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ZERO CYCLES ON ABELIAN VARIETIES, SOMEKAWA K-GROUPS AND LOCAL SYMBOLS

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To my parents with love and great admiration

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ABSTRACT

This thesis consists of two main parts. The first part concerns zero cycles on abelian varieties and their relation to some Milnor type K-groups. In chapter 1 we recall some basic properties of Milnor K-groups and their generalizations, the Somekawa K-groups. The main result of the first part is presented in chapter 2, where we construct, for an abelian variety A over a field k, a decreasing filtration $\{F^r\}_{r\geq 0}$ of the group $CH_0(A)$ having the property that the successive quotients F^r/F^{r+1} are isomorphic after $\otimes\mathbb{Z}[\frac{1}{r!}]$ to a Somekawa type K-group. We then focus on the case when the base field is a finite extension of $\mathbb{Q}_p$. Using the above filtration, we prove some results of arithmetic interest about the structure of the albanese kernel, the kernel of the cycle map to étale cohomology and the Brauer-Manin pairing. The results of this chapter are gathered in one paper, [14].

Chapter 3 serves as a bridge between the first and the second part of this thesis. In this chapter we work with smooth quasi-projective varieties, introducing Suslin's singular homology group and Wiesend's tame class group. The latter group is a first generalization in higher dimensions of the generalized Jacobian varieties of a smooth projective curve. Using these two geometric invariants, we generalize the main theorem of chapter 2 for semi-abelian varieties. We close the chapter by providing some motivation towards a more general reciprocity theory.

The second part concerns a newly developed theory about reciprocity functors introduced by Ivorra and Rülling in [20]. This theory generalizes the theory of Rosenlicht-Serre about local symbols on commutative algebraic groups. In particular, we will see that every reciprocity functor $\mathcal{M}$ has local symbols corresponding to any smooth complete curve C over a field k. These local symbols induce a complex $(\underline{\underline{C}})$. In chapter 4 we focus on the case of a smooth complete curve C over an algebraically closed field k and we compute under two assumptions the homology of the local symbol complex in terms of K-groups of reciprocity functors. We then close the thesis by providing important examples where the assumptions are satisfied. The results of this chapter are gathered in one paper, [15].

CHAPTER 1

INTRODUCTION-BACKGROUND

1.1 Zero cycles

1.1.1 Definition of CH_0 and first properties

Let X be a smooth variety over a field k. The group of zero cycles $Z_0(X)$ is the free abelian group $\bigoplus_{x\in X}\mathbb{Z}$ on all closed points of X. My main object of study is a quotient of this group, namely the group $CH_0(X)$ of zero cycles modulo rational equivalence. In particular, $CH_0(X)$ is the quotient of $Z_0(X)$ modulo the subgroup generated by divisors of functions, $\operatorname{div}(f)$, where f is a function in the function field of some closed integral curve $C \hookrightarrow X$. The group $CH_0(X)$ is a fundamental geometric invariant of the variety with many remarkable properties. Here we mention only a few.

1. The group $CH_0(X)$ is covariant functorial with respect to proper maps and contravariant functorial with respect to finite flat maps.

2. In particular, if X is smooth and proper variety over k, the push forward of the structure morphism $X \to \operatorname{Spec} k$ induces a degree map $\deg : CH_0(X) \to \mathbb{Z}$ that sends the class $[x]$ of a closed point $x \in X$ to the degree of the residue extension $[k(x) : k]$. We denote by $A_0(X)$ the subgroup of degree zero cycles.

3. The group $CH_0(X)$ is a birational invariant and therefore it provides a tool to show that certain varieties are not isomorphic to the projective space.

4. When X is a smooth complete curve over k, the group $CH_0(X)$ coincides with the Picard group, $\operatorname{Pic}(X)$, of isomorphism classes of line bundles on X. When X has a k-rational point, the celebrated Abel-Jacobi map gives an isomorphism $\operatorname{Pic}(X)^0 \xrightarrow{\simeq} J_X(k)$ between the degree zero subgroup of $\operatorname{Pic}(X)$ and the k-rational points of the

Jacobian of X. In particular, we see that the group $CH_0(X) = \mathrm{Pic}(X)$ recovers the genus.

1.1.2 The Albanese map

There is a higher dimensional analogue of the Abel-Jacobi map. Let X be a smooth projective variety over k having a k rational point P. There is a unique abelian variety Alb_X, called the Albanese variety of X, and a unique morphism $\varphi : X \to Alb_X$ taking P to the zero element of Alb_X, satisfying the following universal property. If $f : X \to B$ is a morphism from X to an abelian variety B taking P to the zero element of B, then f factors uniquely through φ. The map φ induces a group homomorphism $\mathrm{alb}_X : A_0(X) \to Alb_X(k)$, not depending on the k-rational point P, called the Albanese map of X.

In higher dimensions the Albanese map is not always injective. The first notable example, constructed by Mumford [29], is a smooth projective surface with positive geometric genus. We will denote $T(X) := \ker(\mathrm{alb}_X)$. Determining the structure of $T(X)$ is a very interesting but rather hard problem that is the subject of many famous conjectures. For example when k is an algebraic number field, the Bloch-Beilinson conjectures predict that $T(X)$ should be finite. On the other hand when $k = \mathbb{C}$ or a p-adic field, $T(X)$ is expected to be enormous in some cases, for example when X is a surface with $p_g(X) > 0$. For a more detailed description of the properties of the group $CH_0(X)$ we refer to [5] for the case of an algebraically closed field and to the following two surveys of Colliot-Thélène, [9] and [10], for arithmetic fields.

We note that $T(X)$ captures almost all the difficulty in determining the structure of $CH_0(X)$. When X has a k-rational point, we obtain a filtration

$$CH_0(X) \supset A_0(X) \supset T(X)$$

with the first two quotients well understood. In most cases it is not at all easy to even find generators for $T(X)$, not to mention relations. One of the methods used in order to approach some of the big conjectures mentioned above, is to extend the filtration $CH_0(X) \supset A_0(X) \supset T(X) \supset F^3 \supset \cdots$ such that the successive quotients have specific generators and relations. In order to illustrate the nature of this method, we give the following example.

Example 1.1.1. Let $X = C_1 \times C_2$ be the product of two smooth projective curves over a field k. The Albanese variety of X is just the product of the two Jacobians, $Alb_X = J_1 \times J_2$. Let $x_0, x_1 \in C_1(k)$ be fixed k-rational points and similarly $y_0, y_1 \in C_2(k)$. It is a simple computation to verify that

$$[(x_1, y_1)] - [(x_1, y_0)] - [(x_0, y_1)] + [(x_0, y_0)] \in T(X).$$

In this case we can therefore describe distinguished generators of $T(X)$. The question that arises though is whether this zero cycle is rational or not. An interesting approach was considered by W. Raskind and M. Spiess in [31]. In this paper they prove that the Albanese kernel is isomorphic to some Milnor-type K-group, namely the Somekawa K-group, $K(k; J_1, J_2)$ attached to the Jacobians J_1, J_2. The advantage of their method is that this K-group has specific generators and relations. They in fact show something much stronger. For $X = C_1 \times \cdots \times C_r$ any product of smooth complete curves, they construct a finite decreasing filtration $\{F^r\}_{r \geq 0}$ of $CH_0(X)$ with all the successive quotients isomorphic to some Somekawa K-group.

The first project of the current thesis is highly inspired by the method introduced by Raskind and Spiess. In chapter 2 we will construct a decreasing filtration $\{F^r\}_{r \geq 0}$ for the group $CH_0(A)$, where A is an abelian variety over a field k. This filtration will satisfy a similar property as the one constructed by Raskind and Spiess.

In the following section we introduce the Milnor K-groups, $K_r^M(k)$, of a field k and their generalization, the Somekawa K-groups.

1.2 Somekawa K-groups

1.2.1 The Milnor K-groups of a field

Let k be a field. For $r \geq 0$, the Milnor K-groups, $K_r^M(k)$, are defined as follows.

$$\begin{aligned}
&K_0^M(k) = \mathbb{Z} \\
&K_1^M(k) = k^\times \\
&K_2^M(k) = \frac{k^\times \otimes k^\times}{< x \otimes (1-x), x \neq 1 >} \\
&\cdots \\
&K_r^M(k) = \frac{\overbrace{k^\times \otimes \cdots \otimes k^\times}^{r}}{< x_1 \otimes \cdots \otimes x_r,\ x_i + x_j = 1,\ 1 \leq i < j \leq r >}
\end{aligned}$$

Notation 1.2.1. The elements of $K_r^M(k)$ are traditionally denoted as symbols. In particular, if $x_i \in k^\times$ for $1 \leq i \leq r$, the class of $x_1 \otimes \cdots \otimes x_r$ in $K_r^M(k)$ is denoted by $\{x_1, \cdots, x_r\}$.

The Milnor K-groups satisfy two important properties.

1. If L/K is a finite field extension, then there is a norm map $N_{L/K} : K_r^M(L) \to K_r^M(K)$ which satisfies the following property. If $E \supset L \supset K$ is a tower of finite extensions and we have elements $x_i \in E^\times$ for some $i \in \{1, \cdots, r\}$ and $x_j \in L^\times$ for every $i \neq j$, then

$$\{x_1, \cdots, N_{E/L}(x_i), \cdots, x_r\} = N_{E/L}(\{\mathrm{res}_{E/L}(x_1), \cdots, x_i, \cdots, \mathrm{res}_{E/L}(x_r)\}),$$

where $N_{E/L} : E^\times \to L^\times$ is the usual norm map of fields and $\mathrm{res}_{E/L} : L^\times \hookrightarrow E^\times$ is the natural restriction. This property is usually referred in the literature as Projection Formula.

2. If C is a smooth complete curve over k, then there is a complex

$$K_{r+1}^M(k(C)) \xrightarrow{\partial} \bigoplus_{x \in C} K_r^M(k(x)) \xrightarrow{\sum_{x \in C} N_{k(x)/k}} K_r^M(k).$$

This second property is usually referred in the literature as Weil reciprocity.

Remark 1.2.2. The map ∂ is called the tame symbol. We will not need its precise definition, but in order to justify the name Weil reciprocity, we define it for $\{f, g\} \in K_2^M(k(C))$, where $f, g \in k(C)^\times$. In this case, the tame symbol ∂_x at a closed point $x \in C$ is defined as

$$\partial_x(\{f, g\}) = ((-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} f^{\mathrm{ord}_x(g)} g^{-\mathrm{ord}_x(f)})(x)$$

Note that the function $h = (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} f^{\mathrm{ord}_x(g)} g^{-\mathrm{ord}_x(f)} \in k(C)^\times$ has the property $\mathrm{ord}_x(h) = 0$, so we can evaluate it at the point x.

Kato and his student Somekawa in [37] made the remarkable observation that the projection formula and the Weil reciprocity "determine" the Milnor K-group. This will become more specific in section 1.2.3. Before that, we need to briefly review some facts about Mackey functors.

1.2.2 Projection Formula-Product of Mackey functors

In this section we will discuss more about the projection formula. A Mackey functor $\mathcal{M}$ is a covariant functor from the category of finitely generated field extensions of k to the category of abelian groups such that for every finite extension L/k there is also a push-forward map $\mathcal{M}(L) \to \mathcal{M}(k)$, satisfying certain functoriality properties. For a precise definition see for example (3.2) in [31].

Let G be a commutative algebraic group over k. Then G induces a Mackey functor. In particular if L/k is a finite extension, then there is a natural restriction map $\mathrm{res}_{L/k} : G(k) \hookrightarrow G(L)$ as well as a trace map $\mathrm{Tr}_{L/k} : G(L) \to G(k)$. We note that if the group law in G is written in multiplicative notation, the trace is replaced with a norm $N_{L/k} : G(L) \to G(k)$. The Mackey functors form an abelian category with a tensor product defined by Kahn [21]. We review the definition of the product here.

