THE UNIVERSITY OF CHICAGO

HOMOGENIZATION OF INTERFACE MOTIONS IN THE PARABOLIC SCALING

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To my grandparents

Was ist der Witz?

— Paul Ehrenfest

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ABSTRACT

This thesis details recent results on the periodic homogenization of interface motions in the parabolic scaling regime.

First, we treat phase transitions in periodic media, developing a variational approach that unifies the point of view of Barles and Souganidis [16] with Aubrey-Mather theory. We show that (possibly non-smooth) pulsating standing waves can be obtained as minimizers of a Percival-type Lagrangian in the spirit of the latter theory. Pulsating standing waves are shown to generate the recurrent plane-like minimizers of the energy and to determine the macroscopic surface tension. In the case of laminar media, these functions are used to demonstrate a number of pathologies, such as the non-differentiability of the surface tension. Additionally, we prove two homogenization results in the sharp-interface limit, one pertaining to laminar media and a restricted class of initial data, and the other, to the Allen-Cahn equation with a periodic dissipation term.

Finally, we study a class of non-variational curvature flows with periodic coefficients. By analyzing a degenerate elliptic equation on the torus, we identify the macroscopic behavior of the moving interface at points where its normal vector is irrational. To analyze the motion near rational normals, we adapt an idea of Feldman and Kim [49] from the study of oscillating boundary value problems. This leads to the conclusion that in dimension $d \geq 3$, the macroscopic interface velocity is a discontinuous function of the normal direction. We prove that the associated interface motion is well-posed in spite of the discontinuities and conclude that homogenization occurs.

CHAPTER 1

INTRODUCTION

This thesis concerns the large-scale behavior of interfaces moving in a heterogeneous medium. The problem is approached through the lens of the homogenization theory for parabolic partial differential equations. A number of new results are proved for variational and non-variational interface motions using techniques from PDE theory, calculus of variations, and ergodic theory. The overarching theme of the work is the codimension-one character of interfaces leads to interesting geometric phenomena and pathologies.

1.1 A Motivating Example: Diffuse Interfaces in a Periodic Medium

1.1.1 The Problem

Let us begin by recalling the classical asymptotics of the Allen-Cahn equation. Given a small parameter $\epsilon > 0$, the problem is to characterize the asymptotic behavior, in the limit $\epsilon \to 0$, of the solution u^ϵ of the Allen-Cahn equation

$$u_t^\epsilon - \Delta u^\epsilon + \epsilon^{-2} W'(u^\epsilon) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty)$$

subject to appropriate initial conditions. For physical reasons, the solution u^ϵ takes values in $[-1, 1]$ and the nonlinearity W' is the derivative of a double-well potential W that attains its global minimum precisely on the set $\{-1, 1\}$.

The motivation for studying the asymptotics of the Allen-Cahn equation comes from materials science. The Allen-Cahn equation is a simplified description of phase transformations in a material coexisting in one of two phases. Unlike, say, the Stefan problem, where the interface that forms between ice and water is modeled as a smooth surface with infinitesimal thickness, the Allen-Cahn equation is a diffuse-interface model, wherein the nonzero thickness of the interface is taken into account. Thus, the solution u^ϵ, referred to as a phase field, indicates the local phase of the material: it is close to 1 in the bulk of one of the phases, close to -1 in the bulk of the other, and the interface corresponds to the region in space where it is away from 1 and -1.

The parameter ϵ describes the width of a typical interface. Notice that the transformation $u^\epsilon(x,t) = u(\epsilon^{-1}x, \epsilon^{-2}t)$ takes a solution u of the unscaled ($\epsilon = 1$) equation to a solution u^ϵ of the scaled equation. Accordingly, the limit $\epsilon \to 0$ describes the evolution of the material at scales that are much lager than the interface width.

Allen and Cahn [3] predicted that the interface between phases moves by mean curvature flow. This has been made precise through the contributions of many mathematicians since then, including de Mottoni and Schatzman [40], Bronsard and Kohn [23], and Chen [30] in the setting of smooth flows and then Evans, Soner, and Souganidis [43], Barles, Soner, and Souganidis [12], and Barles and Souganidis [16] in the general case.

The result, stated roughly, is as follows: if the solution u^ϵ is such that $u^\epsilon(\cdot,0) \to 1$ inside some smooth open set E_0 and $u^\epsilon(\cdot,0) \to -1$ in $\mathbb{R}^d \setminus \overline{E_0}$, and if $(E_t)_{t\geq 0}$ are open sets with boundaries $(\partial E_t)_{t\geq 0}$ moving by mean curvature flow, then

$$u^\epsilon(\cdot,t) \to 1 \text{ in } E_t \quad \text{and} \quad u^\epsilon(\cdot,t) \to -1 \text{ in } \mathbb{R}^d \setminus \overline{E_t}.$$

Put simply, at large (or macroscopic) scales, the interface between phases looks like a smooth boundary moving by mean curvature flow. Since phase fields are replaced by sets in the limit, this type of convergence is referred to as a *sharp-interface limit*.

The Allen-Cahn equation is the L^2 gradient flow of the following energy functional

$$\mathcal{F}^{AC}(u^\epsilon;\Omega) = \int_\Omega \left(\frac{\epsilon}{2}\|Du^\epsilon\|^2 + W(u^\epsilon) \right) dx$$

so energy dissipation is the basic mechanism driving the evolution of the solution. As is pointed out in [19], in principle, the coefficients in the energy should vary in space in general to reflect the changes in the material or impurities embedded within it. This consideration leads to more general energies such as the following one:

$$\mathcal{F}(u^\epsilon;\Omega) = \int_\Omega \left(\frac{\epsilon}{2}\langle a(\epsilon^{-1}x)Du^\epsilon, Du^\epsilon\rangle + W(\epsilon^{-1}x, u^\epsilon)\right)\, dx. \tag{1.1}$$

Here a is a symmetric matrix-valued field and $W(y,\cdot)$ has a double-well shape for each $y \in \mathbb{R}^d$, the spatial variation of a and W modeling the heterogeneity of the medium. The L^2 gradient flow of this energy results in an Allen-Cahn equation with variable coefficients

$$u^\epsilon_t - \operatorname{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W_u(\epsilon^{-1}x, u^\epsilon) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty). \tag{1.2}$$

The question is now to characterize the sharp-interface limit of such an equation.

In order for the solution of (1.2) to converge, some assumptions need to be made on the structure of the coefficients a and W. Throughout this work, we will assume that they are periodic functions, that is,

$$a(y+k) = a(y) \quad \text{and} \quad W(y+k,\cdot) = W(y,\cdot) \quad \text{for each } y \in \mathbb{R}^d,\ k \in \mathbb{Z}^d.$$

The problem is then to determine to what extent averaging effects lead solutions of (1.2) to converge, as in the spatially homogeneous (constant-coefficient) case.

1.1.2 Pulsating Standing Waves

The starting point for our treatment of (1.2) is work by Barles and Souganidis [16]. They showed that if certain assumptions on a and W are satisfied, then the sharp-interface limit of (1.2) is described by macroscopic interfaces that evolve with normal velocity given by

$$V_{\partial E_t} = \bar{M}(n_{\partial E_t})^{-1}\mathrm{tr}\left(\bar{\mathcal{S}}(n_{\partial E_t})A_{\partial E_t}\right). \tag{1.3}$$

Here $n_{\partial E_t}$ is the normal vector and $A_{\partial E_t}$, the second fundamental form of E_t, and the coefficients $\bar{\mathcal{S}}$ and $\bar{M}$ are determined by a and W through averaging effects.

The key assumption in the analysis of (1.2) in [16] is the existence and smoothness of certain special functions, called pulsating standing waves. These are used in an asymptotic expansion for the solution u^ϵ. (The asymptotic expansion employed in [16] is reviewed in Section 4.1.)

Given an $e \in S^{d-1}$, a pulsating standing wave U_e in the e direction is a solution of the PDE

$$\begin{cases} \mathcal{D}_e^*(a(y)\mathcal{D}_e U_e) + W_u(y, U_e) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^d, \\ \lim_{s \to \pm\infty} U_e(s, y) = \pm 1, \quad \partial_s U_e \geq 0, \end{cases} \tag{1.4}$$

where the differential operator $\mathcal{D}_e$ is defined by $\mathcal{D}_e = e\partial_s + D_y$ and $\mathcal{D}_e^*$ is its L^2 adjoint. Equations of this type were introduced by Xin [91].

Notice that the pulsating standing wave equation (1.4) is a degenerate, time-stationary Allen-Cahn equation in a $(d+1)$-dimensional cylinder, whereas the equation of interest to us, (1.2), is posed in d space dimensions. The connection between them is through the family of functions $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ in $\mathbb{R}^d$ defined by

$$u^e_\zeta(x) = U_e(\langle x, e\rangle - \zeta, x). \tag{1.5}$$

A computation shows that if U_e is a solution of (1.4), then, for each $\zeta \in \mathbb{R}$, the function u_ζ is a critical point of the energy $\mathcal{F}$, that is,

$$-\mathrm{div}(a(x)Du^e_\zeta) + W_u(x, u^e_\zeta) = 0 \quad \text{in } \mathbb{R}^d.$$

Furthermore, these functions are plane-like and ordered in the sense that

$$\lim_{\langle x,e\rangle\to\pm\infty} u^e_\zeta(x) = \pm 1 \quad \text{and} \quad u^e_\zeta \leq u^e_{\zeta'} \quad \text{if } \zeta > \zeta'.$$

Thus, the existence of a solution U_e of the pulsating standing wave equation implies the existence of a continuous one-parameter family of plane-like critical points of $\mathcal{F}$.

Pulsating standing waves play a central role in the analysis of (1.2) in [16]. Accordingly, one of the questions treated below is the existence of pulsating waves.

Question 1. *Do pulsating standing waves exist in general? Are they smooth?*

1.1.3 The Surface Tension and Einstein Relation

Using the theory of Γ-convergence, it is possible to argue that the energy $\mathcal{F}$, suitably rescaled, converges to a limiting energy. This was first made precise by Modica and Mortola [69] in the spatially homogeneous setting.

The periodic setting has been studied by Ansini, Braides, and Chiadò-Piat [5] and Cristoferi, Fonseca, Hagerty, and Popovici [35, 36]. To capture the large scale behavior of $\mathcal{F}$, we define the rescaled functional $\mathcal{F}_\epsilon$ for $\epsilon > 0$ by

$$\mathcal{F}_\epsilon(u^\epsilon;\Omega) = \int_\Omega \left(\frac{\epsilon}{2}\langle a(\epsilon^{-1}x)Du^\epsilon, Du^\epsilon\rangle + \epsilon^{-1}W(\epsilon^{-1}x, u^\epsilon)\right) dx. \tag{1.6}$$

The results of [5, 35, 36] show that, under suitable assumptions on a and W, there is a Finsler norm $\bar{\sigma}$ and an anisotropic perimeter functional $\bar{\mathcal{F}}$ of the form

$$\bar{\mathcal{F}}(E;\Omega) = \int_{\partial E\cap\Omega} \bar{\sigma}(n_{\partial E}(\xi))\, \mathcal{H}^{d-1}(d\xi)$$

such that if $u^\epsilon \to 1$ in E and $u^\epsilon \to -1$ in $\Omega \setminus \overline{E}$, then

$$\bar{\mathcal{F}}(E;\Omega) \leq \liminf_{\epsilon \to 0^+} \mathcal{F}_\epsilon(u^\epsilon;\Omega).$$

Further, it is possible to choose $(u^\epsilon)_{\epsilon > 0}$ so that the inequality is an equality and the liminf is a limit. In the statistical mechanics literature, the surface energy density $\bar{\sigma}$ is referred to as the *surface tension*.

Following the discussion in Spohn [87] and related results on the Lebowitz-Penrose functional by Bellettini, Buttà, and Presutti [17], it is natural to expect that the effective interface motion that arises in the limit of (1.2) has the form

$$V_{\partial E_t} = \bar{M}(n_{\partial E_t})^{-1} \text{tr}\left(D^2\bar{\sigma}(n_{\partial E_t}) A_{\partial E_t}\right). \tag{1.7}$$

Here $\bar{\sigma}$ is the surface tension obtained from the Γ-limit of $\mathcal{F}$ and $\bar{M}$ is a mobility coefficient. In the mathematical physics literature, (1.7) is referred to as an Einstein relation since it shows that the homogenized velocity is proportional to the homogenized energy gradient. Mathematically, $\text{tr}(D^2\bar{\sigma}(n)A)$ is the second variation of $\bar{\mathcal{F}}$ so (1.7) implies that "homogenization and gradient flow commute" up to the determination of the mobility coefficient $\bar{M}$, which determines the metric.

The preceding discussion motivates the next question treated in this thesis.

Question 2. *Does the Einstein relation* (1.7) *describe the sharp-interface limit of the Allen-Cahn-like equation* (1.2)*? Is it consistent with the results of [16]?*

1.1.4 A Variational Interpretation of Pulsating Standing Waves

As we argued above, there is reason to believe that the homogenization of the Allen-Cahn-like equation (1.2) will be related to the Γ-limit of the associated energy $\mathcal{F}$. Considering that [16] proves homogenization using pulsating standing waves, that leaves the question whether or not these waves are also relevant from a variational point of view.

Notice that the equation $\mathcal{D}_e^*(a(y)\mathcal{D}_e U) + W_u(y, U) = 0$ in (1.4) is the Euler-Lagrange equation of the energy

$$\mathscr{T}_e(V) = \int_{\mathbb{R}\times\mathbb{T}^d} \left(\frac{1}{2}\langle a(y)\mathcal{D}_e V, \mathcal{D}_e V\rangle + W(y, V)\right) ds\, dy.$$

This suggests it may be possible to study pulsating standing waves using variational techniques.

Question 3. *Is it possible to obtain pulsating standing waves by minimizing $\mathscr{T}_e$? Is a solution of* (1.4) *necessarily a minimizer?*

1.2 A Variational Approach to Pulsating Standing Waves

1.2.1 Assumptions

Throughout the paper, we denote by $\mathbb{T}^d$ the d-torus, defined as a quotient space $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Hence we will treat a and W as functions defined in $\mathbb{T}^d$. More precisely, we make the following assumptions:

$$a : \mathbb{T}^d \to \mathcal{S}_d \quad \text{smooth},$$
$$W : \mathbb{T}^d \times \mathbb{R} \to [0, \infty) \quad \text{smooth}.$$

We take a to be uniformly elliptic. Specifically, we assume there are constants $\lambda, \Lambda > 0$ such that

$$\lambda \mathrm{Id} \le a(y) \le \Lambda \mathrm{Id} \quad \text{for each } y \in \mathbb{T}^d.$$

Concerning the potential W, we assume that there are continuous functions $\overline{W}, \underline{W} : \mathbb{R} \to$

$[0,\infty)$ such that

$$\underline{W}(u) \leq W(y,u) \leq \overline{W}(u) \quad \text{for each } (y,u) \in \mathbb{T}^d \times \mathbb{R}.$$

Further, we assume that the zero sets of these functions are precisely $\{-1,1\}$, that is,

$$\{u \in \mathbb{R} \mid \overline{W}(u) = 0\} = \{u \in \mathbb{R} \mid \underline{W}(u) = 0\} = \{-1,1\}.$$

Finally, we assume that $\overline{W}$ and $\underline{W}$ are convex near 1 and -1, that is,

$$\min\{\overline{W}''(1), \overline{W}''(-1), \underline{W}''(1), \underline{W}''(-1)\} > 0.$$

1.2.2 Results

First, we develop a variational approach to the existence and uniqueness of pulsating standing waves. As was already observed above, the degenerate Allen-Cahn equation $\mathcal{D}_e^*(a(y)\mathcal{D}_e U)) + W_u(y,U) = 0$ is the Euler-Lagrange equation of the non-coercive energy functional $\mathscr{T}_e$ defined by

$$\mathscr{T}_e(V) = \int_{\mathbb{R}\times\mathbb{T}^d} \left(\frac{1}{2}\langle a(y)\mathcal{D}_e V, \mathcal{D}_e V\rangle + W(y,V) \right) dy\, ds.$$

Below we prove that a weak solution of (1.4) can be obtained through minimization of $\mathscr{T}_e$.

To state the result more precisely, let us define the competition class $\mathscr{X}$ and minimal energy $\mathscr{E}(e)$ by

$$\mathscr{E}(e) = \inf\left\{\mathscr{T}_e(V) \mid V \in \mathscr{X}\right\}, \tag{1.8}$$
$$\mathscr{X} = \{V \in L^\infty(\mathbb{R}\times\mathbb{T}^d) \mid -1 \leq V \leq 1, \lim_{s\to\pm\infty} V(s,y) = \pm 1 \text{ in } L^1_{\text{loc}}(\mathbb{R}\times\mathbb{T}^d)\}.$$

Theorem 1 ([73]). *(i) For each $e \in S^{d-1}$, there is a $U_e \in \mathscr{X}$ that is a minimizer for the variational problem $\mathscr{E}(e)$, that is, such that $\mathscr{T}_e(U_e) = \mathscr{E}(e)$. Further, U_e can be chosen in* *such a way that $\partial_s U_e \geq 0$.*

(ii) A minimizer as in (i) is a weak solution of the pulsating standing wave equation (1.4)*. If U_e is a continuous weak solution of* (1.4)*, then $U_e \in \mathscr{X}$ and $\mathscr{T}_e(U_e) = \mathscr{E}(e)$.*

(iii) It is possible to choose a, W, and e so that no minimizer of $\mathscr{E}(e)$ is a continuous function in $\mathbb{R} \times \mathbb{T}^d$.

The theorem shows that pulsating standing waves can be obtained as minimizers of the functional $\mathscr{T}_e$, and that any smooth (or even continuous) pulsating standing wave is necessarily in this class. At the same time, contrary to the assumptions in [16], it shows that smooth pulsating standing waves need not exist in general.

The minimizing property of the pulsating standing waves in Theorem 1 leads to a similar property for the functions $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ determined by the transformation (1.5). Furthermore, it turns out that the minimal value of $\mathscr{T}_e$ is precisely the surface tension $\bar{\sigma}(e)$ obtained through Γ-convergence. This is the subject of the next corollary.

Corollary 1. *(i) If U_e is a minimizer of $\mathscr{E}(e)$, then each function $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ in the family determined by* (1.5) *is a strongly Birkhoff plane-like minimizer of $\mathcal{F}$ (see Definition 9 below).*

(ii) If U_e is a minimizer of $\mathscr{E}(e)$, then $\mathscr{T}_e(U_e) = \bar{\sigma}(e)$.

(iii) For each $\zeta \in \mathbb{R}$, the function u^e_ζ has average energy equal to $\bar{\sigma}(e)$.

The previous results show that pulsating standing waves exist, even though they may not be smooth, and that they generate plane-like minimizers of $\mathcal{F}$ that can be used to compute the surface tension $\bar{\sigma}$. The question remains whether or not there are situations in which the approach of [16] goes through and homogenization holds.

In the next result, we give an affirmative answer. For the sake of generality, we will consider the gradient flow

$$\begin{cases} m(\epsilon^{-1}x, \epsilon Du^\epsilon)u^\epsilon_t - \operatorname{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W_u(\epsilon^{-1}x, u^\epsilon) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.9}$$

where the local mobility $m : \mathbb{T}^d \times \mathbb{R}^d \to (0, \infty)$ is smooth and bounded above and below, i.e.,

$$0 < \inf \left\{ m(y,p) \mid (y,p) \in \mathbb{T}^d \times \mathbb{R}^d \right\} \leq \sup \left\{ m(y,p) \mid (y,p) \in \mathbb{T}^d \times \mathbb{R}^d \right\} < \infty. \quad (1.10)$$

With the mobility term, (1.9) becomes the gradient flow of $\mathcal{F}$ with respect to the L^2-Riemannian metric given by

$$\langle v, w \rangle_{u^\epsilon} = \int_{\mathbb{R}^d} v(x) w(x) m(\epsilon^{-1} x, \epsilon D u^\epsilon) \, dx.$$

We will prove homogenization of (1.9) for specific choices of the initial datum u_0 in the context of laminar media. Here by laminar media we mean the case when the coefficients do not depend on the dth coordinate, or, in other words,

$$a_{y_d} \equiv 0, \quad W_{y_d} \equiv 0, \quad \text{and} \quad m_{y_d} \equiv 0. \quad (1.11)$$

In this setting, it turns out there are smooth pulsating standing waves U_e in every direction $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$. Using these pulsating waves in the asymptotic expansion approach of [16], we show that homogenization holds provided the initial interface is a graph over the dth coordinate.

Theorem 2. *Assume that a, W, and m are independent of the dth coordinate, that is,* (1.11) *holds. Suppose, in addition, that $u_0 \in UC(\mathbb{R}^d; [-1,1])$ satisfies the following two conditions:*

(i) There is a $\mathcal{U} \in UC(\mathbb{R}^{d-1})$ and a $q \in \mathbb{R}^{d-1}$ such that

$$\sup \left\{ |\mathcal{U}(x') - \langle q, x' \rangle| \mid x' \in \mathbb{R}^{d-1} \right\} < \infty$$

and

$$\{x \in \mathbb{R}^d \mid u_0(x) = 0\} = \{(x', x_d) \in \mathbb{R}^d \mid x_d = \mathcal{U}(x')\}.$$

(ii) There is a $\delta_0 > 0$ *and an* $R > 0$ *such that*

$$d_{\mathcal{H}}(\{x \in \mathbb{R}^d \mid u_0(x) = 0\}, \{x \in \mathbb{R}^d \mid |u_0(x)| < \delta_0\}) < R.$$

Under these assumptions, there is a continuous function $h : \mathbb{R}^{d-1} \times [0, \infty) \to \mathbb{R}$ *with* $h(\cdot, 0) = \mathcal{U}$ *such that if* $(u^\epsilon)_{\epsilon>0}$ *are the solutions of* (1.9) *with initial datum* $u^\epsilon(\cdot, 0) = u_0$ *and* $(E_t)_{t\geq 0}$ *are the epigraphs defined by* $E_t = \{x \in \mathbb{R}^d \mid x_d > h(x', t)\}$, *then*

$$\lim_{\epsilon \to 0^+} u^\epsilon = \begin{cases} 1, & \textit{locally uniformly in } \bigcup_{t>0} E_t, \\ -1, & \textit{locally uniformly in } \bigcup_{t>0}(\mathbb{R}^d \setminus \overline{E}_t). \end{cases}$$

Furthermore, there is a mobility coefficient $\bar{\mathcal{M}}(m, \cdot)$ *such that the sets* $(E_t)_{t\geq 0}$ *evolve with normal velocity given by*

$$V_{\partial E_t} = \bar{\mathcal{M}}(m, n_{\partial E_t})^{-1} tr(D^2 \bar{\sigma}(n_{\partial E_t}) A_{\partial E_t}). \tag{1.12}$$

Interestingly, the theorem shows not only that homogenization holds for some choices of initial datum, but also that the limiting interface velocity is determined by an Einstein relation as described by Spohn [87] and Bellettini, Buttà, and Presutti [17].

From the point of view of calculus of variations, the velocity law (1.12) shows that homogenization and commute, at least in this very special case. Indeed, the term $\mathrm{tr}(D^2 \bar{\sigma}(n_{\partial E}) A_{\partial E})$ is the anisotropic curvature of the surface ∂E with respect to $\bar{\sigma}$, which is the second variation of $\bar{\mathcal{F}}$. Hence, at least formally, (1.12) corresponds to the gradient flow of $\bar{\mathcal{F}}$ with respect to a metric determined by $\bar{\mathcal{M}}(m, \cdot)$.

In the laminar setting, there are pathologies lingering in the background that explain our restriction to graphical interfaces. The next result describes some of the known pathologies that can occur in this setting, all of which are related to the fact that there need not be a smooth pulsating standing wave in the directions perpendicular to e_d.

Theorem 3. *In the laminar setting, it is possible to find coefficients a and W such that*

(i) The surface tension $\bar{\sigma}$ is not C^1,

(ii) The mobility coefficient $\bar{\mathcal{M}}(m, \cdot)$ is unbounded, independent of the choice of m,

(iii) The interface velocity can become arbitrarily small relative to the second fundamental form: more precisely,

$$\inf \left\{ \frac{\|D^2\bar{\sigma}(e')\|}{\mathcal{M}(m, e')} \mid e' \in S^{d-1} \setminus \langle e_d \rangle^{\perp} \right\} = 0.$$

1.3 Homogenization of the Allen-Cahn Equation with a Periodic Mobility Coefficient

In our study of the gradient flow (1.9), the key ingredient was the existence of pulsating standing waves, which are entirely determined by the energy $\mathcal{F}$ through a and W. That leaves the question: what happens if $\mathcal{F}$ is spatially homogeneous (a and W are constant) and only the local mobility m is allowed to vary? In other words, the question is to study the asymptotics of the Allen-Cahn equation with a periodic mobility coefficient:

$$\begin{cases} m(\epsilon^{-1}x, \epsilon Du^{\epsilon})u^{\epsilon}_t - \Delta u^{\epsilon} + \epsilon^{-2}W'(u^{\epsilon}) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u^{\epsilon} = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.13}$$

Here we assume that the potential W is a smooth non-negative function, $W^{-1}(\{0\}) = \{-1, 1\}$, and $W''(-1) \wedge W''(1) > 0$.

In the next result, we prove that, in the limit $\epsilon \to 0$, effective interfaces form and move according to mean curvature flow with an effective mobility coefficient $\overline{m}$ that is determined by m through averaging effects.

Theorem 4. *Given any smooth $m : \mathbb{T}^d \times \mathbb{R}^d \to (0, \infty)$ such that (1.10) holds, if $u_0 \in UC(\mathbb{R}^d; [-1, 1])$ and $(u^\epsilon)_{\epsilon>0}$ are the solutions of (1.13), then*

$$u^\epsilon(\cdot, t) \to 1 \quad \text{in } E_t \quad \text{and} \quad u^\epsilon(\cdot, t) \to -1 \quad \text{in } \mathbb{R}^d \setminus \overline{E_t},$$

where $(E_t)_{t\geq 0}$ is a family of open sets with $E_0 = \{u_0 > 0\}$, evolving with normal velocity

$$V_{\partial E_t} = \overline{m}(n_{\partial E_t})^{-1} tr\left(A_{\partial E_t}\right) \tag{1.14}$$

for some continuous, positive function $\overline{m} : S^{d-1} \to (0, \infty)$ determined entirely by m and W.

The proof builds on the approach from [16]. Since the energy $\mathcal{F}$ is spatially homogeneous, there is a smooth family of pulsating standing waves available for use in the asymptotic expansion. The expansion has the form

$$u^\epsilon(x, t) = q\left(\frac{d(x, t)}{\epsilon}\right) + \epsilon P_{Dd(x,t)}\left(\frac{d(x, t)}{\epsilon}, \frac{x}{\epsilon}\right) + \dots$$

Here q is the (centered) standing wave solution of the Allen-Cahn equation in one-dimension; d is the signed distance function to the interface ∂E_t; and $\{P_e\}_{e\in S^{d-1}}$ are correctors. While the e-independence of the standing wave q eliminates any concerns about the existence and smoothness of pulsating waves, the correctors $\{P_e\}_{e\in S^{d-1}}$ do not necessarily exist in every direction.

Given an $e \in S^{d-1}$, the cell problem for the correctors $\{P_e\}_{e\in S^{d-1}}$ has the form

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e P_e) + W''(q(s))P_e = -m(y, \dot{q}(s))\dot{q}(s) + \overline{m}(e)\dot{q}(s) \quad \text{in } \mathbb{R} \times \mathbb{T}^d$$

In general, it is not possible to find solutions P_e of this equation. Rational directions on the sphere, that is, $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, are particularly problematic in this regard. However, in irrational directions $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, it is always possible to construct approximate solutions.

The non-existence of approximate correctors in some directions is a difficulty since the approach in [16] is based on the construction of global sub- and super-solutions. These functions are associated with macroscopic interfaces that are compact hypersurfaces, hence correctors are needed in every direction.

To circumvent this issue, the proof of Theorem 4 proceeds through the development of local sub- and super-solutions, which are localized so that only the vicinity of the surface near the contact point is relevant. These sub- and super-solutions make it possible to show that the effective interface associated with (1.13) moves according to the velocity law (1.14) near points where the normal vector $n_{\partial E_t}$ is irrational. That leaves the question how to show that an interface moving with a prescribed normal velocity in irrational directions is actually moving that way everywhere.

Below we prove that it is, in fact, enough to understand the behavior of the effective interface near points where the normal vector is irrational. The main idea is that (1.13) contains additional structure enabling us to glean some extra information about what happens at points where the normal vector is rational. This information, which amounts to a very weak regularity estimate, allows us to conclude that the effective interface is a genuine solution of (1.14).

1.4 Homogenization of a Nonvariational Interface Motion

In the homogenization problem for the Allen-Cahn equation with a periodic mobility coefficient, we saw that the local behavior of the macroscopic interface was more difficult to analyze in rational directions than irrational ones. The next problem we consider makes the distinction between rational and irrational directions even more apparent.

1.4.1 Curvature-Driven Interfaces in a Periodic Environment

Let us consider an even simpler model of phase interfaces evolving in a periodic medium. We fix a periodic, uniformly elliptic matrix field $a : \mathbb{T}^d \times S^{d-1} \to \mathcal{S}_d$ and a positive function $m : \mathbb{T}^d \times S^{d-1} \to (0,\infty)$, which model the heterogeneity of the medium. Given a small parameter $\epsilon > 0$, consider the evolution of an open set E_t^ϵ with prescribed normal velocity given by

$$V_{\partial E_t^\epsilon} = m(\epsilon^{-1}x, n_{\partial E_t^\epsilon})^{-1}\mathrm{tr}\left(a(\epsilon^{-1}x, n_{\partial E_t^\epsilon})A_{\partial E_t^\epsilon}\right). \tag{1.15}$$

As in the discussion of the Allen-Cahn equation above, this geometric flow is scaled so that it captures the behavior of the interface at spatiotemporal scales that are much larger than the period of the medium. Accordingly, this is a homogenization problem and the question is whether or not averaging results in an effective interface motion in the limit $\epsilon \to 0$.

Below we prove that, indeed, the motion of the interface averages near points where the normal vector $n_{\partial E_t^\epsilon}$ points in an irrational direction, that is, $n_{\partial E_t^\epsilon} \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. Precisely, we obtain effective coefficients $\bar{a} : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to \mathcal{S}_d$ and $\overline{m} : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to (0,\infty)$ such that, in the limit $\epsilon \to 0$, the motion is described by the law

$$V_{\partial E_t} = \overline{m}(n_{\partial E_t})^{-1}\mathrm{tr}\left(\bar{a}(n_{\partial E_t})A_{\partial E_t}\right) \quad \text{if } n_{\partial E_t} \notin \mathbb{R}\mathbb{Z}^d. \tag{1.16}$$

As in Theorem 4 above, this leads to the question whether or not knowledge of the effective motion in irrational directions is enough to characterize the global motion of the surface.

However, unlike Theorem 4, this problem is more subtle because, due to the strong anisotropic character of the motion, it is not obvious a priori that $\bar{a}$ or $\overline{m}$ can be extended to continuous functions in the entire sphere S^{d-1}. In fact, it turns out that, in dimensions $d \geq 3$, the effective coefficients are generically discontinuous at every rational direction. That motivates a study of the effective motion (1.16), where the normal velocity is only prescribed in irrational directions and the coefficients need not be continuous at rational directions. Below we develop a level-set formulation of this motion, which has a comparison principle (hence existence and uniqueness) in spite of the discontinuities.

The discussion above leads to the next theorem. Before stating it, let us make precise the assumptions on the coefficients a an m. We assume that

$$a : \mathbb{T}^d \times S^{d-1} \to \mathcal{S}_d \quad \text{smooth},$$
$$m : \mathbb{T}^d \times S^{d-1} \to (0, \infty) \quad \text{smooth},$$

and we fix $\lambda, \Lambda > 0$ such that

$$\lambda \mathrm{Id} \leq a(y, e) \leq \Lambda \mathrm{Id} \quad \text{for each } (y, e) \in \mathbb{T}^d \times S^{d-1}.$$

Theorem 5. *Given any open set $E_0 \subseteq \mathbb{R}^d$ and $\epsilon > 0$, let $(E_t^\epsilon)_{t \geq 0}$ be the solution of* (1.15) *with $E_0^\epsilon = E_0$. There is a family of open sets $(E_t)_{t \geq 0}$ such that*

(i) $(E_t)_{t \geq 0}$ is the unique solution of (1.16) *with initial condition E_0,*

(ii) $E_t^\epsilon \to E_t$ locally in the Hausdorff metric as $\epsilon \to 0$ for each $t > 0$.

We warn the reader that the theorem above is written somewhat imprecisely. In particular, uniqueness of solutions of (1.15) and (1.16) is complicated by the possibility of "fattening" (cf. [12] or [16]). However, it is possible to get around the non-uniqueness issue by working with the level-set formulation. Accordingly, below the reader will find precise statements in terms of the level-set PDE rather than the geometric flows themselves.

Concerning the discontinuities of a and m, we prove the following result:

Theorem 6. *Consider the case when $a = a(y)$ and $m = m(y)$ for simplicity and fix a dimension $d \geq 3$. For a generic (in the sense of Baire category) a and m satisfying the assumptions above, the effective coefficients $\bar{a} : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to \mathcal{S}_d$ and $\overline{m} : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to$*

$(0,\infty)$ *fail to have well-defined limits at any rational direction* $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$*. In particular,* $\bar{a}$ *and* $\overline{m}$ *cannot be extended to continuous functions in any open subset of* S^{d-1}*.*

We should emphasize that discontinuities are only possible in dimensions $d \geq 3$. We prove below that $\bar{a}$ and $\overline{m}$ extend continuously to S^1 in dimension $d = 2$.

1.4.2 Linear Response

We next consider a problem that is closely related to the previous one, and which once again leads to some surprising pathological behavior. Fix an $e \in S^{d-1}$ and an $\alpha \in \mathbb{R} \setminus \{0\}$. With a and m as above, consider the open sets $(E_t^\epsilon)_{t \geq 0}$ started at the half-space $E_0 = \{x \in \mathbb{R}^d \mid \langle x, e \rangle > 0\}$ and evolving with normal velocity given by

$$V_{\partial E_t^\epsilon} = m(\epsilon^{-1}x, n_{\partial E_t})^{-1} \left(\text{tr} \left(a(\epsilon^{-1}x, n_{\partial E_t^\epsilon}) A_{\partial E_t^\epsilon} \right) - \alpha \right). \tag{1.17}$$

As before, the problem is to identify the asymptotic behavior of E_t^ϵ in the limit $\epsilon \to 0$.

In the physics terminology, this is a question about linear response. The expectation is that, in the limit $\epsilon \to 0$, the set E_t^ϵ converges to a half-space moving with normal velocity

$$V_{\partial E_t} = \overline{m}(n_{\partial E_t})^{-1} \left(\text{tr} \left(\bar{a}(n_{\partial E_t}) A_{\partial E_t} \right) - \alpha \right) = -\overline{m}(n_{\partial E_t})^{-1} \alpha = -\overline{m}(e)^{-1} \alpha.$$

Hence the effective mobility $\overline{m}(e)$ obtained above measures the response of the medium to an applied external force α when the interface is a plane with normal e.

In fact, this is true, except $\overline{m}$ is not the right response coefficient in general. Instead, the asymptotics are described by a function $\overline{m}_{\text{pl}} : S^{d-1} \to (0,\infty)$, which agrees with $\overline{m}$ at irrational directions but need not coincide with it in rational directions. This is stated precisely in the next result.

Theorem 7. *There is a function* $\overline{m}_{pl} : S^{d-1} \to (0,\infty)$ *such that, given* $\alpha \in \mathbb{R} \setminus \{0\}$ *and*

$e \in S^{d-1}$, if $(E_t^\epsilon)_{t\geq 0}$ are the solutions of (1.17) with $E_0^\epsilon = \{x \in \mathbb{R}^d \mid \langle x, e\rangle > 0\}$, then $E_t^\epsilon \to E_t$ locally in the Hausdorff metric as $\epsilon \to 0$, where $(E_t)_{t\geq 0}$ is determined by

$$E_t = \{x \in \mathbb{R}^d \mid \langle x, e\rangle > -\alpha \overline{m}_{pl}(e)^{-1} t\}.$$

The response coefficient $\overline{m}_{pl}$ has the following properties:

(i) $\overline{m}_{pl}(e) = \overline{m}(e)$ if $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$,

(ii) For a generic choice of m, $\overline{m}_{pl}$ is discontinuous at every rational direction and, in dimension $d = 2$, $\overline{m}_{pl}(e) \neq \lim_{\mathbb{R}\mathbb{Z}^2 \notin e' \to e} \overline{m}(e')$ for each $e \in S^1 \cap \mathbb{R}\mathbb{Z}^2$.

The theorem shows that the effective mobility $\overline{m}$ obtained in the purely curvature-driven homogenization problem (1.15) can be interpreted as a linear response coefficient in irrational directions, but this interpretation turns out to fail in rational directions generically. Furthermore, in all dimension $d \geq 2$, the true response coefficient $\overline{m}_{\mathrm{pl}}$ is generically discontinuous at rational directions.

1.5 Future Directions

The results above only scratch the surface where the homogenization theory for interface motions is concerned. In this section, we discuss a few directions for future work.

1.5.1 Laminar Media

In the context of laminar media, we proved that when the initial interface is a graph that crosses the laminations, solutions of the gradient flow of the Allen-Cahn-type energy $\mathcal{F}$ homogenize. The question remains: what if the initial interface is not a graph?

In general, that question is as hard as the original, non-laminar homogenization problem. Indeed, if u_0 is laminar, that is, if $u_0(x', x_d) = u_0(x', 0)$, then the solution u^ϵ of (1.9) is also laminar by uniqueness. Hence we are essentially studying a PDE in $\mathbb{R}^{d-1} \times (0, \infty)$ with non-laminar coefficients — the advantage of the laminarity assumption is apparently lost.

However, this issue does not really arise in dimension $d = 2$. In that case, if the initial datum is laminar, that means it is a function of one variable, and the PDE in $\mathbb{R} \times (0, \infty)$ really is simpler than the one in $\mathbb{R}^2 \times (0, \infty)$. (Curvature is trivial in one dimension.) Thus, in dimension $d = 2$, the homogenization problem in laminar media need not be as difficult as the general case.

In fact, it is possible to show that the effective motion (1.12) is well-posed in dimension $d = 2$, even though $\bar{\sigma}$ need not be C^2 in S^1 and $\bar{\mathcal{M}}(m, \cdot)$ may not be finite everywhere. Well-posedness follows from results on level-set PDE with coefficients that are discontinuous in at most finitely many directions [58, 76, 59] and bounds on the coefficient $\bar{\mathcal{M}}(m, \cdot)^{-1} \|D^2 \bar{\sigma}\|$ (see Proposition 43 below). Furthermore, if we argue by analogy with Theorems 4 and 5, it should be possible to characterize the effective motion using only very weak information about what happens near points where the surface lines up with the laminations.

Stated simply, the discussion above leads to the following problem:

Problem 1. *In the laminar, two-dimensional setting, prove that* (1.9) *homogenizes for any* $u_0 \in UC(\mathbb{R}^2; [-1, 1])$.

Finally, in the proof of Theorem 2, the laminarity assumption (1.11) allows for the construction of pulsating waves, which are entirely determined by a and W. The mobility m only enters the picture through the map $m \mapsto \bar{\mathcal{M}}(m, \cdot)$, which is determined by a formula that makes sense whether or not m is laminar (see Section 4.1). Thus, it is reasonable to expect that Theorem 2 can be improved to cover general, non-laminar mobilities m, as in Theorem 4.

Problem 2. *Extend Theorem 2 to include a non-laminar local mobility m, i.e., replacing* (1.11) *by the assumption that only $a_{y_d} \equiv 0$ and $W_{y_d} \equiv 0$.*

1.5.2 Pinning

Recent work with Feldman [51] shows that effective interfaces need not move in general. Precisely, that work provides examples where homogenization occurs and the effective interfaces $(E_t)_{t \geq 0}$ have trivial normal velocity

$$V_{\partial E_t} = 0,$$

hence $E_t = E_0$ for all times $t \geq 0$. This behavior is called pinning.

The arguments in [51] suggest that pinning is related to the non-smoothness of pulsating waves. Here is the precise result:

Theorem 8 ([51])**.** *There are examples of coefficients a and W such that, independent of m, if $u_0 \in UC(\mathbb{R}^d; [-1, 1])$ and $(u^\epsilon)_{\epsilon > 0}$ are the solutions of* (1.9)*, then, for all $t > 0$,*

$$u^\epsilon(\cdot, t) \to 1 \quad \text{in } \{u_0 > 0\} \quad \text{and} \quad u^\epsilon(\cdot, t) \to -1 \quad \text{in } \{u_0 < 0\}.$$

Furthermore, in these examples, for any $e \in S^{d-1}$, there is no continuous solution of the pulsating standing wave equation (1.4)*.*

The examples in [51] have a very specific form that makes it possible to construct a barrier that is tailored to any initial datum u_0. Once these barriers are constructed, the fact that interfaces cannot move follows directly.

However, the argument that pulsating standing waves cannot be continuous is simpler. All that is required is a strict, time-stationary sub-solution $v : \mathbb{R}^d \to [-2, 2]$ of (1.9) with a bump-like shape. More precisely, v should be such that $\{v > -1\}$ is bounded and nonempty. Via a sliding argument, this suffices to prove that the functions $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ generated by a pulsating standing wave U_e cannot form a foliation, and, thus, U_e itself cannot be continuous. This leads to the question: is the existence of bump-like sub-solutions enough to conclude that pinning holds?

In general, it would be interesting to develop weaker sufficient conditions under which pinning holds. This leads to our next problem:

Problem 3. *Identify sufficient conditions that imply that the effective interfaces associated with* (1.9) *are pinned in the limit* $\epsilon \to 0$.

1.5.3 Random Media

In all of the results described above, it would be interesting to consider generalizations where the underlying heterogeneities are randomly distributed in space rather than periodically. Mathematically, the question is to generalize these results from the class of periodic coefficients to stationary, ergodic coefficients.

In the context of the Allen-Cahn-like equation (1.2), the analysis of random media will likely require the development of new techniques. While we know how to determine the surface tension $\bar{\sigma}$ in this generality (cf. [71]), it is not clear to what extent pulsating standing waves exist or how they would be used to analyze the sharp-interface limit.

The non-variational setting of (1.15) appears to be more promising in the short term. Many of the ingredients used in the proof of Theorems 5 and 7 have random analogues that are well-understood. Nonetheless, many technical problems arise when trying to generalize the proof to the stationary, ergodic setting. Accordingly, this leads to the next problem:

Problem 4. *Prove analogues of Theorem 5 and 7 when the coefficients a and m are stationary and ergodic, but not necessarily periodic.*

1.5.4 Linear Response, Traveling Waves, and Rates of Convergence

It turns out that the linear response coefficient $\overline{m}_{\text{pl}}$ equals (the inverse of) the derivative of the wave speeds associated with the following hyperbolically scaled motion

$$V_{\partial E_t^\epsilon} = m(\epsilon^{-1}x, n_{\partial E_t^\epsilon})^{-1} \left(\epsilon \text{tr} \left(a(\epsilon^{-1}x, n_{\partial E_t^\epsilon}) A_{\partial E_t^\epsilon} \right) + \alpha \right). \tag{1.18}$$

In this scaling regime, Caffarelli and Monneau [26] proved that, in dimension $d = 2$, the interface E_t^ϵ converges to an effective interface E_t moving with velocity

$$V_{\partial E_t} = \lambda_\alpha(n_{\partial E_t}),$$

where the function $\lambda_\alpha : S^{d-1} \to (0,\infty)$ encodes the average speed of planar interfaces in each direction.

Below we prove the following result, which relates the derivative of λ_α to $\overline{m}_{\mathrm{pl}}$.

Theorem 9. *For each $e \in S^{d-1}$,*

$$\overline{m}_{pl}(e)^{-1} = \lim_{\alpha \to 0} \alpha^{-1} \lambda_\alpha(e).$$

The argument of [26] involves certain special solutions of (1.18), called pulsating traveling waves. These can be described using solutions of the following degenerate elliptic PDE in the torus $\mathbb{T}^d$:

$$\lambda_\alpha(e) m \left(y, \frac{e + D\mathcal{V}_{e,\alpha}}{\|e + D\mathcal{V}_{e,\alpha}\|} \right) - \mathrm{tr} \left(A \left(y, \frac{e + D\mathcal{V}_{e,\alpha}}{\|e + D\mathcal{V}_{e,\alpha}\|} \right) D^2 \mathcal{V}_{e,\alpha} \right) - \alpha \|e + D\mathcal{V}_{e,\alpha}\| = 0 \quad \text{in } \mathbb{T}^d.$$

It is shown in [26] that this equation has a solution $\mathcal{V}_{e,\alpha} \in C(\mathbb{T}^d)$ for each $e \in S^{d-1}$. Once such a function has been obtained, the associated pulsating traveling wave is defined by setting $E_t^{e,\alpha} = \{x \in \mathbb{R}^d \mid \lambda_\alpha(e) t + \mathcal{V}_{e,\alpha}(x) > 0\}$. Note that this defines a solution of (1.18).

Since the results above prove that $\alpha \mapsto \lambda_\alpha(e)$ is differentiable at $\alpha = 0$ (see Section 6.4), it is natural to ask what can be said about the regularity of the function $\mathcal{V}_{e,\alpha}$ as $\alpha \to 0$.

Problem 5. *Is the function $\alpha \mapsto \mathcal{V}_{e,\alpha}$ differentiable at $\alpha = 0$? How is this affected by the choice of e?*

Finally, in Propositions 5 and 9 below, we show that the convergence in Theorem 7 can be quantified, and we obtain the optimal rate when e is rational. However, the optimal rate in the irrational case is still unclear.

Problem 6. *Determine the optimal rate of convergence in Theorem 7 (cf. Propositions 5 and 9).*

1.6 Notation

In this section, we catalogue notation used throughout the paper.

1.6.1 General

If $a, b \in \mathbb{R}$, we define $a \vee b$ and $a \wedge b$ by $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

We define $\text{sgn}(s) = \frac{s}{|s|}$ if $s \neq 0$.

If X is a metric space with metric d, we denote by $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$. The Hausdorff distance $d_{\mathcal{H}}(A, B)$ between two sets A, B contained in X is defined by

$$d_{\mathcal{H}}(A, B) = \inf \left\{ \epsilon \geq 0 \mid A \subseteq \bigcup_{b \in B} B(b, \epsilon),\ B \subseteq \bigcup_{a \in A} B(a, \epsilon) \right\}.$$

1.6.2 Euclidean Space

Given $v \in \mathbb{R}^d$, we write $\langle v \rangle = \{\alpha v \mid \alpha \in \mathbb{R}\}$.

The Euclidean inner product between two vectors $\xi, \zeta \in \mathbb{R}^d$ is denoted by $\langle \xi, \zeta \rangle$. If $A \subseteq \mathbb{R}^d$, then $A^\perp = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 0 \text{ if } a \in A\}$.

We write $\| \cdot \|$ for the norm induced by the inner product $\langle \cdot, \cdot \rangle$. S^{d-1} is the $(d-1)$-dimensional sphere in $\mathbb{R}^d$, that is, $S^{d-1} = \{e \in \mathbb{R}^d \mid \|e\| = 1\}$.

In $\mathbb{R}^d$, we denote by $\{e_1, \dots, e_d\}$ the standard orthonormal basis given by

$$e_1 = (1, 0, 0, \dots, 0),\ e_2 = (0, 1, 0, \dots, 0), \dots,\ e_d = (0, \dots, 0, 1).$$

If $e \in S^{d-1}$, $A \subseteq \langle e \rangle^{\perp}$, and $E \subseteq \mathbb{R}$, then we define $A \oplus_e E$ by

$$A \oplus_e E = \{a + \alpha e \mid a \in A,\ \alpha \in E\}.$$

Frequently we will be interested in cubes contained in some hyperplane of $\mathbb{R}^d$. In this case, given $e \in S^{d-1}$, we fix an orthonormal basis $\{v_1, \ldots, v_{d-1}\}$ of $\langle e \rangle^{\perp}$ and, for each $R > 0$, let $Q^e(0, R)$ be the cube in $\langle e \rangle^{\perp}$ determined by this basis, of side length R, and centered at 0. In other words,

$$Q^e(0, R) = \left\{ y \in \langle e \rangle^{\perp} \mid \max\{|\langle y, v_1 \rangle|, \ldots, |\langle y, v_{d-1} \rangle|\} \leq R/2 \right\}.$$

The specific choice of basis is irrelevant where the results of the paper are concerned.

From $Q^e(0, R)$, we form a d-dimensional cube $\mathbf{Q}^e(0, R)$ by extending in the e direction, that is,

$$\mathbf{Q}^e(0, R) = Q^e(0, R) \oplus_e (-R/2, R/2).$$

We denote by $\mathbb{R}\mathbb{Z}^d$ the set of all vectors that are parallel to some integer vector, that is,

$$\mathbb{R}\mathbb{Z}^d = \{\alpha k \mid k \in \mathbb{Z}^d,\ \alpha \in \mathbb{R}\}.$$

1.6.3 Linear Algebra

M_d is the space of real $d \times d$-matrices. $\mathcal{S}_d$ is the subspace consisting of symmetric matrices.

If $A, B \in \mathcal{S}_d$, we write $A \leq B$ if $\langle (A - B)\xi, \xi \rangle \leq 0$ for all $\xi \in \mathbb{R}^d$.

Given $A \in M_d$, we denote its transpose by A^*.

Given $\xi, \zeta \in \mathbb{R}^d$, the tensor product $\xi \otimes \zeta$ is the linear operator on $\mathbb{R}^d$ defined by $(\xi \otimes \zeta)(v) = \langle \zeta, v \rangle \xi$. Given matrices $A, B \in M_d$, the tensor product $A \otimes B$ is the linear operator on M_d defined by $(A \otimes B)(v \otimes w) = Av \otimes Bw$ and extended to the entire space by linearity.

1.6.4 The Torus

The d-dimensional torus is denoted by $\mathbb{T}^d$. This is the quotient space $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ where $x, y \in \mathbb{R}^d$ are equivalent if and only if $x - y \in \mathbb{Z}^d$.

The quotient map is denoted by $\pi_{\mathbb{Z}^d} : \mathbb{R}^d \to \mathbb{T}^d$. In particular, given $x \in \mathbb{R}^d$, the point $\pi_{\mathbb{Z}^d}(x)$ is the unique equivalence class in $\mathbb{T}^d$ containing x.

Given a finite Borel measure μ in $\mathbb{T}^d$, we define the Fourier transform $\hat{\mu} : \mathbb{Z}^d \to \mathbb{C}$ by

$$\hat{\mu}(k) = \int_{\mathbb{T}^d} e^{-i2\pi\langle k,y\rangle}\, \mu(dy).$$

The Fourier transform of Lebesgue measurable functions in $\mathbb{T}^d$ is defined completely analogously.

1.6.5 Functions

Given a family of functions $(f^\epsilon)_{\epsilon>0}$, each defined on a metric space X with metric d, we define the upper and lower half-relaxed limits $\limsup^* f^\epsilon$ and $\liminf_* f^\epsilon$, respectively, by

$$\limsup{}^* f^\epsilon(x) = \lim_{\delta\to 0^+} \sup\left\{f^\epsilon(y) \mid d(x,y) + \epsilon < \delta\right\}, \tag{1.19}$$

$$\liminf{}_* f^\epsilon(x) = \lim_{\delta\to 0^+} \inf\left\{f^\epsilon(y) \mid d(x,y) + \epsilon < \delta\right\}. \tag{1.20}$$

1.6.6 Measure Theory

We denote by $\mathcal{L}^d$ the Lebesgue measure in $\mathbb{R}^d$, and by $\mathcal{H}^{d-1}$, the $(d-1)$-dimensional Hausdorff measure, normalized to coincide with surface area. (We abuse notation and also use these symbols for the corresponding measures in $\mathbb{T}^d$.)

We denote by $\mathscr{P}(X)$ the space of Borel probability measures on a metric space X.

Given a finite measure μ on a measurable space $(X, \mathscr{B})$, we use the notation $⨍_X (\cdot)\, \mu(dx)$

to denote averaging with respect to μ, that is,

$$\fint_X f(x)\,\mu(dx) = \frac{1}{\mu(X)} \int_X f(x)\,\mu(dx) \quad \text{for } f \in L^1(X,\mu).$$

If $(X, \mathscr{B}, \mu)$ is a measure space and $\tau : X \to X$ is a measurable transformation, we define the pushforward measure $\tau_{\#}\mu$ on $\mathscr{B}$ by

$$(\tau_{\#}\mu)(A) = \mu(\tau^{-1}(A)) \quad \text{for } A \in \mathscr{B}.$$

1.6.7 Calculus

We frequently work in the cylinder $\mathbb{R} \times \mathbb{T}^d$, denoting points by (s, y) with $s \in \mathbb{R}$ and $y \in \mathbb{T}^d$. In this space, we write ∂_s for the derivative operator with respect to s and D_y for the derivative with respect to y.

In $\mathbb{R} \times \mathbb{T}^d$, given an $e \in S^{d-1}$, we denote by $\mathcal{D}_e$ the differential operator $\mathcal{D}_e = e\partial_s + D_y$, which takes a function in $\mathbb{R} \times \mathbb{T}^d$ and returns a vector field taking values in $\{0\} \times \mathbb{R}^d$. The L^2 adjoint of $\mathcal{D}_e$ is denoted by $\mathcal{D}_e^*$.

We will use the same notation for classical, weak, and distributional derivatives. In particular, ∂_s is often intended in the distributional sense.

In $\mathbb{R}^d$, for a given $e \in S^{d-1}$, we define the differential operators D_e and D_e^2 by

$$D_e\varphi(x) = (\mathrm{Id} - e \otimes e)D\varphi(x), \quad D_e^2\varphi(x) = (\mathrm{Id} - e \otimes e)D^2\varphi(x)(\mathrm{Id} - e \otimes e) \quad \text{for } \varphi \in C^\infty(\mathbb{R}^d).$$

We abuse notation somewhat by also writing D_e and D_e^2 for the corresponding differential operator for functions on $\mathbb{T}^d$.

1.7 Notes

1.7.1 Diffuse-Interface Models

For physically motivated discussions of diffuse-interface models, we refer the reader to the lecture notes of Langer [64] and the article by Kobayashi [63]. A mathematical physicist's perspective can be found in the textbook by Presutti [80] as well as the articles by Spohn [87] and Bellettini, Buttà, and Presutti [17].

1.7.2 Surface Tension and Γ*-Convergence*

The convergence of the rescaled Allen-Cahn functional $\mathcal{F}^{AC}$ to the perimeter functional is one of the basic examples in the theory of Γ-convergence. It was first proved by Modica and Mortola [69] (cf. [68]). See also

For an introduction to Γ-convergence, see the textbooks by Dal Maso [37] and Braides [22]. Expository accounts of the convergence result for the Allen-Cahn functional and related problems can be found in the book by Braides [21] and the lecture notes of Alberti [1].

In Chapter 2, we prove the existence of the surface tension $\bar{\sigma}$, but we do not treat the Γ-convergence of $\mathcal{F}_\epsilon$ here. The references [5] and [35] prove Γ-convergence of $\mathcal{F}_\epsilon$ in the limit $\epsilon \to 0$ under assumptions more restrictive than those used here: basically, W is assumed independent of the spatial variable y in the former, while, in the latter, a is independent of y. It is expected that the approach of either reference can be used in conjunction with Theorem 16 to prove Γ-convergence of $\mathcal{F}_\epsilon$ under the assumptions on a and W considered here.

1.7.3 Geometric Flows and the Level-Set Method

Throughout this work, we analyze geometric flows using the level-set method. The method was first introduced by Ohta, Jasnow, and Kawaski [77], Osher and Sethian [78], and Sethian

[86]. Its first rigorous formulations were presented by Barles [10] in a first-order context and Evans and Spruck [44] and Chen, Giga, and Goto [31] for mean curvature flow. The state-of-the-art can be found in the paper by Barles and Souganidis [16]; see also the earlier work of Barles, Soner, and Souganidis [12].

1.7.4 The Aubry-Mather and Moser-Bangert Theories

Aubry-Mather theory concerns the study of variational problems with periodic coefficients; see, for instance, Bangert's notes [8] for an introduction. The classical examples involved discrete models in two dimensions.

The consideration of continuum models in higher dimensions constitutes Moser-Bangert theory, which began with the paper by Moser [74]. The prototypical set-up involves an energy such as

$$\mathcal{E}(u;\Omega) = \int_\Omega \left(\frac{1}{2}\langle a(y)Du, Du\rangle + \theta(y)\sin(2\pi u) \right) dy$$

with $a : \mathbb{T}^d \to \mathcal{S}_d$ uniformly elliptic and $\theta \in C(\mathbb{T}^d)$. In contrast to the plane-like minimizers considered here, in this context, one of the basic questions is the existence and properties of so-called WSI solutions, which are critical points u of the energy with graphs that are a bounded distance from a hyperplane in $\mathbb{R}^{d+1}$.

In fact, it is possible to study the plane-like minimizers considered in the Allen-Cahn context within the framework of the heteroclinic solutions studied by Bangert in [9]. This is explained in the book of Rabinowitz and Stredulinsky [82] and the article by Junginger-Gestrich and Valdinoci [60].

1.7.5 Related Models

In the evolutionary equations considered here, we always analyze the parabolic scaling limit. In fact, there are many works that treat the hyperbolic scaling, though open problems remain. A non-exhaustive list of references in this area includes Caffarelli and Monneau [26], Gao and Kim [53], Caffarelli, Lee, and Mellet [25], and Lin and Zlatoš [65].

While our results show that pulsating standing waves for Allen-Cahn-like equations are determined by the plane-like minimizers of $\mathcal{F}$, in other contexts, one would like to understand pulsating *traveling* waves. For instance, if the potential W is replaced by $W(y, u) + \alpha u$ for some small $\alpha \in \mathbb{R} \setminus \{0\}$, then one would like to know whether or not the associated equation has traveling wave solutions. The result of Xin [91] suggests this is the case provided the spatial oscillation of the coefficients a and W is small relative to α. However, even thirty years later, Xin's perturbative result is still among the most general treatments of this question in multiple dimensions; see also Ducrot [41], Rossi and Giletti [57], and Ducrot, Giletti, and Matano [42].

CHAPTER 2

THE SURFACE TENSION, THE MOBILITY, AND ELEMENTS OF ERGODIC THEORY

2.1 Introduction

The aim of this chapter is to introduce some of the fundamental concepts investigated here in their simplest possible setting. The main focus will be on the interpretation of the effective mobility as a linear response coefficient in the context of non-variational interface motions. The ideas involved also come up naturally in the computation of the surface tension in the context of diffuse-interface models, hence we will also discuss that here.

Certain elements of ergodic theory arise naturally in this setting. A self-contained treatment of those facts we will need is provided below.

2.2 Linear Response

In this section, our objective is to compute the effective mobility of the nonvariational interface motion (1.15). More precisely, given a uniformly elliptic matrix field a, a local mobility coefficient m, and a direction e, we would like to understand the limiting behavior of the following interface motion with oscillating coefficients in the limit $\epsilon \to 0$:

$$V_{\partial E_t^\epsilon} = m(\epsilon^{-1} x, n_{\partial E_t^\epsilon})^{-1} \left(\operatorname{tr} \left(a(\epsilon^{-1} x) A_{\partial E_t^\epsilon} \right) + \alpha \right), \quad E_0^\epsilon = \{ x \in \mathbb{R}^d \mid \langle x, e \rangle > 0 \}.$$

Via the level-set formulation (cf. [16]), this is equivalent to studying the asymptotics of the following non-linear diffusion equation:

$$\begin{cases} m(\epsilon^{-1} x, \widehat{Du^\epsilon}) u_t^\epsilon - \operatorname{tr} \left(A(\epsilon^{-1} x, \widehat{Du^\epsilon}) D^2 u^\epsilon \right) - \alpha \| Du^\epsilon \| = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \\ u^\epsilon(x, 0) = \langle x, e \rangle \quad \text{for each } x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

Here, as in the introduction, A is the matrix derived from a by the formula

$$A(y,e) = (\mathrm{Id} - e \otimes e)a(y,e)(\mathrm{Id} - e \otimes e).$$

We will see that this question can quickly be reduced to a homogenization problem for certain elliptic operators with quasi-periodic coefficients.

The main result of this section is stated next. We assume throughout that a and m satisfy the assumptions of Section 1.4.

Theorem 10. *For each $e \in S^{d-1}$, there is a constant $\overline{m}_{pl}(e) > 0$ such that if $(u^\epsilon)_{\epsilon > 0}$ are the solutions of* (2.1)*, then*

$$\lim_{\epsilon \to 0^+} u^\epsilon(x,t) = \langle x, e \rangle + \alpha \overline{m}_{pl}(e)^{-1} t \quad \textit{locally uniformly in } \mathbb{R}^d \times [0, \infty).$$

The proof of the theorem begins with a formal two-scale expansion. Suppose that there is a function $V_e : \mathbb{T}^d \to \mathbb{R}$ such that, up to error terms that vanish to order ϵ^2, we can write u^ϵ as

$$u^\epsilon(x,t) = \langle x, e \rangle + \alpha \overline{m}(e)^{-1} t + \alpha \overline{m}(e)^{-1} \epsilon^2 V_e(\epsilon^{-1} x) + \dots.$$

Plugging the expansion into (2.1) and grouping common terms, we are led to the following problems characterizing V_e:

$$-\mathrm{tr}\left(A\left(y,e\right) D^2 V_e\right) = -m\left(y,e\right) + \overline{m}(e) \quad \text{in } \mathbb{T}^d.$$

In order to analyze this equation, it is useful to bring a back into the picture and note that if we define the differential operator D_e^2 by

$$D_e^2 F = (\mathrm{Id} - e \otimes e) D^2 F (\mathrm{Id} - e \otimes e) \quad \text{for } F \in C^\infty(\mathbb{T}^d),$$

then we can write

$$\operatorname{tr}\left(A(y,e)D^2V_e\right) = \operatorname{tr}\left(a(y,e)D_e^2V_e\right).$$

Hence the equation for V_e becomes

$$-\operatorname{tr}\left(a(y,e)D_e^2V_e\right) = -m(y,e) + \overline{m}(e) \quad \text{in } \mathbb{T}^d. \tag{2.2}$$

This is reminiscent of the ergodic problems arising in the study of diffusions in periodic media (cf. [18, Chapter 3]), the only difference being that the second order term is degenerate elliptic. Nonetheless, the degeneracy has a very specific form.

A change of perspective distinguishes (2.2) as the "probability space lift" (or abstract cell problem) for a uniformly elliptic homogenization problem with quasi-periodic coefficients. To see this, suppose that V_e is a smooth solution of (2.2) and define functions $(\tilde{V}_y^e)_{y\in\mathbb{T}^d}$ in $\langle e\rangle^{\perp}$, the subspace orthogonal to e, by

$$\tilde{V}_y^e(x') = V_e(x'+y) \quad \text{for each } x' \in \langle e\rangle^{\perp}.$$

A direct computation shows that these functions are solutions of the PDE

$$-\operatorname{tr}\left(a(x'+y,e)D^2\tilde{V}_y^e\right) = -m(x'+y,e) + \overline{m}(e) \quad \text{in } \langle e\rangle^{\perp}. \tag{2.3}$$

The matrix field $x' \mapsto a(x'+y,e)$ is quasi-periodic, that is, it equals a linear transformation composed with a periodic function. Varying y changes the coefficient, but in a way that preserves the overall pattern. (Thinking of this in terms of stochastic homogenization, y plays the role of the sample, or what is typically denoted by ω).

In homogenization theory, the solution $\tilde{V}_y^e$ of (2.3) is referred to as a corrector. Thus, we conclude that the solution V_e of (2.2) generates correctors for the linear, uniformly elliptic PDE generated by a in $\langle e\rangle^{\perp}$. As will be shown below, the converse is also true to a certain extent: if we understand the homogenization of quasi-periodic, uniformly elliptic PDE, then we will be able to construct approximate solutions of (2.3), which then give rise to approximate solutions of (2.2)

2.3 Homogenization of Linear, Uniformly Elliptic Equations with Quasi-Periodic Coefficients

For the reader that is not familiar with stochastic homogenization, let us make the previous discussion more precise. It will be useful to consider the following approximation of (2.3). Given $\delta > 0$, well-known arguments show that there are functions $(\tilde{V}_y^{e,\delta})_{y \in \mathbb{T}^d}$ in $\langle e \rangle^{\perp}$ solving the PDE

$$\delta \tilde{V}_y^{e,\delta} - \mathrm{tr}\left(a(y + x', e) D^2 \tilde{V}_y^{e,\delta}\right) = -m(y + x', e) \quad \text{in } \langle e \rangle^{\perp}. \tag{2.4}$$

While it will not be possible to solve (2.3) in general, it would suffice for our purposes if we could prove that there is a constant $\overline{m}(e)$ such that, for each $y \in \mathbb{T}^d$,

$$\delta \tilde{V}_y^{e,\delta} \to -\overline{m}(e) \quad \text{uniformly in } \langle e \rangle^{\perp} \text{ as } \delta \to 0^+.$$

This is a question about averaging. One way to see that is to note that if we define $\tilde{V}_y^{\epsilon}$ by

$$\tilde{V}_y^{\epsilon}(x') = \epsilon^2 \tilde{V}_y^{e,\epsilon^2}(\epsilon^{-1} x')$$

then $\tilde{V}_y^{\epsilon}$ is the solution of

$$\tilde{V}_y^{\epsilon} - \mathrm{tr}\left(a(y + \epsilon^{-1} x') D^2 \tilde{V}_y^{\epsilon}\right) = -m(y + \epsilon^{-1} x') \quad \text{in } \langle e \rangle^{\perp}. \tag{2.5}$$

Notice that from this perspective, the question of the convergence of $\delta \tilde{V}_y^{e,\delta}$ is equivalent to the convergence of $\tilde{V}_y^{\epsilon}$. The coefficients in (2.5) are highly oscillatory as $\epsilon \to 0$ so some mechanism is needed to generate cancellations. Since $a(\cdot, e)$ and $m(\cdot, e)$ are periodic functions, it may not be surprising at this point to learn that averaging is precisely the mechanism involved here.

However, since the problems are posed in $\langle e \rangle^{\perp}$, some work is required to show that (2.5) homogenizes as $\epsilon \to 0$. For the sake of clarity, let us make some of the notions at play here precise. As was mentioned already in the previous section, (2.3), (2.4), and (2.5) are equations with quasi-periodic coefficients.

Definition 1. *Given $m \in \mathbb{N}$, a function $f : \mathbb{R}^m \to \mathbb{R}$ is called* quasi-periodic *if there is an $N \in \mathbb{N}$, a linear map $L : \mathbb{R}^m \to \mathbb{R}^N$, and a periodic function $F : \mathbb{T}^N \to \mathbb{R}$ such that*

$$f(x') = F(L(x')).$$

In our setting, the equations (2.3), (2.4), and (2.5) have quasi-periodic coefficients since, for any fixed $y \in \mathbb{T}^d$, if we let $P_e : \langle e \rangle^{\perp} \to \mathbb{R}^d$ denote the inclusion map, then P_e is linear, $\langle e \rangle^{\perp}$ is isomorphic to $\mathbb{R}^{d-1}$, and

$$a(x' + y) = a(P_e(x') + y).$$

Quasi-periodic functions are a particularly nice class of almost periodic functions, which are frequently studied in homogenization. The definition of almost periodic function that will be most relevant in this work is given next.

Definition 2. *Given $m \in \mathbb{N}$, function $f : \mathbb{R}^m \to \mathbb{R}$ is called* (uniformly) almost periodic *if it is bounded and uniformly continuous and the set of translates*

$$\{f(\cdot + x') \mid x' \in \mathbb{R}^m\}$$

is pre-compact in $BUC(\mathbb{R}^m)$ with the uniform norm topology.

An easy exercise proves that if f is a quasi-periodic function obtained by composing a linear function and a continuous periodic function, then f is (uniformly) almost periodic.

Proposition 1. *Given $m, N \in \mathbb{N}$, if $F \in C(\mathbb{T}^N)$ and $L : \mathbb{R}^m \to \mathbb{R}^N$ is linear, then the function $f : \mathbb{R}^m \to \mathbb{R}$ given by $f(x') = F(L(x'))$ is (uniformly) almost periodic.*

So far, we have shown that (2.5) is a homogenization problem with quasi-periodic coefficients. Hence since quasi-periodic implies almost periodic, it belongs to the class of equations with stationary coefficients commonly considered in stochastic homogenization.

To apply results from that theory, we should be slightly more precise. In what follows, we will consider $\mathbb{T}^d$ as a probability space equipped with some Borel probability measure, typically $\mathcal{L}^d$.

Given $e \in S^{d-1}$, consider the group of translations $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$ acting on $\mathbb{T}^d$ via the formula

$$\tau_{x'}(y) = y + x'.$$

This is a group under composition: $\tau_{x'_1 + x'_2} = \tau_{x'_1} \circ \tau_{x'_2}$ and $\tau_0 = \mathrm{Id}$. Furthermore, it preserves the Lebesgue measure $\mathcal{L}^d$ on the torus:

$$\tau_{x'\#}\mathcal{L}^d = \mathcal{L}^d \quad \text{for each } x' \in \langle e \rangle^\perp.$$

Hence in the language of stochastic homogenization, (2.5) is an equation with stationary coefficients in the sense that we can write

$$a(x' + y) = a(\tau_{x'}(y)) \quad \text{for each } y \in \mathbb{T}^d, \ x' \in \langle e \rangle^\perp.$$

The previous considerations show that (2.5) fits into the typical framework of stochastic homogenization, as, for example, in [29]. The only ingredient that is missing is ergodicity.

Definition 3. *Given $e \in S^{d-1}$, we say that a probability measure $\mu \in \mathscr{P}(\mathbb{T}^d)$ is* ergodic *under the action of* $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$ *if, for each Borel set* $A \subseteq \mathbb{T}^d$ *such that* $0 < \mu(A) < 1$*, there is an* $x' \in \langle e \rangle^\perp$ *such that* $\tau_{x'}(A) \neq A$*.*

As we will see below, if $\mathcal{L}^d$ is ergodic under the action of $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$, then V_y^ϵ does indeed converge to a constant $-\overline{m}(e)$ in the limit $\epsilon \to 0$. However, as we will discuss next, $\mathcal{L}^d$ is not necessarily ergodic. In the non-ergodic case, more work will be necessary to derive the asymptotics of (2.1).

2.4 Averaging in Codimension One

The previous discussion suggests a study of the ergodic theory of codimension-one translations in $\mathbb{T}^d$. More precisely, given an $e \in S^{d-1}$, when is $\mathcal{L}^d$ ergodic under the action of $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$?

The answer to this question lies in the arithmetic properties of the normal vector e. Toward that end, let us define the *group of resonances* M_e associated with a direction e by

$$M_e = \{k \in \mathbb{Z}^d \mid \langle k, e \rangle = 0\}. \tag{2.6}$$

As we will see below, the next theorem offers one way to understand how arithmetic properties of e influence the ergodic behavior of $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$. In the statement, $\mathbb{R}\mathbb{Z}^d$ denotes the set of all vectors in $\mathbb{R}^d$ that are parallel to some nonzero integer vector.

Theorem 11. *Given* $e \in S^{d-1}$*, the following are equivalent:*

(i) $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$*,*

(ii) M_e *is a subgroup of* $\mathbb{Z}^d$ *of rank* $d-1$*,*

(iii) $\{\langle k, e \rangle \mid k \in \mathbb{Z}^d\}$ *is a discrete subgroup of* $\mathbb{R}$ *(and, thus, has rank one).*

Since the proof is a matter of elementary algebra, it is relegated to Appendix A.

We will see below that there are fundamental differences between the averaging properites of $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ depending on whether or not $e \in \mathbb{R}\mathbb{Z}^d$. Following the general practice in quasi-periodic dynamics and homogenization theory, we will say that e is *rational* if $e \in S^{d-1}\cap\mathbb{R}\mathbb{Z}^d$; otherwise, we will say it is *irrational*.

As a direct consequence of the previous theorem, we characterize the ergodic properties of $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$. Here we will change tacks slightly. Instead of trying to prove that $\mathcal{L}^d$ is an ergodic measure, we will instead look for all the possible ergodic invariant measures.

Given $e \in S^{d-1}$, define the set of invariant Borel probability measures $\mathcal{I}(e)$ for $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ by

$$\mathcal{I}(e) = \bigcap_{x'\in\langle e\rangle^\perp} \{\mu \in \mathscr{P}(\mathbb{T}^d) \mid \tau_{x'\#}\mu = \mu\}$$

As we observed in the previous section, $\mathcal{L}^d \in \mathcal{I}(e)$, no matter the choice of e. We will say that $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ is *uniquely ergodic* if this is the only invariant measure, i.e., $\mathcal{I}(e) = \{\mathcal{L}^d\}$. As the name suggests, if $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ is uniquely ergodic, then $\mathcal{L}^d$ is ergodic. Further, in this setting, $\mathcal{L}^d$ is never ergodic if there are other invariant measures. Hence unique ergodicity is the key issue.

The goal of the remainder of the section is the proof of the following two theorems.

Theorem 12. *Given any $e \in S^{d-1}$, the group of transformations $(\tau_{x'})_{x'\in\mathbb{T}^d}$ is uniquely ergodic if and only if $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$.*

Theorem 13. *$\mathcal{L}^d$ is ergodic under the action of $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ if and only if $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ is uniquely ergodic.*

Let us start with Theorem 12. The "only if" direction follows more-or-less directly from Theorem 11, as will be shown shortly. For the "if" direction, we will use the following construction. First, define $r_e > 0$ by

$$r_e = \inf\left\{\langle k, e\rangle \mid k \in \mathbb{Z}^d\right\} \cap (0,\infty). \tag{2.7}$$

Given $r \in \mathbb{R}$, define the sub-torus $\mathbb{T}_e^{d-1}(r)$ by

$$\mathbb{T}_e^{d-1}(r) = \pi_{\mathbb{Z}^d}\left(\{x \in \mathbb{R}^d \mid \langle x, e\rangle = r\}\right). \tag{2.8}$$

It is straightforward to show that $\{\mathbb{T}_e^{d-1}(r)\}_{r\in\mathbb{R}}$ is a collection of smooth submanifolds that foliate $\mathbb{T}^d$. Further, these are invariant under integer translation in the following sense:

$$\mathbb{T}_e^{d-1}(r + \langle k, e\rangle) = \mathbb{T}_e^{d-1}(r) \quad \text{for } r \in \mathbb{R},\ k \in \mathbb{Z}^d. \tag{2.9}$$

It turns out $\mathbb{T}_e^{d-1}(r)$ is compact if and only if $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$.

