THE UNIVERSITY OF CHICAGO

COMPUTABILITY THEORY AND REVERSE MATHEMATICS: MAKING USE OF THE OVERLAPS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
SARAH REITZES

CHICAGO, ILLINOIS
AUGUST 2022

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

First, and foremost, I am greatly indebted to my advisor, Denis Hirschfeldt, for his support, insight, and guidance throughout my PhD. This dissertation would not be possible without him. His insights, both the mathematical ones and the more philosophical ones, have been essential to my research. Maryanthe Malliaris, my secondary advisor, has also been extremely helpful throughout this process. She has brought new perspectives to everything we have discussed. Thank you also to Denis and to Damir Dzhafarov for allowing me to include our joint work in this dissertation. I would also like to acknowledge that I was partially supported by a Focused Research Group grant from the National Science Foundation of the United States, DMS-1854279 (Chicago) and by DGE-1746045.

I am very grateful to all of the mathematicians I have talked to about this work over the years and who have given me feedback and direction, as well as those who have given me the opportunity to talk about this work, including but not limited to Wesley Calvert, Peter Cholak, Dzhafarov, Jun Le Goh, Carl Jockusch, Liling Ko, Steffen Lempp, Tyler Markkanen, Ludovic Patey, Arno Pauly, Gabriela Pinto, Isabella Scott, Reed Solomon, and Linda Westrick. I am very grateful for the undergraduate professors who inspired me to pursue a PhD in mathematics, most notably my undergraduate advisor, Todd Quinto, as well as Moon Duchin, Fulton Gonzalez, and Montserrat Teixidor. I also must thank Ben Hescott, who gave me my first exposure to computability theory, from the theoretical computer science perspective, as an undergraduate.

I would not have survived my PhD without the support of my parents, Jody and Jim Reitzes, as well as that of my brother, Andrew Reitzes. I have had many other friends and supporters throughout the past six years who I must thank, including Nicole, Charlotte, Leah, Cindy, Dolores, Nixia, Zack, Sarah F, Isabella, Erin, Grace, Kailee, Mariya, Bingjin, Shuo, Sarah, Jill, and Danielle. I am thankful to them for standing by my side throughout it all, and helping me to become the mathematician, as well as the person, who I am today.

ABSTRACT

In this dissertation we look at the overlaps between computability theory and reverse mathematics. We start off by examining the reverse mathematical strength of a variation of Hindman's Theorem (HT), constructed by essentially combining HT with the Thin Set Theorem (TS) to obtain a principle which we call thin-HT. thin-HT says that every coloring $c : \mathbb{N} \to \mathbb{N}$ has an infinite set $S \subseteq \mathbb{N}$ whose finite sums are thin for c, meaning that there is an i with $c(s) \neq i$ for all finite sums s of distinct elements from S.

Next, we look at the two-player game introduced by Hirschfeldt and Jockusch in 2016 in which winning strategies for one or the other player precisely correspond to implications and non-implications between Π^1_2 principles over ω-models of RCA_0. They also introduced a version of this game that similarly captures provability over RCA_0. We generalize and extend this game-theoretic framework to other formal systems, and establish a certain compactness result that shows that if an implication $\mathrm{Q} \to \mathrm{P}$ between two principles holds, then there exists a winning strategy that achieves victory in a number of moves bounded by a number independent of the specific run of the game. We also demonstrate how this framework leads to a new kind of analysis of the logical strength of mathematical problems that refines both that of reverse mathematics and that of computability-theoretic notions such as Weihrauch reducibility, allowing for a kind of fine-structural comparison between Π^1_2 principles that has both computability-theoretic and proof-theoretic aspects, and can help us distinguish between these, for example by showing that a certain use of a principle in a proof is "purely proof-theoretic", as opposed to relying on its computability-theoretic strength. We give examples of this analysis to a number of principles at the level of $\mathrm{B}\Sigma^0_2$, uncovering new differences between their logical strengths.

Finally, we further explore the notions of Weihrauch and generalized Weihrauch reducibility over subsystems of second-order arithmetic Γ that we have defined. We look specifically at $\Gamma = \mathrm{RCA}_0$ and $\Gamma = \mathrm{RCA}^*_0$. We focus on particular formulations of Σ^0_1-induction, Δ^0_2-

induction, and Σ^0_2-bounding as Π^1_2-principles, and how these principles relate to each other using this novel method of comparison. In particular, we prove an analogue of Slaman's 2004 result about the equivalence of Σ^0_n-bounding and Δ^0_n-induction in models of $\mathrm{PA}^- + \mathrm{I}\Delta^0_1 + \exp$, for $n = 2$, in our setting. We present specific results as well as a more general metatheorem.

CHAPTER 1

INTRODUCTION

This dissertation is divided into three chapters, not including this one. In the first chapter, we use computability theory and reverse mathematics in a more "typical" way, that is, to prove related theorems, but the actual usage of methods from each domain is separate. Then in the second chapter, we introduce a way to essentially combine computability-theoretic and reverse-mathematical techniques, via reduction games. Finally, in the last chapter, we show how to apply this new method of comparison to see the nuanced differences between problems, as well as to isolate what is purely computability theoretically true from what is proof theoretically true.

Computability theory gives us the language and foundations to compare problems that lie above what is computable, i.e. what is solvable by a computer in any finite amount of time. On the other hand, reverse mathematics gives us a way to compare the relative strength of theorems by establishing implications and nonimplications over a weak subsystem of second-order arithmetic, typically RCA_0, which corresponds roughly to computable mathematics. We will assume some familiarity with reverse mathematics and computability theory. Standard resources for each of these areas separately include [60] and [62], respectively, while [33] is a useful resource for both areas and how they interrelate.

In approaching problems in these fields, one often sees attempts to separate the computability theory from the reverse math. We will refrain from doing that here, because this work, particularly Chapters 3 and 4, came about precisely because these two fields are so intertwined. Instead, we will present an introduction to computability theory and reverse math together. We will then discuss where various versions of Ramsey's Theorem lie in this universe. Ramsey's Theorem can be seen as a classical example of how these fields interrelate. We will finish off our introductory section by discussing some other related principles, and give a taste of how our different methods of comparison compare to one another.

Chapter 2 is based on joint work with Denis R. Hirschfeldt in [36]. We look at a variant of Hindman's Theorem and the Thin Set Theorem, and show that this variant implies ACA_0 over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$ (see Chapters 2 and 3 for definitions of these terms). Chapter 3 is based on joint work with Damir D. Dzhafarov and Hirschfeldt in [25]. Here we introduce the notion of Weihrauch reduction over a subsystem of second-order arithmetic Γ, look at the merits of such a notion, and give examples of the type of analysis it allows for. Finally, in Chapter 4 we apply this notion to a set of principles, particularly different versions of bounding and induction.

1.1 Introduction to computability theory and reverse mathematics

In computability theory, we are interested in the world of the non-computable. By *computable*, we mean "effectively calculable by a human being," or equivalently, able to be computed by a *Turing machine*, which is essentially the most primitive of computers. For more on this, see [62]. Turing machines are given by finite sets of quintuples, which denote the steps the machine takes, the input being read, and the input being printed. Such a quintuple is called a *Turing program*, and a *partial computable function* is any function that can be computed by such a program. Note that each Turing program is a finite object, so there is an effective listing of all such programs. We typically denote this listing by $P_0, P_1, \ldots, P_e, \ldots$. There is thus a corresponding effective listing of all partial computable functions, which we denote by $\varphi_0, \varphi_1, \ldots, \varphi_e, \ldots$. Throughout this dissertation, we will use ω to denote the standard natural numbers.

The Enumeration Theorem is a classic result from computability theory that gives us the notion of a *universal Turing machine*.

Theorem 1.1.1 (The Enumeration Theorem)**.** *There is a partial computable function of two* *variables* $\psi(e,x)$ *such that* $\psi(e,x) = \varphi_e(x)$*. Therefore there is some* i *such that* $\varphi_i(e,x) = \psi(e,x)$*.*

φ_i for the i given by the theorem is called a universal Turing machine. This machine gives us a convenient way of talking about partial computable functions.

Definition 1.1.2. We say that a set A is *computably enumerable*, and abbreviate this as *c.e.*, if A is the domain of some partial computable function.

It follows that there is an effective listing of the c.e. sets, denoted by $W_0, W_1, \ldots, W_e, \ldots$. An *oracle Turing machine* is a Turing machine which has access to the characteristic function of some set A, called the *oracle*. Essentially, the oracle Turing machine is allowed to query whether any element is in A to help it solve problems. An *oracle Turing program* is the Turing program of an oracle Turing machine. We list the oracle Turing programs as $\tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_e, \ldots$. If $\tilde{P}_e$ is an oracle Turing program with oracle A and halts on input x with output y, we write $\Phi_e^A(x) = y$, and we call Φ_e^A a *Turing functional*.

Definition 1.1.3. We say that a partial function θ is *Turing computable in* A and write $\theta \leqslant_{\mathrm{T}} A$ if there exists an e such that

$$\Phi_e^A(y)\downarrow = y \iff \theta(x) = y. \tag{1.1}$$

We then say that a set B is *Turing reducible* to A and write $B \leqslant_{\mathrm{T}} A$ if the characteristic function of B, χ_B, is such that $\chi_B \leqslant_{\mathrm{T}} A$. More colloquially, we say that A *computes* B, or that B is A-computable.

We use $\emptyset'$ to denote the *Halting Problem*, that is, the non-computable problem of whether the partial computable function φ_x halts on input x. Note that the enumeration theorem *relativizes*, that is, it holds with Turing functionals instead of partial computable functions. We state this relativized version now for clarity.

Theorem 1.1.4 (Relativized Enumeration Theorem)**.** *There exists a $z \in \omega$ such that for any set $A \subseteq \omega$ and all $x, y \in \omega$, the function $\Phi_z^A(x, y)$ satisfies $\Phi_z^A(x, y) = \Phi_x^A(y)$.*

Definition 1.1.5. For a problem A, we call the relativization of the Halting Problem to A, that is, $\{x \mid \Phi_x^A(x)\downarrow\}$, the *jump* of A and denote it by A'.

Note that $\emptyset'$ is precisely the jump of $\emptyset$. We can take the jump n times for any finite $n \in \omega$. (In fact, we can do even more than this, but for the purposes of this dissertation, we need only concern ourselves with finite jumps.) We write $A^{(n)}$ for the n^{th} jump of A if $A \neq \emptyset$, or if $A = \emptyset$ and $n \geqslant 2$.

There is a very nice way to characterize sets based on how they compare to the various jumps of $\emptyset$. To state this characterization, we need some terminology. *Bounded quantifiers* are quantifiers of the form $\forall x < t$ or $\exists x < t$, where t is a term in which x does not appear. A Σ_n^0 *formula* is a formula of the form $\exists x_1 \forall x_2 \exists x_3 \forall x_4 \dots Q x_n \varphi$, where φ is a bounded quantifier formula, and Q is $\exists$ if n is odd and $\forall$ if n is even. A Π_n^0 *formula* is a formula of the form $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \dots Q x_n \varphi$, where φ is a bounded quantifier formula, and Q is $\forall$ if n is odd and $\exists$ if n is even. Note that φ is Σ_n^0 iff $\neg\varphi$ is Π_n^0 and vice versa. If φ is both Σ_n^0 and Π_n^0, then we say that φ is Δ_n^0. We say that a set is Σ_n^0, Π_n^0, or Δ_n^0 if it is definable by a Σ_n^0, Π_n^0, or Δ_n^0 formula, respectively. We say that a formula or set is *arithmetic* if it is Σ_n^0 or Π_n^0 for some n. As with our previous notions, the notions of being Σ_n^0, Π_n^0, and Δ_n^0 relativize.

Theorem 1.1.6 (Post's Theorem)**.** *A set A is Σ_{n+1}^0 iff A is c.e. in $\emptyset^{(n)}$. A set A is Δ_{n+1}^0 iff $A \leqslant_{\mathrm{T}} \emptyset^{(n)}$.*

Definition 1.1.7. We say that a set A is *limit-computable* if there is a computable sequence of sets $A_0, A_1, \dots, A_s, \dots$ such that for all x, $A(x) = \lim_s A_s(x)$.

Lemma 1.1.8 (Limit Lemma, Shoenfield 1959, see section 3.6 of [62])**.** *The following are equivalent statements about a set A:*

1. *A is limit-computable;*

2. A *is* Δ^0_2*;*

3. $A \leqslant_{\mathrm{T}} \emptyset'$*.*

We will make use of the equivalence (1) $\iff$ (2) of the Limit Lemma frequently in this dissertation, particularly in Chapter 4, when we leave the world of the standard natural numbers and must be much more careful in how we define what it means for a set to be Δ^0_2.

In reverse math, we work in subsystems of *second-order arithmetic.* Two of the major systems with which we are concerned are RCA_0, the usual weak base system for reverse mathematics, which corresponds roughly to computable mathematics; and ACA_0, which corresponds roughly to arithmetic mathematics (mathematics with arithmetic sets).

Second-order arithmetic is a two-sorted first order language with number variables (lowercase letters) and set variables (uppercase letters), equality, and the symbols $+, \times, \leqslant, 0, 1$, and $\in$. For more about languages and first-order arithmetic, see [15] and [27]. In second-order arithmetic, our atomic formulas are of the form $t = u$, $t \leqslant u$, and $t \in X$ for first-order terms t and u. Note that $X = Y$ is not a formula. However, we can express it by the formula $\forall n[n \in X \leftrightarrow n \in Y]$. We call a formula *arithmetic* if there is no quantification over set variables (although free set variables are permitted). Note that this is the same as our earlier notion of arithmetic, just adapted to the model-theoretic setting.

For principles P of the form $(\forall X)\,[\Phi(X) \to (\exists Y)\,\Psi(X, Y)]$ where Φ and Ψ are arithmetic, we call any X such that $\Phi(X)$ holds an *instance* of P, and any Y such that $\Psi(X, Y)$ holds a *solution* to X. We call such a principle a Π^1_2*-principle*, or a Π^1_2*-problem*. The term "problem" reflects a computability-theoretic view that sees such a sentence as a process of finding a suitable Y given X.

We mentioned that we typically work over the base theory RCA_0. Formally, RCA_0 consists of the first order axioms for a discrete ordered commutative semiring, together with the Δ^0_1-*comprehension scheme*

$$\forall n[\varphi(n) \leftrightarrow \psi(n)] \to \exists X \forall n[n \in X \leftrightarrow \varphi(n)]$$

for all Σ^0_1 formulas φ and Π^0_1 formulas ψ in which X is not free, and the Σ^0_1-*induction scheme*

$$(\varphi(0) \wedge \forall n[\varphi(n) \to \varphi(n+1)]) \to \forall n \varphi(n)$$

for all Σ^0_1 formulas φ. We will also have occasion to consider RCA_0^*, which is roughly RCA_0 where Σ^0_1-induction is weakened to Σ^0_0-induction. We will formally define RCA_0^* later when we have established the appropriate terminology.

We often refer to Σ^0_1-induction as $\mathrm{I}\Sigma^0_1$. Note that we can similarly define Σ^0_n-induction, or $\mathrm{I}\Sigma^0_n$, for all n. Likewise, we can define the principles $\mathrm{I}\Pi^0_n$ and $\mathrm{I}\Delta^0_n$ for all n. We will also frequently come across Σ^0_n-bounding and Π^0_n-bounding, or $\mathrm{B}\Sigma^0_n$ and $\mathrm{B}\Pi^0_n$, respectively. $\mathrm{B}\Sigma^0_n$ is expressed by the axiom scheme consisting of

$$\forall n[(\forall i < n)(\exists k)\varphi(i,k) \to (\exists b)(\forall i < n)(\exists k < b)\varphi(i,k)] \tag{1.2}$$

for each Σ^0_n formula φ. $\mathrm{B}\Pi^0_n$ is defined analogously. We have the following useful chain of implications given by Paris and Kirby; the proof that the implications are strict can be found in [30].

Theorem 1.1.9 (Paris and Kirby [49], Hájek and Pudlak [30])**.** *Over* RCA_0*, we have the following implications, all of which are strict:*

$$\mathrm{I}\Sigma^0_1 \leftarrow \mathrm{B}\Sigma^0_2 \leftarrow \mathrm{I}\Sigma^0_2 \leftarrow \mathrm{B}\Sigma^0_3 \leftarrow \mathrm{I}\Sigma^0_3 \leftarrow \ldots. \tag{1.3}$$

Of these principles, only $\mathrm{I}\Sigma^0_1$ *is provable in* RCA_0*.*

Note that Paris and Kirby actually proved that these implications hold in the first order context, but they remain true over RCA_0.

We often work in ACA_0 as well, which formally consists of the first order axioms for a discrete ordered commutative semiring, together with *arithmetic comprehension*, i.e. set comprehension in the same vein as Δ^0_1- comprehension, but for each arithmetic set. ACA_0 corresponds to the Turing jump existence problem

$$\forall X \exists Y [Y = X'].$$

A *model* or *structure* in the language of second-order arithmetic is of the form $\mathcal{N} = (N, S, +_N, \times_N, 0_N, 1_N, \leqslant_N)$ where $(N, +_N, \times_N, 0_N, 1_N, \leqslant_N)$ is a model of first-order arithmetic and S is a subset of the power set of N. The $\in$ symbol is always interpreted to mean set membership.

Definition 1.1.10 (Simpson [60])**.** Let $\mathrm{L}_2(\exp)$ be L_2, the language of second order arithmetic, augmented by a binary operation symbol $\exp(m, n) = m^n$ intended to denote exponentiation. We take $\exp(t_1, t_2) = t_1^{t_2}$ as a new kind of numerical term, and for each $k < \omega$ we define the Σ^0_k and Σ^1_k formulas of $\mathrm{L}_2(\exp)$ accordingly. We define RCA_0^* to be the $\mathrm{L}_2(\exp)$-theory consisting of RCA_0 minus Σ^0_1 induction plus Σ^0_0 induction plus the exponentiation axioms: $m^0 = 1, m^{n+1} = m^n \cdot m$.

We present the above definition in full for completeness, but for the purposes of this dissertation, it is not necessary to distinguish between L_2-theories and $\mathrm{L}_2(\exp)$-theories.

Definition 1.1.11. For a model $\mathcal{M}$ of second-order arithmetic, if its first-order part $\mathcal{N}$ is *standard*, i.e. equal to ω with the usual interpretations of $+$, $\times$, $\leqslant$, 0, and 1, then we say that $\mathcal{M}$ is an *ω-model* and identify it with its second-order part S.

Definition 1.1.12. For two problems P and Q, we say that P is *ω-reducible* to Q and write $\mathrm{P} \leqslant_\omega \mathrm{Q}$ if every ω-model of $\mathrm{RCA}_0 + \mathrm{Q}$ is a model of P.

Note that if $\mathrm{RCA}_0 + \mathrm{Q} \vdash \mathrm{P}$, then $\mathrm{P} \leqslant_\omega \mathrm{Q}$, but not necessarily vice versa. An ω-model satisfies RCA_0 iff it is a Turing ideal, that is, iff it is closed under Turing reducibility and finite joins. An ω-model satisfies ACA_0 iff it is a jump ideal, that is, iff it is closed under the Turing jump.

Definition 1.1.13. Let P and Q be Π^1_2-problems. We say that P is *computably reducible* to Q, and write $\mathrm{P} \leqslant_{\mathrm{c}} \mathrm{Q}$, if for every instance X of P, there is an X-computable instance $\hat{X}$ of Q such that, for every solution $\hat{Y}$ to $\hat{X}$, there is an $X \oplus \hat{Y}$-computable solution to X.

Definition 1.1.14. Let P and Q be Π^1_2-problems. We say that P is *Weihrauch reducible* to Q, and write $\mathrm{P} \leqslant_{\mathrm{W}} \mathrm{Q}$, if there are Turing functionals Φ and Ψ such that, for every instance X of P, the set $\hat{X} = \Phi^X$ is an instance of Q, and for every solution $\hat{Y}$ to $\hat{X}$, the set $Y = \Psi^{X \oplus \hat{Y}}$ is a solution to X.

In computability theory, our main interest is in comparing problems based on different notions of how difficult they are. We attempt to draw pictures of the computability-theoretic universe by discerning relationships between principles. ω-reducibility, computable reducibility, and Weihrauch reducibility are three tools that we use to do this, and provability over RCA_0 is a fourth. In Chapter 2, we will illustrate how this is done. Then in Chapters 3 and 4, we will expand our repertoire of methods of comparison by generalizing and combining them.

1.2 An example: Ramsey Theory

We now introduce some relevant combinatorial principles. For example, the following versions of Ramsey's Theorem are Π^1_2-problems that have been extensively studied in reverse mathematics and computability theory, and will be useful sources of examples for us as well. (We often state Π^1_2-problems in ways that make mention of objects other than natural numbers and sets of natural numbers. We assume these are coded in an appropriate way.

For combinatorial objects like the ones below, these codings are straightforward and do not affect the analysis of these problems.) For more on the computability theory and reverse mathematics of these principles, see [33].

We write $[X]^n$ for the collection of n-element subsets of X. A *k-coloring* of $[X]^n$ is a map $c : [X]^n \to k$. A coloring of $[X]^2$ is *stable* if $\lim_{y \in X} c(x, y)$ exists for all $x \in X$. $H \subseteq X$ is *homogeneous* for c if there exists an i such that $c(s) = i$ for all $s \in [H]^n$. $L \subseteq X$ is *limit-homogeneous* for $c : [X]^2 \to k$ if there exists an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

Definition 1.2.1. RT^n_k is the Π^1_2 principle that every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

Definition 1.2.2. $\mathrm{RT}^n_{<\infty}$ is the Π^1_2 principle that $\forall k \mathrm{RT}^n_k$.

Definition 1.2.3. SRT^2_k is the Π^1_2 principle that every stable k-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

Definition 1.2.4. D^2_k is the Π^1_2 principle that every stable k-coloring of $[\mathbb{N}]^2$ has an infinite limit-homogeneous set.

We present a series of results about these principles, exhibiting how proof-theoretic and computability-theoretic implications give us different distinctions between them. We begin with a seminal result from Jockusch in [40].

Theorem 1.2.5 (Jockusch [40])**.** *Let $n \geqslant 2$. Then:*

1. *Every computable instance of* $\mathrm{RT}^n_{<\infty}$ *has a* Π^0_n *solution.*
2. *There is a computable instance of* RT^n_2 *with no* Σ^0_2 *solution.*
3. *There is a computable instance of* RT^n_2 *such that every solution computes* $\emptyset^{(n-2)}$*.*

Corollary 1.2.6 (Simpson [60])**.** RT^n_k *and* $\mathrm{RT}^n_{<\infty}$ *are equivalent to* ACA_0 *for* $n \geqslant 3$.

Simpson's corollary follows because we know that ACA_0 is equivalent to the Turing jump existence problem, and by the theorem, RT^n_2 for all $n \geqslant 3$, hence RT^n_k for all $k \geqslant 2$ and $\mathrm{RT}^n_{<\infty}$, have a computable instance that computes $\emptyset'$. Moreover, in analyzing Jockusch's proof, one can see that no induction or comprehension axioms outside of those provided by RCA_0 are needed. Therefore $\mathrm{RT}^n_k + \mathrm{RCA}_0 \vdash \mathrm{ACA}_0$ and $\mathrm{RT}^n_{<\infty} + \mathrm{RCA}_0 \vdash \mathrm{ACA}_0$, and the other direction follows from the first statement in the theorem.

On the other hand, we have the following result from Seetapun and Slaman.

Theorem 1.2.7 (Seetapun and Slaman [58])**.** $\mathrm{ACA}_0 \not\leqslant_\omega \mathrm{RT}^2_{<\infty}$.

It follows that $\mathrm{RT}^2_{<\infty}$ cannot be equivalent to ACA_0 like $\mathrm{RT}^n_{<\infty}$ is for $n > 2$. It can be shown that $\mathrm{RT}^2_{<\infty} \leqslant_\omega \mathrm{RT}^2_2$ (see [33]). However,

Theorem 1.2.8 (Cholak, Jockusch, and Slaman [16])**.** $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^2_{<\infty}$.

This gives us an example of the strength of provability over RCA_0 as compared with ω-reducibility, and helps us begin to understand where RT^2_2 lies in the computability-theoretic universe. There is still much we don't know, or have recently learned. The question of whether SRT^2_2 implies RT^2_2 motivated a great deal of research since being raised by Cholak, Jockusch, and Slaman [16]. Chong, Slaman, and Yang put this to rest in [18].

Theorem 1.2.9 (Chong, Slaman, and Yang [18])**.** $\mathrm{RCA}_0 \nvdash \mathrm{SRT}^2_2 \to \mathrm{RT}^2_2$.

The proof of this theorem makes essential use of non-ω-models. Recently, Monin and Patey [46] have also shown the following.

Theorem 1.2.10 (Monin and Patey [46])**.** $\mathrm{RT}^2_2 \not\leqslant_\omega \mathrm{SRT}^2_2$.

The relationship between SRT^2_2 and D^2_2 is also interesting, and will be discussed in Section 3.6 of Chapter 3. For $n = 1$, note that we have that $\mathrm{RCA}_0 \vdash \mathrm{RT}^1_k$ for each k, since RT^1_k is simply the Pigeonhole Principle. Therefore $\mathrm{RT}^1_{<\infty}$ is true in every ω-model of RCA_0. However,

Theorem 1.2.11 (Hirst [38]). $\mathrm{RCA}_0 \nvdash \mathrm{RT}^1_{<\infty}$.

We also have the following theorem, as proved in [64].

Theorem 1.2.12. *If*

$$\mathrm{ACA}_0 \vdash \forall X[\Theta(X) \to \exists Y \Delta(X,Y)]$$

where Θ *and* Δ *are arithmetic, then there is an* $n \in \omega$ *such that*

$$\mathrm{ACA}_0 \vdash \forall X[\Theta(X) \to \exists Y \in \Sigma^{0,X}_n \Delta(X,Y)].$$

Applying this to the results we have discussed, we obtain the following.

Corollary 1.2.13. $\mathrm{RT} \leqslant_\omega \mathrm{ACA}_0$ *but* $\mathrm{ACA}_0 \nvdash \mathrm{RT}$.

Equivalently, $\mathrm{RT} \leqslant_\omega \mathrm{RT}^3_2$ but $\mathrm{RCA}_0 + \mathrm{RT}^3_2 \nvdash \mathrm{RT}$. In terms of Weihrauch reducibility, three groups of researchers, Hirschfeldt and Jockusch, Brattka and Rakotoniaina, and Patey, independently showed the following.

Theorem 1.2.14 (Hirschfeldt and Jockusch [34] / Brattka and Rakotoniaina [12] / Patey [53]). $\mathrm{RT}^n_k <_{\mathrm{W}} \mathrm{RT}^n_{k+1}$ *for all* $n \geqslant 1$ *and* $k \geqslant 2$.

Hirschfeldt and Jockusch went further, defining a notion known as generalized Weihrauch reducibility that we will explore in Chapter 3. This allowed them to make the following conclusion about how the RT^n_k's compare to one another in terms of ω-reducibility. Here, for problems P and Q, we write that $\mathrm{P} \leqslant^j_\omega \mathrm{Q}$ if there is an ω-reduction $\mathrm{P} \leqslant_\omega \mathrm{Q}$ that can always be done in at most j steps (we will make precise what we mean by a reduction in j steps in Chapter 3).

Theorem 1.2.15 (Hirschfeldt and Jockusch [34]). *Let* $n \geqslant 3$, $j \geqslant 1$, *and* m *be such that* $n+(j-1)(n-2) < m \leqslant n+j(n-2)$. *Then* $\mathrm{RT}^m_k \leqslant^{j+1}_{\mathrm{gW}} \mathrm{RT}^n_k$, *but* $\mathrm{RT}^m_k \nleqslant^j_\omega \mathrm{RT}^n_k$. *Therefore* $\mathrm{RT} \nleqslant^j_\omega \mathrm{RT}^3_2$ *for all* j, *although* $\mathrm{RT} \leqslant_\omega \mathrm{RT}^3_2$.

Patey showed in [52] that for $n \geqslant 3$, the $\mathrm{RT}^n_k \leqslant^2_\omega \mathrm{RT}^n_j$ for $j < k$, but $\mathrm{RT}^n_k \nleqslant^1_\omega \mathrm{RT}^n_j$. For $n = 2$, the least m such that $\mathrm{RT}^n_k \leqslant^m_\omega \mathrm{RT}^n_j$ approaches infinity as k increases.

1.3 Other relevant examples

We now define another set of principles, the relationships among which further illustrate the differences between our various tools of comparison.

Definition 1.3.1. $\Pi^0_1\mathrm{G}$ is the principle that for any X and any uniformly $\Pi^{0,X}_1$ collection $D_0, D_1, \ldots$ of dense properties of binary strings, there is a G such that $\forall i \exists m[G \upharpoonright m \in D_i]$.

Definition 1.3.2. $\Pi^0_1\mathrm{GA}$ is the principle that for any X and any uniformly $\Pi^{0,X}_1$ collection $D_0, D_1, \ldots$ of dense properties of binary strings, there is a sequence $g_0, g_1, \ldots$ of sets such that $\forall i \exists m \exists t \forall u > t[g_u \upharpoonright m = g_t \upharpoonright m \in D_i]$.

Definition 1.3.3. WKL is the principle of *Weak König's Lemma*, which says that every infinite binary tree has an infinite path.

WKL is a principle of much importance in reverse mathematics. Another subsystem of second-order arithmetic over which we frequently work is called WKL_0, which consists of precisely the axioms of RCA_0 together with the statement of WKL.

Definition 1.3.4. The *Atomic Model Theorem (*AMT*)* is the principle that every complete atomic theory has an atomic model.

Theorem 1.3.5 (Conidis [19])**.** $\mathrm{RCA}_0 + \mathrm{AMT}$ *and* $\mathrm{RCA}_0 + \Pi^0_1\mathrm{G}$ *have the same ω-models.*

On the other hand, since $\Pi^0_1\mathrm{GA}$ is trivial as a principle, we have that $\Pi^0_1\mathrm{GA} \leqslant_\mathrm{W} \mathrm{WKL}$.

Theorem 1.3.6 (Hirschfeldt, Lange, and Shore [35])**.** $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \Pi^0_1\mathrm{GA} \vdash \mathrm{I}\Sigma^0_2$.

From this together with results from Hájek that can be found in [29], we get the following corollary:

Corollary 1.3.7. $\mathrm{RCA}_0 + \mathrm{WKL} \nvdash \Pi^0_1\mathrm{GA}$.

This gives an example of the relative strength of Weihrauch reducibility as compared to provability over RCA_0. We also have the following theorem, proved by Hirschfeldt, Shore, and Slaman in [37].

Theorem 1.3.8 (Hirschfeldt, Shore, and Slaman [37])**.** AMT *does not imply* Π^0_1G *over* RCA_0.

Another principle that will be useful to keep in mind is Bound, which is a convenient way of stating Σ^0_1-bounding as a Π^1_2 problem.

Definition 1.3.9 (Pauly, Fouché, and Davie [55])**.** Bound is the Π^1_2 problem where an instance is an enumeration of a bounded set F, and a solution is a bound on the elements of F.

The definition of the *finite parallelization* of Bound, where we run finitely many instances of Bound in parallel (see, for instance, [10]), and which we write as Bound*, is then as follows.

Definition 1.3.10. Bound* is the Π^1_2 problem where an instance is a simultaneous enumeration of a finite family $F_0, \dots, F_k$ of bounded sets, and a solution to this instance consists of a bound for each F_i, or, equivalently, a bound b on $\cup_{i \leqslant k} F_k$.

Another principle we will have occasion to consider is $\mathrm{C}_\mathbb{N}$, choice on the natural numbers. This principle has been extensively studied in Weihrauch reducibility (see e.g. [10] and [11]).

Definition 1.3.11. $\mathrm{C}_\mathbb{N}$ is the Π^1_2 problem where an instance is an enumeration of the complement of a nonempty set X, and a solution is an element of X.

We can strengthen $\mathrm{C}_\mathbb{N}$ to Δ^0_2 sets and restrict to compact sets (in the Heine-Borel sense), as well as consider compact Δ^0_2 sets.

Definition 1.3.12. $\mathrm{C}\Delta^0_2$ is the Π^1_2 problem that a nonempty Δ^0_2 set has an element.

Definition 1.3.13. $\mathrm{K}_{\mathbb{N}}$ is the Π^1_2 problem where an instance is an element $b \in \omega$ and an enumeration of some numbers $x \in \omega$ with $x < b$, and a solution is any $x \in \omega$ with $x < b$ such that x is not enumerated.

Definition 1.3.14. $\mathrm{K}\Delta^0_2$ is the Π^1_2 problem where an instance is an approximation to a Δ^0_2 set of numbers $x \in \omega$ with $x < b$ for some $b \in \omega$, and a solution is any $x \in \omega$ with $x < b$ such that x is not enumerated.

CHAPTER 2

THIN SET VERSIONS OF HINDMAN'S THEOREM

2.1 Introduction

This chapter, based on joint work with Denis R. Hirschfeldt, is part of a line of research on the computability-theoretic and reverse-mathematical strength of versions of Hindman's Theorem [31] that began with the work of Blass, Hirst, and Simpson [6], and has seen considerable interest recently.

The Thin Set Theorem is a variant of Ramsey's Theorem that has been studied from this perspective. It follows easily from Ramsey's Theorem itself.

Definition 2.1.1. *Thin Set Theorem* (TS): For every n and every coloring $c : [\mathbb{N}]^n \to \mathbb{N}$, there is an infinite set $T \subseteq \mathbb{N}$ and an i such that $c(s) \neq i$ for all $s \in [T]^n$. We call such a set T a *thin set* for c. TS^n is the restriction of TS to colorings of $[\mathbb{N}]^n$.

As mentioned in the introduction, Jockusch [40] showed that there is a computable instance of RT^3_2 such that any solution computes the halting problem $\emptyset'$. As shown by Simpson [60], Jockusch's construction can also be used to prove that RT^3_2 (and hence RT) implies ACA_0 over RCA_0. Wang [65] showed that TS, on the other hand, does not have this much power. Indeed, it has a property known as strong cone avoidance, which implies in particular that for every coloring $c : [\mathbb{N}]^n \to \mathbb{N}$ and every noncomputable X, there is an infinite thin set for c that does not compute X. It also follows from strong cone avoidance that TS does not imply ACA_0 over RCA_0.

As shown by Seetapun [58], RT^2_k also fails to imply ACA_0. Indeed, Liu [44, 45] showed that it does not imply the weaker system WKL_0, which consists of RCA_0 together with Weak König's Lemma, or the even weaker system WWKL_0 consisting of RCA_0 together with Weak Weak König's Lemma. Patey [50] showed that the same is true of TS.

We now turn to Hindman's Theorem. For a set $S \subseteq \mathbb{N}$, let $fs(S)$ be the set of sums of nonempty finite sets of distinct elements of S.

Definition 2.1.2. *Hindman's Theorem* (HT): For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite set $S \subseteq \mathbb{N}$ such that all elements of $fs(S)$ have the same color.

Blass, Hirst, and Simpson [6] showed that such an S can always be computed in the $(\omega + 1)$st jump of the coloring, and that there is a computable coloring such that every such S computes $\emptyset'$. By analyzing these proofs they showed that HT is provable in ACA_0^+ (the system consisting of RCA_0 together with the statement that ωth jumps exist) and implies ACA_0 over RCA_0. The exact computability-theoretic and reverse-mathematical strength of HT remains open.

There has recently been interest in studying restricted versions of HT such as the following. (See e.g. [13].)

Definition 2.1.3. $\mathrm{HT}^{\leqslant n}$ is HT restricted to sums of at most n many elements, and $\mathrm{HT}^{=n}$ is HT restricted to sums of exactly n many elements. $\mathrm{HT}^{\leqslant n}_k$ and $\mathrm{HT}^{=n}_k$ are the corresponding restrictions to colorings with k many colors.

Dzhafarov, Jockusch, Solomon, and Westrick [26] showed that $\mathrm{HT}^{\leqslant 3}_3$ implies ACA_0 over RCA_0. Carlucci, Kołodzieczyk, Lepore, and Zdanowski [14] did the same for $\mathrm{HT}^{\leqslant 2}_4$. These principles are also complex in a more heuristic sense: There is no known way to prove even $\mathrm{HT}^{\leqslant 2}_2$ other than to give a proof of the full HT, which has led Hindman, Leader, and Strauss [32] to ask whether every proof of $\mathrm{HT}^{\leqslant 2}$ is also a proof of HT. This question can be formalized by asking whether $\mathrm{HT}^{\leqslant 2}$ (or $\mathrm{HT}^{\leqslant 2}_2$) implies HT, say over RCA_0. A related open question is whether $\mathrm{HT}^{\leqslant 2}_2$ is provable in ACA_0.

The principle $\mathrm{HT}^{=2}$ is quite different, as $\mathrm{HT}^{=2}_k$ follows easily from RT^2_k. Indeed, it was not clear even whether this principle is computably true until the work of Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick [20], who showed that it is not, and that indeed there is a computable instance of $\mathrm{HT}_2^{=2}$ with no Σ^0_2 solutions. (The same had been shown for RT^2_2 by Jockusch [40], who also showed that every computable instance of RT^2_2 has a Π^0_2 solution, which implies that the same is true of $\mathrm{HT}_2^{=2}$.) They also showed that there is a computable instance of $\mathrm{HT}_2^{=2}$ such that every solution has DNC degree relative to $\emptyset'$, and adapted this proof to show that $\mathrm{HT}_2^{=2}$ implies the principle RRT^2_2, a version of the Rainbow Ramsey Theorem, over RCA_0. (See Section 2.3 for definitions.)

In this section, we study further versions of Hindman's Theorem, obtained by combining HT and its variants with the Thin Set Theorem.

Definition 2.1.4. thin-HT: For every coloring $c : \mathbb{N} \to \mathbb{N}$, there is an infinite set $S \subseteq \mathbb{N}$ such that $\mathit{fs}(S)$ is thin for c. We definite restrictions such as thin-$\mathrm{HT}^{\leqslant n}$ analogously.

In Section 2.2, we give similar lower bounds on the complexity of thin-HT as Blass, Hirst, and Simpson [6] gave for HT, which suggests that thin-HT behaves like HT at least to some extent. Indeed, it seems possible that thin-HT is equivalent to HT over RCA_0. The situation for restricted versions is different, however. Clearly, thin-$\mathrm{HT}^{=n}$ follows from TS^n, but in fact so does thin-$\mathrm{HT}^{\leqslant n}$, due to the following fact.

Lemma 2.1.5. *For each n and k, the following holds in $\mathrm{RCA}_0 + \mathrm{TS}^n$: Given $c_i : [\mathbb{N}]^{m_i} \to \mathbb{N}$ for $i \leqslant k$, with $m_i \leqslant n$ for all $i \leqslant k$, there is a single infinite set T and a j such that $c_i(s) \neq j$ for each c_i and each $s \in [T]^{m_i}$ with $i \leqslant k$.*

Proof. We use the fact that TS^n implies TS^m for each $m < n$, and proceed by external induction to prove the stronger assertion that for each $j \leqslant k$, $\mathrm{RCA}_0 + \mathrm{TS}^n$ proves that there is an infinite set T and an infinite set C such that $c_i(s) \notin C$ for each c_i and each $s \in [T]^{m_i}$ with $i \leqslant j$.

We do the base and inductive cases simultaneously. For $j+1 > 0$, assume that that the assertion holds for j and let T and C be as above. For $j+1 = 0$, let $T = C = \mathbb{N}$. Define $d : [T]^{m_{j+1}} \to \mathbb{N}$ as follows. Partition C into infinitely many infinite sets $A_0, A_1, \ldots$. Let

$d(s) = 0$ if either $c_{j+1}(s) \in A_0$ or $c_{j+1}(s) \notin C$, and for $i > 0$, let $d(s) = i$ if $c_{j+1}(s) \in A_i$. By $\mathrm{TS}^{m_{j+1}}$, there is an infinite $U \subseteq T$ that is thin for d. Let $i \notin d([U]^{m_{j+1}})$ and let $D = A_i$. Then U and D are infinite sets such that $c_i(s) \notin D$ for each c_i and each $s \in [U]^{m_i}$ with $i \leqslant j+1$. □

This lemma allows us to get thin-$\mathrm{HT}^{\leqslant n}$ from TS^n by taking a coloring $c : \mathbb{N} \to \mathbb{N}$ and considering the colorings that map $\{a_0, \ldots, a_j\}$ to $c(a_0 + \cdots + a_j)$ for each $j < n$.

There are also differences that have nothing to do with computability theory and reverse mathematics between thin-$\mathrm{HT}^{\leqslant n}$ on the one hand, and thin-HT and $\mathrm{HT}^{\leqslant n}$ on the other. The former remains true if we allow sums of non-distinct elements, but it is not difficult to show that the latter two do not. Similarly, the former remains true for colorings $S \to \mathbb{N}$, where $S \subseteq \mathbb{N}$ is any infinite set, while the latter two again do not.

Nevertheless, even thin-$\mathrm{HT}^{=2}$ still has a significant level of complexity. In Section 2.3, we show that all of the lower bounds mentioned above obtained in [20] for $\mathrm{HT}^{=2}$ still hold for thin-$\mathrm{HT}^{=2}$.

In Section 2.4 we mention some open questions arising from our results, and briefly discuss version of HT obtained by combining it with thin set theorems for colorings with finitely many colors.

2.2 Encoding $\emptyset'$ into thin-HT

In this section, we show how to build on the proof of Theorem 2.2 of Blass, Hirst, and Simpson [6], which shows that there is a computable instance of HT such that every solution computes $\emptyset'$, to show that the same is true of thin-HT. We then derive a reverse-mathematical consequence of our proof.

Theorem 2.2.1. *There is a computable instance of* thin-HT *such that every solution computes $\emptyset'$.*

Proof. As in the proof of Theorem 2.2 of [6], we write each number $x > 0$ as $2^{n_0} + \cdots + 2^{n_k}$ with $n_0 < \cdots < n_k$, and define $\lambda(x) = n_0$ and $\mu(x) = n_k$. A set S *has* 2*-apartness* if for every $x, y \in S$ with $x < y$, we have $\mu(x) < \lambda(y)$. Lemma 4.1 of [6] shows that from any infinite S we can compute an infinite set T with 2-apartness such that $\mathit{fs}(T) \subseteq \mathit{fs}(S)$ (and hence if $\mathit{fs}(S)$ is thin for a coloring, so is $\mathit{fs}(T)$).

Let $x = 2^{n_0} + \cdots + 2^{n_k}$ with $n_0 < \cdots < n_k$. Say that (n_i, n_{i+1}) is a *short gap* in x if there is an $m < n_i$ such that $m \notin \emptyset'[n_{i+1}]$ but $m \in \emptyset'$. Say that (n_i, n_{i+1}) is a *very short gap* in x if there is an $m < n_i$ such that $m \notin \emptyset'[n_{i+1}]$ but $m \in \emptyset'[n_k]$. Let $\mathit{sg}(x)$ and $\mathit{vsg}(x)$ be the numbers of short gaps and very short gaps in x, respectively. Note that sg is not a computable function, but vsg is.

Fix a bijection between $\mathbb{N}$ and the set of pairs (p, i) where p is prime and $1 \leqslant i < p$, and identify $\mathbb{N}$ with this set via this bijection. Define the coloring c by letting $c(x) = (p, i)$ where p is the least prime that does not divide $\mathit{vsg}(x)$ and $\mathit{vsg}(x) = i \bmod p$. We say that x *has color* (p, i) if $c(x) = (p, i)$, and we also say that x *has color* $(p, 0)$ or (p, p) if it has color (q, i) for some $q > p$, i.e., if every prime less than or equal to p divides $\mathit{vsg}(x)$.

Let Y be such that $\mathit{fs}(Y)$ is an infinite thin set for c. We can assume that Y has 2-apartness, by Lemma 4.1 of [6], as mentioned above. This condition ensures that if $x, y \in \mathit{fs}(Y)$ and $\mu(x) < \lambda(y)$, and we express x and y as sums of sets F and G of distinct elements of Y, respectively, then F and G are disjoint, and hence $x + y \in \mathit{fs}(Y)$. Say that $S \subseteq \mathit{fs}(Y)$ is λ*-bounded* if there is a bound on the values of $\lambda(x)$ for $x \in S$ (which includes the case $S = \emptyset$). Note that $\mathit{fs}(Y)$ itself is not λ-bounded. Note also that the union of finitely many λ-bounded sets is λ-bounded. Say that a color j is *almost absent* from $\mathit{fs}(Y)$ if the set of $x \in \mathit{fs}(Y)$ that have color j is λ-bounded. (This definition includes the case $j = (p, 0)$, or equivalently $j = (p, p)$.)

Lemma 2.2.2. *There are p and $0 \leqslant i < p$ such that $(p, i+1)$ is almost absent from $\mathit{fs}(Y)$ but (p, i) is not.*

Proof. Let p be least such that there is a j for which (p, j) is almost absent from $\mathit{fs}(Y)$, which exists since $\mathit{fs}(Y)$ is thin. If $p = 2$ then $(p, j+1)$ cannot be almost absent, since every number has color (p, j) or $(p, j+1)$. Now suppose that $p > 2$ and q is the preceding prime. Since $(q, 0)$ is not almost absent from $\mathit{fs}(Y)$ and every number that has color $(q, 0)$ has color (p, j) for some j, there is some k such that (p, k) is not almost absent. In either case, since having color $(p, 0)$ is the same as having color (p, p), the lemma follows. □

Fix p and i as in the above lemma.

Lemma 2.2.3. *Let* $1 \leqslant j < p$*. Then* $S = \{x \in \mathit{fs}(Y) : \mathit{sg}(x) = j \bmod p\}$ *is* λ*-bounded.*

Proof. Suppose S is not λ-bounded. Let $q_0 < \cdots < q_{m-1}$ be the primes less than p. Since there are only finitely many sequences $(k_0, \ldots, k_{m-1})$ with $k_i < q_i$, there is such a sequence for which $T = \{x \in S : (\forall \ell < m)\ \mathit{sg}(x) = k_\ell \bmod q_\ell\}$ is not λ-bounded.

Since $j \neq 0 \bmod p$, and hence $q_0 \cdots q_{m-1} j \neq 0 \bmod p$, there is a multiple n of $q_0 \cdots q_{m-1}$ such that $nj = 1 \bmod p$ (where $q_0 \cdots q_{m-1} = 1$ if $p = 2$). Since T is not λ-bounded, there are $x_0 < \cdots < x_{n-1} \in T$ such that each $\lambda(x_{k+1})$ is sufficiently large relative to $\mu(x_k)$ to ensure that $(\mu(x_k), \lambda(x_{k+1}))$ is not a short gap. Then the short gaps in $x_0 + \cdots + x_{n-1}$ are exactly the short gaps in $x_0, \ldots, x_{n-1}$, so $\mathit{sg}(x_0 + \cdots + x_{n-1}) = \mathit{sg}(x_0) + \cdots + \mathit{sg}(x_{n-1})$. The latter is equal to $nj \bmod p = 1 \bmod p$, since each x_ℓ is in S, and is also equal to $nk_\ell \bmod q_\ell$ for each $\ell < m$, and hence equal to $0 \bmod q_\ell$ for each $\ell < m$, since $n = 0 \bmod q_\ell$.

Since (p, i) is not almost absent from $\mathit{fs}(Y)$, there is a $y \in \mathit{fs}(Y)$ that has color (p, i) such that $\lambda(y) > \mu(x_{n-1})$, and every number less than $\mu(x_{n-1})$ that is in $\emptyset'$ is already in $\emptyset'[\lambda(y)]$. Note that $\mathit{vsg}(y) = 0 \bmod q_\ell$ for each $\ell < m$, as otherwise $c(y)$ would be of the form (q_ℓ, k) for some $1 \leqslant k < q_\ell$. Now $\mathit{vsg}(x_0 + \cdots + x_{n-1} + y) = \mathit{vsg}(y) + \mathit{sg}(x_0 + \cdots + x_{n-1})$, which is equal to $i+1 \bmod p$, and to $0 \bmod q_\ell$ for all $\ell < m$. So $x_0 + \cdots + x_{n-1} + y$ has color $(p, i+1)$. As we can choose x_0 so that $\lambda(x_0)$ is arbitrarily large, $(p, i+1)$ is not almost absent from $\mathit{fs}(Y)$, contradicting the choice of i. □

So by removing finitely many elements from Y if needed, we can assume that p divides $sg(x)$ for all $x \in fs(Y)$. We can now argue as in the proof of Claim 2 in the proof Theorem 2.2 of [6] to compute $\emptyset'$ from Y: Given n, find $x, y \in Y$ such that $x < y$ and $n < \mu(x)$. The short gaps in $x + y$ are the ones in x, the ones in y, and possibly $(\mu(x), \lambda(y))$. But if the latter is a short gap, then $sg(x + y) = sg(x) + sg(y) + 1$, which is impossible since p divides all three numbers. Thus $n \in \emptyset'$ iff $n \in \emptyset'[\lambda(y)]$. □

The above proof can be carried out in relativized form in RCA_0 except for two issues: One is that in RCA_0 we cannot show that the union of finitely many λ-bounded sets is λ-bounded, which in general requires the Π^0_1-bounding principle. Another is that being almost absent is a Σ^0_2 condition, so we cannot conclude in RCA_0 that there is a least p such that there is a j for which (p, j) is almost absent from $fs(Y)$. Since Π^0_1-bounding follows from Σ^0_2-induction over RCA_0, adding the latter to RCA_0 is sufficient to get around these issues, so we have the following.

