (2019), 215-237.

[EG19] Emerton M., Gee T., *Moduli stacks of étale (ϕ, Γ)-modules and the existence of crystalline lifts*, arXiv:1908.07185. 2019.

[EG21] Emerton M., Gee T., *"Scheme-theoretic images" of certain morphisms of stacks*, Algebraic Geometry 8 (2021), no. 1, 1–132.

[EGH22] Emerton M., Gee T., Hellmann E.,*"An introduction to the categorical p-adic Langlands program"*, notes for the IHES summer school on the Langlands program (2022).

[Ga19] Gao H., "*Wach models and overconvergence of étale (φ, Γ)-modules,* Journal of Number Theory, Volume 205, (2019), 277-299.

[KL11] Kedlaya K, Liu R. *On families of (φ, Γ)-modules.* Algebra & Number Theory, no. 7 (2011): 943-967.

[Sta] The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu.