THE UNIVERSITY OF CHICAGO

TOWARDS A NONSTANDARD FOURIER ANALYSIS IN AUTOMORPHIC FORMS: SOME RESULTS ON TWO TOY EXAMPLES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
SHUYANG CHENG

CHICAGO, ILLINOIS
JUNE 2016

To Carrie and Isaac

TABLE OF CONTENTS

ABSTRACT

This thesis consists of two main parts.

In the first part an integral transform between spaces of nonstandard test functions on the affine space of dimension n is constructed. The integral transform satisfies a summation formula of Poisson type, which is derived from an analogue of the Arthur-Selberg trace formula for the Lie algebra of $n \times n$ matrices.

In the second part a proof of Conjecture 9.12 of Braverman–Kazhdan in their article *γ-functions of representations and lifting* on the acyclicity of their ℓ-adic γ-sheaves over certain affine spaces is given for $\mathrm{GL}(n)$.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to his advisor Ngô Bao Châu, without whose continual support and encouragement this work would not have been possible. He also wishes to thank Bill Casselman, Pierre-Henri Chaudouard, Jayce Getz, Eddie Herman, Robert Kottwitz and Yiannis Sakellaridis for illuminating conversations and valuable suggestions. Finally I would like to thank Carrie and Isaac.

CHAPTER 1

INTRODUCTION

1.1 Background and motivation

Classical Fourier analysis has found application in modular forms since the creation of the theory in the early nineteenth century. As early as 1828, by directly applying the Poisson summation formula Jacobi has shown that his theta function satisfies a modular identity, a result which he attributes to Poisson. The modularity of Jacobi's theta function has been used by Riemann in his seminal 1859 paper to derive the functional equation of the zeta function which now bears his name. The argument of Riemann has subsequently been adapted to establish the functional equations of L-functions twisted by Dirichlet characters and Dedekind zeta functions. The class of such L-functions which satisfy functional equations has been completed by Hecke in 1920 with the introduction of his Größencharakters and the resultant Hecke theta functions and Hecke L-functions.

The development of class field theory in the early twentieth century has lead to the introduction of idèles by Chevalley in 1936. It has since been understood that Hecke's Größencharakters are precisely unitary characters of idèle class groups, or retrospectively automorphic forms on GL(1). In his 1950 thesis, by systematically applying local and global Fourier analysis on the adèles Tate has derived the functional equations of principal L-functions on GL(1) directly from his adèlic Poisson summation formula, circumventing the modularity of the associated theta series. In 1972 Godement and Jacquet have extended Tate's method to principal L-functions of degree n by embedding $\mathrm{GL}(n)$ into the algebra $\mathrm{M}(n)$ of $n \times n$ matrices and applying the higher dimensional adèlic Poisson summation formula for the vector space $\mathrm{M}(n)$.

The notion of a general automorphic L-function has been conceived by Langlands in his 1967 letter to Weil. Such L-functions $L(s, \pi, r)$ depend on an automorphic representation π of a reductive group G and dually a complex representation r of its dual group ${}^LG$, the principal

L-functions being the special case when r is the standard n-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$. Conjecturally all the automorphic L-functions could be reduced to principal ones by functoriality lifting along r, conversely suitable analytic properties including a functional equation of such automorphic L-functions imply the functoriality conjecture for r by the Converse theorem of Cogdell and Piatetski-Shapiro.

Recently in the work of Braverman and Kazhdan, Ngô, Lafforgue and Sakellaridis, a far-reaching generalization of the argument of Godement and Jacquet to $L(s, \pi, r)$ for certain nonstandard r has been proposed. The main idea is to construct a monoidal open embedding of G into a reductive algebraic monoid $M(r)$ generalizing the embedding of $\mathrm{GL}(n)$ into $\mathrm{M}(n)$. Conjecturally there exist r-analogues of local and global Fourier transforms on $M(r)$, the functional equation of $L(s, \pi, r)$ would then follow from a summation formula of Poisson type on $M(r)$. In general $M(r)$ is nonlinear and possibly singular, hence the conjectural Poisson summation formula would involve nonstandard Fourier transforms and spaces of nonstandard test functions. In his 2013 preprint Lafforgue has shown that such summation formulae are in fact equivalent to the existence of functoriality liftings to $\mathrm{GL}(r)$.

1.2 About this thesis

This thesis represents the author's first steps towards a Fourier analysis on the monoid $M(r)$ for nonstandard r, in the form of a collection of results organized around two toy examples.

In the second chapter a toy example of such a summation formula of Poisson type is constructed on the space $\mathcal{A}_n$ of characteristic polynomials of $n \times n$ matrices. In analogy with the theory of Godement and Jacquet, this consists of the construction of nonstandard Schwartz spaces $\mathcal{S}_0(\mathcal{A}_n(\mathbb{Q}_p))$ and $\mathcal{S}_1(\mathcal{A}_n(\mathbb{Q}_p))$ on $\mathcal{A}_n(\mathbb{Q}_p)$ where $\mathbb{Q}_v$ is either $\mathbb{Q}_p$ or $\mathbb{R}$ such that $\mathcal{S}_0(\mathcal{A}_n(\mathbb{Q}_v))$ is dense in $L^2(\mathcal{A}_n(\mathbb{Q}_v))$, together with an invertible integral transform $\mathcal{H}_v$ from $\mathcal{S}_0(\mathcal{A}_n(\mathbb{Q}_v))$ to $\mathcal{S}_1(\mathcal{A}_n(\mathbb{Q}_v))$ such that the following holds.

Proposition (Poisson summation formula) *For each point X in $\mathcal{A}_n(\mathbb{Q})$ there exists a nonzero* *constant $a(X)$ such that for all*

$$\varphi = \varphi_2 \otimes \varphi_3 \otimes \cdots \otimes \varphi_p \otimes \cdots \otimes \varphi_\infty \quad \in \quad \bigotimes_p \mathcal{S}_0(\mathcal{A}_n(\mathbb{Q}_p)) \otimes \mathcal{S}_0(\mathcal{A}_n(\mathbb{R}))$$

equal to a certain $\phi_{0,v}$ almost everywhere and satisfying a certain local cuspidality condition at two places,

$$\sum_{X \in \mathcal{A}_n(\mathbb{Q})} a(X) \cdot \varphi(X) \quad = \quad \sum_{X \in \mathcal{A}_n(\mathbb{Q})} a(X) \cdot \mathcal{H}\varphi(X)$$

where

$$\mathcal{H}\varphi \quad = \quad \mathcal{H}_2\varphi_2 \otimes \mathcal{H}_3\varphi_3 \otimes \cdots \otimes \mathcal{H}_p\varphi_p \otimes \cdots \otimes \mathcal{H}_\infty\varphi_\infty.$$

In general one of the difficulties of constructing the Poisson summation formula on $M(r)$ lies in the existence of singularities on $M(r) - G$. In the toy example, even though the space $\mathcal{A}_n$ is isomorphic to the affine space, it is understood as the quotient space $\mathrm{gl}(n)/\mathrm{GL}(n)$ which has a natural open locus $\mathcal{A}_n^{\mathrm{reg}}$ parametrizing regular semisimple orbits, whose complement contains the singularities of the quotient map from $\mathrm{gl}(n)$ to $\mathrm{gl}(n)/\mathrm{GL}(n)$.

In the third chapter the r-analogue of local Fourier transforms for nonstandard r is considered over finite fields. This toy example has been investigated earlier by Braverman and Kazhdan, their main result is the construction of an irreducible ℓ-adic perverse sheaf $\Phi_{\psi,r}$ on G, generalizing the ℓ-adic Artin–Schreier sheaf $\mathcal{L}_\psi$ on an affine space. They have conjectured that $\Phi_{\psi,r}$ corresponds to the r-analogue of the Fourier kernel function over finite fields under Grothendieck's function-sheaf dictionary, generalizing the relation between $\mathcal{L}_\psi$ and the Fourier–Deligne transform. In the same paper they have also given a partial proof of the previous conjecture assuming the acyclicity of $\Phi_{\psi,r}$ over certain affine spaces, generalizing the acyclicity of $\mathcal{L}_\psi$. It is this last acyclicity conjecture in the case when G is $\mathrm{GL}(n)$ that is established in the third chapter.

CHAPTER 2

HARISH-CHANDRA TRANSFORM ON THE SPACE OF CHARACTERISTIC POLYNOMIALS

2.1 Preliminaries

2.1.1 Notations and definitions

(1.1.1) Let G be a connected reductive group defined over the field of rational numbers $\mathbb{Q}$, let $\mathfrak{g}$ be its Lie algebra equipped with the adjoint action of G from the right. More generally an algebraic group will be denoted by a capital roman letter, its Lie algebra by the corresponding lowercase fraktur letter, except for $\mathfrak{a}$ which is reserved for a Euclidean vector space.

Denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ at a place v. If S is a finite set of places, denote by $\mathbb{Q}_S$ the direct product of $\mathbb{Q}_v$ for all v in S. Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$. Define S-local and global norms by

$$\forall x \in \mathbb{Q}_S \quad |x|_S = \prod_{v \in S} |x|_v, \qquad \forall x \in \mathbb{A} \quad |x|_{\mathbb{A}} = \lim_{S} |x|_S. \tag{1.1.1.1}$$

The local norms are normalized in such a way that

$$\forall x \in \mathbb{Q} - \{0\} \qquad |x|_{\mathbb{A}} = 1. \tag{1.1.1.2}$$

The groups $\mathrm{G}(\mathbb{Q}_v)$, $\mathrm{G}(\mathbb{Q}_S)$ and $\mathrm{G}(\mathbb{A})$ of $\mathbb{Q}_v$, $\mathbb{Q}_S$ and $\mathbb{A}$-valued points of G are locally compact with respect to the analytic topology. The group $\mathrm{G}(\mathbb{Q})$ of $\mathbb{Q}$-valued points of G is a discrete subgroup of $\mathrm{G}(\mathbb{A})$ with respect to the analytic topology.

(1.1.2) Fix a minimal parabolic subgroup P_0 of G. Fix a Levi subgroup M_0 of P_0 with split component A_0. A parabolic subgroup P of G is said to be *standard* if P contains P_0.

Denote by $\mathrm{N_P}$ the unipotent radical of P, by $\mathrm{M_P}$ the unique Levi subgroup of P containing $\mathrm{M_0}$, by $\mathrm{A_P}$ the split component of $\mathrm{M_P}$. Such a Levi subgroup $\mathrm{M_P}$ is said to be *standard*. Denote by $\overline{\mathrm{P}}$ the parabolic subgroup group opposite to P, by $\overline{\mathrm{N}}_{\mathrm{P}}$ its unipotent radical. To simplify notations the standard Levi, split and unipotent components of P_i will be denoted by M_i, A_i and N_i where i is a natural number.

Let M and L be Levi subgroups of G such that M is contained in L, denote by $\mathcal{F}^{\mathrm{L}}(\mathrm{M})$ the set of parabolic subgroups of L that contain M, by $\mathcal{P}^{\mathrm{L}}(\mathrm{M})$ the set of parabolic subgroups of L whose Levi component is M, by $\mathcal{L}^{\mathrm{L}}(\mathrm{M})$ the set of Levi subgroups of L that contain M. To simplify notations denote by $\mathcal{F}(\mathrm{M})$ the set $\mathcal{F}^{\mathrm{G}}(\mathrm{M})$, by $\mathcal{F}^{\mathrm{L}}$ the set $\mathcal{F}^{\mathrm{L}}(\mathrm{M_0})$, by $\mathcal{F}$ the set $\mathcal{F}^{\mathrm{G}}(\mathrm{M_0})$. Similar notations apply to $\mathcal{P}$ and $\mathcal{L}$.

Let P be a parabolic subgroup of G, denote by $X(\mathrm{M_P})$ the group of rational characters of $\mathrm{M_P}$

$$(1.1.2.1) \qquad X(\mathrm{M_P}) \;=\; \mathrm{Hom}_{\mathrm{Grp}/\mathbb{Q}}(\mathrm{M_P}, \mathrm{GL}(1,\mathbb{Q})).$$

Let $\mathfrak{a}_{\mathrm{P}}$ denote the real vector space

$$(1.1.2.2) \qquad \mathfrak{a}_{\mathrm{P}} \;=\; \mathrm{Hom}_{\mathbb{Z}}(X(\mathrm{M_P}), \mathbb{R}),$$

let $\mathfrak{a}_{\mathrm{P}}^*$ denote the dual space

$$(1.1.2.3) \qquad \mathfrak{a}_{\mathrm{P}}^* \;=\; X(\mathrm{M_P}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let

$$(1.1.2.4) \qquad \Phi_{\mathrm{P}},\ \Delta_{\mathrm{P}} \subset \mathfrak{a}_{\mathrm{P}}^*, \qquad \Phi_{\mathrm{P}}^{\vee},\ \Delta_{\mathrm{P}}^{\vee} \subset \mathfrak{a}_{\mathrm{P}}$$

denote respectively the set of *roots*, *simple roots*, *coroots*, *simple coroots* of $\mathrm{A_P}$ in $\mathfrak{g}$ and $\mathfrak{n}_{\mathrm{P}}$.

The quadruple $(X(\mathrm{M}_0), \Phi_0, X(\mathrm{M}_0)^*, \Phi_0^\vee)$ is called the *root datum* of G.

Let P_1 and P_2 be parabolic subgroups of G with P_1 contained in P_2, denote

$$\mathrm{N}_1^2 = \mathrm{N}_1 \cap \mathrm{M}_2, \qquad \overline{\mathrm{N}}_1^2 = \overline{\mathrm{N}}_1 \cap \mathrm{M}_2. \tag{1.1.2.5}$$

Let Δ_1^2 and $\Delta_1^{2,\vee}$ be the set of simple roots and coroots of A_1 in $\mathfrak{n}_1^2$. There are canonical splittings

$$\mathfrak{a}_1 = \mathfrak{a}_1^2 \oplus \mathfrak{a}_2, \qquad \mathfrak{a}_1^* = \mathfrak{a}_1^{2,*} \oplus \mathfrak{a}_2^*. \tag{1.1.2.6}$$

The sets Δ_1^2 and $\Delta_1^{2,\vee}$ form bases of $\mathfrak{a}_1^{2,*}$ and $\mathfrak{a}_1^2$. The respective dual bases are called the *coweights* and *weights* and denoted by $\hat{\Delta}_1^2$ and $\hat{\Delta}_1^{2,\vee}$.

Let $\mathrm{W}_0^{\mathrm{G}}$ be the *Weyl group* of the pair $(\mathrm{G}, \mathrm{A}_0)$. Let M_1 and M_2 be two Levi subgroups of G, define the *Weyl set* $\mathrm{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ to be the set of linear isomorphisms from $\mathfrak{a}_1$ to $\mathfrak{a}_2$ obtained by restricting the action of elements of the Weyl group. The group $\mathrm{W}_0^{\mathrm{G}}$ operates on the root datum of G, hence on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$. Fix Euclidean inner products on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ which are compatible with each other and the underlying root datum, hence invariant under the Weyl group action. The inner product on $\mathfrak{a}_0$ induces an inner product on $\mathfrak{a}_1^2$.

Let τ_1^2 denote the characteristic function on $\mathfrak{a}_0$ of the points that are positive with respect to every element of Δ_1^2, let $\hat{\tau}_1^2$ denote the characteristic function on $\mathfrak{a}_0$ of the set of points that are positive with respect to every element of $\hat{\Delta}_1^{2,\vee}$:

$$\tau_1^2 = \mathbb{I}_{\left\{H \in \mathfrak{a}_0 \colon \alpha(H) > 0, \ \forall \alpha \in \Delta_1^2\right\}}, \qquad \hat{\tau}_1^2 = \mathbb{I}_{\left\{H \in \mathfrak{a}_0 \colon \varpi(H) > 0, \ \forall \varpi \in \hat{\Delta}_1^{2,\vee}\right\}}. \tag{1.1.2.7}$$

To simplify notations denote by τ_1 the set τ_1^{G}, by $\hat{\tau}_1$ the set $\hat{\tau}_1^{\mathrm{G}}$.

(1.1.3) Define $\mathrm{G}(\mathbb{A})^1$ to be the subgroup of $\mathrm{G}(\mathbb{A})$ consisting of elements g of $\mathrm{G}(\mathbb{Q})$ such that

$$(1.1.3.1) \qquad \forall \chi \in X(\mathrm{G}) \qquad |\chi(g)|_{\mathbb{A}} = 1.$$

Fix an admissible maximal compact subgroup

$$(1.1.3.2) \qquad \mathrm{K} \; = \; \prod_v \mathrm{K}_v$$

of $\mathrm{G}(\mathbb{A})$ such that the Iwasawa decomposition

$$(1.1.3.3) \qquad \begin{aligned} \mathrm{G}(\mathbb{A}) &= \mathrm{P}(\mathbb{A})\mathrm{K} \\ &= \mathrm{M}_{\mathrm{P}}(\mathbb{A})^1 \exp(\mathfrak{a}_{\mathrm{P}}) \mathrm{N}_{\mathrm{P}}(\mathbb{A})\mathrm{K} \end{aligned}$$

holds. Denote by

$$H_{\mathrm{P}} \; : \; \mathrm{G}(\mathbb{A}) \quad \longrightarrow \quad \mathfrak{a}_{\mathrm{P}}$$

the natural projection.

The *Tamagawa measure* on $\mathrm{G}(\mathbb{A})$ is the measure induced from the choice of a basis of rational 1-forms on G, which is well-defined by the product formula (1.1.1.2). The Euclidean vector space $\mathfrak{a}_{\mathrm{G}}$ has a translation invariant measure, which without loss of generality assigns the coweight lattice

$$(1.1.3.4) \qquad \mathrm{Hom}_{\mathbb{Z}}(X(\mathrm{G}), \mathbb{Z}) \;\subset\; \mathfrak{a}_{\mathrm{G}}$$

covolume one.

The various measures are compatible under the Iwasawa decomposition in the sense that

$$(1.1.3.5)\qquad \begin{aligned}&\int_{\mathrm{G}(\mathbb{A})} f(g)\,\mathrm{d}g\\ =\;& \int_{\mathrm{M_P}(\mathbb{A})^1}\int_{\mathfrak{a}_\mathrm{P}}\int_{\mathrm{N_P}(\mathbb{A})}\int_{\mathrm{K}} f(mank)\,\mathrm{d}k\mathrm{d}n\mathrm{d}a\mathrm{d}m\\ =\;& \int_{\mathrm{M_P}(\mathbb{A})^1}\int_{\mathfrak{a}_\mathrm{P}}\int_{\mathrm{N_P}(\mathbb{A})}\int_{\mathrm{K}} f(nmak)e^{-2\rho_\mathrm{P}(H_0(a))}\,\mathrm{d}k\mathrm{d}n\mathrm{d}a\mathrm{d}m\end{aligned}$$

where $\mathrm{d}n$ is the Tamagawa measure on $\mathrm{N_P}(\mathbb{A})$ which could also be characterized by assigning $\mathrm{N_P}(\mathbb{Q})$ covolume one in $\mathrm{N_P}(\mathbb{A})$, the point ρ_P in $\mathfrak{a}_0^*$ is the *Weyl vector* defined as the half sum of the roots of $\mathfrak{n}_\mathrm{P}$. The choices of measures on $\mathrm{G}(\mathbb{A})$ and $\mathfrak{a}_\mathrm{G}$ determine a Haar measure on $\mathrm{G}(\mathbb{A})^1$, hence a measure on the automorphic quotient $\mathbb{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$.

Let T' be a point in $\mathfrak{a}_0$, let ω be a compact subset of $\mathrm{N}_0(\mathbb{A})\mathrm{M}_0(\mathbb{A})^1$. The *Siegel set* $\mathfrak{S}(T',\omega)$ is the subset of $\mathrm{G}(\mathbb{A})^1$ defined as

$$(1.1.3.6)\qquad \Big\{x = pak:\ p\in\omega,\ a\in\exp(\mathfrak{a}_0),\ k\in\mathrm{K},\ \beta(H_0(a)-T')\geq 0\ \forall\beta\in\Delta_0\Big\}.$$

A Siegel set is said to be a *Siegel domain* if it contains a fundamental domain for $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$:

$$(1.1.3.7)\qquad \mathrm{G}(\mathbb{A})^1 \;=\; \mathrm{G}(\mathbb{Q})\mathfrak{S}(T',\omega).$$

Let T be a sufficiently positive point in $\mathfrak{a}_0$, such a T is said to be a *truncation parameter.* Let $\mathfrak{S}(T',\omega)$ be a Siegel domain. The *truncated Siegel domain* $\mathfrak{S}^T(T',\omega)$ is defined as

$$(1.1.3.8)\qquad \Big\{x\in\mathfrak{S}(T',\omega):\ \varpi(H_0(x)-T)\leq 0\ \forall\varpi\in\hat{\Delta}_0\Big\}.$$

Proposition (Borel, Harish-Chandra)

There exist a point T' and a compact set ω such that $\mathfrak{S}(T',\omega)$ is a Siegel domain.

Proof. See §13 of [Bo69]. $\square$

The Siegel domain $\mathfrak{S}(T', \omega)$, therefore $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$, has finite volume. The truncated Siegel domain $\mathfrak{S}^T(T', \omega)$ is compact and exhausts $\mathfrak{S}(T', \omega)$ as T approaches infinity. Fix a pair (T', ω) such that $\mathfrak{S}(T', \omega)$ is a Siegel domain, denote by $F^{\mathrm{G}}(x, T)$ the characterstic function on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$ of $\mathfrak{S}^T(T', \omega)$. There are analogues $F^{\mathrm{P}}(x, T)$ on $\mathrm{P}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$.

(1.1.4) The space $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ of *Schwartz functions* on $\mathfrak{g}(\mathbb{A})$ is defined as the tensor product

(1.1.4.1) $$\bigotimes_{p}^{\mathrm{res}} C_c^{\infty}(\mathfrak{g}(\mathbb{Q}_p)) \otimes \mathcal{S}(\mathfrak{g}(\mathbb{R}))$$

restricted at all but finitely many finite primes with respect to the unit vector $\mathbb{I}_{\mathfrak{g}(\mathbb{Z}_p)}$ in $C_c^{\infty}(\mathfrak{g}(\mathbb{Q}_p))$, equipped with the final topology with respect to

(1.1.4.2) $$\mathcal{S}(\mathfrak{g}(\mathbb{A})) \quad = \quad \varinjlim_{S} \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)).$$

Fix a nondegenerate G-invariant rational bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$, fix a global additive unitary character ψ on $\mathbb{A}$

(1.1.4.3) $$\psi : \mathbb{A} \to \mathrm{U}(1)$$

such that

(1.1.4.4) $$\forall x \in \mathbb{Q} \qquad \psi(x) = 1,$$

and $\psi(\langle\ ,\ \rangle)$ identifies $\mathbb{Q}$ and $\mathbb{A}/\mathbb{Q}$ as Pontryagin duals of each other. The *Fourier transform* on $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ is defined by

(1.1.4.5) $$f\hat{\ }(X) \quad = \quad \int_{\mathfrak{g}(\mathbb{A})} f(Y)\psi(\langle X, Y\rangle)\, \mathrm{d}Y.$$

The global Fourier transform on $\mathfrak{g}(\mathbb{A})$ factorizes as the tensor product of the local Fourier transforms on $\mathfrak{g}(\mathbb{Q}_v)$ with respect to compatible choices of $\psi_v(\langle\ ,\ \rangle)$. Denote by $\vee$ the inverse Fourier transform.

Proposition (Poisson summation formula, Tate)
For every Schwartz function f on $\mathfrak{g}(\mathbb{A})$,

$$\sum_{X\in\mathfrak{g}(\mathbb{Q})} f(X) \quad = \quad \sum_{X\in\mathfrak{g}(\mathbb{Q})} f\hat{\ }(X), \tag{1.1.4.6}$$

the sums are the absolutely convergent.

Proof. See §4.2 of [Ta50]. □

(1.1.5) Define an equivalence relation $\sim$ on $\mathfrak{g}(\mathbb{Q})$ by

$$X \sim Y \quad \text{if} \quad \exists g \in \mathrm{G}(\mathbb{Q})\ Y_{\mathrm{ss}} = X_{\mathrm{ss}} \cdot \mathrm{ad}(g). \tag{1.1.5.1}$$

In general $\sim$ is weaker than conjugacy by $\mathrm{G}(\mathbb{Q})$. A typical equivalence class will be denoted by $\mathfrak{o}$.

Let D be the *discriminant function* on $\mathfrak{g}(\mathbb{Q})$. Let X be an element of $\mathfrak{g}(\mathbb{Q})$, define $D(X)$ to be the coefficient of the characteristic polynomial of $\mathrm{ad}(X)$, acting on $\mathfrak{g}(\mathbb{Q})$ as a linear endomorphism, in degree r, the absolute rank of G:

$$r \quad = \quad \mathrm{rank}_{\overline{\mathbb{Q}}}(\mathrm{G} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}). \tag{1.1.5.2}$$

Alternatively D could be defined as the product of the roots of $\mathfrak{g}$ over an algebraically closed field. Denote by D^{M} the discriminant function on $\mathfrak{m}$ for a Levi subgroup M of G.

Let X be a semisimple element of $\mathfrak{g}(\mathbb{Q})$. Then X is said to be *regular* if $D(X)$ does not vanish. Denote by $\mathfrak{g}_{\mathrm{reg.ss}}$ the locus of regular semisimple points on $\mathfrak{g}$. If an equivalence class $\mathfrak{o}$ contains a regular semisimple element, the set $\mathfrak{o}$ is a $\mathrm{G}(\mathbb{Q})$-orbit consisting of regular semisimple elements. Such an $\mathfrak{o}$ is said to be *regular*, otherwise $\mathfrak{o}$ is said to be *singular*. A semisimple element X is said to be $\mathbb{Q}$-*elliptic* if it is stablized under the adjoint action by a maximal torus that is anisotropic over $\mathbb{Q}$ modulo the center of G.

Denote by G_X the centralizer of X in G, let G_X^0 be the connected component of the identity of G_Σ, let $\pi_0(\mathrm{G}_X)$ be the group of connected components of G_X. There is an exact sequence

$$(1.1.5.3) \qquad 1 \longrightarrow \mathrm{G}_X^0 \longrightarrow \mathrm{G}_X \longrightarrow \pi_0(\mathrm{G}_X) \longrightarrow 1\ .$$

(1.1.6) Let S be a finite set of places of $\mathbb{Q}$, denote by G_S the base change $\mathrm{G} \otimes_{\mathbb{Q}} \mathbb{Q}_S$ of G. Let v be a place of $\mathbb{Q}$, denote by G_v the base change $\mathrm{G} \otimes_{\mathbb{Q}} \mathbb{Q}_v$ of G. Similar notations apply to the Lie algebra $\mathfrak{g}$. The underlying topological groups of $\mathrm{G}(\mathbb{Q}_S)$ and $\mathrm{G}_S(\mathbb{Q}_S)$ are the same, however

$$(1.1.6.1) \qquad \begin{aligned} X(\mathrm{G}) &= \mathrm{Hom}_{\mathrm{Grp}/\mathbb{Q}}(\mathrm{G}, \mathrm{GL}(1,\mathbb{Q})), \\ X(\mathrm{G}_S) &= \mathrm{Hom}_{\mathrm{Grp}/\mathbb{Q}_S}(\mathrm{G}_S, \mathrm{GL}(1,\mathbb{Q}_S)) \end{aligned}$$

are in general different.

All the constructions above generalize to the local and S-local settings with similar caveats:

- Fix a minimal parabolic subgroup $\mathrm{P}_{S,0}$ of G_S contained in $\mathrm{P}_{0,S}$, fix a Levi subgroup $\mathrm{M}_{S,0}$ contained in $\mathrm{M}_{0,S}$ with split component $\mathrm{A}_{S,0}$ containing $\mathrm{A}_{0,S}$. The choice of $\mathrm{P}_{S,0}$ is equivalent to a choice of a minimal parabolic $\mathrm{P}_{v,0}$ of G_v for each v in S, similarly for $\mathrm{M}_{S,0}$ and $\mathrm{A}_{S,0}$.

- Let M_S and L_S be Levi subgroups of G_S such that M_S is contained in L_S, denote by $\mathcal{F}^{\mathrm{L}_S}(\mathrm{M}_S)$, $\mathcal{P}^{\mathrm{L}_S}(\mathrm{M}_S)$ and $\mathcal{L}^{\mathrm{L}_S}(\mathrm{M}_S)$ the analogous sets of parabolic and Levi subgroups.

The Levi subgroups M_S and L_S determine local Levi subgroups M_v and L_v for each v in S. There are bijections

(1.1.6.2)
$$\begin{aligned}\mathcal{F}^{\mathrm{L}_S}(\mathrm{M}_S) &= \prod_{v\in S}\mathcal{F}^{\mathrm{L}_v}(\mathrm{M}_v),\\ \mathcal{P}^{\mathrm{L}_S}(\mathrm{M}_S) &= \prod_{v\in S}\mathcal{P}^{\mathrm{L}_v}(\mathrm{M}_v),\\ \mathcal{L}^{\mathrm{L}_S}(\mathrm{M}_S) &= \prod_{v\in S}\mathcal{L}^{\mathrm{L}_v}(\mathrm{M}_v).\end{aligned}$$

If M_S and L_S are the base change of Levi subgroups M and L of G from $\mathbb{Q}$ to $\mathbb{Q}_S$, there are diagonal inclusions

(1.1.6.3)
$$\begin{aligned}\mathcal{F}^{\mathrm{L}}(\mathrm{M}) &\subset \mathcal{F}^{\mathrm{L}_S}(\mathrm{M}_S),\\ \mathcal{P}^{\mathrm{L}}(\mathrm{M}) &\subset \mathcal{P}^{\mathrm{L}_S}(\mathrm{M}_S),\\ \mathcal{L}^{\mathrm{L}}(\mathrm{M}) &\subset \mathcal{L}^{\mathrm{L}_S}(\mathrm{M}_S).\end{aligned}$$

- Let P_S be a parabolic subgroup of G_S, define real vector spaces

(1.1.6.4)
$$\mathfrak{a}_{\mathrm{P}_S} = \mathrm{Hom}_{\mathbb{Z}}(X(\mathrm{M}_{\mathrm{P}_S}),\mathbb{R}), \qquad \mathfrak{a}^*_{\mathrm{P}_S} = X(\mathrm{M}_{\mathrm{P}_S})\otimes_{\mathbb{Z}}\mathbb{R}.$$

If P_S is the base change of a parabolic subgroup P of G from $\mathbb{Q}$ to $\mathbb{Q}_S$, there are diagonal inclusions

(1.1.6.5)
$$\mathfrak{a}_{\mathrm{P}} \subset \mathfrak{a}_{\mathrm{P}_S} = \bigoplus_{v\in S}\mathfrak{a}_{\mathrm{P}_v}, \qquad \mathfrak{a}^*_{\mathrm{P}} \subset \mathfrak{a}^*_{\mathrm{P}_S} = \bigoplus_{v\in S}\mathfrak{a}^*_{\mathrm{P}_v}.$$

- A maximal torus T_S of G_S is equivalent to the choice of a maximal torus T_v of G_v for each v in S. The associated Cartan subalgebra $\mathfrak{t}_S$, which is a free $\mathbb{Q}_S$-module, is equal as an abelian group to the direct sum of the $\mathbb{Q}_v$-vector spaces $\mathfrak{t}_v$ for all v in S.

  A maximal torus T_S is said to be *elliptic* in G_S if it is anisotropic modulo $\mathrm{A}_{\mathrm{G}_S}$ over

$\mathbb{Q}_S$. A maximal torus T_S is elliptic in G_S if and only if T_v is elliptic in G_v for each v in S. If this is the case, $\mathrm{T}_S(\mathbb{Q}_S)$ is compact modulo $\mathrm{A}_{\mathrm{G}_S}(\mathbb{Q}_S)$ in the analytic topology. Denote by $\mathcal{T}_{\mathrm{ell}}(\mathrm{G}_S)$ the set of conjugacy classes of elliptic maximal tori of G_S.

- Let $\mathrm{W}_{S,0}^{\mathrm{G}_S}$ be the Weyl group of the pair $(\mathrm{G}_S, \mathrm{A}_{S,0})$, there is a bijection

$$\mathrm{W}_{S,0}^{\mathrm{G}_S} = \prod_{v\in S} \mathrm{W}_{v,0}^{\mathrm{G}_v}, \tag{1.1.6.6}$$

the linear representation of $\mathrm{W}_{S,0}^{\mathrm{G}_S}$ on $\mathfrak{a}_{S,0}$ is the direct sum of the local representations. Denote by $\mathrm{W}(\mathrm{G}_S, \mathrm{T}_S)$ the Weyl group of the pair $(\mathrm{G}_S(\mathbb{Q}_S), \mathrm{T}_S(\mathbb{Q}_S))$. There is a bijection

$$\mathrm{W}(\mathrm{G}_S, \mathrm{T}_S) = \prod_{v\in S} \mathrm{W}(\mathrm{G}_v, \mathrm{T}_v). \tag{1.1.6.7}$$

- The Schwartz space $\mathcal{S}(\mathfrak{g}_S(\mathbb{Q}_S))$, the Fourier transform $\wedge$, the discriminant function D^{G_S}, and the regular semisimple locus $\mathfrak{g}_{S,\mathrm{reg.ss}}(\mathbb{Q}_S)$ are unchanged under base change from $\mathbb{Q}$ to $\mathbb{Q}_S$.

Proposition (Weyl integration formula)
If f_S is a Schwartz function on $\mathfrak{g}_S(\mathbb{Q}_S)$, then

$$\int_{\mathfrak{g}_S(\mathbb{Q}_S)} f_S(X)\ \mathrm{d}X \tag{1.1.6.8}$$

$$= \sum_{\mathrm{M}_S\in\mathcal{L}^{\mathrm{G}_S}} |\mathrm{W}_{S,0}^{\mathrm{M}_S}||\mathrm{W}_{S,0}^{\mathrm{G}_S}|^{-1} \sum_{\mathrm{T}_S\in\mathcal{T}_{\mathrm{ell}}(\mathrm{M}_S)} |\mathrm{W}(\mathrm{M}_S, \mathrm{T}_S)|^{-1} \times$$

$$\times \int_{\mathfrak{t}_S(\mathbb{Q}_S)} |D^{\mathrm{G}_S}(X)|_S \int_{\mathrm{A}_{\mathrm{M}_S}(\mathbb{Q}_S)\backslash \mathrm{G}_S(\mathbb{Q}_S)} f_S(X\cdot \mathrm{ad}(x))\ \mathrm{d}x\mathrm{d}X.$$

Proof. For the p-adic case see §7.11 of [Ko05]. For the real case see Lemma 2 on page 35 of [Va77] and the references therein. $\square$

Definition Let X be an element of $\mathfrak{g}_{S,\mathrm{reg.ss}}(\mathbb{Q}_S)$. Define the S-local *orbital integral* $I_{\mathrm{G}}^{\mathrm{G}}(X,\ )$ to be the invariant distribution on $\mathfrak{g}_S(\mathbb{Q}_S)$ such that

(1.1.6.9)
$$\forall f_S \in \mathcal{S}(\mathfrak{g}_S(\mathbb{Q}_S))$$
$$I_{\mathrm{G}}^{\mathrm{G}}(X, f_S) = |D^{\mathrm{G}}(X)|_S^{1/2} \int_{\mathrm{G}_{S,X}^0(\mathbb{Q}_S)\backslash \mathrm{G}_S(\mathbb{Q}_S)} f_S(X \cdot \mathrm{ad}(x))\ \mathrm{d}x$$

where $\mathrm{G}_{S,X}^0$ denotes the connected component of the identity of the stablizer subgroup of X in G_S.

2.2 The non-invariant trace formula

In this section a preliminary version of the trace formula for the reductive Lie algebra $\mathfrak{g}$ established in Chaudouard [Ch02a] is recalled.

2.2.1 A motivating example

(2.1.1) Definition Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let $\mathfrak{o}$ be a $\sim$ equivalence class on $\mathfrak{g}(\mathbb{Q})$. Define the *kernel functions* $K(x, f)$ and $K_{\mathfrak{o}}(x, f)$ by

(2.1.1.1)
$$\forall x \in \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}) \qquad K(x, f) = \sum_{X\in\mathfrak{g}(\mathbb{Q})} f(X \cdot \mathrm{ad}(x)),$$
$$K_{\mathfrak{o}}(x, f) = \sum_{X\in\mathfrak{o}} f(X \cdot \mathrm{ad}(x)).$$

(2.1.2) Remark By the Poisson summation formula (1.1.4.6), the function $K(x, f)$ satisfies the functional equation

$$K(x, f) \;=\; K(x, f\hat{\ }). \tag{2.1.2.1}$$

(2.1.3) Proposition *Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$. If G is anisotropic over $\mathbb{Q}$, then*

$$\begin{aligned} &\lim_{S} \sum_{\substack{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim \\ \textit{regular}}} \mathrm{Vol}(\mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{Q})\backslash \mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{A})) \cdot I^{\mathrm{G}}_{\mathrm{G}}(X_\mathfrak{o}, f_S) \\ &\quad + \sum_{\substack{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim \\ \textit{singular}}} \mathrm{Vol}(\mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{Q})\backslash \mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{A})) \cdot I_\mathfrak{o}(f) \\ = \; &\lim_{S} \sum_{\substack{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim \\ \textit{regular}}} \mathrm{Vol}(\mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{Q})\backslash \mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{A})) \cdot I^{\mathrm{G}}_{\mathrm{G}}(X_\mathfrak{o}, f\hat{\ }_S) \\ &\quad + \sum_{\substack{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim \\ \textit{singular}}} \mathrm{Vol}(\mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{Q})\backslash \mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{A})) \cdot I_\mathfrak{o}(f\hat{\ }), \end{aligned} \tag{2.1.3.1}$$

where for each class $\mathfrak{o}$ choose an element $X_\mathfrak{o}$ of $\mathfrak{o}$, for each singular class $\mathfrak{o}$ define

$$I_\mathfrak{o}(f) \;=\; \int_{\mathrm{G}^0_{X_\mathfrak{o}}(\mathbb{A})\backslash \mathrm{G}(\mathbb{A})} f(X_\mathfrak{o} \cdot \mathrm{ad}(x))\, \mathrm{d}x. \tag{2.1.3.2}$$

Proof. The function $K(x, f)$ is continuous. By assumption G is anisotropic, so $\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})$ is compact. Therefore $K(x, f)$ is absolutely integrable, hence

$$\int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})} K(x, f)\, \mathrm{d}x \tag{2.1.3.3}$$

$$= \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})} \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} K_{\mathfrak{o}}(x,f)\ \mathrm{d}x$$

$$= \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})} K_{\mathfrak{o}}(x,f)\ \mathrm{d}x$$

$$= \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \mathrm{Vol}\big(\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{Q})\backslash\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{A})\big) \int_{\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} f(X_{\mathfrak{o}}\cdot \mathrm{ad}(x))\ \mathrm{d}x$$

$$(2.1.3.4) \qquad = \lim_{S} \sum_{\substack{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim \\ \text{regular}}} \mathrm{Vol}\big(\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{Q})\backslash\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{A})\big)\cdot I^{\mathrm{G}}_{\mathrm{G}}(X_{\mathfrak{o}}, f_S)$$

$$+ \sum_{\substack{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim \\ \text{singular}}} \mathrm{Vol}\big(\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{Q})\backslash\mathrm{G}^0_{X_{\mathfrak{o}}}(\mathbb{A})\big)\cdot I_{\mathfrak{o}}(f\hat{\ }),$$

the equality (2.1.3.4) follows from

$$(2.1.3.5) \qquad \forall X_{\mathfrak{o}} \in \mathfrak{g}(\mathbb{Q})\ \exists S\ \forall S' \supset S \qquad |D(X_{\mathfrak{o}})|_{S'} = 1$$

by the product formula (1.1.1.2). The proposition follows from the functional equation (2.1.2.1). □

2.2.2 The non-invariant trace formula of Chaudouard

(2.2.1) Definition Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let $\mathfrak{o}$ be a $\sim$ equivalence class on $\mathfrak{g}(\mathbb{Q})$, let T be a truncation parameter. Define the *truncated kernel function* $k^T_{\mathfrak{o}}(\ ,f)$ on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$ by

(2.2.1.1) $\forall x \in \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$

$$k^T_{\mathfrak{o}}(x,f) = \sum_{\substack{\mathrm{P}\in\mathcal{F} \\ \text{standard}}} (-1)^{\dim(\mathrm{A_P}/\mathrm{A_G})} \sum_{\delta\in\mathrm{P}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})} \hat{\tau}_{\mathrm{P}}(H_0(\delta x)-T) K_{\mathrm{P},\mathfrak{o}}(\delta x)$$

where

$$(2.2.1.2) \qquad K_{\mathrm{P},\mathfrak{o}}(x,f) \;=\; \sum_{X\in\mathfrak{m}_{\mathrm{P}}(\mathbb{Q})\cap\mathfrak{o}} \int_{\mathfrak{n}_{\mathrm{P}}(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(x))\,\mathrm{d}N.$$

Define distributions $J_{\mathfrak{o}}^T$ and J^T on $\mathfrak{g}(\mathbb{A})$ by

$$(2.2.1.3) \qquad \forall f\in\mathcal{S}(\mathfrak{g}(\mathbb{A})) \qquad J_{\mathfrak{o}}^T(f) = \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} k_{\mathfrak{o}}^T(x,f)\,\mathrm{d}x,$$

$$J^T(f) = \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} k_{\mathfrak{o}}^T(x,f)\,\mathrm{d}x.$$

(2.2.2) Proposition *Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let T be a truncation parameter, then*

$$(2.2.2.1) \qquad \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \left|k_{\mathfrak{o}}^T(x,f)\right|\,\mathrm{d}x \;<\; \infty.$$

Proof. This is Théorème 3.1 of [Ch02a]. The following is a sketch of the argument, for the details please refer to loc. cit.

The proof depends on the following lemma.

Lemma (Combinatorial lemma of Langlands)
Let P_1 *and* P_3 *be parabolic subgroups of* G *such that* P_1 *is contained in* P_3*, then*

$$(2.2.2.2) \qquad \sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \mathrm{P}_1\subset\mathrm{P}_2\subset\mathrm{P}_3}} (-1)^{\dim(\mathrm{A}_2/\mathrm{A}_3)}\tau_1^2(H)\hat{\tau}_2^3(H) \;=\; \begin{cases} 1 & \textit{if } \mathrm{P}_1=\mathrm{P}_3, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. See §6 of [Ar78]. □

By (2.2.2.2)

$$\begin{aligned}
&\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}\sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim}\left|k_{\mathfrak{o}}^{T}(x,f)\right|\,\mathrm{d}x \\
&\leq \sum_{\substack{\mathrm{P}_1,\mathrm{P}_4\in\mathcal{F}\\ \text{standard}\\ \mathrm{P}_1\subset\mathrm{P}_4}}\int_{\mathrm{P}_1(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}\sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim}F^{\mathrm{P}_1}(x,T)\sigma_1^4(H_0(x)-T)\times \\
&\qquad\times\left|\sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_1\subset\mathrm{P}_3\subset\mathrm{P}_4}}(-1)^{\dim(\mathrm{A}_3/\mathrm{A}_\mathrm{G})}K_{\mathrm{P}_3,\mathfrak{o}}(x,f)\right|\,\mathrm{d}x
\end{aligned}\tag{2.2.2.3}$$

where

$$\sigma_1^4(H) = \sum_{\substack{\mathrm{P}_5\in\mathcal{F}\\ \mathrm{P}_4\subset\mathrm{P}_5}}(-1)^{\dim(\mathrm{A}_4/\mathrm{A}_5)}\tau_1^5(H)\hat{\tau}_5(H).\tag{2.2.2.4}$$

The second factor of the integrand of the right hand side of (2.2.2.3) satisfies the inequality

$$\begin{aligned}
&\left|\sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_1\subset\mathrm{P}_3\subset\mathrm{P}_4}}(-1)^{\dim(\mathrm{A}_3/\mathrm{A}_\mathrm{G})}K_{\mathrm{P}_3,\mathfrak{o}}(x,f)\right| \\
&\leq \sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \mathrm{P}_1\subset\mathrm{P}_2\subset\mathrm{P}_4}}\left|\sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_3\subset\mathrm{P}_4}}(-1)^{\dim(\mathrm{A}_3/\mathrm{A}_\mathrm{G})}\sum_{X\in\mathfrak{m}_1^2(\mathbb{Q})'\cap\mathfrak{o}}\right. \\
&\qquad\left.\times\sum_{Y\in\mathfrak{n}_2^3(\mathbb{Q})}\int_{\mathfrak{n}_3(\mathbb{A})}f((X+Y+N)\cdot\mathrm{ad}(x))\,\mathrm{d}N\right|
\end{aligned}\tag{2.2.2.5}$$

$$\begin{aligned}
&= \left|\sum_{X\in\mathfrak{m}_1^4(\mathbb{Q})'\cap\mathfrak{o}}\int_{\mathfrak{n}_4(\mathbb{A})}f((X+N)\cdot\mathrm{ad}(x))\,\mathrm{d}N\right| + \\
&\qquad+\sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \mathrm{P}_1\subset\mathrm{P}_2\subsetneq\mathrm{P}_4}}\left|\sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_3\subset\mathrm{P}_4}}(-1)^{\dim(\mathrm{A}_3/\mathrm{A}_\mathrm{G})}\sum_{X\in\mathfrak{m}_1^2(\mathbb{Q})'\cap\mathfrak{o}}\right.
\end{aligned}\tag{2.2.2.6}$$

$$\times \sum_{\overline{Y} \in \overline{\mathfrak{n}}_2^3(\mathbb{Q})'} \int_{\mathfrak{n}_2(\mathbb{A})} f((X+N)\cdot \mathrm{ad}(x))\psi(\langle N, \overline{Y}\rangle)\, \mathrm{d}N \Bigg|$$

where $\mathfrak{m}_{\mathrm{P}}^{\mathrm{Q}}(\mathbb{Q})'$ denotes the set of points of $\mathfrak{m}_{\mathrm{P}}^{\mathrm{Q}}(\mathbb{Q})$ not contained in any proper parabolic subalgebra of $\mathfrak{q}$. Similar notation applies to $\mathfrak{n}$. The equality (2.2.2.6) follows from the Poisson summation formula applied to $\mathfrak{n}_2^3(\mathbb{Q})$ as a lattice in $\mathfrak{n}_2^3(\mathbb{A})$.

By the inclusion-exclusion principle applied to

$$\sum_{\substack{\mathrm{P}_3 \in \mathcal{F} \\ \mathrm{P}_2 \subset \mathrm{P}_3 \subset \mathrm{P}_4}} (-1)^{\dim(\mathrm{A}_3/\mathrm{A}_\mathrm{G})} \tag{2.2.2.7}$$

the last expression in (2.2.2.5) reduces to a majorant of the form

$$\prod_{\alpha \in \Delta_1^4} \int_0^\infty (1+t_\alpha)^{p_\alpha} e^{-q_\alpha t_\alpha}\, \mathrm{d}t_\alpha \tag{2.2.2.8}$$

for some natural numbers p_α and q_α, which is finite. □

(2.2.3) Definition Let P_2 be a standard parabolic subgroup of G, let T be a truncation parameter in $\mathfrak{a}_0$. Define the *geometric gamma′ function* $\Gamma'_{\mathrm{P}_2}(\ ,T)$ on $\mathfrak{a}_0$ by

$$\forall H \in \mathfrak{a}_0 \qquad \Gamma'_{\mathrm{P}_2}(H,T) = \sum_{\substack{\mathrm{P}_3 \in \mathcal{F} \\ \mathrm{P}_2 \subset \mathrm{P}_3}} (-1)^{\dim(\mathrm{A}_3/\mathrm{A}_\mathrm{G})} \tau_2^3(H) \hat{\tau}_3(H-T). \tag{2.2.3.1}$$

(2.2.4) Remark For each parabolic subgroup P_1 of G the geometric gamma′ functions satisfy the identity

$$\hat{\tau}_1(H-T) \;=\; \sum_{\substack{\mathrm{P}_2 \in \mathcal{F} \\ \mathrm{P}_1 \subset \mathrm{P}_2}} (-1)^{\dim(\mathrm{A}_2/\mathrm{A}_\mathrm{G})} \hat{\tau}_1^2(H) \Gamma'_{\mathrm{P}_2}(H,T). \tag{2.2.4.1}$$

For a proof see page 13 of [Ar81].

(2.2.5) Proposition *Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let $\mathfrak{o}$ be a $\sim$ equivalence class, let T be a truncation parameter. Then $J_{\mathfrak{o}}^T(f)$ and $J^T(f)$ are polynomials in T of degree at most* $\dim(\mathrm{A}_0/\mathrm{A}_{\mathrm{G}})$.

Proof. This is Théorème 4.2 of [Ch02a]. The following is a sketch of the argument for $J_{\mathfrak{o}}^T(f)$, the argument for $J^T(f)$ is the same. For the details please refer to loc. cit.

Fix a point T' in $\mathfrak{a}_0$ that is sufficiently positive and assume that T dominates T'. By (2.2.4.1)

$$(2.2.5.1)\qquad J_{\mathfrak{o}}^T(f)$$

$$= \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}\Bigg(\sum_{\substack{\mathrm{P}_1\in\mathcal{F}\\ \text{standard}}}(-1)^{\dim(\mathrm{A}_1/\mathrm{A}_{\mathrm{G}})}\times$$

$$\times\sum_{\delta\in\mathrm{P}_1(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})}\hat{\tau}_1(H_0(\delta x)-T)\cdot K_{\mathrm{P}_1,\mathfrak{o}}(\delta x)\Bigg)\,\mathrm{d}x$$

$$= \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}\Bigg(\sum_{\substack{\mathrm{P}_1,\mathrm{P}_2\in\mathcal{F}\\ \text{standard}\\ \mathrm{P}_1\subset\mathrm{P}_2}}(-1)^{\dim(\mathrm{A}_1/\mathrm{A}_2)}\times$$

$$\times\sum_{\delta\in\mathrm{P}_1(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})}\hat{\tau}_1^2(H_{\mathrm{P}_1}(\delta x)-T')\Gamma'_{\mathrm{P}_2}(H_{\mathrm{P}_2}(\delta x)-T',T-T')\times$$

$$\times\sum_{X\in\mathfrak{m}_1(\mathbb{Q})\cap\mathfrak{o}}\int_{\mathfrak{n}_1(\mathbb{A})}f((X+N)\cdot\mathrm{ad}(\delta x))\,\mathrm{d}N\Bigg)\,\mathrm{d}x$$

$$= \sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \text{standard}}}\int_{\mathrm{M}_2(\mathbb{Q})\backslash\mathrm{M}_2(\mathbb{A})^1}\Bigg(\sum_{\substack{\mathrm{P}_1\in\mathcal{F}\\ \text{standard}\\ \mathrm{P}_1\subset\mathrm{P}_2}}(-1)^{\dim(\mathrm{A}_1/\mathrm{A}_2)}\times$$

$$\times\sum_{\delta\in\mathrm{P}_1(\mathbb{Q})\cap\mathrm{M}_2(\mathbb{Q})\backslash\mathrm{M}_2(\mathbb{Q})}\hat{\tau}_1^2(H_{\mathrm{P}_1}(\delta x)-T')\sum_{X\in\mathfrak{m}_1(\mathbb{Q})\cap\mathfrak{o}}\int_{\mathfrak{n}_1^2(\mathbb{A})}$$

$$\times\left(\int_{\mathrm{K}}\int_{\mathfrak{n}_2(\mathbb{A})} f((X+N)\cdot \mathrm{ad}(\delta x k) + N'\cdot \mathrm{ad}(k))\ \mathrm{d}N'\mathrm{d}k\right)\ \mathrm{d}N\Bigg)\ \mathrm{d}x\ \times$$

$$\times\int_{\mathfrak{a}_2^{\mathrm{G}}} \Gamma'_{\mathrm{P}_2}(H-T', T-T')\ \mathrm{d}H$$

$$= \sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \text{standard}}} J_{\mathfrak{o}}^{\mathrm{M}_2,T'}(f_{\mathrm{P}_2})\int_{\mathfrak{a}_2^{\mathrm{G}}} \Gamma'_{\mathrm{P}_2}(H, T-T')\ \mathrm{d}H$$

where $J_{\mathfrak{o}}^{\mathrm{M}}$ is defined to be the sum of $J_{\mathfrak{o}'}^{\mathrm{M}}$ over all the $\mathrm{M}(\mathbb{Q})$ $\sim$ equivalence classes $\mathfrak{o}'$ contained in $\mathfrak{o}$ and f_{P} is defined by

$$\text{(2.2.5.2)} \qquad f_{\mathrm{P}}(X) \ = \ \int_{\mathrm{K}}\int_{\mathfrak{n}_{\mathrm{P}}(\mathbb{A})} f((X+N)\cdot \mathrm{ad}(k))\ \mathrm{d}N\mathrm{d}k.$$

Because $\int \Gamma'_{\mathrm{P}}(H, T-T')\ \mathrm{d}H$ is a polynomial in T which is homogeneous of degree $\dim(\mathrm{A}_{\mathrm{P}}/\mathrm{A}_{\mathrm{G}})$, Proposition (2.2.5) follows by induction. ☐

(2.2.6) Proposition *Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let T be a truncation parameter. For every positive ϵ*

$$\text{(2.2.6.1)} \quad \left| J^T(f) - \int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T)\sum_{X\in\mathfrak{g}(\mathbb{Q})} f(X\cdot \mathrm{ad}(x))\ \mathrm{d}x\right| \ = \ O(e^{-\epsilon\|T\|})$$

where $\|\ \|$ denotes the Euclidean norm on $\mathfrak{a}_0$, as T approaches infinity such that T is uniformly bounded away from the walls of the positive chamber.

Proof. This is a corollary of the proof of Lemma 3.2.2.1. ☐

(2.2.7) Proposition (Non-invariant trace formula of Chaudouard)

Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, then

(2.2.7.1)
$$\sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} J_{\mathfrak{o}}^{T}(f) \;=\; \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} J_{\mathfrak{o}}^{T}(f\hat{\ })$$

holds as an equality between polynomials in T.

Proof. By (2.1.2.1)

(2.2.7.2)
$$\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T)\sum_{X\in\mathfrak{g}(\mathbb{Q})} f(X\cdot\mathrm{ad}(x))\;\mathrm{d}x$$
$$= \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T)K(x,f)\;\mathrm{d}x$$
$$= \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T)K(x,f\hat{\ })\;\mathrm{d}x.$$

By (2.2.6.1) the difference between $J^{T}(f)$ and $J^{T}(f\hat{\ })$ converges to zero as T approaches infinity. By Proposition (2.2.5) $J^{T}(f)$ and $J^{T}(f\hat{\ })$ are both polynomials in T, hence equal to each other. ☐

(2.2.8) Lemma *There exists a unique point T_0 in $\mathfrak{a}_0^{\mathrm{G}}$ such that*

(2.2.8.1)
$$\forall s\in \mathrm{W}_0^{\mathrm{G}} \qquad H_0(w_s^{-1}) = T_0 - s^{-1}T_0$$

where w_s denotes a representative of s in $\mathrm{G}(\mathbb{Q})$.

Proof. See Lemma 1.1 of [Ar81]. ☐

(2.2.9) Definition Define distributions J and $J_{\mathfrak{o}}$ on $\mathfrak{g}(\mathbb{A})$ by

(2.2.9.1)
$$\forall f\in\mathcal{S}(\mathfrak{g}(\mathbb{A})) \qquad J(f) = J^{T_0}(f),\;\; J_{\mathfrak{o}}(f) = J_{\mathfrak{o}}^{T_0}(f).$$

(2.2.10) Remark The coefficients of the polynomials $J^T(f)$ and $J^T_{\mathfrak{o}}(f)$ in positive degrees depend only on the orbital integrals of f_{P} along proper Levi subalgebras of $\mathfrak{g}$. Therefore the constant terms $J(f)$ and $J_{\mathfrak{o}}(f)$ contain the essential information. The choice of T_0 implies that the distributions J and $J_{\mathfrak{o}}$ are independent of the choice of the minimal parabolic subgroup P_0.

(2.2.11) Proposition *Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$. If $\mathfrak{o}$ is the conjugacy class of a regular semisimple element X which is contained in a standard parabolic $\mathfrak{p}$ and elliptic in its Levi component $\mathfrak{m}$, then*

$$(2.2.11.1) \quad J_{\mathfrak{o}}(f) = |\pi_0(\mathrm{G}_X)|^{-1}\mathrm{Vol}(\mathrm{M}^0_X(\mathbb{Q})\backslash\mathrm{M}^0_X(\mathbb{A})^1)\int_{\mathrm{G}^0_X(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} f(X\cdot\mathrm{ad}(x))\ v_{\mathrm{M}}(x)\mathrm{d}x$$

where the weight factor $v_{\mathrm{M}}(x)$ is defined to be the volume of the convex hull of

$$(2.2.11.2) \qquad \Big\{-H_{\mathrm{P}}(x):\ \mathrm{P}\in\mathcal{P}(\mathrm{M})\Big\}\ \subset\ \mathfrak{a}^{\mathrm{G}}_{\mathrm{M}}.$$

Proof. This is a corollary of Theorem 3.3.11.1. See also (5.4) of [Ch02a]. □

(2.2.12) Proposition *Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let $\mathfrak{o}$ be a $\sim$ equivalence class, then*

$$(2.2.12.1) \qquad \forall x\in\mathrm{G}(\mathbb{A})^1 \qquad J_{\mathfrak{o}}(f\circ\mathrm{ad}(x)) = \sum_{\mathrm{P}\in\mathcal{F}}|\mathrm{W}^{\mathrm{M_P}}_0||\mathrm{W}^{\mathrm{G}}_0|^{-1}J^{\mathrm{M_P}}_{\mathfrak{o}}(f_{\mathrm{P},x})$$

where $J^{\mathrm{M}}_{\mathfrak{o}}$ is defined to be the sum of $J^{\mathrm{M}}_{\mathfrak{o}'}$ over all the $\mathrm{M}(\mathbb{Q})$ $\sim$ equivalence classes $\mathfrak{o}'$ contained in $\mathfrak{o}$, and the function $f_{\mathrm{P},x}$ on $\mathfrak{m}_{\mathrm{P}}(\mathbb{A})$ is defined as

$$(2.2.12.2) \qquad f_{\mathrm{P},x}(X) \ = \ \int_{\mathrm{K}}\int_{\mathfrak{n}_{\mathrm{P}}(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(k))\ v'_{\mathrm{P}}(kx)\mathrm{d}N\mathrm{d}k$$

where the weight factor $v'_{\mathrm{P}}(x)$ *is defined by*

$$(2.2.12.3) \qquad v'_{\mathrm{P}}(x) \;=\; \int_{\mathfrak{a}_{\mathrm{P}}^{\mathrm{G}}} \Gamma'_{\mathrm{P}}(H, -H_{\mathrm{P}}(x))\, \mathrm{d}H.$$

Proof. By definition

$$(2.2.12.4) J_{\mathfrak{o}}^{T}(f \circ \mathrm{ad}(x)) \;=\; \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} \Bigg(\sum_{\substack{\mathrm{P}_1\in\mathcal{F}\\ \text{standard}}} (-1)^{\dim(\mathrm{A}_1\mathrm{A}_\mathrm{G})} \times \\ \times \sum_{\delta\in\mathrm{P}_1(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})} \hat{\tau}_1(H_0(\delta yx) - T)\cdot K_{\mathrm{P}_1,\mathfrak{o}}(\delta yx)\, \mathrm{d}y \Bigg).$$

By (2.2.4.1)

$$(2.2.12.5) \qquad \hat{\tau}_1(H_{\mathrm{P}_1}(\delta yx) - T) \;=\; \sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \mathrm{P}_1\subset\mathrm{P}_2}} (-1)^{\dim(\mathrm{A}_2/\mathrm{A}_\mathrm{G})} \hat{\tau}_1^2(H_{\mathrm{P}_1}(\delta y) - T) \times \\ \times \Gamma'_{\mathrm{P}_2}(H_{\mathrm{P}_1}(\delta y) - T, -H_{\mathrm{P}_1}(kx))$$

where k is a K component of δy under the decomposition of $\mathrm{G}(\mathbb{A})$ as $\mathrm{P}_1(\mathbb{A})\mathrm{K}$, the point $H_{\mathrm{P}_1}(kx)$ is independent of the choice of the element k. Hence the right hand side of (2.2.12.4) is equal to

$$(2.2.12.6) \qquad \sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \text{standard}}} \int_{\mathrm{M}_2(\mathbb{Q})\backslash\mathrm{M}_2(\mathbb{A})^1} \Bigg(\sum_{\substack{\mathrm{P}_1\in\mathcal{F}\\ \text{standard}\\ \mathrm{P}_1\subset\mathrm{P}_2}} (-1)^{\dim(\mathrm{A}_1/\mathrm{A}_2)} \times \\ \times \sum_{\delta\in\mathrm{P}_1(\mathbb{Q})\cap\mathrm{M}_2(\mathbb{Q})\backslash\mathrm{M}_2(\mathbb{Q})} \hat{\tau}_1^2(H_{\mathrm{P}_1}(\delta y) - T) \sum_{X\in\mathfrak{m}_1(\mathbb{Q})\cap\mathfrak{o}} \int_{\mathfrak{n}_1^2(\mathbb{A})} \\ \times \Bigg(\int_{\mathrm{K}} \int_{\mathfrak{n}_2(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(\delta yx) + N'\cdot\mathrm{ad}(kx)) \times$$

$$\times\Big(\int_{\mathfrak{a}_2^{\mathrm{G}}} \Gamma'_{\mathrm{P}_2}(H-T, -H_{\mathrm{P}_2}(kx))\, \mathrm{d}H\Big)\, \mathrm{d}N'\mathrm{d}k \;\Big)\, \mathrm{d}N\Big)\, \mathrm{d}y.$$

Substituting T_0 for T, the expression (2.2.12.6) becomes

(2.2.12.7) $$\sum_{\substack{\mathrm{P}_2\in\mathcal{F}\\ \text{standard}}} J_{\mathfrak{o}}^{\mathrm{M}_2}(f_{\mathrm{P}_2,x}) \;=\; \sum_{\mathrm{P}_2\in\mathcal{F}} |\mathrm{W}_0^{\mathrm{M}_2}||\mathrm{W}_0^{\mathrm{G}}|^{-1} J_{\mathfrak{o}}^{\mathrm{M}_2}(f_{\mathrm{P}_2,x})$$

since the distribution $J_{\mathfrak{o}}^{\mathrm{M}}$ is independent of the choice of the minimal parabolic subgroup P_0. □

2.3 Refined expansions

In this section the distribution $J_{\mathfrak{o}}$ is decomposed as a linear combination of weighted orbital integrals following the methods of Arthur [Ar85] [Ar86].

2.3.1 Weighted orbital integrals

(3.1.1) Definition Let M be a standard Levi subgroup of G. A collection of complex-valued functions

(3.1.1.1) $$\Big\{ c_{\mathrm{P}} \in C^{\infty}(i\mathfrak{a}_{\mathrm{M}}^*) :\ \mathrm{P}\in\mathcal{P}(\mathrm{M}) \Big\}$$

is said to be a (G,M)-*family* if for each pair of adjacent parabolic subgroups P and P′ in $\mathcal{P}(\mathrm{M})$, the functions c_{P} and $c_{\mathrm{P}'}$ agree on the hyperplane spanned by the common wall of the positive chambers of $i\mathfrak{a}_{\mathrm{M}}^*$ defined by P and P′.

Let (c_{P}) be a (G,M)-family, define the function c_{M} on the complement of the coroot hyperplanes in $i\mathfrak{a}_{\mathrm{M}}^*$ by

(3.1.1.2) $$\forall\lambda\in i\mathfrak{a}_{\mathrm{M}}^* \qquad c_{\mathrm{M}}(\lambda) = \sum_{\mathrm{P}\in\mathcal{P}(\mathrm{M})} c_{\mathrm{P}}(\lambda)\theta_{\mathrm{P}}(\lambda)^{-1}$$

where

$$(3.1.1.3) \qquad \theta_{\mathrm{P}}(\lambda) \ = \ \mathrm{Vol}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}/\mathbb{Z}(\Delta_{\mathrm{P}}^{\vee})^{-1}) \prod_{\alpha\in\Delta_{\mathrm{P}}} \langle \check{\alpha}, \lambda\rangle.$$

The function c_{M} extends smoothly over $i\mathfrak{a}_{\mathrm{M}}^{*}$. Denote by c_{M} its value at the origin of $i\mathfrak{a}_{\mathrm{M}}^{*}$.

(3.1.2) Definition Let Q be a parabolic subgroup in $\mathcal{F}(\mathrm{M})$ with Levi component L, let (c_{P}) be a (G,M)-family. Let

$$(3.1.2.1) \qquad i_{\mathrm{Q}}^{\mathrm{G}} \ : \ \mathcal{P}^{\mathrm{L}}(\mathrm{M}) \to \mathcal{P}^{\mathrm{G}}(\mathrm{M})$$

be the map that sends a parabolic subgroup P in $\mathcal{P}^{\mathrm{L}}(\mathrm{M})$ to the unique parabolic subgroup in $\mathcal{P}^{\mathrm{G}}(\mathrm{M})$ that is contained in Q whose intersection with L is P. Denote by $(c_{\mathrm{P}}^{\mathrm{Q}})$ the (L,M)-family

$$(3.1.2.2) \qquad \left\{ c_{i_{\mathrm{Q}}^{\mathrm{G}}(\mathrm{P})} : \ \mathrm{P}\in\mathcal{P}^{\mathrm{L}}(\mathrm{M}) \right\}.$$

Let

$$(3.1.2.3) \qquad j_{\mathrm{M}}^{\mathrm{L}} \ : \ i\mathfrak{a}_{\mathrm{L}}^{*} \to i\mathfrak{a}_{\mathrm{M}}^{*}$$

be the natural inclusion map. Denote by the (c_{P}) the (G,L)-family

$$(3.1.2.4) \qquad \left\{ j_{\mathrm{M}}^{\mathrm{L},*}(c_{\mathrm{P}'}) : \ \mathrm{P}\in\mathcal{P}^{\mathrm{G}}(\mathrm{L}) \right\}$$

where P' is a parabolic subgroup in $\mathcal{P}^{\mathrm{G}}(\mathrm{M})$ contained in P, and the function $j_{\mathrm{M}}^{\mathrm{L},*}(c_{\mathrm{P}'})$ is independent of the choice of P' in $\mathcal{P}^{\mathrm{G}}(\mathrm{M})$.

Let (c_{P}) and (d_{P}) be two (G,M)-families, denote by $((cd)_{\mathrm{P}})$ the product of (c_{P}) and (d_{P}), which is a (G,M)-family. For each parabolic subgroup Q in $\mathcal{F}(\mathrm{M})$ there exists a function c'_{Q}

on $i\mathfrak{a}_{\mathrm{M}}^{*}$ such that

$$\forall \lambda \in i\mathfrak{a}_{\mathrm{M}}^{*} \qquad (cd)_{\mathrm{M}}(\lambda) = \sum_{\mathrm{Q} \in \mathcal{F}(\mathrm{M})} c'_{\mathrm{Q}}(\lambda) d_{\mathrm{M}}^{\mathrm{Q}}(\lambda). \tag{3.1.2.5}$$

Denote by c'_{Q} the value $c'_{\mathrm{Q}}(0)$.

(3.1.3) Definition Let M be a standard Levi subgroup of G. A collection of points

$$\mathcal{Y}_{\mathrm{M}} = \left\{ Y_{\mathrm{P}} \in \mathfrak{a}_{\mathrm{M}} : \ \mathrm{P} \in \mathcal{P}(\mathrm{M}) \right\} \tag{3.1.3.1}$$

is said to be a (G, M)-*orthogonal set* if for each pair of adjacent parabolic subgroups P and P′ in $\mathcal{P}(\mathrm{M})$, the vector

$$Y_{\mathrm{P}} - Y_{\mathrm{P}'} \in \mathfrak{a}_{\mathrm{M}} \tag{3.1.3.2}$$

is orthogonal to the hyperplane spanned by the common wall of the positive chambers defined by P and P′.

Let $\alpha_{\mathrm{P}'}^{\mathrm{P}}{}^{\vee}$ be the unique coroot that is positive for P and negative for P′. A (G, M)-orthogonal set $\mathcal{Y}_{\mathrm{M}}$ is *positive* if

$$\exists t > 0 \qquad Y_{\mathrm{P}} - Y_{\mathrm{P}'} = t\alpha_{\mathrm{P}'}^{\mathrm{P}}{}^{\vee}. \tag{3.1.3.3}$$

(3.1.4) Remark Let $\mathcal{Y}_{\mathrm{M}}$ be a positive (G, M)-orthogonal set, then the collection of functions

$$\left\{ v_{\mathrm{P}}(\mathcal{Y}_{\mathrm{M}})(\lambda) = e^{\langle \lambda, Y_{\mathrm{P}} \rangle} : \ \mathrm{P} \in \mathcal{P}(\mathrm{M}) \right\} \tag{3.1.4.1}$$

forms a (G, M)-family. The associated constant $v_{\mathrm{M}}(\mathcal{Y}_{\mathrm{M}})$ as in (3.1.1.2) is equal to the volume of the convex hull of $\mathcal{Y}_{\mathrm{M}}$ in $\mathfrak{a}_{\mathrm{M}}$.

(3.1.5) Definition Let M be a standard Levi subgroup of G, let x be an element of $\mathrm{G}(\mathbb{Q}_S)$. The collection of points

$$\mathcal{Y}_{\mathrm{M}}(x) \;=\; \Big\{ -H_{\mathrm{P}}(x) : \;\mathrm{P} \in \mathcal{P}(\mathrm{M}) \Big\} \tag{3.1.5.1}$$

forms a positive (G, M)-orthogonal set. Define the *weight factor* $v_{\mathrm{M}}(x)$ to be the associated constant

$$v_{\mathrm{M}}(x) \;=\; v_{\mathrm{M}}(\mathcal{Y}_{\mathrm{M}}(x)). \tag{3.1.5.2}$$

Let X be a point in $\mathfrak{m}(\mathbb{Q}_S)$ such that $\mathrm{G}^0_X(\mathbb{Q}_S)$ is contained in $\mathrm{M}(\mathbb{Q}_S)$, define the *weighted orbital integral* $J^{\mathrm{G}}_{\mathrm{M}}(X, \;)$ to be the distribution on $\mathfrak{g}(\mathbb{Q}_S)$ such that

$$\begin{aligned} &\forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)) \\ &J^{\mathrm{G}}_{\mathrm{M}}(X, f) = |D^{\mathrm{G}}(X)|^{1/2}_S \int_{\mathrm{G}^0_X(\mathbb{Q}_S)\backslash \mathrm{G}(\mathbb{Q}_S)} f(X \cdot \mathrm{ad}(x)) \; v_{\mathrm{M}}(x) \mathrm{d}x. \end{aligned} \tag{3.1.5.3}$$

For a general element X in $\mathfrak{m}(\mathbb{Q}_S)$ define the *weighted orbital integral* $J^{\mathrm{G}}_{\mathrm{M}}(X, \;)$ to be the distribution on $\mathfrak{g}(\mathbb{Q}_S)$ such that

$$\begin{aligned} &\forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)) \\ &J^{\mathrm{G}}_{\mathrm{M}}(X, f) = \lim_{A \to 0} \sum_{\mathrm{L} \in \mathcal{L}(\mathrm{M})} r^{\mathrm{L}}_{\mathrm{M}}(\exp(X_{\mathrm{nil}}), \exp(A)) J^{\mathrm{G}}_{\mathrm{L}}(X + A, f) \end{aligned} \tag{3.1.5.4}$$

where A is a sequence of $\mathbb{Q}_S$-points of $\mathrm{Lie}(\mathrm{A}_{\mathrm{M}})$ such that

$$\mathrm{G}^0_{X+A}(\mathbb{Q}_S) \;\subset\; \mathrm{M}(\mathbb{Q}_S) \tag{3.1.5.5}$$

and $r^{\mathrm{L}}_{\mathrm{P}}(x, a)$ is an auxiliary (L, M)-family constructed by Arthur in §5 of [Ar88a], see also §2.4 of [HW13] and §III.2 of [Wa95].

(3.1.6) Lemma *Let* M *be a standard Levi subgroup of* G. *Let* X *be an element of* $\mathfrak{m}(\mathbb{Q}_S)$, *let* x *be an element of* $\mathrm{G}(\mathbb{Q}_S)$. *Let* f *be a Schwartz function on* $\mathcal{S}(\mathfrak{g}(\mathbb{Q}_S))$. *Then*

$$J_{\mathrm{M}}^{\mathrm{G}}(X, f \circ \mathrm{ad}(x)) \;=\; \sum_{\mathrm{P} \in \mathcal{F}(\mathrm{M})} J_{\mathrm{M}}^{\mathrm{M_P}}(X, f_{\mathrm{P},x}) \tag{3.1.6.1}$$

where $f_{\mathrm{P},x}$ *is the Schwartz function on* $\mathfrak{m}_{\mathrm{P}}(\mathbb{Q}_S)$ *defined as*

$$f_{\mathrm{P},x}(X) \;=\; \int_{\mathrm{K}} \int_{\mathfrak{n}_{\mathrm{P}}(\mathbb{A})} f((X+N)\cdot \mathrm{ad}(k))\; v'_{\mathrm{P}}(kx)\mathrm{d}N\mathrm{d}k \tag{3.1.6.2}$$

where for each y *in* $\mathrm{G}(\mathbb{Q}_S)$ *and for each parabolic subgroup* Q *in* $\mathcal{F}(\mathrm{M})$ *the constant* $v'_{\mathrm{Q}}(y)$ *is the constant* v'_{Q} *intervening in 3.1.2.5 associated to the* (G,M)*-orthogonal set* $\mathcal{Y}_{\mathrm{M}}(x)$ *defined in 3.1.5.1.*

Proof. See III.3.(f) of [Wa95]. □

2.3.2 Nilpotent orbits

(3.2.1) Definition Let $\mathfrak{g}_{\mathrm{nil}}$ be the $\sim$ equivalence class of the origin in $\mathfrak{g}(\mathbb{Q})$, hence the nilpotent locus of $\mathfrak{g}$. Let J_{nil} and J^T_{nil} denote respectively $J_{\mathfrak{g}_{\mathrm{nil}}}$ and $J^T_{\mathfrak{g}_{\mathrm{nil}}}$. Let $J^{\mathrm{G}}_{\mathrm{nil}}$ denote J_{nil} for convenience in inductive arguments involving Levi subgroups.

(3.2.2) Lemma *There exists a continuous seminorm* $\| \;\|$ *on* $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ *such that for every truncation parameter* T *in* $\mathfrak{a}_0$

$$\forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{A})) \qquad \left| J^T_{\mathrm{nil}}(f) - \int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T) \sum_{X \in \mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})} f(X\cdot \mathrm{ad}(x))\;\mathrm{d}x \right| \le \|f\| e^{-\frac{\mathrm{d}(T)}{2}} \tag{3.2.2.1}$$

where $\mathrm{d}(T)$ *denotes the distance from* T *to the root hyperplanes.*

Proof. The argument is valid for a general class $\mathfrak{o}$ and the sum of all the classes $\mathfrak{o}$, hence contains 2.2.6.1 as a special case. The necessary estimates are established by Chaudouard in the proof of Proposition 4.4 of [Ch02a]. The following is a sketch of the argument, for the details please refer to loc. cit. See also Theorem 3.1 of [Ar85].

Following the first part of the proof of (2.2.2.1), by the combinatorial lemma of Langlands, expand $J^T_{\mathrm{nil}}(f)$ as a sum indexed by triples of nested standard parabolic subgroups whose leading term corresponding to $(\mathrm{G},\mathrm{G},\mathrm{G})$ is

$$(3.2.2.2) \qquad \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T) \sum_{X\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})} f(X\cdot\mathrm{ad}(x))\,\mathrm{d}x.$$

Hence the left hand side of the inequality in (3.2.2.1) is bounded by

$$(3.2.2.3) \sum_{\substack{\mathrm{P}_1,\mathrm{P}_2,\mathrm{P}_3\in\mathcal{F}\\ \text{standard}\\ \mathrm{P}_1\subset\mathrm{P}_2\subsetneq\mathrm{P}_3}} \int_{\mathrm{P}_1(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_1}(x,T)\sigma_1^3(H_0(x)-T) \sum_{X\in\mathfrak{m}_1^2(\mathbb{Q})'\cap\mathfrak{o}} \left| \sum_{\overline{Y}\in\overline{\mathfrak{n}}_2^3(\mathbb{Q})'} \Phi_X(x,\overline{Y}) \right| \mathrm{d}x$$

where Φ denotes the partial Fourier transform of f

$$(3.2.2.4) \qquad \Phi_X(x,\overline{Y}) \;=\; \int_{\mathfrak{n}_2(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(x))\cdot\psi(\langle N,\overline{Y}\rangle)\,\mathrm{d}N.$$

Changing variables each summand of (3.2.2.3) is bounded by

$$(3.2.2.5) \qquad \sup_{y\in\Gamma}\Bigg(\int_{\mathrm{A}_1^{\mathrm{G}}(\mathbb{R})_0}\int_{\mathrm{A}_{0,T'}^{1,T}(\mathbb{R})} \delta_0^2(a_1a)^{-1}\sigma_1^3(H_{\mathrm{P}_1}(a_1)-T)\times$$

$$\times \sum_{X\in\mathfrak{m}_1^2(\mathbb{Q})'\cap\mathfrak{o}} \sum_{\overline{Y}\in\overline{\mathfrak{n}}_2^3(\mathbb{Q})'} |\Phi_{X\cdot\mathrm{ad}(a_1a)}(y,\overline{Y}\cdot\mathrm{ad}(a_1a))|\,\mathrm{d}a\mathrm{d}a_1\Bigg)$$

where Γ is a fixed compact subset of $\mathrm{M}_0(\mathbb{A})^1$ which is independant of the truncation parameter T, the subset $\mathrm{A}_{0,T'}^{1,T}(\mathbb{R})$ of $\mathrm{A}_0^1(\mathbb{R})$ is the T',T-truncated part as in (1.1.3.8), and δ_0^2 is the modulus function of P_2.

Let n be a natural number, let $\mathcal{D}$ be an invariant differential operator on $\mathfrak{g}(\mathbb{R})$ of degree n, denote by $\Phi^{\mathcal{D}}$ the partial Fourier transform of $\mathcal{D}f$ where $\mathcal{D}$ operates on f via its archimedean component f_∞, then

$$\left|\Phi_{X\cdot\mathrm{ad}(a_1a)}(y, \overline{Y}\cdot\mathrm{ad}(a_1a))\right| = C(y)\cdot\left|\overline{Y}\cdot\mathrm{ad}(a_1a)\right|^{-n}\cdot\left|\Phi^{\mathcal{D}}_{X\cdot\mathrm{ad}(a_1a)}(y, \overline{Y}\cdot\mathrm{ad}(a_1a))\right| \tag{3.2.2.6}$$

for some constant $C(y)$ that depends continuously on y. For a Schwartz function f on $\mathfrak{g}(\mathbb{A})$ define $N(f)$ to be the smallest natural number N such that f is supported on $\frac{1}{N}\widehat{\mathbb{Z}}\times\mathbb{R}$ where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. Let the natural number n be large enough so that

$$\sum_{\overline{Y}\in\bar{\mathfrak{n}}_2^3(\frac{1}{N(f)}\mathbb{Z})'}\left|\overline{Y}\cdot\mathrm{ad}(a_1a)\right|^{-n} \;\leq\; C\prod_{\alpha\in\Delta_0^3} e^{-k_\alpha\alpha(H_0(a_1a))} \tag{3.2.2.7}$$

for some constant C and natural numbers k_α. Define the seminorm $\|\;\|'$ by

$$\|f\|' \;=\; \sup_{X\in\mathfrak{g}(\mathbb{A})}|\mathcal{D}f(X)|. \tag{3.2.2.8}$$

A Schwartz function on the $\mathbb{A}$-valued points of a rational vector space is bounded by a product of Schwartz functions on each coordinate, hence for Z in $\mathfrak{p}_2(\mathbb{A})$

$$|\mathcal{D}f(Z\cdot\mathrm{ad}(y))| \;\leq\; \Big(\prod_{\mu\in\Phi_0-\Phi_2}\phi_\mu(Z_\mu)\Big)\phi_{\mathfrak{n}_2}(Z_{\mathfrak{n}_2}) \tag{3.2.2.9}$$

where Z_μ and $Z_{\mathfrak{n}_2}$ are the components of Z on the weight space of μ and $\mathfrak{n}_2$, and $\phi_\bullet$ are positive Schwartz functions. If a_0 is an element of $\mathrm{A}_0(\mathbb{R})$, denote by $\Psi(a_0)$ for the sum

$$\Psi(a_0) \;=\; \sum_{X\in\mathfrak{m}_1^2(\frac{1}{N(f)}\mathbb{Z})'\cap\mathfrak{o}}\Big(\prod_{\mu\in\Phi_0-\Phi_2}\phi_\mu(\mu(a_0)^{-1}X_\mu)\Big). \tag{3.2.2.10}$$

Then (3.2.2.5) is bounded by

$$\sup_{y\in\Gamma} C(y)\int_{\mathrm{A}_1^{\mathrm{G}}(\mathbb{R})}\int_{\mathrm{A}_{0,T'}^{1,T}(\mathbb{R})}\Big(\delta_0^2(a_1a)^{-1}\sigma_1^2(H_{\mathrm{P}_1}(a_1)-T)\times \tag{3.2.2.11}$$

$$\times\Psi(a_1a)\cdot C\prod_{\alpha\in\Delta_0^2} e^{-k_\alpha\alpha(H_0(a_1a))}\Big)\ \mathrm{d}a\mathrm{d}a_1,$$

which only depends on the Schwartz function f via its componentwise bounds $\phi_\bullet$ and the lattice $\overline{\mathfrak{n}}_2^3(\frac{1}{N(f)}\mathbb{Z})$, hence is proportional to the seminorm $\| \ \|$ defined by

$$\forall f\in\mathcal{S}(\mathfrak{g}(\mathbb{A})) \qquad \|f\| = \sup_{y\in\Gamma} C(y)\cdot C\cdot\|f\|'\cdot N(f)^n. \tag{3.2.2.12}$$

The constant $\sup_{y\in\Gamma} C(y)\cdot C$ is independant of both the Schwartz function f and the truncation parameter T.

Substituting the definition of the seminorm $\| \ \|$, the majorant (3.2.2.11) reduces to

$$\|f\|\cdot\mathrm{Vol}(\mathrm{A}_{0,T'}^{1,T}(\mathbb{R}))\prod_{\alpha\in\Delta_1^3}\left(e^{-\alpha(T)}\int_0^\infty p(t_\alpha)e^{-t_\alpha}\ \mathrm{d}t_\alpha\right) \tag{3.2.2.13}$$

where $p(t)$ is a polynomial. Because

$$\prod_{\alpha\in\Delta_1^3} e^{-\alpha(T)}\int_0^\infty p(t)e^{-t}\ \mathrm{d}t \ \ \leq \ \ e^{-\mathrm{d}(T)}, \tag{3.2.2.14}$$

and $\mathrm{Vol}(\mathrm{A}_{0,T'}^{1,T}(\mathbb{R}))$ is of polynomial growth in T, the majorant (3.2.2.13) reduces to

$$\|f\|e^{-\frac{\mathrm{d}(T)}{2}}. \tag{3.2.2.15}$$

□

(3.2.3) Definition Let v be a place of $\mathbb{Q}$, let β_v be a bump function on $\mathbb{Q}_v$, let ν be a G-orbit contained in $\mathfrak{g}_{\text{nil}}$, let $\{p_1, \ldots, p_l\}$ be a collection of polynomials rational with rational coefficients cutting out the Zariski closure $\overline{\nu}$. Let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let ϵ be a positive real number. Define the *truncated function* $f^{\epsilon}_{\nu,v}$ on $\mathfrak{g}(\mathbb{A})$ as in [Ar85] by

$$(3.2.3.1) \qquad \forall X \in \mathfrak{g}(\mathbb{A}) \qquad f^{\epsilon}_{\nu,v}(X) = f(X)\beta_v(\epsilon^{-1}|p_1(X)|_v)\ldots\beta_v(\epsilon^{-1}|p_l(X)|_v).$$

(3.2.4) Lemma *Let the place v, the bump function β_v, the orbit ν and the polynomials $\{p_1, \ldots, p_l\}$ be as in (3.2.3), then there exists a natural number m and another seminorm $\| \, \|_1$ for which the inequality (3.2.2.1) holds such that*

$$(3.2.4.1) \qquad \forall \epsilon \text{ such that } 0 < \epsilon < 1 \; \forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{A})) \qquad \|f^{\epsilon}_{\nu,v}\| \leq \epsilon^{-ml}\|f\|_1.$$

Proof. It is enough to consider the case when there is a single polynomial p. There are two cases:

- If v is the archimedean place then

  $$(3.2.4.2) \qquad N(f^{\epsilon}_{\nu,v}) \;=\; N(f),$$

  so by properties of the derivative

  $$(3.2.4.3) \qquad \|f^{\epsilon}_{\nu,v}\|' \;\leq\; \epsilon^{-m}C\|f\|'.$$

  Define $\|f\|'_1$ to be $C\|f\|$.

- If v is finite then

  $$(3.2.4.4) \qquad \|f^{\epsilon}_{\nu,v}\|' = \|f\|',$$

so

$$N(f^{\epsilon}_{\nu,v}) \leq \epsilon^{-m} N(f) \tag{3.2.4.5}$$

since $N(f)$ only depends on the support of f_{finite} and the assignment

$$f \mapsto f^{\epsilon}_{\nu,v} = f \prod \beta_v(\epsilon^{-1}|p|_v) \tag{3.2.4.6}$$

shrinks the support of f by a factor of ϵ^m, upto a multiplicative constant. Define $\|f\|'$ to be $\|f\|$.

Hence (3.2.4.1) follows since

$$\|f\| = \|f\|' N(f). \tag{3.2.4.7}$$

□

(3.2.5) Lemma *Let the place v, the bump function β_v, the orbit ν and the polynomials $\{p_1, \ldots, p_l\}$ be as in (3.2.3), then there exists a positve real number r such that*

(3.2.5.1) $\forall \epsilon > 0 \; \forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{A}))$

$$\int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T) \sum_{X \in \mathfrak{g}(\mathbb{Q}) - \overline{\nu}(\mathbb{Q})} \left| f^{\epsilon}_{\nu,v}(X \cdot \mathrm{ad}(x)) \right| \, \mathrm{d}x \leq \|f\| \epsilon^r (1 + |T|)^{d_0}$$

where d_0 is $\dim(\mathrm{A}_0/\mathrm{A}_\mathrm{G})$, *the split rank of* G.

Proof. This is Lemma 4.1 of [Ar85]. □

(3.2.6) Proposition *Let T be a truncation parameter. For each nilpotent orbit ν there exists a distribution J^T_ν on $\mathfrak{g}(\mathbb{A})$ such that for each Schwartz function f on $\mathfrak{g}(\mathbb{A})$, the expres-*

sion $J_\nu^T(f)$ is a polynomial in T of degree at most d_0, and

$$J_{\mathrm{nil}}^T(f) \quad = \sum_{\substack{\nu\subset\mathfrak{g}_{\mathrm{nil}}\\ \mathit{orbit}}} J_\nu^T(f). \tag{3.2.6.1}$$

There exists a continuous seminorm $\|\ \|$ on $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ and a positive real number ϵ such that

$$\forall f\in\mathcal{S}(\mathfrak{g}(\mathbb{A}))$$
$$\left|J_\nu^T(f)-\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}F^{\mathrm{G}}(x,T)\sum_{X\in\nu(\mathbb{Q})}f(X\cdot\mathrm{ad}(x))\,\mathrm{d}x\right|\le\|f\|e^{-\epsilon\mathrm{d}(T)}. \tag{3.2.6.2}$$

Proof. Define the polynomials recursively by the formula

$$J_{\overline{\nu}}^T(f) \quad = \quad \lim_{\epsilon\to0}J_{\mathrm{nil}}^T(f_{\nu,v}^\epsilon) \tag{3.2.6.3}$$

where $J_{\overline{\nu}}^T$ denotes the sum of $J_{\nu'}^T$ for those orbits ν' contained in $\overline{\nu}$.

The limit of $J_{\mathrm{nil}}^T(f_{\nu,v}^\epsilon)$ as ϵ approaches 0 exists because

$$\sum_{X\in\overline{\nu}(\mathbb{Q})}f(X\cdot\mathrm{ad}(x)) \quad = \quad \sum_{X\in\overline{\nu}(\mathbb{Q})}f_{\nu,v}^\epsilon(X\cdot\mathrm{ad}(x)) \tag{3.2.6.4}$$

implies that

$$\left|J_{\mathrm{nil}}^T(f_{\nu,v}^\epsilon)-\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}F^{\mathrm{G}}(x,T)\sum_{X\in\overline{\nu}(\mathbb{Q})}f(X\cdot\mathrm{ad}(x))\,\mathrm{d}x\right| \tag{3.2.6.5}$$

$$\le\quad\left|J_{\mathrm{nil}}^T(f_{\nu,v}^\epsilon)-\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathrm{A})^1}F^{\mathrm{G}}(x,T)\sum_{X\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})}f_{\nu,v}^\epsilon(X\cdot\mathrm{ad}(x))\,\mathrm{d}x\right|+$$

$$+\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathrm{A})^1}F^{\mathrm{G}}(x,T)\sum_{X\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})-\overline{\nu}(\mathbb{Q})}\left|f_{\nu,v}^\epsilon(X\cdot\mathrm{ad}(x))\right|\,\mathrm{d}x.$$

By (3.2.2.1) the first summand of the right hand side of (3.2.6.5) is bounded by

$$\|f^{\epsilon}_{\nu,v}\|e^{-\frac{\mathrm{d}(T)}{2}} \;\leq\; \epsilon^{-lm}\cdot\|f\|_1\cdot e^{-\frac{\mathrm{d}(T)}{2}} \tag{3.2.6.6}$$

which follows from (3.2.4.1).

The second summand of the right hand side of (3.2.6.5) is bounded by summing over the complement of $\overline{\nu}(\mathbb{Q})$ in $\mathfrak{g}(\mathbb{Q})$ instead of in $\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})$, hence by (3.2.5.1) is bounded by

$$\|f\|_1\cdot\epsilon^r\cdot(1+|T|)^{d_0}. \tag{3.2.6.7}$$

Therefore the right hand side of (3.2.6.5) is bounded by

$$\|f\|_1\Big(\epsilon^{-lm}e^{-\frac{\mathrm{d}(T)}{2}}+\epsilon^r(1+|T|)^{d_0}\Big). \tag{3.2.6.8}$$

It suffices to take δ^n as ϵ with δ bounded strictly between 0 and 1 and n a sequence natural numbers approaching infinity. Then the majorant (3.2.6.8) satisfies the inequality

$$\|f\|_1\Big(e^{|\log(\delta)|lmn-\frac{\mathrm{d}(T)}{2}}+\delta^{rn}(1+|T|)^{d_0}\Big) \;\leq\; \|f\|\cdot\delta^{rn}\cdot(1+|T|)^{d_0} \tag{3.2.6.9}$$

provided $\mathrm{d}(T)$ is bounded below by $C|\log(\delta)|n$ for some constant C and $\|\ \|$ is another continuous seminorm with the same properties.

Therefore for every natural number n and every T in the open subcone

$$\Big\{T:\mathrm{d}(T)>C|\log(\delta)|(n+1)\Big\} \tag{3.2.6.10}$$

of the positive chamber, the following inequality holds

$$|J^T_{\mathrm{nil}}(f^{\delta^n}_{\nu,v})-J^T_{\mathrm{nil}}(f^{\delta^{n+1}}_{\nu,v})| \;\leq\; 2\|f\|(1+|T|)^{d_0}\delta^{rn}. \tag{3.2.6.11}$$

The left hand side is a polynomial in T of degree at most d_0, hence the polynomial extrapolation lemma in Lemma 5.2 of [Ar82] applies:

There exists a constant A such that for all T,

$$|J^T_{\text{nil}}(f^{\delta^n}_{\nu,v}) - J^T_{\text{nil}}(f^{\delta^{n+1}}_{\nu,v})| \quad \leq \quad A \cdot \|f\| \cdot (1+|T|)^{d_0} \cdot \big(|\log(\delta)|(n+1)\big)^{d_0} \cdot \delta^{rn}. \tag{3.2.6.12}$$

Since

$$\sum_{n=0}^{\infty} \big(|\log(\delta)|(n+1)\big)^{d_0} \delta^{rn} \quad < \quad \infty, \tag{3.2.6.13}$$

by telescoping the series

$$\sum_{n=0}^{\infty} J^T_{\text{nil}}(f^{\delta^n}_{\nu,v}) - J^T_{\text{nil}}(f^{\delta^{n+1}}_{\nu,v}) \tag{3.2.6.14}$$

the sequence $J^T_{\text{nil}}(f^{\delta^n}_{\nu,v})$ has a limit as n approaches infinity. The limit is a polynomial in T of degree at most d_0, denoted by $J^T_{\overline{\nu}}(f)$.

By construction

$$\left| J^T_{\overline{\nu}}(f) - \int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T) \sum_{X \in \overline{\nu}(\mathbb{Q})} f(X \cdot \mathrm{ad}(x))\, \mathrm{d}x \right| \tag{3.2.6.15}$$

$$\leq \quad \left| J^T_{\text{nil}}(f^{\delta^n}_{\nu,v}) - J^T_{\overline{\nu}}(f) \right| + \left| J^T_{\text{nil}}(f^{\delta^n}_{\nu,v}) - \int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T) \sum_{X \in \overline{\nu}(\mathbb{Q})} f(X \cdot \mathrm{ad}(x))\, \mathrm{d}x \right|$$

$$\leq \sum_{n=0}^{\infty} A\|f\|(1+|T|)^{d_0} \big(|\log(\delta)|(n+1)\big)^{d_0} \delta^{rn} + \|f\|\delta^{rn}(1+|T|)^{d_0} \tag{3.2.6.16}$$

where (3.2.6.16) holds for a fixed δ, a sufficiently positive T and n the largest natural number such that

$$\mathrm{d}(T) \quad \geq \quad C|\log(\delta)|n. \tag{3.2.6.17}$$

Therefore it is possible choose a new seminorm $\| \ \|$ and a new constant $\epsilon > 0$ such that

$$(3.2.6.18)\left| J^T_{\overline{\nu}}(f) - \int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1} F^{\mathrm{G}}(x,T) \sum_{X \in \overline{\nu}(\mathbb{Q})} f(X \cdot \mathrm{ad}(x)) \, \mathrm{d}x \right| \ \leq \ \|f\| e^{-\epsilon \mathrm{d}(T)}.$$

The corresponding statements for ν instead of $\overline{\nu}$ follow recursively by setting

$$(3.2.6.19) \qquad J^T_{\overline{\nu}}(f) \ = \ \sum_{\substack{\nu' \subset \overline{\nu} \\ \text{orbit}}} J^T_{\nu'}(f).$$

$\square$

(3.2.7) Proposition *Let S be a finite set of places of $\mathbb{Q}$ containing the archimedean place, let f be a Schwartz function on $\mathfrak{g}(\mathbb{A})$ such that f_p is the characteristic function of the standard lattice $\mathfrak{g}(\mathbb{Z}_p)$ whenever p is not contained in S. Let* M *be a standard Levi subgroup of* G*. Denote by $\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}$ the set of* $\mathrm{M}(\mathbb{Q}_S)$*-conjugacy classes in $\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})$. For each nilpotent conjugacy class ν in $\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}$ here exists a constant $a^{\mathrm{M}}(S,\nu)$ such that*

$$(3.2.7.1) \qquad J^{\mathrm{G}}_{\mathrm{nil}}(f) \ = \ \sum_{\mathrm{M} \in \mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{\nu \in \mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}} a^{\mathrm{M}}(S,\nu) J^{\mathrm{G}}_{\mathrm{M}}(\nu, f_S).$$

Remark By Lemma 7.1 of [Ar85], the sets $\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}$ and $\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q}_S)/\mathrm{M}(\mathbb{Q}_S)^1$ are in natural bijection, hence the expression $J^{\mathrm{G}}_{\mathrm{M}}(\nu, f_S)$ makes sense.

Proof. The argument is based on the following two lemmas:

Lemma *For every x in* $\mathrm{G}(\mathbb{A})^1$

$$(3.2.7.2) \qquad J^{\mathrm{G}}_{\mathrm{nil}}(f \circ \mathrm{ad}(x)) \ = \ \sum_{\mathrm{Q} \in \mathcal{F}} |\mathrm{W}_0^{\mathrm{M_Q}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} J^{\mathrm{M_Q}}_{\mathrm{nil}}(f_{\mathrm{Q},x}),$$

where the function $f_{\mathrm{Q},x}$ on $\mathfrak{m}_{\mathrm{Q}}(\mathbb{A})$ is defined as in (2.2.12.2).

Proof. This is a special case of (2.2.12.1). ☐

Lemma *Let* M *be a standard Levi subgroup. For every point* x *in* $\mathrm{G}(\mathbb{Q}_S)^1$ *and* ν *in* $\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q}_S)/\mathrm{M}(\mathbb{Q}_S)^1$

$$(3.2.7.3) \qquad J_{\mathrm{M}}^{\mathrm{G}}(\nu, f_S \circ \mathrm{ad}(x)) \;=\; \sum_{\mathrm{Q}\in\mathcal{F}(\mathrm{M})} J_{\mathrm{M}}^{\mathrm{M}_{\mathrm{Q}}}(\nu, f_{S,\mathrm{Q},x}).$$

Proof. This is a special case of (3.1.6.1). ☐

Argue by induction. Assume that the constants $a^{\mathrm{L}}(S,\nu)$ and the identities

$$(3.2.7.4) \qquad J_{\mathrm{nil}}^{\mathrm{L}}(f) \;=\; \sum_{\mathrm{M}\in\mathcal{L}^{\mathrm{L}}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{L}}|^{-1} \sum_{\nu\in\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}} a^{\mathrm{M}}(S,\nu) J_{\mathrm{M}}^{\mathrm{L}}(\nu, f_S)$$

are known for every proper Levi subgroup L of G.

Define the distribution T^{G} on $\mathfrak{g}(\mathbb{Q}_S)$ by

$$(3.2.7.5) \qquad \forall f_S \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S))$$

$$T^{\mathrm{G}}(f_S) \;=\; J_{\mathrm{nil}}^{\mathrm{G}}\Big(f_S \otimes \bigotimes_{p\notin S} \mathbb{I}_{\mathfrak{g}(\mathbb{Z}_p)}\Big) - \sum_{\substack{\mathrm{M}\in\mathcal{L}\\ \mathrm{M}\neq\mathrm{G}}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \times$$

$$\times \sum_{\nu\in\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}} a^{\mathrm{M}}(S,\nu) J_{\mathrm{M}}^{\mathrm{G}}(\nu, f_S).$$

The distribution T^{G} is supported on $\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q}_S)$. By (3.2.7.2) and (3.2.7.3)

$$(3.2.7.6) \qquad T^{\mathrm{G}}(f_S \circ \mathrm{ad}(x)) - T^{\mathrm{G}}(f_S)$$

$$=\; \Bigg(\sum_{\mathrm{Q}\in\mathcal{F}} |\mathrm{W}_0^{\mathrm{M}_{\mathrm{Q}}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} J_{\mathrm{nil}}^{\mathrm{M}_{\mathrm{Q}}}\Big((f_S \otimes \bigotimes_{p\notin S} \mathbb{I}_{\mathfrak{g}(\mathbb{Z}_p)})_{\mathrm{Q},x}\Big)$$

$$- \sum_{\substack{\mathrm{M}\in\mathcal{L}\\ \mathrm{M}\neq\mathrm{G}}} \sum_{\mathrm{Q}\in\mathcal{F}(\mathrm{M})} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{\nu\in\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}} a^{\mathrm{M}}(S,\nu) J_{\mathrm{M}}^{\mathrm{M_Q}}(\nu, f_{S,\mathrm{Q},x})\Bigg)$$

$$-\Bigg(J_{\mathrm{nil}}^{\mathrm{G}}\Big(f_S \otimes \bigotimes_{p\notin S} \mathbb{I}_{\mathfrak{g}(\mathbb{Z}_p)}\Big) - \sum_{\substack{\mathrm{M}\in\mathcal{L}\\ \mathrm{M}\neq\mathrm{G}}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \times$$

$$\times \sum_{\nu\in\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}} a^{\mathrm{M}}(S,\nu) J_{\mathrm{M}}^{\mathrm{G}}(\nu, f_S)\Bigg)$$

$$= \sum_{\substack{Q\in\mathcal{F}\\ \mathrm{Q}\neq\mathrm{G}}} |\mathrm{W}_0^{\mathrm{M_Q}}||\mathrm{W}_0^{\mathrm{G}}|^{-1}\Bigg(J_{\mathrm{nil}}^{\mathrm{M_Q}}\Big((f_S \otimes \bigotimes_{p\notin S} \mathbb{I}_{\mathfrak{g}(\mathbb{Z}_p)})_{\mathrm{Q},x}\Big)$$

$$- \sum_{\mathrm{M}\in\mathcal{L}^{\mathrm{M_Q}}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{M_Q}}|^{-1} \sum_{\nu\in\mathfrak{m}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{M},S}} a^{\mathrm{M}}(S,\nu) J_{\mathrm{M}}^{\mathrm{M_Q}}(\nu, f_{S,\mathrm{Q},x})\Bigg)$$

which vanishes by the inductive hypothesis that T^{L} is invariant applied to the Levi subgroup $\mathrm{M_Q}$ with Q a proper parabolic subgroup of G. Therefore T^{G} is invariant under the action of $\mathrm{G}(\mathbb{Q}_S)^1$.

Construct the constants $a^{\mathrm{G}}(S,\nu)$ subject to the new identity

$$T^{\mathrm{G}}(f_S) = \sum_{\nu\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{G},S}} a^{\mathrm{G}}(S,\nu) J_{\mathrm{G}}^{\mathrm{G}}(\nu, f_S). \tag{3.2.7.7}$$

Stratify the nilpotent locus $\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q}_S)$ equivariantly by codimension. More precisely define for each natural number d the open set

$$\mathfrak{g}_{\mathrm{nil},d}(\mathbb{Q}_S) = \bigcup_{\substack{\nu\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{G},S}\\ \mathrm{codim}(\nu)\leq d}} \nu(\mathbb{Q}_S). \tag{3.2.7.8}$$

Denote by T_d^{G} the distribution obtained by restricting T^{G} to $\mathfrak{g}_{\mathrm{nil},d}(\mathbb{Q}_S)$.

The open set $\mathfrak{g}_{\mathrm{nil},0}(\mathbb{Q}_S)$ is the regular nilpotent orbit. Since T_0^{G} is invariant it is equal to a multiple of $J_{\mathrm{G}}^{\mathrm{G}}(\nu_{\mathrm{reg}},\ )$, where ν_{reg} denotes the regular nilpotent orbit. Define $a^{\mathrm{G}}(S,\nu_{\mathrm{reg}})$ to be the constant of proportionality.

The other constants $a^{\mathrm{G}}(S,\nu)$ are constructed by induction on the codimension d which ranges among $0,1,2,\dots,\dim(\mathfrak{g}_{\mathrm{nil}})$. Let $T^{\mathrm{G},d}$ be the distribution on the complement of $\mathfrak{g}_{\mathrm{nil},d-1}(\mathbb{Q}_S)$ in $\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q}_S)$ defined by

$$T^{\mathrm{G},d}(f_S) \;=\; T^{\mathrm{G}}(f_S) - \sum_{\substack{\nu\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{G},S}\\ \mathrm{codim}(\nu)<d}} a^{\mathrm{G}}(S,\nu)J_{\mathrm{G}}^{\mathrm{G}}(\nu,f_S). \tag{3.2.7.9}$$

Denote by $T_d^{\mathrm{G},d}$ its restriction to the complement of $\mathfrak{g}_{\mathrm{nil},d-1}(\mathbb{Q}_S)$ in $\mathfrak{g}_{\mathrm{nil},d}(\mathbb{Q}_S)$.

Since the complement of $\mathfrak{g}_{\mathrm{nil},d-1}(\mathbb{Q}_S)$ in $\mathfrak{g}_{\mathrm{nil},d}(\mathbb{Q}_S)$ has an open partition by the nilpotent orbits of codimension d, and the distribution $T_d^{\mathrm{G},d}$ is invariant, there exist constants $a^{\mathrm{G}}(S,\nu)$, one for each ν of codimension d, such that

$$T_d^{\mathrm{G},d}(f_S) \;=\; \sum_{\substack{\nu\in\mathfrak{g}_{\mathrm{nil}}(\mathbb{Q})_{\mathrm{G},S}\\ \mathrm{codim}(\nu)=d}} a^{\mathrm{G}}(S,\nu)J_{\mathrm{G}}^{\mathrm{G}}(\nu,f_S). \tag{3.2.7.10}$$

The constants $a^{\mathrm{G}}(S,\nu)$ are required to satisfy (3.2.7.7), which is equivalent to

$$T^{\mathrm{G}}(f_S) \;=\; T_0^{\mathrm{G},0}(f_S)+T_1^{\mathrm{G},1}(f_S)+T_2^{\mathrm{G},2}(f_S)+\cdots+T_{\dim(\mathfrak{g}_{\mathrm{nil}})}^{\mathrm{G},\dim(\mathfrak{g}_{\mathrm{nil}})}(f_S). \tag{3.2.7.11}$$

Let $v\in S$ be a place of $\mathbb{Q}_S$, let ν be a nilpotent orbit, define the function $f_{S,\nu,v}^{\epsilon}$ by the same formula (3.2.3.1) as for $f_{\nu,v}^{\epsilon}$. Let ν_d denote the complement of $\mathfrak{g}_{\mathrm{nil},d-1}$ in $\mathfrak{g}_{\mathrm{nil},d}$, the union of the nilpotent orbits ν of codimension d. Then the expression $f_{S,\nu_d,v}^{\epsilon}$ makes sense, and

$$T_d^{\mathrm{G},d}(f_S) \;=\; \lim_{\epsilon\to0}T^{\mathrm{G}}(f_{S,\nu_d,v}^{\epsilon}) - \lim_{\epsilon\to0}T^{\mathrm{G}}(f_{S,\nu_{d+1},v}^{\epsilon}). \tag{3.2.7.12}$$

Therefore

$$T_0^{\mathrm{G},0}(f_S) + T_1^{\mathrm{G},1}(f_S) + T_2^{\mathrm{G},2}(f_S) + \cdots + T_{\dim(\mathfrak{g}_{\mathrm{nil}})}^{\mathrm{G},\dim(\mathfrak{g}_{\mathrm{nil}})}(f_S) \tag{3.2.7.13}$$

$$= \lim_{\epsilon \to 0} \Bigg(\underbrace{T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_0,v}}_{\text{this is } T^{\mathrm{G}}(f_S)}) - T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_1,v}) + T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_1,v}) - T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_2,v}) + T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_2,v}) - \ldots$$

$$\cdots - T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_{\dim(\mathfrak{g}_{\mathrm{nil}})},v}) + T^{\mathrm{G}}(f^{\epsilon}_{S,\nu_{\dim(\mathfrak{g}_{\mathrm{nil}})},v}) - T^{\mathrm{G}}\underbrace{(f^{\epsilon}_{S,\nu_{\dim(\mathfrak{g}_{\mathrm{nil}})+1},v})}_{\text{this is void}} \Bigg)$$

$$= T^{\mathrm{G}}(f_S).$$

□

(3.2.8) Remark If ν is the orbit consisting of the origin, the coefficient $a^{\mathrm{M}}(S,\nu)$ is independent of S and equal to the corresponding Tamagawa number

$$a^{\mathrm{M}}(S,0) = \mathrm{Vol}(\mathrm{M}(\mathbb{Q})\backslash\mathrm{M}(\mathbb{A})^1). \tag{3.2.8.1}$$

2.3.3 General orbits

(3.3.1) Definition Let $\mathfrak{o}$ be a $\sim$ equivalence class on $\mathfrak{g}(\mathbb{Q})$. Let P_1 be a parabolic subgroup of G, let M_1 be the Levi component of P_1 which is standard. Fix a semisimple element Σ in $\mathfrak{o}$ such that Σ is contained in the Levi subalgebra $\mathfrak{m}_1$, but not in any proper parabolic subalgebra of $\mathfrak{p}_1$. The group $\mathrm{P}^0_{1,\Sigma}$ is a minimal parabolic subgroup of G^0_Σ with minimal Levi component $\mathrm{M}^0_{1,\Sigma}$. Denote by $\mathcal{F}^\Sigma$ the set of parabolic subgroups of G^0_Σ containing $\mathrm{M}^0_{1,\Sigma}$. A parabolic subgroup Q in $\mathcal{F}^\Sigma$ is said to be standard if Q contains $\mathrm{P}^0_{1,\Sigma}$. Fix a maximal compact subgroup K_Σ of $\mathrm{G}^0_\Sigma(\mathbb{A})$ that is admissible with respect to $\mathrm{M}^0_{1,\Sigma}$ such that for each parabolic subgroup Q in $\mathcal{F}^{\Sigma}$ there is the associated function

$$(3.3.1.1) \qquad H_{\mathrm{Q}} : \quad \mathrm{G}^{0}_{\Sigma}(\mathbb{A}) \ \to \ \mathfrak{a}_{\mathrm{Q}}.$$

There is a unique point $T_{\Sigma,1}$ in $\mathfrak{a}_1$ modulo $\mathfrak{a}_{\mathrm{G}^0_{\Sigma}}$ defined in the same manner as T_0 in $\mathfrak{a}_0^{\mathrm{G}}$ that satisfies an identity analogous to (2.2.8.1). Let L be a Levi subgroup of G^0_{Σ} containing $\mathrm{M}^0_{1,\Sigma}$, denote by $\mathrm{W}_1^{\mathrm{L}}$ the Weyl group of L with respect to the split torus A_1.

Let π_{Σ} be the surjection from $\mathcal{F}(\mathrm{M}_1)$ onto $\mathcal{F}^{\Sigma}$ defined by

$$(3.3.1.2) \qquad \forall \mathrm{P} \in \mathcal{F}(\mathrm{M}_1) \qquad \pi_{\Sigma}(\mathrm{P}) = \mathrm{P}^0_{\Sigma}.$$

Let Q be a parabolic subgroup in $\mathcal{F}^{\Sigma}$. Let $\mathcal{F}_{\mathrm{Q}}(\mathrm{M}_1)$ be the inverse image $\pi_{\Sigma}^{-1}(\mathrm{Q})$. Define subsets $\mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)$ and $\bar{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)$ of $\mathcal{F}(\mathrm{M}_1)$ by

$$(3.3.1.3) \qquad \begin{aligned} \mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1) &= \Big\{ \mathrm{P} \in \mathcal{F}(\mathrm{M}_1) : \ \pi_{\Sigma}(\mathrm{P}) = \mathrm{Q}, \ \mathfrak{a}_{\mathrm{P}} = \mathfrak{a}_{\mathrm{Q}} \Big\} \\ \bar{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1) &= \Big\{ \mathrm{P} \in \mathcal{F}(\mathrm{M}_1) : \ \pi_{\Sigma}(\mathrm{P}) \supset \mathrm{Q} \Big\}. \end{aligned}$$

There are inclusions

$$(3.3.1.4) \qquad \forall \mathrm{Q} \in \mathcal{F}^{\Sigma} \qquad \mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1) \subset \mathcal{F}_{\mathrm{Q}}(\mathrm{M}_1) \subset \bar{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1) \subset \mathcal{F}(\mathrm{M}_1).$$

(3.3.2) Definition Let P_1, M_1, $\mathfrak{o}$, Σ be as in (3.3.1). Let Q be a parabolic subgroup in $\mathcal{F}^{\Sigma}$. Let $\mathcal{Y}$ be a collection of points

$$(3.3.2.1) \qquad \mathcal{Y} \ = \ \Big\{ Y_{\mathrm{P}} \in \mathfrak{a}_0 : \ \mathrm{P} \in \mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1) \Big\}$$

satisfying the compatibility conditions defining a $(\mathrm{G}, \mathrm{M}_1)$-orthogonal set in (3.1.3), namely that for each pair of adjacent parabolic subgroups P and P′ in $\mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)$ the difference between

Y_{P} and $Y_{\mathrm{P}'}$ is orthogonal to the common wall of the positive chambers defined by P and P′.

The collection $\mathcal{Y}$ has a unique extension from $\mathring{\bar{\mathcal{F}}}_{\mathrm{Q}}(\mathrm{M}_1)$ to $\bar{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)$. Let Q′ be a parabolic subgroup in $\mathcal{F}^{\Sigma}$ containing Q, denote by $\mathcal{Y}_{\mathrm{Q}'}$ the collection of points

$$(3.3.2.2) \qquad \mathcal{Y}_{\mathrm{Q}'} \;=\; \Big\{ Y_{\mathrm{P}} : \; \mathrm{P} \in \mathcal{F}_{\mathrm{Q}'}(\mathrm{M}_1) \Big\}.$$

Let Q_3 be a parabolic subgroup in $\mathcal{F}^{\Sigma}$. Define the *gamma′ function* $\Gamma'_{\mathrm{Q}_3}(\ ,\mathcal{Y}_{\mathrm{Q}_3})$ on $\mathfrak{a}_1$ by

$$(3.3.2.3) \qquad \forall H \in \mathfrak{a}_1$$

$$\Gamma'_{\mathrm{Q}_3}(H, \mathcal{Y}_{\mathrm{Q}_3}) = \sum_{\substack{\mathrm{Q}_4 \in \mathcal{F}^{\Sigma} \\ \mathrm{Q}_4 \supset \mathrm{Q}_3}} \tau_3^4(H) \Bigg(\sum_{\mathrm{P} \in \mathcal{F}_{\mathrm{Q}_4}(\mathrm{M}_1)} (-1)^{\dim(\mathrm{A}_{\mathrm{P}}/\mathrm{A}_{\mathrm{G}})} \hat{\tau}_{\mathrm{P}}(H - Y_{\mathrm{P}}) \Bigg).$$

(3.3.3) Remark The function $\Gamma'_{\mathrm{Q}_3}(\ ,\mathcal{Y}_{\mathrm{Q}_3})$ factorizes through the projection from $\mathfrak{a}_1$ onto $\mathfrak{a}_3^{\mathrm{G}}$ and depends continuously on $\mathcal{Y}$. For each parabolic subgroup Q_2 in $\mathcal{F}^{\Sigma}$

$$(3.3.3.1) \sum_{\mathrm{P} \in \mathcal{F}_{\mathrm{Q}_2}(\mathrm{M}_1)} (-1)^{\dim(\mathrm{A}_{\mathrm{P}}/\mathrm{A}_{\mathrm{G}})} \hat{\tau}_{\mathrm{P}}(H - Y_{\mathrm{P}}) = \sum_{\substack{\mathrm{Q}_3 \in \mathcal{F}^{\Sigma} \\ \mathrm{Q}_3 \supset \mathrm{Q}_2}} (-1)^{\dim(\mathrm{A}_2/\mathrm{A}_3)} \hat{\tau}_2^3(H) \Gamma'_{\mathrm{Q}_3}(H, \mathcal{Y}_{\mathrm{Q}_3}).$$

See §4 of [Ar86].

(3.3.4) Lemma *Let* Q *be a parabolic subgroup in* $\mathcal{F}^{\Sigma}$*, let* $\mathcal{Y}$ *be a collection of points as in (3.3.2.2). The function* $\Gamma'_{\mathrm{Q}}(\ ,\mathcal{Y}_{\mathrm{Q}})$ *is compactly supported as a function on* $\mathfrak{a}_{\mathrm{Q}}^{\mathrm{G}}$.

Let (c_{P}) *be the* $(\mathrm{G}, \mathrm{M}_1)$*-family associated with a* $(\mathrm{G}, \mathrm{M}_1)$*-orthogonal set that contains* $\mathcal{Y}_{\mathrm{Q}}$ *as in (3.1.4.1). Let* c'_{P} *be the functions intervening in (3.1.2.5). Let* $\Gamma'_{\mathrm{Q}}\hat{}\,(\ ,\mathcal{Y}_{\mathrm{Q}})$ *denote the Fourier transform of* $\Gamma'_{\mathrm{Q}}(\ ,\mathcal{Y}_{\mathrm{Q}})$*. Then*

$$(3.3.4.1) \qquad \forall \lambda \in i\mathfrak{a}_{\mathrm{Q}}^{\mathrm{G},*} \qquad \Gamma'_{\mathrm{Q}}\hat{}\,(\lambda, \mathcal{Y}_{\mathrm{Q}}) = \sum_{\mathrm{P} \in \mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)} c'_{\mathrm{P}}(\lambda).$$

Proof. See Lemma 4.1 of [Ar86]. □

(3.3.5) Remark The main ingredient of the proof is the following generalization of (2.2.2.2):

Let Q be a parabolic subgroup in $\mathcal{F}^{\Sigma}$, let P be a parabolic subgroup in $\bar{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)$, then

$$(3.3.5.2) \sum_{\substack{\mathrm{P}'\in\bar{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M}_1)\\ \mathrm{P}'\subset\mathrm{P}}} (-1)^{\dim(\mathrm{A}_{\mathrm{P}'}/\mathrm{A}_{\mathrm{P}})}\tau_{\mathrm{Q}}^{\pi_\Sigma(\mathrm{P}')}(H)\hat{\tau}_{\mathrm{P}'}^{\mathrm{P}}(H) = \begin{cases} 1 & \text{if } \mathrm{P}\in\mathcal{F}_{\mathrm{Q}}(\mathrm{M}_1) \text{ and } H\in\mathfrak{a}_{\mathrm{P}}, \\ \\ 0 & \text{otherwise.}\end{cases}$$

This is Lemma 4.2 of [Ar86].

(3.3.6) Lemma (Global semisimple descent)

Let P_1, M_1, $\mathfrak{o}$, Σ *be as in (3.3.1), let* f *be a Schwartz function on* $\mathfrak{g}(\mathbb{A})$*, then*

$$(3.3.6.1)\, J_{\mathfrak{o}}(f) = |\pi_0(\mathrm{G}_\Sigma)|^{-1}\int_{\mathrm{G}_\Sigma^0(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})}\left(\sum_{\mathrm{Q}\in\mathcal{F}^{\Sigma}}|\mathrm{W}_1^{\mathrm{M}_{\mathrm{Q}}}||\mathrm{W}_1^{\mathrm{G}_\Sigma^0}|^{-1}J_{\mathrm{nil}}^{\mathrm{M}_{\mathrm{Q}}}(\Phi_{\mathrm{Q},x}^{T_0-T_{\Sigma,1}})\right)\mathrm{d}x$$

where for a truncation parameter T *in* $\mathfrak{a}_0$ *the function* $\Phi_{\mathrm{Q},x}^{T}$ *on* $\mathfrak{m}_{\mathrm{Q}}(\mathbb{A})$ *is defined by*

$$(3.3.6.2) \quad \forall X\in\mathfrak{m}_{\mathbb{Q}}(\mathbb{A})$$

$$\Phi_{\mathrm{Q},x}^{T}(X) = \int_{\mathrm{K}_\Sigma}\int_{\mathfrak{n}_{\mathrm{Q}}(\mathbb{A})} f\Big(\big(\Sigma+(X+N)\cdot\mathrm{ad}(k)\big)\cdot\mathrm{ad}(x)\Big)\ v'_{\mathrm{Q}}(kx,T)\mathrm{d}N\mathrm{d}k$$

where the weight factor v'_{Q} *is defined as*

$$(3.3.6.3) \qquad v'_{\mathrm{Q}}(kx,T) \;=\; \int_{\mathfrak{a}_{\mathrm{Q}}^{\mathrm{G}}}\Gamma'_{\mathrm{Q}}(H,\mathcal{Y}_{\mathrm{Q}}^{T}(k,x))\,\mathrm{d}H$$

where $\mathcal{Y}^{T}(k,x)$ *is the collection of points defined by*

$$(3.3.6.4) \qquad \forall\mathrm{P}\in\mathcal{F}_{\mathrm{Q}}(\mathrm{M}_1) \qquad Y_{\mathrm{P}}^{T}(k,x) = -H_{\mathrm{P}}(kx)-T_\Sigma+T$$

where T_Σ is a truncation parameter in $\mathfrak{a}_1$ such that $T_\Sigma - T_{\Sigma,1}$ is the projection of $T - T_0$.

Proof. This argument follows the proof of Lemma 6.2 of [Ar86].

Let T be a truncation parameter in $\mathfrak{a}_0$. Define a second truncated kernel function $j^T_{\mathfrak{o}}(\ ,f)$ on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$ by

$$(3.3.6.5)\qquad \forall x \in \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$$

$$j^T_{\mathfrak{o}}(x,f) = \sum_{\substack{\mathrm{P}\in\mathcal{F}\\ \text{standard}}} (-1)^{\dim(\mathrm{A_P}/\mathrm{A_G})} \sum_{\delta\in\mathrm{P}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})} \hat{\tau}_{\mathrm{P}}(H_0(\delta x)-T) J_{\mathrm{P},\mathfrak{o}}(\delta x)$$

where

$$(3.3.6.6) J_{\mathrm{P},\mathfrak{o}}(x,f) = \sum_{X\in\mathfrak{m}_{\mathrm{P}}(\mathbb{Q})\cap\mathfrak{o}} \ \sum_{\eta\in\mathrm{N}_{\mathrm{P},X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_{\mathrm{P}}(\mathbb{Q})} \int_{\mathfrak{n}_{\mathrm{P},X_{\mathrm{ss}}}(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(\eta x))\ \mathrm{d}N.$$

Lemma *The function $j^T_{\mathfrak{o}}(\ ,f)$ is integrable on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$, and*

$$(3.3.6.7)\qquad J^T_{\mathfrak{o}}(f) \ = \ \int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} j^T_{\mathfrak{o}}(x,f)\ \mathrm{d}x.$$

Proof. Following the proof of (2.2.2.1), by the combinatorial lemma of Langlands (2.2.2.2), it suffices to prove that

$$(3.3.6.8)\qquad \sum_{\substack{\mathrm{P}_2,\mathrm{P}_5\in\mathcal{F}\\ \text{standard}\\ \mathrm{P}_2\subset\mathrm{P}_5}} \int_{\mathrm{P}_2(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\times$$

$$\times\left| \sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}} (-1)^{\dim(\mathrm{A}_4/\mathrm{A_G})} J_{\mathrm{P}_4,\mathfrak{o}}(x,f)\right| \mathrm{d}x$$

is finite. Decompose the sum defining $J_{\mathrm{P}_4,\mathfrak{o}}(x,f)$ over the set of parabolic subgroups contained in P_4,

$$J_{\mathrm{P}_4,\mathfrak{o}}(x,f) \tag{3.3.6.9}$$

$$= \sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_3\subset\mathrm{P}_4}} \sum_{X\in\mathfrak{m}_2^3(\mathbb{Q})'\cap\mathfrak{o}} \sum_{Y\in\mathfrak{n}_3^4(\mathbb{Q})} \times \sum_{\eta\in\mathrm{N}_{4,(X+Y)_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_4(\mathbb{Q})} \int_{\mathfrak{n}_{4,(X+Y)_{\mathrm{ss}}}(\mathbb{A})} f\big((X+Y+N)\cdot\mathrm{ad}(\eta x)\big)\ \mathrm{d}N$$

$$= \sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_3\subset\mathrm{P}_4}} \sum_{X\in\mathfrak{m}_2^3(\mathbb{Q})'\cap\mathfrak{o}} \sum_{Z\in\mathfrak{n}^4_{3,X_{\mathrm{ss}}}(\mathbb{Q})} \sum_{\delta\in\mathrm{N}^4_{3,X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_3^4(\mathbb{Q})} \times \sum_{\eta\in\mathrm{N}_{4,(X+Z)_{\mathrm{ss}}\cdot\mathrm{ad}(\delta)}(\mathbb{Q})\backslash\mathrm{N}_4(\mathbb{Q})} \times \int_{\mathfrak{n}_{4,(X+Z)_{\mathrm{ss}}\cdot\mathrm{ad}(\delta)}(\mathbb{A})} f\Big(\big(X+Z+N\cdot\mathrm{ad}(\delta^{-1})\big)\cdot\mathrm{ad}(\delta\eta x)\Big)\ \mathrm{d}N \tag{3.3.6.10}$$

$$= \sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_3\subset\mathrm{P}_4}} \sum_{X\in\mathfrak{m}_2^3(\mathbb{Q})'\cap\mathfrak{o}} \sum_{Z\in\mathfrak{n}^4_{3,X_{\mathrm{ss}}}(\mathbb{Q})} \sum_{\eta\in\mathrm{N}_{3,X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_3(\mathbb{Q})} \times \int_{\mathfrak{n}_{4,(X+Z)_{\mathrm{ss}}}(\mathbb{A})} f\big((X+Z+N)\cdot\mathrm{ad}(\eta x)\big)\ \mathrm{d}N$$

$$= \sum_{\substack{\mathrm{P}_3\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_3\subset\mathrm{P}_4}} \sum_{X\in\mathfrak{m}_2^3(\mathbb{Q})'\cap\mathfrak{o}} \sum_{\eta\in\mathrm{N}_{3,X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_3(\mathbb{Q})} \times \sum_{\overline{Z}\in\overline{\mathfrak{n}}^4_{3,X_{\mathrm{ss}}}(\mathbb{Q})} \int_{\mathfrak{n}_{3,X_{\mathrm{ss}}}(\mathbb{A})} f\big((X+N)\cdot\mathrm{ad}(\eta x)\big)\cdot\psi(\langle N,\overline{Z}\rangle)\ \mathrm{d}N \tag{3.3.6.11}$$

where (3.3.6.10) follows from Corollary 2.4 of [Ch02a] which states that

$$(3.3.6.12) \qquad \sum_{Z\in\mathfrak{n}_{X_{\mathrm{ss}}}(\mathbb{Q})} \sum_{\delta\in\mathrm{N}_{X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}(\mathbb{Q})} f((X+Z)\cdot\mathrm{ad}(\delta)) \quad = \quad \sum_{Y\in\mathfrak{n}(\mathbb{Q})} f(X+Y)$$

and (3.3.6.11) follows from the Poisson summation formula. The right hand side of (3.3.6.11) is independant of P_4, hence the alternating sum in P_4 cancels by the inclusion-exclusion principle, therefore the same estimates in the proof of (2.2.2.1) implies the integrability of $j_{\mathfrak{o}}^T(\ ,f)$.

For the integral representation of $J_{\mathfrak{o}}^T(f)$, consider the $(\mathrm{P}_2,\mathrm{P}_5)$ summand of the integral of $j_{\mathrm{o}}^T(\ ,f)$ over $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$:

$$(3.3.6.13) \qquad \int_{\mathrm{P}_2(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\times$$

$$\times \sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}} (-1)^{\dim(\mathrm{A}_4/\mathrm{A}_\mathrm{G})} j_{\mathrm{P}_4,\mathfrak{o}}(x,f)\ \mathrm{d}x$$

$$= \int_{\mathrm{M}_2(\mathbb{Q})\mathrm{N}_2(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\times$$

$$\times \sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}} (-1)^{\dim(\mathrm{A}_4/\mathrm{A}_\mathrm{G})} \sum_{X\in\mathfrak{m}_4(\mathbb{Q})\cap\mathfrak{o}} \Bigg(\int_{\mathrm{N}_2(\mathbb{Q})\backslash\mathrm{N}_2(\mathbb{A})} \sum_{\eta\in\mathrm{N}_{4,X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_4(\mathbb{Q})}$$

$$\times\bigg(\int_{\mathfrak{n}_{4,X_{\mathrm{ss}}}(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(\eta n_2 x))\ \mathrm{d}N\bigg)\mathrm{d}n_2\Bigg)\mathrm{d}x$$

$$(3.3.6.14) \quad = \int_{\mathrm{M}_2(\mathbb{Q})\mathrm{N}_2(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\times$$

$$\times \sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}} (-1)^{\dim(\mathrm{A}_4/\mathrm{A}_\mathrm{G})} \sum_{X\in\mathfrak{m}_4(\mathbb{Q})\cap\mathfrak{o}} \Bigg(\int_{\mathrm{N}_2(\mathbb{Q})\backslash\mathrm{N}_2(\mathbb{A})} \int_{\mathrm{N}_{4,X_{\mathrm{ss}}}(\mathbb{Q})\backslash\mathrm{N}_4(\mathbb{A})}$$

$$\times\bigg(\int_{\mathfrak{n}_{4,X_{\mathrm{ss}}}(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(nn_2x))\ \mathrm{d}N\bigg)\mathrm{d}n\mathrm{d}n_2\bigg)\mathrm{d}x$$

$$(3.3.6.15)\quad = \int_{\mathrm{M}_2(\mathbb{Q})\mathrm{N}_2(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}}(-1)^{\dim(\mathrm{A}_4/\mathrm{A}_\mathrm{G})}\times$$

$$\times\bigg(\sum_{X\in\mathfrak{m}_4(\mathbb{Q})\cap\mathfrak{o}}\int_{\mathrm{N}_2(\mathbb{Q})\backslash\mathrm{N}_2(\mathbb{A})}\int_{\mathfrak{n}_4(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(n_2x))\ \mathrm{d}N\mathrm{d}n_2\bigg)\mathrm{d}x.$$

The equality (3.3.6.14) holds since $\mathrm{N}_4(\mathbb{Q})\backslash\mathrm{N}_4(\mathbb{A})$ has volume 1, and the equality (3.3.6.15) follows from Corollary 2.5 of [Ch02a] which is the integral analogue of (3.3.6.12). Reversing the combinatorial manipulations to the right hand side of (3.3.6.15) skipping the step (3.3.6.14),

$$(3.3.6.16)\quad \int_{\mathrm{M}_2(\mathbb{Q})\mathrm{N}_2(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}}(-1)^{\dim(\mathrm{A}_4/\mathrm{A}_\mathrm{G})}\times$$

$$\times\bigg(\sum_{X\in\mathfrak{m}_4(\mathbb{Q})\cap\mathfrak{o}}\int_{\mathrm{N}_2(\mathbb{Q})\backslash\mathrm{N}_2(\mathbb{A})}\int_{\mathfrak{n}_4(\mathbb{A})} f((X+N)\cdot\mathrm{ad}(n_2x))\ \mathrm{d}N\mathrm{d}n_2\bigg)\mathrm{d}x$$

$$= \int_{\mathrm{P}_2(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1} F^{\mathrm{P}_2}(x,T)\sigma_2^5(H_0(x)-T)\times$$

$$\times\sum_{\substack{\mathrm{P}_4\in\mathcal{F}\\ \mathrm{P}_2\subset\mathrm{P}_4\subset\mathrm{P}_5}}(-1)^{\dim(\mathrm{A}_4/\mathrm{A}_\mathrm{G})}K_{\mathrm{P}_4,\mathfrak{o}}(x,f)\ \mathrm{d}x,$$

which is the $(\mathrm{P}_2,\mathrm{P}_5)$ summand of the integral of $k_{\mathrm{o}}^T(\ ,f)$ over $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$, hence $J_{\mathfrak{o}}^T(f)$. $\square$

By (3.3.6.7) it is enough to consider the function $J_{\mathrm{P},\mathfrak{o}}(\ ,f)$ instead of $K_{\mathrm{P},\mathfrak{o}}(\ ,f)$ for a parabolic subgroup P of G containing P_1 for the proof of (3.3.6.1). Every element X in $\mathfrak{m}_\mathrm{P}(\mathbb{Q})\cap\mathfrak{o}$ is conjugate under the adjoint action of $\mathrm{G}(\mathbb{Q})$ to the sum of Σ and N_Σ for some

N_Σ in $\mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q})$. More precisely

$$(3.3.6.17) \qquad \exists \mathrm{P}_1' \in \mathcal{F},\ \mathrm{P}_1' \text{ standard},\ \mathrm{P}_1' \subset \mathrm{P} \quad \exists s \in \mathrm{W}(\mathfrak{a}_{\mathrm{P}_1}, \mathfrak{a}_{\mathrm{P}_1'}) \quad \exists \mu \in \mathrm{M}_\mathrm{P}^0(\mathbb{Q})$$
$$\exists N_\Sigma \in \mathfrak{m}_\mathrm{P}(\mathbb{Q}) \cdot \mathrm{ad}(w_s) \cap \mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q}) \quad X = (\Sigma + N_\Sigma) \cdot \mathrm{ad}(w_s^{-1}\mu)$$

where

- the element w_s in $\mathrm{G}(\mathbb{Q})$ is a representative of s;
- the double coset

  $$(3.3.6.18) \qquad [s] \ \in \ \mathrm{W}_0^{\mathrm{M}_\mathrm{P}} \Big\backslash \mathrm{W}(\mathfrak{a}_{\mathrm{P}_1}, \mathfrak{a}_{\mathrm{P}_1'}) \Big/ \mathrm{W}_1^{\mathrm{G}_\Sigma^0}$$

  is uniquely determined;
- for a fixed choice of the element s, the coset

  $$(3.3.6.19) \qquad [\mu] \ \in \ \mathrm{M}_\mathrm{P}(\mathbb{Q}) \cap w_s \mathrm{G}_\Sigma(\mathbb{Q}) w_s^{-1} \Big\backslash \mathrm{M}_\mathrm{P}(\mathbb{Q})$$

  is uniquely determined;
- for a fixed choice of the representative w_s and the element μ, the element N_Σ is uniquely determined.

Let $\mathrm{W}(\mathfrak{a}_1; \mathrm{M}_\mathrm{P}^+, \mathrm{G}_\Sigma^{0,+})$ be the subset of the Weyl group of G defined by

$$(3.3.6.20) \qquad \mathrm{W}(\mathfrak{a}_1; \mathrm{M}_\mathrm{P}^+, \mathrm{G}_\Sigma^{0,+})$$
$$= \bigcup_{\substack{\mathrm{P}_1' \in \mathcal{F} \\ \text{standard} \\ \mathrm{P}_1' \subset \mathrm{P}}} \Big\{ s \in \mathrm{W}(\mathfrak{a}_{\mathrm{P}_1}, \mathfrak{a}_{\mathrm{P}_1'}) :\ \forall \alpha \in \Delta_{\mathrm{P}_1'}^{\mathrm{P}}\ s^{-1}\alpha > 0,\ \forall \beta \in \Delta_1^{\mathrm{G}_\Sigma^0}\ s\beta > 0 \Big\}.$$

The map

$$(3.3.6.21)\qquad \Big(\mathrm{W}(\mathfrak{a}_1;\mathrm{M}_{\mathrm{P}}^{+},\mathrm{G}_{\Sigma}^{0,+})\Big)\times\Big(\mathrm{M}_{\mathrm{P}}(\mathbb{Q})\cap w_s\mathrm{G}_{\Sigma}^{0}(\mathbb{Q})w_s^{-1}\Big\backslash \mathrm{M}_{\mathrm{P}}(\mathbb{Q})\Big)\times$$
$$\times\Big(\mathfrak{m}_{\mathrm{P}}(\mathbb{Q})\cdot\mathrm{ad}(w_s)\cap\mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q})\Big)$$
$$\longrightarrow\ \mathfrak{m}_{\mathrm{P}}(\mathbb{Q})\cap\mathfrak{o}$$

defined by

$$(3.3.6.22)\qquad (s,\mu,N_\Sigma)\ \mapsto\ (\Sigma+N_\Sigma)\cdot\mathrm{ad}(w_s^{-1}\mu)$$

is surjective and the group $\pi_0(\mathrm{G}_\Sigma)$ operates simply transitively on each fiber. Hence

$$(3.3.6.23)\qquad \forall x\in\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$$
$$J_{\mathrm{P},\mathfrak{o}}(x,f)$$

$$=\sum_{s\in\mathrm{W}(\mathfrak{a}_1;\mathrm{M}_{\mathrm{P}}^{+},\mathrm{G}_{\Sigma}^{0,+})}\ \sum_{\mu\in\mathrm{M}_{\mathrm{P}}(\mathbb{Q})\cap w_s\mathrm{G}_{\Sigma}^{0}(\mathbb{Q})w_s^{-1}\big\backslash\mathrm{M}_{\mathrm{P}}(\mathbb{Q})}$$
$$\times\sum_{N_\Sigma\in\mathfrak{m}_{\mathrm{P}}(\mathbb{Q})\cdot\mathrm{ad}(w_s)\cap\mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q})}$$
$$\times|\pi_0(\mathrm{G}_\Sigma)|^{-1}\sum_{\eta\in\mathrm{N}_{\mathrm{P},\Sigma\cdot\mathrm{ad}(w_s^{-1}\mu)}(\mathbb{Q})\backslash\mathrm{N}_{\mathrm{P}}(\mathbb{Q})}\int_{\mathfrak{n}_{\mathrm{P},\Sigma\cdot\mathrm{ad}(w_s^{-1}\mu)}(\mathbb{A})}$$
$$\times f\Big(\big((\Sigma+N_\Sigma)\cdot\mathrm{ad}(w_s^{-1}\mu)+N\big)\cdot\mathrm{ad}(\eta x)\Big)\,\mathrm{d}N$$

$$(3.3.6.24)\qquad =\ |\pi_0(\mathrm{G}_\Sigma)|^{-1}\sum_{s\in\mathrm{W}(\mathfrak{a}_1;\mathrm{M}_{\mathrm{P}}^{+},\mathrm{G}_{\Sigma}^{0,+})}\ \sum_{N_\Sigma\in\mathfrak{m}_{\mathrm{P}}(\mathbb{Q})\cdot\mathrm{ad}(w_s)\cap\mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q})}$$

$$\times \sum_{\pi \in \mathrm{P}(\mathbb{Q}) \cap w_s \mathrm{G}_\Sigma^0(\mathbb{Q}) w_s^{-1} \big\backslash \mathrm{P}(\mathbb{Q})} \int_{\mathfrak{n}_{\mathrm{P},\Sigma}(\mathbb{A}) \cdot \mathrm{ad}(w_s)}$$

$$\times f\Big(\big(\Sigma + N_\Sigma + N\big) \cdot \mathrm{ad}(w_s^{-1} \pi x)\Big) \, \mathrm{d}N$$

where (3.3.6.24) follows from the change of variables

$$\begin{aligned} &\mathrm{M}_\mathrm{P}(\mathbb{Q}) \cap w_s \mathrm{G}_\Sigma^0(\mathbb{Q}) w_s^{-1} \big\backslash \mathrm{M}_\mathrm{P}(\mathbb{Q}) \times \mathrm{N}_{\mathrm{P},\Sigma \cdot \mathrm{ad}(w_s^{-1} \mu)}(\mathbb{Q}) \backslash \mathrm{N}_\mathrm{P}(\mathbb{Q}) \\ \xrightarrow{\sim} \quad & \mathrm{P}(\mathbb{Q}) \cap w_s \mathrm{G}_\Sigma^0(\mathbb{Q}) w_s^{-1} \big\backslash \mathrm{P}(\mathbb{Q}). \end{aligned} \tag{3.3.6.25}$$

Substitute (3.3.6.23) into the formula (3.3.6.7),

(3.3.6.26) $J_\mathfrak{o}^T(f)$

$$= \quad |\pi_0(\mathrm{G}_\Sigma)|^{-1} \int_{\mathrm{G}(\mathbb{Q}) \backslash \mathrm{G}(\mathbb{A})^1} \Bigg(\sum_{\substack{\mathrm{P} \in \mathcal{F} \\ \text{standard}}} (-1)^{\dim(\mathrm{A}_\mathrm{P}/\mathrm{A}_\mathrm{G})} \sum_{\delta \in \mathrm{P}(\mathbb{Q}) \backslash \mathrm{G}(\mathbb{Q})}$$

$$\times \sum_{s \in \mathrm{W}(\mathfrak{a}_1; \mathrm{M}_\mathrm{P}^+, \mathrm{G}_\Sigma^{0,+})} \quad \sum_{N_\Sigma \in \mathfrak{m}_\mathrm{P}(\mathbb{Q}) \cdot \mathrm{ad}(w_s) \cap \mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q})} \quad \sum_{\pi \in \mathrm{P}(\mathbb{Q}) \cap w_s \mathrm{G}_\Sigma^0(\mathbb{Q}) w_s^{-1} \big\backslash \mathrm{P}(\mathbb{Q})}$$

$$\times \int_{\mathfrak{n}_{\mathrm{P},\Sigma}(\mathbb{A}) \cdot \mathrm{ad}(w_s)} f\Big(\big(\Sigma + N_\Sigma + N\big) \cdot \mathrm{ad}(w_s^{-1} \pi \delta x)\Big) \, \mathrm{d}N \times$$

$$\times \hat{\tau}_\mathrm{P}(H_\mathrm{p}(\delta x) - T)\Bigg) \, \mathrm{d}x$$

(3.3.6.27) $$|\pi_0(\mathrm{G}_\Sigma)|^{-1} \int_{\mathrm{G}(\mathbb{Q}) \backslash \mathrm{G}(\mathbb{A})^1} \Bigg(\sum_{\substack{\mathrm{P} \in \mathcal{F} \\ \text{standard}}} (-1)^{\dim(\mathrm{A}_\mathrm{P}/\mathrm{A}_\mathrm{G})} \sum_{s \in \mathrm{W}(\mathfrak{a}_1; \mathrm{M}_\mathrm{P}^+, \mathrm{G}_\Sigma^{0,+})}$$

$$\times \sum_{\xi \in \mathrm{Q}(\mathbb{Q}) \backslash \mathrm{G}(\mathbb{Q})} \quad \sum_{N_\Sigma \in \mathfrak{m}_{\mathrm{Q},\mathrm{nil}}(\mathbb{Q})} \int_{\mathfrak{n}_\mathrm{Q}(\mathbb{A})} f((\Sigma + N_\Sigma + N) \cdot \mathrm{ad}(\xi x)) \, \mathrm{d}N \times$$

$$\times\hat{\tau}_{\mathrm{P}}(H_{\mathrm{p}}(w_s\xi x)-T)\Big)\,\mathrm{d}x$$

$$= \ |\pi_0(\mathrm{G}_\Sigma)|^{-1}\int_{\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1}\Big(\sum_{\substack{\mathrm{Q}\in\mathcal{F}^\Sigma\\ \text{standard}}}\ \sum_{\xi\in\mathrm{Q}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})}$$

$$\times\sum_{N_\Sigma\in\mathfrak{m}_{\mathrm{Q},\mathrm{nil}}(\mathbb{Q})}\int_{\mathfrak{n}_{\mathrm{Q}}(\mathbb{A})} f((\Sigma+N_\Sigma+N)\cdot\mathrm{ad}(\xi x))\,\mathrm{d}N\ \times$$

$$\times\sum_{\substack{\mathrm{P}\in\mathcal{F},\ \text{standard}\\ s\in\mathrm{W}(\mathfrak{a}_1;\mathrm{M}_{\mathrm{P}}^{+},\mathrm{G}_\Sigma^{0,+})\\ w_s^{-1}\mathrm{P}w_s\cap\mathrm{G}_\Sigma^0=\mathrm{Q}}}(-1)^{\dim(\mathrm{A}_{\mathrm{P}}/\mathrm{A}_{\mathrm{G}})}\hat{\tau}_{\mathrm{P}}(H_{\mathrm{p}}(w_s\xi x)-T)\Big)\,\mathrm{d}x.$$

where in (3.3.6.27) ξ denotes the product $w_s^{-1}\pi\delta$ and Q denotes the standard parabolic subgroup of G_Σ^0 defined by

(3.3.6.28) $$\mathrm{Q} \ = \ w_s^{-1}\mathrm{P}w_s\cap\mathrm{G}_\Sigma^0$$

with Levi decomposition

(3.3.6.29) $$\mathfrak{m}_{\mathrm{Q}}=\mathfrak{m}_{\mathrm{P}}\cdot\mathrm{ad}(w_s)\cap\mathfrak{g}_\Sigma,\qquad \mathfrak{n}_{\mathrm{Q}}=\mathfrak{n}_{\mathrm{P}}\cdot\mathrm{ad}(w_s)\cap\mathfrak{g}_\Sigma.$$

The assignment

(3.3.6.30) $$(\mathrm{P},s)\ \mapsto\ \mathrm{P}'=w_s^{-1}\mathrm{P}w_s$$

defines a bijection

(3.3.6.31) $$\Big\{\mathrm{P}\in\mathcal{F},\ s\in\mathrm{W}(\mathfrak{a}_1;\mathrm{M}_{\mathrm{P}}^{+},\mathrm{G}_\Sigma^{0,+}):\ \mathrm{P}\ \text{standard},\ w_s^{-1}\mathrm{P}w_s\cap\mathrm{G}_\Sigma^0=\mathrm{Q}\Big\}$$

$$\xrightarrow{\sim}\mathcal{F}_{\mathrm{Q}}(\mathrm{M}_1),$$

hence

$$(3.3.6.32)\qquad J^T_{\mathfrak{o}}(f)$$

$$= |\pi_0(\mathrm{G}_\Sigma)|^{-1}\int_{\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})^1}\Bigg(\sum_{\substack{\mathrm{Q}\in\mathcal{F}^\Sigma\\ \text{standard}}}\ \sum_{\xi\in \mathrm{Q}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{Q})}$$

$$\times \sum_{N_\Sigma\in\mathfrak{m}_{\mathrm{Q},\mathrm{nil}}(\mathbb{Q})}\int_{\mathfrak{n}_\mathrm{Q}(\mathbb{A})} f((\Sigma+N_\Sigma+N)\cdot \mathrm{ad}(\xi x))\ \mathrm{d}N \times$$

$$\times \sum_{\mathrm{P}'\in\mathcal{F}_\mathrm{Q}(\mathrm{M}_1)}(-1)^{\dim(\mathrm{A}_{\mathrm{P}'}/\mathrm{A}_\mathrm{G})}\hat{\tau}_{\mathrm{P}'}(H_{\mathrm{P}'}(\xi x)-s^{-1}(T-T_0)-T_0)\Bigg)\ \mathrm{d}x$$

where s denotes the second component of the inverse image of P' under the map defined in (3.3.6.30).

Let T_Σ be a truncation parameter for the triple $(\mathrm{G}_\Sigma(\mathbb{A})^0, \mathrm{M}_{1,\Sigma}(\mathbb{A}), \mathrm{K}_\Sigma)$ in $\mathfrak{a}_1$ such that $T_\Sigma - T_{\Sigma,1}$ is the projection of $T-T_0$. By (3.3.3.1)

$$(3.3.6.33)\qquad \sum_{\mathrm{P}\in\mathcal{F}_\mathrm{Q}(\mathrm{M}_1)}(-1)^{\dim(\mathrm{A}_\mathrm{P}/\mathrm{A}_\mathrm{G})}\hat{\tau}_\mathrm{P}\big(H_\mathrm{P}(\delta xy)-s^{-1}(T-T_0)-T_0\big)$$

$$= \sum_{\mathrm{P}\in\mathcal{F}_\mathrm{Q}(\mathrm{M}_1)}(-1)^{\dim(\mathrm{A}_\mathrm{P}/\mathrm{A}_\mathrm{G})}\hat{\tau}_\mathrm{P}\big((H_\mathrm{Q}(\delta x)-T_\Sigma)-Y^T_\mathrm{P}(\delta x,y)\big)$$

$$= \sum_{\substack{\mathrm{Q}'\in\mathcal{F}^\Sigma\\ \mathrm{Q}'\supset\mathrm{Q}}}(-1)^{\dim(\mathrm{A}_\mathrm{Q}/\mathrm{A}_{\mathrm{Q}'})}\hat{\tau}^{\mathrm{Q}'}_\mathrm{Q}\big(H_\mathrm{Q}(\delta x)-T_\Sigma\big)\times$$

$$\times\Gamma'_{\mathrm{Q}'}\big(H_{\mathrm{Q}'}(\delta x)-T_\Sigma,\mathcal{Y}^T_{\mathrm{Q}'}(\delta x,y)\big)$$

where the family $\mathcal{Y}^T_{\mathrm{Q}'}(\delta x,y)$ is defined by

$$(3.3.6.34)\qquad \forall \mathrm{P}\in\mathcal{F}_\mathrm{Q}(\mathrm{M}_1)\qquad Y^T_\mathrm{P}(\delta x,y)=-H_\mathrm{P}(ky)+s^{-1}(T-T_0)-T_\Sigma+T_0,$$

where k is the K_Σ component of δx under the Iwasawa decomposition with respect to P_Σ and s is the second component of the inverse image of P under the map defined in (3.3.6.30). On the right hand side of (3.3.6.32) make the change of variables

$$\begin{aligned}(3.3.6.35)\qquad & \mathrm{Q}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{Q})\times\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1\\ & \quad\overset{\sim}{\longrightarrow}\quad \mathrm{Q}(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{Q})\times\mathrm{G}^0_\Sigma(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{A})\cap\mathrm{G}(\mathbb{A})^1\times\mathrm{G}^0_\Sigma(\mathbb{A})\backslash\mathrm{G}(\mathbb{A}).\end{aligned}$$

Then

$$\begin{aligned}(3.3.6.36)\quad & J^T_{\mathfrak{o}}(f)\\ = \ & |\pi_0(\mathrm{G}_\Sigma)|^{-1}\int_{\mathrm{G}^0_\Sigma(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})}\int_{\mathrm{G}^0_\Sigma(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{A})\cap\mathrm{G}(\mathbb{A})^1}\Bigg(\sum_{\substack{\mathrm{Q}\in\mathcal{F}^\Sigma\\ \text{standard}}}\ \sum_{\substack{\mathrm{Q}'\in\mathcal{F}^\Sigma\\ \mathrm{Q}'\supset\mathrm{Q}}}\\ & \times\sum_{\delta\in\mathrm{Q}(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{Q})}\ \sum_{N_\Sigma\in\mathfrak{m}_{\mathrm{Q},\mathrm{nil}}(\mathbb{Q})}(-1)^{\dim(\mathrm{A}_\mathrm{Q}/\mathrm{A}_{\mathrm{Q}'})}\times\\ & \times\int_{\mathfrak{n}_\mathrm{Q}(\mathbb{A})}f((\Sigma+N_\Sigma+N)\cdot\mathrm{ad}(\delta xy))\,\mathrm{d}N\times\\ & \times\hat{\tau}^{\mathrm{Q}'}_{\mathrm{Q}}\big(H_\mathrm{Q}(\delta x)-T_\Sigma\big)\Gamma'_{\mathrm{Q}'}\big(H_{\mathrm{Q}'}(\delta x)-T_\Sigma,\mathcal{Y}^T_{\mathrm{Q}'}(\delta x,y)\big)\Bigg)\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$\begin{aligned}(3.3.6.37) = \ & |\pi_0(\mathrm{G}_\Sigma)|^{-1}\int_{\mathrm{G}^0_\Sigma(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})}\Bigg(\sum_{\substack{\mathrm{Q}'\in\mathcal{F}^\Sigma\\ \text{standard}}}\int_{\mathrm{K}_\Sigma}\int_{\mathrm{A}_{\mathrm{Q}'}(\mathbb{R})\cap\mathrm{G}(\mathbb{A})^1}\int_{\mathrm{M}_{\mathrm{Q}'}(\mathbb{Q})\backslash\mathrm{M}_{\mathrm{Q}'}(\mathrm{A})^1}\\ & \times\Bigg(\sum_{\substack{\mathrm{Q}\in\mathcal{F}^\Sigma\\ \text{standard}\\ \mathrm{Q}\subset\mathrm{Q}'}}\ \sum_{\mu\in\mathrm{Q}(\mathbb{Q})\cap\mathrm{M}_{\mathrm{Q}'}(\mathbb{Q})\backslash\mathrm{M}_{\mathrm{Q}'}(\mathbb{Q})}\ \sum_{N_\Sigma\in\mathfrak{m}_{\mathrm{Q},\mathrm{nil}}(\mathbb{Q})}(-1)^{\dim(\mathrm{A}_\mathrm{Q}/\mathrm{A}_{\mathrm{Q}'})}\times\\ & \times\int_{\mathfrak{n}_\mathrm{Q}(\mathbb{A})\cap\mathfrak{m}_{\mathrm{Q}'}(\mathbb{A})}\Phi^T_{\mathrm{Q}',a,k,y}\big((N_\Sigma+N)\cdot\mathrm{ad}(\mu m)\big)\,\mathrm{d}N\times\end{aligned}$$

$$\times\hat{\tau}_{\mathrm{Q}}^{\mathrm{Q}'}\left(H_{\mathrm{Q}}(\mu m)-T_{\Sigma}\right)\Big)\,\mathrm{d}m\mathrm{d}a\mathrm{d}k\Bigg)\,\mathrm{d}y$$

$$= \;|\pi_0(\mathrm{G}_\Sigma)|^{-1}\int_{\mathrm{G}^0_\Sigma(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})}\sum_{\substack{\mathrm{Q}'\in\mathcal{F}^\Sigma\\ \text{standard}}}$$

$$\times\Bigg(\int_{\mathrm{K}_\Sigma}\int_{\mathrm{A}_{\mathrm{Q}'}(\mathbb{R})\cap\mathrm{G}(\mathbb{A})^1} J_{\mathrm{nil}}^{\mathrm{M}_{\mathrm{Q}'},T_\Sigma}(\Phi^T_{\mathrm{Q}',a,k,y})\,\mathrm{d}a\mathrm{d}k\Bigg)\,\mathrm{d}y.$$

where on the right hand side of (3.3.6.37) $\Phi^T_{\mathrm{Q}',a,k,y}$ denotes the function on $\mathfrak{m}_{\mathrm{Q}'}(\mathbb{A})$ defined by

$$\text{(3.3.6.38)}\qquad \forall X\in\mathfrak{m}_{\mathrm{Q}'}(\mathbb{A})$$

$$\Phi^T_{\mathrm{Q}',a,k,y}(X) \;=\; \int_{\mathfrak{m}_{\mathrm{Q}'}(\mathbb{A})} f((\Sigma+X+N)\cdot\mathrm{ad}(ky))\,\mathrm{d}N\times$$

$$\times\Gamma'_{\mathrm{Q}'}\big(H_{\mathrm{Q}'}(a)-T_\Sigma,\mathcal{Y}^T_{\mathrm{Q}'}(k,y)\big),$$

and the equality (3.3.6.37) follows from the changes of variables

$$\text{(3.3.6.39)}\qquad \mathrm{Q}(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{Q})\times\mathrm{G}^0_\Sigma(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{A})\cap\mathrm{G}(\mathbb{A})^1$$

$$\xrightarrow{\sim}\quad \mathrm{Q}(\mathbb{Q})\cap\mathrm{M}_{\mathrm{Q}'}(\mathbb{Q})\backslash\mathrm{M}_{\mathrm{Q}'}(\mathbb{Q})\times\mathrm{Q}'(\mathbb{Q})\backslash\mathrm{G}^0_\Sigma(\mathbb{A})\cap\mathrm{G}(\mathbb{A})^1$$

and

$$\text{(3.3.6.40)}\qquad \mathfrak{n}_{\mathrm{Q}}(\mathbb{A})\;\xrightarrow{\sim}\;\mathfrak{n}_{\mathrm{Q}}(\mathbb{A})\cap\mathfrak{m}_{\mathrm{Q}'}(\mathbb{A})\times\mathfrak{n}_{\mathrm{Q}'}(\mathbb{A})$$

and the Iwasawa decomposition for $\mathrm{G}^0_\Sigma(\mathbb{A})\cap\mathrm{G}(\mathbb{A})^1$.

Evaluate T at T_0 and T_Σ at $T_{\Sigma,1}$, and apply (3.3.4.1), then

$$J_{\mathfrak{o}}(f) \;=\; |\pi_0(\mathrm{G}_\Sigma)|^{-1} \int_{\mathrm{G}^0_\Sigma(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} \sum_{\substack{\mathrm{Q}'\in\mathcal{F}^\Sigma \\ \text{standard}}} J_{\mathrm{nil}}^{\mathrm{M}_{\mathrm{Q}'}}(\Phi_{\mathrm{Q}',y}^{T_0-T_{\Sigma,1}})\,\mathrm{d}y. \tag{3.3.6.41}$$

By the defining property (2.2.8.1) of the points T_0 and $T_{\Sigma,1}$,

$$J_{\mathrm{nil}}^{\mathrm{M}_{\mathrm{Q}'}}(\Phi_{\mathrm{Q}',y}^{T_0-T_{\Sigma,1}}) \;=\; J_{\mathrm{nil}}^{\mathrm{M}_{\mathrm{Q}''}}(\Phi_{\mathrm{Q}'',y}^{T_0-T_{\Sigma,1}}) \tag{3.3.6.42}$$

whenever Q' and Q'' are parabolic subgroups in $\mathcal{F}^\Sigma$ conjugate under the Weyl group of G^0_Σ, hence

$$\begin{aligned} &J_{\mathfrak{o}}(f) \\ &\quad=\; |\pi_0(\mathrm{G}_\Sigma)|^{-1} \int_{\mathrm{G}^0_\Sigma(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} \left(\sum_{\mathrm{Q}\in\mathcal{F}^{\mathrm{G}_\Sigma}} |\mathrm{W}_1^{\mathrm{M}_\mathrm{Q}}||\mathrm{W}_1^{\mathrm{G}^0_\Sigma}|^{-1} J_{\mathrm{nil}}^{\mathrm{M}_\mathrm{Q}}(\Phi_{\mathrm{Q},x}^{T_0-T_{\Sigma,1}}) \right) \mathrm{d}x. \end{aligned} \tag{3.3.6.43}$$

□

(3.3.7) Definition Fix a finite set of places S. Let M be a standard Levi subgroup of G, let Ξ be a semisimple element of $\mathfrak{m}(\mathbb{Q})$, let ν be a nilpotent element of $\mathfrak{m}_\Xi(\mathbb{Q}_S)$ defined upto the adjoint action of $\mathrm{M}_\Xi(\mathbb{Q}_S)$. The element

$$X = \Xi + \nu \;\in\; \mathfrak{m}(\mathbb{Q}_S) \tag{3.3.7.1}$$

is well-defined modulo $\mathrm{M}(\mathbb{Q}_S)$. Denote by D^{G} the discriminant function on $\mathfrak{g}$.

Let T be a point in $\mathfrak{a}_\mathrm{M}$, let x be an element of $\mathrm{G}(\mathbb{Q}_S)$. Let $(v_\mathrm{P}(x,T))$ be the (G,M)-family defined by

$$\forall \mathrm{P}\in\mathcal{F}(\mathrm{M})\ \forall\lambda\in i\mathfrak{a}^*_\mathrm{M} \qquad v_\mathrm{P}(x,T)(\lambda) = v_\mathrm{P}(x)(\lambda)\times e^{\langle\lambda,T\rangle}. \tag{3.3.7.2}$$

Let Q be a parabolic subgroup of G^{0}_{Ξ} containing M^{0}_{Ξ}. Define the *weight factor* v'_{Q} by

$$v'_{\mathrm{Q}}(x,T) \;=\; \sum_{\mathrm{P}\in\mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M})} v'_{\mathrm{P}}(x,T) \tag{3.3.7.3}$$

where the set $\mathring{\mathcal{F}}_{\mathrm{Q}}(\mathrm{M})$ is defined with respect to the semisimple element Ξ. Let $T_{\Xi,\mathrm{M}}$ be the point in $\mathfrak{a}_{\mathrm{M}^{0}_{\Xi}}$ modulo $\mathfrak{a}_{\mathrm{G}^{0}_{\Xi}}$ defined in the same manner as T_0 in $\mathfrak{a}_0^{\mathrm{G}}$ that satisfies an identity analogous to (2.2.8.1).

(3.3.8) Lemma *Let* L *be a Levi subgroup in* $\mathcal{L}(\mathrm{M})$*, let* T *be a point in* $\mathfrak{a}_{\mathrm{M}}$*, then*

$$\forall x \in \mathrm{G}(\mathbb{Q}_S)\ \forall y \in \mathrm{G}^{0}_{\Xi}(\mathbb{Q}_S) \qquad v_{\mathrm{L}}(yx) = \sum_{\mathrm{Q}\in\mathcal{F}^{\Xi}(\mathrm{M}^{0}_{\Xi})} v^{\mathrm{Q}}_{\mathrm{L}^{0}_{\Xi}}(y)v'_{\mathrm{Q}}(kx,T) \tag{3.3.8.1}$$

where $v_{\mathrm{L}}(x)$ *is the weight factor defined in* (3.1.5.2) *and* k *is the* K_{Ξ}*-component of* y *under the Iwasawa decomposition of* $\mathrm{G}^{0}_{\Xi}(\mathbb{Q}_S)$ *with respect to the parabolic subgroup* Q^{0}_{Ξ}.

Proof. This is Corollary 8.4 of [Ar88a]. □

(3.3.9) Lemma (Local semisimple descent)

Let f_S *be a Schwartz function on* $\mathfrak{g}(\mathbb{Q}_S)$*, let* X *and* ν *be as in* (3.3.7.1)*, then*

$$J^{\mathrm{G}}_{\mathrm{M}}(X,f_S) \;=\; |D^{\mathrm{G}}(\Xi)|_S^{1/2}\int_{\mathrm{G}^{0}_{\Xi}(\mathbb{Q}_S)\backslash\mathrm{G}(\mathbb{Q}_S)} \Bigg(\sum_{\mathrm{Q}\in\mathcal{F}^{\Xi}(\mathrm{M}^{0}_{\Xi})} J^{\mathrm{M}_{\mathrm{Q}}}_{\mathrm{M}^{0}_{\Xi}}(\nu,\Phi^{T_0-T_{\Xi,\mathrm{M}}}_{S,\mathrm{Q},x})\Bigg)\,\mathrm{d}x \tag{3.3.9.1}$$

where $\Phi^{T}_{S,\mathrm{Q},x}$ *is the function on* $\mathfrak{m}_{\mathrm{Q}}(\mathbb{Q}_S)$ *defined by*

$$\begin{aligned} &\forall Y \in \mathfrak{m}_{\mathbb{Q}}(\mathbb{Q}_S)\\ &\quad \Phi^{T}_{S,\mathrm{Q},x}(Y)\\ &\quad = \int_{\mathrm{K}_{\Xi}}\int_{\mathfrak{n}_{\mathrm{Q}}(\mathbb{Q}_S)} f_S\Big(\big(\Xi + (Y+N)\cdot\mathrm{ad}(k)\big)\cdot\mathrm{ad}(x)\Big)\; v'_{\mathrm{Q}}(kx,T)\mathrm{d}N\mathrm{d}k \end{aligned} \tag{3.3.9.2}$$

where the weight factor $v'_{\mathrm{Q}}(x,T)$ *is defined as in* (3.3.7.3)*.*

Proof. This argument follows the proof of Corollary 8.7 of [Ar88a], see also the discussion in §2.6 of [HW13].

Since the point $T_0 - T_{\Xi,\mathrm{M}}$ lies in $\mathfrak{a}_{\mathrm{M}}$, the identity (3.3.8.1) is valid for the weight factor $v'_{\mathrm{Q}}(x, T_0 - T_{\Xi,\mathrm{M}})$, hence

$$
\begin{aligned}
&(3.3.9.3) \qquad J^{\mathrm{G}}_{\mathrm{M}}(X, f_S) \\
&= \lim_{A\to 0}\Bigg(|D^{\mathrm{G}}(X+A)|_S^{1/2} \int_{\mathrm{G}^0_{X_{\mathrm{ss}}}(\mathbb{Q}_S)\backslash \mathrm{G}(\mathbb{Q}_S)} \int_{\mathrm{G}^0_{X+A}(\mathbb{Q}_S)\backslash \mathrm{G}^0_{X_{\mathrm{ss}}}(\mathbb{Q}_S)} \\
&\qquad \times f_S((X+A)\cdot \mathrm{ad}(yx)) \times \\
&\qquad\quad \times\Big(\sum_{\mathrm{L}\in\mathcal{L}(\mathrm{M})} r^{\mathrm{L}}_{\mathrm{M}}\big(\exp(X_{\mathrm{nil}}), \exp(A)\big) v_{\mathrm{L}}(yx)\Big)\mathrm{d}y\mathrm{d}x\Bigg) \\
&= \lim_{A\to 0}\Bigg(|D^{\mathrm{G}}(X+A)|_S^{1/2} \int_{\mathrm{G}^0_{\underline{\Xi}}(\mathbb{Q}_S)\backslash \mathrm{G}(\mathbb{Q}_S)} \int_{\mathrm{M}^0_{X}(\mathbb{Q}_S)\backslash \mathrm{G}^0_{\underline{\Xi}}(\mathbb{Q}_S)} \\
&\qquad \times f_S((X+A)\cdot \mathrm{ad}(yx)) \times \\
&\qquad\quad \times\Big(\sum_{\mathrm{Q}\in\mathcal{F}^{\Xi}(\mathrm{M}^0_{\underline{\Xi}})} \sum_{\mathrm{L}\in\mathcal{L}^{\mathrm{M}_{\mathrm{Q}}}(\mathrm{M}^0_{\underline{\Xi}})} r^{\mathrm{L}}_{\mathrm{M}^0_{\underline{\Xi}}}\big(\exp(X_{\mathrm{nil}}), \exp(A)\big) \times \\
&\qquad\qquad \times v^{\mathrm{Q}}_{\mathrm{L}}(y) v'_{\mathrm{Q}}(kx, T_0 - T_{\Xi,\mathrm{M}})\Big)\mathrm{d}y\mathrm{d}x\Bigg) \\
&(3.3.9.4) \qquad = \lim_{A\to 0}\Bigg(|D^{\mathrm{G}}(X+A)|_S^{1/2}\, |D^{\mathrm{G}_{\Xi}}(X+A)|_S^{-1/2} \int_{\mathrm{G}^0_{\underline{\Xi}}(\mathbb{Q}_S)\backslash \mathrm{G}(\mathbb{Q}_S)}
\end{aligned}
$$

$$\times\Big(\sum_{\mathrm{Q}\in\mathcal{F}^{\Xi}(\mathrm{M}^0_{\Xi})}\ \sum_{\mathrm{L}\in\mathcal{L}^{\mathrm{M_Q}}(\mathrm{M}_{\Xi})} r^{\mathrm{L}}_{\mathrm{M}^0_{\Xi}}\big(\exp(X_{\mathrm{nil}}),\exp(A)\big)\times$$

$$\times J^{\mathrm{M_Q}}_{\mathrm{L}}(X_{\mathrm{nil}}+A,\Phi^{T_0-T_{\Xi,\mathrm{M}}}_{S,\mathrm{Q},y})\Big)\mathrm{d}y\Big)$$

$$=\ |D^{\mathrm{G}}(\Xi)|^{1/2}_{S}\int_{\mathrm{G}^0_{\Xi}(\mathbb{Q}_S)\backslash\mathrm{G}(\mathbb{Q}_S)}\Big(\sum_{\mathrm{Q}\in\mathcal{F}^{\Xi}(\mathrm{M}^0_{\Xi})} J^{\mathrm{M_Q}}_{\mathrm{M}_{\Xi}}(X_{\mathrm{nil}},\Phi^{T_0-T_{\Xi,\mathrm{M}}}_{S,\mathrm{Q},y})\Big)\mathrm{d}y$$

where the equality (3.3.9.4) follows from the change of variables

(3.3.9.5) $$\mathrm{M}^0_X(\mathbb{Q}_S)\backslash\mathrm{G}^0_{\Xi}(\mathbb{Q}_S)\ \xrightarrow{\sim}\ \mathrm{M}_X(\mathbb{Q}_S)\backslash\mathrm{M}_{\mathrm{Q}^0_{\Xi}}(\mathbb{Q}_S)\times\mathrm{N_Q}(\mathbb{Q}_S)\times\mathrm{K}_{\Xi}$$

for each parabolic subgroup Q in $\mathcal{F}^{\Xi}(\mathrm{M}^0_{\Xi})$. □

(3.3.10) Definition Let $\equiv$ be the equivalence relation on $\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}$ defined by

(3.3.10.1) $\forall X\in\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}\quad\forall Y\in\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}\quad X\equiv Y\quad$ if

$$\exists\delta\in\mathrm{M}(\mathbb{Q})\quad\exists\eta\in\mathrm{M}^0_{X_{\mathrm{ss}}}(\mathbb{Q}_S)\quad X_{\mathrm{ss}}=Y_{\mathrm{ss}}\cdot\mathrm{ad}(\delta)\quad X_{\mathrm{nil}}=Y_{\mathrm{nil}}\cdot\mathrm{ad}(\delta\eta).$$

Let $(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S}$ denote the collection of $\equiv$ equivalence classes in $\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}$. By abuse of notation denote the $\equiv$ equivalence class of an element X in $\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}$ by the same symbol X.

(3.3.11) Proposition *Let $\mathfrak{o}$ be a $\sim$ equivalence class on $\mathfrak{g}(\mathbb{Q})$. Let $S_{\mathfrak{o}}$ be a sufficiently large finite set of places of $\mathbb{Q}$. Then for each standard Levi subgroup* M *of* G *and for each $\equiv$ equivalence class X in $\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}$ there exists a constant $a^{\mathrm{M}}(S_{\mathfrak{o}},X)$ such that*

(3.3.11.1) $\forall f\in\mathcal{S}(\mathfrak{g}(\mathbb{A}))$

$$J_{\mathfrak{o}}^{\mathrm{G}}(f) = \sum_{\mathrm{M}\in\mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S_{\mathfrak{o}}}} a^{\mathrm{M}}(S_{\mathfrak{o}},X) J_{\mathrm{M}}^{\mathrm{G}}(X,f_{S_{\mathfrak{o}}}).$$

Proof. Retain the notations of (3.3.6.1) and (3.3.9.1). Let Ξ be equal to Σ, hence

$$X = \Sigma + \nu \quad \in \quad \mathfrak{m}(\mathbb{Q}_S)/\mathrm{ad}(\mathrm{M}(\mathbb{Q}_S)), \tag{3.3.11.2}$$

and

$$T_0 - T_{\Xi,\mathrm{M}} \quad = \quad T_0 - T_{\Sigma,1}. \tag{3.3.11.3}$$

Without loss of generality assume that the finite set $S_{\mathfrak{o}}$ is large enough such that

(3.3.11.4) $\forall p \notin S_{\mathfrak{o}}$ $\bullet \Sigma \in \mathfrak{g}(\mathbb{Z}_p)$;

$\bullet |D(\Sigma)|_p = 1$;

$\bullet \mathfrak{g}(\mathbb{Z}_p) \cdot \mathrm{ad}(\mathrm{K}_p) = \mathfrak{g}(\mathbb{Z}_p)$;

$\bullet \mathrm{K}_p \cap \mathrm{G}_{\Sigma}^0(\mathbb{Q}_p) = \mathrm{K}_{\Sigma,p}$ is hyperspecial in $\mathrm{G}_{\Sigma}^0(\mathbb{Q}_p)$;

$\bullet \forall x_p \in \mathrm{G}(\mathbb{Q}_p)\ (\Sigma + \mathfrak{g}_{\Sigma,\mathrm{nil}}(\mathbb{Q}_p)) \cdot \mathrm{ad}(x_p) \cap \mathfrak{g}(\mathbb{Z}_p) \neq \emptyset$

$\Rightarrow x_p \in \mathrm{G}_{\Sigma}^0(\mathbb{Q}_p)\mathrm{K}_p$.

The last condition in (3.3.11.4) is satisfied for large enough $S_{\mathfrak{o}}$ by Lemma 6.1 of [Ar86]. Since

$$\Big(H \mapsto \Gamma'_{\mathrm{Q}}\big(H - T_{\Sigma,1}, \mathcal{Y}_{\mathrm{Q}}^{T_0}(k,x)\big)\Big)^{\wedge} \quad = \quad v'_{\mathrm{Q}}(kx, T_0 - T_{\Sigma,1})(\lambda) \tag{3.3.11.5}$$

the conditions (3.3.11.4) imply that

$$\Phi_{\mathrm{Q},x}^{T_0-T_{\Sigma,1}} \quad = \quad \Phi_{S_{\mathfrak{o}},\mathrm{Q},x_{S_{\mathfrak{o}}}}^{T_0-T_{\Sigma,1}} \otimes \bigotimes_{p\notin S_{\mathfrak{o}}} \mathbb{I}_{\mathrm{G}_{\Sigma}^0(\mathbb{Q}_p)\mathrm{K}_p}(x) \cdot \mathbb{I}_{\mathfrak{m}_{\mathrm{Q}}(\mathbb{Z}_p)}. \tag{3.3.11.6}$$

By(3.3.6.1)) and (3.2.7.1),

$$(3.3.11.7) \qquad J^{\mathrm{G}}_{\mathfrak{o}}(f)$$

$$= |\pi_0(\mathrm{G}_\Sigma)|^{-1} \int_{\mathrm{G}^0_\Sigma(\mathbb{Q}_{S_\mathfrak{o}})\backslash \mathrm{G}(\mathbb{Q}_{S_\mathfrak{o}})} \Bigg(\sum_{\mathrm{Q}\in\mathcal{F}^\Sigma} \sum_{\mathrm{L}\in\mathcal{L}^{\mathrm{M_Q}}(\mathrm{M}^0_{1,\Sigma})} |\mathrm{W}^{\mathrm{L}}_1||\mathrm{W}^{\mathrm{G}^0_\Sigma}_1|^{-1} \times$$

$$\times \sum_{\nu\in(\mathfrak{l}_{\mathrm{nil}}(\mathbb{Q}))_{\mathrm{L},S_\mathfrak{o}}} a^{\mathrm{L}}(S_\mathfrak{o},\nu) J^{\mathrm{M_Q}}_{\mathrm{L}}(\nu, \Phi^{T_0-T_{\Sigma,1}}_{S_\mathfrak{o},\mathrm{Q},x}) \Bigg) \mathrm{d}x$$

$$(3.3.11.8) \quad = |\pi_0(\mathrm{G}_\Sigma)|^{-1} \sum_{\mathrm{M}\in\mathring{\mathcal{L}}_\Sigma(\mathrm{M}_1)} |\mathrm{W}^{\mathrm{M}^0_\Sigma}_1||\mathrm{W}^{\mathrm{G}^0_\Sigma}_1|^{-1} \times$$

$$\times \sum_{\nu\in(\mathfrak{m}_{\Sigma,\mathrm{nil}}(\mathbb{Q}))_{\mathrm{M}^0_\Sigma,S_\mathfrak{o}}} a^{\mathrm{M}^0_\Sigma}(S_\mathfrak{o},\nu) J^{\mathrm{G}}_{\mathrm{M}}(\Sigma+\nu, f_{S_\mathfrak{o}})$$

where on the right hand side of (3.3.11.8) the symbol $\mathring{\mathcal{L}}_\Sigma(\mathrm{M}_1)$ denotes the set

$$(3.3.11.9) \qquad \mathring{\mathcal{L}}_\Sigma(\mathrm{M}_1) = \Big\{ \mathrm{M}\in\mathcal{L}(\mathrm{M}_1): \ \mathrm{A_M} = \mathrm{A}_{\mathrm{M}^0_\Sigma} \Big\}.$$

The equality (3.3.11.8) follows from (3.3.9.1) and the bijection

$$(3.3.11.10) \qquad \pi_\Sigma : \quad \mathring{\mathcal{L}}_\Sigma(\mathrm{M}_1) \ \overset{\sim}{\longrightarrow}\ \mathcal{L}^{\mathrm{G}^0_\Sigma}(\mathrm{M}^0_{1,\Sigma})$$

defined by

$$(3.3.11.11) \qquad \forall \mathrm{M}\in\mathring{\mathcal{L}}_\Sigma(\mathrm{M}_1) \qquad \pi_\Sigma(\mathrm{M}) = \mathrm{M}^0_\Sigma = \mathrm{L}$$

and the equation

$$(3.3.11.12) \qquad |D(\Sigma)|_{S_\mathfrak{o}} = 1$$

which follows from (3.3.11.4).

Define the constant $a^{\mathrm{M}}(S_{\mathfrak{o}}, X)$ by

$$a^{\mathrm{M}}(S_{\mathfrak{o}}, X) \;=\; |\pi_0(\mathrm{M}_\Sigma)|^{-1} \sum_{\substack{\nu\in(\mathfrak{m}_{\Sigma,\mathrm{nil}})_{\mathrm{M}^0_\Sigma, S_{\mathfrak{o}}} \\ \Sigma+\nu\equiv X \,\mathrm{mod}(\mathrm{M}, S_{\mathfrak{o}})}} a^{\mathrm{M}^0_\Sigma}(S_{\mathfrak{o}}, \nu) \tag{3.3.11.13}$$

if X_{ss} is $\mathbb{Q}$-elliptic in $\mathfrak{m}(\mathbb{Q})$ and zero otherwise. For a fixed Σ and let M vary in $\mathcal{L}(\mathrm{M}_1)$, then Σ is $\mathbb{Q}$-elliptic in $\mathfrak{m}(\mathbb{Q})$ if and only if A_{M} and A_{M_Σ} are equal, hence by (3.3.11.7)

$$\begin{aligned} J^{\mathrm{G}}_{\mathfrak{o}}(f) \;=\; & \sum_{\mathrm{M}\in\mathcal{L}(\mathrm{M}_1)} |\mathrm{W}_1^{\mathrm{M}^0_\Sigma}||\mathrm{W}_1^{\mathrm{G}^0_\Sigma}|^{-1}|\pi_0(\mathrm{M}_\Sigma)||\pi_0(\mathrm{G}_\Sigma)|^{-1} \times \\ & \times \sum_{\substack{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S_{\mathfrak{o}}} \\ X_{\mathrm{ss}}=\Sigma}} a^{\mathrm{M}}(S_{\mathfrak{o}}, X) J^{\mathrm{G}}_{\mathrm{M}}(X, f_{S_{\mathfrak{o}}}). \end{aligned} \tag{3.3.11.14}$$

The set

$$\left\{ (\mathrm{M},\sigma) : \begin{array}{l} \mathrm{M}\in\mathcal{L}, \sigma \text{ is a semisimple } \mathrm{M}(\mathbb{Q})\text{-} \\ \text{-orbit in } \mathfrak{m}(\mathbb{Q}),\ \Sigma \text{ is conjugate} \\ \text{to a point in } \sigma \text{ under } \mathrm{G}(\mathbb{Q}). \end{array} \right\} \tag{3.3.11.15}$$

admits a natural action by $\mathrm{W}^{\mathrm{G}}_0$ which preserves $a^{\mathrm{M}}(S_{\mathfrak{o}}, X)$ and $J^{\mathrm{G}}_{\mathrm{M}}(X, f_{S_{\mathfrak{o}}})$, where each pair (M,σ) has stablizer $\mathrm{W}^{\mathrm{M}}_0$. Denote by $\mathrm{W}_1^{\mathrm{G}_\Sigma}$ the quotient of the normalizer of A_1 in G_Σ by $\mathrm{M}_{1,\Sigma}$. There is an exact sequence

$$1 \longrightarrow \mathrm{W}_1^{\mathrm{G}^0_\Sigma} \longrightarrow \mathrm{W}_1^{\mathrm{G}_\Sigma} \longrightarrow \pi_0(\mathrm{G}_\Sigma) \longrightarrow 1\,. \tag{3.3.11.16}$$

Similar notations apply to M_Σ. The subset of (3.3.11.15)

$$\left\{ (\mathrm{M},\sigma) :\ \mathrm{M}\in\mathcal{L}(\mathrm{M}_1) \text{ and } \sigma = \Sigma\cdot\mathrm{ad}(\mathrm{M}(\mathbb{Q})) \right\} \tag{3.3.11.17}$$

admits a natural action of $\mathrm{W}_1^{\mathrm{G}_\Sigma}$, where each pair (M,σ) has stablizer $\mathrm{W}_1^{\mathrm{M}_\Sigma}$. The quotients

$$(3.3.11.18) \qquad \left\{ (\mathrm{M},\sigma) : \begin{array}{l} \mathrm{M}\in\mathcal{L}, \sigma \text{ is a semisimple } \mathrm{M}(\mathbb{Q})\text{-} \\ \text{-orbit in } \mathfrak{m}(\mathbb{Q}),\ \Sigma \text{ is conjugate} \\ \text{to a point in } \sigma \text{ under } \mathrm{G}(\mathbb{Q}). \end{array} \right\} \Big/ \mathrm{W}_0^{\mathrm{G}}$$

and

$$(3.3.11.19) \qquad \left\{ (\mathrm{M},\sigma) :\ \mathrm{M}\in\mathcal{L}(\mathrm{M}_1) \text{ and } \sigma = \Sigma\cdot\mathrm{ad}(\mathrm{M}(\mathbb{Q})) \right\} \Big/ \mathrm{W}_1^{\mathrm{G}_\Sigma}$$

are in bijection, therefore

$$(3.3.11.20) \qquad J_{\mathfrak{o}}^{\mathrm{G}}(f)$$

$$= \sum_{\mathrm{M}\in\mathcal{L}(\mathrm{M}_1)} |\mathrm{W}_1^{\mathrm{M}_\Sigma}||\mathrm{W}_1^{\mathrm{G}_\Sigma}|^{-1} \sum_{\substack{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S_{\mathfrak{o}}} \\ X_{\mathrm{ss}}=\Sigma}} a^{\mathrm{M}}(S_{\mathfrak{o}},X) J_{\mathrm{M}}^{\mathrm{G}}(X, f_{S_{\mathfrak{o}}})$$

$$= \sum_{\mathrm{M}\in\mathcal{L}} \frac{|\mathrm{W}_0^{\mathrm{M}}|}{|\mathrm{W}_0^{\mathrm{G}}|} \left(\frac{|\mathrm{W}_1^{\mathrm{M}_\Sigma}|}{|\mathrm{W}_1^{\mathrm{G}_\Sigma}|}\right)^{-1} |\mathrm{W}_1^{\mathrm{M}_\Sigma}||\mathrm{W}_1^{\mathrm{G}_\Sigma}|^{-1} \times$$

$$\times \sum_{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S_{\mathfrak{o}}}} a^{\mathrm{M}}(S_{\mathfrak{o}},X) J_{\mathrm{M}}^{\mathrm{G}}(X, f_{S_{\mathfrak{o}}})$$

$$= \sum_{\mathrm{M}\in\mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S_{\mathfrak{o}}}} a^{\mathrm{M}}(S_{\mathfrak{o}},X) J_{\mathrm{M}}^{\mathrm{G}}(X, f_{S_{\mathfrak{o}}}).$$

□

(3.3.12) Remark For a semisimple element X in $\mathfrak{o}$,

$$(3.3.12.1) \qquad a^{\mathrm{M}}(S_{\mathfrak{o}},X) \;=\; |\pi_0(\mathrm{M}_X)|^{-1}\mathrm{Vol}\big(\mathrm{M}_X^0(\mathbb{Q})\backslash\mathrm{M}_X^0(\mathbb{A})^1\big)$$

if X is $\mathbb{Q}$-elliptic in $\mathfrak{m}(\mathbb{Q})$, and vanishes otherwise. If X is in addition assumed to be regular, the identity (3.3.11.1) reduces to (2.2.11.1).

2.3.4 The refined trace formula

(3.4.1) Proposition (Refined trace formula)

For each sufficiently large finite set S of places of $\mathbb{Q}$, for each $\sim$ equivalence class $\mathfrak{o}$ in $\mathfrak{g}(\mathbb{Q})$, for each $\equiv$ equivalence class X in $(\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o})_{\mathrm{M},S}$, there exists a constant $a^{\mathrm{M}}(S,X)$ such that

(3.4.1.1) $\forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{A}))$

$$\lim_S \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \sum_{\mathrm{M}\in\mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S}} a^{\mathrm{M}}(S,X) J_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$$

$$= \lim_S \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \sum_{\mathrm{M}\in\mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{X\in(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{M},S}} a^{\mathrm{M}}(S,X) J_{\mathrm{M}}^{\mathrm{G}}(X, f\hat{\ }_S).$$

Proof. By (2.2.7.1) it is enough to show that the left hand side of (3.4.1.1) is equal to

(3.4.1.2)
$$J(f) \;=\; \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} J_{\mathfrak{o}}(f).$$

For each individual class $\mathfrak{o}$ by (3.3.11.1)

(3.4.1.3)
$$J_{\mathfrak{o}}(f) \;=\; \lim_S \sum_{\mathrm{M}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_X a^{\mathrm{M}}(S,X) J_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$$

where the limit stablizes as S grows large enough.

Lemma *Let Γ be a compact subset of $\mathfrak{g}(\mathbb{A})$. Then there are only finitely many classes $\mathfrak{o}$ such that $\mathfrak{o} \cdot \mathrm{ad}(\mathrm{G}(\mathbb{A}))$ intersects Γ.*

Proof. The lemma follows from Corollary A.2 of [Ar86] which states that there exists a compact set G_Γ contained in $\mathrm{G}(\mathbb{A})^1$ such that

$$\forall x \in \mathrm{G}(\mathbb{A})^1 - \mathrm{G}(\mathbb{Q})\mathrm{G}_\Gamma \qquad \mathfrak{g}(\mathbb{Q})' \cdot \mathrm{ad}(x) \cap \Gamma = \emptyset \tag{3.4.1.4}$$

where $\mathfrak{g}(\mathbb{Q})'$ denotes the set of points of $\mathfrak{g}(\mathbb{Q})$ not contained in any proper parabolic subalgebra. This is established by a reduction theory argument.

To prove the lemma first consider a class $\mathfrak{o}$ contained in $\mathfrak{g}(\mathbb{Q})'$. There exists a point X in $\mathfrak{o}$ such that $X \cdot \mathrm{ad}(\mathrm{G}_\Gamma)$ is contained in Γ, hence there are only finitely many such X, hence finitely many such $\mathfrak{o}$.

Next for an arbitrary class $\mathfrak{o}$ there is some Levi subalgebra $\mathfrak{m}$ such that $\mathfrak{o}$ intersects $\mathfrak{m}(\mathbb{Q})'$. Then by the Iwasawa decomposition

$$\begin{aligned} & \mathfrak{o} \cdot \mathrm{ad}(\mathrm{G}(\mathbb{A})^1) \\ = \; & \mathfrak{o} \cdot \mathrm{ad}(\mathrm{M}(\mathbb{A})^1) \cdot \mathrm{ad}(\mathrm{N}(\mathbb{Q})\backslash\mathrm{N}(\mathbb{A})) \cdot \mathrm{ad}(\mathrm{K}) \\ = \; & (\mathfrak{o} \cap \mathfrak{m}(\mathbb{Q})') \cdot \mathrm{ad}(\mathrm{M}_{\Gamma\cap\mathfrak{m}(\mathbb{A})}) \cdot \mathrm{ad}(\mathrm{N}(\mathbb{Q})\backslash\mathrm{N}(\mathbb{A})) \cdot \mathrm{ad}(\mathrm{K}), \end{aligned} \tag{3.4.1.5}$$

the lemma follows by induction. □

The Schwartz function f is compactly supported if its component at infinity f_∞ is compactly supported. For such an f the limit on the left hand side of (3.4.1.1) stablizes for every S such that

$$S \;\supset \bigcup_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} S_{\mathfrak{o}} \tag{3.4.1.6}$$

where the union is supported on a finite set of $\mathfrak{o}$. Hence for a compactly supported f the left hand side of (3.4.1.1) is equal to $J(f)$.

However the function spaces $C_c^\infty(\mathfrak{g}(\mathbb{R}))$ and hence $C_c^\infty(\mathfrak{g}(\mathbb{Q}_S))$ are dense in $\mathcal{S}(\mathfrak{g}(\mathbb{R}))$ and $\mathcal{S}(\mathfrak{g}(\mathbb{Q}_S))$ respectively, therefore $C_c^\infty(\mathfrak{g}(\mathbb{A}))$ is dense in $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ which by (1.1.4.2) is defined as

$$(3.4.1.7) \qquad \mathcal{S}(\mathfrak{g}(\mathbb{A})) \;=\; \varinjlim \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S))$$

equipped with the final topology.

Since $J(f)$ extends continuously to all Schwartz functions, the limit on the left hand side of (3.4.1.1) exists, and the interchangeability of the operations of taking the limit as S approaches infinity and taking the sum over all classes $\mathfrak{o}$ extends from $C_c^\infty(\mathfrak{g}(\mathbb{A}))$ to $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$. This establishes the identity (3.4.1.1) for all Schwartz functions f on $\mathfrak{g}(\mathbb{A})$. □

2.4 A simple invariant trace formula

In this section following the arguments in §7 of [Ar88c] the refined trace formula (3.4.1.1) is reduced to an identity between invariant distributions on $\mathfrak{g}(\mathbb{A})$ for a suitably restricted class of test functions.

2.4.1 Parabolic descent and parabolic induction

(4.1.1) Definition Let P be a parabolic subgroup in $\mathcal{F}$ with Levi component M and unipotent radical N. Let f_S be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_S)$. Define $f_{S,\mathrm{P}}$ to be the Schwartz function on $\mathfrak{m}(\mathbb{Q}_S)$ by *parabolic descent* along P

$$(4.1.1.1) \qquad \forall X \in \mathfrak{m}(\mathbb{Q}_S)$$

$$f_{S,\mathrm{P}}(X) = \int_{\mathrm{K}_S} \int_{\mathfrak{n}(\mathbb{Q}_S)} f_S((X+N)\cdot \mathrm{ad}(k))\, \mathrm{d}N \mathrm{d}k.$$

For each parabolic subgroup P_S in $\mathcal{F}^{\mathrm{G}_S}$ define the Schwartz function f_{S,P_S} on $\mathfrak{m}_S(\mathbb{Q}_S)$ analogously.

(4.1.2) Remark Parabolic descent preserves orbital integrals, i.e.

$$\text{(4.1.2.1)} \qquad \forall X \in \mathfrak{m}_{\mathrm{reg,ss}}(\mathbb{Q}_S) \qquad I_{\mathrm{G}}^{\mathrm{G}}(X, f_S) = I_{\mathrm{M}}^{\mathrm{M}}(X, f_{S,\mathrm{P}}).$$

See §13.12 of [Ko05]. Hence the orbital integrals of $f_{S,\mathrm{P}}$ on $\mathfrak{m}_{\mathrm{reg,ss}}(\mathbb{Q}_S)$ is independent of the choice of the parabolic subgroup P in $\mathcal{P}(\mathrm{M})$. In this case denote $f_{S,\mathrm{P}}$ by the alternative notation $f_{S,\mathrm{M}}$ such that

$$\text{(4.1.2.2)} \qquad \forall X \in \mathfrak{m}_{\mathrm{reg,ss}}(\mathbb{Q}_S) \qquad I_{\mathrm{M}}^{\mathrm{M}}(X, f_{S,\mathrm{M}}) = I_{\mathrm{M}}^{\mathrm{M}}(X, f_{S,\mathrm{P}})$$

is well-defined.

(4.1.3) Remark With compatible choices of Fourier transforms on $\mathfrak{g}(\mathbb{Q}_S)$ and $\mathfrak{m}(\mathbb{Q}_S)$, parabolic descent intertwines the Fourier transforms, i.e.

$$\text{(4.1.3.1)} \qquad (f_S\hat{\ })_{\mathrm{P}} \quad = \quad (f_{S,\mathrm{P}})\hat{\ }.$$

See §13.13 of [Ko05].

(4.1.4) Definition Let P be a parabolic subgroup in $\mathcal{F}$ with Levi component M and unipotent radical N. Let X be an element in $\mathfrak{m}(\mathbb{Q}_S)$. Define $\mathrm{Ind}_{\mathrm{M,P}}^{\mathrm{G}}(X)$ to be the $\mathrm{ad}(\mathrm{G}(\mathbb{Q}_S))$-invariant subset of $\mathfrak{g}(\mathbb{Q}_S)$ by *parabolic induction* along P

$$\text{(4.1.4.1)} \qquad \mathrm{Ind}_{\mathrm{M,P}}^{\mathrm{G}}(X) \quad = \quad \Big(\big(X \cdot \mathrm{ad}(\mathrm{M}) + \mathfrak{n}\big) \cdot \mathrm{ad}(\mathrm{G})\Big)_{\mathrm{reg}}(\mathbb{Q}_S)$$

where the subscript reg denotes the regular locus, the set of points with minimal dimensional isotropy group in G.

(4.1.5) Remark With a fixed choice of M, parabolic induction along P is independent of the choice of the parabolic subgroup P in $\mathcal{P}(\mathrm{M})$, hence adopt the alternative notation

$\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X)$. For a proof see Satz 2.6 of [Bo81]. If the inducing Levi subgroup M is understood tacitly, denote $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X)$ by the alternative notation X^{G}.

(4.1.6) Remark If X is regular semisimple then $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X)$ is equal to the $\mathrm{ad}(\mathrm{G}(\mathbb{Q}_S))$-orbit of X. In general $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X)$ is a finite union of $\mathrm{ad}(\mathrm{G}(\mathbb{Q}_S))$-orbits which are geometrically conjugate to each other, but $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X)$ is still called the *induced orbit* of X. See page 255 of [Ar88a] and §2.1 of [Bo81]. If L is a Levi subgroup in $\mathcal{L}(\mathrm{M})$, define the invariant weighted orbital integral along the induced orbit X^{L} by

$$(4.1.6.1) \qquad \forall f_S \in \mathfrak{g}(\mathbb{Q}_S) \qquad J_{\mathrm{L}}^{\mathrm{G}}(X^{\mathrm{L}}, f_S) = \sum_{\substack{\Omega \subset X^{\mathrm{L}} \\ \mathrm{ad}(\mathrm{L}(\mathbb{Q}_S))\text{-orbit}}} J_{\mathrm{L}}^{\mathrm{G}}(\Omega, f_S).$$

(4.1.7) Remark Parabolic induction is transitive in nested chains of Levi subgroups, i.e.

$$(4.1.7.1) \qquad \forall \mathrm{L} \in \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \qquad \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X) = \mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{M}}^{\mathrm{L}}(X)).$$

See §2.3 of [Bo81].

(4.1.8) Remark Parabolic induction is compatible with Jordan decomposition, i.e.

$$(4.1.8.2) \qquad \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(X) \quad = \quad \Big(X_{\mathrm{ss}} + \mathrm{Ind}_{\mathrm{M}^0_{X_{\mathrm{ss}}}}^{\mathrm{G}^0_{X_{\mathrm{ss}}}}(X_{\mathrm{nil}})\Big) \cdot \mathrm{ad}(\mathrm{G}(\mathbb{Q}_S)).$$

See §2.4 of [Bo81]. Also see Lemma 2 of [Ho13].

(4.1.9) Lemma (Descent and splitting of (G, M)-families)

Let M *be a Levi subgroup in* $\mathcal{L}$. *There exist a function* $d_{\mathrm{M}}^{\mathrm{G}}(\ ,\ )$

$$(4.1.9.1) \qquad d_{\mathrm{M}}^{\mathrm{G}} : \quad \mathcal{L}(\mathrm{M}) \times \mathcal{L}(\mathrm{M}) \quad \longrightarrow \quad \mathbb{R},$$

and a partially defined map $s(\ ,\ )$

$$(4.1.9.2) \qquad s: \quad \mathcal{L}(\mathrm{M}) \times \mathcal{L}(\mathrm{M}) \longrightarrow \mathcal{F}(\mathrm{M}) \times \mathcal{F}(\mathrm{M})$$

whose domain contains the pairs $(\mathrm{L}_1, \mathrm{L}_2)$ *for which* $d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2)$ *is nonzero, such that*

- *if* $(\mathrm{L}_1, \mathrm{L}_2)$ *is contained in the domain of* s*, then*

  $$(4.1.9.3) \qquad s(\mathrm{L}_1, \mathrm{L}_2) \in \mathcal{P}(\mathrm{L}_1) \times \mathcal{P}(\mathrm{L}_2);$$

- *if* (c_{P}) *is a* (G, M)*-family and* L *is a Levi subgroup in* $\mathcal{L}(\mathrm{M})$*, then*

  $$(4.1.9.4) \qquad c_{\mathrm{L}} = \sum_{\mathrm{L}' \in \mathcal{L}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}, \mathrm{L}') c_{\mathrm{M}}^{\mathrm{Q}'}$$

  where Q' *denotes the second component of* $s(\mathrm{L}, \mathrm{L}')$*;*

- *if* (c_{P}) *and* (d_{P}) *are* (G, M)*-families, then*

  $$(4.1.9.5) \qquad (cd)_{\mathrm{M}} = \sum_{\mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2) c_{\mathrm{M}}^{\mathrm{Q}_1} d_{\mathrm{M}}^{\mathrm{Q}_2}$$

  where

  $$(4.1.9.6) \qquad (\mathrm{Q}_1, \mathrm{Q}_2) = s(\mathrm{L}_1, \mathrm{L}_2).$$

Analogous results hold for the groups G_v *and* G_S*.*

Proof. See §7 of [Ar88b]. □

(4.1.10) Remark The constant $d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2)$ is defined to be the volume in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$ of the image of a fundamental parallelotope in the direct sum of $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}_1}$ and $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}_2}$ under the natural map

(4.1.10.1)
$$\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}_1} \oplus \mathfrak{a}_{\mathrm{M}}^{\mathrm{L}_2} \longrightarrow \mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$$

if (4.1.10.1) is an isomorphism, and zero otherwise. In the former case $d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2)$ is equal to the volume in $\mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}}$ of the image of a fundamental parallelotope in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}_1}$ under the natural isomorphism

(4.1.10.2)
$$\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}_1} \longrightarrow \mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}}.$$

(4.1.11) Remark The map s depends on the choice of a vector ξ in general position in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$. Let L_1 and L_2 be Levi subgroups in $\mathcal{L}(\mathrm{M})$ such that (4.1.10.1) is an isomorphism, then

(4.1.11.1)
$$\exists \xi_1 \in \mathfrak{a}_{\mathrm{L}_1}^{\mathrm{G}} \ \exists \xi_2 \in \mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}} \qquad \xi = \frac{\xi_1}{2} - \frac{\xi_2}{2}.$$

Define $s(\mathrm{L}_1, \mathrm{L}_2)$ to be $(\mathrm{Q}_1, \mathrm{Q}_2)$ where Q_i is the parabolic subgroup in $\mathcal{P}(\mathrm{L}_i)$ whose corresponding positive chamber in $\mathfrak{a}_{\mathrm{L}_i}^{\mathrm{G}}$ contains the vector ξ_i where the index i is 1 or 2.

(4.1.12) Lemma (Descent and splitting of weighted orbital integrals)
Let M *be a Levi subgroup in* $\mathcal{L}$*, let* X *be an element of* $\mathfrak{m}(\mathbb{Q}_S)$*. Let* f_S *be a Schwartz function on* $\mathfrak{g}(\mathbb{Q}_S)$*. Let* ξ *be a vector in general position in* $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$*, let*

(4.1.12.1)
$$\begin{aligned} d_{\mathrm{M}}^{\mathrm{G}} &: \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \times \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \longrightarrow \mathbb{R} \\ s &: \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \times \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \longrightarrow \mathcal{F}^{\mathrm{G}}(\mathrm{M}) \times \mathcal{F}^{\mathrm{G}}(\mathrm{M}) \end{aligned}$$

be defined as in Remark (4.1.10) and Remark (4.1.11) with respect to ξ*.*

- *Let* L *be a Levi subgroup in* $\mathcal{L}^{\mathrm{G}}(\mathrm{M})$*, then*

$$J_{\mathrm{L}}^{\mathrm{G}}(X^{\mathrm{L}}, f_S) = \sum_{\mathrm{L}' \in \mathcal{L}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}, \mathrm{L}') J_{\mathrm{M}}^{\mathrm{L}'}(X, f_{S,\mathrm{Q}'}) \tag{4.1.12.2}$$

  where Q' *denotes the second component of* $s(\mathrm{L}, \mathrm{L}')$*.*

- *Let* S *be the set* $\{v_1, v_2\}$*. Let* f_S *be of the form* $f_{v_1} \otimes f_{v_2}$ *where* f_{v_i} *is a Schwartz function on* $\mathfrak{g}(\mathbb{Q}_{v_i})$ *where the index* i *is 1 or 2. Then*

$$J_{\mathrm{M}}^{\mathrm{G}}(X, f_S) = \sum_{\mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2) J_{\mathrm{M}}^{\mathrm{L}_1}(X, f_{v_1,\mathrm{Q}_1}) J_{\mathrm{M}}^{\mathrm{L}_2}(X, f_{v_2,\mathrm{Q}_2}) \tag{4.1.12.3}$$

  where

$$(\mathrm{Q}_1, \mathrm{Q}_2) = s(\mathrm{L}_1, \mathrm{L}_2). \tag{4.1.12.4}$$

Local identities analogous to (4.1.12.2) also hold for G_v*.*

Proof. For a regular semisimple X this follows from (4.1.9.4) and (4.1.9.5). For a general X this follows from the limit formula (3.1.5.4). □

(4.1.13) Corollary (Descent and splitting of orbital integrals)
Let X *be an element of* $\mathfrak{g}(\mathbb{Q}_S)$*. Let* f_S *be a Schwartz function on* $\mathfrak{g}(\mathbb{Q}_S)$*.*

- *Let* L *be a Levi subgroup in* $\mathcal{L}$ *such that* X *is contained in* $\mathfrak{l}(\mathbb{Q}_S)$*, then*

$$I_{\mathrm{G}}^{\mathrm{G}}(X^{\mathrm{G}}, f_S) = I_{\mathrm{L}}^{\mathrm{L}}(X, f_{S,\mathrm{L}}). \tag{4.1.13.1}$$

- *Let* S *be the set* $\{v_1, v_2\}$*. Let* f_S *be of the form* $f_{v_1} \otimes f_{v_2}$ *where* f_{v_i} *is a Schwartz* *function on $\mathfrak{g}(\mathbb{Q}_{v_i})$ where the index i is 1 or 2. Then*

$$(4.1.13.2) \qquad I_{\mathrm{G}}^{\mathrm{G}}(X, f_{v_1} \otimes f_{v_2}) \ = \ I_{\mathrm{G}}^{\mathrm{G}}(X, f_{v_1}) I_{\mathrm{G}}^{\mathrm{G}}(X, f_{v_2}).$$

Local identities analogous to (4.1.13.1) also hold for G_v.

Proof. The identity (4.1.13.1) follows from (4.1.12.2) where the only nonzero summand on the right hand side corresponds to the Levi subgroup L in $\mathcal{L}(\mathrm{L})$.

The identity (4.1.13.2) follows from (4.1.12.3) where the only summand on the right hand side corresponds to the pair (G, G) in $\mathcal{L}(\mathrm{G}) \times \mathcal{L}(\mathrm{G})$. □

(4.1.14) Remark If X is a regular semisimple element of $\mathfrak{m}(\mathbb{Q})$ then the local analogue of (4.1.12.2) becomes

$$(4.1.14.1) \qquad J_{\mathrm{L}_v}^{\mathrm{G}_v}(X, f_v) \ = \ \sum_{\mathrm{L}'_v \in \mathcal{L}^{\mathrm{G}_v}(\mathrm{M}_v)} d_{\mathrm{M}_v}^{\mathrm{G}_v}(\mathrm{L}_v, \mathrm{L}'_v) J_{\mathrm{M}_v}^{\mathrm{L}'_v}(X, f_{v,\mathrm{Q}'_v}).$$

Repeatedly applying the identities (4.1.12.3) and (4.1.14.1) reduces a regular semisimple weighted orbital integral $J_{\mathrm{M}}^{\mathrm{G}}(X, \ )$ that appears generically in the refined trace formula (3.4.1.1) to a linear combination of products of local weighted orbital integrals $J_{\mathrm{M}'_v}^{\mathrm{G}_v}(X, \ )$ where M'_v is contained in M_v and X is elliptic in $\mathfrak{m}'_v(\mathbb{Q}_v)$.

2.4.2 A simple form of the trace formula

(4.2.1) Definition Let v be a place of $\mathbb{Q}$, let f_v be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_v)$. The function f_v is said to be *cuspidal* if it is a finite linear combination of Schwartz functions $f_{v,i}$ on $\mathfrak{g}(\mathbb{Q}_v)$ such that

$$(4.2.1.1) \qquad \forall \text{index } i \quad \forall \mathrm{P}_v \in \mathcal{F}^{\mathrm{G}_v} - \{\mathrm{G}_v\} \qquad f_{v,i,\mathrm{P}_v} = 0.$$

Denote the space of cuspidal Schwartz functions on $\mathfrak{g}(\mathbb{Q}_v)$ by $\mathcal{S}_{\mathrm{cusp}}(\mathfrak{g}(\mathbb{Q}_v))$.

(4.2.2) Remark A Schwartz function f_v on $\mathfrak{g}(\mathbb{Q}_v)$ is cuspidal if

$$I_{\mathrm{G}}^{\mathrm{G}}(X, f_v) \;=\; 0 \tag{4.2.2.1}$$

whenever X is a regular semisimple element of $\mathfrak{g}(\mathbb{Q}_v)$ which is not $\mathbb{Q}_v$-elliptic.

(4.2.3) Remark The space $\mathcal{S}_{\mathrm{cusp}}(\mathfrak{g}(\mathbb{Q}_v))$ is stable under the Fourier transform on $\mathfrak{g}(\mathbb{Q}_v)$ by (4.1.3.1).

(4.2.4) Proposition (Simple invariant trace formula)

For each sufficiently large finite set S of places of $\mathbb{Q}$, for each $\sim$ equivalence class $\mathfrak{o}$ in $\mathfrak{g}(\mathbb{Q})$, for each $\equiv$ equivalence class X in $(\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o})_{\mathrm{M},S}$, there exists a constant $a^{\mathrm{M}}(S, X)$ such that for each Schwartz function f on $\mathfrak{g}(\mathbb{A})$ which is cuspidal at two distinct places of $\mathbb{Q}$

$$\lim_S \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \; \sum_{X\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{G},S}} a^{\mathrm{G}}(S,X) I_{\mathrm{G}}^{\mathrm{G}}(X, f_S) \tag{4.2.4.1}$$

$$= \lim_S \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \; \sum_{X\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{G},S}} a^{\mathrm{G}}(S,X) I_{\mathrm{G}}^{\mathrm{G}}(X, f\hat{}_S).$$

Proof. This argument follows the proof of Theorem 7.1(b) of [Ar88c].

Let v_1 and v_2 denote two places at which f is cuspidal. Without loss of generality the set S contains both v_1 and v_2, and there exist two subsets S_1 and S_2 which partition S such that S_i contains v_i where the index i is 1 or 2.

With a choice of ξ as in Remark (4.1.11), it follows from (4.1.12.3) that

$$J_{\mathrm{M}}^{\mathrm{G}}(X, f_S) = \sum_{\mathrm{L}_1,\mathrm{L}_2\in\mathcal{L}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2) J_{\mathrm{M}}^{\mathrm{L}_1}(X, f_{S_1,\mathrm{Q}_1}) J_{\mathrm{M}}^{\mathrm{L}_2}(X, f_{S_2,\mathrm{Q}_2}) \tag{4.2.4.2}$$

where

$$(4.2.4.3) \qquad (\mathrm{Q}_1, \mathrm{Q}_2) \;=\; s(\mathrm{L}_1, \mathrm{L}_2).$$

Since f is cuspidal at v_1 and v_2, each summand on the right hand side of (4.2.4.2) vanishes unless the parabolic subgroups Q_1 and Q_2 are both equal to G. Hence the Levi subgroups M_1 and M_2 are both equal to G, which implies that

$$(4.2.4.4) \qquad d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{L}_1, \mathrm{L}_2) \;=\; d_{\mathrm{M}}^{\mathrm{G}}(\mathrm{G}, \mathrm{G})$$

vanishes unless the Levi subgroup M is equal to G by Remark (4.1.10).

Then (4.2.4.1) follows from the refined trace formula (3.4.1.1) and Remark (4.2.3). □

(4.2.5) Remark Each distribution that appears in the summands of the simple invariant trace formula (4.2.4.1) factorizes into local distributions that are invariant under the adjoint action of G on $\mathfrak{g}$.

2.5 The Harish-Chandra transform on the space of characteristic polynomials

In this section an integral transform on the space of characteristic polynomials satisfying a summation formula of Poisson type is constructed.

2.5.1 Preliminaries

(5.1.1) In this chapter G denotes the general linear group $\mathrm{GL}(n, \mathbb{Q})$ for some natural number n, with the standard choice of the minimal Levi subgroup M_0 and the Borel subgroup B to be the subgroup consisting of the diagonal matrices and the upper triangular matrices respectively.

(5.1.2) Let $\mathcal{A}_{\mathrm{G}}$ denote the affine space of characterstic polynomials of $n \times n$ matrices over $\mathbb{Q}$, i.e.

(5.1.2.1) $$\mathcal{A}_{\mathrm{G}} \;=\; \mathrm{gl}(n, \mathbb{Q}) /\!\!/ \mathrm{GL}(n, \mathbb{Q})$$

where $/\!\!/$ denotes the affine quotient and $\mathrm{GL}(n, \mathbb{Q})$ acts on $\mathrm{gl}(n, \mathbb{Q})$ from the right by conjugation. The discriminant function D on $\mathrm{gl}(n, \mathbb{Q})$ descends to a polynomial on $\mathcal{A}_{\mathrm{G}}$, denote by $\mathcal{A}_{\mathrm{G,reg}}$ the open subset where D does not vanish.

Let M be a standard Levi subgroup of G. Denote by $\mathcal{A}_{\mathrm{M}}$ the affine quotient of $\mathfrak{m}$ by the adjoint action of M, then $\mathcal{A}_{\mathrm{M}}$ is an affine space and there exists a partition $(n_1, n_2, \ldots, n_r)$ of n such that

(5.1.2.2) $$\mathcal{A}_{\mathrm{M}} \;=\; \mathcal{A}_{\mathrm{G}_1} \times \mathcal{A}_{\mathrm{G}_2} \times \cdots \times \mathcal{A}_{\mathrm{G}_r}.$$

The embedding of $\mathfrak{m}$ into $\mathfrak{g}$ induces a map

(5.1.2.3) $$\pi_{\mathrm{M}} : \quad \mathcal{A}_{\mathrm{M}} \;\longrightarrow\; \mathcal{A}_{\mathrm{G}}$$

which is finite of degree $|\mathrm{W}_0^{\mathrm{M}}|^{-1}|\mathrm{W}_0^{\mathrm{G}}|$ and étale over $\mathcal{A}_{\mathrm{G,reg}}$.

(5.1.3) Let v be a place of $\mathbb{Q}$. Equip $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ with the measure

(5.1.3.1) $$|D(X_v)|_v^{-1/2}\mathrm{d}X_v$$

where $\mathrm{d}X_v$ denotes the standard translation invariant measure on the affine space $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$. This is equal to the pushforward of the translation invariant measure on $\mathfrak{m}_0(\mathbb{Q}_v)$ along the Chevalley morphism

(5.1.3.2) $$\pi_{\mathrm{M}_0} : \quad \mathfrak{m}_0 = \mathcal{A}_{\mathrm{M}_0} \;\longrightarrow\; \mathfrak{m}_0 /\!\!/ \mathrm{W}_0^{\mathrm{G}} = \mathcal{A}_{\mathrm{G}}.$$

The complement of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ in $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ is a null set.

For each standard Levi subgroup M of G equip $\mathcal{A}_{\mathrm{M}}(\mathbb{Q}_v)$ with the product measure

$$\prod_{i=1}^{r} |D^{\mathrm{G}_i}(X_{i,v})|_v^{-1/2} \mathrm{d}X_{i,v} \tag{5.1.3.3}$$

where $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_r$ are related to M as in (5.1.2.2).

Denote by $\mathcal{A}_{\mathrm{M,reg}}(\mathbb{Q}_v)_{\mathrm{ell}}$ the subset of $\mathcal{A}_{\mathrm{M,reg}}(\mathbb{Q}_v)$ consisting of the images of the $\mathbb{Q}_v$-elliptic elements in $\mathfrak{m}(\mathbb{Q}_v)$. Then

$$\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \quad = \quad \coprod_{\mathrm{M}\in\mathcal{L}} \pi_{\mathrm{M}}(\mathcal{A}_{\mathrm{M,reg}}(\mathbb{Q}_v)_{\mathrm{ell}}). \tag{5.1.3.4}$$

This decomposition is compatible with the measures on $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ and $\mathcal{A}_{\mathrm{M}}(\mathbb{Q}_v)$.

The local measures induce S-local and global measures on $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_S)$ and $\mathcal{A}_{\mathrm{G}}(\mathbb{A})$.

(5.1.4) Definition Let S be a finite set of places of $\mathbb{Q}$, let M be a standard Levi subgroup of G, let X be an element of $\mathfrak{m}(\mathbb{Q}_S)$. Define the *invariant weighted orbital integral* $I_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ to be the distribution on $\mathfrak{g}(\mathbb{Q}_S)$ such that

$$\forall f_S \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)) \tag{5.1.4.1}$$

$$\begin{aligned} &I_{\mathrm{M}}^{\mathrm{G}}(X, f_S) \\ &\quad = J_{\mathrm{M}}^{\mathrm{G}}(X, f_S) - \sum_{\substack{\mathrm{L}\in\mathcal{L}(\mathrm{M}) \\ \mathrm{L}\neq\mathrm{G}}} \Bigg(\sum_{\mathrm{L}'_S\in\mathcal{L}^{\mathrm{L}_S}} |\mathrm{W}_{S,0}^{\mathrm{L}'_S}||\mathrm{W}_{S,0}^{\mathrm{L}_S}|^{-1} \times \\ &\qquad \times \sum_{\mathrm{T}_S\in\mathcal{T}_{\mathrm{ell}}(\mathrm{L}'_S)} |\mathrm{W}(\mathrm{L}'_S, \mathrm{T}_S)|^{-1} \times \\ &\qquad\quad \times \int_{\mathfrak{t}_S(\mathbb{Q}_S)} J_{\mathrm{L}}^{\mathrm{G}}(Y, f_S\check{\ }) \Big(I_{\mathrm{M}}^{\mathrm{L}}(X,\ )\hat{\ }(Y) \Big)\ |D^{\mathrm{L}}(Y)|_S^{1/2} \mathrm{d}Y \Bigg) \end{aligned}$$

where $I_{\mathrm{M}}^{\mathrm{M}}(X,\ )$ denotes the standard orbital integral on $\mathfrak{m}(\mathbb{Q}_S)$ defined by (1.1.6.9).

(5.1.5) Remark The invariant weighted orbital integral $I_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ is well-defined.

- The distribution $I_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ does not depend on the choice of the Fourier transforms on $\mathfrak{g}(\mathbb{Q}_S)$ and its Levi subalgebras $\mathfrak{l}(\mathbb{Q}_S)$ as long as these are compatible. See [Wa95] Lemme VI.5 for the p-adic case, the same argument also works for the real and S-local cases.

- The integral in (5.1.4.1) converges since the distribution $I_{\mathrm{M}}^{\mathrm{L}}(X,\ )\hat{}$ is represented by a smooth function supported on $\mathfrak{g}_{\mathrm{reg.ss}}(\mathbb{Q}_S)$ such that the function

  $$Y \ \mapsto\ \Big(I_{\mathrm{M}}^{\mathrm{L}}(X,\ )\hat{}(Y)\Big)|D^{\mathrm{L}}(Y)|_S^{1/2} \tag{5.1.5.1}$$

  is locally

  $$O\Big(\max\{1, -\log(|D^{\mathrm{G}}(Y)|_S)^N\}\Big) \tag{5.1.5.2}$$

  on $\mathfrak{t}(\mathbb{Q}_S)$ for some natural number N and tempered at infinity, and the function

  $$Y \mapsto J_{\mathrm{L}}^{\mathrm{G}}(Y, f\check{}_S) \tag{5.1.5.3}$$

  is locally

  $$O\Big(\max\{1, -\log(|D^{\mathrm{G}}(Y)|_S)^M\}\Big) \tag{5.1.5.4}$$

  on $\mathfrak{t}(\mathbb{Q}_S)$ for some natural number M and rapidly decreasing at infinity. For the p-adic case see Lemme VI.3(iv) and Corollaire III.6 of [Wa95]. For the real case see Proposition 9 on page 108 of [Va77] and Corollary 7.4 of [Ar76].

(5.1.6) Proposition *Let* M *be a standard Levi subgroup of* G*, let* X *be an element of* $\mathfrak{m}(\mathbb{Q})$*. Then* $I_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ *is an invariant distribution on* $\mathfrak{g}(\mathbb{Q}_S)$*, i.e.*

$$\forall x \in \mathrm{G}(\mathbb{Q}_S)\ \forall f_S \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)) \quad I_{\mathrm{M}}^{\mathrm{G}}(X, f_S \circ \mathrm{ad}(x)) = I_{\mathrm{M}}^{\mathrm{G}}(X, f_S). \tag{5.1.6.1}$$

Proof. For the p-adic case see Proposition VI.1 of [Wa95], the same argument also works for the S-local case. □

(5.1.7) Remark To quote Waldspurger from §Introduction of [Wa95]:

“ *On dispose de deux ensembles de distributions invariantes sur* G*:*

(1) les intégrales orbitales associées aux éléments semi-simples de G*;*

(2) les caractères de représentations tempérées irréductibles de G.

(. . . . . .)

Remplaçons G par son algèbre de Lie $\mathfrak{g}$ *et considérons l'espace de distributions invariantes par l'action adjointe de G. L'ensemble (1) a un analogue évident: les intégrales orbitales associées aux éléments semi-simples de* $\mathfrak{g}$*. Le seul but de cet article est de fournir un support un peu consistant à l'idée, d'ailleurs banale, que l'analogue de (2) est l'ensemble des transformées de Fourier des intégrales orbitales précédents (invariantes).* ”

The duality between the orbital integrals and the Fourier transforms of the invariant weighted orbital integrals is embodied in the local trace formulae on the Lie algebra $\mathfrak{g}$.

(5.1.8) Proposition (Local invariant trace formula of Waldspurger)

Let v *be a place of* $\mathbb{Q}$*, let* f_v *and* g_v *be Schwartz functions on* $\mathfrak{g}_v(\mathbb{Q}_v)$*, then*

$$\sum_{\mathrm{M}_v \in \mathcal{L}^{\mathrm{G}_v}} |\mathrm{W}_{v,0}^{\mathrm{M}_v}||\mathrm{W}_{v,0}^{\mathrm{G}_v}|^{-1} \sum_{\mathrm{T}_v \in \mathcal{T}_{\mathrm{ell}}(\mathrm{M}_v)} |\mathrm{W}(\mathrm{M}_v, \mathrm{T}_v)|^{-1} \times \tag{5.1.8.1}$$

$$\times \int_{\mathfrak{t}_v(\mathbb{Q}_v)} (-1)^{\dim(\mathrm{A}_{\mathrm{M}_v}/\mathrm{A}_{\mathrm{G}_v})} I^{\mathrm{G}_v}_{\mathrm{M}_v}(X_v, f_v\hat{}) I^{\mathrm{G}_v}_{\mathrm{G}_v}(X_v, g_v)\, \mathrm{d}X_v$$

$$= \sum_{\mathrm{M}_v \in \mathcal{L}^{\mathrm{G}_v}} |\mathrm{W}^{\mathrm{M}_v}_{v,0}||\mathrm{W}^{\mathrm{G}_v}_{v,0}|^{-1} \sum_{\mathrm{T}_v \in \mathcal{T}_{\mathrm{ell}}(\mathrm{M}_v)} |\mathrm{W}(\mathrm{M}_v, \mathrm{T}_v)|^{-1} \times$$

$$\times \int_{\mathfrak{t}_v(\mathbb{Q}_v)} (-1)^{\dim(\mathrm{A}_{\mathrm{M}_v}/\mathrm{A}_{\mathrm{G}_v})} I^{\mathrm{G}_v}_{\mathrm{M}_v}(X_v, g_v\hat{}) I^{\mathrm{G}_v}_{\mathrm{G}_v}(X_v, f_v)\, \mathrm{d}X_v.$$

Proof. For the p-adic case see Théorème VII.1 of [Wa95]. The argument only uses combinatorial properties of $(\mathrm{G}_v, \mathrm{M}_v)$-families and standard results from harmonic analysis on a $\mathbb{Q}_v$-vector space, hence works equally well in the real case. □

(5.1.9) Definition Let S be a finite set of places of $\mathbb{Q}$, let M be a Levi subgroup in $\mathcal{L}$, let X be an element of $\mathfrak{m}(\mathbb{Q}_S)$. Define the *vector-valued orbital integral* $\mathcal{I}^{\mathrm{G}}_{\mathrm{M}}(X,\ )$ to be the vector-valued distribution on $\mathfrak{g}(\mathbb{Q}_S)$ taking values in $\bigoplus_{\mathrm{Q} \in \mathcal{F}(\mathrm{M})} \mathbb{C}$ such that

(5.1.9.1) $$\forall f_S \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)$$

$$\mathcal{I}^{\mathrm{G}}_{\mathrm{M}}(X, f_S) = \left((-1)^{\dim(\mathrm{A}_\mathrm{M}/\mathrm{A}_{\mathrm{M}_\mathrm{Q}})} I^{\mathrm{M}_\mathrm{Q}}_{\mathrm{M}}(X, f_{S,\mathrm{Q}}) \right)_{\mathrm{Q} \in \mathcal{F}(\mathrm{M}).}$$

(5.1.10) Lemma (Induction and splitting)

Let S be a finite set of places of $\mathbb{Q}$, let M *be a Levi subgroup in $\mathcal{L}$, let X be an element of $\mathfrak{m}(\mathbb{Q}_S)$, let f_S be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_S)$.*

- *If* L *is a Levi subgroup in $\mathcal{L}$*(M)*, then $\mathcal{I}^{\mathrm{G}}_{\mathrm{L}}(X, f_S)$ is completely determined by $\mathcal{I}^{\mathrm{G}}_{\mathrm{M}}(X, f_S)$.*
- *If S is the set $\{v_1, v_2, \ldots, v_r\}$ for some natural number r and*

(5.1.10.1) $$\forall i = 1, 2, \ldots, r\ \exists f_{v_i} \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_{v_i}))$$

$$f_S = f_{v_1} \otimes f_{v_2} \otimes \cdots \otimes f_{v_r},$$

then $\mathcal{I}_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$ *is completely determined by* $\mathcal{I}_{\mathrm{M}}^{\mathrm{G}}(X_{v_i}, f_{v_i})$ *where* i *ranges among* $1, 2, \ldots, r$.

Proof. This follows from analogues of the descent and splitting identities (4.1.12.2) and (4.1.12.3) for the invariant weighted orbital integrals $I_{\mathrm{M}}^{\mathrm{G}}(X,\ )$, which could be deduced by the same arguments as in §7 of [Ar88b]. □

(5.1.11) Lemma (Local trace formula)

Let v *be a place of* $\mathbb{Q}$, *let* f_v *and* g_v *be Schwartz functions on* $\mathfrak{g}(\mathbb{Q}_v)$, *then*

(5.1.11.1)
$$\sum_{\mathrm{M}\in\mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{\mathrm{T}_v\in\mathcal{T}_{\mathrm{ell}}(\mathrm{M})} |\mathrm{W}(\mathrm{M},\mathrm{T}_v)|^{-1} \times$$
$$\times \int_{\mathfrak{t}_v(\mathbb{Q}_v)} \mathcal{I}_{\mathrm{M}}^{\mathrm{G}}(X_v, f_v\hat{\ })I_{\mathrm{G}}^{\mathrm{G}}(X_v, g_v)\ \mathrm{d}X_v$$
$$= \sum_{\mathrm{M}\in\mathcal{L}} |\mathrm{W}_0^{\mathrm{M}}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \sum_{\mathrm{T}_v\in\mathcal{T}_{\mathrm{ell}}(\mathrm{M})} |\mathrm{W}(\mathrm{M},\mathrm{T}_v)|^{-1} \times$$
$$\times \int_{\mathfrak{t}_v(\mathbb{Q}_v)} \mathcal{I}_{\mathrm{M}}^{\mathrm{G}}(X_v, g_v\hat{\ })I_{\mathrm{G}}^{\mathrm{G}}(X_v, f_v)\ \mathrm{d}X_v.$$

Proof. This follows from (5.1.8.1).

(5.1.12) Lemma (Global trace formula)

Let f *be a Schwartz function on* $\mathfrak{g}(\mathbb{A})$ *which is cuspidal at two distinct places of* $\mathbb{Q}$, *then*

(5.1.12.1)
$$\lim_{S} \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \ \sum_{X\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{G},S}} a^{\mathrm{G}}(S,X)I_{\mathrm{G}}^{\mathrm{G}}(X, f_S)$$
$$= \lim_{S} \sum_{\mathfrak{o}\in\mathfrak{g}(\mathbb{Q})/\sim} \ \sum_{X\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o})_{\mathrm{G},S}} a^{\mathrm{G}}(S,X)\mathcal{I}_{\mathrm{G}}^{\mathrm{G}}(X, f\hat{\ }_S)$$

where the limit is taken over finite sets S *of places of* $\mathbb{Q}$, *and the coefficients* $a^{\mathrm{M}}(S,X)$ *are defined as in* (3.3.11.13).

Proof. This follows from (4.2.4.1). ☐

2.5.2 The local and global Schwartz spaces

(5.2.1) Definition Let v be a place of $\mathbb{Q}$. Let X_v be an element of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$. Define the *maximal orbital integral* $\mathcal{I}^{\mathrm{G}}_{\max}(X_v,\ )$ to be the vector-valued distribution on $\mathfrak{g}(\mathbb{Q}_v)$ such that

$$(5.2.1.1) \qquad \forall f_v \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_v)) \qquad \mathcal{I}^{\mathrm{G}}_{\max}(X_v, f_v) = \mathcal{I}^{\mathrm{G}}_{\mathrm{M}[\widetilde{X}_v\text{-ell}]}(\widetilde{X}_v, f_v)$$

where $\widetilde{X}_v$ is a regular semisimple element in $\mathfrak{g}(\mathbb{Q}_v)$ lifting X_v, and $\mathrm{M}[\widetilde{X}_v\text{-ell}]$ is a standard Levi subgroup of G such that $\mathfrak{m}[\widetilde{X}_v\text{-ell}](\mathbb{Q}_v)$ contains $\widetilde{X}_v$ as a $\mathbb{Q}_v$-elliptic element.

(5.2.2) Definition Let v be a place of $\mathbb{Q}$. Define the *local Schwartz space* $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ to be the space of complex-valued functions on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ such that

$$(5.2.2.1) \qquad \begin{aligned} & \varphi_v \in \mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \\ \Leftrightarrow\ & \exists f_v \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_v))\ \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \\ & \varphi_v(X_v) = I^{\mathrm{G}}_{\mathrm{G}}(X_v, f_v). \end{aligned}$$

Define the *local Schwartz space* $\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ to be the space of vector-valued functions on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ such that

$$(5.2.2.2) \qquad \begin{aligned} & \varphi_v \in \mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \\ \Leftrightarrow\ & \exists f_v \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_v))\ \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \\ & \varphi_v(X_v) = \mathcal{I}^{\mathrm{G}}_{\max}(X_v, f_v). \end{aligned}$$

(5.2.3) Definition Let v be a place of $\mathbb{Q}$. If v is p-adic, denote by Λ_0 the standard lattice

$$(5.2.3.1) \qquad \Lambda_0 = \mathrm{gl}(n,\mathbb{Z})(\mathbb{Z}_p) \;\subset\; \mathrm{gl}(n,\mathbb{Q})(\mathbb{Q}_p)$$

and denote by $\mathbb{I}_{\Lambda_0}$ its characteristic function. If v is archimedean, denote by $\mathcal{E}$ the Gaussian function on $\mathfrak{g}(\mathbb{R})$ which is self-dual with respect to the unitary Fourier transform, i.e.

$$\forall M \in \mathfrak{g}(\mathbb{R}) \qquad \mathcal{E}(M) = e^{-\pi \mathrm{Tr}(M^{\mathrm{T}}M)}$$

where M^{T} denotes the transpose of M and Tr denotes the trace of an $n \times n$ matrix.

Define the *basic function* $\phi_{0,v}$ to be the element of $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ such that

$$(5.2.3.2) \qquad \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$$

$$\phi_{0,v}(X_v) = \begin{cases} I_{\mathrm{G}}^{\mathrm{G}}(X_v, \mathbb{I}_{\Lambda_0}) & \text{if } v \text{ is } p\text{-adic,} \\ \\ I_{\mathrm{G}}^{\mathrm{G}}(X_v, \mathcal{E}) & \text{if } v \text{ is archimedean.} \end{cases}$$

Define the *basic function* $\phi_{1,v}$ to be the element of $\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ such that

$$(5.2.3.3) \qquad \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$$

$$\phi_{1,v}(X_v) = \begin{cases} \mathcal{I}_{\max}^{\mathrm{G}}(X_v, \mathbb{I}_{\Lambda_0}) & \text{if } v \text{ is } p\text{-adic,} \\ \\ \mathcal{I}_{\max}^{\mathrm{G}}(X_v, \mathcal{E}) & \text{if } v \text{ is archimedean.} \end{cases}$$

(5.2.4) Lemma If v is a p-adic place, then a complex-valued function φ_v on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ belongs to $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ if and only if the following conditions are satisfied:

- φ_v is a locally constant function on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$;
- after extending by zero to $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$, φ_v is compactly supported;
- for each singular point Z_v in the complement of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_S)$ in $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$, there exists an open neighborhood U_{φ_v,Z_v} of Z_v in $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ such that

  $$\varphi_v(\ )\big|_{U_{\varphi_v,Z_v}} \in \mathrm{span}\left\{ \Gamma^{\mathrm{G}}_{\mathrm{G}}(\ ,\nu)\big|_{U_{\varphi_v,Z_v}} : \begin{array}{c} \nu \subset \mathfrak{g}_{\widetilde{Z}_v}(\mathbb{Q}_v) \\ \text{nilpotent orbit} \end{array} \right\} \tag{5.2.4.1}$$

  where $\varphi\big|_U$ denotes the restriction of the function φ to the open subset U, the point $\widetilde{Z}_v$ is a regular semisimple element in $\mathfrak{g}(\mathbb{Q}_v)$ lifting Z_v, and $\Gamma^{\mathrm{G}}_{\mathrm{G}}(\ ,\nu)$ denotes the Shalika germ at the nilpotent orbit ν, see §17 of [Ko05].

Proof. Away from the singular locus the function φ_v is a linear combination of the characteristic functions of small compact open sets, and these all have lifts in $\mathcal{S}(\mathfrak{g}(\mathbb{Q}_v))$.

Near the singular locus the theory of Shalika germs implies that the asymptotic behavior of φ_v near the singular point Z_v is necessary. It remains to show that all the Shalika germs appear in $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$, in other words there is no nontrivial linear relation among the possible asymptotes of $I^{\mathrm{G}}_{\mathrm{G}}(X_v, f_v)$ as f_v ranges over $\mathcal{S}(\mathfrak{g}(\mathbb{Q}_v))$. The Shalika germ associated to the nilpotent orbit ν is homogeneous of degree equal to the codimension of ν in the nilpotent locus, hence the only possible linear relations are among the Shalika germs associated to the nilpotent orbits of the same dimension. But such nilpotent orbits are separated, so there are Schwarts functions f_v on $\mathfrak{g}(\mathbb{Q}_v)$ that vanish on all but one of the nilpotent orbits of a given dimension, hence these too are linearly independent. $\square$

(5.2.5) Remark Let v be a p-adic place. It is conjectured by Jacquet for a general reductive Lie algebra and proven by Waldspurger for gl(n) that a Schwartz function f_v on

$\mathfrak{g}(\mathbb{Q}_v)$ has the property that

$$(5.2.5.1) \qquad \exists \lambda \in \mathbb{C} \ \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$$

$$I_{\mathrm{G}}^{\mathrm{G}}(X_v, f_v) = \lambda \cdot \phi_{0,v}(X_v)$$

if and only if $I_{\mathrm{G}}^{\mathrm{G}}(X_v, f_v)$ and $I_{\mathrm{G}}^{\mathrm{G}}(X_v, f_v\hat{\ })$ are both supported on the subset of $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ consisting of the characteristic polynomials with coefficients in $\mathbb{Z}_v$.

(5.2.6) Definition Define the *global Schwartz spaces* $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A}))$ and $\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{A}))$ to be the tensor products of local Schwartz spaces as defined in (5.2.2.1) and (5.2.2.2)

$$(5.2.6.1) \qquad \begin{aligned} \mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A})) &= \bigotimes_v^{\mathrm{res}} \mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \\ \mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{A})) &= \bigotimes_v^{\mathrm{res}} \mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \end{aligned}$$

restricted with respect to the basic functions $\phi_{0,v}$ and $\phi_{1,v}$ as defined in (5.2.3.2) and (5.2.3.3).

2.5.3 The Harish-Chandra transform

(5.3.1) Definition Let v be a place of $\mathbb{Q}$. For an element φ_v in $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$, choose a Schwartz function f_v on $\mathfrak{g}(\mathbb{Q}_v)$ such that

$$(5.3.1.1) \qquad \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \qquad \varphi_v(X_v) = I_{\mathrm{G}}^{\mathrm{G}}(X_v, f_v).$$

Let $f_v\hat{\ }$ be the Fourier transform of f_v, denote by $\mathcal{H}_v(\varphi_v)$ the element in $\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ such that

$$(5.3.1.2) \qquad \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \qquad \mathcal{H}_v(\varphi_v)(X_v) = \mathcal{I}_{\mathrm{max}}^{\mathrm{G}}(X_v, f_v\hat{\ }).$$

Define the *local Harish-Chandra transform* to be the linear operator

$$\mathcal{H}_v : \quad \mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \ \longrightarrow \ \mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \tag{5.3.1.3}$$

defined by

$$\mathcal{H}_v : \quad \varphi_v \ \mapsto \ \mathcal{H}_v(\varphi_v). \tag{5.3.1.4}$$

(5.3.2) Definition Let v be a place of $\mathbb{Q}$. If X_v is an element of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})$, then the orbital integral $\mathcal{I}^{\mathrm{G}}_{\mathrm{max}}(X_v,\ )$ is a vector-valued tempered distribution on $\mathfrak{g}(\mathbb{Q}_v)$. Denote by $\mathcal{I}^{\mathrm{G}}_{\mathrm{max}}(X_v,\ )\hat{}$ its componentwise Fourier transform, which is represented by a conjugation invariant function on $\mathfrak{g}(\mathbb{Q}_v)$, hence descends to a function on $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ denoted by

$$Y_v \ \mapsto \ \Big(\mathcal{I}^{\mathrm{G}}_{\mathrm{max}}(X_v,\ )\hat{}\Big)(Y_v). \tag{5.3.2.1}$$

Define the *local Harish-Chandra kernel function* $\mathcal{K}_v(\ ,\ )$ to be the vector-valued bivariate function on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ such that

$$\begin{gathered}\forall X_v, Y_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)\\ \mathcal{K}_v(X_v, Y_v) = |D(Y_v)|_v^{1/2}\Big(\mathcal{I}^{\mathrm{G}}_{\mathrm{max}}(X_v,\ )\hat{}\Big)(Y_v).\end{gathered} \tag{5.3.2.2}$$

(5.3.3) Remark In the case when v is a p-adic place, each component of the vector-valued kernel function $\mathcal{K}_v$ is denoted by $\hat{i}^{\mathrm{G}}_{\mathrm{M}}(X_v, Y_v)$ for some Levi subgroup M by Waldspurger in [Wa95].

(5.3.4) Lemma *Let v be a place of $\mathbb{Q}$. The local Harish-Chanrda transform $\mathcal{H}_v$ is an integral operator with integral kernel $\mathcal{K}_v$, i.e.*

$$\forall \varphi_v \in \mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \ \forall X_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \tag{5.3.4.1}$$

$$\mathcal{H}_v(\varphi_v)(X_v) = \int_{\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)} \mathcal{K}_v(X_v, Y_v)\varphi_v(Y_v)\ |D(Y_v)|_v^{1/2}\mathrm{d}Y_v.$$

In particular the operator $\mathcal{H}_v$ is well-defined.

Proof. Every X_v in $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ has a δ-sequence in $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$. More precisely there exists a sequence of functions $\delta_{X_v,1}, \delta_{X_v,2}, \delta_{X_v,3}, \ldots$ in $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ such that

$$\lim_{i\to\infty} \delta_{X_v,i} \ = \ \delta_{X_v} \tag{5.3.4.2}$$

as distributions on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$, where δ_{X_v} denotes the Dirac distribution at X_v. For each natural number i choose a Schwartz function $\widetilde{\delta}_{X_v,i}$ on $\mathfrak{g}(\mathbb{Q}_v)$ such that

$$\forall Y_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \qquad \delta_{X_v,i}(Y_v) = I_{\mathrm{G}}^{\mathrm{G}}(Y_v, \widetilde{\delta}_{X_v,i}). \tag{5.3.4.3}$$

Let f_v be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_v)$ such that

$$\forall Y_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \qquad \varphi_v(Y_v) = I_{\mathrm{G}}^{\mathrm{G}}(Y_v, f_v). \tag{5.3.4.4}$$

By the local trace formula (5.1.11.1)

$$\int_{\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)} \mathcal{I}_{\max}^{\mathrm{G}}(Y_v, f_v\hat{\ })I_{\mathrm{G}}^{\mathrm{G}}(Y_v, \widetilde{\delta}_{X_v,i})\ |D(Y_v)|_v^{-1/2}\mathrm{d}Y_v \tag{5.3.4.5}$$

$$= \int_{\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)} \mathcal{I}_{\max}^{\mathrm{G}}(Y_v, \widetilde{\delta}_{X_v,i}\hat{\ })I_{\mathrm{G}}^{\mathrm{G}}(Y_v, f_v)\ |D(Y_v)|_v^{-1/2}\mathrm{d}Y_v,$$

hence

$$\mathcal{H}_v(\varphi_v)(X_v) \tag{5.3.4.6}$$

$$= \int_{\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)} \mathcal{H}_v(\varphi_v)(Y_v)\delta_{X_v}(Y_v)\ |D(Y_v)|_v^{-1/2}\mathrm{d}Y_v$$

$$= \lim_{i\to\infty}\int_{\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)} \mathcal{H}_v(\varphi_v)(Y_v)\delta_{X_v,i}(Y_v)\ |D(Y_v)|_v^{-1/2}\mathrm{d}Y_v$$

$$= \lim_{i\to\infty}\int_{\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)} \mathcal{I}^{\mathrm{G}}_{\max}(Y_v, \widetilde{\delta}_{X_v,i}\hat{\ })\varphi_v(Y_v)\ |D(Y_v)|_v^{-1/2}\mathrm{d}Y_v.$$

Hence $\mathcal{H}_v$ is an integral operator with integral kernel

$$\lim_{i\to\infty}\mathcal{I}^{\mathrm{G}}_{\max}(Y_v, \widetilde{\delta}_{X_v,i}\hat{\ }) \ = \ |D(Y_v)|_v^{1/2}\left(\mathcal{I}^{\mathrm{G}}_{\max}(X_v,\ )\hat{\ }\right)(Y_v) \tag{5.3.4.7}$$

which is independent of the choice of the sequence $\left(\widetilde{\delta}_{X_v,i}\right)_{i=1}^{\infty}$ and equal to the local Harish-Chandra kernel $\mathcal{K}_v(X_v, Y_v)$. □

(5.3.5) Remark The local Harish-Chandra transform preserves the basic functions, i.e.

$$\mathcal{H}_v : \quad \phi_{0,v} \ \mapsto \ \phi_{1,v}. \tag{5.3.5.1}$$

(5.3.6) Definition For a finite set S of places of $\mathbb{Q}$, define the *S-local Harish-Chandra transform* $\mathcal{H}_S$ to be the tensor product

$$\mathcal{H}_S = \bigotimes_{v\in S}\mathcal{H}_v : \quad \bigotimes_{v\in S}\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \ \longrightarrow \ \bigotimes_{v\in S}\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)). \tag{5.3.6.1}$$

By (5.3.5.1) the limit of $\mathcal{H}_S$ as S approaches infinity defines a linear operator

$$\lim_{S}\mathcal{H}_S : \quad \overset{\mathrm{res}}{\bigotimes_{v}}\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)) \ \longrightarrow \ \overset{\mathrm{res}}{\bigotimes_{v}}\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)), \tag{5.3.6.2}$$

which is defined to be the *global Harish-Chandra transform*

$$(5.3.6.3)\qquad \mathcal{H}:\quad \mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A})) \;\longrightarrow\; \mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{A})).$$

2.5.4 The Poisson summation formula

(5.4.1) Definition Let Z be a singular point in the complement of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})$ in $\mathcal{A}_{\mathrm{G}}(\mathbb{Q})$. Denote by $\mathfrak{o}_Z$ the $\sim$ equivalence class in $\mathfrak{g}(\mathbb{Q})$ which is the fiber of Z. Let S be a finite set of places of $\mathbb{Q}$, let φ_S be an element of $\bigotimes_{v\in S}\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$, let f_S be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_S)$ such that

$$(5.4.1.1)\qquad \forall X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_S)\qquad \varphi_S(X)=I_{\mathrm{G}}^{\mathrm{G}}(X,f_S)$$

By (3.1.5.4), if Z' is a $\equiv$ equivalence class in $(\mathfrak{m}(\mathbb{Q})\cap\mathfrak{o}_Z)_{\mathrm{M},S}$, then $I_{\mathrm{G}}^{\mathrm{G}}(Z',f_S)$ is determined by $\varphi_S(X)$ where X ranges over the elements of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_S)$ close to Z'. Denote

$$(5.4.1.2)\qquad \varphi_S(Z') \;=\; I_{\mathrm{G}}^{\mathrm{G}}(Z',f_S).$$

(5.4.2) Definition Let X be an element of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})$, let $\widetilde{X}$ be a regular semisimple element of $\mathfrak{g}(\mathbb{Q})$ lifting X, then by (3.3.12.1) the constant $a^{\mathrm{G}}(S,\widetilde{X})$ as defined in (3.3.11.13) is independent of the finite set of places S and determined by X. Denote

$$(5.4.2.1)\qquad a(X) \;=\; a^{\mathrm{G}}(S,\widetilde{X}).$$

(5.4.3) Definition Let v be a place of $\mathbb{Q}$, let φ_v be an element in $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$, then φ_v is said to be *cuspidal* if

$$(5.4.3.1)\qquad \varphi_v(X_v) \;=\; 0$$

whenever X_v is the image of a point $\widetilde{X}_v$ in $\mathfrak{g}(\mathbb{Q}_v)$ which is regular semisimple and not $\mathbb{Q}_v$-elliptic.

(5.4.4) Proposition (Poisson summation formula)

Let φ be an element of $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A}))$ which is cuspidal at two distinct places of $\mathbb{Q}$, then

$$\begin{aligned}
(5.4.4.1) \qquad & \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\varphi(X) + \\
& + \sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} \lim_{S}\left(\sum_{Z'\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o}_Z)_{\mathrm{G},S}} a^{\mathrm{G}}(S,Z')\varphi_S(Z')\right) \\
= & \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\mathcal{H}(\varphi)(X) + \\
& + \sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} \lim_{S}\left(\sum_{Z'\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o}_Z)_{\mathrm{G},S}} a^{\mathrm{G}}(S,Z')\mathcal{H}_S(\varphi_S)(Z')\right).
\end{aligned}$$

Proof. This follows from the global trace formula (5.1.12.1) and Remark (4.2.2). □

(5.4.5) Remark The Poisson summation formula (5.4.4.1) has the general form

$$\begin{aligned}
(5.4.5.1) \qquad & \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\varphi(X) + \sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} (\ldots.) \\
& = \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\mathcal{H}(\varphi)(X) + \sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} (\ldots.).
\end{aligned}$$

(5.4.6) Corollary *Let v and w be two places of $\mathbb{Q}$. Let φ be an element of $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A}))$ which is cuspidal at two distinct places of $\mathbb{Q}$ such that φ_v vanishes on a neighborhood of the complement of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ in $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ and $\mathcal{H}_w(\varphi_w)$ vanishes on a neighborhood of the* *complement of* $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_w)$ *in* $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_w)$*. Then*

$$\sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\varphi(X) \;=\; \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\mathcal{H}(\varphi)(X). \tag{5.4.6.1}$$

Proof. This follows from (5.4.5.1). □

(5.4.7) Corollary (Poisson summation formula for $\mathcal{H}_0$)

There exists an endomorphism $\mathcal{H}_0$ *of* $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A}))$*, defined similarly as* $\mathcal{H}$ *as a tensor product of local endomorphisms* $\mathcal{H}_{0,v}$*, such that for each Schwartz function* φ *on* $\mathcal{A}_{\mathrm{G}}(\mathbb{A})$ *which is cuspidal at two distinct places of* $\mathbb{Q}$

$$\begin{aligned}(5.4.7.1)\qquad & \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\varphi(X)+\\ &\quad+\sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} \lim_{S}\left(\sum_{Z'\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o}_Z)_{\mathrm{G},S}} a^{\mathrm{G}}(S,Z')\varphi_S(Z')\right)\\ =\;& \sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\mathcal{H}_0(\varphi)(X)+\\ &\quad+\sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} \lim_{S}\left(\sum_{Z'\in(\mathfrak{g}(\mathbb{Q})\cap\mathfrak{o}_Z)_{\mathrm{G},S}} a^{\mathrm{G}}(S,Z')\mathcal{H}_{0,S}(\varphi_S)(Z')\right).\end{aligned}$$

Proof. The local endomorphisms $\mathcal{H}_{0,v}$ are defined by sending the orbital integral $I_{\mathrm{G}}^{\mathrm{G}}(X,f_v)$ to the orbital integral of the Fourier transform $I_{\mathrm{G}}^{\mathrm{G}}(X,f_v\hat{\ })$ for a Schwartz function f_v on $\mathfrak{g}(\mathbb{Q}_v)$. If v is a p-adic place, the endomorphism $\mathcal{H}_{0,v}$ is well-defined due to the density of regular semisimple orbital integrals. See §27 of [Ko05]. If v is archimedean, then $\mathcal{H}_{0,v}$ is well-defined due to the second last identity on page 104 of [Va77]. The local basic function $\phi_{0,v}$ is preserved by $\mathcal{H}_{0,v}$, hence the global endomorphism $\mathcal{H}_0$ is also well-defined.

The Poisson summation formula (5.4.7.1)follows from (5.4.4.1) since the cuspidality condition on φ implies that only those X for which the orbital integrals $I_{\mathrm{G}}^{\mathrm{G}}(X,f_v)$ and

$\mathcal{I}^{\mathrm{G}}_{\mathrm{max}}(X, f_v)$ are equal, where f_v denotes a local Schwartz function on $\mathfrak{g}(\mathbb{Q}_v)$ whose orbital integral is equal to φ_v, contribute to the identity. $\square$

(5.4.8) Remark The endomorphism $\mathcal{H}_0$ is the Harish-Chandra transform originally considered by Jacquet. If v is a p-adic place, then it is conjectured by Jacquet for a general reductive Lie algebra and proven by Waldspurger for gl(n) that a Schwartz function φ_v on $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ is proportional to $\phi_{0,v}$ if and only if φ_v and $\mathcal{H}_{0,v}(\varphi_v)$ are both supported on the subset of $\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v)$ consisting of the characteristic polynomials with coefficients in $\mathbb{Z}_v$.

(5.4.9) Corollary *Let v be a place of $\mathbb{Q}$. The local Harish-Chandra transform $\mathcal{H}_v$ is a bijection from $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ onto $\mathcal{S}_1(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$.*

Proof. The following argument is suggested by Sakellaridis.

The local Harish-Chandra transform $\mathcal{H}_v$ is surjective by definition. For injectivity assume for contradiction that there exists an element φ_v in $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_v))$ and a point X_v in $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$ such that

$$\forall Y_v \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v) \qquad \mathcal{H}_v(\varphi_v)(Y_v) = 0 \tag{5.4.9.1}$$

and

$$\varphi_v(X_v) \neq 0. \tag{5.4.9.2}$$

Since φ_v is smooth on $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_v)$, without loss of generality X_v lies in the dense subset $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})$. By parabolic descent along a suitable parabolic subgroup, the argument is reduced to the special case that X_v is the image of a $\mathbb{Q}_v$-elliptic element of $\mathfrak{g}(\mathbb{Q}_v)$.

Let φ be en element of $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{A}))$ which is cuspidal at two other places and whose local component at v is equal to φ_v, then by the Poisson summation formula (5.4.4.1)

$$\sum_{X\in\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})} a(X)\varphi(X)+\sum_{Z\in\mathcal{A}_{\mathrm{G}}(\mathbb{Q})-\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})}\begin{pmatrix}\text{contribution from}\\ \text{the singular locus}\end{pmatrix} = 0 \tag{5.4.9.3}$$

since $\mathcal{H}(\varphi)$ vanishes identically at the place v. There are two cases:

- If v is p-adic, then there exists an integer N such that X_v lies in the image of $\mathfrak{g}_{\text{reg.ss}}(N^{-1}\mathbb{Z})$ in $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})$ under the natural projection.

  At a finite place w distinct from v, denote by $\Lambda_{N,w}$ the lattice

  $$\Lambda_{N,w} \quad = \quad \mathfrak{g}(N^{-1}\mathbb{Z}_w) \ \subset \ \mathfrak{g}(\mathbb{Q}_w). \tag{5.4.9.4}$$

  Let φ_w be the element of $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_w))$ such that

  $$\forall Y_w \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_w) \qquad \varphi_w(Y_w) = I_{\mathrm{G}}^{\mathrm{G}}(Y_w, \mathbb{I}_{\Lambda_{N,w}}) \tag{5.4.9.5}$$

  where $\mathbb{I}_{\Lambda_{N,w}}$ denotes the characteristic function of $\Lambda_{N,w}$.

  At infinity, the set of rational points where $\bigotimes_{w<\infty}\varphi_w$ is nonzero is contained in the discrete subset

  $$\pi_\infty(\mathfrak{g}_{\text{reg.ss}}(N^{-1}\mathbb{Z})) \ \subset \ \mathcal{A}_{\mathrm{G,reg}}(\mathbb{R}) \tag{5.4.9.6}$$

  where π_∞ denotes the natural projection from $\mathfrak{g}(\mathbb{R})$ to $\mathcal{A}_{\mathrm{G}}(\mathbb{R})$. Choose φ_∞ to be a bump function supported away from the complement of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{R})$ in $\mathcal{A}_{\mathrm{G}}(\mathbb{R})$ such that X_v is the only point contained in

  $$\operatorname{supp}(\varphi_\infty) \ \cap \ \pi_\infty(\mathfrak{g}_{\text{reg.ss}}(N^{-1}\mathbb{Z})). \tag{5.4.9.7}$$

Choose φ to be $\bigotimes_w \varphi_w$, then the left hand side of (5.4.9.3) is equal to

$$a(X_v)\varphi(X_v) \quad = \quad \mathrm{Vol}\left(\mathrm{T}_{X_v}(\mathbb{Q})\backslash\mathrm{T}_{X_v}(\mathbb{A})^1\right)\lim_S\left(\prod_{w\in S}\varphi_w(X_v)\right) \tag{5.4.9.8}$$

for some torus T_{X_v}, which is nonzero.

- If v is archimedean, then choose a regular semisimple element $\widetilde{X}_\infty$ of $\mathfrak{g}(\mathbb{Q})$ lifting X_∞ and choose a Schwartz function f_∞ on $\mathfrak{g}(\mathbb{A})$ such that

$$\forall Y_\infty \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{R}) \qquad \varphi_\infty(Y_\infty) = I_{\mathrm{G}}^{\mathrm{G}}(Y_\infty, f_\infty). \tag{5.4.9.9}$$

If N is an integer, denote by $\Lambda_{N,\infty}$ the lattice

$$\Lambda_{N,\infty} \quad = \quad \mathfrak{g}(N\mathbb{Z}) \quad \subset \quad \mathfrak{g}(\mathbb{R}). \tag{5.4.9.10}$$

Since f_∞ is a Schwartz function on $\mathfrak{g}(\mathbb{R})$, the quantity $\varphi_\infty(Y_\infty)$ is rapidly decreasing as Y_∞ approaches infinity in $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{R})$ in such a way that $|D(Y_\infty)|$ is uniformly bounded below, hence for every positive real numbers ϵ and r there exists a natural number N_ϵ such that

$$\forall N \geq N_\epsilon \qquad \sum_{\substack{Y\in\pi_\infty(\widetilde{X}_\infty+\Lambda_{N,\infty}) \\ Y\neq X_\infty,\ |D(Y)|\geq 1}} |\varphi_\infty(Y)| < \epsilon N^{-r} \tag{5.4.9.11}$$

where π_∞ denotes the natural projection from $\mathfrak{g}(\mathbb{R})$ to $\mathcal{A}_{\mathrm{G}}(\mathbb{R})$. Choose

$$\epsilon \quad = \quad \frac{1}{2(n!)^{n\cdot n!}}|a(X_\infty)\varphi_\infty(X_\infty)| \tag{5.4.9.12}$$

where n is the rank of G and choose r to be $2\dim(\mathfrak{g})$. Let N be a natural number greater than N_ϵ such that

$$(5.4.9.13) \qquad \forall Y \in \pi_\infty(\widetilde{X}_\infty + \Lambda_{N,\infty}) \qquad D(Y) \neq 0.$$

At a finite place w, denote by $\Lambda_{N,w}$ the lattice

$$(5.4.9.14) \qquad \Lambda_{N,w} \quad = \quad \mathfrak{g}(N\mathbb{Z}_w) \quad \subset \quad \mathfrak{g}(\mathbb{Q}_w).$$

Let φ_w be the element of $\mathcal{S}_0(\mathcal{A}_{\mathrm{G}}(\mathbb{Q}_w))$ such that

$$(5.4.9.15) \qquad \forall Y_w \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}_w) \qquad \varphi_w(Y_w) = I_{\mathrm{G}}^{\mathrm{G}}(Y_w, \mathbb{I}_{\widetilde{X}_\infty + \Lambda_{N,w}})$$

where $\mathbb{I}_{\widetilde{X}_\infty + \Lambda_{N,w}}$ denotes the characteristic function of the translation of $\Lambda_{N,w}$ by $\widetilde{X}_\infty$. Choose φ to be $\bigotimes_w \varphi_w$, then the left hand side of (5.4.9.3) is equal to

$$\begin{aligned}(5.4.9.16) \qquad & a(X_\infty) \prod_{w<\infty} \varphi_w(X_\infty) \cdot \varphi_\infty(X_\infty) + \\ & + \sum_{\substack{Y \in \pi_\infty(\widetilde{X}_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty,\ |D(Y)| \geq 1}} a(Y) \prod_{w<\infty} \varphi_w(Y) \cdot \varphi_\infty(Y) \\ = \; & C \cdot \Bigg(a(X_\infty)\varphi_\infty(X_\infty) + \\ & + \sum_{\substack{Y \in \pi_\infty(\widetilde{X}_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty,\ |D(Y)| \geq 1}} \left(a(Y) \frac{\prod_{w<\infty} \varphi_w(Y)}{\prod_{w<\infty} \varphi_w(X_\infty)} \right) \varphi_\infty(Y) \Bigg)\end{aligned}$$

where C is a nonzero constant. Let Y be an element of $\mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q})$. By definition

$$(5.4.9.17) \qquad a(Y) \quad = \quad \pm|\pi_0(\mathrm{M}_Y)|^{-1}\mathrm{Vol}(\mathrm{T}_Y(\mathbb{Q})\backslash\mathrm{T}_Y(\mathbb{A})^1)$$

where M is a Levi subgroup in $\mathcal{L}$ and T_Y is a maximal torus in G. By the Main theorem in §5 of [On63],

$$\text{(5.4.9.18)} \qquad \mathrm{Vol}(\mathrm{T}_Y(\mathbb{Q})\backslash\mathrm{T}_Y(\mathbb{A})^1) \;\leq\; |H^1_{\mathrm{Gal}}(\mathbb{Q}, \widehat{\mathrm{T}}_Y)|$$

where H^1_{Gal} denotes the first Galois cohomology group and $\widehat{\mathrm{T}}_Y$ denotes the Galois module of algebraic characters of T_Y. Let F be the splitting field of T_Y, denote by Γ the Galois group of F over $\mathbb{Q}$. Then by the inflation-restriction exact sequence

$$\text{(5.4.9.19)} \qquad \begin{tikzcd} 0 \arrow[r] & H^1(\Gamma, \widehat{\mathrm{T}}_Y) \arrow[r] & H^1_{\mathrm{Gal}}(\mathbb{Q}, \widehat{\mathrm{T}}_Y) \arrow[r] & H^1_{\mathrm{Gal}}(F, \widehat{\mathrm{T}}_Y)^\Gamma \arrow[d, equal] \\ & & & \mathrm{Hom}(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}^n)^\Gamma \end{tikzcd}$$

and the compactness of $\mathrm{Gal}(\overline{F}/F)$ which implies that the only continuous homomorphism from $\mathrm{Gal}(\overline{F}/F)$ to $\mathbb{Z}^n$ is the trivial homomorphism,

$$\text{(5.4.9.20)} \qquad |H^1(\Gamma, \widehat{\mathrm{T}}_Y)| \;=\; |H^1_{\mathrm{Gal}}(\mathbb{Q}, \widehat{\mathrm{T}}_Y)|.$$

The group $H^1(\Gamma, \widehat{\mathrm{T}}_Y)$ is annihilated by the order of Γ which is bounded by the factorial of n since the $\widehat{\mathrm{T}}_Y$ splits over the splitting field of the characteristic polynomial represented by Y which is of degree n. By the bar resolution, the group $H^1(\Gamma, \widehat{\mathrm{T}}_Y)$ is a subquotient of $\bigoplus_{g\in\Gamma} \mathbb{Z}^n$ which is generated by at most $n \cdot n!$ elements, hence by (5.4.9.17), (5.4.9.18) and (5.4.9.20)

$$\text{(5.4.9.21)} \qquad \forall Y \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}) \qquad |a(Y)| \leq (n!)^{n \cdot n!}.$$

As N approaches infinity, either $\prod_{w<\infty} \varphi_w(Y)$ vanishes or

$$\text{(5.4.9.22)} \qquad N^{-\dim(\mathfrak{g})} \leq \prod_{w<\infty} \varphi_w(Y) \leq 1,$$

and $\prod_{w<\infty} \varphi_w(X_\infty)$ is nonzero. Hence without loss of generality N is large enough so that

$$\forall Y \in \mathcal{A}_{\mathrm{G,reg}}(\mathbb{Q}) \qquad \frac{\prod_{w<\infty} \varphi_w(Y)}{\prod_{w<\infty} \varphi_w(X_\infty)} \leq N^r. \tag{5.4.9.23}$$

By (5.4.9.21) and (5.4.9.23),

$$\begin{aligned}
(5.4.9.24) \qquad & \sum_{\substack{Y \in \pi_\infty(\widetilde{X}_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty,\ |D(Y)| \geq 1}} \left| a(Y) \frac{\prod_{w<\infty} \varphi_w(Y)}{\prod_{w<\infty} \varphi_w(X_\infty)} \varphi_\infty(Y) \right| \\
& \leq \ N^r (n!)^{n \cdot n!} \sum_{\substack{Y \in \pi_\infty(\widetilde{X}_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty,\ |D(Y)| \geq 1}} |\varphi_\infty(Y)| \\
(5.4.9.25) \qquad & < \ N^r (n!)^{n \cdot n!} \epsilon N^{-r} \\
(5.4.9.26) \qquad & = \ (n!)^{n \cdot n!} \frac{1}{2(n!)^{n \cdot n!}} |a(X_\infty) \varphi_\infty(X_\infty)| \\
& = \ \frac{1}{2} |a(X_\infty) \varphi_\infty(X_\infty)|
\end{aligned}$$

where the inequality (5.4.9.25) follows from (5.4.9.11) and the equality (5.4.9.26) follows from (5.4.9.12). Hence the right hand side of (5.4.9.16) is nonzero.

In either case the left hand side of (5.4.9.3) is nonzero, which is a contradiction. □

(5.4.10) Remark Injectivity of $\mathcal{H}_v$ is analogous to the classical result of Harish-Chandra on density of regular semisimple orbital integrals which states that for a p-adic reductive Lie algebra $\mathfrak{g}$, if f is a Schwartz function on $\mathfrak{g}$ such that all the regular semisimple orbital integrals of f vanish, then $\mathcal{D}(f)$ vanishes for every invariant distribution $\mathcal{D}$ on $\mathfrak{g}$. See §27 of [Ko05].

2.A Appendix: Scissors congruence and orbital integrals

In this appendix weighted orbital integrals on $\mathfrak{g}$ are interpreted as scissors congruence classes of polyhedra. Similar results also hold for invariant weighted orbital integrals.

(A.1) Definition Let $\mathbb{E}^n$ denote the n-dimensional Euclidean space. A polytope in $\mathbb{E}^n$ that is closed with nonempty interior is said to be *proper*. Let P and Q be proper convex polytopes in $\mathbb{E}^n$. Then P and Q are said to be *translational scissors congruent* if there exist convex polytopes $P_1, P_2, \ldots, P_l$ and $Q_1, Q_2, \ldots, Q_l$ in $\mathbb{E}^n$ such that

$$P = \bigcup_{i=1}^{l} P_i \qquad Q = \bigcup_{i=1}^{l} Q_i \tag{A.1.1}$$

and P_i is a translation of Q_i for each index i among $1, 2, \ldots, l$. Define the *scissors group* of $\mathbb{E}^n$, denoted by $\mathbb{S}(\mathbb{E}^n)$, to be the quotient of the free abelian group generated by the proper convex polytopes in $\mathbb{E}^n$ modulo translational scissors congruence.

(A.2) Definition A flag Φ of linear subspaces in $\mathbb{E}^n$ is said to be *strict of length l* if

$$\Phi \;=\; V_0 \supset V_1 \supset \cdots \supset V_l \tag{A.2.1}$$

where V_i has codimension i. Let Φ be a strict flag of length l. A *rigging* $\mathbf{r}$ of Φ is a collection

$$\mathbf{r} \;=\; \Big\{ r_1, r_2, \ldots, r_l \Big\} \tag{A.2.2}$$

where r_i is a real linear functional on V_{i-1} with kernel V_i for each i among $1, 2, \ldots, l$. Two riggings $\mathbf{r}$ and $\mathbf{r}'$ are *equivalent* if r_i and r_i' are positive multiples of each other for each i among $1, 2, \ldots, l$. Denote by $\mathcal{R}ig(\Phi)$ the collection of equivalence classes of riggings of Φ. A *rigged flag* $\Phi^{\mathbf{r}}$ is defined to be a strict flag Φ together with a choice of an element $\mathbf{r}$ in $\mathcal{R}ig(\Phi)$.

An *orientation* of $\mathbb{E}^n$ is an ordered basis of $\mathbb{E}^n$ defined upto a linear transformation with positive determinant. The product of an orientation of $\mathbb{E}^n$ with a translation invariant measure on $\mathbb{E}^n$ is equal to a volume form on $\mathbb{E}^n$. Fix an ordered basis $\mathcal{B}$

$$\mathcal{B} = \Big(b_1, b_2, \ldots, b_n\Big) \tag{A.2.3}$$

of $\mathbb{E}^n$ such that

$$\mathcal{B}' = \Big(b_{l+1}, b_{l+2}, \ldots, b_n\Big) \tag{A.2.4}$$

is an ordered basis of V_l. Let $\mathbf{r}$ be a rigging of Φ, choose vectors

$$c_1 \in V_0, \quad c_2 \in V_1, \quad \ldots, \quad c_l \in V_{l-1} \tag{A.2.5}$$

such that

$$\forall i = 1, 2, \ldots, l \qquad r_i(c_i) > 0. \tag{A.2.6}$$

Denote by $\mathcal{B}^{\mathbf{r}}$ the ordered basis

$$\mathcal{B}^{\mathbf{r}} = \Big(c_1, c_2, \ldots, c_l, b_{l+1}, b_{l+2}, \ldots, b_n\Big) \tag{A.2.7}$$

of $\mathbb{E}^n$. Define the *sign* of $\mathbf{r}$ by

$$\operatorname{sign}(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathcal{B} \text{ and } \mathcal{B}^{\mathbf{r}} \text{ define} \\ & \text{the same orientation of } \mathbb{E}^n, \\ & \\ -1 & \text{if } \mathcal{B} \text{ and } \mathcal{B}^{\mathbf{r}} \text{ define} \\ & \text{opposite orientations of } \mathbb{E}^n. \end{cases} \tag{A.2.8}$$

Let $\Phi^{\mathbf{r}}$ be a rigged flag of length l in $\mathbb{E}^n$. Let P be a proper convex polytope in $\mathbb{E}^n$. Then the $\Phi^{\mathbf{r}}$-*boundary* $\partial_{\Phi^{\mathbf{r}}}$ of P is defined by

$$\text{(A.2.9)} \qquad \partial_{\Phi^{\mathbf{r}}} P \;=\; r_l^{\min}\bigl(r_{l-1}^{\min}\bigl(\ldots\bigl(r_2^{\min}\bigl(r_1^{\min}(P)\bigr)\ldots\bigr)$$

where for each subset S of V_{i-1}

$$\text{(A.2.10)} \qquad r_i^{\min}(S) \;=\; \begin{cases} v_i + r_i^{-1}\Bigl(\min\limits_{s\in S} r_i(s)\Bigr) & \text{if } v_i \text{ is a vector in } V_{i-1} \\ & \text{such that} \\ & \quad v_i + r_i^{-1}\Bigl(\min\limits_{s\in S} r_i(s)\Bigr) \\ & \text{is a subset of } V_i \text{ with} \\ & \text{nonempty interior,} \\ & \\ \emptyset & \text{if no such } v_i \text{ exists,} \end{cases}$$

which is a subset of V_i defined upto translation. Fix a translation invariant measure on the Euclidean space V_l. This determines the volume of the convex polytope $\partial_{\Phi^{\mathbf{r}}} P$. Then define the *Hadwiger invariant* Had_Φ of P with respect to Φ by

$$\text{(A.2.11)} \qquad \mathrm{Had}_\Phi(P) \;=\; \sum_{\mathbf{r}\in\mathcal{R}ig(\Phi)} \mathrm{sign}(\mathbf{r})\mathrm{Vol}(\partial_{\Phi^{\mathbf{r}}} P).$$

(A.3) Remark Each Hadwiger invariant defines a real-valued additive function on $\mathbb{S}(\mathbb{E}^n)$.

(A.4) Lemma *Let P and Q be proper convex polytopes in $\mathbb{E}^n$. Then P are Q are translational scissors congruent if and only if for every l among $0, 1, 2, \ldots, n$, for each strict flag Φ of length l,*

$$\text{(A.4.1)} \qquad \mathrm{Had}_\Phi(P) \;=\; \mathrm{Had}_\Phi(Q).$$

Proof. See Corollary 2 in §4 of [Mo93a]. □

(A.5) Lemma *Let (H_Φ) be a collection of real numbers indexed by the set of all strict flags Φ in $\mathbb{E}^n$ which vanishes for all but finitely many Φ. Then there exists an element $[P]$ in the scissors group $\mathbb{S}(\mathbb{E}^n)$ such that*

$$\forall \text{ strict flag } \Phi \text{ in } \mathbb{E}^n \qquad H_\Phi = \mathrm{Had}_\Phi([P]) \tag{A.5.1}$$

if and only if for every l among $0, 1, 2, \ldots, n$, for each strict flag

$$\Phi \;=\; V_0 \supset V_1 \supset \cdots \supset V_l \tag{A.5.2}$$

of length l, for every i among $1, 2, \ldots, l-1$,

$$\sum_{\Phi' \in \mathcal{F}_{\mathrm{s}}(\Phi, i)} H_{\Phi'} \;=\; 0 \tag{A.5.3}$$

where $\mathcal{F}_{\mathrm{s}}(\Phi, i)$ is the set defined by

$$\mathcal{F}_{\mathrm{s}}(\Phi, i) \tag{A.5.4}$$

$$= \;\left\{\Phi' \text{ strict flag in } \mathbb{E}^n : \; \Phi' = V_0 \supset \cdots \supset V_{i-1} \supset U_i \supset V_{i+1} \supset \cdots \supset V_l\right\},$$

and

$$\sum_{\Phi' \in \mathcal{F}_{\mathrm{s}}(\Phi, l)} H_{\Phi'} \wedge \mathbf{d}U_l \;=\; 0 \tag{A.5.5}$$

where

$$\Phi' \;=\; V_0 \supset V_1 \supset \cdots \supset V_{l-1} \supset U_l \tag{A.5.6}$$

and $\mathbf{d}U_l$ *is the element of* $\bigwedge^{n-l}\mathbb{E}^{n,*}$ *defined as the product of the fixed orientation and translation invariant measure on* U_l*.*

Proof. See Corollary 3 in §4 of [Mo93a]. $\square$

(A.6) Remark The scissors group $\mathbb{S}(\mathbb{E}^n)$ has the structure of a real vector space.

(A.7) Definition Let M be a Levi subgroup in $\mathcal{L}$. Let $\mathcal{Y}_{\mathrm{M}}$ and $\mathcal{Z}_{\mathrm{M}}$ be positive (G,M)-orthogonal sets. Then $\mathcal{Y}_{\mathrm{M}}$ and $\mathcal{Z}_{\mathrm{M}}$ are said to be *scissors congruent as* (G,M)-*orthogonal sets* if the convex hull of $\mathcal{Y}_{\mathrm{M}}$ in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$ is translational scissors congruent to the convex hull of $\mathcal{Z}_{\mathrm{M}}$ in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$. Define the *scissors group* of $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$, denoted by $\mathbb{S}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})$, to be the quotient of the free abelian group generated by the positive (G,M)-orthogonal sets modulo translational scissors congruence in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$.

(A.8) Remark The Hadwiger invariants of $\mathcal{Y}_{\mathrm{M}}$ are supported on the strict flags Φ in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$ of the form

$$\Phi \;=\; \mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^0} \supset \mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^1} \supset \cdots \supset \mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^l} \tag{A.8.1}$$

where

$$\mathrm{L}^0 \supset \mathrm{L}^1 \supset \cdots \supset \mathrm{L}^l \tag{A.8.2}$$

is a nested chain of Levi subgroups in $\mathcal{L}(\mathrm{M})$. Denote the collection of such Φ by $\mathcal{F}_{\mathrm{s}}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})$.

A rigging $\mathbf{r}$ of Φ is equivalent to a nested chain of parabolic subgroups

$$\mathbf{r} \;=\; \mathrm{Q}^0 \supset \mathrm{Q}^1 \supset \cdots \supset \mathrm{Q}^l \tag{A.8.3}$$

such that Q^i is a parabolic subgroup in $\mathcal{P}(\mathrm{L}^i)$ for each i among $0, 1, 2, \ldots, l$.

Fix a vector ξ in general position in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$. Then ξ defines a total order on $\Delta_{\mathrm{L}}^{\mathrm{G}}$ for each Levi subgroup L in $\mathcal{L}(\mathrm{M})$, which induces a consistent choice of signs for all rigged flags $\Phi^{\mathbf{r}}$ with Φ in $\mathcal{F}_{\mathrm{s}}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})$. More precisely choose an element s of the Weyl group that stablizes M such that the parabolic subgroup $s\mathrm{Q}^{l}$ is standard. The nested chain

$$s\mathrm{Q}^{0} \supset s\mathrm{Q}^{1} \supset \cdots \supset s\mathrm{Q}^{l} \tag{A.8.4}$$

determines a sequence of positive roots α^{i} where i ranges among $1, 2, \ldots, l$. Let σ be the permutation on l letters such that $\alpha^{\sigma(i)}$ is strictly increasing with respect to the total order determined by ξ. Then the sign of the rigging $\mathbf{r}$ is equal to

$$\operatorname{sign}(\mathbf{r}) \;=\; \operatorname{sign}\big(\operatorname{Det}(s)\big)\operatorname{sign}\big(\sigma\big). \tag{A.8.5}$$

Each subspace $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}}$ of $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$ is equipped with the translation invariant measure determined by the coweight lattice. This choice of orientations and measures determines the numerical values of the Hadwiger invariants $\mathrm{Had}_{\Phi}(\mathcal{Y}_{\mathrm{M}})$.

(A.9) Definition Let M be a Levi subgroup in $\mathcal{L}$, let X be an element of $\mathfrak{m}(\mathbb{Q}_{S})$. Define the *scissors-congruence-valued orbital integral*, or *orbital integrohedron* $\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ to be the vector-valued distribution on $\mathfrak{g}(\mathbb{Q}_{S})$ taking values in $\bigoplus_{\Phi\in\mathcal{F}_{\mathrm{s}}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})}\mathbb{C}$ such that

$$\forall f_{S} \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_{S})) \tag{A.9.1}$$

$$\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X, f_{S}) = \left(\sum_{\mathbf{r}\in\mathcal{R}ig(\Phi)} \operatorname{sign}(\mathbf{r}) J_{\mathrm{M}}^{\mathrm{L}^{l}}(X, f_{S,\mathrm{Q}^{l}})\right)_{\Phi\in\mathcal{F}_{\mathrm{s}}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})}$$

where Φ, $\mathbf{r}$, L^{l} and Q^{l} are related as in (A.8.1), (A.8.2) and (A.8.3).

For each Levi subgroup M_{S} in $\mathcal{L}^{\mathrm{G}_{S}}$ define $\mathbb{J}_{\mathrm{M}_{S}}^{\mathrm{G}_{S}}(X,\ )$ by the analogous formula.

(A.10) Lemma *Let* M *be a Levi subgroup in* $\mathcal{L}$*, let* X *be an element of* $\mathfrak{m}(\mathbb{Q}_S)$*, let* f_S *be a real-valued Schwartz function on* $\mathfrak{g}(\mathbb{Q}_S)$*. Then* $\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$ *defines a unique element of* $\mathbb{S}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})$.

Proof. The Schwartz function f_S is real-valued, so $\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$ is a collection of real numbers indexed by the strict flags Φ in $\mathcal{F}_{\mathrm{s}}(\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}})$. It suffices to verify (A.5.3) and (A.5.5).

The left hand side of (A.5.3) is equal to

(A.10.1)
$$\sum_{\Phi' \in \mathcal{F}_{\mathrm{s}}(\Phi, i)} \sum_{\mathbf{r}' \in \mathcal{R}ig(\Phi')} \mathrm{sign}(\mathbf{r}') J_{\mathrm{M}}^{\mathrm{L}^{l\prime}}(X, f_{S,\mathrm{Q}^{l\prime}})$$

$$= \sum_{\substack{\mathrm{Q}^{0\prime} \supset \mathrm{Q}^{1\prime} \supset \cdots \supset \mathrm{Q}^{l\prime} \\ \forall j=0,1,2,\ldots,l \ \mathrm{Q}^{j\prime} \in \mathcal{P}(\mathrm{L}^{j\prime}) \\ \forall j=0,1,2,\ldots,l \ j \neq i \Rightarrow \mathrm{L}^{j\prime} = \mathrm{L}^{j}}} \mathrm{sign}(\mathbf{r}') J_{\mathrm{M}}^{\mathrm{L}^{l\prime}}(X, f_{S,\mathrm{Q}^{l\prime}})$$

where Φ, L^j and Φ', $\mathbf{r}'$, $\mathrm{L}^{j\prime}$, $\mathrm{Q}^{j\prime}$ are related as in (A.8.1), (A.8.2) and (A.8.3). The summands of the right hand side of (A.10.1) with a fixed minimal term $\mathrm{Q}^{l\prime}$ are in (1,1) correspondence with the sequences of roots (α^j) that are positive with respect to $\mathrm{Q}^{l\prime}$ such that

(A.10.2)
$$\alpha^1, \alpha^2, \ldots, \alpha^{i-1}, \alpha^{i+2}, \alpha^{i+3}, \ldots, \alpha^l$$

are determined by Φ. Hence the summand

(A.10.3)
$$J_{\mathrm{M}}^{\mathrm{L}^{l\prime}}(X, f_{S,\mathrm{Q}^{l\prime}})$$

appears twice on the right hand side of (A.10.1) with opposite signs, so (A.10.1) vanishes. Hence $\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$ satisfies (A.5.3).

The left hand side of (A.5.5) is equal to

(A.10.4)
$$\sum_{\Phi' \in \mathcal{F}_{\mathrm{s}}(\Phi, l)} \sum_{\mathbf{r}' \in \mathcal{R}ig(\Phi')} \mathrm{sign}(\mathbf{r}') J_{\mathrm{M}}^{\mathrm{L}^{l\prime}}(X, f_{S,\mathrm{Q}^{l\prime}}) \wedge \mathbf{d}\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l\prime}}$$

$$= \sum_{\substack{\mathrm{Q}^0 \supset \mathrm{Q}^1 \supset \cdots \supset \mathrm{Q}^{l-1} \supset \mathrm{Q}^{l'} \\ \mathrm{Q}^{l'} \in \mathcal{P}(\mathrm{L}^{l'})}} \mathrm{sign}(\mathbf{r}') J_{\mathrm{M}}^{\mathrm{L}^{l'}}(X, f_{S,\mathrm{Q}^{l'}}) \wedge \mathbf{d}\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l'}}$$

where Φ, Q^j and Φ', $\mathbf{r}'$, $\mathrm{L}^{l'}$, $\mathrm{Q}^{l'}$ are related as in (A.8.1), (A.8.2) and (A.8.3). It suffices to show that (A.10.4) vanishes as a differential form on $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l-1}}$. Let L be a Levi subgroup in $\mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M})$ such that

$$\text{(A.10.5)} \qquad \dim(\mathrm{A}_{\mathrm{M}}/\mathrm{A}_{\mathrm{L}}) \;=\; 1,$$

denote by $\mathbf{d}\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}}$ the differential form on $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l-1}}$ defined by pulling back the volume form on $\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}}$ along the natural projection

$$\text{(A.10.6)} \qquad \mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l-1}} \;\longrightarrow\; \mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}}.$$

Then the orthogonal projection of (A.10.4) onto the one dimensional subspace

$$\text{(A.10.7)} \qquad \mathrm{span}(\mathbf{d}\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}}) \;\subset\; \bigwedge^{\dim(\mathrm{A}_{\mathrm{M}}/\mathrm{A}_{\mathrm{G}})-l} \mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l-1},*}$$

is equal to

$$\text{(A.10.8)} \qquad \sum_{\substack{\mathrm{Q}^0 \supset \mathrm{Q}^1 \supset \cdots \supset \mathrm{Q}^{l-1} \supset \mathrm{Q}^{l'} \\ \mathrm{Q}^{l'} \in \mathcal{P}(\mathrm{L}^{l'})}} \mathrm{sign}(\mathbf{r}') d_{\mathrm{M}}^{\mathrm{L}^{l-1}}(\mathrm{L}, \mathrm{L}^{l'}) J_{\mathrm{M}}^{\mathrm{L}^{l'}}(X, f_{S,\mathrm{Q}^{l'}}) \wedge \mathbf{d}\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}}$$

by the definition of the constant $d_{\mathrm{M}}^{\mathrm{L}^{l-1}}(\mathrm{L}, \mathrm{L}^{l'})$ in Remark (4.1.10). The summation in (A.10.8) is taken over the set

$$\text{(A.10.9)} \qquad \left\{ (\mathrm{L}^{l'}, \mathrm{Q}^{l'} \cap \mathrm{L}^{l-1}) : \begin{array}{l} \mathrm{L}^{l'} \in \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M}),\ \dim(\mathfrak{a}_{\mathrm{L}^{l'}}^{\mathrm{G}}) = l, \\ \mathrm{Q}^{l'} \in \mathcal{P}(\mathrm{L}^{l'}),\ \mathrm{Q}^{l'} \subset \mathrm{Q}^{l-1} \end{array} \right\}$$

$$\subset \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M}) \times \mathcal{F}^{\mathrm{L}^{l-1}}(\mathrm{M})$$

Let ξ^{l-1} be the projection of the vector ξ in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$ used to define the orientation onto $\mathfrak{a}_{\mathrm{M}}^{\mathrm{L}^{l-1}}$. Then ξ^{l-1} determines a partial map

$$s^{l-1}: \quad \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M}) \times \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M}) \;\longrightarrow\; \mathcal{F}^{\mathrm{L}^{l-1}}(\mathrm{M}) \times \mathcal{F}^{\mathrm{L}^{l-1}}(\mathrm{M}) \tag{A.10.10}$$

as in Remark (4.1.11) which is positive in the sense that

$$\mathrm{sign}(\mathbf{r}) \;=\; 1 \tag{A.10.11}$$

if the rigging $\mathbf{r}$ corresponds to the element $(\mathrm{L}^{l\prime}, \mathrm{Q}^{l,+})$ in the set (A.10.9) where $\mathrm{Q}^{l,+}$ denotes the second component of $s^{l-1}(\mathrm{L}, \mathrm{L}^{l\prime})$. Let $\mathrm{Q}^{l,-}$ denote the opposite parabolic of $\mathrm{Q}^{l,+}$, then (A.10.9) is equal to the disjoint union

$$\begin{aligned} &\Big\{(\mathrm{L}^{l\prime}, \mathrm{Q}^{l,+}) : \ \mathrm{L}^{l\prime} \in \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M})\Big\} \coprod \\ &\quad \coprod \Big\{(\mathrm{L}^{l\prime}, \mathrm{Q}^{l,-}) : \ \mathrm{L}^{l\prime} \in \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M})\Big\}, \end{aligned} \tag{A.10.12}$$

hence (A.10.8) is equal to

$$\begin{aligned} &\Bigg(\sum_{\mathrm{L}^{l\prime} \in \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{L}^{l-1}}(\mathrm{L}, \mathrm{L}^{l\prime}) J_{\mathrm{M}}^{\mathrm{L}^{l\prime}}(X, f_{S,\mathrm{Q}^{l,+}}) \\ &\quad - \sum_{\mathrm{L}^{l\prime} \in \mathcal{L}^{\mathrm{L}^{l-1}}(\mathrm{M})} d_{\mathrm{M}}^{\mathrm{L}^{l-1}}(\mathrm{L}, \mathrm{L}^{l\prime}) J_{\mathrm{M}}^{\mathrm{L}^{l\prime}}(X, f_{S,\mathrm{Q}^{l,-}}) \Bigg) \wedge \mathbf{d}\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}} \end{aligned} \tag{A.10.13}$$

$$= \Big(J_{\mathrm{L}}^{\mathrm{L}^{l-1}}(X^{\mathrm{L}}, f_{S,\mathrm{Q}^{l-1}}) - J_{\mathrm{L}}^{\mathrm{L}^{l-1}}(X^{\mathrm{L}}, f_{S,\mathrm{Q}^{l-1}}) \Big) \wedge \mathbf{d}\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^{l-1}} \tag{A.10.14}$$

$$= \quad 0$$

where the equality (A.10.14) follows from the S-local version of (4.1.12.2) and the fact that $\mathrm{Q}^{l,-}$ is the second component of $s^{l-1,-}(\mathrm{L}, \mathrm{L}^{l\prime})$ where $s^{l-1,-}(\ ,\ )$ is the partial map determined as in Remark (4.1.11) by the vector $-\xi^{l-1}$. Hence (A.10.8) vanishes for every L, so (A.10.4) vanishes as a differential form on $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$. Hence $\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X, f_S)$ satisfies (A.5.5). $\square$

(A.11) Remark In §23 of [Ko05] Kottwitz defined weight factors and weighted orbital integrals taking values in the complexified K-group of the toric variety of the fan of the root hyperplanes in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G},*}$. In §4 of [Mo93b] Morelli proved that this K-group is the additive group of translational scissors congruent classes of positive (G, M)-orthogonal sets whose vertices are contained in the coweight lattice in $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$.

(A.12) Definition Let M be a Levi subgroup in $\mathcal{L}$. Define the *total scissors ring* of $\mathfrak{a}_{\mathrm{M}}^{\mathrm{G}}$, denoted by $\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})$, to be the direct sum

(A.12.1)
$$\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}}) \quad = \bigoplus_{\mathrm{L} \in \mathcal{L}^{\mathrm{G}}(\mathrm{M})} \mathbb{S}(\mathfrak{a}_{\mathrm{L}}^{\mathrm{G}}).$$

Define a bilinear product $\boxtimes$ on $\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})$ by

(A.12.2)
$$\forall \mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}^{\mathrm{G}}(\mathrm{M})\ \forall [\mathcal{Y}_{\mathrm{L}_i}] \in \mathbb{S}(\mathfrak{a}_{\mathrm{L}_i}^{\mathrm{G}}) \text{ where } i = 1, 2$$

$$[\mathcal{Y}_{\mathrm{L}_1}] \boxtimes [\mathcal{Y}_{\mathrm{L}_2}]$$

$$= \begin{cases} j^*\Big([\mathcal{Y}_{\mathrm{L}_1} \times \mathcal{Y}_{\mathrm{L}_2}]\Big) & \text{if the natural map} \\ & \quad j:\ \mathfrak{a}_{\mathrm{L}_1 \cap \mathrm{L}_2}^{\mathrm{G}} \longrightarrow \mathfrak{a}_{\mathrm{L}_1}^{\mathrm{G}} \oplus \mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}} \\ & \text{is an isomorphism,} \\ & \\ 0 & \text{otherwise,} \end{cases}$$

where the $(\mathrm{G} \times \mathrm{G}, \mathrm{L}_1 \times \mathrm{L}_2)$-family $\mathcal{Y}_{\mathrm{L}_1} \times \mathcal{Y}_{\mathrm{L}_2}$ is well-defined upto translational scissors congruence in $\mathfrak{a}_{\mathrm{L}_1}^{\mathrm{G}} \oplus \mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}}$, and j^* is the homomorphism

$$\text{(A.12.3)} \qquad j^* : \quad \mathbb{S}(\mathfrak{a}_{\mathrm{L}_1}^{\mathrm{G}} \oplus \mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}}) \longrightarrow \mathbb{S}(\mathfrak{a}_{\mathrm{L}_1 \cap \mathrm{L}_2}^{\mathrm{G}})$$

induced by j.

(A.13) Remark The total scissors ring $\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})$ is graded by

$$\text{(A.13.1)} \qquad \forall n \in \mathbb{N} \qquad \Big(\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})\Big)_n = \bigoplus_{\substack{\mathrm{L} \in \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \\ \dim(\mathfrak{a}_{\mathrm{L}}^{\mathrm{G}}) = n}} \mathbb{S}(\mathfrak{a}_{\mathrm{L}}^{\mathrm{G}}).$$

The grading (A.13.1) has a refinement into the $\mathcal{L}^{\mathrm{G}}(\mathrm{M})$-grading defined by

$$\text{(A.13.2)} \qquad \forall \mathrm{L} \in \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \qquad \Big(\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})\Big)_{\mathrm{L}} = \mathbb{S}(\mathfrak{a}_{\mathrm{L}}^{\mathrm{G}})$$

where the monoid structure on $\mathcal{L}^{\mathrm{G}}(\mathrm{M})$ is defined by intersection.

Each Levi subgroup L in $\mathcal{L}^{\mathrm{G}}(\mathrm{M})$ defines a homogeneous ideal

$$\text{(A.13.3)} \qquad \bigoplus_{\substack{\mathrm{L}' \in \mathcal{L}^{\mathrm{G}}(\mathrm{M}) \\ \mathrm{L}' \not\supset \mathrm{L}}} \mathbb{S}(\mathfrak{a}_{\mathrm{L}'}^{\mathrm{G}}) \subset \mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}}),$$

and taking the quotient of $\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})$ modulo (A.13.3) defines the homomorphism

$$\text{(A.13.4)} \qquad q_{\mathrm{M}}^{\mathrm{L}} : \quad \mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}}) \longrightarrow \mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{L})}^{\mathrm{G}}).$$

The map $q_{\mathrm{M}}^{\mathrm{G}}$ is the quotient modulo the augmentation ideal onto the graded component $\big(\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})\big)_{\mathrm{G}}$, which is a copy of $\mathbb{R}$.

(A.14) Definition Let M be a Levi subgroup in $\mathcal{L}$, let X be an element of $\mathfrak{m}(\mathbb{Q}_S)$. Define the *total orbital integrohedron* $\mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ to be the $\mathbb{S}(\mathfrak{a}_{\mathcal{L}(\mathrm{M})}^{\mathrm{G}})\otimes\mathbb{C}$-valued distribution on $\mathfrak{g}(\mathbb{Q}_S)$ such that

(A.14.1) $\forall f_S \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_S)$

$$\mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,f_S)=\Big((-1)^{\dim(\mathrm{A_L}/\mathrm{A_G})}|\mathrm{W}_0^{\mathrm{L}}||\mathrm{W}_0^{\mathrm{G}}|^{-1}\mathbb{J}_{\mathrm{L}}^{\mathrm{G}}(X^{\mathrm{L}},f_S)\Big)_{\mathrm{L}\in\mathcal{L}^{\mathrm{G}}(\mathrm{M})}$$

For each Levi subgroup M_S in $\mathcal{L}^{\mathrm{G}_S}$ define $\mathcal{J}_{\mathrm{M}_S}^{\mathrm{G}_S}(X,\ )$ by the analogous formula.

(A.15) Lemma (Induction and splitting of orbital integrohedra)
Let M be a Levi subgroup in $\mathcal{L}$, let X be an element of $\mathfrak{m}(\mathbb{Q}_S)$, let f_S be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_S)$.

- *Let L be a Levi subgroup in $\mathcal{L}(\mathrm{M})$, then*

  (A.15.1) $$\mathcal{J}_{\mathrm{L}}^{\mathrm{G}}(X^{\mathrm{L}},f_S) = q_{\mathrm{M}}^{\mathrm{L}}\Big(\mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,f_S)\Big).$$

- *Let S be the set $\{v_1,v_2\}$, let f_S be of the form $f_{v_1}\otimes f_{v_2}$ where f_{v_i} is a Schwartz function on $\mathfrak{g}(\mathbb{Q}_{v_i})$ where the index i is 1 or 2, then*

  (A.15.2) $$\mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,f_{v_1}\otimes f_{v_2}) = \mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,f_{v_1})\boxtimes\mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,f_{v_2}).$$

Local identities analogous to (A.15.1) *also hold for* G_v.

Proof. The induction identity (A.15.1) follows from the definition (A.14.1).

The splitting identity (A.15.2) follows by comparing the Hadwiger invariants of the two sides of (A.15.2). Retain the notations of (A.9.1). Let L be a Levi subgroup in $\mathcal{L}(\mathrm{M})$, let

Φ be a strict flag in $\mathcal{F}_{\mathrm{S}}(\mathfrak{a}_{\mathrm{L}}^{\mathrm{G}})$. It suffices to show that the Φ-components of the Hadwiger invariants of the L-components of the two sides of (A.15.2) are equal.

The L-component of the right hand side of (A.15.2) is

$$\text{(A.15.3)} \qquad \sum_{\substack{\mathrm{L}_1,\mathrm{L}_2\in\mathcal{L}(\mathrm{L}) \\ \mathfrak{a}_{\mathrm{L}}^{\mathrm{G}}\to\mathfrak{a}_{\mathrm{L}_1}^{\mathrm{G}}\oplus\mathfrak{a}_{\mathrm{L}_2}^{\mathrm{G}} \\ \text{isomorphism}}} (-1)^{\dim(\mathrm{A}_{\mathrm{L}}/\mathrm{A}_{\mathrm{G}})}|\mathrm{W}_0^{\mathrm{L}_1}||\mathrm{W}_0^{\mathrm{G}}|^{-1}|\mathrm{W}_0^{\mathrm{L}_2}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \times$$

$$\times\mathbb{J}_{\mathrm{L}_1}^{\mathrm{G}}(X^{\mathrm{L}_1}, f_{v_1}) \boxtimes \mathbb{J}_{\mathrm{L}_2}^{\mathrm{G}}(X^{\mathrm{L}_2}, f_{v_2}),$$

whose Hadwiger invariant has Φ-component equal to

$$\text{(A.15.4)} \qquad \sum_{\mathrm{L}_1,\mathrm{L}_2\in\mathcal{L}(\mathrm{L})} (-1)^{\dim(\mathrm{A}_{\mathrm{L}}/\mathrm{A}_{\mathrm{G}})}|\mathrm{W}_0^{\mathrm{L}_1}||\mathrm{W}_0^{\mathrm{G}}|^{-1}|\mathrm{W}_0^{\mathrm{L}_2}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \times$$

$$\times d_{\mathrm{L}}^{\mathrm{L}^l}(\mathrm{L}_1^l, \mathrm{L}_2^l)\Bigg(\sum_{\mathbf{r}_1\in\mathcal{R}ig(\Phi_1)} \operatorname{sign}(\mathbf{r}_1) J_{\mathrm{L}_1}^{\mathrm{L}_1^l}(X^{\mathrm{L}_1}, f_{v_1,\mathrm{Q}_1^l})\Bigg) \times$$

$$\times\Bigg(\sum_{\mathbf{r}_2\in\mathcal{R}ig(\Phi_2)} \operatorname{sign}(\mathbf{r}_2) J_{\mathrm{L}_2}^{\mathrm{L}_2^l}(X^{\mathrm{L}_2}, f_{v_2,\mathrm{Q}_2^l})\Bigg)$$

where Φ_i denotes the strict flag in $\mathfrak{a}_{\mathrm{L}_i}^{\mathrm{G}}$ induced by Φ, and Φ_i, $\mathbf{r}_i$, L_i^l and Q_i^l are related as in (A.8.1), (A.8.2) and (A.8.3) where the index i is 1 or 2. Combine $\mathbf{r}_1$ and $\mathbf{r}_2$ into a rigging $\mathbf{r}$ of Φ, the expression (A.15.4) becomes

$$\text{(A.15.5)} \qquad \sum_{\mathrm{L}_1,\mathrm{L}_2\in\mathcal{L}(\mathrm{L})} (-1)^{\dim(\mathrm{A}_{\mathrm{L}}/\mathrm{A}_{\mathrm{G}})}|\mathrm{W}_0^{\mathrm{L}_1}||\mathrm{W}_0^{\mathrm{G}}|^{-1}|\mathrm{W}_0^{\mathrm{L}_2}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \times$$

$$\times \sum_{\mathrm{Q}_1^l\in\mathcal{P}(\mathrm{L}_1^l)} \sum_{\mathrm{Q}_2^l\in\mathcal{P}(\mathrm{L}_2^l)} \operatorname{sign}(\mathbf{r}) d_{\mathrm{L}}^{\mathrm{L}^l}(\mathrm{L}_1^l, \mathrm{L}_2^l) \times$$

$$\times J_{\mathrm{L}_1}^{\mathrm{L}_1^l}(X^{\mathrm{L}_1}, f_{v_1,\mathrm{Q}_1^l}) J_{\mathrm{L}_2}^{\mathrm{L}_2^l}(X^{\mathrm{L}_2}, f_{v_2,\mathrm{Q}_2^l})$$

$$
\begin{aligned}
= \sum_{\mathrm{L}_1,\mathrm{L}_2\in\mathcal{L}(\mathrm{L})} & (-1)^{\dim(\mathrm{A}_\mathrm{L}/\mathrm{A}_\mathrm{G})}|\mathrm{W}_0^{\mathrm{L}_1}||\mathrm{W}_0^{\mathrm{G}}|^{-1}|\mathrm{W}_0^{\mathrm{L}_2}||\mathrm{W}_0^{\mathrm{G}}|^{-1} \times \\
& \times \sum_{\mathrm{Q}_1^l\in\mathcal{P}(\mathrm{L}_1^l)} \sum_{\mathrm{Q}_2^l\in\mathcal{P}(\mathrm{L}_2^l)} \operatorname{sign}(\mathbf{r}) d_\mathrm{L}^{\mathrm{L}^l}(\mathrm{L}_1^l,\mathrm{L}_2^l) \times \\
& \quad \times \sum_{\mathrm{L}_1^\circ\in\mathcal{L}^{\mathrm{L}_1^l}(\mathrm{L})} d_\mathrm{L}^{\mathrm{L}_1^l}(\mathrm{L}_1,\mathrm{L}_1^\circ) J_\mathrm{L}^{\mathrm{L}_1^\circ}(X^\mathrm{L}, f_{v_1,\mathrm{Q}_1^{l,\circ}}) \times \\
& \quad\quad \times \sum_{\mathrm{L}_2^\circ\in\mathcal{L}^{\mathrm{L}_2^l}(\mathrm{L})} d_\mathrm{L}^{\mathrm{L}_2^l}(\mathrm{L}_2,\mathrm{L}_2^\circ) J_\mathrm{L}^{\mathrm{L}_2^\circ}(X^\mathrm{L}, f_{v_2,\mathrm{Q}_2^{l,\circ}})
\end{aligned}
$$

by (4.1.12.2), where $\mathrm{Q}_i^{l,\circ}$ is the second component of $s_{\mathrm{Q}_i^l}(\mathrm{L}_i,\mathrm{L}_i^\circ)$ for the choice of a collection of partial maps $(s_{\mathrm{Q}_i^l})_{\mathrm{Q}_i^l\in\mathcal{P}(\mathrm{L}_1')}$ such that

$$
\text{(A.15.6)} \qquad \forall \mathrm{Q}_i^l \in \mathcal{P}(\mathrm{L}_i^l) \qquad \mathrm{Q}_i^{l,\circ} \subset \mathrm{Q}_i^l
$$

where the index i is 1 or 2. The choice of such a collection is equivalent to the choice of a collection of vectors $(\xi_{\mathrm{Q}_i^l})_{\mathrm{Q}_i^l\in\mathcal{P}(\mathrm{L}_1')}$

$$
\text{(A.15.7)} \qquad \forall \mathrm{Q}_i^l \in \mathcal{P}(\mathrm{L}_i^l) \qquad \xi_{\mathrm{Q}_i^l} \in \mathfrak{a}_\mathrm{L}^{\mathrm{L}_i^l}
$$

in general position and positive with respect to Q_i^l. By definition

$$
\text{(A.15.8)} \qquad d_\mathrm{L}^{\mathrm{L}^l}(\mathrm{L}_1^l,\mathrm{L}_2^l) d_\mathrm{L}^{\mathrm{L}_1^l}(\mathrm{L}_1,\mathrm{L}_1^\circ) d_\mathrm{L}^{\mathrm{L}_2^l}(\mathrm{L}_2,\mathrm{L}_2^\circ) = d_\mathrm{L}^{\mathrm{L}^l}(\mathrm{L}_1^\circ,\mathrm{L}_2^\circ),
$$

hence the right hand side of (A.15.5) is equal to

$$
\text{(A.15.9)} \qquad (-1)^{\dim(\mathrm{A}_\mathrm{L}/\mathrm{A}_\mathrm{G})} \sum_{\mathbf{r}\in\mathcal{R}ig(\Phi)} \operatorname{sign}(\mathbf{r}) \times
$$

$$\times \sum_{\mathrm{L}_1,\mathrm{L}_2\in\mathcal{L}(\mathrm{L})} \sum_{\substack{\mathrm{L}_1^\circ\in\mathcal{L}^{\mathrm{L}_1^l}(\mathrm{L})\\ \mathrm{L}_2^\circ\in\mathcal{L}^{\mathrm{L}_2^l}(\mathrm{L})}} |\mathrm{W}_0^{\mathrm{L}_1}||\mathrm{W}_0^{\mathrm{G}}|^{-1}|\mathrm{W}_0^{\mathrm{L}_2}||\mathrm{W}_0^{\mathrm{G}}|^{-1}\times$$

$$\times d_{\mathrm{L}}^{\mathrm{L}^l}(\mathrm{L}_1^\circ,\mathrm{L}_2^\circ)J_{\mathrm{L}}^{\mathrm{L}_1^\circ}(X^{\mathrm{L}},f_{v_1,\mathrm{Q}_1^{l,\circ}})J_{\mathrm{L}}^{\mathrm{L}_2^\circ}(X^{\mathrm{L}},f_{v_2,\mathrm{Q}_2^{l,\circ}})$$

where $(\mathrm{Q}_1^{l,\circ},\mathrm{Q}_2^{l,\circ})$ is the image of $(\mathrm{L}_1^\circ,\mathrm{L}_2^\circ)$ under the partial map $s_{\mathrm{Q}_1^l,\mathrm{Q}_2^l}$ determined by the vector

$$\frac{\xi_{\mathrm{Q}_1^l}}{2}-\frac{\xi_{\mathrm{Q}_2^l}}{2} \;\in\; \mathfrak{a}_{\mathrm{L}}^{\mathrm{L}^l}=\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}_1^l}\oplus\mathfrak{a}_{\mathrm{L}}^{\mathrm{L}_2^l}. \tag{A.15.10}$$

For each fixed L and fixed Φ with minimal term L^l, the map

$$\left\{(\mathrm{L}_1,\mathrm{L}_2,\mathrm{L}_1^\circ,\mathrm{L}_2^\circ):\begin{array}{l}\mathrm{L}_1\in\mathcal{L}(\mathrm{L}),\ \mathrm{L}_2\in\mathcal{L}(\mathrm{L}),\\ \mathrm{L}_1^\circ\in\mathcal{L}^{\mathrm{L}_1^l}(\mathrm{L}),\ \mathrm{L}_2^\circ\in\mathcal{L}^{\mathrm{L}_2^l}(\mathrm{L})\end{array}\right\} \tag{A.15.11}$$

$$\longrightarrow \quad \left\{(\mathrm{L}_1^\circ,\mathrm{L}_2^\circ):\ \mathrm{L}_1^\circ\in\mathcal{L}^{\mathrm{L}^l}(\mathrm{L}),\ \mathrm{L}_2^\circ\in\mathcal{L}^{\mathrm{L}^l}(\mathrm{L})\right\}$$

defined by

$$(\mathrm{L}_1,\mathrm{L}_2,\mathrm{L}_1^\circ,\mathrm{L}_2^\circ)\ \mapsto\ (\mathrm{L}_1^\circ,\mathrm{L}_2^\circ) \tag{A.15.12}$$

is surjective. Fix Levi subgroups L_1 and L_2 in $\mathcal{L}(\mathrm{L})$ and parabolic subgroups P_1 in $\mathcal{P}(\mathrm{L}_1)$ and P_2 in $\mathcal{P}(\mathrm{L}_2)$, and let $\mathrm{P}_{1,2}$ be the parabolic subgroup in $\mathcal{P}(\mathrm{L})$ whose unipotent radical $\mathrm{N}_{\mathrm{P}_{1,2}}$ contains $\mathrm{N}_{\mathrm{P}_1}$ and $\mathrm{N}_{\mathrm{P}_2}$, then the map (A.15.12) restricts to a bijection between Levi subgroups L_1° and L_2° that are standard with respect to P_1 and P_2 or $\mathrm{P}_{1,2}$. Hence each fiber of the map (A.15.12) containing the pair $(\mathrm{L}_1,\mathrm{L}_2)$ has cardinality

$$\frac{|\mathrm{W}_0^{\mathrm{G}}|/|\mathrm{W}_0^{\mathrm{L}}|}{\Big(|\mathrm{W}_0^{\mathrm{G}}|/|\mathrm{W}_0^{\mathrm{L}_1}|\Big)\Big(|\mathrm{W}_0^{\mathrm{G}}|/|\mathrm{W}_0^{\mathrm{L}_2}|\Big)}, \tag{A.15.13}$$

hence (A.15.9) is equal to

$$\text{(A.15.14)}\qquad (-1)^{\dim(\mathrm{A_L}/\mathrm{A_G})}|\mathrm{W}_0^{\mathrm{L}}||\mathrm{W}_0^{\mathrm{G}}|^{-1}\sum_{\mathbf{r}\in\mathcal{R}ig(\Phi)}\mathrm{sign}(\mathbf{r})\times$$

$$\times\sum_{\mathrm{L}_1^\circ,\mathrm{L}_2^\circ\in\mathcal{L}^{\mathrm{L}^l}(\mathrm{L})} d_{\mathrm{L}}^{\mathrm{L}^l}(\mathrm{L}_1^\circ,\mathrm{L}_2^\circ)J_{\mathrm{L}}^{\mathrm{L}_1^\circ}(X^{\mathrm{L}},f_{v_1,\mathrm{Q}_1^{l,\circ}})J_{\mathrm{L}}^{\mathrm{L}_2^\circ}(X^{\mathrm{L}},f_{v_2,\mathrm{Q}_2^{l,\circ}})$$

$$= (-1)^{\dim(\mathrm{A_L}/\mathrm{A_G})}|\mathrm{W}_0^{\mathrm{L}}||\mathrm{W}_0^{\mathrm{G}}|^{-1}\sum_{\mathbf{r}\in\mathcal{R}ig(\Phi)}\mathrm{sign}(\mathbf{r})J_{\mathrm{L}}^{\mathrm{L}^l}(X^{\mathrm{L}},f_{v_1,\mathrm{Q}^l}\otimes f_{v_2,\mathrm{Q}^l})$$

by (4.1.12.3) where Φ, $\mathbf{r}$, L^l and Q^l are related as in (A.8.1), (A.8.2) and (A.8.3), which is equal to the Φ-component of the Hadwiger invariant of the L-component of the left hand side of (A.15.2). □

(A.16) Remark The orbital integrohedra $\mathbb{J}_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ and $\mathcal{J}_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ could be alternatively defined as distributions on $\mathfrak{g}(\mathbb{Q}_S)$ obtained by integrating against certain polyhedron-valued weight factors with a formula analogous to (3.1.5.3) if X is regular semisimple.

2.B Appendix: The example of GL(2)

In this appendix some explicit computations for the group GL(2) are carried out. Similar local computations could be found in [Ev98].

(B.1) Definition In this appendix v denotes an odd rational prime p, and $|\ |$ denotes the p-adic absolute value. Let G denote the group $\mathrm{GL}(2,\mathbb{Q}_p)$, let M denote the minimal Levi subgroup of G consisting of the diagonal matrices, then M is the Levi component of the Borel subgroup B consisting of the upper triangular matrices. The groups G and M are the only standard Levi subgroups of G.

Let $\mathfrak{g}'$ denote the Lie algebra $\mathrm{sl}(2,\mathbb{Q}_p)$, equipped with the adjoint action of G from the right. Let $\mathfrak{m}'$ denote the intersection of $\mathfrak{m}$ and $\mathfrak{g}'$. Denote by $\mathcal{A}$ the affine quotient $\mathfrak{g}'/\!\!/\mathrm{G}$

which is identified with the affine line $\mathbb{L}$ via the negative determinant map

$$-\det : \quad \mathfrak{g}' \longrightarrow \mathbb{L} = \mathcal{A}. \tag{B.1.1}$$

(B.2) Remark Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$ consisting of the diagonal matrices on which G operates trivially. Since

$$\mathfrak{g} \;=\; \mathfrak{g}' \oplus \mathfrak{z} \tag{B.2.1}$$

as representations of G, for computing the local basic functions and the local Harish-Chandra transforms it suffices to consider $\mathfrak{g}'$ instead of $\mathfrak{g}$.

(B.3) Definition The discriminant function D on $\mathcal{A}$ is equal to $4X$ where X denotes the coordinate function on $\mathcal{A}$. Let η be an element of $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$ which is identified with the subset of units $\mathbb{Q}_p^\times$ of $\mathbb{Q}_p$. Then η is said to be

- *split* if η has a square root in $\mathbb{Q}_p$;
- *unramified elliptic* if η does not have a square root in $\mathbb{Q}_p$ but the p-adic valuation of η is even;
- *ramified elliptic* if the p-adic valuation of η is odd.

The image of $\mathfrak{m}'_{\mathrm{reg.ss}}(\mathbb{Q}_p)$ in $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$ is equal to the subset of split elements.

(B.4) Definition The algebraic differential form

$$\mathrm{d}_\eta X \;=\; \frac{\mathrm{d}x \wedge \mathrm{d}y}{y} = \frac{\mathrm{d}y \wedge \mathrm{d}z}{2x} = \frac{\mathrm{d}z \wedge \mathrm{d}x}{z} \tag{B.4.1}$$

on the orbit

$$(\text{B.4.2}) \qquad \mathfrak{g}'_\eta \;=\; \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in \mathfrak{g}' :\ x^2 + yz = \eta \right\}$$

where η lies in $\mathcal{A}_{\mathrm{reg}}$ is invariant under the action of G.

(B.5) Lemma *Let f be a Schwartz function on $\mathfrak{g}'(\mathbb{Q}_p)$, then*

$$(\text{B.5.1}) \qquad \forall \eta \in \mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p) \qquad I_{\mathrm{G}}^{\mathrm{G}}(\eta, f) = \frac{1}{1+p^{-1}} \int_{\mathfrak{g}'_\eta(\mathbb{Q}_p)} f(X)\ |\mathrm{d}_\eta X|.$$

Proof. There exists a multiplicative constant λ such that

$$(\text{B.5.2}) \qquad \forall f \in \mathcal{S}(\mathfrak{g}'(\mathbb{Q}_p))\ \forall \eta \in \mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$$

$$I_{\mathrm{G}}^{\mathrm{G}}(\eta, f) = \lambda \cdot \int_{\mathfrak{g}'_\eta(\mathbb{Q}_p)} f(X)\ |\mathrm{d}_\eta X|,$$

hence the constant λ is determined by

$$(\text{B.5.3}) \qquad I_{\mathrm{G}}^{\mathrm{G}}(1, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}) \;=\; \lambda \cdot \int_{\mathfrak{g}'_1(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}(X)\ |\mathrm{d}_1 X|.$$

By parabolic descent along the Borel subgroup B, the left hand side of (B.5.3) is equal to

$$(\text{B.5.4}) \qquad I_{\mathrm{M}}^{\mathrm{M}}(1, (\mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)})_{\mathrm{B}}) \;=\; I_{\mathrm{M}}^{\mathrm{M}}(1, \mathbb{I}_{\mathfrak{m}'(\mathbb{Z}_p)})$$

which is equal to one. Hence by (B.5.3) the reciprocal of λ is equal to

$$(\text{B.5.5}) \qquad \int_{\mathfrak{g}'_1(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}\left(\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) \left| \frac{\mathrm{d}x \wedge \mathrm{d}y}{y} \right|$$

$$
\begin{aligned}
&= \int_{\mathbb{Z}_p}\left(\int_{\left\{y\in\mathbb{Z}_p:\ z=\frac{1-x^2}{y}\in\mathbb{Z}_p\right\}}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x\\
&= \int_{|1+x|<1}\left(\int_{|1-x^2|\le|y|\le1}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x+\\
&\quad +\int_{|1-x|<1}\left(\int_{|1-x^2|\le|y|\le1}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x+\\
&\quad\quad +\int_{\substack{|1+x|=1\\|1-x|=1}}\left(\int_{|1-x^2|\le|y|\le1}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x\\
&= \sum_{i=1}^{\infty}\left(\int_{|1+x|=p^{-i}}\mathrm{d}x\right)\left(\int_{p^{-i}\le|y|\le1}\frac{\mathrm{d}y}{|y|}\right)+\\
&\quad +\sum_{j=1}^{\infty}\left(\int_{|1-x|=p^{-j}}\mathrm{d}x\right)\left(\int_{p^{-j}\le|y|\le1}\frac{\mathrm{d}y}{|y|}\right)+\\
&\quad\quad +\left(\int_{\substack{|1+x|=1\\|1-x|=1}}\mathrm{d}x\right)\left(\int_{|y|=1}\frac{\mathrm{d}y}{|y|}\right)\\
&= \sum_{i=1}^{\infty}\left(\frac{1}{p^i}-\frac{1}{p^{i+1}}\right)\left(\frac{(i+1)(p-1)}{p}\right)+\\
&\quad +\sum_{j=1}^{\infty}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\left(\frac{(j+1)(p-1)}{p}\right)+\\
&\quad\quad +\left(\frac{p-2}{p}\right)\left(\frac{p-1}{p}\right)\\
&= \frac{p-1}{p}\left(2\cdot\left(\frac{1}{p}+\frac{1}{p-1}\right)+\frac{p-2}{p}\right)\\
&= 1+p^{-1}.
\end{aligned}
$$

□

(B.6) Lemma *Let η be an element of $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$, then*

$$\phi_{0,v}(\eta) = \begin{cases} 1 & \text{if } \eta \text{ is split} \\ & \quad \text{and contained in } \mathbb{Z}_p - \{0\}, \\ 1 - \dfrac{2|\eta|^{1/2}}{1+p} & \text{if } \eta \text{ is unramified elliptic} \\ & \quad \text{and contained in } \mathbb{Z}_p - \{0\}, \\ 1 - p^{-1/2}|\eta|^{1/2} & \text{if } \eta \text{ is ramified elliptic} \\ & \quad \text{and contained in } \mathbb{Z}_p - \{0\}, \\ 0 & \text{if } \eta \text{ does not lie in } \mathbb{Z}_p - \{0\}. \end{cases} \tag{B.6.1}$$

Proof. If η does not lie in $\mathbb{Z}_p - \{0\}$, then the orbit $\mathfrak{g}'_\eta(\mathbb{Q}_p)$ is disjoint from the support of the characteristic function $\mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}$, hence $\phi_{0,v}(\eta)$ vanishes.

If η is split and contained in $\mathbb{Z} - \{0\}$, then by (B.5.4) the value $\phi_{0,v}(\eta)$ is equal to one.

If η is unramified elliptic and contained in $\mathbb{Z}_p - \{0\}$, then by (B.5.1)

$$\phi_{0,v}(\eta) = \frac{1}{1+p^{-1}} \int_{\mathfrak{g}'_\eta(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}(X)\ |\mathrm{d}_\eta X| \tag{B.6.2}$$

$$= \frac{1}{1+p^{-1}} \int_{\mathfrak{g}'_\eta(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}\left(\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) \left| \frac{\mathrm{d}x \wedge \mathrm{d}y}{y} \right|$$

$$= \frac{1}{1+p^{-1}} \int_{|x|\le 1} \left(\int_{|\eta - x^2| \le |y| \le 1} \frac{\mathrm{d}y}{|y|} \right) \mathrm{d}x \tag{B.6.3}$$

Decompose the integral on the right hand side of (B.6.3) over two disjoint regions:

- If $|x^2|$ is strictly less than $|\eta|$, then $|\eta - x^2|$ is equal to $|\eta|$, hence

$$\int_{|x^2|<|\eta|}\left(\int_{|\eta-x^2|\leq|y|\leq 1}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x \tag{B.6.4}$$

$$= \left(\int_{|x^2|<|\eta|}\mathrm{d}x\right)\left(\int_{|\eta|\leq|y|\leq 1}\frac{\mathrm{d}y}{|y|}\right)$$

$$= \left(p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}\right)\left(\frac{(\mathrm{val}(\eta)+1)(p-1)}{p}\right)$$

where $\mathrm{val}(\eta)$ denotes the p-adic valuation of η.

- If $|x^2|$ is greater than or equal to $|\eta|$, then $|\eta - x^2|$ is equal to $|x^2|$, hence

$$\int_{|\eta|\leq|x^2|\leq 1}\left(\int_{|\eta-x^2|\leq|y|\leq 1}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x \tag{B.6.5}$$

$$= \int_{|\eta|\leq|x^2|\leq 1}\left(\int_{|x^2|\leq|y|\leq 1}\frac{\mathrm{d}y}{|y|}\right)\mathrm{d}x$$

$$= \int_{|\eta|\leq|x^2|\leq 1}(2\mathrm{val}(x)+1)\left(\frac{p-1}{p}\right)\mathrm{d}x$$

$$= \frac{p-1}{p}\left(1-p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}+2\int_{|\eta|\leq|x^2|\leq 1}\mathrm{val}(x)\ \mathrm{d}x\right)$$

$$= \frac{p-1}{p}\left(1-p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}+2\sum_{i=0}^{\frac{\mathrm{val}(\eta)}{2}} i\left(\frac{1}{p^i}-\frac{1}{p^{i+1}}\right)\right)$$

$$= \frac{p-1}{p}\Bigg(1-p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}+$$

$$+2\cdot\left(-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}+\frac{1-p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}}{p-1}\right)\Bigg).$$

Hence the right hand side of (B.6.3) is equal to

$$\begin{aligned}
&\text{(B.6.6)} \qquad \frac{1}{1+p^{-1}} \cdot \frac{p-1}{p}\Bigg(\Big(\mathrm{val}(\eta)+1\Big)p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)} + \\
&\qquad -\Big(\mathrm{val}(\eta)+2\Big)p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)} + \Big(1-p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)}\Big)\Big(1+\frac{2}{p-1}\Big)\Bigg) \\
&= \frac{p-1}{p+1}\Bigg(-p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)} - \Big(\frac{p+1}{p-1}\Big)p^{-\left(\frac{\mathrm{val}(\eta)}{2}+1\right)} + \frac{p+1}{p-1}\Bigg) \\
&= 1 - \frac{2}{p+1}p^{-\frac{\mathrm{val}(\eta)}{2}}.
\end{aligned}$$

If η is ramified elliptic and contained in $\mathbb{Z}_p - \{0\}$, then by (B.5.1)

$$\text{(B.6.7)} \qquad \phi_{0,v}(\eta) = \frac{1}{1+p^{-1}} \int_{|x|\le 1}\Bigg(\int_{|\eta-x^2|\le|y|\le 1}\frac{\mathrm{d}y}{|y|}\Bigg)\,\mathrm{d}x$$

Decompose the integral on the right hand side of (B.6.7) over two disjoint regions:

- If $|x^2|$ is less than $|\eta|$, then $|\eta - x^2|$ is equal to $|\eta|$, hence

$$\begin{aligned}
&\text{(B.6.8)} \qquad \int_{|x^2|<|\eta|}\Bigg(\int_{|\eta-x^2|\le|y|\le 1}\frac{\mathrm{d}y}{|y|}\Bigg)\,\mathrm{d}x \\
&= \Bigg(\int_{|x^2|<|\eta|}\mathrm{d}x\Bigg)\Bigg(\int_{|\eta|\le|y|\le 1}\frac{\mathrm{d}y}{|y|}\Bigg) \\
&= \Bigg(p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}\Bigg)\Bigg(\frac{(\mathrm{val}(\eta)+1)(p-1)}{p}\Bigg).
\end{aligned}$$

- If $|x^2|$ is greater than $|\eta|$, then $|\eta - x^2|$ is equal to $|x^2|$, hence

$$\text{(B.6.9)} \qquad \int_{|\eta|\le|x^2|\le 1}\Bigg(\int_{|\eta-x^2|\le|y|\le 1}\frac{\mathrm{d}y}{|y|}\Bigg)\,\mathrm{d}x$$

$$= \int_{|\eta|\leq|x^2|\leq 1}\Big(\int_{|x^2|\leq|y|\leq 1}\frac{\mathrm{d}y}{|y|}\Big)\ \mathrm{d}x$$

$$= \int_{|\eta|\leq|x^2|\leq 1}(2\mathrm{val}(x)+1)\Big(\frac{p-1}{p}\Big)\ \mathrm{d}x$$

$$= \frac{p-1}{p}\bigg(1-p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}+2\int_{|\eta|\leq|x^2|\leq 1}\mathrm{val}(x)\ \mathrm{d}x\bigg)$$

$$= \frac{p-1}{p}\bigg(1-p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}+2\sum_{i=0}^{\frac{\mathrm{val}(\eta)-1}{2}} i\Big(\frac{1}{p^i}-\frac{1}{p^{i+1}}\Big)\bigg)$$

$$= \frac{p-1}{p}\bigg(1-p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}+$$

$$+2\cdot\bigg(-\Big(\frac{\mathrm{val}(\eta)+1}{2}\Big)p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}+\frac{1-p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}}{p-1}\bigg)\bigg).$$

Hence the right hand side of (B.6.7) is equal to

$$\text{(B.6.10)}\qquad \frac{1}{1+p^{-1}}\cdot\frac{p-1}{p}\bigg(\Big(\mathrm{val}(\eta)+1\Big)p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}+$$

$$-\Big(\mathrm{val}(\eta)+1\Big)p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}+\Big(1-p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}\Big)\Big(1+\frac{2}{p-1}\Big)\bigg)$$

$$= \frac{p-1}{p+1}\Big(1-p^{-\left(\frac{\mathrm{val}(\eta)+1}{2}\right)}\Big)\Big(\frac{p+1}{p-1}\Big)$$

$$= 1-p^{-1/2}p^{-\frac{\mathrm{val}(\eta)}{2}}.$$

□

(B.7) Definition The weighted orbital integral $J_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta},\ )$ where $\widetilde{\eta}$ is an element of $\mathfrak{g}'_{\mathrm{reg.ss}}(\mathbb{Q}_p)$ lifting η is defined is η is split. For such an η choose a square root $\sqrt{\eta}$ of η in $\mathbb{Q}_p$ and fix the lift $\widetilde{\eta}$ to be

$$\widetilde{\eta} \;=\; \begin{pmatrix} \sqrt{\eta} & \\ & -\sqrt{\eta} \end{pmatrix}. \tag{B.7.1}$$

(B.8) Remark With respect to the choice of B as the Borel subgroup and $\mathrm{G}(\mathbb{Z}_p)$ as the maximal compact subgroup, the weight factor $v(\ )$ appearing in the definition of $J_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta},\ )$ is the function on $\mathrm{G}(\mathbb{Q}_p)$ such that $v(\ )$ is invariant under left translation by $\mathrm{M}(\mathbb{Q}_p)$ and right translation by $\mathrm{G}(\mathbb{Z}_p)$ and

$$\forall u \in \mathbb{Q}_p \tag{B.8.1}$$

$$v\Big(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\Big) = \begin{cases} 0 & \text{if } u \text{ is contained in } \mathbb{Z}_p, \\ & \\ -\mathrm{val}(u) & \text{otherwise.} \end{cases}$$

See §12.1 of [Ko05].

(B.9) Lemma *Let η be a split element of $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$, then*

$$J_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}) \;=\; \begin{cases} \dfrac{\mathrm{val}(\eta)}{2} - \dfrac{1}{p-1} + \dfrac{|\eta|^{1/2}}{p-1} & \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\ & \\ 0 & \text{otherwise.} \end{cases} \tag{B.9.1}$$

Proof. By definition

$$J_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}) \tag{B.9.2}$$

$$= \; |D(\eta)|^{1/2} \int_{\mathrm{M}(\mathbb{Q}_p)\backslash \mathrm{G}(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}(\widetilde{\eta} \cdot \mathrm{ad}(g)) \; v(g)\mathrm{d}g$$

$$\text{(B.9.3)} \quad = \; |\eta|^{1/2} \int_{\mathbb{Q}_p} \int_{\mathrm{G}(\mathbb{Z}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}\Bigg(\begin{pmatrix} \sqrt{\eta} & \\ & -\sqrt{\eta} \end{pmatrix} \cdot \mathrm{ad}\Bigg(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \Bigg) \mathrm{ad}(k) \Bigg) \times$$

$$\times v\Bigg(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \Bigg) \mathrm{d}k\mathrm{d}u$$

$$= \; |\eta|^{1/2} \int_{\mathbb{Q}_p} \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}\Bigg(\begin{pmatrix} \sqrt{\eta} & 2u\sqrt{\eta} \\ & -\sqrt{\eta} \end{pmatrix} \Bigg) \; v\Bigg(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \Bigg) \mathrm{d}u$$

$$= \; |\eta|^{1/2} \int_{1<|u|\leq |2\sqrt{\eta}|^{-1}} -\mathrm{val}(u) \; \mathrm{d}u$$

$$= \; |\eta|^{1/2} \sum_{i=1}^{\frac{\mathrm{val}(\eta)}{2}} i \Big(\frac{1}{p^{-i}} - \frac{1}{p^{-i+1}} \Big)$$

$$= \; |\eta|^{1/2} \Bigg(\frac{\mathrm{val}(\eta)}{2} p^{\frac{\mathrm{val}(\eta)}{2}} - \frac{p^{\frac{\mathrm{val}(\eta)}{2}} - 1}{p-1} \Bigg)$$

where the equality (B.9.3) follows from the Iwasawa decomposition. □

(B.10) Lemma *Let η be a split element of $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$, then*

$$\text{(B.10.1)} \quad I^{\mathrm{G}}_{\mathrm{M}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}) \; = \; \begin{cases} \dfrac{\mathrm{val}(\eta)}{2} + \dfrac{|\eta|^{1/2}}{p-1} - \dfrac{2p+1}{p^2-1} & \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\ \\ \dfrac{|\eta|^{-1}}{1-p^{-2}} - \dfrac{|\eta|^{-1/2}}{1-p^{-1}} & \text{otherwise.} \end{cases}$$

Proof. By definition

$$\text{(B.10.2)} \qquad I_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)})$$

$$= J_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}) - \int_{\mathfrak{m}'(\mathbb{Q}_p)} J_{\mathrm{M}}^{\mathrm{G}}(X, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)})\psi\big(\mathrm{Tr}(\widetilde{\eta}^{\mathrm{T}}X)\big)\ \mathrm{d}X$$

$$= J_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}) - \int_{\mathbb{Q}_p} J_{\mathrm{M}}^{\mathrm{G}}\Bigg(\begin{pmatrix} x & \\ & -x \end{pmatrix}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)}\Bigg)\psi(2\sqrt{\eta}x)\ \mathrm{d}x$$

where ψ denotes the additive character of $\mathbb{Q}_p$ involved in the definition of the Fourier transform on a p-adic vector space, the superscript T denotes the transpose of a 2×2 matrix and Tr denotes the trace of a 2×2 matrix.

The Fourier transform of $\mathrm{val}(\ )\mathbb{I}_{\mathbb{Z}_p}(\ )$ on $\mathbb{Q}_p$ is the function such that

$$\text{(B.10.3)} \qquad \forall x \in \mathbb{Q}_p \qquad \Big(\mathrm{val}(\ )\mathbb{I}_{\mathbb{Z}_p}(\ )\Big)\hat{\ }(x)$$

$$= \Big(\sum_{i=1}^{\infty} \mathbb{I}_{p^i\mathbb{Z}_p}(\ )\Big)\hat{\ }(x)$$

$$= \sum_{i=1}^{\infty} \frac{1}{p^i}\mathbb{I}_{p^{-i}\mathbb{Z}_p}(x)$$

$$= \sum_{i=\max\{1,-\mathrm{val}(x)\}}^{\infty} \frac{1}{p^i}$$

$$= \frac{p^{-\max\{1,-\mathrm{val}(x)\}}}{1-p^{-1}}$$

$$= \min\Big\{\frac{p^{-1}}{1-p^{-1}}, \frac{|x|^{-1}}{1-p^{-1}}\Big\}.$$

The Fourier transform of $| \, |\mathbb{I}_{\mathbb{Z}_p}(\ )$ on $\mathbb{Q}_p$ is the function such that

$$\text{(B.10.4)} \qquad \forall x \in \mathbb{Q}_p \qquad \Big(| \, |\mathbb{I}_{\mathbb{Z}_p}(\ )\Big)\hat{\ }(x)$$

$$= \Big(\mathbb{I}_{\mathbb{Z}_p}(\ ) + \sum_{i=1}^{\infty}\Big(\frac{1}{p^i} - \frac{1}{p^{i-1}}\Big)\mathbb{I}_{p^i\mathbb{Z}_p}(\ )\Big)\hat{\ }(x)$$

$$= \mathbb{I}_{\mathbb{Z}_p}(x) + \sum_{i=1}^{\infty}\Big(\frac{1}{p^{2i}} - \frac{1}{p^{2i-1}}\Big)\mathbb{I}_{p^{-i}\mathbb{Z}_p}(x)$$

$$= \mathbb{I}_{\mathbb{Z}_p}(x) + \sum_{i=\max\{1,-\mathrm{val}(x)\}}^{\infty}\Big(\frac{1}{(-p)^{2i}} + \frac{1}{(-p)^{2i-1}}\Big)$$

$$= \mathbb{I}_{\mathbb{Z}_p}(x) + \frac{(-p)^{-\big(2\max\{1,-\mathrm{val}(x)\}-1\big)}}{1+p^{-1}}$$

$$= \mathbb{I}_{\mathbb{Z}_p}(x) - \min\Big\{\frac{p^{-1}}{1+p^{-1}}, \frac{p|x|^{-2}}{1+p^{-1}}\Big\}.$$

Hence by (B.9.1) and (B.10.2) the invariant weighted orbital integral $I_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathfrak{g}'(\mathbb{Z}_p)})$ is equal to

$$\text{(B.10.5)} \qquad \Big(\frac{\mathrm{val}(\eta)}{2} - \frac{1}{p-1} + \frac{|\eta|^{1/2}}{p-1}\Big)\mathbb{I}_{\mathbb{Z}_p-\{0\}}(\eta)$$

$$-\Big(\Big(\mathrm{val}(\ ) - \frac{1}{p-1} + \frac{|\ |}{p-1}\Big)\mathbb{I}_{\mathbb{Z}_p}(\ )\Big)\hat{\ }(2\sqrt{\eta})$$

$$= \Big(\frac{\mathrm{val}(\eta)}{2} - \frac{1}{p-1} + \frac{|\eta|^{1/2}}{p-1}\Big)\mathbb{I}_{\mathbb{Z}_p-\{0\}}(\eta) + \frac{1}{p-1}\mathbb{I}_{\mathbb{Z}_p}(\eta)$$

$$-\Big(\mathrm{val}(\ )\mathbb{I}_{\mathbb{Z}_p}(\ )\Big)\hat{\ }(2\sqrt{\eta}) - \frac{1}{p-1}\Big(| \, |\mathbb{I}_{\mathbb{Z}_p}(\ )\Big)\hat{\ }(2\sqrt{\eta})$$

$$
\text{(B.10.6)} \qquad = \begin{cases} \dfrac{\mathrm{val}(\eta)}{2} + \dfrac{|\eta|^{1/2}}{p-1} & \\[2ex] -\dfrac{p^{-1}}{1-p^{-1}} - \dfrac{1}{p-1}\Big(1 - \dfrac{p^{-1}}{1+p^{-1}}\Big) & \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\[2ex] -\dfrac{|2\sqrt{\eta}|^{-1}}{1-p^{-1}} - \dfrac{1}{p-1}\Big(- \dfrac{p|2\sqrt{\eta}|^{-2}}{1+p^{-1}}\Big) & \text{otherwise,} \end{cases}
$$

$$
= \begin{cases} \dfrac{\mathrm{val}(\eta)}{2} + \dfrac{|\eta|^{1/2}}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{2+p^{-1}}{1+p^{-1}}\Big) & \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\[2ex] -\dfrac{|\eta|^{-1/2}}{1-p^{-1}} + \dfrac{|\eta|^{-1}}{(1-p^{-1})(1+p^{-1})} & \text{otherwise,} \end{cases}
$$

where the equality (B.10.6) follows from (B.10.3) and (B.10.4). □

(B.11) Remark If η is a split element of $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$, then $\phi_{1,v}(\eta)$ has three components indexed by the parabolic subgroups B, $\overline{\mathrm{B}}$ and G, and

$$
\text{(B.11.1)} \qquad \phi_{1,v}(\eta)_{\mathrm{B}} = \phi_{1,v}(\eta)_{\overline{\mathrm{B}}} = 1 \qquad \phi_{1,v}(\eta)_{\mathrm{G}} = -I_{\mathrm{M}}^{\mathrm{G}}(\widetilde{\eta}, \mathbb{I}_{\mathbb{Z}_p}).
$$

If η is an elliptic element of $\mathcal{A}_{\mathrm{reg}}(\mathbb{Q}_p)$, then $\phi_{1,v}(\eta)$ has one component corresponding to the parabolic subgroup G which is equal to $\phi_{0,v}(\eta)$.

(B.12) Remark: Towards an invariant trace formula The simple invariant trace formula (4.2.4.1) holds only for test functions which are cuspidal at two distinct places of $\mathbb{Q}$. An invariant trace formula, once extended to full generality for all Schwartz functions on $\mathfrak{g}(\mathbb{A})$, will likely involve more intricate distributions similar to the invariant weighted orbital integrals $I_{\mathrm{M}}^{\mathrm{G}}(X,\ )$ which are absent from (4.2.4.1) due to the cuspidality condition. One such identity could be established when $\mathfrak{g}$ is gl(2) following the arguments of Langlands in [La80].

Fix a sufficiently large finite set S of places of $\mathbb{Q}$, let f_S be a Schwartz function on $\mathfrak{g}(\mathbb{Q}_S)$.

It follows from the non-invariant trace formula (3.4.1.1) that

$$\text{(B.12.1)}\qquad \sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in\mathfrak{o}_{\mathrm{G},S}} a^{\mathrm{G}}(S,X)I_{\mathrm{G}}^{\mathrm{G}}(X,f_S)+$$

$$+\frac{1}{2}\sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in(\mathfrak{m}\cap\mathfrak{o})_{\mathrm{M},S}} a^{\mathrm{M}}(S,X)J_{\mathrm{M}}^{\mathrm{G}}(X,f_S)$$

$$=\sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in\mathfrak{o}_{\mathrm{G},S}} a^{\mathrm{G}}(S,X)I_{\mathrm{G}}^{\mathrm{G}}(X,f\hat{\ }_S)+$$

$$+\frac{1}{2}\sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in(\mathfrak{m}\cap\mathfrak{o})_{\mathrm{M},S}} a^{\mathrm{M}}(S,X)J_{\mathrm{M}}^{\mathrm{G}}(X,f\hat{\ }_S).$$

The weighted orbital integral $J_{\mathrm{M}}^{\mathrm{G}}(X,f_S)$, considered as a function in the variable X on $\mathfrak{m}(\mathbb{Q}_S)$, has at worst logarithmic singularity along the discriminant locus. There is a natural way to subtract from $J_{\mathrm{M}}^{\mathrm{G}}(X,f_S)$ an invariant function in X such that the remainder extends to a continuous function on $\mathfrak{m}(\mathbb{Q}_S)$ which is denoted by $J_{\mathrm{M}}^{\mathrm{G,reg}}(X,f_S)$. In general $J_{\mathrm{M}}^{\mathrm{G,reg}}(X,f_S)$ is not smooth, as could be seen for instance from (B.9.1). However $J_{\mathrm{M}}^{\mathrm{G,reg}}(X,f_S)$, when restricted to a function on the torus $\mathrm{M}(\mathbb{Q}_S)$, has integrable Mellin transform, which follows from the estimates in §9 of [La80]. By a suitable adelic extension of the arguments of Ferrar in [Fe37], it could be shown that the Poisson summation formula applies to the function $J_{\mathrm{M}}^{\mathrm{G,reg}}(X,f_S)$ and implies that

$$\text{(B.12.2)}\qquad \sum_{X\in\mathfrak{m}(\mathbb{Q})} J_{\mathrm{M}}^{\mathrm{G,reg}}(X,f_S) = \sum_{X\in\mathfrak{m}(\mathbb{Q})} J_{\mathrm{M}}^{\mathrm{G,reg}}(X,f_S)\hat{\ }$$

where the Fourier transform is taken over $\mathfrak{m}(\mathbb{Q}_S)$.

By definition the coefficients $a^{\mathrm{M}}(S,X)$ are independent of the point X, hence a suitable linear combination of (B.12.1) and (B.12.2) gives the identity

$$\text{(B.12.3)} \qquad \sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in\mathfrak{o}_{\mathrm{G},S}} a^{\mathrm{G}}(S,X) I^{\mathrm{G}}_{\mathrm{G}}(X,f_S) +$$

$$+\frac{1}{2}\sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in(\mathfrak{m}\cap\mathfrak{o})_{\mathrm{M},S}} a^{\mathrm{M}}(S,X)\Big(J^{\mathrm{G}}_{\mathrm{M}}(X,f_S) - J^{\mathrm{G,reg}}_{\mathrm{M}}(X,f^{\vee}{}_S)^{\wedge}\Big)$$

$$= \sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in\mathfrak{o}_{\mathrm{G},S}} a^{\mathrm{G}}(S,X) I^{\mathrm{G}}_{\mathrm{G}}(X,f^{\wedge}{}_S) +$$

$$+\frac{1}{2}\sum_{\mathfrak{o}\in\mathfrak{g}/\sim}\ \sum_{X\in(\mathfrak{m}\cap\mathfrak{o})_{\mathrm{M},S}} a^{\mathrm{M}}(S,X)\Big(J^{\mathrm{G}}_{\mathrm{M}}(X,f^{\wedge}{}_S) - J^{\mathrm{G,reg}}_{\mathrm{M}}(X,f_S)^{\wedge}\Big).$$

Since

$$\text{(B.12.4)} \qquad I^{\mathrm{G}}_{\mathrm{G}}(X,f_S) \quad \text{and} \quad J^{\mathrm{G}}_{\mathrm{M}}(X,f_S) - J^{\mathrm{G,reg}}_{\mathrm{M}}(X,f^{\vee}{}_S)^{\wedge}$$

are invariant under the adjoint action, the identity (B.12.3) is an invariant trace formula for gl(2) which is valid for all Schwartz functions on $\mathfrak{g}(\mathbb{Q}_S)$.

In order to extend this construction to a general reductive Lie algebra $\mathfrak{g}$, the singularities of higher rank weighted orbital integrals need to be separated, and the general invariant trace formula for each proper Levi subalgebra $\mathfrak{m}$ needs to be extended to test functions beyond the Schwartz space $\mathcal{S}(\mathfrak{m}(\mathbb{Q}_S))$.

(B.13) Remark Arthur's work [Ar88b] and [Ar88c] on the invariant trace formula for reductive groups G follows an approach which is dual to the approach taken in the previous remark. Instead of working with the geometric side of the trace formula, Arthur devised the cancellation of singularities on the spectral side of the trace formula, utilizing the underlying complex structure and in particular the powerful techniques of residue calculus. However such an alternative is unavailable in the Lie algebra case since both sides of the trace formula are geometric in nature.

(B.14) Remark Distributions similar to $J_{\mathrm{M}}^{\mathrm{G,reg}}(X, \ )$ has been constructed by Chaudouard in [Ch02b]. He has shown that his version of the weighted orbital integral $J_{\mathrm{M}}^{\mathrm{G},b}(X, \ )$ remains bounded as X traverses $\mathfrak{m}(\mathbb{Q}_S)$ while also enjoying many other nice properties expected of the usual weighted orbital integrals.

(B.15) Remark There has been some important recent work on extending the Arthur-Selberg trace formula beyond the usual space of Schwartz functions. See [FL11] and [Ma11].

CHAPTER 3

ON A CONJECTURE OF BRAVERMAN–KAZHDAN

3.1 Preliminaries on ℓ-adic γ-sheaves

In this section some preliminary results on ℓ-adic γ-sheaves established by Braverman–Kazhdan in [BK00] and [BK02] are recalled.

(1.1) Notation Let k be an algebraic closure of a finite field k_0 with q elements of sufficiently large characteristic p. Let ℓ be a prime number which is distinct from p, let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of the field of ℓ-adic numbers.

If X is a k-scheme, let $D^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$ denote the derived category of complexes of ℓ-adic étale sheaves on X with bounded constructible cohomology, let $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ denote the abelian subcategory of ℓ-adic perverse sheaves on X. If d is an integer, let $[d]$ denote the dth translation functor on $D^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$.

If f is a k-linear morphism of k-schemes, the six functors f^*, f_*, $f_!$, $f^!$, $\otimes_X$, and $\mathcal{H}om_X$ are understood in the derived sense. If j is the morphism of inclusion of an open k-subscheme, let $j_{!*}$ denote the intermediate extension functor of Goresky–MacPherson for ℓ-adic perverse sheaves (see [BBD82]).

Let $\mathbb{G}_{\mathrm{a},k}$ denote the k-additive group, let $\mathbb{G}_{\mathrm{m},k}$ denote the k-multiplicative group. Let ψ be a nontrivial $\overline{\mathbb{Q}}_\ell$-valued additive character of k_0, let $\mathcal{L}_\psi$ denote the Artin–Schreier sheaf on $\mathbb{G}_{\mathrm{a},k}$ defined as the twisted product of the Artin–Schreier covering of $\mathbb{G}_{\mathrm{a},k}$ by the character ψ (see [De77s]).

(1.2) Notation If n is a nonzero natural number, let $\mathrm{GL}_{n,k}$ denote the k-algebraic group of invertible $n \times n$ matrices under multiplication. Let $\mathrm{B}_{n,k}$ denote the Borel subgroup consisting of the upper triangular matrices, let $\mathrm{U}_{n,k}$ denote the unipotent radical of $\mathrm{B}_{n,k}$. Let $\mathrm{T}_{n,k}$ denote the maximal torus consisting of the diagonal matrices, let $\mathrm{N}_{n,k}$ denote the normalizer of $\mathrm{T}_{n,k}$ in $\mathrm{GL}_{n,k}$, let W_n denote the Weyl group which is the finite group underlying the constant finite k-algebraic group $\mathrm{N}_{n,k}/\mathrm{T}_{n,k}$.

Let $\mathrm{T}_{n,k}/\mathrm{W}_n$ denote the affine quotient of $\mathrm{T}_{n,k}$ by W_n which is the space of characteristic polynomials of invertible $n \times n$ matrices, let q_n denote the quotient morphism

$$\mathrm{q}_n : \quad \mathrm{T}_{n,k} \longrightarrow \mathrm{T}_{n,k}/\mathrm{W}_n. \tag{1.2.1}$$

Let $\mathrm{GL}^{\mathrm{reg}}_{n,k}$ denote the open k-subscheme of $\mathrm{GL}_{n,k}$ consisting of the regular matrices (see [St65]). Let j_n denote the inclusion morphism

$$\mathrm{j}_n : \quad \mathrm{GL}^{\mathrm{reg}}_{n,k} \longrightarrow \mathrm{GL}_{n,k}. \tag{1.2.2}$$

Let p_n denote the morphism which assigns to each regular invertible $n \times n$ matrix its characteristic polynomial

$$\mathrm{p}_n : \quad \mathrm{GL}^{\mathrm{reg}}_{n,k} \longrightarrow \mathrm{T}_{n,k}/\mathrm{W}_n. \tag{1.2.3}$$

Let $\widetilde{\mathrm{GL}}_{n,k}$ denote the total space of the Grothendieck–Springer resolution whose set of A-valued points $\widetilde{\mathrm{GL}}_{n,k}(A)$ is equal to

$$\left\{ \Big(g, h\mathrm{B}_{n,k}(A)\Big) : \begin{array}{cl} g & \in \mathrm{GL}_{n,k}(A), \\ h\mathrm{B}_{n,k}(A) & \in \mathrm{GL}_{n,k}(A)/\mathrm{B}_{n,k}(A), \\ h^{-1}gh & \in \mathrm{B}_{n,k}(A). \end{array} \right\} \tag{1.2.4}$$

for each k-algebra A. Let $\tilde{\mathrm{q}}_n$ denote the morphism

$$\tilde{\mathrm{q}}_n : \quad \widetilde{\mathrm{GL}}_{n,k} \longrightarrow \mathrm{GL}_{n,k} \tag{1.2.5}$$

whose induced map $\tilde{\mathrm{q}}_n(A)$ on the A-valued points is defined by

$$(1.2.6) \qquad \begin{aligned} \tilde{\mathrm{q}}_n(A)\Big(g, h\mathrm{B}_{n,k}(A)\Big) &= g \\ &\in \mathrm{GL}_{n,k}(A) \end{aligned}$$

for each k-algebra A. Let $\tilde{\mathrm{p}}_n$ denote the morphism

$$(1.2.7) \qquad \tilde{\mathrm{p}}_n : \quad \widetilde{\mathrm{GL}}_{n,k} \longrightarrow \mathrm{T}_{n,k}$$

whose induced map $\tilde{\mathrm{p}}_n(A)$ on the A-valued points is defined by

$$(1.2.8) \qquad \begin{aligned} \tilde{\mathrm{p}}_n(A)\Big(g, h\mathrm{B}_{n,k}(A)\Big) &= h^{-1}gh\mathrm{U}_{n,k}(A) \\ &\in \mathrm{B}_{n,k}(A)/\mathrm{U}_{n,k}(A) \\ &\simeq \mathrm{T}_{n,k}(A) \end{aligned}$$

for each k-algebra A.

If M is a standard Levi subgroup of $\mathrm{GL}_{n,k}$, then there exists a partition $(n_1, n_2, \ldots, n_m)$ of n such that

$$(1.2.9) \qquad \mathrm{M} \simeq \mathrm{GL}_{n_1,k} \times \mathrm{GL}_{n_2,k} \times \cdots \times \mathrm{GL}_{n_m,k}$$

consisting of the block diagonal matrices of size $(n_1, n_2, \ldots, n_m)$. Denote analogously

$$(1.2.10) \qquad \begin{aligned} \mathrm{W}_{\mathrm{M}} &\simeq \mathrm{W}_{n_1} \times \mathrm{W}_{n_2} \times \cdots \times \mathrm{W}_{n_m} \\ \mathrm{M}^{\mathrm{reg}} &\simeq \mathrm{GL}^{\mathrm{reg}}_{n_1,k} \times \mathrm{GL}^{\mathrm{reg}}_{n_2,k} \times \cdots \times \mathrm{GL}^{\mathrm{reg}}_{n_m,k} \\ \widetilde{\mathrm{M}} &\simeq \widetilde{\mathrm{GL}}_{n_1,k} \times \widetilde{\mathrm{GL}}_{n_2,k} \times \cdots \times \widetilde{\mathrm{GL}}_{n_m,k}, \end{aligned}$$

and

$$(1.2.11) \qquad \mathrm{q}_{\mathrm{M}} = \Pi \mathrm{q}_{n_i} : \quad \mathrm{T}_{n,k} \simeq \prod_i \mathrm{T}_{n_i,k} \longrightarrow \mathrm{T}_{n,k}/\mathrm{W}_{\mathrm{M}} \simeq \prod_i \mathrm{T}_{n_i,k}/\mathrm{W}_{n_i}$$

$$\mathrm{j}_{\mathrm{M}} = \Pi \mathrm{j}_{n_i} : \quad \mathrm{M}^{\mathrm{reg}} \simeq \prod_i \mathrm{GL}^{\mathrm{reg}}_{n_i,k} \longrightarrow \mathrm{M} \simeq \prod_i \mathrm{GL}_{n_i,k}$$

$$\mathrm{p}_{\mathrm{M}} = \Pi \mathrm{p}_{n_i} : \quad \mathrm{M}^{\mathrm{reg}} \simeq \prod_i \mathrm{GL}^{\mathrm{reg}}_{n_i,k} \longrightarrow \mathrm{T}_{n,k}/\mathrm{W}_{\mathrm{M}} \simeq \prod_i \mathrm{T}_{n_i,k}/\mathrm{W}_{n_i}$$

$$\tilde{\mathrm{q}}_{\mathrm{M}} = \Pi \tilde{\mathrm{q}}_{n_i} : \quad \widetilde{\mathrm{M}} \simeq \prod_i \widetilde{\mathrm{GL}}_{n_i,k} \longrightarrow \mathrm{M} \simeq \prod_i \mathrm{GL}_{n_i,k}$$

$$\tilde{\mathrm{p}}_{\mathrm{M}} = \Pi \tilde{\mathrm{p}}_{n_i} : \quad \widetilde{\mathrm{M}} \simeq \prod_i \widetilde{\mathrm{GL}}_{n_i,k} \longrightarrow \mathrm{T}_{n,k} \simeq \prod_i \mathrm{T}_{n_i,k}.$$

(1.3) Notation The ℓ-adic dual group of $\mathrm{GL}_{n,k}$ is the ℓ-adic Lie group $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Let $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ denote the standard maximal torus consisting of the diagonal matrices which is the ℓ-adic dual torus of $\mathrm{T}_{n,k}$. For each standard Levi subgroup $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ consisting of block diagonal matrices, let W_{M} denote the Weyl group of $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ identified with the subgroup of $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ consisting of the permutation matrices.

If ρ_{M} is a representation of a standard Levi subgroup $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ on an r-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V, choose an ordered basis $\mathcal{B}$ of V consisting of weight vectors under the action of $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ which is stable under the restriction of ρ_{M} to W_{M}. Identify the group of $\overline{\mathbb{Q}}_\ell$-linear automorphisms of V with $\mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ as ℓ-adic Lie groups via the choice of $\mathcal{B}$.

Let ρ_{T} denote the restriction of ρ_{M} to the diagonal matrices

$$(1.3.1) \qquad \rho_{\mathrm{T}} : \quad \check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell) \longrightarrow \check{\mathrm{T}}_r(\overline{\mathbb{Q}}_\ell),$$

let $\hat{\rho}_{\mathrm{T}}$ denote the morphism of k-tori

$$(1.3.2) \qquad \hat{\rho}_{\mathrm{T}} : \quad \mathrm{T}_{r,k} \longrightarrow \mathrm{T}_{n,k}$$

which is dual to ρ_{T}.

Let $\rho_{\mathrm{W_M}}$ denote the restriction of ρ_{M} to the permutation matrices

$$(1.3.3) \qquad \rho_{\mathrm{W_M}} : \quad \mathrm{W_M} \longrightarrow \mathrm{W}_r.$$

Let ε_{M} denote the restriction of the $\overline{\mathbb{Q}}_\ell$-valued sign character ε_n of W_n to the subgroup $\mathrm{W_M}$.

(1.4) Definition If $\mathcal{K}$ and $\mathcal{L}$ are objects in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{GL}_{n,k}, \overline{\mathbb{Q}}_\ell)$, define the *convolution product* $\mathcal{K} * \mathcal{L}$ to be the object

$$(1.4.1) \qquad \mathcal{K} * \mathcal{L} \quad = \quad \mu_!(\mathcal{K} \boxtimes \mathcal{L})$$

in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{GL}_{n,k}, \overline{\mathbb{Q}}_\ell)$, where $\boxtimes$ denotes the external tensor product and μ denotes the morphism of matrix multiplication

$$(1.4.2) \qquad \mu : \quad \mathrm{GL}_{n,k} \times \mathrm{GL}_{n,k} \longrightarrow \mathrm{GL}_{n,k}.$$

Identify $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell)$ with a full subcategory of $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{GL}_{n,k}, \overline{\mathbb{Q}}_\ell)$ by extension by zero which is stable under convolution. Let $*$ denote the restriction of the convolution product to $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell)$ as well.

(1.5) Definition If M is a standard Levi subgroup of $\mathrm{GL}_{n,k}$, fix notations as in the diagram

(1.5.1)
$$\begin{tikzcd}
\widetilde{\mathrm{M}} \arrow[r, "\tilde{\mathrm{p}}_{\mathrm{M}}"] \arrow[d, "\tilde{\mathrm{q}}_{\mathrm{M}}"'] & \mathrm{T}_{n,k} \\
\mathrm{M}. &
\end{tikzcd}$$

Define the *parabolic induction functor*

(1.5.2)
$$\mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{M}} : \quad D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{M}, \overline{\mathbb{Q}}_\ell)$$

to be the composite functor

(1.5.3)
$$\mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{M}} \;=\; \tilde{\mathrm{q}}_{\mathrm{M},!} \circ \tilde{\mathrm{p}}_{\mathrm{M}}{}^{*}[d]$$

where

(1.5.4)
$$d \;=\; \dim_k(\mathrm{M}) - n.$$

(1.6) Definition If P is a standard parabolic subgroup of $\mathrm{GL}_{n,k}$, let M denote its Levi component, fix notations as in the diagram

(1.6.1)
$$\begin{tikzcd}
\mathrm{P} \arrow[r, "\mathrm{i}"] \arrow[d, "\mathrm{q}"'] & \mathrm{GL}_{n,k} \\
\mathrm{M} &
\end{tikzcd}$$

where the arrows denote the inclusion and quotient morphisms. Define the *parabolic restriction functor*

(1.6.2)
$$\mathrm{Res}_{\mathrm{M}}^{\mathrm{GL}_{n,k}} : \quad D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{GL}_{n,k}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{M}, \overline{\mathbb{Q}}_\ell)$$

to be the composite functor

$$\mathrm{Res}_{\mathrm{M}}^{\mathrm{GL}_{n,k}} \;=\; \mathrm{q}_! \circ \mathrm{i}^*. \tag{1.6.3}$$

(1.7) Proposition *Let* $\mathcal{K}$ *be an object in* $D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell)$*, let* Φ *be an object in* $\mathrm{Perv}(\mathrm{GL}_{n,k}, \overline{\mathbb{Q}}_\ell)$ *which is equivariant with respect to the adjoint action. Let* q *denote the quotient morphism*

$$\mathrm{q}: \quad \mathrm{GL}_{n,k} \longrightarrow \mathrm{GL}_{n,k}/\mathrm{U}_{n,k}. \tag{1.7.1}$$

If the object $\mathrm{q}_!(\Phi)$ *in* $D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{GL}_{n,k}/\mathrm{U}_{n,k}, \overline{\mathbb{Q}}_\ell)$ *is supported on the* k*-subscheme* $\mathrm{B}_{n,k}/\mathrm{U}_{n,k}$*, then*

$$\Phi * \mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{GL}_{n,k}}(\mathcal{K}) \;\simeq\; \mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{GL}_{n,k}}\big(\mathrm{Res}_{\mathrm{T}_{n,k}}^{\mathrm{GL}_{n,k}}(\Phi) * \mathcal{K}\big). \tag{1.7.2}$$

Proof. This is Proposition 2.9 in [BK02]. □

(1.8) Definition If ρ_{M} is a $\overline{\mathbb{Q}}_\ell$-linear representation of a standard Levi subgroup $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, then define ρ_{M} to be *positive* if the restriction of ρ_{M} to the center of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is a direct sum of $\overline{\mathbb{Q}}_\ell$-valued characters χ_i of $\overline{\mathbb{Q}}_\ell^{\times}$ of the form

$$\chi_i(t) \;=\; t^{\alpha_i} \tag{1.8.1}$$

where the indices α_i are positive integers.

(1.9) Proposition *Let ρ_{T} be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ with a choice of $\mathcal{B}$, fix notations as in the diagram*

(1.9.1)
$$\begin{tikzcd}
\mathrm{T}_{r,k} \arrow[r, "\mathrm{tr}"] \arrow[d, "\hat{\rho}_{\mathrm{T}}"'] & \mathbb{G}_{\mathrm{a},k} \\
\mathrm{T}_{n,k}.
\end{tikzcd}$$

If ρ_{T} is positive, then the forget support morphism

(1.9.2)
$$\hat{\rho}_{\mathrm{T},!}\big(\mathrm{tr}^*(\mathcal{L}_\psi)[n]\big) \longrightarrow \hat{\rho}_{\mathrm{T},*}\big(\mathrm{tr}^*(\mathcal{L}_\psi)[n]\big)$$

defines an isomorphism between irreducible objects in $\mathrm{Perv}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell)$ *which is independent of the choice of $\mathcal{B}$ upto an isomorphism.*

Proof. This is Theorem 4.2 in [BK02]. □

(1.10) Definition If ρ_{T} is a $\overline{\mathbb{Q}}_\ell$-linear representation of $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ which is positive, define the *ℓ-adic hypergeometric sheaf of Braverman–Kazhdan* $\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}}$ to be the object

(1.10.1)
$$\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \;=\; \hat{\rho}_{\mathrm{T},!}\big(\mathrm{tr}^*(\mathcal{L}_\psi)[n]\big)$$

in $\mathrm{Perv}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell)$ (see [GL96]).

(1.11) Proposition *Let ρ_{M} be a $\overline{\mathbb{Q}}_\ell$-linear representation of a standard Levi subgroup* $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ *of* $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *with a choice of $\mathcal{B}$, fix notations as in the diagram*

(1.11.1)
$$\begin{tikzcd}
 & \mathrm{M}^{\mathrm{reg}} \arrow[r, "\mathrm{j_M}"] \arrow[d, "\mathrm{p_M}"'] & \mathrm{M} \\
\mathrm{T}_{n,k} \arrow[r, "\mathrm{q_M}"] & \mathrm{T}_{n,k}/\mathrm{W_M}.
\end{tikzcd}$$

If ρ_{M} is positive, then

$$\mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{M}}(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}}) \tag{1.11.2}$$

is a semisimple object in $\mathrm{Perv}(\mathrm{M},\overline{\mathbb{Q}}_\ell)$ *which is isomorphic to the Goresky–MacPherson intermediate extension of its restriction to* $\mathrm{GL}_{n,k}^{\mathrm{reg}}$, *hence*

$$\mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{M}}(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}}) \;\simeq\; \mathrm{j}_{\mathrm{M},!*}\Big(\mathrm{p}_{\mathrm{M}}{}^{*}\big(\mathrm{q}_{\mathrm{M},!}(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\big)[d]\Big) \tag{1.11.3}$$

where

$$d \;=\; \dim_k(\mathrm{M}) - n \tag{1.11.4}$$

as W_r*-equivariant objects in* $\mathrm{Perv}(\mathrm{M},\overline{\mathbb{Q}}_\ell)$ *where r denotes the dimension of ρ_{M} over $\overline{\mathbb{Q}}_\ell$.*

Let W_{M} *operate on* (1.11.2) *via* $\rho_{\mathrm{W}_{\mathrm{M}}}$, *then*

$$\mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{M}}\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{n,k}}{\otimes} \big(\varepsilon_{\mathrm{M}} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{\mathrm{M}}})\big)\Big) \tag{1.11.5}$$

is independent of the choice of $\mathcal{B}$ upto an isomorphism as a W_{M}*-equivariant object in* $\mathrm{Perv}(\mathrm{M},\overline{\mathbb{Q}}_\ell)$.

Proof. This is Proposition 6.2 and Proposition 6.4 in [BK02]. □

(1.12) Definition If ρ_{M} is an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of a standard Levi subgroup $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$ of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is positive, define the *ℓ-adic γ-sheaf of Braverman–Kazhdan* $\Phi_{\psi,\rho_{\mathrm{M}}}$ to be the W_{M}-invariant summand

$$\Phi_{\psi,\rho_{\mathrm{M}}} \;=\; \Big(\mathrm{Ind}_{\mathrm{T}_{n,k}}^{\mathrm{M}}\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{n,k}}{\otimes} \big(\varepsilon_{\mathrm{M}} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{\mathrm{M}}})\big)\Big)\Big)^{\mathrm{W}_{\mathrm{M}}} \tag{1.12.1}$$

as an object in $\mathrm{Perv}(\mathrm{M},\overline{\mathbb{Q}}_\ell)$.

(1.13) Proposition *Let* ρ *be a* $\overline{\mathbb{Q}}_\ell$*-linear representation of* $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$*, let* P *be a standard parabolic subgroup of* $\mathrm{GL}_{n,k}$ *with Levi component* M*, let* ρ_{M} *denote the restriction of* ρ *to the standard Levi subgroup* $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$*.*

If ρ *is positive, then* $\Phi_{\psi,\rho_{\mathrm{M}}}$ *is an irreducible object in* $\mathrm{Perv}(\mathrm{M},\overline{\mathbb{Q}}_\ell)$ *which is equivariant with respect to the adjoint action of* M*, and*

$$\Phi_{\psi,\rho_{\mathrm{M}}} \simeq \mathrm{Res}_{\mathrm{M}}^{\mathrm{GL}_{n,k}}(\Phi_{\psi,\rho}). \tag{1.13.1}$$

Proof. This is Proposition 6.4 and Theorem 6.6 in [BK02]. □

(1.14) Conjecture (Conjecture 9.12 in [BK00])
Let ρ *be a* $\overline{\mathbb{Q}}_\ell$*-linear representation of* $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *which is positive. Let* g *be an element in* $\mathrm{GL}_{n,k}(k)$*, let* i *denote the inclusion morphism*

$$\mathrm{i}: \quad g\mathrm{U}_{n,k} \longrightarrow \mathrm{GL}_{n,k} \tag{1.14.1}$$

of the left coset $g\mathrm{U}_{n,k}$*.*

If g *is not contained in* $\mathrm{B}_{n,k}(k)$*, then*

$$H_{\mathrm{c}}^{\bullet}\big(g\mathrm{U}_{n,k}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \simeq 0. \tag{1.14.2}$$

3.2 A motivating example: GL(2)

In this section a proof of Conjecture 9.12 in [BK00] for GL(2) due to Braverman–Kazhdan in [BK02] is recalled.

(2.1) Lemma A *Let ρ be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of* $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$*, fix notations as in the diagram*

(2.1.1)
$$\mathrm{T}_{2,k} \xrightarrow{\mathrm{q}_2} \mathrm{T}_{2,k}/\mathrm{W}_2 \xrightarrow{\det} \mathbb{G}_{\mathrm{m},k}.$$

If ρ is positive, then

(2.1.2)
$$\det{}_!\Big(\mathrm{q}_{2,!}\Big(\mathrm{Hyp}_{\psi,\rho_\mathrm{T}} \underset{\mathrm{T}_{2,k}}{\otimes} \big(\varepsilon_2 \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_2})\big)\Big)^{\mathrm{W}_2}\Big) \;\simeq\; 0$$

as objects in $D^\mathrm{b}_\mathrm{c}(\mathbb{G}_{\mathrm{m},k}, \overline{\mathbb{Q}}_\ell)$.

Proof. The left-hand side of (2.1.2) is a direct summand of

(2.1.3)
$$(\det \circ \, \mathrm{q}_2)_!(\mathrm{Hyp}_{\psi,\rho_\mathrm{T}}).$$

Since the representation ρ is positive, the object (2.1.3) in $D^\mathrm{b}_\mathrm{c}(\mathbb{G}_{\mathrm{m},k}, \overline{\mathbb{Q}}_\ell)$ is a generalized Kloosterman sheaf in the sense of [De77s], which by Théorème 7.8 in [De77s] is an irreducible perverse sheaf. Hence the left-hand side of (2.1.2) is isomorphic to either (2.1.3) or zero.

By Proposition 7.20 in [De77s] the symmetric group W_r operates on (2.1.3) via ε_r, hence the actions of W_2 on the left-hand side of (2.1.2) and (2.1.3) are incompatible. Hence the isomorphism (2.1.2) holds. □

(2.2) Lemma B *Let g be an element in* $\mathrm{GL}_{2,k}(k)$*, fix notations as in the diagram*

(2.2.1)
$$\begin{tikzcd}
g\mathrm{U}_{2,k} \arrow[r, dashed, "\mathrm{i}^\mathrm{reg}"] \arrow[d, equal] & \mathrm{GL}^\mathrm{reg}_{2,k} \arrow[r, "\mathrm{p}_2"] \arrow[d, "\mathrm{j}_2"'] & \mathrm{T}_{2,k}/\mathrm{W}_2 \arrow[dd, "\det"'] \\
g\mathrm{U}_{2,k} \arrow[r, "\mathrm{i}"] \arrow[d, "\det"'] & \mathrm{GL}_{2,k} & \\
\mathrm{Spec}(k) \arrow[rr, "\det(g)"] & & \mathbb{G}_{\mathrm{m},k}
\end{tikzcd}$$

where i *denotes the inclusion morphism,* $\mathrm{i}^{\mathrm{reg}}$ *denotes the lift of* i *through* j_2 *if such a lift exists, and* $\det(g)$ *denotes the inclusion morphism of the point* $\det(g)$ *in* $\mathbb{G}_{\mathrm{m},k}$.

If g *is not contained in* $\mathrm{B}_{2,k}(k)$*, then* i *lifts through* j_2*, and* $\mathrm{p}_2 \circ \mathrm{i}^{\mathrm{reg}}$ *defines an isomorphism between* $g\mathrm{U}_{2,k}$ *and the fiber*

$$g\mathrm{U}_{2,k} \;\simeq\; \mathrm{T}_{2,k}/\mathrm{W}_2 \underset{\mathbb{G}_{\mathrm{m},k}}{\times} \mathrm{Spec}(k). \tag{2.2.2}$$

Proof. By assumption g is not contained in $\mathrm{B}_{2,k}(k)$, hence

$$g \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.2.3}$$

where c is nonzero, hence for each k-algebra A the coset $g\mathrm{U}_{2,k}(A)$ consists of the matrices with coefficients in A of the form

$$\begin{pmatrix} a & b+au \\ c & d+cu \end{pmatrix} \tag{2.2.4}$$

where u is an element of A, which is contained in $\mathrm{GL}_{2,k}^{\mathrm{reg}}(A)$ whose complement in $\mathrm{GL}_{2,k}(A)$ is the set of scalar matrices.

Since c is nonzero, taking trace defines an isomorphism between $g\mathrm{U}_{2,k}$ and $\mathbb{G}_{\mathrm{a},k}$, hence the p_2 restricts to an isomorphism between $g\mathrm{U}_{2,k}$ and its image in $\mathrm{T}_{2,k}/\mathrm{W}_2$ which is the fiber of $\mathrm{T}_{2,k}/\mathrm{W}_2$ at the point $\det(g)$. □

(2.3) Proposition C *Let* ρ *be a* $\overline{\mathbb{Q}}_\ell$*-linear representation of* $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ *which is positive. Let* g *be an element in* $\mathrm{GL}_{2,k}(k)$*, let* i *denote the inclusion morphism*

$$\mathrm{i}: \quad g\mathrm{U}_{2,k} \longrightarrow \mathrm{GL}_{2,k} \tag{2.3.1}$$

of the left coset $g\mathrm{U}_{2,k}$.

If g *is not contained in* $\mathrm{B}_{2,k}(k)$*, then*

$$H_{\mathrm{c}}^{\bullet}\big(g\mathrm{U}_{2,k}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \;\simeq\; 0. \tag{2.3.2}$$

Proof. The following argument is due to Braverman–Kazhdan in [BK02].

By Proposition (1.11) and Lemma B (2.2)

$$\mathrm{i}^*(\Phi_{\psi,\rho}) \;\simeq\; \mathrm{i}^*\Big(\mathrm{p_2}^*\Big(\mathrm{q}_{2,!}\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{2,k}}{\otimes} \big(\varepsilon_2 \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_2})\big)\Big)^{\mathrm{W}_2}\Big)\Big), \tag{2.3.3}$$

hence by Lemma A (2.1)

$$\begin{aligned} &H_{\mathrm{c}}^{\bullet}\big(g\mathrm{U}_{2,k}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \\ \simeq\;& \mathrm{det}_!\Big(\mathrm{q}_{2,!}\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{2,k}}{\otimes} \big(\varepsilon_2 \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_2})\big)\Big)^{\mathrm{W}_2}\Big)\Big|_{\det(g)} \\ \simeq\;& 0 \end{aligned} \tag{2.3.4}$$

where $|_{\det(g)}$ denotes the stalk at $\det(g)$. □

3.3 Lemmas on mirabolic groups

In this section some lemmas on mirabolic groups needed for the proof of Conjecture 9.12 in [BK00] for GL(n) are established.

(3.1) Notation Let V_n denote the standard n-dimensional vector space over k with standard basis

$$\Big\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\Big\} \tag{3.1.1}$$

and standard dual basis

$$\left\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n\right\}. \tag{3.1.2}$$

Let H_n denote the subspace

$$H_n \;=\; \hat{\mathbf{e}}_n{}^{\perp} \tag{3.1.3}$$

of V_n.

Let $\mathrm{Q}_{n,k}$ denote the *general mirabolic subgroup* of $\mathrm{GL}_{n,k}$ defined to be the parabolic subgroup which preserves the hyperplane H_n, let $\mathrm{H}_{n,k}$ denote the unipotent radical of $\mathrm{Q}_{n,k}$ which is isomorphic to $\mathbb{G}_{\mathrm{a},k}{}^{n-1}$.

If $\vec{u}$ is a vector

$$\vec{u} \;=\; \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ 0 \end{pmatrix} \tag{3.1.4}$$

in H_n, let u denote the $n \times n$ matrix

$$\begin{aligned} u \;&=\; \left(\begin{array}{cccc|c} 1 & & & & u_1 \\ & 1 & & & u_2 \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1} \\ \hline & & & & 1 \end{array}\right) \\ &=\; I_n + (\quad|\vec{u}) \end{aligned} \tag{3.1.5}$$

where I_n denotes the $n \times n$ identity matrix. The assignment of u to $\vec{u}$ defines an isomorphism between H_n and $\mathrm{H}_{n,k}(k)$ as k-vector spaces.

(3.2) Lemma *Let m be a natural number, let Λ^m denote the mth exterior power functor. Let g be an element in $\mathrm{GL}_{n,k}(k)$.*

Let Λ^m_g be the map

$$\Lambda^m_g : \quad H_n \longrightarrow \mathrm{End}_k\big(\Lambda^m(V_n)\big) \tag{3.2.1}$$

which assigns to each vector $\vec{u}$ in H_n the k-linear endormorphism of $\Lambda^m(V_n)$

$$\Lambda^m_g(\vec{u}) \quad = \quad \Lambda^m(gu) - \Lambda^m(g), \tag{3.2.2}$$

then Λ^m_g is a k-linear transformation.

Proof. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be m vectors in V, then

$$\begin{aligned}
&\Lambda^m_g(\vec{u})\big(\vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\big) \qquad (3.2.3)\\
&= \Lambda^m(gu)\big(\vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\big) - \Lambda^m(g)\big(\vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\big)\\
&= \Lambda^m(g)\Big(\Lambda^m(u)\big(\vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\big) - \vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\Big)\\
&= \Lambda^m(g)\Big((u\vec{v}_1) \wedge (u\vec{v}_2) \wedge \cdots \wedge (u\vec{v}_m) - \vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\Big)\\
&= \Lambda^m(g)\Big(\big(\vec{v}_1 + \langle \hat{\mathbf{e}}_n, \vec{v}_1\rangle \vec{u}\big) \wedge \big(\vec{v}_2 + \langle \hat{\mathbf{e}}_n, \vec{v}_2\rangle \vec{u}\big) \wedge \ldots\\
&\qquad \cdots \wedge \big(\vec{v}_m + \langle \hat{\mathbf{e}}_n, \vec{v}_m\rangle \vec{u}\big) - \vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_m\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Lambda^m(g)\Big(\langle \hat{\mathbf{e}}_n, \vec{v}_1\rangle \vec{u}\wedge \vec{v}_2 \wedge \dots \wedge \vec{v}_m + \langle \hat{\mathbf{e}}_n, \vec{v}_2\rangle \vec{v}_1 \wedge \vec{u} \wedge \dots \wedge \vec{v}_m + \dots \\
&\qquad \dots + \langle \hat{\mathbf{e}}_n, \vec{v}_m\rangle \vec{v}_1 \wedge \vec{v}_2 \wedge \dots \wedge \vec{u}\Big) \\
&= \Lambda^m(g)\Big(\vec{u}\wedge \Big(\langle \hat{\mathbf{e}}_n, \vec{v}_1\rangle \vec{v}_2 \wedge \dots \wedge \vec{v}_m - \langle \hat{\mathbf{e}}_n, \vec{v}_2\rangle \vec{v}_1 \wedge \vec{v}_3 \wedge \dots \wedge \vec{v}_m + \dots \\
&\qquad \dots \pm \langle \hat{\mathbf{e}}_n, \vec{v}_m\rangle \vec{v}_1 \wedge \vec{v}_2 \wedge \dots \wedge \vec{v}_{m-1}\Big)\Big),
\end{aligned}
$$

which depends k-linearly on $\vec{u}$. □

(3.3) Lemma *Let g be an element in $\mathrm{GL}_{n,k}(k)$, let $\vec{u}$ be a vector in H_n. Then g and gu have the same characteristic polynomial if and only if*

$$
\langle \hat{\mathbf{e}}_n, g\vec{u}\rangle,\ \langle \hat{\mathbf{e}}_n, g^2\vec{u}\rangle,\ \langle \hat{\mathbf{e}}_n, g^3\vec{u}\rangle,\ \langle \hat{\mathbf{e}}_n, g^4\vec{u}\rangle,\ \langle \hat{\mathbf{e}}_n, g^5\vec{u}\rangle,\ \dots \tag{3.3.1}
$$

all vanish.

Proof. Since the characteristic of k is sufficiently large, the matrices g and gu have the same characteristic polynomial if and only if

$$
\mathrm{tr}(g) = \mathrm{tr}(gu),\ \mathrm{tr}(g^2) = \mathrm{tr}((gu)^2),\ \mathrm{tr}(g^3) = \mathrm{tr}((gu)^3),\ \dots \tag{3.3.2}
$$

By the binomial expansion for noncommutative variables

$$
\begin{aligned}
\mathrm{tr}((gu)^m) &= \mathrm{tr}\Big(\big(g + g(\quad|\vec{u})\big)^m\Big) \\
&= \mathrm{tr}\Big(g^m + g^{m-1}g(\quad|\vec{u}) + \dots + \big(g(\quad|\vec{u})\big)^m\Big)
\end{aligned} \tag{3.3.3}
$$

$$= \quad \mathrm{tr}(g^m) + \mathrm{tr}\big(g^m(\quad|\vec{u})\big) + \cdots + \mathrm{tr}\big((g(\quad|\vec{u}))^m\big).$$

Since the trace of a product of square matrices is invariant under cyclic permutations, each summand except for the first one on the right hand side of (3.3.3) is equal to an expression of the form

$$\mathrm{tr}\Big(g^{\alpha_1}(\quad|\vec{u})g^{\alpha_2}(\quad|\vec{u})g^{\alpha_3}(\quad|\vec{u})\cdots g^{\alpha_l}(\quad|\vec{u})\Big) \tag{3.3.4}$$

$$= \quad \mathrm{tr}\Big((\quad|g^{\alpha_1}\vec{u})(\quad|g^{\alpha_2}\vec{u})(\quad|g^{\alpha_3}\vec{u})\cdots(\quad|g^{\alpha_l}\vec{u})\Big)$$

$$= \quad \langle\hat{\mathbf{e}}_n, g^{\alpha_1}\vec{u}\rangle \cdot \langle\hat{\mathbf{e}}_n, g^{\alpha_2}\vec{u}\rangle \cdot \langle\hat{\mathbf{e}}_n, g^{\alpha_3}\vec{u}\rangle \cdots \langle\hat{\mathbf{e}}_n, g^{\alpha_l}\vec{u}\rangle$$

where each index α_i is nonnegative and

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_l \quad = \quad m. \tag{3.3.5}$$

Hence the lemma follows by induction on m. □

(3.4) Lemma *Let n_1 and n_2 be two natural numbers which sum to n. Let g be a block upper triangular matrix*

$$g \quad = \quad \left(\begin{array}{c|c} g_1 & g_3 \\ \hline & g_2 \end{array}\right) \tag{3.4.1}$$

in $\mathrm{GL}_{n,k}(k)$ *of size* (n_1, n_2). *Let* U *denote the unipotent subgroup of* $\mathrm{GL}_{n,k}$ *consisting of the matrices of the form*

$$\left(\begin{array}{c|c} I_{n_1} & * \\ \hline & I_{n_2} \end{array}\right) \tag{3.4.2}$$

where I_r *denotes the* $r \times r$ *identity matrix. If* m *is a natural number which is less than or equal to* n, *let* U_m *denote the* k*-subgroup of* U *consisting of the matrices which are of the form*

$$\left(\begin{array}{c|c} I_m & * \\ \hline & I_{n-m} \end{array}\right) \tag{3.4.3}$$

as well.

If g_1 *is of the form*

$$\left(\begin{array}{c|c} * & * \\ \hline I_{n_1-1} & * \end{array}\right), \tag{3.4.4}$$

then there exists a k*-subgroup of* U *which is isomorphic to* ${\mathbb{G}_{\mathrm{a},k}}^{n_2}$ *and operates freely on* $g\mathrm{U}$ *by conjugation, such that the resultant* ${\mathbb{G}_{\mathrm{a},k}}^{n_2}$*-principal bundle admits the subcoset* $g\mathrm{U}_{n_1-1}$ *as a cross section.*

If g_2 *is of the form*

$$\left(\begin{array}{c|c} * & * \\ \hline I_{n_2-1} & * \end{array}\right), \tag{3.4.5}$$

then there exists a k*-subgroup of* U *which is isomorphic to* ${\mathbb{G}_{\mathrm{a},k}}^{n_1\times(n_2-1)}$ *and operates freely on* $g\mathrm{U}$ *by conjugation, such that the resultant* ${\mathbb{G}_{\mathrm{a},k}}^{n_1\times(n_2-1)}$*-principal bundle admits the subcoset* $g\mathrm{U}_{n-1}$ *as a cross section.*

Proof. Let A be a k-algebra. Let

$$u = \left(\begin{array}{cccc|c} 1 & & & & \\ & \ddots & & & \\ & & 1 & & {}^{\mathrm{T}}\vec{u}_{n_1-1} \\ & & & 1 & \\ \hline & & & & I_{n_2} \end{array}\right) \tag{3.4.6}$$

be an element of $\mathrm{U}(A)$ where T denotes matrix transposition, let

$$g' = \left(\begin{array}{ccccc|c} * & * & \cdots & * & * & * \\ 1 & & & & * & * \\ & \ddots & & & \vdots & \vdots \\ & & 1 & & * & * \\ & & & 1 & * & {}^{\mathrm{T}}\vec{a}_{n_1} \\ \hline & & & & & g_2 \end{array}\right) \tag{3.4.7}$$

be an element of $g\mathrm{U}(A)$, then $u^{-1}g'u$ is equal to

$$\left(\begin{array}{ccccc|c} * & * & \cdots & * & * & * \\ 1 & & & & * & * \\ & \ddots & & & \vdots & \vdots \\ & & 1 & & * & * \\ & & & 1 & * & {}^{\mathrm{T}}\vec{a}_{n_1} - {}^{\mathrm{T}}\vec{u}_{n_1-1} \\ \hline & & & & & g_2 \end{array}\right). \tag{3.4.8}$$

Hence the second statement in the lemma follows from the fact that

$${}^{\mathrm{T}}\vec{a}_{n_1} - {}^{\mathrm{T}}\vec{u}_{n_1-1} \tag{3.4.9}$$

attains each value in A^{n_2} once and only once as ${}^{\mathrm{T}}\vec{u}_{n_1-1}$ traverses A^{n_2}.

Reset notations in the first half of the proof. Let A be a k-algebra. Let

$$u = \left(\begin{array}{c|cccc} I_{n_1} & & \vec{u}_2 & \cdots & \vec{u}_{n_2} \\ \hline & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{array}\right) \tag{3.4.10}$$

be an element of $\mathrm{U}(A)$, let

$$g' = \left(\begin{array}{c|ccccc} g_1 & \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{n_2-1} & * \\ \hline & * & * & \cdots & * & * \\ & 1 & & & & * \\ & & 1 & & & * \\ & & & \ddots & & \vdots \\ & & & & 1 & * \end{array}\right) \tag{3.4.11}$$

be an element of $g\mathrm{U}(A)$, then $u^{-1}g'u$ is equal to

$$\left(\begin{array}{c|ccccc} g_1 & \vec{a}_1 - \vec{u}_2 & \vec{a}_2 + g_1\vec{u}_2 - \vec{u}_3 & \cdots & \vec{a}_{n_2-1} + g_1\vec{u}_{n_2-1} - \vec{u}_{n_2} & * \\ \hline & * & * & \cdots & * & * \\ & 1 & & & & * \\ & & 1 & & & * \\ & & & \ddots & & \vdots \\ & & & & 1 & * \end{array}\right). \tag{3.4.12}$$

Hence the second statement in the lemma follows from the fact that the first $n_2 - 1$ columns of the right upper block of (3.4.12)

$$\begin{cases} \vec{a}_1 - \vec{u}_2 \\ \vec{a}_2 + g_1\vec{u}_2 - \vec{u}_3 \\ \vec{a}_3 + g_1\vec{u}_3 - \vec{u}_4 \\ \cdots \\ \vec{a}_{n_2-1} + g_1\vec{u}_{n_2-1} - \vec{u}_{n_2} \end{cases} \tag{3.4.13}$$

attains each value in $A^{n_1\times(n_2-1)}$ once and only once as $(\vec{u}_2, \ldots, \vec{u}_{n_2})$ traverses $A^{n_1\times(n_2-1)}$.

□

3.4 A proof for $\mathrm{GL}(n)$

In this section a proof of Conjecture 9.12 in [BK00] together its extension to parabolic subgroups is given for $\mathrm{GL}(n)$.

(4.1) Lemma A† *Let n_1, n_2, n_3 be three natural numbers which sum to n such that n_2 is greater than one, let M denote the standard Levi subgroup of $\mathrm{GL}_{n,k}$ consisting of the block diagonal matrices of size (n_1, n_2, n_3). Let ρ_{M} be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$, fix notations as in the diagram*

$$\begin{tikzcd} \mathrm{T}_{n,k} \simeq \mathrm{T}_{n_1,k} \times \mathrm{T}_{n_2,k} \times \mathrm{T}_{n_3,k} \arrow[r, "\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I}"] & \mathrm{T}_{n_1,k} \times \mathrm{T}_{n_2,k}/\mathrm{W}_{n_2} \times \mathrm{T}_{n_3,k} \arrow[d, "\mathrm{I}\times\det\times\mathrm{I}"'] \\ & \mathrm{T}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_3,k} \end{tikzcd} \tag{4.1.1}$$

where I denotes the identity morphism.

If ρ_{M} is positive, then

$$(\mathrm{I}\times\det\times\mathrm{I})_!\Big((\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})_!\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{n,k}}{\otimes} \big(\varepsilon_{n_2} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{n_2}})\big)\Big)^{\mathrm{W}_{n_2}}\Big) \tag{4.1.2}$$

vanishes as an object in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{T}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_3,k}, \overline{\mathbb{Q}}_\ell)$*, where* $\rho_{\mathrm{W}_{n_2}}$ *denotes the restriction of* $\rho_{\mathrm{W}_{\mathrm{M}}}$ *to the second direct factor of* W_{M} *with respect to the direct factorization*

$$(4.1.3) \qquad \mathrm{W}_{\mathrm{M}} \ \simeq \ \mathrm{W}_{n_1} \times \mathrm{W}_{n_2} \times \mathrm{W}_{n_3}.$$

Proof. The left-hand side of (4.1.2) is a direct summand of

$$(4.1.4) \qquad \big((\mathrm{I} \times \det \times \mathrm{I}) \circ (\mathrm{I} \times \mathrm{q}_{n_2} \times \mathrm{I})\big)_!(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}}) \ \simeq \ \mathrm{Hyp}_{\psi,\rho'_{\mathrm{T}}}[d],$$

where

$$(4.1.5) \qquad d \ = \ n_2 - 1,$$

and ρ'_{T} denotes the restriction of ρ_{T} to the subtorus

$$(4.1.6) \qquad \begin{aligned} \check{\mathrm{T}}_{n_1}(\overline{\mathbb{Q}}_\ell) \times \overline{\mathbb{Q}}_\ell^{\times} \times \check{\mathrm{T}}_{n_3}(\overline{\mathbb{Q}}_\ell) \ &\subset \ \check{\mathrm{T}}_{n_1}(\overline{\mathbb{Q}}_\ell) \times \check{\mathrm{T}}_{n_2}(\overline{\mathbb{Q}}_\ell) \times \check{\mathrm{T}}_{n_3}(\overline{\mathbb{Q}}_\ell) \\ &\simeq \ \check{\mathrm{T}}_{n}(\overline{\mathbb{Q}}_\ell) \end{aligned}$$

where $\overline{\mathbb{Q}}_\ell^{\times}$ embeds into $\check{\mathrm{T}}_{n_2}(\overline{\mathbb{Q}}_\ell)$ diagonally, which is positive since ρ_{M}, hence ρ_{T}, is positive.

By Proposition (1.9) the object (4.1.4) in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{T}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_3,k}, \overline{\mathbb{Q}}_\ell)$ is a shifted irreducible perverse sheaf, hence the left-hand side of (4.1.2) is isomorphic to either (4.1.4) or zero.

By Proposition 7.20 in [De77s] the symmetric group W_r operates on (4.1.4) via ε_r. Since n_2 is greater than one, the character ε_{n_2} in nontrivial, hence the actions of W_{n_2} on the left-hand side of (4.1.2) and (4.1.4) are incompatible. Hence the isomorphism (4.1.2) holds. □

(4.2) Lemma B† *Let* n_1*,* n_2*,* n_3 *be three natural numbers which sum to* n*, let* M *denote the standard Levi subgroup of* $\mathrm{GL}_{n,k}$ *consisting of the block diagonal matrices of size*

$(n_1, 1, \ldots, 1, n_3)$. *Let* ρ_{M} *be an* r*-dimensional* $\overline{\mathbb{Q}}_\ell$*-linear representation of* $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$*, fix notations as in the diagram*

(4.2.1)
$$\begin{tikzcd}
 & \mathrm{GL}_{n_1,k} \times \mathrm{T}_{n_2,k} \times \mathrm{GL}_{n_3,k} \simeq \mathrm{M} \arrow[d, "\mathrm{I}\times \mathrm{q}_{n_2}\times \mathrm{I}"'] \\
 & \mathrm{GL}_{n_1,k} \times \mathrm{T}_{n_2,k}/\mathrm{W}_{n_2} \times \mathrm{GL}_{n_3,k} \arrow[d, "\mathrm{I}\times \det\times \mathrm{I}"'] \\
\mathrm{GL}^{\mathrm{reg}}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}^{\mathrm{reg}}_{n_3,k} \arrow[r, "\mathrm{j}_{n_1}\times \mathrm{I}\times \mathrm{j}_{n_3}"] & \mathrm{GL}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}_{n_3,k}
\end{tikzcd}$$

where I *denotes the identity morphism.*

If ρ_{M} *is positive, then*

(4.2.2)
$$\big((\mathrm{I} \times \det \times \mathrm{I}) \circ (\mathrm{I} \times \mathrm{q}_{n_2} \times \mathrm{I})\big)_!(\Phi_{\psi,\rho_{\mathrm{M}}})[-d],$$

where

(4.2.3)
$$d \;=\; n_2 - 1,$$

is a semisimple object in $\mathrm{Perv}(\mathrm{GL}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}_{n_3,k}, \overline{\mathbb{Q}}_\ell)$ *which is isomorphic to the Goresky–MacPherson intermediate extension of its restriction to*

(4.2.4)
$$\mathrm{GL}^{\mathrm{reg}}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}^{\mathrm{reg}}_{n_3,k}.$$

Proof. By taking W_{M}-invariant summands it suffices to verify the lemma for

(4.2.5)
$$\mathrm{Ind}^{\mathrm{M}}_{\mathrm{T}_{n,k}}(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})$$

instead of $\Phi_{\psi,\rho_{\mathrm{M}}}$. Fix notations as in the Cartesian diagram

(4.2.6)

$$\begin{tikzcd}
 & \mathrm{T}_{n_1,k} \times \mathrm{T}_{n_2,k} \times \mathrm{T}_{n_3,k} \simeq \mathrm{T}_{n,k} \arrow[d, "(\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})"] \\
\widetilde{\mathrm{M}} \simeq \widetilde{\mathrm{GL}}_{n_1,k} \times \mathrm{T}_{n_2,k} \times \widetilde{\mathrm{GL}}_{n_3,k} \arrow[ur, "\tilde{\mathrm{p}}_{\mathrm{M}}"] \arrow[d, "(\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})"'] \arrow[dr, "\tilde{\mathrm{q}}_{\mathrm{M}}"] & \mathrm{T}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_3,k} \\
\widetilde{\mathrm{GL}}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \widetilde{\mathrm{GL}}_{n_3,k} \arrow[ur, "\tilde{\mathrm{p}}_{n_1}\times\mathrm{I}\times\tilde{\mathrm{p}}_{n_3}"] \arrow[dr, "\tilde{\mathrm{q}}_{n_1}\times\mathrm{I}\times\tilde{\mathrm{q}}_{n_3}"'] & \mathrm{GL}_{n_1,k} \times \mathrm{T}_{n_2,k} \times \mathrm{GL}_{n_3,k} \simeq \mathrm{M} \arrow[d, "(\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})"] \\
 & \mathrm{GL}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}_{n_3,k}.
\end{tikzcd}$$

By definition

(4.2.7)

$$\begin{aligned}
& \big((\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})\big)_!\big(\mathrm{Ind}^{\mathrm{M}}_{\mathrm{T}_{n,k}}(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\big) \\
\simeq\ & \big((\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})\big)_!\Big(\tilde{\mathrm{q}}_{\mathrm{M},!}\big(\tilde{\mathrm{p}}_{\mathrm{M}}{}^*(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\big)[e]\Big) \\
\simeq\ & (\tilde{\mathrm{q}}_{n_1}\times\mathrm{I}\times\tilde{\mathrm{q}}_{n_3})_!\Big(\big((\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})\big)_!\big(\tilde{\mathrm{p}}_{\mathrm{M}}{}^*(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\big)\Big)[e] \\
\simeq\ & (\tilde{\mathrm{q}}_{n_1}\times\mathrm{I}\times\tilde{\mathrm{q}}_{n_3})_!\Big((\tilde{\mathrm{p}}_{n_1}\times\mathrm{I}\times\tilde{\mathrm{p}}_{n_3})^*\Big(\\
& \quad \big((\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})\big)_!(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\Big)\Big)[e] \\
\simeq\ & \mathrm{Ind}^{\mathrm{GL}_{n_1,k}\times\mathbb{G}_{\mathrm{m},k}\times\mathrm{GL}_{n_3,k}}_{\mathrm{T}_{n_1,k}\times\mathbb{G}_{\mathrm{m},k}\times\mathrm{T}_{n_3,k}}\Big(\big((\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times\mathrm{q}_{n_2}\times\mathrm{I})\big)_!(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\Big)
\end{aligned}$$

where

$$e \;=\; {n_1}^2 - n_1 + {n_3}^2 - n_3. \tag{4.2.8}$$

Let ρ'_{T} denote the restriction of ρ_{T} to the subtorus

$$\begin{aligned} \check{\mathrm{T}}_{n_1}(\overline{\mathbb{Q}}_\ell) \times {\overline{\mathbb{Q}}_\ell}^{\times} \times \check{\mathrm{T}}_{n_3}(\overline{\mathbb{Q}}_\ell) \;&\subset\; \check{\mathrm{T}}_{n_1}(\overline{\mathbb{Q}}_\ell) \times \check{\mathrm{T}}_{n_2}(\overline{\mathbb{Q}}_\ell) \times \check{\mathrm{T}}_{n_3}(\overline{\mathbb{Q}}_\ell) \\ &\simeq\; \check{\mathrm{T}}_{n}(\overline{\mathbb{Q}}_\ell) \end{aligned} \tag{4.2.9}$$

where ${\overline{\mathbb{Q}}_\ell}^{\times}$ embeds into $\check{\mathrm{T}}_{n_2}(\overline{\mathbb{Q}}_\ell)$ diagonally. Since ρ_{M} is positive, so are ρ_{T} and ρ'_{T}, hence

$$\big((\mathrm{I} \times \det \times \mathrm{I}) \circ (\mathrm{I} \times \mathrm{q}_{n_2} \times \mathrm{I})\big)_!(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}}) \;\simeq\; \mathrm{Hyp}_{\psi,\rho'_{\mathrm{T}}}[d], \tag{4.2.10}$$

hence

$$\begin{aligned} &\big((\mathrm{I} \times \det \times \mathrm{I}) \circ (\mathrm{I} \times \mathrm{q}_{n_2} \times \mathrm{I})\big)_!\big(\mathrm{Ind}^{\mathrm{M}}_{\mathrm{T}_{n,k}}(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}})\big)[-d] \\ \simeq\;& \mathrm{Ind}^{\mathrm{GL}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}_{n_3,k}}_{\mathrm{T}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_3,k}}(\mathrm{Hyp}_{\psi,\rho'_{\mathrm{T}}}) \end{aligned} \tag{4.2.11}$$

which is a semisimple object in $\mathrm{Perv}(\mathrm{GL}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}_{n_3,k}, \overline{\mathbb{Q}}_\ell)$ isomorphic to the Goresky–MacPherson intermediate extension of its restriction to

$$\mathrm{GL}^{\mathrm{reg}}_{n_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}^{\mathrm{reg}}_{n_3,k} \tag{4.2.12}$$

by Proposition (1.11) since ρ'_{T} is positive. □

(4.3) Proposition C† *Let ρ be a $\overline{\mathbb{Q}}_\ell$-linear representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is positive. Let n_1 and n_2 be two natural numbers which sum to n such that n_1 is greater than one. Let*

U *denote the unipotent subgroup of* $\mathrm{GL}_{n,k}$ *consisting of the matrices of the form*

$$(4.3.1) \qquad \left(\begin{array}{c|c} I_{n_1-1} & * \\ \hline & I_{n_2+1} \end{array}\right)$$

where I_r *denotes the* $r \times r$ *identity matrix, let* g *be a block upper triangular matrix*

$$(4.3.2) \qquad g \;=\; \left(\begin{array}{c|c} g_1 & g_3 \\ \hline & g_2 \end{array}\right)$$

in $\mathrm{GL}_{n,k}(k)$ *of size* (n_1, n_2), *let* i *denote the inclusion morphism*

$$(4.3.3) \qquad \mathrm{i}: \quad g\mathrm{U} \longrightarrow \mathrm{GL}_{n,k}$$

of the left coset $g\mathrm{U}$.

If g_1 *is not contained in* $\mathrm{Q}_{n_1,k}(k)$, *then*

$$(4.3.4) \qquad H^{\bullet}_{\mathrm{c}}\big(g\mathrm{U}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \;\simeq\; 0.$$

Proof. By Lemma (3.2) the map

$$(4.3.5) \qquad \mathrm{tr}(\Lambda^{\bullet}_{g_1}): \quad H_{n_1} \longrightarrow k^{n_1},$$

which assigns to each vector $\vec{u}$ in H_{n_1} the difference between the characteristic polynomials of the $n_1 \times n_1$ matrices $g_1 u$ and g, is a k-linear transformation. By Lemma (3.3) the kernel of $\mathrm{tr}(\Lambda^{\bullet}_{g_1})$ is stable under g_1, which descends to a k-linear endomorphism

$$(4.3.6) \qquad [g_1] \;\in\; \mathrm{End}_k\big(V_{n_1}/\ker(\mathrm{tr}(\Lambda^{\bullet}_{g_1}))\big)$$

whose transpose ${}^{\mathrm{T}}[g_1]$ admits the covector

$$\hat{\mathbf{e}}_{n_1} \;\in\; \big(V_{n_1}/\ker(\mathrm{tr}(\Lambda^{\bullet}_{g_1}))\big)^* \tag{4.3.7}$$

as a cyclic vector. Hence g_1 is conjugate under the action of $\mathrm{Q}_{n_1,k}(k)$ to a block upper triangular matrix

$$h \;=\; \left(\begin{array}{c|c} h_1 & h_3 \\ \hline & h_2 \end{array}\right) \tag{4.3.8}$$

in $\mathrm{GL}_{n_1,k}(k)$ of size (m_1, m_2) such that h_2 is of the form

$$\left(\begin{array}{c|c} * & * \\ \hline I_{m_2-1} & * \end{array}\right), \tag{4.3.9}$$

where m_1 and m_2 are two natural numbers which sum to n_1 such that m_2 is greater than one since g_1 is not contained in $\mathrm{Q}_{n_1,k}(k)$ by assumption. Let M denote the standard Levi subgroup of $\mathrm{GL}_{n,k}$ consisting of the block diagonal matrices of size (m_1, m_2, n_2), let ρ_{M} denote the restriction of ρ to $\mathrm{M}(\overline{\mathbb{Q}}_\ell)$.

The characteristic polynomial morphism on $\mathrm{GL}_{m_2,k}$ is smooth at each point in $h_2\mathrm{H}_{m_2,k}$ by the Jacobian criterion, hence $h_2\mathrm{H}_{m_2,k}$ is contained in $\mathrm{GL}^{\mathrm{reg}}_{m_2,k}$. Fix notations as in the diagram

(4.3.10)
$$\begin{tikzcd}
h_2\mathrm{H}_{m_2,k} \arrow[r, dashed, "h_1^{\mathrm{reg}}\times \mathrm{i}_{\mathrm{H}}\times g_2^{\mathrm{reg}}"] \arrow[dd, equal] & \mathrm{GL}_{m_1,k}^{\mathrm{reg}} \times \mathrm{GL}_{m_2,k}^{\mathrm{reg}} \times \mathrm{GL}_{n_2,k}^{\mathrm{reg}} \simeq \mathrm{M}^{\mathrm{reg}} \arrow[d, "\mathrm{I}\times \mathrm{p}_{m_2}\times \mathrm{I}"'] \\
 & \mathrm{GL}_{m_1,k}^{\mathrm{reg}} \times \mathrm{T}_{m_2,k}/\mathrm{W}_{m_2} \times \mathrm{GL}_{n_2,k}^{\mathrm{reg}} \arrow[d, "\mathrm{j}_{m_1}\times \mathrm{I}\times \mathrm{j}_{n_2}"'] \\
h_2\mathrm{H}_{m_2,k} \arrow[r, "h_1\times(\mathrm{p}_{m_2}\circ \mathrm{i}_{\mathrm{H}})\times g_2"] \arrow[d, "\det"'] & \mathrm{GL}_{m_1,k} \times \mathrm{T}_{m_2,k}/\mathrm{W}_{m_2} \times \mathrm{GL}_{n_2,k} \arrow[d, "\mathrm{I}\times \det \times \mathrm{I}"'] \\
\mathrm{Spec}(k) \arrow[r, "h_1\times \det(h_2)\times g_2"] & \mathrm{GL}_{m_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{GL}_{n_2,k}
\end{tikzcd}$$

where I denotes the identity morphism, i_{H} denotes the inclusion morphism of $h_2\mathrm{H}_{m_2,k}$ in $\mathrm{GL}_{m_2,k}^{\mathrm{reg}}$, y denotes the inclusion morphism of the point y in $\mathrm{GL}_{r,k}$, and y^{reg} denotes the lift of y through j_r if such a lift exists. By construction $h_1 \times (\mathrm{p}_{m_2} \circ \mathrm{i}_{\mathrm{H}}) \times g_2$ defines an isomorphism between $h_2\mathrm{H}_{m_2,k}$ and the fiber

$$(4.3.11)\qquad (\mathrm{GL}_{m_1,k} \times \mathrm{T}_{m_2,k}/\mathrm{W}_{m_2} \times \mathrm{GL}_{n_2,k}) \underset{\mathrm{GL}_{m_1,k}\times\mathbb{G}_{\mathrm{m},k}\times\mathrm{GL}_{n_2,k}}{\times} \mathrm{Spec}(k).$$

If h_1 is contained in $\mathrm{GL}_{m_1,k}^{\mathrm{reg}}(k)$ and g_2 is contained in $\mathrm{GL}_{n_2,k}^{\mathrm{reg}}(k)$, fix notations as in the diagram

(4.3.12)
$$\begin{tikzcd}
h_2\mathrm{H}_{m_2,k} \arrow[r, "h_1^{\mathrm{reg}}\times \mathrm{i}_{\mathrm{H}}\times g_2^{\mathrm{reg}}"] & \mathrm{M}^{\mathrm{reg}} \arrow[d, "\mathrm{p}_{\mathrm{M}}"'] \\
\mathrm{T}_{n,k} \arrow[r, "\mathrm{q}_{\mathrm{M}}"] \arrow[dd, "(\mathrm{I}\times\det\times\mathrm{I})\circ(\mathrm{I}\times \mathrm{q}_{m_2}\times \mathrm{I})"] & \mathrm{T}_{n,k}/\mathrm{W}_{\mathrm{M}} \arrow[d, equal, "\simeq"'] \\
 & \mathrm{T}_{m_1,k}/\mathrm{W}_{m_1} \times \mathrm{T}_{m_2,k}/\mathrm{W}_{m_2} \times \mathrm{T}_{n_2,k}/\mathrm{W}_{n_2} \arrow[d, "\mathrm{I}\times\det\times\mathrm{I}"'] \\
\mathrm{T}_{m_1,k} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_2,k} \arrow[r, "\mathrm{q}_{m_1}\times \mathrm{I}\times \mathrm{q}_{n_2}"] & \mathrm{T}_{m_1,k}/\mathrm{W}_{m_1} \times \mathbb{G}_{\mathrm{m},k} \times \mathrm{T}_{n_2,k}/\mathrm{W}_{n_2},
\end{tikzcd}$$

then by Proposition (1.11)

$$(4.3.13) \qquad H_{\mathrm{c}}^{\bullet}\big(h_2\mathrm{H}_{m_2,k}, (h_1 \times \mathrm{i}_{\mathrm{H}} \times g_2)^*(\Phi_{\psi,\rho_{\mathrm{M}}})\big)$$

$$\simeq \; H_{\mathrm{c}}^{\bullet}\Big(h_2\mathrm{H}_{m_2,k}, (h_1^{\mathrm{reg}} \times \mathrm{i}_{\mathrm{H}} \times g_2^{\mathrm{reg}})^*\big({\mathrm{j}_{\mathrm{M}}}^*(\Phi_{\psi,\rho_{\mathrm{M}}})\big)\Big)$$

$$\simeq \; H_{\mathrm{c}}^{\bullet}\Big(h_2\mathrm{H}_{m_2,k}, (h_1^{\mathrm{reg}} \times \mathrm{i}_{\mathrm{H}} \times g_2^{\mathrm{reg}})^*\Big($$

$${\mathrm{p}_{\mathrm{M}}}^*\Big(\mathrm{q}_{\mathrm{M},!}\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{n,k}}{\otimes} \big(\varepsilon_{\mathrm{M}} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{\mathrm{M}}})\big)\Big)^{\mathrm{W}_{\mathrm{M}}}\Big)[d]\Big)\Big)$$

$$\simeq \; (\mathrm{I} \times \det \times \mathrm{I})_!\Big(\mathrm{q}_{\mathrm{M},!}\Big(\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{n,k}}{\otimes} \big(\varepsilon_{\mathrm{M}} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{\mathrm{M}}})\big)\Big)^{\mathrm{W}_{\mathrm{M}}}\Big)[d]\Big|_{\mathrm{p}(g)}$$

$$\simeq \; \Big((\mathrm{q}_{m_1} \times \mathrm{I} \times \mathrm{q}_{n_2})_!\Big(\big((\mathrm{I} \times \det \times \mathrm{I}) \circ (\mathrm{I} \times \mathrm{q}_{m_2} \times \mathrm{I})\big)_!\Big($$

$$\mathrm{Hyp}_{\psi,\rho_{\mathrm{T}}} \underset{\mathrm{T}_{n,k}}{\otimes} \big(\varepsilon_{\mathrm{M}} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{\mathrm{M}}})\big)\Big)^{\mathrm{W}_{m_2}}\Big)^{\mathrm{W}_{m_1} \times \mathrm{W}_{n_2}}\Big)[d]\Big|_{\mathrm{p}(g)}$$

where

$$(4.3.14) \qquad d \;=\; {m_1}^2 + {m_2}^2 + {n_2}^2 - n,$$

W_{m_2} and $\mathrm{W}_{m_1} \times \mathrm{W}_{n_2}$ operate as direct factors of W_{M} with respect to the direct factorization

$$(4.3.15) \qquad \mathrm{W}_{\mathrm{M}} \;\simeq\; \mathrm{W}_{m_1} \times \mathrm{W}_{m_2} \times \mathrm{W}_{n_2},$$

and $|_{\mathrm{p}(g)}$ denotes the stalk at $(\mathrm{p}_{m_1}(h_1), \det(h_2), \mathrm{p}_{n_2}(g_2))$, which vanishes by Lemma $\mathrm{A}^{\dagger}$ (4.1). If h_1 is not necessarily contained in $\mathrm{GL}_{m_1,k}^{\mathrm{reg}}(k)$ or g_2 is not necessarily contained in

$\mathrm{GL}^{\mathrm{reg}}_{n_2,k}(k)$, then

$$(4.3.16) \qquad H^\bullet_{\mathrm{c}}\big(h_2\mathrm{H}_{m_2,k}, (h_1 \times \mathrm{i}_\mathrm{H} \times g_2)^*(\Phi_{\psi,\rho_\mathrm{M}})\big)$$

$$\simeq \Big(\big((\mathrm{I} \times \det \times \mathrm{I}) \circ (\mathrm{I} \times \mathrm{q}_{m_2} \times \mathrm{I})\big)_!\Big(\Phi_{\psi,\rho_\mathrm{L}} \underset{\mathrm{L}}{\otimes} \big(\varepsilon_{m_2} \otimes (\varepsilon_r \circ \rho_{\mathrm{W}_{m_2}})\big)$$

$$\Big)^{\mathrm{W}_{m_2}}\Big)[e]\Big|_{\det(g)}$$

where

$$(4.3.17) \qquad e \;=\; {m_2}^2 - m_2,$$

$|_{\det(g)}$ denotes the stalk at $(h_1, \det(h_2), g_2)$, and ρ_L denotes the restriction of ρ to $\mathrm{L}(\overline{\mathbb{Q}}_\ell)$ where L denotes the standard Levi subgroup of $\mathrm{GL}_{n,k}$ consisting of the block diagonal matrices of size $(m_1, 1, \ldots, 1, n_2)$, which vanishes by perverse continuation by Lemma B† (4.2).

Let P denote the standard parabolic subgroup of $\mathrm{GL}_{n,k}$ consisting of the block upper triangular matrices of size (m_1, m_2, n_2), fix notations as in the diagram

$$(4.3.18) \qquad \begin{tikzcd} \mathrm{P} \underset{\mathrm{M}}{\times} h_2\mathrm{H}_{m_2,k} \arrow[r, "\mathrm{I}\times(h_1\times\mathrm{i}_\mathrm{H}\times g_2)"] \arrow[d, "\mathrm{q}\times\mathrm{I}"'] & \mathrm{P} \arrow[r, "\mathrm{i}_\mathrm{P}"] \arrow[d, "\mathrm{q}"'] & \mathrm{GL}_{n,k} \\ h_2\mathrm{H}_{m_2,k} \arrow[r, "h_1\times\mathrm{i}_\mathrm{H}\times g_2"] & \mathrm{M} & \end{tikzcd}$$

where the arrows denote the inclusion and quotient morphisms, then by Proposition (1.13)

$$(4.3.19) \qquad H^\bullet_{\mathrm{c}}\big(h_2\mathrm{H}_{m_2,k}, (h_1 \times \mathrm{i}_\mathrm{H} \times g_2)^*(\Phi_{\psi,\rho_\mathrm{M}})\big)$$

$$\simeq H^\bullet_{\mathrm{c}}\Big(h_2\mathrm{H}_{m_2,k}, (h_1 \times \mathrm{i}_\mathrm{H} \times g_2)^*\big(\mathrm{Res}_{\mathrm{M}}^{\mathrm{GL}_{n,k}}(\Phi_{\psi,\rho})\big)\Big)$$

$$\simeq \; H_{\mathrm{c}}^{\bullet}\Big(h_2\mathrm{H}_{m_2,k}, (h_1 \times \mathrm{i}_{\mathrm{H}} \times g_2)^*\big(\mathrm{q}_!\big(\mathrm{i}_{\mathrm{P}}{}^*(\Phi_{\psi,\rho})\big)\big)\Big),$$

hence by the Leray spectral sequence

$$H_{\mathrm{c}}^{\bullet}\Big(\mathrm{P} \underset{\mathrm{M}}{\times} h_2\mathrm{H}_{m_2,k}, \big(\mathrm{i}_{\mathrm{P}} \circ (\mathrm{I} \times (h_1 \times \mathrm{i}_{\mathrm{H}} \times g_2))\big)^*(\Phi_{\psi,\rho})\Big) \;\simeq\; 0. \tag{4.3.20}$$

The fibered product $\mathrm{P} \times_{\mathrm{M}} h_2\mathrm{H}_{m_2,k}$ is isomorphic to the k-subscheme of $\mathrm{GL}_{n,k}$ consisting of the block upper triangular matrices of the form

$$\left(\begin{array}{c|c|c} h_1 & * & * \\ \hline & h_2\mathrm{H}_{m_2,k} & * \\ \hline & & g_2 \end{array}\right), \tag{4.3.21}$$

where by construction each element in $h_2\mathrm{H}_{m_2,k}(A)$ is of the form

$$\left(\begin{array}{c|c} * & * \\ \hline I_{m_2-1} & * \end{array}\right) \tag{4.3.22}$$

for each k-algebra A. Hence by Lemma (3.4), for each $m_2 \times n_2$ matrix g_4 and for each $m_1 \times n_2$ matrix g_5 with coefficients in k, the k-subscheme

$$\left(\begin{array}{c|c|c} h_1 & h_3 & g_5 \\ \hline & h_2 & g_4 \\ \hline & & g_2 \end{array}\right)\mathrm{U} \;\subset\; \mathrm{P} \underset{\mathrm{M}}{\times} h_2\mathrm{H}_{m_2,k} \tag{4.3.23}$$

is a cross section of $\mathrm{P} \times_{\mathrm{M}} h_2\mathrm{H}_{m_2,k}$ as a principal bundle induced by a unipotent subgroup of P operating freely by conjugation. By construction there exists an element

$$s \;\in\; \left(\begin{array}{c|c} \mathrm{Q}_{n_1,k}(k) & \\ \hline & I_{n_2} \end{array}\right) \tag{4.3.24}$$

such that there exist an $m_2 \times n_2$ matrix g_4 and an $m_1 \times n_2$ matrix g_5 with coefficients in k such that

$$\left(\begin{array}{c|c|c} h_1 & h_3 & g_5 \\ \hline & h_2 & g_4 \\ \hline & & g_2 \end{array}\right) \quad = \quad s^{-1}gs. \tag{4.3.25}$$

Let π denote the projection morphism

$$\pi : \quad \mathrm{P} \underset{\mathrm{M}}{\times} h_2\mathrm{H}_{m_2,k} \longrightarrow s^{-1}gs\mathrm{U} \tag{4.3.26}$$

of the trivial principal bundle $\mathrm{P} \times_{\mathrm{M}} h_2\mathrm{H}_{m_2,k}$. By Proposition (1.13) $\Phi_{\psi,\rho}$ is equivariant with respect to the adjoint action ad of $\mathrm{GL}_{n,k}$, which contains the structure group of $\mathrm{P} \times_{\mathrm{M}} h_2\mathrm{H}_{m_2,k}$, on the right, hence

$$\begin{aligned} & H^\bullet_{\mathrm{c}}\Big(s^{-1}gs\mathrm{U}, \pi_!\big((\mathrm{i}_{\mathrm{P}} \circ (\mathrm{I} \times (h_1 \times \mathrm{i}_{\mathrm{H}} \times g_2)))^*(\Phi_{\psi,\rho})\big)[f]\Big) \\ \simeq \quad & H^\bullet_{\mathrm{c}}\Big(s^{-1}gs\mathrm{U}, \big(\mathrm{ad}(s) \circ \mathrm{i} \circ \mathrm{ad}(s)^{-1}\big)^*(\Phi_{\psi,\rho})\Big) \\ \simeq \quad & H^\bullet_{\mathrm{c}}\big(g\mathrm{U}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \end{aligned} \tag{4.3.27}$$

since U is stable under conjugation by s, where

$$f \quad = \quad m_1(m_2 - 1) + n_2, \tag{4.3.28}$$

which vanishes by the Leray spectral sequence. □

(4.4) Corollary (Conjecture 9.12 in [BK00])

Let ρ be a $\overline{\mathbb{Q}}_\ell$-linear representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is positive. Let g be an element in

$\mathrm{GL}_{n,k}(k)$, *let* i *denote the inclusion morphism*

$$\mathrm{i} : \quad g\mathrm{U}_{n,k} \longrightarrow \mathrm{GL}_{n,k} \tag{4.4.1}$$

of the left coset $g\mathrm{U}_{n,k}$.

If g *is not contained in* $\mathrm{B}_{n,k}(k)$, *then*

$$H^{\bullet}_{\mathrm{c}}\big(g\mathrm{U}_{n,k}, \mathrm{i}^{*}(\Phi_{\psi,\rho})\big) \;\; \simeq \;\; 0. \tag{4.4.2}$$

Proof. Let P be a standard parabolic subgroup of $\mathrm{GL}_{n,k}$ consisting of the block upper triangular matrices of size (n_1, n_2) such that

$$g \;\; = \;\; \left(\begin{array}{c|c} g_1 & g_3 \\ \hline & g_2 \end{array}\right) \tag{4.4.3}$$

is contained in $\mathrm{P}(k)$ where g_1 is not contained in $\mathrm{Q}_{n_1,k}(k)$ which implies that n_1 is greater than one, let U denote the unipotent subgroup of $\mathrm{GL}_{n,k}$ consisting of the matrices of the form

$$\left(\begin{array}{c|c} I_{n_1-1} & * \\ \hline & I_{n_2+1} \end{array}\right) \tag{4.4.4}$$

where I_r denotes the $r \times r$ identity matrix.

Since g is contained in $\mathrm{P}(k)$, so is each element

$$g' \;\; = \;\; \left(\begin{array}{c|c} g'_1 & g'_3 \\ \hline & g'_2 \end{array}\right) \tag{4.4.5}$$

in $g\mathrm{U}_{n,k}(k)$ where g_1' is not contained in $\mathrm{Q}_{n_1,k}(k)$ since neither is g_1, hence

$$H^\bullet_{\mathrm{c}}\big(g\mathrm{U}_{n,k}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \;\simeq\; 0 \tag{4.4.6}$$

by Proposition $\mathrm{C}^\dagger$ (4.3) applied to each subcoset $g'\mathrm{U}$ in $g\mathrm{U}_{n,k}$. □

(4.5) Corollary *Let ρ be a $\overline{\mathbb{Q}}_\ell$-linear representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is positive, let $\mathcal{K}$ be an object in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{T}_{n,k}, \overline{\mathbb{Q}}_\ell)$, then*

$$\Phi_{\psi,\rho} * \mathrm{Ind}^{\mathrm{GL}_{n,k}}_{\mathrm{T}_{n,k}}(\mathcal{K}) \;\simeq\; \mathrm{Ind}^{\mathrm{GL}_{n,k}}_{\mathrm{T}_{n,k}}\big(\mathrm{Res}^{\mathrm{GL}_{n,k}}_{\mathrm{T}_{n,k}}(\Phi_{\psi,\rho_{\mathrm{T}}}) * \mathcal{K}\big). \tag{4.5.1}$$

Proof. This follows from Corollary (4.4) and Proposition (1.7). □

(4.6) Corollary *Let ρ be a $\overline{\mathbb{Q}}_\ell$-linear representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is positive. Let* P *be a standard maximal parabolic subgroup of $\mathrm{GL}_{n,k}$, let* U *denote the unipotent radical of* P*. Let g be an element in $\mathrm{GL}_{n,k}(k)$, let* i *denote the inclusion morphism*

$$\mathrm{i}: \quad g\mathrm{U} \longrightarrow \mathrm{GL}_{n,k} \tag{4.6.1}$$

of the left coset gU.

If g is not contained in $\mathrm{P}(k)$, then

$$H^\bullet_{\mathrm{c}}\big(g\mathrm{U}, \mathrm{i}^*(\Phi_{\psi,\rho})\big) \;\simeq\; 0. \tag{4.6.2}$$

Proof. Since P is a maximal parabolic subgroup of $\mathrm{GL}_{n,k}$, there exist two natural numbers n_1 and n_2 which sum to n such that P consists of the block upper triangular matrices of size (n_1, n_2). Argue by induction on n_2.

If n_2 is equal to one, then the corollary follows from Proposition $\mathrm{C}^\dagger$ (4.3). Otherwise n_2 is greater than one. Let P' denote the standard parabolic subgroup of $\mathrm{GL}_{n,k}$ consisting of the block upper triangular matrices of size (n_1+1, n_2-1), let U' denote the unipotent radical of P'. Since g is not contained in $\mathrm{P}(k)$, there exists an element

$$s \;\in\; \left(\begin{array}{c|c} \mathrm{GL}_{n_1,k}(k) & \\ \hline & \mathrm{GL}_{n_2,k}(k) \end{array}\right) \tag{4.6.3}$$

such that

$$\begin{aligned} s^{-1}gs &= g' \\ &= \left(\begin{array}{c|c} g'_1 & g'_4 \\ \hline g'_3 & g'_2 \end{array}\right), \end{aligned} \tag{4.6.4}$$

where the $n_2 \times n_1$ matrix g'_3 is of the form

$$\left(\begin{array}{c|c} & 1 \\ \hline * & * \end{array}\right) \tag{4.6.5}$$

where at least one of the matrices $*$ have nonzero coefficients such that g' is not contained in $\mathrm{P}'(k)$. Let i' denote the inclusion morphism

$$\mathrm{i}' : \quad g'\mathrm{U}' \longrightarrow \mathrm{GL}_{n,k} \tag{4.6.6}$$

of the left coset $g'\mathrm{U}'$, then by the induction hypothesis applied to $n_2 - 1$

$$H^{\bullet}_{\mathrm{c}}\big(g'\mathrm{U}', \mathrm{i}'^{,*}(\Phi_{\psi,\rho})\big) \;\simeq\; 0. \tag{4.6.7}$$

Fix notations as in the Cartesian diagram

$$\begin{tikzcd}
\mathrm{U}' \underset{\mathrm{GL}_{n,k}}{\times} \mathrm{U} \arrow[r, "\mathrm{I}\times\mathrm{i}_{\mathrm{U}}"] \arrow[d, "\mathrm{i}'_{\mathrm{U}}\times\mathrm{I}"'] & \mathrm{U}' \arrow[d, "\mathrm{i}'_{\mathrm{U}}"] \\
\mathrm{U} \arrow[r, "\mathrm{i}_{\mathrm{U}}"] & \mathrm{GL}_{n,k}
\end{tikzcd} \tag{4.6.8}$$

where the arrows denote the inclusion morphisms and I denotes the identity morphism. By Lemma (3.4) the subcoset

$$g'\Big(\mathrm{U}' \underset{\mathrm{GL}_{n,k}}{\times} \mathrm{U}\Big) \;\subset\; g'\mathrm{U}' \tag{4.6.9}$$

is a cross section of $g'\mathrm{U}'$ as a principal bundle induced by a k-subgroup of U' operating freely by conjugation. Let π denote the projection morphism

$$\pi :\quad g'\mathrm{U}' \longrightarrow g'\Big(\mathrm{U}' \underset{\mathrm{GL}_{n,k}}{\times} \mathrm{U}\Big) \tag{4.6.10}$$

of the trivial principal bundle $g'\mathrm{U}'$. By Proposition (1.13) $\Phi_{\psi,\rho}$ is equivariant with respect to the adjoint action of $\mathrm{GL}_{n,k}$, which contains the structure group of $g'\mathrm{U}'$, on the right, hence

$$H_{\mathrm{c}}^{\bullet}\Big(g'\Big(\mathrm{U}' \underset{\mathrm{GL}_{n,k}}{\times} \mathrm{U}\Big), \pi_!\big(\mathrm{i}'^{,*}(\Phi_{\psi,\rho})\big)[f]\Big) \tag{4.6.11}$$

$$\simeq\; H_{\mathrm{c}}^{\bullet}\Big(g'\Big(\mathrm{U}' \underset{\mathrm{GL}_{n,k}}{\times} \mathrm{U}\Big), \big(\mathrm{i}'\circ \mathrm{l}(g')\circ(\mathrm{I}\times\mathrm{i}_{\mathrm{U}})\circ \mathrm{l}(g')^{-1}\big)^{*}(\Phi_{\psi,\rho})\Big)$$

where l denotes the multiplication action of $\mathrm{GL}_{n,k}$ on itself on the left and

$$f \;=\; n_2 - 1, \tag{4.6.12}$$

which vanishes by the Leray spectral sequence. Since g' is not contained in $\mathrm{P}'(k)$, neither is any element g'' in $g'\mathrm{U}(k)$, hence

$$H_{\mathrm{c}}^{\bullet}\Big(g'\mathrm{U}, \big(\mathrm{l}(g') \circ \mathrm{i}_{\mathrm{U}} \circ \mathrm{l}(g')^{-1}\big)^{*}(\Phi_{\psi,\rho})\Big) \tag{4.6.13}$$

$$\simeq \quad H_{\mathrm{c}}^{\bullet}\Big(g'\mathrm{U}, \big(\mathrm{ad}(s) \circ \mathrm{i} \circ \mathrm{ad}(s)^{-1}\big)^{*}(\Phi_{\psi,\rho})\Big)$$

$$\simeq \quad H_{\mathrm{c}}^{\bullet}\big(g\mathrm{U}, \mathrm{i}^{*}(\Phi_{\psi,\rho})\big)$$

since U is stable under conjugation by s, which vanishes by the acyclicity of $\Phi_{\psi,\rho}$ over each subcoset $g''(\mathrm{U}' \times_{\mathrm{GL}_{n,k}} \mathrm{U})$ in $g'\mathrm{U}$. □

(4.7) Corollary *Let ρ be a $\overline{\mathbb{Q}}_\ell$-linear representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is positive. Let* P *be a standard parabolic subgroup of $\mathrm{GL}_{n,k}$, let* U *denote the unipotent radical of* P*. Let g be an element in $\mathrm{GL}_{n,k}(k)$, let* i *denote the inclusion morphism*

$$\mathrm{i}: \quad g\mathrm{U} \longrightarrow \mathrm{GL}_{n,k} \tag{4.7.1}$$

of the left coset $g\mathrm{U}$.

If g is not contained in $\mathrm{P}(k)$, then

$$H_{\mathrm{c}}^{\bullet}\big(g\mathrm{U}, \mathrm{i}^{*}(\Phi_{\psi,\rho})\big) \;\simeq\; 0. \tag{4.7.2}$$

Proof. Let Q be a standard maximal parabolic subgroup of $\mathrm{GL}_{n,k}$ containing P such that g is not contained in $\mathrm{Q}(k)$, let V denote the unipotent radical of Q. Since g is not contained in $\mathrm{Q}(k)$, neither is any element g' in $g\mathrm{U}(k)$, hence

$$H_{\mathrm{c}}^{\bullet}\big(g\mathrm{U}_{n,k}, \mathrm{i}^{*}(\Phi_{\psi,\rho})\big) \;\simeq\; 0 \tag{4.7.3}$$

by Corollary (4.6) applied to each subcoset $g'\mathrm{V}$ in $g\mathrm{U}$. □

3.A Appendix: Smoothness of hypergeometric sheaves

In this appendix the hypergeometric sheaves of Braverman–Kazhdan are shown to be lisse perverse sheaves under the positivity condition.

(A.1) Notation Let $\mathbb{P}^1_k$ denote the k-projective line, let $\mathrm{j_m}$ and $\mathrm{j_a}$ denote the inclusion morphisms

(A.1.1)
$$\mathrm{j_m}: \quad \mathbb{G}_{\mathrm{m},k} \longrightarrow \mathbb{P}^1_k$$

$$\mathrm{j_a}: \quad \mathbb{G}_{\mathrm{a},k} \longrightarrow \mathbb{P}^1_k,$$

let π denote the completed Artin–Schreier covering

(A.1.2)
$$\pi: \quad \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$$

defined by the equation

(A.1.3)
$$\pi\big([X:Y]\big) \quad = \quad [X^q - XY^{q-1} : Y^q]$$

with respect to the homogeneous coordinates $[X:Y]$ on $\mathbb{P}^1_k$.

If ρ is an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ with a choice of $\mathcal{B}$, let u denote the universal morphism

(A.1.4)
$$\mathrm{u}: \quad \mathrm{T}_{r,k} \longrightarrow \mathrm{T}_{n,k} \times (\mathbb{P}^1_k)^r$$

induced by the universal property of the Cartesian product from the diagram

(A.1.5)
$$\begin{tikzcd}
\mathrm{T}_{r,k} \simeq (\mathbb{G}_{\mathrm{m},k})^r \arrow[r, "(\mathrm{j}_{\mathrm{m}})^r"] \arrow[d, "\hat{\rho}"'] & (\mathbb{P}^1_k)^r \\
\mathrm{T}_{n,k} &
\end{tikzcd}$$

where $\mathrm{T}_{r,k}$ and $(\mathbb{G}_{\mathrm{m},k})^r$ are identified via the choice of $\mathcal{B}$, let j and $\overline{\rho}$ denote the inclusion and projection morphisms

(A.1.6)
$$\begin{tikzcd}
\mathrm{T}_{r,k} \arrow[r, "\mathrm{j}"] \arrow[dr, "\hat{\rho}"'] & \overline{\mathrm{T}}_{r,k} \arrow[d, "\overline{\rho}"'] \\
& \mathrm{T}_{n,k}
\end{tikzcd}$$

where $\overline{\mathrm{T}}_{r,k}$ denotes the closure of the image of u in $\mathrm{T}_{n,k} \times (\mathbb{P}^1_k)^r$.

(A.2) Lemma *Let ρ be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of the torus $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$, fix notations as in the diagram*

(A.2.1)
$$\begin{tikzcd}
\big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \arrow[r, "\mathrm{I}\times\mathrm{i}"] \arrow[d, "(\pi^r\times\mathrm{I})\times\mathrm{I}"'] & (\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k} \arrow[d, "\pi^r\times\mathrm{I}"'] \\
\overline{\mathrm{T}}_{r,k} \arrow[r, "\mathrm{i}"] \arrow[d, "\overline{\rho}"'] & (\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k} \\
\mathrm{T}_{n,k} &
\end{tikzcd}$$

where the square is Cartesian, i *denotes the inclusion morphism and* I *denotes the identity morphism.*

If $\hat{\rho}$ is surjective, then the composite morphism

(A.2.2)
$$\overline{\rho} \circ \big((\pi^r \times \mathrm{I}) \times \mathrm{I}\big)$$

is universally locally acyclic with respect to the constant ℓ-adic sheaf $\overline{\mathbb{Q}}_\ell$ *on* $((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}) \times_{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}} \overline{\mathrm{T}}_{r,k}$.

Proof. It suffices to verify the analogous universal local acyclicity property for $\overline{\rho}$ and $(\pi^r \times \mathrm{I}) \times \mathrm{I}$.

The argument involves the theory of toric varieties (see [Fu93]). If T is a k-torus, let X(T) denote the free abelian group of k-linear cocharacters

$$\gamma : \quad \mathbb{G}_{\mathrm{m},k} \longrightarrow \mathrm{T} \tag{A.2.3}$$

under pointwise multiplication. If $\overline{\mathrm{T}}$ is a T-toric variety, let $\Sigma(\overline{\mathrm{T}})$ denote the pair

$$\Sigma(\overline{\mathrm{T}}) \;=\; \big(\mathrm{X}(\mathrm{T}), \Sigma\big) \tag{A.2.4}$$

where Σ denotes the fan of convex rational polyhedral cones in $\mathrm{X}(\mathrm{T}) \otimes \mathbb{R}$ associated with $\overline{\mathrm{T}}$.

Let $\mathrm{X}(\hat{\rho})$ denote the homomorphism

$$\mathrm{X}(\hat{\rho}) : \quad \mathrm{X}(\mathrm{T}_{r,k}) \longrightarrow \mathrm{X}(\mathrm{T}_{n,k}) \tag{A.2.5}$$

of free abelian groups, let

$$\mathrm{graph}\big(\mathrm{X}(\hat{\rho})\big) \;\subset\; \mathrm{X}(\mathrm{T}_{n,k}) \times \mathrm{X}(\mathrm{T}_{r,k}) \tag{A.2.6}$$

denote its graph, let

$$\ker\big(\mathrm{X}(\hat{\rho})\big) \;=\; \mathrm{graph}\big(\mathrm{X}(\hat{\rho})\big) \cap \big(\{\vec{0}\} \times \mathrm{X}(\mathrm{T}_{r,k})\big) \tag{A.2.7}$$

denote its kernel, let

$$\mathrm{im}\big(\mathrm{X}(\hat{\rho})\big) \;\subset\; \mathrm{X}(\mathrm{T}_{n,k}) \tag{A.2.8}$$

denote its image, hence the complex

$$(\text{A.2.9}) \qquad 0 \longrightarrow \ker\bigl(\mathrm{X}(\hat{\rho})\bigr) \longrightarrow \mathrm{graph}\bigl(\mathrm{X}(\hat{\rho})\bigr) \longrightarrow \mathrm{im}\bigl(\mathrm{X}(\hat{\rho})\bigr) \longrightarrow 0$$

is exact and split.

Since $\hat{\rho}$ is surjective by assumption, there exists an isogeny ι of k-tori through which $\overline{\rho}$ lifts

$$(\text{A.2.10}) \qquad \begin{tikzcd} & \widetilde{\mathrm{T}}_{n,k} \arrow[d, "\iota"'] \\ \overline{\mathrm{T}}_{r,k} \arrow[ur, "\tilde{\rho}"] \arrow[r, "\overline{\rho}"] & \mathrm{T}_{n,k} \end{tikzcd}$$

such that

$$(\text{A.2.11}) \qquad \mathrm{im}\bigl(\mathrm{X}(\hat{\rho})\bigr) \quad = \quad \mathrm{X}(\widetilde{\mathrm{T}}_{n,k}).$$

Let Σ be a fan in $\mathbb{R}^r$ such that

$$(\text{A.2.12}) \qquad \Sigma\bigl((\mathbb{P}^1_k)^r\bigr) \quad = \quad (\mathbb{Z}^r, \Sigma),$$

then

$$(\text{A.2.13}) \qquad \Sigma\bigl(\mathrm{T}_{n,k} \times (\mathbb{P}^1_k)^r\bigr) \quad = \quad \bigl(\mathbb{Z}^n \times \mathbb{Z}^r, \{\vec{0}\} \times \Sigma\bigr),$$

hence

$$(\text{A.2.14}) \qquad \begin{aligned} \Sigma(\overline{\mathrm{T}}_{r,k}) \quad &= \quad \Bigl(\mathrm{graph}\bigl(\mathrm{X}(\hat{\rho})\bigr), \bigl(\{\vec{0}\} \times \Sigma\bigr)\big|_{\mathrm{graph}(\mathrm{X}(\hat{\rho}))\otimes\mathbb{R}}\Bigr) \\ &= \quad \Bigl(\mathrm{im}\bigl(\mathrm{X}(\hat{\rho})\bigr) \times \ker\bigl(\mathrm{X}(\hat{\rho})\bigr), \{\vec{0}\} \times \bigl(\Sigma|_{\ker(\mathrm{X}(\hat{\rho}))\otimes\mathbb{R}}\bigr)\Bigr) \end{aligned}$$

where $|$ denotes the restriction of a fan to a subspace, hence there exists a Cartesian square of k-schemes

$$\begin{tikzcd} \overline{\mathrm{T}}_{r,k} \arrow[r] \arrow[d, "\tilde{\rho}"'] & \overline{\mathrm{Ker}_0}(\hat{\rho}) \arrow[d] \\ \widetilde{\mathrm{T}}_{n,k} \arrow[r] & \mathrm{Spec}(k) \end{tikzcd} \tag{A.2.15}$$

where $\overline{\mathrm{Ker}_0}(\hat{\rho})$ denotes the closure of $\mathrm{Ker}_0(\hat{\rho})$ in $\overline{\mathrm{T}}_{r,k}$ where

$$\mathrm{Ker}_0(\hat{\rho}) \quad \subset \quad \mathrm{T}_{r,k} \tag{A.2.16}$$

denotes the identity component of the kernel of $\hat{\rho}$. By Corollaire 2.16 in [De77f] $\tilde{\rho}$ is universally locally acyclic with respect to the constant ℓ-adic sheaf $\overline{\mathbb{Q}}_\ell$ on $\overline{\mathrm{T}}_{r,k}$. Since the characteristic of k is sufficiently large, the isogeny ι is étale, hence $\overline{\rho}$, which is the composite of $\tilde{\rho}$ and ι, is universally locally acyclic with respect to the constant ℓ-adic sheaf $\overline{\mathbb{Q}}_\ell$ on $\overline{\mathrm{T}}_{r,k}$.

Fix notations as in the Cartesian diagram

$$\begin{tikzcd} \big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \mathrm{T}_{r,k} \arrow[r, "\mathrm{I}\times\mathrm{j}"] \arrow[d, "(\pi^r\times\mathrm{I})\times\mathrm{I}"'] & \big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \arrow[d, "(\pi^r\times\mathrm{I})\times\mathrm{I}"'] \\ \mathrm{T}_{r,k} \arrow[r, "\mathrm{j}"] & \overline{\mathrm{T}}_{r,k}, \end{tikzcd} \tag{A.2.17}$$

then $(\pi^r \times \mathrm{I}) \times \mathrm{I}$ is locally acyclic with respect to the constant ℓ-adic sheaf $\overline{\mathbb{Q}}_\ell$ on $((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}) \times_{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}} \mathrm{T}_{r,k}$ by smoothness.

The complement

$$\big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} - (\mathrm{I} \times \mathrm{j})\Big(\big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \mathrm{T}_{r,k}\Big) \tag{A.2.18}$$

is covered by open k-subschemes of the form

$$\begin{tikzcd}
\big(\widetilde{\mathrm{U}} \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \arrow[r] \arrow[d, "\mathrm{I}\times\mathrm{i}"'] & \big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \arrow[d, "\mathrm{I}\times\mathrm{i}"'] \\
\widetilde{\mathrm{U}} \times \mathrm{T}_{n,k} \arrow[r] \arrow[d, "\mathrm{p}"'] & (\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k} \arrow[d, "\mathrm{p}"'] \\
\widetilde{\mathrm{U}} \arrow[r] \arrow[d, "\pi^r"'] & (\mathbb{P}^1_k)^r \arrow[d, "\pi^r"'] \\
\mathrm{U} \arrow[r] & (\mathbb{P}^1_k)^r
\end{tikzcd} \tag{A.2.19}$$

where p denotes the projection morphism, the squares are Cartesian, and U is an open k-subscheme of $(\mathbb{P}^1_k)^r$ where at least one of the coordinates is zero or infinity. By (A.2.15) it suffices to consider U which intersect

$$\overline{\mathrm{Ker}}_0(\hat{\rho}) - \mathrm{Ker}_0(\hat{\rho}) \quad \subset \quad (\mathbb{P}^1_k)^r - \mathrm{T}_{r,k}, \tag{A.2.20}$$

hence without loss of generality U is a repeated Cartesian product of factors of the form

$$\mathbb{G}_{\mathrm{a},k} \quad \text{or} \quad \mathbb{P}^1_k - \mathrm{Spec}(k) \quad \subset \quad \mathbb{P}^1_k \tag{A.2.21}$$

where $\mathrm{Spec}(k)$ denotes the point zero in $\mathbb{P}^1_k$, such that the image of $\mathrm{Ker}_0(\hat{\rho})$ under the projection from U to each Cartesian factor of the form $\mathbb{P}^1_k - \mathrm{Spec}(k)$ is dense.

Let X_i denote a coordinate on a Cartesian factor of the form $\mathbb{G}_{\mathrm{a},k}$, let $1/Y_j$ denote a coordinate on a Cartesian factor of the form $\mathbb{P}^1_k - \mathrm{Spec}(k)$, then for each index j_0 and for each β in $\mathbb{G}_{\mathrm{m},k}$ the point in U with coordinates

$$\Big(X_1, X_2, \ldots, \frac{1}{Y_1}, \frac{1}{Y_2}, \ldots, \frac{1}{Y_{j_0}}, \ldots\Big) \tag{A.2.22}$$

is equivalent under the action of $\mathrm{Ker}_0(\hat{\rho})$ to a point with coordinates

$$\Big(X_1', X_2', \ldots, \frac{1}{Y_1'}, \frac{1}{Y_2'}, \ldots, \frac{\beta}{Y_{j_0}}, \ldots\Big). \tag{A.2.23}$$

Let $\widetilde{X}_i$ denote a coordinate on $\mathbb{G}_{\mathrm{a},k}$, let $\widetilde{Y}_j$ denote a coordinate on

$$\mathbb{P}^1_k - \underline{k_0} \tag{A.2.24}$$

where $\underline{k_0}$ denotes the constant finite k-subscheme of $\mathbb{G}_{\mathrm{a},k}$ defined by the polynomial

$$X^q - X, \tag{A.2.25}$$

such that

$$\Bigg(\widetilde{X}_1, \widetilde{X}_2, \ldots, \frac{1}{\widetilde{Y}_1}, \frac{1}{\widetilde{Y}_2}, \ldots, \hat{\rho}\Big(\widetilde{X}_i{}^q - \widetilde{X}_i, \frac{1}{\widetilde{Y}_j{}^q} - \frac{1}{\widetilde{Y}_j}\Big)_{i,j=1,2,\ldots}\Bigg) \tag{A.2.26}$$

are coordinates on $(\widetilde{\mathrm{U}} \times \mathrm{T}_{n,k}) \times_{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}} \overline{\mathrm{T}}_{r,k}$. With respect to the coordinates (A.2.26) the restriction of $(\pi^r \times \mathrm{I}) \times \mathrm{I}$ to

$$\big(\widetilde{\mathrm{U}} \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \;\subset\; \big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \tag{A.2.27}$$

is defined by the polynomial

$$\Bigg(\widetilde{X}_i{}^q - \widetilde{X}_i, \frac{1}{\widetilde{Y}_j{}^q/(1 - \widetilde{Y}_j{}^{q-1})}, \hat{\rho}\Big(\widetilde{X}_i{}^q - \widetilde{X}_i, \frac{1}{\widetilde{Y}_j^q} - \frac{1}{\widetilde{Y}_j}\Big)\Bigg)_{i,j=1,2,\ldots}. \tag{A.2.28}$$

Since (A.2.28) is equal to the composite of the étale morphisms

$$\pi(\widetilde{X}_i) \;=\; \widetilde{X}_i{}^q - \widetilde{X}_i, \tag{A.2.29}$$

the radicial morphisms

$$\sigma(\widetilde{Y}_j) \;=\; \widetilde{Y}_j{}^{q}, \tag{A.2.30}$$

and the changes of variables (A.2.23) with

$$\beta \;=\; 1 - \widetilde{Y}_j{}^{q-1}, \tag{A.2.31}$$

each of which is universally locally acyclic with respect to the constant ℓ-adic sheaf $\overline{\mathbb{Q}}_\ell$ on its source, so is the composite (A.2.28). Hence $(\pi^r \times \mathrm{I}) \times \mathrm{I}$ is universally locally acyclic with respect to the constant ℓ-adic sheaf $\overline{\mathbb{Q}}_\ell$ on $((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}) \times_{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}} \overline{\mathrm{T}}_{r,k}$. Hence the lemma follows by composition. □

(A.3) Lemma *Let ρ be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of the torus $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$, fix notations as in the diagram*

$$\begin{tikzcd} & \big((\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}\big) \underset{(\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k}}{\times} \overline{\mathrm{T}}_{r,k} \arrow[d, "(\pi^r \times \mathrm{I}) \times \mathrm{I}"'] \\ \mathrm{T}_{r,k} \arrow[r, "\mathrm{j}"] \arrow[d, "\mathrm{tr}"'] & \overline{\mathrm{T}}_{r,k} \\ \mathbb{G}_{\mathrm{a},k} & \end{tikzcd} \tag{A.3.1}$$

where I *denotes the identity morphism.*

If ρ is positive, then

$$\mathrm{j}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big) \tag{A.3.2}$$

is isomorphic to a direct summand of

$$\big((\pi^r \times \mathrm{I}) \times \mathrm{I}\big)_!\big(\overline{\mathbb{Q}}_\ell[r]\big) \tag{A.3.3}$$

as objects in $\mathrm{Perv}(\overline{\mathrm{T}}_{r,k}, \overline{\mathbb{Q}}_\ell)$.

Proof. Fix notations as in the commutative diagram

$$\begin{tikzcd}
 & & & \overline{\mathrm{T}}_{r,k} \arrow[r, "\mathrm{i}"] & (\mathbb{P}^1_k)^r \times \mathrm{T}_{n,k} \arrow[d, "\mathrm{p}"'] \\
 & & & & (\mathbb{P}^1_k)^r \\
\mathrm{T}_{r,k} \simeq (\mathbb{G}_{\mathrm{m},k})^r \arrow[uurrr, "\mathrm{j}"] \arrow[rr] \arrow[dr, "\mathrm{tr}"'] & & (\mathbb{G}_{\mathrm{a},k})^r \arrow[uurr, "(\mathrm{j}_\mathrm{a})^r"] \arrow[drr, "\mathrm{sum}"] & & \\
 & \mathbb{G}_{\mathrm{a},k} \arrow[rrr, equal] & & & \mathbb{G}_{\mathrm{a},k}
\end{tikzcd} \tag{A.3.4}$$

where p denotes the projection morphism, the horizontal arrow denotes the inclusion morphism, sum denotes the summation morphism, and $\mathrm{T}_{r,k}$ and $(\mathbb{G}_{\mathrm{m},k})^r$ are identified via a choice of $\mathcal{B}$.

By the decomposition theorem

$$\pi_!(\overline{\mathbb{Q}}_\ell[1]) \;\simeq\; \overline{\mathbb{Q}}_\ell[1] \oplus \bigoplus_{\psi'} \mathrm{j}_{\mathrm{a},!*}(\mathcal{L}_{\psi'}[1]) \tag{A.3.5}$$

as objects in $\mathrm{Perv}(\mathbb{P}^1_k, \overline{\mathbb{Q}}_\ell)$, where ψ' traverses the set of nontrivial $\overline{\mathbb{Q}}_\ell$-valued additive characters of k_0. Since π is totally ramified at infinity which is the complement of $\mathbb{G}_{\mathrm{a},k}$ in $\mathbb{P}^1_k$, the stalk of $\pi_!(\overline{\mathbb{Q}}_\ell[1])$ at infinity is one-dimensional, hence

$$\mathrm{j}_{\mathrm{a},!*}(\mathcal{L}_{\psi'}[1]) \;\simeq\; \mathrm{j}_{\mathrm{a},!}(\mathcal{L}_{\psi'}[1]) \tag{A.3.6}$$

for each nontrivial ψ'. Hence by the Künneth theorem the object

$$(\mathrm{A.3.7}) \qquad (\mathrm{j_a})^r{}_!\big(\mathrm{sum}^*(\mathcal{L}_\psi)[r]\big) \;\simeq\; \big(\mathrm{j}_{\mathrm{a},!}(\mathcal{L}_\psi[1])\big)\boxtimes\cdots\boxtimes\big(\mathrm{j}_{\mathrm{a},!}(\mathcal{L}_\psi[1])\big)$$

in $\mathrm{Perv}((\mathbb{P}^1_k)^r,\overline{\mathbb{Q}}_\ell)$ is isomorphic to a direct summand of

$$(\mathrm{A.3.8}) \qquad \pi^r{}_!(\overline{\mathbb{Q}}_\ell[r]) \;\simeq\; \big(\pi_!(\overline{\mathbb{Q}}_\ell[1])\big)\boxtimes\cdots\boxtimes\big(\pi_!(\overline{\mathbb{Q}}_\ell[1])\big).$$

Since ρ is positive by assumption, the k-subscheme

$$(\mathrm{A.3.9}) \qquad (\mathrm{p}\circ\mathrm{i})\big(\overline{\mathrm{T}}_{r,k}-\mathrm{j}(\mathrm{T}_{r,k})\big) \;\subset\; (\mathbb{P}^1_k)^r$$

is contained in

$$(\mathrm{A.3.10}) \qquad (\mathbb{P}^1_k)^r-(\mathrm{j_a})^r\big((\mathbb{G}_{\mathrm{a},k})^r\big),$$

hence the object

$$(\mathrm{A.3.11}) \qquad \mathrm{j}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big)$$

in $\mathrm{Perv}(\overline{\mathrm{T}}_{r,k},\overline{\mathbb{Q}}_\ell)$ is isomorphic to a direct summand of

$$(\mathrm{A.3.12}) \qquad \big((\pi^r\times\mathrm{I})\times\mathrm{I}\big)_!\big(\overline{\mathbb{Q}}_\ell[r]\big) \;\simeq\; \mathrm{i}^*\big((\pi^r\times\mathrm{I})_!(\overline{\mathbb{Q}}_\ell)\big)[r]$$

by the proper base change theorem. □

(A.4) Proposition *Let ρ be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ such that $\hat{\rho}$ is surjective.*

If ρ is positive, then the morphism $\overline{\rho}$ is universally locally acyclic with respect to the object

$$\mathrm{j}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big) \tag{A.4.1}$$

in $\mathrm{Perv}(\overline{\mathrm{T}}_{r,k},\overline{\mathbb{Q}}_\ell)$.

Proof. The following argument is essentially due to Katz–Laumon in [KL85].

Since $(\pi^r \times \mathrm{I}) \times \mathrm{I}$ is finite, by Lemma (A.2) and the proper base change theorem $\overline{\rho}$ is universally locally acyclic with respect to the object

$$\big((\pi^r \times \mathrm{I}) \times \mathrm{I}\big)_!\big(\overline{\mathbb{Q}}_\ell[r]\big) \tag{A.4.2}$$

in $\mathrm{Perv}(\overline{\mathrm{T}}_{r,k},\overline{\mathbb{Q}}_\ell)$, hence by Lemma (A.3) $\overline{\rho}$ is universally locally acyclic with respect to the object

$$\mathrm{j}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big) \tag{A.4.3}$$

in $\mathrm{Perv}(\overline{\mathrm{T}}_{r,k},\overline{\mathbb{Q}}_\ell)$. □

(A.5) Corollary *Let ρ be an r-dimensional $\overline{\mathbb{Q}}_\ell$-linear representation of $\check{\mathrm{T}}_n(\overline{\mathbb{Q}}_\ell)$ which is positive, then $\mathrm{Hyp}_{\psi,\rho}$ is an ℓ-adic perverse local system on the image of $\hat{\rho}$.*

Proof. Since ρ is positive by assumption, by Proposition (A.4) $\overline{\rho}$ is universally locally acyclic with respect to the object

$$\mathrm{j}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big) \tag{A.5.1}$$

in $\mathrm{Perv}(\overline{\mathrm{T}}_{r,k},\overline{\mathbb{Q}}_\ell)$ when restricted to the image of $\overline{\rho}$ which is equal to the image of $\hat{\rho}$. Hence by Théorèm 5.3.1 in [De77a] each cohomology sheaf of the ℓ-adic complex

$$\begin{aligned} \text{(A.5.2)} \qquad \mathrm{Hyp}_{\psi,\rho} &\simeq \hat{\rho}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big) \\ &\simeq \overline{\rho}_*\Big(\mathrm{j}_!\big(\mathrm{tr}^*(\mathcal{L}_\psi)[r]\big)\Big) \end{aligned}$$

is an ℓ-adic local system on the image of $\hat{\rho}$, hence $\mathrm{Hyp}_{\psi,\rho}$ is an ℓ-adic perverse local system by Proposition (1.9). □

REFERENCES

[Ar76] J. Arthur, *The characters of discrete series as orbital integrals*, Inv. Math. 32 (1976), 205-261.

[Ar78] J. Arthur, *A trace formula for reductive groups I: terms associated to classes in* G($\mathbb{Q}$), Duke Math. J. 245 (1978), 911-952.

[Ar81] J. Arthur, *The trace formula in invariant form*, Ann. of Math. 114 (1981), 1-74.

[Ar82] J. Arthur, *On a family of distributions obtained from Eisenstein series I: Application of the Paley-Wiener theorem*, Amer. J. Math. 104 (1982), 1243-1288.

[Ar85] J. Arthur, *A measure on the unipotent variety*, Canad. J. Math. 37 (1985), 1237-1274.

[Ar86] J. Arthur, *On a family of distributions obtained from orbits*, Canad. J. Math. 38 (1986), 179-214.

[Ar88a] J. Arthur, *The local behaviour of weighted orbital integrals*, Duke Math. J. 56 (1988), 223-293.

[Ar88b] J. Arthur, *The invariant trace formula I. Local theory*, J. Amer. Math. Soc. 1 (1988), 323-383.

[Ar88c] J. Arthur, *The invariant trace formula II. Global theory*, J. Amer. Math. Soc. 1 (1988), 501-554.

[Ar05] J. Arthur, *An introduction to the trace formula*, in *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Math. Proc. 4 (2005), Amer. Math. Soc., Providence, RI, 1-263.

[Bo81] W. Borho, *Über Schichten halbeinfacher Lie-Algebren*, Inv. Math. 65 (1981), 283-371.

[BBD82] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, in Analysis and topology on singular spaces, I (Luminy 1981), Astérisque, 100 (1982), 5-171.

[BK00] A. Braverman and D. Kazhdan, *γ-functions of representations and lifting* (with an appendix by V. Vologodsky), Geom. Funct. Anal. 2000, Special Volume, Part I, 237-278.

[BK02] A. Braverman and D. Kazhdan, *γ-sheaves on reductive groups*, in Studies in memory of Issai Schur (Chevaleret/Rehovot 2000), Progr. Math., 210 (2003), 27-47.

[Bo69] A. Borel, *Introduction aux groupes arithmétiques*, Actualitées Sci. Ind. No. 1341, Hermann, Paris, 1969.

[Ch02a] P.-H. Chaudouard, *La formule des traces pour les algèbres de Lie*, Math. Ann. 322 (2002), 347-382.

[Ch02b] P.-H. Chaudouard, *Intégrales orbitales pondérées sur les algèbres de Lie : le cas p-adique* Canad. J. Math. 54 (2002), 263-302.

[De77a] P. Deligne, *Cohomologie étale: les points de départ*, in Cohomologie étale (SGA 4 1/2), Lecture Notes in Math., 569 (1977), 4-75.

[De77s] P. Deligne, *Applications de la formule des traces aux sommes trigonométriques*, in Cohomologie étale (SGA 4 1/2), Lecture Notes in Math., 569 (1977), 168-232.

[De77f] P. Deligne, *Théorèmes de finitude en cohomologie ℓ-adique*, in Cohomologie étale (SGA 4 1/2), Lecture Notes in Math., 569 (1977), 233-261.

[Ev98] U. Everling, *An example of Fourier transforms of orbital integrals and their endoscopic transfer*, New York J. Math. 4 (1998), 17-29.

[Fe37] W. L. Ferrar, *Summation formulae and their relation to Dirichlet's series II*, Compos. Math. 4 (1937), 394-405.

[FL11] T. Finis and E. Lapid, *On the Arthur-Selberg trace formula for* GL(2), Groups Geom. Dyn. 5 (2011), No. 2, Special issue on the occasion of Fritz Grunewald's 60th birthday, 367-391.

[Fu93] W. Fulton, *Introduction to toric varieties*, Ann. of Math. Stud., 131 (1993).

[GL96] O. Gabber and F. Loeser, *Faisceaux pervers ℓ-adiques sur un tore*, Duke Math. J., 83 (1996), 501-606.

[Ho13] W. Hoffmann, *Induced conjugacy classes, prehomogeneous varieties, and canonical parabolic subgroups*, Preprint, 2013, arXiv:1206.3068.

[HW13] W. Hoffmann and S. Wakatsuki, *On the geometric side of the Arthur trace formula for the symplectic group of rank 2*, Preprint, 2013, arXiv:1310.0541.

[KL85] N. Katz and G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Publ. Math. IHES, 62 (1985), 145-202.

[Ko05] R. Kottwitz, *Harmonic analysis on reductive p-adic groups and Lie algebras*, in *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Math. Proc., Vol. 4, Amer. Math. Soc., Providence, RI, 2005, 391-522.

[La80] R. Langlands, *Base Change for* GL(2), Annals of Math. Studies 96, Princeton U. P. 1980.

[Ma11] J. Matz, *Arthur's trace formula for* GL(2) *and* GL(3) *and non-compactly supported test functions*, Dissertation, Düsseldorf, 2011, `http://www.math.uni-bonn.de/people/matz/`.

[Mo93a] R. Morelli, *Translation scissors congruence*, Adv. Math. 100 (1993), 1-27.

[Mo93b] R. Morelli, *The K theory of a toric variety*, Adv. Math. 100 (1993), 154-182.

[On63] T. Ono, *On the Tamagawa number of algebraic tori*, Ann. of Math. 78 (1963), 47-73.

[St65] R. Steinberg, *Regular elements of semisimple algebraic groups*, Publ. Math. IHES, 25 (1965), 49-80.

[Ta50] J. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, in *Algebraic Number Theory*, Academic Press, New York, 1967, 305-347.

[Va77] V. S. Varadarajan, *Harmonic analysis on real reductive groups, Part one: Invariant analysis on a real reductive Lie algebra*, in *Harmonic analysis on real reductive groups*, Lecture Notes in Math., Vol. 576, Springer, New York, 1977, 1-174.

[Wa95] J.-L. Waldspurger, *Une formule des traces locale pour les algèbres de Lie p-adiques*, J. Reine Angew. Math. 465 (1995), 41-99.