Theorem 2.2.4. thin-HT *implies* ACA_0 *over* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$.

We do not know whether the use of $\mathrm{I}\Sigma^0_2$ in this theorem can be removed.

2.3 Hard Instances of thin-HT$^{=2}$

In this section, we show that all the lower bounds on the complexity of $\mathrm{HT}^{=2}_2$ obtained by Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick [20] still hold for thin-HT$^{=2}$. (Of course, all upper bounds on the complexity of $\mathrm{HT}^{=2}_2$ automatically hold for thin-HT$^{=2}$, as the latter follows easily from the former.) As in that paper, we use the computable version of the Lovász Local Lemma due to Rumyantsev and Shen [56, 57]. In particular, we use the following consequence of Corollary 7.2 in [57] given in [20], with an addendum on uniformity as noted at the end of Section 4 of [20]. This uniformity, which in [20] is used only to obtain results on Weihrauch reducibility, will be essential in all our results, as their proofs will require applying Theorem 2.3.1 infinitely often.

Theorem 2.3.1 (essentially Rumyantsev and Shen [57]). *For each $q \in (0,1)$ there is an M such that the following holds. Let $F_0, F_1, \ldots$ be a computable sequence of finite sets, each of size at least M. Suppose that for each $m \geqslant M$ and n, there are at most 2^{qm} many j such that $|F_j| = m$ and $n \in F_j$, and that there is a computable procedure P for determining the set of all such j given m and n. Then there is a computable $c : \mathbb{N} \to 2$ such that for each j the set F_j is not homogeneous for c. Furthermore, c can be obtained uniformly computably from $F_0, F_1, \ldots$ and P (for a fixed q).*

We will also rely in this section on arguments in [20] when they carry through in this case in an entirely analogous way.

We now introduce a notion of largeness that will be key to our iterated applications of Theorem 2.3.1. As in [20], we will be diagonalizing against Σ^0_2 sets, so this notion will be defined in terms of sets that are c.e. relative to $\emptyset'$. For a set A and a number s, we write $s + A$ for the set $\{s + a : a \in A\}$. We write W_e for the eth enumeration operator. Given e and s, for each $x \in W_e^{\emptyset'}[s]$, let t_x be the least t such that $x \in W_e^{\emptyset'}[u]$ for all $u \in [t, s]$. (I.e., t_x measures how long x has been in $W_e^{\emptyset'}$.) Order the elements of $W_e^{\emptyset'}[s]$ by letting $x \prec y$ if either $t_x < t_y$ or both $t_x = t_y$ and $x < y$. Let $E_e^n[s]$ be the set consisting of the least n many elements of $W_e^{\emptyset'}[s]$ under this ordering, or $E_e^n[s] = [0, n)$ if $W_e^{\emptyset'}[s]$ has fewer than n many elements. If there is an s such that $E_e^n[t] = E_e^n[s]$ for all $t > s$ then let $E_e^n = E_e^n[s]$.

Definition 2.3.2. For a binary function f, say that a set D is *f-large* if for all e and k such that $E_e^{f(e,k)}$ is defined, we have $|D \cap (s + E_e^{f(e,k)})| \geqslant k$ for all sufficiently large s.

Note that $\mathbb{N}$ is g-large for the function $g(e, k) = k$, and that f-largeness is preserved under finite difference. The following lemma captures the key property of this notion of largeness.

Lemma 2.3.3. *From a binary function f and an f-large set D, we can uniformly compute a binary function $\widehat{f}$ and a splitting $D = D^0 \sqcup D^1$ such that each D^i is $\widehat{f}$-large.*

Before proving this lemma, let us derive some of its consequences, beginning with computability-theoretic lower bounds on the complexity of thin-$\mathrm{HT}^{=2}$. A function f is *diagonally noncomputable* (*DNC*) relative to an oracle X if $f(e) \neq \Phi_e^X(e)$ for all e such that $\Phi_e^X(e)$ is defined, where Φ_e is the eth Turing functional. A degree is *DNC* relative to X if it computes a function that is DNC relative to X. An infinite set A is *effectively immune* relative to X if there is an X-computable function f such that if $W_e^X \subseteq A$ then $|W_e^X| < f(e)$.

Theorem 2.3.4 (Jockusch [41])**.** *A degree is DNC relative to X if and only if it computes a set that is effectively immune relative to X.*

The proof of the following theorem shows how to obtain a hard computable instance of thin-$\mathrm{HT}^{=2}$ from Lemma 2.3.3.

Theorem 2.3.5. *There is a computable instance of* thin-$\mathrm{HT}^{=2}$ *such that any solution is effectively immune relative to $\emptyset'$, and hence has DNC degree relative to $\emptyset'$.*

Proof. Let $D_0 = \mathbb{N}$ and $f_0(e,k) = k$. Given D_n and f_n, let $\widehat{f_n}$ and D_n^i be as in Lemma 2.3.3, let $f_{n+1} = \widehat{f_n}$, and let $D_{n+1} = D_n^1$. Note that the D_n are uniformly computable. Let $c(x)$ be the largest $n \leqslant x$ such that $x \in D_n$. Then c is a computable coloring of $\mathbb{N}$. If $c(x) = n$ and $x > n$ then $x \in D_n$ but $x \notin D_m$ for $m > n$, so $x \in D_n^0$. Thus for each n, we have that the difference between $c^{-1}(n)$ and D_n^0 is finite, and hence $c^{-1}(n)$ is f_n-large.

Let S be a solution to c as an instance of thin-$\mathrm{HT}^{=2}$, and let n be such that $c(x+y) \neq n$ for all distinct $x, y \in S$. For any e, if $|W_e^{\emptyset'}| \geqslant f_n(e,1)$ then $E_e^{f_n(e,1)} \subseteq W_e^{\emptyset'}$ is defined, and hence $c^{-1}(n) \cap (s + E_e^{f_n(e,1)}) \neq \emptyset$ for all sufficiently large s. In other words, if s is sufficiently large then there is an $x \in E_e^{f_n(e,1)}$ such that $c(x+s) = n$. It follows that $E_e^{f_n(e,1)} \nsubseteq S$, and hence $W_e^{\emptyset'} \nsubseteq S$, since $E_e^{f_n(e,1)} \subseteq W_e^{\emptyset'}$. Thus we conclude that if $W_e^{\emptyset'} \subseteq S$

then $|W_e^{\emptyset'}| < f_n(e,1)$. Since $f_n(e,1)$ is computable as a function of e, it follows that S is effectively immune relative to $\emptyset'$, and hence has DNC degree relative to $\emptyset'$. □

No infinite Σ_2^0 set can be effectively immune relative to $\emptyset'$, so we have the following.

Corollary 2.3.6. *There is a computable instance of* thin-HT$^{=2}$ *with no* Σ_2^0 *solution.*

It follows that thin-HT is not provable in WKL_0, since the latter has ω-models consisting entirely of Δ_2^0 sets. It was noted in [20] that $\mathrm{HT}_2^{=2}$ does not imply WKL_0, and hence neither does thin-HT$^{=2}$. Thus thin-HT$^{=2}$ and WKL_0 are incomparable over RCA_0. In fact, as mentioned in the introduction, Patey [50] showed that TS does not imply WKL_0, or even WWKL_0, and we can easily adapt the proof of Theorem 2.3.5 to thin-HT$^{=n}$ for any $n > 2$, so we have the following.

Corollary 2.3.7. *For each $n > 1$, both* thin-HT$^{=n}$ *and* thin-HT$^{\leqslant n}$ *are incomparable with* (W)WKL_0 *over* RCA_0*.*

Arguing as in the proof of Corollary 3.6 of [20], we have the following.

Corollary 2.3.8. *There is a computable instance of* thin-HT$^{=2}$ *such that all solutions are hyperimmune.*

The reverse-mathematical analog of the existence of degrees that are DNC over the jump is the principle 2-DNC, defined e.g. in Section 4 of [20]. Miller [unpublished] showed that 2-DNC is equivalent, both over RCA_0 and in the sense of Weihrauch reducibility, to the following version of the Rainbow Ramsey Theorem, which was shown by Patey [51] to be strictly weaker than TS^2.

Definition 2.3.9. RRT_2^2: Let $c : [\mathbb{N}]^2 \to \mathbb{N}$ be such that $|c^{-1}(i)| \leqslant 2$ for all i. Then there is an infinite set R such that c is injective on $[R]^2$.

As discussed in [20], the proof of Theorem 2.3.1 carries through in RCA_0, from which it will follow that so does the proof of Lemma 2.3.3 that we will give below. Thus the proof of

Theorem 2.3.5 also carries through in RCA_0, except for one issue: Having $|W_e^{\emptyset'}| \geqslant m$ does not necessarily imply in RCA_0 that E_e^m is defined. (The issue is that RCA_0 does not imply the Π_1^0-bounding principle.) However, we can get around this problem exactly as in Section 4 of [20], by using the principle 2-EI defined there, thus obtaining the following.

Theorem 2.3.10. thin-$\mathrm{HT}^{=2}$ *implies* RRT_2^2 *over* RCA_0.

We can also obtain a Weihrauch reduction from RRT_2^2 to a version of thin-$\mathrm{HT}^{=2}$ as in the final paragraph of Section 4 of [20], but we have to be a bit careful because in the proof of Theorem 2.3.5, the function witnessing that S is effectively immune relative to $\emptyset'$ is obtained uniformly not from S, but from an n such that $c(x+y) \neq n$ for all distinct $x, y \in S$. Let strong thin-$\mathrm{HT}^{=2}$ be the version of thin-$\mathrm{HT}^{=2}$ where a solution to an instance c consists of both a solution S to c as an instance of thin-$\mathrm{HT}^{=2}$ and an n as above. Then we have the following.

Theorem 2.3.11. RRT_2^2 *is Weihrauch-reducible to* strong thin-$\mathrm{HT}^{=2}$.

We do not know, however, whether this theorem remains true if we replace strong thin-$\mathrm{HT}^{=2}$ by thin-$\mathrm{HT}^{=2}$.

None of the above results depend on the addition function in particular, and can be adapted as in [20] to any function $f : [\mathbb{N}]^2 \to \mathbb{N}$ that is *addition-like*, which means that

1. f is computable,

2. there is a computable function g such that $f(\{x, y\}) > n$ for all $y > g(x, n)$, and

3. there is a b such that for all $x \neq y$, there are at most b many z's for which $f(\{x, z\}) = f(\{x, y\})$.

We finish this section by proving Lemma 2.3.3.

Proof of Lemma 2.3.3. Let f be a binary function and D an f-large set. We will apply Theorem 2.3.1 to obtain a computable $c : \mathbb{N} \to 2$. We then define $D^i = \{n \in D : c(n) = i\}$. The value of q will not matter here, so let us fix $q = \frac{1}{2}$. Let M be as in Theorem 2.3.1.

Let g be a computable injective binary function with computable image such that $kg(e,k) \leqslant 2^{\frac{g(e,k)}{2}}$ and $g(e,k) \geqslant M$ for all e and k.

Say that s is *acceptable for* e, k if $|D \cap (s + E_e^{f(e,kg(e,k))}[s])| \geqslant kg(e,k)$ and for every $t < s$ such that $(s + E_e^{f(e,kg(e,k))}[s]) \cap (t + E_e^{f(e,kg(e,k))}[t]) \neq \emptyset$, we have $E_e^{f(e,kg(e,k))}[s] = E_e^{f(e,kg(e,k))}[t]$. If s is acceptable for e, k then let $F_{e,k,s,0}$ be the first $g(e,k)$ many elements of $s + E^{f(e,kg(e,k))}[s]$, let $F_{e,k,s,1}$ be the next $g(e,k)$ many elements of $s + E^{f(e,kg(e,k))}[s]$, and so on, until $F_{e,k,s,k-1}$.

Let $\mathcal{F}$ consist of all $F_{e,k,s,j}$ for all e, k, all s acceptable for e, k, and all $j < k$. Then we can arrange the elements of $\mathcal{F}$ into a computable sequence of finite sets, each of size at least M. Fix x and m. If m is not in the image of g then there are no elements of $\mathcal{F}$ of size m. Otherwise, there is a unique pair e, k such that $m = g(e,k)$, and all elements of $\mathcal{F}$ of size m that contain x are of the form $F_{e,k,s,j}$ for some $s \leqslant x$. We can computably determine all such sets from m and x, and the definition of acceptability means that there are at most $kg(e,k) \leqslant 2^{\frac{m}{2}}$ many such sets.

Thus the hypotheses of Theorem 2.3.1 hold, and hence there is a c, obtained uniformly computably from f and D, such that none of the sets in $\mathcal{F}$ are homogeneous for c. Let $\widehat{f}(e,k) = f(e,kg(e,k))$ and let $D^i = \{n \in D : c(n) = i\}$. Fix e and k such that $E_e^{\widehat{f}(e,k)}$ is defined. If s is sufficiently large then s is acceptable for e, k, and $F_{e,k,s,j} \subseteq s + E_e^{\widehat{f}(e,k)}$ for all $j < k$. For each $j < k$ and $i < 2$, there is at least one $x \in F_{e,k,s,j}$ such that $c(x) = i$. Since the $F_{e,k,s,j}$ are disjoint, $|D^i \cap (s + E_e^{\widehat{f}(e,k)})| \geqslant k$. Thus D^i is $\widehat{f}$-large. $\square$

2.4 Open Questions

In this section, we collect a few open questions and possible directions for further work arising from the above results.

Open Question 2.4.1. Does thin-HT imply ACA_0 over RCA_0 (i.e., without assuming $\mathrm{I}\Sigma^0_2$)?

Of course, one way to give a positive answer to this question would be to show that thin-HT implies $\mathrm{I}\Sigma^0_2$ over RCA_0. If that is not the case, then it could be interesting to try to determine the first-order part of thin-HT.

Open Question 2.4.2. Is thin-HT provable in ACA_0?

Open Question 2.4.3. Does thin-HT imply HT, say over RCA_0?

In the spirit of Hindman, Leader, and Strauss [32], we can also ask the less formal question of whether there is a proof of thin-HT that is not already a proof of HT.

Open Question 2.4.4. Is RRT^2_2 Weihrauch-reducible to thin-$\mathrm{HT}^{=2}$ (as opposed to strong thin-$\mathrm{HT}^{=2}$)?

Open Question 2.4.5. What is the exact relationship between thin-$\mathrm{HT}^{=2}$ and each of TS^2, RRT^2_2, and $\mathrm{HT}^{=2}$?

There are also versions of the Thin Set Theorem for colorings with finitely many colors. For example, an instance of TS^n_k is a coloring c of $[\mathbb{N}]^n$ with k many colors, and a solution to this instance is an infinite set T such that $|c([T]^n)| < k$. This principle and RT^n_k form the two ends of a spectrum of principles $\mathrm{RT}^n_{k,j}$ for $1 \leqslant j < k$, where an instance is a coloring c of $[\mathbb{N}]^n$ with k many colors, and a solution to this instance is an infinite set T such that $|c([T]^n)| \leqslant j$. It would be interesting to pursue versions of HT based on these principles. One might hope to show, for instance, that there is a boundary between principles that "behave like HT", e.g. $\mathrm{HT}^{\leqslant 2}_4$, which as mentioned in the introduction was shown to imply ACA_0 in [14]; and those that "behave like versions of TS / RT", e.g. the thin version of $\mathrm{HT}^{\leqslant 2}_4$, which can easily be shown to follow from $\mathrm{RT}^2_{4,2}$.

CHAPTER 3

REDUCTION GAMES, PROVABILITY, AND COMPACTNESS

3.1 Introduction

This chapter is based on joint work Damir D. Dzhafarov and Denis R. Hirschfeldt (see [25]). In many cases, nonimplications over RCA_0 are proved using ω-models. Implication over RCA_0 and ω-reducibility are not fine enough for some purposes, so other notions of computability-theoretic reduction between theorems have been extensively studied. These are particularly well-adapted to Π^1_2-problems.

Hirschfeldt and Jockusch [34] gave characterizations of both $\mathrm{P} \leqslant_\omega \mathrm{Q}$ and $\mathrm{RCA}_0 \vdash \mathrm{Q} \to \mathrm{P}$ for Π^1_2-problems P and Q in terms of winning strategies in certain games. In this chapter, we study further aspects of the latter characterization and generalizations of it, in particular establishing a compactness theorem that shows that certain winning strategies can always be chosen to win in a number of moves bounded by a number independent of the instance of P being considered. As explained below, this theorem can be seen as a generalization of a metatheorem about ACA_0. This metatheorem has been used, for instance, to translate computability-theoretic results of Jockusch [40] into a proof that $\mathrm{ACA}_0 \nvdash \mathrm{RT}$.

The difference between the two game-theoretic characterizations in [34] is that for ω-reducibility, the games are played over the standard natural numbers, while for provability over RCA_0 they are played over possibly nonstandard models of Σ^0_1-PA (the first-order part of RCA_0). We hope to show in this chapter that there is a rich theory to be obtained by generalizing computability-theoretic reductions between Π^1_2-problems to models of subsystems of second-order arithmetic with possibly nonstandard first-order parts, and to begin its systematic development. In particular, this theory allows us to conduct a fine-structural comparison between such problems that has both computability-theoretic and proof-theoretic aspects, and can help us distinguish between these, for example by showing that a certain use of a principle in a proof is “purely proof-theoretic”, as opposed to relying on its computability-theoretic strength.

Computable reducibility and Weihrauch reducibility are two of the most widely-studied notions of computability-theoretic reducibility between Π^1_2-problems. The latter (in a more general form) has a long history, particularly in computable analysis (see e.g. [10]), while the former was introduced by Dzhafarov [22]. These two reducibilities allow us to use only a single instance of Q in solving an instance of P. To generalize these notions to allow multiple instances of Q to be used, Hirschfeldt and Jockusch [34] defined the following game.

Definition 3.1.1. Let P and Q be Π^1_2-problems. The *reduction game* $G(\mathrm{Q} \to \mathrm{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays an instance X_0 of P. Then Player 2 either plays an X_0-computable solution to X_0 and wins, or plays an X_0-computable instance Y_1 of Q.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q. Then Player 2 either plays an $(X_0 \oplus \cdots \oplus X_{n-1})$-computable solution to X_0 and wins, or plays an $(X_0 \oplus \cdots \oplus X_{n-1})$-computable instance Y_n of Q.

(5) If the game never ends then Player 1 wins.

A winning strategy for Player 2 in this game is a form of generalized computable reduction. Hirschfeldt and Jockusch [34] showed that if $\mathrm{P} \leqslant_\omega \mathrm{Q}$ then Player 2 has a winning strategy for $G(\mathrm{Q} \to \mathrm{P})$, while otherwise Player 1 has a winning strategy for $G(\mathrm{Q} \to \mathrm{P})$, so generalized computable reducibility is actually the same as ω-reducibility. They then defined an analogous notion of generalized Weihrauch reducibility, where $\mathrm{P} \leqslant_{\mathrm{gW}} \mathrm{Q}$ if Player 2 has a uniformly computable winning strategy for $G(\mathrm{Q} \to \mathrm{P})$. (See [34] for the details of this definition.) Neumann and Pauly [48] gave an equivalent definition in terms of an operator $^\diamond$ on the Weihrauch degrees. (See also [66] for some more recent discussion of, and results about, the $^\diamond$ operator.)

We can generalize the notions of instance and solution of a Π^1_2-problem $\mathrm{P} \equiv \forall X\,[\Theta(X) \to \exists Y\,\Psi(X,Y)]$ to possibly nonstandard structures in the language of first-order arithmetic in a natural way. We denote the languages of first- and second-order arithmetic by L_1 and L_2, respectively. Let M be an L_1-structure. We denote the domain of M by $|M|$. For $S \subseteq |M|$, we denote the L_2-structure with first-order part M and second-order part S by (M,S). For an L_1-structure M, an *M-instance of* P is an $X \subseteq |M|$ such that $(M,\{X\}) \vDash \Theta(X)$, and a *solution* to this instance is a $Y \subseteq |M|$ such that $(M,\{X,Y\}) \vDash \Psi(X,Y)$.

Hirschfeldt and Jockusch [34, Section 4.5] noted that reduction games can be extended to possibly nonstandard countable models of Σ^0_1-PA (i.e., first-order parts of models of RCA_0), with Δ^0_1-definability playing the role of computability as follows. For $X_0,\dots,X_n \subseteq |M|$, we denote by $M[X_0,\dots,X_n]$ the L_2-structure with first-order part M and second-order part consisting of all $X \subseteq |M|$ that are Δ^0_1-definable over $|M| \cup \{X_0,\dots,X_n\}$, which means that there are Σ^0_1 formulas $\varphi_0(x)$ and $\varphi_1(x)$ with parameters from $|M| \cup \{X_0,\dots,X_n\}$ such that $(M,\{X_0,\dots,X_n\}) \vDash \forall x\,(\varphi_0(x) \leftrightarrow \neg\varphi_1(x))$ and $X = \{n \in |\mathcal{M}| : (M,\{X_0,\dots,X_n\}) \vDash \varphi_0(n)\}$.

Definition 3.1.2. Let P and Q be Π^1_2-problems. The *RCA_0-reduction game $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$* is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a countable L_1-structure M and an M-instance X_0 of P such that $M[X_0] \vDash \mathrm{RCA}_0$. Then Player 2 either plays a solution to X_0 in $M[X_0]$ and

wins, or plays an M-instance Y_1 of Q in $M[X_0]$.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q such that $M[X_0, \ldots, X_{n-1}] \models \mathrm{RCA}_0$. Then Player 2 either plays a solution to X_0 in $M[X_0, \ldots, X_{n-1}]$ and wins, or plays an M-instance Y_n of Q in $M[X_0, \ldots, X_{n-1}]$.

(5) If the game never ends then Player 1 wins.

This definition allows us to capture provability over RCA_0 in terms of winning strategies.

Proposition 3.1.3 (Hirschfeldt and Jockusch [34]). *Let* P *and* Q *be* Π^1_2*-problems. If* $\mathrm{RCA}_0 \vdash \mathrm{Q} \to \mathrm{P}$ *then Player 2 has a winning strategy for* $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$*. Otherwise, Player 1 has a winning strategy for* $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$*.*

The proof of this proposition is essentially the same as that of the analogous result for games over the standard natural numbers and ω-reducibility in [34, Proposition 4.2]. We will prove a stronger version in Proposition 3.2.4.

However, it might be that the above definition is not quite the best one. In Section 3.2, we will discuss a modified game. We will define it for arbitrary subsystems of second-order arithmetic, but in the case of RCA_0, the modified game $\widehat{G}^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$ is defined as above, except that on its first move, Player 1 must play not only a countable L_1-structure M, but a model $\mathcal{M}$ of RCA_0 with countable first-order part (but possibly uncountable second-order part); and from then on, its moves $X_0, X_1, \ldots$ must all come from $\mathcal{M}$. This game makes intuitive sense in that if Player 1 is trying to claim that $\mathrm{RCA}_0 \nvdash \mathrm{Q} \to \mathrm{P}$, then it should be prepared to propose a model of RCA_0 within which to witness this fact. This idea was not noticed in [34] because in the ω-model case after which the original RCA_0-reduction game was modeled, there is really no issue, since Player 1 always automatically plays within a particular model of RCA_0, namely $(\omega, \mathcal{P}(\omega))$, where $\mathcal{P}(\omega)$ is the full power set of ω. (For a nonstandard model M, of course, the full power set will include a cut, so we cannot add it to M to obtain a model of RCA_0.)

As we will see in Section 3.2, Proposition 3.1.3 still holds for this modified game, indeed with the same proof. But we will also be able to prove a stronger version that shows that a certain kind of compactness theorem holds in this case: As shown in [34], for a game $G(\mathrm{Q} \to \mathrm{P})$ over the standard natural numbers, it is possible that Player 2 has a winning strategy but there is no n such that Player 2 has a winning strategy that is guaranteed to win in at most n many moves. As we will show in Section 3.3, for our modified games over possibly nonstandard models, this will no longer be the case, which makes sense given that these games capture notions of provability, and a proof of $\mathrm{Q} \to \mathrm{P}$ is a finite object.

Theorem 3.1.4. *Let* P *and* Q *be* Π^1_2*-problems. If* $\mathrm{RCA}_0 \vdash \mathrm{Q} \to \mathrm{P}$ *then there is an* n *such that Player 2 has a winning strategy for* $\widehat{G}^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$ *that ensures victory in at most* n *many moves. Otherwise, Player 1 has a winning strategy for* $\widehat{G}^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$*.*

We do not know whether the first part of this result holds for the game $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$ as well.

Theorem 3.1.4, whose proof will in fact use the compactness theorem for first-order logic, can be seen as a generalization of the following fact, which appears in Wang [64], where it is said that it is "almost certainly a known theorem in proof theory." For a model-theoretic proof using compactness due to Jockusch, see [33, Section 6.3].

Theorem 3.1.5 (see Wang [64])**.** *Let* $\mathrm{P} \equiv \forall X\, [\Theta(X) \to \exists Y\, \Psi(X,Y)]$ *be a* Π^1_2*-problem. If* P *is provable in* ACA_0*, then there is an* $n \in \omega$ *such that* ACA_0 *proves* $\forall X\, [\Theta(X) \to \exists Y \in \Sigma^{0,X}_n\, \Psi(X,Y)]$*.*

As mentioned above, this theorem implies for instance that $\mathrm{ACA}_0 \nvdash \mathrm{RT}$, because Jockusch [40] showed that for each $n \geqslant 2$, there is an instance of RT^n_2 (and hence of RT) with no Σ^0_n solutions. On the other hand, Jockusch also showed that every instance of RT^n_k has a Π^0_n solution, which implies that every ω-model of ACA_0 is a model of RT.

Notice that if we take Q to be the statement that for each X, the Turing jump X' exists, then the provability of P in ACA_0 is equivalent to the provability of $\mathrm{Q} \to \mathrm{P}$ in RCA_0. As part of the proof of Theorem 3.1.4 in Section 3.3, we will prove a theorem that is a direct generalization of Theorem 3.1.5. Montalbán and Shore [47] also generalized this theorem, in a different way that is particularly suited to problems where each instance has a unique solution, and is indeed equivalent to ours in that case, but is not strong enough for our purposes.

As an example of the application of Theorem 3.1.4, we will obtain a simple proof that RT^2_2 does not imply $\mathrm{RT}^2_{<\infty}$, even over RCA_0 together with all Π^1_1 formulas true over the natural numbers.

Let Γ be a class of formulas. Recall that $\mathrm{I}\Gamma$ is the axiom scheme stating that induction holds for formulas in Γ. Recall also that the Γ-*bounding axiom scheme* $\mathrm{B}\Gamma$ consists of all formulas of the form

$$\forall n\,[\forall i < n\,\exists k\,\varphi(i,k)\;\rightarrow\;\exists b\,\forall i < n\,\exists k \leqslant b\,\varphi(i,k)]$$

for each formula φ in Γ such that b is not free in φ. Note that φ is allowed to have parameters. The system $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$, which is strictly intermediate between RCA_0 and $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$, has been particularly prominent in reverse mathematics. (In most cases, it is actually $\mathrm{B}\Pi^0_1$ that is used, but $\mathrm{B}\Pi^0_1$ and $\mathrm{B}\Sigma^0_2$ are easily seen to be equivalent over RCA_0.) For example, Hirst [38] showed that $\mathrm{RT}^1_{<\infty}$ is equivalent to $\mathrm{B}\Sigma^0_2$ over RCA_0.

In Section 3.4, we will consider computable winning strategies and the notion of generalized Weihrauch reducibility over possibly nonstandard models. There is an intriguing connection here with analogs of RCA_0 for intuitionistic logic, first noted in work of Kuyper [43]. We will comment on this connection briefly in that section, but leave further work in this direction for follow-up work. In Section 3.5 we will consider single-instance reducibilities such as computable and Weihrauch reducibility in this context. Our results throughout will apply not only to RCA_0 but also to other systems at the level of computable mathematics, including extensions of RCA_0 by first-order principles, such as $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ or $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n$, and also restrictions such as RCA^*_0, which roughly speaking is RCA_0 with Σ^0_1-induction replaced by Σ^0_0-induction.

In Sections 3.6 and 3.7, we will undertake a case study in the analysis of mathematical principles under Weihrauch and generalized Weihrauch reducibility over possibly nonstandard models, by considering several principles that are equivalent over RCA_0 to Σ^0_2-bounding. We will see how this framework allows us to uncover some hitherto hidden differences between quite similar principles.

3.2 Reduction games and provability

In this section, we generalize Definition 3.1.2 from RCA_0 to other axiom systems Γ, modify it as described above, and prove a more general version of Proposition 3.1.3. Of course, we cannot in general require Player 1's moves to result in models of Γ, since it might be the case that no structure of the form $M[X_0, \ldots, X_{n-1}]$ is a model of Γ. However, we can require that Player 1 never make it impossible for the model built by its moves to be extendable to a model of Γ. Say that an L_2-structure $\mathcal{M}$ is *consistent with* Γ if it is contained in a model $\mathcal{N}$ of Γ with the same first-order part. (Note that if $\mathcal{M}$ is countable, then we can require $\mathcal{N}$ to be countable as well without changing the notion.)

The systems Γ for which we will prove that winning strategies for the following game correspond to provability over Γ will actually have the property that every structure consistent with Γ is in fact a model of Γ. The reason we give the definition in the more general setting is that, when analyzing the provability of $\mathrm{Q} \to \mathrm{P}$ in Γ, we will also want to consider games over $\Gamma + \mathrm{Q}$. We will see that doing so makes no difference in the case of general winning strategies, but does in the case of computable winning strategies.

Definition 3.2.1. Let Γ be a set of L_2-formulas and let P and Q be Π^1_2-problems. The Γ-*reduction game* $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a countable L_1-structure M and an M-instance X_0 of P such that $M[X_0]$ is consistent with Γ. Then Player 2 either plays a solution to X_0 in $M[X_0]$ and wins, or plays an M-instance Y_1 of Q in $M[X_0]$.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q such that $M[X_0, \ldots, X_{n-1}]$ is consistent with Γ. Then Player 2 either plays a solution to X_0 in $M[X_0, \ldots, X_{n-1}]$ and wins, or plays an M-instance Y_n of Q in $M[X_0, \ldots, X_{n-1}]$.

(5) If the game never ends then Player 1 wins.

We modify this game as follows.

Definition 3.2.2. Let Γ be a set of L_2-formulas consistent with Δ^0_1-comprehension, and let P and Q be Π^1_2-problems. The *modified* Γ*-reduction game* $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ is a two-player game played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a model (M, S) of Γ such that M is countable and S is closed under Δ^0_1-comprehension, and an M-instance X_0 of P in S. Then Player 2 either plays a solution to X_0 in $M[X_0]$ and wins, or plays an M-instance Y_1 of Q in $M[X_0]$.

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q in S. Then Player 2 either plays a solution to X_0 in $M[X_0, \ldots, X_{n-1}]$ and wins, or plays an M-instance Y_n of Q in $M[X_0, \ldots, X_{n-1}]$.

(5) If the game never ends then Player 1 wins.

If Γ is consistent with Δ^0_1-comprehension and $\Gamma \nvdash \mathrm{Q} \to \mathrm{P}$, then Player 1 has winning strategies in both of these games (as we will see in the second part of the proof of Proposition 3.2.4 below), but we cannot hope in general that the same is the case for Player 2 if $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$, because of that player's restriction to playing computably. However, if Γ is sufficiently well-behaved, then this is no longer an obstacle, and we can obtain a generalization of Proposition 3.1.3 with essentially the same proof. The key property here is that all axioms of Γ other than Δ^0_1-comprehension be Π^1_1. Of course, this property holds of RCA_0, as well as commonly-studied first-order extensions such as $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ and $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n$, and restrictions such as RCA^*_0.

In the proof, we will actually use the following properties, but it is not difficult to show that they are equivalent to saying that Γ is a consistent set of L_2-formulas consisting of Δ^0_1-comprehension together with a set of Π^1_1 formulas.

1. Γ is a consistent set of L_2-formulas that includes all instances of Δ^0_1-comprehension.
2. If an L_2-structure is closed under Δ^0_1-definability and is consistent with Γ, then it is a model of Γ.
3. For every countable L_1-structure M and $X_0, X_1, \ldots \subseteq |M|$, if each $M[X_0, \ldots, X_n]$ is a model of Γ, then so is their union $M[X_0, X_1, \ldots]$.

The following simple but important result follows from these properties.

Lemma 3.2.3. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas. Let* P *and* Q *be Π^1_2-problems. Let M be an L_1-structure and $X_0, \ldots, X_n \subseteq |M|$ be sets set such that $M[X_0, \ldots, X_n]$ is consistent with Γ. If $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$, then either every instance of* P *in $M[X_0, \ldots, X_n]$ has a solution in $M[X_0, \ldots, X_n]$, or else* Q *has an instance in $M[X_0, \ldots, X_n]$.*

Proof. Fix M and $X_0, \ldots, X_n$. Since the L_2-structure $M[X_0, \ldots, X_n]$ is closed under Δ^0_1-comprehension it is in fact a model of Γ, as noted above. If Q has no instance in $M[X_0, \ldots, X_n]$, then $M[X_0, \ldots, X_n]$ trivially satisfies Q. Hence, by assumption, $M[X_0, \ldots, X_n]$ also satisfies P. So every instance of P in $M[X_0, \ldots, X_n]$ has a solution in $M[X_0, \ldots, X_n]$. □

Later on, when we prove a generalization of Theorem 3.1.4, we will also need to assume that Γ is strong enough to prove the existence of a universal Σ^0_1 formula, but of course that holds of all systems we normally study in reverse mathematics.

We should also expect Γ and $\Gamma + \mathrm{Q}$ to behave similarly here, since there is no difference between saying that $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ and saying that $\Gamma + \mathrm{Q} \vdash \mathrm{Q} \to \mathrm{P}$. This fact will be of interest below when we consider computable winning strategies.

Proposition 3.1.3 can be generalized as follows. Notice that of the four games $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$, $G^{\Gamma+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$, $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$, and $\widehat{G}^{\Gamma+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$, the first is the hardest one for Player 2 to win, while the last is the hardest one for Player 1 to win.

Proposition 3.2.4. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas. Let* P *and* Q *be Π^1_2-problems. If $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ then Player 2 has a winning strategy for $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ (and hence for each of the three other games above). Otherwise, Player 1 has a winning strategy for $\widehat{G}^{\Gamma+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$ (and hence for each of the three other games above).*

Proof. If $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ then Player 2 can play according to the following strategy. Let M be the L_1-structure played by Player 1 on its first move. At the nth move, if Player 2 has a legal winning move, Player 2 makes that move. Otherwise, it lets $Y_{n,0}, Y_{n,1}, \ldots$ be all M-instances of Q in $M[X_0, \ldots, X_{n-1}]$, where $X_0, \ldots, X_{n-1}$ are Player 1's first n moves. For the least pair $\langle m, i \rangle$ with $m \leqslant n$ for which Player 2 has not yet acted, it then acts by playing $Y_{m,i}$ (to which Player 1 must reply with a solution to $Y_{m,i}$). Note that Player 2 always has some legal move, by Lemma 3.2.3. Suppose Player 2 never has a winning move, and Player 1 never fails to have a legal move. By our assumptions on Γ, each $M[X_0, \ldots, X_{n-1}]$ is a model of Γ, and hence so is their union $M[X_0, X_1, \ldots]$. But Player 2's strategy ensures that this structure is also a model of Q, so it must also be a model of P, and hence must contain a solution to X_0. This solution is in $M[X_0, \ldots, X_{n-1}]$ for some n, which gives Player 2 a winning nth move.

If $\Gamma \nvdash \mathrm{Q} \to \mathrm{P}$ then let (M, S) be a model of $\Gamma + \mathrm{Q} + \neg\mathrm{P}$ and let X_0 be an M-instance of P in S with no solution in S. Since (M, S) is a model of Γ, it is closed under Δ^0_1-definability, so as long as Player 1's moves stay inside S, so must Player 2's moves. Furthermore, the fact that (M, S) is a model of Q implies that, as long as Player 2's moves stay inside S, Player 1 will always be able to reply with moves that stay inside S. So Player 1 can simply begin by playing (M, S) and X_0, and then keep playing elements of S, which ensures that the game never ends (unless Player 2 cannot make its first move, in which case it loses immediately). □

Remark 3.2.5. We can extend the above framework beyond extensions of Δ^0_1-comprehension by Π^1_1 formulas. Let us consider ACA_0, for instance. If we redefine $M[X_0, \ldots, X_{n-1}]$ by replacing Δ^0_1-definability by arithmetic definability, then use this new definition in the definitions of the Γ-reduction game and the modified Γ-reduction game, then Proposition 3.2.4 carries through essentially unchanged.

There is nothing particularly special about this $\Gamma = \mathrm{ACA}_0$ case. All we need is the existence of a smallest model $M[X_0, \ldots, X_{n-1}]$ of Γ with first-order part M containing $X_0, \ldots, X_{n-1} \subseteq |M|$ (if there is any such model at all), and the requirement that then $\bigcup_n M[X_0, \ldots, X_{n-1}]$ is also a model of Γ (which will happen if Γ is Π^1_2-axiomatizable). For systems Γ that do not have such minimal models, such as WKL_0, we can still extend these ideas by redefining our games in a way that does not affect our results when applied to systems that do have minimal models. For example, $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ can now be played according to the following rules.

(1) If at any point a player cannot make a move, the opponent wins.

(2) If one of the players wins, the game ends.

(3) On the first move, Player 1 plays a model (M, S) of Γ with M countable, an M-instance X_0 of P in S, and a submodel (M, S_0) of (M, S) containing X_0. Then Player 2 either

plays a solution to X_0 in (M, S_0) and wins, or plays an M-instance Y_1 of Q in (M, S_0).

(4) For $n > 1$, on the nth move, Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q in S and a submodel (M, S_{n-1}) of (M, S) containing X_{n-1}. Then Player 2 either plays a solution to X_0 in (M, S_{n-1}) and wins, or plays an M-instance Y_n of Q in (M, S_{n-1}).

(5) If the game never ends then Player 1 wins.

Theorem 3.3.1 below remains true for ACA_0, for instance, since in Theorem 3.3.4 we can replace the eth Turing functional by the eth arithmetical functional. It is not clear how generally Theorem 3.3.1 holds for other systems, but we will not pursue this further generalization of our framework here.

3.3 Reduction games and compactness

As mentioned in the introduction, we can improve on Proposition 3.2.4 by showing that a certain kind of compactness theorem holds, with the very mild extra assumption that Γ proves the existence of a universal Σ^0_1 formula, i.e., that there is a Σ^0_1 formula $\theta(e, n, X)$ such that for every Σ^0_1 formula $\varphi(e, n, X)$, we have $\Gamma \vdash \forall e\, \exists i\, \forall n\, \forall X\, (\theta(i, n, X) \leftrightarrow \varphi(e, n, X))$. In this case, we assume we have fixed such a θ and a bijective pairing function $\langle \cdot, \cdot \rangle$, and write $Y = \Phi^X_e$ to mean that for $e = \langle i, j \rangle$, we have $\forall n\, [\theta(i, n, X) \leftrightarrow \neg\theta(j, n, X)]$ and $\forall n\, [n \in Y \leftrightarrow \theta(i, n, X)]$.

The following result, which we will prove in this section, has Theorem 3.1.4 as a special case.

Theorem 3.3.1. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas that proves the existence of a universal Σ^0_1 formula. Let* P *and* Q *be Π^1_2-problems. If $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ then there is an n such that Player 2 has a winning strategy for $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ (and hence for $\widehat{G}^{\Gamma+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$) that ensures victory in at most n many moves. Otherwise, Player 1 has a winning strategy for $\widehat{G}^{\Gamma+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$ (and hence for $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$).*

Notice that if the formulas added to Δ^0_1-comprehension to obtain Γ are true over the standard natural numbers, then a winning strategy for Player 2 for $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most n many moves also yields a winning strategy for Player 2 for $G(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most n many moves, since a run of the latter game is a special case of a run of the former game in which Player 1 begins by playing the model $(\omega, \mathcal{P}(\omega))$. Thus it is not a coincidence that all the examples we have of situations in which $G(\mathrm{Q} \to \mathrm{P})$ can be won by Player 2, but not in a number of moves bounded ahead of time, are ones in which $\mathrm{P} \leqslant_\omega \mathrm{Q}$ but $\mathrm{RCA}_0 \nvdash \mathrm{Q} \to \mathrm{P}$. In fact, the following stronger fact holds, where, as defined in [34], $\mathrm{P} \leqslant^n_\omega \mathrm{Q}$ means that Player 2 has a winning strategy for $G(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most $n + 1$ many moves.

Corollary 3.3.2. *Let* Γ *consist of* RCA_0 *together with all* Π^1_1 *formulas true over the natural numbers. If* $\mathrm{P} \nleqslant^n_\omega \mathrm{Q}$ *for all* n*, then* $\Gamma \nvdash \mathrm{Q} \to \mathrm{P}$.

Notice that the Γ in this corollary includes full arithmetical induction. An interesting example of the application of this corollary is to take Q to be RT^2_2 and P to be $\mathrm{RT}^2_{<\infty}$. Cholak, Jockusch, and Slaman [16] showed that $\mathrm{RCA}_0 \nvdash \mathrm{RT}^2_k \to \mathrm{RT}^2_{<\infty}$ for all k, but the proof relies on a difference between the first-order parts of these two principles, and hence does not work if we add arithmetical induction to RCA_0. (Note that, with full induction, $\mathrm{RT}^2_{<\infty}$ does in fact follow from RT^2_2.) Patey [52] showed that $\mathrm{RT}^2_{<\infty} \nleqslant^n_\omega \mathrm{RT}^2_k$ for all n and k, so we have the following.

Corollary 3.3.3. *Let* Γ *consist of* RCA_0 *together with all* Π^1_1 *formulas true over the natural numbers. Then* $\Gamma \nvdash \mathrm{RT}^2_k \to \mathrm{RT}^2_{<\infty}$ *for all* k.

We learned from Yokoyama [personal communication] that he and Slaman have recently noticed that this corollary can also be obtained by a more direct model-theoretic argument, still using Patey's result.

The proof of Theorem 3.3.1 will use the following result, which is of independent interest as a generalization of Theorem 3.1.5.

Theorem 3.3.4. *Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1 formulas that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems. For $n \in \omega$, let $\Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n)$ be a formula asserting that*

$$\begin{gathered}
\text{if } X_0 \text{ is a P-instance then } (Y_0 = \Phi^{X_0}_{e_0} \wedge (\text{either } Y_0 \text{ is a solution to } X_0 \text{ or} \\
(Y_0 \text{ is a Q-instance and if } X_1 \text{ is a solution to } Y_0 \text{ then } (Y_1 = \Phi^{X_0 \oplus X_1}_{e_1} \wedge \\
(\text{either } Y_1 \text{ is a solution to } X_0 \text{ or} \\
(Y_1 \text{ is a Q-instance and if } X_2 \text{ is a solution to } Y_1 \text{ then } (Y_2 = \Phi^{X_0 \oplus X_1 \oplus X_2}_{e_2} \wedge \\
(\text{either } Y_2 \text{ is a solution to } X_0 \text{ or } \ldots \\
\vdots \\
\ldots (Y_n = \Phi^{X_0 \oplus \cdots \oplus X_n}_{e_n} \wedge Y_n \text{ is a solution to } X_0)) \cdots),
\end{gathered}$$

and let Δ_n be

$$\forall X_0\, \exists e_0, Y_0\, \forall X_1\, \exists e_1, Y_1 \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

If $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$, then there exists an $n \in \omega$ such that $\Gamma \vdash \Delta_n$.

Proof. Suppose that $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ but $\Gamma \vdash \neg\Delta_n$ for all n. Extend L_2 to include a function symbol f from first-order objects to second-order objects. Call this new language L_2'. Let $\langle \cdot, \ldots, \cdot \rangle$ be a fixed numbering scheme for finite tuples of numbers.

For each n, there is a model $\mathcal{M} = (M, S)$ of $\Gamma + \neg\Delta_n$. We can turn $\mathcal{M}$ into an L_2'-structure by defining the interpretation $f^{\mathcal{M}}$ by recursion as follows.

There is an $X_0 \in S$ such that

$$\mathcal{M} \vDash \forall e_0, Y_0\, \exists X_1\, \forall e_1, Y_1 \cdots \exists X_n\, \forall e_n, Y_n\, \neg\Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Let $f^{\mathcal{M}}(\langle\rangle) = X_0$.

Assume we have defined $f^{\mathcal{M}}(\langle e_0, \ldots, e_{j-1}\rangle)$, where $j < n$, and have also defined $Y_{\langle e_0\rangle}, Y_{\langle e_0,e_1\rangle}, \ldots, Y_{\langle e_0,\ldots,e_{j-1}\rangle} \in S$ so that

$$\begin{aligned}\mathcal{M} \vDash \forall e_j, Y_j\, \exists X_{j+1}\, \forall e_{j+1}, Y_{j+1} \exists X_{j+2} \cdots \exists X_n\, \forall e_n, Y_n\, \neg\Theta_n(e_0, \ldots, e_n,\\ f^{\mathcal{M}}(\langle\rangle), f^{\mathcal{M}}(\langle e_0\rangle), \ldots, f^{\mathcal{M}}(\langle e_0, \ldots, e_{j-1}\rangle), X_{j+1} \ldots, X_n,\\ Y_{\langle e_0\rangle}, Y_{\langle e_0,e_1\rangle}, \ldots, Y_{\langle e_0,\ldots,e_{j-1}\rangle}, Y_j, \ldots, Y_n).\end{aligned}$$

Given $e_j \in M$, let $Y_{\langle e_0,\ldots,e_j\rangle} = \Phi_{e_j}^{(f^{\mathcal{M}}(\langle\rangle)\oplus f^{\mathcal{M}}(\langle e_0\rangle)\oplus\cdots\oplus f^{\mathcal{M}}(\langle e_0,\ldots,e_{j-1}\rangle))} \in S$. Then there is an $X_{j+1} \in S$ such that

$$\begin{aligned}\mathcal{M} \vDash \forall e_{j+1}, Y_{j+1}\, \exists X_{j+2}\, \forall e_{j+2}, Y_{j+2} \exists X_{j+3} \cdots \exists X_n\, \forall e_n, Y_n\, \neg\Theta_n(e_0, \ldots, e_n,\\ f^{\mathcal{M}}(\langle\rangle), f^{\mathcal{M}}(\langle e_0\rangle), \ldots, f^{\mathcal{M}}(\langle e_0, \ldots, e_{j-1}\rangle), X_{j+1} \ldots, X_n,\\ Y_{\langle e_0\rangle}, Y_{\langle e_0,e_1\rangle}, \ldots, Y_{\langle e_0,\ldots,e_j\rangle}, Y_{j+1}, \ldots, Y_n).\end{aligned}$$

Let $f^{\mathcal{M}}(\langle e_0, \ldots, e_j\rangle) = X_{j+1}$.

Having defined $f^{\mathcal{M}}$ on all $\langle e_0, \ldots, e_i\rangle$ for $i \leqslant n$, let $f^{\mathcal{M}}(x) = \emptyset$ for all other $x \in M$.

Let

$$\begin{aligned}\Psi_k \equiv \forall e_0, Y_0 \cdots \forall e_k, Y_k\, \neg\Theta_k(e_0, \ldots, e_k,\\ f(\langle\rangle), f(\langle e_0\rangle), \ldots, f(\langle e_0, \ldots, e_k\rangle), Y_0, \ldots, Y_k).\end{aligned}$$

Then $(\mathcal{M}; f^{\mathcal{M}}) \vDash \Psi_n$ by the definition of $f^{\mathcal{M}}$. It is easy to see that this fact implies that $(\mathcal{M}; f^{\mathcal{M}}) \vDash \Psi_k$ for all $k \leqslant n$.

Thus every set $\Gamma \cup \{\Psi_0, \ldots, \Psi_n\}$ is satisfiable, and hence so is the union $\Gamma \cup \{\Psi_0, \Psi_1, \ldots\}$. Let $\mathcal{N} = (N, T)$ be a model of this set. Now we have a winning strategy for Player 1 for

$G^\Gamma(\mathrm{Q} \to \mathrm{P})$: Player 1 begins by playing N and $f^{\mathcal{N}}(\langle\rangle)$, and if $e_0, \ldots, e_{n-1}$ are indices for Player 2's first n moves, then Player 1 plays $f^{\mathcal{N}}(\langle e_0, \ldots, e_{n-1}\rangle)$ on its next move. By the definition of Ψ_n, Player 2 can never play a solution to $f^{\mathcal{N}}(\langle\rangle)$.

But by Proposition 3.2.4 and our assumption that $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$, Player 2 must have a winning strategy for $G^\Gamma(\mathrm{Q} \to \mathrm{P})$, so we have a contradiction. □

Proof of Theorem 3.3.1. We use the notation of Theorem 3.3.4. By Proposition 3.2.4, it is enough to show that if $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ then there is an n such that Player 2 has a winning strategy for $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most n many moves. So suppose that $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$. Let n be as in Theorem 3.3.4.

Player 2 can play as follows. Let $\mathcal{M} = (M, S)$ be the model of Γ played by Player 1 on its first move. Since $\mathcal{M}$ is a model of Γ, it is also a model of Δ_n. Let X_0 be Player 1's first move. Since X_0 is in S, there are $e_0 \in M$ and $Y_0 \in S$ such that $\mathcal{M}$ satisfies

$$\forall X_1\, \exists e_1, Y_1\, \forall X_2\, \exists e_2, Y_2 \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Now Player 2 plays Y_0. Let X_1 be Player 1's next move. Then there are $e_1 \in M$ and $Y_1 \in S$ such that $\mathcal{M}$ satisfies

$$\forall X_2\, \exists e_2, Y_2\, \forall X_3\, \exists e_3, Y_3 \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Now Player 2 plays Y_1.

Continuing in this way, by the definition of Δ_n, some Y_i with $i \leqslant n$ must be a solution to X_0, and thus this strategy ensures victory by Player 2 in at most $n+1$ many moves. □

We do not know whether Theorem 3.3.1 holds for $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ in general, but normally, if $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ then the proof allows us to obtain a winning strategy for Player 2 in $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ (and even in $G^\Gamma(\mathrm{Q} \to \mathrm{P})$) that is relatively easy to describe. (The special case of computable winning strategies will be discussed in Section 3.4.) In such cases, we can show that there is an n such that this particular winning strategy allows Player 2 to win in at most n many moves, not just in $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ but in fact in $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$. Here we are thinking of strategies that are first-order definable, but we need to take into account the possibility that there might not be a unique choice of move at a given point (keeping in mind that the idea of choosing the least among the indices of equally good moves is not always available when working over nonstandard models).

Definition 3.3.5. Let Γ be a consistent set of L_2-formulas and let $\Lambda(X, n, e)$ be an arithmetic formula. Say that Player 2 plays a run of $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ or $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ *according to* Λ if given Player 1's first n moves, M (or (M, S)) and $X_0, \dots, X_{n-1} \subseteq M$, Player 2 plays $\Phi_e^{X_0 \oplus \cdots \oplus X_{n-1}}$ for some $e \in M$ such that $M[X_0, \dots, X_{n-1}] \vDash \Lambda(X_0 \oplus \cdots \oplus X_{n-1}, n-1, e)$.

Theorem 3.3.6. *Let Γ be a consistent extension of Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula. Let* P *and* Q *be Π^1_2-problems and Λ be an arithmetic formula such that Player 2 wins any run of $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ that it plays according to Λ. Then there is an n such that Player 2 wins any run of $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ that it plays according to Λ in at most n many moves.*

Proof. Let Θ_n be as in Theorem 3.3.4. Let Ξ_n be a formula asserting that, for all $i \leqslant n$, if X_0 is a P-instance and no Y_j with $j < i$ is a solution to X_0, then $\Lambda(X_0 \oplus \cdots \oplus X_i, i, e_i)$. Let $\widehat{\Theta}_n$ be $\Xi_n \to \Theta_n$, and let Ω_n be

$$\forall X_0\, \forall e_0\, \exists Y_0\, \forall X_1\, \forall e_1\, \exists Y_1 \cdots \forall X_n\, \forall e_n\, \exists Y_n \qquad \widehat{\Theta}_n(e_0, \dots, e_n, X_0, \dots, X_n, Y_0, \dots, Y_n).$$

Suppose there is a run of $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ such that Player 2 plays according to Λ but does not win within n moves. Let M and $X_0, \dots, X_{n-1}$ be Player 1's first n moves in that run. Then

$M[X_0, \ldots, X_{n-1}]$ can be extended to a model (M, S) of Γ, and in that model, Ω_{n-1} does not hold. Thus, to establish the theorem, it is enough to show that $\Gamma \vdash \Omega_n$ for some n.

Assume for a contradiction that $\Gamma \nvdash \Omega_n$ for all n. Expand L_2 by adding first-order constant symbols $c_0, c_1, \ldots$ and second-order constant symbols $C_0, C_1, \ldots$. Then a compactness argument as in the proof of Theorem 3.3.4 shows that there is a model $\mathcal{M}$ of Γ and interpretations $c_0^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots$ and $C_0^{\mathcal{M}}, C_1^{\mathcal{M}}, \ldots$ such that each $\Phi_{c_n^{\mathcal{M}}}^{C_0^{\mathcal{M}} \oplus \cdots \oplus C_n^{\mathcal{M}}}$ is total in $\mathcal{M}$, and $\mathcal{M}$ together with these interpretations satisfies

$$\neg\widehat{\Theta}_n(c_0, \ldots, c_n, C_0, \ldots, C_n, \Phi_{c_0}^{C_0}, \ldots, \Phi_{c_n}^{C_0 \oplus \ldots \oplus C_n})$$

for all n. But then there is a run of $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ in which Player 2 plays according to Λ but does not win, namely the one in which Player 1 begins by playing $\mathcal{M}$, then at each move plays $C_n^{\mathcal{M}}$, and Player 2 responds with $\Phi_{c_n^{\mathcal{M}}}^{C_0^{\mathcal{M}} \oplus \cdots \oplus C_n^{\mathcal{M}}}$, which contradicts our hypothesis. □

For Γ is as in Theorem 3.3.1, write $\Gamma \vdash^n \mathrm{Q} \to \mathrm{P}$ to mean that Player 2 has a winning strategy for $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most $n+1$ many moves. Then the first part of the theorem can be restated as $\Gamma \vdash \mathrm{Q} \to \mathrm{P} \;\Rightarrow\; \exists n\, [\Gamma \vdash^n \mathrm{Q} \to \mathrm{P}]$. The idea behind this notation is that we can see the least n such that $\Gamma \vdash^n \mathrm{Q} \to \mathrm{P}$ as a measure of the number of applications of Q needed to prove P over Γ. The $n = 0$ case is equivalent to $\Gamma \vdash \mathrm{P}$. We will discuss the $n = 1$ case in Section 3.5, but make the following remark for now.

Remark 3.3.7. Recall that $\mathrm{P} \leqslant_{\omega}^{n} \mathrm{Q}$ means that Player 2 has a winning strategy for $G(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most $n+1$ many moves. Hirschfeldt and Jockusch [34] stated that $\mathrm{P} \leqslant_{\omega}^{1} \mathrm{Q}$ is equivalent to $\mathrm{P} \leqslant_{\mathrm{c}} \mathrm{Q}$, but that is not quite correct, because if P is computably true (i.e., if $\mathrm{P} \leqslant_{\omega}^{0} \mathrm{Q}$) but has an instance that does not compute any instance of Q, then $\mathrm{P} \leqslant_{\omega}^{1} \mathrm{Q}$ but $\mathrm{P} \nleqslant_{\mathrm{c}} \mathrm{Q}$. (The same point was made in the context of Weihrauch reducibility by Brattka, Gherardi, and Pauly [10, Section 3].) As this fairly uninteresting case is the only in which the two notions differ, however, we can generally ignore the distinction. We mention it, and make the following remarks, only because an analogous situation will be relevant below.

We can define $\mathrm{P} \leqslant_{\omega}^{=n} \mathrm{Q}$ to mean that Player 2 has a winning strategy for $G(\mathrm{Q} \to \mathrm{P})$ that ensures victory in exactly $n+1$ many moves. Then $\mathrm{P} \leqslant_{\mathrm{c}} \mathrm{Q}$ is equivalent to $\mathrm{P} \leqslant_{\omega}^{=1} \mathrm{Q}$. This definition is not otherwise very useful, though, because if Player 2 can win $G(\mathrm{Q} \to \mathrm{P})$ in $m \geqslant 2$ many moves, then it can also win that game in k many moves for any $k > m$, simply by repeating its first move until it is ready to win, except in the case in which Player 2's first move is an instance of Q with no solution (and in this context we are generally not interested in problems that are false over ω as statements of second-order arithmetic).

Note also that $\mathrm{P} \leqslant_{\omega}^{n} \mathrm{Q}$ is not quite equivalent to $\exists m \leqslant n\, [\mathrm{P} \leqslant_{\omega}^{=m} \mathrm{Q}]$, again because of 1-move runs. For example, let P be the Π^1_2-problem whose instances are $\emptyset$ and $\emptyset'$, with unique solutions $\emptyset$ and $\emptyset''$, respectively; and let Q be the Π^1_2-problem whose only instance is $\emptyset'$, with unique solution $\emptyset''$. If Player 1 begins by playing $\emptyset'$, then Player 2 cannot win immediately, but can play $\emptyset'$, to which Player 1 must reply with $\emptyset''$, at which point Player 2 wins by playing $\emptyset''$. So in this case, Player 2 wins in 2 moves. However, if Player 1 plays $\emptyset$, then Player 2 has only one legal move, namely the winning move $\emptyset$. Thus $\mathrm{P} \leqslant_{\omega}^{1} \mathrm{Q}$, but the first case shows that $\mathrm{P} \nleqslant_{\omega}^{=0} \mathrm{Q}$, while the second case shows that $\mathrm{P} \nleqslant_{\omega}^{=1} \mathrm{Q}$.

Similar considerations hold for the notion of $\mathrm{P} \leqslant_{\mathrm{gW}}^{n} \mathrm{Q}$ introduced in [34], and for $\Gamma \vdash^{n} \mathrm{Q} \to \mathrm{P}$. One way around these issues is to replace Q with the problem $\widehat{\mathrm{Q}}$ where an instance is either $\{0\} \cup \{n+1 : n \in X\}$ for an instance X of Q, with a solution to this instance being any solution to X; or $\emptyset$, with the only solution being $\emptyset$ (although if we allow problems Q that have instances with no solutions, we might still have $\mathrm{P} \leqslant_{\mathrm{gW}}^{n} \widehat{\mathrm{Q}}$ but not have $\exists m \leqslant n\, [\mathrm{P} \leqslant_{\mathrm{gW}}^{=m} \widehat{\mathrm{Q}}]$, because a computable winning strategy might not be able to tell when it is about to play an instance of Q with no solution, and thus instantly win).

The definition of $\Gamma \vdash^{n} \mathrm{Q} \to \mathrm{P}$ was made in [34] (for $\Gamma = \mathrm{RCA}_0$), but with $G^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ in place of $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$. We have chosen our definition in light of Theorem 3.3.1, but at least in natural cases, there should be no difference, as shown by the following fact.

Proposition 3.3.8. *Let* Γ *be a consistent extension of* Δ^0_1*-comprehension that proves the existence of a universal* Σ^0_1 *formula. Let* P *and* Q *be* Π^1_2*-problems and* Λ *be an arithmetic formula such that Player 2 wins any run of* $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ *that it plays according to* Λ *in at most* n *many moves. Then Player 2 wins any run of* $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ *that it plays according to* Λ *in at most* n *many moves.*

Proof. In the notation of the proof of Theorem 3.3.6, it is easy to see that $\Gamma \vdash \Omega_{n-1}$, and hence Player 2 has a winning strategy for $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ that ensures victory in at most n many moves as in that proof. □

Remark 3.3.9. Hirst and Mummert [39] discussed a different potential form of instance-counting, based on a notion of proving a Π^1_2 principle P with one typical use of another Π^1_2 principle Q in a system Γ. While the definition of that notion in their paper is not quite correct [Hirst and Mummert, personal communication], its main significance is that it allowed them to conclude that, in cases of interest, Γ then proves that for every instance X of P, there is an instance Y of Q such that if Y has a solution then so does X. While their paper is mostly concerned with intuitionistic logic, they also gave examples showing that this notion does not seem useful in the context of classical logic. In particular they showed how RT^2_4 can be obtained with one typical use of RT^2_2 over RCA_0, contrary both to our intuition and to the fact that $\mathrm{RCA}_0 \nvdash^1 \mathrm{RT}^2_2 \to \mathrm{RT}^2_4$, which follows from Patey's result [53] that $\mathrm{RT}^2_4 \nleq_{\mathrm{c}} \mathrm{RT}^2_2$. In fact, as conjectured by J. Miller [Hirst and Mummert, personal communication], this phenomenon is not a particularity of this and other examples mentioned in [39], but is in fact completely general. Indeed, in classical logic, if $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$ then we can always argue in Γ as follows: Let X be an instance of P. Then there are i and Y such that either $i = 0$ and Y is a solution to X, or $i = 1$ and Y is an instance of Q with no solution. If $i = 1$ then we get a contradiction from one use of Q, so $i = 0$ and hence Y is a solution to X.

Perhaps more satisfying than the above argument is the following one, which is directly in the style of the one given in [39] for RT^2_2 and RT^2_4. Let Γ be as in Theorem 3.3.1, and let P and Q be Π^1_2-problems such that $\Gamma \vdash \mathrm{Q} \to \mathrm{P}$. Let Θ_n and Δ_n be as in Theorem 3.3.4. By that theorem, there is an n such that $\Gamma \vdash \Delta_n$. The following proof can be carried out in Γ.

Let X_0 be an instance of P. For each $k = 0, \ldots, n$ in turn, proceed as follows. Given $X_0, \ldots, X_k$, $e_0, \ldots, e_{k-1}$, and $Y_0, \ldots, Y_{k-1}$, let e_k and Y_k be such that

$$\forall X_{k+1}\, \exists e_{k+1}, Y_{k+1} \cdots \forall X_n\, \exists e_n, Y_n\, \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

If Y_k is a solution to X_0 then let $Y = Y_k$ and let $i = 0$. Otherwise, Y_k is an instance of Q. Either that instance has a solution or not. If it does not then let $Y = Y_k$ and let $i = 1$. If it does, then let X_{k+1} be such a solution.

By the definition of Θ_n, we must eventually define Y, i, and j. If $i = 1$ then Y is an instance of Q with no solution. But with one application of Q, we can obtain a solution to Y, so we must have $i = 0$, and hence Y is a solution to X_0.

3.4 Computable winning strategies

We now turn to the notion of generalized Weihrauch reducibility for games over possibly nonstandard models. Let Γ be a set of L_2-formulas consistent with Δ^0_1-comprehensiaon that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems. A *computable strategy* for Player 2 in $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ or $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ consists of Player 2 playing according to the formula $e = \Phi_k(n-1)$ (in the sense of Definition 3.3.5) for some $k \in \omega$.

Remark 3.4.1. To be precise, in the above definition we also need to have a mechanism to distinguish computably when Player 2 has played a winning move. Formally, we can simply slightly alter our games so that a move by Player 2 is either $\{n+1 : n \in Y\}$ where Y is a Q-instance or $\{0\} \cup \{n+1 : n \in Y\}$ where Y is a solution to Player 1's first move X_0.

Combining Theorem 3.3.6 and Proposition 3.3.8 gives us the following.

Proposition 3.4.2. *Let* Γ *be a consistent extension of* Δ^0_1*-comprehension that proves the existence of a universal* Σ^0_1 *formula, and let* P *and* Q *be* Π^1_2*-problems. Then the following are equivalent.*

(1) *Player 2 has a computable winning strategy for* $G^\Gamma(\mathrm{Q} \to \mathrm{P})$*.*

(2) *Player 2 has a computable winning strategy for* $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$*.*

(3) *There is an* $n \in \omega$ *such that Player 2 has a computable strategy for* $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ *that ensures victory in at most* n *many moves.*

(4) *There is an* $n \in \omega$ *such that Player 2 has a computable strategy for* $\widehat{G}^\Gamma(\mathrm{Q} \to \mathrm{P})$ *that ensures victory in at most* n *many moves.*

Furthermore, n *witnesses* (3) *iff it witnesses* (4)*.*

If the conditions in this proposition hold, then we say that P is *generalized Weihrauch reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{gW}} \mathrm{Q}$. We can of course define an instance-counting version of this notion, writing $\mathrm{P} \leqslant^{\Gamma,n}_{\mathrm{gW}} \mathrm{Q}$ if $n+1$ witnesses that item (3) above holds.

As an example of the application of Proposition 3.4.2, we can obtain an analog of Corollary 3.3.3, using the fact that Hirschfeldt and Jockusch [34, Theorem 4.21] showed that $\mathrm{RT}^1_{<\infty} \nleqslant^n_{\mathrm{gW}} \mathrm{RT}^1_k$ for all n, while Patey [52, Theorem 6.0.1] showed that the same holds for higher exponents. (Notice that Corollary 3.3.3 itself works only for exponent 2, since $\mathrm{RT}^1_{<\infty}$ is provable in $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$, while RT^n_k for $k > 1$ and $\mathrm{RT}^n_{<\infty}$ are both equivalent to ACA_0 over RCA_0 for $n > 2$, as shown by Simpson [59] using work of Jockusch [40].)

Corollary 3.4.3. *Let* Γ *consist of* RCA_0 *together with all* Π^1_1 *formulas true over the natural numbers. Then* $\mathrm{RT}^n_{<\infty} \nleqslant^\Gamma_{\mathrm{gW}} \mathrm{RT}^n_k$ *for all* n *and* k*.*

Kuyper [43] studied a notion closely related to this kind of instance-counting (though he considered only the case where Γ is RCA_0). We give a slightly different definition that is easily seen to be equivalent to his.

Definition 3.4.4. Let P and Q be Π^1_2-problems. Say that P *Weihrauch-reduces to the composition of* n *many copies of* Q *via* $e_0, \ldots, e_n$ if for every $X_0, \ldots, X_n$,

if X_0 is a P-instance then

$$\Phi_{e_0}^{X_0} \text{ is a Q-instance and if } X_1 \text{ is a solution to } \Phi_{e_0}^{X_0} \text{ then}$$

$$\Phi_{e_1}^{X_0 \oplus X_1} \text{ is a Q-instance and if } X_2 \text{ is a solution to } \Phi_{e_1}^{X_0 \oplus X_1} \text{ then}$$

$$\vdots$$

$$\Phi_{e_{n-1}}^{X_0 \oplus \cdots \oplus X_{n-1}} \text{ is a Q-instance and if } X_n \text{ is a solution to } \Phi_{e_{n-1}}^{X_0 \oplus \cdots \oplus X_{n-1}} \text{ then}$$

$$\Phi_{e_n}^{X_0 \oplus \cdots \oplus X_n} \text{ is a solution to } X_0.$$

(Note that in the $n = 0$ case, this statement becomes

$$\text{if } X_0 \text{ is a P-instance then } \Phi_{e_0}^{X_0} \text{ is a solution to } X_0.)$$

Kuyper considered the situation where there are $n \in \omega$ and $e_0, \ldots, e_n \in \omega$ such that RCA_0 proves that P Weihrauch-reduces to the composition of n many copies of Q via $e_0, \ldots, e_n$. For a fixed n, it is not difficult to see that this condition is equivalent to saying that Player 2 has a a computable winning strategy for $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ that ensures victory in exactly $n+1$ many moves, unless it wins earlier by playing an instance of Q with no solution. One might think that this is the same as saying that there is an n such that $\mathrm{P} \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0, n} \mathrm{Q}$, and hence by Proposition 3.4.2 to $\mathrm{P} \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{Q}$, but Remark 3.3.7 applies here as well. The example given there shows that it is possible to have $\mathrm{P} \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0, 1} \mathrm{Q}$ but not have Kuyper's condition hold. However, Kuyper's condition is equivalent to $\mathrm{P} \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \widehat{\mathrm{Q}}$ for the modified problem $\widehat{\mathrm{Q}}$

defined in that remark, so we we will express it in this form.

Kuyper [43] claimed that his condition is equivalent to a form of intuitionistically provable implication. Uftring [63, **?**] found a counterexample that shows that Kuyper's argument is flawed. Kuyper (see [63, **?**]) proposed fixing his proof by replacing the condition $\mathrm{P} \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \widehat{\mathrm{Q}}$ with $\mathrm{P} \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0 + \mathrm{Q}} \widehat{\mathrm{Q}}$. Uftring's example shows that it is possible for Player 2 to have a computable winning strategy for $G^{\mathrm{RCA}_0+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$ but not for $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$, in contrast with the case for general winning strategies in Proposition 3.2.4, so we present a version of it now. We will give another example with the same properties in Section 3.6.

Example 3.4.5 (Uftring [63, **?**])**.** The proof of Gödel's Incompleteness Theorem shows that there is a primitive recursive predicate G such that $G(n)$ holds for all $n \in \omega$ but RCA_0 cannot prove $\forall x\, G(x)$. For $X \neq \emptyset$, write μX for the least element of X. Let

$$\mathrm{P} \equiv \forall X\, \exists Y\, \forall x\, G(x)$$

and

$$\mathrm{Q} \equiv \forall X\, [X \neq \emptyset \to \exists Y\, G(\mu X)].$$

In $G^{\mathrm{RCA}_0+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$, Player 1's first move M and X_0 must be such that $M[X_0]$ is consistent with Q, so $M[X_0] \vDash \forall x\, G(x)$, and hence Player 2 can play, say, $\emptyset$ on its first move and win. In $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$, however, Player 1 can play an $M \vDash \neg\forall x\, G(x)$, together with, say, $X_0 = \emptyset$. Then this instance of P has no solution, so the only way for Player 2 to win is eventually to play an M-instance of Q with no solution, that is, an X such that $M \vDash \neg G(\mu X)$.

For any model M of Σ^0_1-PA, we can consider a run in which Player 1 plays M and then keeps playing $\emptyset$ until Player 2 either declares victory or wins by playing an M-instance of Q with no solution. (Notice that we can computably determine if the latter case holds, since the condition $G(\mu X)$ is computable.) If Player 2 has a computable winning strategy for $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$, then there is a computable procedure that, over any model M of Σ^0_1-PA, simulates the above run, making Player 2's moves according to this procedure, outputting 0 if Player 2 declares victory, and outputting μX if Player 2 plays the M-instance X of Q with no solution. The output of this procedure is 0 iff $M \vDash \forall x\, G(x)$. Since this procedure works for any model M of Σ^0_1-PA, we have an existential first-order sentence that is provably equivalent to $\forall x\, G(x)$ over RCA_0, which is a contradiction, because any existential first-order sentence true in the standard natural numbers is provable in RCA_0.

For some Π^1_2-problems Q, on the other hand, there is no difference between $G^{\mathrm{RCA}_0+\mathrm{Q}}(\mathrm{Q} \to \mathrm{P})$ and $G^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$ because every countable model of RCA_0 can be extended to a countable model of $\mathrm{RCA}_0 + \mathrm{Q}$ with the same first-order part, and hence the notion of consistency used in Definition 3.2.1 is the same for RCA_0 and $\mathrm{RCA}_0 + \mathrm{Q}$. (Showing that this is the case for a given Q is typically done to show that Q is Π^1_1-conservative over RCA_0.) Examples include WKL, as shown by Harrington (see [60, Theorem IX.2.1]), COH, as shown by Cholak, Jockusch, and Slaman [16], and AMT, as shown by Hirschfeldt, Shore, and Slaman [37].

As highlighted by the work of Kuyper and Uftring, the connections with intuitionistic provability are rather subtle, and we believe that generalized Weihrauch reducibility over possibly nonstandard models can be useful in clarifying them. However, as the methods and issues are rather different from the ones used here, we leave this for future work.

3.5 Single-instance reductions

As noted in Remark 3.3.7, $\mathrm{P} \leqslant_{\mathrm{c}} \mathrm{Q}$ iff Player 2 has a strategy for $G(\mathrm{Q} \to \mathrm{P})$ that ensures victory in exactly two moves. Similarly, $\mathrm{P} \leqslant_{\mathrm{W}} \mathrm{Q}$ iff Player 2 has a computable strategy for $G(\mathrm{Q} \to \mathrm{P})$ that ensures victory in exactly two moves. We can define the analogous notions for games over possibly nonstandard models. Let us explicitly define these analogs for computable and Weihrauch reducibilities, and then look at several examples involving them. Although we will not work with them here, we also define the analogs of several related notions of computability-theoretic reduction between Π^1_2-problems.

Definition 3.5.1. Let Γ be a set of L_2-formulas consistent with Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula and let P and Q be Π^1_2-problems.

1. We say that P is *computably reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{c}} \mathrm{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in $M[X]$ such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[X, \widehat{Y}]$.

2. We say that P is *Weihrauch reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{W}} \mathrm{Q}$, if there are $e, i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, the set $\widehat{X} = \Phi^X_e$ is an M-instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi^{X \oplus \widehat{Y}}_i$ is a solution to X.

3. We say that P is *strongly computably reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{sc}} \mathrm{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in $M[X]$ such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[\widehat{Y}]$.

4. We say that P is *strongly Weihrauch reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{sW}} \mathrm{Q}$, if there are $e, i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, the set $\widehat{X} = \Phi^X_e$ is an M-instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi^{\widehat{Y}}_i$ is a solution to X.

5. We say that P is *omnisciently computably reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{oc}} \mathrm{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[X, \widehat{Y}]$.

6. We say that P is *omnisciently Weihrauch reducible over* Γ to Q, and write $\mathrm{P} \leqslant^\Gamma_{\mathrm{oW}} \mathrm{Q}$, if there is an $i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed

under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi_i^{X \oplus \widehat{Y}}$ is a solution to X.

7. We say that P is *strongly omnisciently computably reducible over* Γ to Q, and write $\mathrm{P} \leqslant^{\Gamma}_{\mathrm{soc}} \mathrm{Q}$, if for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, there is a solution to X in $M[\widehat{Y}]$.

8. We say that P is *strongly omnisciently Weihrauch reducible over* Γ to Q, and write $\mathrm{P} \leqslant^{\Gamma}_{\mathrm{soW}} \mathrm{Q}$, if there is an $i \in \omega$ such that for every model (M, S) of Γ with M countable and S closed under Δ^0_1-comprehension, and every M-instance X of P in S, there is an M-instance $\widehat{X}$ of Q in S such that for every solution $\widehat{Y}$ to $\widehat{X}$ in S, the set $\Phi_i^{\widehat{Y}}$ is a solution to X.

Remark 3.5.2. In light of comments made above, it might be more natural to consider versions corresponding to games in which Player 2 can always win in one or two moves, rather than exactly two moves (even if in natural cases, there will be no difference). Rather than introduce more terminology and notation, however, that can be done simply by replacing Q with the problem $\widehat{\mathrm{Q}}$ from Remark 3.3.7 in the above definitions.

The study of Weihrauch reducibility in this extended setting seems particularly promising, given the extensive theory that has been developed for Weihrauch reducibility over the standard natural numbers. In particular, there are several operators on the Weihrauch degrees whose analogs in this setting should be of interest. One example is the finite parallelization: For a problem P, the problem P^* is the one whose instances consist of finitely many instances $X_0, \dots, X_k$ of P, with a solution consisting of one solution to each X_i. Clearly, $\mathrm{P}^* \leqslant_{\mathrm{gW}} \mathrm{P}$ for any Π^1_2-problem P, but this fact does not hold in our setting, because given an instance $X_0, \dots, X_k$ of P^*, the obvious reduction strategy for Player 2 takes $k + 1$ many moves, and

k might be nonstandard. The following example will be relevant in the next section.

Example 3.5.3. Recall that Pauly, Fouché, and Davie [55] defined Bound as follows: An instance is an enumeration of a bounded set F, and a solution is a bound on the elements of F. An instance of Bound^* is then a simultaneous enumeration of a finite family $F_0, \ldots, F_k$ of bounded sets, and a solution to this instance consists of a bound for each F_i, or, equivalently, a bound b on $\bigcup_{i \leqslant k} F_k$. (This is basically the principle FUF studied by Frittaion and Marcone [28].) It is easy to see that Bound and Bound^* are Weihrauch-equivalent, but that is no longer the case for Weihrauch-equivalence (or even provable equivalence) over RCA_0, since as statements in second-order arithmetic, Bound is trivially true, while Bound^* is a way to state $\mathrm{B}\Pi^0_1$, and hence is equivalent to $\mathrm{B}\Sigma^0_2$ over RCA_0, as we further discuss in the following section. Thus $\mathrm{RCA}_0 \nvdash \mathrm{Bound} \to \mathrm{Bound}^*$, and hence $\mathrm{Bound}^* \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Bound}$.

It is not clear what the correct generalization of the $^\diamond$ operator of Neumann and Pauly [48] to this setting is. However, one would expect that it would still have the property that P^* is reducible to $\mathrm{P}^\diamond$, and hence, by the above example, that it would no longer be equivalent to gW-reducibility.

On the other hand, it is clear that, as for standard Weihrauch reducibility, if $\mathrm{P} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Q}$ then $\mathrm{P}^* \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Q}^*$. It is also not difficult to see that, more generally, if $\mathrm{P} \leqslant^{\mathrm{RCA}_0, n}_{\mathrm{gW}} \mathrm{Q}$ then $\mathrm{P}^* \leqslant^{\mathrm{RCA}_0, n}_{\mathrm{gW}} \mathrm{Q}^*$. Thus, by Proposition 3.4.2, if $\mathrm{P} \leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Q}$ then $\mathrm{P}^* \leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Q}^*$. (The same holds for other appropriate systems in place of RCA_0, of course.)

An important point here is that while the principles we consider in reverse mathematics are typically true—in the sense that they hold in $(\omega, \mathcal{P}(\omega))$, or equivalently for Π^1_2-problems, that every instance (over the standard natural numbers) has at least one solution—many of them have nontrivial first-order parts. For example, if $\mathrm{B}\Sigma^0_2$ fails in M, then M cannot be the first-order part of a model of $\mathrm{RCA}_0 + \mathrm{RT}^1_{<\infty}$ (or of $\mathrm{RCA}_0 + \mathrm{RT}^n_k$ for any $n, k \geqslant 2$). Furthermore, for any such M there is an instance of $\mathrm{RT}^1_{<\infty}$ (i.e., a $k \in |M|$ together with a function $c : |M| \to \{j \in |M| : j <^M k\}$) with no solutions. The same is true of Bound^*, to give another example.

We want to use notions such as Weihrauch reducibility over RCA_0 and other systems to study these kinds of principles (as we will do in the next two sections), so it is important that our definitions above do not assume that every instance of a problem has a solution. This fact is particularly worth noting for Weihrauch reducibility, because we usually think of (classical) Weihrauch reducibility between Π^1_2-problems as a special case of the general notion from computable analysis, which is defined using partial multifunctions between represented spaces. (See for instance Brattka, Gherardi, and Pauly [10] or Brattka and Pauly [11].) This point is a bit subtle, and was missed, e.g., in the paper Dorais, Dzhafarov, Hirst, Mileti, and Shafer [21], where a proof is given in Corollaries A.3 and A.4 establishing a correspondence between Π^1_2 principles on the one hand and certain classes of partial multifunctions on the other. Indeed, the proof there works only if the Π^1_2 principles in question are assumed to be true, which is not explicitly mentioned.

There is more than one way to formalize the notion of a partial multifunction between spaces X and Y. One is to say that it is simply a relation $R \subseteq X \times Y$. Then the domain of the multifunction is $\{x \in X : \exists y\,(x, y) \in R\}$. Another is to say that it is a (possibly partial) function from X to the power set of Y. In this case, the domain of the multifunction can include elements that are mapped to no values at all. The first formalization is the one normally used in the definition of Weihrauch reducibility in computable analysis, which is convenient in particular because of the need to use choice functions in working with represented spaces. And indeed, a true Π^1_2-problem P corresponds to the partial multifunction $F : \subseteq 2^\omega \rightrightarrows 2^\omega$ in this sense whose domain is the set of instances of P, and which maps any such instance X to the solutions to X.

This correspondence breaks down for a Π^1_2-problem that has instances with no solutions, however, unless we move to the second formalization of the notion of multifunction, or allow a multifunction to consist of a relation $R \subseteq X \times Y$ together with a set D such that

$\{x \in X : \exists y\,(x, y) \in R\} \subseteq D \subseteq X$, where D represents the domain of the function. This distinction operates even at the level of the Weihrauch degrees (equivalence classes under Weihrauch reducibility), because a problem in which some instance has no solutions can never be Weihrauch reducible to one in which every instance has a solution, and if P has a computable instance with no solutions, then every problem is Weihrauch reducible to P. As discussed in [10], and in more detail in [11], this top degree is usually added to the lattice of Weihrauch degrees as a formal object.

The distinction between the two approaches is also relevant to the notion of *extended Weihrauch reducibility* investigated by Bauer [2] (see also [3]), following work by Bauer and Yoshimura [4, 5]. The focus in that work is on comparing universally quantified statements in the setting of constructive mathematics, using a notion called *instance reducibility*, which can also be understood as an extension of the Weihrauch degrees that in particular allows for "questions that do not have an answer" but that are still "valid" for the purposes of considering whether or not they are reducible to other questions (Bauer [1]).

3.6 Limit-homogeneous sets

In this section and the next, we give some examples of comparisons of Π^1_2-problems using W- and gW-reducibility over possibly nonstandard models, focusing on versions of $\mathrm{B}\Sigma^0_2$. A natural way to think of $\mathrm{B}\Pi^0_1$ as a Π^1_2-problem is to identify a Π^0_1 formula $\varphi(i, k)$ with a simultaneous enumeration of the sets $\{m : \forall k < m\, \neg\varphi(i, k)\}$ for $i < n$. Then a b as in the definition of $\mathrm{B}\Pi^0_1$ is the same as a common bound for these sets. Thus we arrive at Bound*, as defined in Example 3.5.3.

Recall also the Π^1_2-problems SRT^2_2 and D^2_2. Clearly, SRT^2_2 implies D^2_2. Cholak, Jockusch, and Slaman [16] claimed that the converse implication also holds over RCA_0, but their proof actually required $\mathrm{B}\Sigma^0_2$. Chong, Lempp, and Yang [17] closed this gap by showing that D^2_2 implies $\mathrm{B}\Sigma^0_2$ over RCA_0.

The argument in [16] also shows that $\mathrm{SRT}^2_2 \leqslant_{\mathrm{c}} \mathrm{D}^2_2$. Dzhafarov [23] and Brattka and Rakotoniaina [12] showed that $\mathrm{SRT}^2_2 \nleqslant_{\mathrm{W}} \mathrm{D}^2_2$. Hirschfeldt and Jockusch [34] noted that $\mathrm{SRT}^2_2 \leqslant^2_{\mathrm{gW}} \mathrm{D}^2_2$, however. To consider this reduction in more detail, we define the following Π^1_2-problem.

Definition 3.6.1. LH: If $c : [\mathbb{N}]^2 \to 2$ is such that $\lim_y c(x, y) = 1$ for all x, then c has an infinite homogeneous set.

This problem is a convenient way to state the principle that for every 2-coloring of pairs, every infinite limit-homogeneous set has an infinite homogeneous subset.

From the reverse-mathematical perspective, LH is equivalent to $\mathrm{B}\Sigma^0_2$.

Proposition 3.6.2. $\mathrm{RCA}_0 \vdash \mathrm{LH} \leftrightarrow \mathrm{B}\Sigma^0_2$.

Proof. First, assume $\mathrm{B}\Sigma^0_2$. Fix an instance c of LH. Let S be the set of all tuples $(x_0, \ldots, x_{n-1}, y)$ such that $x_0 < \cdots < x_{n-1} < y$ and $c(x_m, y) = 1$ for all $m < n$. We claim that for all $x_0 < \cdots < x_{n-1}$, there is a y such that $(x_0, \ldots, x_{n-1}, y) \in S$. For each $m < n$ there is a $b_m > x_{n-1}$ such that $c(x_m, y) = 1$ for all $y > b_m$. By $\mathrm{B}\Sigma^0_2$ (or really $\mathrm{B}\Pi^0_1$), there is a $b > x_{n-1}$ such that $c(x_m, y) = 1$ for all $m < n$ and $y > b$. Then $(x_0, \ldots, x_{n-1}, b + 1) \in S$, which proves our claim. Now we can define a homogeneous set H for c by primitive recursion: Let $h_0 = 0$, let h_{n+1} be the least y such that $(h_0, \ldots, h_n, y) \in S$, and let $H = \{h_0, h_1, \ldots\}$.

Now assume LH. We prove $\mathrm{RT}^1_{<\infty}$. Assume for a contradiction that $d : \mathbb{N} \to k$ has no infinite homogeneous set. Then for each $i < k$ there is a b such that $d(x) \neq i$ for all $x > b$. Define $c : [\mathbb{N}]^2 \to 2$ by letting $c(x, y) = 0$ if $d(x) = d(y)$ and letting $c(x, y) = 1$ otherwise. Then $\lim_y c(x, y) = 1$ for all x, so by LH, c has an infinite homogeneous set H. Let $x_0 < \cdots < x_k \in H$. Then for all $m < n \leqslant k$, we have that $c(x_m, x_n) = 1$ and hence $d(x_m) \neq d(x_n)$. But then $\{d(x_0), \ldots, d(x_k)\}$ has cardinality $k + 1$, which is impossible. ☐

However, the first part of the above proof shows that LH is computability-theoretically trivial, and indeed uniformly computably true, so that $\mathrm{LH} \leqslant^0_{\mathrm{gW}} \mathrm{P}$ for any P, or equivalently

$\mathrm{LH} \leqslant_{\mathrm{W}} 1$, where 1 is the identity problem for which an instance is any X and the only solution to this instance is X itself. We can obtain SRT^2_2 from D^2_2 as follows: Given a stable coloring $c : \mathbb{N} \to 2$, use D^2_2 to obtain a limit-homogeneous set L. Now an application of RT^1_2 (which is Weihrauch-reducible to D^2_2) yields an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$. We can think of c restricted to L as a coloring of $\mathbb{N}$ by identifying the nth element of L with n. If $i = 0$, we can also replace c by the coloring whose value at (x, y) is $1 - c(x, y)$. We can then apply LH to obtain an infinite homogeneous set for c. Since LH is Weihrauch-trivial, this procedure shows that $\mathrm{SRT}^2_2 \leqslant^2_{\mathrm{gW}} \mathrm{D}^2_2$. (Since the use of RT^1_2 is computably trivial, it also shows that $\mathrm{SRT}^2_2 \leqslant_{\mathrm{c}} \mathrm{D}^2_2$, as mentioned above.)

Over nonstandard models, however, things are different. In the presence of $\mathrm{B}\Sigma^0_2$, the first part of the proof of Proposition 3.6.2 shows that LH is still Weihrauch-trivial, i.e., $\mathrm{LH} \leqslant^{\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2}_{\mathrm{W}} 1$, and hence $\mathrm{SRT}^2_2 \leqslant^{\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2,\, 2}_{\mathrm{gW}} \mathrm{D}^2_2$. Of course, if P does not imply $\mathrm{B}\Sigma^0_2$ over RCA_0, then we cannot have $\mathrm{LH} \leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{P}$. But what if we take P to be some form of $\mathrm{B}\Sigma^0_2$? A natural choice is Bound*, as it is essentially the form of $\mathrm{B}\Pi^0_1$ used in the first part of the proof of Proposition 3.6.2.

We will show that $\mathrm{LH} \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}}$ Bound*, but we can actually obtain a stronger result by considering the contrapositive form of $\mathrm{B}\Pi^0_1$: Given a simultaneous enumeration of sets $F_0, \ldots, F_{n-1}$ with no common bound, there is an $i < n$ such that F_i is infinite. Given such an enumeration, we can define an n-coloring c of $\mathbb{N}$ as follows: for each m, wait until a number greater than m is enumerated into some F_i, then give m the color i. From an infinite homogeneous set for c, we can obtain an $i < n$ such that F_i is infinite. Conversely, given an n-coloring c of $\mathbb{N}$, the sets $F_i = \{m : c(m) = i\}$ for $i < n$ have no common bound, and from an $i < n$ such that F_i is infinite, we can obtain an infinite homogeneous set for c. Both of these processes can be carried out over RCA_0, so up to Weihrauch equivalence over RCA_0, the contrapositive form of $\mathrm{B}\Pi^0_1$ is $\mathrm{RT}^1_{<\infty}$, in the form in which it is usually stated as a Π^1_2-problem, in which an instance consists of a k-coloring of $\mathbb{N}$ together with the number

k.

Remark 3.6.3. The above argument (which we heard from Pauly [personal communication]) also gives a simple proof of Hirst's result from [38] (see also [33, Theorem 6.81]) that $\mathrm{B}\Sigma^0_2$ and $\mathrm{RT}^1_{<\infty}$ are equivalent over RCA_0.

There is a stronger form of $\mathrm{RT}^1_{<\infty}$, which we will call $\mathrm{stRT}^1_{<\infty}$, in which the number of colors is not part of the instance. That is, an instance consists of a function $\mathbb{N} \to \mathbb{N}$ with bounded range (and a solution is still an infinite homogeneous set). As shown by Brattka and Rakotoniaina [12], and also noted by Hirschfeldt and Jockusch [34], $\mathrm{RT}^1_{<\infty} <_{\mathrm{W}} \mathrm{stRT}^1_{<\infty}$. In this section, we show that $\mathrm{LH} \not\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{stRT}^1_{<\infty}$. We will show in Proposition 3.7.6 that $\mathrm{Bound}^* \leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{stRT}^1_{<\infty}$, so this result implies that $\mathrm{LH} \not\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Bound}^*$, but we also give a direct proof of the latter fact, which uses the same technique but is simpler.

Both proofs will use the following notion of forcing.

Definition 3.6.4. Let N be an L_1-structure. We define a notion of forcing P_N as follows. (If N is the standard natural numbers then we denote this notion by P_ω.) Write $[m]^2$ for the set of $(x, y) \in [|N|]^2$ such that $x, y <^N m$. A condition is an N-finite function of the form $p : [m]^2 \to 2$ for some $m \in |N|$. Say that a condition q extends such a p if q extends p as a function and $q(i, j) = 1$ for all $i <^N m$ and $j \geqslant^N m$ on which it is defined. Define the notion of $c : [|N|]^2 \to 2$ extending p in the same way. (Notice that if for every $m \in |N|$ there is a condition $p : [m]^2 \to 2$ such that c extends p, then c is an N-instance of LH.)

We will also use the following fact. (A 1-*elementary extension* of a structure N is an extension of N that satisfies exactly the same existential sentences with parameters from N.)

Lemma 3.6.5. *There is a* 1*-elementary extension M of the standard natural numbers such that for the collection S of all subsets of $|M|$ that are Δ^0_1-definable over M,*

1. *(M, S) is a model of* RCA_0 *and*

2. *for any condition p for the notion of forcing P_M, there is an M-instance of* LH *in S that extends p (in the sense of Definition 3.6.4) and has no solution in S.*

Proof. Let N be any nonstandard elementary extension of the standard natural numbers, and let $a \in N$ be a nonstandard element. Then in particular $N \vDash \mathrm{I}\Sigma^0_2$, and so

$$M = \{x \in N : x \text{ is } \Sigma^0_2\text{-definable in } (N, a)\}$$

is a model of $\mathrm{I}\Sigma^0_1 + \neg\mathrm{B}\Sigma^0_2$ which is a 1-elementary (in fact, 2-elementary) substructure of N. (See Hájek and Pudlak [30, Theorem IV.1.33] or Kossak [42, p. 223].) Thus, M is a 1-elementary extension of the standard model, and for S as in the statement, (M, S) is a model of RCA_0. Since $\mathrm{B}\Sigma^0_2$ fails in M, it follows by Proposition 3.6.2 that LH fails in (M, S). Fix an instance $c : [M]^2 \to 2$ of LH in S with no solution in S. Then given a condition $p : [m]^2 \to 2$ for P_M, we can define $d : [M]^2 \to 2$ by

$$d(x,y) = \begin{cases} p(x,y) & \text{if } x, y < m, \\ 1 & \text{if } x < m \text{ and } y \geqslant m, \\ c(x,y) & \text{otherwise.} \end{cases}$$

Clearly, d is in S and is an instance of LH that extends p. But if H is any solution to d then $\{x \in H : x \geqslant m\}$ is a solution to c, so d cannot have any solution in S. □

Proposition 3.6.6. LH $\not\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}}$ Bound*.

Proof. Assume for a contradiction that LH $\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}}$ Bound*. By Proposition 3.4.2, there is an $n \in \omega$ such that Player 2 has a computable strategy for $\widehat{G}^{\mathrm{RCA}_0}(\text{Bound}^* \to \text{LH})$ that ensures victory in at most n many moves. Fix such a strategy.

For a condition $p : [j]^2 \to 2$ for the notion of forcing P_ω, we can consider what happens when our fixed strategy for Player 2 is applied to a run in which Player 1 plays $(\omega, \mathcal{P}(\omega))$ and

p as a partial first move. Unless the strategy declares victory on its first move, it must play part of an instance of Bound*, which is just a simultaneous enumeration of a finite family of sets. We may assume by the usual convention on uses that no number greater than j is enumerated. Let b_0^p be the least bound on the set of all numbers enumerated in this way. Now, if Player 1 plays b_0^p, then unless our strategy declares victory on its second move, it again must play part of an instance of Bound*, yielding an analogous bound b_1^p. Continuing in this way, we obtain numbers $b_0^p, b_1^p, \ldots, b_k^p$ for some $k < n$. Let $b_i^p = 0$ for $k < i < n$.

For $i < n$ and $m \in \omega$, let $D_{i,m}$ be the set of conditions p such that $b_i^p \geqslant m$. If some $D_{0,m}$ is not dense then let m_0 be the least such m. In this case, there is a condition $p_0 \in D_{0,m_0-1}$ with no extension in D_{0,m_0}. Notice that $b_0^q = m_0 - 1$ for all extensions q of p_0. Now, if some $D_{1,m}$ is not dense below p_0 then let m_1 be the least such m. In this case, there is an extension of p_0 in D_{1,m_1-1} with no extension in D_{1,m_1}. Proceeding in this way, we obtain a condition p such that either m_i is defined for every $i < n$, or there is a $k < n$ such that m_i is defined for all $i < k$ and every $D_{k,m}$ is dense below p. In either case, $b_i^q = m_i - 1$ for all extensions q of p and all i such that m_i is defined.

We claim that the latter case cannot hold. Suppose otherwise. Let c be an instance of LH that extends p and meets every $D_{k,m}$ (i.e., every $D_{k,m}$ contains a q such that c extends q). Then Player 1 can play $(\omega, \mathcal{P}(\omega))$ and c on its first move, and if Player 2 follows our fixed strategy, then the moves $m_0, m_1, \ldots, m_{k-1}$ will be legal for Player 1 (as otherwise some finite portion of c is a condition q extending p with $b_i^q > m_i$ for some $i < k$). But then Player 2's $(k+1)$st move is not an instance of Bound*.

Thus each m_i for $i < n$ is defined, and we have the following for our fixed condition p:

$$\forall q \, \forall i < n \, [\text{if } q \text{ extends } p \text{ then } b_i^q = m_i - 1]. \tag{3.1}$$

Now let M and S be as in Lemma 3.6.5. Then p is also a condition for P_M, so there is an M-instance d of LH in S that extends p and has no solution in S. But it is easy to check that (3.1) is a Π^0_1 statement, so since M is a 1-elementary extension of the standard natural numbers, it also holds over M. So Player 1 can play (M, S) and d on its first move, and if Player 2 follows our fixed strategy, then the moves $m_0, m_1, \ldots, m_{n-1}$ will be legal for Player 1 (as otherwise some finite portion of d is a condition q extending p with $b^q_i >^M m_i$ for some $i < n$). But then Player 2 has not won the game by the nth move (since the only way for Player 2 to win this run of the game is to play an M-instance of Bound* with no solution), contrary to assumption. □

Thus LH and Bound* constitute a natural example of the phenomenon witnessed by Uftring's Example 3.4.5.

We can also interpret the fact that LH $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0 + \mathrm{Bound}^*} 1$ but LH $\nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0}$ Bound* as saying that the use of Bound* in the first part of the proof of Proposition 3.6.2 is "purely proof-theoretic". It neither requires a further "computability-theoretic application" of Bound* nor can be replaced by one or more such applications (in the uniform setting). Uncovering this kind of information seems to be a promising aspect of this approach to calibrating the logical strength of Π^1_2-problems.

Proposition 3.6.6 does not show that $\mathrm{SRT}^2_2 \nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{D}^2_2$, but it suggests that this might well be the case, which would provide an even more natural version of Example 3.4.5, and show that the proof of SRT^2_2 from D^2_2 necessarily makes both computability-theoretic and further proof-theoretic use of D^2_2. Indeed, it even seems possible that LH $\nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{D}^2_2$.

Open Question 3.6.7. Is $\mathrm{SRT}^2_2 \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{D}^2_2$? Is LH $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{D}^2_2$?

We now strengthen Proposition 3.6.6 as described above.

Proposition 3.6.8. LH $\nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{stRT}^1_{<\infty}$.

Proof. Assume for a contradiction that LH $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{stRT}^1_{<\infty}$. By Proposition 3.4.2, there is an $n \in \omega$ such that Player 2 has a computable strategy for $\widehat{G}^{\mathrm{RCA}_0}(\mathrm{stRT}^1_{<\infty} \to \mathrm{LH})$ ensuring victory in at most n many moves. There is then also a strategy that ensures victory in exactly

n many moves, since Player 2 can extend the length of any game by playing computable (Δ^0_1-definable) instances of $\mathrm{stRT}^1_{<\infty}$ on all its moves from some point on. Fix such a strategy, and for notational convenience, assume $n > 1$.

We begin as in the previous proof by considering games over the standard natural numbers. Note that if Player 2 plays according to its strategy and does not declare victory on some move, then it has to play an instance of $\mathrm{stRT}^1_{<\infty}$ only provided all of Player 1's moves so far have been legal. However, since every set can be viewed as a coloring $\omega \to \omega$ (not necessarily with bounded range), we can always assume that Player 2 plays such a coloring. This coloring may be partial, however, in which case by usual use conventions we can assume it is defined on a finite initial segment of ω.

Fix a condition p for the notion of forcing P_ω. For each $\alpha \in \omega^{\leqslant n-2}$, we define a coloring f^p_α of a finite initial segment of ω. Having done so, we let $H^p_{\alpha^\frown v}$ for each $v \in \omega$ be the set of all x such that $f^p_\alpha(x) = v$. We start with α equal to λ, the empty string. As in the proof of Proposition 3.6.6, suppose Player 1 plays $(\omega, \mathcal{P}(\omega))$ and p as a partial first move. Since $n > 1$, the strategy for Player 2 makes it play a coloring of a finite initial segment of ω as its partial first move. Let f^p_λ be this coloring. Now, suppose f^p_α has been defined for some α with $|\alpha| < n-2$, and fix $v \in \omega$. Suppose Player 1 plays $(\omega, \mathcal{P}(\omega))$ and p as a partial first move, and for $0 < k \leqslant |\alpha| + 1$, plays $H^p_{(\alpha^\frown v)\upharpoonright k}$ as a partial $(k+1)$st move, with Player 2 playing according to its fixed strategy. Since $n > |\alpha| + 2$, the strategy for Player 2 makes it again play a coloring of an initial segment as its partial $(|\alpha| + 2)$nd move. Let $f^p_{\alpha^\frown v}$ be this coloring.

We now define a finitely branching subtree T of $\omega^{\leqslant n-2}$, and for each $\alpha \in T$, a condition p_α, such that the following properties hold:

1. For all $\alpha, \beta \in T$, if β length-lexicographically precedes α then p_α extends p_β.

2. For every $\alpha^\frown v \in T$ and for every $m \in \omega$, the set of conditions p with $f^p_\alpha(x) = v$ for some $x \geqslant m$ is dense below p_α.

3. For every $\alpha \in T$ and every v such that $\alpha^\frown v \in \omega^{n-1} \backslash T$, if $f^p_\alpha(x) = v$ for some condition p extending p_α and some x, then $x \in \operatorname{dom} f^{p_\alpha}_\alpha$.

We put strings α into T and define p_α simultaneously. Initially, put $\lambda \in T$ and let p_λ be the empty condition. Notice that properties 1–3 hold vacuously at this point.

Next, assume we are at a point in the definition of T at which properties 1–3 hold, and consider the length-lexicographically least $\alpha \in T$ with $|\alpha| < n-2$ such that we have not yet put $\alpha^\frown v$ into T for any v. Let $\beta \in T$ be length-lexicographically largest such that p_β has been defined. Let $c : \omega \to \omega$ extend p and be sufficiently generic for the forcing notion P_ω. If Player 1 plays $(\omega, \mathcal{P}(\omega))$ and c on its first move, then the strategy for Player 2 makes it play an instance f_0 of $\mathrm{stRT}^1_{<\infty}$ in response. By property 2 and the genericity of c, the set of x such that $f_0(x) = \alpha(0)$ is infinite, so $H_0 = \{x : f_0(x) = \alpha(0)\}$ is a legal second move for Player 1. Then the strategy for Player 2 makes it play another instance f_1 of $\mathrm{stRT}^1_{<\infty}$ on its second move, and the set $H_1 = \{x : f_1(x) = \alpha(1)\}$ will be infinite and hence a legal third move for Player 1. Since $n > |\alpha| + 2$, if we continue in this way we analogously define $f_0, \ldots, f_{|\alpha|}$ and $H_0, \ldots, H_{|\alpha|-1}$, with f_k played by Player 2 on its $(k+1)$st move for all $k \leqslant |\alpha|$, and H_k played by Player 1 on its $(k+2)$nd move for all $k < |\alpha|$. Since the strategy for Player 2 is computable and hence continuous, it is easy to see by induction that if q is any condition extended by c then $f^q_{\alpha \upharpoonright k}$ is an initial segment of f_k, and $H^p_{\alpha \upharpoonright (k+1)}$ is an initial segment of H_k. Now, as $f_{|\alpha|}$ is an instance of $\mathrm{stRT}^1_{<\infty}$, there must be a condition q_0 extended by c and a $b \in \omega$ such that $f^r_\alpha(x) < b$ for all x and all r extending q_0.

We now decide for which $v < b$ to add $\alpha^\frown v$ to T and define $p_{\alpha^\frown v}$. Fix v, and suppose we have already decided this for all $w < v$. For notational convenience, assume we have also defined an auxiliary condition q_v extending q_0. If there is a condition r extending q_v such that for every $m \in \omega$, every extension of r has a further extension s such that $f^s_\alpha(x) = v$ for some $x \geqslant m$, then let $\alpha^\frown v \in T$ and let $p_{\alpha^\frown v} = q_{v+1} = r$. Otherwise, there is an extension r of q_v such that for every extension s of r, if $f^s_\alpha(x) = v$ for some x then x is in the domain of f^r_α, and we let $q_{v+1} = r$ and let $\alpha^\frown v \notin T$. It is readily seen that this process adds $\alpha^\frown v$ to T for at least one v, and for only finitely many v, and that properties 1, 2, and 3 are preserved.

Let $p^* = p_\beta$ for the length-lexicographically largest $\beta \in T$. Let M be as given by Lemma 3.6.5, and let S be the set of subsets of $|M|$ that are Δ^0_1-definable over M. Every condition for P_ω is also a condition for P_M. So let c be an instance of LH in S that extends p^* and has no solution in S. For every node $\alpha \in T$, let G_α be the following run of a game. Player 1 plays (M, S) and c as its first move, and Player 2 plays according to its strategy. On its $(k+1)$st move for $0 < k \leqslant |\alpha|$, Player 1 always plays the set of all $x \in M$ that are colored $\alpha(k-1)$ by the coloring played by Player 2 on its previous move (assuming it played a total coloring and not just a partial one). We claim that there is an $\alpha \in T$ of length $n-2$ such that Player 1's moves in G_α are all legal. We argue by induction (along the standard number $n-1$) that for each $k < n-1$ there is such an $\alpha \in T$ of length k. Suppose that for some $\alpha \in T$ of length $k-1$, Player 1's moves in G_α are all legal. Then on its $(|\alpha|+1)$st move in G_α, Player 2 plays an instance f of $\mathrm{stRT}^1_{<\infty}$. Now, property 3 in the definition of T is a Π^0_1 statement of arithmetic, so since M is a 1-elementary extension of ω, it must also hold in M. Thus, all the $v \in M$ such that $f^{-1}(v)$ is unbounded in M must be among those for which $\alpha^\frown v \in T$. Since there are only standardly many such v, there must be at least one for which $f^{-1}(v)$ really is unbounded in M, so Player 1's moves in $G_{\alpha^\frown v}$ will all be legal. This establishes the claim. To complete the proof, fix such an α of length $n-2$. All sets played by Player 1 are clearly in S, so when Player 2 declares victory on its nth (i.e., $(|\alpha|+2)$nd) move in G_α it must play a solution to c in S. But there is no such solution by hypothesis, which is a contradiction. □

3.7 Versions of Π^0_1-bounding

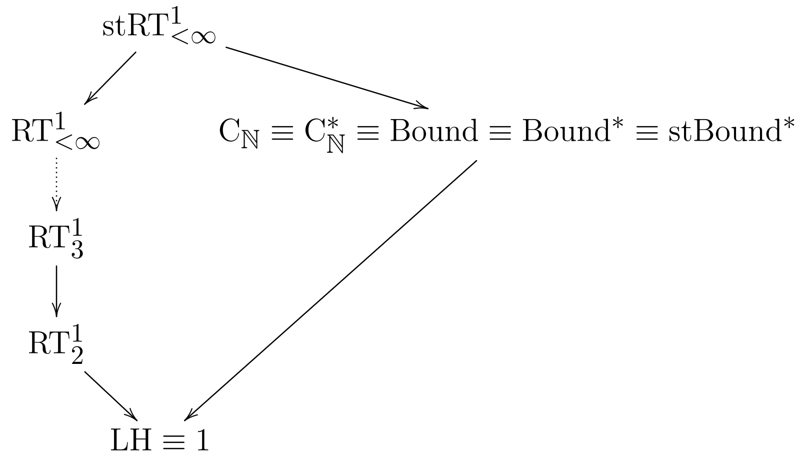
In this section we fill out the picture of implications between versions of $\mathrm{B}\Pi^0_1$ and related principles.

As with $\mathrm{RT}^1_{<\infty}$, we can define a strong form stBound* of Bound* by having the number of sets not be part of the instance. A convenient way to express this problem is to say that an instance is an enumeration of a subset X of $\mathbb{N} \times \mathbb{N}$ such that $\{n : \exists k\,(n,k) \in X\}$ is bounded, and for each n, so is the set $\{k : (n,k) \in X\}$; and a solution is a bound on $\{k : \exists n\,(n,k) \in X\}$. It is easy to see that stBound* $\equiv_{\mathrm{W}}$ Bound, but we will see that this equivalence no longer holds in our setting.

Another problem worth mentioning in this connection is $\mathrm{C}_{\mathbb{N}}$. The finite parallelization $\mathrm{C}^*_{\mathbb{N}}$ is yet another equivalent of $\mathrm{B}\Sigma^0_2$: In one direction, we can enumerate the sets $\{m : \forall k < m\, \neg\varphi(i,k)\}$ for a given Π^0_1 formula $\varphi(i,k)$, and from a tuple containing an element of the complement of each of these sets, obtain a common bound on the sets. In the other direction, given simultaneous enumerations of the complements of the nonempty sets $F_0, \dots, F_n$, by $\mathrm{B}\Pi^0_1$, there is a b such that each F_i has an element less than b. Now bounded Π^0_1-comprehension, which holds in RCA_0, gives us the set of all tuples $(a_0, \dots, a_j)$ with $j \leqslant n$ and $a_i \in F_i$ for all $i \leqslant j$, and set induction shows that there must be such a tuple with $j = n$.

It is easy to see that $\mathrm{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathrm{C}^*_{\mathbb{N}}$, and Pauly, Fouché, and Davie [55] showed that Bound $\equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$, using the Weihrauch equivalence between $\mathrm{C}_{\mathbb{N}}$ and its restriction $\mathrm{UC}_{\mathbb{N}}$ to enumerations of complements of singleton sets, which was proved by Brattka, de Brecht, and Pauly [7]. Brattka and Rakotoniaina [12] showed that $\mathrm{C}_{\mathbb{N}} \mid_{\mathrm{W}} \mathrm{RT}^1_{<\infty}$ and $\mathrm{C}_{\mathbb{N}} <_{\mathrm{W}} \mathrm{stRT}^1_{<\infty}$. Indeed, it is even the case that $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$; we will prove a stronger version of this fact below. It is also worth noting that $\mathrm{RT}^1_2 <_{\mathrm{W}} \mathrm{RT}^1_3 <_{\mathrm{W}} \cdots$, as shown by Brattka and Rakotoniaina [12] and Hirschfeldt and Jockusch [34]. Thus we have the following picture for Weihrauch reducibility:

Figure 3.1: How our principles compare with respect to $\leqslant_{\mathrm{W}}$



(3.2)

Hirschfeldt and Jockusch [34, Proposition 4.7] showed that $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{gW}} \mathrm{RT}^1_2$, but their proof in fact shows that $\mathrm{stRT}^1_{<\infty} \leqslant_{\mathrm{gW}} \mathrm{RT}^1_2$. On the other hand, we have the following.

Proposition 3.7.1. $\mathrm{RT}^1_2 \nleqslant_{\mathrm{gW}} \mathrm{C}_{\mathbb{N}}$.

Proof. Suppose that $\mathrm{RT}^1_2 \leqslant_{\mathrm{gW}} \mathrm{C}_{\mathbb{N}}$ via a computable strategy P for Player 2. As Player 1, we can begin to build a coloring c by coloring numbers in order, initially giving each number the color 0, and simulate the action of P. We can assume that, even when provided with inputs that do not correspond to a run of $G(\mathrm{C}_{\mathbb{N}} \to \mathrm{RT}^1_2)$, if P does not declare victory at a given move, then it outputs an enumeration of the complement of some set, though in that case the set might be empty.

Let A_i be the set whose complement is being enumerated by P as its $(i+1)$st move (if P has not declared victory at or before that move). We guess at each stage that the least number k_i currently in A_i is a solution to the corresponding instance of $\mathrm{C}_{\mathbb{N}}$ and play that as our $(i+2)$nd move in the simulation. If we ever find that k_i is not in A_i, we restart the simulation (but do not change c on the numbers at which we have already defined it). For the least such i, say that i causes the simulation to restart. If the current simulation is not restarted, then eventually P must declare victory at some move, and declare some number

Figure 3.2: How our principles compare with respect to $\leqslant_{\mathrm{gW}}$

$$\begin{array}{c} \mathrm{stRT}^1_{<\infty} \equiv \mathrm{RT}^1_{<\infty} \equiv \mathrm{RT}^1_2 \\ \downarrow \\ \mathrm{C}_{\mathbb{N}} \equiv \mathrm{C}^*_{\mathbb{N}} \equiv \mathrm{Bound} \equiv \mathrm{Bound}^* \equiv \mathrm{stBound}^* \\ \downarrow \\ \mathrm{LH} \equiv 1 \end{array} \tag{3.3}$$

m to be in the set it outputs at that move. We then start to give our numbers the color $1 - c(m)$. If we were to do this forever, then m could not be part of a solution to c, so our current simulation cannot be a true run of the game, and hence eventually some i must cause it to restart.

Thus the simulation is restarted infinitely often. There are now two cases.

If there is a least i that causes the simulation to restart infinitely often, then, by induction, $k_0, \ldots, k_{i-1}$ have final values, and if we play c on our first move, and then play these values in turn, we produce a run of our game in which P's $(i+1)$st move is an enumeration of $\mathbb{N}$, and hence is not an instance of $\mathrm{C}_{\mathbb{N}}$, which is a contradiction.

Otherwise, again by induction, all k_i's have final values, and if we play c on our first move, and then play these values in turn, we produce a run of our game in which P never declares victory, which is again a contradiction. □

So for gW-reducibility, we have the following simpler picture:

It is easy to check that all the Weihrauch reductions in Diagram (3.2) still work over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$, so that diagram also reflects the relationships between these principles with respect to $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2}$ (or $\leqslant_{\mathrm{W}}^{\Gamma}$ for any extension Γ of $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ by formulas true over the natural numbers). Diagram (3.3), however, does change if we work over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$. We still have the equivalence between RT^1_j and RT^1_k for $j, k \geqslant 2$ (which holds even over RCA_0, with the usual proof), but Corollary 3.4.3 shows that $\mathrm{RT}^1_{<\infty} \not\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2} \mathrm{RT}^1_k$ for all n and k. Similarly, we have the following.

Proposition 3.7.2. $\mathrm{C}_{\mathbb{N}} \nleqslant^n_{\mathrm{gW}} \mathrm{RT}^1_{<\infty}$ *for all* n*, so if we let* Γ *consist of* RCA_0 *together with all* Π^1_1 *formulas true over the natural numbers then* $\mathrm{C}_{\mathbb{N}} \nleqslant^{\Gamma}_{\mathrm{gW}} \mathrm{RT}^1_{<\infty}$.

Proof. Suppose that $\mathrm{C}_{\mathbb{N}} \leqslant^n_{\mathrm{gW}} \mathrm{RT}^1_{<\infty}$ via a computable strategy P for Player 2. We can assume that, even when provided with inputs that do not correspond to a run of $G(\mathrm{RT}^1_{<\infty} \to \mathrm{C}_{\mathbb{N}})$, if P does not declare victory at a given move, then its output at that move, if nonempty, is a number k together with a possibly partial $c : \mathbb{N} \to k$.

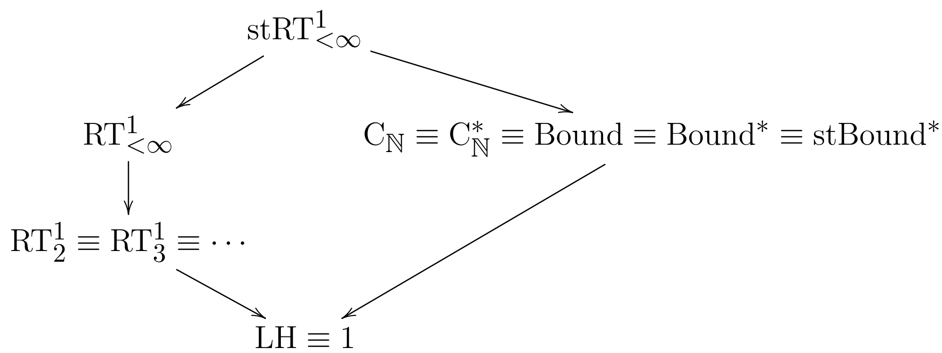
For a possibly partial $c : \mathbb{N} \to k$, let $H_c = \{c^{-1}(0), \ldots, c^{-1}(k-1)\}$. Note that if c is total then at least one element of H_c is a solution to c as an instance of $\mathrm{RT}^1_{<\infty}$. We can start building an instance E of $\mathrm{C}_{\mathbb{N}}$ by initially not enumerating any numbers, and running simulations of possible runs of $G(\mathrm{RT}^1_{<\infty} \to \mathrm{C}_{\mathbb{N}})$ beginning with E, where each time P plays some c, we play a simulation for each possible move for Player 1 in H_c. (Notice that c might not actually be an instance of $\mathrm{RT}^1_{<\infty}$ because this simulation might not correspond to an actual run of the game, but H_c is still finite. This is the reason we could not work with $\mathrm{stRT}^1_{<\infty}$ here, because in that case P would be able to play functions with unbounded range during simulations that do not correspond to actual runs.)

Whenever in any of these simulations P declares victory at or before the $(n+1)$st move with a purported solution m, we enumerate m into E. Since each H_c is finite, and we consider only finitely many c's during this construction, we enumerate only finitely many numbers into E, and this strategy ensures that there is a run of $G(\mathrm{RT}^1_{<\infty} \to \mathrm{C}_{\mathbb{N}})$ beginning with E in which either P does not declare victory by its $(n+1)$st move, or it does so with a purported solution m that is enumerated into E, and hence is not in fact a solution to E. In either case we have a contradiction.

The second part of the proposition now follows from Proposition 3.4.2. □

Thus we have the following picture for gW-reducibility over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ (or over any extension of $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ by Π^1_1 formulas true over the natural numbers):

Figure 3.3: How our principles compare with respect to $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2}$



(3.4)

When working over RCA_0, things change even further. We do still have Bound $\equiv_{\mathrm{W}}^{\mathrm{RCA}_0}$ $\mathrm{C}_{\mathbb{N}}$, $\mathrm{Bound}^* \equiv_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{C}^*_{\mathbb{N}}$, and $\mathrm{C}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{stRT}^1_{<\infty}$, with essentially the same proofs. The only parts that require a bit of care are $\mathrm{C}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ Bound and $\mathrm{C}^*_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{Bound}^*$. We prove the latter, as the former is similar but simpler. We argue in RCA_0. Given an enumeration of the complements of nonempty sets $A_0, \dots, A_n$, constituting an instance of $\mathrm{C}^*_{\mathbb{N}}$, we define enumerations of sets $F_0, \dots, F_n$ by putting s into F_i whenever the least element m^i_s of A_i at stage s of the enumeration of its complement leaves A_i at that stage. If F_i were unbounded, then so would be the set of numbers m^i_s, since the map taking F_i to this set is injective and computable. But then A_i would be empty. So each F_i is bounded, and hence our enumeration of $F_0, \dots, F_n$ is an instance of Bound^*. If s is a solution to this instance then for each $i \leqslant n$, the least element of A_i at stage s must be in A_i, so from s we obtain a solution to our instance of $\mathrm{C}^*_{\mathbb{N}}$.

However, every instance of $\mathrm{C}_{\mathbb{N}}$ and Bound in every model of RCA_0 has a solution, while this is not the case for $\mathrm{C}^*_{\mathbb{N}}$ and Bound^*, which are equivalent to $\mathrm{B}\Sigma^0_2$ over RCA_0 as statements of second-order arithmetic. So $\mathrm{C}_{\mathbb{N}}$ is strictly below $\mathrm{C}^*_{\mathbb{N}}$ under both $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ and $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0}$, and similarly for Bound and Bound^*.

We also no longer have a Weihrauch-reduction of $\mathrm{stBound}^*$ to Bound^*, but do have one in two steps, because an instance of Bound^* (or even Bound) can be used to determine the number of sets being enumerated in an instance of stBound*, allowing us to solve that instance with a second application of Bound*.

Proposition 3.7.3. stBound$^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0,2}$ Bound* *but* stBound$^* \nleqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ Bound**.*

Proof. Given an instance X of stBound*, we can first build an instance of Bound by enumerating n whenever X enumerates (n,k) for some k. Given a solution b to this instance, we can build an instance of Bound* consisting of enumerations of sets $F_0, \ldots, F_{b-1}$ by enumerating k into F_n whenever X enumerates (n,k). A solution to this instance is also a solution to X.

For the second part, suppose that stBound$^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ Bound* via Φ_e and Φ_i. An enumeration E of $\emptyset$ is an instance of stBound*, so Φ_e^E must be an instance of Bound*. This instance has a fixed number of sets k, which must be the same standard natural number no matter what model of RCA_0 we are working in, because the convergent computation over the standard natural numbers still exists in any such model. Now let (M,S) be a model of RCA_0 that contains an M-instance D of stBound* with no solution. We can delay D to define a new M-instance $\widehat{D}$ of stBound* that enumerates the same set as D but agrees with E up to the use of the part of the computation of Φ_e^E that fixes the number of sets at k. Then $\widehat{D}$ has no solution, but $\Phi_e^{\widehat{D}}$ is an instance of Bound* with a standard number of sets, and hence must have a solution b. But then Φ_i should be able to compute a solution to $\widehat{D}$ from $\widehat{D}$ and b, which is a contradiction. □

We can make the first part of this proposition a bit more precise by using the compositional product from the theory of Weihrauch reducibility: stBound$^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ Bound$^* \star$Bound.

The second part of the proposition easily generalizes to establish the following useful principle (which we state for RCA_0 but of course applies to other systems as well).

Proposition 3.7.4. *Let* P *and* Q *be* Π^1_2*-problems such that*

1. P *has an* ω*-instance* X *such that for any finite initial segment* σ *of* X*, there is a model* (M,S) *of* RCA_0 *and an* M*-instance* Y *of* P *in* S *that extends* σ *and has no solution*

in S; and

2. *every instance X of* Q *includes a parameter* $k_X \in \mathbb{N}$ *such that for every model* (M, S) *of* RCA_0 *and every M-instance X of* Q *in S, if* k_X *is a standard natural number, then X has a solution in S.*

Then $\mathrm{P} \not\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{Q}$.

As an example of the application of this principle, we have the following.

Corollary 3.7.5. $\mathrm{LH} \not\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{RT}^1_{<\infty}$.

We also have the following other example of a W-reducibility that becomes a gW-reducibility in two steps when generalized to models of RCA_0.

Proposition 3.7.6. $\mathrm{stBound}^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0,2} \mathrm{stRT}^1_{<\infty}$ *but* $\mathrm{Bound}^* \not\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{stRT}^1_{<\infty}$.

Proof. For the first part, we argue in RCA_0. Given an instance X of $\mathrm{stBound}^*$, let $E_{i,n}$ be the set of k such that (i, k) has been enumerated into X by stage n, and let i_n be the least i that maximizes $\max E_{i,n}$ (which exists because the function taking i to $\max E_{i,n}$ is computable). We first produce an instance of $\mathrm{stRT}^1_{<\infty}$ by giving n the color i_n. Given a solution H to this instance, let i be the color of the elements of H. Now apply Bound (which is W-reducible over RCA_0 to $\mathrm{stRT}^1_{<\infty}$) to obtain a bound b on $\{k : (i, k) \in X\}$. This bound must be a solution to X, because if $(j, k) \in X$ for some j and $k > b$, then once (j, k) is enumerated into X at some stage m, we cannot have $i_n = i$ for $n \geqslant m$.

Now suppose that $\mathrm{Bound}^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{stRT}^1_{<\infty}$ via Φ_e and Φ_i. We work over a model M of Σ^0_1-PA that satisfies Σ^0_2-bounding but not Σ^0_3-bounding. Then there is a Δ^0_2 M-instance $c : |M| \to k$ of $\mathrm{RT}^1_{<\infty}$ with no solution. Say that sets $F_0, \ldots, F_{k-1}$ are acceptable if $c(n) = i$ for every $i < k$ and $n \in F_i$. Notice that in this case, each F_i is bounded, so an enumeration of an acceptable family of sets is an M-instance of Bound^*.

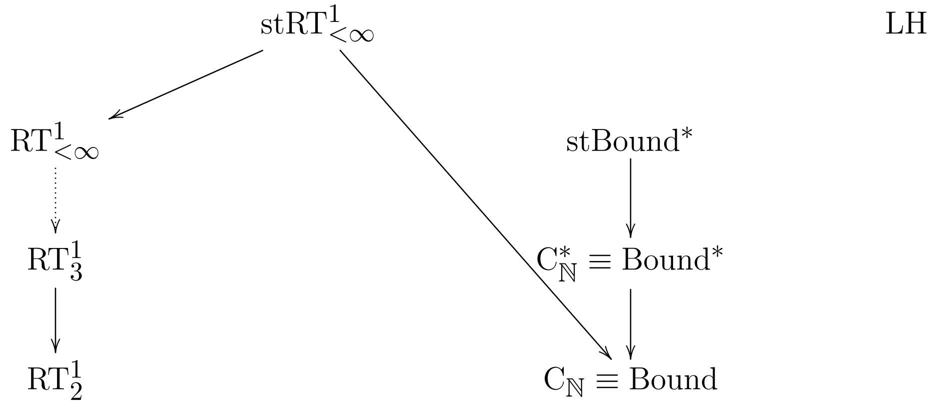
Thinking of M-finite enumerations of acceptable families as a notion of forcing, suppose that for each $j \in M$, the set of such enumerations E for which some element greater than j is in the range of Φ_e^E is dense. Then we can computably build an enumeration D of an acceptable family such that Φ_e^D has unbounded range, and is thus not an instance of $\mathrm{stRT}^1_{<\infty}$. As this situation cannot happen, there must be a $j \in M$ and an M-finite enumeration E of an acceptable family such that for every enumeration D of an acceptable family extending E, the range of Φ_e^D is bounded by j.

Now we start building such a D by monitoring $\Phi_i^{D \oplus H_p}$ for each $H_p = \{n : \Phi_e^D(n) = p\}$ with $p <^M j$. Whenever we see $\Phi_i^{D \oplus H_p}$ return a number m_p, we enumerate $m_p + 1$ into $F_{c(m_p+1)}$, where $F_0, \ldots, F_{k-1}$ is the family that D is enumerating. The set of $p <^m j$ such that m_p is ever defined is a bounded Σ^0_1 set, and the map taking each p in this set to m_p is computable, so the set of m_p's is M-finite. But then the restriction of c to this set is also M-finite, because the fact that M satisfies Σ^0_2-bounding implies that the intersection of a Δ^0_2 set with an M-finite set is M-finite. So D is an M-finite extension of the M-finite enumeration E, and hence is itself M-finite, and thus Φ_e^D is a computable instance of $\mathrm{stRT}^1_{<\infty}$, and hence must have a solution. But then some H_p with $p <^M j$ must be such a solution, and hence $\Phi_i^{D \oplus H_p}$ must be a solution to D. But we ensured that this is not the case, so we have a contradiction. □

The first part of this proof shows more precisely that $\mathrm{stBound}^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{Bound} \star \mathrm{stRT}^1_{<\infty}$ and that $\mathrm{Bound}^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{Bound} \star \mathrm{RT}^1_{<\infty}$.

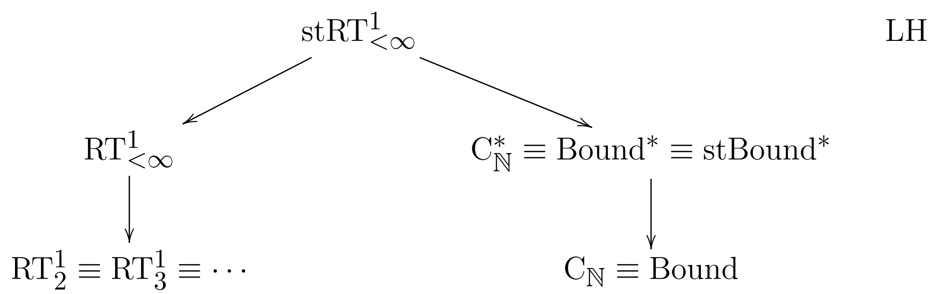
Combining the results above with Proposition 3.6.8 gives us the following pictures of the $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ and $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0}$ cases, respectively.

Figure 3.4: How our principles compare with respect to $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$



(3.5)

Figure 3.5: How our principles compare with respect to $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0}$



(3.6)

CHAPTER 4

EXPLORING THE LANDSCAPE OF WEIHRAUCH AND GENERALIZED WEIHRAUCH REDUCTION OVER RCA_0 AND RCA_0^*

4.1 Introduction

In the previous chapter, Dzhafarov, Hirschfeldt, and Reitzes combined methods from reverse mathematics and computability theory to produce the notion of Weihrauch reducibility over subsystems of second-order arithmetic. The authors went on to begin establishing relationships between principles in this realm, particularly in terms of Weihrauch reducibility and generalized Weihrauch reducibility over RCA_0. Our aim in this chapter is to further explore the structure of the relationships between principles in this setting, in particular considering Weihrauch reducibility over RCA_0^*.

Recall that a Π^1_2-problem is a sentence

$$\forall X[\Theta(X) \to \exists Y \Psi(X, Y)]$$

of second-order arithmetic such that Θ and Ψ are arithmetic. We say that an $X \subseteq \omega$ such that $\Theta(X)$ holds is an *instance* of this problem, and a *solution* is a $Y \subseteq \omega$ such that $\Psi(X, Y)$ holds. We typically denote Π^1_2-problems by P and Q.

We were motivated in particular by Slaman's paper [61] to see if we could translate his results about the equivalence of Σ^0_n-bounding and Δ^0_n-induction in models of $\mathrm{PA}^- + \mathrm{I}\Delta^0_1 + \mathrm{exp}$, to analogous results in terms of (generalized) Weihrauch reducibility over RCA_0 or even over RCA_0^*. This work is by no means exhaustive; the aim is simply to get started with this line of research introduced in the previous chapter (see [25]). To this end, we started by considering Δ^0_2-induction, and whether it can be proven to be equivalent to Σ^0_2-bounding. However, this seemed complex initially, and we wanted to get a better handle of how induction principles behave in this setting. This led to the introduction of the principles of the form $\mathrm{F}\Sigma^0_n$ for $n \in \omega$, which are a natural way of thinking about Σ^0_n-induction as a Π^1_2 problem. We will formalize this definition in the next section, but for now, it is enough to think of $\mathrm{F}\Sigma^0_n$ as the principle that for every Σ^0_n set A with nonempty complement, there exists an $a \in \overline{A}$ such that either $a = 0$ or a is the successor of an element of A. That is, given a Σ^0_n set, a solution is a place where Σ^0_n-induction fails.

Our purpose in furthering this line of study is to get more information about classical results, and delineate relationships that aren't obvious from previous results. For example, in the previous chapter, we showed that although $\mathrm{stBound}^* \equiv_{\mathrm{W}} \mathrm{Bound}^*$, $\mathrm{stBound}^* \not\leq^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}^*$. We aim to find more results like this.

In Section 4.2 of this chapter, we give some background, in particular introducing the majority of the principles that we will be studying. We begin in Section 4.3 with the introduction of some specific results delineating the relationships between the principles. One of our goals with this chapter is to begin to make a distinction between what is true in the proof-theoretic sense and what is true in the computability-theoretic sense. For instance, in Section 4.4, we do a complete examination of the proofs in Slaman's paper [61] in our setting. We obtain generalized Weihrauch reductions rather than Weihrauch reductions, because we must apply the induction principle multiple times to get the corresponding bounding principle, and vice versa. However, we do not have the same number of applications as in Slaman's original proofs, as some of those applications are purely proof-theoretic. Indeed, open questions still remain about whether all of the applications of induction/bounding in our proofs are computability-theoretic, or whether we can in fact get Weihrauch reductions through some clever observation or combination of applications.

As another example, in Section 4.5, we obtain many non-reductions between the principles we are considering, through purely proof-theoretic means. On the other hand, in Section

4.6, we introduce the notion of regularity to motivate the proof of a metatheorem. This metatheorem allows us to group all of the principles we consider essentially as "strong" and "weak", thereby producing a series of non-reductions from each of the weak principles to each of the strong ones (the weak do not imply the strong). The determination of strength and weakness is based on purely structural properties of the principles; there is nothing proof-theoretic here. In Section 4.7, we return to more specific results, in particular now turning to those involving the regularity principles and Δ_2^0 principles we have introduced. In Section 4.8, we discuss some related results about Weak Weak König's Lemma. We conclude the chapter with the open questions that remain, further explorations, and some relevant diagrams.

4.2 Background

We first present a more formal definition of the problem $\mathrm{F}\Sigma_1^0$.

Definition 4.2.1. Let $\mathrm{F}\Sigma_1^0$ denote the Π_2^1 problem for every Σ_1^0 set A with nonempty complement, there exists an $a \in \overline{A}$ such that either $a = 0$ or $a = S(b)$ for some $b \in A$, where S denotes the successor function. An instance of $\mathrm{F}\Sigma_1^0$ is the enumeration of a set of ordered pairs B, where we let $A = \{x \mid \exists y(y, x) \in B\}$ and a solution is an a such that for all y, $(y, a) \notin B$ where either $a = 0$ or there exists some b such that $\exists y(y, b) \in B$ and $a = S(b)$.

We can also define the principle $\mathrm{F}\Sigma_n^0$ for $n > 1$ as the natural extension of $\mathrm{F}\Sigma_1^0$ to Σ_n^0 sets, as referenced in the introduction. However, our focus in this chapter will be on the problem $\mathrm{F}\Sigma_1^0$. As explained more generally in the introduction, $\mathrm{F}\Sigma_1^0$ gives us a natural way to think about Σ_1^0 induction as a Π_2^1 problem; given a Σ_1^0 set, a solution is a place where Σ_1^0 induction fails for this set. Although this principle has not been studied in this form in the context of Weihrauch reducibility, it can easily be seen to be Weihrauch equivalent to the principle $\mathrm{UC}_{\mathbb{N}}$ of unique closed choice on the natural numbers studied by Brattka, de Brecht, and Pauly in section 6 of [7]. Although $\mathrm{UC}_{\mathbb{N}}$ is a restriction of the principle $\mathrm{C}_{\mathbb{N}}$ which we will see below, the authors show in that paper that the two are in fact Weihrauch-equivalent. $\mathrm{F}\Sigma^0_1$ can also be seen to be Weihrauch equivalent to the principle of Π^0_1-choice on the natural numbers, as in [8].

We can also consider the corresponding principle for Π^0_1 sets.

Definition 4.2.2. Let $\mathrm{F}\Pi^0_1$ denote the Π^1_2 problem for every Π^0_1 set A with nonempty complement, there exists an $a \in \overline{A}$ such that either $a = 0$ or $a = S(b)$ for some $b \in A$, where S denotes the successor function. An instance of $\mathrm{F}\Pi^0_1$ is the enumeration of a set of ordered pairs B, where we let $A = \{x \mid \forall y(y, x) \in B\}$ and a solution is an a such that there is a y with $(y, a) \notin B$ where either $a = 0$ or there exists some b such that $\forall y(y, b) \in B$ and $a = S(b)$.

As with $\mathrm{F}\Sigma^0_1$, we can extend $\mathrm{F}\Pi^0_1$ to the principles $\mathrm{F}\Pi^0_n$ for $n > 1$ in the natural way, but in this chapter, we will focus on $\mathrm{F}\Pi^0_1$. We will see that $\mathrm{F}\Pi^0_1$ is weaker than its Σ^0_1 counterpart, in large part because we can enumerate the complement of a Π^0_1 set, thereby giving us knowledge of an element of the complement without having to apply any other principles. In the classical Weihrauch setting, $\mathrm{F}\Pi^0_1$ can be seen to be Weihrauch-equivalent to the principle LPO^*, which has been extensively studied in that realm (see, e.g. [11]). This equivalence is shown by Pauly in [54].

Remark 4.2.3. In this chapter we will frequently discuss Σ^0_1 and Δ^0_2 sets while working in RCA^*_0. Unless otherwise specified, by a Σ^0_1 set we mean a collection X of pairs of the form (x, y), where we think of the set as $\{x \mid \exists y((x, y) \in X)\}$. By a Δ^0_2 set, we mean a collection of elements X such that $(\forall x)(\exists t)((\forall s > t)(\langle x, s\rangle \in X) \vee (\forall s > t)(\langle x, s\rangle \notin X))$.

Later on, we will extend our method of looking at induction for Σ^0_1 and Π^0_1 sets to Δ^0_2 sets. It is therefore relevant that Slaman showed in [61] that Σ^0_n-bounding, or $\mathrm{B}\Sigma^0_n$, is equivalent over RCA_0 to Δ^0_n induction. In particular, we are interested here in Σ^0_2-bounding. As in the previous chapter, we look at Σ^0_2-bounding in the convenient form of Bound*. We will also consider two versions of bounded Σ^0_1-comprehension, a weak version in which the bound is given ahead of time, and a strong version in which it is not.

Definition 4.2.4. $\mathrm{B}\Sigma^0_1\mathrm{CA}$ is the Π^1_2 problem of bounded Σ^0_1-comprehension: given an enumeration of a set X and a bound b on the set, a solution is $\{x \in X \mid x < b\}$.

Definition 4.2.5. $\mathrm{stB}\Sigma^0_1\mathrm{CA}$ is the Π^1_2 problem of strong bounded Σ^0_1-comprehension: given an enumeration of a bounded set X, where the bound on X is not given, a solution is the set X itself.

Note that $\mathrm{B}\Sigma^0_1\mathrm{CA}$ can also be shown to be equivalent to the problem LPO^* in the Weihrauch setting, so we have that $\mathrm{F}\Sigma^0_1 \equiv_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$.

Remark 4.2.6. We could also define analogous bounded Π^0_1-comprehension problems $\mathrm{B}\Pi^0_1\mathrm{CA}$ and $\mathrm{stB}\Pi^0_1\mathrm{CA}$. In the weak case, the corresponding Σ^0_1 and Π^0_1 notions are equivalent; this is not necessarily true for the strong versions.

Definition 4.2.7. In this section, we take $\mathrm{RT}^1_{<\infty}$ to be the Π^1_2 problem where an instance is a coloring $c : \omega \to k$ for some $k \in \omega$, where k is given, and a solution is an $i < k$ such that $\{x : c(x) = i\}$ is an infinite homogeneous set for c.

Remark 4.2.8. Note that in [24] the aforementioned version of Ramsey's Theorem is referred to as $\mathrm{rt}^1_{<\infty}$. $\mathrm{RT}^1_{<\infty}$ is reserved for the Π^1_2 problem where an instance is a coloring $c : \omega \to k$, for some $k \in \omega$, where k is given, and a solution is an infinite homogeneous set for c, rather than the color to which the set is homogeneous. The two versions of Ramsey's Theorem are Weihrauch equivalent, so we need not distinguish between them here, but are not equivalent in the context of strong Weihrauch reducibility.

Definition 4.2.9. For our purposes, we take $\mathrm{stRT}^1_{<\infty}$ to be the Π^1_2 problem where an instance is a coloring $c : \omega \to k$ for some $k \in \omega$, where k is not given, and a solution is an $i < k$ such that $\{x : c(x) = i\}$ is an infinite homogeneous set for c.

Remark 4.2.10. Recall from the previous chapter that we say that an L_2-structure $\mathcal{M}$ is *consistent with* an axiom system Γ if $\mathcal{M}$ is contained in a model $\mathcal{N} \models \Gamma$ such that $\mathcal{M}$ and $\mathcal{N}$ have the same first-order part. If $\mathcal{M}$ is countable, we may require $\mathcal{N}$ to be countable as well without changing the notion In this chapter, every axiom system Γ that we consider will have the property that every model consistent with Γ is actually a model of Γ. However, when considering whether Γ proves $\mathrm{Q} \to \mathrm{P}$, we sometimes want to look at games over $\Gamma + \mathrm{Q}$, for which we need to distinguish between being consistent with $\Gamma + \mathrm{Q}$ and being a model of $\Gamma + \mathrm{Q}$.

For the rest of this chapter, we will limit ourselves to $\Gamma \in \{\mathrm{RCA}_0, \mathrm{RCA}_0^*, \mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2\}$.

Remark 4.2.11. For convenience of notation, we will frequently write in this chapter $\Psi^{A \oplus n}$ for a set A and a number n, where we mean that Ψ has oracle access to the set A and the number n.

4.3 A first set of specific results

We begin by considering bounded Σ^0_1-comprehension. Our first goal is to find some relationship between bounded Σ^0_1-comprehension and Σ^0_1-induction, where we view Σ^0_1-induction as the principle $\mathrm{F}\Sigma^0_1$ given in Definition 4.2.1 and work in the setting of Weihrauch and generalized Weihrauch reducibility over RCA_0, or better yet, RCA_0^*. This will serve as a building block to looking at the results from [61] in our setting. We begin with the following theorem from Simpson in [60].

Theorem 4.3.1 (Simpson [60])**.** RCA_0 *proves* $\mathrm{B}\Sigma^0_1\mathrm{CA}$.

Proof. Suppose we have an enumeration of a set X. Given a number b, suppose that there is no finite set Y such that $\forall i(i \in Y \leftrightarrow (i < b \wedge i \in X))$. Then by Lemma II.3.7 in [60], there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall m(f(m) < b \wedge f(m) \in X)$. In particular the restriction of f to $\{0, \ldots, b-1, b\}$ is a finite one-to-one function from $\{0, \ldots, b-1, b\}$

into $\{0, \ldots, b-1\}$. But no finite function can have these properties. This contradiction completes the proof. □

Lemma 4.3.2 (Paris and Kirby [49]). *If φ is Σ^0_n then $(\forall x < y)\varphi$ is equivalent in* $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n$ *to a Σ^0_n formula.*

Proof. Work in $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n$. We will proceed by induction on n. Certainly, this is true for $n = 0$, so suppose it's true for $n-1$. Let φ be $(\exists z)\theta(x, y, z)$ for some $\theta \in \Pi^0_{n-1}$. Then

$$\mathrm{B}\Sigma^0_n \vdash (\forall x < y)\varphi \leftrightarrow (\exists t)(\forall x < y)(\exists z < t)\theta. \tag{4.1}$$

By the induction hypothesis, $(\forall z < t)\neg\theta$ is equivalent to a Σ^0_{n-1} formula, so $(\exists z < t)\theta$ is equivalent to a Π^0_{n-1} formula. Therefore by (4.1), $(\forall x < y)\varphi$ is equivalent to a Σ^0_n formula, as desired. □

To say something about the relationship between bounded Σ^0_1-comprehension and Σ^0_1-induction, we will need to make use of $\mathrm{C}_{\mathbb{N}}$.

Proposition 4.3.3. $\mathrm{C}_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound.

Proof. We argue in a model (M, S) of RCA_0. Let the enumeration of A be a $\mathrm{C}_{\mathbb{N}}$-instance, so $\overline{A}$ is nonempty. Define the enumeration of a set F by putting s into F whenever the least element m_s of $\overline{A}$ at stage s of the enumeration of A enters A at stage s. If F were unbounded, then the set of numbers m_s would be as well, since the map taking F to this set is injective and computable. But then A would be empty. Therefore F is bounded, and therefore the enumeration of F is an instance of Bound. If s is a solution to this Bound-instance, then the least element of A at stage s must be in $\overline{A}$, and therefore is a solution to our $\mathrm{C}_{\mathbb{N}}$-instance. □

It follows that $\mathrm{C}_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound* as well, since clearly Bound $\leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}}$ Bound*. From the previous chapter, we also have $\mathrm{C}^*_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound*. We can now take the first step in determining the relationship between $\mathrm{F}\Sigma^0_1$ and (st)$\mathrm{B}\Sigma^0_1$CA.

Proposition 4.3.4. $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA} \star \mathrm{C}_{\mathbb{N}}$.

Proof. Suppose $(M, S) \models \mathrm{RCA}^*_0$. Suppose that $B \subseteq S$ is an $\mathrm{F}\Sigma^0_1$-instance. Let $A = \{x \mid (\exists y)(y, x) \in B\}$. Then B is a $\mathrm{C}_{\mathbb{N}}$-instance, so looking at $\overline{A}$ as a Π^0_1 set in the usual way, we can find some a such that $a \in \overline{A}$. By $\mathrm{B}\Sigma^0_1\mathrm{CA}$, we can use an enumeration of A and a to get $\{x \in A \mid x < a\} = \{x \in M \mid x < a \wedge (\exists y)(y, x) \in B\}$. This set exists in the model, so for each x in the set, we can compute whether $S(x)$ is in the set. If we encounter some x for which this is not the case, then $S(x)$ is a solution to our $\mathrm{F}\Sigma^0_1$-instance. Otherwise, a must be a solution to our $\mathrm{F}\Sigma^0_1$-instance. Note that this procedure is uniform because we can computably figure out which option we must choose for our solution. □

Proposition 4.3.5. $\mathrm{stB}\Sigma^0_1\mathrm{CA} \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{F}\Sigma^0_1$.

Proof. We will follow the idea of the proof of Theorem II.3.9. in [60]. Suppose that $(M, S) \models \mathrm{RCA}^*_0$ and X is an instance of $\mathrm{stB}\Sigma^0_1\mathrm{CA}$, bounded by some number b. Let $\varphi(k)$ be the principle "there exists a finite one-to-one function with domain k whose image is a subset of X". Note that φ is Σ^0_1. Let $B = \{k : \varphi(k)\}$. Then there is an enumeration A of B in S, and $b + 1 \notin B$, so A is an $\mathrm{F}\Sigma^0_1$-instance. Let k be a solution to A as an $\mathrm{F}\Sigma^0_1$-instance, and note that we must have $0 \in B$, so $k > 0$ and therefore $k - 1 \in B$ and $k \in \overline{B}$. Let f be a finite one-to-one function with domain $k - 1$ whose universe is a subset of X. Note that we can construct f in M. If the range of f were a strict subset of X, then we could add 1 element to the range of f and map k to it, getting $k \in B$, a contradiction. Therefore the range of f must be all of X. Then the range of f is a solution to X as an $\mathrm{stB}\Sigma^0_1\mathrm{CA}$-instance. □

Proposition 4.3.6. $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{stB}\Sigma^0_1\mathrm{CA}$.

Proof. Suppose $(M, S) \models \mathrm{RCA}^*_0$. Suppose that $B \subseteq S$ is an $\mathrm{F}\Sigma^0_1$-instance. Let $A = \{x \mid (\exists y)(y, x) \in B\}$. We construct a $\mathrm{stB}\Sigma^0_1\mathrm{CA}$-instance X in stages $s \in \omega$. At stage $s = 0$, do nothing. At stage $s > 0$, for each $u < s$, enumerate u into X only if for all $v \leqslant u$, v has been enumerated into A. Note that X is bounded because B is a $\mathrm{F}\Sigma^0_1$-instance, so there exists some $a \in \overline{A}$, and X is bounded by that a. As the enumeration of X is our $\mathrm{stB\Sigma^0_1CA}$-instance, our solution is the set X itself. Although we don't know a, from X, we can get a solution to B as an $\mathrm{F\Sigma^0_1}$-instance by taking the largest element of X. □

Proposition 4.3.7. $\mathrm{B\Sigma^0_1CA} \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{Bound}^*$.

Proof. Work in a model (M, S) of RCA^*_0. Suppose we have a $\mathrm{B\Sigma^0_1CA}$-instance that is the enumeration of a set X with bound b given by $Y = \{(x, s) \mid x \text{ enters } X \text{ at stage } s\}$. For each $i < b$, let $S_i = \{s \mid (i, s) \in Y\}$ (some of the sets S_i may be empty). Then each S_i is bounded because Y converges to the bounded set X, so the enumeration of the sets $S_0, S_1, \ldots, S_{b-1}$ comprise a Bound^*-instance. Given a solution s to $S_0, S_1, \ldots, S_{b-1}$ as a Bound^*-instance, by construction Y has converged to X by stage s, we can just enumerate Y for s stages and then we will have reconstructed X, thereby obtaining a solution to our $\mathrm{B\Sigma^0_1CA}$-instance. □

We are now able to say something about the relationship between $\mathrm{F\Sigma^0_1}$ and Bound^*, a first step in our goal of determining the relationship between Δ^0_2-induction and Σ^0_2-bounding in this setting. Throughout, we will follow the idea of the proofs in [61], with adaptations as necessary to the computability-theoretic setting.

Proposition 4.3.8. $\mathrm{F\Sigma^0_1} \leqslant^{\mathrm{RCA}_0,3}_{\mathrm{gW}} \mathrm{Bound}^*$.

Proof. Suppose $(M, S) \models \mathrm{RCA}_0$. Suppose that $B \subseteq S$ is an $\mathrm{F\Sigma^0_1}$-instance. Let $A = \{x \mid (\exists y)(y, x) \in B\}$. Then B is a $\mathrm{C}_{\mathbb{N}}$-instance and by Proposition 4.3.3, we have $\mathrm{C}_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}^*$, so we can use $\mathrm{C}_{\mathbb{N}}$ to find some a such that $a \in \overline{A}$.

We will construct an instance $\mathcal{F} = \{F_1, F_2, \ldots, F_a\}$ of Bound^* with a many sets as follows. Enumerate the elements of A. Whenever an element $x < a$ enters A at stage s_x, we enumerate s_x into the set F_x. Then each set F_x is either empty or a singleton, and hence clearly bounded, so $\mathcal{F}$ is indeed a Bound^*-instance. Let b be a solution to this Bound^*-instance. Then if we enumerate A for b steps, we will have enumerated every element of A

below a. Then we know $\{x \in A \mid x < a\}$, so by the previous proposition, we can apply $\mathrm{B}\Sigma^0_1\mathrm{CA}$ to obtain a solution to our $\mathrm{F}\Sigma^0_1$-instance. $\square$

Of course, we would like to improve upon the previous result by getting a Weihrauch reduction, not just a generalized Weihrauch reduction. However, in order to get a Weihrauch reduction, we must do away with the initial application of $\mathrm{C}_{\mathbb{N}}$ to find an element $a \in \overline{A}$. It turns out that in losing this information about our $\mathrm{F}\Sigma^0_1$-instance, we must also lose some information about our Bound*-instance. As a a result, when we shift from a generalized Weihrauch reduction to a Weihrauch reduction, we go from reducing to Bound* to reducing to stBound*.

Proposition 4.3.9. $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}}$ stBound*.

Proof. Suppose $(M, S) \models \mathrm{RCA}^*_0$. Suppose that $B \subseteq S$ is an $\mathrm{F}\Sigma^0_1$-instance. Let $A = \{x \mid (\exists y)(y, x) \in B\}$. We will build a stBound*-instance using the following procedure. We proceed by stages $s \in \omega$. Begin to enumerate A. At stage $s = 0$, set $g = 0$. At stage $s > 0$, look at the least element u not enumerated into A such that every element $v < u$ is enumerated into A. We guess that u is a solution to B by setting $g = u + 1$ if $g < u + 1$ currently and enumerating the pair (s, g) into a set C if we have changed the value of g. Then C is either empty or a singleton, so in particular C is bounded. Let $C^s = \{u \mid (u, s) \in C\}$ and let $C_u = \{s \mid (u, s) \in C\}$. Clearly each set C^s is finite. On the other hand, each set C_u is also finite because it must have 0 or 1 element. Then we can apply stBound* to C to get a bound on the collection of sets C_u. This gives us a bound t on the stage s at which elements $v < u$ are enumerated into A, for each u that we guess to be a solution to B. Then at stage t, the least element u that hasn't been enumerated into A is either $u = 0 \in \overline{A}$ or is such that $u \in \overline{A}$ and $u + 1 \in A$, and therefore u is our solution to B as an $\mathrm{F}\Sigma^0_1$-instance. $\square$

Note that in the above proof, knowing that there exists an $a \in \overline{A}$ but not knowing the value of a is key. If we knew the value of a, then we would have a reduction to Bound* instead of stBound*. In fact, it turns out that we cannot get a Weihrauch reduction from $\mathrm{F}\Sigma^0_1$ to Bound* at all over RCA_0^*; a generalized Weihrauch reduction is the best that we can do.

To show this, we need the following lemma, which is a generalization of Proposition 3.7.4. It was noted before that Proposition 3.7.4 could be generalized, but here we explicitly state the generalization and give its proof for clarity.

Lemma 4.3.10. *Let Γ be a consistent extension of Δ^0_1-comprehension that proves the existence of a universal Σ^0_1 formula. Let* P *and* Q *be Π^1_2-problems such that*

1. P *has an ω-instance X such that for any finite initial segment σ of X, there is a model (M, S) of Γ and an M-instance Y of* P *in S that extends σ and has no solution in S; and*

2. *every instance X of* Q *includes a parameter $k_X \in \mathbb{N}$ such that for every model (M, S) of Γ and every M-instance X of* Q *in S, if k_X is a standard natural number, then X has a solution in S.*

Then $\mathrm{P} \not\leqslant^{\Gamma}_{\mathrm{W}} \mathrm{Q}$.

Proof. Suppose that $\mathrm{P} \leqslant^{\Gamma}_{\mathrm{W}} \mathrm{Q}$ via Φ_e and Φ_i. Let X be an ω-instance of P such that for any finite initial segment σ of X, there is a model $(M, S) \models \Gamma$ and an M-instance Y of P in S that extends σ and has no solution in S. Then Φ_e^X must be an instance of Q. This instance has a fixed parameter k, which must be the same standard natural number no matter what model of Γ we are working in, because the convergent computation over the standard natural numbers still exists in any such model. Now let (M, S) be a model of Γ that contains an M-instance Y of P with no solution. We can modify Y to define a new M-instance $\hat{Y}$ of P that agrees with X up to the use of the part of the computation of Φ_e^X that fixes the parameter k, and then agrees with Y after that. Since $\hat{Y}$ only differs finitely from Y, $\hat{Y}$ has no solution, but $\Phi_e^{\hat{Y}}$ is an instance of Q with k a standard natural number, and hence must have a solution x. But then Φ_i should be able to compute a solution to $\hat{Y}$ from $\hat{Y}$ and x, which is a contradiction. □

From this lemma, we obtain the desired corollary.

Corollary 4.3.11. $\mathrm{F}\Sigma^0_1 \not\leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{Bound}^*$.

Proof. Note that the empty enumeration is an ω-instance of $\mathrm{F}\Sigma^0_1$ any finite initial segment of which can be extended to an $\mathrm{F}\Sigma^0_1$-instance with no solution, in particular in any model of RCA^*_0 where Σ^0_1 induction fails. Moreover, every instance X of Bound^* includes a fixed number of sets $k \in \mathbb{N}$ such that for every model (M, S) of RCA^*_0 and every M-instance X of Bound^* in S, if k is a standard natural number, then X has a solution in S. The non-reduction then follows by Lemma 4.3.10. □

We can also look at what happens in the other direction. We obtain the following results.

Proposition 4.3.12. $\mathrm{Bound} \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{F}\Sigma^0_1$.

Proof. Work in a model of RCA^*_0. Let the enumeration of a set F be a Bound-instance. Consider the set $A = \{n \mid (\exists x \in F)(n \leqslant x)\} \cup \{0\}$. An enumeration of A exists in RCA^*_0 and A is Σ^0_1. Moreover, $\overline{A}$ is nonempty because F is bounded. Therefore the enumeration of A is a $\mathrm{F}\Sigma^0_1$-instance. Let a be a solution to A as an $\mathrm{F}\Sigma^0_1$-instance. Since $0 \in A$, we must have that $a \in \overline{A}$ and $a = S(b)$ for some $b \in A$. Then for all $x \in F$ with $x \neq 0$, $a > x$, and $a > 0$ by construction. Therefore a is a solution to F as a Bound-instance. □

Proposition 4.3.13. $\mathrm{stBound}^* \leqslant_{\mathrm{W}} \mathrm{F}\Sigma^0_1$, *and hence* $\mathrm{Bound}^* \leqslant_{\mathrm{W}} \mathrm{F}\Sigma^0_1$.

Proof. Let the enumeration of a set F of pairs be a $\mathrm{stBound}^*$-instance, so $\{n \mid (\exists k)((n, k) \in F)\}$ is bounded and for each n, so is the set $\{k \mid (n, k) \in F\}$. We are looking for a bound on $\{k \mid (\exists n)((n, k) \in F\}$. Let $A = \{k \mid (\exists x)(\exists n)((n, x) \in F \text{ and } k \leqslant x)\} \cup \{0\}$. Then an enumeration of A exists, and A is Σ^0_1.

$\overline{A}$ is nonempty because $\{k \mid (\exists n)((n,k) \in F\}$ is bounded. Therefore the enumeration of A is a $\mathrm{F}\Sigma^0_1$-instance. Since $0 \in A$, we must have that $a \in \overline{A}$ and $a = S(b)$ for some $b \in A$. Since $a \in \overline{A}$, for all x and all n, if $(n,x) \in F$, then $a > x$. That is, a is a bound on $\{k \mid (\exists n)((n,k) \in F\}$, as desired. □

Note that we previously used that $\mathrm{C}_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound. Therefore we have $\mathrm{C}_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Sigma^0_1$. We now look at the other direction. We will give a direct argument, but this can be proved indirectly, such as in the proof of Lemma 2.3 in [55]. In the proof of that Lemma, Bound and $\mathrm{C}_{\mathbb{N}}$ are also shown to be equivalent to the principle of unique closed choice, $\mathrm{UC}_{\mathbb{N}}$.

Proposition 4.3.14. Bound $\leqslant_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.

Proof. Let the enumeration of a bounded set A be a Bound-instance. Let $B = \{b \mid (\exists a \in A)(b \leqslant a)\}$. Then B is Σ^0_1, and $\overline{B}$ is nonempty because A is bounded. Therefore the enumeration of B is a $\mathrm{C}_{\mathbb{N}}$-instance. Let b be a solution to B as a $\mathrm{C}_{\mathbb{N}}$-instance, so $b \in \overline{B}$. Then for every $a \in A$, $b > a$. Therefore b is a solution to A as a Bound-instance. □

It is also natural to be curious about the relationship between $\mathrm{F}\Pi^0_1$ and $\mathrm{F}\Sigma^0_1$. As discussed in Section 3, intuitively, $\mathrm{F}\Pi^0_1$ is a weaker principle than $\mathrm{F}\Sigma^0_1$. Thus it is natural to first look at the $\mathrm{F}\Pi^0_1$ to $\mathrm{F}\Sigma^0_1$ direction of the relationship. We obtain the following result.

Proposition 4.3.15. $\mathrm{F}\Pi^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{F}\Sigma^0_1$.

Proof. Suppose $(M,S) \models \mathrm{RCA}^*_0$. Let $\overline{B}$ be an $\mathrm{F}\Pi^0_1$-instance. As in the previous proofs, we may assume that $0 \in B$. By the definition of a $\mathrm{F}\Pi^0_1$-instance, we have some $b \in \overline{B}$. Consider $C := \{x \leqslant b : b - x \in \overline{B}\}$. Note that $b \notin C$. Therefore C is an $\mathrm{F}\Sigma^0_1$-instance, so there is some $c \in \overline{C}$ such that either $c = 0$ or $c - 1 \in C$. Since $b \in \overline{B}$, $0 \in C$, so we must have $c > 0$ and $c - 1 \in C$. Then $b - c \in B$ but $b - c - 1 \in \overline{B}$. Therefore $b - c$ is a solution to $\overline{B}$ as an $\mathrm{F}\Pi^0_1$-instance.

□

Alternatively, note that given an instance A of $\mathrm{F}\Pi^0_1$, we can find an element $a \in \overline{A}$, and by $\mathrm{B}\Sigma^0_1\mathrm{CA}$, we get a solution to A as an $\mathrm{F}\Pi^0_1$-instance. We will look at the other direction in Section 6.

Consider the version of $\mathrm{F}\Sigma^0_1$ where we add to the instance a place where induction fails, i.e. an element a of $\overline{A}$. Note that the difference between $\mathrm{F}\Sigma^0_1$ as originally stated and $\mathrm{F}\Pi^0_1$ is that this a is not given for $\mathrm{F}\Sigma^0_1$, but we can find it for $\mathrm{F}\Pi^0_1$. Therefore $\mathrm{F}\Pi^0_1$ is equivalent to this stronger version of $\mathrm{F}\Sigma^0_1$. Then making use of the compositional product operation, $\star$, between Weihrauch degrees (see, e.g. [10]), and the structure of the proof of Corollary 4.3.8, we obtain the following additional corollary.

Corollary 4.3.16. $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Pi^0_1 \star \mathrm{Bound}^* \star \mathrm{C}_{\mathbb{N}}$.

Just as we have been looking at the relationship in terms of (generalized) Weihrauch reduction over RCA_0 and RCA^*_0 between the various induction principles, it is natural to also look at the relationship in this arena between different choice principles. To start, we look at $\mathrm{C}\Delta^0_2$ and $\mathrm{C}_{\mathbb{N}}$.

Proposition 4.3.17. $\mathrm{C}\Delta^0_2 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.

Proof. Work in a model (M, S) of RCA^*_0. Let X be (an approximation in RCA^*_0 to) a nonempty Δ^0_2 set. Begin to approximate X in stages and form a set Y as follows: at stage s, whenever $n \in \overline{X}$, enumerate all pairs (n, t) for $t \leqslant s$ into $\overline{Y}$. If later $n \in X$ at stage s_1, do not enumerate pairs (n, t) for $s < t \leqslant s_1$ into $\overline{Y}$. In general, whenever $n \in \overline{X}$ at stage s, we enumerate all of the pairs (n, t) for $t \leqslant s$ into $\overline{Y}$, and whenever $n \in X$ at stage s, we stop enumerating pairs into $\overline{Y}$. Y is a Π^0_1 set because the process by which we build Y is a Π^0_1 process. Moreover, Y is nonempty because X is. Therefore the enumeration of $\overline{Y}$ is a $\mathrm{C}_{\mathbb{N}}$-instance. Given a solution to $\overline{Y}$ as a $\mathrm{C}_{\mathbb{N}}$-instance, that is, an element of Y, the element is of the form (n, s) for some $n \in X$, so n is a solution to our $\mathrm{C}\Delta^0_2$-instance. □

On the other hand, it is clear that $\mathrm{C}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{C}\Delta_2^0$ because $\mathrm{C}_{\mathbb{N}}$ is a special case of $\mathrm{C}\Delta_2^0$. Therefore

Corollary 4.3.18. $\mathrm{C}_{\mathbb{N}} \equiv_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{C}\Delta_2^0$.

With this equivalence in mind, we will refer only to $\mathrm{C}_{\mathbb{N}}$ going forward. Another choice principle of interest is $\mathrm{K}_{\mathbb{N}}$. Since $\mathrm{K}_{\mathbb{N}}$ is a special case of $\mathrm{C}_{\mathbb{N}}$, we clearly have $\mathrm{K}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{C}_{\mathbb{N}}$. In fact, Brattka, Gherardi, and Marcone showed in [9] that $\mathrm{K}_{\mathbb{N}} <_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$. We will show a slightly stronger separation between the two principles in Section 7.

Proposition 4.3.19. $\mathrm{K}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{F}\Pi_1^0$.

Proof. Work in a model (M, S) of RCA_0. Suppose we have a $\mathrm{K}_{\mathbb{N}}$-instance that is the enumeration of a set X with bound b. We will construct a Π_1^0 set A. We start off with $A = M$ and $\overline{A} = \emptyset$, we define $c = b+1$, and we enumerate c into $\overline{A}$. Whenever we enumerate a new element x into X, we decrement c by 1 and enumerate c into $\overline{A}$. Then clearly $\overline{A}$ is Σ_1^0 and nonempty, so it follows that A is an $\mathrm{F}\Pi_1^0$-instance. Let a be a solution to A as an $\mathrm{F}\Pi_1^0$-instance, so $a \in \overline{A}$. By construction, we cannot have $a = 0$, because the smallest possible element of $\overline{A}$ is $b + 1 - b = 1$. Then we must have $a > 0$ with $a - 1 \in A$. By construction, a must then be the smallest element of $\overline{A}$, which means that it is of the form $b + 1 - i$, where i is the number of elements in X. But then we can solve our $\mathrm{K}_{\mathbb{N}}$-instance by simply enumerating Y until i elements have appeared. Therefore $\mathrm{K}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{F}\Pi_1^0$, as desired.

□

Proposition 4.3.20. $\mathrm{F}\Pi_1^0 \nleqslant_{\mathrm{W}} \mathrm{K}_{\mathbb{N}}$.

Proof. Suppose, to the contrary, that we had a Weihrauch reduction $\mathrm{F}\Pi_1^0 \leqslant_{\mathrm{W}} \mathrm{K}_{\mathbb{N}}$, given by Turing functionals Φ and Ψ. Note that for any x, the enumeration of the complement of the singleton $\{x\}$ is a $\mathrm{F}\Pi_1^0$-instance. For concreteness, take $x = 1$. Then $\Phi^{\omega \setminus \{1\}}$ must be a

$K_{\mathbb{N}}$-instance, corresponding to some bound b. Wait until a stage s at which, for each $x < b$ such that x currently looks like a solution to our $K_{\mathbb{N}}$-instance, $\Psi^{(\omega\setminus\{1\})\oplus x}\downarrow = 1$. Once this convergence has occurred, we enumerate 0 into the complement of our $F\Pi^0_1$-instance. This gives us a contradiction, because then 1 can no longer be a solution to our $F\Pi^0_1$-instance, yet, $\Psi^{(\omega\setminus\{0,1\})\oplus x}$ has already converged to 1 for every possible solution x to our $K_{\mathbb{N}}$-instance. □

Proposition 4.3.21. $K_{\mathbb{N}} \not\leqslant_W RT^1_2$.

Proof. Suppose, to the contrary, that we had a Weihrauch reduction $K_{\mathbb{N}} \leqslant_W RT^1_2$, given by Turing functionals Φ and Ψ. We will build a $K_{\mathbb{N}}$-instance given by the enumeration of some set X smaller than some bound b. Start by letting X be the enumeration of $\emptyset$, and let $b = 3$. Then we look at $\Phi^{\langle X,b\rangle}$, which must give us a RT^1_2-instance, because X and b already comprise a valid $K_{\mathbb{N}}$-instance. $\Phi^{\langle X,b\rangle}$ has two possible solutions, 0 and 1. Then if we let $x_i = \Psi^{\langle X,b\rangle\oplus i}$ for $i = 0, 1$, we can diagonalize against both possible solutions by enumerating x_0 and x_1 into X. Then X is still a valid $K_{\mathbb{N}}$-instance, with solutions the numbers between 0 and 3 that have not yet been enumerated (there may be more than one such number if one of the x_i's is undefined). However, we have ensured that $\Psi^{\langle X,b\rangle\oplus 0}$ and $\Psi^{\langle X,b\rangle\oplus 1}$ cannot converge to these solutions, giving us a contradiction. □

Using essentially the same proof, we can also show that $K_{\mathbb{N}} \not\leqslant_W RT^1_k$ for any k. The previous argument is also an example of the following more general phenomenon. First, we make a definition to help narrow the scope of problems we will consider.

Definition 4.3.22. We say that a Π^1_2 problem P is *first-order* and write $P \in \mathcal{F}$ if the codomain of P is $\mathbb{N}$.

Theorem 4.3.23. *Fix $k \geqslant 1 \in \omega$. Suppose that* P *is a first-order Π^1_2 principle such that for any initial segment σ of any* P*-instance X and any $x_0, x_1, \ldots, x_{k-1} \in \omega$, there is another* P*-instance $\hat{X}$ extending σ for which none of $x_0, x_1, \ldots, x_{k-1}$ are solutions. Then* $P \not\leqslant_W RT^1_k$.

Proof. Suppose, to the contrary, that we had a Weihrauch reduction $\mathrm{P} \leqslant_{\mathrm{W}} \mathrm{RT}^1_k$, given by Turing functionals Φ and Ψ. Let σ be such that for any $n < k$, if $\Psi^{\sigma \oplus n}$ diverges, then $\Psi^{\tau \oplus n}$ also diverges, for any τ extending σ such that τ is an initial segment of some P-instance. Let $x_0, x_1, \ldots, x_m$ be the values of $\Psi^{\sigma \oplus 0}$, $\Psi^{\sigma \oplus 1}$, $\ldots$, $\Psi^{\sigma \oplus k-1}$ for those that converge. By assumption, there is a P-instance $\hat{X}$ extending σ for which none of $x_0, x_1, \ldots, x_m$ are solutions. Take $\hat{X}$ to be our P-instance, and look at the RT^1_2-instance $\Phi^{\hat{X}}$. But whenever $\Psi^{\hat{X} \oplus i}$ converges, it converges to some x_j, and no x_j can be a solution to $\hat{X}$, giving us a contradiction. □

Corollary 4.3.24. *For* $\mathrm{P} \in \{\mathrm{F}\Sigma^0_1, \mathrm{F}\Pi^0_1, \mathrm{Bound}^*, \mathrm{stBound}^*, \mathrm{Bound}, \mathrm{K}\Delta^0_2, \mathrm{B}\Sigma^0_1\mathrm{CA}\}$*,* $\mathrm{P} \not\leqslant_{\mathrm{W}} \mathrm{RT}^1_2$.

Note that with some generalization, we could also obtain that $\mathrm{P} \not\leqslant^n_{\mathrm{gW}} \mathrm{RT}^1_2$ for any fixed $n \in \omega$. Since we have seen that $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Pi^0_1 \star \mathrm{Bound}^* \star \mathrm{C}_{\mathbb{N}}$, we next examine the relationship between $\mathrm{B}\Sigma^0_1\mathrm{CA}$ and Bound.

Proposition 4.3.25. $\mathrm{B}\Sigma^0_1\mathrm{CA} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}$.

Proof. Work in a model (M, S) of RCA_0. Suppose we have a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance that is the enumeration of a set X with bound b given by $Y = \{(x, s) \mid x \text{ enters } X \text{ at stage } s\}$. Let $S = \{s \mid (\exists x)(x, s) \in Y\}$. Then S is bounded because Y converges to the bounded set X, so the enumeration of S is a Bound-instance. Given a solution s to S as a Bound-instance, by construction Y has converged to X by stage s, so we can just enumerate Y for s stages and then we will have reconstructed X, thereby obtaining a solution to our $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance. □

Note that the above argument uses that $\mathrm{B}\Sigma^0_1\mathrm{CA}$ is true in RCA_0, and would not go through over RCA^*_0. We will look at what happens in the other direction in Sections 6 and 7. Now, consider the following extension of $\mathrm{B}\Sigma^0_1\mathrm{CA}$ to Δ^0_2 sets.

Definition 4.3.26. $\mathrm{B}\Delta^0_2\mathrm{CA}$ is the Π^1_2 principle where an instance is a C such that

$$(\forall x)(\exists t)((\forall s > t)(\langle x, s\rangle \in C) \vee (\forall s > t)(\langle x, s\rangle \notin C)) \\ \wedge (\exists b)(\forall x)((\exists s)(\langle x, s\rangle \in C) \to (x \leqslant b)). \tag{4.2}$$

A solution is $\{x \mid (\forall y)(\exists s > y)(\langle x, s\rangle \in C \wedge x < b)\}$.

Here, we define $\mathrm{B}\Delta^0_2\mathrm{CA}$ based on the weak notion of Δ^0_2 sets in RCA^*_0. We could also define a principle $\mathrm{stB}\Delta^0_2\mathrm{CA}$ using the strong notion of Δ^0_2 sets in RCA^*_0, but we do not do so to avoid analogizing with $\mathrm{stB}\Sigma^0_1\mathrm{CA}$ (recall that we called this principle strong because the bound on the instance is not given). Of course, we could also define separate versions of $\mathrm{B}\Sigma^0_1\mathrm{CA}$ based on Σ^0_1 sets in the strong and weak sense, but we will not consider those here.

Proposition 4.3.27. $\mathrm{B}\Delta^0_2\mathrm{CA} \not\leqslant_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$.

Proof. Suppose, to the contrary, that there exists a Weihrauch reduction $\mathrm{B}\Delta^0_2\mathrm{CA} \leqslant_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$, given by Turing functionals Φ and Ψ. We will build a $\mathrm{B}\Delta^0_2\mathrm{CA}$-instance given by the approximation $\hat{X}$ of a set X and a bound b.

Let $b = 1$. We build $\hat{X}$ in stages $s \in \omega$. At stage $s = 0$, do nothing. For $s > 0$, at stage s enumerate the pair $(0, s)$ into $\hat{X}$. Continue until we reach a stage t at which $\Phi^{\langle \hat{X}, b\rangle}$ converges to an initial segment of a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance σ with bound n and $\Psi^{\langle \hat{X}, b\rangle \oplus \{x<n \mid x \text{ has been enumerated by } \sigma\}}$ converges to a set of numbers $\hat{Y}$ below b. Then we must have either $\hat{Y} = \emptyset$ or $\hat{Y} = \{0\}$. Then for all $t' > t$, start enumerating $(1, t')$ into $\hat{X}$ (and stop enumerating $(0, t')$ into $\hat{X}$). Continue until we reach a stage t_1 at which $\Phi^{\langle \hat{X}, b\rangle}$ converges to an initial segment of a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance σ_1 with bound n and $\Psi^{\langle \hat{X}, b\rangle \oplus \{x<n \mid x \text{ has been enumerated by } \sigma_1\}}$ converges to an initial segment τ_1 of a solution to $\hat{X}$. We proceed in the same way as before, diagonalizing against τ_1 by now approximating that $1 \notin X$ and $0 \in X$. Note that by the definition of a Σ^0_1 set together with the Limit Lemma, we can switch whether an element is in X any finite number of times, but our

$\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance can only change at most once for each of its n possible elements, and hence can change at most n times. Therefore this process is guaranteed to produce a contradiction.

□

We now turn our attention back to the relationship between $\mathrm{C}_{\mathbb{N}}$ and Bound.

Proposition 4.3.28. Bound $\leqslant^{\mathrm{RCA}_0^*}_{\mathrm{W}}$ $\mathrm{C}_{\mathbb{N}}$. *Therefore* $\mathrm{C}_{\mathbb{N}} \equiv^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound, *and* $\mathrm{C}^*_{\mathbb{N}} \equiv^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound*.

Proof. Let B be a Bound-instance. Let $A = \{n \mid (\forall m \in B)(m \leqslant n)\}$, so A is the set of possible solutions to B as a Bound-instance. The enumeration of $\overline{A}$ is a $\mathrm{C}_{\mathbb{N}}$-instance because $\overline{A} = \{n \mid (\exists m \in B)(m > n)\}$ is computable from B, hence is Σ^0_1 and A is nonempty because B is bounded. Let n be a solution to $\overline{A}$ as a $\mathrm{C}_{\mathbb{N}}$-instance, so $n \in A$. Then n is a solution to B as a Bound-instance, thereby completing the reduction. □

It remains open whether this equivalence also holds over RCA^*_0.

Proposition 4.3.29. $\mathrm{K}_{\mathbb{N}} \leqslant_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$.

Proof. Let the enumeration of a set X be a $\mathrm{K}_{\mathbb{N}}$-instance, bounded by a given number $b \in \omega$. Then the enumeration of X is also a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance, with solution $\{x \in X \mid x < b\}$. Note that since the enumeration of X is a $\mathrm{K}_{\mathbb{N}}$-instance, X has nontrivial complement below b. Since our solution tells us the elements of $\{x \in X \mid x < b\}$, we can computably determine the elements of $\overline{X}$ below b, thereby yielding a solution to our $\mathrm{K}_{\mathbb{N}}$-instance (say, take the least element of $\overline{X}$ below b). □

From [12] we have $\mathrm{K}_{\mathbb{N}} \not\leqslant_{\mathrm{W}} \mathrm{RT}^1_2$. It follows that $\mathrm{B}\Sigma^0_1\mathrm{CA} \not\leqslant_{\mathrm{W}} \mathrm{RT}^1_2$ also.

Proposition 4.3.30. $\mathrm{B}\Sigma^0_1\mathrm{CA} \not\leqslant_{\mathrm{W}} \mathrm{K}_{\mathbb{N}}$.

Proof. Suppose, to the contrary, that $\mathrm{B}\Sigma^0_1\mathrm{CA} \leqslant_{\mathrm{W}} \mathrm{K}_{\mathbb{N}}$. Let Φ and Ψ be such that for any set X with bound b such that the enumeration of X together with b is a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance,

$\Phi^{\langle X,b\rangle}$ is a $\mathrm{K}_{\mathbb{N}}$ instance and given a solution a to $\Phi^{\langle X,b\rangle}$, $\Psi^{\langle X,b\rangle\oplus a}$ is a solution to $\Phi^{\langle X,b\rangle}$ as a $\mathrm{K}_{\mathbb{N}}$-instance.

We build a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance in stages $s \in \omega$. Our $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance will be of the form $\{(n,s) \mid n \text{ is enumerated at stage } s\}$. We will use A to denote our enumeration, and b to denote our corresponding bound. We begin by not enumerating any elements into A, and letting $b = 1$. Then we look at $\Phi^{\langle A,b\rangle}$ and wait until s is large enough that $\Phi^{\langle A,b\rangle}$ converges to an initial segment σ of a $\mathrm{K}_{\mathbb{N}}$-instance at stage s. Now let $x_1, x_2, \dots, x_{m_s}$ denote all possible solutions to all $\mathrm{K}_{\mathbb{N}}$-instances extending σ at stage s, and for each i with $1 \leqslant i \leqslant m_s$, look at $\Psi^{\langle A,b\rangle\oplus x_i}$. Note that as s increases, some possible solutions disappear. Wait until s is large enough that each $\Psi^{\langle A,b\rangle\oplus x_i}$ converges to the empty set. If this never happens, then we have a contradiction, because if our $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance is the enumeration of the empty set with bound 1, then the only solution is the string 0. So suppose at some stage $t \in \omega$ that $\Psi^{\langle A,0\rangle\oplus x_i}$ converges to the string 0 for each i. Then we enumerate an element into A, say we enumerate 0 into A. Then any initial segment of the set we are enumerating must start with 1. This contradiction implies that our Weihrauch reduction cannot exist, as desired. □

We now introduce an auxiliary principle $\lim_2$ to help us delineate the relationship between $\mathrm{K}\Delta^0_2$ and $\mathrm{K}_{\mathbb{N}}$.

Definition 4.3.31. $\lim_2$ is the Π^1_2 problem where an instance is an infinite binary sequence with a limit, and a solution is the limit such a sequence.

Proposition 4.3.32. $\lim_2 \not\leqslant_{\mathrm{gW}} \mathrm{K}_{\mathbb{N}}$.

Proof. Suppose, to the contrary, that there is a generalized Weihrauch reduction $\lim_2 \leqslant_{\mathrm{gW}} \mathrm{K}_{\mathbb{N}}$. By assumption, we are given Turing functionals $\Phi_1, \Phi_2, \dots$ such that if Player 1 plays the initial segment $(x_0, x_1, \dots, x_k)$ of a $\lim_2$-instance on their 1^{st} move for some $k \in \omega$, and a solution y_j to Player 2's $(j-1)^{st}$ move on their j^{th} move for $1 \leqslant j \leqslant i-1$, then $\Phi_i^{(x_0,x_1,\dots,x_k)\oplus y_1 \oplus \dots \oplus y_{i-1} \oplus y_i}$ is an initial segment of Player 2's i^{th} move. We work in stages

$s \in \omega$ to build a $\lim_2$-instance. At stage s, we define the value of x_s. Start by setting $x_0 = 0$, $x_1 = 0, \ldots$. Then currently, the only solution to our $\lim_2$-instance is 0. Then if we run the assumed generalized Weihrauch reduction, we will obtain a tree of possible gameplay, based on the possible moves that Player 2 makes and Player 1's possible responses, given that Player 1 starts by playing $(x_0, x_1, \ldots, x_k)$. The j^{th} level of this tree will correspond to Player 1's j^{th} move in $G(\mathrm{K}_{\mathbb{N}} \to \lim_2)$. If this is indeed a valid generalized Weihrauch reduction, then the tree will be completely finished at the n^{th} level for some fixed n. This is because the property of being a solution to a $\mathrm{K}_{\mathbb{N}}$-instance is co-c.e., and there are in fact only finitely many possible solutions to a given $\mathrm{K}_{\mathbb{N}}$-instance.

Since being a solution to a $\mathrm{K}_{\mathbb{N}}$-instance is co-c.e., if we start with some number of possible solutions, as more elements are enumerated into our $\mathrm{K}_{\mathbb{N}}$-instance, we can lose solutions, but never gain them. Then eventually, we will reach a stage at which all of the remaining solutions are true solutions, so the tree is finite, essentially giving us a sort of compactness. Therefore, we may assume that if y is a solution to all $\mathrm{K}_{\mathbb{N}}$-instances extending the initial segment $\Phi_n^{(x_0,x_1,\ldots,x_k)\oplus y_1 \oplus \ldots \oplus y_{n-1}}$ that Player 2 could play and declare victory in the game, i.e. y is a leaf on the tree of gameplay, then we have a Turing functional Ψ such that $\Psi^{(x_0,x_1,\ldots,x_k)\oplus y}$ gives a solution to any valid $\lim_2$-instance extending $(x_0, x_1, \ldots, x_k)$. By construction, there is a stage s by which every path remaining on the tree ends on a leaf that points to 0 (i.e. Ψ is 0 on every such path), because 0 is the only possible solution to our $\lim_2$-instance after n moves. Once we reach this stage s, we let $x_{s+1} = 1$, $x_{s+2} = 1, \ldots$. But then the only solution to $\{x_1, x_2, \ldots\}$ as a $\lim_2$-instance is 1, and our reduction has already determined that 0 is the only possible solution. This contradiction implies the non-reduction. □

Remark 4.3.33. In the above proof, we make essential use of the fact that if a problem has finitely many solutions, and we consider the tree of gameplay given by a supposed generalized Weihrauch reduction to this principle (see the below proof for the specific construction of such a tree), this tree will eventually be finite. Moreover, if the problem is co-c.e., meaning that the complement of every instance is Σ^0_1, or even if the problem is Δ^0_2, all of its proposed solutions will eventually settle, in that every false solution will be seen to be false.

It is easy to see that $\lim_2 \leqslant_{\mathrm{W}} \mathrm{B}\Delta^0_2\mathrm{CA}$ and $\lim_2 \leqslant_{\mathrm{W}} \mathrm{K}\Delta^0_2$. We therefore obtain the following corollary.

Corollary 4.3.34. $\mathrm{B}\Delta^0_2\mathrm{CA} \nleqslant_{\mathrm{gW}} \mathrm{K}_{\mathbb{N}}$ *and* $\mathrm{K}\Delta^0_2 \nleqslant_{\mathrm{gW}} \mathrm{K}_{\mathbb{N}}$.

Note that the proof of Proposition 4.3.27 also shows that $\lim_2 \nleqslant_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$.

Now, we prove that $\mathrm{F}\Pi^0_1$ implies $\mathrm{B}\Sigma^0_1\mathrm{CA}$ in terms of Weihrauch reducibility over RCA^*_0, using a proof similar to that of Proposition 4.3.19.

Proposition 4.3.35. $\mathrm{B}\Sigma^0_1\mathrm{CA} \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{F}\Pi^0_1$.

Proof. Work in a model (M, S) of RCA_0. Suppose we have a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance that is the enumeration of a set X with bound b given by $Y = \{(x, s) \mid x \text{ enters } X \text{ at stage } s\}$. We will construct a Π^0_1 set A. We start off with $A = M$ and $\overline{A} = \emptyset$, we define $c = b^{2b+1} + 1$, and we enumerate c into $\overline{A}$. Whenever we enumerate some (x, s) into Y, we decrement c by 1 and enumerate c into $\overline{A}$. Then clearly $\overline{A}$ is Σ^0_1 and nonempty, so it follows that A is an $\mathrm{F}\Pi^0_1$-instance. Let a be a solution to A as an $\mathrm{F}\Pi^0_1$-instance, so $a \in \overline{A}$. By construction, we cannot have $a = 0$, because the smallest possible element of $\overline{A}$ is $b^{2b+1} + 1 - b$, which is at least $0^{2(0)+1} + 1 - 0 = 1$. Then we must have $a > 0$ with $a - 1 \in A$. By construction, a must then be the smallest element of $\overline{A}$, which means that it is of the form $b^{2b+1} + 1 - i$, where i is the number of elements of in Y (or equivalently, the number of elements in X). But then we can solve our $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance by simply enumerating Y into i elements have appeared. Therefore $\mathrm{B}\Sigma^0_1\mathrm{CA} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Pi^0_1$, as desired. ∎

Proposition 4.3.36. $\mathrm{F}\Pi^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$.

Proof. Work in a model (M, S) of RCA^*_0. Suppose we have an $\mathrm{F}\Pi^0_1$-instance given by a Π^0_1 set A with nonempty complement. Then $\overline{A}$ is Σ^0_1, so we can enumerate some element $x \in \overline{A}$.

Consider $C = \{a \in \overline{A} \mid a < x\}$. Then the enumeration of C is a $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance with bound x. A solution to this $\mathrm{B}\Sigma^0_1\mathrm{CA}$-instance is the set C. Therefore $C \in S$, so by set induction there is a least element $c \in C$. Then $c \in \overline{A}$ with $c < x$, and for all $b < c$, $b \in A$. Therefore c is a solution to our $\mathrm{F}\Pi^0_1$-instance, as desired.

□

Therefore $\mathrm{F}\Pi^0_1 \equiv^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$. For consistency, we will refer only to $\mathrm{F}\Pi^0_1$ going forward. Note that from the above argument, we in fact get that $\mathrm{L}\Pi^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$, where $\mathrm{L}\Pi^0_1$ denotes the Π^0_1 least-number principle. More generally, we see from this argument that any principle where a bound on the instance is part of the instance is implied by any bounded comprehension principle. Thus the fact that $\mathrm{F}\Pi^0_1 \equiv^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$ tells us that $\mathrm{F}\Pi^0_1$ is essentially the ultimate bounded level 1 principle.

Proposition 4.3.37. $\mathrm{F}\Pi^0_1 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{K}\Delta^0_2$.

Proof. Work in a model (M, S) of RCA_0. Suppose we have an $\mathrm{F}\Pi^0_1$-instance given by a Π^0_1 set A with nonempty complement. Let a be an element of $\overline{A}$, which we are given because $\overline{A}$ is Σ^0_1. Let $C = \{(x, s) \mid (x = 0 \text{ and } x \in \overline{A}) \vee (x \text{ has been enumerated into } \overline{A} \text{ by stage } s \text{ but } x - 1 \text{ has not been})\}$. Then C approximates a Δ^0_2 set, and if we let $\hat{C}$ be the restriction to the elements of C below a, then both $\hat{C}$ and $\overline{\hat{C}}$ approximate bounded Δ^0_2 sets. In particular, $\overline{\hat{C}}$ constitutes a $\mathrm{K}\Delta^0_2$-instance because it must be nonempty, since we are working over RCA_0. A solution to this $\mathrm{K}\Delta^0_2$-instance is an element smaller than a in the complement of the set that $\overline{\hat{C}}$ approximates, that is, in the set that $\hat{C}$ approximates. But this is precisely an x such that either $x = 0$ and $x \in \overline{A}$ or $x > 0$ and $x \in \overline{A}$ but $x - 1 \in A$. Therefore this x is a solution to our $\mathrm{F}\Pi^0_1$-instance. □

We will see in Section 4.5 that the above result does not hold over RCA^*_0.

Remark 4.3.38. Note that Section 3.7 outlines the relationships between many of the principles we have considered here in terms of Weihrauch and generalized Weihrauch reducibility over RCA_0. These include $\mathrm{stRT}^1_{<\infty}$, $\mathrm{RT}^1_{<\infty}$, RT^1_2, $\mathrm{stBound}^*$, Bound^*, and Bound. We include all of the relationships in our diagrams at the end of this chapter.

The below propositions can be shown with the same proof as that of the corresponding fact over RCA_0.

Proposition 4.3.39. $\mathrm{stBound}^* \leqslant^{\mathrm{RCA}_0^*,2}_{\mathrm{gW}} \mathrm{stRT}^1_{<\infty}$.

Proposition 4.3.40 (Dzhafarov, Hirschfeldt, and Reitzes [25])**.** $\mathrm{stBound}^* \leqslant^{\mathrm{RCA}_0^*,2}_{\mathrm{gW}} \mathrm{Bound}^*$ *but* $\mathrm{stBound}^* \nleqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}^*$.

One principle that we have noticeably omitted considering so far is the least number principle for Π^0_n sets. For Π^0_1 sets, we can interpret the least number principle $\mathrm{L}\Pi^0_1$ as the following problem min.

Definition 4.3.41. min is the Π^1_2 principle where an instance is an infinite sequence p in $\mathbb{N}^\mathbb{N}$ that is not surjective and a solution is the least $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $p(k) \neq n$.

The following two results are extensions of the result from [12] that $\mathrm{C}_\mathbb{N} \equiv_{\mathrm{sW}} \min$. In [12], Brattka and Rakotoniaina also argue that the n^{th} jump of min, $\min^{(n)}$, corresponds easily to the least number principle for Π^0_{n+1} sets $\mathrm{L}\Pi^0_{n+1}$.

Proposition 4.3.42. $\mathrm{C}_\mathbb{N} \leqslant^{\mathrm{RCA}_0^*}_{\mathrm{sW}} \min$.

Proof. We work in a model of RCA_0^* and let p be a $\mathrm{C}_\mathbb{N}$-instance, that is, an enumeration of the complement of a nonempty set A. Then we can also view p as a min-instance, and given a solution n to p as a min-instance, we must have $n \in A$. □

Note that $\min \nleqslant^{\mathrm{RCA}_0^*}_{\mathrm{sW}} \mathrm{C}_\mathbb{N}$, since $\mathrm{C}_\mathbb{N}$ is true in every model of RCA_0^*, while min is not. We will see more arguments of this kind in Section 4.5.

Proposition 4.3.43. $\min \leqslant^{\mathrm{RCA}_0}_{\mathrm{sW}} \mathrm{C}_\mathbb{N}$.

Proof. We work in a model of RCA_0 and let p be a min-instance, so $p \in \mathbb{N}^{\mathbb{N}}$. We will enumerate the complement of a set A in stages $s \in \omega$. At stage i, let $k_i = \min\{n \mid (\forall j \leqslant i)(p(j) \neq n)\}$. Note that this minimum exists in RCA_0 because the set is computable. At stage i, we remove all $\langle n, m\rangle$ from A for $n, m \leqslant i$ and $m \neq k_i$. Then our final set A is such that $A \subseteq \{\langle n, k\rangle \in \mathbb{N} : n \in \mathbb{N}, k = \min(\mathbb{N}\backslash \mathrm{range}(p))\}$, and A is nonempty. Given a number $\langle n, k\rangle \in A$, we can easily find $k = \min(\mathbb{N}\backslash \mathrm{range}(p))$. That is, given a solution to our $\mathrm{C}_{\mathbb{N}}$-instance, we can easily find a solution to our min-instance, completing the reduction. □

4.4 Extension to Δ^0_2 sets

We now extend some of the notions we have been considering to Δ^0_2 sets, and introduce some new notions. A natural place to start is with the Δ^0_2-least number principle.

Definition 4.4.1. We say that a set X is Δ^0_2 *in the strong sense* if we have that $(\forall x)(\forall y < x)(\exists t)((\forall s > t)(\langle y, s\rangle \in X) \vee (\forall s > t)(\langle y, s\rangle) \notin X))$.

Definition 4.4.2. We say that a set X is Δ^0_2 *in the weak sense* if we have that $(\forall x)(\exists t)((\forall s > t)(\langle x, s\rangle \in X) \vee (\forall s > t)(\langle x, s\rangle) \notin X))$.

We present two versions, a strong one and a weak one, of the Δ^0_2-least number principle $\mathrm{L}\Delta^0_2$, corresponding to our weak and strong notions, respectively, of a Δ^0_2 set.

Definition 4.4.3. $\mathrm{L}\Delta^0_2$ is the Π^1_2 principle where an instance is a nonempty set C such that

$$(\forall x)(\exists t)((\forall s > t)(\langle x, s\rangle \in C) \vee (\forall s > t)(\langle x, s\rangle \notin C)). \tag{4.3}$$

A solution is an a such that for all sufficiently large s, $\langle a, s\rangle \in C$ and for all $b < a$ and all sufficiently large u, $\langle b, u\rangle \notin C$.

Definition 4.4.4. $\mathrm{wkL}\Delta^0_2$ is the Π^1_2 principle where an instance is a nonempty set C such that

$$(\forall x)(\exists t)(\forall y < x)((\forall s > t)(\langle y, s\rangle \in C) \vee (\forall s > t)(\langle y, s\rangle \notin C)). \tag{4.4}$$

A solution is an a such that for all sufficiently large s, $\langle a, s\rangle \in C$ and for all $b < a$ and all sufficiently large u, $\langle b, u\rangle \notin C$.

In both of the above definitions, by sufficiently large s (or u) we mean large enough so that either for all $s' \geqslant s$ (or $u' \geqslant u$), $\langle a, s'\rangle \in C$ (or $\langle b, u'\rangle \in C$) or for all $s' \geqslant s$ (or $u' \geqslant u$), $\langle a, s'\rangle \notin C$ (or $\langle b, u'\rangle \notin C$). In other words, sufficiently large denotes a stage at which C has made a decision about whether it contains the corresponding element.

Note that $\mathrm{L}\Delta^0_2$ can also be used to obtain the greatest element of a set when we know a bound on that set, for instance by taking the least number of the form $a - x$ for some fixed a. In this chapter we will mainly be concerned with $\mathrm{L}\Delta^0_2$, but we can also extend this principle more generally to Δ^0_n sets.

Proposition 4.4.5. $\mathrm{L}\Delta^0_2 \leqslant_{\mathrm{W}}$ Bound.

Proof. Let C be an $\mathrm{L}\Delta^0_2$-instance. Then C is an approximation to a nonempty set, and without loss of generality, we may assume that C is defined in such a way that there is a least x such that $\langle x, s\rangle \in C$. We construct a set B in stages $s \in \omega$ as follows. For each s, let x_s be the least x such that $\langle x, s\rangle \in C$. At stage $s = 0$, enumerate 0 into B. For each stage $s > 0$, if $x_s \neq x_{s-1}$, enumerate s into B. Then B is the enumeration of a set which is bounded because the least element of the set approximated by C must eventually stabilize, and therefore the sequence $x_0, x_1, \ldots$ must eventually converge. Let b be a solution to B as a Bound-instance, so for all $s \in B$, $s \leqslant b$. Then $\langle x_b, s\rangle \in C$ for every $s \geqslant b$. If $x_b = 0$, then it follows that x_b is a solution to C as a $\mathrm{L}\Delta^0_2$-instance. Otherwise, we must have $\langle x_b, s\rangle \in C$ for all $s \geqslant b$, by the definition of b. Therefore in both cases x_b is a solution to our $\mathrm{L}\Delta^0_2$-instance, thereby completing the reduction.