Lemma 1. *Given any $e \in S^{d-1}$, we have:*

(i) $\mathbb{T}_e^{d-1}(0)$ *is compact if and only if* $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$.

(ii) If $e \in \mathbb{R}\mathbb{Z}^d$, *then the number* r_e *defined by* (2.7) *is positive and hence the foliation* $\{\mathbb{T}_e^{d-1}(r)\}_{r\in\mathbb{R}}$ *is periodic with fundamental period* r_e, *that is,*

$$\mathbb{T}_e^{d-1}(r + r_e) = \mathbb{T}_e^{d-1}(r) \quad \textit{for each } r \in \mathbb{R}.$$

(iii) If $e \notin \mathbb{R}\mathbb{Z}^d$, *then each leaf of the foliation* $\{\mathbb{T}_e^{d-1}(r)\}_{r\in\mathbb{R}}$ *is dense in* $\mathbb{T}^d$.

(iv) If $e \in \mathbb{R}\mathbb{Z}^d$, *then the normalized* $(d-1)$*-dimensional Hausdorff measure on* $\mathbb{T}_e^{d-1}(r)$ *is an invariant probability measure for the group* $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$.

It is not hard to show that the measures in (iv) are the extremal elements of $\mathcal{I}(e)$ when e is rational. More precisely, if $e \in \mathbb{R}\mathbb{Z}^d$ and $\mu \in \mathcal{I}(e)$, then there is a Borel probability measure κ_μ on $[0, r_e)$ such that

$$\int_{\mathbb{T}^d} f(y)\,\mu(dy) = \int_0^{r_e} \left(\mathcal{H}^{d-1}(\mathbb{T}_e^{d-1}(r))^{-1} \int_{\mathbb{T}_e^{d-1}(r)} f(y')\,\mathcal{H}^{d-1}(dy')\right) \kappa_\mu(dr).$$

Proof. We begin with (i). First, suppose that $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$ and fix $r \in \mathbb{R}$. We will show that $\mathbb{T}_e^{d-1}(r)$ is compact by proving it is homeomorphic to $\mathbb{T}^{d-1}$.

By Theorem 11, M_e has rank $d-1$. Let $\{k_1, \dots, k_{d-1}\}$ denote a basis of M_e. Since $\{k_1, \dots, k_{d-1}\}$ forms a basis of $\langle e \rangle^{\perp}$ over $\mathbb{R}$, it follows that, for any $x \in \mathbb{R}^d$, we can write

$$x = \langle x, e \rangle e + \sum_{i=1}^{d-1} a_i(x) k_i,$$

for some linear functionals $\{a_1, \dots, a_{d-1}\}$ determined by $\{k_1, \dots, k_{d-1}\}$.

Let $L(r) = \{x \in \mathbb{R}^d \mid \langle x, e \rangle = r\}$ so that $\mathbb{T}_e^{d-1}(r) = \pi_{\mathbb{Z}^d}(L(r))$. By elementary point set topology, to prove that $\mathbb{T}_e^{d-1}(r)$ is homeomorphic to $\mathbb{T}^{d-1}$, it suffices to observe that the map $p_e : L(r) \to \mathbb{R}^{d-1}$ given by

$$p_e(x) = \sum_{i=1}^{d-1} a_i(x) e_i \tag{2.10}$$

is a homeomorphism and

$$\pi_{\mathbb{Z}^{d-1}}(p_e(x)) = \pi_{\mathbb{Z}^{d-1}}(p_e(y)) \quad \text{if} \quad \pi_{\mathbb{Z}^d}(x) = \pi_{\mathbb{Z}^d}(y).$$

(Here $\{e_1, \dots, e_{d-1}\}$ is the standard orthonormal basis of $\mathbb{R}^{d-1}$.)

Conversely, suppose that $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ and fix $r \in \mathbb{R}$. By Theorem 11, the set $\{\langle k, e \rangle \mid k \in \mathbb{Z}^d\}$ is a non-discrete subgroup of $\mathbb{R}$. In particular, it is dense in $\mathbb{R}$. Thus, given any $s \in \mathbb{R}$, we can find a sequence $(k_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}^d$ such that

$$\lim_{n \to \infty} (r + \langle k_n, e \rangle) = s.$$

In view of (2.9), this shows that every point in $\mathbb{T}_e^{d-1}(s)$ is a limit point of $\mathbb{T}_e^{d-1}(r)$. Since s was arbitrary, $\mathbb{T}_e^{d-1}(r)$ is a proper dense subset of $\mathbb{T}^d$, hence it cannot be compact.

Next, we prove (ii). Concerning the fundamental period r_e defined by (2.7) in the case that $e \in \mathbb{R}\mathbb{Z}^d$, Theorem 11 tells us that the group $\{\langle k, e\rangle \mid k \in \mathbb{Z}^d\}$ is discrete. Hence it cannot accumulate at zero, and the positivity of r_e follows. The translation property (2.9) then implies that $\mathbb{T}_e^{d-1}(r + r_e) = \mathbb{T}_e^{d-1}(r)$.

(iii) was established during the proof of (ii).

Finally, we prove (iv). We claim that, for each $r \in [0, r_e)$, the surface measure $\nu_e^r := \mathcal{H}^{d-1} \restriction_{\mathbb{T}_e^{d-1}(r)}$ is invariant under $(\tau_{x'})_{x' \in \langle e\rangle^\perp}$. Observe that if $x \in \mathbb{R}^d$, then the pushforward $\tau_{x\#}\nu_e^r$ is given by

$$\tau_{x\#}\nu_e^r = \nu_e^{r+\langle x,e\rangle}.$$

Thus, the probability measure $\nu_e^r(\mathbb{T}^d)^{-1}\nu_e^r$ is invariant under $(\tau_{x'})_{x' \in \langle e\rangle^\perp}$. □

Proof of Theorem 12. First, suppose that $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. Lemma 1 implies that, for each $r \in [0, r_e)$, the group $(\tau_{x'})_{x' \in \langle e\rangle^\perp}$ has an invariant probability measure supported on $\mathbb{T}_e^{d-1}(r)$. Since those surfaces are disjoint, we conclude that $(\tau_{x'})_{x' \in \langle e\rangle^\perp}$ is not uniquely ergodic.

Next, suppose $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. We claim that $\mathcal{I}(e) = \{\mathcal{L}^d\}$, hence $(\tau_{x'})_{x' \in \langle e\rangle^\perp}$ is uniquely ergodic. Indeed, suppose that $\mu \in \mathcal{I}(e)$ and choose a $k \in \mathbb{Z}^d$ such that $\hat{\mu}(k) \neq 0$. Since $\mu \in \mathcal{I}(e)$, we can compute

$$\hat{\mu}(k) = (\tau_{x'\#}\mu)(k) = e^{-i2\pi\langle k,x'\rangle}\hat{\mu}(k) \quad \text{for each } x' \in \langle e\rangle^\perp.$$

It follows that the function $x' \mapsto e^{-i2\pi\langle k,x'\rangle}$ is identically one in $\langle e\rangle^\perp$. By connectedness, that implies that the linear functional $\ell : \langle e\rangle^\perp \to \mathbb{R}$ given by $\ell(x') = \langle k, x'\rangle$ is constant. We know that $\ell(0) = 0$ so $\ell \equiv 0$, or, in other words, $k \in \langle e\rangle$. Since $e \notin \mathbb{R}\mathbb{Z}^d$, we can only conclude that $k = 0$. Therefore, at the level of the Fourier transform, we have $\hat{\mu} = \delta_0$, which implies that $\mu = \mathcal{L}^d$ after Fourier inversion. □

Finally, we prove Theorem 13, making precise our previous comments concerning the connection between unique ergodicity and the ergodicity of $\mathcal{L}^d$.

Proof of Theorem 13. First, we prove the "if" direction. Suppose that the group $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ is uniquely ergodic, hence $\mathcal{I}(e) = \{\mathcal{L}^d\}$. To see that $\mathcal{L}^d$ is ergodic, we need to show that any invariant Borel set has measure zero or one. More precisely, suppose that $A \subseteq \mathbb{T}^d$ is Borel and $\tau_{x'}(A) = A$ for each $x' \in \langle e\rangle^\perp$. We need to prove that $\mathcal{L}^d(A) \in \{0, 1\}$.

Of course, if $\mathcal{L}^d(A) = 0$, then we are done. Hence assume $\mathcal{L}^d(A) > 0$. Define the probability measure $\mu_A \in \mathscr{P}(\mathbb{T}^d)$ by defining

$$\int_{\mathbb{T}^d} f(y)\mu_A(dy) = \frac{1}{\mathcal{L}^d(A)} \int_A f(y)\, dy \quad \text{for each } f \in C(\mathbb{T}^d).$$

We claim that $\mu_A \in \mathcal{I}(e)$. Indeed, given $x' \in \langle e\rangle^\perp$ and $f \in C(\mathbb{T}^d)$, we have

$$\begin{aligned}\int_{\mathbb{T}^d} f(y)(\tau_{x'}\mu_A)(dy) &= \frac{1}{\mathcal{L}^d(A)} \int_{\mathbb{T}^d} f(y + x')\chi_A(y)\, dy \\ &= \frac{1}{\mathcal{L}^d(A)} \int_{\mathbb{T}^d} f(y + x')\chi_A(y + x')\, dy = \int_{\mathbb{T}^d} f(y)\, \mu_A(dy).\end{aligned}$$

This proves $\mu_A \in \mathcal{I}(e)$. Since $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ is uniquely ergodic by assumption, we conclude that $\mathcal{L}^d = \mu_A$ and, thus, $\mathcal{L}^d(A) = 1$.

Conversely, suppose that $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$ is not uniquely ergodic. By Theorem 12, it follows that $e \in \mathbb{R}\mathbb{Z}^d$. Define $A \subseteq \mathbb{T}^d$ by

$$A = \left\{\pi_{\mathbb{Z}^d}(se + x') \mid x' \in \langle e\rangle^\perp,\ 0 < s < \frac{r_e}{2}\right\},$$

where r_e is defined by (2.7). Observe that, by Fubini's Theorem, we have

$$\mathcal{L}^d(A) = \frac{r_e}{2} \cdot \mathcal{H}^{d-1}(\mathbb{T}_e^{d-1}(0)) < r_e\mathcal{H}^{d-1}(\mathbb{T}_e^{d-1}(0)) = \mathcal{L}^d(\mathbb{T}^d).$$

In particular, $0 < \mathcal{L}^d(A) < 1$. At the same time, a straightforward computation shows that $\tau_{x'}(A) = A$ for all $x' \in \langle e\rangle^\perp$. This proves $\mathcal{L}^d$ is not ergodic under the action of

$(\tau_{x'})_{x' \in \langle e \rangle^\perp}$. □

For later use, we record an important consequence of the ergodicity of $\mathcal{L}^d$ in the irrational case.

Proposition 2. *Given $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ and $f \in L^1(\mathbb{T}^d)$, there is a measurable set $E \subseteq [0, r_e)$ such that $\mathcal{L}^1(E) = r_e$ and, for each $s \in E$,*

$$\lim_{R \to \infty} \frac{1}{R^{d-1}} \int_{Q^e(0,R)} f(se + x') \, \mathcal{H}^{d-1}(dx') = \int_{\mathbb{T}^d} f(y) \, dy. \tag{2.11}$$

Further, if $f \in C(\mathbb{T}^d)$, then the convergence is uniform, that is,

$$\lim_{R \to \infty} \sup \left\{ \left| \frac{1}{R^{d-1}} \int_{Q^e(0,R)} f(se + x') \, \mathcal{H}^{d-1}(dx') - \int_{\mathbb{T}^d} f(y) \, dy \right| \mid s \in [0, r_e) \right\} = 0.$$

Proof. By the ergodic theorem, there is a Lebesgue measurable, $\{\tau_{x'}\}_{x' \in \langle e \rangle^\perp}$-invariant set $G \subseteq \mathbb{R}^d$ such that $\mathcal{L}^d(\mathbb{R}^d \setminus G) = 0$ and

$$\lim_{R \to \infty} R^{1-d} \int_{Q^e(0,R)} f(\bar{y} + x') \, \mathcal{H}^{d-1}(dx') = \int_{\mathbb{T}^d} f(y) \, dy \quad \text{for each} \quad \bar{y} \in G.$$

Define $E \subseteq \mathbb{R}$ by

$$E = \left\{ s \in \mathbb{R} \mid se + x' \in G \text{ for } \mathcal{H}^{d-1}\text{-almost every } x' \in \langle e \rangle^\perp \right\}.$$

We claim that $\mathcal{L}^1(\mathbb{R} \setminus E) = 0$. Indeed, by Fubini's Theorem, we can write

$$0 = \mathcal{L}^d(\mathbb{R}^d \setminus G) = \int_{\mathbb{R} \setminus E} \mathcal{H}^{d-1}(\{x' \in \langle e \rangle^\perp \mid se + x' \notin G\}) \, ds.$$

We are left to conclude that $\mathcal{L}^1(\mathbb{R} \setminus E) = 0$.

Now we claim that if $s \in E$, then (2.11) holds. To see this, fix $s \in E$ and observe that there is a $x' \in \langle e\rangle^\perp$ so that $se+x' \in G$. Thus, by $\{\tau_{x'}\}_{x'\in\langle e\rangle^\perp}$-invariance, $se \in G$. Therefore, (2.11) follows.

Finally, to prove the uniform convergence for $f \in C(\mathbb{T}^d)$, we use the Krylov-Bogoliubov trick. For $(s,R) \in [0,r_e] \times (0,\infty)$, define the probability measure $\mu_{(s,R)}$ by

$$\int_{\mathbb{T}^d} f(y)\mu_{(s,R)}(dy) = \frac{1}{R^{d-1}} \int_{Q^e(0,R)} f(se+x')\,\mathcal{H}^{d-1}(dx').$$

Fix $f \in C(\mathbb{T}^d)$. What we aim to prove can be reformulated as:

$$\lim_{R\to\infty} \sup \left\{ \left| \int_{\mathbb{T}^d} f(y)\,\mu_{(s,R)}(dy) - \int_{\mathbb{T}^d} f(y)\,dy \right| \;\mid\; s \in [0,r_e) \right\} = 0.$$

If this were false, then we could find a sequence $(s_n,R_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} R_n = \infty$ and a $\nu > 0$ such that

$$\left| \int_{\mathbb{T}^d} f(y)\,\mu_{(s_n,R_n)}(dy) - \int_{\mathbb{T}^d} f(y)\,dy \right| \geq \nu \quad \text{for each } n \in \mathbb{N}.$$

To see this is impossible, notice that, up to passing to a subsequence, there is no loss of generality assuming that there is a $\underline{\mu} \in \mathscr{P}(\mathbb{T}^d)$ such that $\mu_{(s_n,R_n)} \overset{*}{\rightharpoonup} \underline{\mu}$ as $n \to \infty$. In particular,

$$\left| \int_{\mathbb{T}^d} f(y)\,\underline{\mu}(dy) - \int_{\mathbb{T}^d} f(y)\,\mu(dy) \right| \geq \nu,$$

hence $\underline{\mu} \neq \mathcal{L}^d$. On the other hand, a straightforward computation shows that $\underline{\mu} \in \mathcal{I}(e)$, contradicting the unique ergodicity of $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$. □

Finally, we will observe the following consequence of what has been proven so far, which is worth noting in its own right and will appear in Chapter 6.

Theorem 14. *Fix $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. If $f \in L^1(\mathbb{T}^d)$ varies only in the e direction, that is, if $(Id - e\otimes e)Df = 0$ in $\mathbb{T}^d$ in the sense of distributions, then f is constant, that is,*

$f = \int_{\mathbb{T}^d} f(y)\, dy$ $\mathcal{L}^d$*-almost everywhere in* $\mathbb{T}^d$*.*

Proof. If $(\mathrm{Id} - e \otimes e)Df = 0$ in $\mathbb{T}^d$ in the sense of distributions, then, by mollification, we can obtain a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{T}^d)$ such that $f_n \to f$ in $L^1(\mathbb{T}^d)$ as $n \to \infty$ and $(\mathrm{Id} - e \otimes e)Df_n = 0$ pointwise in $\mathbb{T}^d$. Hence there is no loss of generality assuming that $f \in C^\infty(\mathbb{T}^d)$.

Since $(\mathrm{Id} - e \otimes e)Df = 0$ in $\mathbb{T}^d$, it follows that $f(y + x') = f(y)$ for each $y \in \mathbb{T}^d$ and $x' \in \langle e\rangle^\perp$. In particular, for each $c \in \mathbb{R}$, the set $A_c = \{y \in \mathbb{T}^d \mid f(y) < c\}$ is $(\tau_{x'})_{x'\in\langle e\rangle^\perp}$-invariant. Therefore, by ergodicity (Theorems 12 and 13), either $\mathcal{L}^d(A_c) = 0$ or $\mathcal{L}^d(A_c) = 1$. Since f is continuous, this can only mean that either $A_c = \mathbb{T}^d$ or else $A_c = \emptyset$. This being true for every $c \in \mathbb{R}$, we are left to conclude that f is a constant function. □

2.5 Proof of Theorem 10

2.5.1 Irrational Directions

We now have most of the necessary ingredients to analyze the asymptotics of the problem (2.4). In this section, we will begin with the case when e is irrational, while the next section will treat the rational case.

For the remainder of this section, suppose that $e \notin \mathbb{R}\mathbb{Z}^d$. Theorems 12 and 13 shows that (2.4) can be understood as a linear, uniformly elliptic PDE with stationary, ergodic coefficients. Accordingly, the results of Caffarelli, Souganidis, and Wang [29] and Caffarelli and Souganidis [28] are applicable.

For later use, it will be worthwhile to consider the more general cell problem for an arbitrary right-hand side $f \in C(\mathbb{T}^d)$. That is, in the next result, our interest is in the functions $(\tilde{V}_y^{e,\delta,f})_{y\in\mathbb{T}^d}$ solving the family of PDE indexed by $y \in \mathbb{T}^d$:

$$\delta \tilde{V}_y^{e,\delta,f} - \mathrm{tr}\left(a(y + x')D^2\tilde{V}_y^{e,\delta,f}\right) = f(y + x') \quad \text{in } \langle e\rangle^\perp. \tag{2.12}$$

Lifting to the torus, this is equivalent to solving the single penalized cell problem

$$\delta V^{e,\delta,f} - \operatorname{tr}\left(a(y)D^2V^{e,\delta,f}\right) = f(y) \quad \text{in } \mathbb{T}^d. \tag{2.13}$$

As in Section 2.2, the two perspectives are related by the transformation

$$\tilde{V}_y^{e,\delta,f}(x') = V^{e,\delta,f}(y+x') \quad \text{for } y \in \mathbb{T}^d,\ x' \in \langle e\rangle^\perp.$$

The next result characterizes the behavior of the solutions $(V^{e,\delta,f})_{\delta>0}$ of the penalized cell problem as $\delta \to 0$.

Proposition 3. *Given $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, there is a probability measure $\bar{\mu}_e \in \mathscr{P}(\mathbb{T}^d)$ with the following property: if $f \in C(\mathbb{T}^d)$ and $(\tilde{V}_y^{e,\delta,f})_{y\in\mathbb{T}^d}$ are the solutions of* (2.12)*, then*

$$\lim_{\delta\to 0^+} \sup\left\{|\delta\tilde{V}_y^{e,\delta,f}(x') - \overline{f}(e)| \mid y \in \mathbb{T}^d,\ x' \in \langle e\rangle^\perp\right\} = 0, \tag{2.14}$$

where $\overline{f}(e) = \int_{\mathbb{T}^d} f(y)\,\bar{\mu}_e(dy)$. In particular, if $V^{e,\delta,f}$ is the solution of the lifted problem (2.13)*, then $\delta V^{e,\delta,f} \to \overline{f}(e)$ uniformly in $\mathbb{T}^d$ as $\delta \to 0$.*

The equation (2.13) has a stochastic representation in terms of the diffusion process X^e determined by the SDE

$$dX_t^e = (\mathrm{Id} - e\otimes e)\sqrt{a(X_t^e)}dB_t.$$

In view of that connection, it is not surprising that the measure $\bar{\mu}_e$ is an invariant measure for X^e. Precisely, if $X_0^e \sim \bar{\mu}_e$ initially, then $X_t^e \sim \bar{\mu}_e$ for all $t > 0$. Invariant measures can also be characterized analytically, as we recall next.

Definition 4. *A probability measure $\mu \in \mathscr{P}(\mathbb{T}^d)$ is said to be an* invariant probability measure *for the linear operator* $-tr(a(y)D_e^2)$ *in* $\mathbb{T}^d$ *if*

$$\int_{\mathbb{T}^d} tr(a(y)D_e^2\psi(y))\,\mu(dy) = 0 \quad \textit{for each } \psi \in C^2(\mathbb{T}^d).$$

The set of all such measures will be denoted by $\mathscr{I}_e^a$.

The representation of the effective mobility $\overline{m}$ in terms of invariant measures will be particularly convenient when it comes time to investigate its continuity properties. The next result completely characterizes $\mathscr{I}_e^a$ when e is irrational.

Theorem 15. *If* $e \notin \mathbb{R}\mathbb{Z}^d$, *then there is a probability measure* $\bar{\mu}_e \in \mathscr{P}(\mathbb{T}^d)$ *such that* $\mathscr{I}_e^a = \{\bar{\mu}_e\}$. *Further,* $\bar{\mu}_e \ll \mathcal{L}^d$ *and there is a constant* $C(d-1, \lambda^{-1}\Lambda) > 0$ *such that*

$$\int_{\mathbb{T}^d} \left| \frac{d\bar{\mu}_e}{d\mathcal{L}^d}(y) \right|^{d-1} dy \leq C(d-1, \lambda^{-1}\Lambda).$$

The proof of the theorem is split between this section and the next. First, in this section, we will invoke results of Caffarelli and Souganidis [28] on the homogenization of uniformly elliptic PDE with almost periodic coefficients. Those results immediately imply the convergence of the solutions of the penalized cell problem (2.4). Later, we will see that homogenization also implies uniqueness of the invariant measure $\bar{\mu}_e$.

Proof of Proposition 3. Since (2.12) is a second-order, uniformly elliptic PDE with almost periodic coefficients, the results of [28, Appendix B] imply that there is a constant $\overline{f}(e) \in \mathbb{R}$ such that

$$\lim_{\delta \to 0^+} \sup \left\{ |\delta \tilde{V}_0^{e,\delta,f}(x') - \overline{f}(e)| \mid x' \in \langle e \rangle^{\perp} \right\} = 0.$$

In terms of the lift $V^{e,\delta,f}$, this becomes

$$\lim_{\delta \to 0^+} \sup \left\{ |\delta V^{e,\delta,f}(x') - \overline{f}(e)| \mid x' \in \langle e \rangle^{\perp} \right\} = 0.$$

At the same time, given $y \in \mathbb{T}^d$, if we denote by $A \subseteq \mathbb{T}^d$ the set appearing in this supremum, that is, if $A = \pi_{\mathbb{Z}^d}(\langle e \rangle^{\perp})$, then A is dense in $\mathbb{T}^d$ by Lemma 1, (iii). Therefore, by continuity of $V^{e,\delta,f}$ for each fixed δ, we have

$$\begin{aligned} 0 &= \lim_{\delta \to 0^+} \sup \left\{ |\delta V^{e,\delta,f}(x') - \overline{f}(e)| \mid x' \in \langle e \rangle^{\perp} \right\} \\ &= \lim_{\delta \to 0^+} \sup \left\{ |\delta V^{e,\delta,f}(y) - \overline{f}(e)| \mid y \in \mathbb{T}^d \right\}. \end{aligned}$$

This proves that $\delta V^{e,\delta} \to \overline{f}(e)$ uniformly in $\mathbb{T}^d$. Note that this immediately implies that (2.14) also holds.

Since the map $f \mapsto V^{e,\delta,f}$ is linear for each fixed (δ, e), it follows that $f \mapsto \overline{f}(e)$ must also be linear. Furthermore, the comparison principle shows that

$$|\overline{f}(e)| \leq \|f\|_{L^\infty(\mathbb{T}^d)}$$

and if f is constant, say, $f \equiv C_0$, then $\tilde{V}_y^{e,\delta,f} \equiv \delta^{-1} C_0$ so $\overline{f}(e) = C_0$. Therefore, by the Riesz Representation Theorem, there is a probability measure $\bar{\mu}_e$ such that $\overline{f}(e) = \int_{\mathbb{T}^d} f(y) \, \bar{\mu}_e(dy)$. □

In the remainder of this section, we prove Theorem 10 through a rigorous interpretation of the formal analysis presented in Section 2.2. In the process, it will be useful to know that the solution $V^{e,\delta,f}$ of (2.13) is as smooth as f. The next theorem provides a basic estimate to that effect.

Proposition 4. *For each $e \in S^{d-1}$ and $\delta > 0$, if $V^{e,\delta,f}$ is the solution of (2.13) and $f \in C^{2,\alpha}(\mathbb{T}^d)$ for some $\alpha > 0$, then*

$$\sup \left\{ \delta \|V^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} + \delta^2 \|DV^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} + \delta^3 \|D^2 V^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} \mid \delta > 0 \right\} < \infty.$$

The proof of the proposition is deferred to the end of the section. For the time being, we use it in conjunction with Proposition 3 to prove Theorem 10 in the irrational case.

Proof of Theorem 10, irrational case. We argue using half-relaxed limits. To start with, we claim that

$$\limsup{}^{*} u_e^{\epsilon}(x,t) \leq \langle x, e\rangle + \alpha \overline{m}(e)^{-1} t. \tag{2.15}$$

To see this, fix $\beta \in (\alpha, \infty)$, let $\epsilon \in (0,1)$, set $\delta(\epsilon) = \epsilon^{\frac{1}{4}}$, and define $v^{\epsilon} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$v^{\epsilon}(x,t) = \langle x, e\rangle + \beta \overline{m}(e)^{-1} \left(t + \epsilon^2 V_e^{\delta(\epsilon)}(\epsilon^{-1} x) \right),$$

where V_e^{δ} is the solution of (2.13) with $f = -m(\cdot, e)$. We claim there is an $\epsilon_0 > 0$ depending only on m and β such that v^{ϵ} is a super-solution of (2.1) if $\epsilon \in (0, \epsilon_0)$.

Indeed, invoking the convergence of $\delta V^{e,\delta}$ to $\overline{m}$, we find

$$\begin{aligned} m(\epsilon^{-1}x, \widehat{Dv^{\epsilon}}) v_t^{\epsilon} - \operatorname{tr}\left(A(\epsilon^{-1}x, \widehat{Dv^{\epsilon}}) D^2 v^{\epsilon} \right) - \alpha \|Dv^{\epsilon}\| \\ &= (\beta - \alpha) + \beta \overline{m}(e)^{-1}(-\delta V_e^{\delta}(\epsilon^{-1}x) - \overline{m}(e)) + O(\epsilon^{\frac{1}{2}}) \\ &= \beta - \alpha + o(1). \end{aligned}$$

Thus, there is an $\epsilon_0 \in (0,1)$ such that if $\epsilon \in (0, \epsilon_0)$, then

$$m(\epsilon^{-1}x, \widehat{Dv^{\epsilon}}) v_t^{\epsilon} - \operatorname{tr}\left(A(\epsilon^{-1}x, \widehat{Dv^{\epsilon}}) D^2 v^{\epsilon} \right) - \alpha \|Dv^{\epsilon}\| \geq \frac{\beta - \alpha}{2} \quad \text{in } \mathbb{R}^d \times \mathbb{R}.$$

Now we prove (2.15). First, notice that, by the choice of $\delta(\epsilon)$,

$$\langle x, e\rangle \leq v^{\epsilon}(x, 0) + \beta \overline{m}(e)^{-1} \|\delta V_e^{\delta}\|_{L^{\infty}(\mathbb{T}^d)} \epsilon^{\frac{7}{4}}$$

Thus, the comparison principle implies

$$u^{\epsilon}(x,t) \leq v^{\epsilon}(x,t) + \beta \overline{m}(e)^{-1} \|m\|_{L^{\infty}(\mathbb{T}^d)} \epsilon^{\frac{7}{4}}.$$

Sending $\epsilon \to 0^+$, we deduce that

$$\limsup^* u^\epsilon(x,t) \leq \langle x, e\rangle + \beta \overline{m}(e)^{-1} t.$$

At the same time, β was an arbitrary number in (α, ∞). Therefore, sending $\beta \to \alpha^+$, we recover (2.15).

Replacing $\beta \in (\alpha, \infty)$ by $\beta \in (-\infty, \alpha)$, we similarly prove that

$$\liminf_* u^\epsilon(x,t) \geq \langle x, e\rangle + \alpha \overline{m}(e)^{-1} t.$$

□

When the cell problem (2.2) has a smooth solution, it is straightforward to deduce that an $\mathcal{O}(\epsilon)$ rate of convergence holds.

Proposition 5. *Fix $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ and $\alpha \in \mathbb{R} \setminus \{0\}$, and let $(u^\epsilon)_{\epsilon > 0}$ denote the solutions of the problem* (2.1). *If there is a $V_e \in C^2(\mathbb{T}^d)$ solving* (2.2)*, then there are constants $C, \epsilon_0 > 0$ such that, for any $t > 0$ and any $\epsilon \in (0, \epsilon_0)$,*

$$\sup \left\{ |u^\epsilon(x,t) - \langle x, e\rangle - \alpha \overline{m}(e)^{-1} t| \mid x \in \mathbb{R}^d \right\} \leq C(1 + |t|)\epsilon,$$
$$d_{\mathcal{H}}(\{u^\epsilon = 0\}, \{x \in \mathbb{R}^d \mid \langle x, e\rangle = \alpha \overline{m}(e)^{-1} t\}) \leq 4C(1 + |t|)\epsilon.$$

The proof is deferred to Section 2.5.3.

Proof of Proposition 4. The estimate $\delta \|V^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} \leq C$ follows from the comparison principle. Further, since a is smooth, by studying the slices $\{\tilde{V}_y^{e,\delta,f}\}_{y \in \mathbb{T}^d}$, which solve a linear uniformly elliptic equation, we obtain the estimate

$$\delta \|D_e V^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} + \delta \|D_e^2 V^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} \leq C$$

for the tangential derivatives

$$D_e V^{e,\delta,f} = (\mathrm{Id} - e \otimes e) D V^{e,\delta,f}, \quad D_e^2 V^{e,\delta,f} = (\mathrm{Id} - e \otimes e) D^2 V^{e,\delta,f} (\mathrm{Id} - e \otimes e).$$

It only remains to estimate the first- and second-order directional derivatives in the e direction.

Formally differentiating (2.13) in the e direction, we see that the function $W^{e,\delta,f} = \langle DV^{e,\delta}, e\rangle$ solves a uniformly elliptic PDE of the form

$$\delta W^{e,\delta,f} - \mathrm{tr}\left(a(y) D^2 W^{e,\delta,f}\right) = F(e,\delta,y) \quad \text{in } \mathbb{T}^d.$$

This can be made rigorous using difference quotients. Note that the estimates on $V^{e,\delta}$ imply that $\|F(e,\delta,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq C\delta^{-1}$. Hence $\delta^2 \|W^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} \leq C$ by comparison.

Passing to the slices $\{\tilde{W}_y^{e,\delta,f}\}_{y \in \mathbb{T}^d}$ defined in $\langle e\rangle^\perp$ by

$$\tilde{W}_y^{e,\delta,f}(x') = W^{e,\delta,f}(y + x'),$$

we see that, for each $y \in \mathbb{T}^d$, $\tilde{W}_y^{e,\delta,f}$ is the solution of the linear uniformly elliptic PDE

$$\delta \tilde{W}_y^{e,\delta,f} - \mathrm{tr}\left(a(y+x') D^2 \tilde{W}_y^{e,\delta,f}\right) = F(e,\delta,y+x').$$

Thus, by Schauder estimates, $\delta^2 \|D\tilde{W}_y^{e,\delta,f}\|_{L^\infty(\langle e\rangle^\perp)} + \delta^2 \|D\tilde{W}_y^{e,\delta,f}\|_{L^\infty(\langle e\rangle^\perp)} \leq C$. In terms of the lift $W^{e,\delta,f}$, this gives $\delta^2(\|D_e W^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} + \|D_e^2 W^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)}) \leq C$. This proves that $\delta^2 \|\langle DV^{e,\delta,f}, e\rangle\|_{L^\infty(\mathbb{T}^d)} \leq C$ and hence $V^{e,\delta,f}$ has a full continuous derivative with estimate $\delta^2 \|DV^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} \leq C$.

After differentiating once more, we similarly find that $\delta^3 \|D^2 V^{e,\delta,f}\|_{L^\infty(\mathbb{T}^d)} \leq C$. $\square$

2.5.2 Invariant Measures

Before proceeding to the analysis of linear response in the case when $e \in \mathbb{R}\mathbb{Z}^d$, we study the invariant measures $\mathscr{I}_e^a$ of the operator $-\text{tr}(a(y)D_e^2)$. We start with the easier rational case before turning to irrational directions and proving Theorem 15.

The main result concerning the structure of $\mathscr{I}_e^a$ when $e \in \mathbb{R}\mathbb{Z}^d$ is stated next.

Proposition 6. *Given any $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, there is a function $\mu_e : \mathbb{R} \to \mathscr{I}_e^a$, $\mu_e : s \mapsto \mu_e^s$, such that*

(i) The support of μ_e^r equals $\mathbb{T}_e^{d-1}(r)$ and $\mu_e^r \ll \mathcal{H}^{d-1} \restriction_{\mathbb{T}_e^{d-1}(r)}$,

(ii) $\mu_e^{r+r_e} = \mu_e^r$,

(iii) $\mathscr{I}_e^a$ is the closed convex hull of $\{\mu_e^r \mid r \in [0, r_e)\}$.

Finally, there is a constant $C(d-1, \lambda^{-1}\Lambda) > 0$ such that, given $r \in [0, r_e)$, if we define $h_e^r : \mathbb{T}^d \to [0, \infty)$ so that

$$\int_{\mathbb{T}^d} f(y)\, \mu_e^r(dy) = \fint_{\mathbb{T}_e^{d-1}(r)} f(y')h_e^r(y')\, \mathcal{H}^{d-1}(dy') \quad \textit{for } f \in C(\mathbb{T}^d),$$

then

$$\fint_{\mathbb{T}_e^{d-1}(r)} \left|h_e^r(y')\right|^{d-1} \mathcal{H}^{d-1}(dy') \leq C(d-1, \lambda^{-1}\Lambda).$$

The proof boils down to decomposing the torus as $\mathbb{T}^d = \cup_{r \in [0, r_e)} \mathbb{T}_e^{d-1}(r)$ and studying the restriction of $-\text{tr}(a(y)D_e^2)$ on each slice.

Proof. See Proposition 75 in the appendix for the existence and uniqueness of the measure μ_e^r and the L^{d-1} estimate on the density h_e^r.

The fact that $\mu_e^{r+r_e} = \mu_e^r$ follows from the identity $\mathbb{T}_e^{d-1}(r + r_e) = \mathbb{T}_e^{d-1}(r)$ and uniqueness.

To deduce that $\mathscr{I}_e^a$ is the closed convex hull of $\{\mu_e^r\}_{r\in[0,r_e)}$, note that it is enough to prove that, for each $\mu \in \mathscr{I}_e^a$, there is a Borel probability measure ν_μ on $[0, r_e)$ such that

$$\int_{\mathbb{T}^d} f(y)\,\mu(dy) = \int_0^{r_e} \left(\int_{\mathbb{T}_e^{d-1}(r)} f(y')\,\mu_e^r(dy') \right) \nu_\mu(dr) \quad \text{for } f \in C(\mathbb{T}^d).$$

This is indeed the case; see [70, Proof of Theorem 3] for the details. □

Combining the previous result and Proposition 3, we readily prove Theorem 15.

Proof of Theorem 15. Fix $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. We begin by proving that $\mathscr{I}_e^a$ is nonempty, then we prove that $\mathscr{I}_e^a = \{\bar{\mu}_e\}$ and $\bar{\mu}_e$ satisfies the L^{d-1} estimate.

To begin with, fix a sequence $(e_n)_{n\in\mathbb{N}} \subseteq S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$ such that $e = \lim_{n\to\infty} e_n$. To simplify the notation, define $(\mu_n)_{n\in\mathbb{N}}$ by $\mu_n = \mu_{e_n}^0$. By Lemma 17 and Proposition 6, there is a subsequence $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ and a $\mu \ll \mathcal{L}^d$ such that $\mu = \lim_{j\to\infty} \mu_{n_j}$ weakly-$*$ and

$$\int_{\mathbb{T}^d} \left| \frac{d\mu}{d\mathcal{L}^d}(y) \right|^{d-1} dy \leq C(d-1, \lambda^{-1}\Lambda). \tag{2.16}$$

It is easy to see that $\mu \in \mathscr{I}_e^a$. Indeed, if $\psi \in C^2(\mathbb{T}^d)$, then

$$\int_{\mathbb{T}^d} \operatorname{tr}\left(a(y) D_e^2 \psi(y)\right) \mu(dy) = \lim_{j\to\infty} \int_{\mathbb{T}^d} \operatorname{tr}\left(a(y) D_{e_{n_j}}^2 \psi(y)\right) \mu_{n_j}(dy) = 0.$$

Next, we show that $\mathscr{I}_e^a = \{\bar{\mu}_e\}$, where $\bar{\mu}_e$ is the measure from Proposition 3. Indeed, if $\mu \in \mathscr{I}_e^a$, $f \in C^{2,\alpha}(\mathbb{T}^d)$, and $V^{e,\delta,f}$ is the associated solution of (2.13), then $V^{e,\delta,f} \in C^{2,\alpha}(\mathbb{T}^d)$ by Proposition 4 and, thus,

$$\int_{\mathbb{T}^d} \delta V^{e,\delta,f}(y)\,\mu(dy) = \int_{\mathbb{T}^d} f(y)\,\mu(dy).$$

Sending $\delta \to 0^+$, we find $\int_{\mathbb{T}^d} f(y)\,\mu(dy) = \overline{f}(e) = \int_{\mathbb{T}^d} f(y)\,\bar{\mu}_e(dy)$. This proves $\mu = \bar{\mu}_e$.

Finally, we proved that there is a $\mu \in \mathscr{I}_e^a$ for which the L^{d-1} estimate (2.16) holds, hence the same estimate applies to $\bar{\mu}_e$ by uniqueness. $\square$

2.5.3 Rational Directions

The analysis of (2.1) is more complicated in the rational setting. The reason goes back to Theorem 12: when $e \in \mathbb{R}\mathbb{Z}^d$, the translation group $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$ is not uniquely ergodic.

When $e \in \mathbb{R}\mathbb{Z}^d$, (2.4) is actually a PDE with periodic coefficients — periodic with respect to the group of resonances M_e. That is, for each $y \in \mathbb{T}^d$,

$$a(y + x' + k') = a(y + x') \quad \text{for each } k' \in M_e.$$

Since M_e has rank $d-1$ by Theorem 11, if we identify $\langle e \rangle^\perp$ with $\mathbb{R}^{d-1}$ and M_e with $\mathbb{Z}^{d-1}$, then (2.4) becomes a linear, uniformly elliptic second-order equation with periodic coefficients. Therefore, classical results in homogenization apply to describe the asymptotics. At the same time, since $(\tau_{x'})_{x' \in \langle e \rangle^\perp}$ is not uniquely ergodic, the asymptotics of $(\tilde{V}_y^{e,\delta})_{\delta > 0}$ will depend on which of the invariant measures is "seen" by y.

Proposition 7. *Fix $f \in C(\mathbb{T}^d)$ and $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. There is an $f_e^\perp \in C(\mathbb{T}^d)$ varying only in the e direction such that if $(\tilde{V}_y^{e,\delta,f})_{y \in \mathbb{T}^d}$ denote the solutions of* (2.12)*, then*

$$\lim_{\delta \to 0^+} \sup \left\{ \left| \delta \tilde{V}_y^{e,\delta}(x') - f_e^\perp(y) \right| \mid y \in \mathbb{T}^d, \ x' \in \langle e \rangle^\perp \right\} = 0. \tag{2.17}$$

Furthermore, $f_e^\perp$ is determined by the formula

$$f_e^\perp(y) = \int_{\mathbb{T}_e^{d-1}(\langle y, e \rangle)} f(y') \, \mu_e^{\langle y, e \rangle}(dy'), \tag{2.18}$$

where $\{\mu_e^r \mid r \in [0, r_e)\}$ are the probability measures from Proposition 6.

Proof. As explained above, (2.12) can be regarded as a second-order, uniformly elliptic PDE with periodic coefficients. Therefore, the existence of $f_e^\perp$ and the uniform convergence with respect to y and x' is classical (cf. [28, Section 7]). A straightforward compactness argument shows that $f_e^\perp$ must be continuous. Furthermore, (2.17) directly implies that $f^\perp$ only varies in the e direction, that is, $f_e^\perp(y) = f_e^\perp(\langle y, e\rangle e)$ for all $y \in \mathbb{T}^d$.

Finally, as in the proof of Proposition 3, when $f \in C^{2,\alpha}(\mathbb{T}^d)$, we now that the solution $V^{e,\delta,f}$ of (2.13) is smooth for any given $\delta > 0$, hence

$$\int_{\mathbb{T}^d} \delta V^\delta(y)\, \mu_e^r(dy) = \int_{\mathbb{T}^d} f(y)\, \mu_e^r(dy) \quad \text{for each } \delta > 0,\ r \in [0, r_e).$$

Sending $\delta \to 0^+$ with r fixed and recalling that μ_e^r is supported on $\mathbb{T}_e^{d-1}(r)$, we conclude that (2.18) holds. $\square$

To complete the proof of Theorem 28 in the rational case, we need to accommodate the fact that the constant $\overline{m}(e)$ from the irrational case has to be replaced by an oscillating function $m_e^\perp$. This can be corrected by replacing the formal asymptotic expansion used in the irrational case by the following one:

$$u^\epsilon(x,t) = \langle x, e\rangle + \alpha \overline{m}_{\mathrm{pl}}(e)^{-1} t + \epsilon \mathcal{V}_e(\epsilon^{-1}\langle x, e\rangle) + \epsilon^2 \alpha \overline{m}_{\mathrm{pl}}(e)^{-1} V_e(\epsilon^{-1} x) + \dots$$

Here we assume that $\mathcal{V}_e$ varies only in the e direction and hence can be treated as a function of one variable. In order for this expansion to generate a solution of (2.1), formal computations suggest that $\mathcal{V}_e$ and V_e must solve the equations

$$\overline{m}_{\mathrm{pl}}(e)^{-1} m_e^\perp(se) - |1 + \mathcal{V}_{e,s}| = 0 \quad \text{in } \mathbb{R}/r_e\mathbb{Z}, \tag{2.19}$$

$$m\,(y,e) - \operatorname{tr}\left(A\,(y,e)\, D^2 V_e\right) = m_e^\perp(\langle y, e\rangle e) \quad \text{in } \mathbb{T}^d. \tag{2.20}$$

The next result makes this argument rigorous.

Proposition 8. *For any $e \in \mathbb{R}\mathbb{Z}^d$ and $\alpha \in \mathbb{R} \setminus \{0\}$, we have:*

(i) The cell problem (2.20) *has a solution* $V_e \in C^{2,\alpha}(\mathbb{T}^d)$ *and* $m_e^{\perp} \in C^{2,\alpha}(\mathbb{T}^d)$.

(ii) There is a unique constant $\overline{m}_{pl}(e) > 0$ *for which* (2.19) *has a* C^2, r_e*-periodic solution* $\mathcal{V}_e : \mathbb{R} \to \mathbb{R}$. *Further,* $\overline{m}_{pl}(e)$ *is given explicitly by*

$$\overline{m}_{pl}(e) = r_e^{-1} \int_0^{r_e} m_e^{\perp}(se)\, ds.$$

(iii) For each $\alpha \in \mathbb{R} \setminus \{0\}$, *there is an* $\epsilon_0^+ > 0$ *and a family* $(\alpha_\epsilon^+)_{\epsilon \in (0,\epsilon_0^+)}$ *such that the functions* $(u^{+,\epsilon})_{\epsilon \in (0,\epsilon_0^+)}$ *defined by*

$$u^{+,\epsilon}(x,t) = \langle x, e\rangle + \epsilon \mathcal{V}_e(\epsilon^{-1}\langle x, e\rangle) + \alpha_\epsilon^+ \overline{m}_{pl}(e)^{-1} \left(\epsilon^2 V_e(\epsilon^{-1}x) + t\right)$$

are super-solutions of (2.1). *Furthermore, there is a constant* $C_+ > 0$ *depending only on* $\mathcal{V}_e$ *and* V_e *such that, for each* $(x,t) \in \mathbb{R}^d \times \mathbb{R}$,

$$|\alpha_\epsilon^+ - \alpha| \leq C_+ \epsilon$$
$$|u^{+,\epsilon}(x,t) - \langle x, e\rangle - \alpha \overline{m}(e)^{-1} t| \leq C_+ \epsilon (1 + |t|)$$

(iv) For each $\alpha \in \mathbb{R} \setminus \{0\}$, *there is an* $\epsilon_0^- > 0$ *and a family* $(\alpha_\epsilon^-)_{\epsilon \in (0,\epsilon_0^-)}$ *such that the functions* $(u^{-,\epsilon})_{\epsilon \in (0,\epsilon_0^-)}$ *defined by*

$$u^{-,\epsilon}(x,t) = \langle x, e\rangle + \epsilon \mathcal{V}_e(\epsilon^{-1}\langle x, e\rangle) + \alpha_\epsilon^- \overline{m}_{pl}(e)^{-1} \left(\epsilon^2 \tilde{V}_e(\epsilon^{-1}x) + t\right)$$

are sub-solutions of (2.1) *in* $\mathbb{R}^d \times \mathbb{R}$. *Furthermore, there is a constant* $C_- > 0$ *depending only on* $\mathcal{V}_e$ *and* V_e *such that, for each* $(x,t) \in \mathbb{R}^d \times \mathbb{R}$,

$$|\alpha_\epsilon^- - \alpha| \leq C_- \epsilon$$
$$|u^{-,\epsilon}(x,t) - \langle x, e\rangle - \alpha \overline{m}(e)^{-1} t| \leq C_- \epsilon (1 + |t|)$$

We refer to [70, Proposition 7] for the proof of part (i) of the proposition. The remainder of this section treats the rest of the proof.

The equation (2.19) describes the pulsating wave solutions of a interface motion in one dimension. Indeed, if $\mathcal{V}_e$ is a solution and we denote by s the spatial variable, then the function $\mathcal{V}(s,t) = s + \mathcal{V}_e(s,t) + \alpha \overline{m}_{\mathrm{pl}}(e)^{-1} t$ is a pulsating wave solution of the PDE

$$m_e^{\perp}(se)\mathcal{V}_t - \alpha|\mathcal{V}_s| = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}. \tag{2.21}$$

The analysis of the interface motion determined by (2.21) is particularly simple. Indeed, since we are interested in plane-like solutions, let us consider the motion started from a plane, which, in one dimension, is simply a point. In the level-set formulation, this can be described through the PDE

$$\begin{cases} m_e^{\perp}(se)\mathcal{U}_t - \alpha|\mathcal{U}_s| = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \mathcal{U}(s,0) = s & \text{for } s \in \mathbb{R}. \end{cases}$$

An easy comparison argument shows that $\mathcal{U}$ is an increasing function of s for all time, hence $|\mathcal{U}_s| = \mathcal{U}_s$, and then the equation becomes linear transport $m_e^{\perp}(se)\mathcal{U}_t - \alpha\mathcal{U}_s = 0$. Put slightly differently, when we consider the motion of a plane (point) in one dimension, it is determined by the ODE

$$\begin{cases} m_e^{\perp}(X_t^s e)\dot{X}_t^s = 1, \\ X_0^s = s, \end{cases}$$

Integration of the equation readily yields the next result.

Lemma 2. *Fix $r > 0$. If $\tilde{m} : \mathbb{R}/r\mathbb{Z} \to (0,\infty)$ is C^1, then, for any $s \in \mathbb{R}$, the solution $X^{s,\tilde{m}}$ of the one-dimensional ODE*

$$\begin{cases} \tilde{m}(X_t^{s,\tilde{m}})\dot{X}_t^{s,\tilde{m}} = 1, \\ X_0^{s,\tilde{m}} = s, \end{cases}$$

is such that

$$\lim_{t\to\infty} \frac{X_t^{s,\tilde{m}} - s}{t} = \frac{1}{\fint_0^r \tilde{m}(u)\,du}.$$

Furthermore, the function $\tilde{P} : \mathbb{R}/r\mathbb{Z} \to \mathbb{R}$ *given by*

$$\tilde{P}(s) = \left(\fint_0^{r_e} \tilde{m}(u)\,du\right)^{-1} \int_0^s \tilde{m}(s)\,ds - 1$$

solves the pulsating wave equation

$$\frac{\tilde{m}(s)}{\fint_0^r \tilde{m}(u)\,du} - |1 + \tilde{P}_s| = 0 \quad \textit{in } \mathbb{R}/r\mathbb{Z}.$$

Proof of Proposition 8. Part (i) readily follows from classical results on the existence of correctors in the periodic setting; see [70, Proposition 7] for a complete proof. Part (ii) is a direct application of Lemma 2. It only remains to prove parts (iii) and (iv). We will only treat (iii) since (iv) follows similarly.

Let $\epsilon > 0$ and $\alpha_\epsilon^+ \in \mathbb{R}$ be free variables for the moment and define $u^{+,\epsilon} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$u^{+,\epsilon}(x,t) = \langle x, e\rangle + \epsilon \mathcal{V}_e(\epsilon^{-1}\langle x, e\rangle) + \alpha_\epsilon \overline{m}_{pl}(e)^{-1}\left(\epsilon^2 V_e(\epsilon^{-1}x) + t\right). \tag{2.22}$$

We will show that if $\epsilon > 0$ is small enough and $\alpha_\epsilon^+ = \alpha + C_0\epsilon$ for some large enough constant $C_0 > 0$, then $u^{+,\epsilon}$ is a super-solution of (2.1).

Let us study the equation for $u^{+,\epsilon}$ term by term. First, the term with the time derivative. To declutter the notation, we define $p_\epsilon : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ by

$$p_\epsilon(x,t) = Du^{+,\epsilon}(x,t) = (1 + \mathcal{V}_{e,s}(\epsilon^{-1}\langle x, e\rangle))e + \epsilon\alpha_\epsilon^+ \overline{m}_{pl}(e)^{-1} DV_e(\epsilon^{-1}x).$$

Observe that since $1+\mathcal{V}_{e,s}\geq 1$, it follows that $\hat{p}_\epsilon = e + O(\epsilon)$. Thus,

$$m(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})u_t^{+,\epsilon} = \alpha_\epsilon^+ \overline{m}_{pl}(e)^{-1} m(\epsilon^{-1}x, e) + O(\epsilon).$$

Next, the curvature term. First, notice that if we write $\mathcal{N}(p) = \hat{p}\otimes\hat{p}$, then, by Taylor expansion,

$$\begin{aligned}
\mathrm{tr}\left(A(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})D^2u^{+,\epsilon}\right) &= \mathrm{tr}\left(a(\epsilon^{-1}x, p_\epsilon)(\mathrm{Id} - \hat{p}_\epsilon\otimes\hat{p}_\epsilon)D^2u^{+,\epsilon}(\mathrm{Id} - \hat{p}_\epsilon\otimes\hat{p}_\epsilon)\right) \\
&= \mathrm{tr}\left(a(\epsilon^{-1}x, e)(\mathrm{Id} - e\otimes e)D^2u^{+,\epsilon}(\mathrm{Id} - e\otimes e)\right) \\
&\quad + \mathrm{tr}\left(D_p a(\epsilon^{-1}x, e)[p_\epsilon - e](\mathrm{Id} - e\otimes e)D^2u^{+,\epsilon}(\mathrm{Id} - e\otimes e)\right) \\
&\quad + \mathrm{tr}\left(a(\epsilon^{-1}x, e)D_p\mathcal{N}(e)[p_\epsilon - e]D^2u^{+,\epsilon}(\mathrm{Id} - e\otimes e)\right) \\
&\quad + \mathrm{tr}\left(a(\epsilon^{-1}x, e)(\mathrm{Id} - e\otimes e)D^2u^{+,\epsilon}D_p\mathcal{N}(e)[p_\epsilon - e]\right) \\
&\quad + \|D^2u^{+,\epsilon}\|_{L^\infty(\mathbb{T}^d)}O(\|\hat{p}_\epsilon - e\|^2)
\end{aligned}$$

Next, observe that $D^2u^{+,\epsilon} = \epsilon^{-1}\mathcal{V}_{e,ss}e\otimes e + \alpha_\epsilon\overline{m}_{pl}(e)^{-1}D^2V_e$ and, thus, in each of the first four terms in the previous expressions, we can replace $D^2u^{+,\epsilon}$ by $\alpha_\epsilon^+\overline{m}_{pl}(e)^{-1}D^2V_e$. This leads to

$$\begin{aligned}
\mathrm{tr}\left(A(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})D^2u^{+,\epsilon}\right) &= \alpha_\epsilon^+\overline{m}_{pl}(e)^{-1}\mathrm{tr}\left(A(\epsilon^{-1}x, e)D^2V_e\right) \\
&\quad + \|D^2V_e\|_{L^\infty(\mathbb{T}^d)}O(\|\hat{p}_\epsilon - e\|) + \|D^2u^{+,\epsilon}\|_{L^\infty(\mathbb{T}^d)}O(\|\hat{p}_\epsilon - e\|^2)
\end{aligned}$$

Since $\|D^2u^{+,\epsilon}\|\leq C\epsilon^{-1}$ and $\|\hat{p}_\epsilon - e\| = O(\epsilon)$, we conclude

$$\mathrm{tr}\left(A(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})D^2u^{+,\epsilon}\right) = \alpha_\epsilon^+\overline{m}_{pl}(e)^{-1}\mathrm{tr}\left(A(\epsilon^{-1}x, e)D^2V_e\right) + O(\epsilon).$$

Finally, we treat the first-order term. Since $\|e\| = 1$, we have

$$\alpha\|Du^{+,\epsilon}\| = \alpha|1 + \mathcal{V}_{e,ss}| + O(\epsilon).$$

Putting it all together and invoking the equations satisfied by V_e and $\mathcal{V}_e$, we find, for some $\overline{C} > 0$,

$$\begin{aligned} m(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})u_t^{+,\epsilon} - \operatorname{tr}\left(A(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})D^2u^{+,\epsilon}\right) - \alpha\|Du^{+,\epsilon}\| \geq \\ \alpha_\epsilon^+ \overline{m}_{pl}(e)^{-1}\left(m(\epsilon^{-1}x, e) - \operatorname{tr}\left(A(\epsilon^{-1}x, e)D^2V_e\right)\right) - \alpha|1 + \mathcal{V}_{e,s}| - (\overline{C} + o(1))\epsilon = \\ \alpha_\epsilon^+ \overline{m}_{pl}(e)^{-1}m_e^\perp(\epsilon^{-1}\langle x, e\rangle) - \alpha\overline{m}_{pl}(e)^{-1}m_e^\perp(\epsilon^{-1}\langle x, e\rangle) - (\overline{C} + o(1))\epsilon \\ = (C_0 - C + o(1))\epsilon\overline{m}_{pl}(e)^{-1}m_e^\perp(\epsilon^{-1}\langle x, e\rangle). \end{aligned}$$

Setting $C_0 = C + 1$, we deduce that there is an $\epsilon_0 > 0$ such that $u^{+,\epsilon}$ is a super-solution of (2.1) in $\mathbb{R}^d \times \mathbb{R}$. □

Finally, notice that the previous result implies a rate of convergence for (2.1) for rational directions e.

Proposition 9. *If $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$ and $(u^\epsilon)_{\epsilon>0}$ are the solutions of (2.1), then there is a constant $C_0 > 0$ and an $\epsilon_0 > 0$ such that, for each $t > 0$ and $\epsilon \in (0, \epsilon_0)$,*

$$\sup\left\{|u^\epsilon(x,t) - \langle x, e\rangle - \alpha\overline{m}(e)^{-1}t| \mid x \in \mathbb{R}^d\right\} \leq C_0(1 + |t|)\epsilon, \tag{2.23}$$

$$d_{\mathcal{H}}(\{u^\epsilon = 0\}, \{x \in \mathbb{R}^d \mid \langle x, e\rangle = \alpha\overline{m}(e)^{-1}t\}) \leq 4C_0(1 + |t|)\epsilon. \tag{2.24}$$

Proof. We will prove that part (iii) of the previous proposition implies an upper bound of the form

$$u^\epsilon(x,t) \leq \langle x, e\rangle + \alpha\overline{m}_{\text{pl}}(e)^{-1}t + C_0(1 + |t|)\epsilon.$$

The corresponding lower bound can be obtained using part (iv). This proves (2.23), from which (2.24) follows easily.

By part (iii) of Proposition 8, there is an $\epsilon_0 > 0$, a family $(\alpha_\epsilon^+)_{\epsilon\in(0,\epsilon_0)}$, and a family of functions $(u^{+,\epsilon})_{\epsilon\in(0,\epsilon_0)}$ given by

$$u^{+,\epsilon}(x,t) = \langle x,e\rangle + \epsilon\mathcal{V}_e(\epsilon^{-1}\langle x,e\rangle) + \alpha_\epsilon^+\overline{m}_{pl}(e)^{-1}\left(\epsilon^2 V_e(\epsilon^{-1}x) + t\right)$$

and such that, for $\epsilon \in (0,\epsilon_0)$,

$$|u^{+,\epsilon}(x,t) - \langle x,e\rangle - \alpha\overline{m}(e)^{-1}t| \leq C_+\epsilon(1+|t|). \tag{2.25}$$

$$m(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})u_t^{+,\epsilon} - \mathrm{tr}\left(A(\epsilon^{-1}x, \widehat{Du^{+,\epsilon}})D^2u^{+,\epsilon}\right) \geq 0 \quad \text{in } \mathbb{R}^d\times(0,\infty).$$

Notice that we can write

$$u^{+,\epsilon}(x,0) \geq \langle x,e\rangle - C\epsilon$$

since $\mathcal{V}_e$ and V_e are both bounded periodic functions. Therefore, by the comparison principle,

$$u^{+,\epsilon}(x,t) + C\epsilon \geq u^\epsilon(x,t) \quad \text{for each } (x,t)\in\mathbb{R}^d\times(0,\infty).$$

Combining this with (2.25), we find

$$u^\epsilon(x,t) \leq \langle x,e\rangle + \alpha\overline{m}_{\mathrm{pl}}(e)^{-1}t + (C + C_+(1+|t|))\epsilon.$$

□

The same proof applies to obtain a rate in the irrational case.

Proof of Proposition 5. We argue exactly as in the proof of Proposition 9, except the first order term $\mathcal{V}_e$ gets set to zero. □

For the sake of completeness, note that the previous proposition implies Theorem 10 holds in the case when $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. Together with the results of Section 2.5.1, this completes the proof of the theorem.

Finally, we show via an example that the rates of convergence obtained above are optimal.

Proposition 10. *For any $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, there are coefficients m and a such that if $(u^\epsilon)_{\epsilon>0}$ are the solutions of* (2.1)*, then*

$$\sup\left\{|u^\epsilon(x,t) - \langle x, e\rangle - \alpha\overline{m}(e)^{-1}t| \mid (x,t) \in \mathbb{R}^d \times (0,\infty)\right\} = c\epsilon$$

for some $c > 0$.

Proof. Given $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, choose any m and a that vary only in the e direction, with m non-constant. To see that such a choice is possible, we can choose a $k \in \mathbb{Z}^d \setminus \{0\}$ such that $e = \|k\|^{-1}k$ and define, for example, $m(y) = 2 + \sin(2\pi\langle k, y\rangle)$ and $a \equiv 1$. (In fact, given any smooth, positive function $f : \mathbb{R}/r_e\mathbb{Z} \to (0,\infty)$, we can set $m(y) = f(\langle y, e\rangle e)$, and similarly for a. Here r_e is defined by (2.7).)

Let $(u^\epsilon)_{\epsilon>0}$ denote the solution of (2.1) with this choice of e and m. Since the coefficients m and a vary only in the e direction, the same can be said of u^ϵ by uniqueness. Thus, $u^\epsilon(x,t) = \mathcal{U}^\epsilon(\langle x, e\rangle, t)$, where $\mathcal{U}^\epsilon$ solves the one-dimensional problem

$$\begin{cases} m(\epsilon^{-1}se)\mathcal{U}^\epsilon_t - \alpha|\mathcal{U}^\epsilon_s| = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ \mathcal{U}^\epsilon(s,0) = s & \text{for } s \in \mathbb{R}. \end{cases}$$

Observe that if we write $\mathcal{U}^\epsilon(s,t) = \epsilon\mathcal{U}_\epsilon(\epsilon^{-1}s, \epsilon^{-2}t)$, then $\mathcal{U}_\epsilon$ is the solution of the unscaled equation with ϵ-dependent forcing:

$$\begin{cases} m(se)\mathcal{U}_{\epsilon,t} - \alpha\epsilon|\mathcal{U}_{\epsilon,s}| = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ \mathcal{U}_\epsilon(s,0) = s & \text{for } s \in \mathbb{R}. \end{cases} \tag{2.26}$$

If $\mathcal{V}_\epsilon$ is the solution of the related transport equation

$$\begin{cases} m(se)\mathcal{V}_{\epsilon,t} - \alpha\epsilon\mathcal{V}_{\epsilon,s} = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \mathcal{V}_\epsilon(s,0) = s & \text{for } s\in\mathbb{R}. \end{cases}$$

then $\mathcal{V}_\epsilon$ is given by

$$\mathcal{V}_\epsilon(s,t) = X_t^{\epsilon,s},$$

where $X^{\epsilon,s}$ is the solution of the ODE $m(X_t^{\epsilon,s})\dot{X}_t^{\epsilon,s} = \alpha\epsilon$ with $X_0^{\epsilon,s} = s$. Since one-dimensional ODE preserve the ordering on the real line, it follows that

$$\mathcal{V}_\epsilon(s_1,t) < \mathcal{V}_\epsilon(s_2,t) \quad \text{if } s_1 < s_2,\ t\in(0,\infty).$$

Therefore, $\mathcal{V}_{\epsilon,s} > 0$ and, thus, $\mathcal{V}_\epsilon$ is also a solution of (2.26). Put another way, by uniqueness, $\mathcal{U}_\epsilon \equiv \mathcal{V}_\epsilon$.

Next, notice that we can integrate the ODE to find

$$M(X_T^{\epsilon,s}) - M(s) = \int_0^T m(X_t^{\epsilon,s})\dot{X}_t^{\epsilon,s}\,dt = \alpha\epsilon T,$$

where M is the anti-derivative of m (considered as a function of one-variable), i.e.

$$M(s) = \int_0^s m(\xi e)\,d\xi.$$

Since m is periodic, we can write $M(s) = \langle m\rangle s + P(s)$, where $\langle m\rangle = r_e^{-1}\int_0^{r_e} m(se)\,ds$ and P is a smooth, r_e-periodic function. Hence

$$\langle m\rangle(X_T^{\epsilon,s} - s) = \alpha\epsilon T - P(X_T^{\epsilon,s}) - P(s)$$

which, upon replacing s by $s\epsilon^{-1}$ and t by $t\epsilon^{-2}$, implies that

$$\begin{aligned}|\mathcal{U}^\epsilon(s,t) - s - \alpha\langle m\rangle^{-1}t| &= |\epsilon X^{\epsilon, s\epsilon^{-1}}_{t\epsilon^{-2}} - s - \alpha\langle m\rangle^{-1}t| \\ &= \epsilon\langle m\rangle^{-1}|P(X^{\epsilon, s\epsilon^{-1}}_{t\epsilon^{-2}}) - P(s\epsilon^{-1})|.\end{aligned}$$

Lastly, we note that, for any $s \in \mathbb{R}$, we have $\lim_{t\to\infty} X^{\epsilon,s}_t = \infty$. Hence, for any $x \in \mathbb{R}^d$,

$$\begin{aligned}&\sup\left\{|u^\epsilon(x,t) - \langle x, e\rangle - \alpha\langle m\rangle^{-1}t| \mid t > 0\right\} \\ &\quad = \sup\left\{|\mathcal{U}^\epsilon(\langle x, e\rangle, t) - \langle x, e\rangle - \alpha\langle m\rangle^{-1}t| \mid t > 0\right\} \\ &\quad = \epsilon\langle m\rangle^{-1}\sup\left\{|P(s') - P(\langle x, e\rangle\epsilon^{-1})| \mid s' \in [0, r_e)\right\}.\end{aligned}$$

This proves that $\overline{m}_{\text{pl}}(e) = \langle m\rangle$ in this case. Since m is non-constant, P is non-constant and it follows that the number c given by

$$c = \langle m\rangle^{-1}\sup\left\{|P(s') - P(s'')| \mid s', s'' \in \mathbb{R}\right\}$$

is positive. Finally, taking the supremum in x, we obtain the desired bound. □

2.5.4 Mean Curvature Flow with a Periodic Mobility Coefficient

In this section, we study the cell problem (2.2) associated with (2.1) in the case when $a \equiv \text{Id}$. In this case, the curvature term becomes mean curvature, and the second-order operator appearing in the cell problem is simply the Laplacian. Accordingly, the analysis of invariant measures and correctors becomes much simpler.

Before beginning the analysis of the cell problem, let us note that, since $a \equiv \text{Id}$, the linear response coefficient $\overline{m}_{\text{pl}}$ can be computed explicitly. To see this, first, notice that if $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then $\mathscr{I}^a_e = \{\mathcal{L}^d\}$. This follows from the fact that $\mathcal{L}^d$ is clearly in $\mathscr{I}^a_e$ and Theorem 15 implies the invariant measure is unique. When $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, a straightforward computation shows that the function $r \mapsto \mu_e^r$ is determined by

$$\mu_e^r = \mathcal{H}^{d-1}(\mathbb{T}_e^{d-1}(r))^{-1}\mathcal{H}^{d-1} \restriction_{\mathbb{T}_e^{d-1}(r)}$$

By the results of Sections 2.5.1 and 2.5.3, it follows that $\overline{m}_{\text{pl}}$ is given by

$$\overline{m}_{\text{pl}}(e) = \int_{\mathbb{T}^d} m(y, e)\, dy \quad \text{for each } e \in S^{d-1}.$$

Note, in particular, that $\overline{m}_{\text{pl}}$ is as smooth as m.

When $e \notin \mathbb{R}\mathbb{Z}^d$, Proposition 5 shows that a $\mathcal{O}(\epsilon)$ rate of convergence holds in Theorem 10 provided the cell problem (2.2) has a smooth solution. In the next result, we show that a smooth solution exists provided $m(\cdot, e)$ is smooth enough and the direction $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ satisfies an arithmetic condition.

Proposition 11. *Fix $e \in S^{d-1}$. If $m(\cdot, e) \in H^s(\mathbb{T}^d)$ for some $s > \frac{d}{2} + \frac{1}{d-1} + 2$ and there is a $C_e \in (0, 1)$ and $\frac{1}{d-1} < \tau < s - \frac{d}{2} - 2$ such that*

$$\|k - \langle k, e\rangle e\| \geq C_e \|k\|^{-\tau} \quad \textit{for each } k \in \mathbb{Z}^d \setminus \{0\}, \tag{2.27}$$

then there is a solution $V_e \in C^2(\mathbb{T}^d)$ of the equation

$$m(\cdot, e) - tr((Id - e \otimes e)D^2 V_e) = \overline{m}(e). \tag{2.28}$$

Furthermore, V_e is the unique such solution among all functions $U \in L^2(\mathbb{T}^d)$ such that

$$\int_{\mathbb{T}^d} U(y)\, dy = \int_{\mathbb{T}^d} V_e(y)\, dy.$$

Concerning the generality of the assumption (2.27), see [54], where it is shown that $\mathcal{H}^{d-1}$-almost every $e \in S^{d-1}$ satisfies such an estimate for any given $\tau > \frac{1}{d-1}$. More precisely, if

$A(C_e, \tau)$ is the set of all such e, then there is a constant $B(d, \tau) > 0$ such that

$$\mathcal{H}^{d-1}(S^{d-1} \setminus A(C_e, \tau)) \leq B(d, \tau) C_e^{d-1}.$$

It should be noted that if e satisfies (2.27), then $e \notin \mathbb{R}\mathbb{Z}^d$.

Proof. We argue using Fourier series. Define $\hat{V}_e : \mathbb{Z}^d \to \mathbb{C}$ by $\hat{V}_e(0) = 0$ and

$$\hat{V}_e(k) = -\frac{\hat{m}(k)}{4\pi^2 \|k - \langle k, e \rangle e\|^2}.$$

Since $m(\cdot, e) \in H^s(\mathbb{T}^d)$ and $s > \tau + \frac{d}{2} + 2$, for each $i \in \{0, 1, 2\}$, we have

$$\sum_{k \in \mathbb{Z}^d} \|k\|^i |\hat{V}_e(k)| \leq C_e^{-2} \left(\sum_{k \in \mathbb{Z}^d} \|k\|^{2(\tau + i - s)} \right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}^d} \|k\|^{2s} |\hat{m}(k, e)|^2 \right)^{\frac{1}{2}} < \infty.$$

Thus, we can define $V_e \in C^2(\mathbb{T}^d)$ by

$$V_e(y) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \hat{V}_e(k) e^{i2\pi \langle k, y \rangle}$$

and then

$$(\mathrm{Id} - e \otimes e) D^2 V_e(y) = -\sum_{k \in \mathbb{Z}^d} 4\pi^2 (k \otimes k - \langle k, e \rangle e \otimes k) \hat{V}_e(k) e^{i2\pi \langle k, y \rangle}.$$

In particular, by construction, V_e is a solution of (2.28). Since $k - \langle k, e \rangle e \neq 0$ for each $k \in \mathbb{Z}^d$, a straightforward argument shows that V is unique up to the addition of a constant. □

Finally, we show that correctors need not exist.

Proposition 12. *For any $d \in \mathbb{N} \setminus \{1\}$, there is an $m = m(y) \in C^\infty(\mathbb{T}^d)$ and an $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ for which there is no C^2 solution of the cell problem* (2.28).

As in the existence result, we will prove non-existence by choosing a direction e with suitable arithmetic properties. To do this, we adapt an argument appearing in the lectures notes of Ghys [55].

Define $\lambda = \sum_{n=1}^{\infty} 10^{-n!}$. The next result shows this number is well-approximated by rationals.

Proposition 13. *There is a sequence $(p_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and a constant $C > 0$ such that*

$$\left|\lambda - p_k 10^{-k!}\right| \leq C 10^{-(k+1)!}, \quad C^{-1} 10^{k!} \leq |p_k| \leq C 10^{k!}$$

Proof. For each $N \in \mathbb{N}$, write $\sum_{n=1}^{N} 10^{-n!} = p_N 10^{-N!}$ for some $p_N \in \mathbb{N}$. The remainder is readily estimated as

$$\begin{aligned}\sum_{n=N+1}^{\infty} 10^{-n!} &= 10^{-(N+1)!} \sum_{j=0}^{\infty} 10^{(N+1)!-(N+j+1)!} \\ &\leq 10^{-(N+1)!} \sum_{j=0}^{\infty} 10^{-(N+j)} \leq C 10^{-(N+1)!}.\end{aligned}$$

Note, in addition, that p_N satisfies the estimates

$$\frac{1}{10} 10^{N!} \leq 10^{N!} \sum_{j=1}^{N} 10^{-j!} = p_N \leq \lambda \cdot 10^{N!}.$$

□

Fix $e \in S^{d-1} \cap \operatorname{span}(\{e_1, e_2\})$ such that $\frac{\langle e, e_2 \rangle}{\langle e, e_1 \rangle} = \lambda$. Note that this uniquely determines e (up to antipodal points).

With $(p_k)_{k\in\mathbb{N}}$ as in the last result and $\{e_1, e_2, \ldots, e_d\} \subseteq \mathbb{Z}^d$ denoting the standard orthonormal basis of $\mathbb{R}^d$, define $(j_k)_{k\in\mathbb{N}} \subseteq \mathbb{Z}^2$ as follows:

$$j_k = 10^{k!} e_1 + p_k e_2.$$

The previous estimate on λ implies the following one:

Proposition 14. *For each $k \in \mathbb{N}$,*

$$k\|j_k - \langle j_k, e\rangle e\|^2 \leq Ck10^{-2k\cdot k!}.$$

Proof. Let $e^{\perp} = -\langle e, e_2\rangle e_1 + \langle e, e_1\rangle e_2$ and notice that $\{e, e^{\perp}\}$ is orthonormal. Using the fact that $j_k \in \mathrm{span}(\{e_1, e_2\}) = \mathrm{span}(\{e, e^{\perp}\})$, we find

$$\begin{aligned}\|j_k - \langle j_k, e\rangle e\|^2 &= \left|\langle j_k, e^{\perp}\rangle\right|^2 \\ &= \left|-10^{k!}\langle e, e_2\rangle + p_k\langle e, e_1\rangle\right|^2 \\ &= 10^{2\cdot k!} \cdot \frac{1}{|\langle e, e_1\rangle|^2} \cdot \left|p_k 10^{-k!} - \frac{\langle e, e_2\rangle}{\langle e, e_1\rangle}\right|^2 \leq C10^{-2k\cdot k!}.\end{aligned}$$

□

Finally, define $m \in C(\mathbb{T}^d)$ by its Fourier series:

$$\hat{m}(k) = \begin{cases} \ell\|j_\ell - \langle j_\ell, e\rangle e\|^2, & \text{if } k \in \{j_\ell, -j_\ell\} \text{ for some } \ell \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Due to the rapid decay of its Fourier coefficients, $m \in C(\mathbb{T}^d)$, and, by symmetry, it is real-valued. In fact, m is smooth, as we note next.

Proposition 15. *For each $\beta > 0$, we have*

$$\sup\left\{\|k\|^{\beta}|\hat{m}(k)| \;\middle|\; k \in \mathbb{Z}^2\right\} < \infty.$$

In particular, $m \in C^{\infty}(\mathbb{T}^d)$.

Proof. Given $\beta > 0$, if $k \in \{-j_\ell, j_\ell\}$ for some $\ell \in \mathbb{N}$, then

$$\begin{aligned}\|k\|^\beta |\hat{m}(k)| &= \ell \|j_\ell\|^\beta \|j_\ell - \langle j_\ell, e\rangle e\|^2 \\ &\leq c\ell 10^{\beta \ell!} 10^{-2\ell \cdot \ell!} = c\ell 10^{-(2\ell-\beta)\cdot \ell!}.\end{aligned}$$

Thus, $\|k\|^\beta |\hat{m}(k)| \to 0$ as $\|k\| \to \infty$. □

Finally, we claim that (2.2) has no solution in the direction e.

Proposition 16. *There is no smooth function $V_e \in C^2(\mathbb{T}^d)$ satisfying* (2.2)*.*

Proof. A classical solution is clearly a distributional solution so it suffices to show that there is no distributional solution $V_e \in L^1(\mathbb{T}^d)$.

Suppose that $V_e \in L^1(\mathbb{T}^d)$ is a distributional solution of (2.2). Applying the Fourier transform to the solution, we find

$$\hat{V}_e(j_\ell) = -\frac{\ell \|j_\ell - \langle j_\ell, e\rangle e\|^2}{4\pi^2 \|j_\ell - \langle j_\ell, e\rangle e\|^2} = -\frac{\ell}{4\pi^2} \quad \text{for each } \ell \in \mathbb{N}.$$

This shows that the Fourier transform $\hat{V}_e$ is unbounded, contradicting the assumption that $V_e \in L^1(\mathbb{T}^d)$. □

It is not hard to show that, in fact, viscosity solutions of (2.2) are distributional solutions. Hence the proof actually shows that a *continuous* corrector need not exist in general.

2.6 The Surface Tension

In this section, we define the (macroscopic) surface tension associated with the energy

$$\mathcal{F}(u;\Omega) = \int_\Omega \left(\frac{1}{2}\langle a(y)Du, Du\rangle + W(y,u) \right) dy.$$

It is worth remarking that the proof given here applies more generally to the stationary ergodic setting (see [71]).

2.6.1 Finite-Volume Surface Tension

Given $e \in S^{d-1}$, we will compute the surface tension $\bar{\sigma}(e)$ by studying the optimal energy among configurations u that connect -1 to 1 along the e direction.

In what follows, it will be convenient to shift the coefficients in the energy. To that end, for $y_0 \in \mathbb{T}^d$, define the shifted functional $\mathcal{F}^{y_0}$ by

$$\mathcal{F}^{y_0}(u;\Omega) = \int_{\Omega} \left(\frac{1}{2} \langle a(y+y_0)Du, Du \rangle + W(y+y_0,u) \right) \, dy.$$

Fix a smooth function $\eta : \mathbb{R} \to [-1,1]$ such that

$$\int_{-\infty}^{\infty} \left(\frac{\Lambda}{2} \eta'(s)^2 + \overline{W}(\eta(s)) \right) \, ds < \infty, \qquad \lim_{s \to \pm\infty} \eta(s) = \pm 1.$$

For $e \in S^{d-1}$, we let $\eta_e : \mathbb{R}^d \to [-1,1]$ be the function defined by $\eta_e(x) = \eta(\langle x, e \rangle)$. Given a bounded open set $A \subseteq \mathbb{R}^d$ and $y \in \mathbb{T}^d$, define $\Sigma_\eta^y(e,A)$ by

$$\Sigma_\eta^y(e,A) = \min \left\{ \mathcal{F}^y(v;A) \mid v \in H^1(A;[-1,1]), \ v - \eta_e \in H_0^1(A) \right\}.$$

We refer to Σ_η^y as the *finite-volume surface tension.*

The main result of this section show that the optimal energy in $\mathbf{Q}^e(0,R)$ for configurations with boundary condition η_e, that is, the quantity $\Sigma_\eta^y(e, \mathbf{Q}^e(0,R))$, converges after a suitable rescaling.

Theorem 16. *There is a positively one-homogeneous convex function* $\bar{\sigma} : \mathbb{R}^d \to [0,\infty)$ *such* *that, for each* $e \in S^{d-1}$ *and every* $y \in \mathbb{T}^d$*, we have*

$$\bar{\sigma}(e) = \lim_{R \to \infty} R^{1-d} \Sigma_\eta^y(e, \mathbf{Q}^e(0, R)). \tag{2.29}$$

Before proceeding further, we record a useful observation. Here and henceforth, we define the constant $C(\eta) > 0$ by

$$C(\eta) = \int_{-\infty}^{\infty} \left(\frac{\Lambda \eta'(s)^2}{2} + \overline{W}(\eta(s)) \right) ds.$$

Proposition 17. *If* $e \in S^{d-1}$*,* $y \in \mathbb{T}^d$*,* $E \subseteq \langle e \rangle^\perp$ *is open and bounded, and* $I \subseteq \mathbb{R}$ *is open and bounded, then*

$$0 \leq \Sigma_\eta^y(e, E \oplus_e I) \leq C(\eta) \mathcal{H}^{d-1}(E).$$

Proof. Using η_e itself as a candidate, we obtain an upper bound:

$$0 \leq \Sigma_\eta^y(e, E \oplus_e I) \leq \mathcal{F}^y(\eta_e; E \oplus_e I).$$

It only remains to estimate $\mathcal{F}^y(\eta_e; E \oplus_e I)$. Since $a \leq \Lambda \mathrm{Id}$, $W(y, u) \leq \overline{W}(u)$, and η_e only varies in the e direction, Fubini's Theorem readily implies

$$\mathcal{F}^y(\eta_e; E \oplus_e I) \leq C(\eta) \mathcal{H}^{d-1}(E).$$

□

2.6.2 Averaging

In what follows, it will be useful to think of the finite-volume surface tension as a function of three variables rather than two.

Definition 5. *Given* η *as above and* $y \in \mathbb{T}^d$*, for* $e \in S^{d-1}$*, open and bounded* $E \subseteq \langle e \rangle^\perp$*,* *and $h>0$, we define the quantity $\tilde{\Sigma}^y_\eta(e,E,h)$ by*

$$\tilde{\Sigma}^y_\eta(e,E,h) = \Sigma^y_\eta(e, E \oplus_e (-h,h)).$$

The next proposition gives the essential properties of $\tilde{\Sigma}^y_\eta$. In particular, in the language of Dal Maso and Modica [38], $\tilde{\Sigma}^y_\eta$ is a sub-additive process.

Proposition 18. *For each fixed $e \in S^{d-1}$ and $h>0$, the function $E \mapsto \tilde{\Sigma}^y_\eta(e,E,h)$ satisfies:*

(i) If $E, E_1, \dots, E_N \subseteq \langle e\rangle^\perp$ are all open and bounded, $\{E_1,\dots,E_N\}$ is pairwise disjoint, $\bigcup_{i=1}^N E_i \subseteq E$, and $\mathcal{H}^{d-1}(E \setminus \bigcup_{i=1}^N E_i) = 0$, then

$$\tilde{\Sigma}^y_\eta(e,E,h) \leq \sum_{i=1}^N \tilde{\Sigma}^y_\eta(e,E_i,h). \tag{2.30}$$

(ii) $\tilde{\Sigma}^y_\eta(e,\cdot,h)$ is uniformly bounded in the following sense:

$$0 \leq \tilde{\Sigma}^y_\eta(e,E,h) \leq C(\eta)\mathcal{H}^{d-1}(E). \tag{2.31}$$

(iii) For each $E \subseteq \langle e\rangle^\perp$ and each $y' \in \langle e\rangle^\perp$, the following equation holds:

$$\tilde{\Sigma}^{y+y'}_\eta(e,E,h) = \tilde{\Sigma}^y_\eta(e,E+y',h).$$

Proof. First, observe that (ii) follows directly from Proposition 17, and (iii) is an immediate consequence of the definitions.

Next, we prove (i). Suppose $E, E_1, \dots, E_N \subseteq \langle e\rangle^\perp$ are given and satisfy the assumptions. For each $i \in \{1,2,\dots,N\}$, pick $u_i \in H^1(E_i \oplus_e (-h,h); [-1,1])$ such that

(i) $\mathcal{F}^y(u_i, E_i \oplus_e (-h,h)) = \tilde{\Sigma}^y_\eta(e,E_i,h)$

(ii) $u_i - \eta_e \in H^1_0(E_i \oplus_e (-h,h))$.

Define a new function $u : E \oplus_e (-h,h) \to [-1,1]$ by

$$u(x) = \begin{cases} u_i(x), & x \in E_i \oplus_e (-h,h) \\ 0, & \text{otherwise.} \end{cases}$$

By the choice of boundary conditions, $u - \eta_e \in H_0^1(E \oplus_e (-h,h))$. Moreover,

$$\begin{aligned} \tilde{\Sigma}^y_\eta(e,E,h) &\leq \mathcal{F}^y(u, E \oplus_e (-h,h)) \\ &= \sum_{i=1}^N \mathcal{F}^y(u_i, E_i \oplus_e (-h,h)) = \sum_{i=1}^N \tilde{\Sigma}^y_\eta(e,E_i,h). \end{aligned}$$

This establishes (i). $\square$

We now use the sub-additive ergodic theorem to average out the variations in the medium in directions perpendicular to e. In what follows, we will say that a Lebesgue measurable function $f : \mathbb{T}^d \to \mathbb{R}$ is $\{\tau_{x'}\}_{x' \in \langle e \rangle^\perp}$-invariant for some $e \in S^{d-1}$ if

$$f(y + x') = f(y) \quad \text{for each } y \in \mathbb{T}^d,\ x' \in \langle e \rangle^\perp.$$

Proposition 19. *For each $e \in S^{d-1}$ and $h > 0$, there is a $\{\tau_{x'}\}_{x' \in \langle e \rangle^\perp}$-invariant function $y \mapsto \tilde{\sigma}^y(e,h)$ and a $\{\tau_{x'}\}_{x' \in \langle e \rangle^\perp}$-invariant Borel set $A_h(e) \subseteq \mathbb{T}^d$ such that $\mathcal{L}^d(A_h(e)) = 1$ and, for each $y \in A_h(e)$,*

$$\tilde{\sigma}^y(e,h) = \lim_{R \to \infty} R^{1-d} \tilde{\Sigma}^y_\eta(e, Q^e(0,R), h).$$

Proof. Apply the multi-parameter sub-additive ergodic theorem to the probability space $(\Omega, \mathbb{P}) = (\mathbb{T}^d, \mathcal{L}^d)$ with measure-preserving transformations $\{\tau_y\}_{y \in \mathbb{T}^d}$ given by translation; see [71, Appendix B]. $\square$

2.6.3 The Limit $h \to \infty$

We now study the limit $h \to \infty$ of $\tilde{\Sigma}^y_\eta(e, E, h)$. We begin by defining the corresponding quantity.

Definition 6. *Given $e \in S^{d-1}$ and a bounded, open $E \subseteq \langle e\rangle^\perp$, we define the function $y \mapsto \tilde{\Sigma}^y_\eta(e, E, \infty)$ by*

$$\tilde{\Sigma}^y_\eta(e, E, \infty) = \min \left\{ \mathcal{F}^y(u; E \oplus_e \mathbb{R}) \mid -1 \leq u \leq 1, \ u = \eta_e \text{ on } \partial E \oplus_e \mathbb{R} \right\}.$$

In the main result of this section, we prove that $\lim_{h\to\infty} \tilde{\Sigma}^y_\eta(e, E, h) = \tilde{\Sigma}^y_\eta(e, E, \infty)$.

Before we proceed further, we remark that $\tilde{\Sigma}^y_\eta(e, E, \infty)$ satisfies its own version of Proposition 18. Therefore, the following analogue of Proposition 19 holds:

Proposition 20. *For each $e \in S^{d-1}$, there is a $\{\tau_{x'}\}_{x' \in \langle e\rangle^\perp}$-invariant function $\tilde{\sigma}(e, \infty)$ and a $\{\tau_{x'}\}_{x' \in \langle e\rangle^\perp}$-invariant Borel set $A_\infty(e) \subseteq \mathbb{T}^d$ such that $\mathcal{L}^d(A_\infty(e)) = 1$ and, for each $y \in A_\infty(e)$,*

$$\tilde{\sigma}^y(e, \infty) = \lim_{R\to\infty} R^{1-d} \tilde{\Sigma}^y_\eta(e, Q^e(0, R), \infty). \tag{2.32}$$

It only remains to analyze the limit $h \to \infty$. We start by observing that $h \mapsto \tilde{\Sigma}^y_\eta(e, E, h)$ is *almost* non-increasing.

Proposition 21. *Fix $e \in S^{d-1}$, $y \in \mathbb{T}^d$, and $E \subseteq \langle e\rangle^\perp$ bounded and open. If $h_1 > h_2$, then*

$$\tilde{\Sigma}^y_\eta(e, E, h_1) \leq \tilde{\Sigma}^y_\eta(e, E, h_2) + \mathcal{H}^{d-1}(E) e(h_2), \tag{2.33}$$

where

$$e(h) = \int_{\{|s|>h\}} \left(\frac{\Lambda \eta'(s)^2}{2} + \overline{W}(\eta(s)) \right) ds.$$

In particular, for each $e \in S^{d-1}$, $R > 0$, and $y \in \mathbb{T}^d$, the limit $\lim_{h\to\infty} \tilde{\Sigma}^y_\eta(e, Q^e(0, R), h)$ exists.

Proof. Fix $\epsilon > 0$. Choose a $u : E \oplus_e (-h_2, h_2) \to [-1, 1]$ such that $u - \eta_e \in H_0^1(E \oplus_e (-h_2, h_2))$ and

$$\mathcal{F}^y(u; E \oplus_e (-h_2, h_2)) = \tilde{\Sigma}_\eta^y(e, E, h_2).$$

Define $\tilde{u} : E \oplus (-h_1, h_1) \to [-1, 1]$ by

$$\tilde{u}(x) = \begin{cases} u(x), & x \in E \oplus_e (-h_2, h_2) \\ \eta_e(x), & \text{otherwise.} \end{cases}$$

Then $\tilde{u} - \eta_e \in H_0^1(E \oplus_e (-h_1, h_1))$ and

$$\begin{aligned} \tilde{\Sigma}_\eta^y(e, E, h_1) &\leq \mathcal{F}^y(\tilde{u}, E \oplus_e (-h_1, h_1)) \\ &= \mathcal{F}^y(u, E \oplus (-h_2, h_2)) \\ &\quad + \mathcal{H}^{d-1}(E) \int_{\{|t| \in (h_2, h_1)\}} \left(\frac{\Lambda}{2} \eta'(t)^2 + \overline{W}(q(t)) \right) dt \\ &\leq \tilde{\Sigma}_\eta^y(e, E, h_2) + \mathcal{H}^{d-1}(E) e(h_2). \end{aligned}$$

Thus, we obtain (2.33).

Finally, sending $h_1 \to \infty$ with h_2 fixed and then sending $h_2 \to \infty$, we find

$$\begin{aligned} \limsup_{h_1 \to \infty} \tilde{\Sigma}_\eta^y(e, E, h_1) &\leq \liminf_{h_2 \to \infty} \left(\tilde{\Sigma}_\eta^y(e, E, h_2) + \mathcal{H}^{d-1}(E) e(h_2) \right) \\ &= \liminf_{h_2 \to \infty} \tilde{\Sigma}_\eta^y(e, E, h_2). \end{aligned}$$

This proves $\lim_{h \to \infty} \tilde{\Sigma}_\eta^y(e, E, h)$ exists. □

Finally, we prove the result that was promised at the beginning of this sub-section:

Proposition 22. *For each $e \in S^{d-1}$ and bounded, open $E \subseteq \langle e \rangle^\perp$, we have*

$$\tilde{\Sigma}_\eta^y(e, E, \infty) = \lim_{h \to \infty} \tilde{\Sigma}_\eta^y(e, E, h). \tag{2.34}$$

Proof. If $h > 0$ and $u \in H^1(E \oplus_e (-h,h);[-1,1])$ equals η_e on $\partial(E \oplus_e (-h,h))$, then the function $\tilde{u} \in H^1_{\mathrm{loc}}(E \oplus_e \mathbb{R})$ given by

$$\tilde{u}(x) = \begin{cases} u(x), & |\langle x, e\rangle| \leq h, \\ \eta_e(x), & \text{otherwise}, \end{cases}$$

satisfies $\tilde{u} - \eta_e \in H^1_0(E \oplus_e \mathbb{R})$. Thus,

$$\tilde{\Sigma}^y_\eta(e, E, \infty) \leq \mathcal{F}^y(\tilde{u}; E \oplus_e \mathbb{R}) \leq \mathcal{F}^y(u; E \oplus_e (-h,h)) + \mathcal{H}^{d-1}(E)e(h).$$

Since u was arbitrary, we deduce that $\tilde{\Sigma}^y_\eta(e, E, \infty) \leq \tilde{\Sigma}^y_\eta(e, E, h) + \mathcal{H}^{d-1}(E)e(h)$. Sending $h \to \infty$, we conclude $\tilde{\Sigma}^y_\eta(e, E, \infty) \leq \lim_{h \to \infty} \tilde{\Sigma}^y_\eta(e, E, h)$.

To obtain the complementary inequality, let $u \in H^1_{\mathrm{loc}}(E \oplus_e \mathbb{R};[-1,1])$ be any function attaining the minimum in (2.34). We will show that it is possible to appropriately truncate u without changing its energy too much.

First, observe that for each $\delta > 0$,

$$\lim_{R \to \infty} \mathcal{L}^d(\{x \in E \oplus_e [R, +\infty) \mid |u(x) - 1| > \delta\}) = 0$$
$$\lim_{R \to \infty} \mathcal{L}^d(\{x \in E \oplus_e (-\infty, -R] \mid |u(x) + 1| > \delta\} = 0.$$

This is a consequence of the Poincaré inequality, which in this setting states

$$\|u - \eta_e\|_{L^2(E \oplus_e \mathbb{R})} \leq \|Du - Dq_e\|_{L^2(E \oplus_e \mathbb{R})}.$$

For each $n \in \mathbb{N}$, fix a smooth function $f_n : \mathbb{R} \to [0,1]$ such that $f_n \equiv 1$ in $[-n,n]$, $f_n \equiv 0$ in $\mathbb{R} \setminus [-(n+1), n+1]$, and $|f_n'| \leq 2$. Let $u_n \in H^1_{\mathrm{loc}}(E \oplus_e \mathbb{R})$ be the function defined by

$$u_n(x) = f_n(\langle x, e\rangle)u(x) + (1 - f_n(\langle x, e\rangle))\eta_e(x).$$

We readily obtain the following bounds on the energy of u_n:

$$\begin{aligned}\mathcal{F}^y(u_n; E \oplus_e (-(n+1), n+1)) \leq \mathcal{F}^y(u; E \oplus_e (-n, n)) \\ + \frac{\Lambda}{2} \int_{E \oplus_e \{n < |s| < n+1\}} |Du(x)|^2 \, dx \\ + \frac{\Lambda}{2} \mathcal{H}^{d-1}(E) \int_{\{n \leq |s| \leq n+1\}} q'(s)^2 \, ds \\ + 2 \int_{E \oplus_e \{n < |s| < n+1\}} |u(x) - \eta_e(x)|^2 \, dx \\ + \int_{E \oplus_e \{n < |s| < n+1\}} W(u_n(x)) \, dx.\end{aligned}$$

Since $u, \eta_e \to \pm 1$ in measure as $\langle x, e \rangle \to \pm\infty$, we find

$$\mathcal{F}^y(u_n; E \oplus_e (-(n+1), n+1)) \leq \tilde{\Sigma}^y_\eta(e, E, \infty) + o(1).$$

as $n \to \infty$. Therefore, since $u_n = \eta_e$ on $E \oplus_e \{-(n+1), n+1\}$, we conclude

$$\lim_{h \to \infty} \tilde{\Sigma}^y_\eta(e, E, h) \leq \lim_{n \to \infty} \mathcal{F}^y(u_n; E \oplus_e (-(n+1), n+1)) \leq \tilde{\Sigma}^y_\eta(e, E, \infty).$$

□

2.6.4 Infinite-Volume Surface Tension

We now identify the infinite-volume surface tension $\bar{\sigma}$. To start with, we treat this as two different functions $\bar{\sigma}^*$ and $\bar{\sigma}_*$ in $\mathbb{T}^d$; later, it will become apparent these functions coincide and are constant.

Definition 7. *The upper and lower infinite-volume surface tensions are the functions* $\bar{\sigma}^*, \bar{\sigma}_*$:

$\mathbb{T}^d \to [0,\infty)$ *given by*

$$\bar{\sigma}^*(e,y) = \limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e,Q^e(0,R),R), \quad \bar{\sigma}_*(e,y) = \liminf_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e,Q^e(0,R),R). \tag{2.35}$$

Our first observation is these quantities are constant functions in $\mathbb{T}^d$:

Theorem 17. *For each $e \in S^{d-1}$ and every $y \in \mathbb{T}^d$, we have*

$$\bar{\sigma}^*(e,y) = \bar{\sigma}^*(e,0) \quad \textit{and} \quad \bar{\sigma}_*(e,y) = \bar{\sigma}_*(e,0).$$

Theorem 17 will be proved in two steps. In the first, we note that $\bar{\sigma}^*(e,\cdot)$ and $\bar{\sigma}_*(e,\cdot)$ are $\{\tau_{x'}\}_{x'\in\langle e\rangle^\perp}$-invariant. This step follows from what we already proved in Section 2.6.2, especially Propositions 18 and 21. In the second step, we show that $\bar{\sigma}^*(e,\cdot)$ and $\bar{\sigma}_*(e,\cdot)$ are invariant under the action of translations in the e direction. The proof of this is very similar to that of Proposition 22.

To simplify the proof of Theorem 17, we begin with the perpendicular directions.

Proposition 23. *For each $e \in S^{d-1}$, $\bar{\sigma}^*(e,\cdot)$ and $\bar{\sigma}_*(e,\cdot)$ are $\{\tau_{x'}\}_{x'\in\langle e\rangle^\perp}$-invariant.*

Proof. We will prove that $\bar{\sigma}_*(e,y+x') = \bar{\sigma}_*(e,y)$ if $y \in \mathbb{T}^d$ and $x \in \langle e\rangle^\perp$. The corresponding proof for $\bar{\sigma}^*$ follows similarly.

Supppose $y \in \mathbb{T}^d$ and $x' \in \langle e\rangle^\perp$. We will show that $\bar{\sigma}_*(e,y+x') \leq \bar{\sigma}_*(e,y)$. Given any $R > 0$, Proposition 18 implies

$$\tilde{\Sigma}^{y+x'}_\eta(e,Q^e(0,R+|x'|_\infty),R+|x'|_\infty) = \tilde{\Sigma}^y_\eta(e,Q^e(x',R+|x'|_\infty),R+|x'|_\infty). \tag{2.36}$$

Since $Q^e(x',R+|x'|_\infty) \supseteq Q^e(0,R)$, we use (2.30) and (2.31) from Proposition 18 to find

$$\tilde{\Sigma}^y_\eta(e,Q^e(x',R+|x'|_\infty),R+|x'|_\infty) \leq \tilde{\Sigma}^y_\eta(e,Q^e(0,R),R+|x'|_\infty) + C(\eta)R^{d-1}o(1). \tag{2.37}$$

as $R \to \infty$. Finally, appealing to (2.33) from Proposition 21 yields

$$\tilde{\Sigma}_\eta^y(e, Q^e(x', R + |x'|_\infty), R + |x'|_\infty) \leq \tilde{\Sigma}_\eta^y(e, Q^e(0, R), R) + R^{d-1}(e(R) + C(\eta)o(1)). \quad (2.38)$$

Combining (2.36) and (2.38), dividing by R^{d-1}, and sending $R \to \infty$, we obtain $\bar{\sigma}_*(e, y + x') \leq \bar{\sigma}_*(e, y)$.

Replacing x' with $-x'$ and y with $y+x'$ yields $\bar{\sigma}_*(e, y) \leq \bar{\sigma}_*(e, y+x')$. Therefore, equality holds. □

2.6.5 The Normal Direction

It remains to consider normal directions. That is, Theorem 17 is proved as soon as we establish that $t \mapsto \bar{\sigma}_*(e, y + te)$ is independent of $t \in \mathbb{R}$. We prove this next using the fundamental estimate of Γ-convergence (cf. [5] or [71, Appendix A]).

Proposition 24. *Given any $e \in S^{d-1}$, $y \in \mathbb{T}^d$, and $t \in \mathbb{R}$, we have*

$$\bar{\sigma}^*(e, y + te) = \bar{\sigma}^*(e, y) \quad \text{and} \quad \bar{\sigma}_*(e, y + te) = \bar{\sigma}_*(e, y).$$

Proof. Fix $y \in \Omega$ and $t \in \mathbb{R}$. We will show that

$$\bar{\sigma}_*(e, y) \leq \bar{\sigma}_*(e, y + te).$$

As in the last proof, this implies that equality actually holds (as we can replace y by $y + te$ and t by $-t$). The arguments for $\bar{\sigma}^*$ are similar, hence omitted.

It is convenient to introduce a free parameter $\alpha \in (0, 1)$. In the final step of the proof, we will send $\alpha \to 1^-$.

For each $R > 0$, let $\tilde{v}_R \in H^1(\mathbf{Q}^e(0, \alpha R); [-1, 1])$ be a minimizer of the functional

$\mathcal{F}^{y+te}(\cdot;\mathbf{Q}^e(0,\alpha R))$ subject to the boundary conditions $\tilde{v}_R = \eta_e$, that is,

$$\tilde{v}_R - \eta_e \in H^1_0(\mathbf{Q}^e(0,\alpha R)), \quad \mathcal{F}^{y+te}(\tilde{v}_R;\mathbf{Q}^e(0,\alpha R)) = \tilde{\Sigma}^{y+te}_\eta(e, Q^e(0,\alpha R),\alpha R).$$

In order to compare $\tilde{\Sigma}^y_\eta(e,Q(0,R),R)$ and $\tilde{\Sigma}^{y+te}_\eta(e,Q(0,\alpha R),\alpha R)$, we shift perspectives by defining the function v_R in $\mathbf{Q}^e(te,\alpha R)$ by

$$v_R(x) = \tilde{v}_R(x - te).$$

Notice that v_R minimizes $\mathcal{F}^y(\cdot;\mathbf{Q}^e(te,\alpha R))$ with the boundary condition $T_{te}\eta_e$ given by $T_{te}\eta_e(x) = \eta_e(x-te)$. We extend v_R to $\mathbb{R}^d$ by setting $v_R(x) = T_{te}\eta_e(x)$ if $x \in \mathbb{R}^d \setminus \mathbf{Q}^e(te,\alpha R)$.

Finally, before we apply the fundamental estimate, it is convenient to move to macroscopic coordinates. We let $\epsilon = R^{-1}$ and define a new function $v^\epsilon : \mathbb{R}^d \to [-1,1]$ by the following rule:

$$v^\epsilon(x) = v_R\left(\frac{x}{\epsilon}\right).$$

The effect of the definition is this: v^ϵ minimizes $\mathcal{F}^y_\epsilon(\cdot;\mathbf{Q}^e(\epsilon te,\alpha))$ subject to the boundary condition $T_{\epsilon te}(\eta^\epsilon_e)$ on $\partial Q^e(\epsilon te,\alpha)$, where $\eta^\epsilon_e(x) = \eta(\epsilon^{-1}\langle x, e\rangle)$.

We will now apply the fundamental estimate with the open sets U, U', and V defined as follows:

$$U = \mathbf{Q}^e(0,\alpha),\ U' = \mathbf{Q}^e\left(0,1\right),\ V = \mathbf{Q}^e(0,1) \setminus \mathbf{Q}^e(0,\alpha).$$

We will work with the functions $(v_\epsilon)_{\epsilon>0}$ and $(\eta^\epsilon_e)_{\epsilon>0}$. Observe that

$$\lim_{\epsilon\to 0^+} \|v^\epsilon - \eta^\epsilon_e\|_{L^1(V)} = 0.$$

Thus, by the fundamental estimate, there is a function $\tilde{e} : (0,\infty) \to (0,\infty)$ such that $\lim_{\epsilon\to 0^+} \tilde{e}(\epsilon) = 0$ and a family of cut-off functions $(\psi_\epsilon)_{\epsilon>0} \subseteq C^\infty_c(U';[0,1])$ satisfying $\psi_\epsilon \equiv 1$

in U such that

$$\mathcal{F}_\epsilon^y(\psi_\epsilon v_\epsilon + (1-\psi_\epsilon)q_e^\epsilon; U \cup V) \leq \mathcal{F}_\epsilon^y(v_\epsilon; U') + \mathcal{F}_\epsilon^y(q_e^\epsilon; V) + \tilde{e}(\epsilon).$$

Now observe that we can make the following simplifications:

$$\begin{aligned}\mathcal{F}_\epsilon^y(v_\epsilon; U') &= R^{1-d}\mathcal{F}^{y+te}(\tilde{v}_R; \mathbf{Q}^e(0,\alpha R)) + R^{1-d}\mathcal{F}^y(v_R; RU' \setminus \mathbf{Q}^e(te, \alpha R)) \\ &= R^{1-d}\tilde{\Sigma}_\eta^{y+te}(0, Q^e(0,\alpha R), \alpha R) + \omega(\alpha) + \eta(\alpha, R),\end{aligned}$$

where

$$\begin{aligned}\omega(\alpha) &\leq (1-d)(1-\alpha)C(\eta) \\ \eta(\alpha, R) &\leq \alpha^{d-1}\int_{\{|s|\geq \alpha R\}} \left(\frac{\Lambda}{2}q'(s)^2 + \overline{W}(q(s))\right) ds.\end{aligned}$$

Similarly, $\mathcal{F}_\epsilon^y(\eta_e^\epsilon; V) = \nu(\alpha) + \gamma(\alpha, R)$, where

$$\begin{aligned}\nu(\alpha) &\leq (1-d)\,(1-\alpha)\,C(\eta), \\ \gamma(\alpha, R) &\leq \alpha^{d-1}\int_{\{|s|\geq \alpha R - |t|\}} \left(\frac{\Lambda}{2}q'(s)^2 + \overline{W}(q(s))\right) ds.\end{aligned}$$

Sending $R \to \infty$ and observing that $\lim_{R\to\infty}(\eta(\alpha, R) + \gamma(\alpha, R)) = 0$, we obtain

$$\begin{aligned}\liminf_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), R) &\leq \liminf_{\epsilon\to 0^+} \mathcal{F}_\epsilon^y(\psi_\epsilon v_\epsilon + (1-\psi_\epsilon)q_e^\epsilon; U \cup V) \\ &\leq \alpha^{d-1}\liminf_{T\to\infty} T^{1-d}\tilde{\Sigma}_\eta^{y+te}(0, Q^e(0,T), T) \\ &\quad + \omega(\alpha) + \nu(\alpha).\end{aligned}$$

Since $\lim_{\alpha \to 1^-} (\omega(\alpha) + \nu(\alpha)) = 0$ and $\alpha \in (0,1)$ was arbitrary, we conclude

$$\begin{aligned} \bar{\sigma}_*(e,y) &= \liminf_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0,R), R) \\ &\leq \liminf_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^{y+te}(0, Q^e(0,R), R) \\ &= \bar{\sigma}_*(e, y+te). \end{aligned}$$

□

2.6.6 Thermodynamic Limit

Now we tie all the pieces together. While we know that the upper and lower surface tensions $\bar{\sigma}^*(e)$ and $\bar{\sigma}_*(e)$ are constant (i.e., independent of shifts in the coefficients), we do not know that they coincide. This is where Propositions and 19 and 20 enter the picture.

The main result of this section follows:

Theorem 18. *Given $e \in S^{d-1}$, there is a Lebesgue measurable set $A(e) \subseteq \mathbb{T}^d$ with full measure $\mathcal{L}^d(A(e)) = 1$ such that, if $y \in A(e)$, then*

$$\lim_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0,R), R) \quad \textit{exists}$$

and, for each $\kappa > 0$,

$$\begin{aligned} \lim_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0,R), \infty) &= \lim_{h \to \infty} \limsup_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0,R), h) \\ &= \lim_{h \to \infty} \liminf_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0,R), h) \\ &= \lim_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0,R), \kappa R). \end{aligned}$$

In particular, $\bar{\sigma}^*(e) = \bar{\sigma}_*(e)$.

In order to highlight the main ideas without delving too far into technicalities, we will only prove the theorem in the case that $e \notin \mathbb{R}\mathbb{Z}^d$. (See the Notes for a discussion of the rational case.)

In what follows, the following lemma will be helpful:

Lemma 3. *Suppose that* $e \in S^{d-1} \backslash \mathbb{R}\mathbb{Z}^d$. *For each* $n, m \in \mathbb{N}$, *denote by* $\tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n), m)$ *and* $\tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n), \infty)$ *the averages*

$$\tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n), m) = \int_{\mathbb{T}^d} \tilde{\Sigma}_\eta^y(e, Q^e(0, 2^n), m)\, dy,$$
$$\tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n, \infty) = \int_{\mathbb{T}^d} \tilde{\Sigma}_\eta^y(e, Q^e(0, 2^n), \infty)\, dy.$$

For each $n, m \in \mathbb{N}$, *we have*

$$\tilde{\Sigma}_\eta^{avg}(e, Q(0, 2^{n+1}), m) \leq 2^{d-1} \tilde{\Sigma}_\eta^{avg}(e, Q(0, 2^n), m)$$

and the following limit holds:

$$\tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n), \infty) = \lim_{m \to \infty} \tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n), m).$$

Furthermore, there is a Lebesgue measurable set $A(e) \subseteq \mathbb{T}^d$ *such that*

(i) $\mathcal{L}^d(A(e)) = 1$

(ii) If $y \in A(e)$, *then*

$$\begin{aligned}
\tilde{\sigma}^y(e, m) &= \lim_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0, R), m) \\
&= \lim_{n \to \infty} 2^{n(1-d)} \tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n), m), \\
\tilde{\sigma}^y(e, \infty) &= \lim_{R \to \infty} R^{1-d} \tilde{\Sigma}_\eta^y(e, Q^e(0, R), \infty) \\
&= \lim_{n \to \infty} 2^{n(1-d)} \tilde{\Sigma}_\eta^{avg}(e, Q^e(0, 2^n). \infty).
\end{aligned}$$

The lemma shows that averaging preserves, and even improves, the sub-additivity properties of the finite-volume surface tension. As we will see, averaging allows us to verify one of the crucial inequalities in the proof of Theorem 18.

For the most part, Lemma 3 follows from what we have already proven and elementary properties of integration. We defer the proof to the end of the sub-section and proceed to the proof of the thermodynamic limit.

Proof of Theorem 18, irrational case. We begin by invoking Lemma 3, thereby obtaining a full-measure set $A(e) \subseteq \mathbb{T}^d$. Define $\tilde{\Sigma}_\eta^{\text{avg}}$ as in the lemma. We will prove that if $y \in A(e)$, then the following limits all exist and satisfy:

$$\lim_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), \infty) = \lim_{h\to\infty} \limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), h), \tag{2.39}$$

$$= \lim_{h\to\infty} \liminf_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), h)$$

$$\lim_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), \infty) = \lim_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), \kappa R). \tag{2.40}$$

Here $\kappa > 0$ is arbitrary. Notice that once this is done, we set $\kappa = 1$ to conclude that each of these quantities equals $\bar{\sigma}^*(e)$ and $\bar{\sigma}_*(e)$, hence $\bar{\sigma}^*(e) = \bar{\sigma}_*(e)$.

Assume henceforth that $y \in A(e)$.

To establish (2.39), we first prove

$$\lim_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), \infty) \leq \lim_{h\to\infty} \liminf_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), h). \tag{2.41}$$

Indeed, by (2.33) and (2.34),

$$R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), \infty) \leq R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0,R), h) + e(h).$$

Thus, sending $R \to \infty$ and then $h \to \infty$, we obtain (2.41).

Next, we prove the inequality complementary to (2.41), that is, we show that

$$\lim_{h\to\infty}\limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), h) \le \lim_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q(0,R), \infty). \tag{2.42}$$

We begin by observing that if $h_1 > h_2$, then

$$\limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), h_1) \le \limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), h_2) + e(h_2).$$

Thus, arguing as in Proposition 21, we see that

$$\lim_{h\to\infty}\limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), h) \quad \text{exists}$$

for every $y \in \mathbb{T}^d$.

Since $y \in A(e)$, the following equations hold:

$$\lim_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), \infty) = \lim_{n\to\infty} 2^{n(1-d)}\tilde{\Sigma}^{\mathrm{avg}}_\eta(e, Q^e(0,2^n), \infty)$$

$$\begin{aligned}\lim_{h\to\infty}\limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q(0,R), h)\\ = \lim_{m\to\infty}\lim_{n\to\infty} 2^{n(1-d)}\tilde{\Sigma}^{\mathrm{avg}}_\eta(e, Q(0,2^n), m)).\end{aligned}$$

Thus, to obtain an inequality between the left-hand sides it suffices to complete the easier task of comparing the right-hand sides.

To proceed, we invoke Lemma 3 to find

$$2^{(n+1)(1-d)}\tilde{\Sigma}^{\mathrm{avg}}_\eta(e, Q^e(0,2^{n+1}), m) \le 2^{n(1-d)}\tilde{\Sigma}^{\mathrm{avg}}_\eta(e, Q^e(0,2^n), m).$$

Consequently, for each fixed $k \in \mathbb{N}$,

$$\lim_{n\to\infty} 2^{n(1-d)}\tilde{\Sigma}_\eta^{\text{avg}}(e, Q^e(0, 2^n), m) \le 2^{k(1-d)}\tilde{\Sigma}_\eta^{\text{avg}}(e, Q^e(0, 2^k), m).$$

Sending $m \to \infty$ while k remains fixed, we use the lemma to conclude

$$\lim_{m\to\infty} \lim_{n\to\infty} 2^{n(1-d)}\tilde{\Sigma}_\eta^{\text{avg}}(e, Q^e(0, 2^n), m) \\ \le 2^{k(1-d)}\tilde{\Sigma}_\eta^{\text{avg}}(e, Q^e(0, 2^k), \infty).$$

Taking $k \to \infty$, this becomes

$$\lim_{m\to\infty} \lim_{n\to\infty} 2^{n(1-d)}\tilde{\Sigma}_\eta^{\text{avg}}(e, Q(0, 2^n), m) \\ \le \lim_{k\to\infty} 2^{k(1-d)}\tilde{\Sigma}_\eta^{\text{avg}}(e, Q^e(0, 2^k), \infty)$$

Appealing to the observation in the previous paragraph, we conclude (2.42) holds. Now (2.39) follows from (2.41) and (2.42).

We proceed to the proof of (2.40). Again, we break the equality into two inequalities. Fix $h > 0$. Recalling (2.33) and sending $R \to \infty$, we find

$$\limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0, R), \kappa R) \le \limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q^e(0, R), h) + e(h).$$

Sending $h \to \infty$ and appealing to (2.39), this becomes

$$\limsup_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q(0, R), \kappa R) \le \lim_{R\to\infty} R^{1-d}\tilde{\Sigma}_\eta^y(e, Q(0, R), \infty).$$

To deduce the opposite inequality, first observe that, by passing to the limit $h \to \infty$ in

(2.33), we obtain

$$R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), \infty) \leq R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), \kappa R) + e(\kappa R).$$

Thus, we conclude, after sending $R \to \infty$,

$$\lim_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), \infty) \leq \liminf_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), \kappa R).$$

Therefore, $\lim_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), \kappa R)$ exists and (2.40) holds. $\square$

Now that Theorem 18 is proved, Theorem 16 follows immediately.

Proof of Theorem 16. Theorem 18 proves that $\bar{\sigma}^*(e) = \bar{\sigma}_*(e)$. Let $\bar{\sigma}(e)$ be this number, i.e., $\bar{\sigma}(e) = \bar{\sigma}^*(e) = \bar{\sigma}_*(e)$.

Equivalently, the equality $\bar{\sigma}^*(e) = \bar{\sigma}_*(e)$ implies that, for each $y \in \mathbb{T}^d$, the quantity $R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), R)$ converges to $\bar{\sigma}(e)$ as $R \to \infty$. Thus, recalling the definition of $\tilde{\Sigma}^y_\eta$, we find

$$\bar{\sigma}(e) = \lim_{R\to\infty} R^{1-d}\tilde{\Sigma}^y_\eta(e, Q^e(0,R), R) = \lim_{R\to\infty} R^{1-d}\Sigma^y_\eta(e, \mathbf{Q}^e(0,R)).$$

Finally, extend $\bar{\sigma}$ to $\mathbb{R}^d$ by one-homogeneous extension, that is,

$$\bar{\sigma}(v) = \|v\|\bar{\sigma}(\|v\|^{-1}v) \quad \text{if } v \neq 0, \quad \text{and} \quad \bar{\sigma}(0) = 0.$$

A classical argument can be used to prove that $\bar{\sigma}$ is convex; see the works of Messenger, Miracle-Solé, and Ruiz [66] and Caffarelli and de la Llave [27]. $\square$

Finally, we prove the lemma.

Proof of Lemma 3. The properties of $\tilde{\Sigma}^{\text{avg}}_\eta$ follow from those of $\tilde{\Sigma}^y_\eta$ as a consequence of elementary integration theory (e.g., the dominated convergence theorem). In particular, here we use Propositions 17–20.

Recall from Propositions 19 and 20 that, for each $m \in \mathbb{N} \cup \{\infty\}$, there is a $\{\tau_{x'}\}_{x' \in \langle e \rangle^{\perp}}$-invariant Borel set $A_m(e)$ such that $\mathcal{L}^d(A_m(e)) = 1$ and, for each $y \in A_m(e)$,

$$\tilde{\sigma}^y(e, m) = \lim_{n \to \infty} 2^{n(1-d)} \tilde{\Sigma}^y_\eta(e, Q^e(0, 2^n), m).$$

Since $A_m(e)$ has full measure, we can integrate to find

$$\int_{\mathbb{T}^d} \tilde{\sigma}^y(e, m)\, dy = \lim_{n \to \infty} 2^{n(1-d)} \tilde{\Sigma}^{\text{avg}}_\eta(e, Q^e(0, 2^n), m).$$

Finally, recall that $\mathcal{L}^d$ is ergodic under the action of $\{\tau_{x'}\}_{x' \in \langle e \rangle^{\perp}}$. Thus, $y \mapsto \tilde{\sigma}^y(e, m)$ is constant $\mathcal{L}^d$-almost everywhere. More precisely, there is a $\{\tau_{x'}\}_{x' \in \langle e \rangle^{\perp}}$-invariant Borel set $C_m(e) \subseteq \mathbb{T}^d$ such that $\mathcal{L}^d(C_m(e)) = 1$ and

$$\tilde{\sigma}^{\underline{y}}(e, m) = \int_{\mathbb{T}^d} \tilde{\sigma}^y(e, m)\, dy \quad \text{for each } \underline{y} \in C_m(e).$$

We conclude by setting $A(e) = \cap_{m \in \mathbb{N} \cup \{\infty\}} A_m(e) \cap C_m(e)$. $\square$

2.7 Notes

2.7.1 Averaging

The ergodic theory results in Section 2.4 will not surprise experts, but it is a somewhat unconventional spin on a classical subject and there does not seem to be a standard reference for this material. Nonetheless, the results of that section should be seen as a higher-dimensional, continuum version of the classical fact that the semi-group generated by the map $x \mapsto x + \alpha$ in the torus $\mathbb{T}$ is ergodic if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

2.7.2 Linear Response

The inspiration to consider the linear response problem (2.1) came from heuristic discussions by Spohn [87] and Bellettini, Buttà, and Presutti [17]. Those references concern interface motions driven by energy dissipation, whereas (2.1) is nonvariational.

In the variational setting, the analogue of (2.1) can lead to a stationary plane in the limit. For instance, if we consider the diffuse-interface setting, then we should replace (2.1) by the equation

$$\begin{cases} m(\epsilon^{-1}x, \widehat{Du^\epsilon})u^\epsilon_t - \text{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}(W_u(\epsilon^{-1}x, u^\epsilon) + \epsilon\alpha) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = \eta(\langle \cdot, e\rangle) & \text{on } \mathbb{R}^d \times \{0\}, \end{cases}$$

where $\eta : \mathbb{R} \to [-1,1]$ is a smooth function with $\lim_{s\to\pm\infty} \eta(s) = \pm 1$. The paper with Feldman [51] shows that it is possible to find coefficients a and W such that, no matter the choice of m, pinning occurs for all $\alpha \in \mathbb{R}$ and all $e \in S^{d-1}$. (In other words, $u^\epsilon(x) \to 1$ if $\langle x, e\rangle > 0$ and $u^\epsilon(x) \to -1$ if $\langle x, e\rangle < 0$.) Put another way, it is possible that $\overline{m}_{\text{pl}} \equiv \infty$.

2.7.3 Surface Tension

Section 2.6 is inspired mainly by Chapter 7 of the book by Presutti [80] and the work of Ansini, Braides, and Chiadò-Piat [5]. The main takeaway is Theorem 18, which shows that the averaging processes occurring separately in tangential and normal directions "commute." It is shown in [71] that this remains true even in stationary, ergodic media, a fact that was not obvious in Presutti's original approach, wherein only spatially homogeneous media was considered. Throughout the arguments, the so-called fundamental estimate of Γ-convergence is a convenient technical tool. This is a standard component of the theory of Γ-convergence (cf. [37, Chapter 18]), which was originally formulated and proved in the diffuse-interface context in [5].

CHAPTER 3

PULSATING STANDING WAVES AND THE SURFACE TENSION

In the last section, we introduced the surface tension $\bar{\sigma}$, which describes the large scale behavior of the energy functional $\mathcal{F}$. Here we show that the surface tension can also be represented via a variational problem posed in $\mathbb{R}\times\mathbb{T}^d$ as described in Section 1.2. This leads to a link between the surface tension, the plane-like minimizers of $\mathcal{F}$ studied in Aubry-Mather theory, and pulsating standing waves.

Recall from the introduction that, given an $e \in S^{d-1}$, we are interested in the functional $\mathscr{T}_e$ given by

$$\mathscr{T}_e(V) = \int_{\mathbb{R}\times\mathbb{T}^d} \left(\frac{1}{2}\langle a(y)\mathcal{D}_e V, \mathcal{D}_e V\rangle + W(y,V)\right) dy\, ds.$$

The problem is to minimize $\mathscr{T}_e$ in the class of functions heteroclinic between -1 and 1 in the cylinder $\mathbb{R}\times\mathbb{T}^d$. Recall that we use the notation

$$\mathscr{E}(e) = \inf\{\mathscr{T}_e(V) \mid V \in \mathscr{X}\},$$
$$\mathscr{X} = \{V \in L^\infty(\mathbb{R}\times\mathbb{T}^d) \mid -1 \le V \le 1,\ \lim_{s\to\pm\infty} V(s,y) = \pm 1 \text{ in } L^1_{\text{loc}}(\mathbb{R}\times\mathbb{T}^d)\}.$$

Henceforth, we denote the set of minimizers by $\mathcal{M}(e)$, that is,

$$\mathcal{M}(e) = \{U \in \mathscr{X} \mid \mathscr{T}_e(U) = \mathscr{E}(e)\}.$$

Note that any minimizer $U_e \in \mathcal{M}(e)$ is a weak solution of the pulsating standing wave equation:

$$\begin{cases} \mathcal{D}_e^*(a(y)\mathcal{D}_e U_e) + W_u(y,U_e) = 0 \quad \text{in } \mathbb{R}\times\mathbb{T}^d, \\ \lim_{s\to\pm\infty} U_e(s,y) = \pm 1, \quad \partial_s U_e \ge 0, \end{cases} \tag{3.1}$$

3.1 Pulsating Standing Waves and Plane-like Minimizers

The goal of this section is to show how the pulsating standing wave equation (3.1) in $\mathbb{R} \times \mathbb{T}^d$ relates back to the original problem in $\mathbb{R}^d$. By the end, we will have proved the existence of minimizers of $\mathscr{E}$ and shown that minimizing $\mathscr{E}$ is equivalent to finding certain plane-like minimizers of $\mathcal{F}$.

We will start with a basic calculation that is well-known in the literature on pulsating waves. Suppose that $U : \mathbb{R} \times \mathbb{T}^d \to [-1, 1]$ is a smooth solution of (3.1) satisfying $\lim_{s \to \pm\infty} U(s, x) = \pm 1$ uniformly in $\mathbb{T}^d$. Observe that if we define functions $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ in $\mathbb{R}^d$ by

$$u_\zeta(x) = U(\langle x, e \rangle - \zeta, x)$$

then, for each $\zeta \in \mathbb{R}$, u_ζ is a critical point of $\mathcal{F}$, that is,

$$-\text{div}(a(y) D u_\zeta) + W_u(y, u_\zeta) = 0 \quad \text{in} \quad \mathbb{R}^d.$$

Furthermore, these critical points are "plane-like" in the sense that

$$\lim_{\epsilon \to 0^+} u_\zeta(\epsilon^{-1} x) = \begin{cases} 1, & \text{if } \langle x, e \rangle > 0, \\ -1, & \text{if } \langle x, e \rangle < 0. \end{cases}$$

Importantly, here ζ is a parameter: whereas in $\mathbb{R} \times \mathbb{T}^d$ we are dealing with a single function U, when we pass to the physical space $\mathbb{R}^d$, we have to consider the whole family of functions $\{u_\zeta\}_{\zeta \in \mathbb{R}}$. In particular, if U is a smooth solution of (3.1) with $\partial_s U > 0$, then $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ is an ordered, continuous one-parameter family of critical points of $\mathcal{F}$. As we will see, such families do not exist in general, and hence smooth pulsating standing waves will not necessarily exist.

3.1.1 An Ergodic Theorem

As we noted already above, the pulsating standing wave equation (3.1) is the Euler-Lagrange equation of $\mathscr{T}_e$. Thus, it will be useful to know how to write $\mathscr{T}_e$ in terms of the functions $\{u_\zeta\}_{\zeta\in\mathbb{R}}$ rather than U. The next theorem shows how this is done.

In the statement of the theorem, for $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, we denote by Q_e a fundamental domain for the action of the group M_e (see (2.6)) on $\langle e\rangle^\perp$. More precisely, given a $\mathbb{Z}$-basis $\{k_1, \ldots, k_{d-1}\}$ of M_e, define Q_e by

$$Q_e = \left\{ \sum_{i=1}^{d-1} \lambda_i k_i \mid (\lambda_1, \ldots, \lambda_{d-1}) \in [0,1)^{d-1} \right\}.$$

Theorem 19. *Suppose that $V \in L^1(\mathbb{R}\times\mathbb{T}^d)$ and define functions $(v_\zeta)_{\zeta\in\mathbb{R}}$ in $L^1_{loc}(\mathbb{R}^d)$ by $v_\zeta(x) = V(\langle x, e\rangle - \zeta, x)$. Given any $e \in S^1$, we have*

$$\int_{\mathbb{R}\times\mathbb{T}^d} V(s,y)\,dy\,ds = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\{ \lim_{R\to\infty} R^{1-d} \int_{Q^e(0,R)\oplus_e\mathbb{R}} v_\zeta(y)\,dy \right\} d\zeta.$$

Furthermore, the following dichotomy holds:

(i) If $e \in \mathbb{R}\mathbb{Z}^d$, then

$$\int_{\mathbb{R}\times\mathbb{T}^d} V(s,y)\,dy\,ds = r_e^{-1}\mathcal{H}^{d-1}(Q_e)^{-1}\int_0^{r_e}\int_{Q_e\oplus_e\mathbb{R}} v_\zeta(y)\,dy\,d\zeta.$$

(ii) If $e \notin \mathbb{R}\mathbb{Z}^d$, then

$$\int_{\mathbb{R}\times\mathbb{T}^d} V(s,y)\,dy\,ds = \lim_{R\to\infty} R^{1-d}\int_{Q^e(0,R)\oplus_e\mathbb{R}} v_\zeta(y)\,dy \quad \textit{for a.e. } \zeta\in\mathbb{R}.$$

(iii) If $e \notin \mathbb{R}\mathbb{Z}^d$ and, in addition, the function $G : \mathbb{T}^d \to \mathbb{R}$ given by

$$G(y) = \int_{-\infty}^{\infty} V(s, se + y)\, ds$$

is continuous, then

$$\lim_{R \to \infty} \sup \left\{ \left| R^{1-d} \int_{Q^e(0,R) \oplus_e \mathbb{R}} v_\zeta(y)\, dy - \int_{\mathbb{R} \times \mathbb{T}^d} V(s, y)\, ds\, dy \right| \mid \zeta \in \mathbb{R} \right\} = 0.$$

The distinction between rational and irrational directions in the theorem stems from the fact that the transformation functions $(v_\zeta)_{\zeta \in \mathbb{R}}$ are periodic when $e \in \mathbb{R}\mathbb{Z}^d$, but only quasi-periodic when $e \notin \mathbb{R}\mathbb{Z}^d$. This is explained in detail in [73, Section 4]. Without going into all the details here, we will need to define the quotient space $\mathbb{T}^{d-1}_e \oplus_e \mathbb{R} = \mathbb{R}^d / M_e$ for $e \in \mathbb{R}\mathbb{Z}^d$, which is in bijective correspondence with $Q_e \oplus_e \mathbb{R}$ and is diffeomorphic to the product $\mathbb{T}^{d-1} \times \mathbb{R}$.

Throughout the remainder of this chapter and the next, we will identify functions $v : \mathbb{T}^{d-1}_e \oplus_e \mathbb{R} \to \mathbb{R}$ with functions $v : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$v(x + k) = v(x) \quad \text{for each } x \in \mathbb{R}^d,\ k \in M_e.$$

In particular, if $(v_\zeta)_{\zeta \in \mathbb{R}}$ are defined as in the theorem for some $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, then it is easy to see that these functions have this M_e-invariance property and, thus, can be regarded as functions defined in $\mathbb{T}^{d-1}_e \oplus_e \mathbb{R}$.

We will only prove parts (ii) and (iii) of Theorem 19. Part (i) follows by similar arguments; the proof is sketched in [73, Appendix A].

Proof of Theorem 19, (ii) and (iii). Let $\zeta \in \mathbb{R}$ be a free parameter. First, observe that we can write

$$\int_{Q^e(0,R)\oplus_e\mathbb{R}} v_\zeta(x)\,dx = \int_{Q^e(0,R)}\int_{-\infty}^{\infty} V(s-\zeta, se+x')\,ds\,dx'$$
$$= \int_{Q^e(0,R)}\int_{-\infty}^{\infty} V(s,(s+\zeta)e+x')\,ds\,dx'.$$

Since $V \in L^1(\mathbb{R}\times\mathbb{T}^d)$, it follows that $y \mapsto \int_{-\infty}^{\infty} V(s,(s+\zeta)e+y)\,ds$ is in $L^1(\mathbb{T}^d)$, no matter the choice of ζ. Therefore, we can invoke Proposition 2 to deduce that, for almost every $\zeta \in \mathbb{R}$, we have

$$\lim_{R\to\infty} R^{1-d}\int_{Q^e(0,R)\oplus_e\mathbb{R}} v_\zeta(x)\,dx = \lim_{R\to\infty} R^{1-d}\int_{Q^e(0,R)}\int_{-\infty}^{\infty} V(s,(s+\zeta)e+x^\perp)\,ds\,dx^\perp$$
$$= \int_{\mathbb{T}^d}\left(\int_{-\infty}^{\infty} V(s,(s+\zeta)e+y)\,ds\right)dy$$
$$= \int_{-\infty}^{\infty}\int_{\mathbb{T}^d} V(s,y)\,dy\,ds.$$

This proves (ii). At the same time, if the function $y \mapsto \int_{-\infty}^{\infty} V(s,se+y)\,ds$ is continuous, then the convergence in Proposition 2 can be upgraded to uniform convergence, proving (iii). □

As an immediate consequence of the previous theorem, we show that the variational problem (1.8) is bounded below by the surface tension $\bar{\sigma}$.

Proposition 25. *For each* $e \in S^{d-1}$, $\bar{\sigma}(e) \leq \mathscr{E}(e)$.

Proof. Let $V \in \mathscr{X}$. If $\mathscr{T}_e(V) = \infty$, there is nothing to prove. Hence let us assume that $\mathscr{T}_e(V) < \infty$.

By the previous theorem, there is a measurable set $E \subseteq \mathbb{R}$ such that $\mathcal{L}^1(\mathbb{R}\setminus E) = 0$ and if we define $(v_\zeta)_{\zeta\in\mathbb{R}}$ by $v_\zeta(x) = V(\langle x, e\rangle - \zeta, x)$, then, for each $\zeta \in E$,

$$f(\zeta) := \lim_{R\to\infty} R^{1-d}\mathcal{F}(v_\zeta; Q^e(0,R)\oplus_e \mathbb{R}) < \infty,$$

$$\lim_{n\to\infty} v_\zeta(nx) = \chi_{\{\langle x,e\rangle>0\}} - \chi_{\{\langle x,e\rangle<0\}} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^d),$$

and

$$\mathscr{T}_e(V) = \lim_{T\to\infty} T^{-1}\int_0^T f(\zeta)\,d\zeta.$$

To prove that $\mathscr{T}_e(V) \geq \bar{\sigma}(e)$, we will show that $f(\zeta) \geq \bar{\sigma}(e)$ for each $\zeta \in E$.

In what follows, fix a smooth function $\eta : \mathbb{R} \to [-1,1]$ such that

$$\lim_{s\to\pm\infty} \eta(s) = \pm 1, \qquad \int_{-\infty}^{\infty} \left(\frac{\Lambda}{2}\eta'(s)^2 + \overline{W}(\eta(s))\right) ds < \infty$$

and define $\eta_e : \mathbb{R}^d \to [-1,1]$ by $\eta_e(x) = \eta(\langle x, e\rangle)$. Using η as boundary condition, let Σ^0_η be the finite-volume surface tension from Section 2.6.1.

Fix $\zeta \in E$. We will apply the fundamental estimate [71, Appendix A]. Fix $\rho > 0$ and define $V_n(\rho) \subseteq \mathbb{R}^d$ by

$$V_n(\rho) = \mathbf{Q}^e(0, \rho^{-1}n) \setminus \mathbf{Q}^e(0,n). \tag{3.2}$$

Observe that

$$\lim_{n\to\infty} n^{-d}\int_{V_n(\rho)} |v_\zeta(y) - \eta_e(y)|\,dy = \lim_{n\to\infty}\int_{V_1(\rho)} |v_\zeta(nx) - \eta(n\langle x,e\rangle)|\,dx = 0.$$

Thus, by the fundamental estimate, there are cut-off functions $(\psi_n)_{n\in\mathbb{N}} \subseteq C^\infty_c(\mathbb{R}^d)$ such that

$$\psi_n = 1 \quad \text{in } \mathbf{Q}^e(0,n), \quad \psi_n = 0 \quad \text{in } \mathbb{R}^d \setminus \mathbf{Q}^e(0,\rho^{-1/2}n),$$

and, as $n \to \infty$,

$$\mathcal{F}(\psi_n v_\zeta + (1-\psi_n) q_e; \mathbf{Q}^e(0,\rho^{-1}n) \leq \mathcal{F}(v_\zeta; \mathbf{Q}^e(0,\rho^{-1/2}n)) + \mathcal{F}(\eta_e; V_n(\rho)) + o(n^{d-1}).$$

Observe that we can estimate the boundary term as follows:

$$\begin{aligned}\mathcal{F}(\eta_e; V_n(\rho)) \leq& \int_{-\infty}^{\infty} \left(\frac{\Lambda}{2}\eta'(s)^2 + \overline{W}(\eta(s))\right) ds \cdot \mathcal{H}^{d-1}(Q^e(0,\rho^{-1}n) \setminus Q^e(0,n)) \\ &+ \int_{|s|\geq \rho^{-1}n} \left(\frac{\Lambda}{2}\eta'(s) + \overline{W}(\eta(s))\right) ds \cdot \mathcal{H}^{d-1}(Q^e(0,n)).\end{aligned}$$

Thus, in the limit $n \to \infty$, we find

$$\begin{aligned}\bar{\sigma}(e)\rho^{d-1} &= \lim_{n\to\infty} n^{1-d}\Sigma^0_\eta(e, \mathbf{Q}^e(0,\rho^{-1}n)) \\ &\leq \lim_{n\to\infty} n^{1-d}\mathcal{F}(\psi_n v_\zeta + (1-\psi_n)q_e; \mathbf{Q}^e(0,\rho^{-1}n)) \\ &\leq \lim_{n\to\infty} n^{1-d}\mathcal{F}(v_\zeta; \mathbf{Q}^e(0,\rho^{-1/2}n)) \\ &= f(\zeta)\rho^{-(d-1)/2}.\end{aligned}$$

Sending $\rho \to 1^-$, we conclude that $\bar{\sigma}(e) \leq f(\zeta)$.

Invoking the definition of $\mathscr{E}(e)$, we conclude that $\mathscr{E}(e) \geq \bar{\sigma}(e)$. □

3.1.2 Plane-Like Minimizers

Theorem 19 allows us to relate minimizers of the Lagrangian $\mathscr{E}$ to plane-like minimizers of $\mathcal{F}$. Let us start by recalling some terminology.

Definition 8. *A function $v \in H^1_{loc}(\mathbb{R}^d; [-1,1])$ is called a* Class A minimizer *of $\mathcal{F}$ if*

$$\mathcal{F}(v; B(0,R)) \leq \mathcal{F}(w; B(0,R)) \quad \textit{for each } R > 0 \textit{ and each } w \in v + H^1_0(B(0,R)).$$

Our interest is in Class A minimizers with the strong Birkhoff property.

Definition 9. *Given $e \in S^{d-1}$, a Class A minimizer u of $\mathcal{F}$ is called a strongly Birkhoff plane-like minimizer of $\mathcal{F}$ in the e direction if, given any $k \in \mathbb{Z}^d$, we have*

$$u(\cdot + k) \leq u \quad \textit{if } \langle k, e\rangle \leq 0 \quad \textit{and} \quad u(\cdot + k) \geq u \quad \textit{if } \langle k, e\rangle \geq 0.$$

Remark 1. *A weaker version of the Birkhoff property requires only that the translate of u are totally ordered under $\leq$, that is, for each $k \in \mathbb{Z}^d$, either $u(\cdot + k) \leq u$ or $u(\cdot + k) \geq u$. For the most part, this will not be relevant here, but see Propostion 52 for an example of a Class A minimizer of $\mathcal{F}$ that is plane-like and that possesses this weaker Birkhoff property but is not strongly Birkhoff according to the previous definition.*

It is straightforward to verify that strongly Birkhoff plane-like minimizers are indeed plane-like:

Proposition 26. *Fix $e \in S^{d-1}$. If u is a strongly Birkhoff plane-like minimizer of $\mathcal{F}$, then there is a constant $C_u > 0$ such that*

$$\sup\{|u(x) - 1| + \|Du(x)\| \mid \langle x, e\rangle \geq R\} \leq e^{-C_u R},$$
$$\sup\{|u(x) + 1| + \|Du(x)\| \mid \langle x, e\rangle \leq -R\} \leq e^{-C_u R}.$$

This can be proved by arguing as in [73, Proposition 22]; hence the proof is omitted.

Valdinoci proved that $\mathcal{F}$ has strongly Birkhoff plane-like minimizers in any direction $e \in S^{d-1}$.

Theorem 20 ([90])**.** *For any $e \in S^{d-1}$, $\mathcal{F}$ has at least one strongly Birkhoff plane-like minimizer u. Furthermore, if u is any strongly Birkhoff plane-like minimizer and $\eta \in (0, 1)$,* *then there is an $M(\eta) > 0$ depending only on a, W, and η and an $S_u \in \mathbb{R}$ such that*

$$\left\{x \in \mathbb{R}^d \mid |u(x)| < \eta\right\} \subseteq \left\{x \in \mathbb{R}^d \mid S_u \leq \langle x, e\rangle \leq S_u + M(\eta)\right\}$$

It turns out that plane-like minimizers have average energy equal to the surface tension.

Theorem 21. *Let $e \in S^{d-1}$. If u is a strongly Birkhoff plane-like minimizer of $\mathcal{F}$ in the e direction, then*

$$\bar{\sigma}(e) = \lim_{R \to \infty} R^{1-d} \int_{Q^e(0,R) \oplus_e \mathbb{R}} \left(\frac{1}{2}\langle a(y)Du, Du\rangle + W(y,u)\right) dy.$$

Proof. Fix a smooth function $\eta : \mathbb{R} \to [-1,1]$ such that $\int_{-\infty}^{\infty} \left(\frac{\Lambda}{2}\eta'(s)^2 + \overline{W}(\eta(s))\right) ds < \infty$ and $\lim_{s\to\pm\infty} \eta(s) = \pm 1$. Denote by $\eta_e : \mathbb{R}^d \to [-1,1]$ the function $\eta_e(x) = \eta(\langle x, e\rangle)$. Using η as boundary condition, let Σ_η^0 be the finite-volume surface tension from Section 2.6.1. The arguments of Proposition 25 show that

$$\liminf_{R\to\infty} R^{1-d} \int_{Q^e(0,R)\oplus_e\mathbb{R}} \left(\frac{1}{2}\langle a(y)Du, Du\rangle + W(y,u)\right) dy \geq \bar{\sigma}(e).$$

In view of Theorem 16, to conclude that equality holds, it is enough to prove that

$$\limsup_{R\to\infty} R^{1-d} \int_{Q^e(0,R)\oplus_e\mathbb{R}} \left(\frac{1}{2}\langle a(y)Du, Du\rangle + W(y,u)\right) dy \leq \lim_{R\to\infty} R^{1-d}\Sigma_\eta^0(e, \mathbf{Q}^e(0,R)). \tag{3.3}$$

To see that (3.3) holds, choose a sequence $(v_n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$,

$$v_n \in H^1(\mathbf{Q}^e(0,n)), \quad v_n - \eta_e \in H_0^1(\mathbf{Q}^e(0,n)),$$
$$\mathcal{F}(v_n; \mathbf{Q}^e(0,n)) = \Sigma_\eta^0(e, \mathbf{Q}^e(0,n)).$$

Fix $\rho \in (0,1)$ and define $V_n(\rho) \subseteq \mathbb{R}^d$ by (3.2). It is convenient to extend v_n to $V_n(\rho)$ by setting it equal to η_e there, that is,

$$v_n = \eta_e \quad \text{in } V_n(\rho).$$

Notice that, by Proposition 26, we have

$$\lim_{n\to\infty} n^{-d} \int_{V_n(\rho)} |v_n(y) - u(y)|\, dy = \lim_{n\to\infty} \int_{V_1(\rho)} |\eta(n\langle x, e\rangle) - u(nx)|\, dx = 0.$$

Thus, we can apply the fundamental estimate [71, Appendix A]. In particular, there are cut-off functions $(\psi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d; [0,1])$ such that

$$\psi_n = 1 \quad \text{in } \mathbf{Q}^e(0,n), \quad \psi_n = 0 \quad \text{in } \mathbb{R}^d \setminus \mathbf{Q}^e(0, \rho^{-1/2} n).$$

and, as $n \to \infty$, we have

$$n^{1-d}\mathcal{F}(\psi_n v_n + (1-\psi_n)u; \mathbf{Q}^e(0,\rho^{-1}n)) \leq n^{1-d}\mathcal{F}(v_n; \mathbf{Q}^e(0,\rho^{-1/2}n)) + n^{1-d}\mathcal{F}(u; V_n(\rho)) + o(1). \tag{3.4}$$

At the same time, since $\psi_n v_n + (1-\psi_n)u \in u + H_0^1(\mathbf{Q}^e(0,\rho^{-1}n))$, we have

$$\mathcal{F}(u; \mathbf{Q}^e(0,\rho^{-1}n)) \leq \mathcal{F}(\psi_n v_n + (1-\psi_n)u; \mathbf{Q}^e(0,\rho^{-1}n)).$$

By Proposition 26 and the boundedness of Du, we have

$$\mathcal{F}(u; V_n(\rho)) \leq e^{-cn}\mathcal{H}^{d-1}(Q^e(0,n)) + C\mathcal{H}^{d-1}(Q^e(0,\rho^{-1}n) \setminus Q^e(0,n)) \tag{3.5}$$

Furthermore, by construction, we can estimate

$$\mathcal{F}(v_n; \mathbf{Q}^e(0,\rho^{-1/2}n)) \leq \Sigma_\eta^0(e, \mathbf{Q}^e(0,n)) + \mathcal{F}(q_e; V_n(\rho^{-1/2})). \tag{3.6}$$

Hence, as in the proof of Proposition 25,

$$\limsup_{n\to\infty} n^{1-d}\mathcal{F}(v_n;\mathbf{Q}^e(0,\rho^{-1/2}n)) \leq \lim_{n\to\infty} n^{1-d}\Sigma^0_\eta(e,\mathbf{Q}^e(0,n)).$$

Sending $n\to\infty$ in (3.4), we thus find

$$\limsup_{n\to\infty} n^{1-d}\mathcal{F}(u;\mathbf{Q}^e(0,\rho^{-1}n)) \leq \lim_{n\to\infty} n^{1-d}\Sigma^0_\eta(e,\mathbf{Q}^e(0,n)).$$

Since $\rho\in(0,1)$ was arbitrary, upon sending $\rho\to 1^-$, we obtain (3.3). $\square$

3.1.3 Minimizers of $\mathscr{T}_e$

We proceed to show that minimizers of $\mathscr{T}_e$ exist. In the rational case, this follows readily from what we learned about plane-like minimizers.

Proposition 27. *For each $e\in S^{d-1}\cap\mathbb{R}\mathbb{Z}^d$, we have*

$$\mathscr{E}(e) = \min\Big\{\mathcal{H}^{d-1}(Q_e)^{-1}\mathcal{F}(v;Q_e\oplus_e\mathbb{R}) \mid v:\mathbb{T}^{d-1}_e\oplus_e\mathbb{R}\to[-1,1], \\ \lim_{\langle x,e\rangle\to\pm\infty} v(x)=\pm 1\Big\}.$$

Furthermore, $\mathscr{E}(e)=\bar{\sigma}(e)$ and $\mathcal{M}(e)$ is non-empty.

Proof. Choose a $k_e\in\mathbb{Z}^d$ such that $\langle k_e,e\rangle = r_e$. Suppose that $v:\mathbb{T}^{d-1}_e\oplus_e\mathbb{R}\to[-1,1]$ satisfies $\lim_{\langle x,e\rangle\to\pm\infty} v(x)=\pm1$ in $L^1_{\text{loc}}(\mathbb{R}^d)$. Define functions $(v_\zeta)_{\zeta\in\mathbb{R}}$ in $\mathbb{T}^{d-1}_e\oplus_e\mathbb{R}$ by

$$v_\zeta(x) = v(x+\ell k_0) \quad \text{with } \ell = \left\lceil \frac{\zeta}{r_e}\right\rceil.$$

Lift these functions to $\mathbb{R}\times\mathbb{T}^d$ via the map $V:\mathbb{R}\times\mathbb{T}^d\to[-1,1]$ defined by

$$V(s,x) = v_{\langle x,e\rangle - s}(x).$$

An exercise shows that $V \in \mathscr{X}$. Furthermore, we can compute

$$\mathscr{E}(e) \leq \mathscr{T}_e(V) = r_e^{-1} \int_0^{r_e} \mathcal{H}^{d-1}(Q_e)^{-1} \mathcal{F}(v_\zeta; Q_e \oplus_e \mathbb{R})\, d\zeta = \mathcal{H}^{d-1}(Q_e)^{-1} \mathcal{F}(v; Q_e \oplus_e \mathbb{R}). \tag{3.7}$$

This proves that

$$\mathscr{E}(e) \leq \min \Big\{ \mathcal{H}^{d-1}(Q_e)^{-1} \mathcal{F}(v; Q_e \oplus_e \mathbb{R}) \ \Big|\ v : \mathbb{T}_e^{d-1} \oplus_e \mathbb{R} \to [-1,1], \\ \lim_{\langle x,e\rangle \to \pm\infty} v(x) = \pm 1 \Big\}.$$

To deduce that equality actually holds, let $v = u$ be a strongly Birkhoff plane-like minimizer and define $U = V$. Notice that $\partial_s U \geq 0$ in the distributional sense since u is a strongly Birkhoff plane-like minimizer. By Theorem 21,

$$\mathcal{H}^{d-1}(Q_e)^{-1} \mathcal{F}(u; Q_e \oplus_e \mathbb{R}) = \bar{\sigma}(e).$$

Thus, by Proposition 25, $\bar{\sigma}(e) \leq \mathscr{E}(e) \leq \mathscr{T}_e(U) = \bar{\sigma}(e)$ so equality holds in (3.7). Moreover, we have shown that $\mathscr{E}(e) = \bar{\sigma}(e)$ and $U \in \mathcal{M}(e)$. □

Remark 2. *In the proof of Proposition 27, we showed that the minimum of $\mathcal{F}(\cdot; Q_e \oplus_e \mathbb{R})$ among functions in $\mathbb{T}_e^{d-1} \oplus_e \mathbb{R}$ heteroclinic between 1 and -1 is achieved by a strongly Birkhoff plane-like minimizer. In fact, any function attaining this minimum is necessarily a strongly Birkhoff plane-like minimizer. For a detailed proof, see [81].*

Next, we treat irrational directions.

Proposition 28. *If $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then $\mathscr{E}(e) = \bar{\sigma}(e)$ and $\mathcal{M}(e)$ is non-empty.*

Proof. Let u be a strongly Birkhoff plane-like minimizer of $\mathcal{F}$ in the e direction. For each $k \in \mathbb{Z}^d$, define $u_{\langle k,e\rangle}$ by

$$u_{\langle k,e\rangle}(x) = u(x-k).$$

Since u is a strongly Birkhoff plane-like minimizer,

$$u_{\langle k,e\rangle} \geq u_{\langle k',e\rangle} \quad \text{if } \langle k,e\rangle \leq \langle k',e\rangle.$$

Thus, since e is irrational, for any $\zeta \in \mathbb{R}$, we can define u_ζ^+ and u_ζ^- by

$$u_\zeta^+(x) = \inf\left\{u_{\langle k,e\rangle}(x) \mid \zeta > \langle k,e\rangle\right\}, \quad u_\zeta^-(x) = \sup\left\{u_{\langle k,e\rangle}(x) \mid \zeta < \langle k,e\rangle\right\}.$$

Notice that $(u_\zeta^+)_{\zeta\in\mathbb{R}}$ and $(u_\zeta^-)_{\zeta\in\mathbb{R}}$ both consist of strongly Birkhoff plane-like minimizers of $\mathcal{F}$ in the e direction. Furthermore, for each $k \in \mathbb{Z}^d$,

$$u^\pm_{\zeta+\langle k,e\rangle}(x) = u^\pm_\zeta(x-k). \tag{3.8}$$

Define $U^+, U^- : \mathbb{R}\times\mathbb{R}^d \to [-1,1]$ by

$$U^\pm(s,y) = u^\pm_{\langle y,e\rangle - s}(y).$$

From (3.8), we deduce that $U^\pm(s, y+k) = U^\pm(s,y)$ for each $k \in \mathbb{Z}^d$. Thus, U^+ and U^- can be seen as functions in $\mathbb{R}\times\mathbb{T}^d$. Since $(u_\zeta^-)_{\zeta\in\mathbb{R}}$ and $(u_\zeta^+)_{\zeta\in\mathbb{R}}$ are two families of strongly Birkhoff plane-like minimizers, a direct application of Theorems 19 and 21 gives

$$\mathscr{T}_e(U^+) = \mathscr{T}_e(U^-) = \bar\sigma(e).$$

Note, furthermore, that $U^\pm(\cdot + s) \to \pm 1$ as $s \to \pm\infty$. Therefore, $\{U^+, U^-\} \subseteq \mathscr{X}$, and we conclude that $\mathscr{E}(e) = \bar\sigma(e)$ and $\{U^+, U^-\} \subseteq \mathcal{M}(e)$. $\square$

Above we showed that plane-like minimizers of $\mathcal{F}$ in $\mathbb{R}^d$ induce minimizers of $\mathscr{T}_e$ in $\mathbb{R}\times\mathbb{T}^d$. It is possible to proceed in the opposite direction, constructing plane-like minimizers in $\mathbb{R}^d$ from minimizers in $\mathscr{T}_e$.

Theorem 22. *If $U_e \in \mathcal{M}(e)$, then there are functions $U_e^+, U_e^- \in \mathcal{M}(e)$ with the following properties:*

(i) $U_e^+ = U_e^- = U_e$ *a.e. in* $\mathbb{R} \times \mathbb{T}^d$.

(ii) If $\{u_\zeta^+\}_{\zeta \in \mathbb{R}}$ (resp. $\{u_\zeta^-\}_{\zeta \in \mathbb{R}}$) denotes the family of functions generated by U_e^+ (resp. U_e^-), then the map $\zeta \mapsto u_\zeta^+$ (resp. $\zeta \mapsto u_\zeta^-$) is right-continuous (resp. left-continuous) with respect to the topology of local uniform convergence.

(iii) For each $\zeta \in \mathbb{R}$, the following limits holds

$$u_\zeta^+ = \lim_{\mu \to \zeta^+} u_\mu^+ = \lim_{\mu \to \zeta^+} u_\mu^-, \quad u_\zeta^- = \lim_{\mu \to \zeta^-} u_\mu^+ = \lim_{\mu \to \zeta^-} u_\mu^-$$

locally uniformly in $\mathbb{R}^d$.

(iv) The set $\mathscr{D} = \{\zeta \in \mathbb{R} \mid u_\zeta^+ \neq u_\zeta^-\}$ is countable.

(v) For each $\zeta \in \mathbb{R}$ and $\bullet \in \{+, -\}$, $\lim_{r \to \pm\infty} u_\zeta^\bullet(re + x^\perp) = \pm 1$ uniformly with respect to $x^\perp \in \langle e \rangle^\perp$.

(vi) If $\mathscr{D}$ is empty, then $U_e^+ = U_e^- \in UC(\mathbb{R} \times \mathbb{T}^d)$ and $\zeta \mapsto u_\zeta$ is continuous in the topology of uniform convergence.

For the proof, see [73, Proposition 21].

Remark 3. *When $a \equiv Id$ and $W(y, u) \equiv W(u)$, there is a function $q : \mathbb{R} \to (-1, 1)$ solving the ODE*

$$-q'' + W'(q) = 0 \quad \text{in } \mathbb{R}, \quad \lim_{s \to \pm\infty} q(s) = \pm 1, \quad q(0) = 0.$$

It is well-known that q is a minimizer of the variational problem

$$\int_{-\infty}^{\infty} \left(\frac{1}{2} q'(s)^2 + W(q(s)) \right) ds = \inf \Bigg\{ \int_{-\infty}^{\infty} \left(\frac{1}{2} \eta'(s)^2 + W(\eta(s)) \right) ds \mid \lim_{s \to \pm\infty} \eta(s) = \pm 1, \\ -1 \leq \eta \leq 1 \Bigg\}.$$

Combining this with a slicing argument, we deduce that the function $q_e : \mathbb{R}^d \to (-1,1)$ defined by $q_e(x) = q(\langle x, e \rangle)$ is a Class A minimizer of $\mathcal{F}$. In fact, it is a strongly Birkhoff plane-like minimizer. It follows that if we define $U_e(s,y) = q(s)$ for any $e \in S^{d-1}$, then $U_e \in \mathcal{M}(e)$.

3.1.4 Uniqueness and Non-Uniqueness

Since $\mathscr{T}_e$ is not coercive, minimizers are not unique (modulo translations) in general. Actually, the arithmetic properties of e are the deciding factor here. Roughly speaking, minimizers of $\mathscr{T}_e$ are unique modulo translations precisely when e is irrational.

Theorem 23. *If $U, \tilde{U} \in \mathcal{M}(e)$ and $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then there is a $\tilde{s} \in \mathbb{R} \times \mathbb{T}^d$ such that*

$$\tilde{U}(s,y) = U(s + \tilde{s}, y) \quad \textit{for a.e. } (s,y) \in \mathbb{R} \times \mathbb{T}^d.$$

Proof. By Theorem 22, there is no loss of generality if we assume that U and $\tilde{U}$ are such that the associated functions $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ and $\{\tilde{u}_\zeta\}_{\zeta \in \mathbb{R}}$ are right-continuous with respect to ζ. Furthermore, [73, Proposition 13] implies that we can find an $\tilde{s} \in \mathbb{R}$ such that

$$\int_{\mathbb{T}^d} U(\tilde{s}, y)\, dy = \int_{\mathbb{T}^d} \tilde{U}(0,y)\, dy.$$

We claim that $U(s + \tilde{s}, y) = \tilde{U}(s,y)$ a.e. in $\mathbb{R} \times \mathbb{T}^d$. Since $U(\cdot + \tilde{s}, \cdot) \in \mathcal{M}(e)$, there is no loss of generality in assuming that $\tilde{s} = 0$. After making this translation, it remains to prove that

$U = \tilde{U}$ a.e.

Define $\overline{U} = U \vee \tilde{U}$ and $\underline{U} = U \wedge \tilde{U}$. By [73, Lemma 2], $\overline{U}, \underline{U} \in \mathcal{M}(e)$. Hence if we define $(\bar{u}_\zeta)_{\zeta \in \mathbb{R}}$ and $(\underline{\mathrm{u}}_\zeta)_{\zeta \in \mathbb{R}}$ by $\bar{u}_\zeta(x) = \overline{U}(\langle x, e\rangle - \zeta, x)$ and $\underline{\mathrm{u}}_\zeta(x) = \underline{U}(\langle x, e\rangle - \zeta, x)$, then every element of the families $\{\bar{u}_\zeta\}_{\zeta \in \mathbb{R}}$ and $\{\underline{\mathrm{u}}_\zeta\}_{\zeta \in \mathbb{R}}$ is a critical point of $\mathcal{F}$. Since $\underline{\mathrm{u}}_\zeta \leq \bar{u}_\zeta$, the strong maximum principle implies that either $\underline{\mathrm{u}}_\zeta = \bar{u}_\zeta$ or $\underline{\mathrm{u}}_\zeta < \bar{u}_\zeta$ in $\mathbb{R}^d$ (cf. [39, Corollary A.3]). To conclude, we will assume that there is a $\zeta' \in \mathbb{R}$ so that $u_{\zeta'} = \underline{\mathrm{u}}_{\zeta'} < \bar{u}_{\zeta'} = \tilde{u}_{\zeta'}$ and show that this leads to a contradiction. (If instead $\tilde{u}_{\zeta'} = \underline{u}_{\zeta'} < \bar{u}_{\zeta'} = u_{\zeta'}$, switch the roles of U and $\tilde{U}$.)

From the identity $u_\zeta < \tilde{u}_\zeta$, we know that

$$u_{\zeta' + \langle k, e\rangle} = u_{\zeta'}(\cdot - k) < \tilde{u}_{\zeta'}(\cdot - k) = \tilde{u}_{\zeta' + \langle k, e\rangle} \quad \text{for each } k \in \mathbb{Z}^d.$$

Thus, since $\{\langle k, e\rangle \mid k \in \mathbb{Z}^d\}$ is dense by Theorem 11 and both $\zeta \mapsto u_\zeta$ and $\zeta \mapsto \tilde{u}_\zeta$ are right-continuous, we deduce that $u_\zeta \leq \tilde{u}_\zeta$ for each $\zeta \in \mathbb{R}$. Of course, by the strong maximum principle, for any given $\zeta \in \mathbb{R}$, we must have either $u_\zeta < \tilde{u}_\zeta$ or $u_\zeta = \tilde{u}_\zeta$. But we know the inequality is strict when $\zeta = \zeta'$ so it must always be strict. (Otherwise, if $u_{\zeta''} = \tilde{u}_{\zeta''}$ for some $\zeta'' \in \mathbb{R}$, we use the density of $\{\zeta'' + \langle k, e\rangle \mid k \in \mathbb{Z}^d\}$ to obtain the contradiction $u_{\zeta'} = \tilde{u}_{\zeta'}$.)

Integrating over $\{0\} \times \mathbb{T}^d$, we obtain

$$\begin{aligned} 0 &< \int_{\mathbb{T}^d} \left(\tilde{u}_{\langle y, e\rangle}(y) - u_{\langle y, e\rangle}(y) \right) dy \\ &= \int_{\mathbb{T}^d} (\tilde{U}(0, y) - U(0, y)) \, dy = 0. \end{aligned}$$

This contradiction shows $u_\zeta = \tilde{u}_\zeta$ for all $\zeta \in \mathbb{R}$, hence $U = \tilde{U}$. □

When $e \in \mathbb{R} \times \mathbb{Z}^d$, then minimizers are no longer translates of each other. A quick way to see this is to consider the spatially homogeneous case when $a \equiv \mathrm{Id}$ and $W(y, u) \equiv W(u)$. Fix an $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. Recall from Remark 3 that there is a $U_e \in \mathcal{M}(e)$ of the form

$U_e(s,y) = q(s)$. The corresponding plane-like minimizers $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ are given by

$$u_\zeta(x) = q(\langle x, e \rangle - \zeta).$$

At the same time, let $h : \mathbb{R} \to \mathbb{R}$ be any strictly increasing function such that

$$h(\zeta + r_e) = h(\zeta) + r_e \quad \text{for each } \zeta \in \mathbb{R}. \tag{3.9}$$

Define $\{u^h_\zeta\}_{\zeta \in \mathbb{R}}$ by

$$u^h_\zeta(x) = u_{h(\zeta)}(x).$$

Notice that $\{u^h_\zeta\}_{\zeta \in \mathbb{R}}$ is simply a reparametrization of $\{u_\zeta\}_{\zeta \in \mathbb{R}}$. Finally, define $U^h : \mathbb{R} \times \mathbb{R}^d \to [-1, 1]$ by

$$U^h(s, y) = u_{\langle y, e \rangle - s}(y).$$

It is an exercise to show that U^h descends to a function in $\mathbb{R} \times \mathbb{T}^d$. A straightforward computation then shows that $U^h \in \mathcal{M}(e)$.

By construction and an explicit computation, U^h is not equal to a translate of U unless h is of the form $h(\zeta) = \zeta + a$ for some $a \in \mathbb{R}$.

The previous discussion can be generalized.

Proposition 29. *Suppose $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$ and let $r_e > 0$ be the constant given by (2.7). Fix $U \in \mathcal{M}(e)$ for which the corresponding functions $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ are right-continuous with respect to ζ as in Theorem 22. If U generates more than one function, that is, if there is a $\zeta_1 \in (0, r_e)$ such that $u_0 \neq u_{\zeta_1} \neq u_{r_e}$, then there is a $\tilde{U} \in \mathcal{M}(e)$ that is not a translate of U.*

See [73, Proposition 25] for the proof.

3.1.5 Non-Smoothness

In general, minimizers of $\mathscr{T}_e$ need not be smooth. The reason for this is that if $U \in \mathcal{M}(e)$ is smooth, then the corresponding plane-like minimizers $\{u_\zeta\}_{\zeta\in\mathbb{R}}$ form a smooth, one-parameter family of minimizers of $\mathcal{F}$. To put it another way, the smoothness of U implies that $\mathcal{F}$ has a family of minimizers whose graphs foliate $\mathbb{R}^d \times (-1, 1)$. In Aubry-Mather theory, the existence and/or non-existence of such foliations is known to be a non-trivial issue.

Aubry-Mather theory therefore suggests that minimizing weak solutions of the pulsating standing wave equation (3.1) will not be smooth in general. We refer to [73] and [51] for examples. Nonetheless, that still leaves the question whether or not there are smooth pulsating standing waves that are not minimizing.

The next theorem rules this out. If U is a smooth solution of the pulsating standing wave equation (1.4), then it is necessarily minimizing. (In fact, continuity is enough.)

Theorem 24. *If $e \in S^{d-1}$ and $U \in C(\mathbb{R} \times \mathbb{T}^d; [-1, 1])$ is a weak solution of the pulsating standing wave equation* (1.4)*, then $U \in \mathcal{M}(e)$ and the critical points $\{u_\zeta\}_{\zeta\in\mathbb{R}}$ of $\mathcal{F}$ generated by U are strongly Birkhoff plane-like minimizers. In particular, this applies to the pulsating standing waves of [16, Equation 6.8].*

We must emphasize that, in the theorem, monotonicity of U in the s variable is part of the hypothesis.

Proof. Assume that $U \in C(\mathbb{R} \times \mathbb{T}^d; [-1, 1])$ satisfies $\mathcal{D}_e^*(a(y)\mathcal{D}_e U) + W_u(y, U) = 0$, $\partial_s U \geq 0$ in the distributional sense in $\mathbb{R} \times \mathbb{T}^d$, and

$$\lim_{s\to\pm\infty} U(\cdot + s, \cdot) = \pm 1 \quad \text{in } L^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}^d).$$

Let $\{u_\zeta\}_{\zeta\in\mathbb{R}}$ be the functions generated by U. Arguing as in the proof of [73, Proposition 20], we find that for a.e. $\zeta \in \mathbb{R}$, the function u_ζ is a distributional solution of $-\text{div}(a(x)Du_\zeta) +$

$W_u(x, u_\zeta) = 0$ in $\mathbb{R}^d$. Since $(x, \zeta) \mapsto u_\zeta(x)$ is bounded and continuous, every member of the family is necessarily a distributional solution, and the smoothness of the coefficients implies that $\{u_\zeta\}_{\zeta \in \mathbb{R}} \subseteq C^{2,\alpha}(\mathbb{R}^d)$.

Fix $\zeta \in \mathbb{R}$. We claim that u_ζ is a Class A minimizer of $\mathcal{F}$. To see this, fix $R > 0$ and pick $w \in H^1(B(0,R); [-1,1])$ such that

$$\mathcal{F}(w; B(0,R)) = \inf \left\{ \mathcal{F}(u_\zeta + f; B(0,R)) \mid f \in C_c^\infty(B(0,R); [-1,1]) \right\},$$
$$w = u_\zeta \text{ on } \partial B(0,R).$$

By elliptic regularity, w extends to a continuous function in $\overline{B(0,R)}$ (cf. [56, Theorem 8.34]), and the strong maximum principle (cf. [39, Corollary A.3]) implies

$$-1 < \min \left\{ w(x) \mid x \in \overline{B(0,R)} \right\} \leq \max \left\{ w(x) \mid x \in \overline{B(0,R)} \right\} < 1.$$

Henceforth, let $\zeta_1, \zeta_2 \in [-\infty, \infty]$ be defined by

$$\zeta_1 = \sup \left\{ \zeta' \in \mathbb{R} \mid u_{\zeta'} > w \text{ in } B(0,R) \right\}, \quad \zeta_2 = \inf \left\{ \zeta' \in \mathbb{R} \mid u_{\zeta'} < w \text{ in } B(0,R) \right\}.$$

Since $u_\zeta \to \pm 1$ locally uniformly as $\zeta \to \pm\infty$, it follows that $-\infty < \zeta_1 \leq \zeta_2 < \infty$.

From the inequality $u_{\zeta_1} \geq u_\zeta$ on $\partial B(0,R)$ and the ordering of $\{u_\zeta\}_{\zeta \in \mathbb{R}}$, we know that $\zeta_1 \leq \zeta$. Similarly, $\zeta \leq \zeta_2$. In particular, this implies $u_{\zeta_2} \leq u_\zeta \leq u_{\zeta_1}$ in $\mathbb{R}^d$.

We claim that $u_{\zeta_1} = u_\zeta = u_{\zeta_2}$ in $\mathbb{R}^d$. To see this, observe that the map $(x, \zeta) \mapsto u_\zeta(x)$ is continuous, and, thus, there is an $x_1 \in \overline{B(0,R)}$ such that $u_{\zeta_1}(x_1) = w(x_1)$. Since $u_{\zeta_1} \geq w$, the strong maximum principle implies that we can assume without loss of generality that $x_1 \in \partial B(0,R)$ (see, e.g., [39, Corollary A.3]). Thus, $u_{\zeta_1}(x_1) = u_\zeta(x_1)$. Since u_ζ and u_{ζ_1} are solutions and $u_{\zeta_1} \geq u_\zeta$ in the whole space, we conclude that $u_\zeta = u_{\zeta_1}$. A similar argument shows $u_{\zeta_2} = u_\zeta$. Since $u_{\zeta_2} \leq w \leq u_{\zeta_1}$ in $\overline{B(0,R)}$ by construction, this implies $u_\zeta = w$.

We showed that if $\zeta \in \mathbb{R}$ and $R > 0$, then

$$\mathcal{F}(u_\zeta; B(0,R)) = \inf\left\{\mathcal{F}(u_\zeta + f; B(0,R)) \mid f \in C_c^\infty(B(0,R); [-1,1])\right\}.$$

Therefore, $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ is a family of Class A minimizers of $\mathcal{F}$.

Finally, we prove that $U \in \mathcal{M}(e)$. To start with, recall that $\partial_s U \geq 0$, hence, for each $\zeta \in \mathbb{R}$ and $k \in \mathbb{Z}^d$, we have

$$u_\zeta(\cdot + k) = u_{\zeta - \langle k, e\rangle} \geq u_\zeta \quad \text{if } \langle k, e\rangle \geq 0, \quad u_\zeta(\cdot + k) = u_{\zeta - \langle k, e\rangle} \leq u_\zeta \quad \text{if } \langle k, e\rangle \leq 0.$$

Since, by what was already proved, $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ consists of Class A minimizers, this proves they are strongly Birkhoff plane-like minimizers. Therefore, by Theorem 21,

$$\lim_{R \to \infty} R^{1-d}\mathcal{F}(u_\zeta; B(0,R)) = \mathscr{E}(e) \quad \text{for each } \zeta \in \mathbb{R}.$$

Thus, by Theorem 19, $\mathscr{T}_e(U) = \mathscr{E}(e)$ and we conclude that $U \in \mathcal{M}(e)$. □

3.2 The Mobility

In this section, we define the mobility $\bar{M}$. Following [16], we would like to define it by $\bar{M}(e) = \|\partial_s U_e\|^2_{L^2(\mathbb{R} \times \mathbb{T}^d)}$ for some $U_e \in \mathcal{M}(e)$ with $\partial_s U_e \in L^2(\mathbb{R} \times \mathbb{T}^d)$. When no such minimizer exists, we set $\bar{M}(e) = \infty$. However, this is not well-defined if $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$.

The difficulty becomes apparent when we recall the construction in Section 3.1.4. Recall the definition of the period r_e in (2.7). Suppose that $U \in \mathcal{M}(e)$ and $\|\partial_s U\|^2_{L^2(\mathbb{R} \times \mathbb{T}^d)} < \infty$. If we fix an increasing $h \in H^1_{\text{loc}}(\mathbb{R})$ such that $h(\zeta + r_e) = h(\zeta) + r_e$ and define $U^h \in \mathcal{M}(e)$ as in Section 3.1.4, then

$$\|\partial_s U^{(h)}\|_{L^2(\mathbb{R} \times \mathbb{T}^d)} = m_e^{-1} \int_0^{m_e} \left\{\mathcal{H}^{d-1}(Q_e)^{-1} \int_{Q_e \oplus_e \mathbb{R}} \partial_\zeta u_\zeta(x)^2 \, dx\right\} h'(\zeta)^2 \, d\zeta.$$

Since the only restriction on the choice of h is that $m_e^{-1}\int_0^{m_e} h'(\zeta)\,d\zeta = 0$, this quantity appears to depend on this choice in general.

Nonetheless, the following definition turns out to be useful. To start with, given $e \in S^{d-1}$, define $\bar{M}(e)$ by

$$\bar{M}(e) = \inf\left\{\|\partial_s U\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)} \mid U \in \mathcal{M}(e)\right\}. \tag{3.10}$$

Let us emphasize that, in this definition, the infimum is only necessary when $e \in \mathbb{R}\mathbb{Z}^d$ since otherwise the minimizer is unique up to translation.

3.2.1 Derivative of the Surface Tension

It turns out that if $\bar{M}(e) < \infty$ in some direction $e \in S^{d-1}$, then $\bar{\sigma}$ is differentiable in that direction.

Theorem 25. *Given any $e \in S^{d-1}$, if $\bar{M}(e) < \infty$, then $\bar{\sigma}$ is differentiable at e. Furthermore, if $U_e \in \mathcal{M}(e)$ is such that $\|\partial_s U_e\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} < \infty$, then*

$$D\bar{\sigma}(e) = \int_{\mathbb{R}\times\mathbb{T}^d} a(y)\mathcal{D}_e U \cdot \partial_s U_e \, dy\, ds. \tag{3.11}$$

Proof. We know that $\bar{\sigma}$ is a convex function. Therefore, it suffices to prove a one-sided Taylor estimate of the form

$$\bar{\sigma}(v) \leq \bar{\sigma}(e) + \langle p, v - e\rangle + o(\|v - e\|)$$

for some $p \in \mathbb{R}^d$. If such an estimate can be obtained, then necessarily $\bar{\sigma}$ is differentiable at e and $D\bar{\sigma}(e) = p$.

Recall the definition of $\mathscr{T}_v$ for arbitrary $v \in \mathbb{R}^d \setminus \{0\}$ from [73, Remark 2]. Given any

$v \in \mathbb{R}^d \setminus \{0\}$, testing with U_e gives

$$\bar{\sigma}(v) \leq \mathscr{T}_v(U_e) = \mathscr{T}_e(U_e) + \int_{\mathbb{R}\times\mathbb{T}^d} \langle a(y)\mathcal{D}_e U_e, v - e\rangle \partial_s U_e \, dy \, ds$$
$$+ \int_{\mathbb{R}\times\mathbb{T}^d} \frac{1}{2}\langle a(y)(v-e), v-e\rangle |\partial_s U_e|^2 \, dy \, ds.$$

Thus, since $\mathscr{T}_e(U_e) = \bar{\sigma}(e)$, we find

$$\bar{\sigma}(v) \leq \bar{\sigma}(e) + \langle p(e), v - e\rangle + \Lambda \|\partial_s U\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)} \|v - e\|^2$$

with $p(e)$ given by the right-hand side of (3.11). □

3.3 Counter-examples to Equipartition of Energy

In this section, we use analysis in $\mathbb{R}\times\mathbb{T}^d$ to address a question raised by Choksi, Fonseca, Lin, and Raghavendran [33] concerning equipartition of energy. More precisely, recall that, in the spatially homogeneous setting, the strongly Birkhoff plane-like minimizers in the direction e are of the form $u_\zeta(x) = q(\langle x, e\rangle - \zeta)$, where q is the solution of the ODE

$$-q'' + W'(q) = 0 \quad \text{in } \mathbb{R}, \qquad \lim_{s\to\pm\infty} q(s) = \pm\infty, \quad q(0) = 0.$$

This is a Hamiltonian ODE, hence the function $s \mapsto \frac{1}{2}q'(s)^2 - W(s)$ is constant. As $|s| \to \infty$, it vanishes. Therefore, so-called equipartition of energy holds: if u_ζ is a strongly Birkhoff plane-like minimizer, then

$$\frac{1}{2}\|Du_\zeta\|^2 = W(u_\zeta) \quad \text{in } \mathbb{R}^d.$$

In plain English, the two terms in $\mathcal{F}$ contribute equally to the energy.

A natural question is whether or not equipartition of energy holds in the periodic setting. In [33], the authors proposed a method for approximating the surface tension based on energy equipartition in the special case where a and W satisfy

$$a \equiv \mathrm{Id}, \quad W(y,u) = \theta(y)W(u),$$

where $\theta : \mathbb{T}^d \to (0,\infty)$ is smooth. Their method involves computing the average energy of the function u_e given by

$$u_e(x) = q(h_e(x)),$$

where h_e is the signed distance to the hyperplane $\{x \in \mathbb{R}^d \mid \langle x, e\rangle = 0\}$ with respect to the Riemannian metric induced by $\sqrt{\theta}$. More precisely, h_e is the unique viscosity solution of the Eikonal-type equation

$$\begin{cases} \|Dh_e\| - \sqrt{\theta(y)} = 0 & \text{in } \{x \in \mathbb{R}^d \mid \langle x, e\rangle > 0\}, \\ h_e = 0 & \text{on } \{x \in \mathbb{R}^d \mid \langle x, e\rangle = 0\}, \\ \sqrt{\theta(y)} - \|Dh_e\| = 0 & \text{in } \{x \in \mathbb{R}^d \mid \langle x, e\rangle < 0\}. \end{cases}$$

Notice that with this choice of h_e, u_e has the equipartition property, namely,

$$\frac{1}{2}\|Du_e\|^2 = \theta(y)W(u_e) \quad \text{in } \mathbb{R}^d.$$

Following [33], let us denote by $\lambda(e)$ the average energy of u_e, that is,

$$\lambda(e) = \lim_{R\to\infty} R^{1-d} \int_{\mathbf{Q}^e(0,R)} \left(\frac{1}{2}\|Du_e(y)\|^2 + \theta(y)W(u_e(y))\right) dy. \tag{3.12}$$

$\lambda(e)$ is proposed as an approximation of $\bar{\sigma}(e)$ in [33], and the question is raised whether or not $\lambda(e) = \bar{\sigma}(e)$ for some choices of e and θ. The next result shows that $\lambda(e)$ is well-defined and $\lambda(e) > \bar{\sigma}(e)$ always holds for non-constant θ.

Theorem 26. *For each $e \in S^{d-1}$, the limit in the definition of $\lambda(e)$ exists. Furthermore, $\lambda(e) > \bar{\sigma}(e)$ holds if θ is non-constant.*

The theorem will be proved in two parts corresponding to rational and irrational directions.

In the proof, it will be useful to know that h_e grows linearly at large scales. Toward that end, the next result is enough for our purposes.

Proposition 30. *Let* $\underline{\theta} = \min\{\theta(y) \mid y \in \mathbb{T}^d\}$ *and* $\overline{\theta} = \max\{\theta(y) \mid y \in \mathbb{T}^d\}$*. For each* $y \in \mathbb{T}^d$*, we have*

$$\underline{\theta}|\langle y, e\rangle| \leq |h_e(y)| \leq \bar{\theta}|\langle y, e\rangle|.$$

The proposition follows directly from the comparison principle, hence the proof is omitted.

3.3.1 Rational Case

The proof of Theorem 26 is easier in the rational case (that is, when $e \in \mathbb{R}\mathbb{Z}^d$) due, in effect, to compactness. Notice, for instance, that h_e descends to a function in $\mathbb{T}^{d-1}_e \oplus_e \mathbb{R}$ by the comparison principle and the periodicity of θ. From this, it follows immediately that

$$\begin{aligned}\lim_{R \to \infty} R^{1-d} &\int_{Q^e(0,R) \oplus_e \mathbb{R}} \left(\frac{1}{2} \|D(q \circ h_e)(y)\|^2 + \theta(y) W((q \circ h_e)(y)) \right) dy \\ &= \mathcal{H}^{d-1}(Q_e)^{-1} \int_{Q_e \oplus_e \mathbb{R}} \left(\frac{1}{2} \|D(q \circ h_e)(y)\|^2 + \theta(y) W((q \circ h_e)(y)) \right) dy.\end{aligned}$$

Notice that this allows us to bound $\lambda(e)$ using Proposition 27.

The proof that follows was pointed out to me by W.M. Feldman.

Proof of Theorem 26, rational case. The discussion just preceding this proof shows that the limit (3.12) is well-defined. It only remains to prove that if θ is non-constant, then $\lambda(e) > \bar{\sigma}(e)$. Let us argue by contrapositive.

Assume that $\lambda(e) = \bar{\sigma}(e)$. We claim that this implies θ is a constant. To prove this, we will show that $-\Delta h_e = 0$ in $\mathbb{R}^d$. Lemma 16 in the appendix then implies that θ must be constant.

Recall that u_e is the plane-like function defined by $u_e(x) = q(h_e(x))$. From the discussion preceding this proof, we know that u_e descends to a function in $\mathbb{T}_e^{d-1} \oplus_e \mathbb{R}$ and

$$\mathcal{H}^{d-1}(Q_e)^{-1}\mathcal{F}(u_e; Q_e \oplus_e \mathbb{R}) = \lambda(e) = \bar{\sigma}(e).$$

Furthermore, Proposition 27 and the limit $\lim_{s\to\pm\infty} q(s) = \pm 1$ together imply that

$$\lim_{s\to\pm\infty} u_e(\cdot + se) = \pm 1 \quad \text{locally uniformly in } \mathbb{R}^d.$$

Thus, by Proposition 27, u_e is a minimizer of $\mathcal{F}(\cdot; Q_e \oplus_e \mathbb{R})$ among periodic functions heteroclinic between -1 and 1, and it satisfies the Euler-Lagrange PDE

$$-\Delta u_e + \theta(y)W'(u_e) = 0 \quad \text{in } \mathbb{R}^d.$$

Since $u_e = q \circ h_e$ and $\|Dh_e\|^2 = \theta$ a.e., the equation above can be rewritten as

$$0 = -q''(h_e)\theta(y) - q'(h_e)\Delta h_e + \theta(y)W'(q(h_e)) \quad \text{a.e. in } \mathbb{R}^d.$$

Recall from the start of this section that $\frac{1}{2}q'(s)^2 = W(q(s))$, hence

$$0 = -q'(h_e)\Delta h_e \quad \text{a.e. in } \mathbb{R}^d.$$

It is a fact that q' is positive everywhere (see, e.g., [72, Appendix C.1]). Thus, $-\Delta h_e = 0$ in $\mathbb{R}^d$, as claimed. $\square$

3.3.2 Irrational Case

In the proof of the theorem below, to lighten the notation, we will write

$$\Sigma_e^\zeta = \left\{ y \in \mathbb{R}^d \mid \langle y, e \rangle = \zeta \right\} \quad \text{for } e \in S^{d-1},\ \zeta \in \mathbb{R}.$$

In addition to h_e, it will be convenient to consider the full family of functions $(h_e^\zeta)_{\zeta \in \mathbb{R}}$ defined so that h_e^ζ is the signed distance to Σ_e^ζ with respect to the Riemannian distance induced by $\sqrt{\theta}$. In other words, for each $\zeta \in \mathbb{R}$, h_e^ζ is the solution of the Eikonal equation:

$$\begin{cases} \|Dh_e^\zeta\| - \sqrt{\theta(y)} = 0 & \text{in } \{y \in \mathbb{R}^d \mid \langle y, e \rangle > 0\}, \\ h_e^\zeta = 0 & \text{on } \Sigma_e^\zeta, \\ \sqrt{\theta(y)} - \|Dh_e^\zeta\| = 0 & \text{in } \{y \in \mathbb{R}^d \mid \langle y, e \rangle < 0\}. \end{cases}$$

The reason we consider $(h_e^\zeta)_{\zeta \in \mathbb{R}}$ is these functions naturally lift to $\mathbb{R} \times \mathbb{T}^d$, which makes analyzing the limit in the definition (3.12) of $\lambda(e)$ easier. The existence of this lift is implied by the next result.

Proposition 31. *(i) If $k \in \mathbb{Z}^d$, then $h_e^{\zeta - \langle k, e \rangle}(x) = h_e^\zeta(x + k)$ for each $x \in \mathbb{R}^d$.*

(ii) $\zeta \mapsto h_e^\zeta$ is uniformly continuous in the following sense:

$$\lim_{\delta \to 0^+} \sup \left\{ |h_e^\zeta(x) - h_e^{\zeta'}(x)| \mid x \in \mathbb{R}^d,\ |\zeta - \zeta'| < \delta \right\} = 0.$$

(iii) For each $\zeta \in \mathbb{R}$ and $y \in \mathbb{R}^d$,

$$\underline{\theta} |\langle y, e \rangle - \zeta| \leq |h_e^\zeta(y)| \leq \overline{\theta} |\langle y, e \rangle - \zeta|.$$

The proposition follows readily from the comparison principle and the periodicity of θ, therefore, the proof is omitted.

In order to regain some compactness and periodicity, it is convenient to lift the functions

$(h_e^\zeta)_{\zeta\in\mathbb{R}}$ to the cylinder $\mathbb{R}\times\mathbb{T}^d$.

Proposition 32. *There is a $H_e \in UC(\mathbb{R}\times\mathbb{T}^d)$ such that*

$$h_e^\zeta(x) = H_e(\langle x, e\rangle - \zeta, x) \quad \textit{if } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}. \tag{3.13}$$

Proof. Define $H_e : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ by

$$H_e(s, y) = h_e^{\langle y,e\rangle - s}(y).$$

Since $(\zeta, y) \mapsto (\langle y, e\rangle - \zeta, y)$ is an involution, (3.13) holds.

Note that Proposition 31, (i) implies that H_e is periodic in its second argument.

Finally, by Proposition 31, (ii) and (iii), H_e is uniformly continuous. □

We are ultimately interested in $q \circ h_e^\zeta$ rather than h_e^ζ. Accordingly, it will be useful to define $Q_e \in UC(\mathbb{R}\times\mathbb{T}^d)$ by

$$Q_e(s, y) = q(H_e(s, y)).$$

Hence the following identity holds:

$$q(h_e^\zeta(x)) = Q_e(\langle x, e\rangle - \zeta, x).$$

The next result links the action of $\mathcal{D}_e$ in $\mathbb{R}\times\mathbb{T}^d$ to differentiation in $\mathbb{R}^d$:

Proposition 33. *For a.e. $(s, y) \in \mathbb{R}\times\mathbb{R}^d$, we have*

$$\begin{aligned}(\mathcal{D}_e H_e)(s, y) &= Dh_e^{\langle y,e\rangle - s}(y),\\ (\mathcal{D}_e Q_e)(s, y) &= q'(h_e^{\langle y,e\rangle - s}(y))Dh_e^{\langle y,e\rangle - s}(y).\end{aligned}$$

Proof. We know that $Dh_e^\zeta(x)$ exists for almost every $(\zeta, x) \in \mathbb{R}\times\mathbb{R}^d$. Accordingly, if E is the set of such points, then the set F given by

$$F = \{(s,y) \in \mathbb{R} \times \mathbb{R}^d \mid (\langle y, e \rangle - s, y) \in E\},$$

has full measure in $\mathbb{R}\times\mathbb{R}^d$. (The transformation $(s,y) \mapsto (\langle y,e\rangle - s, y)$ is measure-preserving.)

Note that if $(s,y) \in F$, then

$$H_e(s + h \cdot \nu, y + h) = h_e^{\langle y,e \rangle - s}(y+h) = h_e^{\langle y,e\rangle - s}(y) + Dh_e^{\langle y,e\rangle - s}(y) \cdot h + o(h) \quad \text{as } \|h\| \to 0.$$

This proves that $\mathcal{D}_e H_e(s,y)$ exists and is given by the indicated formula.

The formula for $\mathcal{D}_e Q_e$ follows from the chain rule. □

We now apply Theorem 19 to Q_e to deduce the well-definedness of λ.

Proposition 34. *For each $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, the number $\lambda(e)$ is well-defined in (3.12). In fact, for each $\zeta \in \mathbb{R}$,*

$$\begin{aligned} \lambda(e) &= \int_{\mathbb{R}\times\mathbb{T}^d} \left(\frac{1}{2}\|\mathcal{D}_e Q_e\|^2 + \theta(y) W(Q_e) \right) dy\, ds \\ &= \lim_{R\to\infty} R^{1-d} \int_{Q^e(0,R)\oplus_e \mathbb{R}} \left(\frac{1}{2}\|D(q \circ h_e^{\zeta})\|^2 + \theta(y) W((q\circ h_e^{\zeta})) \right) dx. \end{aligned} \tag{3.14}$$

Proof. The strategy is to define $\lambda(e)$ by

$$\lambda(e) = \int_{\mathbb{R}\times\mathbb{T}^d} \left(\frac{1}{2}\|\mathcal{D}_e Q_e\|^2 + \theta(y) W(Q_e) \right) dy\, ds \tag{3.15}$$

and then to show that this agrees with (3.12) by invoking Theorem 19.

Integrability: We start by showing that $\frac{1}{2}\|\mathcal{D}_e Q_e\|^2 + \theta(y)W(Q_e)$ is integrable. To start with, recall that $\frac{1}{2}q'(s)^2 = W(q(s))$ for each $s \in \mathbb{R}$. Thus, by Proposition 33,

$$\frac{1}{2}\|D_e Q_e\|^2 + \theta(y) W(Q_e) = \theta(y) q'(H_e)^2 \quad \text{a.e. in } \mathbb{R}\times\mathbb{R}^d.$$

We know that $0 < q'(s) \leq e^{-c|s|}$ for some $c > 0$ (see, e.g., [72, Appendix C.1]). Furthermore, if $\underline{\theta} = \min\{\theta(y) \mid y \in \mathbb{T}^d\}$ and $\overline{\theta} = \max\{\theta(y) \mid y \in \mathbb{T}^d\}$, then the comparison principle implies that

$$\underline{\theta}|\langle x, e\rangle - \zeta\rangle| \leq h_e^{\zeta}(x) \leq \overline{\theta}|\langle x, e\rangle - \zeta|.$$

From this, we deduce that

$$\int_{\mathbb{R}\times\mathbb{T}^d} \left(\frac{1}{2}\|D_e Q_e\|^2 + \theta(y)W(Q_e)\right) dy\, ds \leq \overline{\theta} \int_{\mathbb{R}\times\mathbb{T}^d} e^{-2c|s|}\, dy\, ds < \infty.$$

Hypotheses of Theorem 19, (iii): We check that the function V given by

$$V(s, y) = \frac{1}{2}\|(\mathcal{D}_e Q_e)(s, y)\|^2 + \theta(y)W(Q_e(s, y)) = \theta(y)q'(H_e(s, y))^2.$$

satisfies the assumptions of Theorem 19, (iii).

The previous part of the proof showed that $V \in L^1(\mathbb{R} \times \mathbb{T}^d)$. Since q' and H_e are both uniformly continuous, $V \in UC(\mathbb{R} \times \mathbb{T}^d)$. Furthermore, if $G : \mathbb{T}^d \to \mathbb{R}$ is given by

$$G(y) = \int_{-\infty}^{\infty} V(s, se + y)\, ds,$$

then

$$0 \leq V(s, se + y) = \theta(y)q'(H_e(s, se + y))^2 \leq \overline{\theta}e^{-2c|s|}$$

so the dominated convergence theorem implies that G is continuous.

Conclusion: We showed that V satisfies all the hypotheses of Theorem 19, (iii). Accordingly, for each $\zeta \in \mathbb{R}$, the limit in (3.14) exists and equals (3.15). □

Proof of Theorem 26, irrational case. Suppose that $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. The previous proposition shows that $\lambda(e)$ is well-defined. It remains to show that $\lambda(e) > \bar{\sigma}(e)$ unless θ is constant.

Let us argue by contrapositive, that is, we will assume that $\lambda(e) = \bar{\sigma}(e)$ and prove that θ must be constant.

Assume that $\lambda(e) = \bar{\sigma}(e)$. It follows from Proposition 34 that Q_e minimizes $\mathscr{T}_e$ (i.e, $Q_e \in \mathcal{M}(e)$). Therefore, Q_e is a weak solution of the pulsating standing wave equation

$$\mathcal{D}_e^* \mathcal{D}_e Q_e + \theta(y) W'(Q_e) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$

(See the proof of [73, Proposition 20].) Recalling that $Q_e = q \circ H_e$ and $\|\mathcal{D}_e H_e\|^2 = \theta$ a.e., this implies that, in the distributional sense,

$$q'(H_e)\mathcal{D}_e^* \mathcal{D}_e H_e - q''(H_e)\theta(y) + \theta(y) W'(q(H_e)) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$

At the same time, we know that q satisfies $-q'' + W'(q) = 0$ in $\mathbb{R}$, so cancellation yields

$$-q'(H_e)\mathcal{D}_e^* \mathcal{D}_e H_e = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$

Thus, arguing as in [73, Proposition 20], we deduce that, for a.e. $\zeta \in \mathbb{R}$,

$$-q'(h_e^\zeta)\Delta h_e^\zeta = 0 \quad \text{in } \mathbb{R}^d.$$

We know that q' is positive everywhere so this implies $-\Delta h_e^\zeta = 0$ in $\mathbb{R}^d$ for all $\zeta \in \mathbb{R}$. Finally, we invoke Lemma 16 in the appendix to conclude that θ is a constant function. $\square$

3.4 Notes

In the proof of the existence of pulsating standing waves, we used the fact that strongly Birkhoff plane-like minimizers exist in every direction. This question was considered by a number of different authors from a variety of viewpoints. This includes contributions of

Alessio, Jeanjean, and Montecchiari [2], Rabinowitz and Stredulinsky [81], and Valdinoci [90]. Later, it was understood that existence also follows from the fundamental work of Bangert [9]. The fact that Allen-Cahn-type functionals fit into the framework of Bangert's work is explained in both the book of Rabinowitz and Stredulinsky [82] and the paper by Junginger-Gestrich and Valdinoci [60].

For a proof of the existence of pulsating standing waves that does not rely on the existence of plane-like minimizers of $\mathcal{F}$, see [73]. The proof given there also does not require any regularity on a or W.

The functional $\mathscr{T}_e$ is closely related to the so-called Percival Lagrangian in the Aubry-Mather and Moser-Bangert theories; see, for instance, Moser [74], Bessi [20], and de la Llave and Su [88].

In Section 3.2.1, we computed the derivative of the surface tension in directions in which there is a sufficiently regular pulsating standing wave. It is expected that the subdifferential of the surface tension can be completely characterized in terms of the structure of plane-like minimizers. Results of this type are known in the Aubry-Mather and Moser-Bangert theories, notably the works of Senn [85, 84] in the latter context.

CHAPTER 4

SHARP-INTERFACE LIMIT IN LAMINAR MEDIA

In this chapter, we prove Theorem 2, a sharp-interface limit for graphical interfaces in laminar media. More precisely, we are interested in the $\epsilon \to 0$ asymptotics of the gradient flow

$$\begin{cases} m(\epsilon^{-1}x, \epsilon Du^\epsilon)u^\epsilon_t - \text{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W_u(\epsilon^{-1}x, u^\epsilon) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{4.1}$$

in the special case when m, a, and W are laminar, which, for concreteness, we take to mean

$$m_{y_d} \equiv 0, \quad a_{y_d} \equiv 0, \quad W_{y_d} \equiv 0. \tag{4.2}$$

In plain English, m, a, and W are independent of the dth spatial coordinate y_d.

There are two major pay-offs here. First, the theorem gives examples where (4.1) homogenizes with a non-trivial effective interface motion emerging in the limit. Second, we show below that an Einstein relation describes the effective velocity. In particular, the homogenized velocity is determined by the formula

$$V_{\partial E_t} = \bar{\mathcal{M}}(m, n_{\partial E_t})^{-1} \text{tr}\left(D^2\bar{\sigma}(n_{\partial E_t})A_{\partial E_t}\right),$$

where $\bar{\sigma}$ is the surface tension, as defined in the last two chapters. Thus, Theorem 2 shows that, at least in this setting, “gradient flow and homogenization commute.”

4.1 Formal Asymptotics

We will not give the full proof of Theorem 2 here; see [73] for all the details. However, in this section, we will describe the formal asymptotics that motivate the proof. Later sections of this chapter are devoted to the analysis of pulsating standing waves and correctors, which is needed to make the formal analysis rigorous. The final section explains why it is possible to tie together these ingredients into a proof in the special case when the interface is a graph crossing the laminations.

The formal analysis described below was originally devised, in the spatially homogeneous setting, by Rubinstein, Keller, and Sternberg [83]; the specific approach used here for the periodic setting was introduced by Barles and Souganidis [16].

4.1.1 Preliminaries

Recall that we expect that there are open sets $\{E_t\}_{t \geq 0}$ such that

$$\{u^\epsilon(\cdot, t) \approx 1\} \to E_t \quad \text{and} \quad \{u^\epsilon(\cdot, t) \approx -1\} \to \mathbb{R}^d \setminus \overline{E}_t \quad \text{as } \epsilon \to 0^+.$$

The question is the identification of $\{E_t\}_{t \geq 0}$. Let us hypothesize that it is governed by a geometric flow of the following form:

$$\bar{\mathcal{M}}(m, n_{\partial E_t}) V_{\partial E_t} = \operatorname{tr}\left(\bar{\mathcal{S}}(n_{\partial E_t}) A_{\partial E_t}\right). \tag{4.3}$$

Here $\bar{\mathcal{S}}$ and $\bar{\mathcal{M}}$ are effective coefficients, which we expect to appear in the sharp-interface limit due to averaging.

Since we are arguing formally, we will assume the sets $\{E_t\}_{t \geq 0}$ have smooth boundaries, which vary smoothly as functions of t.

To relate the solution u^ϵ of (4.1) to the macroscopic interface ∂E_t, we will use the signed distance function $d : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ to $\{E_t\}_{t \geq 0}$, defined by

$$d(x,t) = \begin{cases} \operatorname{dist}(x, \partial E_t), & \text{if } x \in E_t, \\ -\operatorname{dist}(x, \partial E_t), & \text{otherwise.} \end{cases}$$

Note that the smoothness of the evolution $\{E_t\}_{t\geq 0}$ implies the smoothness of d locally near points on the interface, and (4.3) holds if and only if d satisfies

$$\bar{\mathcal{M}}(m, Dd)d_t - \operatorname{tr}\left(\bar{\mathcal{S}}(Dd)D^2 d\right) = 0 \quad \text{on} \bigcup_{t\geq 0} \partial E_t \times \{t\}. \tag{4.4}$$

4.1.2 Asymptotic Expansion

Following [16], let us formally expand the solution u^ϵ of (4.1) as follows:

$$\begin{aligned} u^\epsilon(x,t) = {} & U_{Dd(x,t)}\left(\epsilon^{-1}d(x,t), \epsilon^{-1}x\right) + \epsilon Q_{Dd(x,t)}^{D^2 d(x,t)}\left(\epsilon^{-1}d(x,t), \epsilon^{-1}x\right) \\ & + \epsilon P_{Dd(x,t)}^{d_t(x,t)}(\epsilon^{-1}d(x,t), \epsilon^{-1}x) + \cdots \end{aligned} \tag{4.5}$$

Here $\{U_e\}_{e\in S^{d-1}}$, $\{Q_e^X\}_{(e,X)\in S^{d-1}\times\mathcal{S}_d}$, and $\{P_e^q\}_{(e,q)\in S^{d-1}\times\mathbb{R}}$ are functions to be determined and d is the signed distance to $\{E_t\}_{t\geq 0}$ as above.

We search for $\{U_e\}$, $\{Q_e^X\}$, and $\{P_e^q\}$ among functions that are periodic in the second variable, reflecting the periodicity of the coefficients m, a, and W. Hence we fix the domain of these functions to be $\mathbb{R}\times\mathbb{T}^d$.

In what follows, we will assume everything is smooth, including not only the evolving sets $\{E_t\}_{t\geq 0}$, but also the functions $\{U_e\}$, $\{Q_e^X\}$, and $\{P_e^q\}$ and the dependence of these functions on the parameters (e, X, q).

For the ansatz (4.5) to produce a solution of (4.1), we require that

$$0 = m(\epsilon^{-1}x, \epsilon Du^\epsilon)u_t^\epsilon - \operatorname{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W_u(\epsilon^{-1}x, u^\epsilon) = \epsilon^{-2}A_1 + \epsilon^{-1}A_2 + \dots, \tag{4.6}$$

where the neglected terms are of lower order in ϵ.

4.1.3 Vanishing to order ϵ^{-2}.

Setting $A_1 = 0$ and substituting $y = \epsilon^{-1}x$ and $e = Dd(x,t)$ leads to the following equations for $\{U_e\}$:

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e U_e) + W_u(y, U_e) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^d. \tag{4.7}$$

At the same time, away from the interface ∂E_t, we know that $u^\epsilon \approx 1$ in E_t and $u^\epsilon \approx -1$ outside. That suggests the limiting condition

$$\lim_{s \to \pm\infty} U_e(s, y) = \pm 1.$$

Finally, it is convenient to add the monotonicity assumption $\partial_s U_e > 0$. Thus, we arrive precisely at the pulsating standing wave equation (3.1) from Chapter 3.

Next, we set $A_2 = 0$ and proceed similarly. At this stage, derivatives of the map $e \mapsto U_e$ appear. In order to obviate the need for calculus on manifolds, it is convenient to extend $\{U_e\}_{e \in S^{d-1}}$ to $\{U_v\}_{v \in \mathbb{R}^d \setminus \{0\}}$ according to the rule (cf. [73, Remark 2])

$$U_v(s, y) = U_e(\|v\| s, y) \quad \text{for } e = \frac{v}{\|v\|}.$$

Next, to simplify the notation, we define the vector-valued functions $\{R_v\}_{v \in \mathbb{R}^d \setminus \{0\}}$ to be the derivative of $v \mapsto U_v$, hence

$$\langle R_v(s, y), \xi \rangle = \lim_{h \to 0} \frac{U_{v + h\xi}(s, y) - U_v(s, y)}{h} \quad \text{for } (s, y) \in \mathbb{R} \times \mathbb{T}^d, \ \xi \in \mathbb{R}^d. \tag{4.8}$$

4.1.4 Vanishing to order ϵ^{-1}.

We now proceed to investigate the consequences of the identity $A_2 = 0$. Making the substitutions

$$y = \epsilon^{-1}x, \quad e = Dd(x,t), \quad X = D^2 d(x,t), \quad q = d_t(x,t),$$

we derive the equation

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e(Q_e^X + P_e^q)) + W_{uu}(y, U_e)(Q_e^X + P_e^q) = G(s, y, e, X, q) \quad \text{in } \mathbb{R} \times \mathbb{T}^d, \tag{4.9}$$

where G is given by

$$\begin{aligned} G(s, y, e, X, q) &= G_1(s, y, e, X) - G_2(s, y, e, q), \\ G_1(s, y, e, X) &= \operatorname{tr}(a(y)X)\, \partial_s U_e + 2\langle a(y)e, X \partial_s R_e(s, y)\rangle \\ &\quad + 2\operatorname{tr}(a(y) D_y R_e(s, y) X) + \langle (\operatorname{div} a)(y), X R_e(s, y)\rangle, \\ G_2(s, y, e, q) &= q m(y, \mathcal{D}_e U_e) \partial_s U_e. \end{aligned}$$

The question now is the solvability of the linear equation (4.9).

Here is where $\bar{\mathcal{M}}$ and $\bar{\mathcal{S}}$ come into the picture. There is a natural solvability condition associated with equations of the form

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e P) = F(s, y) \quad \text{in } \mathbb{R} \times \mathbb{T}^d.$$

To see this, first, notice that differentiating (4.7) with respect to s shows that the function $V_e := \partial_s U_e$ solves the linear PDE

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e V_e) + W_{uu}(y, U_e) V_e = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^d.$$

Hence, multiplying the previous equation by V_e and integrating by parts, we obtain

$$0 = \int_{\mathbb{R}\times\mathbb{T}^d} (\mathcal{D}_e^*(a(y)\mathcal{D}_e V_e) + W_{uu}(y, U_e)V_e)\, P \, dy\, ds = \int_{\mathbb{R}\times\mathbb{T}^d} F(s, y) V_e \, dy \, ds.$$

Due to this solvability condition, we are led to the following equations for Q_e^X and P_e^q:

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e Q_e^X) + W_{uu}(y, U_e)Q_e^X = G_1(s, y, e, X) - \overline{G}_1(e, X)\partial_s U_e \quad \text{in } \mathbb{R} \times \mathbb{T}^d, \tag{4.10}$$

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e P_e^q) + W_{uu}(y, U_e)P_e^q = -\left[G_2(s, y, e, q) - \overline{G}_2(e, q)\partial_s U_e\right] \quad \text{in } \mathbb{R} \times \mathbb{T}^d. \tag{4.11}$$

These equations are solvable only for the constants $\overline{G}_1$ and $\overline{G}_2$ given by:

$$\begin{aligned}\overline{G}_1(e, X) = \|V_e\|^{-2}_{L^2(\mathbb{R}\times\mathbb{T}^d)} \int_{\mathbb{R}\times\mathbb{T}^d} V_e \big(&\mathrm{tr}(a(y)X)V_e + 2\langle a(y)e, X\partial_s R_e\rangle + 2\mathrm{tr}(a(y)D_y R_e X) \\ &+\langle (\operatorname{div} a)(y), XR_e\rangle\big)\, dy\, ds, \\ \overline{G}_2(e, q) = q\|V_e\|^{-2}_{L^2(\mathbb{R}\times\mathbb{T}^d)} \int_{\mathbb{R}\times\mathbb{T}^d} &m(y, \mathcal{D}_e U_e)|V_e|^2\, dy\, ds.\end{aligned}$$

By linearity in X and q, we can fix a symmetric matrix-valued function $\bar{\mathcal{S}}$ and a positive function $\bar{\mathcal{M}}$ such that

$$\mathrm{tr}(\bar{\mathcal{S}}(e)X) = \overline{G}_1(e, X)\|V_e\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)}, \quad \bar{\mathcal{M}}(m, e)q = \overline{G}_2(e, q)\|V_e\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Finally, in order for the sum $P_e^X + P_e^q$ to solve (4.9), we require that $\overline{G}_1(e, X) - \overline{G}_2(e, q) = 0$, which, rewritten in terms of d, yields the equation

$$\bar{\mathcal{M}}(m, Dd(x, t))d_t(x, t) - \mathrm{tr}(\bar{\mathcal{S}}(Dd(x, t))D^2 d(x, t)) = 0,$$

which is precisely the PDE we sought to derive.

4.2 Smooth Pulsating Waves in Laminar Media

In the remainder of this chapter, we will indicate how the formal asymptotics presented in the previous section can be made rigorous in the setting of Theorem 2. To begin, we will show that, in laminar media, smooth pulsating standing waves always exist in directions that cross the laminations. Furthermore, these pulsating waves can be parametrized so that they vary smoothly with respect to the direction, just as was assumed above.

To motivate what follows, observe that it is not unreasonable to postulate that a pulsating standing wave U_e should have the same symmetries as the coefficients. Hence if (4.2) holds, then we might expect U_e to have the same property, namely, $(U_e)_{y_d} \equiv 0$. In the case that $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$, when we consider the plane-like minimizers $\{u_\zeta\}_{\zeta\in\mathbb{R}}$ generated by U_e, we find that

$$\begin{aligned} u_\zeta(x) &= U_e(\langle x, e\rangle - \zeta, x) \\ &= U_e\left(\langle x, e\rangle - \zeta, x - \zeta\langle e_d, e\rangle^{-1} e_d\right) \\ &= u_0(x - \zeta\langle e_d, e\rangle^{-1} e_d). \end{aligned} \tag{4.12}$$

Thus, $\partial_\zeta u_\zeta = \langle e_d, e\rangle^{-1}(u_\zeta)_{y_d}$ and so elliptic regularity implies that $\partial_\zeta u_\zeta$ is smooth. In particular, this makes U_e smooth in the s variable, hence in all the variables.

4.2.1 Laminar Pulsating Standing Waves

Let us show that, indeed, when the coefficients are laminar (i.e., (4.2) holds), it is possible to find a laminar pulsating standing wave in directions that cross the laminations.

Proposition 35. *Given any $e \in S^{d-1} \setminus \langle e_d\rangle^\perp$, if the laminarity assumption* (4.2) *holds, then there is a unique $U_e \in \mathcal{M}(e)$ such that $(U_e)_{y_d} \equiv 0$ and $\int_{\mathbb{T}^d} U_e(0, y)\, dy = 0$.*

For the rest of the chapter, we will always denote by U_e the laminar minimizer from this proposition.

To prove the proposition, we will use the following observation. In general, for an arbitrary $e \in S^{d-1}$, the function $(a, p) \mapsto \|ae + p\|^2$ is not coercive in $\mathbb{R} \times \mathbb{R}^d$, the tangent space of $\mathbb{R} \times \mathbb{T}^d$. However, if we fix an $e' \in S^{d-1} \setminus \langle e_d\rangle^\perp$ and restrict attention to the subspace $\mathbb{R} \times \langle e_d\rangle^\perp$ of $\mathbb{R} \times \mathbb{R}^d$, then there is a continuous function $\mu : S^{d-1} \setminus \langle e_d\rangle^\perp \to (0, \infty)$ such that

$$\|ae' + p\|^2 = |\langle e', e_d\rangle|^2 a^2 + \|(e' - \langle e', e_d\rangle e_d)a + p\|^2 \geq \mu(e')(a^2 + \|p\|^2). \tag{4.13}$$

Notice that this implies that if $V \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}^d)$ and $V_{y_d} \equiv 0$, then

$$\|\mathcal{D}_{e'} V\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)} \geq \mu(e') \left(\|\partial_s V\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \|D_y V\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)} \right). \tag{4.14}$$

Put simply, the semi-norm $\|\mathcal{D}_{e'} V\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)}$ is coercive on laminar functions V.

In light of the previous considerations, it is useful to think of functions $V \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}^d)$ with $V_{y_d} \equiv 0$ as functions in $\mathbb{R} \times \mathbb{T}^{d-1}$. Indeed, given such a function V, there is a $\tilde{V} \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{d-1})$ such that

$$V(s, (y_1, \dots, y_{d-1}, y_d)) = \tilde{V}(s, (y_1, \dots, y_{d-1})) \quad \text{for a.e. } (s, (y_1, \dots, y_{d-1}, y_d)) \in \mathbb{R} \times \mathbb{T}^d.$$

Hence we may as well treat V as a function in $\mathbb{R} \times \mathbb{T}^{d-1}$.

This is useful in the context of Proposition 35 since if $V \in \mathcal{M}(e)$ and $V_{y_d} \equiv 0$, then V is a solution of the *uniformly elliptic* PDE

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e V) + W_u(y, V) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^{d-1}.$$

The uniform ellipticity of this PDE is a consequence of (4.13).

Proof of Proposition 35. To start with, suppose that $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ and fix $U \in \mathcal{M}(e)$. We claim that $U_{y_d} \equiv 0$. Indeed, due to the laminarity assumption (4.2), given any $\alpha \in \mathbb{R}$, the function $U^{(\alpha)}$ given by $U^{(\alpha)}(s, y) = U(s, y + \alpha e_d)$ satisfies $U^{(\alpha)} \in \mathcal{M}(e)$. Furthermore, by translation invariance,

$$\int_{\mathbb{T}^d} U^{(\alpha)}(s, y)\, dy = \int_{\mathbb{T}^d} U(s, y)\, dy$$

Therefore, by uniqueness (Theorem 23), it follows that $U^{(\alpha)} = U$ a.e. in $\mathbb{R} \times \mathbb{T}^d$. Since α was arbitrary, we deduce that $U_{y_d} \equiv 0$. We therefore obtain U_e by setting $U_e(s, y) = U(s + s_0, y)$ with $s_0 \in \mathbb{R}$ chosen so that $\int_{\mathbb{T}^d} U(s_0, y)\, dy = 0$.

Finally, if $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, then there is a sequence $(\nu_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ such that $e = \lim_{n \to \infty} \nu_n$. Notice that $(\nu_n)_{n \geq N} \subseteq \mathbb{R}^d \setminus \langle e_d \rangle^{\perp}$ provided N is large enough.

By coercivity (4.14), $(U_{\nu_n})_{n \in \mathbb{N}}$ converges along subsequences in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}^d)$ and pointwise a.e. That follows from the bound

$$\mu(\nu_n) \left(\|\partial_s U_{\nu_n}\|^2_{L^2(\mathbb{R} \times \mathbb{T}^d)} + \|D_y U_{\nu_n}\|^2_{L^2(\mathbb{R} \times \mathbb{T}^d)} \right) \leq \mathscr{T}_{\nu_n}(U_{\nu_n}) = \bar{\sigma}(\nu_n).$$

Suppose, then, that $V = \lim_{j \to \infty} U_{\nu_{n_j}}$ along some subsequence $(n_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$. By Fatou's Lemma,

$$\mathscr{T}_e(V) \leq \liminf_{j \to \infty} \mathscr{T}_{\nu_{n_j}}(U_{\nu_{n_j}}) = \bar{\sigma}(e).$$

At the same time, we know that $\int_{\mathbb{T}^d} V(0, y)\, dy = 0$ so V is not constant, and $\partial_s V \geq 0$ so $\lim_{s \to \pm\infty} V(s, y) = \pm 1$ must hold. In particular, $V \in \mathscr{X}$ and Proposition 27 implies that $V \in \mathcal{M}(e)$.

Finally, let us prove that the function U_e as above is necessarily unique. Suppose that $\tilde{U}_e$ is another such function, that is, $\tilde{U}_e \in \mathcal{M}(e)$, $(\tilde{U}_e)_{y_d} \equiv 0$, and $\int_{\mathbb{T}^d} \tilde{U}_e(0, y)\, dy = 0$. Since these functions are laminar, we can consider them as functions in $\mathbb{R} \times \mathbb{T}^{d-1}$ by neglecting the dth coordinate. We know that U_e and $\tilde{U}_e$ are both solutions of the Euler-Lagrange PDE

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e U) + W_u(y, U) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^{d-1}.$$

By coercivity (4.13), this equation is uniformly elliptic in $\mathbb{R} \times \mathbb{T}^{d-1}$. In particular, it has a strong maximum principle. It is straightforward to show that $U_{\min} = U_e \wedge \tilde{U}_e$ and $U_{\max} = U_e \vee \tilde{U}_e$ are both still minimizers, and $U_{\min} \leq U_{\max}$ obviously holds. Hence, by the strong maximum principle, $U_{\min} < U_{\max}$ or $U_{\min} \equiv U_{\max}$. The latter case implies $U_e \equiv \tilde{U}_e$ as desired. Otherwise, we either have $U_e < \tilde{U}_e$ or $U_e > \tilde{U}_e$ in $\mathbb{R} \times \mathbb{T}^{d-1}$ (since the Euler-Lagrange equation, being uniformly elliptic, makes U_e and $\tilde{U}_e$ continuous in $\mathbb{R} \times \mathbb{T}^{d-1}$). However, neither of these are possible since $\int_{\mathbb{T}^d} U_e(0, y)\, dy = \int_{\mathbb{T}^d} \tilde{U}_e(0, y)\, dy = 0$. □

We showed above that U_e can be considered as the solution of a uniformly elliptic PDE in $\mathbb{R} \times \mathbb{T}^{d-1}$. It follows by elliptic regularity that U_e is smooth. Let us record this fact here for completeness:

Proposition 36. *For any $e \in S^{d-1} \setminus \langle e_d \rangle^{\perp}$, the minimizer U_e of Proposition 35 is in $C^{\infty}(\mathbb{R} \times \mathbb{T}^{d-1})$.*

4.3 Derivatives of U_e

By invoking the bounded width property of plane-like minimizers, we obtain

Proposition 37. *Given $e \in S^{d-1} \setminus \langle e_d \rangle^{\perp}$, let U_e be the minimizer of Proposition 35 and let $(u^e_{\zeta})_{\zeta \in \mathbb{R}}$ be the plane-like minimizers its generates. There are constants $C, \nu > 0$ independent of e such that*

$$|u^e_{\zeta}(x) - 1| \leq C e^{-\nu(\langle x, e \rangle - \zeta)},$$
$$|u^e_{\zeta}(x) + 1| \leq C e^{\nu(\langle x, e \rangle - \zeta)}.$$

Proof. To start with, fix a $\delta \in (0, 1)$ such that $W_{uu}(y, u) \geq \frac{\alpha}{2}$ for each $u \in (-1, -1 + \delta) \cup (1 - \delta, 1)$ and every $y \in \mathbb{T}^d$.

Observe that the normalization $\int_{\mathbb{T}^{d-1}} U_e(0, y)\, dy$ furnishes a $\zeta \in [0, \sqrt{d}]$ and a $\tilde{y} \in [0, 1)^{d-1}$ such that $u_{\zeta}(\tilde{y}) \in (-1 + \delta, 1 - \delta)$. By Theorem 20, there is an $M_{\delta} > 0$ depending on δ but not e such that

$$\{-1 + \delta \leq u^e_{\zeta} \leq 1 - \delta\} \subseteq \left\{ x \in \mathbb{R}^d \mid |\langle x - \tilde{y}, e \rangle| \leq M_{\delta} \right\}.$$

From this, an exercise involving the transformation $U_e \mapsto (u^e_\zeta)_{\zeta \in \mathbb{R}}$ implies that there is a $K > 0$ depending only on δ and e such that

$$\begin{cases} U_e(s,y) \geq 1 - \delta & \text{if } s > K, \\ U_e(s,y) \leq -1 + \delta & \text{if } s < -K. \end{cases}$$

Now notice that $\Psi = 1 - U_e$ satisfies

$$\begin{cases} \mathcal{D}_e^*(a(y)\mathcal{D}_e \Psi) + \frac{\alpha}{2}\Psi \leq 0 & \text{in } \{s > K\}, \\ \Psi \leq \delta & \text{on } \{s = K\}, \\ \lim_{s \to \infty} \Psi = 0 & \text{uniformly in } \mathbb{T}^{d-1}. \end{cases}$$

Let $\bar{\Psi}(s,y) = \delta e^{-\nu(s-K)}$ and observe that $\bar{\Psi}$ is a super-solution of the same equation provided $\nu < \frac{\alpha}{2(\Lambda + d\text{Lip}(a))}$. Thus, by the maximum principle,

$$1 - U_e(s,y) = \Psi(s,y) \leq \delta e^{-\nu(s-K)} \quad \text{if } s \geq K.$$

Arguing similarly, one can show that

$$U_e(s,y) + 1 \leq \delta e^{\nu(s+K)} \quad \text{if } s \leq -K.$$

Putting the estimates together with the trivial bound $|U_e| \leq 1$ and the change-of-variables $\langle x, e \rangle - \zeta = s$, we obtain the desired conclusion. □

By linearizing the equation around U_e, we obtain estimates on $\partial_s U_e$.

Proposition 38. *For each $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$, we have $\partial_s U_e \in L^2(\mathbb{R} \times \mathbb{T}^{d-1})$. Moreover, for each $\delta > 0$, there is a constant C_δ depending on δ, λ, Λ, and W and a constant $\beta > 0$*

depending on Lip(a), d, α, and Λ such that if $\|e - e_d\| \geq \delta$*, then*

$$0 < \partial_s U_e(s,x) \leq C_\delta \|\partial_s U_e\|_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})} e^{-\beta|s|} \quad \textit{for each } (s,y) \in \mathbb{R}\times\mathbb{T}^{d-1}.$$

Proof. The estimate (4.14) directly implies that $\partial_s U_e \in L^2(\mathbb{R}\times\mathbb{T}^{d-1})$.

Now observe that $V_e := \partial_s U_e$ is a weak solution of the uniformly elliptic PDE

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e V_e) + W_{uu}(y, U_e)V_e = 0 \quad \text{in } \mathbb{R}\times\mathbb{T}^{d-1}.$$

By Proposition 37, there is an $M > 0$ (independent of e) such that $W_{uu}(y, U_e) > \frac{\alpha}{2}$ if $|s| \geq M$. Hence, arguing as in Proposition 37, the exponential decay of V_e follows.

Finally, the strict positivity of V_e is a consequence of the Harnack inequality. □

4.3.1 Analysis of $\mathcal{L}_e$

In this section, we analyze the operator $\mathcal{L}_e$ obtained by linearizing the pulsating wave equation around U_e. More precisely, we define the unbounded operator $\mathcal{L}_e$ in $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$ as follows:

$$\begin{cases} D(\mathcal{L}_e) = H^2(\mathbb{R}\times\mathbb{T}^{d-1}), \\ \mathcal{L}_e\Phi = \mathcal{D}_e^*(a(y)\mathcal{D}_e\Phi) + W_{uu}(y,U_e)\Phi. \end{cases}$$

Throughout the remainder of this section, we will write $V_e := \partial_s U_e$ for convenience.

To start with, we prove a useful representation of the quadratic form determined by $\mathcal{L}_e$:

Proposition 39. *If* $\Phi \in H^2(\mathbb{R}\times\mathbb{T}^{d-1})$ *and* $\Psi = V_e^{-1}\Phi$*, then*

$$\int_{\mathbb{R}\times\mathbb{T}^{d-1}} \left(\langle a(y)\mathcal{D}_e\Phi, \mathcal{D}_e\Phi\rangle + W_{uu}(y,U_e)\Phi^2\right) dx\, ds = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \langle a(y)\mathcal{D}_e\Psi, D_e\Psi\rangle V_e^2 \, dx\, ds.$$

Proof. When Φ is smooth, this is a classical argument involving integration-by-parts and the fact that V_e is a positive eigenfunction of $\mathcal{L}_e$. The general case follows by approximation. □

Finally, we will need the following result to construct the correctors used in the analysis of the sharp-interface limit:

Proposition 40. *$\mathcal{L}_e$ is closed, self-adjoint, and Ker$(\mathcal{L}_e) = \langle V_e \rangle$. Moreover, Ran$(\mathcal{L}_e) = \langle V_e \rangle^{\perp}$.*

Proof. Define $\tilde{\alpha} : \mathbb{R} \setminus \{0\} \to (0, \infty)$ by $\tilde{\alpha}(s, y) = W_{uu}(y, \operatorname{sgn}(s))$ and let $\mathcal{L}_\alpha$ on $L^2(\mathbb{R} \times \mathbb{T}^{d-1})$ be the unbounded operator with domain $H^2(\mathbb{R} \times \mathbb{T}^{d-1})$ given by

$$\mathcal{L}_\alpha \Phi = \mathcal{D}_e^*(a(y)\mathcal{D}_e \Phi) + \tilde{\alpha}(s, y)\Phi.$$

By uniform ellipticity, $\mathcal{L}_\alpha$ is a closed operator. Indeed, by L^2 estimates for uniformly elliptic equations (cf. [56, Theorem 9.11] or [45, Section 6.3.1]),

$$\|\Phi\|^2_{H^2([n,n+1]\times\mathbb{T}^{d-1})} \leq C\left(\|\Phi\|^2_{L^2([n-1,n+2]\times\mathbb{T}^{d-1})} + \|\mathcal{L}_\alpha \Phi\|^2_{L^2([n-1,n+2]\times\mathbb{T}^{d-1})}\right).$$

Summing over n, we find

$$\|\Phi\|^2_{H^2(\mathbb{R}\times\mathbb{T}^{d-1})} \leq C\left(\|\Phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})} + \|\mathcal{L}_\alpha \Phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})}\right).$$

Thus, the graph of $\mathcal{L}_\alpha$ is closed in $L^2(\mathbb{R}\times\mathbb{T}^{d-1}) \times L^2(\mathbb{R}\times\mathbb{T}^{d-1})$, and $\mathcal{L}_\alpha$ is a closed operator.

Since $W_{uu}(y, 1) \wedge W_{uu}(y, -1) \geq \underline{W}''(1) \wedge \underline{W}''(-1) > 0$ by assumption, the operator $\mathcal{L}_\alpha^{-1} : L^2(\mathbb{R} \times \mathbb{T}^{d-1}) \to H^2(\mathbb{R} \times \mathbb{T}^{d-1})$ exists and is bounded.

Observe that we can write $\mathcal{L}_e = \mathcal{L}_\alpha + M_\alpha$, where $M_\alpha \Phi = (W_{uu}(y, U_e) - \tilde{\alpha}(s, y))\Phi$ is a bounded linear operator on $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$. In particular, $\mathcal{L}_e = (\mathrm{Id} + M_\alpha \mathcal{L}_\alpha^{-1})\mathcal{L}_\alpha$. Since $\mathcal{L}_\alpha^{-1}$ takes $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$ continuously into $H^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $W_{uu}(y, U_e) - \tilde{\alpha}(s, y) \to 0$ uniformly as $|s| \to \infty$, it follows that $M_\alpha \mathcal{L}_\alpha^{-1}$ is compact. Therefore, by the Fredholm alternative, $\mathrm{Id} + M_\alpha \mathcal{L}_\alpha^{-1}$ is a closed operator with closed range. Since $\mathcal{L}_e = (\mathrm{Id} + M_\alpha \mathcal{L}_\alpha^{-1})\mathcal{L}_\alpha$, we deduce that $\mathcal{L}_e$ is also.

$\mathcal{L}_e$ is clearly symmetric. Therefore, to prove it is self-adjoint, it is only necessary to show that $D(\mathcal{L}_e^*) = D(\mathcal{L}_e)$. This follows, for example, by mollification.

The previous proposition showed $\mathrm{Ker}(\mathcal{L}_e) = \langle V_e \rangle$. Finally, since $\mathcal{L}_e$ is self-adjoint with closed range, the identity $\mathrm{Ran}(\mathcal{L}_e) = \langle V_e \rangle^\perp$ follows. □

4.3.2 Differentiating U_e

Since we are differentiating $\bar{\sigma}$, it is convenient to follow the approach of [73, Remark 2]. Let us define, for each $v \in \mathbb{R}^d \setminus \langle e_d \rangle^\perp$, the pulsating standing wave U_v by

$$U_v(s,x) = U_{\|v\|^{-1}v}(\|v\| s, x).$$

If $\mathscr{T}_v$ is the functional defined in [73, Remark 2], then U_v is a minimizer and $\mathscr{T}_v(U_v) = \bar{\sigma}(v)$. In particular, U_v satisfies

$$\mathcal{D}_v^*(a(y)\mathcal{D}_v U_v) + W_u(y, U_v) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^{d-1}. \tag{4.15}$$

Now we differentiate U_e with respect to e. To start with, we fix $\xi \in \mathbb{R}^d$ and define $R^\xi_{e,h}$ by

$$R^\xi_{e,h}(s,y) = \frac{U_{e+h\xi}(s,y) - U_e(s,y)}{h}.$$

The following result follows from a direct manipulation of the equations (4.15) satisfied by $U_{e+h\xi}$ and U_e.

Proposition 41. *$R^\xi_{e,h}$ satisfies the PDE*

$$\begin{cases} \mathcal{L}_e R^\xi_{e,h} = B_h R^\xi_{e,h} + K_h & \text{in } \mathbb{R} \times \mathbb{T}^{d-1}, \\ \int_{\mathbb{T}^{d-1}} R^\xi_{e,h}(0,x)\, dx = 0, \end{cases}$$

where B_h and K_h are given by

$$B_h = -\int_0^1 \{W_{uu}(y, U_e + t(U_{e+h\xi} - U_e)) - W_{uu}(y, U_e)\}\, dt,$$

$$K_h = \langle \xi, a(y)\mathcal{D}_e V_{e+h\xi}\rangle + \langle \xi, a(y)\mathcal{D}_{e+h\xi} V_{e+h\xi}\rangle + \langle div\, a, \xi\rangle V_{e+h\xi}.$$

The sequence $(K_h)_{h\in(-1,1)}$ is uniformly bounded in $C_0(\mathbb{R}\times\mathbb{T}^{d-1})$ and

$$\lim_{h\to 0} \|B_h\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})} = 0.$$

Now we use uniform ellipticity to pass to the limit $h \to 0$.

Proposition 42. *The limit $R_e^\xi = \lim_{h\to 0} R_{e,h}^\xi$ exists in $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$. Moreover, R_e^ξ is the unique solution of the equation*

$$\begin{cases} \mathcal{L}_e R_e^\xi = 2\langle a(y)\xi, \mathcal{D}_e V_e\rangle + \langle div\, a, \xi\rangle V_e \quad in\ \mathbb{R}\times\mathbb{T}^{d-1}, \\ \int_{\mathbb{R}\times\mathbb{T}^{d-1}} R_e^\xi(0,y)\, dy = 0. \end{cases}$$

In the proof, we will use Schauder estimates for linear elliptic equations (cf. [56, Theorem 6.2]).

Proof. The main technicality in the proof is the need to work around the kernel of $\mathcal{L}_e$. Since $(R_{e,h}^\xi)_{h\in\mathbb{R}} \subseteq L^2(\mathbb{R}\times\mathbb{T}^{d-1})$, we can fix $(Q_{e,h}^\xi)_{h\in\mathbb{R}} \subseteq \langle V_e\rangle^\perp$ and $(c_h)_{h\in\mathbb{R}}$ such that

$$R_{e,h}^\xi = c_h V_e + Q_{e,h}^\xi.$$

Since $U_{e+h\eta} \to U_e$ uniformly as $h \to 0$, we can fix a $\delta > 0$ such that

$$\|B_h\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})} \leq \frac{\alpha}{2} < W_{uu}(y,1) \wedge W_{uu}(y,-1) \quad \text{if } |h| < \delta. \tag{4.16}$$

We will use this to show that $(R_{e,h}^\xi)_{h\in(-\delta,\delta)}$ satisfies an exponential estimate similar to the one derived for V_e in Proposition 38.

We claim that $(Q^{\xi}_{e,h})_{h\in(-\delta,\delta)}$ is pre-compact in $C_0(\mathbb{R}\times\mathbb{T}^{d-1})$ and $(c_h)_{h\in(-\delta,\delta)}$ is bounded in $\mathbb{R}$. To see this, we first prove that $\limsup_{h\to 0}\|R^{\xi}_{e,h}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}<\infty$.

Assume to the contrary that there is a sequence $(h_n)_{n\in\mathbb{N}}\subseteq(-\delta,\delta)$ such that

$$\lim_{n\to\infty}\|R^{\xi}_{e,h_n}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}=\infty.$$

Define $(\tilde{R}_n)_{n\in\mathbb{N}}$ by $\tilde{R}_n=\|R^{\xi}_{e,h_n}\|^{-1}_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}R^{\xi}_{e,h_n}$. Notice that $\tilde{R}_n$ satisfies the PDE

$$\mathcal{L}_e\tilde{R}_n=B_{h_n}\tilde{R}_n+\|R^{\xi}_{e,h_n}\|^{-1}_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}K_{h_n}. \tag{4.17}$$

Thus, Schauder estimates (cf. [56, Theorem 6.2]) imply $(\tilde{R}_n)_{n\in\mathbb{N}}$ is bounded in $C^{2,\mu}_0(\mathbb{R}\times\mathbb{T}^{d-1})$ for some $\mu\in(0,1)$.

We claim that $(\tilde{R}_n)_{n\in\mathbb{N}}$ is pre-compact in $C^2_0(\mathbb{R}\times\mathbb{T}^{d-1})$. Let us write

$$\begin{aligned}
\tilde{R}_n &= \tilde{Q}_n+\tilde{c}_nV_e,\\
\tilde{Q}_n &= \|R^{\xi}_{e,h_n}\|^{-1}_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}Q^{\xi}_{e,h_n},\\
\tilde{c}_n &= \|R^{\xi}_{e,h_n}\|^{-1}_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}c_{h_n}.
\end{aligned}$$

Notice that $\|\tilde{R}_n\|^2_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})}=\|\tilde{Q}_n\|^2_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})}+|\tilde{c}_n|^2\|V_e\|^2_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})}$. Moreover, in view of Proposition 38, Schauder estimates for the equation satisfied by V_e, and the uniform bound $\|\tilde{R}_n\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})}=1$, we can use (4.16) to argue as in Proposition 38 that there are constants $C,\gamma>0$ such that

$$|\tilde{R}_n(s,y)|\leq Ce^{-\gamma|s|}. \tag{4.18}$$

This bound and Schauder estimates imply $(\tilde{R}_n)_{n\in\mathbb{N}}$ is pre-compact in both $C^2_0(\mathbb{R}\times\mathbb{T}^{d-1})$ and $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$. From this, we deduce that $(\tilde{c}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$.

By compactness, we can assume without loss of generality (i.e. by passing to a subsequence) that there is an $\tilde{R} \in C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and a $\tilde{c} \in \mathbb{R}$ such that $\tilde{R}_n \to \tilde{R}$ in $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $\tilde{c}_n \to \tilde{c}$. Passing to the limit in (4.17) and recalling that $B_h \to 0$ uniformly, we find $\mathcal{L}_e\tilde{R} = 0$. Thus, $\tilde{R} = \tilde{c}V_e$ by Proposition 40. On the other hand,

$$\tilde{c}\int_{\mathbb{T}^{d-1}} V_e(0,y)\,dy = \lim_{n\to\infty}\int_{\mathbb{T}^{d-1}} \tilde{R}_n(0,y)\,dy = 0.$$

Since $V_e > 0$ in $\mathbb{R}\times\mathbb{T}^{d-1}$, we conclude that $\tilde{c} = 0$, which gives $\tilde{R} \equiv 0$. This is a contradiction, however, since $\|\tilde{R}\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})} = \lim_{n\to\infty}\|\tilde{R}_n\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{d-1})} = 1$.

From the preceding discussion, we deduce that $(R^{\xi}_{e,h})_{h\in(-\delta,\delta)}$ is bounded in $C_0(\mathbb{R}\times\mathbb{T}^{d-1})$. By Schauder estimates, it is actually bounded in $C_0^{2,\mu}(\mathbb{R}\times\mathbb{T}^{d-1})$. In view of the estimate (4.18), $(R^{\xi}_{e,h})_{h\in(-\delta,\delta)}$ is pre-compact in both $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$, which implies the real numbers $(c_h)_{h\in(-\delta,\delta)}$ are also bounded. Thus, $(Q^{\xi}_{e,h})_{h\in(-\delta,\delta)}$ is pre-compact in $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$ as well.

Pick a sequence $(h_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $h_n \to 0$ as $n \to \infty$. Without loss of generality, we can assume there is a $\bar{Q} \in C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and a $\bar{c} \in \mathbb{R}$ such that $\bar{Q} = \lim_{n\to\infty}\tilde{Q}^{\xi}_{e,h_n}$ in $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $\bar{c} = \lim_{n\to\infty} c_{h_n}$. Passing to the limit in the equations satisfied by $(Q^{\xi}_{e,h})_{h\in(-\delta,\delta)}$, we find

$$\mathcal{L}_e\bar{Q} = \bar{K}, \tag{4.19}$$

where $\bar{K} = \lim_{h\to\infty} K_h = 2\langle\xi, a\mathcal{D}_eV_e\rangle + \langle\operatorname{div} a, \xi\rangle V_e$.

Notice that there is at most one solution of (4.19) in $H^2(\mathbb{R}\times\mathbb{T}^{d-1}) \cap \langle V_e\rangle^\perp$. Indeed, if $\tilde{Q} \in H^2(\mathbb{R}\times\mathbb{T}^{d-1}) \cap \langle V_e\rangle^\perp$ is another solution, then $\mathcal{L}_e(\tilde{Q} - \bar{Q}) = 0$. In particular, by Proposition 40, $\tilde{Q} - \bar{Q} \in \langle V_e\rangle \cap \langle V_e\rangle^\perp = \{0\}$.

The previous paragraph shows that the limiting function $\bar{Q}$ did not depend on the sequence $(h_n)_{n\in\mathbb{N}}$. Furthermore, integrating on $\{0\}\times\mathbb{T}^{d-1}$, we obtain

$$0 = \int_{\mathbb{T}^{d-1}} \bar{Q}(0,x)\,dx + \bar{c}\int_{\mathbb{T}^{d-1}} V_e(0,x)\,dx.$$

Thus, $\bar{c}$ is also uniquely determined, independently of $(h_n)_{n\in\mathbb{N}}$.

Putting it all together, we conclude that there is a unique $Q_e^\xi \in C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and a unique $c\in\mathbb{R}$ such that $Q_e^\xi + cV_e = \lim_{h\to 0} R_{e,h}^\xi$ in $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ and $L^2(\mathbb{R}\times\mathbb{T}^{d-1})$. Furthermore, Q_e^ξ is the unique solution of $\mathcal{L}_e Q_e^\xi = 2\langle\xi, a\mathcal{D}_e V_e\rangle + \langle \operatorname{div} a, \xi\rangle V_e$ in $\langle V_e\rangle^\perp$ and c is determined by the requirement that $\int_{\mathbb{T}^{d-1}}(Q_e^\xi(0,y) + cV_e(0,y))\,dy = 0$. $\square$

Remark 4. *Notice that*

$$\frac{V_{e+h\xi} - V_e}{h} = \partial_s R_{e,h}^\xi,$$
$$\frac{D_x U_{e+h\xi} - D_x U_e}{h} = D_x R_{e,h}^\xi.$$

Thus, the $C_0^2(\mathbb{R}\times\mathbb{T}^{d-1})$ convergence just proved implies $\partial_s R_e^\xi = \lim_{h\to 0}\frac{V_{e+h\xi}-V_e}{h}$ and $D_x R_e^\xi = \lim_{h\to 0}\frac{D_x U_{e+h\xi}-D_x U_e}{h}$ in $C_0(\mathbb{R}\times\mathbb{T}^{d-1})$. Appealing to Schauder estimates and the uniform exponential decay of $(R_{e,h}^\xi)_{h\in(-1,1)}$ as $|s|\to\infty$, we can show this convergence also holds in $L^p(\mathbb{R}\times\mathbb{T}^{d-1})$ for any $p\in[1,\infty)$.

4.4 The Einstein Relation

In this section, we show that the coefficient $\bar{\mathcal{S}}$ predicted in the formal computations of Section 4.1 equals the second derivative of the surface tension in the laminar setting. More precisely, we show that

$$\bar{\mathcal{S}}(e) = D^2\bar{\sigma}(e) \quad \text{for each } e \in S^{d-1}\setminus\langle e_d\rangle^\perp.$$

In the process, we prove that $\bar{\sigma}$ is C^2 in $\mathbb{R}^d\setminus\langle e_d\rangle^\perp$.

4.4.1 Second Derivative of the Surface Tension

Proposition 43. $\bar{\sigma} \in C^2(\mathbb{R}^d \setminus \langle e_d \rangle^\perp)$. *In fact, if for each* $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$ *and* $\xi \in \mathbb{R}^d$, *we define* Ψ_e^ξ *in* $\mathbb{R} \times \mathbb{T}^{d-1}$ *by* $\Psi_e^\xi = V_e^{-1} R_e^\xi$, *then*

$$\langle D^2\bar{\sigma}(e)\xi, \xi\rangle = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \langle a(y)(\xi + \mathcal{D}_e\Psi_e^\xi), \xi + \mathcal{D}_e\Psi_e^\xi\rangle V_e^2 \, dy \, ds. \tag{4.20}$$

Moreover, the following bound holds:

$$\bar{M}(e)^{-1} D^2\bar{\sigma}(e) \leq \Lambda(Id - e \otimes e) \quad \text{in } S^{d-1} \setminus \langle e_d \rangle^\perp.$$

Proof. Fix $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$. Recall that the formula (3.11) from Theorem 25 is applicable. Therefore, differentiating under the integral sign using Proposition 42 and Remark 4, we obtain

$$\begin{aligned}
D^2\bar{\sigma}(e) &= \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \left(V_e^2 a(y) + a(y)\mathcal{D}_e U_e \otimes \partial_s R_e + V_e a(y)\mathcal{D}_e R_e\right) dy\, ds \\
&= \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \left(V_e^2 a(y) - a(y)\mathcal{D}_e V_e \otimes R_e + \mathcal{D}_e^*(a(y)V_e)R_e\right) dy\, ds \\
&= \int_{\mathbb{R}\times\mathbb{T}^{d-1}} V_e^2 a(y)\, dy\, ds - \int_{\mathbb{R}\times\mathbb{T}^{d-1}} (2a(y)\mathcal{D}_e V_e - V_e \operatorname{div} a) \otimes R_e \, dy\, ds \\
&= \int_{\mathbb{R}\times\mathbb{T}^{d-1}} V_e^2 a(y)\, dy\, ds - \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \mathcal{L}_e R_e \otimes R_e \, dy\, ds.
\end{aligned}$$

Integration-by-parts then gives

$$\begin{aligned}
\langle D^2\bar{\sigma}(e)\xi, \xi\rangle = &\int_{\mathbb{R}\times\mathbb{T}^{d-1}} \langle a(y)\xi, \xi\rangle V_e^2 \, dx\, ds \\
&- \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \left(\langle a(y)\mathcal{D}_e R_e^\xi, \mathcal{D}_e R_e^\xi\rangle + W_{uu}(y, U_e)|R_e^\xi|^2\right) dy\, ds.
\end{aligned}$$

Since $\mathcal{L}_e$ is a non-negative operator by Proposition 39, the right-most term is non-positive and we deduce from this that

$$\langle D^2\bar{\sigma}(e)\xi,\xi\rangle \leq \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \langle a(y)\xi,\xi\rangle V_e^2\,dy\,ds \leq \Lambda\bar{M}(e).$$

Now we substitute Ψ_e^ξ for R_e^ξ and use the equation satisfied by R_e^ξ to obtain

$$\langle D^2\bar{\sigma}(e)\xi,\xi\rangle = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \langle a(y)(\xi+\mathcal{D}_e\Psi_e^\xi),\xi+\mathcal{D}_e\Psi_e^\xi\rangle V_e^2\,dy\,ds.$$

Observe that the last relation implies $\langle D^2\bar{\sigma}(e)\xi,\xi\rangle > 0$ if $\xi \in \mathbb{R}^d \setminus \langle e\rangle^\perp$. Indeed, if it vanished, we would be left to conclude that $\xi + \mathcal{D}_e\Psi_e^\xi = 0$ in $\mathbb{R}\times\mathbb{T}^{d-1}$. However, from the definition of $\mathcal{D}_e$, this would yield

$$0 = \int_{\mathbb{T}^{d-1}} (\xi + \mathcal{D}_e\Psi_e^\xi(0,x))\,dx = \xi + e\int_{\mathbb{T}^{d-1}} \partial_s\Psi_e^\xi(0,x)\,dx,$$

which is impossible unless $\xi \in \langle e\rangle$. □

4.4.2 Revisiting the Mobility

In this section, we will show that the mobility $\bar{M}(e)$ is precisely the energy of $\partial_s U_e$.

Proposition 44. *For each $e \in S^{d-1}\setminus\langle e_d\rangle^\perp$, we have*

$$\bar{M}(e) = \|\partial_s U_e\|^2_{L^2(\mathbb{R}\times\mathbb{T}^{d-1})}.$$

To do this, we begin with an elementary version of the maximum principle.

Proposition 45. *Given $e \in S^{d-1}\cap\mathbb{R}\mathbb{Z}^d$, if $v_1, v_2 : \mathbb{T}^{d-1}\oplus_e\mathbb{R} \to [-1,1]$ are minimizers of $\mathcal{F}(\cdot;Q_e\oplus_e\mathbb{R})$, that is,*

$$\mathcal{F}(v_1;Q_e\oplus_e\mathbb{R}) = \mathcal{F}(v_2;Q_e\oplus_e\mathbb{R}) = \bar{\sigma}(e)\mathcal{H}^{d-1}(Q_e),$$

then $v_1 \leq v_2$ *or* $v_2 \leq v_1$.

Proof. Since v_1 and v_2 are both minimizers, we know that $v_1 \wedge v_2$ and $v_1 \vee v_2$ are minimizers as well. The Euler-Lagrange equation $-\text{div}(a(y)Dv) + W_u(y, v) = 0$ has a strong maximum principle. Thus, either $v_1 \wedge v_2 < v_1 \vee v_2$ or $v_1 \equiv v_2$ in $\mathbb{R}^d$. □

Proof of Proposition 44. If $e \notin \mathbb{R}\mathbb{Z}^d$, then the equation follows by uniqueness. Thus, assume that $e \in \mathbb{R}\mathbb{Z}^d$.

Suppose that $U \in \mathcal{M}(e)$ and let $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ be the minimizers generated by U, i.e., the functions $u_\zeta(x) = U(\langle x, e \rangle - \zeta, x)$. Note that these functions are minimizers of $\mathcal{F}(\cdot; Q_e \oplus_e \mathbb{R})$. Denote by $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ the minimizers generated by U_e. By Proposition 45, there is an increasing $h : \mathbb{R} \to \mathbb{R}$ such that $h(\zeta + r_e) = h(\zeta) + r_e$ and

$$u_\zeta(x) = u^e_{h(\zeta)}(x).$$

In particular,

$$U(s, y) = u^e_{h(\langle y,e \rangle - s)}(y) = U_e(\langle y, e \rangle - h(\langle y, e \rangle - s), y).$$

From this, we deduce that h is differentiable a.e. and Theorem 19 implies that

$$\|\partial_s U\|^2_{L^2(\mathbb{R} \times \mathbb{T}^d)} = r_e^{-1} \mathcal{H}^{d-1}(Q_e)^{-1} \int_0^{r_e} \left(\int_{Q_e \oplus_e \mathbb{R}} u^e_{h(\zeta)}(x)^2 \, dx \right) h'(\zeta)^2 \, d\zeta.$$

At the same time, by (4.12), the function $\zeta \mapsto \int_{Q_e \oplus_e \mathbb{R}} u^e_\zeta(x)^2 \, dx$ is constant. Therefore, we can write

$$\|\partial_s U\|^2_{L^2(\mathbb{R} \times \mathbb{T}^d} = \mathcal{H}^{d-1}(Q_e)^{-1} \int_{Q_e \oplus_e \mathbb{R}} u^e_0(x)^2 \, dx \cdot r_e^{-1} \int_0^{r_e} h'(\zeta)^2 \, d\zeta.$$

We know that $h(\zeta + r_e) = h(\zeta) + r_e$ for all $\zeta \in \mathbb{R}$, from which we deduce the trivial bound on the Dirichlet energy

$$r_e^{-1} \int_0^{r_e} h'(\zeta)^2 \, d\zeta \geq 1$$

and it follows that

$$\|\partial_s U\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)} \geq \mathcal{H}^{d-1}(Q_e)^{-1} \int_{Q_e \oplus_e \mathbb{R}} u_0^e(x)^2 \, dx = \|\partial_s U_e\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

This proves $\bar{M}(e) = \|\partial_s U_e\|^2_{L^2(\mathbb{R}\times\mathbb{T}^d)}$ by definition (Section 3.2). □

4.4.3 Computation of $\bar{\mathcal{S}}$

Proposition 46. *If $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$, then the matrix $\bar{\mathcal{S}}$ of Section 4.1 is given by $\bar{\mathcal{S}}(e) = D^2\bar{\sigma}(e)$.*

Proof. Recall that in the very first computation in the previous proof, we obtained

$$D^2\bar{\sigma}(e) = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \Big(V_e^2 a(y) + a(y)\mathcal{D}_e U_e \otimes \partial_s R_e + V_e a(y) \mathcal{D}_e R_e \Big) \, dy \, ds.$$

Writing $\mathcal{D}_e U_e = e\partial_s U_e + D_x U_e$ and integrating both terms by parts, we arrive at

$$D^2\bar{\sigma}(e) = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} \big(V_e^2 a(y) + V_e \cdot \operatorname{div} a \otimes R_e + 2V_e a(y) D_x R_e \\ + 2V_e a(y) e \otimes \partial_s R_e \big) \, dy \, ds.$$

To finish the proof, first, recall that symmetric matrices are determined by their quadratic forms. Therefore, it only remains to show that $\langle D^2\bar{\sigma}(e)\xi, \xi\rangle = \langle \bar{\mathcal{S}}(e)\xi, \xi\rangle$ independently of the choice of $\xi \in \mathbb{R}^d$. Additionally, recall that if $w, v \in \mathbb{R}^d$, then

$$\langle (w \otimes v)\xi, \xi\rangle = \langle \xi, w\rangle\langle v, \xi\rangle = \langle (v \otimes w)\xi, \xi\rangle. \tag{4.21}$$

Using (4.21), we find

$$\langle D^2\bar{\sigma}(e)\xi,\xi\rangle = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} V_e \Big(V_e\langle a(y)\xi,\xi\rangle + \langle \operatorname{div} a,\xi\rangle R_e^{\xi} + 2\langle a(y)D_x R_e^{\xi},\xi\rangle \tag{4.22}$$
$$+2\langle a(y)e,\xi\rangle\partial_s R_e^{\xi}\Big)\, dy\, ds$$
$$= \int_{\mathbb{R}\times\mathbb{T}^{d-1}} V_e\left(\langle (a(y)e\otimes\partial_s R_e)\xi,\xi\rangle + \langle(\partial_s R_e\otimes a(y)e)\xi,\xi\rangle\right.$$
$$+V_e\langle a(y)\xi,\xi\rangle + 2\langle a(y)D_x R_e\xi,\xi\rangle + \frac{1}{2}\left(\langle(\operatorname{div} a\otimes R_e)\xi,\xi\rangle\right.$$
$$\left.\left.+\langle(R_e\otimes \operatorname{div} a)\xi,\xi\rangle\right)\right)\, dy\, ds.$$

Thus, by our previous observation,

$$D^2\bar{\sigma}(e) = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} V_e\Bigg(a(y)e\otimes\partial_s R_e + \partial_s R_e\otimes a(y)e + V_e a(y)$$
$$+ 2a(y)D_x R_e + \frac{1}{2}\left(\operatorname{div} a\otimes R_e + R_e\otimes \operatorname{div} a\right)\Bigg)\, dy\, ds = \bar{\mathcal{S}}(e).$$

□

Remark 5. *We proved above that $\bar{\mathcal{S}} = D^2\bar{\sigma}$ in $S^{d-1}\setminus\langle e_d\rangle^{\perp}$ when a and W are laminar. In fact, at a purely formal level, if pulsating standing waves exist and are smooth in the variables (s,y,e), then the identity $\bar{\mathcal{S}} = D^2\bar{\sigma}$ must hold. This can be shown via the computations in the last proof — the laminarity assumption plays no role. This applies, in particular, to the setting considered in [16, Section 6].*

4.5 Correctors

In this section, we prove the existence and smoothness of the correctors $\{Q_e^X\}$ and $\{P_e^q\}$ when $e\in S^{d-1}\setminus\langle e_d\rangle^{\perp}$.

The next result concerns the properties of the corrector Q_e^X when the symmetric matrix X has the form $X = \xi\otimes\xi$ for some $\xi\in\mathbb{R}^d$.

Proposition 47. *Given any $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$ and $\xi \in \mathbb{R}^d$, there is a unique $Q_e^\xi \in H^2(\mathbb{R} \times \mathbb{T}^{d-1})$ solving the PDE*

$$\begin{cases} \mathcal{L}_e Q_e^\xi = F_{\xi,e} \quad \textit{in } \mathbb{R} \times \mathbb{T}^{d-1}, \\ \lim_{|s| \to \infty} Q_e^\xi(s,x) = 0 \quad \textit{uniformly in } \mathbb{T}^{d-1}, \\ \int_{\mathbb{T}^{d-1}} Q_e^\xi(0,y)\, dy = 0, \end{cases}$$

where $F_{\xi,e} : \mathbb{R} \times \mathbb{T}^{d-1} \to \mathbb{R}$ is given by

$$F_{\xi,e} = -\bar{M}(e)^{-1} \langle D^2 \bar{\sigma}(e)\xi, \xi \rangle V_e + \langle a(y)\xi, \xi \rangle V_e + \langle div\, a, \xi \rangle R_e^\xi$$
$$+ 2\langle a(y)\xi, D_x R_e^\xi \rangle + 2\langle a(y)e, \xi \rangle \partial_s R_e^\xi.$$

If, for each fixed ξ, the function $Q^\xi : S^{d-1} \setminus \langle e_d \rangle^\perp \to H^2(\mathbb{R} \times \mathbb{T}^d)$ given by $Q^\xi(e) = Q_e^\xi$ is twice continuously differentiable with respect to both the $H^2(\mathbb{R} \times \mathbb{T}^d)$ and $C_0(\mathbb{R} \times \mathbb{T}^d)$ topologies.

Proof. Concerning existence, notice that $F_{\xi,e} \in \langle V_e \rangle^\perp$ by (4.22) and the definition of $\bar{M}(e)$. Thus, Proposition 40 provides the existence of Q_e^ξ. Arguing as in the analysis of the function R_e^ξ, we see that $Q_e^\xi(s,y) \to 0$ at an exponential rate as $|s| \to \infty$.

The proof that $e \mapsto Q_e^\xi$ is twice differentiable in e proceeds exactly as in the corresponding proof for U_e. □

Now that we define Q_e^X for elementary tensors X, we can extend to arbitrary matrices by linearity. Given $A \in \mathcal{S}_d$, if we expand A as $A = \sum_{i=1}^d \lambda_i \xi_i^A \otimes \xi_i^A$, where $\{\xi_1^A, \dots, \xi_d^A\}$ is an orthonormal basis, then we can define $Q_e^A \in H^2(\mathbb{R} \times \mathbb{T}^{d-1}) \cap C_0^2(\mathbb{R} \times \mathbb{T}^{d-1})$ by

$$Q_e^A = \sum_{i=1}^d \lambda_i Q_e^{\xi_i^A}. \tag{4.23}$$

Notice that Q_e^A satisfies the equation

$$\mathcal{L}_e Q_e^A = F_{A,e} \quad \text{in } \mathbb{R} \times \mathbb{T}^{d-1}, \quad \int_{\mathbb{T}^{d-1}} P_e^A(0,x)\,dx = 0, \tag{4.24}$$

where $F_{A,e}$ is given by

$$\begin{aligned} F_{A,e} = &-\bar{M}(e)^{-1}\mathrm{tr}(D^2\overline{\sigma}(e)A)V_e + \mathrm{tr}(a(y)A)V_e + \mathrm{tr}([\mathrm{div}\, a \otimes R_e]A) \\ &+ 2\mathrm{tr}([a(y) \otimes D_x R_e]A) + 2\mathrm{tr}([a(y)e \otimes \partial_s R_e]A). \end{aligned}$$

Observe that, by uniqueness of solutions of (4.24), the map $A \mapsto Q_e^A$ is linear. Hence as a consequence of Proposition 47, we have

Proposition 48. *There is a C^2 map $Q : S^{d-1} \setminus \langle e_d \rangle^\perp \to C_0^2(\mathbb{R} \times \mathbb{T}^{d-1}; \mathcal{S}_d)$ such that if $e \in S^{d-1} \setminus \langle e_d \rangle^\perp$, $A \in \mathcal{S}_d$, and Q_e^A is given by (4.23), then*

$$Q_e^A(s,y) = tr(Q_e(s,y)A) \quad \text{for each } (s,y) \in \mathbb{R} \times \mathbb{T}^{d-1}.$$

Furthermore, for each $K \subset\subset S^{d-1} \setminus \langle e_d \rangle^\perp$, there is a constant $C(K) > 0$ such that

$$\sqrt{tr(Q_e(s,y)^2)} \leq C(K) \exp\left(-C(K)^{-1}|s|\right) \quad \text{for } (s,y) \in \mathbb{R} \times \mathbb{T}^{d-1},\ e \in K.$$

Finally, we prove the existence and smoothness of the correctors $\{P_e^q\}$.

Proposition 49. *There is a C^2 map $P : S^{d-1} \setminus \langle e_d \rangle^\perp \to C_0^2(\mathbb{R} \times \mathbb{T}^{d-1})$ such that, for each $e \in S^{d-1}$, the function $P_e = P(e)$ is the solution of the PDE*

$$\mathcal{D}_e^*(a(y)\mathcal{D}_e P_e) + W_{uu}(y, U_e)P_e = -m(y, \mathcal{D}_e U_e)\partial_s U_e + \bar{M}(e)^{-1}\bar{\mathcal{M}}(m,e)\partial_s U_e \quad \text{in } \mathbb{R} \times \mathbb{T}^{d-1},$$

where the constant $\bar{\mathcal{M}}(m,e) \in \mathbb{R}$ *is given by*

$$\bar{\mathcal{M}}(m,e) = \int_{\mathbb{R}\times\mathbb{T}^{d-1}} m(y, \mathcal{D}_e U_e)|\partial_s U_e|^2 \, dy \, ds.$$

Moreover, for each $K \subset\subset S^{d-1} \setminus \langle e_d \rangle^{\perp}$*, there is a constant* $C(K) > 0$ *such that, for each* $e \in K$*,*

$$|P_e(s,y)| \leq C(K) \exp\left(-C(K)^{-1}|s|\right) \quad \text{for } (s,y) \in \mathbb{R} \times \mathbb{T}^{d-1}.$$

By linearity, the function P_e can be used to define the correctors $\{P_e^q\}$ from Section 4.1 by the rule $P_e^q(s,y) = qP_e(s,y)$ for $q \in \mathbb{R}$.

4.6 The Sharp-Interface Limit

This section describes the proof of Theorem 2 in earnest. The key idea is smooth pulsating waves and correctors exist in directions $e \in S^{d-1} \setminus \langle e_d \rangle^{\perp}$. These can be used to characterize the effective interface motion near points where the normal vector is in $S^{d-1} \setminus \langle e_d \rangle^{\perp}$. When the interface is a graph over the dth coordinate, these are all the possible values of the normal vector, so we have identified the limit.

Recall that our interest is in the interface motion with prescribed normal velocity given by

$$V_{\partial E_t} = \bar{\mathcal{M}}(m,e)^{-1} \mathrm{tr}\left(D^2\bar{\sigma}(n_{\partial E_t}) A_{\partial E_t}\right).$$

Postulating that the interface is a graph over the dth coordinate, or, more precisely, that there is a function $u : \mathbb{R}^{d-1} \times (0,\infty) \to \mathbb{R}$ such that

$$E_t = \left\{(x', x_d) \in \mathbb{R}^d \mid x_d > v(x',t)\right\},$$

we obtain the following PDE for the evolving graph

$$\bar{\mu}(m, Dv)v_t - \text{tr}\left(\bar{\mathcal{G}}(Dv)(D^2v)\right) = 0 \quad \text{in } \mathbb{R}^{d-1} \times (0, \infty), \tag{4.25}$$

where $\bar{\mu}$ and $\bar{\mathcal{G}}$ are determined by

$$\bar{\mu}(m,p) = \bar{\mathcal{M}}\left(m, (1+\|p\|^2)^{-1/2}(-p,1)\right), \quad \bar{\mathcal{G}}(p) = \pi \circ D^2\bar{\sigma}\left((1+\|p\|^2)^{-1/2}(-p,1)\right) \circ \pi^*$$

and $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the orthogonal projection $\pi(x', x_d) = x'$. Since $\bar{\sigma}$ is convex and $D^2\bar{\sigma}$ is continuous away from $\langle e_d \rangle^\perp$, this equation has a unique, continuous solution for a given initial datum $\mathcal{U}_0 \in UC(\mathbb{R}^{d-1})$ with at most linear growth.

To obtain the function u, we begin as in [16] by defining sets $\{\Omega_t^1\}_{t>0}$ and $\{\Omega_t^2\}_{t>0}$ by

$$\Omega_t^1 = \{x \in \mathbb{R}^d \mid \liminf_* u^\epsilon(x,t) = 1\}, \quad \Omega_t^2 = \{x \in \mathbb{R}^d \mid \limsup^* u^\epsilon(x,t) = -1\}.$$

We then define the minimal supergraph and maximal subgraph of Ω_t^1 and Ω_t^2, respectively, through the functions η and ν given by

$$\eta(x',t) = \inf\left\{a \in \mathbb{R} \mid \{x'\} \times (a, \infty) \subseteq \Omega_t^1\right\}, \tag{4.26}$$

$$\nu(x',t) = \sup\left\{a \in \mathbb{R} \mid \{x'\} \times (-\infty, a) \subseteq \Omega_t^2\right\}. \tag{4.27}$$

Theorem 2 is then equivalent to the following statement:

Theorem 27. *Under the assumptions of Theorem 2, if $v : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ is the unique viscosity solution of the PDE* (4.25) *with initial condition $v(\cdot, 0) = v_0$, then*

$$v = \eta = \nu \quad \textit{in } \mathbb{R}^{d-1} \times (0, \infty).$$

See [73, Section 9] for the proof.

4.7 Pathological Examples in Dimension Two

In this section, we discuss a particular class of laminar examples in dimension $d = 2$. These exhibit a number of behaviors that are pathological compared to the spatially homogeneous case.

In the remainder of the section, in addition to the laminarity assumption (4.2), we assume that a takes values in the diagonal matrices, that is,

$$a(y) = \langle a(y)e_1, e_1\rangle e_1 \otimes e_1 + \langle a(y)e_2, e_2\rangle e_2 \otimes e_2 \quad \text{for each } y \in \mathbb{T}^d. \tag{4.28}$$

The next result describes pathologies that occur whenever $\mathcal{M}(e_1)$ contains no continuous pulsating waves.

Theorem 28. *In dimension $d = 2$, there are coefficients (a, W) for which (4.2) and (4.28) holds and such that $\mathcal{M}(e_1)$ contains no continuous pulsating waves. Given such coefficients (a, W), the following observations hold:*

(a) $\bar{\sigma}$ is not differentiable at e_1.

(b) There is a constant $C \geq 1$ such that

$$C^{-1} \leq \liminf_{e \to e_1} |\langle e, e_2\rangle| \bar{M}(e) \leq \limsup_{e \to e_1} |\langle e, e_2\rangle| \bar{M}(e) \leq C.$$

(c) The interface velocity gets arbitrarily small near e_1 in the following sense:

$$\liminf_{e \to e_1} \frac{\|D^2\bar{\sigma}(e)\|}{\bar{M}(e)} = 0.$$

Note that if (c) holds, then, for an arbitrary smooth m satisfying assumption (1.10), the interface velocity from (1.12) gets arbitrarily small in the e_1 direction in the sense that

$$\liminf_{e \to e_1} \frac{\|D^2 \bar{\sigma}(e)\|}{\bar{\mathcal{M}}(m,e)} = 0. \tag{4.29}$$

This follows from the fact that $c^{-1}\bar{M}(e) \leq \bar{\mathcal{M}}(m,e) \leq c\bar{M}(e)$ for some constant c depending only on m. The asymptotics (4.29) shows that, unlike the spatially homogeneous case, in periodic media, the effective interface velocity can be an arbitrarily small multiple of the curvature.

4.7.1 Preliminaries

In this section, to prepare for the proof of Theorem 28, we start with some preliminaries on plane-like minimizers in the laminar setting. To start with, consider the one-dimensional variational problem given by

$$\mathscr{E}' = \min \left\{ \int_{-\infty}^{\infty} \left(\frac{1}{2} \langle a(se_1, 0)e_1, e_1 \rangle v'(s)^2 + W((se_1, 0), v(s)) \right) ds \ \middle| \ v : \mathbb{R} \to [-1, 1], \right.$$
$$\left. \lim_{s \to \pm\infty} v(s) = \pm 1 \right\}.$$

Due to the laminarity assumption, this problem characterizes the strongly Birkhoff plane-like minimizers in the e_1 direction, as we prove next. It will be convenient to define the set of minimizers $\mathcal{M}'$ by

$$\mathcal{M}' = \left\{ v : \mathbb{R} \to [-1, 1] \ \middle| \ \lim_{s \to \pm\infty} v(s) = \pm 1, \right.$$
$$\left. \mathcal{E}' = \int_{-\infty}^{\infty} \left(\frac{1}{2} \langle a(se_1, 0)e_1, e_1 \rangle v'(s)^2 + W((se_1, 0), v(s)) \right) ds \right\}.$$

Proposition 50. *Suppose that $d = 2$ and a and W satisfy the laminarity assumption* (4.2). *If $U \in \mathcal{M}(e_1)$, then the plane-like minimizers $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ generated by U vary only in the e_1*

direction and $\{u_\zeta\}_{\zeta\in\mathbb{R}} \subseteq \mathcal{M}'$.

Proof. To start with, for any $\zeta \in \mathbb{R}$, u_ζ is a minimizer of the variational problem in $\mathbb{T}^{d-1}_{e_1} \oplus_{e_1} \mathbb{R}$ from Proposition 27. At the same time, for any $t \in \mathbb{Q}$, the function $x \mapsto u_\zeta(x+te_2) =: u_\zeta^{(t)}(x)$ remains a minimizer by (4.2). Therefore, the functions $\underline{u} = u_\zeta \wedge u_\zeta^{(t)}$ and $\overline{u} = u_\zeta \vee u_\zeta^{(t)}$ are also minimizers, and then the strong maximum principle implies that $u_\zeta < u_\zeta^{(t)}$, $u_\zeta > u_\zeta^{(t)}$, or $u_\zeta \equiv u_\zeta^{(t)}$.

If $u_\zeta < u_\zeta^{(t)}$, then $u_\zeta < u_\zeta^{(nt)}$ for each $n \in \mathbb{N}$ by induction. On the other hand, by assumption, $Nt \in \mathbb{N}$ for some $N \in \mathbb{N}$, hence, by periodicity, $u_\zeta \equiv u_\zeta(\cdot + Nte_2) = u_\zeta^{(Nt)}$, a contradiction. We similarly conclude that u_ζ cannot lie above $u_\zeta^{(t)}$. Therefore, $u_\zeta \equiv u_\zeta^{(t)}$.

Since $u_\zeta^{(t)} \equiv u_\zeta$ for any $t \in \mathbb{Q}$, it follows that $(u_\zeta)_{x_2} \equiv 0$, that is, u_ζ depends only on the first variable.

Lastly, given any $v : \mathbb{R} \to [-1, 1]$ with $\lim_{s\to\pm\infty} v(s) = \pm 1$, we can define a $u : \mathbb{T}^{d-1}_e \oplus_e \mathbb{R} \to [-1, 1]$ by $u(x_1, x_2) = v(x_1)$, and this is a competitor for the variational problem of Proposition 27. Therefore,

$$\mathscr{E}(e_1) \leq \mathcal{F}(u; Q_e \oplus_e \mathbb{R}) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\langle a(se_1, 0)e_1, e_1\rangle v'(s)^2 + W((se_1, 0), v(s)) \right) ds.$$

At the same time, by Theorem 19,

$$\begin{aligned}\mathscr{E}(e_1) &= r_e^{-1} \int_0^{r_e} \int_{-\infty}^{\infty} \left(\frac{1}{2}\langle a(se_1, 0)e_1, e_1\rangle (u_\zeta)_{x_1}(se_1, 0)^2 + W((se_1, 0), u_\zeta(se_1, 0)) \right) ds\, d\zeta \\ &\geq \mathscr{E}'.\end{aligned}$$

Therefore, $\mathscr{E}(e_1) = \mathscr{E}'$ and $\{u_\zeta^e\}_{\zeta\in\mathbb{R}} \subseteq \mathcal{M}'$. □

Notice that elements of $\mathcal{M}'$ are ordered, that is, if $v_1, v_2 \in \mathcal{M}'$, then $v_1 < v_2$, $v_1 > v_2$, or $v_1 \equiv v_2$. This follows from the same maximum principle argument used in the previous proof. It follows that the graphs of minimizers in $\mathcal{M}'$ are pairwise disjoint.

We will say that $\mathcal{M}'$ forms a foliation if the graphs of the minimizers in $\mathcal{M}'$ foliate $\mathbb{R} \times (-1, 1)$. From the previous proposition, we observe that $\mathcal{M}'$ forms a foliation if and only if $\mathcal{M}(e_1)$ contains a continuous element.

Proposition 51. *Suppose that $d = 2$ and a and W satisfy the laminarity assumption* (4.2)*. $\mathcal{M}(e_1)$ contains a continuous element if and only if $\mathcal{M}'$ forms a foliation.*

In particular, if $\mathcal{M}'$ forms a foliation, then, for each $s \in \mathbb{R}$, there is a $\bar{u} \in (-1, 1)$ such that, for each $v \in \mathcal{M}'$, $v(s) \neq \bar{u}$.

Proof. If $U \in \mathcal{M}(e_1)$ is continuous in $\mathbb{R} \times \mathbb{T}^2$ and $\{u_\zeta\}_{\zeta \in \mathbb{R}}$ are the minimizers generated by U, then the map $\zeta \mapsto u_\zeta$ is continuous. It follows easily that, for any $(x, \bar{u}) \in \mathbb{R}^2 \times (-1, 1)$, there is a $\bar{\zeta} \in \mathbb{R}$ such that $u_{\bar{\zeta}}(x) = \bar{u}$. By the previous proposition, $\{u_\zeta\}_{\zeta \in \mathbb{R}} \subseteq \mathcal{M}'$. Hence we proved that each point in $\mathbb{R} \times (-1, 1)$ is contained in one of the graphs of an element of $\mathcal{M}'$, and it follows that $\mathcal{M}'$ forms a foliation.

Now we prove the converse. Suppose that $\mathcal{M}'$ forms a foliation. Fix $v_0 \in \mathcal{M}'$. Since $\mathcal{M}'$ is ordered and $\lim_{s \to \infty} v(s) = 1$, it is not hard to show that $v_0(\cdot - 1) < v_0$. Given that $\mathcal{M}'$ forms a foliation, for each $\zeta \in [0, 1]$, we can fix a $v_\zeta \in \mathcal{M}'$ such that

$$v_\zeta(0) = (1 - \zeta) v_0(0) + \zeta v_0(-1).$$

Given $m \in \mathbb{N}$, extend to $[m, m + 1)$ via the rule

$$v_\zeta = v_{\zeta - m}(\cdot - m) \quad \text{if } m \leq \zeta < m + 1.$$

The reader can check that with this definition $\zeta \mapsto v_\zeta$ is a continuous map from $\mathbb{R}$ into $\mathcal{M}'$ with the topology of uniform convergence, and $v_{\zeta + m} = v(\cdot - m)$ for each $m \in \mathbb{N}$. Thus, the function $V : \mathbb{R} \times \mathbb{T}^2 \to [-1, 1]$ given by $V(s, y) = v_{y_1 - s}(y_1)$ is well-defined and the previous proposition implies that $V \in \mathcal{M}(e_1)$. $\square$

Remark 6. *In view of the previous result, in the proof of Theorem 28, there is no loss of generality in assuming the following:*

$$\textit{There is no } v \in \mathcal{M}' \textit{ such that } v(1/4) = 0. \tag{4.30}$$

Above we observed that the laminarity assumption (4.2) implies that strongly Birkhoff plane-like minimizers in the e_1 direction are all one-dimensional. By symmetry, the same observation also applies to minimizers in the $-e_1$ direction. In the rest of this section, we investigate the properties of plane-like minimizers in directions $e \notin \langle e_1 \rangle$.

Given $e \in S^1 \setminus \{e_1, -e_1\}$, let U_e be the pulsating standing wave from Proposition 35, and let $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ be the plane-like minimizers generated by U_e. Notice that $u^e_\zeta(x, y) = U_e(x\langle e, e_1\rangle + y\langle e, e_2\rangle - \zeta, x)$ since U_e has the same laminar symmetry as the coefficients a and W. From this, we see that

$$u^e_\zeta(x, y) = u^e_0(x, y - \langle e, e_2\rangle^{-1}\zeta).$$

In particular, the one-parameter family $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ is generated by translation in the y variable.

Next, observe that $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ is periodic with respect to a finer lattice than the module M_e defined in Section 2.4. In this simple, two-dimensional setting, we can simply observe that

$$u^e_\zeta\left(x + 1, y - \frac{\langle e, e_1\rangle}{\langle e, e_2\rangle}\right) = U_e(x\langle e, e_1\rangle + y\langle e - \zeta, e_2\rangle, x) = u^e_\zeta(x, y).$$

From this, it is convenient to define I_e (analogous to Q_e in Section 2.4) by

$$I_e = \left\{ s\left(1, -\langle e, e_2\rangle^{-1}\langle e, e_1\rangle\right) \mid s \in [0, 1] \right\}. \tag{4.31}$$

4.7.2 Non-Differentiability

Now we show that $\bar{\sigma}(e)$ is not differentiable at e_1 if $\mathcal{M}(e_1)$ has no continuous elements and (4.2) holds. To start with, it will be useful in what follows to utilize so-called heteroclinic minimizers located inside the gaps of the one-dimensional ones. Since we are working in a laminar medium in $\mathbb{R}^2$, the structure of these heteroclinic solutions is particularly simple. A much more general treatment can be found in [9].

In the rest of this section, we will write (x, y) for points in $\mathbb{R}^2$ with $\langle (x, y), e_1 \rangle = x$ and $\langle (x, y), e_2 \rangle = y$. Moreover, for $e \in S^1 \setminus \{e_1, -e_1\}$, we will let $\{u^e_\zeta\}_{\zeta \in \mathbb{R}}$ denote the family of functions generated by U_e, where U_e is the laminar pulsating wave from Proposition 35.

Proposition 52. *Suppose that assumptions* (4.2)*,* (4.28)*, and* (4.30) *all hold. If* $(\nu_n)_{n\in\mathbb{N}} \subseteq S^1 \setminus \{e_1, -e_1\}$ *and* $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *satisfy, for each* $n \in \mathbb{N}$*,*

(i) $\langle \nu_n, e_2 \rangle > 0$ *(resp.* $\langle \nu_n, e_2 \rangle < 0$*),*

(ii) $u^{\nu_n}_{\zeta_n}(\frac{1}{4}, 0) = 0$*,*

and if $\lim_{n\to\infty} \nu_n = e_1$*, then there is a subsequence* $(n_j)_{j\in\mathbb{N}}$ *and a Class A minimizer* u *of* $\mathcal{F}$ *such that* $u = \lim_{j\to\infty} u^{\nu_{n_j}}_{\zeta_{n_j}}$ *locally uniformly in* $\mathbb{R}^2$ *and*

$$u(x + ke_1, y) \geq u(x, y) \quad \textit{(resp. } u(x + ke_1, y) \leq u(x, y)\textit{)} \quad \textit{if } k \in \mathbb{N},$$

$$u(x, y + \delta) > u(x, y) \quad \textit{(resp. } u(x, y + \delta) < u(x, y)\textit{)} \quad \textit{if } \delta > 0,$$

$$\lim_{x\to\pm\infty} u(x, y) = \pm 1 \quad \textit{(resp. } \lim_{x\to\pm\infty} u(x, y) = \mp 1\textit{)}.$$

Proof. To start with, assume that $\langle \nu_n, e_2 \rangle > 0$ independently of n. By the Arzelà-Ascoli Theorem and elliptic regularity, if $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ is any subsequence, then there is a further subsequence $(n_{j_k})_{k\in\mathbb{N}}$ and a Class A minimizer u of $\mathcal{F}$ such that $u = \lim_{k\to\infty} u^{\nu_{n_{j_k}}}_{\zeta_{n_{j_k}}}$ locally uniformly.

Since strongly Birkhoff plane-like minimizers have bounded width (Theorem 20), it is not hard to show that $\lim_{\langle x, e_1\rangle \to \pm\infty} u(x) = \pm 1$ uniformly in $\langle e_1 \rangle^{\perp}$.

If $k \in \mathbb{N}$, then $\langle k e_1, \nu_n \rangle > 0$ for large enough n. Thus, $u(x + k e_1, y) \geq u(x, y)$. Similarly, if $k \in -\mathbb{N}$, then $u(x+k, y) \leq u(x,y)$. In view of what was proved in the previous paragraph, the inequality is strict if $k \neq 0$.

Next, recall that we can write

$$u^{\nu_n}_{\zeta_n}(x,y) = U_{\nu_n}(x\langle \nu_n, e_1\rangle + y\langle \nu_n, e_2\rangle - \zeta, x).$$

Thus, $(u^{\nu_n}_{\zeta_n})_y = \langle \nu_n, e_2 \rangle \partial_s U_{\nu_n} > 0$ in $\mathbb{R}^2$. Therefore, since $u^{\nu_n}_{\zeta_n} \to u$ locally uniformly, it follows that $u_y \geq 0$. Finally, observe that $v = u_y$ satisfies $-\text{div}(a(x)Dv) + W_{uu}(y,u)v = 0$ in $\mathbb{R}^2$. Thus, by the strong maximum principle (cf. [39, Corollary A.3]), either $v > 0$ or $v \equiv 0$.

If $u_y \equiv 0$, then $u = u(x)$ and then the fact that $u(\frac{1}{4}) = 0$ and u is a Class A minimizer heteroclinic between 1 and -1 would contradict (4.30). Therefore, $u_y > 0$ in $\mathbb{R}^2$. □

Now we show that $\bar{\sigma}$ is not differentiable at e_1 in this set-up:

Proposition 53. *If we define $e_\theta = \cos(\theta) e_1 + \sin(\theta) e_2$, then*

$$\lim_{\theta \to 0^+} \langle D\bar{\sigma}(e_\theta), e_2 \rangle > 0, \quad \lim_{\theta \to 0^-} \langle D\bar{\sigma}(e_\theta), e_2\rangle < 0.$$

In particular, $\bar{\sigma}$ is not differentiable at e_1.

Proof. Suppose $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$ and fix $\zeta_\theta \in \mathbb{R}$ such that $u^{e_\theta}_{\zeta_\theta}$ satisfies $u^{e_\theta}_{\zeta_\theta}(\frac{1}{4}, 0) = 0$. Recalling the discussion at the end of Section 4.7.1 and the equation (3.11) for $D\bar{\sigma}(e_\theta)$, we compute

$$\begin{aligned}D\bar{\sigma}(e_\theta) &= \int_{\mathbb{R}\times\mathbb{T}^d} a(y)\mathcal{D}_{e_\theta}U_{e_\theta}\partial_s U_{e_\theta}\,dy\,ds \\ &= \langle e_\theta, e_2\rangle^{-1}\mathcal{H}^1(I_{e_\theta})^{-1}\int_{I_{e_\theta}\oplus_{e_\theta}\mathbb{R}} a(x)Du^{e_\theta}_{\zeta_\theta}(x,y)\partial_y u^{e_\theta}_{\zeta_\theta}(x,y)\,dx\,dy \\ &= \operatorname{sgn}(\langle e_\theta, e_2\rangle)\int_{I_{e_\theta}\oplus_{e_\theta}\mathbb{R}} a(x)Du^{e_\theta}_{\zeta_\theta}(x,y)\partial_y u^{e_\theta}_{\zeta_\theta}(x,y)\,dx\,dy.\end{aligned}$$

In particular, since a takes values in the diagonal matrices,

$$\langle D\bar{\sigma}(e_\theta), e_2\rangle = \operatorname{sgn}(\langle e_\theta, e_2\rangle)\int_{I_{e_\theta}\oplus_{e_\theta}\mathbb{R}} a_2(x)\partial_y u^{e_\theta}_{\zeta_\theta}(x,y)^2\,dx\,dy.$$

This shows that $\langle D\bar{\sigma}(e_\theta), e_2\rangle > 0$ if $\langle e_\theta, e_2\rangle > 0$ and $\langle D\bar{\sigma}(e_\theta), e_2\rangle < 0$ if $\langle e_\theta, e_2\rangle < 0$.

Since we are working in dimension two, note that $\partial\bar{\sigma}(e_1)$ is either a singleton or a line segment. Thus, $\lim_{\theta\to 0^+} D\bar{\sigma}(e_\theta)$ and $\lim_{\theta\to 0^-} D\bar{\sigma}(e_\theta)$ both exist and converge to either $D\bar{\sigma}(e_1)$ or the (distinct) boundary endpoints of $\partial\bar{\sigma}(e_1)$. Thus, from the previous paragraph, we see that $D\bar{\sigma}(e_1)$ exists only if $\lim_{\theta\to 0}\langle D\bar{\sigma}(e_\theta), e_2\rangle = 0$. That means that to prove non-differentiability, we only need to show that

$$\liminf_{\theta\to 0} |\langle D\bar{\sigma}(e_\theta), e_2\rangle| > 0.$$

We will proceed arguing by contradiction. Suppose that, in contrast to what we wish to prove, $\lim_{\theta\to 0^+}\langle D\bar{\sigma}(e_\theta), e_2\rangle = 0$. Appealing to our previous computations and the positivity of a_2, we find

$$\lim_{\theta\to 0^+}\int_{I_{e_\theta}\oplus\mathbb{R}} \partial_y u^{e_\theta}_{\zeta_\theta}(x,y)^2\,dx\,dy = 0. \tag{4.32}$$

We claim this is impossible.

Indeed, if (4.32) were true, then we would deduce that $(u^{e_\theta}_{\zeta_\theta})_y \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^2)$. Passing to a sub-sequence $\theta_n \to 0^+$, we can assume that there is a Class A minimizer u satisfying the conclusions of Proposition 52 such that $u^{e_{\theta_n}}_{\zeta_{\theta_n}} \to u$ locally uniformly. From the local uniform convergence, we deduce that $(u^{e_{\theta_n}}_{\zeta_{\theta_n}})_y \rightharpoonup u_y$ in $L^2_{\text{loc}}(\mathbb{R}^2)$. We are left to conclude that $u_y \equiv 0$, which contradicts the fact that u is strictly increasing in the y variable. □

4.7.3 Asymptotics of the Mobility

Proposition 54. *If (a, W) are such that (4.2) holds and $\mathcal{M}(e_1)$ contains no continuous elements, then, in the notation of Proposition 53, there are positive constants $C_+, C_- > 0$ such that*

$$C_\pm = \lim_{\theta \to 0^\pm} |\langle e_\theta, e_2 \rangle| \int_{\mathbb{R}\times\mathbb{T}} a_2(y) \partial_s U_{e_\theta}(s,y)^2 \, ds \, dy.$$

Proof. In the previous proof, we showed that $\lim_{\theta\to 0^\pm} |\langle D\bar{\sigma}(e_\theta), e_2\rangle| > 0$.

On the other hand, using the identity $\langle \mathcal{D}_{e_\theta} U_{e_\theta}, e_2 \rangle = \langle e_\theta, e_2 \rangle \partial_s U_{e_\theta}$, we find

$$\begin{aligned} |\langle D\bar{\sigma}(e_\theta), e_2\rangle| &= \left| \int_{\mathbb{R}\times\mathbb{T}} \langle a(y) \mathcal{D}_{e_\theta} U_{e_\theta}(s,y), e_2 \rangle \partial_s U_{e_\theta}(s,y) \, dy \, ds \right| \\ &= |\langle e_\theta, e_2\rangle| \int_{\mathbb{R}\times\mathbb{T}} a_2(y) \partial_s U_{e_\theta}(s,y)^2 \, dy \, ds. \end{aligned}$$

□

4.7.4 Proof of Theorem 28

Proof of Theorem 28. Propositions 53 and 54 prove (i) and (ii), respectively. It only remains to prove (iii). We proceed by appealing to the fact that $D^2\bar{\sigma}$ is a Radon measure in $\mathbb{R}^2$.

From (ii), we know there is a $C > 0$ such that

$$C^{-1} |\langle e, e_2 \rangle|^{-1} \le \bar{M}(e) \le C |\langle e, e_2\rangle|^{-1}$$

For convenience, extend $\bar{M}$ to $\mathbb{R}^d \setminus \{0\}$ by $\bar{M}(v) = \bar{M}(\|v\|^{-1} v)$. Let Q be the cone $Q =$

$\{x \in \mathbb{R}^2 \mid |\langle x, e_2\rangle| \leq \frac{1}{2}\langle x, e_1\rangle\}$. Integrating over $Q \cap \{1/2 \leq \|v\| \leq 2\}$, we find

$$
\begin{aligned}
C^{-1} \int_{\{\frac{1}{2} \leq \|v\| \leq 2\} \cap Q} \left(\frac{\|D^2\bar{\sigma}(v)\|}{\bar{M}(v)} \right) |\langle v, e_2\rangle|^{-1} \, dv \\
&\leq \int_{\{\frac{1}{2} \leq \|v\| \leq 2\} \cap Q} \left(\frac{\|D^2\bar{\sigma}(v)\|}{\bar{M}(v)} \right) \bar{M}(v) \, dv \\
&\leq \|D^2\bar{\sigma}\| \left(\left\{ \frac{1}{2} \leq \|v\| \leq 2 \right\} \cap Q \right) < \infty.
\end{aligned}
$$

Since $e \mapsto |\langle e, e_2\rangle|^{-1}$ is not integrable in any arc of S^1 containing e_1 and $v \mapsto \bar{\sigma}(v)$ is 1-homogeneous, we conclude $\liminf_{e \to \pm e_1} \frac{\|D^2\bar{\sigma}(e)\|}{\bar{M}(e)} = 0$. □

4.8 Notes

It is increasingly becoming a well-known fact that geometric homogenization problems in laminar media behave better than those in general periodic media. The main inspiration to consider diffuse interfaces in a laminar medium came from the paper by Barles, Cesaroni, and Novaga on anisotropic curvature flow [13]. Additionally, in Moser-Bangert theory, Moser highlighted the usefulness of laminar examples in his lecture notes [75].

In [87], Spohn posits the physically relevant form of the effective interface velocity $V(n) = \bar{M}(n)^{-1}\mathrm{tr}(D^2\bar{\sigma}(n)A)$. In the context of interface motions, the only example where this is really well understood is in the gradient flows of the Lebowitz-Penrose functional arising in the study of the Ising model, as studied by Katsoulakis and Souganidis [61, 62], Buttà [24], and Bellettini, Buttà, and Presutti [17]. It should be emphasized that the Lebowitz-Penrose functional is translationally-invariant — it models a spatially homogeneous medium.

CHAPTER 5

THE ALLEN-CAHN EQUATION WITH A PERIODIC MOBILITY COEFFICIENT

We begin with a study of the gradient flow when the energy is spatially homogeneous and only the mobility accounts for heterogeneities in the medium. Specifically, we are interested in the asymptotics of the solution u^ϵ of the PDE

$$\begin{cases} m(\epsilon^{-1}x, \epsilon Du^\epsilon)u^\epsilon_t - \Delta u^\epsilon + \epsilon^{-2}W'(u^\epsilon) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{5.1}$$

We prove that the effective interface motion is determined by the law

$$V_{\partial E_t} = \overline{m}(n_{\partial E_t})^{-1}\text{tr}(A_{\partial E_t}), \tag{5.2}$$

where $\overline{m}$ is determined by the formula

$$\overline{m}(e) = c_W^{-1} \int_{\mathbb{R}\times\mathbb{T}^d} m(y, \dot{q}(s)e)\dot{q}(s)^2 \, dy \, ds, \quad c_W = \int_{-\infty}^{\infty} \dot{q}(s)^2 \, ds. \tag{5.3}$$

Above q is the (centered) standing wave associated with the Allen-Cahn equation; see Remark 3. Notice that $\overline{m}$ is a continuous, positive function in S^{d-1}.

Theorem 29. *Let $\overline{m} : S^{d-1} \to [\theta, \Theta]$ denote the effective mobility defined by* (5.3)*. If $u_0 \in UC(\mathbb{R}^d; [-1,1])$, $(u^\epsilon)_{\epsilon>0} \subseteq C(\mathbb{R}^d \times [0,\infty); [-1,1])$ are the viscosity solutions of* (5.1)*, and $\bar{u} : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is the unique viscosity solution of the effective level set PDE*

$$\begin{cases} \overline{m}(\widehat{D\bar{u}})\bar{u}_t - tr\left(\left(Id - \widehat{D\bar{u}} \otimes \widehat{D\bar{u}}\right) D^2\bar{u}\right) = 0 & in \ \mathbb{R}^d \times (0,\infty), \\ \bar{u} = u_0 & on \ \mathbb{R}^d \times \{0\}, \end{cases} \tag{5.4}$$

then $u^\epsilon \to \pm 1$ locally uniformly in $\{\pm \bar{u} > 0\}$.

The proof of the theorem follows the general strategy provided by the work of Barles and Souganidis [16]. The basic difficulty is, as we have already seen in Chapter 3, rational directions are not well-behaved. To get around this, we introduce a notion of solutions of (5.4) that requires less information at contact points where the gradient points in a rational direction.

5.1 Solutions in Irrational Directions

For the sake of generality, we will consider in this section level-set PDE more general than (5.4). The discussion here will allow for discontinuous operators in order to apply to the situations encountered later in Chapter 6. In particular, this section concerns viscosity inequalities of the form

$$u_t - \overline{F}(Du, D^2u) \leq 0, \quad u_t - \underline{F}(Du, D^2u) \geq 0,$$

where $\overline{F}$ and $\underline{F}$ are semi-continuous, geometric differential operators.

Throughout this section, we fix operators $\overline{F}, \underline{F} : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ and a sequence of "bad" directions $\{\nu_n\}_{n \in \mathbb{N}} \subseteq S^{d-1}$. Here are the assumptions on $\overline{F}$ and $\underline{F}$:

(i) (Geometric) If $F \in \{\overline{F}, \underline{F}\}$, $(p, X) \in \mathbb{R}^d \times \mathcal{S}_d$, $\mu \in \mathbb{R}$, and $\kappa > 0$, then

$$F(\kappa p, \kappa X + \mu p \otimes p) = \kappa F(p, X).$$

(ii) (Degenerate elliptic) If $F \in \{\overline{F}, \underline{F}\}$, $p \in \mathbb{R}^d \setminus \{0\}$, $X, Y \in \mathcal{S}_d$, and $Y \geq 0$, then

$$0 \leq F(p, X + Y) - F(p, X).$$

(iii) (Stationary planes) $\overline{F}(e, 0) = \underline{F}(e, 0) = 0$ for each $e \in S^{d-1}$.

(iv) (Semi-continuity) $\overline{F}$ is upper semi-continuous and $\underline{F}$ is lower semi-continuous.

In the examples of interest here, the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ will be precisely $S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. If instead of integer periodicity, we were to replace $\mathbb{Z}^d$ by some other lattice Λ in $\mathbb{R}^d$, then $\{\nu_n\}_{n\in\mathbb{N}}$ would instead be $S^{d-1} \cap \mathbb{R}\Lambda$.

Definition 10. *Given an open set $U \subseteq \mathbb{R}^d \times (0, \infty)$, we say that a locally bounded, upper semi-continuous function $v : U \to \mathbb{R}$ satisfies $v_t - \overline{F}(Dv, D^2v) \leq 0$ in "good" directions in U if there is a constant $K(v) > 0$ such that, given any smooth function $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ and any point $(x_0, t_0) \in U$ at which the difference $v - \varphi$ has a strict local maximum, the following conditions are met:*

(a) If $D\varphi(x_0, t_0) \neq 0$ and $\widehat{D\varphi}(x_0, t_0) \in S^{d-1} \setminus \{\nu_n\}_{n\in\mathbb{N}}$, then

$$\varphi_t(x_0, t_0) - \overline{F}(D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leq 0.$$

(b) If $D\varphi(x_0, t_0) \neq 0$ and $\widehat{D\varphi}(x_0, t_0) \in \{\nu_n\}_{n\in\mathbb{N}}$, then

$$\varphi_t(x_0, t_0) \leq K(v) \left\| \left(Id - \widehat{D\varphi}(x_0, t_0) \otimes \widehat{D\varphi}(x_0, t_0) \right) D^2\varphi(x_0, t_0) \right\|.$$

(c) If $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$, then

$$\varphi_t(x_0, t_0) \leq 0.$$

Similarly, a locally bounded, lower semi-continuous function $w : U \to \mathbb{R}$ satisfies $w_t - \underline{F}(Dw, D^2w) \geq 0$ in "good" directions in U if there is a constant $K(w) > 0$ such that, given any smooth function $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ and any point $(x_0, t_0) \in U$ at which the difference $w - \varphi$ has a strict local minimum, the following conditions are met:

(a) If $D\varphi(x_0,t_0) \neq 0$ *and* $\widehat{D\varphi}(x_0,t_0) \notin \{\nu_n\}_{n\in\mathbb{N}}$*, then*

$$\varphi_t(x_0,t_0) - \underline{F}(D\varphi(x_0,t_0), D^2\varphi(x_0,t_0)) \geq 0.$$

(b) If $D\varphi(x_0,t_0) \neq 0$ *and* $\widehat{D\varphi}(x_0,t_0) \in \{\nu_n\}_{n\in\mathbb{N}}$*, then*

$$\varphi_t(x_0,t_0) \geq -K(v) \left\| \left(Id - \widehat{D\varphi}(x_0,t_0) \otimes \widehat{D\varphi}(x_0,t_0) \right) D^2\varphi(x_0,t_0) \right\|.$$

(c) If $\|D\varphi(x_0,t_0)\| = \|D^2\varphi(x_0,t_0)\| = 0$*, then*

$$\varphi_t(x_0,t_0) \geq 0.$$

When $\{\nu_n\}_{n\in\mathbb{N}} = S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, we will simply say "irrational directions" instead of "good directions."

The next result shows that sub- and super-solutions in "good" directions are equivalent to classical viscosity sub- and super-solutions.

Theorem 30. *Given any open set* $U \subseteq \mathbb{R}^d \times (0,\infty)$*, if* $v : U \to \mathbb{R}$ *satisfies* $v_t - \overline{F}(Dv, D^2v) \leq 0$ *in "good" directions in* U*, then* v *satisfies* $v_t - \overline{F}(Dv, D^2v) \leq 0$ *in the viscosity sense in* U*.*

Similarly, if $w : U \to \mathbb{R}$ *satisfies* $w_t - \underline{F}(Dw, D^2w) \geq 0$ *in "good" directions in* U*, then* w *satisfies* $w_t - \underline{F}(Dw, D^2w) \geq 0$ *in the viscosity sense in* U*.*

The proof is deferred to Chapter 7.

5.2 Proof of Theorem 29

Henceforth $u_0 \in UC(\mathbb{R}^d; [-1,1])$ is fixed and $(u^\epsilon)_{\epsilon>0}$ are the solutions of (5.1). The existence and uniqueness of these solutions follows from [].

The macroscopic phases that develop in the sharp-interface limit are described by the following open sets, parametrized by $t > 0$:

$$\Omega_t^{(1)} = \left\{ x \in \mathbb{R}^d \mid \liminf_* u^\epsilon(x,t) = 1 \right\},$$
$$\Omega_t^{(2)} = \left\{ x \in \mathbb{R}^d \mid \limsup{}^* u^\epsilon(x,t) = -1 \right\}.$$

Recall that the half-relaxed limits in the definition are given by

$$\liminf_* u^\epsilon(x,t) = \lim_{\delta \to 0^+} \inf \left\{ u^\epsilon(y,s) \mid \epsilon + \|x-y\| + |t-s| \leq \delta \right\},$$
$$\limsup{}^* u^\epsilon(x,t) = \lim_{\delta \to 0^+} \sup \left\{ u^\epsilon(y,s) \mid \epsilon + \|x-y\| + |t-s| \leq \delta \right\}.$$

We proceed by proving that $(\Omega_t^{(1)})_{t>0}$ and $(\Omega_t^{(2)})_{t>0}$ define super- and sub-flows, respectively, in the sense of [16]. It will not be necessary to know what that means in this paper. Instead, we associate phase indicator functions $\chi_*, \chi^* : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ to the sets $(\Omega_t^{(1)})_{t>0}$ and $(\Omega_t^{(2)})_{t>0}$ and and prove these are respectively discontinuous super- and sub-solutions of (5.4). χ_* and χ^* are defined for $(x,t) \in \mathbb{R}^d \times (0,\infty)$ via the formulae

$$\chi_*(x,t) = \begin{cases} 1, & \text{if } x \in \Omega_t^{(1)}, \\ -1, & \text{otherwise}, \end{cases} \qquad \chi^*(x,t) = \begin{cases} 1, & \text{if } x \in \mathbb{R}^d \setminus \Omega_t^{(2)}, \\ -1, & \text{otherwise}, \end{cases}$$

and at $t = 0$ by

$$\chi_*(x,0) = \lim_{\delta \to 0^+} \inf \left\{ \chi_*(y,s) \mid \|x-y\| + s < \delta,\ s > 0 \right\},$$
$$\chi^*(x,0) = \lim_{\delta \to 0^+} \sup \left\{ \chi^*(y,s) \mid \|x-y\| + s < \delta,\ s > 0 \right\}.$$

The next result is the main step in the proof of Theorem 29:

Proposition 55. χ^* *(resp. χ_*) is a sub-solution (resp. super-solution) of* (5.1) *in irrational* *directions in* $\mathbb{R}^d \times (0,\infty)$*. Furthermore,* $\chi^*(\cdot,0) = -1$ *in* $\{u_0 < 0\}$ *and* $\chi_*(\cdot,0) = 1$ *in* $\{u_0 > 0\}$*.*

Proof. Where χ_* is concerned, the first statement follows from Propositions 57, 62, and 64, and the second, from Proposition 65 below. The statements concerning χ^* then follow by replacing u^ϵ by $-u^\epsilon$, W by $u \mapsto W(-u)$, χ^* by $-\chi^*$, etc. since this has the effect of transforming super-solutions into sub-solutions. □

In view of Theorem 30, we can remove the "irrational directions" qualifier and instead treat χ^* and χ_* as viscosity sub- and super-solutions. It only remains to prove that, even though these functions are discontinuous, χ^* and χ_* can still be compared to $\bar{u}$. This part is classical.

Theorem 31. *If* $\bar{u}$ *is the solution of* (5.4)*, then* $\chi^* \leq -\chi_{\{\bar{u}<0\}}$ *and* $\chi_* \geq \chi_{\{\bar{u}>0\}}$ *in* $\mathbb{R}^d \times [0,\infty)$*.*

Proof. Due to the positivity and continuity of $\overline{m}$, the existence and uniqueness of $\bar{u}$ is standard (see [12]). Since u_0 is uniformly continuous and the equation is translationally invariant, a well-known (e.g., approximation) argument shows that $\bar{u}$ is uniformly continuous in both variables. Thus, [16, Proposition 2.1] applies, giving the desired conclusion. □

Finally, notice that Theorem 31 implies Theorem 29 by definition of χ^* and χ_*. Therefore, it only remains to prove Proposition 55 and Theorem 30.

5.3 Approximate Correctors

The purpose of this section is to prove the existence of approximate correctors in irrational directions. More precisely, given an $e \notin \mathbb{R}\mathbb{Z}^d$ and a $\nu > 0$, we find a constant $\overline{m}(e) > 0$ and a $\tilde{P}_e^\nu \in C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d)$ such that

$$\left| [m(y, \dot{q}(s)e) - \overline{m}(e)]\dot{q}(s) + \mathcal{D}_e^* \mathcal{D}_e \tilde{P}_e^\nu + W''(q(s))\tilde{P}_e^\nu \right| \leq \nu \dot{q}(s) \quad \text{in } \mathbb{R} \times \mathbb{T}^d. \tag{5.5}$$

When $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, this is possible because the diffusion in the $\langle e \rangle^{\perp}$ directions explores the entire torus. We will show below that the same strategy does not work in rational directions precisely because this is no longer the case. In Remark 9 below, we show that when m is sufficiently regular, (2.3) has solutions in certain Diophantine directions; Remark 8 shows that there is an obstruction when $e \in \mathbb{R}\mathbb{Z}^d$.

Recall that the effective mobility $\overline{m}(e)$ is given by

$$\overline{m}(e) = c_W^{-1} \int_{\mathbb{R} \times \mathbb{T}^d} m(y, \dot{q}(s)e)\dot{q}(s)^2 \, dy \, ds, \quad c_W = \int_{-\infty}^{\infty} \dot{q}(s)^2 \, ds. \tag{5.6}$$

The main result of this section concerning existence of approximate correctors is stated next.

Theorem 32. *Fix $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. Given any $\nu > 0$, there is a $\tilde{P}_e^{\nu} \in C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d)$ such that* (5.5) *holds.*

To prove the theorem, we start by regularizing m: given $\mu \in (0,1)$, fix $\tilde{m} \in C^{\mu}(\mathbb{R} \times \mathbb{T}^d)$ such that

$$\sup \left\{ |m(y, \dot{q}(s)e) - \tilde{m}(s,y)| \mid (s,y) \in \mathbb{R} \times \mathbb{T}^d \right\} \leq \frac{1}{3}\nu, \tag{5.7}$$

$$\|\tilde{m}\|_{C^{\mu}(\mathbb{R} \times \mathbb{T}^d)} + \|D_y \tilde{m}\|_{C^{\mu}(\mathbb{R} \times \mathbb{T}^d)} + \|D_y^2 \tilde{m}\|_{C^{\mu}(\mathbb{R} \times \mathbb{T}^d)} < \infty. \tag{5.8}$$

We decompose $\tilde{m}$ in the following way:

$$\tilde{m}(s,y) = \int_{\mathbb{T}^d} \tilde{m}(s,y') \, dy' + \left(\tilde{m}(s,y) - \int_{\mathbb{T}^d} \tilde{m}(s,y') \, dy' \right) =: \tilde{m}_1(s) + \tilde{m}_2(s,y). \tag{5.9}$$

Correspondingly, we define a corrector $\overline{P}_e$ and penalized correctors $(P_2^{\delta})_{\delta > 0}$ solving the following PDE:

$$\tilde{m}_1(s)\dot{q}(s) - \ddot{\overline{P}}_e + W''(q(s))\overline{P}_e = \overline{\tilde{m}}(e)\dot{q}(s) \quad \text{in } \mathbb{R}, \tag{5.10}$$

$$\tilde{m}_2(s,y)\dot{q}(s) + \delta P_2^\delta + \mathcal{D}_e^*\mathcal{D}_e P_2^\delta + W''(q(s))P_2^\delta = 0 \quad \text{in } \mathbb{R}\times\mathbb{T}^d, \tag{5.11}$$

$$\overline{\tilde{m}}(e) = c_W^{-1}\int_{\mathbb{R}\times\mathbb{T}^d} \tilde{m}(s,y)\dot{q}(s)^2\,dy\,ds. \tag{5.12}$$

Here, as above, $\mathcal{D}_e = e\partial_s + D_y$ and $\mathcal{D}_e^*$ is its L^2 adjoint.

The existence and regularity of $\overline{P}_e$ and $(P_e^\delta)_{\delta>0}$ is discussed in [72, Appendix C]. Theorem 32 is proved as soon as we establish the following:

Proposition 56. $\|\dot{q}^{-1}(\delta P_2^\delta)\|_{L^\infty(\mathbb{R}\times\mathbb{T}^d)} \to 0$ *as* $\delta \to 0$*. In particular, given* $\nu > 0$*, if* $\delta > 0$ *is small enough, then* $\tilde{P}_e^\nu = \overline{P}_e + P_e^\delta$ *satisfies* (5.5).

5.3.1 Convergence of δP_2^δ

Since $(P_2^\delta)_{\delta>0} \subseteq C^{2,\mu}(\mathbb{R}\times\mathbb{T}^d)$, straightforward manipulations show that the functions $(v_\zeta^\delta)_{\zeta\in\mathbb{R}}$ obtained by the rule

$$v_\zeta^\delta(x) = V_2^\delta(\langle x, e\rangle - \zeta, x), \quad V_2^\delta(s,y) = \dot{q}(s)^{-1}P_2^\delta(s,y)$$

are solutions of the following family of PDE:

$$\tilde{m}_2(\langle x,e\rangle - \zeta, x) + \delta v_\zeta^\delta - \Delta v_\zeta^\delta - \frac{2\ddot{q}(\langle x,e\rangle - \zeta)}{\dot{q}(\langle x,e\rangle - \zeta)}\langle e, Dv_\zeta^\delta\rangle = 0 \quad \text{in } \mathbb{R}^d.$$

Thus, the asymptotic behavior of $(\delta P_2^\delta)_{\delta>0}$ is captured by that of $(\delta v_\zeta^\delta)_{\delta>0}$.

Notice that if we define $(\tilde{v}_\zeta^\delta)_{\zeta\in\mathbb{R}}$ by

$$\tilde{v}_\zeta^\delta(x) = v_\zeta^\delta(x + \zeta e), \tag{5.13}$$

then the functions $(\tilde{v}^\delta_\zeta)_{\zeta\in\mathbb{R}}$ satisfy the "centered" family of PDE:

$$\tilde{m}_2(\langle x, e\rangle, x + \zeta e) + \delta\tilde{v}^\delta_\zeta - \Delta\tilde{v}^\delta_\zeta - \frac{2\ddot{q}(\langle x, e\rangle)}{\dot{q}(\langle x, e\rangle)}\langle e, D\tilde{v}^\delta_\zeta\rangle = 0 \quad \text{in } \mathbb{R}^d. \tag{5.14}$$

Therefore, letting $\tilde{p} : \mathbb{R}\times\mathbb{R}\times(0,\infty) \to (0,\infty)$ and $g : \langle e\rangle^\perp \times \langle e\rangle^\perp \times (0,\infty) \to (0,\infty)$ denote the fundamental solutions of the operators $\partial_t - \partial_{ss} + W''(q(s))$ and $\partial_t - \Delta_{\langle e\rangle^\perp}$, respectively, we find

$$\begin{aligned} \tilde{v}^\delta_\zeta(x) &= -\int_0^\infty e^{-\delta t} U_\zeta(x,t)\,dt, \\ U_\zeta(x,t) &:= \int_{-\infty}^\infty Q^{\tilde{s}}_\zeta(x + (\tilde{s} - \langle x, e\rangle)e, t)\tilde{p}(\langle x, e\rangle, \tilde{s}, t)d\tilde{s}, \\ Q^{\tilde{s}}_\zeta(x,t) &:= \int_{\langle e\rangle^\perp} \tilde{m}_2(\tilde{s}, x + y' + \zeta e) g(0, y', t)\,\mathcal{H}^{d-1}(dy'). \end{aligned} \tag{5.15}$$

(Above $\Delta^\perp_{\langle e\rangle}$ is the Laplacian in the $\langle e\rangle^\perp$ directions, that is, $\Delta_{\langle e\rangle^\perp} = \text{tr}((\text{Id} - e\otimes e)D^2)$. Thus, $g(0,y',t) = (4\pi t)^{-\frac{(d-1)}{2}} \exp\left(-(4t)^{-1}\|y'\|^2\right)$.)

We will prove that $\delta\tilde{v}^\delta_\zeta \to 0$ uniformly using the averaging induced by the diffusion in $\langle e\rangle^\perp$. Toward that end, the following observation will play a decisive role.

Lemma 4. *If $\mathcal{F}$ is a compact subset of $C(\mathbb{T}^d)$ in the uniform norm topology, then there is a modulus $\eta : [0,\infty) \to [0,\infty)$ with $\lim_{\delta\to 0^+}\eta(\delta) = 0$ such that*

$$\sup\left\{\left|\int_{\langle e\rangle^\perp} u(x+y')g(0,y',t)\,dt - \int_{\mathbb{T}^d} u(y)\,dy\right| \mid x\in\mathbb{T}^d,\ u\in\mathcal{F}\right\} \leq \eta(t^{-1}).$$

Proof. We only need to prove uniform convergence at a given $u \in C(\mathbb{T}^d)$. The uniformity in $\mathcal{F}$ then follows by the Arzelà-Ascoli Theorem.

To start with, assume that $u \in C(\mathbb{T}^d)$ is such that its Fourier series is summable, that is, $\sum_{k \in \mathbb{Z}^d} |\hat{u}(k)| < \infty$. If we define $Q : \mathbb{T}^d \times (0, \infty) \to \mathbb{R}$ by

$$Q(x,t) = \int_{\langle e \rangle^\perp} u(x + y') g(0, y', t) \, \mathcal{H}^{d-1}(dy'),$$

then an elementary computation shows that

$$\hat{Q}(k,t) = \hat{u}(k) e^{-4\pi^2 \|k - \langle k, e \rangle e\|^2 t}.$$

Thus, the fact that $k \neq \langle k, e \rangle e$ for all $k \in \mathbb{Z}^d$ readily implies

$$\lim_{t \to \infty} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{Q}(k,t)| = 0.$$

In particular,

$$\lim_{t \to \infty} \sup \left\{ \left| \int_{\langle e \rangle^\perp} u(x + y') g(0, y', t) \, \mathcal{H}^{d-1}(dy') - \int_{\mathbb{T}^d} u(y) \, dy \right| \ \mid \ x \in \mathbb{T}^d \right\} = 0.$$

Finally, the general case follows by approximation. □

With Lemma 4 in hand, Proposition 56 follows readily:

Proof of Proposition 56. Recall that $V_2^\delta = \dot{q}^{-1} P_2^\delta$. Hence our previous computations yield

$$\begin{aligned} \|\dot{q}^{-1}(\delta P_2^\delta)\|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)} &= \sup \left\{ |\delta \tilde{v}_\zeta^\delta(x)| \ \mid \ x \in \mathbb{R}^d, \ \zeta \in \mathbb{R} \right\} \\ &\leq \int_0^\infty \delta e^{-\delta t} \|Q_\zeta^{\tilde{s}}(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} \, dt. \end{aligned}$$

Observe that $\{\tilde{m}_2(\tilde{s}, \cdot + \zeta e) \ \mid \ (\tilde{s}, \zeta) \in \mathbb{R}^2\}$ is relatively compact in $C(\mathbb{T}^d)$ by the choice of $\tilde{m}_2$. Thus, by Lemma 4, there is a modulus $\eta : [0, \infty) \to [0, \infty)$ with $\lim_{\delta \to 0^+} \eta(\delta) = 0$ such that

$$\sup \left\{ \|Q_\zeta^{\tilde{s}}(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} \ \mid \ (\tilde{s}, \zeta) \in \mathbb{R}^2 \right\} \leq \eta(t^{-1}).$$

Putting it all together, we conclude by observing that

$$\limsup_{\delta \to 0^+} \|\dot{q}^{-1}(\delta P_2^\delta)\|_{L^\infty(\mathbb{R}\times\mathbb{T}^d)} \leq \lim_{\delta\to 0^+} \int_0^\infty \delta e^{-\delta t} \eta(t^{-1})\, dt = 0.$$

□

Now that Theorem 32 is proved, a few remarks are in order.

Remark 7. *The diffusion in $\langle e\rangle^\perp$ directions is needed in Proposition 56. More precisely, the same is no longer true if the forcing only depended on s.*

Here is an example. Set $W(u) = (1-u^2)^2$ and consider the penalized correctors solving the following:

$$q(s)\dot{q}(s) + \delta P^\delta - \ddot{P}^\delta + W''(q(s))P^\delta = 0 \quad \text{in } \mathbb{R}.$$

In this case, q is an even function so $q\dot{q}$ is orthogonal to $\dot{q}$. Thus, one can show that $\delta P^\delta \to 0$ uniformly as $\delta \to 0$. However, this is no longer true when we renormalize by $\dot{q}$.

In this case, $v^\delta = \dot{q}^{-1}P^\delta$ solves the ODE:

$$q(s) + \delta v^\delta - \ddot{v}^\delta + 2\tanh(s)\dot{v}^\delta = 0 \quad \text{in } \mathbb{R}.$$

For a given $\delta > 0$, as $\bar{s} \to \infty$, we find that $v^\delta(\cdot + \bar{s})$ converges to the bounded solution v_+^δ of

$$1 + \delta v_+^\delta - \ddot{v}_+^\delta + 2\dot{v}_+^\delta = 0 \quad \text{in } \mathbb{R}.$$

Since the coefficients are constant, this gives $-\delta v_+^\delta \equiv 1$. In particular,

$$\left\|\dot{q}^{-1}(\delta P^\delta)\right\|_{L^\infty(\mathbb{R})} = \sup\left\{\delta|v^\delta(s)| \ \mid \ s \in \mathbb{R}\right\} \geq 1 \quad \text{for all } \delta > 0.$$

It is worth noting that, in the previous example, any bounded solution P of $q(s)\dot{q}(s) - \ddot{P} + W''(q(s))P = 0$ in $\mathbb{R}$ necessarily grows much faster than $\dot{q}$ as $s \to \pm\infty$. That is, in this *case, the function* $\dot{q}^{-1}P$ *is an unbounded solution of the associated ODE.*

Not only was the diffusion in orthogonal directions necessary in the proof of Theorem 32, but irrationality of e was also.

Remark 8. *If* $e \in \mathbb{R}\mathbb{Z}^d$*, then* $(\dot{q}^{-1}\delta P_2^{\delta})_{\delta>0}$ *converges uniformly to a non-constant function in general. To see this, notice that, if* $e \in \mathbb{R}\mathbb{Z}^d$*, then, in Lemma 4, the conclusion changes to the following one:*

$$\lim_{t\to\infty} \int_{\langle e\rangle^{\perp}} u(x+y',t)g(0,y',t)\,\mathcal{H}^{d-1}(dy') = \fint_{\mathbb{T}_e^{d-1}(\langle x,e\rangle)} u(\xi)\,\mathcal{H}^{d-1}(d\xi).$$

Here the sub-tori $\{\mathbb{T}_e^{d-1}(r)\}_{r\in\mathbb{R}}$ *are defined as in Section 2.4.*

As a consequence of the previous observation, we see that if $(P^{\delta})_{\delta>0}$ *are the bounded solutions of* $\tilde{m}(s,y) + \delta P^{\delta} + \mathcal{D}_e^*\mathcal{D}_e P^{\delta} + W''(q(s))P^{\delta} = 0$ *in* $\mathbb{R}\times\mathbb{T}^d$ *and* $e \in \mathbb{R}\mathbb{Z}^d$*, then*

$$\lim_{\delta\to 0^+} \dot{q}(s)^{-1}(\delta P^{\delta}(s,y)) = \underline{\tilde{m}}_e(\langle y,e\rangle - s) \quad \textit{uniformly in } \mathbb{R}\times\mathbb{T}^d,$$

where $\underline{\tilde{m}}_e : [0,r_e) \to \mathbb{R}$ *is given by*

$$\underline{\tilde{m}}_e(\zeta) = c_W^{-1}\int_{-\infty}^{\infty}\fint_{\mathbb{T}_e^{d-1}(\zeta)} \tilde{m}(s,\xi)\dot{q}(s)^2\,\mathcal{H}^{d-1}(d\xi)\,ds.$$

While $\underline{\tilde{m}}_e$ *certainly extends to a periodic function in* $\mathbb{R}$*, it need not be constant.*

Similarly, (2.3) *cannot have a (e.g., weak) solution unless* $\underline{\tilde{m}}_e$ *is constant, and one can show that* (5.5) *cannot hold unless* 2ν *is larger than the oscillation of* $\underline{\tilde{m}}_e$*.*

Finally, we describe a situation where (2.3) does have solutions in certain directions.

Remark 9. *If we impose enough regularity assumptions on* m *and arithmetic conditions on* e*, and if* $\{Q_{\zeta}^{\tilde{s}}\}$ *are defined as above, then it is possible to show the following estimate*

$$\sup\left\{\int_0^{\infty} \|Q_{\zeta}^{\tilde{s}}(\cdot,t)\|_{L^{\infty}(\mathbb{T}^d)}\,dt \;\mid\; (\tilde{s},\zeta)\in\mathbb{R}\right\} < \infty.$$

This can be made precise by following the proof of Proposition 11 above. With this estimate, we use (5.15) *to see that* $\|v^{\delta}_{\zeta}\|_{L^{\infty}(\mathbb{R}^d)}$ *is bounded independently of* (δ, ζ)*. Therefore, we can send* $\delta \to 0$ *to obtain a solution of* (2.3)*.*

5.4 Irrational Contact Points

This section is devoted to the proof of Proposition 55. This section establishes that the phase indicator function χ_* satisfies condition (a) in the definition of a super-solution in irrational directions (see Definition 10).

Put another way, the goal of this section is to prove the following:

Proposition 57. *If* φ *is a smooth function in* $\mathbb{R}^d \times (0,\infty)$*;* $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$ *is a point where* $\chi_* - \varphi$ *has a strict local minimum; and* $D\varphi(x_0,t_0) \in \mathbb{R}^d \setminus \mathbb{R}\mathbb{Z}^d$*, then*

$$\overline{m}(\widehat{D\varphi}(x_0,t_0))\varphi_t(x_0,t_0) - tr\left(\left(Id - \widehat{D\varphi}(x_0,t_0) \otimes \widehat{D\varphi}(x_0,t_0)\right) D^2\varphi(x_0,t_0)\right) \geq 0. \quad (5.16)$$

The proof of Proposition 57 proceeds by contradiction and is divided into three steps. The first step involves the construction of a suitable local sub-solution of (5.4). The second step, the so-called initialization step, shows that the solutions $(u^{\epsilon})_{\epsilon>0}$ develop a relatively sharp interface around the level surface $\{\varphi = \varphi(x_0,t_0)\}$ after a short macroscopic time. As in [16], this initial step allows us to convert the macroscopic sub-solution of the first step into a sub-solution of (5.1). This conversion is precisely the third step. If φ does not satisfy (5.16), these sub-solutions slip underneath the solutions $(u^{\epsilon})_{\epsilon>0}$ and force (x_0,t_0) to be an interior point of the evolution $t \mapsto \Omega^{(1)}_t$, a contradiction.

5.4.1 Macroscopic Sub-Solution

Here we recall some useful observations that follow from the assumption that $D\varphi(x_0,t_0) \neq 0$. It will be useful to introduce some notation.

Throughout the section, we let $e = \widehat{D\varphi}(x_0, t_0)$. Let $\{e_1, \dots, e_{d-1}\}$ be an orthonormal basis for $\mathbb{R}^{d-1}$ and $\mathcal{O}_e : \mathbb{R}^{d-1} \to \mathbb{R}^d$ be a linear isometry with $\mathcal{O}_e(\mathbb{R}^{d-1}) = \langle e \rangle^{\perp}$. Given $R > 0$, we define the open cube $Q(0, R) \subseteq \mathbb{R}^{d-1}$ by

$$Q(0,R) = \{x' \in \mathbb{R}^{d-1} \mid \max\{|\langle x', e_1\rangle|, \dots, |\langle x', e_{d-1}\rangle|\} < R/2\}.$$

For an $\tilde{x}' \in \mathbb{R}^{d-1}$, we set $Q(\tilde{x}', R) = \tilde{x}' + Q(0, R)$.

Using the coordinates determined by $\mathcal{O}_e$, we define open rectangular prisms $Q^e(0, R, \rho)$ of base length $R > 0$ and height ratio ρ by

$$Q^e(0, R, \rho) = \left\{\mathcal{O}_e(x') + se \mid x' \in Q(0, R),\ s \in (-\rho R/2, \rho R/2)\right\}.$$

Given $x \in \mathbb{R}^d$, we define $Q^e(x, R, \rho) = x + Q^e(0, R, \rho)$.

Finally, an arbitrary point $x \in \mathbb{R}^d$ will frequently be written as $x = (x_e, x')$ with the understanding that $x_e \in \mathbb{R}$ and $x' \in \mathbb{R}^{d-1}$ are such that $x = \mathcal{O}_e(x') + x_e e$. In particular, we will write $x_0 = (x_{0,e}, x_0')$.

We will use the fact that $\{\varphi = 0\}$ is locally a one-parameter family of graphs near (x_0, t_0). Specifically, we have

Proposition 58. *There are constants $\rho, \nu, S, V > 0$ and a smooth function $g : Q(x_0', S) \times (t_0 - \nu, t_0 + \nu) \to \mathbb{R}$ such that*

(i) $\varphi(x, t) > 0$ (resp. $\varphi(x, t) \leq 0$) for some $(x, t) \in Q^e(x_0, S, \rho) \times (t_0 - \nu, t_0 + \nu)$ if and only if $x_e > g(x', t)$ (resp. $x_e \leq g(x', t)$).

(ii) $|g(x_1', t) - g(x_2', t)| \leq \frac{1}{2}\rho(d-1)^{-\frac{1}{2}}\|x_1' - x_2'\|$ no matter the choice of $x_1', x_2' \in Q(x_0', S)$ or $t \in (t_0 - \nu, t_0 + \nu)$.

(iii) $|g(x', t) - g(x', s)| \leq V|t - s|$ for all $x' \in Q(x_0', S)$ and $t, s \in (t_0 - \nu, t_0 + \nu)$.

Further, we can assume that $\rho < 1$ *and* (x_0, t_0) *is a strict local minimum of* $\chi_* - \varphi$ *in* $Q^e(x_0, S, \rho) \times (t_0 - \nu, t_0 + \nu)$.

Proof. The construction of g is a classical application of the implicit function theorem. The fact that $D\varphi(x_0, t_0) = \|D\varphi(x_0, t_0)\| e$ implies that $Dg(x_0, t_0) = 0$, and hence the existence of $\rho < 1$. □

By assumption, there is an $\tilde{\alpha} > 0$ such that

$$\overline{m}(\widehat{D\varphi}(x_0, t_0))\varphi_t(x_0, t_0) - \operatorname{tr}\left(\left(\mathrm{Id} - \widehat{D\varphi}(x_0, t_0) \otimes \widehat{D\varphi}(x_0, t_0)\right) D^2\varphi(x_0, t_0)\right)$$
$$\leq -10\tilde{\alpha}\|D\varphi(x_0, t_0)\|.$$

In particular, since g is smooth, this implies there is an $0 < S_1 < S$ and a $0 < \nu_1 < \nu$ such that

$$\overline{m}\left(\frac{e - Dg}{\sqrt{1 + \|Dg\|^2}}\right) g_t - \operatorname{tr}\left(\left(\mathrm{Id} - \frac{Dg \otimes Dg}{1 + \|Dg\|^2}\right) D^2 g\right) \geq 9\tilde{\alpha} \quad \text{in } Q(x_0', S_1) \times (t_0 - \nu_1, t_0 + \nu_1). \tag{5.17}$$

Next, given a free variable $c > 0$ to be determined, define $\tilde{g} : Q(x_0', S) \times (t_0 - \nu, t_0 + \nu) \to \mathbb{R}$ by

$$\tilde{g}(x', t) = g(x', t) + \frac{c\|x' - x_0'\|^2}{2}.$$

Notice that there is a $c_1 > 0$ such that if $c \in (0, c_1)$, then

$$\overline{m}\left(\frac{e - D\tilde{g}}{\sqrt{1 + \|D\tilde{g}\|^2}}\right) \tilde{g}_t - \operatorname{tr}\left(\left(\mathrm{Id} - \frac{D\tilde{g} \otimes D\tilde{g}}{1 + \|D\tilde{g}\|^2}\right) D^2 \tilde{g}\right) \geq 8\tilde{\alpha} \quad \text{in } Q(x_0', S_1) \times (t_0 - \nu_1, t_0 + \nu_1). \tag{5.18}$$

Finally, let $d : Q^e(x_0, S_1, \rho) \times (t_0 - \nu_1, t_0 + \nu_1) \to \mathbb{R}$ and $d_c : Q^e(x_0, S_1, \rho) \times (t_0 - \nu_1, t_0 + \nu_1) \to \mathbb{R}$ be the signed distance functions to $\{x_e = g(x', t)\}$ and $\{x_e = \tilde{g}(x', t)\}$, respectively.

Specifically, we define d by

$$d(x,t) = \begin{cases} \operatorname{dist}(x, \{\tilde{x} \in Q^e(x_0, S_1, \rho) \mid \tilde{x}_e = g(\tilde{x}', t)\}), & \text{if } x_e > g(x', t), \\ -\operatorname{dist}(x, \{\tilde{x} \in Q^e(x_0, S_1, \rho) \mid \tilde{x}_e = g(\tilde{x}', t)\}), & \text{otherwise}, \end{cases}$$

and we define d_c similarly, but with g replaced by $\tilde{g}$.

Let $\alpha_0 > 0$ be another free variable. Arguing as in [73, Appendix C], we deduce the following facts about d_c:

Lemma 5. *Making ν_1 smaller if necessary, there is a constant $\gamma > 0$ depending on c_1, φ, and S_1 but not on c such that*

$$M := \sup \Big\{ \|\partial_t^j D^i d_c(x,t)\| + \|\partial_t^j D^i d(x,t)\| \mid x \in Q^e(x_0, S_1/2, \rho),$$
$$t \in (t_0 - \nu_1, t_0 + \nu_1),\ |d_c(x,t)| < \gamma,\ i + j \leq 4 \} < \infty.$$

Furthermore, we can choose $0 < S_1' < S_1$ and $0 < \nu_1' < \nu_1$ in such a way that (making γ smaller, if necessary)

$$\overline{m}(Dd_c) d_{c,t} - \Delta d_c \leq -7\tilde{\alpha} \quad \text{in } Q^e(x_0, S_1', \rho) \times (t_0 - \nu_1', t_0 - \nu_1') \cap \{|d| < \gamma\}.$$

and

$$|\overline{m}(Dd_c) - \overline{m}(e)| \leq \alpha_0 \quad \text{in } Q^e(x_0, S_1', \rho) \times (t_0 - \nu_1', t_0 - \nu_1') \cap \{|d| < \gamma\}.$$

Finally, we let $\eta, \beta > 0$ be free variables to be determined and pick $0 < S_2 < S_1'$ and $0 < \nu_2 < \nu_1'$ and define $\tilde{d}_c : Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2) \to \mathbb{R}$ by

$$\tilde{d}_c(x,t) = d_c(x + \eta(t - (t_0 - \nu_2))e, t).$$

Notice that $\tilde{d}_c(\cdot, t)$ is the signed distance to the surface $\{x_e = \tilde{g}(x', t) - \eta(t - (t_0 - \nu_2))\}$, and it is well-defined by making S_2 and ν_2 smaller if necessary.

$\tilde{d}_c$ is the sub-solution of (5.4) that will be used in the sequel. Its key properties are summarized next. In the statement, we use the notation ∂_p for the parabolic boundary. Specifically, for a space-time domain $Q \times (a, b)$ that means

$$\partial_p[Q \times (a, b)] = \partial Q \times (a, b] \cup \overline{Q} \times \{a\}.$$

Proposition 59. *If $\eta \in (0, \Theta^{-1}\tilde{\alpha})$, $\beta \in (0, \frac{1}{2}\eta\nu_2 \wedge 1 \wedge \frac{1}{2}S_2)$, $c \in (0, c_1)$, $\nu_2 \in (0, \nu_1')$, and $S_2 \in (0, S_1'/2)$ satisfy the inequalities*

$$(3\eta + 2V)\nu_2 < \frac{\rho S_2}{4} + \frac{cS_2^2}{8}, \tag{5.19}$$

then

$$\overline{m}(D\tilde{d}_c)\tilde{d}_{c,t} - \Delta\tilde{d}_c \le -6\tilde{\alpha}, \ |\overline{m}(D\tilde{d}_c) - \overline{m}(e)| \le \alpha_0 \quad \textit{in } \{|\tilde{d}_c| < \gamma\},$$
$$\chi_{\{\tilde{d}_c \ge \beta\}} \le \chi_{\{d \ge \beta\}} \quad \textit{on } \partial_p[Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2)].$$

Proof. The inequality $2\eta\nu_2 < \frac{\rho S_2}{2} < \frac{\rho(S_1' - S_2)}{2}$ implies that $x + \eta(t - (t_0 - \nu_2))e \in Q^e(x_0, S_1', \rho)$ whenever $x \in Q^e(x_0, S_2, \rho)$. Thus, the first statement follows from Lemma 5 and the inequality $\overline{m} \le \Theta$.

Concerning the second statement, we know that $\tilde{d}_c(\cdot, t_0 - \nu_2) = d_c(\cdot, t_0 - \nu_2)$. Therefore, the ordering between g and $\tilde{g}$ implies $\tilde{d}_c \le d$ on the surface $\{t = t_0 - \nu_2\}$.

Next, we check the remaining inequalities, namely,

$$\chi_{\{\tilde{d}_c \ge 2\beta\}} \le \chi_{\{d \ge 2\beta\}} \quad \text{on } \partial Q^e(x_0, S_2, \rho) \times [t_0 - \nu_2, t_0 + \nu_2].$$

We will start by examining points $x = (x_e, x')$ with $x' \in \partial Q(x_0, S_2)$.

Assume that $\tilde{d}_c(x, t) \ge \beta$ and $x' \in \partial Q(x_0, S_2)$. To show that $d(x, t) \ge \beta$, we proceed point by point. We claim that if $d(y,t)=0$, then $\|y-x\| \geq \beta$. If $\|y'-x'\| \geq \beta$, we are done. Therefore, assume that $\|y'-x'\| \leq \beta$.

By the definition of $\tilde{d}_c$ and d_c, we have

$$g(x',t) + \frac{cS_2^2}{8} - 2\eta\nu_2 \leq g(x',t) + \frac{c\|x'-x_0'\|^2}{2} - \eta(t-(t_0-\nu_2)) < x_e. \tag{5.20}$$

At the same time, $|g(y',t) - g(x',t)| \leq \frac{1}{2}\rho\beta$. Therefore, since $y_e = g(y',t)$, we find

$$y_e \leq g(x',t) + \frac{\rho\beta}{2} < x_e - \frac{cS_2^2}{8} + 2\eta\nu_2 + \frac{\beta}{2}.$$

Recalling that $2\beta < \eta\nu_2$ and appealing to (5.19), we conclude

$$\|y-x\| \geq |y_e - x_e| = x_e - y_e \geq \frac{cS_2^2}{8} - \left(2\eta\nu_2 + \frac{\beta}{2}\right) \geq \beta.$$

Hence $|d(x,t)| \geq \beta$. Similarly, (5.20) shows that $g(x',t) < x_e$ so $d(x,t) > 0$. Thus, $d(x,t) \geq \beta$ as claimed.

Finally, we consider points (x,t) with x on the top or bottom of the box $Q^e(x_0, S_2, \rho)$, that is, points for which $|x_e - x_{0,e}| = \rho S_2/2$. To start with, observe that if $x_e = x_{0,e} - \rho S_2/2$, then $\tilde{d}_c(x,t) < 0 < \beta$. Indeed, it suffices to show that $x_e + \eta(t-(t_0-\nu_2)) < \tilde{g}(x',t)$, which is true since, by (5.19),

$$\begin{aligned} x_e + \eta(t-(t_0-\nu_2)) &\leq g(x_0', t_0) - \frac{\rho S_2}{2} + 2\eta\nu_2 \\ &\leq \tilde{g}(x',t) + 2(V+\eta)\nu_2 - \frac{\rho S_2}{4} < \tilde{g}(x',t). \end{aligned}$$

It remains to consider the case when $x_e = x_{0,e} + \rho S_2/2$. We claim that $d(x,t) \geq \beta$ in this case. To see this, we first show that if $d(y,t)=0$ and $y \in Q^e(x_0, S_1, \rho)$, then $\|y-x\| \geq \beta$.

Indeed, in case $y \in Q^e(x_0, S_2 + 2\beta, \rho)$, we apply (5.19) to find

$$\begin{aligned} \|y - x\| &\geq |y_e - x_e| \\ &= \left|(x_e - x_{0,e}) + (g(x_0, t_0) - g(y', t))\right| \\ &\geq \frac{\rho S_2}{2} - |g(x_0, t_0) - g(y', t)| \\ &\geq \frac{\rho S_2}{2} - \frac{\rho(S_2 + 2\beta)}{4} - 2V\nu_2 \geq \beta. \end{aligned}$$

On the other hand, if $y \in Q^e(x_0, S_1, \rho) \setminus Q^e(x_0, S_2 + 2\beta, \rho)$, then we can consider two cases: (i) $y' \in Q(x_0', S_1) \setminus Q(x_0', S_2 + 2\beta)$ or (ii) $\rho(S_2 + 2\beta)/2 \leq |y_e| < \rho S_1/2$. In case (i), $\|x - y\| \geq \|x' - y'\| \geq \beta$ follows immediately. On the other hand, in case (ii), we can assume that both $|y_e - x_{0,e}| \geq \rho(S_2 + 2\beta)/2$ and $y' \in Q(x_0', S_2 + 2\beta)$, but then this contradicts Proposition 58 and $2V\nu_2 < \rho S_2/2$. So only case (i) is possible and then $|d(x, t)| \geq \beta$ follows.

Similarly, we find that $d(x, t) > 0$ so $d(x, t) \geq \beta$. □

In the remainder of this section, we will adjust the constants if necessary so that the hypotheses of Proposition 59 hold. In addition, we impose the following constraint on η:

$$\eta\nu_2 < \gamma.$$

The justification for this restriction comes in the remark that follows. Henceforth, η, c, ν_2, and S_2 are fixed. We reserve the right to make $\beta > 0$ smaller later. Also note that α_0 remains undetermined at this stage and so far no restrictions have been imposed on it.

Remark 10. *Notice that the boundary inequality in Proposition 59 has the following (trivial) consequence: for each $\epsilon > 0$,*

$$(1-\beta\epsilon)\chi_{\{\tilde{d}_c \geq \beta\}} - \chi_{\{\tilde{d}_c < \beta\}} \leq (1-\beta\epsilon)\chi_{\{d \geq \beta\}} - \chi_{\{d < \beta\}} \quad \text{on } \partial_p[Q(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2)].$$

Further, notice that since $Dd_c(x_0, t_0) = e$ and $\eta\nu_2 < \gamma$, it follows that

$$\tilde{d}_c(x_0, t_0) = d_c(x_0 + \eta\nu_2 e, t_0) = \eta\nu_2 > 2\beta.$$

In particular, $\{\tilde{d}_c > 2\beta\}$ contains a neighborhood of (x_0, t_0).

5.4.2 Initialization

In this section, we prove an initialization result that shows that the solutions $(u^\epsilon)_{\epsilon>0}$ develop a sharp transition along the interface $\{\varphi = 0\}$ when $\epsilon > 0$ is sufficiently small. Here we follow [16] using the result of [72, Appendix A].

Proposition 60. *Given $\delta \in (0, \frac{1}{2})$, there is a $\tau = \tau(\delta, \beta, \varphi) > 0$ and an $\epsilon_0 = \epsilon_0(\beta, \varphi, u_0) > 0$ such that if $\epsilon \in (0, \epsilon_0)$ and $t \in [t_0 - \nu_2, t_0 + \nu_2]$, then*

$$u^\epsilon(\cdot, t + \tau\epsilon^2 \log(\epsilon^{-1})) \geq (1 - \beta\epsilon)\chi_{\{d(\cdot,t)\geq\beta\}} - \chi_{\{d(\cdot,t)\leq\beta\}} \quad \textit{in } \overline{Q^e(x_0, S_2, \rho)}.$$

Proof. First of all, since $\{d \geq \beta\} \subseteq \{\chi_* = 1\}$, there is an $\epsilon_0 > 0$ such that

$$\{d \geq \beta\} \cap (Q^e(x_0, S, \rho) \times [t_0 - \nu, t_0 - \nu]) \subseteq \{u^\epsilon > 1 - \delta\} \quad \text{if } \epsilon \in (0, \epsilon_0). \tag{5.21}$$

For the rest of the proof, fix $t \in [t_0 - \nu_2, t_0 + \nu_2]$.

Let $\underline{\psi} : \mathbb{R}^d \to \mathbb{R}$ be the function given by

$$\underline{\psi}(x) = \begin{cases} 1 - \delta, & x \in Q^e(x_0, (S_1 + S_2)/2, \rho) \cap \{d(\cdot, t) \geq \beta/2\}, \\ -1, & \text{otherwise.} \end{cases}$$

Fix a smooth, symmetric non-negative function ρ such that $\rho = 0$ in $\mathbb{R}^d \setminus B(0, 1)$ and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. Let $(\rho_\nu)_{\nu>0}$ be the mollifying family in $\mathbb{R}^d$ given by $\rho_\nu(x) = \nu^{-d}\rho(\nu^{-1}x)$.

In view of Lemma 5, there is a $\overline{\nu} > 0$ small and independent of t with the following property:

$$x \in \{d \geq \beta\} \cap Q^e(x_0, S_2, \rho) \implies B(x, \overline{\nu}) \subseteq \{d(\cdot, t) \geq \beta/2\} \cap Q^e(x_0, (S_1 + S_2)/2, \rho),$$
$$x \in \{d \leq 0\} \cap Q^e(x_0, S_2, \rho) \implies B(x, \overline{\nu}) \subseteq \{d(\cdot, t) \leq \beta/2\} \cap Q^e(x_0, (S_1 + S_2)/2, \rho).$$

Define $\psi = \rho_{\overline{\nu}} * \underline{\psi}$. Recall that

$$\|D\psi\|_{L^\infty(\mathbb{R}^d)} \leq C_\rho \overline{\nu}^{-1}, \tag{5.22}$$
$$\|D^2\psi\|_{L^\infty(\mathbb{R}^d)} \leq C_\rho \overline{\nu}^{-2}. \tag{5.23}$$

Notice that, by the choice of $\overline{\nu}$, if $x \in Q^e(x_0, S_2, \rho)$ and $d(x, t) \geq \beta$, then

$$\psi(x) = \int_{B(x,\overline{\nu})} \psi(y) \rho_{\overline{\nu}}(x - y)\, dy = 1 - \delta.$$

Similarly, if $x \in Q^e(x_0, S_2, \rho)$ and $d(x, t) \leq 0$, then $\psi(x) = -1$. In summary,

$$\{d \geq \beta\} \cap Q^e(x_0, S_2, \rho) \subseteq \{\psi = 1 - \delta\}, \quad \{d \leq 0\} \cap Q^e(x_0, S_2, \rho) \subseteq \{\psi = -1\} \tag{5.24}$$

Let $\tilde{\chi}^\epsilon$ be the function from [72, Lemma 12]; let $K > 0$ be a free variable; and define a family $(\underline{u}^\epsilon)_{\epsilon>0}$ in $\mathbb{R}^d \times [t, \infty)$ by

$$\underline{u}^\epsilon(x, t') = \tilde{\chi}^\epsilon(\psi(x) - \epsilon^{-1}K(t' - t), \epsilon^{-1}(t' - t)).$$

The construction of [72, Appendix A] gives that $\tilde{\chi}^\epsilon_s \leq -\overline{f}(\tilde{\chi}^\epsilon)$, where $\overline{f}$ is defined in [72, Equation 51]. Thus, by arguing as in [16, Lemma 4.1], we find that $\underline{u}^\epsilon$ is a sub-solution of (5.1) if K is large enough. (Notice that K depends only on $\overline{\nu}$ through (5.22) and (5.23) and, thus, is independent of t). Further, $\underline{u}^\epsilon(\cdot, 0) = \psi \leq u^\epsilon(\cdot, t)$ by (5.21). Therefore,

$$\underline{u}^\epsilon(x, t') \leq u^\epsilon(x, t' + t) \quad \text{if } (x, t') \in \mathbb{R}^d \times [0, \infty).$$

Now the conclusion follows from the properties of χ arguing exactly as in [16]. Note that τ depends only on K and so is independent of t. $\square$

5.4.3 Mesoscopic Sub-Solutions

Finally, we use the macroscopic sub-solution of Section 5.4.1, namely $\tilde{d}_c$, to build mesoscopic sub-solutions of (5.1) that converge to 1 in the sets $\{\tilde{d}_c > 2\beta\}$. Appealing to Remark 10, we will then conclude that $u^\epsilon \to 1$ uniformly in a neighborhood of (x_0, t_0), a patent contradiction.

Recall that $e = \widehat{D\varphi}(x_0, t_0)$. Let α_1 be one last free variable, for convenience. Invoking Theorem 32, we fix an approximate corrector $P_e = \overline{P}_e + P_2^\delta \in C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d)$ such that (5.5) holds with $\nu = \alpha_1$.

Define a family $(v^\epsilon)_{\epsilon > 0}$ in $\{(x, t) \in Q^e(x_0, S_2, \rho) \times [t_0 - \nu_2, t_0 + \nu_2] \mid |\tilde{d}_c(x, t)| < \gamma\}$ by

$$v^\epsilon(x, t) = q\left(\frac{\tilde{d}_c(x, t) - 2\beta}{\epsilon}\right) + \epsilon\left(\tilde{d}_{c,t}(x, t) P_e\left(\frac{\tilde{d}_c(x, t) - 2\beta}{\epsilon}, \frac{x}{\epsilon}\right) - 2\beta\right).$$

We show below that, provided α_0, α_1, and β are chosen appropriately, v^ϵ is a sub-solution of (5.1) as soon as $\epsilon > 0$ is small enough.

In order to invoke the comparison principle, we extend $(v^\epsilon)_{\epsilon > 0}$ to $Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2)$. The construction again follows [16]. First, we define $(\overline{v}^\epsilon)_{\epsilon > 0}$ by

$$\overline{v}^\epsilon(x, t) = \begin{cases} \max\{v^\epsilon(x, t), -1\}, & \text{if } \tilde{d}_c(x, t) \geq -(\gamma + 2\beta)/2, \\ -1, & \text{if } \tilde{d}_c(x, t) < -(\gamma + 2\beta)/2. \end{cases}$$

Finally, we fix a smooth, non-decreasing function $f : \mathbb{R} \to [0,1]$ such that

$$f(\xi) = 1 \quad \text{if} \quad \xi \geq \frac{7\gamma}{8} + \frac{\beta}{4}, \quad f(\xi) = 0 \quad \text{if} \quad \xi \leq \frac{3\gamma}{4} + \frac{\beta}{2}$$

and define $(w^\epsilon)_{\epsilon > 0}$ by

$$w^\epsilon(x,t) = (1 - f(\tilde{d}_c(x,t)))\overline{v}^\epsilon(x,t) + f(\tilde{d}_c(x,t))(1 - \beta\epsilon).$$

Here is the main result we will need to proceed:

Proposition 61. *There is an $\epsilon_1 > 0$ and a choice of the parameters α_0, α_1, and β such that w^ϵ satisfies*

$$\begin{cases} m(\epsilon^{-1}x, \widehat{Dw^\epsilon})w^\epsilon_t - \Delta w^\epsilon + \epsilon^{-2}W'(w^\epsilon) \leq 0 & \text{in } Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0], \\ w^\epsilon \leq (1 - \beta\epsilon)\chi_{\{\tilde{d}_c \geq \beta\}} - \chi_{\{\tilde{d}_c < \beta\}} & \text{on } \partial_p[Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0]]. \end{cases}$$

Furthermore, if $(x,t) \in Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0]$ satisfies $\tilde{d}_c(x,t) > 2\beta$, then

$$\liminf{}_* w^\epsilon(x,t) = 1.$$

Before proving Proposition 61, let us use it to prove Proposition 57.

Proof of Proposition 57. By Propositions 60 and 61 and Remark 10, for each $\epsilon \in (0, \epsilon_0 \wedge \epsilon_1)$, we have

$$w^\epsilon \leq u^\epsilon(\cdot, \cdot + \tau\epsilon^2 \log(\epsilon^{-1})) \quad \text{on } \partial_p[Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0)].$$

According to Remark 10 and Proposition 61, there is a small $r > 0$ such that

$$1 = \liminf{}_* w^\epsilon(x,t) \quad \text{if } \|x - x_0\| + |t - t_0| < r.$$

Therefore, for such points (x,t),

$$\liminf_* u^\epsilon(x,t) \geq \liminf_* w^\epsilon(x,t) = 1.$$

Hence $\chi_* = 1$ in a neighborhood of (x_0,t_0). This contradicts the assumption that $D\varphi(x_0,t_0)$ is nonzero. □

Now we proceed with the proof of Proposition 61. The proof will be presented through a series of lemmas. The first deals with v^ϵ near the interface.

Lemma 6. *There is a choice of β, α_1, and α_0 such that: (i) β is small enough to satisfy the constraints of the previous section and (ii) there is a constant $\nu(\beta,\tilde{\alpha})$ such that, for all $\epsilon > 0$ small enough, we have*

$$m(\epsilon^{-1}x, \widehat{Dv^\epsilon})v_t^\epsilon - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) \leq -\frac{\nu(\beta,\tilde{\alpha})}{3\epsilon} \quad \textit{in } \{|d| < \gamma\}.$$

The selection of β, α_1, and α_0 below is a little delicate. The reason is, at some stage, the fact that $D\tilde{d}_c$ is not constant introduces errors. Let ω_e be a modulus of continuity for m and $\overline{m}$ at e, that is,

$$\omega_e(\chi) = \sup\left\{|m(y,v) - m(y,e)| + |\overline{m}(v) - \overline{m}(e)| \;\middle|\; \|v - e\| \leq \chi,\ y \in \mathbb{T}^d\right\}.$$

The errors $D\tilde{d}_c$ are proportional to $\omega_e(\alpha_0)$ with proportionality constants depending on the choice of α_1 through the magnitudes of the derivatives of P_e. Therefore, to control these errors, we need to choose α_1 before α_0. Given that β depends on α_0 through Proposition 59, it has to be chosen last.

Proof. In what follows, to declutter the notation, it will be convenient to define $s = s(x,t) =$

$\tilde{d}_c(x,t) - 2\beta$ and p_ϵ by

$$p_\epsilon(x,t) = \epsilon \mathcal{D}_{Dd(x,t)} P_e(\epsilon^{-1}s(x,t), \epsilon^{-1}x) + \epsilon^2 D\tilde{d}_{c,t}(x,t) P_e(\epsilon^{-1}s(x,t), \epsilon^{-1}x).$$

By definition of v^ϵ, the regularity of P_e^δ, and the definition of M in Lemma 5, we have

$$\begin{aligned} m(\epsilon^{-1}x, \epsilon Dv^\epsilon) v_t^\epsilon &= \epsilon^{-1}\dot{q}(\epsilon^{-1}s) m(\epsilon^{-1}x, \dot{q}(\epsilon^{-1}s) D\tilde{d}_c(x,t) + p_\epsilon) \tilde{d}_{c,t}(x,t) + O(1) \\ &\leq \epsilon^{-1}\dot{q}(\epsilon^{-1}s) \left(m(\epsilon^{-1}x, \dot{q}(\epsilon^{-1}s)e) \tilde{d}_{c,t}(x,t) + M\omega_e(\alpha_0 + O(\epsilon)) \right) + O(1). \end{aligned}$$

Note, in addition, that, no matter the choice of $e' \in S^{d-1}$, the function $P_2^\delta = V_2^\delta \dot{q}$ defined in (5.11) satisfies

$$\mathcal{D}_{e'}^* \mathcal{D}_{e'} P_2^\delta + W''(q(s)) P_2^\delta = \dot{q}(s) \left(\mathcal{D}_{e'}^* \mathcal{D}_{e'} V_2^\delta - \frac{2\ddot{q}(s)}{\dot{q}(s)} \langle e', \mathcal{D}_{e'} V_2^\delta \rangle \right) \quad \text{in } \mathbb{R} \times \mathbb{T}^d.$$

Combining this with the estimate $\|D\tilde{d}_c - e\| \leq \alpha_0$ from Proposition 59, we find

$$\begin{aligned} -\Delta v^\epsilon &= -\epsilon^{-2}\ddot{q}(\epsilon^{-1}s) - \epsilon^{-1}\dot{q}(\epsilon^{-1}s)\Delta\tilde{d}_c(x,t) \\ &\quad + \epsilon^{-1}\mathcal{D}^*_{D\tilde{d}_c(x,t)} \mathcal{D}_{D\tilde{d}_c(x,t)} P_e(\epsilon^{-1}s, \epsilon^{-1}x)\tilde{d}_{c,t}(x,t) + O(1) \\ &\leq -\epsilon^{-2}\ddot{q}(\epsilon^{-1}s) + \epsilon^{-1}\dot{q}(\epsilon^{-1}s) \left([\overline{m}(e) - m(\epsilon^{-1}x, \dot{q}(\epsilon^{-1}s)e)]\tilde{d}_{c,t}(x,t) - \Delta\tilde{d}_c(x,t) \right. \\ &\quad \left. + M\alpha_1 + O(\alpha_0) \right) - \epsilon^{-1}W''(q(\epsilon^{-1}s))P_e + O(1). \end{aligned}$$

Therefore, after observing that we can write

$$\epsilon^{-2}W'(v^\epsilon) = \epsilon^{-2}W'(q) + \epsilon^{-1}W''(q)P^\delta - 2\beta\epsilon^{-1}W''(q) + O(1)$$

we find

$$m(\epsilon^{-1}x,\epsilon Dv^\epsilon)v_t^\epsilon - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) = \epsilon^{-1}\Big\{\dot{q}(\epsilon^{-1}s,\epsilon^{-1}x)\Big(\overline{m}(D\tilde{d}_c(x,t))\tilde{d}_{c,t}(x,t) - \Delta\tilde{d}_c(x,t) + 2M\omega_e(\alpha_0+O(\epsilon)) + O(\alpha_0) + M\alpha_1\Big) - 2\beta W''(q(\epsilon^{-1}s))\Big\} + O(1).$$

Thus, appealing once more to Proposition 59,

$$m(\epsilon^{-1}x,\epsilon Dv^\epsilon)v_t^\epsilon - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) \leq \epsilon^{-1}\Big\{-(6\tilde{\alpha} + 2M\omega_e(\alpha_0+O(\epsilon)) + O(\alpha_0) + M\alpha_1)\dot{q}(\epsilon^{-1}s) - 2\beta\epsilon W''(q(\epsilon^{-1}s))\Big\} + O(1).$$

Finally, we choose α_1, α_0, and β, in that order. To start with, choose α_1 so that $M\alpha_1 < \tilde{\alpha}$. Next, choose α_0 so that, in the expression above, as soon as ϵ is small enough, we have

$$2M\omega_e(\alpha_0+O(\epsilon)) + O(\alpha_0) \leq 2M\omega_e(2\alpha_0) + O(\alpha_0) \leq \tilde{\alpha}.$$

Note that this choice depends on α_1 through the magnitude of the derivatives of V_2^δ, which contribute to the $O(\alpha_0)$ term. However, it does not depend on any of the parameters introduced in Section 5.4.1 (and, in particular, introduces no new restrictions on β) so there is no risk of circular reasoning.

By [16, Lemma 4.3], there is a $\overline{\beta}(\tilde{\alpha}) > 0$ such that if $\beta \in (0,\overline{\beta}(\tilde{\alpha}))$, then

$$\nu(\beta,\tilde{\alpha}) := \sup\big\{3\tilde{\alpha}\dot{q}(s) + 2\beta W''(q(s)) \mid s \in \mathbb{R}\big\} > 0. \tag{5.25}$$

At last, fix such a β consistently with the restrictions of Section 5.4.1. Note that, with this choice of $(\alpha_0, \alpha_1, \beta)$, for small enough $\epsilon > 0$, we find

$$m(\epsilon^{-1}x, \epsilon Dv^\epsilon)v_t^\epsilon - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) \leq -\frac{\nu(\beta, \tilde{\alpha})}{2\epsilon} + O(1) \quad \text{in } \{|\tilde{d}_c| < \gamma\}.$$

□

Henceforth, we assume that β, α_0, and α_1 have been chosen so that Lemma 6 holds. These three parameters will remain fixed throughout the rest of this section.

Next, we show that the functions $(\overline{v}^\epsilon)_{\epsilon>0}$ are sub-solutions away from $\{\tilde{d}_c > 0\}$.

Lemma 7. *If $\epsilon > 0$ is small enough, then $\overline{v}^\epsilon$ is a sub-solution of (5.1) in $\{\tilde{d}_c < \gamma\}$.*

Proof. It is clear that $\overline{v}^\epsilon$ is a sub-solution in $\{-\frac{(\gamma+2\beta)}{2} < \tilde{d}_c < \gamma\}$, being the maximum of two sub-solutions there. At the same time, we claim that $\overline{v}^\epsilon$ is a sub-solution in $\{\tilde{d}_c < -(\frac{3\beta}{2} + \frac{\gamma}{4})\}$ as soon as $\epsilon > 0$ is small enough. Indeed, if $\tilde{d}_c(x,t) < -(\frac{3\beta}{2} + \frac{\gamma}{4})$, then the exponential estimates of [72, Appendix C] imply that

$$v^\epsilon(x,t) \leq q\left(-\frac{\gamma}{4\epsilon}\right) + \epsilon\left(M\left|P_e\left(\frac{\tilde{d}_c(x,t) - 2\beta}{\epsilon}\right)\right| - 2\beta\right) \leq -1 + C\exp\left(-\frac{\gamma}{4C\epsilon}\right) - 2\beta\epsilon.$$

Hence $\overline{v}^\epsilon = -1$ in $\{\tilde{d}_c < -(\frac{3\beta}{2} + \frac{\gamma}{4})\}$ as soon as ϵ is sufficiently small. In particular, that makes $\overline{v}^\epsilon$ is a sub-solution in $\{\tilde{d}_c < \gamma\}$. □

Finally, we verify that $(w^\epsilon)_{\epsilon>0}$ remains a sub-solution inside $\{\tilde{d}_c > 0\}$ and has the right boundary behavior.

Lemma 8. *If $\epsilon > 0$ is small enough, then w^ϵ is a sub-solution of (5.1) in $Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2)$ and*

$$w^\epsilon \leq (1 - \beta\epsilon)\chi_{\{\tilde{d}_c \geq \beta\}} - \chi_{\{\tilde{d}_c < \beta\}} \quad \text{on } \partial_p[Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0]].$$

Proof. Arguing as in Lemma 7, we see that $\overline{v}^\epsilon = v^\epsilon$ in $\{\tilde{d}_c > \frac{\gamma+2\beta}{2}\}$ as soon as $\epsilon > 0$ is sufficiently small. In fact, we can assume that $1 - v^\epsilon \leq 2\beta\epsilon$ in $\{\tilde{d}_c > \frac{\gamma+2\beta}{2}\}$.

Plugging w^ϵ into the equation in $\{\frac{\gamma+2\beta}{2} < \tilde{d}_c < \gamma\}$, we find

$$\begin{aligned} m\left(\epsilon^{-1}x, \widehat{Dw^\epsilon}\right) w_t^\epsilon - \Delta w^\epsilon + \epsilon^{-2} W'(w^\epsilon) = m\left(\epsilon^{-1}x, \widehat{Dw^\epsilon}\right) w_t^\epsilon - (1 - f(\tilde{d}(x,t)))\Delta v^\epsilon \\ + 2f'(\tilde{d}(x,t))\langle D\tilde{d}, Dv^\epsilon\rangle + \epsilon^{-2}W'(w^\epsilon) \\ + \left(f'(\tilde{d}(x,t))\Delta\tilde{d} + 2f''(\tilde{d}(x,t))\right) \\ \times (v^\epsilon(x,t) - (1-\beta\epsilon)) \\ = (I) + (II) + (III) + (IV) + (V) + (VI), \end{aligned}$$

where, in view of the exponential estimates of [72, Appendix C], the error terms can be estimated as follows:

$$
\begin{aligned}
(I) &= (1-f(\tilde{d}(x,t)))\{m(\epsilon^{-1}x,\widehat{Dv^\epsilon})v_t^\epsilon - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon)\} \\
&\le -\frac{1}{3}(1-f(\tilde{d}(x,t)))\nu(\beta,\tilde{\alpha})\epsilon^{-1}, \\
(II) &= f(\tilde{d}(x,t))\epsilon^{-2}W'(1-\beta\epsilon) \le -f(\tilde{d}(x,t))\beta W''(1)\epsilon^{-1} + O(1), \\
(III) &= \epsilon^{-2}W'((1-f)v^\epsilon + f(1-\beta\epsilon)) - (1-f)\epsilon^{-2}W'(v^\epsilon) - f\epsilon^{-2}W'(1-\beta\epsilon) \\
&= \epsilon^{-2}O(|v^\epsilon - 1|^2 + \epsilon^2) = O(1), \\
(IV) &= (1-f(\tilde{d}(x,t)))(m(\epsilon^{-1}x,\widehat{Dw^\epsilon}) - m(\epsilon^{-1}x,\widehat{Dv^\epsilon}))v_t^\epsilon \\
&\le C\left[\epsilon^{-1}\dot{q}\left(\frac{\tilde{d}(x,t)-2\beta}{\epsilon}\right) + \left|\partial_s P_e\left(\frac{\tilde{d}(x,t)-2\beta}{\epsilon}\right)\right| + \epsilon\right] \\
&\le C\exp\left(-(C\epsilon)^{-1}\left(\frac{\gamma-2\beta}{2}\right)\right), \\
(V) &= (1-f(\tilde{d}(x,t)))m(\epsilon^{-1}x,\widehat{Dw^\epsilon})(w_t^\epsilon - v_t^\epsilon) \\
&\le C\exp\left(-(C\epsilon)^{-1}\left(\frac{\gamma-2\beta}{2}\right)\right) + Cf'(\tilde{d}(x,t))\tilde{d}_t|v^\epsilon(x,t) - (1-\beta\epsilon)| \\
&\le C\left[\exp\left(-(C\epsilon)^{-1}\left(\frac{\gamma-2\beta}{2}\right)\right) + \epsilon\right], \\
(VI) &= 2f'(\tilde{d}(x,t))\langle D\tilde{d}, Dv^\epsilon\rangle + \left(f'(\tilde{d}(x,t))\Delta\tilde{d} + 2f''(\tilde{d}(x,t))\right)(v^\epsilon(x,t) - (1-\beta\epsilon)) \\
&\le C\left(\left(\epsilon^{-1}+1\right)\exp\left(-(C\epsilon)^{-1}\left(\frac{\gamma-2\beta}{2}\right)\right)\right) + C\epsilon.
\end{aligned}
$$

In particular, we find, in the limit $\epsilon \to 0^+$,

$$
m\left(\epsilon^{-1}x,\widehat{Dw^\epsilon}\right)w_t^\epsilon - \Delta w^\epsilon + \epsilon^{-2}W'(w^\epsilon) \le -\min\left\{\frac{1}{3}\nu(\beta,\tilde{\alpha}), \beta W''(1)\right\}\epsilon^{-1} + O(1).
$$

Thus, w^ϵ is a sub-solution in the domain $\{\frac{\gamma+2\beta}{2} < \tilde{d}_c < \gamma\}$ as soon as ϵ is small enough. At the same time, $w^\epsilon = \overline{v}_\epsilon$ in $\{\tilde{d}_c < \frac{3\gamma}{4} + \frac{\beta}{2}\}$ and $w^\epsilon = 1 - \beta\epsilon$ in $\{\frac{7\gamma}{8} + \frac{\beta}{4} < \tilde{d}_c\}$ so w^ϵ is actually a sub-solution in $Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2]$.

Finally, we check the boundary condition. We claim that, for all $\epsilon > 0$ small enough,

$$v^\epsilon \leq (1-\beta\epsilon)\chi_{\{\tilde{d}_c \geq \beta\}} - \chi_{\{\tilde{d}_c < \beta\}} \quad \text{in } Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2). \tag{5.26}$$

To see this, first, choose $\kappa > 0$ such that

$$\max\{\dot{q}(s) \mid s \geq \kappa\} < \beta\Theta^{-1}.$$

Now notice that if $\tilde{d}_c(x,t) \leq 2\beta + \kappa\epsilon$, then

$$v^\epsilon(x,t) \leq q(\kappa) + \epsilon\Theta\|\dot{q}\|_{L^\infty(\mathbb{R})} - 2\beta\epsilon,$$

while $\tilde{d}_c(x,t) > 2\beta + \kappa\epsilon$ implies, by the choice of κ,

$$v^\epsilon(x,t) \leq 1 - \beta\epsilon.$$

Thus, there is an $\bar{\epsilon} > 0$ such that, for each $\epsilon \in (0, \bar{\epsilon})$,

$$v^\epsilon \leq 1 - \beta\epsilon \quad \text{in } Q^e(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2).$$

Finally, if $\tilde{d}_c(x,t) < \beta$, then, making $\bar{\epsilon} > 0$ smaller if necessary, we find, for each $\epsilon \in (0, \bar{\epsilon})$,

$$v^\epsilon(x,t) \leq -1 + C\exp\left(-\frac{\beta}{C\epsilon}\right) - 2\beta\epsilon \leq -1.$$

This completes the proof of (5.26). Since $f(\xi) = 0$ if $\xi \leq 2\beta$, the claimed boundary behavior off w^ϵ follows. □

5.5 Rational Contact Points

In this section, we prove the analogue of Proposition 57 for rational directions assuming in addition that the level set of φ is nearly flat at the contact point. That is, we tackle condition (b) in Definition 10. The main result is stated below:

Proposition 62. *Fix $\delta \in (0,1)$. If φ is a smooth function in $\mathbb{R}^d \times (0,\infty)$; $(x_0,t_0) \in \mathbb{R}^d \times (0,\infty)$ is a point where $\chi_* - \varphi$ has a strict local minimum; $D\varphi(x_0,t_0) \in \mathbb{R}\mathbb{Z}^d \setminus \{0\}$; and the level set of φ has is δ-flat at (x_0,t_0) in the following sense*

$$\left\| \left(Id - \widehat{D\varphi}(x_0,t_0) \otimes \widehat{D\varphi}(x_0,t_0) \right) D^2\varphi(x_0,t_0) \right\| \leq \delta \|D\varphi(x_0,t_0)\|,$$

then

$$\varphi_t(x_0,t_0) \geq -10\theta^{-1} d\delta \|D\varphi(x_0,t_0)\|. \tag{5.27}$$

The proof of Proposition 62 is a minor modification of the proof of Proposition 57. Let us summarize the details.

Again, proceed by contradiction. If (5.27) fails, then we can construct $\tilde{d}_c$ once more in such a way that

$$\tilde{d}_{c,t} \leq -9\theta^{-1} d\delta \quad \text{in } \{|\tilde{d}_c| < \gamma\}.$$

Further, by continuity, we can make S_2, ν_2, c, and γ so small that

$$|\Delta \tilde{d}_c| \leq 2d\delta \quad \text{in } \{|\tilde{d}_c| < \gamma\}.$$

Hence, with these changes, the conclusions of Proposition 59 still hold except $-6\tilde{\alpha}$ should be replaced by $-6d\delta$ and the mobility $\overline{m}$ by the constant θ.

The construction of mesoscopic sub-solutions proceeds as before, except this time v^ϵ is simply given by

$$v^{\epsilon}(x,t) = q\left(\frac{\tilde{d}_c(x,t) - 2\beta}{\epsilon}\right) - 2\beta\epsilon.$$

When it comes time to check that v^{ϵ} is a sub-solution, we use

$$m(\epsilon^{-1}x, D\tilde{d}_c)\tilde{d}_{c,t} - \Delta\tilde{d}_c \leq \theta\tilde{d}_{c,t} - \Delta\tilde{d}_c \leq -6d\delta \quad \text{in } \{|\tilde{d}_c| < \gamma\}.$$

The remainder of the construction goes through exactly as before.

5.6 Shrinking Sub-Solutions

In this section, we construct mesoscopic sub-solutions of (5.1) that approximate characteristic functions of shrinking balls. Using these, we prove that the limiting evolution satisfies the remaining differential inequality in Definition 10, namely condition (c). Employing similar ideas, we also prove that the phase indicator functions χ_* and χ^* are compatible with the initial datum.

5.6.1 Finite Speed of Shrinking

As shown in [16], to prove χ_* satisfies the right differential inequality when the gradient vanishes, it suffices to check that a ball contained in $\{\chi_* = 1\}$ cannot shrink too fast. Toward that end, we begin by proving the next result:

Proposition 63. *Fix $R > 0$ and $t_0 \geq 0$ and assume that $B(x_0, R) \subseteq \Omega^{(1)}_{t_0}$. Given $\underline{\theta} \in (0, \theta)$, there is an $h > 0$ depending continuously on R (and independent of (x_0, t_0)) such that*

$$B\left(x_0, \sqrt{R^2 - 2\underline{\theta}^{-1}(d-1)s}\right) \subseteq \Omega^{(1)}_{t_0+s} \quad \textit{if } s \in [0, h).$$

By invoking the proposition, we can prove

Theorem 33. *Fix $R > 0$ and $t_0 > 0$ and assume that $B(x_0, R) \subseteq \Omega_t^{(1)}$. If $\underline{\theta} < \theta$, then*

$$B\left(x_0, \sqrt{R^2 - 2\underline{\theta}^{-1}(d-1)s}\right) \subseteq \Omega_{t_0+s}^{(1)} \quad \text{for each } s \in \left[0, \frac{\underline{\theta} R^2}{2(d-1)}\right].$$

Proof. For each $s \in [0, \frac{\underline{\theta} R^2}{2(d-1)}]$, define $R : (0, \infty) \to [0, \infty)$ by

$$R(s) = \sup\left\{r \geq 0 \mid B(x_0, r) \subseteq \Omega_{t_0+s}^{(1)}\right\}.$$

Note that the definition of R implies that $B(x_0, R(s)) \subseteq \Omega_{t_0+s}^{(1)}$. Moreover, by assumption, $R(0) \geq R$. Let $T = \inf\{s > 0 \mid R(s) = 0\}$.

We claim that $s \mapsto R(s)$ is a lower semi-continuous viscosity super-solution of the ODE

$$\underline{\theta}\dot{R} + \frac{(d-1)}{R} \geq 0 \quad \text{in } (0, T).$$

The lower semi-continuity follows from the fact that χ_* is lower semi-continuous.

Notice that, given an $s \in (0, T)$, Proposition 63 yields an $h > 0$ such that if $s' \in (s-h, s)$, then

$$R(s) \geq \sqrt{R(s')^2 - \frac{2(d-1)(s-s')}{\underline{\theta}}}.$$

From this, it follows easily that if φ is a smooth function and $s' \mapsto R(s') - \varphi(s')$ has a local minimum at s, then

$$\underline{\theta}\dot{\varphi}(s) + \frac{(d-1)}{R(s)} \geq 0.$$

By the comparison principle for viscosity solutions, we deduce that $s \mapsto R(s)$ is at least as large as the solution of the ODE with initial condition $R(0)$. In particular,

$$R(s) \geq \sqrt{R(0) - \frac{2(d-1)s}{\underline{\theta}}} \geq \sqrt{R - \frac{2(d-1)s}{\underline{\theta}}} \quad \text{in } (0, T).$$

Note, in addition, that this inequality yields $T \geq \frac{\underline{\theta} R^2}{2(d-1)}$. □

Now we prove Proposition 63. First, observe that the function $d : \mathbb{R}^d \times [0, \frac{\underline{\theta} R^2}{2(d-1)}] \to \mathbb{R}$ given by

$$d(x,t) = \sqrt{R^2 - 2\underline{\theta}^{-1}(d-1)t} - \|x\|$$

satisfies

$$\theta d_t - \mathrm{tr}\left(\left(\mathrm{Id} - \widehat{Dd} \otimes \widehat{Dd}\right) D^2 d\right) = -\frac{\theta(d-1)}{\underline{\theta}\sqrt{R^2 - 2\underline{\theta}^{-1}(d-1)(t-t_0)}} + \frac{(d-1)}{\|x\|}.$$

Note, in addition, that $d_t \leq 0$. A direction computation yields the following lemma:

Lemma 9. *Fix $R > 0$ and $t_0 > 0$. For each $\rho \in (0,1)$ and $\nu \in (0, \frac{\theta}{\underline{\theta}} - 1)$, the function d above satisfies*

$$\theta d_t - tr\left(\left(Id - \widehat{Dd} \otimes \widehat{Dd}\right) D^2 d\right) \leq -\frac{1}{R}\left[\frac{\theta}{\underline{\theta}} - 1 - \nu\right] \quad in\ A_{\rho,\nu} \times \left(0, \frac{R^2\underline{\theta}}{2(d-1)}\right),$$

where $A_{\rho,\nu} = B(0,(1-\rho)^{-1}R) \setminus \overline{B(0,(1+\nu)^{-1}R)}$.

We use d to construct global mesoscopic sub-solutions arguing as in Section 5.4. To start with, define $v^\epsilon : A_{\rho,\nu} \times \left(0, \frac{R^2\underline{\theta}}{2(d-1)}\right) \to \mathbb{R}$ by

$$v^\epsilon(x,t) = q\left(\frac{d(x,t) - 2\beta}{\epsilon}\right) - 2\beta\epsilon.$$

Observe that, using the sign of d_t, we can compute

$$\begin{aligned} m(\epsilon^{-1}x, \widehat{Dv^\epsilon})v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) &= \epsilon^{-1}m(\epsilon^{-1}x, Dd(x,t))\dot{q}d_t - \epsilon^{-2}\ddot{q} - \epsilon^{-1}\dot{q}\Delta d \\ &\quad + \epsilon^{-2}W'(q) - 2\beta\epsilon^{-1}W''(q) \\ &\leq \epsilon^{-1}\dot{q}\left(\theta d_t - \Delta d\right) - 2\beta\epsilon^{-1}W''(q) \\ &\leq -\epsilon^{-1}\left(C_R\dot{q} + 2\beta W''(q)\right), \end{aligned}$$

where $C_R = \frac{1}{R}\left[\frac{\theta}{\theta'} - 1 - \nu\right] > 0$. As in [16], we can choose $\overline{\beta} = \overline{\beta}(\nu) > 0$ so that, for each $\beta \in (0, \overline{\beta})$,

$$\mu_\beta := \min\left\{C_R\dot{q}(s) + 2\beta W''(q(s)) \mid s \in \mathbb{R}\right\} > 0.$$

This gives

$$m(\epsilon^{-1}x, \widehat{Dv^\epsilon})v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) \leq -\mu_\beta\epsilon^{-1} \quad \text{in } A_{\rho,\nu} \times \left(0, \frac{R^2\underline{\theta}}{2(d-1)}\right).$$

We will not be able to proceed in the entire time interval $\left(0, \frac{R^2\underline{\theta}}{2(d-1)}\right)$ since the interface $\{d = 0\}$ does not remain in $A_{\rho,\nu}$. Therefore, we restrict attention to $\mathbb{R}^d \times (0,T)$ for some $T > 0$ and choose $\gamma > 0$ so that

$$\{(x,t) \in \mathbb{R}^d \times (0,T) \mid |d(x,t)| < \gamma\} \subseteq A_{\rho,\nu} \times [0,T].$$

Clearly, it is possible to do this by continuity. A concrete choice of T and γ is

$$T = \frac{\underline{\theta}}{4(d-1)} \cdot \frac{R^2\nu(\nu+2)}{(\nu+1)^2}, \quad \gamma = \left[\frac{R\nu}{2(\nu+1)}\right] \wedge \left[\frac{\rho R}{2(1-\rho)}\right].$$

Notice that, for a fixed $(\underline{\theta}, \rho, \nu)$, T and γ depend continuously on R.

Next, we define $(\overline{v}^\epsilon)_{\epsilon>0}$ and $(w^\epsilon)_{\epsilon>0}$ in $\mathbb{R}^d \times [0,T]$ as before with the choice of γ just selected. To get things started, we need the following variant of Lemma 8:

Lemma 10. *Under the assumptions of Proposition 63, given $\beta > 0$, there are constants $\tau, \epsilon_0 > 0$ depending only on β such that, for each $\epsilon \in (0, \epsilon_0)$,*

$$u^\epsilon(\cdot, t_0 + \tau\epsilon^2 \log(\epsilon^{-1})) \geq (1 - \beta\epsilon)\chi_{\{\|x-x_0\| \leq R-\beta\}} - \chi_{\{\|x-x_0\|>R-\beta\}} \quad \text{in } \mathbb{R}^d.$$

The proof follows by arguing exactly in [16, Lemma 4.1], replacing the function χ used there by the same function $\tilde{\chi}^\epsilon$ used in the proof of Proposition 60.

Comparing u^ϵ and w^ϵ as in Section 5.4, we find

$$w^\epsilon(x, t) \leq u^\epsilon(x, t + \tau\epsilon^2 \log(\epsilon^{-1})) \quad \text{if } (x, t) \in \mathbb{R}^d \times [0, T].$$

Combined with the fact that $\liminf_* w^\epsilon(x, t) = 1$ if $d(x, t) \geq 2\beta$, this gives

$$\{d(\cdot, t) \geq 2\beta\} \subseteq \Omega_t^{(1)} \quad \text{if } t \in [0, T].$$

Sending $\beta \to 0$, we obtain the conclusion of Proposition 63 with $h = T$.

5.6.2 Super-Solution Property at Flat Parts

The finite shrinking speed of the previous section is intimately related to the final differential inequality in Definitions 10 and 12. In fact, it implies it, as shown in the next result.

Proposition 64. *If $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ is smooth, $\chi_* - \varphi$ has a strict local minimum at $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$, and $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$, then*

$$\varphi_t(x_0, t_0) \geq 0.$$

The proof below is based on an insight from [16].

Proof. First, notice that if $\chi_*(x_0, t_0) = 1$, then $\chi_* = 1$ in a neighborhood of (x_0, t_0), and this implies $\varphi_t(x_0, t_0) = 0$ directly.

Assume now that $\chi_*(x_0, t_0) = -1$ and, without loss of generality, that $\varphi(x_0, t_0) = 0$. It follows that there is an open ball $B \subseteq \mathbb{R}^d \times (0, \infty)$ containing (x_0, t_0) such that

$$\chi_*(x, t) - \varphi(x, t) \geq -1 \quad \text{if } (x, t) \in B.$$

In particular, since $\varphi(x_0, t_0) = 0$, this gives

$$\chi_*(x, t) - \varphi_t(x_0, t_0)(t - t_0) + o(\|x - x_0\|^2 + |t - t_0|) \geq -1 \quad \text{if } (x, t) \in B. \tag{5.28}$$

Let $C = \frac{4(d-1)}{\underline{\theta}}$. We claim that there is a sequence $((x_n, t_n))_{n \in \mathbb{N}}$ such that

$$(x_0, t_0) = \lim_{n \to \infty} (x_n, t_n), \quad t_n < t_0,$$
$$\|x_n - x_0\|^2 \leq C|t_n - t_0|, \quad \chi_*(x_n, t_n) = -1.$$

Assuming the claim is true for now, we set $(x, t) = (x_n, t_n)$ in (5.28) to find

$$\varphi_t(x_0, t_0)(t_0 - t_n) + o(|t_n - t_0|) \geq 0.$$

Dividing by $t_0 - t_n$ and sending $n \to \infty$, this yields

$$\varphi_t(x_0, t_0) \geq 0.$$

It remains to prove the claim. We argue by contradiction, assuming that it is false. We can then fix an $s \in (0, t_0)$ such that $B(x_0, \sqrt{C(t_0 - s)}) \subseteq \Omega_s^{(1)}$. Now Theorem 33 implies that

$$B\left(x_0, \sqrt{C(t_0 - s) - \frac{2(d-1)(t - s)}{\underline{\theta}}}\right) \subseteq \Omega_t^{(1)} \quad \text{if } t \in \left[0, \frac{C\underline{\theta}(t_0 - s)}{2(d-1)}\right] + s.$$

At the same time, notice that, by the choice of C,

$$s + \frac{C\underline{\theta}(t_0 - s)}{2(d-1)} = s + 2(t_0 - s) > t_0.$$

Thus, we deduce that

$$x_0 \in B\left(x_0, \sqrt{C(t_0 - s) - \frac{2(d-1)(t_0 - s)}{\underline{\theta}}}\right) \subseteq \Omega^{(1)}_{t_0},$$

but this contradicts the assumption that $\chi_*(x_0, t_0) = -1$. □

5.6.3 Initial Datum

The proof of Proposition 63 can be modified slightly to prove that $\{\chi_*(\cdot, 0) = 1\} \supseteq \{u_0 > 0\}$, as claimed in Proposition 55.

Proposition 65. $\chi_*(\cdot, 0) = 1$ *in* $\{u_0 > 0\}$.

To prove this, we will use the following variant of Lemma 10.

Lemma 11. *Given* $\beta, r \in (0, 1)$ *and* $x_0 \in \mathbb{R}^d$*, if* $B(x_0, r) \subseteq \{u_0 > 0\}$*, then there is a* $\tau > 0$ *depending only on* β *and* r *and* $\epsilon_0 \in (0, 1)$ *such that, for each* $\epsilon \in (0, \epsilon_0)$*,*

$$u^\epsilon(\cdot, \tau\epsilon^2 \log(\epsilon^{-1})) \geq (1 - \beta\epsilon)\chi_{\{\|x - x_0\| \leq r - \beta\}} - \chi_{\{\|x - x_0\| > r - \beta\}}.$$

Now we prove the proposition.

Proof of Proposition 65. Fix $x_0 \in \{u_0 > 0\}$. We need to prove that $\chi_*(x_0, 0) = 1$. Toward this end, first, fix an $r > 0$ such that $B(x_0, r) \subseteq \{u_0 > 0\}$.

Arguing exactly as in the proof of Proposition 63 with Lemma 11 in place of Lemma 10 and $\underline{\theta} = 2^{-1}\theta$, we find an $h > 0$ such that

$$B\left(x_0, \sqrt{r^2 - 4\theta^{-1}(d-1)t}\right) \subseteq \Omega_t^{(1)} \quad \text{for each } t \in (0, h).$$

It follows that there is an $h' > 0$ such that $B\left(x_0, 2^{-1}r\right) \subseteq \Omega_t^{(1)}$ for each $t \in (0, h')$. By the definition of $\Omega_t^{(1)}$ and χ_*, this implies that

$$\begin{aligned}\chi_*(x_0, 0) &= \lim_{\delta \to 0^+} \inf \{\chi_*(y, s) \mid \|x - y\| + s \leq \delta,\ s > 0\} \\ &\geq \inf \left\{\chi_*(y, s) \mid y \in B(x_0, 2^{-1}r),\ s \in (0, h')\right\} = 1.\end{aligned}$$

Since $\chi_* \leq 1$, the proof is complete. □

5.7 Notes

The inspiration to consider the Allen-Cahn equation with a mobility coefficient initially came from the work of Taylor and Cahn [89]. Equations with a mobility coefficient as in (5.1) also appear in the study of the Ising model (cf. [62]), and the homogenization of a gradient flow-type ODE with periodic dissipation has recently been studied by Mielke, Montefusco, and Peletier [67].

CHAPTER 6

HOMOGENIZATION OF NONVARIATIONAL INTERFACE MOTIONS

This chapter is devoted to the homogenization of nonvariational interface motions driven by curvature effects alone. More precisely, the focus is the analysis of the asymptotics of the interface motion given by

$$V_{\partial E_t^\epsilon} = m(\epsilon^{-1}x, n_{\partial E_t^\epsilon})^{-1}\mathrm{tr}\left(a(\epsilon^{-1}x, n_{\partial E_t^\epsilon})A_{\partial E_t^\epsilon}\right).$$

As in Chapter 2, we study this motion using by appealing to the level-set formulation. Hence we are interested in the asymptotics as $\epsilon \to 0$ of the nonlinear diffusion equation given by:

$$\begin{cases} m(\epsilon^{-1}x, \widehat{Du^\epsilon})u_t^\epsilon - \mathrm{tr}\left(A(\epsilon^{-1}x, \widehat{Du^\epsilon})D^2u^\epsilon\right) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{6.1}$$

Here, as in the previous chapters, A is obtained from the coefficient field a by the formula

$$A(y,e) = (\mathrm{Id} - e \otimes e)a(y,e)(\mathrm{Id} - e \otimes e). \tag{6.2}$$

Recall from the introduction that we assume that a and m are smooth and a is uniformly elliptic, hence there are constants $\lambda, \Lambda > 0$ such that

$$\lambda \mathrm{Id} \leq a(y,e) \leq \Lambda \mathrm{Id} \quad \text{for each } (y,e) \in \mathbb{T}^d \times S^{d-1}. \tag{6.3}$$

We prove that (6.1) homogenizes in the limit $\epsilon \to 0$. Specifically, the solutions u^ϵ converges to the solution $\bar{u}$ of the effective equation

$$\begin{cases} \overline{m}(\widehat{D\bar{u}})\bar{u}_t - \operatorname{tr}\left(\bar{A}(\widehat{D\bar{u}})D^2\bar{u}\right) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ \bar{u} = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{6.4}$$

As before, $\bar{A}$ is related to an effective diffusion matrix $\bar{a}$ as in (6.2).

The most interesting aspect of this homogenization result is the effective diffusion matrix $\bar{a}$ and effective mobility $\overline{m}$ are, in general, discontinuous functions in dimensions $d \geq 3$. As a consequence, part of the proof requires an understanding of existence, uniqueness, and stability for second-order level-set PDE with discontinuous coefficients. In fact, the discontinuities of $\bar{a}$ and $\overline{m}$ are mild enough that (6.4) has a comparison principle. The proof of the comparison principle is deferred to the next chapter.

The main result of this chapter is stated next.

Theorem 34. *There are effective coefficients* $\overline{m} : S^{d-1}\backslash\mathbb{R}\mathbb{Z}^d \to (0,\infty)$ *and* $\bar{a} : S^{d-1}\backslash\mathbb{R}\mathbb{Z}^d \to \mathcal{S}_d$ *such that if* $u_0 \in UC(\mathbb{R}^d)$ *and* $(u^\epsilon)_{\epsilon>0}$ *are the solutions of* (6.1)*, then:*

(i) There is a unique viscosity solution $\bar{u} : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ *of* (6.4).

(ii) $u^\epsilon \to \bar{u}$ *locally uniformly as* $\epsilon \to 0^+$.

Concerning the continuity properties of the effective coefficients $\bar{a}$ and $\overline{m}$, we will show that continuity holds in dimension $d = 2$, but only directional limits exist in dimensions $d \geq 3$. This is related to a fundamental question concerning diffusion processes with quasi-periodic coefficients.

Given $e \in S^{d-1}$, consider the diffusion process X^e determined by the SDE

$$dX^e_t = (\mathrm{Id} - e \otimes e)\sqrt{a(X^e_t)}\, dB_t. \tag{6.5}$$

Since a is $\mathbb{Z}^d$-periodic, we can consider X^e as a process in $\mathbb{T}^d$. Let $\mathscr{I}^a_e$ denote the collection of probability measures that are invariant under (6.5), which coincides with the analytical definition in Chapter 2.

The next theorem describes the structure of $\mathscr{I}_e^a$ and its connection to the effective coefficients $\bar{a}$ and $\overline{m}$.

Theorem 35. *The following statements all hold:*

(i) If $e \notin \mathbb{R}\mathbb{Z}^d$, then there is a unique probability measure $\bar{\mu}_e$ such that $\mathscr{I}_e^a = \{\bar{\mu}_e\}$. Furthermore, $\bar{\mu}_e \ll \mathcal{L}^d$.

(ii) If $e \in \mathbb{R}\mathbb{Z}^d$, then there is an r_e-periodic function $\mu_e : \mathbb{R} \to \mathscr{P}(\mathbb{T}^d)$, $\mu_e : r \mapsto \mu_e^r$, such that $\mathscr{I}_e^a$ equals the closed convex hull of $\{\mu_e^r \mid r \in \mathbb{R}\}$. For each $r \in [0, r_e)$, we have $\mu_e^r \ll \mathcal{H}^{d-1} \restriction_{\mathbb{T}_e^{d-1}(r)}$.

(iii) For each $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, $\bar{a}$ and $\overline{m}$ are given by

$$\bar{a}(e) = \int_{\mathbb{T}^d} a(y,e)\,\bar{\mu}_e(dy), \quad \overline{m}(e) = \int_{\mathbb{T}^d} m(y,e)\,\bar{\mu}_e(dy). \tag{6.6}$$

It is not hard to show that the functions $\bar{a}$ and $\overline{m}$ are continuous at irrational directions by compactness. The story is more complicated in rational directions.

First, we need a digression on analysis in S^{d-1} because it turns out that the limiting behavior of $e \mapsto \mathscr{I}_e^a$ at a rational direction depends on the direction of approach. To make that precise, notice that if $(e_n)_{n\in\mathbb{N}} \subseteq S^{d-1}$ converges to e as $n \to \infty$, then there is necessarily a a subsequence $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ and an $\eta \in S^{d-1} \cap \langle e \rangle^{\perp}$ such that

$$-\eta = \lim_{j\to\infty} \frac{e_{n_j} - e}{\|e_{n_j} - e\|}.$$

Geometrically, that means e_{n_j} approximately approaches e along the great circle given by $\theta \mapsto \cos(\theta)e + \sin(\theta)\eta$, or, put simply, $e_{n_j} \to e$ along the η direction. It is necessary to take account of the direction η when studying the continuity properties of $e \mapsto \mathscr{I}_e^a$.

Finally, in the statement, it is convenient to use the metrizability of $\mathscr{P}(\mathbb{T}^d)$. Toward that end, we fix here and henceforth a metric $D : \mathscr{P}(\mathbb{T}^d) \times \mathscr{P}(\mathbb{T}^d) \to [0, \infty)$ inducing the weak-$*$ topology (e.g. Wasserstein distance).

Theorem 36. *For each $e \in \mathbb{R}\mathbb{Z}^d$, if we define the function $\tilde{\mu}_e : S^{d-1} \cap \langle e \rangle^{\perp} \to \mathscr{P}(\mathbb{T}^d)$, $\tilde{\mu}_e : \eta \mapsto \tilde{\mu}_e^{\eta}$, by*

$$\tilde{\mu}_e^{\eta} = \left(\int_0^{r_e} \langle a_e^{\perp}(se)\eta, \eta \rangle^{-1} \, ds \right)^{-1} \int_0^{r_e} \langle a_e^{\perp}(se)\eta, \eta \rangle^{-1} \mu_e^s \, ds, \tag{6.7}$$

$$a_e^{\perp}(y) = \int_{\mathbb{T}^d} a(y', e) \mu_e^{\langle y, e \rangle}(dy'), \tag{6.8}$$

then this function describes the continuity properties of $e' \mapsto \mathscr{I}_{e'}^a$ at e in the sense that

$$\lim_{\delta \to 0^+} \sup \left\{ D(\mu, \tilde{\mu}_e^{\eta}) \mid \mu \in \mathscr{I}_{e'}^a, \ \|e' - e\| + \left\| \frac{e' - e}{\|e' - e\|} + \eta \right\| < \delta \right\} = 0.$$

When the dimension $d = 2$, it turns out that the theorem implies that $\bar{a}$ and $\overline{m}$ both extend continuously to the entire sphere S^{d-1}; see Theorem 37. Continuity, however, does *not* hold in general when $d \geq 3$.

By using the explicit form of the limiting measure $\tilde{\mu}_e^{\eta}$ from the previous theorem, we readily deduce that $\bar{a}$ and $\overline{m}$ are generically discontinuous in dimensions $d \geq 3$. In the next result, we denote by $\mathcal{S}_d(\lambda, \Lambda)$ the set of symmetric matrices with eigenvalues strictly between λ and Λ, that is,

$$\mathcal{S}_d(\lambda, \Lambda) = \{ A \in \mathcal{S}_d \mid \lambda \mathrm{Id} < A < \Lambda \mathrm{Id} \}. \tag{6.9}$$

Corollary 2. *Assume $d \geq 3$. There is a residual set $\mathscr{C}_d \subseteq C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda))$ such that if a is independent of the e variable and $a \in \mathscr{C}_d$, then the following statements hold:*

(a) If $\tilde{\mu}$ is the function defined in Theorem 36, then

$$\forall e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d \quad \#\{ \tilde{\mu}_e^{\eta} \mid \eta \in S^{d-1} \cap \langle e \rangle^{\perp} \} = \infty.$$

(b) The effective coefficient $\bar{a} : S^{d-1} \backslash \mathbb{R}\mathbb{Z}^d \to \mathcal{S}_d(\lambda, \Lambda)$ *defined in Theorem 35 has infinitely many distinct directional limits at each* $e \in \mathbb{R}\mathbb{Z}^d$*.*

A similar argument shows that the linear response coefficient $\overline{m}_{\text{pl}}$ obtained in Chapter 2 is generically discontinuous in *all* dimension $d \geq 2$. In what follows, we denote by $\mathscr{X}_d$ the natural space of e-independent coefficients (a, m) given by

$$\mathscr{X}_d = C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda)) \times C^{2,\alpha}(\mathbb{T}^d; (0, \infty))$$

Recall that $\mathcal{S}_d(\lambda, \Lambda)$ is the set of symmetric matrices bounded between λ and Λ (see (6.9)). We endow $\mathscr{X}_d$ with the $C^{2,\alpha}$ norm, which makes it an open subset of a Banach space.

Corollary 3. *If* $d \geq 2$*, then there is a residual set* $\mathcal{A}_d \subseteq \mathscr{X}_d$ *such that if* m *and* a *are independent of* e *and* $(a, m) \in \mathcal{A}_d$*, then, for each* $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$*, the following hold:*

(i) There is an $\eta \in S^{d-1} \cap \langle e \rangle^{\perp}$ *such that* $\overline{m}_{pl}(e) \neq \tilde{m}_e^{\eta}$*.*

(ii) If $d \geq 3$*, then* $\overline{m} : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to (0, \infty)$ *does not have a limit at* e*.*

In particular, if $(a, m) \in \mathcal{A}_d$*, then* $\overline{m}_{pl}$ *is not a continuous function.*

Since Corollary 3 follows from the same kinds of arguments as Corollary 2, its proof is omitted.

6.1 Homogenization in Irrational Directions

In this section, we show that the asymptotics of the homogenization problem (6.1) are controlled by the effective equation (6.4). More precisely, defining half-relaxed limits $\bar{u}^*$ and

$\bar{u}_*$ by

$$\bar{u}^*(x,t) = \lim_{\delta \to 0^+} \sup \left\{ u^\epsilon(y,s) \ \mid \ \|x-y\| + |t-s| + \epsilon \leq \delta \right\},$$

$$\bar{u}_*(x,t) = \lim_{\delta \to 0^+} \inf \left\{ u^\epsilon(y,s) \ \mid \ \|x-y\| + |t-s| + \epsilon \leq \delta \right\},$$

we will show that $\bar{u}^*$ and $\bar{u}_*$ are, respectively, sub- and super-solution of (6.4). As in Chapter 5, the proof will be based on the notion of viscosity solutions in irrational directions.

At contact points where the normal vector is irrational, we will use the approximate correctors of the previous section and Evans's perturbed test function method [45]. Due to the flexibility provided by Definition 10, the necessary viscosity inequalities in rational directions follow directly from the structure of the PDE.

The precise result we aim to prove is stated next.

Proposition 66. *If* $(u^\epsilon)_{\epsilon > 0}$ *are the solutions of* (6.1) *for some fixed* $u_0 \in UC(\mathbb{R}^d)$ *and* $\bar{u}^* = \limsup^* u^\epsilon$ *and* $\bar{u}_* = \liminf_* u^\epsilon$ *are the corresponding half-relaxed limits, then* $\bar{u}^*$ *is a sub-solution of* (6.4) *in irrational directions in* $\mathbb{R}^d \times (0,\infty)$ *and* $\bar{u}_*$ *is a super-solution of* (6.4) *in irrational directions in* $\mathbb{R}^d \times (0,\infty)$*. Furthermore,* $\bar{u}^*(\cdot, 0) \leq u_0 \leq \bar{u}_*(\cdot, 0)$*.*

Proof. We break the proof down into three steps, one step for each condition in Definition 10 and a final one for the initial condition. Since the argument for $\bar{u}^*$ is analogous, we restrict attention to $\bar{u}_*$.

We begin with the viscosity inequalities in $\mathbb{R}^d \times (0,\infty)$. Assume that $\varphi : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ is a smooth function and $(x_0, t_0) \in \mathbb{R}^d \times (0,\infty)$ is a point where $\bar{u}_* - \varphi$ attains a strict local minimum.

Step 1: Irrational Normal

Assume that $D\varphi(x_0,t_0) \notin \mathbb{R}\mathbb{Z}^d$ and let $\delta > 0$. Define $e = \widehat{D\varphi}(x_0,t_0)$ and $X =$

$\|D\varphi(x_0,t_0)\|^{-1}D^2\varphi(x_0,t_0)$. Define $f : \mathbb{T}^d \to \mathbb{R}$ by

$$f(y) = \mathrm{tr}\,(A(y,e)X) - m(y,e)\varphi_t(x_0,t_0).$$

Let $V^{e,\delta,f}$ be the solution of the penalized cell problem (2.13). Recall from Proposition 3 that the following limiting behavior holds

$$\lim_{\delta\to 0^+} \sup\left\{|V^{e,\delta,f}(y) - \overline{f}(e)| \mid y \in \mathbb{T}^d\right\} = 0,$$

where $\overline{f}(e) = \int_{\mathbb{T}^d} f(y)\,\bar{\mu}_e(dy)$. From the definition of $\bar{a}(e)$ and $\overline{m}(e)$, the identity $\overline{f}(e) = \mathrm{tr}(\overline{A}(e)X) - \overline{m}(e)\varphi_t(x_0,t_0)$ holds with $\overline{A}(e) = (\mathrm{Id} - e\otimes e)\bar{a}(e)(\mathrm{Id} - e\otimes e)$.

For each $\epsilon > 0$, define the perturbed test function φ_ϵ by

$$\varphi_\epsilon(x,t) = \varphi(x,t) + \epsilon^2 V^{e,\delta,f}(\epsilon^{-1}x).$$

Since $V^{e,\delta,f}$ is bounded, there is a sequence $(\epsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ and points $(x_n,t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}(\epsilon_n, x_n, t_n) = (0, x_0, t_0)$$

and, for each $n \in \mathbb{N}$, $u^\epsilon - \varphi_\epsilon$ has a local minimum at (x_n,t_n).

Since u^ϵ is a viscosity solution of (6.1) and $V^{e,\delta,f} \in C^2(\mathbb{T}^d)$, we can compute

$$\begin{aligned}
0 &\leq m(\epsilon_n^{-1}x_n, \widehat{D\varphi_{\epsilon_n}}(x_n,t_n))\varphi_{\epsilon_n,t}(x_n,t_n) - \mathrm{tr}\left(A(\epsilon_n^{-1}x_n, \widehat{D\varphi_{\epsilon_n}}(x_n,t_n))D^2\varphi_{\epsilon_n}(x_n,t_n)\right)\\
&= m(\epsilon_n^{-1}x_n, D\widehat{\varphi(x_0,t_0)})\varphi_t(x_0,t_0) - \mathrm{tr}\left(A(\epsilon_n^{-1}x_n, \widehat{D\varphi}(x_0,t_0))D^2\varphi(x_0,t_0)\right)\\
&- \mathrm{tr}\left(a(\epsilon_n^{-1}x_n, e)D_e^2 V^{e,\delta,f}(\epsilon_n^{-1}x_n)\right) + o(1)\\
&\leq \sup\left\{-\delta V^{e,\delta,f}(y) \mid y \in \mathbb{T}^d\right\} + o(1).
\end{aligned}$$

Sending first $n \to \infty$ and then $\delta \to 0^+$, this becomes

$$0 \leq \overline{m}(e)\varphi_t(x_0, t_0) - \operatorname{tr}\left(\overline{A}(e)X\right) = \overline{m}(\widehat{D\varphi}(x_0, t_0))\varphi_t(x_0, t_0) - \operatorname{tr}\left(\overline{A}(e)D^2\varphi(x_0, t_0)\right).$$

Step 2: Rational Normal, Flat Level Set

Next, we assume that $D\varphi(x_0, t_0) \neq 0$ and define δ by

$$\delta = \|D\varphi(x_0, t_0)\|^{-1} \left\| \left(\mathrm{Id} - \widehat{D\varphi}(x_0, t_0) \otimes \widehat{D\varphi}(x_0, t_0)\right) D^2\varphi(x_0, t_0) \right\|.$$

Since $\bar{u}_* - \varphi$ has a strict local minimum at (x_0, t_0), we can fix a sequence $\epsilon_n \to 0^+$ and points $(x_{\epsilon_n}, t_{\epsilon_n}) \to (x_0, t_0)$ such that $u^{\epsilon_n} - \varphi$ has a local minimum at (x_n, t_n) for each n. Using the equation directly, this implies that

$$\begin{aligned} 0 &\leq \varphi_t(x_{\epsilon_n}, t_{\epsilon_n}) - \operatorname{tr}\left(A(\epsilon_n^{-1}x_n, \widehat{D\varphi}(x_n, t_n))D^2\varphi(x_n, t_n)\right), \\ &\leq \varphi_t(x_{\epsilon_n}, t_{\epsilon_n}) + \Lambda\|D^2_{e_n}\varphi(x_n, t_n)\|, \end{aligned}$$

where $e_n = \widehat{D\varphi}(x_n, t_n)$. Sending $n \to \infty$, we recover

$$\varphi_t(x_0, t_0) \geq -\Lambda\|D^2_e\varphi(x_0, t_0)\| \geq -\Lambda\delta\|D\varphi(x_0, t_0)\|.$$

We conclude that $\bar{u}_*$ satisfies condition (b) in Definition 10 with $K = \Lambda$.

Step 3: Vanishing Normal

Finally, if $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$, then we can find $\epsilon_n \to 0^+$ and $(x_n, t_n) \to (x_0, t_0)$ such that

$$0 \leq \varphi_t(x_n, t_n) - \operatorname{tr}\left(A(\epsilon_n^{-1}x_n, \widehat{D\varphi}(x_n, t_n))D^2\varphi(x_n, t_n)\right)$$

In the limit $n \to \infty$, we use the uniform ellipticity of a to conclude that $\varphi_t(x_0, t_0) \geq 0$. This proves $\bar{u}_*$ satisfies condition (c) in Definition 10.

Step 4: Initial Condition

Here we proceed by approximating u_0 by smooth functions. First, observe that if $D^2 u_0 \in BC(\mathbb{R}^d)$, then, by comparison with sub- and super-solutions of the form $u_0(x) \pm Ct$, we deduce that, independent of ϵ, the solution u^ϵ satisfies the following Lipschitz estimate in time

$$|u^\epsilon(x,t) - u^\epsilon(x,t')| \leq C|t - t'|.$$

Accordingly, in this case, we invoke the definition of $\bar{u}_*$ to deduce that, for any $x \in \mathbb{R}^d$, we have

$$\bar{u}_*(x,0) \geq \lim_{\delta \to 0^+} (u_0(x) - C\delta) = u_0(x).$$

This proves that $\bar{u}_* \geq u_0$.

Finally, even if u_0 does not have a continuous and bounded second derivative, nonetheless the assumption that $u_0 \in UC(\mathbb{R}^d)$ implies that we can find a sequence $(u_0^{(n)})_{n \in \mathbb{N}} \subseteq UC(\mathbb{R}^d$ such that $u_0^{(n)} \to u_0$ uniformly as $n \to \infty$ and $D^2 u_0^{(n)} \in BC(\mathbb{R}^d)$ for each fixed n. By the comparison principle, if $u^{\epsilon,(n)}$ denotes the solution of (6.1) with initial condition $u^{\epsilon,(n)} = u_0^{(n)}$, then

$$\sup\left\{|u^\epsilon(x,t) - u^{\epsilon,(n)}(x,t)| \mid (x,t) \in \mathbb{R}^d \times (0,\infty)\right\} \leq \sup\left\{|u_0(x) - u^{(n)}(x)| \mid x \in \mathbb{R}^d\right\}.$$

From this, we fix $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, send $\epsilon \to 0^+$, and use the assumption that $D^2 u_0^{(n)} \in BC(\mathbb{R}^d)$ to find

$$\begin{aligned} \bar{u}_*(x,0) &\geq \lim_{\delta \to 0^+} \inf\left\{u^{\epsilon,(n)}(y,s) \mid \epsilon + \|x - y\| + s \leq \delta\right\} - \|u_0^{(n)} - u_0\|_{L^\infty(\mathbb{R}^d)} \\ &\geq u_0^{(n)}(x) - \|u_0^{(n)} - u_0\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

Sending $n \to \infty$, we deduce that $\bar{u}_*(x,0) \geq u_0(x)$. Since x was arbitrary, we conclude that

$\bar{u}_* \geq u_0$. □

6.2 Continuity and Discontinuity of Homogenized Coefficients

In this section, we study the continuity properties of the effective coefficients $\overline{m}$ and $\bar{a}$. The main results are Theorem 36 and Corollary 2 concerning the (dis)continuity properties of $\overline{m}$ and $\bar{a}$.

6.2.1 Strategy of Proof

Before entering into the details, let us briefly give a heuristic explanation of the strategy of the proof and the core technical issues that arise. The main idea of the proof and the discussion that follows are inspired by the strategy introduced by Feldman and Kim [49].

Suppose $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. In the proof of Theorem 36, we proceed by studying the behavior of $(\bar{\mu}_{e_n})_{n\in\mathbb{N}}$ along a sequence of irrational directions $(e_n)_{n\in\mathbb{N}}$ with $e_n \to e$ and $\frac{e_n - e}{\|e_n - e\|} \to -\eta$.

$\bar{\mu}_{e_n}$ captures the long-time asymptotics of the diffusion process governed by (6.5). Therefore, it is natural to pass to the diffusions X^{e_n} and consider their behavior. Further, in light of the structure of $\mathscr{I}_e^a$, the only question is the e marginal of any limit point of $(\bar{\mu}_{e_n})_{n\in\mathbb{N}}$, that is, we only need to study $f(\langle X_t^{e_n}, e\rangle)$, where f is an r_e-periodic function of one variable and $t > 0$ is large.

We may as well assume that $\langle X_0^{e_n}, e_n\rangle = 0$, which implies that $\langle X_t^{e_n}, e_n\rangle = 0$ for all $t \geq 0$. Therefore, if we write $e_n = \cos(\theta_n)e - \sin(\theta_n)\eta_n$ for some $\theta_n \in (-\pi, \pi]$ and $\eta_n \in S^{d-1} \cap \langle e\rangle^\perp$, we have

$$\langle X_t^{e_n}, e\rangle = \langle X_t^{e_n}, e - e_n\rangle = (1 - \cos(\theta_n)\langle X_t^{e_n}, e\rangle + O(\theta_n^2).$$

Hence to recover anything meaningful from $f(\langle X_t^{e_n}, e\rangle)$, we need to wait until $\|X_t^{e_n}\| \approx \theta_n^{-1}$. Since this takes a time proportional to θ_n^{-2}, we should study $f(\langle X_{\theta_n^{-2}T}^{e_n}, e\rangle)$ in the simultaneous limit $n, T \to \infty$.

Put another way, at the level of the PDE, we would like to understand the behavior as $n \to \infty$ and $\delta \to 0^+$ of the penalized correctors $(V_n^\delta)_{(\delta,n)\in(0,\infty)\times\mathbb{N}}$ solving

$$\delta V_n^\delta - \text{tr}\left(a(\theta_n^{-1}x, e_n)D_e^2 V_n^\delta\right) = f(\theta_n^{-1}\langle x, e\rangle) \quad \text{in } \langle e_n\rangle^\perp.$$

A simple homogenization argument shows that, for a fixed $\delta > 0$, $V_n^\delta \to \tilde{V}^\delta$ as $n \to \infty$, where $\tilde{V}^\delta$ is the solution of the problem

$$\delta\tilde{V}^\delta - \text{tr}\left(a_e^\perp(\langle \eta, x\rangle e)D_e^2\tilde{V}^\delta\right) = f(\langle \eta, x\rangle e) \quad \text{in } \langle e\rangle^\perp.$$

It turns out that extracting the limit of $\delta\tilde{V}^\delta$ as $\delta \to 0^+$ leads to the correct limit of $\int_{\mathbb{T}^d} f(\langle y, e\rangle)\,\bar{\mu}_{e_n}(dy)$. The question is simply how to show that the limits $n \to \infty$ and $\delta \to 0^+$ commute.

The argument below shows how to do this working at the level of the obstacle problems introduced in [29] rather than the penalized correctors. Working with penalized correctors is difficult since it requires understanding the rate at which δV_n^δ converges as $\delta \to 0^+$, independently of n. We circumvent this by passing to the obstacle problem approach of [29] and leveraging an upper semi-continuity property proved there. In this way, it is possible to first send $n \to \infty$ and then $\delta \to 0^+$ without explicitly quantifying the rates of convergence of the associated almost periodic homogenization problems.

6.2.2 Preliminaries

We start by introducing some notation that will be used later.

Henceforth, given an arbitrary $f \in C(\mathbb{T}^d)$, it will be convenient to define the family of shifted functions $\{f^y\}_{y \in \mathbb{T}^d}$ by

$$f^y(\underline{y}) = f(\underline{y} + y) \quad \text{for } y, \underline{y} \in \mathbb{T}^d.$$

The reader familiar with stochastic homogenization can think of the shift y like an element ω of a probability space $(\Omega, \mathbb{P})$. In our case, $\Omega = \mathbb{T}^d$ and $\mathbb{P} = \mathcal{L}^d$, although we will also have occasion to work with the surface area measures on the sub-tori $\{\mathbb{T}^{d-1}_e(0)\}_{e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d}$.

Given an $f \in C(\mathbb{T}^d)$, $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, and $\eta \in S^{d-1} \cap \langle e \rangle^{\perp}$, we define the constant $\tilde{f}^{\eta}_e$ by

$$\tilde{f}^{\eta}_e = \int_{\mathbb{T}^d} f(y)\, \tilde{\mu}^{\eta}_e(dy), \tag{6.10}$$

where $\tilde{\mu}^{\eta}_e$ is the probability measure defined by (6.7).

To prove Theorem 36, observe that it suffices to prove the following preliminary result. Here and in what follows, for a given $f \in C(\mathbb{T}^d)$, we define $f^{\perp}_e$ and $\overline{f}(e)$ as in Section 2.5 (see (2.18) and Proposition 3).

Proposition 67. *Fix $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, $\eta \in S^{d-1} \cap \langle e \rangle^{\perp}$, and $f \in C^{2,\alpha}(\mathbb{T}^d)$. If $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$ satisfies*

$$\lim_{n \to \infty} \left(\|e_n - e\| + \left\| \frac{e_n - e}{\|e_n - e\|} + \eta \right\| \right) = 0. \tag{6.11}$$

then

$$\lim_{n \to \infty} \sup \left\{ |f^{\perp}_{e_n}(y) - \tilde{f}^{\eta}_e| \ \mid \ y \in \mathbb{T}^d \right\} = 0.$$

Similarly, if $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ satisfies (6.11), then

$$\lim_{n \to \infty} |\overline{f}(e_n) - \tilde{f}^{\eta}(e)| = 0.$$

Notice that when when the dimension $d = 2$, the set $S^{d-1} \cap \langle e \rangle^{\perp}$ contains just two antipodal directions, $S^{d-1} \cap \langle e \rangle^{\perp} = \{e^{\perp}, -e^{\perp}\}$. Hence, by inspection of the formula for $\tilde{f}^{\eta}(e)$, we deduce that the function $e \mapsto \overline{f}(e)$ extends continuously to rational directions in this dimension.

Theorem 37. *If $d = 2$, then the function $e \mapsto \overline{f}(e)$ can be extended continuously from $S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ to S^{d-1}. In particular, this applies to the effective coefficients $\bar{a}$ and $\overline{m}$ in Theorem 34.*

Since the theorem follows directly from the argument above, the proof is omitted.

The main thrust of the remainder of the section is the proof of Proposition 67. For the rest of the section, fix an e, η, $(e_n)_{n \in \mathbb{N}}$, and f as in that proposition.

By assumption, we can fix $(\eta_n)_{n \in \mathbb{N}} \subseteq S^{d-1}$ and $(\theta_n)_{n \in \mathbb{N}} \subseteq (-\pi, \pi]$ such that

$$e_n = \cos(\theta_n) e - \sin(\theta_n) \eta_n.$$

We will assume without loss of generality that $(\theta_n)_{n \in \mathbb{N}} \subseteq (-\pi, \pi) \setminus \{0\}$.

For each $n \in \mathbb{N}$, let $O_n : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal transformation such that

$$O_n(e_n) = e, \quad O_n(\sin(\theta_n) e + \cos(\theta_n) \eta_n) = \eta_n, \quad O_n \restriction_{\langle e, e_n \rangle^{\perp}} = \mathrm{Id} \restriction_{\langle e, e_n \rangle^{\perp}}$$

Notice that $\lim_{n \to \infty} O_n = \mathrm{Id}$ and $O_n(\langle e_n \rangle^{\perp}) = \langle e \rangle^{\perp}$.

Finally, we let $\{v_1, \dots, v_{d-1}\}$ be an orthonormal basis of $\langle e \rangle^{\perp}$ and define, for each $R > 0$, the cube $Q^*_R \subseteq \langle e \rangle^{\perp}$ by

$$Q^*_R = \left\{ y \in \langle e \rangle^{\perp} \ \mid \ |\langle y, v_1 \rangle| \vee \dots \vee |\langle y, v_{d-1} \rangle| < R/2 \right\}.$$

Given $n \in \mathbb{N}$, we define the analogous cubes $\{Q^{(n)}_R \ \mid \ R > 0\}$ in $\langle e_n \rangle^{\perp}$ by $Q^{(n)}_R = O_n^{-1}(Q^*_R)$.

6.2.3 *Obstacle Problems*

It is technically very convenient to replace the penalized correctors $\{V_n^\delta\}_{(\delta,n)}$ of the previous discussion by solutions of a related family of obstacle problems. In this section, we describe the set-up.

Given $n \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $R > 0$, and $y \in \mathbb{T}^d$, we set $\nu = (n, \gamma, R)$ and define the obstacle sub- and super-solutions $u^{\nu,y}$ and $u_{\nu,y}$ as the solutions of the equations

$$\begin{cases} \max\left\{-\text{tr}\left(a^y(\theta_n^{-1}y')D^2_{e_n}u^{\nu,y}\right) - f^y(\theta_n^{-1}y') - \gamma, u^{\nu,y}\right\} = 0 & \text{in } Q_R^{(n)}, \\ u^{\nu,y} = 0 & \text{on } \partial Q_R^{(n)}, \end{cases} \tag{6.12}$$

$$\begin{cases} \min\left\{-\text{tr}\left(a^y(\theta_n^{-1}y')D^2_{e_n}u_{\nu,y}\right) - f^y(\theta_n^{-1}y') - \gamma, u_{\nu,y}\right\} = 0 & \text{in } Q_R^{(n)}, \\ u_{\nu,y} = 0 & \text{on } \partial Q_R^{(n)}. \end{cases} \tag{6.13}$$

Existence and uniqueness for these problems can be proved using a comparison principle or through a penalization procedure as in [29].

When $\nu = (*, \gamma, R)$, $u^{\nu,y}$ and $u_{\nu,y}$ are the solutions of the homogenized problems

$$\begin{cases} \max\left\{-\text{tr}\left(a_e^\perp(y + \langle y', \eta\rangle e)D^2_e u^{\nu,y}\right) - f_e^\perp(y + \langle y', \eta\rangle e) - \gamma, u^{\nu,y}\right\} = 0 & \text{in } Q_R^*, \\ u^{\nu,y} = 0 & \text{on } \partial Q_R^*, \end{cases} \tag{6.14}$$

$$\begin{cases} \min\left\{-\text{tr}\left(a_e^\perp(y + \langle y', \eta\rangle e)D^2_e u_{\nu,y}\right) - f_e^\perp(y + \langle y', \eta\rangle e) - \gamma, u_{\nu,y}\right\} = 0 & \text{in } Q_R^*, \\ u_{\nu,y} = 0 & \text{on } \partial Q_R^*. \end{cases} \tag{6.15}$$

Here $a_e^\perp$ and $f_e^\perp$ are the functions defined in (6.8) and (2.18). For these equations, the regularity of $a_e^\perp$ and $f_e^\perp$ proved in Proposition 8 is more than enough to guarantee well-posedness.

Next, for $n \in \mathbb{N} \cup \{*\}$, we define the contact sets $K^{\nu,y}$ and $K_{\nu,y}$ by

$$K^{\nu,y} = \{y' \in Q_R^{(n)} \mid u^{\nu,y}(y') = 0\}, \quad K_{\nu,y} = \{y' \in Q_R^{(n)} \mid u_{\nu,y}(y') = 0\}.$$

By the results of [29], for each $n \in \mathbb{N} \cup \{*\}$, there are functions $\bar{\ell}_n^{\perp}, \underline{\ell}_n^{\perp} : \mathbb{R} \times \mathbb{T}^d \to [0,1]$ such that, for each $\gamma \in \mathbb{R}$, $\nu(R) = (n, \gamma, R)$, and $y \in \mathbb{T}^d$, we have

$$\bar{\ell}_n^{\perp}(\gamma, y) = \lim_{R \to \infty} R^{1-d} \mathcal{H}^{d-1}(K^{\nu_n(R),y})$$
$$\underline{\ell}_n^{\perp}(\gamma, y) = \lim_{R \to \infty} R^{1-d} \mathcal{H}^{d-1}(K_{\nu_n(R),y}).$$

Notice that, by analogy with Proposition 7, for each $n \in \mathbb{N}$, $\bar{\ell}_n^{\perp}(\gamma, \cdot)$ and $\underline{\ell}_n^{\perp}(\gamma, \cdot)$ vary only in the e_n direction. (In particular, if $e_n \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then $\bar{\ell}_n^{\perp}(\gamma, \cdot)$ and $\underline{\ell}_n^{\perp}(\gamma, \cdot)$ are constant functions by Theorem 14.)

Next, note that these functions satisfy the following variational principles: if $e_n \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then the unique ergodicity of the associated group of translations (i.e., Theorem 12) implies that, for each $y \in \mathbb{T}^d$,

$$\bar{\ell}_n^{\perp}(\gamma, y) = \inf \left\{ R^{1-d} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_n(R),\underline{y}}) \, d\underline{y} \mid R > 0 \right\}, \tag{6.16}$$
$$\underline{\ell}_n(\gamma, y) = \inf \left\{ R^{1-d} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K_{\nu_n(R),\underline{y}}) \, d\underline{y} \mid R > 0 \right\},$$

while in the case $e_n \in \mathbb{R}\mathbb{Z}^d$, we instead can only say that, for each $y \in \mathbb{T}^d$,

$$\overline{\ell}_n(\gamma, y) = \inf \left\{ R^{1-d} \fint_{\mathbb{T}_{e_n}^{d-1}(\langle y, e \rangle)} \mathcal{H}^{d-1}(K^{\nu_n(R),\xi}) \, \mathcal{H}^{d-1}(d\xi) \mid R > 0 \right\}, \tag{6.17}$$
$$\underline{\ell}_n(\gamma, y) = \inf \left\{ R^{1-d} \fint_{\mathbb{T}_{e_n}^{d-1}(\langle y, e \rangle)} \mathcal{H}^{d-1}(K_{\nu_n(R),\xi}) \, \mathcal{H}^{d-1}(d\xi) \mid R > 0 \right\}.$$

All of this follows from the sub-additive ergodic theorem (cf. [29, Appendix A]).

Finally, notice that, when $n = *$, we can argue as in the case when $n \in \mathbb{N}$ and $e_n \in \mathbb{R}\mathbb{Z}^d$. Since the coefficients of the associated obstacle problems vary only in the η direction, it follows that the integrals over $\mathbb{T}_e^{d-1}(s)$ so obtained do not depend on s. Therefore, as in the case when $n \in \mathbb{N}$ and $e_n \notin \mathbb{R}\mathbb{Z}^d$, we find, for each $y \in \mathbb{T}^d$,

$$\bar{\ell}_*^{\perp}(\gamma, y) = \inf \left\{ R^{1-d} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_*(R),\underline{y}})\, d\underline{y} \ \mid \ R > 0 \right\}, \tag{6.18}$$
$$\underline{\ell}_*^{\perp}(\gamma, y) = \inf \left\{ R^{1-d} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K_{\nu_*(R),\underline{y}})\, d\underline{y} \ \mid \ R > 0 \right\}.$$

6.2.4 Homogenization of Obstacle Problems

By a direct analogy to the discussion of the penalized correctors above, we now show that the obstacle problems (6.12) and (6.13) homogenize to the problems (6.14) and (6.15) as $n \to \infty$.

Proposition 68. *Given $(\gamma, R) \in \mathbb{R} \times (0, \infty)$ and $y \in \mathbb{T}^d$, if $\nu_n = (n, \gamma, R)$ and $\nu_* = (*, \gamma, R)$, and if $(y_n)_{n \in \mathbb{N}} \subseteq \mathbb{T}^d$ is such that $y = \lim_{n \to \infty} y_n$, then*

$$0 = \lim_{\delta \to 0^+} \sup \left\{ |u^{\nu_n, y_n}(y_1') - u^{\nu_*, y}(y_2')| \ \mid \ (y_1', y_2') \in Q_R^{(n)} \times Q_R^*, \ n^{-1} + \|y_1' - y_2'\| < \delta \right\},$$
$$0 = \lim_{\delta \to 0^+} \sup \left\{ |u_{\nu_n, y_n}(y_1) - u_{\nu_*, y}(y_2)| \ \mid \ (y_1', y_2') \in Q_R^{(n)} \times Q_R^*, \ n^{-1} + \|y_1' - y_2'\| < \delta \right\}.$$

Proof. To prove this, we will work with half-relaxed limits. The proof for super-solutions follows by analogous arguments so we will restrict attention to sub-solutions.

Define $\bar{u}^*$ and $\bar{u}_*$ in $\overline{Q_R^*}$ by

$$\bar{u}^*(y') = \lim_{\delta \to 0^+} \sup \left\{ u^{\nu_n, y_n}(\underline{y}') \ \mid \ \underline{y}' \in Q_R^{(n)}, \ n^{-1} + \|y' - \underline{y}'\| < \delta \right\},$$
$$\bar{u}_*(y') = \lim_{\delta \to 0^+} \inf \left\{ u^{\nu_n, y_n}(\underline{y}') \ \mid \ \underline{y}' \in Q_R^{(n)}, \ n^{-1} + \|y' - \underline{y}'\| < \delta \right\}.$$

It suffices to show that $\bar{u}^* = \bar{u}_* = u^{\nu_*, y}$. To do this, we only need to prove that $\bar{u}^*$ and $\bar{u}_*$

are sub- and super-solution of (6.14) and apply the comparison principle. Since the proof for $\bar{u}^*$ is almost identical, we will restrict attention to $\bar{u}_*$.

Assume that $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a smooth function and $y_0' \in Q_R^*$ is a point where $\bar{u}_* - \varphi$ has a strict local minimum. We claim that

$$\max \left\{ -\mathrm{tr} \left(a_e^{\perp}(y + \langle y_0', \eta \rangle e) D_e^2 \varphi(y_0') \right) - f_e^{\perp}(y + \langle y_0', \eta \rangle e) - \gamma, \bar{u}_*(y_0') \right\} \geq 0. \tag{6.19}$$

Clearly, we can assume that $\bar{u}_*(y_0') < 0$.

Let us argue by contradiction. If (6.19) fails to hold, then there is a $\zeta > 0$ such that

$$- \mathrm{tr} \left(a_e^{\perp}(y + \langle y_0', \eta \rangle e) D_e^2 \varphi(y_0') \right) - f_e^{\perp}(y + \langle y_0', \eta \rangle e) - \gamma < -\zeta. \tag{6.20}$$

Define $g \in C(\mathbb{T}^d)$ by

$$g(\underline{y}) = f(y + \underline{y}) + \mathrm{tr} \left(a(y + \underline{y}) D_e^2 \varphi(y_0') \right).$$

Let $V^{e,\delta,g} \in C^2(\mathbb{T}^d)$ be the solution of (2.13) with $\delta = \delta(\zeta) > 0$ chosen so small that

$$\left| g_e^{\perp}(y + \underline{y}) - \mathrm{tr} \left(a^y(\underline{y}) D_e^2 V^{e,\delta,g}(\underline{y}) \right) - g(y + \underline{y}) \right| \leq \zeta \quad \text{for each } \underline{y} \in \mathbb{T}^d.$$

For each $n \in \mathbb{N}$, pick a point y_n' where the function $y' \mapsto u^{\nu_n, y_n}(y') - \varphi(y') - \theta_n^2 V(\theta_n^{-1} y')$ attains its minimum in the closure of $Q_R^{(n)}$. By classical arguments, we can pass to a subsequence $(y_{n_j}')_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} y_{n_j}' = y_0', \quad \lim_{j \to \infty} u^{\nu_{n_j}, y_{n_j}}(y_{n_j}') = \bar{u}_*(y_0') < 0.$$

There is no loss of generality in assuming that both $y_{n_j}' \in Q_R^{(n_j)}$ and $u^{\nu_{n_j}, y_{n_j}}(y_{n_j}') < 0$

for all $j \in \mathbb{N}$. Thus, the equation satisfied by $u^{\nu_{n_j}, y_{n_j}}$ gives

$$
\begin{aligned}
0 &\leq -\mathrm{tr}\left(a^{y_{n_j}}(\theta_{n_j}^{-1} y'_{n_j})[D^2_{e_{n_j}}\varphi(y'_{n_j}) + D^2_{e_{n_j}} V(\theta_{n_j}^{-1} y'_{n_j})]\right) - f^{y_{n_j}}(\theta_{n_j}^{-1} y'_{n_j}) - \gamma \\
&= -\mathrm{tr}\left(a^{y}(\theta_{n_j}^{-1} y'_{n_j}) D^2_e V(\theta_{n_j}^{-1} y_{n_j})\right) - g(\theta_{n_j}^{-1} y'_{n_j}) - \gamma + o(1) \\
&\leq g_e^{\perp}(\theta_{n_j}^{-1} y'_{n_j}) + \zeta + o(1).
\end{aligned}
$$

Since $y'_{n_j} \in \langle e_{n_j} \rangle^{\perp}$, we have

$$
\theta_{n_j}^{-1}\langle y'_{n_j}, e\rangle = \theta_{n_j}^{-1}\langle y'_{n_j}, e - e_{n_j}\rangle = \left(\frac{1-\cos(\theta_{n_j})}{\theta_{n_j}}\right)\langle e, y'_{n_j}\rangle + \left(\frac{\sin(\theta_{n_j})}{\theta_{n_j}}\right)\langle \eta_{n_j}, y'_{n_j}\rangle.
$$

Thus, $\theta_{n_j}^{-1}\langle y'_{n_j}, e\rangle \to \langle y'_0, \eta\rangle$ and we find, in the limit $n \to \infty$,

$$
-\zeta \leq g_e^{\perp}(\langle y'_0, \eta\rangle e) - \gamma = -\mathrm{tr}\left(a_e^{\perp}(y + \langle y'_0, \eta\rangle e) D^2_e \varphi(y'_0)\right) - f_e^{\perp}(y + \langle y'_0, \eta\rangle e) - \gamma.
$$

This directly contradicts (6.20).

We deduce that $\bar{u}_*$ is a super-solution of (6.14) in the interior of Q^*_R. Using barriers, it is not hard to show that $\bar{u}_* \geq 0$ in ∂Q^*_R. Thus, $\bar{u}_*$ is a super-solution. □

6.2.5 Densities of Contact Sets

Once we state some properties of the densities $\bar{\ell}_n^{\perp}$ and $\underline{\ell}_n^{\perp}$, we will have all the tools necessary to prove Theorem 36.

The following result follows by arguing as in [29] (also see [7]). In the statement, we write $e_* = e$.

Proposition 69. *For each $y \in \mathbb{T}^d$, there is a sequence $(\gamma_n(y))_{n \in \mathbb{N} \cup \{*\}} \subseteq \mathbb{R}$ such that, for each $n \in \mathbb{N} \cup \{*\}$, the following statements hold:*

(i) $\bar{\ell}_n^{\perp}(\gamma, y) = 0$ if $\gamma < \gamma_n(y)$ and $\bar{\ell}_n^{\perp}(\gamma, y) \geq c|\gamma - \gamma_n(y)|^{d-1}$ if $\gamma > \gamma_n(y)$.

(ii) $\underline{\ell}_n^{\perp}(\gamma, y) = 0$ *if* $\gamma > \gamma_n(y)$ *and* $\underline{\ell}_n^{\perp}(\gamma, y) \geq c|\gamma - \gamma_n(y)|^{d-1}$ *if* $\gamma < \gamma_n(y)$.

(iii) If $n \in \mathbb{N}$ *and, for each* $\nu = (n, R)$, $v^{\nu,y}$ *is the solution of the Dirichlet problem*

$$\begin{cases} -tr\left(a^y(y')D^2_{e_n}v^{\nu,y}\right) - f^y(y') = \gamma_n(y) & in\ Q_R^{(n)}, \\ v^{\nu,y} = 0 & on\ \partial Q_R^{(n)}, \end{cases}$$

then

$$\lim_{R\to\infty} \sup\left\{R^{-2}|v^{(n,R),y}(y')| \mid y' \in Q_R^{(n)}\right\} = 0.$$

(iv) If, for each $\nu = (*, R)$, $v^{\nu,y}$ *is the solution of the Dirichlet problem*

$$\begin{cases} -tr\left(a_e^{\perp}(y+y')D^2_{e}v^{\nu,y}\right) - f_e^{\perp}(y+y') = \gamma_*(y) & in\ Q_R^{(n)}, \\ v^{\nu,y} = 0 & on\ \partial Q_R^{(n)}, \end{cases}$$

then

$$\lim_{R\to\infty} \sup\left\{R^{-2}|v^{(*,R),y}(y')| \mid y' \in Q_R^{(n)}\right\} = 0.$$

(The constant $c > 0$ *depends on* λ, Λ, *and* d, *but not on* n.*)*

In addition, we will need the following fact, adapted from [29], that follows from the homogenization result above:

Proposition 70. *For each* $(\gamma, y) \in \mathbb{R} \times \mathbb{T}^d$, *we have*

$$\bar{\ell}_*^{\perp}(\gamma, y) \geq \lim_{n\to\infty} \sup\left\{\bar{\ell}_n^{\perp}(\gamma, \underline{y}) \mid \underline{y} \in \mathbb{T}^d\right\},$$
$$\underline{\ell}_*^{\perp}(\gamma, y) \geq \lim_{n\to\infty} \sup\left\{\underline{\ell}_n^{\perp}(\gamma, \underline{y}) \mid \underline{y} \in \mathbb{T}^d\right\}.$$

The proof uses the upper semi-continuity properties of the contact sets $\{K^{\nu,x}\}$. When $(e_n)_{n\in\mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, this can be proved with Fatou's Lemma by adapting the idea of [29] directly to our setting using Proposition 68.

When $(e_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}\mathbb{Z}^d$, the situation is more delicate since the probability measures in the variational principle (6.17) depend on n. As pointed out by W.M. Feldman, the argument from the irrational case still applies provided we replace Fatou's Lemma by the generalization due to Feinberg, Kasyanov, and Zadoianchuk [46].

Proof of Proposition 70. We give the details for $(\bar{\ell}_n^{\perp})_{n \in \mathbb{N}}$; the same basic idea also applies to $(\underline{\ell}_n^{\perp})_{n \in \mathbb{N}}$.

Step 1: Property of Contact Sets

Given $(\gamma, R) \in \mathbb{R} \times (0, \infty)$ and $y \in \mathbb{T}^d$, let $\nu_n = (n, \gamma, R)$ and $\nu_* = (*, \gamma, R)$, and assume that $(y_n)_{n \in \mathbb{N}} \subseteq \mathbb{T}^d$ is such that $\lim_{n \to \infty} y_n = y$. We claim that

$$\mathcal{H}^{d-1}(K^{\nu_*, y}) \geq \limsup_{n \to \infty} \mathcal{H}^{d-1}(K^{\nu_n, y_n}). \tag{6.21}$$

To see this, first, define $\tilde{K} \subseteq Q_R^*$ by

$$\tilde{K} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \left\{ y' \in Q_R^* \mid u^{\nu_m, y_m}(O_m^{-1}(y')) = 0 \right\}.$$

Notice that Proposition 68 implies that $\tilde{K} \subseteq K^{\nu_*, y}$. Therefore, using the measure preserving property of the orthogonal transformations $\{O_m\}_{m \in \mathbb{N}}$, we find

$$\begin{aligned} \mathcal{H}^{d-1}(K^{\nu_*, y}) &\geq \mathcal{H}^{d-1}(\tilde{K}) \\ &= \lim_{n \to \infty} \mathcal{H}^{d-1} \left(\bigcup_{m=n}^{\infty} \left\{ y' \in Q_R^* \mid u^{\nu_m, y_m}(O_m^{-1}(y)) = 0 \right\} \right) \\ &\geq \limsup_{n \to \infty} \mathcal{H}^{d-1}(K^{\nu_n, y_n}). \end{aligned}$$

Step 2: Convenient Reformulation

The result of the previous step can be reformulated slightly. Define Borel functions

$\{f_n\}_{n\in\mathbb{N}}$ and f_* in $\mathbb{T}^d$ by

$$f_n(y) = \mathcal{H}^{d-1}(K^{\nu_n,y}), \quad f_*(y) = \mathcal{H}^{d-1}(K^{\nu_*,y}).$$

By Step 1, for each $y \in \mathbb{T}^d$, we have

$$f_*(y) \geq \lim_{\delta\to 0^+} \sup\left\{f_n(\underline{y}) \mid n^{-1} + |y - \underline{y}| < \delta\right\}. \tag{6.22}$$

Step 2: Conclusion

We conclude the proof by considering two cases: (a) $(e_n)_{n\in\mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ and (b) $(e_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}\mathbb{Z}^d$. Note that there is no loss of generality in assuming that either (a) or (b) holds since we can always pass to subsequences if necessary.

In either case, we start by fixing an $\epsilon > 0$. By invoking (6.18), we can fix an $R > 0$ such that

$$\bar{\ell}_*^{\perp}(\gamma, y) \geq R^{1-d} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_*,\underline{y}})\, d\underline{y} - \epsilon.$$

Now we consider cases (i) and (ii) separately.

In case (i), Fatou's Lemma, the conclusion of Step 1, and (6.16) combine to give

$$\begin{aligned}\int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_*,y})\, dy &\geq \limsup_{n\to\infty} \int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_n,y})\, dy \\ &\geq \limsup_{n\to\infty} R^{d-1} \sup\left\{\bar{\ell}_n(\gamma, y) \mid y \in \mathbb{T}^d\right\}.\end{aligned}$$

Thus,

$$\bar{\ell}_*(\gamma, y) \geq \limsup_{n\to\infty} \sup\left\{\bar{\ell}_n(\gamma, \underline{y}) \mid \underline{y} \in \mathbb{T}^d\right\} - \epsilon.$$

In case (ii), we can combine (6.17), (6.22), Lemma 17 in the appendix, and the generalization of Fatou's Lemma in [46, Theorem 1.1] to find

$$\int_{\mathbb{T}^d} \mathcal{H}^{d-1}(K^{\nu_*,y})\,dy \geq \limsup_{n\to\infty} \fint_{\mathbb{T}^{d-1}_{e_n}(\langle y_n, e\rangle)} \mathcal{H}^{d-1}(K^{\nu_n,\xi})\,\mathcal{H}^{d-1}(d\xi)$$
$$\geq \limsup_{n\to\infty} R^{d-1}\overline{\ell}_n(\gamma, y_n).$$

Since $(y_n)_{n\in\mathbb{N}}$ was arbitrary, we conclude

$$\overline{\ell}_*(\gamma) \geq \limsup_{n\to\infty} \sup\left\{\overline{\ell}_n(\gamma, y) \mid y \in \mathbb{T}^d\right\} - \epsilon.$$

In any case, the arbitrariness of $\epsilon > 0$ gives the desired result. □

Finally, combining Propositions 69 and 70, we obtain

Proposition 71. *For each $y \in \mathbb{T}^d$, we have*

$$\lim_{n\to\infty} \sup\left\{|\gamma_n(\underline{y}) - \gamma_*(y)| \mid \underline{y} \in \mathbb{T}^d\right\} = 0.$$

Proof. Note that it is enough to prove the following two inequalities:

$$\gamma_*(y) \geq \limsup_{n\to\infty} \sup\left\{\gamma_n(\underline{y}) \mid \underline{y} \in \mathbb{T}^d\right\},$$
$$\gamma_*(y) \leq \liminf_{n\to\infty} \inf\left\{\gamma_n(\underline{y}) \mid \underline{y} \in \mathbb{T}^d\right\}.$$

We will only prove the first inequality since the second one follows by a similar argument in which $(\overline{\ell}_n^{\perp})_{n\in\mathbb{N}}$ replaces $(\underline{\ell}_n^{\perp})_{n\in\mathbb{N}}$.

Choose a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{T}^d$ such that

$$\limsup_{n\to\infty} \sup\left\{\gamma_n(\underline{y}) \mid \underline{y} \in \mathbb{T}^d\right\} = \lim_{n\to\infty} \gamma_n(y_n).$$

To obtain the desired result, we will show that if $\gamma < \lim_{n\to\infty} \gamma_n(y_n)$, then $\gamma \leq \gamma_*(y)$.

To see this, suppose that $\gamma < \lim_{n \to \infty} \gamma_n(y_n)$. By Propositions 69 and 70,

$$\underline{\ell}_*^{\perp}(\gamma, y) \geq \limsup_{n \to \infty} \underline{\ell}_n^{\perp}(\gamma, y_n) \geq c \limsup_{n \to \infty} (\gamma_n(y_n) - \gamma)^{d-1} > 0.$$

From this, Proposition 69 with $n = *$ yields $\gamma \leq \gamma_*(y)$. □

6.2.6 Proof of Proposition 67

The results of the previous section directly imply Proposition 67, as we now show.

To start with, we identify the functions $(\gamma_n)_{n \in \mathbb{N}}$ of Proposition 69.

Lemma 12. *For each $n \in \mathbb{N}$ and $y \in \mathbb{T}^d$, we have*

$$\gamma_n(y) = \begin{cases} -f_{e_n}^{\perp}(y), & \text{if } e_n \in \mathbb{R}\mathbb{Z}^d, \\ -\overline{f}(e_n), & \text{otherwise.} \end{cases}$$

Proof. Fix $n \in \mathbb{N}$ and $y \in \mathbb{T}^d$. By Proposition 69 and rescaling, if we let $(v_\epsilon)_{\epsilon > 0}$ be the solutions of the Dirichlet problem

$$\begin{cases} -\mathrm{tr}\left(a^y(\epsilon^{-1}x') D_{e_n}^2 v_\epsilon\right) - f^y(\epsilon^{-1}x') = \gamma_n(X) & \text{in } Q_1^{(n)}, \\ v_\epsilon = 0 & \text{on } \partial Q_1^{(n)}, \end{cases}$$

then $v_\epsilon \to 0$ uniformly in $Q_1^{(n)}$ as $\epsilon \to 0^+$.

At the same time, if $e_n \in \mathbb{R}\mathbb{Z}^d$, this is a periodic elliptic homogenization problem in $\langle e_n \rangle^{\perp}$ so we know from classical results and the definitions of $f_{e_n}^{\perp}$ and $a_{e_n}^{\perp}$ (see (2.18) and (6.8)) that $v_\epsilon \to \bar{v}$, where $\bar{v}$ is the solution of the constant coefficient equation

$$\begin{cases} -\mathrm{tr}\left(a_{e_n}^{\perp}(y) D_{e_n}^2 \bar{v}\right) - f_{e_n}^{\perp}(y) = \gamma_n(y) & \text{in } Q_1^{(n)} \\ \bar{v} = 0 & \text{on } \partial Q_1^{(n)} \end{cases}$$

The previous paragraph says that $\bar{v} \equiv 0$ in $Q_1^{(n)}$. Therefore, $\gamma_n(y) = -f_{e_n}^{\perp}(y)$.

If, on the other hand, $e_n \notin \mathbb{R}\mathbb{Z}^d$, then the results of [29] again imply homogenization with $a_{e_n}^{\perp}(y)$ replaced by $\bar{a}(e_n)$ and $f_{e_n}^{\perp}(y)$ by $\overline{f}(e_n)$. Hence, by the same argument as in the rational case, we can only conclude that $\gamma_n(y) = -\overline{f}(e_n)$. □

Next, we show that $-\gamma_*$ equals the constant $\tilde{f}_e^{\eta}$ (see (6.10)).

Lemma 13. *For each $y \in \mathbb{T}^d$,*

$$-\gamma_*(y) = \tilde{f}_e^{\eta}. \tag{6.23}$$

Proof. Fix $y \in \mathbb{T}^d$. As in Lemma 12, $-\gamma_*(y)$ equals the average of $f_e^{\perp}$ against the invariant measure associated with the operator $-\mathrm{tr}\left(a_e^{\perp}(\epsilon^{-1}\langle y', \eta\rangle e)\right)$. Since the coefficient of the operator varies only in the η direction, a symmetry argument shows that the homogenized coefficients are the same as if we consider the one-dimensional problem. In particular, by a well-known computation, the coefficient $-\gamma_*(y)$ is given by

$$-\gamma_*(y) = \left(\int_0^{r_e} \langle a_e^{\perp}(se)\eta, \eta\rangle^{-1}\, ds\right)^{-1} \int_0^{r_e} f_e^{\perp}(se)\langle a_e^{\perp}(se)\eta, \eta\rangle^{-1}\, ds.$$

We conclude by comparing the definitions of $f_e^{\perp}$ and $\tilde{f}_e^{\eta}$ (see (2.18), (6.10), and (6.7)). □

6.2.7 Proof of Theorem 36

Finally, we put all the ingredients together to prove the theorem.

Proof of Theorem 36. Notice that to obtain the conclusion of Theorem 36, it suffices to show that if $(e_n)_{n\in\mathbb{N}} \subseteq S^{d-1}$ is such that $e_n \to e$ and $\frac{e_n - e}{\|e_n - e\|} \to -\eta$ as $n \to \infty$; $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathscr{P}(\mathbb{T}^d)$ is such that $\mu_n \in \mathscr{I}_{e_n}^{a}$ for each n; and $f \in C(\mathbb{T}^d)$, then

$$\lim_{n\to\infty} \int_{\mathbb{T}^d} f(y)\mu_n(dy) = \int_{\mathbb{T}^d} f(y)\tilde{\mu}_e^{\eta}(dy).$$

Passing to a subsequence if necessary, there is no loss of generality splitting into cases: (a) $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ and (b) $(e_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}\mathbb{Z}^d$.

In case (a), we know $\int_{\mathbb{T}^d} f(y) \, \mu_n(dy) = \overline{f}(e_n)$ by the uniqueness of the invariant measure. Hence, by Proposition 71, Lemma 13, and the definition of $\tilde{f}_e^{\eta}$ (see (6.10)),

$$\lim_{n \to \infty} \int_{\mathbb{T}^d} f(y) \, \mu_n(dy) = - \lim_{n \to \infty} \gamma_n(y) = -\gamma_*(y) = \tilde{f}_e^{\eta} = \int_{\mathbb{T}^d} f(y) \, \tilde{\mu}_e^{\eta}(dy).$$

In case (b), we know that $\mu_n = \mu_{e_n}^{s_n}$ for some $s_n \in [0, r_{e_n})$. Thus, by Proposition 71 and (2.18),

$$\lim_{n \to \infty} \int_{\mathbb{T}^d} f(y) \, \mu_n(dy) = \lim_{n \to \infty} f_{e_n}^{\perp}(s_n e_n) = - \lim_{n \to \infty} \gamma_n(s_n e_n) = -\gamma_*(y) = \tilde{f}_e^{\eta}.$$

□

6.2.8 Generic Discontinuities

In light of the formulas obtained for the limiting measures in the previous section, it is natural to expect that $\bar{a}$ and $\overline{m}$ are generically discontinuous at some rational directions. In this section, we prove that, in fact, $\bar{a}$ and $\overline{m}$ are generically discontinuous at every rational direction when $d \geq 3$.

Proof of Corollary 2. To start with, since $d \geq 3$, we can fix $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$ and let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence of points in $S^{d-1} \cap \langle e \rangle^{\perp}$ with $\eta_n \notin \{\eta_m, -\eta_m\}$ for all $n \neq m$. The goal is to prove that, for each $n, m \in \mathbb{N}$ with $n \neq m$, the following sets are open and dense in $C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda))$ in the $C^{2,\alpha}$ norm topology:

$$\mathcal{U}_e(n,m) = \left\{ a \in C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda)) \mid \tilde{\mu}_e^{\eta_n} \neq \tilde{\mu}_e^{\eta_m} \right\},$$
$$\mathcal{V}_e(n,m) = \left\{ a \in C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda)) \mid \tilde{a}_e^{\eta_n} \neq \tilde{a}_e^{\eta_m} \right\}.$$

That these sets are open is immediate. It only remains to show they are dense.

We start with $\mathcal{U}_e(n,m)$. It turns out that we only need to understand the derivative of the map $a \mapsto a_e^\perp$. Toward that end, for each $n \in \mathbb{N}$, define $u_n \in C^{2,\alpha}(\mathbb{T}^d)$ to be the solution of the cell problem (2.20) with $f(y) = \langle a(y)\eta_n, \eta_n\rangle$. Notice that the oscillating function $f_e^\perp$ equals $\langle a_e^\perp \eta_n, \eta_n\rangle$ in this case.

Step 1: Perturb so that $D_e^2 u_n \neq D_e^2 u_m$

To start with, we claim that there is no loss of generality in assuming that $D_e^2 u_n \neq D_e^2 u_m$. Indeed, if $D_e^2 u_n = D_e^2 u_m$, then (2.20) implies $\langle [a - a_e^\perp]\eta_n, \eta_n\rangle = \langle [a - a_e^\perp]\eta_m, \eta_m\rangle$. That is,

$$\langle a(y)\eta_n, \eta_n\rangle - \langle a(y)\eta_m, \eta_m\rangle = \langle a_e^\perp(\langle y, e\rangle e)\eta_n, \eta_n\rangle - \langle a_e^\perp(\langle y, e\rangle e)\eta_m, \eta_m\rangle \quad \text{if } y \in \mathbb{T}^d.$$

In particular, the left-hand side varies only in the e direction. This symmetry is easily broken, for instance, by replacing a by $y \mapsto a(y) + \nu \cos(2\pi\langle k, y\rangle)\eta_n \otimes \eta_n$ for some $k \in \mathbb{Z}^d \setminus \langle e\rangle$ and $\nu \in \mathbb{R}$ sufficiently small.

Step 2: Restrict attention to $a_e^\perp$

First, observe that $\tilde{\mu}_e^{\eta_n} = \tilde{\mu}_e^{\eta_m}$ if and only if there is a $C > 0$ such that $\langle a_e^\perp \eta_n, \eta_n\rangle = C\langle a_e^\perp \eta_m, \eta_m\rangle$ in $\mathbb{T}^d$. This is a direct consequence of the formula (6.7). We claim that if $\langle a_e^\perp \eta_n, \eta_n\rangle = C\langle a_e^\perp \eta_m, \eta_m\rangle$ for some $C > 0$, then this symmetry is broken by some arbitrarily small perturbation of a.

To see this, we will differentiate the function $a \mapsto a_e^\perp$. Given $a_* \in C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d)$ and $h \in \mathbb{R}$ small enough, define $a_h = a + h a_*$ and let $(a_h)_e^\perp$ be the associated averaged tensor.

Employing arguments similar to those in the proof of [70, Proposition 7], we see that, for each $r \in [0, r_e)$, $h \in \mathbb{R}$, and $j \in \{n, m\}$, if $\tilde{U}_{j,re}^h$ is the solution of the equation

$$\begin{cases} -\mathrm{tr}\left(a_h(re + x')D^2 \tilde{U}_{j,re}^h\right) = \langle a_h(re + x')\eta_j, \eta_j\rangle - \langle (a_h)_e^\perp(re)\eta_j, \eta_j\rangle & \text{in } \mathbb{T}_e^{d-1}(0) \\ \tilde{U}_{j,re}^h(0) = 0 & \end{cases}$$

then there are functions $\{\tilde{U}_{n,re}, \tilde{U}_{m,re}\} \in C(\mathbb{T}_e^{d-1}(0))$ and a function $a_*^\perp \in C(\mathbb{T}^d)$ varying only in the e direction such that, for $j \in \{n, m\}$ and $r \in [0, r_e)$,

$$\tilde{U}_{j,re} = \lim_{h \to 0} \frac{\tilde{U}^h_{j,re} - \tilde{U}^0_{j,re}}{h} \quad \text{uniformly in } \mathbb{T}_e^{d-1}(0),$$
$$a_*^\perp = \lim_{h \to 0} \frac{(a_h)_e^\perp - a_e^\perp}{h} \quad \text{uniformly in } \mathbb{T}^d.$$

Furthermore, $\tilde{U}_{j,re}$ is a solution of the equation

$$-\mathrm{tr}\left(a(re + x')D_e^2 \tilde{U}_{j,re}\right) = \mathrm{tr}\left(a_*(re + x')D_e^2 u_j(re + x')\right) - \langle a_*^\perp(re)\eta_j, \eta_j\rangle \quad \text{in } \mathbb{T}_e^{d-1}(0).$$

Notice that $a_*^\perp(y) = \int_{\mathbb{T}^d} \mathrm{tr}\left(a_*(y')D_e^2 u_j(y')\right) \mu_e^{\langle y, e\rangle}(dy')$.

By the previous step, there is no loss in generality assuming that $D_e^2 u_n \neq D_e^2 u_m$. In fact, since these functions are C^α, we can fix $y_1, y_2 \in \mathbb{T}^d$ so that $\langle y_1, e\rangle \neq \langle y_2, e\rangle$ and $D_e^2 u_n(y_i) \neq D_e^2 u_m(y_i)$ for $i \in \{1, 2\}$. Since μ_e^s is supported on $\mathbb{T}_e^{d-1}(s)$ for each $s \in [0, r_e)$, we can we fix $a_* \in C^\infty(\mathbb{T}^d; \mathcal{S}_d)$ so that

$$\int_{\mathbb{T}^d} \mathrm{tr}\left(a_*(y')D_e^2 u_n(y')\right) \mu_e^{\langle y_1, e\rangle}(dy') > 0 > \int_{\mathbb{T}^d} \mathrm{tr}\left(a_*(y')D_e^2 u_m(y')\right) \mu_e^{\langle y_1, e\rangle}(dy'),$$
$$\int_{\mathbb{T}^d} \mathrm{tr}\left(a_*(y')D_e^2 u_n(y')\right) \mu_e^{\langle y_2, e\rangle}(dy') < 0 < \int_{\mathbb{T}^d} \mathrm{tr}\left(a_*(y')D_e^2 u_m(y')\right) \mu_e^{\langle y_2, e\rangle}(dy').$$

From this, we find that, for all $h > 0$ sufficiently small,

$$\frac{\langle (a_h)_e^\perp(y_1)\eta_m, \eta_m\rangle}{\langle (a_h)_e^\perp(y_1)\eta_n, \eta_n\rangle} < C < \frac{\langle (a_h)_e^\perp(y_2)\eta_m, \eta_m\rangle}{\langle (a_h)_e^\perp(y_2)\eta_n, \eta_n\rangle}.$$

Therefore, $\langle (a_h)_e^\perp \eta_n, \eta_n\rangle$ is not a constant multiple of $\langle (a_h)_e^\perp \eta_m, \eta_m\rangle$ even while we have the approximation estimate $\|a - a_h\|_{C^{2,\alpha}(\mathbb{T}^d)} \leq Ch$.

We conclude from the preceding that $\mathcal{U}_e(n, m)$ is dense in $C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda, \Lambda))$. It remains to prove the same thing for $\mathcal{V}_e(n, m)$.

Step 3: Ensure $\tilde{a}_e^{\eta_n} \neq \tilde{a}_e^{\eta_m}$

We want to show that $\mathcal{V}_e(n,m)$ is dense. Given that $\tilde{a}_e^{\eta} = \int_{\mathbb{T}^d} a(y)\, \tilde{\mu}_e^{\eta}(dy)$, this is intuitively clear in light of what we just proved.

The previous arguments show that we can assume that $a \in \mathcal{U}_e(n,m) \setminus \mathcal{V}_e(n,m)$ to start with. We will show that there is an $m \in C^{2,\alpha}(\mathbb{T}^d)$ varying only in the e direction such that the function $a_h = (1+hm)^{-1}a$ is in $\mathcal{V}_e(n,m)$ for all $h \in \mathbb{R}$ sufficiently small.

To start with, notice that, by arguing as in [70, Section 4.4], we see that, for each $\eta \in S^{d-1} \cap \langle e \rangle^{\perp}$, the limiting coefficient $\tilde{a}_{h,e}^{\eta}$ associated with a_h is given by

$$\tilde{a}_{h,e}^{\eta} = \frac{\tilde{a}_e^{\eta}}{1 + h \int_{\mathbb{T}^d} m(y) \tilde{\mu}_e^{\eta}(dy)}. \tag{6.24}$$

Since m varies only in the e direction, the integral in the denominator becomes

$$\int_{\mathbb{T}^d} m(y) \tilde{\mu}_e^{\eta}(dy) = \frac{\int_0^{r_e} m(se) \langle a_e^{\perp}(se)\eta, \eta \rangle^{-1}\, ds}{\int_0^{r_e} \langle a_e^{\perp}(se)\eta, \eta \rangle^{-1}\, ds}.$$

Given that $a \in \mathcal{U}_e(n,m)$, we know that the function $s \mapsto \langle a_e^{\perp}(se)\eta_n, \eta_n \rangle$ does not equal a multiple of $s \mapsto \langle a_e^{\perp}(se)\eta_m, \eta_m \rangle$, and, therefore, we can choose m so that $\int_{\mathbb{T}^d} m(\langle y, e \rangle e) \tilde{\mu}_e^{\eta_n}(dy) \neq \int_{\mathbb{T}^d} m(\langle y, e \rangle e) \tilde{\mu}_e^{\eta_m}(dy)$. Thus, since $\tilde{a}_e^{\eta_n} = \tilde{a}_e^{\eta_m}$, (6.24) implies $\tilde{a}_{h,e}^{\eta_n} \neq \tilde{a}_{h,e}^{\eta_m}$ for all h small enough. This proves a is a limit point of $\mathcal{V}_e(n,m)$.

Conclusion

We showed that $\mathcal{U}_e(n,m)$ and $\mathcal{V}_e(n,m)$ are both open and dense in $C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda,\Lambda))$ in the $C^{2,\alpha}$ norm topology. Define $\mathscr{C}_d$ by

$$\mathscr{C}_d = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N} \setminus \{n\}} \mathcal{U}_e(n,m) \cap \mathcal{V}_e(n,m).$$

This set is residual, being a countable intersection of open, dense sets. Further, since $C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d(\lambda,\Lambda))$ is an open subset of the Banach space $C^{2,\alpha}(\mathbb{T}^d; \mathcal{S}_d)$, the Baire Category

Theorem implies that $\mathscr{C}_d$ is itself dense. □

6.3 Proof of Theorem 34

In this section, we show how the results of the previous two sections imply Theorem 34. To begin with, since the effective coefficients are not continuous in general, we define upper and lower semi-continuous envelopes $\overline{F}, \underline{F} : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ by

$$\overline{F}(p, X) = \lim_{\delta \to 0^+} \sup \left\{ \overline{m}(\|p'\|^{-1}p')^{-1} \text{tr} \left(\bar{A}(\|p'\|^{-1}p')X \right) \mid p' \notin \mathbb{R}\mathbb{Z}^d, \ \|p' - p\| \leq \delta \right\}, \tag{6.25}$$

$$\underline{F}(p, X) = \lim_{\delta \to 0^+} \inf \left\{ \overline{m}(\|p'\|^{-1}p')^{-1} \text{tr} \left(\bar{A}(\|p'\|^{-1}p')X \right) \mid p' \notin \mathbb{R}\mathbb{Z}^d, \ \|p' - p\| \leq \delta \right\}, \tag{6.26}$$

where, as before, $\bar{A}(e) = (\text{Id} - e \otimes e)\bar{a}(e)(\text{Id} - e \otimes e)$.

The operators $\overline{F}$ and $\underline{F}$ allow us to define viscosity solutions of (6.4) in spite of the possible discontinuity of $\overline{m}$ and $\bar{a}$.

Definition 11. *We say that an upper semi-continuous function $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a viscosity sub-solution of* (6.4) *if $v_t - \overline{F}(Dv, D^2v) \leq 0$ in the viscosity sense in $\mathbb{R}^d \times (0, \infty)$ and $v \leq u_0$ on $\mathbb{R}^d \times \{0\}$.*

Similarly, a lower semi-continuous function $w : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a viscosity super-solution of (6.4) *if $w_t - \underline{F}(Dw, D^2w) \geq 0$ in the viscosity sense in $\mathbb{R}^d \times (0, \infty)$ and $w \geq u_0$ on $\mathbb{R}^d \times \{0\}$.*

Finally, a continuous function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a viscosity solution of (6.4) *if it is both a sub-solution and a super-solution.*

In Section 6.1, we showed that the half-relaxed limits $\bar{u}^*$ and $\bar{u}_*$ satisfy

$$\bar{u}^*_t - \overline{F}(D\bar{u}^*, D^2\bar{u}^*) \leq 0 \quad \text{and} \quad \bar{u}_{*,t} - \underline{F}(D\bar{u}_*, D^2\underline{u}_*) \geq 0$$

in irrational directions in $\mathbb{R}^d \times (0,\infty)$. The results of Section 7.1 below imply that "in irrational directions" can be removed — these are sub- and super-solutions in the standard viscosity sense.

To prove that $\bar{u}^* = \bar{u}_*$, we will invoke Theorem 38 below, which is a comparison principle for second-order level-set PDE with discontinuities in countably many directions. To do so, we need to check that $\overline{F}$ and $\underline{F}$ satisfy the assumptions of that theorem. That is the subject of the next two results.