□

Note that it is not possible to get $\mathrm{L}\Delta^0_2 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound, because Bound is true in every model of RCA_0, while $\mathrm{L}\Delta^0_2$ is not. We will see more results like this in the next section. However, we do get $\mathrm{L}\Delta^0_2 \leqslant^{\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2}_{\mathrm{W}}$ Bound by the above argument.

Proposition 4.4.6. Bound$^* \leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{L}\Delta^0_2$, *and* Bound$^* \leqslant^{\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2}_{\mathrm{W}} \mathrm{L}\Delta^0_2$.

Proof. We show the generalized Weihrauch reduction. Work in a model (M, S) of RCA_0. Let $\mathcal{F} = \{F_0, F_1, \ldots, F_n\}$ be a Bound*-instance for some given $n \in M$. Consider $A := \{i < n \mid$ the least upper bound for F_i is bigger than the least upper bound for all subsequent $F_j\}$. We write A as a set for convenience, but in fact in RCA_0, A is really a predicate. Note that A is Δ^0_2 because there exists a stage where elements stop being enumerated into F_i, since F_i is a bounded set. At that stage, if $i \notin A$ then i will not later enter A, but if $i \in A$, i could move out of A. This implies that A is Δ^0_2.

Now suppose we have a solution i to A as an $\mathrm{L}\Delta^0_2$-instance. We claim that the largest element of F_i is a solution to $\mathcal{F}$ as a Bound*-instance. Call this least upper bound u. By an application of $\mathrm{L}\Delta^0_2$, we can obtain u from i. Suppose, to the contrary, that there is some $j < i$ such that F_j has a larger least upper bound than F_i. Certainly, there can't be such a $j > i$, by the definition of i. Then consider $\{k < i \mid (\forall y < u)(y \text{ is not an upper bound for } F_k)\}$. We can write this as

$$\{k < i \mid (\forall y < u)(\exists x \in F_k)(x > y)\}, \tag{4.5}$$

so this is a Σ^0_1 set. Then by $\mathrm{L}\Sigma^0_1$ it has a largest element k. But then $k \in A$ and $k < i$, which is a contradiction. Therefore we must have that u is a solution to $\mathcal{F}$ as a Bound*-instance as desired.

Note that we have Bound$^* \leqslant^{\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2}_{\mathrm{W}}$ Bound, and Bound $\leqslant^{\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2}_{\mathrm{W}} \mathrm{L}\Delta^0_2$, since the condition of being an upper bound is Δ^0_2, so our reduction becomes a Weihrauch reduction in this case. □

On the other hand, from Proposition 4.3.11 we deduce that $\mathrm{L}\Delta^0_2 \nleqslant^{\mathrm{RCA}_0}_{\mathrm{W}}$ Bound*, because clearly $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{L}\Delta^0_2$.

We next extend the notions of $\mathrm{F}\Sigma^0_2$ and $\mathrm{F}\Pi^0_2$ to Δ^0_2 sets. While we could try to extend the notion in full generality to Δ^0_n sets, there is ambiguity in what we might mean by a Δ^0_n set in this setting.

Definition 4.4.7. $\mathrm{wkF}\Delta^0_2$ is the Π^1_2 principle where an instance is a set C such that

$$(\forall x)(\forall y < x)(\exists t)((\forall s > t)(\langle y, s\rangle \in C) \vee (\forall s > t)(\langle y, s\rangle \notin C)) \wedge (\exists z)(\exists t)(\forall s > t)(\langle z, s\rangle \notin C). \tag{4.6}$$

A solution is an a such that for all sufficiently large y, $\langle a, y\rangle \notin C$ where either $a = 0$ or there exists some b such that for all sufficiently large y, $\langle b, y\rangle \in C$ and $a = S(b)$.

Definition 4.4.8. $\mathrm{F}\Delta^0_2$ is the Π^1_2 principle where an instance is a set C such that

$$(\forall x)(\exists t)((\forall s > t)(\langle x, s\rangle \in C) \vee (\forall s > t)(\langle x, s\rangle \notin C)) \wedge (\exists z)(\exists t)(\forall s > t)(\langle z, s\rangle \notin C). \tag{4.7}$$

A solution is an a such that for all sufficiently large y, $\langle a, y\rangle \notin C$ where either $a = 0$ or there exists some b such that for all sufficiently large y, $\langle b, y\rangle \in C$ and $a = S(b)$.

In other words, $\mathrm{F}\Delta^0_2$ is an extension of the principles $\mathrm{F}\Sigma^0_n$ to Δ^0_2 sets, and $\mathrm{wkF}\Delta^0_2$ is an extension of the principles $\mathrm{F}\Sigma^0_n$ to sets that are Δ^0_2 in the strong sense. Note that $\mathrm{F}\Delta^0_2$ is a special case of $\mathrm{L}\Delta^0_2$.

Thinking of $\mathrm{F}\Delta^0_2$ as Δ^0_2-induction, a natural first question arises: what is the relationship between $\mathrm{F}\Delta^0_2$ and Σ^0_2-bounding? It is well-known that if M is a model of $\mathrm{PA}^- + \mathrm{I}\Sigma^0_0$, then $M \models \mathrm{B}\Sigma^0_2 \implies M \models \mathrm{I}\Delta^0_2$ (see, e.g. [49]). In our setting, we can ask, does $\mathrm{F}\Delta^0_2 \leqslant^{\Gamma}_{\mathrm{(g)W}} \mathrm{Bound}^*$ for $\Gamma = \mathrm{RCA}_0$ or $\Gamma = \mathrm{RCA}^*_0$?

Proposition 4.4.9. $\mathrm{F}\Delta^0_2 \leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Bound}^*$.

Proof. Let (M, S) be a model of RCA_0. Let H be an $\mathrm{F}\Delta^0_2$-instance. Since $\mathrm{C}\Delta^0_2 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{C}_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}^*$, we can find some a such that for some $z \in \overline{J}$, $a > z$. We may assume that H is of the form $H = \{\langle x, s\rangle \mid x \in J \text{ at stage } s\}$ for a Δ^0_2 set J. For each $x < a$, let H_x be the possibly empty set of stages s such that $\langle x, s\rangle \in H$ but $\langle x, s-1\rangle \notin H$, or $\langle x, s\rangle \notin H$ but $\langle x, s-1\rangle \in H$. Then $\mathcal{H} = \{H_0, H_1, \ldots, H_{a-1}\}$ comprises a Bound*-instance. Let b be a solution to this Bound*-instance. Then by stage b, every element below a in J has stabilized. Therefore if we approximate J by H for b steps, then we can find an element below a which will be a solution to J as an $\mathrm{F}\Delta^0_2$-instance. □

Corollary 4.4.10. $\mathrm{F}\Delta^0_2 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}^* \star \mathrm{C}\Delta^0_2$.

However, from Proposition 4.3.11 we deduce that $\mathrm{F}\Delta^0_2 \nleqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{Bound}^*$.

Definition 4.4.11. We say that an approximation C to a set X is Σ^0_2 if whenever $x \in X$, we have

$$(\exists t)((\forall s > t)(\langle x, s\rangle \in C)), \tag{4.8}$$

and whenever $x \in \overline{X}$, we have

$$(\forall t)((\exists s > t)(\langle x, s\rangle \notin C)). \tag{4.9}$$

Note that we could also define a stronger form of Σ^0_2 than above, where our definition is with respect to initial segments, in a similar manner to the way we define weak and strong versions of Δ^0_2. However, for our purposes, we do not need the stronger version of Σ^0_2.

As usual, let PA^- denote the axioms for the nonnegative part of a discretely ordered ring, letting $\mathrm{I}\Sigma_0$ denote the Σ^0_0-induction principle, and letting exp denote the exponentiation function. Slaman showed in [61] that if M is a model of $\mathrm{PA}^- + \mathrm{I}\Sigma_0 + \exp$, then $M \models \mathrm{I}\Delta^0_2 \implies M \models \mathrm{B}\Sigma^0_2$. In the realm of (generalized) Weihrauch reduction over subsystems of second-order arithmetic, the question becomes, does $\mathrm{Bound}^* \leqslant^{\Gamma}_{\mathrm{(g)W}} \mathrm{F}\Delta^0_2$ for $\Gamma = \mathrm{RCA}_0$ or $\Gamma = \mathrm{RCA}^*_0$?

Theorem 4.4.12. $\mathrm{Bound}^* \leqslant^{\mathrm{RCA}_0,2}_{\mathrm{gW}} \mathrm{F}\Delta^0_2$.

Proof. We follow the idea of the proof of Theorem 2.1 in [61]. For this proof, we work in a model (M, S) of RCA_0. Suppose that $\mathcal{F} = \{F_0, F_1, \ldots, F_n\}$ is an instance of Bound* in S. From $\mathcal{F}$, we will build an $\mathrm{F}\Delta^0_2$-instance using several auxiliary objects outside of the model. We follow the structure of the proof from [61].

Lemma 4.4.13. *There exists a function $f : [0, n+1) \to M$, such that f is injective and the graph of f is Δ^0_2 (in the strong sense) in M. Moreover, f is obtained uniformly from $\mathcal{F}$. If there does not exist a global bound on $\mathcal{F}$, then the range of f is unbounded in M.*

Proof. Define $f := \{(x, \langle x, s_x \rangle) : x < n + 1\}$, where s_x is the least s such that s is an upper bound for F_x. While we call f a function, we view f not as a function in the sense of our model, but as a formula. We can think of f as mapping each set in our Bound*-instance to itself together with its least bound. As in [61], it is clear that f is injective. It is also clear that f is obtained uniformly from $\mathcal{F}$. The graph of f is Δ^0_2, in the strong sense, in M because it is a function on the known domain $[0, n + 1)$. Therefore, we can approximate the graph of f, because if we start approximating one of the bounds s_i by its current least upper bound as we enumerate F_i, as we enumerate more elements into F_i, our approximation to s_i can only increase. We have three possible cases when trying to determine if a number y is currently the least upper bound for one of the sets F_x. In the first case, we initially say y is too small to be a least upper bound, and as we see more elements enter F_x, we'll never change our mind about that. In the second case, we initially say that y is the least upper bound, and then we can either continue to say that, or decide it's too small after larger elements are enumerated. In the third case, we initially say y is too big to be a least upper bound, in which case we might always say that, or we might eventually say it is the least upper bound, and after that point either continue to say that it's the least upper bound or decide that it's too small. Therefore if we are trying to determine whether y is our current least upper bound, by the case analysis, we can only change our mind at most twice.

We can generalize this to finitely many y_k at once using another case analysis. Note that this generalization is necessary to show that the graph of f is Δ^0_2 in the strong sense; otherwise we have only shown that the graph of f is Δ^0_2 in the weaker sense. Given finitely many tuples $(0, \langle 0, y_0 \rangle), (1, \langle 1, y_1 \rangle), \ldots, (k, \langle k, y_k \rangle)$, for some $k < n+1$, if we want to know whether each of the tuples lies in the graph of f, we can start enumerating the corresponding sets F_i simultaneously. Then it will either turn out that each $y_i = s_i$, in which case there is some stage after which this has been seen for each $i < k$ (because we are working in a model of RCA_0), or else there is at least one $y_i \neq s_i$. In the latter case, there are two separate sub-cases: either some $y_i > s_i$, in which case we never think that $(i, \langle i, y_i \rangle)$ is in the graph of f, or every $y_i \leqslant s_i$ but at least one $y_i < s_i$, in which case we will eventually see that $y_i \neq s_i$. This case analysis shows that $graph(f)$ is Δ^0_2 in the strong sense, as desired.

Note that if there exists a global bound on $\mathcal{F}$, then there exists an $i \leqslant n$ such that any bound for F_i is a global bound for $\mathcal{F}$. Moreover, if there does not exist a global bound on $\mathcal{F}$, then there cannot be a bound on $\{s_x : x < n+1\}$. It therefore follows that the range of f must be unbounded in M. □

Lemma 4.4.14. *There is a Σ^0_2 subset I of M (not necessarily an element of S) and a function g such that the following conditions hold:*

1. *$I \subseteq [0, n+1)$;*

2. *$g : I \to M$;*

3. *g is obtained uniformly from $\mathcal{F}$;*

4. *The graph of g is Σ^0_2 in M;*

5. *For each $i \in I$, $g(i)$ is the code for a sequence $\langle m_j : j < i \rangle$ such that for all unequal j_1 and j_2 less than i, $m_{j_1} \neq m_{j_2}$;*

6. *For each $i_1 < i_2 \in I$, if $g(i_1)$ codes $\langle m_j : j < i_2 \rangle$ and $g(i_2)$ codes $\langle n_j : j < i_2 \rangle$, then for all $j < i_1$, $m_j = n_j$;*

7. *For each $m < n$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$.*

If there does not exist a global bound on $\mathcal{F}$, then I is in fact a non-principal Σ^0_2-cut. If there exists a global bound on $\mathcal{F}$, then $\mathcal{F}$ has a solution as a Bound*-*instance in M, and we have that $I = [0, n]$. Therefore in this case g induces a permutation of $[0, n]$.*

Proof. Let f be as in Lemma 4.4.13. Note that the set of numbers less than n is a coded finite set in the sense of M. Then we can think of m as being enumerated into this set at stage $f(m)$. Since f is injective, during any stage, at most one number is enumerated. Then at each stage $s \in M$, we can define the sequence of numbers that have been enumerated so far and order them according to the order in which they were enumerated by f. In essence, we can reorder the numbers ℓ and m that are smaller than n so that ℓ comes before m iff $f(\ell) < f(m)$.

Now define g so that $g(i)$ is a code for the sequence $\langle m_j : j < i \rangle$ such that for each $j < i$, m_j is the j^{th} number enumerated by f (where we mean being enumerated by f in the sense explained in the previous paragraph). We can think of g as building a sequence of numbers that orders the sets in our Bound*-instance based on their least bounds given by f. More formally, $g(i) = \langle m_j : j < i \rangle$ iff there is an s such that

1. $\{m : (\exists y < s)(f(m) = y)\} = \{m : (\exists j < i)(m = m_j)\}$, and

2. for all j_1 and j_2 less than i, $(j_1 < j_2) \leftrightarrow (f(m_{j_1}) < f(m_{j_2}))$.

Informally, condition (1) says that at a given stage everything that's been enumerated so far by f is accounted for by g, and condition (2) says that g is in the desired order.

We first show that g is well-defined, by showing that the two conditions hold of exactly one sequence, and that sequence is as specified in the informal definition above. Indeed, if conditions (1) and (2) hold, then they completely determine the elements of a sequence where they hold: such a sequence must have length i, and if there were s_1 and s_2 such that $\{m : (\exists y < s_1)(f(m) = y)\} = \{m : (\exists y < s_2)(f(m) = y)\}$, then one of these sets would have to be a superset of the other, a contradiction. Moreover, by (2), any sequence where these conditions hold is in order. Therefore if the conditions hold, then they can only hold of one sequence. Furthermore, note that for all $j < i$, the j^{th} number enumerated by f is enumerated at some stage s_j, and $\{s_j \mid j < i\}$ is Σ^0_1, and therefore bounded, since we are working in a model of RCA_0. If we let t be a bound on this set, then we get $\{m : (\exists y < t)(f(m) = y)\} = \{m : (\exists j < i)(m = m_j)\}$ as desired.

Note that g is obtained uniformly from $\mathcal{F}$ by construction. To show that the graph of g is Σ^0_2, we note that by (1), any point in the graph of g is enumerable from the graph of f, which by the previous lemma is Δ^0_2 in the strong sense. Let $I = \operatorname{dom} g$. Since I is the domain of g, I is Σ^0_2 as well.

By the definition of g, for each $i \in I$, there is an $s \in M$ such that $|\{m \in M : f(m) < s\}| \geqslant i$. For such an s and any $j < i$, there is an $s_j < s$ such that $\{m \in M : f(m) < s_j\} = g(j)$, so therefore I is an initial segment of M. Moreover, the map $i \mapsto m_{i-1}$ is an order isomorphism between $I \backslash \{0\}$ and the range of f, and for each $i \in I \backslash \{0\}$, the restriction of this map to $\{j : 0 < j \leqslant i\}$ is coded within M. Therefore ordering $I \backslash \{0\}$ and the range of f by the ordering of M produces isomorphic order types.

Now suppose that $i \in M$ and $i > n$. If $i \in I$, then $g(i)$ would code a sequence of length greater than n containing elements less than n, and with no repetitions. This is impossible, so I must be a subset of $[0, n+1)$.

Note that for every $m \in \operatorname{dom} f$ and every $i \in I$ that is sufficiently large, m appears in the sequence coded by $g(i)$. Then since $\operatorname{dom} f = [0, n+1)$, for each $m < n+1$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$.

Suppose that there does not exist a global bound on $\mathcal{F}$, so by the previous lemma, the range of f is unbounded in M. Then since I has the same order type as the range of f, I cannot have a greatest element. Therefore I is a proper nonprincipal cut in M.

On the other hand, suppose that there exists a global bound on $\mathcal{F}$, so $\mathcal{F}$ does have a solution in M. Then by the previous lemma, the range of f is bounded, meaning there is a stage at which all elements of f, and hence all elements of g, have been enumerated. Then I is an initial segment in M, with a last element. Then g has domain $I = [0, n]$, because otherwise since $I \subseteq [0, n+1)$, g would have domain $[0, k]$ for some $k < n$, contradicting that for each $m < n+1$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$. □

Now as in [61], let g and I be fixed as in the previous lemma and let $m^* := \langle m_i^* : i \in I \rangle$ be the sequence of length I such that for each $i \in I$, m_i^* is equal to the i^{th} element of $g(i+1)$. In other words, m^* is the sequence given by the limit of the range of g. That is, m^* is the limit of the range of the sequences that order the sets in our Bound*-instance based on their least bound. m^* encodes this ordering into numbers. Note that $g(i)$ is an ordered sequence of values m_j^*, not just a single value m_j^*.

Lemma 4.4.15. *Suppose that $c \in M$, $p = \langle p_j : j < c \rangle$ is coded in M, and p is a sequence of elements of M that are less than n. Then, either c is not an upper bound for I, or there is an $i \in I$ such that $p_i \neq m_i^*$.*

Proof. Let c and $p = \langle p_i : i < c \rangle$ be given as above, and suppose that c is an upper bound for I. Suppose, to the contrary, that for all $i \in I$, $m_i^* = p_i$. Then consider $J = \{j : (\forall i < j)(p_i \neq p_j)\}$. For each $i \in I$, the sequence coded by $g(i)$ has no repeated values, so $I \subseteq J$. On the other hand, every element of M that is less than n appears in m^*. Therefore, if $j < c$ and $j \notin I$, then there is an $i \in I$ such that $m_i^* = p_j$. But then $p_i = p_j$ and so $j \notin J$. This contradiction implies that $J = I$. But then J is a proper Σ_0^0-cut, contradicting that $M \models \mathrm{RCA}_0$. □

We can now prove Theorem 4.4.12. We'll define a collection of intervals $[c_i, d_i]$ for $i \in I$. If $i = 0$, then let $c_0 := 0$ and $d_0 := n^n$. For $i > 0$, we define

$$c_i := \sum_{j<i} m_j^* n^{n-(j+1)} \tag{4.10}$$

and $d_i := c_i + n^{n-i}$.

Now since $m_j^* < n$ for each j, for each $i \in I$, we have $[c_{i+1}, d_{i+1}] \subseteq [c_i, d_i]$. Let

$$J := \{x : (\exists i)(x \leqslant c_i)\} \tag{4.11}$$

and

$$K := \{x : (\exists i)(x \geqslant d_i)\}. \tag{4.12}$$

Note that the sequences of c_i's and d_i's are each Σ_2^0, since g is Σ_2^0. Moreover, J and K are both obtained with one existential quantifier from the c_i's. Therefore, it is not too difficult to show that J and K are Σ_2^0 as well.

Recall our Bound*-instance $\mathcal{F} = F_0, F_1, \ldots, F_n$. Suppose that Player 1 plays $\mathcal{F}$ as their first move in the game $G^{\mathrm{RCA}_0}(\text{Bound}^* \to \mathrm{F}\Delta_2^0)$. We claim that J is a valid move for Player 2 to play as their first move in response. To prove this claim, we must show that J is indeed an $\mathrm{F}\Delta_2^0$-instance. Note first that J has nonempty complement because each m_i^* is smaller than n, so we can crudely bound each c_i by n^{n+1}. Therefore J is bounded by n^{n+1}. To show that J is indeed a Δ_2^0 set, we will argue in cases.

First, suppose that there exists an $i \leqslant n$ such that any bound for F_i is a global bound for $\mathcal{F}$. Then in particular, $\mathcal{F}$ has a solution in M, so by Lemma 4.4.14, $I = [0, n]$. Therefore g has domain $[0, n+1)$, so $c_n + 1 = d_n$. Then in this case J and K are complements, so since J is Σ_2^0 as explained above, J is in fact Δ_2^0. In fact, J is a finite initial segment, with largest element, c_n. Note that the c_i's encode $\sum_{j<i} m_j^*$, and J is the downward closure of the c_i's, so from c_n, we can tell which j is such that m_j^* is largest: it is the last element of c_n. But then we know which j is such that F_j has the largest least upper bound, which must be a global upper bound on the Bound*-instance $\mathcal{F}$ by construction.

Now suppose that there is no $i \leqslant n$ such that any bound for F_i is a global bound for $\mathcal{F}$. Then by Lemma 4.4.14, I is a non-principal Σ_2^0 cut. Note that J is closed downward in M, and K is closed upward. By construction, for every $x \in J$ and $y \in K$, $y > x$. Now suppose that p is an element of M that is neither an element of J nor one of K. Since J is closed downward, for every $x \in J$, $p > x$, and likewise since K is closed upward, for every $y \in K$, we must have $p < y$. Then in base-n, letting p_i be the coefficient of the n^{n-i-1} term in this representation, for every $i \in I$ we have

$$\sum_{j<i} m_j^* n^{n-(j+1)} < p = \sum_{j<i} p_j n^{n-(j+1)} < \sum_{j<i} m_j^* n^{n-(j+1)} + n^{n-i}. \tag{4.13}$$

Then for each $i \in I$, $p_i = m_i^*$, and by Δ_1^0-induction, it follows that $\langle n_j : j < n\rangle$ is coded by an element of M. This contradicts Lemma 4.4.15. Therefore we must have $J \cup K = M$. Note that J is Σ_2^0 as explained above. Then J is a Σ_2^0 cut and the Σ_2^0 set K is its complement in M.

Therefore in both cases, the reduction goes through. Note that the nonuniformity only comes in showing that J is an $\mathrm{F}\Delta_2^0$-instance, and this need not be uniform. However, in both cases, the presentation of J as a Δ_2^0 set is the same, since we have shown that in both cases J is a Σ_2^0 set with Σ_2^0 complement K. Note that the two notions of a Δ_2^0 set actually coincide in the case of J, because J is downward closed.

Now in order to complete the reduction, we must obtain an actual bound that is a solution to the Bound*-instance; it's not enough just to know that a bound on one of the sets F_i is sufficient. For this, we apply Bound to F_i, where i is the index such that any bound for F_i is a global bound for $\mathcal{F}$. We can do this because we know that Bound $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ $\mathrm{F}\Sigma_1^0$ $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$ $\mathrm{F}\Delta_2^0$ by Proposition 4.3.12. Therefore we have a generalized Weihrauch reduction Bound* $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0,2}$ $\mathrm{F}\Delta_2^0$ as desired.

□

Note that we could substitute our first application of $\mathrm{F}\Delta_2^0$ with an application of $\mathrm{L}\Delta_2^0$, by using the least element of K instead of the largest element of J to determine the index $j \in I$

such that m_j^* is largest. With this substitution, we could more precisely get the reduction $\mathrm{Bound}^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{Bound} \star \mathrm{L}\Delta_2^0$.

From Proposition 3.7.3, we have that $\mathrm{stBound}^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0,2} \mathrm{Bound}^*$. The below corollary then follows.

Corollary 4.4.16. $\mathrm{stBound}^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0,4} \mathrm{F}\Delta_2^0$.

So we have shown that Slaman's results from [61] hold up in our setting, with some limitations. In particular, the proof of Lemma 4.4.13 and the use of Δ_1^0-induction in the application of Lemma 4.4.15 prevent our argument from going through in RCA_0^*. While studying these results in the realm of Weihrauch reduction over RCA_0 and RCA_0^* was our initial motivation, in getting here we have come across a series of principles, which must be interrelated in some way. For the rest of this chapter, we will continue to delineate these relationships.

4.5 Proof-theoretic non-reductions

In this section, we will use proof theory to our advantage to obtain a series of non-reductions between the principles we have been considering. We will make use of several proof theoretic facts about RCA_0 and RCA_0^*, most of which are well known and can be seen in more detail in [33] and [30].

Remark 4.5.1. Note that every instance of $\mathrm{F}\Sigma_1^0$ has a solution in RCA_0, by definition. However, as Bound^* and $\mathrm{stBound}^*$ are versions of Σ_2^0-bounding, not every instance of either of these principles has a solution in RCA_0. Therefore $\mathrm{stBound}^* \nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Sigma_1^0$ and $\mathrm{Bound}^* \nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Sigma_1^0$. On the other hand, there is a model of RCA_0 that is not a model of $\mathrm{B}\Delta_2^0\mathrm{CA}$, since this can be shown over RCA_0 to require $\mathrm{B}\Sigma_2^0$. Therefore $\mathrm{B}\Delta_2^0\mathrm{CA} \nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Sigma_1^0$ as well.

Similarly, $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{C}_{\mathbb{N}}$ because $\mathrm{C}_{\mathbb{N}}$ is true in every model of RCA^*_0 (or equivalently, $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}}$ Bound), while $\mathrm{F}\Sigma^0_1$ is not true in RCA^*_0, precisely because in RCA^*_0 Σ^0_1-induction is weakened to Σ^0_0-induction. For the same reason, $\mathrm{F}\Pi^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{C}_{\mathbb{N}}$. We also get that $\mathrm{F}\Pi^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}}$ Bound. It follows that $\mathrm{K}_{\mathbb{N}}$ is also true in every model of RCA^*_0, and that $\mathrm{F}\Delta^0_2$ is not true in every model of RCA^*_0. By the result in [61], we also have that $\mathrm{F}\Delta^0_2$ is not true in every model of RCA_0. Similarly, we get that $\mathrm{L}\Delta^0_2$ is not true in every model of RCA_0 either. On the other hand, we know that RT^1_2 holds in every model of RCA_0.

Putting all of this together, we get the following corollaries.

Corollary 4.5.2. *In terms of generalized Weihrauch reducibility over* RCA_0*, letting* $X = \{\mathrm{Bound}^*, \mathrm{stBound}^*, \mathrm{C}^*_{\mathbb{N}}, \mathrm{F}\Delta^0_2, \mathrm{L}\Delta^0_2, \mathrm{B}\Delta^0_2\mathrm{CA}\}$ *and* $Y = \{\mathrm{F}\Sigma^0_1, \mathrm{F}\Pi^0_1, \mathrm{Bound}, \mathrm{C}_{\mathbb{N}}, \mathrm{K}_{\mathbb{N}}, \mathrm{K}\Delta^0_2, \mathrm{RT}^1_2\}$*, for all* $\mathrm{P} \in X$ *and* $\mathrm{Q} \in Y$*, we have* $\mathrm{P} \not\leq^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Q}$.

Corollary 4.5.3. *We also have the following weaker results, which hold only in terms of generalized Weihrauch reducibility over* RCA^*_0*:*

1. $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}}$ Bound*;*
2. $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{C}_{\mathbb{N}}$*;*
3. $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{K}_{\mathbb{N}}$*;*
4. $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{K}\Delta^0_2$*;*
5. $\mathrm{F}\Sigma^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{RT}^1_2$*;*
6. $\mathrm{F}\Pi^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}}$ Bound*;*
7. $\mathrm{F}\Pi^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{C}_{\mathbb{N}}$*;*
8. $\mathrm{F}\Pi^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{K}_{\mathbb{N}}$*;*
9. $\mathrm{F}\Pi^0_1 \not\leq^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{K}\Delta^0_2$*;*

10. $\mathrm{F}\Pi^0_1 \nleqslant^{\mathrm{RCA}_0^*}_{\mathrm{gW}} \mathrm{RT}^1_2$.

This serves as our initial pass for obtaining non-reductions among the principles we have been considering in some sort of uniform way. In the next section, we expand our knowledge of non-reductions with the help of a metatheorem that gives a sufficient condition to obtain $\mathrm{Q} \nleqslant^n_{\mathrm{gW}} \mathrm{P}$ for any fixed $n \in \omega$, and hence by Theorem 3.3.1, that $\mathrm{Q} \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{P}$ and $\mathrm{Q} \nleqslant^{\mathrm{RCA}_0^*}_{\mathrm{gW}} \mathrm{P}$ as well.

4.6 Metatheorem and related results

To motivate our metatheorem, we look at regularity (see [30] for the formal proof-theoretic definition of regularity).

Definition 4.6.1. The *regularity principle* $\mathrm{R}\Sigma^0_1$ is the Π^1_2 principle where an instance is a Σ^0_1 set A and a u such that $(\forall w)(\exists x > w)(\exists y \leqslant u)((x, y) \in A)$ and a solution is a y such that $(y \leqslant u) \wedge (\forall w)(\exists x > w)((x, y) \in A)$.

As with previous principles, we can define $\mathrm{R}\Sigma^0_n$ for $n > 1$ as the natural extension of $\mathrm{R}\Sigma^0_1$ to Σ^0_n sets. However, this is beyond the scope of this dissertation. Note that for any $\mathrm{R}\Sigma^0_1$-instance A, where A is the enumeration of a set of pairs (x, n), if for each such pair we instead enumerate (s, n), where s is the stage that (x, n) is enumerated into A, we obtain a computable instance of $\mathrm{R}\Sigma^0_1$ with the same solutions as our original instance. However, this is not the case in RCA_0^*.

We present two more versions of the regularity principle because in thinking about what regularity should look like as a Π^1_2 problem, it is first of all unclear whether the parameter u should be taken as known or not. Moreover, we will later see that $\mathrm{R}\Sigma^0_1$ is equivalent to $\mathrm{RT}^1_{<\infty}$. However, the natural strong version of regularity for Σ^0_1 sets, which we call $\mathrm{stR}\Sigma^0_1$ and define in the same way as $\mathrm{R}\Sigma^0_1$ but with the parameter u not given, is not equivalent to

$\mathrm{stRT}^1_{<\infty}$. To obtain equivalence with $\mathrm{stRT}^1_{<\infty}$ we need an additional constraint on our Σ^0_1 set, yielding in general the third middle version $\mathrm{medR}\Sigma^0_1$ of regularity.

Definition 4.6.2. The *medium regularity principle* $\mathrm{medR}\Sigma^0_1$ is the Π^1_2 principle where an instance is a Σ^0_1 set A for which there is a u such that $(\forall w)(\exists x > w)(\exists y \leqslant u)((x, y) \in A)$, and for all $y > u$, (x, y) is never enumerated, and a solution is a y such that $(y \leqslant u) \wedge (\forall w)(\exists x > w)((x, y) \in A)$.

Definition 4.6.3. The *strong regularity principle* $\mathrm{stR}\Sigma^0_1$ is the Π^1_2 principle where an instance is a Σ^0_1 set A for which there is a u such that $(\forall w)(\exists x > w)(\exists y)((x, y) \in A)$ and a solution is a y such that $(y \leqslant u) \wedge (\forall w)(\exists x > w)((x, y) \in A)$.

While there is a natural way to extend the weakest regularity principle $\mathrm{R}\Sigma^0_1$ from Σ^0_1 sets to Σ^0_n sets, and the same is true for the strong regularity principle $\mathrm{stR}\Sigma^0_1$, it is not immediately clear how to do so for the medium regularity principle $\mathrm{medR}\Sigma^0_1$.

Proposition 4.6.4. $\mathrm{medR}\Sigma^0_1 \equiv^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{stRT}^1_{<\infty}$.

Proof. We work in a model of RCA^*_0 for both directions.

For $\mathrm{medR}\Sigma^0_1 \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{stRT}^1_{<\infty}$, let A be a Σ^0_1 set such that $(\forall w)(\exists x > w)(\exists y \leqslant u)((x, y) \in A)$, for some $u \in \omega$ which is not given. By the definition of a $\mathrm{medR}\Sigma^0_1$-instance, we may assume u is chosen so that for all $y > u$, (x, y) is never enumerated. Construct a coloring c in stages $s \in \omega$ as follows. Let $n = 1$. Enumerate the n^{th} element (x, y) of A, and search for an $s > 0$ with $x > s$, if such an s exists. Then let $c(s) = y$ if applicable, and in any case increase n by 1 and repeat the procedure. Since we have a $\mathrm{medR}\Sigma^0_1$-instance, we will enumerate pairs (x, y) for infinitely many x, so it follows that c will be an instance of Ramsey's Theorem for singletons, but since we don't know u, we don't know the range of c, so we will in particular get an instance of $\mathrm{stRT}^1_{<\infty}$. Let y be such that $\{s \mid c(s) = y\}$ is an infinite homogeneous set for c. Then $(\forall w)(\exists x > w)((x, y) \in A)$, and we must have $y \leqslant u$ since no (x, y) with $y > u$ is ever enumerated. Therefore y is a solution to A as an $\mathrm{medR}\Sigma^0_1$-instance as desired.

For $\mathrm{stRT}^1_{<\infty} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{medR}\Sigma^0_1$, note that if we think of a coloring c as a set of tuples $A = \{\langle x, y\rangle \mid c(x) = y\}$, then A is a $\mathrm{medR}\Sigma^0_1$-instance because for each $x \in \omega$, there is at most one y such that $(x, y) \in A$, so in particular, there is a u such that for all $y > u$, (x, y) is never enumerated. Furthermore, given a solution y to A, we get that $\{x \mid c(x) = y\}$ is an infinite homogeneous set for c. □

Proposition 4.6.5. $\mathrm{R}\Sigma^0_1 \equiv_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{RT}^1_{<\infty}$.

Proof. We work in a model of RCA_0^* for both directions. We proceed similarly to the previous proof, with some minor modifications.

For $\mathrm{R}\Sigma^0_1 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{RT}^1_{<\infty}$, let A be a Σ^0_1 set and u be a natural number such that $(\forall w)(\exists x > w)(\exists y \leqslant u)((x, y) \in A)$. We may assume that A is enumerated without repetitions. Construct a coloring c in stages $s \in \omega$ as follows. Let $n = 1$. Enumerate the n^{th} element (x, y) of A, and if $y \leqslant u$, search for an $s > 0$ with $x > s$, if such an s exists. Then let $c(s) = y$ if applicable, and in any case increase n by 1 and repeat the procedure. Since we have a $\mathrm{R}\Sigma^0_1$-instance, we will enumerate pairs (x, y) for infinitely many x, so it follows that c will be an instance of $\mathrm{RT}^1_{<\infty}$. Let y be such that $\{s \mid c(s) = y\}$ is an infinite homogeneous set for c. Then $(\forall w)(\exists x > w)((x, y) \in A)$, and by construction $y \leqslant u$. Therefore y is a solution to A as an $\mathrm{R}\Sigma^0_1$-instance as desired.

For $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{R}\Sigma^0_1$, note that if we think of a k-coloring c as a set of tuples $A = \{\langle x, y\rangle \mid c(x) = y\}$, then A together with the number k comprise an $\mathrm{R}\Sigma^0_1$-instance, and given a solution y to A, we get that $\{x \mid c(x) = y\}$ is an infinite homogeneous set for c. □

Remark 4.6.6. Recall that Hirst showed in [38] that $\mathrm{RT}^1_{<\infty}$ is equivalent to $\mathrm{B}\Sigma^0_2$ over RCA_0. It follows that $\mathrm{RT}^1_{<\infty}$, and hence $\mathrm{R}\Sigma^0_1$, is not true in every model of RCA_0. Therefore there also exist models of RCA_0 where the stronger versions $\mathrm{medR}\Sigma^0_1$ and $\mathrm{stR}\Sigma^0_1$ also do not hold. We then get the following follow-up to Corollary 4.5.2.

Corollary 4.6.7. *Let* $X = \{\mathrm{R}\Sigma^0_1, \mathrm{medR}\Sigma^0_1, \mathrm{stR}\Sigma^0_1\}$ *and let* $Y = \{\mathrm{F}\Sigma^0_1, \mathrm{F}\Pi^0_1, \mathrm{Bound}, \mathrm{C}_{\mathbb{N}}, \mathrm{K}_{\mathbb{N}}, \mathrm{K}\Delta^0_2, \mathrm{RT}^1_2\}$. *Then for all* $\mathrm{P} \in X$ *and* $\mathrm{Q} \in Y$, *we have* $\mathrm{P} \not\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{Q}$.

Clearly, $R\Sigma^0_1 \leqslant^{RCA_0^*}_W stR\Sigma^0_1$. But suppose we are trying to show that $stR\Sigma^0_1 \nleqslant^{RCA_0}_W R\Sigma^0_1$. Intuitively, it seems like this should be true, since the parameter u is not given with a $stR\Sigma^0_1$-instance, but the corresponding parameter (we'll call it u', to avoid confusion) must be given in order to have a $R\Sigma^0_1$-instance. Let's construct a Σ^0_1 set A such that the enumeration of A constitutes an $stR\Sigma^0_1$-instance, so $(\forall w)(\exists x > w)(\exists y \leqslant u')((x, y) \in A)$ for some v. Let our supposed reduction be given by Turing functionals Φ and Ψ.

Start enumerating all pairs in the order $(0, 0), (1, 0), (2, 0)$, and so forth into A until Φ^A gives us a value for u. Then for each $z \leqslant u$, wait until $\Psi^{A\oplus z}$ converges to some value y_z. Then we ensure that y_z is not a solution to A as an $stR\Sigma^0_1$-instance by putting $(x, y_z) \in \overline{A}$ for sufficiently large x, and putting all other pairs $(x, y) \in A$. By construction, A is a $stR\Sigma^0_1$-instance. Let $z \leqslant u$ be a solution to Φ^A as an $R\Sigma^0_1$-instance. Then y_z must eventually be found. Now $\Psi^{A\oplus z} \downarrow= y_z$, but $(x, y_z) \in \overline{A}$ for sufficiently large x, so y_z is not a solution to A as an $stR\Sigma^0_1$ instance, giving the desired contradiction.

Note that in this argument, we are not using any properties specific to $stR\Sigma^0_1$ and $R\Sigma^0_1$. Rather, we are exploiting the fact that each $R\Sigma^0_1$-instance has a fixed numerical parameter u that bounds one of (in fact, all of) its solutions. This, combined with the fact that given u numbers, we can build a $stR\Sigma^0_1$-instance, consistent with the initial segment of a $stR\Sigma^0_1$-instance that we started with, that avoids each of these u numbers, is what gives us the non-reduction. This method can be generalized to the generalized Weihrauch case.

We make this generalization precise in the following metatheorem.

Theorem 4.6.8. *Suppose we have Π^1_2 principles* P *and* Q*, where* P *and* Q *are first-order, and* Q *has computable instances. Suppose that for any* P*-instance X, there exists a computable procedure for computing a number k from X such that X has a solution between 0 and k. Suppose additionally that there exists a non-empty, infinite set S of strings such that S is a tree, and for each $\sigma \in S$ and for any finite k and any $n_0, n_1, \ldots, n_k \in \omega$, there is a path extending σ that is a* Q*-instance Y for which $n_0, n_1, \ldots, n_k$ are not solutions and such that*

Y is a path on S. Then $\mathrm{Q} \not\leqslant^n_{\mathrm{gW}} \mathrm{P}$ *for any fixed* $n \in \omega$. *In particular,* $\mathrm{Q} \not\leqslant_{\mathrm{W}} \mathrm{P}$.

Moreover, by Theorem 3.3.1, for such problems P *and* Q, *we get* $\mathrm{Q} \not\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{P}$ *and* $\mathrm{Q} \not\leqslant^{\mathrm{RCA}_0^*}_{\mathrm{gW}} \mathrm{P}$ *as well.*

Note that it need not be the case that every path through S is a Q-instance. In fact, in many cases, S can just be taken to be the set of initial segments of instances of Q. Furthermore, we include as a hypothesis that Q must have computable instances to ensure that we don't wind up in a case where the game could be winnable in $\leqslant n+1$ moves but not in exactly $n+1$ moves.

Proof. Suppose for a contradiction that $\mathrm{Q} \leqslant^n_{\mathrm{gW}} \mathrm{P}$ for some fixed $n \in \omega$ via a computable strategy for Player 2 where Player 2 wins in at most $n+1$ moves. As mentioned above, by the hypothesis that Q must have computable instances we may assume that Player 2 wins in exactly $n+1$ moves. Suppose further that Θ is a Turing functional that allows us to compute the parameter k from the corresponding P-instance. We will construct a winning strategy for Player 1, that is, a Q-instance for them to play initially. We envision the possible plays of the game as a tree with $n+1$ levels. Since strings are partial P-instances, we can look at Θ^σ for strings σ. Θ^σ may or may not converge, but for our purposes, we can ignore the strings on which Θ does not converge (since our procedure need not be computable).

Start defining initial segments of Q-instances. Let $X, x_1, \ldots, x_n$ represent Player 1's moves, where X is a Q-instance and $x_1, \ldots, x_n$ are solutions to the P-instances $X_1, \ldots, X_n$ played by Player 2. For σ a string and $1 \leqslant i \leqslant n-1$, let Φ_i be the Turing functional such that $\Phi_i^{\sigma \oplus x_1 \oplus \ldots \oplus x_{i-1}}$ is an initial segment of Player 2's i^{th} move in response to σ and $x_1, \ldots, x_{i-1}$. We are given the Φ_i's by the assumption that $\mathrm{Q} \leqslant^n_{\mathrm{gW}} \mathrm{P}$. We are also given a functional Ψ such that if Player 1 initially plays the Q-instance X, and if $x_1, \ldots, x_n$ are indeed valid moves for Player 1 to play, then $\Psi^{X \oplus x_1 \oplus \ldots \oplus x_n}$ converges and is a solution to X.

We then form our tree. First, let S be as in the theorem statement and let $\sigma \in S$ be such that $\Theta^{\Phi_1^\sigma}$ converges to some value $k_\emptyset$. This convergence is guaranteed because there exists a Q-instance Y of which every initial segment lies in S, so there is some $\sigma \in S$ that is a valid initial segment of a first move for Player 1, and is such that Φ_1^σ converges to a valid initial segment of a first move in response to σ for Player 2. Moreover, we can choose σ to be a long enough initial segment so that $\Theta^{\Phi_1^\sigma}$ converges as well. We denote the value it converges to by $k_\emptyset$ in reference to the position of the parent node, which is simply the root of the tree. Then we let the first level of nodes of our tree consist, in order from left to right, of nodes labelled $0, 1, 2, \ldots, k_\emptyset$.

Let $\sigma_\emptyset = \sigma$. For each i with $0 \leqslant i \leqslant k_\emptyset$, look for an initial segment $\sigma_i \in S$ extending σ_{i-1} (where by σ_{-1} we mean $\sigma_\emptyset$) such that $\Theta^{\Phi_2^{\sigma_i \oplus i}}$ converges. If there is such an initial segment σ_i in S, then the node on the second level of the tree labelled i will have children corresponding to nodes labelled $0, 1, 2, \ldots k_i$, where $k_i = \Theta^{\Phi_2^{\sigma_i \oplus i}}$. Otherwise, let $\sigma_i = \sigma_{i-1}$ and we give the corresponding node one child labelled 0 and define the corresponding $k_i = 0$.

To build the third level of the tree, for each i with $0 \leqslant i \leqslant k_\emptyset$ and each j with $0 \leqslant j \leqslant k_i$, look for an initial segment $\sigma_{ij} \in S$ extending $\sigma_{i(j-1)}$ (or extending $\sigma_{(i-1)k_{i-1}}$ if $i \geqslant 1$ and $j = 0$, and extending $\sigma_{k_\emptyset}$ if $i = 0$ and $j = 0$) such that $\Theta^{\Phi_3^{\sigma_{ij} \oplus i \oplus j}}$ converges. If there is such an initial segment σ_{ij} in S, then the node on the third level of the tree labelled j will have children corresponding to nodes labelled $0, 1, 2, \ldots, k_{ij}$, where $k_{ij} = \Theta^{\Phi_3^{\sigma_{ij} \oplus i \oplus j}}$. Otherwise, let $\sigma_{ij} = \sigma_{i(j-1)}$ if $j \geqslant 1$, let $\sigma_{ij} = \sigma_{(i-1)k_{i-1}}$ if $i \geqslant 1$ and $j = 0$, and let $\sigma_{ij} = \sigma_{k_\emptyset}$ if $i = 0$ and $j = 0$. In each of these cases we give the corresponding node one child labelled 0 and define the corresponding k value to be $k_{ij} = 0$.

In general, to build the m^{th} level of the tree, where $m \leqslant n$, we order the strings α with $|\alpha| = m - 1$ lexicographically as $\alpha_0, \alpha_1, \ldots, \alpha_\ell$. We look for an initial segment $\sigma_{\alpha_i \wedge j} \in S$, for $0 \leqslant i \leqslant \ell$ and $0 \leqslant j \leqslant k_{\alpha_i}$, extending the string $\sigma_{\alpha_i \wedge j-1}$ (or, if $i \geqslant 1$ and $j = 0$, extending the string $\sigma_{\alpha_{i-1} \wedge k_{\alpha_{i-1}}}$, or if $i = 0$ and $j = 0$, extending the last string on the previous level of the tree), such that $\Theta^{\Phi_{m+1}^{\sigma_{\alpha_i} \oplus \alpha_i[0] \oplus \alpha_i[1] \oplus \ldots \oplus \alpha_i[m-2] \oplus j}}$ converges. If there is such an initial segment $\sigma_{\alpha_i \wedge j}$ in S, then the node on the $(m+1)^{st}$ level of the tree labelled j will have the children $0, 1, 2, \ldots, k_{\alpha_i \wedge j}$ where $k_{\alpha_i \wedge j} = \Theta^{\Phi_{m+1}^{\sigma_{\alpha_i} \oplus \alpha_i[0] \oplus \alpha_i[1] \oplus \ldots \oplus \alpha_i[m-2] \oplus j}}$. Otherwise, we let $\sigma_{\alpha_i \wedge j} = \sigma_{\alpha_{i-1} \wedge j}$ if $i \geqslant 1$, $\sigma_{\alpha_i \wedge j} = \sigma_{\alpha_{i-1} \wedge k_{\alpha_{i-1}}}$ if $i \geqslant 1$ and $j = 0$, or let $\sigma_{\alpha_i \wedge j}$ equal the last string on the previous level of the tree if $i = 0$ and $j = 0$, and in all of these cases we give the corresponding node one child labelled 0 and define the corresponding k value to be 0.

For the last step, we order the strings β with $|\beta| = n-1$ lexicographically as $\beta_0, \beta_1, \ldots, \beta_p$. We look for an initial segment $\sigma_{\beta_i \wedge j} \in S$, for $0 \leqslant i \leqslant p$ and $0 \leqslant j \leqslant k_{\beta_i}$, extending the string $\sigma_{\beta_i \wedge j-1}$ (or, if $i \geqslant 1$ and $j = 0$, extending the string $\sigma_{\beta_{i-1} \wedge k_{\beta_{i-1}}}$, or if $i = 0$ and $j = 0$, extending the last string on the previous level of the tree), such that $\Psi^{\sigma_{\beta_i} \oplus \beta_i[0] \oplus \beta_i[1] \oplus \ldots \oplus \beta_i[n-2] \oplus j}$ converges. If there is such an initial segment $\beta_{\alpha_i \wedge j}$ in S, then the node on the $(n+1)^{st}$ level of the tree labelled j will have the children corresponding to nodes labelled $0, 1, 2, \ldots, y_{\beta_i \wedge j}$ where $y_{\beta_i \wedge j} = \Psi^{\sigma_{\beta_i} \oplus \beta_i[0] \oplus \beta_i[1] \oplus \ldots \oplus \beta_i[n-2] \oplus j}$. Otherwise, we let $\sigma_{\beta_i \wedge j} = \sigma_{\beta_{i-1} \wedge j}$ if $i \geqslant 1$, $\sigma_{\beta_i \wedge j} = \sigma_{\beta_{i-1} \wedge k_{\beta_{i-1}}}$ if $i =\geqslant 1$ and $j = 0$, or let $\sigma_{\beta_i \wedge j}$ equal the last string on the previous level of the tree if $i = 0$ and $j = 0$, and in all of these cases we define the corresponding y value to be 0.

Consider all of the y values that we have defined: call them $y_0, y_1, \ldots, y_q$. There are finitely many of them. Therefore by hypothesis, there is a path extending $\sigma_{\beta_p \wedge k_{\beta_p}}$ that is a Q-instance Y for which $y_0, y_1, \ldots, y_q$ are not solutions, and such that every initial segment of Y lies in S. Note that by construction, every defined string σ_γ corresponding to a node in the tree is extended by $\sigma_{\beta_p \wedge k_{\beta_p}}$.

Definition 4.6.9. We say that we *get stuck* in building the branch corresponding to the string α at the m^{th} level of the tree if the node labelled $\alpha[m-1]$ has one child labelled 0, and no other children.

Lemma 4.6.10. *If Player 1 plays legally throughout the game starting with the initial move Y, it is not possible to get stuck (on the branch corresponding to the length m initial segment* *of Y) at any level in building this tree.*

Proof. The main idea of the proof of this lemma is that if we got stuck building this tree on the branch corresponding to the length m initial segment of Y, then this would correspond to a set of moves for Player 2 and responses from Player 1 that would cause Player 2 to not be able to move at some point before the end of the game, contradicting the definitions of the Φ_i and Ψ.

Suppose, for a contradiction, that Player 1 plays legally throughout the game with the initial move Y yet we get stuck, for the first time, when building the m^{th} level of the tree for some $m < n-1$. Note that the labels of the nodes on the m^{th} level of the tree are bounded locally by some value k_γ. When we say that Player 1 plays legally throughout the game, this means that Player 1 must play a value that is smaller than the corresponding k_γ-value, and is actually a solution to the P-instance played by Player 2 on their most recent move. Since we get stuck at building the m^{th} level of the tree, $\Theta^{\Phi_{m+1}^{Y \oplus Y[0] \oplus Y[1] \oplus \ldots \oplus Y[m-1]}}$ must not have converged. Recall that $\Phi_{m+1}^{Y \oplus Y[0] \oplus Y[1] \oplus \ldots \oplus Y[m-1]}$ denotes Player 2's $(m+1)^{st}$ move in response to Y. Note that this should converge provided that Y is a valid Q-instance. By hypothesis, Player 1 has played legally. Therefore $\Phi_{m+1}^{Y \oplus Y[0] \oplus Y[1] \oplus \ldots \oplus Y[m-1]}$ converges to a partial P-instance. Then the error must be in the convergence of Θ. But by choice of Y, this is impossible, giving us the desired contradiction. □

So then Player 1 must reach the last level of the tree in its moves, traversing a path of nodes whose labels, when concatenated, form some string γ. Since Player 1 reaches the bottom level of the tree, we must have that y_γ is defined, so y_γ must be a solution to Y as a Q-instance. But by the definition of Y, this cannot be.

□

From our metatheorem, we can deduce the following.

Corollary 4.6.11. *Let* $X = \{\mathrm{F}\Sigma^0_1$, $\mathrm{medR}\Sigma^0_1$, $\mathrm{stR}\Sigma^0_1$, Bound, Bound*, stBound*, $\mathrm{C}_\mathbb{N}$, $\mathrm{C}^*_\mathbb{N}$,

$\mathrm{F}\Delta^0_2$, $\mathrm{L}\Delta^0_2\}$ *and let* $Y = \{\mathrm{R}\Sigma^0_1, \mathrm{B}\Delta^0_2\mathrm{CA}, \mathrm{K}_{\mathbb{N}}, \mathrm{K}\Delta^0_2, \mathrm{F}\Pi^0_1\}$. *Then for any* $\mathrm{Q} \in X$ *and any* $\mathrm{P} \in Y$, *by Theorem* 4.6.8 *we have* $\mathrm{Q} \nleqslant^n_{\mathrm{gW}} \mathrm{P}$ *for any fixed* $n \in \omega$, *so in particular* $\mathrm{Q} \nleqslant_{\mathrm{W}} \mathrm{P}$, *and also* $\mathrm{Q} \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{P}$ *(which implies that* $\mathrm{Q} \nleqslant^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{P}$*).*

In fact, we can take RT^n_k for any n and k, any bounded comprehension axiom, and any compact choice principle to be part of the set Y, since for all principles of these forms, the parameter k is part of the instance.

Proof. First, note that for all $\mathrm{Q} \in X$ and for all $\mathrm{P} \in Y$, Q and P are first-order Π^1_2 principles or can be encoded as such. Furthermore, for all $\mathrm{Q} \in X$, Q has computable instances.

For $\mathrm{P} = \mathrm{R}\Sigma^0_1$, note that for any $\mathrm{R}\Sigma^0_1$-instance A, we are given a u such that $(\forall w)(\exists x > w)(\exists y \leqslant u)((x, y) \in A)$. Since a solution to A is a y such that $(y \leqslant u) \wedge (\forall w)(\exists x > w)((x, y) \in A)$, clearly u works as the parameter k specified in the statement of Theorem 4.6.8.

For $\mathrm{P} = \mathrm{B}\Delta^0_2\mathrm{CA}$, for any $\mathrm{B}\Delta^0_2\mathrm{CA}$-instance given by an approximation to a set X with bound b, a solution to X is $\{x \in X \mid x < b\}$. As in the previous case, we can encode $\{x \in X \mid x < b\}$ as an integer $i \in \omega$, and we can choose a system of encoding that guarantees that we have $i \leqslant 2^b$. Then we can take 2^b to be the parameter k specified in the statement of Theorem 4.6.8.

For $\mathrm{P} = \mathrm{K}_{\mathbb{N}}$, since a $\mathrm{K}_{\mathbb{N}}$-instance is some $b \in \omega$ together with an enumeration of some numbers $x < b$, and a solution is any $x \in \omega$ with $x < b$ that is not enumerated, we can take b to be our parameter k.

For $\mathrm{P} = \mathrm{K}\Delta^0_2$, since a $\mathrm{K}_{\mathbb{N}}$-instance is some $b \in \omega$ together with an approximation to a Δ^0_2 set below b, and a solution is any $x \in \omega$ with $x < b$ that is not in the Δ^0_2 set, we can take b to be our parameter k.

For $\mathrm{P} = \mathrm{F}\Pi^0_1$, note that for any $\mathrm{F}\Pi^0_1$-instance A, we can computably obtain some $k \in \overline{A}$, as $\overline{A}$ is by definition a nonempty Σ^0_1 set. By the definition of $\mathrm{F}\Pi^0_1$, A must have a solution between 0 and k.

For P $= \mathrm{RT}^1_2$, we can take $k = 2$, since RT^1_2-instances are 2-colorings, and then by definition we will always have a solution between 0 and 2.

Now for each Q $\in X$, we can take S to be the set of all initial segments of Q-instances, and S is clearly a tree. Of course, what an initial segment of a Q-instance is depends on our encoding, so without loss of generality, we assume that our encodings are such that in all of the below cases, any string can be an initial segment of a Q-instance. This is not generally true but can be done in all of the cases discussed here. We will show below for each Q $\in Y$ that for any $\sigma \in S$, any finite k, and any $n_0, n_1, \ldots, n_k \in \omega$, there is a path extending σ that is a Q-instance B for which $n_0, n_1, \ldots, n_k$ are not solutions, and trivially, every initial segment of B lies in S.

For Q $= \mathrm{F}\Sigma^0_1$, it is easy to see how to construct B: simply enumerate elements extending σ so that for each i with $0 \leqslant i \leqslant k$, n_i is enumerated into B.

For Q $= \mathrm{medR}\Sigma^0_1$, we can first enumerate the elements specified by σ into B, and then for each i with $0 \leqslant i \leqslant k$ and all sufficiently large x, keep (x, n_i) in $\overline{B}$, thereby ensuring that no n_i is a solution to B as an $\mathrm{medR}\Sigma^0_1$-instance. Now let $m = \max\{n_1 + 1, n_2 + 1, \ldots, n_k + 1\}$. Then finally, we enumerate all pairs of the form (j, m), for $j \in \omega$, into B. This maintains that none of the n_i are solutions to B, while ensuring that B is a $\mathrm{medR}\Sigma^0_1$-instance.

For Q $= \mathrm{stR}\Sigma^0_1$, we can first enumerate the elements specified by σ into B, and then for each i with $0 \leqslant i \leqslant k$ and all sufficiently large x, keep (x, n_i) in $\overline{B}$, thereby ensuring that no n_i is a solution to B as an $\mathrm{stR}\Sigma^0_1$-instance. Finally, we enumerate all other pairs into B. Then such a B is clearly a $\mathrm{stR}\Sigma^0_1$-instance.

For Q $=$ Bound, we let B be an extension of σ obtained by the following procedure: for each i with $0 \leqslant i \leqslant k$, enumerate $n_i + 1$ into B.

For Q $=$ Bound*, to construct our Bound*-instance $B = (B_1, \ldots, B_m)$, let $\sigma = (\sigma_1, \ldots, \sigma_m)$ and we let B_i be an extension of σ_i obtained by the following procedure: for each j with $0 \leqslant j \leqslant k$, enumerate $n_j + 1$ into B_i for each i.

For Q = stBound*, to construct our stBound*-instance B, which by the definition of stBound* is an enumeration of a set X such that $\{n : \exists k(n,k) \in X\}$ is bounded, and for each n, so is the set $\{k : (n,k) \in X\}$, we let B be an extension of σ obtained by the following procedure: for each j with $0 \leqslant j \leqslant k$, enumerate $(n_j + 1, n_j + 1)$ into X. Then it is clear that $\{n : \exists k(n,k) \in X\}$ is bounded, and so is $\{k : (n,k) \in X\}$, but we must have that $\{k : \exists n(n,k) \in X\}$ contains $n_j + 1$ for each j, so therefore no n_j is a solution to B as a stBound*-instance.

For Q = $\mathrm{C}_{\mathbb{N}}$, we let B be an extension of σ obtained by the following procedure: for each i with $0 \leqslant i \leqslant k$, enumerate n_i into B.

Note that if Q is a principle involving Δ^0_2 sets, since we are showing a non-reduction, it's enough to show that the non-reduction holds for the strong notion of a Δ^0_2 set.

For Q = $\mathrm{F}\Delta^0_2$, we can approximate B by a set $C = \{\langle x, s\rangle \mid x \text{ is in } B \text{ at stage } s\}$. We can enumerate elements extending σ into $\overline{C}$, and for each i with $0 \leqslant i \leqslant k$, enumerate $\langle n_i, s\rangle$ into C for all s.

For Q = $\mathrm{L}\Delta^0_2$, we can we can approximate B by a set $C = \{\langle x, s\rangle \mid x \text{ is in } B \text{ at stage } s\}$, and for each i with $0 \leqslant i \leqslant k$, enumerate $\langle n_i + 1, s\rangle$ into C for all s.

□

While this corollary gives us a lot of non-reductions, it does not cover every situation. For instance, consider the relationship between $\mathrm{stR}\Sigma^0_1$ and $\mathrm{medR}\Sigma^0_1$, which are both principles that lie in the set X from the corollary. We have:

Proposition 4.6.12. $\mathrm{stR}\Sigma^0_1 \nleqslant_{\mathrm{W}} \mathrm{medR}\Sigma^0_1$.

Proof sketch. We will construct a Σ^0_1 set A such that the enumeration of A constitutes an $\mathrm{stR}\Sigma^0_1$-instance, so $(\forall w)(\exists x > w)(\exists y \leqslant v)((x,y) \in A)$ for some v. Suppose for a contradiction that there is such a reduction, given by Turing functionals Φ and Ψ.

Start enumerating all pairs of the form $(0,0), (1,0), (2,0)$, and so forth into A. We want to determine a u such that $(\forall w)(\exists x > w)(\exists y \leqslant u)((x,y) \in \Phi^A)$ and for all $y > u$, (x,y) is never enumerated into Φ^A, so that Φ^A is a $\mathrm{medR}\Sigma^0_1$-instance. First, we guess that $u = 0$. As we enumerate pairs into A, we look at Φ^A. We stick with our guess that $u = 0$ until we see some pair $(i_1, j_1) \in \Phi^A$ with $j_1 > 0$. Then we guess that $u = j_1$. We stick with our guess that $u = j_1$ until we see some pair $(i_2, j_2) \in \Phi^A$ with $j_2 > j_1$. We continue in this fashion; we keep looking for an initial segment σ of a $\mathrm{stR}\Sigma^0_1$-instance extending what we have already constructed, such that Φ^σ forces us to increase our guess for u. If we ever fail to find such a σ, we can essentially apply the proof of Theorem 4.6.8 in this special case and conclude as desired. Otherwise, we can always find such a σ, that is, we can always add elements to A that force some element (x, y) into Φ^A with $y > u$, forcing us to continually increase our guess for u. Then in particular, if we add $(n, 0)$ to A for the smallest n such that $(n, 0) \notin A$, we get that A is an instance of $\mathrm{stR}\Sigma^0_1$. However, Φ^A violates the definition of an instance of $\mathrm{medR}\Sigma^0_1$, a contradiction.

□

We included the previous proof sketch for clarity, but note that the statement of that proposition is in fact covered by the following one.

Proposition 4.6.13. $\mathrm{stR}\Sigma^0_1 \not\leqslant^n_{\mathrm{gW}} \mathrm{medR}\Sigma^0_1$ *for any fixed* $n \in \omega$. *Therefore by Theorem 3.3.1, we get* $\mathrm{stR}\Sigma^0_1 \not\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{medR}\Sigma^0_1$ *and* $\mathrm{stR}\Sigma^0_1 \not\leqslant^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \mathrm{medR}\Sigma^0_1$ *as well.*

Proof. Suppose for a contradiction that $\mathrm{stR}\Sigma^0_1 \leqslant^n_{\mathrm{gW}} \mathrm{medR}\Sigma^0_1$ for some fixed $n \in \omega$ via a computable strategy for Player 2 where Player 2 wins in at most n+1 moves. As in previous proofs, we can assume that Player 2 wins in exactly $n + 1$ moves. We will construct a winning strategy for Player 1, meaning a $\mathrm{stR}\Sigma^0_1$-instance for them to play initially, via a forcing argument. We envision the possible plays of the game as a tree with n levels.

For (σ, τ) a pair of strings of natural numbers where σ and τ have the same length, and $1 \leqslant i \leqslant n - 1$, let Φ_i be the Turing functional such that $\Phi_i^{(\sigma,\tau)\oplus y_1 \oplus \ldots \oplus y_{i-1}}$ is an initial segment of Player 2's i^{th} move in response to (σ, τ) and $y_1, \ldots, y_{i-1}$. We are given the Φ_i's by the assumption that $\mathrm{stR}\Sigma^0_1 \leqslant^n_{\mathrm{gW}} \mathrm{medR}\Sigma^0_1$. We are also given a functional Ψ such that if Player 1 initially plays the stRΣ^0_1-instance Y, and if $y_1, \ldots, y_n$ are indeed valid moves for Player 2 to play, then $\Psi^{Y \oplus y_1 \oplus \ldots \oplus y_n}$ converges and is a solution to Y.

For each finite string α with length at most n, we have a parameter k_α, which may or may not be defined. To start, we leave all of the parameters k_α undefined.

For each pair of strings (σ, τ) where σ and τ have the same length, we define the tree with root (σ, τ). The root node of the tree is labelled (σ, τ). To determine the first level of the tree, we look at the initial segment of a medRΣ^0_1-instance given by $\Phi_1^{(\sigma,\tau)}$. Look at the maximum of the range of $\Phi_1^{(\sigma,\tau)}$ (i.e. the least upper bound). Suppose it is possible to extend (σ, τ) to some (σ', τ') such that the maximum of the range of $\Phi_1^{(\sigma',\tau')}$ is larger than the maximum of the range of $\Phi_1^{(\sigma,\tau)}$. Then we leave the value of k_1 undefined, and we do not add any nodes to the tree. If this is not possible, then we define k_1 to be the maximum of the range of $\Phi_1^{(\sigma,\tau)}$. We then add $k_1 + 1$ children labelled $0, 1, \ldots, k_1$ to the root node.

In the case where it is possible to extend (σ, τ) to some (σ', τ') such that the maximum of the range of $\Phi_1^{(\sigma',\tau')}$ is larger than the maximum of the range of $\Phi_1^{(\sigma,\tau)}$, our tree is done; it consists solely of the root node. But in the latter case, we continue building the tree. For each j with $0 \leqslant j \leqslant k_1$, we look at $\Phi_2^{(\sigma,\tau)\oplus j}$, each of which is a new initial segment of a medRΣ^0_1-instance. For each j, we suppose it is possible to extend (σ, τ) to some (σ', τ') such that the maximum of the range of $\Phi_2^{(\sigma',\tau')\oplus j}$ is larger than the maximum of the range of $\Phi_2^{(\sigma,\tau)\oplus j}$, and if this is indeed the case, then we leave the corresponding value $k_{2,j}$ undefined, and we do not add any children to the node labelled j. However, in the case where this is not possible, we define the corresponding value $k_{2,j}$ to be the maximum of the range of $\Phi_2^{(\sigma,\tau)\oplus j}$ and add $k_{2,j} + 1$ children to the node labelled j. We label these children $0, 1, \ldots, k_{2,j}$.

We continue in this manner until we are either in a case where we are done adding children to any node before the n^{th} level, or until we reach the n^{th} level. At the n^{th} level of the tree, we consider the paths α of length $n - 1$ consisting of labels of nodes from the root to each node at the n^{th} level, not including the label (σ, τ) on the root. Then for each of these nodes, we look at $\Psi^{(\sigma,\tau)\oplus\alpha[0]\oplus\alpha[1]\oplus\ldots\oplus\alpha[n-2]}$. This gives us some solution to a $\mathrm{stR}\Sigma^0_1$-instance extended by (σ,τ). Our goal is to avoid this proposed solution, provided that $\Psi^{(\sigma,\tau)\oplus\alpha[0]\oplus\alpha[1]\oplus\ldots\oplus\alpha[n-2]}$ converges. Suppose that we have ℓ of these solutions to be avoided, call them $a_0,\ldots,a_{\ell-1}$.

Now naturally, if (σ',τ') extends (σ,τ), we want the tree with root (σ,τ) to be a subset of the tree with root (σ',τ'). We will accomplish this through two means: first in how we define extension, which we will discuss further on, and second in how we define the tree with root (σ,τ). To this end, once we find our $a_0,\ldots,a_\ell$ values, if they exist, we go back through the tree we have built, starting with the root, and now we stipulate that in order to define k_α, the maximum of the range of $\Phi_i^{(\sigma,\tau)\oplus\alpha[0]\oplus\alpha[1]\oplus\ldots\oplus\alpha[i-2]}$ must capture the maximum of the range of $\Phi_i^{(\sigma',\tau')\oplus\alpha[0]\oplus\alpha[1]\oplus\ldots\oplus\alpha[i-2]}$ for all (σ',τ') extending (σ,τ), for whichever level i of the tree we are looking at, such that (σ',τ') avoids the solutions $a_0,\ldots,a_{\ell-1}$, meaning that the part of (σ',τ') that is different from (σ,τ) does not contain instances of the solutions $a_0,\ldots,a_{\ell-1}$ (it is okay if (σ,τ) contains instances of these solutions, since we don't need to avoid them entirely; we just want to prevent them from occurring infinitely often). Note that this is a more restrictive condition than we had in the first iteration of building the tree, so all this can change is that some undefined leaves may become defined and lead to new paths, that possibly reach the n^{th} level, thereby potentially yielding more forbidden solutions. If so, we take these new forbidden solutions and add them to the list $a_0,\ldots,a_{\ell-1}$, and call these new solutions $a_\ell,\ldots,a_m$.

Then we go back through the tree and stipulate that in order to define k_α, the maximum of the range of $\Phi_i^{(\sigma,\tau)\oplus\alpha[0]\oplus\alpha[1]\oplus\ldots\oplus\alpha[i-2]}$ must capture the maximum of the range of $\Phi_i^{(\sigma',\tau')\oplus\alpha[0]\oplus\alpha[1]\oplus\ldots\oplus\alpha[i-2]}$ for all (σ',τ') extending (σ,τ), for whichever level i of the tree we are looking at, such that (σ',τ') respects the solutions $a_0,\ldots,a_{\ell-1},a_\ell,\ldots,a_m$. Note that again, this can only add to the tree. We continue this process until no more forbidden solutions arise. We must reach a point where no more forbidden solutions arise, because otherwise our finitely branching finite depth tree would be infinite.

Definition 4.6.14. We say that the tree we obtain at the end of this process is *correct*. For completeness, we also say that any tree built in this way that does not reach the n^{th} level is also correct.

We now define a notion of forcing. Our conditions are sets (strings) (σ, τ) of pairs (n, k), for $n, k \in \omega$, where σ and τ have the same length, i.e. if $\sigma = n_0 n_1 \ldots n_\ell$ and $\tau = m_0 m_1 \ldots m_\ell$, then $(\sigma, \tau) = (n_0, m_0)(n_1, m_1) \ldots (n_\ell, m_\ell)$. We say that a pair of strings (σ', τ') extends a pair of strings (σ, τ) if σ' extends σ as strings and τ' extends τ as strings, if the tree with root (σ', τ') avoids all of the solutions that the tree with root (σ, τ) avoids, and if whenever k_α is defined in the latter tree, it is defined and takes on the same value in the former tree, so that the tree with root (σ', τ') is a superset of the tree with root (σ, τ).

Definition 4.6.15. We say that a tree is *maximal correct* if there is no correct tree properly containing it, as well as no tree properly containing it with a larger number of forbidden solutions.

Now consider a sufficiently generic object G with respect to our notion of forcing. Note that again by finiteness, there exists a maximal correct tree corresponding to some (σ, τ) along the generic G. Now there are only finitely many forbidden solutions for this maximal correct tree. Fix some u that is not a forbidden solution. Then for every k, the property of a condition containing k many instances of (n, u) is dense for every k. Then G must contain infinitely many instances of (n, u). Therefore our sufficiently generic object G satisfies the requirements to be a $\mathrm{stR}\Sigma^0_1$-instance, with solution u. Then if Player 2 plays as its moves $\Phi_1^{(\sigma,\tau)}, \Phi_2^{(\sigma,\tau)\oplus y_1}, \Phi_3^{(\sigma,\tau)\oplus y_1 \oplus y_2}, \ldots, \Phi_{n-1}^{(\sigma,\tau)\oplus y_1 \oplus y_2 \oplus \ldots \oplus y_{n-1}}, \Psi^{G \oplus y_1 \oplus \ldots \oplus y_n}$, where $y_1, y_2, \ldots, y_n$ are the solutions played by Player 1, and these are valid moves for Player 2 to play, then $\Psi^{G \oplus y_1 \oplus \ldots \oplus y_n}$ converges and is a solution to G. Now for each i with $1 \leqslant i \leqslant n$, if y_i is a leaf at the n^{th} level of the maximal correct tree with root (σ, τ), then it is clearly avoided by G. Otherwise, y_i corresponds to a branch in the tree that does not reach the n^{th} level, so not all k_α-values are defined for this proposed solution. Let α be the length-lexicographically first string such that k_α is undefined. Then the branch on which k_α lies does not correspond to a valid $\mathrm{medR}\Sigma^0_1$-instance, because if k_α is undefined, then the range at that point on the branch is unbounded. More precisely, for any potential bound b, the set of conditions where we've forced the approximation to k_α (since k_α is never actually defined in this case) to be larger than b is dense. This is because given a condition (σ, τ), we can change it finitely (i.e. finitely increase one of its k_α values) without changing what its corresponding forbidden potential solutions are. Therefore by the genericity of G, we cannot get a $\mathrm{medR}\Sigma^0_1$-instance. This contradiction implies the desired non-reduction.

□

After reading through the above proof, one may wonder: would the proof of Theorem 4.6.8 benefit from introducing a notion of forcing? However, note that the metatheorem case is actually simpler, as the notions of correctness and maximal correctness are not needed; we just have a tree, and a notion of extension. Therefore a notion of forcing is not necessary for clarity in the way that it is for the previous proof.

4.7 Specific results involving regularity and Δ^0_2 principles

We now turn back to more specific relationships between the principles we have been looking at, in particular incorporating the newer principles we have introduced involving regularity and Δ^0_2 sets. Our goal in this section is to illuminate more of the relationships between these principles. We begin with regularity.

Proposition 4.7.1. Bound $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*}$ $\mathrm{medR}\Sigma^0_1$.

Proof. Work in a model (M, S) of RCA_0^*. Suppose we have a Bound-instance given by the enumeration of some bounded set F. Let $A = \{(x, y) \mid y$ is the largest element that has been enumerated into F by stage $x\}$. Then A is a computable set, and if u is an upper bound for F, then $(\forall w)(\exists x > w)(\exists y \leqslant u)((x,y) \in A)$, and for all $y > u$, (x,y) is never enumerated. Let y be a solution to A as a $\text{medR}\Sigma^0_1$-instance, so $y \leqslant u$ and $(\forall w)(\exists x > w)((x,y) \in A)$. Then by construction, we must have that y is an upper bound for F. □

Proposition 4.7.2. $\text{F}\Sigma^0_1 \leqslant^{\text{RCA}^*_0}_{\text{W}} \text{medR}\Sigma^0_1$.

Proof. Work in a model (M,S) of RCA^*_0. Suppose the enumeration of a set A is an instance of $\text{F}\Sigma^0_1$. Let $S = \{(s,n) \mid n \notin A[s]$, but for every $m < n, m \in A[s]\}$. Since the enumeration of A is an $\text{F}\Sigma^0_1$-instance, there is some element $u \in \overline{A}$. Then for all $y > u$, $(x,y) \notin S$ for any x. Then S is an instance of $\text{medR}\Sigma^0_1$ because for all w, there exists an $x > w$ and a $y \leqslant u$ with $(x,y) \in S$; in particular, for all w, there exists an $x > w$ with $(x,u) \in S$. Now let y be a solution to S as a $\text{medR}\Sigma^0_1$-instance. Then $y \leqslant u$, where u is such that for all $z > u$, (x,z) is never enumerated in S, and for all w, there exists an $x > w$ such that $(x,y) \in S$. Therefore every $m < y$ is in $A[x]$ for infinitely many x, hence every $m < y$ is in A, but $y \in \overline{A}[x]$ for infinitely many x, so $y \notin A$. Therefore y is a solution to A as an $\text{F}\Sigma^0_1$-instance. □

In particular, this implies that $\text{F}\Sigma^0_1 \leqslant^{\text{RCA}^*_0}_{\text{W}} \text{stR}\Sigma^0_1$ as well.

Proposition 4.7.3. $\text{F}\Pi^0_1 \leqslant^{\text{RCA}^*_0}_{\text{W}} \text{R}\Sigma^0_1$.

Proof. Work in a model (M,S) of RCA^*_0. Suppose we have a $\text{F}\Pi^0_1$-instance given by a set $\overline{A}$, where A is a Σ^0_1 set. Then we can enumerate some element $a \in A$. Let $B = \{(s,n) \mid n > 0$ and has been enumerated into A by stage s but $n-1$ has not been, or $n = 0$ and n has been enumerated into A by stage $s\}$. Then B is Σ^0_1 and $(\forall w)(\exists s > w)(\exists n \leqslant a)((s,n) \in B)$, because $a \in A$, and either there exists some $x \leqslant a$ with $x \in A$ and $x - 1 \in \overline{A}$, hence $(s,x) \in B$ for all sufficiently large x, or else $0 \in \overline{A}$ and hence $(s,0) \in B$ for all s. Therefore B and a comprise a $\text{R}\Sigma^0_1$-instance. Let y be a solution to this $\text{R}\Sigma^0_1$-instance, so $y \leqslant a$ and $(\forall w)(\exists s > w)((s,y) \in B)$. Then by definition of B, y is a solution to our $\text{F}\Pi^0_1$-instance. □

Proposition 4.7.4. $\text{K}_{\mathbb{N}} \leqslant^{\text{RCA}^*_0}_{\text{W}} \text{R}\Sigma^0_1$.

Proof. Work in a model (M, S) of RCA_0^*. Suppose that the enumeration of a set X together with a bound b comprise an instance of $\mathrm{K}_\mathbb{N}$. Let $S = \{(s, x) \mid x$ has not been enumerated into X by stage $s\}$. Then S is a Σ_1^0 set and since $\overline{X} \cap \{x \mid x < b\}$ must be nonempty, then $(\forall w)(\exists x > w)(\exists y \leqslant b - 1)((x, y) \in S)$ by the definition of a $\mathrm{K}_\mathbb{N}$-instance. Therefore S is a $\mathrm{R}\Sigma_1^0$-instance. Let y be a solution to S as a $\mathrm{R}\Sigma_1^0$-instance. Then $y < b$ and there are infinitely many s such that y has not been enumerated into X by stage s. Then $y \in \overline{X}$, and therefore y is a solution to X as a $\mathrm{K}_\mathbb{N}$-instance. □

Proposition 4.7.5. $\mathrm{C}_\mathbb{N} \leqslant_\mathrm{W}^{\mathrm{RCA}_0^*} \mathrm{stR}\Sigma_1^0$.

Proof. Work in a model (M, S) of RCA_0^*. Suppose we have a nonempty set X, so the enumeration of the complement of X is a $\mathrm{C}_\mathbb{N}$-instance. Let $S = \{(s, x) \mid x$ has not been enumerated into $\overline{X}$ by stage $s\}$. Then S is a Σ_1^0 set and since there is some $u \in X$, we have $(\forall w)(\exists x > w)(\exists y \leqslant u)((x, y) \in S)$. Therefore S is a $\mathrm{stR}\Sigma_1^0$-instance. Let y be a solution to S as a $\mathrm{stR}\Sigma_1^0$-instance, so $y \leqslant u$ and there are infinitely many stages s such that $(s, y) \in S$. That is, there are infinitely many stages s such that y has not been enumerated into $\overline{X}$ by stage s, so $y \in X$. Therefore y is a solution to the enumeration of $\overline{X}$ as a $\mathrm{C}_\mathbb{N}$-instance, as desired. □

Proposition 4.7.6. $\mathrm{B}\Delta_2^0\mathrm{CA} \leqslant_\mathrm{W} \mathrm{R}\Sigma_1^0$.

Proof. Let the approximation to a set X together with a bound b comprise a $\mathrm{B}\Delta_2^0\mathrm{CA}$-instance. For this construction, we will identify binary strings of length b with numbers less than 2^b. At each stage $s \in \omega$, we have approximated some initial segment of a $\mathrm{B}\Delta_2^0\mathrm{CA}$-instance, which we can code by a number x_s. We can code our initial segments in such a way that $x_s \leqslant 2^b$ for all $s \in \omega$. Let A be the enumeration of $\{(s, x_s) \mid s \in \omega\}$. Note that as $s \to \infty$, our initial segment will stabilize, and therefore the sequence $\{x_s \mid s \in \omega\}$ will converge. Then we have $(\forall w)(\exists s > w)(\exists y \leqslant x_s)((s, y) \in A)$, and therefore $(\forall w)(\exists s > w)(\exists y \leqslant 2^b)((s, y) \in A)$. Then A is a $\mathrm{R}\Sigma_1^0$-instance. Let y be a solution to A as a $\mathrm{R}\Sigma_1^0$-

instance. Then we must have $y = \lim_{s\to\infty} x_s$, which gives us a solution to our $\mathrm{B}\Delta^0_2\mathrm{CA}$-instance, as desired. □

Note that we cannot do better than a Weihrauch reduction with this argument, since it falls apart over RCA_0. Now consider the following principle $\lim_{\mathbb{N}}$ (see [10] for more about this principle).

Definition 4.7.7. $\lim_{\mathbb{N}}$ is the Π^1_2-principle where an instance is an infinite sequence $(p_0, p_1, \ldots)$ that has a limit and a solution is $\lim_{n\to\infty} p_n$.

Clearly, $\mathrm{B}\Delta^0_2\mathrm{CA} \leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$, because thinking of an infinite sequence with a limit as a Δ^0_2 set via the Limit Lemma, the only difference between the two principles is the existence of a bound. It is also clear that $\mathrm{F}\Delta^0_2 \leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$, and that $\lim_{\mathbb{N}} \equiv_{\mathrm{W}}$ Bound. Also, we have that $\lim_{\mathbb{N}} \equiv_{\mathrm{gW}} \lim_2$ by the following argument: given a $\lim_{\mathbb{N}}$ instance, we can think of it as a $\lim_2$ instance by breaking it into the case where the number is 0 and > 0. Then if the solution is > 0, we consider the $\lim_2$ instances with cases 1 and > 1, and so forth, until we get the solution k for the $k, > k$ split. Then k is a solution to our original $\lim_{\mathbb{N}}$-instance.

In terms of proof theory, it is evident that $\lim_{\mathbb{N}}$ holds in every model of RCA^*_0, hence in every model of RCA_0. Therefore we get that for $\mathrm{P} \in X = \{$Bound*, stBound*, $\mathrm{F}\Delta^0_2$, $\mathrm{L}\Delta^0_2$, $\mathrm{B}\Delta^0_2\mathrm{CA}$, $\mathrm{R}\Sigma^0_1$, $\mathrm{medR}\Sigma^0_1$, $\mathrm{stR}\Sigma^0_1\}$ that $\mathrm{P} \not\leqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \lim_{\mathbb{N}}$. We also get that for $\mathrm{P} \in Y = \{\mathrm{F}\Sigma^0_1$, $\mathrm{F}\Pi^0_1\}$, that $\mathrm{P} \not\leqslant^{\mathrm{RCA}^*_0}_{\mathrm{gW}} \lim_{\mathbb{N}}$. Furthermore, since $\mathrm{B}\Delta^0_2\mathrm{CA}$ and $\mathrm{F}\Delta^0_2$ are both "strong" with respect to Theorem 4.6.8, it follows that $\lim_{\mathbb{N}}$ is as well.

Proposition 4.7.8. $\mathrm{R}\Sigma^0_1 \not\leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$.

Proof. Suppose, to the contrary, that there exists a Weihrauch reduction $\mathrm{R}\Sigma^0_1 \leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$ given by Turing functionals Φ and Ψ. We will build an $\mathrm{R}\Sigma^0_1$-instance consisting of a Σ^0_1 set A and a bound u. For simplicity, we let $u = 1$. Start enumerating pairs $(n, 0)$ into A for $n = 0, 1, 2, \ldots$. Then Φ^A will look like an initial segment of a $\lim_{\mathbb{N}}$-instance, that is, a finite sequence of numbers, say $\Phi^A = (n_0, n_1, \ldots, n_{k_0})$ for some k_0. Then we can guess that n_{k_0} is a solution to this Φ^A instance, and look at $\Psi^{A\oplus n_{k_0}}$. Right now the only possible solution to a $\mathrm{R}\Sigma^0_1$-instance extending A is 0, so if $\Psi^{A\oplus n_{k_0}}$ eventually converges, it must converge to 0. If it never converges, then we must have a $\lim_{\mathbb{N}}$-instance with no solution corresponding to a $\mathrm{R}\Sigma^0_1$-instance with no solution, so suppose it converges. Then continue to enumerate pairs $(n, 0)$ for larger and larger n into A until we see this convergence at some stage s_0, and once we do, we start enumerating pairs $(n, 1)$ into A for $n = s_0+1, s_0+2, \ldots$. Again, we enumerate enough pairs so that if $\Phi^A = (n_0, n_1, \ldots, n_{k_1})$, then $\Psi^{A\oplus n_{k_1}}$ converges. Then since the only possible solution to a $\mathrm{R}\Sigma^0_1$-instance extending A right now is 1, it must converge to 1 at some stage $s_1 > s_0$. Then we start enumerating pairs $(n, 0)$ again for $n = s_1 + 1, s_1 + 2, \ldots$ into A. Continuing in this manner, since Φ^A is an initial segment of a $\lim_{\mathbb{N}}$-instance, eventually, it must converge, that is, there is some i such that for all $j \geqslant i$, $n_j = n_i$. Then at that point, the value of $\Psi^{A\oplus n_{k_i}}$ will not change, so once we start enumerating pairs of the form (n, s_i+1) into A for s_i larger than all of the previous s values and large enough that $\Psi^{A\oplus n_{k_i}}$ converges (to 0 or 1 depending on whether i is odd or even, respectively), we obtain a contradiction, as desired. □

Corollary 4.7.9. $\mathrm{R}\Sigma^0_1 \nleqslant_{\mathrm{W}} \mathrm{B}\Delta^0_2\mathrm{CA}$.

Corollary 4.7.10. $\mathrm{R}\Sigma^0_1 \nleqslant_{\mathrm{W}} \mathrm{F}\Delta^0_2$.

We can make a similar argument to get that $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}} \mathrm{F}\Delta^0_2$, thinking of our $\mathrm{R}\Sigma^0_1$-instance as a coloring rather than an enumeration of a Σ^0_1 set. Then since $\mathrm{F}\Pi^0_1 \leqslant_{\mathrm{W}} \mathrm{F}\Delta^0_2$ and $\mathrm{F}\Sigma^0_1 \leqslant_{\mathrm{W}} \mathrm{F}\Delta^0_2$, we obtain the following corollaries.

Corollary 4.7.11. $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}} \mathrm{F}\Pi^0_1$.

Corollary 4.7.12. $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}} \mathrm{F}\Sigma^0_1$.

Note that since $\mathrm{F}\Pi^0_1 \leqslant_{\mathrm{W}} \mathrm{B}\Delta^0_2\mathrm{CA}$, we have that $\mathrm{R}\Sigma^0_1 \nleqslant_{\mathrm{W}} \mathrm{B}\Sigma^0_1\mathrm{CA}$ also. Again, we can make a similar argument to obtain the following as well.

Corollary 4.7.13. $\mathrm{RT}^1_2 \not\leqslant_{\mathrm{W}} \mathrm{B}\Delta^0_2\mathrm{CA}$.

We now consider where $\lim_{\mathbb{N}}$ lies in our universe of principles.

Proposition 4.7.14. Bound $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \lim_{\mathbb{N}}$.

Proof. Work in a model (M, S) of RCA_0^*. Let the enumeration of a set F be a Bound-instance. We will define a $\lim_{\mathbb{N}}$-instance $(x_0, x_1, \ldots, x_n, \ldots)$. To define x_0, we begin to enumerate elements of F, and we define x_0 to be the first such element enumerated. For $n > 0$, to define x_n, we enumerate F for n additional steps and look at whether x_{n-1} is a bound for the elements enumerated thus far. If so, then we define $x_n := x_{n-1}$. Otherwise, we let x_n be the least number greater than n that currently bounds the elements enumerated so far. If this sequence does not stabilize, then $(x_0, x_1, \ldots, x_n, \ldots)$ is not a $\lim_{\mathbb{N}}$-instance, but then F is unbounded, hence not a Bound-instance. Otherwise, the sequence stabilizes, and is therefore a $\lim_{\mathbb{N}}$-instance. Given a solution to this $\lim_{\mathbb{N}}$-instance, by construction, it must also be a solution to F as a Bound-instance, thereby completing the reduction. □

From [10], we have that $\mathrm{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \lim_{\mathbb{N}}$. It therefore follows that $\mathrm{K}\Delta^0_2 \leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$.

4.8 Weak Weak König's Lemma

We now turn to some results in our setting involving variants of Weak Weak König's Lemma. We begin with some definitions.

Definition 4.8.1. WWKL is the principle that every subtree T of $2^{<\omega}$ such that

$$\liminf_{n} \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n} > 0 \tag{4.14}$$

has an infinite path.

Definition 4.8.2. Let $q < 1$ be a positive rational. q-WWKL is the principle that every subtree T of $2^{<\omega}$ such that

$$\frac{|\{\sigma \in 2^n : \sigma \in T\}|}{2^n} \geqslant q \tag{4.15}$$

for all n has an infinite path. In this case we say that T has *measure* $\geqslant q$.

Dorais, Dzhafarov, Hirst, Mileti, and Shafer proved the following relationship between this principle for different values of p in [21].

Proposition 4.8.3 (Dorais, Dzhafarov, Hirst, Mileti, and Shafer [21])**.** *For all positive rationals $p < q < 1$, p-WWKL $\nleqslant_W$ q-WWKL.*

Recall that WWKL* denotes the finite parallelization of WWKL, where finitely many instances of WWKL are run in parallel. We use $\widehat{\text{WWKL}}$ to denote the infinite parallelization of WWKL, where infinitely many instances of WWKL are run in parallel. Note that WWKL* $\leqslant_W$ WWKL by an interleaving argument: given a finite number of trees, we can weave them together in such a way that a path in the interleaved tree can be untangled into paths in each of the original trees. However,

Proposition 4.8.4. $\widehat{\text{WWKL}} \nleqslant_W$ WWKL*.*

Proof. We follow the proof that p-WWKL $\nleqslant_W$ q-WWKL in [21].

Suppose $\widehat{\text{WWKL}} \leqslant_W$ WWKL, and let Φ and Ψ witness a Weihrauch reduction from $\widehat{\text{WWKL}}$ to WWKL. We build a sequence of computable trees $\mathcal{T} = \{T_0, T_1, \ldots\}$. For each $i \in \omega$, let $q_i = \frac{1}{i+1}$, so $\lim\limits_{i\to\infty} q_i = 0$. Our construction will guarantee that each T_i has measure at least q_i, but such that $\Phi^{\mathcal{T}}$ has measure less than $2q_i$ for each i, hence $\Phi^{\mathcal{T}}$ must have measure 0, the desired contradiction.

We shall regard each partial computable function as defining an initial segment of a computable subtree of $2^{<\omega}$, with each new convergence giving an entire new level of the tree, and only strings of maximal length at the previous level being extended. Let φ_j be the

j^{th} such tree. Since the construction of $\mathcal{T}$ is computable, there is some index j that we use to produce $\mathcal{T}$. In fact, the paths through $\mathcal{T}$ that this reduction produces are really $\Psi^{\varphi_j \oplus \mathcal{T}}$. In the end we produce some tree $\mathcal{T} = \varphi_k$, where k may or may not be the index j that we started out with, since $\mathcal{T}$ may have changed during the construction. But the construction is uniform, so there is a computable function f such that $f(j) = k$. By the recursion theorem, there is some j such that $\varphi_{f(j)} = \varphi_j$. So, to produce $\mathcal{T}$, we will use this index j. Therefore we do not need to consider $\mathcal{T}$ in the oracle for Ψ. By the same argument, it follows that we do not need to consider $\mathcal{T}$ in the oracle for Φ either, so that we do not need to worry about $\mathcal{T}$ changing during our construction.

For the construction, we fix a sequence of positive numbers $\{a_i : i \in \omega\}$ such that $q_i < 2^{-a_i} < 2q_i$ for each i. At stage s of the construction, for each $i \leqslant s$, we shall define $T_{i,s} = T_i \cap 2^{\leqslant s}$, starting with $T_{0,0} = \{\emptyset\}$. For $i > s$, we define $T_{i,s} = 2^{\leqslant s}$, because T_i must be defined to some point for every i in order to apply Φ. Let n_s be the height of $\Phi^{T_{0,s} \oplus T_{1,s} \oplus \cdots}$. Assume, without loss of generality, that $n_s \leqslant s$ for all s.

At stage $s+1$, for each $i \leqslant s+1$, choose at least a_i many numbers $x_{i,0} < \ldots < x_{i,a_i - 1}$ that we have not yet acted for for T_i, as defined below. Assume inductively that for each $i \leqslant s+1$ and each $\alpha_i \in 2^{a_i}$, there is a string $\sigma_i \in T_{i,s}$ of length s with $\sigma_i(x_{i,j}) = \alpha_i(j)$ for all j with $x_{i,j} < s$. We consider two cases.

Note that when we apply Ψ to an initial segment τ of $\Phi^{T_{0,s} \oplus T_{1,s} \oplus \cdots}$, we get an object that we can think of as an infinite sequence of initial segments of paths $(b_0, b_1, \ldots)$, where b_i is an initial segment of a path in T_i, with all but finitely many of these initial segments empty. We write $\Psi^{\tau_i}(x_{i,j})$ to denote the $x_{i,j}^{th}$ position in the path b_i (which may or may not be defined).

We check which of the following cases we are in and act accordingly for each $i \leqslant s+1$ in order. After we finish the procedure for $i = s+1$, we set $T_{i,s+1} = 2^{\leqslant s+1}$ for $i > s+1$. We then move to stage $s+2$, and go through the procedure for each $i \leqslant s+2$ in order, set

$T_{i,s+2} = 2^{\leqslant s+2}$ for all $i > s+2$, and so forth.

Case 1_i: If any of the following apply:

- $\Phi^{T_{0,s} \oplus T_{1,s} \oplus \cdots}$ contains fewer than $2^{n_i+1} q_s$ many strings of length n_s;
- $x_{i,a_i-1} \geqslant s$;
- $x_{i,a_i-1} < s$ but $\Psi^{\tau_i}(x_{i,j}) \uparrow$ for some $\tau \in \Phi^{T_{0,s} \oplus T_{1,s} \oplus \cdots}$ of length n_i, and some $j < a_i$;

then we obtain $T_{i,s+1}$ from $T_{i,s}$ by adding $\sigma_i 0$ and $\sigma_i 1$ for each $\sigma_i \in T_{i,s}$ of length s.

Case 2_i: Otherwise, choose $\alpha_i \in 2^{a_i}$ so that $\Psi^{\tau_i}(x_{i,j}) \downarrow = \alpha_i(j)$ for all $j < a_i$ for at least 2^{-a_i} many strings $\tau \in \Phi^{T_{0,s} \oplus T_{1,s} \oplus \cdots}$ of length n_i. Then, we obtain $T_{i,s+1}$ from $T_{i,s}$ by adding $\sigma_i 0$ and $\sigma_i 1$ for each $\sigma_i \in T_{i,s}$ of length s with $\sigma_i(x_{i,j}) \neq \alpha_i(j)$ for some $j < a_i$. Say we have acted for $x_{i,0}, \ldots, x_{i,a_i-1}$.

For the verification, clearly, each T_i is a computable subtree of $2^{<\omega}$. Note that the measure of T_i is cut down only when the construction enters Case 2_i, at which point it is cut down by a factor of precisely 2^{-a_i}. Whenever the construction enters case 2_i for some i, the measure of $\Phi^{\mathcal{T}}$ is cut down by at least a factor of 2^{-a_i}. We claim that for each i, there is a stage s_i such that $\Phi^{T_{0,s_i} \oplus T_{1,s_i} \oplus \cdots}$ contains fewer than $2^{n_{s_i}+1} q_i$ many strings of length s_i, so that the measure of $\Phi^{\mathcal{T}}$ is less than $2q_i$. Therefore the measure of $\Phi^{\mathcal{T}}$ is less than $2q_i$ for each i, and hence $\mathcal{T}$ has measure 0, as desired. Moreover, if we fix the least such s_i for each i, since $2^{-a_i} < 2q_i$, it follows that each T_{i,s_i} contains at least $2^{s_i} q_i$ strings of length s_i. Since the construction can never enter Case 2_i at any stage after s_i, it follows that the measure of T_i is at least q_i.

To prove the claim, fix i, and let t_i be any stage such that $\Phi^{T_{0,t_i} \oplus T_{1,t_i} \oplus \cdots}$ contains at least $2^{n_{t_i}+1} q_i$ many strings of length n_{t_i}. Fix the least $x_{i,0} < \ldots < x_{i,a_i-1}$ not yet acted for for T_i prior to stage $t_i + 1$. For each path B through $\Phi^{\mathcal{T}}$, we have that $\Psi^{B_i}(x_{i,j}) \downarrow$ for all $i \leqslant t_i$ and all $j < a_i - 1$, so by compactness, there is an $s > \max\{t_i, x_{a_i-1}\}$ such that $\Psi^{\tau_i}(x_{i,j}) \downarrow$ for all $j < a_i$ and all $\tau \in \Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots \oplus T_{i,s} \cdots}$ of length n_s. Fix the least such

s. Then the construction never enters Case 2_i strictly between stages t_i and s because T_i satisfies the third bullet point until stage s, so $T_{i,s}$ contains at least $2^{n_i+1}q_i$ many strings of length n_s. Since $x_{a_i-1} < s$ by definition, it follows that at stage s, the second (and third) bullet points of Case 1_i are not satisfied. Therefore Case 2_i applies at stage s. So we have shown that the construction continues to enter Case 2_i until the measure of $\Phi^{\mathcal{T}}$ has been sufficiently cut down. This proves the claim. $\square$

Having determined the relationship between WWKL and its infinite parallelization, it is natural to turn next to its finite parallelization. We have

Proposition 4.8.5. WWKL$^* \not\leq_{\mathrm{W}}^{\mathrm{RCA}_0}$ WWKL.