Proposition 72. *The operators $\overline{F}$ and $\underline{F}$ given by* (6.25) *and* (6.26) *satisfy the assumptions (i)-(v) of Theorem 38.*

Proof. The definitions of $\overline{F}$ and $\underline{F}$ directly imply that assumption (i), (iii), and (iv) hold.

Concerning assumption (ii), notice that $\lambda \mathrm{Id} \leq \bar{a}(e) \leq \Lambda \mathrm{Id}$ for each $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ by (6.6) and (6.3). Thus, the coefficient $\bar{A}(e)$ satisfies

$$\lambda(\mathrm{Id} - e \otimes e) \leq \bar{A}(e) \leq \Lambda(\mathrm{Id} - e \otimes e) \quad \text{for each } e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d,$$

and this implies $\overline{F}$ and $\underline{F}$ satisfy the ellipticity assumption (ii).

Next, we claim that (v) holds. Fix a point $(p, X) \in (\mathbb{R}^d \setminus \{0\}) \times \mathcal{S}_d$ for which $\hat{p} \notin \{\nu_n\}_{n \in \mathbb{N}}$. Define $e = \hat{p}$. To see that $\overline{F}(p, X) = \underline{F}(p, X)$, first, observe that if $(e_n)_{n \in \mathbb{N}} \subseteq S^{d-1}$ is such that $\lim_{n \to \infty} e_n = e$ and $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathscr{P}(\mathbb{T}^d)$ are such that $\mu_n \in \mathscr{I}^a_{e_n}$ for each n, then $\mu_n \rightharpoonup \bar{\mu}_e$. Indeed, by compactness, there is no loss of generality assuming that $(\mu_n)_{n \in \mathbb{N}}$ converges to some probability measure μ. If $\psi \in C^2(\mathbb{T}^d)$, then

$$\int_{\mathbb{T}^d} \mathrm{tr}\left(a(y) D^2_e \psi(y)\right) \, \mu(dy) = \lim_{n \to \infty} \int_{\mathbb{T}^d} \mathrm{tr}\left(a(y) D^2_{e_n} \psi(y)\right) \, \mu_n(dy) = 0.$$

Hence, by uniqueness (Theorem 15), $\mu = \bar{\mu}_e$.

From the preceding considerations and (6.6), we see that

$$\lim_{\delta\to 0^+} \sup\left\{\|\bar{a}(e') - \bar{a}(e)\| + |\overline{m}(e') - \overline{m}(e)| \ \mid \ e' \in S^{d-1} \setminus \mathbb{RZ}^d, \ \|e' - e\| \leq \delta\right\} = 0.$$

From this, it follows easily that $\overline{F}(p, X) = \underline{F}(p, X)$. □

Finally, we treat (vi). By analogy with similar results in homogenization and Aubry-Mather theory (cf. [47, Lemma 3.1] and [84, Theorem 3]), we expect that there is a modulus $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{\delta \to 0^+} \omega(\delta) = 0$ such that, for each $e \in S^{d-1} \cap \mathbb{RZ}^d$, the following estimate holds:

$$\sup\left\{\frac{\overline{F}(e, X) - \underline{F}_*(e, X)}{1 + \|X\|} \ \mid \ X \in \mathcal{S}_d\right\} \leq \omega(r_e). \tag{6.27}$$

When $d = 2$, it is not hard to show that there is a constant $A > 0$ such that

$$\sup\left\{|a_e^{\perp}(y) - a_e^{\perp}(\hat{y})| + |m_e^{\perp}(y) - m_e^{\perp}(\hat{y})| \ \mid \ y, \hat{y} \in \mathbb{T}^d\right\} \leq A r_e.$$

If such an estimate were to hold in higher dimensions (possibly with $A r_e$ replaced by $\omega(r_e)$), then it would imply (6.27). However, this remains to be seen.

Instead, we employ a soft argument pointed out by I.C. Kim:

Lemma 14. *The pair $(\overline{F}, \underline{F})$ satisfies (vi) with $\{\nu_n\}_{n \in \mathbb{N}}$ any enumeration of $S^{d-1} \cap \mathbb{RZ}^d$.*

Proof. We claim that

$$\lim_{N \to \infty} \sup\left\{\text{diam}(\mathscr{I}^a_{e_n}) \ \mid \ n \geq N\right\} = 0.$$

Here $\text{diam}(\mathscr{I}^a_e)$ is the diameter of $\mathscr{I}^a_e$ with respect to D, the metric on $\mathscr{P}(\mathbb{T}^d)$ chosen just prior to Theorem 36. Notice that, in view of Theorem 36 and the definition of $\overline{F}$ and $\underline{F}$ through $\bar{a}$ and $\overline{m}$, the claim implies (vi) holds.

To prove it, we argue by contradiction, exploiting the compactness of S^{d-1}. If (vi) fails, then we can find $\zeta > 0$, $(e_n)_{n\in\mathbb{N}} \subseteq S^{d-1}$, and sequences $(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\inf\left\{D(\mu_{e_n}^{s_n}, \mu_{e_n}^{t_n}) \mid n \in \mathbb{N}\right\} \geq \zeta. \tag{6.28}$$

(Recall that D is some fixed metric on $\mathscr{P}(\mathbb{T}^d)$, as in Theorem 36.) To see this is impossible, note that, up to extraction, we can assume that there is an $e \in S^{d-1}$ such that $\lim_{n\to\infty} e_n = e$.

If $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then, as in the previous proof, any accumulation point of $(\mu_{e_n}^{s_n})_{n\in\mathbb{N}}$ or $(\mu_{e_n}^{t_n})_{n\in\mathbb{N}}$ is in $\mathscr{I}_e^a$, hence must equal $\bar{\mu}_e$. In particular, $\mu_{e_n}^{s_n} \overset{*}{\rightharpoonup} \bar{\mu}_e$ and $\mu_{e_n}^{t_n} \overset{*}{\rightharpoonup} \bar{\mu}_e$, contradicting (6.28).

On the other hand, if $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$, then, passing to another subsequence if necessary, we can assume that there is an $\eta \in S^{d-1} \cap \langle e \rangle^\perp$ such that $\frac{e_n - e}{\|e_n - e\|} \to -\eta$ as $n \to \infty$. By Theorem 36, this implies $\mu_{e_n}^{s_n} \overset{*}{\rightharpoonup} \tilde{\mu}_e^\eta$ and $\mu_{e_n}^{t_n} \overset{*}{\rightharpoonup} \tilde{\mu}_e^\eta$, another contradiction. □

Finally, Theorem 34 is proved.

Proof of Theorem 34. We begin with the uniqueness of a solution of the effective equation

$$\overline{m}(\widehat{D\bar{u}})\bar{u}_t - \operatorname{tr}(\bar{A}(\widehat{D\bar{u}})D^2\bar{u}) = 0.$$

As explained above, we use $\overline{F}$ and $\underline{F}$ to study this problem since the coefficients may be discontinuous. By Proposition 72 and Lemma 14, these operators satisfy the assumptions of Theorem 38. Therefore, if a solution exists, it is unique.

Next, recall from Proposition 66 and Theorem 30 that the half-relaxed limits $\bar{u}^*$ and $\bar{u}_*$ satisfy $\bar{u}^*_t - \overline{F}(D\bar{u}^*, D^2\bar{u}^*) \leq 0$ in $\mathbb{R}^d \times (0, \infty)$ and $\bar{u}_{*,t} - \underline{F}(D\bar{u}_*, D^2\bar{u}_*) \geq 0$ in $\mathbb{R}^d \times (0, \infty)$ and, in addition, $\bar{u}^*(\cdot, 0) \leq u_0 \leq \bar{u}_*(\cdot, 0)$ in $\mathbb{R}^d$. Therefore, Theorem 38 implies that $\bar{u}^* \leq \bar{u}_*$ in $\mathbb{R}^d \times (0, \infty)$. At the same time, the definitions of the half-relaxed limits already guarantee that $\bar{u}_* \leq \bar{u}^*$. Hence if we define $\bar{u} : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ by $\bar{u} = \bar{u}_* = \bar{u}^*$, then $\bar{u}$ is a continuous function and it satisfies both $\bar{u}_t - \overline{F}(D\bar{u}, D^2\bar{u}) \leq 0$ and $\bar{u}_t - \underline{F}(D\bar{u}, D^2\bar{u}) \geq 0$. This proves the existence of a viscosity solution of (6.4).

Finally, an exercise shows that the equality $\bar{u}_* = \bar{u}^*$ implies that $u^\epsilon \to \bar{u}$ locally uniformly in $\mathbb{R}^d \times [0, \infty)$ as $\epsilon \to 0^+$. $\square$

6.4 Derivatives of Front Speeds in Dimension Two

This section treats the proof of Theorem 9, i.e., we prove that in dimension $d = 2$, the linear response coefficient $\overline{m}_{\text{pl}}$ is (the inverse of) the derivative of the front speed with respect to the applied forced. Precisely, in the level set formulation, we start with the hyperbolically scaled PDE

$$\begin{cases} m(\delta^{-1}x, \widehat{Du^\delta})u^\delta_t - \delta \text{tr}\left(A(\delta^{-1}x, \widehat{Du^\delta})D^2u^\delta\right) - \alpha\|Du^\delta\| = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u^\delta = u_0 & \text{on } \mathbb{R}^2 \times \{0\}. \end{cases} \tag{6.29}$$

Caffarelli and Monneau [26] proved that this problem homogenizes in the limit $\delta \to 0$ to a first-order, geometric PDE of the form

$$\begin{cases} \bar{u}_t + \lambda_\alpha(\widehat{D\bar{u}})\|D\bar{u}\| = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \bar{u} = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases}$$

Here $\lambda_\alpha : S^{d-1} \to \mathbb{R}$ encodes the average speed of planar solutions of the interface motion associated with (6.29). Below we prove that the derivative of this function is $\overline{m}_{\text{pl}}(e)^{-1}$:

$$\lim_{\alpha \to 0} \alpha^{-1}\lambda_\alpha(e) = \overline{m}_{\text{pl}}(e)^{-1}.$$

In the proof, we use the fact that there is a pulsating wave solution of (6.29). This is constructed using a viscosity solution $\mathcal{V}_{e,\alpha} \in C(\mathbb{T}^d)$ of the equation

$$\lambda_e(\alpha) m\left(y, \frac{e+D\mathcal{V}_{e,\alpha}}{\|e+D\mathcal{V}_{e,\alpha}\|}\right) - \operatorname{tr}\left(A\left(y, \frac{e+D\mathcal{V}_{e,\alpha}}{\|e+D\mathcal{V}_{e,\alpha}\|}\right) D^2\mathcal{V}_{e,\alpha}\right) - \alpha\|e+D\mathcal{V}_{e,\alpha}\| = 0 \quad \text{in } \mathbb{T}^d. \tag{6.30}$$

Since $d = 2$, the existence of $\mathcal{V}_{e,\alpha}$ follows from [26].

Proof of Theorem 9. As in Section 2.5, the proof is neater depending on whether or not (2.2) has a smooth solution. We only give the arguments in the case when $e \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ without assuming that (2.2) has a smooth solution.

Fix $\beta \in (1, \infty)$. Let $v^\epsilon(x,t) = \langle x, e\rangle + \beta\overline{m}(e)^{-1}(\epsilon^2 V_e^{\delta(\epsilon)}(\epsilon^{-1}x) + t)$ for $\delta(\epsilon) = \epsilon^{\frac{1}{4}}$, where V_e^δ is the solution of (2.13) with $f = -m(\cdot, e)$. Arguing as in the proof of Theorem 10, we see that there is an $\epsilon_0 \in (0,1)$ such that, for each $\epsilon \in (0, \epsilon_0)$, v^ϵ is a super-solution of the first equation in (2.1) with $\alpha = 1$.

Let $\mathcal{V}_{e,\epsilon}$ be a solution of (6.30) with $\alpha = \epsilon$ and set $u^\epsilon(y,t) = \langle y, e\rangle + \mathcal{V}_{e,\epsilon}(y) + \lambda_\epsilon(e)t$. Notice that this is a viscosity solution of (6.29) with $\delta = 1$ and $\alpha = \epsilon$. Hence $(x,t) \mapsto \epsilon u^\epsilon(\epsilon^{-1}x, \epsilon^{-2}t)$ is a viscosity solution of the first equation in (2.1) with $\alpha = 1$. Since $\mathcal{V}_{e,\epsilon}$ is bounded,

$$\epsilon u^\epsilon(\epsilon^{-1}x, 0) \leq v^\epsilon(x, 0) + \epsilon\|\mathcal{V}_{e,\epsilon}\|_{L^\infty(\mathbb{T}^d)} + C\beta\epsilon^2\|V_e^{\delta(\epsilon)}\|_{L^\infty(\mathbb{T}^d)}$$

and, thus, the comparison principle implies that, for each $(x,t) \in \mathbb{R}^d \times (0, \infty)$,

$$\begin{aligned}
\langle x, e\rangle + \epsilon\mathcal{V}_{e,\epsilon}(\epsilon^{-1}x) + \epsilon^{-1}\lambda_\epsilon(e)t &= \epsilon u^\epsilon(\epsilon^{-1}x, \epsilon^{-2}t) \\
&\leq v^\epsilon(x,t) + \epsilon\|\mathcal{V}_{e,\epsilon}\|_{L^\infty(\mathbb{T}^d)} + C\beta\epsilon^2\|V_e^{\delta(\epsilon)}\|_{L^\infty(\mathbb{T}^d)} \\
&= \langle x, e\rangle + \beta\overline{m}(e)^{-1}\epsilon^2 V_e^{\delta(\epsilon)}(\epsilon^{-1}x) + \epsilon\|\mathcal{V}_{e,\epsilon}\|_{L^\infty(\mathbb{T}^d)} \\
&\quad + C\beta\epsilon^2\|V_e^{\delta(\epsilon)}\|_{L^\infty(\mathbb{T}^d)} + \beta\overline{m}(e)^{-1}t.
\end{aligned}$$

Dividing by t and sending $t \to \infty$, we deduce that $\epsilon^{-1}\lambda_\epsilon(e) \leq \beta\overline{m}(e)^{-1}$ for all $\epsilon > 0$ small enough. Finally, sending first $\epsilon \to 0^+$ and then $\beta \to 1^+$, we conclude

$$\limsup_{\epsilon \to 0^+} \epsilon^{-1} \lambda_\epsilon(e) \leq \overline{m}(e)^{-1}.$$

Arguing using sub-solutions instead of super-solutions, we similarly deduce that

$$\liminf_{\epsilon \to 0^+} \epsilon^{-1} \lambda_\epsilon(e) \geq \overline{m}(e)^{-1}.$$

It remains to show that $\alpha^{-1}\lambda_\alpha(e) \to \overline{m}(e)^{-1}$ as $\alpha \to 0^-$. To do this, we repeat the previous proof, simply replacing $\alpha = 1$ in (2.1) by $\alpha = -1$. □

Remark 11. *When e is rational or e is irrational and* (2.2) *has a smooth solution, it is possible to prove the following rate: $|\epsilon^{-1}\lambda_\epsilon(e) - \overline{m}(e)^{-1}| \leq C_e \epsilon$ for some $C_e > 0$.*

6.5 Notes

The formal expansion used in this chapter and Section 2.5 was inspired by similar ones developed by Barles and Souganidis [16] and Barles, Cesaroni, and Novaga [13].

As we saw in this chapter and in Chapter 2, the homogenization of nonvariational interface motions in the parabolic scaling limit turns out to be related to the homogenization theory for uniformly elliptic equations in nondivergence form. The main reference for this in the fully nonlinear setting is by Caffarelli, Souganidis, and Wang [29]. We have also relied on the representation of the effective coefficients in terms of invariant measures in the linear case, which is well-known in periodic media (cf. [18]) and follows from the work of Papanicolaou and Varadhan [79] in stationary ergodic (e.g., almost periodic) media.

A major difficulty that arises in the interfacial setting is the discontinuity of the effective coefficients. As we have seen, this is ultimately related to the rational/irrational dichotomy, which results in a loss of ergodicity and hence non-uniqueness of invariant measures. The same issue appears in other periodic homogenization problems involving averaging in codimension one.

This is particularly true in the the homogenization of boundary value problems, such as

$$\begin{cases} -F(\epsilon^{-1}x, D^2u^\epsilon) = 0 & \text{in } \Omega, \\ u^\epsilon = f(\epsilon^{-1}x) & \text{on } \partial\Omega. \end{cases} \tag{6.31}$$

If F is uniformly elliptic and periodic in the spatial variable and if $f \in C(\mathbb{T}^d)$, then Feldman [48] showed it is possible to prove homogenization for nice-enough domains Ω. The effective equation has the form

$$\begin{cases} -\overline{F}(D^2\bar{u}) = 0 & \text{in } \Omega, \\ \bar{u} = \bar{f}(n_{\partial\Omega}) & \text{on } \partial\Omega, \end{cases}$$

for some effective nonlinearity $\overline{F}$ and an effective boundary condition $\bar{f} = \bar{f}(n)$, which is a function of the normal vector $n_{\partial\Omega}$ to the boundary. $\bar{f}$ will be continuous in irrational directions, but discontinuous at rational ones in general.

In the determination of the directional limits of the invariant measures (and hence also the effective coefficients), we used a strategy introduced by Feldman and Kim [49] in the context of (6.31). For further results on oscillating Dirichlet and Neumann problems, we refer the reader to the work of Barles, Da Lio, Lions, and Souganidis [11], Barles and Mironescu [15], Gérard-Varet and Masmoudi [54], Choi and Kim [32], Feldman and Zhang [52], and Feldman, Kim, and Souganidis [50].

In Section 6.4, we showed that the linear response coefficient $\overline{m}_{\text{pl}}$ is the inverse of the derivative of the average front speed λ_α at $\alpha = 0$. Even though the interface motion here is in nondivergence form, nonetheless this is completely consistent with the situation in the analysis of certain variational interface motions, specifically the Allen-Cahn equation (cf. [87, Section 4]) and the gradient flow of the Lebowitz-Penrose functional in statistical mechanics (cf. [24, 62]).

CHAPTER 7

LEVEL-SET PDE WITH DISCONTINUOUS COEFFICIENTS

7.1 Solutions in "Irrational" Directions

Recall that in Chapter 5 we were interested in functions that are only known to satisfy partial differential inequalities when the gradient points in an irrational direction. Recall that we made this precise in Section 5.1 in the more general context where the differential inequalities are determined by two nonlinearities $\overline{F}, \underline{F} : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ and the "bad" directions form a sequence $\{\nu_n\}_{n \in \mathbb{N}} \subseteq S^{d-1}$, which in the applications of interest to us is precisely $S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. The goal of this section is to prove Theorem 30, which shows that sub- and super-solutions in "good" directions are the same as sub- and super-solutions in the viscosity sense.

To prove the theorem, it will be convenient to use the following notion of viscosity sub- and super-solutions, which is specific to level set PDE and was introduced by Barles and Georgelin [14].

Definition 12. *Given an open set $U \subseteq \mathbb{R}^d \times (0, \infty)$, a locally bounded, upper semi-continuous function $v : U \to \mathbb{R}$ satisfies $v_t - \overline{F}(Dv, D^2 v) \leq 0$ in U if, given any smooth function $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ and any point $(x_0, t_0) \in U$ at which the difference $v - \varphi$ has a strict local maximum, the following conditions are met:*

(a) If $D\varphi(x_0, t_0) \neq 0$, then

$$\varphi_t(x_0, t_0) - \overline{F}(D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leq 0.$$

(b) If $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$, then

$$\varphi_t(x_0, t_0) \leq 0.$$

Similarly, a locally bounded, lower semi-continuous function $w : U \to \mathbb{R}$ satisfies $w_t - \underline{F}(Dw, D^2 w) \geq 0$ in the viscosity sense in U if $v = -w$ satisfies $v_t + \underline{F}(-Dv, -D^2 v) \leq 0$ in the viscosity sense in U.

With Definition 12 in mind, formally, the reason Theorem 30 is true is if the solution in question were smooth, then it would be clear. Indeed, we only need to check points where $Du(x_0, t_0) \neq 0$ since otherwise Definition 12, (b) and Definition 10, (iii) are in agreement. If $Du(x_0, t_0) \in \{\nu_n\}_{n \in \mathbb{N}}$, then either $Du(x, t) \notin \{\nu_n\}_{n \in \mathbb{N}}$ for some (x, t) arbitrarily close to (x_0, t_0), in which case the necessary differential inequality follows by continuity, or $Du(x, t) = Du(x_0, t_0)$ in a neighborhood of (x_0, t_0). In the latter case, differentiation shows

$$\left(\mathrm{Id} - \widehat{Du}(x_0, t_0) \otimes \widehat{Du}(x_0, t_0)\right) D^2 u(x_0, t_0) = 0$$

and now we are in the purview of Definition 10, (ii), which, in view of the assumption that $\underline{F}(\cdot, 0) \equiv \overline{F}(\cdot, 0) \equiv 0$, is consistent with Definition 12.

The sub- and super-solutions we work with in the proof are discontinuous, being indicator functions of open sets, and, thus, far from smooth. To circumvent this, we show that the sketch above is correct when u is semi-convex or semi-concave and then use sup- and inf-convolutions to pass to the general case.

In order to make the previous sketch rigorous in the case of a semi-convex/semi-concave function, we will invoke properties of the derivatives of such functions.

7.1.1 Preliminaries

First, we recall that a semi-convex/semi-concave function has a derivative in BV_{loc}.

Lemma 15. *If $\Omega \subseteq \mathbb{R}^d$ is a bounded open set and $u : \Omega \to \mathbb{R}$ is semi-convex or semi-concave, then $Du \in BV_{loc}(\Omega; \mathbb{R}^d)$ and the absolutely continuous part of the derivative of Du coincides with $D^2 u$ $\mathcal{L}^d$-almost everywhere in Ω.*

Next, we show that the differentiation step in the sketch can be made rigorous even in the semi-convex/semi-concave case. In the lemma below, we have in mind that V is the derivative of a semi-convex/semi-concave function.

Proposition 73. *Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded open set and $V \in BV_{loc}(\Omega; \mathbb{R}^m)$ for some $m \in \mathbb{N}$. Let $D^{ac}V \in L^1_{loc}(\Omega; \mathbb{R}^{d\times m})$ denote the Radon-Nikodym derivative of the Radon measure DV with respect to $\mathcal{L}^d$. Given any $v \in \mathbb{R}^d$ and $e \in S^{d-1}$,*

$$D^{ac}V = 0 \quad \mathcal{L}^d\text{-a.e. in } \{V = v\}$$

and

$$\left(Id - \widehat{V} \otimes \widehat{V}\right) D^{ac}V = 0 \quad \mathcal{L}^d\text{-a.e. in } \{\widehat{V} = e\}.$$

Proof. Let $\mathcal{D}_V$ denote the set of approximate differentiability points of V, that is, $x \in \mathcal{D}_V$ if and only if there is a linear map $A_x : \mathbb{R}^d \to \mathbb{R}^m$ such that

$$\lim_{r\to 0^+} r^{-d} \int_{B(x,r)} \frac{\|V(y) - V(x) - A_x(y - x)\|}{r}\, dy = 0.$$

Since $V \in BV_{\text{loc}}(\Omega; \mathbb{R}^m)$, it follows that $\mathcal{L}^d(\Omega \setminus \mathcal{D}_V) = 0$ and $A_x = D^{\text{ac}}V(x)$ for a.e. $x \in \Omega$ (see [4, Theorem 3.83]).

A straightforward computation shows $D^{\text{ac}}V = 0$ a.e. in $\{V = v\}$ (see also [4, Proposition 3.73]).

Define $f : \mathbb{R}^d \to \mathbb{R}^d$ by $f(p) = \widehat{p}$ if $p \neq 0$ and $f(0) = 0$. It is not hard to see that each $x \in \mathcal{D}_V$ with $A_x \neq 0$ is an approximate differentiability point of $f(V)$. Furthermore, a straightforward computation shows its approximate derivative is given by $\|V\|^{-1} \left(\text{Id} - \widehat{V} \otimes \widehat{V}\right) D^{\text{ac}}V$ a.e. (Both statements can be found in [4, Proposition 3.71]). As in the case of $\{V = v\}$, it is not hard to see that the approximate derivative of $f(V)$ has to vanish a.e. in $\{f(V) = e\} = \{\widehat{V} = e\}$. □

7.1.2 *Proof of Theorem 30*

Now we are prepared for the

Proof of Theorem 30. The theorem has two halves, one for sub-solutions and the other, for super-solutions. We will only prove the half concerning super-solutions since the other half follows by very similar arguments.

Let $U \subseteq \mathbb{R}^d \times (0, \infty)$ be an open set and suppose that $w : U \to \mathbb{R}$ satisfies $w_t - \underline{F}(Dw, D^2w) \geq 0$ in "good" directions in U. We need to prove that $w_t - \underline{F}(Dw, D^2w) \geq 0$ in the viscosity sense in U.

Let V be a bounded, open subset of U such that $\overline{V}$ is compactly contained in U. For convenience, pick a $T > 0$ such that $V \subseteq \mathbb{R}^d \times (0, T)$.

For each $\eta > 0$, define the inf-convolution w_η of u by

$$w_\eta(x, t) = \inf \left\{ w(y, s) + \frac{\|y - x\|^2}{2\eta} + \frac{(t - s)^2}{2\eta} \mid (y, s) \in U \cap (\mathbb{R}^d \times (0, T)) \right\}.$$

A classical argument shows that w_η is a semi-concave function in $U \cap (\mathbb{R}^d \times (0, T))$.

Another well-known argument shows that there is an $\eta_0 > 0$ such that if $0 < \eta < \eta_0$, then, for each $(x, t) \in V$, there is a $(y, s) \in U \cap (\mathbb{R}^d \times (0, T))$ such that

$$w_\eta(x, t) = w(y, s) + \frac{\|y - x\|^2}{2\eta} + \frac{(t - s)^2}{2\eta}.$$

From this, we can argue as in [34] to show that w_η satisfies $(w_\eta)_t - \underline{F}(Dw_\eta, D^2w_\eta) \geq 0$ in "good" directions in V. In fact, we claim that, for each $\eta \in (0, \eta_0)$,

$$(w_\eta)_t - \underline{F}(Dw_\eta, D^2w_\eta) \geq 0 \text{ in the viscosity sense in } V. \tag{7.1}$$

Here we remind the reader that we will work with the (equivalent) definition of viscosity sub-solution provided by Definition 12.

Henceforth, fix $\eta \in (0, \eta_0)$ and let us proceed to the proof of (7.1). Assume that φ is smooth and $w_\eta - \varphi$ has a strict local minimum at $(x_0, t_0) \in V$. If $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$ or $\widehat{D\varphi}(x_0) \notin \{\nu_n\}_{n\in\mathbb{N}}$, then there is nothing to show since w_η is a supersolution in "good" directions in V. Thus, it remains to consider the case when $D\varphi(x_0) \neq 0$ and $\widehat{D\varphi}(x_0) \in \{\nu_n\}_{n\in\mathbb{N}}$.

In what follows (subtracting a constant from φ if necessary), let $s > 0$ be such that $B((x_0, t_0), s) \subseteq V$, $w_\eta(x, t) > \varphi(x, t)$ for $x \in B((x_0, t_0), s) \setminus \{(x_0, t_0)\}$, and $w_\eta(x_0, t_0) = \varphi(x_0, t_0)$.

Since w_η is semi-concave and φ is smooth, it follows that $w_\eta - \varphi$ is semi-concave in $B((x_0, t_0), s)$. Thus, for each $\delta > 0$ small enough, we can apply Jensen's Lemma [34, Lemma A.3], thereby obtaining a set $K_\delta \subseteq B((x_0, t_0), s)$ such that $\mathcal{L}^{d+1}(K_\delta) > 0$ and, for each $(x, t) \in K_\delta$,

(i) w_η is twice punctually differentiable at (x, t).

(ii) There is an $(a_{(x,t)}, p_{(x,t)}) \in B(0, \delta)$ such that the function $(y, r) \mapsto w_\eta(y, r) - \varphi(y, r) - \langle p_{(x,t)}, y\rangle - a_{(x,t)} r$ has a local minimum in $B((x_0, t_0), s)$.

We claim that we can find an $(x_1, t_1) \in K_\delta$ such that

$$- \delta \leq \varphi_t(x_1, t_1) - \inf \left\{ \underline{F}(p, D^2\varphi(x_1, t_1)) \mid \|p - D\varphi(x_1, t_1)\| \leq \delta \right\} \tag{7.2}$$

Making δ and s smaller if necessary, we can assume that

$$Dw_\eta(x_1, t_1) = D\varphi(x_1, t_1) + p_{(x_1,t_1)} \neq 0 \quad \text{for all} \quad (x_1, t_1) \in K_\delta.$$

We will prove (7.2) by studying the structure of the spatial derivatives of w_η in K_δ. We only need to consider the following two cases:

(a) For $\mathcal{L}^d$-almost every $(x, t) \in K_\delta$, $\widehat{Dw_\eta}(x, t) \in \{\nu_n\}_{n\in\mathbb{N}}$.

(b) There is a measurable $A_\delta \subseteq K_\delta$ such that $\widehat{Du_\eta}(x,t) \in S^{d-1} \setminus \{\nu_n\}_{n\in\mathbb{N}}$ for $\mathcal{L}^d$-almost every $(x,t) \in A_\delta$ and $\mathcal{L}^{d+1}(A_\delta) > 0$.

The easier case is (b). If (b) holds, then we can fix an $(x_1,t_1) \in A_\delta$ and invoke the super-solution property of w_η at (x_1,t_1) to find

$$0 \leq (w_\eta)_t(x_1,t_1) - \underline{F}(Dw_\eta(x_1,t_1), D^2 w_\eta(x_1,t_1)).$$

Now we recall that, by (ii), the following relations hold:

$$\begin{aligned} &Dw_\eta(x_1,t_1) = D\varphi(x_1,t_1) + p_{(x_1,t_1)}, \quad D^2 w_\eta(x_1,t_1) \geq D^2\varphi(x_1,t_1), \\ &w_{\eta,t}(x_1,t_1) = \varphi_t(x_1,t_1) + a_{(x_1,t_1)}. \end{aligned}$$

Since $\underline{F}$ is elliptic, this gives (7.2).

Next, we turn to case (a). Given $t \in (0,T)$, let $U_t = \{x \in \mathbb{R}^d \mid (x,t) \in U\}$. Recall from Proposition 73 that the map $Dw_\eta(\cdot,t) \in BV_{\text{loc}}(U_t;\mathbb{R}^d)$ for each fixed t and $D^{\text{ac}}(Dw_\eta(\cdot,t)) = D^2 w_\eta(\cdot,t)$ a.e. Let us define $\{B_n\}_{n\in\mathbb{N}}$ by

$$B_n = \left\{(x,t) \in K_\delta \mid \frac{Dw_\eta(x,t)}{\|Dw_\eta(x,t)\|} = \nu_n\right\}.$$

Since we assumed (a) holds, it follows that $\sum_n \mathcal{L}^{d+1}(B_n) > 0$.

An immediate application of Lemma 15, Proposition 73, and Fubini's Theorem shows that

$$\left(\text{Id} - \widehat{Dw_\eta} \otimes \widehat{Dw_\eta}\right) D^2 w_\eta = 0 \quad \text{a.e. in} \bigcup_{n\in\mathbb{N}} B_n.$$

Thus, we can fix a point $(x_1, t_1) \in \bigcup_e B_e$ such that

$$\left(\mathrm{Id} - \widehat{Dw_\eta}(x_1, t_1) \otimes \widehat{Dw_\eta}(x_1, t_1)\right) D^2 w_\eta(x_1, t_1) = 0.$$

Since $\bigcup_{n\in\mathbb{N}} B_n \subseteq K_\delta$ and w_η satisfies $(w_\eta)_t - \underline{F}(Dw_\eta, D^2 w_\eta) \geq 0$ in "good" directions in V, we have

$$0 \leq w_{\eta,t}(x_1, t_1) \leq \varphi_t(x_1, t_1) + \delta - \underline{F}(D\varphi(x_1, t_1), D^2\varphi(x_1, t_1)).$$

This implies (7.2).

We conclude that, no matter which of cases (a) or (b) occur, there is an $(x_1, t_1) \in K_\delta$ such that (7.2) holds. Next, by recalling that (x_0, t_0) is a strict local minimum of $w_\eta - \varphi$ in $B((x_0, t_0), s)$ and $K_\delta \subseteq B((x_0, t_0), s)$, a straightforward argument shows that $(x_1, t_1) \to (x_0, t_0)$ as $\delta \to 0^+$. Thus, sending $\delta \to 0^+$ in (7.2) and recalling that $\underline{F}$ is lower semi-continuous, we obtain the desired inequality:

$$0 \leq \varphi_t(x_0, t_0) - \underline{F}(D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)).$$

Since φ was arbitrary, we proved that (7.1) holds as long as $\eta \in (0, \eta_0)$. At the same time, we know that, for each $(x, t) \in V$,

$$w(x, t) = \lim_{\delta \to 0^+} \inf \left\{ w_\eta(y, s) \mid \|x - y\| + |t - s| + \eta \leq \delta \right\}.$$

Thus, the stability properties of viscosity solutions (see [34, Section 6]) imply that $w_t - \underline{F}(Dw, D^2 w) \geq 0$ in the viscosity sense in V. Since V was arbitrary, we conclude that $w_t - \underline{F}(Dw, D^2 w) \geq 0$ holds in the viscosity sense in U. □

7.2 Comparison Principle for Level-Set PDE with Discontinuities

In addition to the encountering functions that only satisfied the effective equation in irrational directions, in Chapter 6, we also derive effective equations that were discontinuous at rational directions. This leads to the question whether or not solutions of the corresponding equations are unique.

As in the last section, here we consider the general case where $\overline{F}, \underline{F} : \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ are two elliptic operators and $\{\nu_n\}_{n \in \mathbb{N}} \subseteq S^{d-1}$ is a sequence of "bad" directions. The idea we have in mind is that there is some operator F, which is only partially defined in $\mathbb{R}^d \times \mathcal{S}_d$ and for which we would like to study the equation

$$u_t - F(Du, D^2 u) = 0. \tag{7.3}$$

$\overline{F}$ and $\underline{F}$ will capture, in some sense, the worst u can be to qualify as a solution of this equation. More precisely, we will recast the equation (7.3) as two differential inequalities

$$u_t - \overline{F}(Du, D^2 u) \leq 0 \quad \text{and} \quad u_t - \underline{F}(Du, D^2 u) \geq 0 \quad \text{in the viscosity sense.} \tag{7.4}$$

We will show below these two inequalities are tantamount to a single equation like (7.3) provided $\overline{F}$ and $\underline{F}$ satisfy some assumptions. The main assumption is that they coincide in "good" directions — hence, in effect, we are asking that the partially-defined operator F in (7.3) be continuous and well-behaved in those directions.

Here are the precise assumptions on $\overline{F}$ and $\underline{F}$:

(i) (Geometric) If $G \in \{\overline{F}, \underline{F}\}$, $(p, X) \in \mathbb{R}^d \times \mathcal{S}_d$, $\mu \in \mathbb{R}$, and $\kappa > 0$, then

$$G(\kappa p, \kappa X + \mu p \otimes p) = \kappa G(p, X).$$

(ii) (Strongly degenerate elliptic) There are constants $\lambda, \Lambda > 0$ such that if $G \in \{\overline{F}, \underline{F}\}$,

$p \in \mathbb{R}^d \setminus \{0\}$, $X, Y \in \mathcal{S}_d$, and $Y \geq 0$, then

$$\lambda \|\tilde{Y}_{\hat{p}}\| \leq G(p, X+Y) - G(p, X) \leq \Lambda \|\tilde{Y}_{\hat{p}}\|.$$

(iii) (Stationary planes) $\overline{F}(e, 0) = \underline{F}(e, 0) = 0$ for each $e \in S^{d-1}$.

(iv) (Semi-continuity) $\overline{F}$ is upper semi-continuous, $\underline{F}$ is lower semi-continuous, and they are upper and lower semi-continuous envelopes of each other in the following sense:

$$\overline{F}(p, X) = \lim_{\delta \to 0^+} \sup \left\{ \underline{F}(p', X') \mid \|p' - p\| + \|X' - X\| < \delta \right\},$$
$$\underline{F}(p, X) = \lim_{\delta \to 0^+} \inf \left\{ \overline{F}(p', X') \mid \|p' - p\| + \|X' - X\| < \delta \right\}.$$

(v) (Continuity at "good" directions) If $(p, X) \in (\mathbb{R}^d \setminus \{0\}) \times \mathcal{S}_d$ and $\hat{p} \notin \{e_n\}_{n \in \mathbb{N}}$, then

$$\overline{F}(p, X) = \underline{F}(p, X).$$

(vi) (Controlled oscillation) The discontinuities of $\overline{F}$ and $\underline{F}$ can be controlled in the following manner:

$$\lim_{N \to \infty} \sup \left\{ \frac{\overline{F}(e_n, X) - \underline{F}(e_n, X)}{1 + \|X\|} \mid X \in \mathcal{S}_d,\ n \geq N \right\} = 0.$$

Notice that the effective equations encountered in Chapter 6 satisfy these assumptions by Proposition 72 and Lemma 14.

The remainder of the section is devoted to the proof of a comparison principle for (7.4), stated next:

Theorem 38. *Assume that $u : \mathbb{R}^d \times (0, T) \to \mathbb{R}$ is a locally bounded, upper semi-continuous function such that $u_t - \overline{F}(Du, D^2 u) \leq 0$ in $\mathbb{R}^d \times (0, T)$ and and $v : \mathbb{R}^d \times (0, T) \to \mathbb{R}$ is a locally* *bounded, lower semi-continuous function such that* $v_t - \underline{F}(Du, D^2u) \geq 0$ *in* $\mathbb{R}^d \times (0, \infty)$. *If (i)-(vi) all hold and* u *and* v *satisfy the following condition*

$$\lim_{\delta \to 0^+} \sup \{u^*(x,0) - v_*(y,0) \mid \|x - y\| < \delta\} \leq 0, \tag{7.5}$$

then $u \leq v$ *in* $\mathbb{R}^d \times (0, T)$.

The idea of the proof is this: assumption (v) implies that, for each $\beta > 0$, $\overline{F}$ and $\underline{F}$ *almost* coincide (up to a β error) except at finitely many rational directions. The papers of Gurtin, Soner, and Souganidis [58], Ohnuma and Sato [76], and Ishii [59] show how to prove a comparison principle in the case when $\overline{F}$ and $\underline{F}$ coincide at all but finitely many directions. Therefore, if we can manage the β error, a comparison principle should hold in our setting as well.

7.2.1 Proof of Theorem 38

As in [59], the proof of Theorem 38 proceeds by replacing the Euclidean norm by some other Finsler norm in a variable-doubling argument. (Recall that $\psi : \mathbb{R}^d \to [0, \infty)$ is a Finsler norm if it is convex, positively one-homogeneous, and positive away from zero.) To improve the result of [59] from finitely many discontinuity points to our setting, we use the following fact:

Proposition 74. *There is a universal constant* $c_0 > 0$ *such that if* $\{e_n\}_{n \in \mathbb{N}} \subseteq S^{d-1}$, *then, for each* $N \in \mathbb{N}$, *there is a Finsler norm* $\psi_N \in C^2(\mathbb{R}^d \setminus \{0\})$ *such that*

$$1 \leq \psi_N(e) \leq \frac{5}{4}, \quad D^2\psi_N(e) \leq c_0(Id - e \otimes e) \quad \text{for each } e \in S^{d-1}$$

and the following property holds: given $p \in \mathbb{R}^d \setminus \{0\}$, *if* $\widehat{D\psi_N}(p) = e_i$ *for some* $i \in \{1, 2, \ldots, N\}$, *then* $D^2\psi_N(p) = 0$.

The proof of Proposition 74 can be found in [70]. The idea is simple: start with the ball $B(0,1)$ and deform it at the points $\{e_1, \dots, e_N\}$ so that it is flat there. The result is a convex set containing zero, and its Minkowski gauge is the desired Finsler norm ψ_N.

With Proposition 74 in hand, the proof of Theorem 38 is achieved by mimicking that in [59] while controlling the errors that arise.

Proof of Theorem 38. By (i), if $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function, then $\varphi(u)$ remains a sub-solution and $\varphi(v)$, a super-solution. Set $u_M = \varphi_M(u)$ and $v_M = \varphi_M(v)$ with $\varphi_M(w) = \max\{\min\{w, M\}, -M\}$. Notice that the pair (u_M, v_M) satisfies (7.5) for each $M > 0$. Therefore, if the theorem holds when u and v are bounded, then $u_M \leq v_M$ in $\mathbb{R}^d \times (0,T)$ independently of M, and in the limit $M \to \infty$, we deduce that $u \leq v$ in the limit $M \to \infty$. Hence there is no loss of generality in assuming that u and v are both bounded.

We argue by contradiction, assuming that the following inequality holds:

$$\sup \left\{ u(x,t) - v(x,t) \ \mid \ (x,t) \in \mathbb{R}^d \times (0,T] \right\} > 0.$$

It follows that we can fix $\sigma > 0$ small enough that

$$\sup \left\{ u(x,t) - v(x,t) - \sigma t \ \mid \ (x,t) \in \mathbb{R}^d \times (0,T] \right\} > 0. \tag{7.6}$$

Let $\zeta, \beta, \gamma > 0$ be free variables. By (v), we can fix an $N = N(\gamma) \in \mathbb{N}$ such that

$$\sup \left\{ \overline{F}(e_i, X) - \underline{F}(e_i, X) \ \mid \ i \in \mathbb{N} \setminus \{1, 2, \dots, N\} \right\} \leq \gamma(1 + \|X\|) \quad \text{if } X \in \mathcal{S}_d.$$

Letting ψ_N be the Finsler norm of Proposition 74 with $\{\nu_n\}_{n \in \mathbb{N}}$ the set of "bad" directions associated with the pair $(\overline{F}, \underline{F})$, define $\Phi = \Phi_{\zeta,\beta} : \mathbb{R}^d \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ by

$$\Phi(x,y,t) = u(x,t) - v(y,t) - \frac{\psi_N(x-y)^4}{4\zeta} - \frac{\beta}{2}\|y\|^2 - \sigma t.$$

Since u and v are bounded, Φ is bounded above and attains its maximum in $\mathbb{R}^d \times \mathbb{R}^d \times [0,T]$. Let $(\bar{x}, \bar{y}, \bar{t}) = (\bar{x}_{\zeta,\beta}, \bar{y}_{\zeta,\beta}, \bar{t}_{\zeta,\beta})$ be such a global maximum, that is,

$$\Phi(\bar{x}, \bar{y}, \bar{t}) = \max \left\{ \Phi(x,y,t) \mid (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0,T] \right\}.$$

By the boundedness of u and v and the lower bound $\psi_N \geq \| \cdot \|$, there is a γ-independent constant $C > 0$ such that

$$\sup \left\{ \frac{\beta \|\bar{y}_{\zeta,\beta}\|^2}{2} + \frac{\|\bar{x}_{\zeta,\beta} - \bar{y}_{\zeta,\beta}\|^4}{\zeta} \mid (\zeta, \beta) \in (0,\infty) \times (0,\infty) \right\} \leq C. \tag{7.7}$$

In view of (7.6) and the assumptions on u and v, there are constants $\zeta_0, \beta_0 > 0$ such that $\bar{t}_{\zeta,\beta} > 0$ for all $(\zeta, \beta) \in (0, \zeta_0) \times (0, \beta_0)$ and all $\gamma \in (0,1)$. Henceforth let $(\zeta, \beta) \in (0, \zeta_0) \times (0, \beta_0)$ and assume $\gamma < 1$.

Since $\bar{t} > 0$, we can invoke [59, Lemma 1] (a variant of the maximum principle for semi-continuous functions) and the equations satisfied by u and v. This gives matrices $X, Y \in \mathcal{S}_d$ and numbers $a, b \in \mathbb{R}$ so that if $\bar{A} = \bar{A}(\bar{x} - \bar{y}) \in \mathcal{S}_d$ and $\bar{p} = \bar{p}(\bar{x} - \bar{y}) \in \mathbb{R}^d$ are defined by

$$\begin{aligned}
\bar{p}(\bar{x} - \bar{y}) &= \zeta^{-1} \psi_N(\bar{x} - \bar{y})^3 D\psi_N(\bar{x} - \bar{y}), \\
\bar{A}(\bar{x} - \bar{y}) &= \zeta^{-1} \psi_N(\bar{x} - \bar{y})^3 D^2\psi_N(\bar{x} - \bar{y}) + 3\zeta^{-1} \psi_N(\bar{x} - \bar{y})^2 D\psi_N(\bar{x} - \bar{y})^{\otimes 2},
\end{aligned}$$

(with $\bar{A}(0) = 0$), then

$$\sigma = a - b, \quad -3 \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -(Y + \beta \mathrm{Id}) \end{pmatrix} \leq 3 \begin{pmatrix} \bar{A} & -\bar{A} \\ -\bar{A} & \bar{A} \end{pmatrix},$$
$$a - \overline{F}(\bar{p}, X) \leq 0, \quad b - \underline{F}(\bar{p} - \beta \bar{y}, Y) \geq 0.$$

Note, in addition, that Proposition 74 and (7.7) yield the following β-independent estimates on $\|\bar{p}\|$ and $\|\bar{A}\|$:

$$\|\bar{p}\| \leq \frac{5}{4}C^{\frac{3}{4}}\zeta^{-\frac{1}{4}}, \quad \|\bar{A}\| \leq \sqrt{C}\zeta^{-\frac{1}{2}}. \tag{7.8}$$

Hence, we can send $\beta \to 0^+$ and invoke (7.7) to obtain $\xi \in \mathbb{R}^d$, $\tilde{p} = \bar{p}(\xi) \in \mathbb{R}^d$, $\tilde{A} = \bar{A}(\xi) \in \mathcal{S}_d$, and $\tilde{X}, \tilde{Y} \in \mathcal{S}_d$ such that

$$\sigma + \underline{F}(\tilde{p}, \tilde{Y}) - \overline{F}(\tilde{p}, \tilde{X}) \leq 0, \quad -3\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \leq \begin{pmatrix} \tilde{X} & 0 \\ 0 & -\tilde{Y} \end{pmatrix} \leq 3\begin{pmatrix} \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{pmatrix}.$$

There are four cases left to consider: (i) $\xi = 0$, (ii) $\widehat{D\psi}_N(\xi) \in \{e_1, \dots, e_N\}$, (iii) $\widehat{D\psi}_N(\xi) \in \{e_{N+1}, e_{N+2}, \dots\}$, and (iv) $\widehat{D\psi}_N(\xi) \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$.

Case (i): $\xi = 0$

In this case, we have $\tilde{p} = 0$ and $\tilde{A} = 0$, hence $\tilde{X} = 0$ and $\tilde{Y} = 0$. This yields the estimate

$$\sigma \leq \sigma + \underline{F}(0, 0) - \overline{F}(0, 0) \leq 0. \tag{7.9}$$

Case (ii): $\widehat{D\psi}_N(\xi) \in \{e_1, e_2, \dots, e_N\}$

In this case, Proposition 74 implies that $D^2\psi_N(\xi) = 0$. Thus, $\tilde{A} = ce_i \otimes e_i$ for some $c > 0$ and $\|\tilde{p}\|^{-1}\tilde{p} = e_i$ so (i) and (vi) give

$$\overline{F}(\tilde{p}, X) \leq \overline{F}(\tilde{p}, 3ce_i \otimes e_i) = 0, \quad \underline{F}(\tilde{p}, \tilde{Y}) \geq \underline{F}(\tilde{p}, -3ce_i \otimes e_i) = 0.$$

Combining these estimates, we obtain

$$\sigma \leq \sigma + \underline{F}(\tilde{p}, \tilde{Y}) - \overline{F}(\tilde{p}, \tilde{X}) \leq 0. \tag{7.10}$$

Case (iii): $\widehat{D\psi}_N(\xi) \in \{e_{N+1}, e_{N+2}, \dots\}$

By the choice of N,

$$\underline{F}(\tilde{p}, \tilde{Y}) \geq \overline{F}(\tilde{p}, \tilde{Y}) - \gamma\|\tilde{Y}\| - \gamma\|\tilde{p}\| \geq \overline{F}(\tilde{p}, \tilde{Y}) - \gamma\|\tilde{p}\| - 3\gamma\|\tilde{A}\|.$$

From this and the inequality $\tilde{X} \leq \tilde{Y}$, we find

$$\sigma - \gamma\|\tilde{p}\| - 3\gamma\|\tilde{A}\| \leq \sigma + \underline{F}(\tilde{p}, \tilde{Y}) - \overline{F}(\tilde{p}, \tilde{X}) \leq 0. \tag{7.11}$$

Case (iv): $\widehat{D\psi}_N(\xi) \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$

Here $\overline{F}(\tilde{p}, \tilde{X}) = \underline{F}(\tilde{p}, \tilde{X})$ and, thus,

$$\underline{F}(\tilde{p}, \tilde{Y}) - \overline{F}(\tilde{p}, \tilde{X}) = \underline{F}(\tilde{p}, \tilde{Y}) - \underline{F}(\tilde{p}, \tilde{X}) \geq 0.$$

This gives our last estimate:

$$\sigma \leq \sigma + \underline{F}(\tilde{p}, \tilde{Y}) - \overline{F}(\tilde{p}, \tilde{X}) \leq 0. \tag{7.12}$$

Combining (7.9), (7.10), (7.11), and (7.12), we conclude that

$$\sigma \leq (\|\tilde{p}\| + 3\|\tilde{A}\|)\gamma. \tag{7.13}$$

However, in view of (7.8), this is a contradiction as soon as γ is small enough compared to ζ and σ. □

7.3 Notes

The proof of the comparison principle for level-set PDE with countably many discontinuities (Theorem 38) was inspired by the approach of Ishii in [59]. See also Gurtin, Soner, and Souganidis [58] and Ohnuma and Sato [76] for related results on level-set PDE.

APPENDIX A

TECHNICAL LEMMATA

A.1 Arithmetic Properties of Rational Hyperplanes

Proof of Theorem 11. We begin by showing that (ii) and (iii) are equivalent. Let $G = \{\langle k, e\rangle \mid k \in \mathbb{Z}^d\}$. Since $\{\langle e_i, e\rangle \mid 1 \leq i \leq d\}$ is a generating set, G is a finitely generated subgroup of $\mathbb{R}$. Thus, it has a basis $\{a_\alpha\}_{\alpha\in\mathcal{A}}$. For each $\alpha \in \mathcal{A}$, fix a $k_\alpha \in \mathbb{Z}^d$ such that $\langle k_\alpha, e\rangle = a_\alpha$. It is easy to see that $\{k_\alpha\}_{\alpha\in\mathcal{A}}$ is independent over $\mathbb{Z}$, hence the index set $\mathcal{A}$ is finite.

Let $\langle k_\alpha \mid \alpha \in \mathcal{A}\rangle$ denote the subgroup of $\mathbb{Z}^d$ generated by $\{k_\alpha\}_{\alpha\in\mathcal{A}}$. By construction, we have a direct sum of Abelian groups $\mathbb{Z}^d = M_e \oplus \langle k_\alpha \mid \alpha \in \mathcal{A}\rangle$. In particular, the ranks of M_e and G sum to d, and so M_e has rank $d-1$ if and only if G has rank one. Hence (ii) and (iii) are equivalent.

Next, we claim that (ii) implies (i). Suppose that (ii) holds. As in the previous paragraph, we can fix $\hat{k} \in \mathbb{Z}^d$ with $\langle \hat{k}, e\rangle \neq 0$ and a basis $\{k_1, \ldots, k_{d-1}\}$ of M_e. To make $\hat{k}$ orthogonal to M_e, we define $\underline{k}$ by

$$\underline{k} = \left(\prod_{i=1}^{d-1} \|k_i\|^2\right) \cdot \left(\hat{k} - \sum_{i=1}^{d-1} \frac{\langle \hat{k}, k_i\rangle}{\|k_i\|^2} k_i\right)$$

Notice that $\underline{k} \in \mathbb{Z}^d \cap M_e^\perp$. Since $\{k_1, \ldots, k_{d-1}\}$ is linearly independent over $\mathbb{R}$ (being linearly independent over $\mathbb{Z}$), it is easy to see that $M_e^\perp$ is a linear subspace of $\mathbb{R}$ of dimension one. Hence $M_e^\perp = \langle e\rangle$, and then it necessarily follows that $e = \rho\|\underline{k}\|^{-1}\underline{k}$ for some $\rho \in \{-1, 1\}$.

Finally, we show that (i) implies (ii), that is, if $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$, then M_e has rank $d-1$. To see this, fix $k \in \mathbb{Z}^d \setminus \{0\}$ such that $e = \|k\|^{-1}k$. Since $k \neq 0$, we can fix $j \in \{1, 2, \ldots, d\}$ such that $\langle k, e_j\rangle \neq 0$. This implies, in particular, that the set $\{k, e_1, \ldots, e_d\} \setminus \{e_j\}$ is linearly independent.

Given $i \in \{1, 2, \dots, d\} \setminus \{j\}$, define a_i by

$$a_i = \|k\|^2 e_i - \langle e_i, k \rangle k.$$

Notice that $\{a_i\}_{i \neq j}$ is an independent subset of M_e with $d-1$ elements. It follows that M_e has rank at least $d-1$. At the same time, since $M_e \subseteq \mathbb{Z}^d \cap \langle e \rangle^\perp$ and $\mathbb{Q}^d \cap \langle e \rangle^\perp$ has dimension $d-1$ over $\mathbb{Q}$, we know M_e has rank at most $d-1$. Therefore, its rank is exactly $d-1$. □

A.2 Harmonic Distance Functions

Lemma 16. *Suppose that $\theta : \mathbb{T}^d \to (0, \infty)$ is a continuous function, $e \in S^{d-1}$, $\zeta \in \mathbb{R}$, and $h : \mathbb{R}^d \to \mathbb{R}$ is the distance to the affine hyperplane $\Sigma_\nu^\zeta := \{y \in \mathbb{R}^d \mid y \cdot \nu = \zeta\}$ with respect to the Riemannian metric induced by $\sqrt{\theta}$, that is, the unique viscosity solution of the Eikonal equation*

$$\begin{cases} \|Dh\| - \sqrt{\theta(y)} = 0 & \text{in } \{y \in \mathbb{R}^d \mid \langle y, e \rangle > \zeta\}, \\ h = 0 & \text{on } \Sigma_\nu^\zeta. \end{cases} \tag{A.1}$$

If $-\Delta h = 0$ in $\{y \in \mathbb{R}^d \mid \langle y, e \rangle > \zeta\}$, then h is a linear function and θ is constant.

Proof. Standard arguments imply that there is a unique constant $\theta^*(e) > 0$ and a continuous function $v : \mathbb{T}^d \to \mathbb{R}$ that is a viscosity solution of the cell problem

$$\|e + Dv\|^2 - \frac{\theta(y)}{\theta^*(e)} = 0 \quad \text{in } \mathbb{T}^d.$$

Define $u : \mathbb{R}^d \to \mathbb{R}$ by

$$u(y) = \theta^*(\nu)^{1/2} \left(\langle y, e \rangle + v(y) \right).$$

This is a viscosity solution of the eikonal equation in the whole space

$$\|Du\| - \sqrt{\theta(y)} = 0 \quad \text{in } \mathbb{R}^d.$$

Notice that since v is periodic, we have

$$\sup\left\{|u(y)-\theta^*(e)^{1/2}\langle y,e\rangle| \mid y\in\mathbb{R}^d\right\}=\theta^*(e)^{1/2}\|v\|_{L^\infty(\mathbb{T}^d)}<\infty.$$

It follows that we can find a constant $C>0$ such that

$$u(y)-C\leq 0\leq u(y)+C \quad \text{if } y\in\Sigma_\nu^\zeta.$$

Thus, by the comparison principle for the Eikonal equation (A.1),

$$u(y)-C\leq h(y)\leq u(y)+C \quad \text{if } \langle y,e\rangle\geq 0.$$

From the previous considerations, we deduce that the function $w:\{y\in\mathbb{R}^d \mid \langle y,e\rangle\geq 0\}\to\mathbb{R}$ given by $w(y)=h(y)-\theta^*(e)^{1/2}\langle y,e\rangle$ is a bounded harmonic function in $\mathbb{R}^d$ that vanishes on Σ_ν^ζ. By Liouville's Theorem, $w\equiv 0$. That is, $h(y)=\theta^*(e)^{1/2}\langle y,e\rangle$ for each $y\in\mathbb{R}^d$ satisfying $\langle y,e\rangle\geq 0$.

We are left to conclude that $\theta(y)=\|Dh\|^2=\theta^*(e)$ in $\{y\in\mathbb{R}^d \mid \langle y,e\rangle\geq 0\}$. Since θ is periodic, this implies $\theta\equiv\theta^*(e)$. □

A.3 Equidistribution of Codimension-One Sub-tori

We will be interested in certain probability measures supported on the sub-tori defined by (2.8). Toward that end, the result that follows is fundamental.

Recall that if μ is a finite measure, we write $⨍_A(\cdot)\,\mu(dy)=\frac{1}{\mu(A)}\int_A(\cdot)\,\mu(dy)$.

Lemma 17. *If $(e_n)_{n\in\mathbb{N}}\subseteq S^{d-1}\cap\mathbb{R}\mathbb{Z}^d$ is any infinite sequence and $(s_n)_{n\in\mathbb{N}}\subseteq[0,\infty)$ satisfies $s_n\in[0,r_{e_n})$ for each $n\in\mathbb{N}$, then*

$$\mathcal{H}^{d-1}(\mathbb{T}^{d-1}_{e_n}(s_n))^{-1}\mathcal{H}^{d-1}\restriction_{\mathbb{T}^{d-1}_e(s_n)}\overset{*}{\rightharpoonup}\mathcal{L}^d \tag{A.2}$$

Furthermore, if $(f_n)_{n\in\mathbb{N}}$ *are* $\mathcal{H}^{d-1}$*-measurable functions in* $\mathbb{T}^d$*,* $p \in [1,\infty]$*, and* $C > 0$ *is such that*

$$\left(\fint_{\mathbb{T}^{d-1}_{e_n}(s_n)} |f_n(\xi)|^p \, \mathcal{H}^{d-1}(d\xi) \right)^{\frac{1}{p}} \leq C$$

and if we define measures $(\mu_n)_{n\in\mathbb{N}}$ *on* $\mathbb{T}^d$ *by*

$$\int_{\mathbb{T}^d} g(y) \, \mu_n(dy) = \fint_{\mathbb{T}^{d-1}_{e_n}(s_n)} g(\xi) f_n(\xi) \, \mathcal{H}^{d-1}(d\xi),$$

then there is a subsequence $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ *and a measure* $\tilde{\mu}$ *such that* $\mu_{n_j} \overset{*}{\rightharpoonup} \tilde{\mu}$ *as* $j \to \infty$*,* $\tilde{\mu} \ll \mathcal{L}^d$*, and*

$$\left\| \frac{d\tilde{\mu}}{d\mathcal{L}^d} \right\|_{L^p(\mathbb{T}^d)} \leq C. \tag{A.3}$$

Proof. First, we prove that the normalized surface measures converge to $\mathcal{L}^d$. Assume that $g \in C(\mathbb{T}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d} |\hat{g}(k)| < \infty$. An exercise in Fourier analysis shows that if $k \in \mathbb{Z}^d$, then

$$\fint_{\mathbb{T}^{d-1}_{e_n}(s_n)} e^{i2\pi\langle k,\xi\rangle} \mathcal{H}^{d-1}(d\xi) = \begin{cases} e^{i2\pi\langle k,e_n\rangle s_n}, & \text{if } k \in \langle e_n\rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \left| \fint_{\mathbb{T}^{d-1}_{e_n}(s_n)} g(\xi) \, \mathcal{H}^{d-1}(d\xi) - \int_{\mathbb{T}^d} g(y) \, dy \right| &= \left| \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{g}(k) \fint_{\mathbb{T}^{d-1}_{e_n}(s_n)} e^{i2\pi\langle k,\xi\rangle} \mathcal{H}^{d-1}(d\xi) \right| \\ &\leq \sum_{k\in\mathbb{Z}^d\cap\langle e_n\rangle\setminus\{0\}} |\hat{g}(k)|. \end{aligned}$$

Since $(e_n)_{n\in\mathbb{N}}$ is infinite, it follows that for each $R > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\mathbb{Z}^d \cap \langle e_n\rangle \setminus \{0\} \subseteq \mathbb{R}^d \setminus B(0,R).$$

Thus,

$$\limsup_{n\to\infty}\left|⨍_{\mathbb{T}^{d-1}_{en}(s_n)} g(\xi)\,\mathcal{H}^{d-1}(d\xi) - \int_{\mathbb{T}^d} g(y)\,dy\right| \leq \lim_{R\to\infty}\sum_{k\in\mathbb{Z}^d\setminus B(0,R)} |\hat{g}(k)| = 0.$$

Recalling that functions with summable Fourier coefficients are dense in $C(\mathbb{T}^d)$, we conclude that (A.2) holds as claimed.

Next, we prove the claim concerning $(\mu_n)_{n\in\mathbb{N}}$. First, observe that Hölder's inequality implies

$$\|\mu_n\|(\mathbb{T}^d) = ⨍_{\mathbb{T}^{d-1}_{en}(s_n)} |f_n(\xi)|\,\mathcal{H}^{d-1}(d\xi) \leq C \quad \text{if } n \in \mathbb{N}.$$

Thus, $(\mu_n)_{n\in\mathbb{N}}$ is pre-compact in $C(\mathbb{T}^d)^*$, which gives the desired sub-sequence $(n_j)_{j\in\mathbb{N}}$ and limit point $\tilde{\mu}$.

Note that if $g \in C(\mathbb{T}^d)$ and $q \in [1,\infty]$ is the conjugate exponent of p (i.e., the q so that $p^{-1} + q^{-1} = 1$), then

$$\left|⨍_{\mathbb{T}^{d-1}_{en}(s_n)} g(\xi) f_n(\xi)\,\mathcal{H}^{d-1}(d\xi)\right| \leq C\left(⨍_{\mathbb{T}^{d-1}_{en}(s_n)} |g(\xi)|^q\,\mathcal{H}^{d-1}(d\xi)\right)^{\frac{1}{q}}.$$

Assume first that $q < \infty$. Since $|g|^q \in C(\mathbb{T}^d)$, we have

$$\left|\int_{\mathbb{T}^d} g(y)\,\tilde{\mu}(dy)\right| \leq C \lim_{n\to\infty}\left(⨍_{\mathbb{T}^{d-1}_{en}(s_n)} |g(\xi)|^q\,\mathcal{H}^{d-1}(d\xi)\right)^{\frac{1}{q}} = C\|g\|_{L^q(\mathbb{T}^d)}.$$

This proves $\tilde{\mu} \ll \mathcal{L}^d$ and (A.3) holds when $q < \infty$. When $q = \infty$, a similar argument applies. □

A.4 Construction of Invariant Measures

In this section, we construct invariant measures for the differential operator $-\mathrm{tr}(a(y)D_e^2)$ when $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. Recall that $\mathscr{I}_e^a$ denotes the family of all such measures.

Intuitively, it is clear this can be done by "slicing" the operator using the decomposition $\mathbb{T}^d = \cup_{r\in[0,r_e)} \mathbb{T}_e^{d-1}(r)$. In each sub-torus $\mathbb{T}_e^{d-1}(r)$, $-\mathrm{tr}(a(y)D_e^2)$ acts as a uniformly elliptic operator, hence it has a unique invariant measure.

Nonetheless, for some e, $\mathbb{T}_e^{d-1}(r)$ can have a very different geometry than $\mathbb{T}^{d-1}$ (e.g., when $d = 2$, it could be a rhombus with very exaggerated angles). Therefore, in this section, we give a detailed proof that the invariant measures have uniform L^{d-1} estimates in spite of the geometry.

Proposition 75. *For each $e \in S^{d-1}$ and $r \in \mathbb{R}$, there is a unique probability measure $\mu_e^r \in \mathscr{I}_e^a$ that is supported in $\mathbb{T}_e^{d-1}(r)$. Further, there is a $\mathcal{H}^{d-1}$-integrable function $h_e^r : \mathbb{T}^d \to [0,\infty)$ such that*

$$\int_{\mathbb{T}^d} f(y)\,\mu_e^r(dy) = \fint_{\mathbb{T}_e^{d-1}(r)} f(y')h_e^r(y')\,\mathcal{H}^{d-1}(dy') \quad \text{for } f \in C(\mathbb{T}^d),$$

and we have an L^{d-1} estimate

$$\fint_{\mathbb{T}_e^{d-1}(r)} |h_e^r(y')|^{d-1}\,\mathcal{H}^{d-1}(dy') \leq C(d-1, \lambda^{-1}\Lambda)$$

for some $C(d-1,\lambda^{-1}\Lambda) > 0$.

To overcome geometric difficulties, we will exploit averaging. Accordingly, let the integer vectors $\{k_1, \ldots, k_{d-1}\}$ be a $\mathbb{Z}$-basis of M_e, as in Section 2.4. We claim that, for any $\alpha \in (0,1)$, we can fix a set $\{\tilde{k}_1(\alpha), \ldots, \tilde{k}_{d-1}(\alpha)\}$ such that

$$\langle \tilde{k}_i(\alpha), \tilde{k}_j(\alpha)\rangle = 1, \quad \alpha \leq \frac{\|\tilde{k}_i(\alpha)\|}{\|\tilde{k}_j(\alpha)\|} \leq \alpha^{-1} \quad \text{for each} \quad i,j \in \{1,2,\ldots,d-1\}. \tag{A.4}$$

Indeed, applying the Gram-Schmidt process to $\{k_1, \ldots, k_{d-1}\}$, we can construct a new set $\{\tilde{k}_1, \ldots, \tilde{k}_{d-1}\} \subseteq M_e$ such that $\langle \tilde{k}_i, \tilde{k}_j \rangle = 0$ for $i \neq j$. When $d = 3$ (the $d = 2$ case being trivial), by choosing $m_1, m_2 \in \mathbb{N}$ appropriately and setting $\tilde{k}_1(\alpha) = m_1 \tilde{k}_1$ and $\tilde{k}_2(\alpha) = m_2 \tilde{k}_2$, we have

$$\alpha \leq \frac{\|\tilde{k}_1(\alpha)\|}{\|\tilde{k}_2(\alpha)\|} = \left(\frac{m_1}{m_2}\right) \frac{\|\tilde{k}_1\|}{\|\tilde{k}_2\|} \leq \alpha^{-1}.$$

This shows we can ensure (A.4) holds when $d \in \{2, 3\}$. An inductive argument shows that this can, in fact, be done for general $d \geq 2$.

Given $r \in \mathbb{R}$, let $L(r) = \{x \in \mathbb{R}^d \mid \langle x, e \rangle = r\}$, and define $\tilde{p}_{e,\alpha} : L(r) \to \mathbb{R}^{d-1}$ by

$$\tilde{p}_{e,\alpha}\left(re + \sum_{i=1}^{d-1} \beta_i \tilde{k}_i(\alpha)\right) = \sum_{i=1}^{d-1} \beta_i e_i.$$

Observe that $\tilde{p}_e$ is an affine map and its derivative $D\tilde{p}_{e,\alpha}(0)$ maps $\tilde{k}_i(\alpha)$ to e_i for each $i \in \{1, 2, \ldots, d-1\}$.

Henceforth we denote by $\tilde{M}_e$ the subgroup of M_e generated by $\{\tilde{k}_1(\alpha), \ldots, \tilde{k}_{d-1}(\alpha)\}$. In other words, $\tilde{M}_e$ is the subset of M_e determined by

$$\tilde{M}_e = \left\{\sum_{i=1}^{d-1} m_i \tilde{k}_i(\alpha) \mid (m_1, \ldots, m_{d-1}) \in \mathbb{Z}^{d-1}\right\}$$

Finally, let the torus $\tilde{T}_e^{d-1}(r)$ be the quotient space $\tilde{T}_e^{d-1}(r) = L(r)/\tilde{M}_e$. Observe that

$$x - y \in M_e \quad \iff \quad \tilde{p}_{e,\alpha}(x) - \tilde{p}_{e,\alpha}(y) = \tilde{p}_{e,\alpha}(x - y) \in \mathbb{Z}^{d-1}.$$

Thus, $\tilde{p}_e$ induces a diffeomorphism between $\tilde{T}_e^{d-1}(r)$ and $\mathbb{T}^{d-1}$.

With these preliminaries out of the way, we are ready to construct invariant measures in rational directions.

Proof of Proposition 75. We begin by proving existence, then uniqueness.

Step 1: Existence

Fix $\alpha \in (0,1)$ and let $\tilde{p}_{e,\alpha}$ and $\tilde{T}_e^{d-1}(r)$ be as constructed above. If $g \in C^\infty(\tilde{T}_e^{d-1}(r))$, then

$$-\text{tr}(a(y')D^2 g(y')) = -\text{tr}(\underline{a}(\tilde{p}_e(y'))D^2 \underline{g}(\tilde{p}_e(y'))),$$

where $\underline{a} : \mathbb{T}^{d-1} \to \mathcal{S}_d$ and $\underline{g} \in C^\infty(\mathbb{T}^{d-1})$ are given by

$$\underline{a}(\xi) = D\tilde{p}_{e,\alpha}(0)a(\tilde{p}_{e,\alpha}^{-1}(\xi))D\tilde{p}_{e,\alpha}(0)^*, \quad \underline{g}(\xi) = g(\tilde{p}_{e,\alpha}^{-1}(\xi)).$$

By construction of $\tilde{p}_{e,\alpha}$, we know that $\underline{a}$ satisfies

$$\lambda\alpha^2\|\zeta\|^2 \leq \langle \underline{a}(\xi)\zeta, \zeta\rangle \leq \Lambda\alpha^{-2}\|\zeta\|^2 \quad \text{for each } \zeta \in \mathbb{R}^{d-1}.$$

By classical results on linear, uniformly elliptic PDE with periodic coefficients (see [18, Chapter 3]), there is a unique measure $\underline{\mu} \in \mathscr{P}(\mathbb{T}^{d-1})$ such that

$$\int_{\mathbb{T}^{d-1}} \text{tr}\left(\underline{a}(\xi)D^2\underline{g}(\xi)\right) \underline{\mu}(d\xi) = 0 \quad \text{if } \underline{g} \in C^\infty(\mathbb{T}^{d-1}).$$

Furthermore, $\underline{\mu} \ll \mathcal{L}^{d-1}$, the support of $\underline{\mu}$ equals $\mathbb{T}^{d-1}$, and

$$\int_{\mathbb{T}^{d-1}} \left|\frac{d\underline{\mu}}{d\mathcal{L}^{d-1}}(\xi)\right|^{d-1} \mathcal{L}^{d-1}(d\xi) \leq C(d-1, \alpha^{-4}\lambda^{-1}\Lambda).$$

For this last estimate, see [79] or [6].

Let us push $\underline{\mu}$ forward onto $\tilde{T}_e^{d-1}(r)$. Define $\mu_e^r \in \mathscr{P}(\tilde{T}_e^{d-1}(r))$ by

$$\mu_e^r(A) = \underline{\mu}(p_e(A)) \quad \text{for Borel } A \subseteq \mathbb{T}^{d-1}.$$

Given $g \in C^\infty(\tilde{T}_e^{d-1}(r))$, integration gives

$$\int_{\tilde{T}_e^{d-1}(r)} \operatorname{tr}\left(a(y')D^2 g(y')\right)\,\mu_e^r(dy') = \int_{\mathbb{T}^{d-1}} \operatorname{tr}\left(\underline{a}(\xi)D^2\underline{g}(\xi)\right)\,\underline{\mu}(d\xi) = 0.$$

We claim that μ_e^r descends to a probability measure in $\mathbb{T}_e^{d-1}(r) = L(r)/M_e$. Indeed, any $k \in M_e$ acts on $\tilde{T}_e^{d-1}(r)$ by translation $\tau_k(y') = y' + k$. If we replace a by $a(\cdot + k)$, then we get another measure $\mu' \in \mathscr{P}(\tilde{T}_e^{d-1}(r))$. By uniqueness of the invariant measure $\underline{\mu}$ constructed above, we must have $\mu' = \tau_{k\#}\mu_e^r$. At the same time, $a(\cdot + k) \equiv a$ so uniqueness implies that $\tau_{k\#}\mu_e^r = \mu_e^r$. It follows that μ_e^r can be regarded as a probability measure on $\mathbb{T}_e^{d-1}(r)$.

From the preceding considerations, we see that if $\psi \in C^\infty(\mathbb{T}^d)$, then, denoting by $g = \psi \restriction_{\mathbb{T}_e^{d-1}(r)}$ we deduce that

$$\begin{aligned}\int_{\mathbb{T}^d} \operatorname{tr}\left(a(y)D_e^2\psi(y)\right)\,\mu_e^r(dy) &= \int_{\mathbb{T}_e^{d-1}(r)} \operatorname{tr}\left(a(y')D^2 g(y')\right)\,\mu_e^r(dy') \\ &= \int_{\tilde{T}_e^{d-1}(r)} \operatorname{tr}\left(a(y')D^2 g(y')\right)\,\mu_e^r(dy') = 0.\end{aligned}$$

We conclude that $\mu_e^r \in \mathscr{I}_e^a$. Furthermore, the support of μ_e^r equals $\mathbb{T}_e^{d-1}(r)$ and $\mu_e^r \ll \mathcal{H}^{d-1} \restriction_{\mathbb{T}_e^{d-1}(r)}$.

If we define the Borel function $h_e^r : \mathbb{T}^d \to [0,\infty)$ so that we have

$$\int_{\mathbb{T}^d} f(y)\,\mu_e^r(dy) = \fint_{\mathbb{T}_e^{d-1}(r)} f(y')h_e^r(y')\,\mathcal{H}^{d-1}(dy') \quad \text{for } f \in C(\mathbb{T}^d),$$

then a change of variables shows that

$$\begin{aligned}\fint_{\mathbb{T}_e^{d-1}(r)} \left|h_e^r(y')\right|^{d-1}\,\mathcal{H}^{d-1}(dy') &= \fint_{\tilde{T}_e^{d-1}(r)} \left|h_e^r(y')\right|^{d-1}\,\mathcal{H}^{d-1}(dy') \\ &= \int_{\mathbb{T}^{d-1}} \left|\frac{d\underline{\mu}}{d\mathcal{L}^{d-1}}(\xi)\right|^{d-1}\,d\xi \leq C(d-1,\alpha^{-4}\lambda^{-1}\Lambda).\end{aligned}$$

Step 2: Uniqueness

If μ is a probability measure on $\mathbb{T}^d$ with support contained in $\mathbb{T}_e^{d-1}(r)$, then, for any $\alpha \in (0,1)$, μ can be regarded as a probability measure in $\tilde{T}_e^{d-1}(r)$. It follows from the uniqueness of the measure $\underline{\mu}$ that $(\tilde{p}_{e,\alpha})_{\#}\mu = \underline{\mu}$. Therefore, $\mu = \mu_e^r$. □

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