Proof. Suppose not, and let Φ and Ψ witness a Weihrauch reduction over RCA_0 from WWKL* to WWKL. We work in a model $(M, S) \models \mathrm{RCA}_0 + \neg\mathrm{B}\Sigma^0_2$. We build a sequence of computable trees $\mathcal{T} = \{T_0, T_1, \ldots, T_m\}$ for some finite m, which may or may not be nonstandard (in fact, necessarily, it must be nonstandard). For each $i \in \omega$, let $q_i = \frac{1}{i+1}$, so $\lim\limits_{i\to\infty} q_i = 0$. Our construction will guarantee that each T_i has measure that is bounded below, but such that $\Phi^{\mathcal{T}}$ has measure less than $2q_i$ for each i, hence $\Phi^{\mathcal{T}}$ must have measure 0, the desired contradiction. We will use the failure of $\mathrm{B}\Sigma^0_2$ in $\mathcal{M}$, and the consequent failure of $\mathrm{RT}^1_{<\infty}$ (see [38]), to ensure that the measures of the T_i's do not get too small.

We shall regard each partial computable function as defining an initial segment of a computable subtree of $2^{<\omega}$, with each new convergence giving an entire new level of the tree, and only strings of maximal length at the previous level being extended. Let φ_j be the j^{th} such tree. Since the construction of $\mathcal{T}$ is computable, there is some index j that we use to produce $\mathcal{T}$. In fact, the paths through $\mathcal{T}$ that this reduction produces are really $\Psi^{\varphi_j \oplus \mathcal{T}}$. In the end we produce some tree $\mathcal{T} = \varphi_k$, where k may or may not be the index j that we started out with, since $\mathcal{T}$ may have changed during the construction. But the construction is uniform, so there is a computable function f such that $f(j) = k$. By the recursion theorem (one can verify that the proof of the recursion theorem holds in general models of RCA_0), there is some j such that $\varphi_{f(j)} = \varphi_j$. So, to produce $\mathcal{T}$, we will use this index j. Therefore we do not need to consider $\mathcal{T}$ in the oracle for Ψ. By the same argument, it follows that we do not need to consider $\mathcal{T}$ in the oracle for Φ either, so that we do not need to worry about $\mathcal{T}$ changing during our construction.

Since $(M, S) \models \mathrm{RCA}_0 + \neg\mathrm{RT}^1_{<\infty}$, there exists an m-coloring of $\mathbb{N}$ for some m such that the coloring does not have an infinite homogeneous set. Let c be such a coloring.

For the construction, without loss of generality, we assume that, $q_{c(i)} \geqslant q_i$, since otherwise we can just start the sequence $\{q_i : i \in \omega\}$ at $i = m$. Fix a sequence of positive numbers $\{a_i : i \in \omega\}$ such that $q_i < 2^{-a_i} < 2q_i$ for each i. At stage s of the construction, for each $i \leqslant \min\{m, s\}$, we shall define $T_{i,s} = T_i \cap 2^{\leqslant s}$, starting with $T_{0,0} = \{\emptyset\}$. For $i \in (s, m]$, we define $T_{i,s} = 2^{\leqslant s}$, because T_i must be defined to some point for every $i \leqslant m$ in order to apply Φ. Let n_s be the height of $\Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots \oplus T_{m,s}}$. Assume, without loss of generality, that $n_s \leqslant s$ for all s.

At stage $s + 1$, choose at least a_{s+1} many numbers $x_0 < \ldots < x_{a_{s+1}-1}$ that we have not yet acted for, as defined below. Assume inductively that for each $\alpha \in 2^{a_{s+1}}$ and each $i \leqslant \min\{m, s + 1\}$ there is a string $\sigma_i \in T_{i,s}$ of length s with $\sigma_i(x_j) = \alpha(j)$ for all j with $x_j < s$. Note that when we apply Ψ to an initial segment τ of $\Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots \oplus T_{m,s}}$, we get a sequence of initial segments of paths $(b_0, b_1, \ldots, b_m)$, where b_i is an initial segment of a path in T_i. We write $\Psi^{\tau_i}(x_j)$ to denote the x_j^{th} position in the path b_i. We consider two cases.

Case 1: If any of the following apply:

- $\Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots, T_{m,s}}$ contains fewer than $2^{n_s+1} q_s$ many strings of length n_s;
- $x_{a_{s+1}-1} \geqslant s$;
- $x_{a_{s+1}-1} < s$ but $\Psi^{\tau_{c(s)}}(x_j) \uparrow$ for some $\tau \in \Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots \oplus T_{m,s}}$ of length n_s, and some $j < a_{s+1}$;

then we obtain each $T_{i,s+1}$ from $T_{i,s}$ by adding $\sigma_i 0$ and $\sigma_i 1$ for each $\sigma_i \in T_{i,s}$ of length s.

For $i \in (s+1, m]$, let $T_{i,s+1} = 2^{\leqslant s+1}$.

Case 2: Otherwise, choose $\alpha \in 2^{a_{s+1}}$ so that $\Psi^{\tau_{c(s)}}(x_j) \downarrow= \alpha(j)$ for all $j < a_{s+1}$ for at least $2^{-a_{s+1}}$ many strings $\tau \in \Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots \oplus T_{m,s}}$ of length n_s. We obtain $T_{c(s),s+1}$ from $T_{c(s),s}$ by adding $\sigma 0$ and $\sigma 1$ for each $\sigma \in T_{c(s),s}$ of length s with $\sigma(x_j) \neq \alpha(j)$ for some $j < a_{s+1}$. For $i < \min\{m, s+1\}$ with $i \neq c(s)$, we obtain each $T_{i,s+1}$ from $T_{i,s}$ by adding $\sigma_i 0$ and $\sigma_i 1$ for each $\sigma_i \in T_{i,s}$ of length s. For $i \in (s+1, m]$, let $T_{i,s+1} = 2^{\leqslant s+1}$.

For the verification, clearly, each T_i is a computable subtree of $2^{<\omega}$. Note that the measure of T_i is cut down only when the construction enters Case 2 and $i = c(s)$ for some $s > i$, at which point it is cut down by a factor of precisely 2^{-a_s}. Whenever the construction enters case 2, the measure of $\Phi^{\mathcal{T}}$ is cut down by at least a factor of 2^{-a_s} for some s. We claim that for each i, there is a stage $s_i \geqslant i$ such that $\Phi^{T_{0,s_i} \oplus T_{1,s_i} \oplus \ldots \oplus T_{m,s_i}}$ contains fewer than $2^{n_{s_i}+1} q_{s_i}$ many strings of length s_i, so that the measure of $\Phi^{\mathcal{T}}$ is less than $2q_{s_i} \leqslant 2q_i$. Therefore the measure of $\Phi^{\mathcal{T}}$ is less than $2q_i$ for each i, and hence $\mathcal{T}$ has measure 0, as desired. Moreover, since $\mathrm{RT}^1_{<\infty}$ fails for c, for each i, $c(j) = i$ for only finitely many j, so that the measure of T_i is cut down only finitely many times. Therefore each T_i has measure bounded below (specifically, by $\prod_{i=1}^{k} 2^{-a_{s_i}}$ for $k \in \omega$ and $s_1 < s_2 < \ldots < s_k \in \omega$), so $\mathcal{T}$ is a valid WWKL*-instance.

To prove the claim, fix i, and let t_i be any stage larger than i such that $\Phi^{T_{0,t_i} \oplus T_{1,t_i} \oplus \ldots \oplus T_{m,t_i}}$ contains at least $2^{n_{t_i}+1} q_{t_i}$ many strings of length n_{t_i}. Fix the least $x_0 < \ldots < x_{a_{t_i+1}-1}$ not yet acted for prior to stage $t_i + 1$. For each path B through $\Phi^{\mathcal{T}}$, we have that $\Psi^{B_{c(t_i)}}(x_j) \downarrow$ for all $j < a_{t_i+1} - 1$, so by compactness, there is an $s > \max\{m, t_i, x_{a_{t_i+1}-1}\}$ such that $\Psi^{\tau_{c(s)}}(x_j) \downarrow$ for all $j < a_{t_i+1} - 1$ and all $\tau \in \Phi^{T_{0,s} \oplus T_{1,s} \oplus \ldots \oplus T_{m,s}}$ of length n_s. Fix the least such s. Then the construction never enters Case 2 strictly between stages t_i and s because T_i satisfies the third bullet point until stage s, so $T_{i,s}$ contains at least $2^{n_s+1} q_s$ many strings of length n_s. Since $x_{a_{t_i+1}-1} < s$ by definition, it follows that at stage s, the second (and third) bullet points of Case 2 are not satisfied. Therefore Case 2 applies at stage s. So we have shown that the construction continues to enter Case 2 until the measure of $\Phi^{\mathcal{T}}$ has been sufficiently cut down. This proves the claim. $\square$

We leave the exploration of how WWKL and its variants fit in with the other principles we have studied for future work.

4.9 Open questions and future directions

We now turn to the relationships that have not been delimited in this chapter, the missing pieces so to speak. Note that the strongest possible positive result would be to obtain a Weihrauch reduction over RCA_0^*, while the strongest possible negative result would be to show the lack of a generalized Weihrauch reduction.

Open Question 4.9.1. Does $\mathrm{F}\Delta^0_2 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{stBound}^*$, or even does $\mathrm{F}\Delta^0_2 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{stBound}^*$? What happens when we look at $\mathrm{L}\Delta^0_2$ instead of $\mathrm{F}\Delta^0_2$? For $\mathrm{L}\Delta^0_2$, the questions of whether there is a (generalized) Weihrauch reduction over RCA_0 or RCA_0^* are all open.

Open Question 4.9.2. What is the relationship between $\mathrm{wkF}\Delta^0_2$ and Bound^* and $\mathrm{stBound}^*$? More specifically, can we get that $\mathrm{wkF}\Delta^0_2 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{stBound}^*$? Can we get that $\mathrm{Bound}^* \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*} \mathrm{wkF}\Delta^0_2$?

Open Question 4.9.3. We know that $\mathrm{Bound}^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Delta^0_2$ and $\mathrm{stBound}^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Delta^0_2$, but is there a Weihrauch reduction over RCA_0? What happens over RCA_0^*? What about $\mathrm{stBound}^*$? What about for $\mathrm{L}\Delta^0_2$ instead of $\mathrm{F}\Delta^0_2$? Again, we know that $\mathrm{Bound}^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{L}\Delta^0_2$, but is there a Weihrauch reduction over RCA_0? What happens over RCA_0^*?

Open Question 4.9.4. We know by Proposition 4.3.40 that there is a generalized Weihrauch reduction over RCA_0 from $\mathrm{stBound}^* \equiv_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Delta^0_2$ to $\mathrm{medR}\Sigma^0_1$ and hence to $\mathrm{stR}\Sigma^0_1$ (and no Weihrauch reduction over RCA_0), but does $\mathrm{F}\Delta^0_2 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{stR}\Sigma^0_1$, or does $\mathrm{F}\Delta^0_2 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{medR}\Sigma^0_1$? Is there a (generalized) Weihrauch reduction over RCA_0^*?

Open Question 4.9.5. How do WKL and WWKL fit in with the other principles?

Open Question 4.9.6. We have mainly considered min as it related to $\mathrm{C}_{\mathbb{N}}$. How does it relate to the other principles?

Open Question 4.9.7. We have used $\lim_2$ mainly as an auxiliary principle. Where does it fit in with the other principles?

Open Question 4.9.8. We know that $\mathrm{C}_{\mathbb{N}} \equiv_{\mathrm{W}}^{\mathrm{RCA}_0}$ Bound. What happens over RCA_0^*? If they are not equivalent, are $\mathrm{C}_{\mathbb{N}}^*$ and Bound*, or even $\mathrm{C}_{\mathbb{N}}^*$ and stBound*?

Open Question 4.9.9. Do we have a reduction $\mathrm{C}_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*}$ stBound*?

Open Question 4.9.10. Can we show that $\mathrm{B}\Delta_2^0\mathrm{CA} \nleqslant_{\mathrm{gW}} \mathrm{F}\Pi_1^0$?

Open Question 4.9.11. We know that $\mathrm{stR}\Sigma_1^0 \nleqslant_{\mathrm{W}} \mathrm{medR}\Sigma_1^0$, and that $\mathrm{stR}\Sigma_1^0 \nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{medR}\Sigma_1^0$, and consequently, the same is true for $\mathrm{R}\Sigma_1^0$ and RT_2^1 replacing $\mathrm{medR}\Sigma_1^0$. However, does $\mathrm{stR}\Sigma_1^0 \leqslant_{\mathrm{gW}} \mathrm{medR}\Sigma_1^0$ (or equivalently, we can replace $\mathrm{medR}\Sigma_1^0$ with $\mathrm{R}\Sigma_1^0$ or RT_2^1)?

Open Question 4.9.12. Can we show that $\mathrm{L}\Delta_2^0 \nleqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{F}\Delta_2^0$ or that $\mathrm{L}\Delta_2^0 \nleqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{F}\Delta_2^0$? Can we show that there is no (generalized) Weihrauch reduction over RCA_0^*?

Open Question 4.9.13. Does $\mathrm{L}\Delta_2^0 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{medR}\Sigma_1^0$? Is there a generalized Weihrauch reduction over RCA_0, or a (generalized) Weihrauch reduction over RCA_0^*? What about $\mathrm{stR}\Sigma_1^0$?

Open Question 4.9.14. Can we show that $\mathrm{R}\Sigma_1^0 \nleqslant_{\mathrm{gW}} \mathrm{F}\Delta_2^0$? What about RT_2^1?

Open Question 4.9.15. Can we show that $\mathrm{stR}\Sigma_1^0 \nleqslant_{\mathrm{gW}} \mathrm{B}\Delta_2^0\mathrm{CA}$? What about $\mathrm{medR}\Sigma_1^0$, $\mathrm{R}\Sigma_1^0$, or RT_2^1 in place of $\mathrm{stR}\Sigma_1^0$?

Open Question 4.9.16. Does $\mathrm{F}\Pi_1^0 \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{RT}_2^1$? What about a generalized Weihrauch reduction over RCA_0^*?

Open Question 4.9.17. Can we show that there is no generalized Weihrauch reduction from Bound (or equivalently, stBound*, Bound*, $F\Sigma^0_1$, $F\Delta^0_2$, or $L\Delta^0_2$) to $F\Pi^0_1$?

Open Question 4.9.18. Does stBound* $\leqslant_{gW}$ $B\Delta^0_2CA$? (Or equivalently, replace stBound* with Bound, Bound*, $F\Sigma^0_1$, $F\Delta^0_2$, or $L\Delta^0_2$)

Open Question 4.9.19. Can we show that $B\Delta^0_2CA \not\leqslant_W$ Bound*? Can we show that there is no generalized Weihrauch reduction? What about stBound*?

Open Question 4.9.20. Can we show that $B\Delta^0_2CA \not\leqslant_{gW}$ $\mathrm{medR}\Sigma^0_1$? Can we even show that $B\Delta^0_2CA \not\leqslant_W \mathrm{medR}\Sigma^0_1$? What about (generalized) Weihrauch reduction over RCA_0 or RCA^*_0? What about $\mathrm{stR}\Sigma^0_1$?

Open Question 4.9.21. Can we show that $B\Delta^0_2CA \not\leqslant^{RCA_0}_W R\Sigma^0_1$? What about a generalized Weihrauch reduction over RCA_0? What about a (generalized) Weihrauch reduction over RCA^*_0?

Open Question 4.9.22. Does $K_{\mathbb{N}} \leqslant^{RCA^*_0}_{gW} RT^1_2$?

Open Question 4.9.23. Can we show that $K\Delta^0_2 \not\leqslant_{gW} RT^1_2$?

Open Question 4.9.24. Does $B\Delta^0_2CA \leqslant_W K\Delta^0_2$? Is there even a generalized Weihrauch reduction?

Open Question 4.9.25. Does $RT^1_2 \leqslant_W K\Delta^0_2$? Is there even a generalized Weihrauch reduction? What happens over RCA_0, or RCA^*_0?

Open Question 4.9.26. Does $R\Sigma^0_1 \leqslant_W K\Delta^0_2$? Is there even a generalized Weihrauch reduction?

Open Question 4.9.27. Does stBound* $\leqslant_{gW} K\Delta^0_2$?

Open Question 4.9.28. Does $\mathrm{stR}\Sigma^0_1 \leqslant_{gW} K\Delta^0_2$?

Open Question 4.9.29. Does $\mathsf{F}\Pi^0_1 \leqslant_{\mathrm{gW}} \mathsf{K}_{\mathbb{N}}$, that is, are $\mathsf{F}\Pi^0_1$ and $\mathsf{K}_{\mathbb{N}}$ equivalent in terms of generalized Weihrauch reduction?

Open Question 4.9.30. We know that $\mathrm{B}\Delta^0_2\mathrm{CA} \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{F}\Sigma^0_1$. However, can we show there is no Weihrauch reduction, or no generalized Weihrauch reduction?

Open Question 4.9.31. Can we show that $\mathrm{F}\Sigma^0_1 \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}}$ Bound?

Open Question 4.9.32. We know that $\mathrm{RT}^1_2 \nleqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Pi^0_1$, but can we show that $\mathrm{RT}^1_2 \nleqslant^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{F}\Pi^0_1$? Can we at least show that there is no generalized Weihrauch reduction over RCA^*_0?

Open Question 4.9.33. Can we show that $\mathrm{Bound}^* \leqslant^{\mathrm{RCA}^*_0}_{\mathrm{W}} \mathrm{medR}\Sigma^0_1$? Can we show a generalized Weihrauch reduction over RCA^*_0?

Open Question 4.9.34. Does $\mathrm{B}\Delta^0_2\mathrm{CA} \leqslant_{\mathrm{W}} \mathrm{F}\Delta^0_2$, or even $\mathrm{L}\Delta^0_2$? Can we show that there is a generalized Weihrauch reduction? What about a (generalized) Weihrauch reduction over RCA_0, or RCA^*_0?

Open Question 4.9.35. We know that $\mathrm{K}_{\mathbb{N}} \leqslant_{\mathrm{W}} \mathrm{F}\Pi^0_1$, but is there a (generalized) Weihrauch reduction over RCA^*_0?

Open Question 4.9.36. Can we show that $\lim_{\mathbb{N}} \nleqslant_{\mathrm{gW}} \mathrm{F}\Sigma^0_1$ (or equivalently, that there is no gW-reduction from $\lim_{\mathbb{N}}$ to $\mathrm{stBound}^*$, Bound^*, Bound, $\mathrm{F}\Delta^0_2$, or $\mathrm{L}\Delta^0_2$)?

Open Question 4.9.37. Does $\mathrm{F}\Sigma^0_1 \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction over RCA_0?

Open Question 4.9.38. Does $\lim_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Sigma^0_1$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA^*_0?

Open Question 4.9.39. Is there a Weihrauch reduction, or even a generalized Weihrauch reduction, from $\lim_{\mathbb{N}}$ to $\mathrm{F}\Pi^0_1$? Does $\lim_{\mathbb{N}} \leqslant^{\mathrm{RCA}_0}_{\mathrm{W}} \mathrm{F}\Pi^0_1$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA^*_0?

Open Question 4.9.40. Does $F\Pi^0_1 \leqslant_W \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction? What happens over RCA_0?

Open Question 4.9.41. Does $\lim_{\mathbb{N}} \leqslant_W^{RCA_0}$ Bound? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA_0^*?

Open Question 4.9.42. Can we show that $\lim_{\mathbb{N}} \nleqslant_W^{RCA_0}$ Bound*? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA_0^*? What happens with stBound*?

Open Question 4.9.43. By Theorem 4.6.8, we know that $\lim_{\mathbb{N}} \nleqslant_W B\Delta^0_2CA$ and $\lim_{\mathbb{N}} \nleqslant_{gW}^{RCA_0} B\Delta^0_2CA$. Does $\lim_{\mathbb{N}} \leqslant_{gW} B\Delta^0_2CA$?

Open Question 4.9.44. By Theorem 4.6.8, we know that $\lim_{\mathbb{N}} \nleqslant_W K\Delta^0_2$ and $\lim_{\mathbb{N}} \nleqslant_{gW}^{RCA_0} K\Delta^0_2$. Does $\lim_{\mathbb{N}} \leqslant_{gW} K\Delta^0_2$? What about $K_{\mathbb{N}}$?

Open Question 4.9.45. Does $K_{\mathbb{N}} \leqslant_W^{RCA_0} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA_0^*?

Open Question 4.9.46. Does $R\Sigma^0_1 \leqslant_{gW} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction over RCA_0 or RCA_0^*? What about RT^1_2?

Open Question 4.9.47. Does stBound* $\leqslant_W^{RCA_0} stR\Sigma^0_1$? Is there a (generalized) Weihrauch reduction over RCA_0^*?

Open Question 4.9.48. Does $stR\Sigma^0_1 \leqslant_W^{RCA_0}$ stBound*? Is there a generalized Weihrauch reduction over RCA_0^*? What about Bound* or $C_{\mathbb{N}}^*$?

Open Question 4.9.49. Does $\lim_{\mathbb{N}} \leqslant_W^{RCA_0} L\Delta^0_2$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA_0^*?

Open Question 4.9.50. Does $L\Delta^0_2 \leqslant_W^{RCA_0} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA_0^*?

Open Question 4.9.51. Does $\lim_{\mathbb{N}} \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \mathrm{F}\Delta^0_2$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA^*_0?

Open Question 4.9.52. Does $\mathrm{F}\Delta^0_2 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction over RCA_0? What happens over RCA^*_0?

Open Question 4.9.53. Does $\mathrm{K}\Delta^0_2 \leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction? What happens over RCA_0, or over RCA^*_0?

Open Question 4.9.54. Does $\lim_{\mathbb{N}} \leqslant_{\mathrm{W}} \mathrm{stR}\Sigma^0_1$? Is there a generalized Weihrauch reduction? What happens over RCA_0, or over RCA^*_0? What about $\mathrm{medR}\Sigma^0_1$?

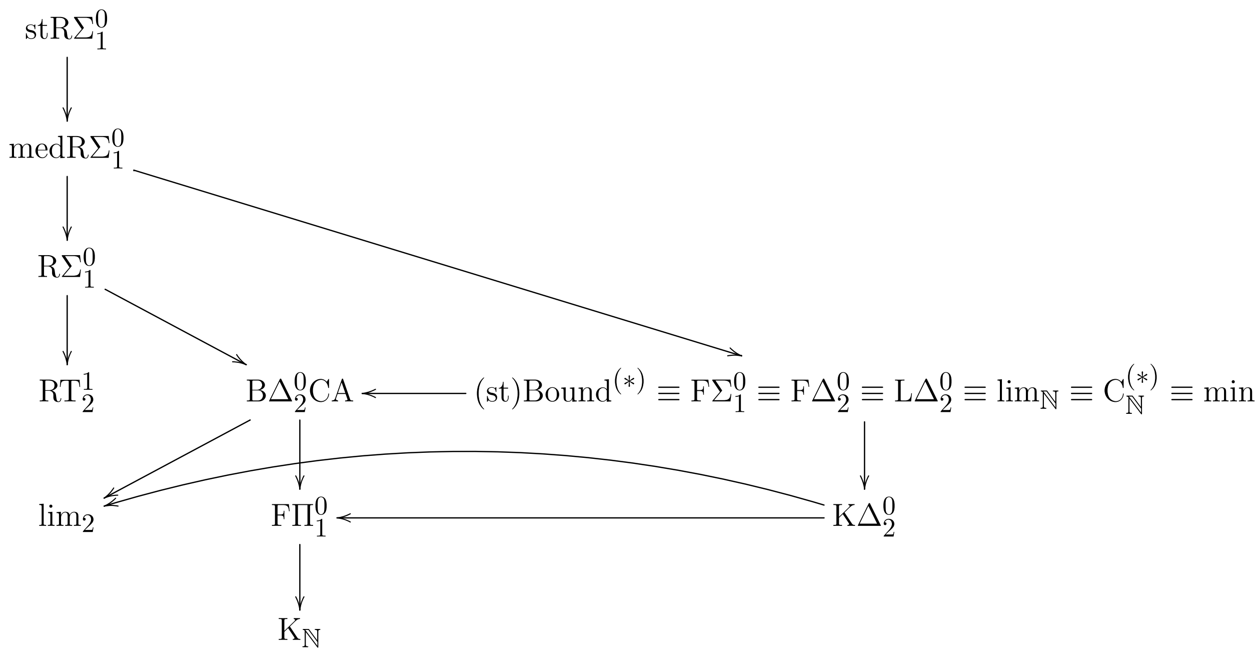
Open Question 4.9.55. Does $\mathrm{stR}\Sigma^0_1 \leqslant_{\mathrm{W}} \lim_{\mathbb{N}}$? Is there a generalized Weihrauch reduction? What about $\mathrm{medR}\Sigma^0_1$?

Open Question 4.9.56. By Theorem 4.6.8, we know that $\lim_{\mathbb{N}} \not\leqslant_{\mathrm{W}} \mathrm{R}\Sigma^0_1$ and $\lim_{\mathbb{N}} \not\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} \mathrm{R}\Sigma^0_1$. Does $\lim_{\mathbb{N}} \leqslant_{\mathrm{gW}} \mathrm{R}\Sigma^0_1$? What about RT^1_2?

This dissertation is by no means exhaustive. There are many other principles that we could have considered, many of which are mentioned in [12], including:

- $\mathrm{BWT}_{\mathbb{N}}$, the Bolzano Weierstrass Theorem on $\mathbb{N}$, which can be seen as a counterpart of $\mathrm{R}\Sigma^0_1$;
- C_2, choice on $\{0, 1\}$;
- C^*_2, the finite parallelization of choice on $\{0, 1\}$, which is equivalent in terms of strong Weihrauch reduction to $\mathrm{K}_{\mathbb{N}}$;
- $\mathrm{K}'_{\mathbb{N}}$, the jump of $\mathrm{K}_{\mathbb{N}}$;
- $\mathrm{RT}_{1,\mathbb{N}}$, which is a way of expressing $\mathrm{RT}^1_{<\infty}$ as a Π^1_2 problem, where the number of colors is left unspecified, which is equivalent to $\mathrm{K}'_{\mathbb{N}}$ in terms of Weihrauch reduction;

Figure 4.1: Here we show the known relationships between the principles with respect to Weihrauch reducibility $\leqslant_W$.



- $\mathrm{SRT}_{1,\mathbb{N}}$, which is a way of expressing stable Ramsey's Theorem $\mathrm{SRT}^1_{<\infty}$ as a Π^1_2 problem, where the number of colors is left unspecified, which is equivalent to $\mathrm{C}_{\mathbb{N}}$ in terms of Weihrauch reduction.

Note that in the below diagrams, we show implications between the principles with respect to various reducibilities. However, we do not make a distinction between known non-implications and cases where the implication is unknown.

Figure 4.2: Here we show the known relationships between the principles with respect to generalized Weihrauch reducibility $\leqslant_{\mathrm{gW}}$.

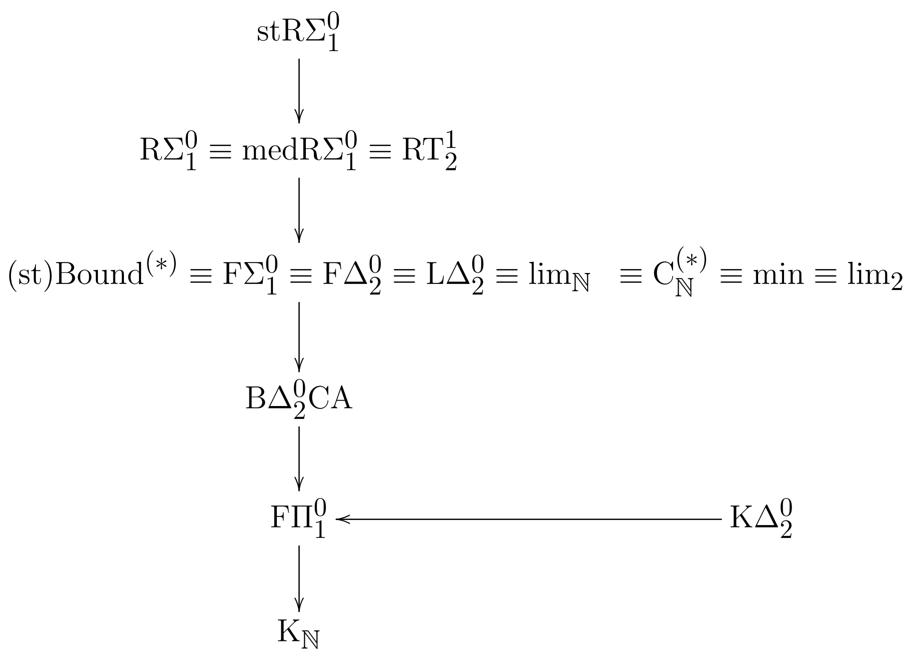


Figure 4.3: Here we show the known relationships between the principles with respect to Weihrauch reducibility over $\mathrm{RCA}_0 \leqslant_{\mathrm{W}}^{\mathrm{RCA}_0}$.

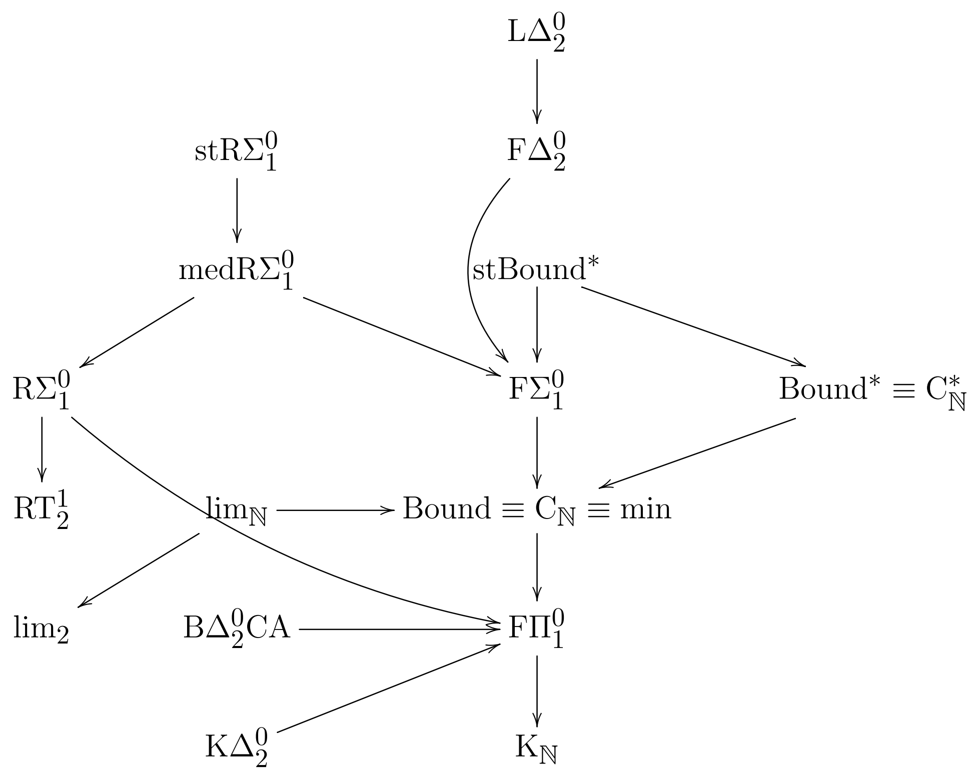


Figure 4.4: Here we show the known relationships between the principles with respect to generalized Weihrauch reducibility over RCA_0 $\leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0}$.

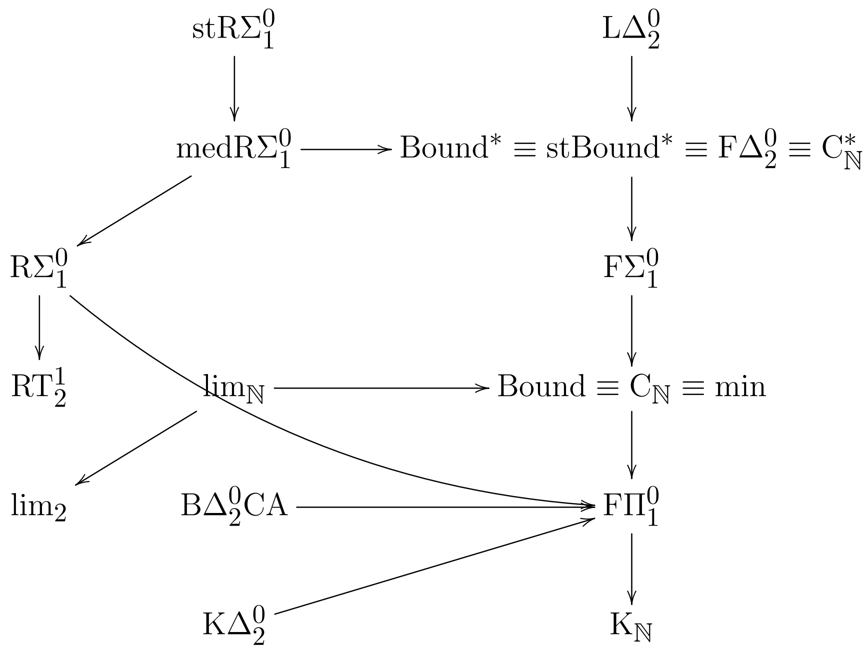


Figure 4.5: Here we show the known relationships between the principles with respect to Weihrauch reducibility over RCA_0^* $\leqslant_{\mathrm{W}}^{\mathrm{RCA}_0^*}$.

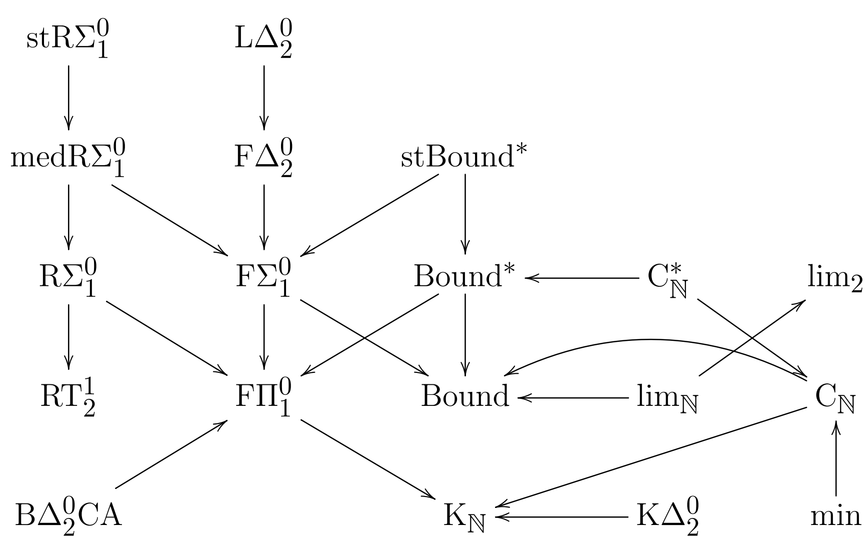
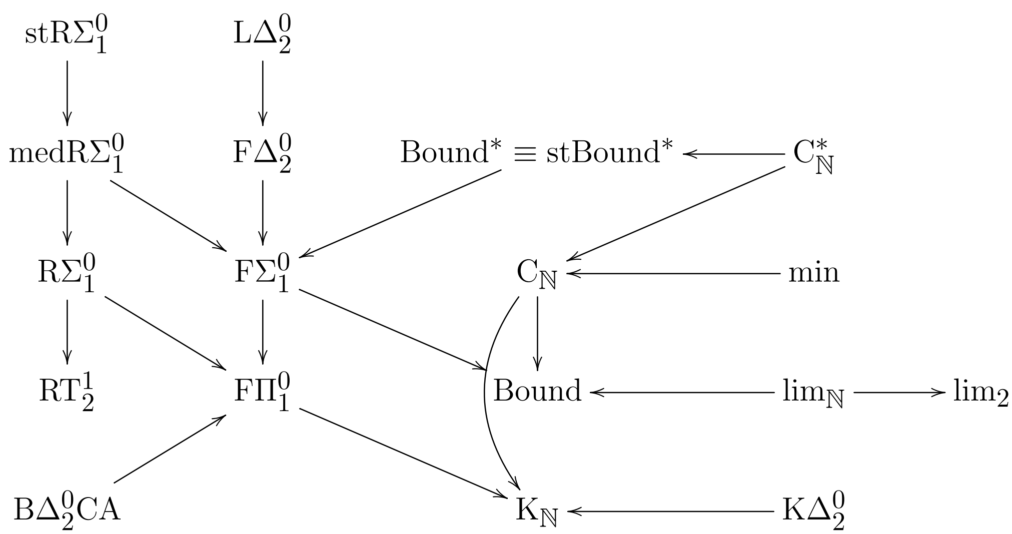


Figure 4.6: Here we show the known relationships between the principles with respect to generalized Weihrauch reducibility over $\mathrm{RCA}_0^* \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0^*}$.



REFERENCES

[1] A. Bauer, *Reductions in computability theory from a constructive point of view*, 2014.

[2] A. Bauer, *Instance reducibility and Weihrauch degrees*, 2021.

[3] A. Bauer, *Instance reducibility and Weihrauch degrees*. `https://www.youtube.com/watch?v=_CoDNlF-zoI`, 2021. Recording of a talk at the Seminar for Foundations of Mathematics and Theoretical Computer Science, University of Ljubljana.

[4] A. Bauer and K. Yoshimura, *The Weihrauch lattice is too small*, in Computability and Complexity in Analysis 2014, 2014.

[5] ——, *Instance reducibility and extended Weihrauch degrees*, Abstracts from Computability, Continuity, Constructivity 2019 - From Logic to Algorithms, (2019), pp. 10–12.

[6] A. R. Blass, J. L. Hirst, and S. G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, Contemporary Math, 65 (1987), pp. 125–156.

[7] V. Brattka, M. de Brecht, and A. Pauly, *Closed choice and a uniform low basis theorem*, Annals of Pure and Applied Logic, 163 (2012), p. 986–1008.

[8] V. Brattka, G. Gherardi, R. Hölzl, H. Nobrega, and A. Pauly, *Borel choice*. in preparation.

[9] V. Brattka, G. Gherardi, and A. Marcone, *The Bolzano–Weierstrass theorem is the jump of weak Kőnig's lemma*, Annals of Pure and Applied Logic, 163 (2012), p. 623–655.

[10] V. Brattka, G. Gherardi, and A. Pauly, *Weihrauch complexity in computable analysis*, in Theory and Applications of Computability, Springer International Publishing, 2021, pp. 367–417.

[11] V. Brattka and A. Pauly, *On the algebraic structure of Weihrauch degrees*, Logical Methods in Computer Science, 14 (2018).

[12] V. BRATTKA and T. RAKOTONIAINA, *On the uniform computational content of Ramsey's theorem*, The Journal of Symbolic Logic, 82 (2017), p. 1278–1316.

[13] L. Carlucci, *Restrictions on Hindman's theorem: an overview*, in Connecting with Computability: CIE 2021, L. D. Mol, A. Weiermann, F. Manea, and D. Fernández-Duque, eds., vol. 12813 of Lecture Notes in Computer Science, Springer, 2021, pp. 94–105.

[14] L. Carlucci, L. A. Kołodzieczyk, F. Lepore, and K. Zdanowski, *New bounds on the strength of some restrictions of Hindman's theorem*, Computability, 9 (2020), pp. 139–153.

[15] C. C. Chang and H. J. Keisler, *Model Theory*, Dover Publications, Inc., Mineola, New York, 2012.

[16] P. A. Cholak, C. G. Jockusch, and T. A. Slaman, *On the strength of Ramsey's theorem for pairs*, The Journal of Symbolic Logic, 66 (2001), pp. 1–55.

[17] C. T. Chong, S. Lempp, and Y. Yang, *On the role of collection principles for σ^0_2 formulas in second-order reverse mathematics*, Proceedings of the American Mathematical Society, 138 (2010), pp. 1093–1100.

[18] C. T. Chong, T. A. Slaman, and Y. Yang, *The metamathematics of stable Ramsey's theorem for pairs*, Journal of the American Mathematical Society, 27 (2014), pp. 863–892.

[19] C. Conidis, *Classifying model-theoretic properties*, Journal of Symbolic Logic, 73 (2008), pp. 885–905.

[20] B. F. Csima, D. D. Dzhafarov, D. R. Hirschfeldt, C. G. Jockusch, Jr., R. Solomon, and L. Brown Westrick, *The reverse mathematics of Hindman's Theorem for sums of exactly two elements*, Computability, 8 (2020), pp. 253–263.

[21] F. G. Dorais, D. D. Dzhafarov, J. L. Hirst, J. R. Mileti, and P. Shafer, *On uniform relationships between combinatorial problems*, Transactions of the American Mathematical Society, 368 (2016), pp. 1321–1359.

[22] D. D. Dzhafarov, *Cohesive avoidance and strong reductions*, Proceedings of the American Mathematical Society, 143 (2015), pp. 869–876.

[23] D. D. Dzhafarov, *Strong reductions between combinatorial principles*, Journal of Symbolic Logic, 81 (2016), pp. 1405–1431.

[24] D. D. Dzhafarov, J. L. Goh, D. R. Hirschfeldt, L. Patey, and A. Pauly, *Ramsey's theorem and products in the Weihrauch degrees*, Computability, 9 (2020), pp. 85–100.

[25] D. D. Dzhafarov, D. R. Hirschfeldt, and S. C. Reitzes, *Reduction games, provability, and compactness*, Journal of Mathematical Logic, to appear.

[26] D. D. Dzhafarov, C. G. Jockusch, R. Solomon, and L. B. Westrick, *Effectiveness of Hindman's theorem for bounded sums*, in Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday, A. Day, M. Fellows, N. Greenberg, B. Khoussainov, A. Melnikov, and F. Rosamond, eds., Springer International Publishing, Cham, 2017, pp. 134–142.

[27] H. B. Enderton, *A Mathematical Introduction to Logic*, Harcourt/Academic Press, Burlington, MA, 2001.

[28] E. Frittaion and A. Marcone, *Linear extensions of partial orders and reverse mathematics*, Mathematical Logic Quarterly, 58 (2012), p. 417–423.

[29] P. HÁJEK, *Interpretability and fragments of arithmetic*, in Arithmetic, proof theory, and computational complexity, vol. 23 of Oxford Logic Guides, Oxford Science Publications, 1991, pp. 185–196.

[30] P. HÁJEK AND P. PUDLÁK, *Metamathematics of first-order arithmetic*, in Perspectives in Mathematical Logic, Spring-Verlag, Berlin, 1993.

[31] N. HINDMAN, *Finite sums from sequences within cells of partitions of N*, Journal of Combinatorial Theory, Series A, 17 (1974), pp. 1–11.

[32] N. HINDMAN, I. LEADER, AND D. STRAUSS, *Open problems in partition regularity*, Combinatorics, Probability & Computing, 12 (2003), pp. 571–583.

[33] D. R. HIRSCHFELDT, *Slicing the truth: On the computable and reverse mathematics of combinatorial principles*, in Lecture Note Series, Institute for Mathematical Sciences, University of Singapore, vol. 28, World Scientific, Singapore, 2014.

[34] D. R. HIRSCHFELDT AND C. G. JOCKUSCH, JR., *On notions of computability-theoretic reduction between π^1_2 principles*, Journal of Mathematical Logic, 16 (2016), p. 69.

[35] D. R. HIRSCHFELDT, K. LANGE, AND R. A. SHORE, *Induction, bounding, weak combinatorial principles, and the homogeneous model theorem*, Memoirs of the American Mathematical Society, 249 (2017), p. 101 pp.

[36] D. R. HIRSCHFELDT AND S. C. REITZES, *Thin set versions of Hindman's theorem.*

[37] R. A. HIRSCHFELDT, DENIS R. SHORE AND T. A. SLAMAN, *The atomic model theorem and type omitting*, Transactions of the American Mathematical Society, 361 (2009), pp. 5805–5837.

[38] J. L. HIRST, *Combinatorics in Subsystems of Second Order Artithmetic*, PhD thesis, 1987.

[39] J. L. Hirst and C. Mummert, *Using Ramsey's theorem once*, Archive for Mathematical Logic, 58 (2019), pp. 857–866.

[40] C. G. Jockusch, *Ramsey's theorem and recursion theory*, The Journal of Symbolic Logic, 37 (1972), pp. 268–280.

[41] ——, *Degrees of functions with no fixed points*, in Logic, Methodology and Philosophy of Science VIII, J. E. Fenstad, I. T. Frolov, and R. Hilpinen, eds., vol. 126 of Studies in Logic and the Foundations of Mathematics, Elsevier, 1989, pp. 191–201.

[42] R. Kossak, *On extensions of models of strong fragments of arithmetic*, in Proceedings of the American Mathematical Society 108, 1990, pp. 223–232.

[43] R. Kuyper, *On Weihrauch reducibility and intuitionistic reverse mathematics*, Journal of Symbolic Logic, to appear.

[44] J. Liu, RT^2_2 *does not imply* WKL_0, The Journal of Symbolic Logic, 77 (2012), pp. 609–620.

[45] L. Liu, *Cone avoiding closed sets*, Transactions of the American Mathematical Society, 367 (2015), pp. 1609–1630.

[46] B. Monin and L. Patey, SRT^2_2 *does not imply* RT^2_2 *in omega-models*, Advances in Mathematics, to appear.

[47] A. Montalbán and R. A. Shore, *Conservativity of ultrafilters over subsystems of second order arithmetic*, Journal of Symbolic Logic, 83 (2018), pp. 740–765.

[48] E. Neumann and A. Pauly, *A topological view on algebraic computation models*, Journal of Complexity, 44 (2018), pp. 1–22.

[49] J. B. Paris and L. Kirby, σ_n*-collection schemas in arithmetic*, in Proceedings of Logic Colloquium, 1977.

[50] L. Patey, *On combinatorial weaknesses of Ramseyan principles.* https://ludovicpatey.com/media/research/combinatorial-weaknesses-draft.pdf.

[51] L. Patey, *Somewhere over the rainbow Ramsey theorem for pairs*, 2015.

[52] L. Patey, *The Reverse Mathematics of Ramsey-Type Theorems*, PhD thesis, 2016.

[53] L. Patey, *The weakness of being cohesive, thin or free in reverse mathematics*, Israel Journal of Mathematics, 216 (2016), pp. 905–955.

[54] A. Pauly, *Methoden zum vergleich der unstetigkeit von funktionen*, 2007.

[55] A. Pauly, G. Davie, and W. L. Fouché, *Weihrauch-completeness for layerwise computability*, Logical Methods in Computer Science, 14 (2018).

[56] A. Rumyantsev, *Infinite computable version of Lovász local lemma*, 2010.

[57] A. Rumyantsev and A. Shen, *Probabilistic constructions of computable objects and a computable version of Lovász local lemma*, Fundamenta Informaticae, 132 (2014), pp. 1–14.

[58] D. Seetapun and T. A. Slaman, *On the strength of Ramsey's theorem*, Notre Dame Journal of Formal Logic, 36 (1995), pp. 570–582.

[59] S. G. Simpson, *Subsystems of Second Order Aritmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, first ed., 1999.

[60] ——, *Subsystems of Second Order Arithmetic*, Perspectives in Logic, Cambridge University Press, Cambridge and Association for Symbolic Logic, Poughkeepsie, NY, second ed., 2009.

[61] T. A. Slaman, *σ_n-bounding and δ_n-induction*, Proceedings of the American Mathematical Society, 132 (2004), pp. 2449–2456.

[62] R. I. Soare, *Turing Computability: Theory and Applications*, Spring-Verlag, Berlin, 2016.

[63] P. UFTRING, *The characterization of WEIHRAUCH reducibility in systems containing E-PA$^\omega$ + QF-AC^{0,0}*, The Journal of Symbolic Logic, 86 (2020), p. 224–261.

[64] H. Wang, *Popular Lectures on Mathematical Logic*, Dover Publications, Inc., New York, revised reprint of the 1981 second edition ed., 1993.

[65] W. Wang, *Some logically weak Ramseyan theorems*, Advances in Mathematics, 261 (2014), p. 1–25.

[66] L. Westrick, *A note on the diamond operator*, Computability, 10 (2021), pp. 107–110.