Definition 1.2.3. Let $\mathcal{M}_1, \cdots, \mathcal{M}_r$ be Mackey functors over k. Let L be a finitely generated extension of k. Then,

$$(\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_r)(L) := (\bigoplus_{L'/L} \mathcal{M}_1(L') \bigotimes \cdots \bigotimes \mathcal{M}_r(L'))/R,$$

where the sum is extended over all finite extensions L' of L and R is the subgroup generated by the following family of elements:

If $L \subset K \subset E$ is a tower of finite field extensions and we have elements $x_i \in \mathcal{M}_i(E)$ for some $i \in \{1, \cdots, r\}$ and $x_j \in \mathcal{M}_j(K)$, for every $j \neq i$, then

$$x_1 \otimes \cdots \otimes \mathrm{Tr}_{E/K}(x_i) \otimes \cdots \otimes x_r - \mathrm{res}_{E/K}(x_1) \otimes \cdots \otimes x_i \otimes \cdots \otimes \mathrm{res}_{E/K}(x_r) \in R$$

Remark 1.2.4. We note here that for a finite field extension L/k and for a commutative algebraic group G, the trace map $\mathrm{Tr}_{L/k}$ is really a trace in the usual sense. For example if L/k is a Galois extension with Galois group G_k, then the trace map $\mathrm{Tr}_{L/k} : G(L) \to G(k)$ is defined as $\mathrm{Tr}_{L/k}(x) = \sum_{g \in G_k} gx$.

Notation 1.2.5. Let $\mathcal{M}_1, \cdots, \mathcal{M}_r$ be Mackey functors. The generators of the product $(\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_r)(k)$ will be denoted as $(m_1, \cdots, m_r)_{L/k}$, where $m_i \in \mathcal{M}_i(L)$.

Since $\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_r$ is again a Mackey functor, for a finite extension L/k there is a trace map, $\mathrm{Tr}_{L/k}$ defined as follows.

$$\begin{aligned} \mathrm{Tr}_{L/k} : \quad & (M_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} M_n)(L) \to (M_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} M_n)(k) \\ & (m_1, \cdots, m_n)_{E/L} \to (m_1, \cdots, m_n)_{E/k}. \end{aligned}$$

Moreover, if $j : k \hookrightarrow L$ is any field extension, then to define the restriction map

$$\operatorname{res}_{L/k}(M_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} M_n)(k) \to (M_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} M_n)(L),$$

we consider a generator $(a_1, \cdots, a_n)_{k'/k}$ of $(M_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} M_n)(k)$, where k'/k is a finite extension. We can write $k' \otimes L = \prod_{i=1}^{m} B_i$, where B_i are Artin local rings over L of length e_i, for $i = 1, \cdots, m$. The residue field L_i of B_i is a finite extension of L, for $i = 1, \cdots, m$ and an extension of k'. We define:

$$\operatorname{res}_{L/k}((a_1, \cdots, a_n)_{k'/k}) = \sum_{i=1}^{m} e_i(\operatorname{res}_{L_i/k'}(a_1), \cdots, \operatorname{res}_{L_i/k'}(a_n))_{L_i/L}.$$

Notice that if L/k is a finite extension of k and $(a_1, \cdots, a_n)_{k'/k} \in (M_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} M_n)(k)$, then we have:

$$\operatorname{Tr}_{L/k}(\operatorname{res}_{L/k}((a_1, \cdots, a_n)_{k'/k})) = [L : k](a_1, \cdots, a_n)_{k'/k}.$$

Remark 1.2.6. Using this new set-up of Mackey functors, we can restate the projection formula of the Milnor K-group $K_r^M(k)$ as follows. For every $r \geq 0$ we have a surjection

$$\overbrace{(\mathbb{G}_m \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathbb{G}_m)}^{r}(k) \twoheadrightarrow K_r^M(k).$$

1.2.3 The Somekawa K-groups

In this section we give an alternative description of the Milnor K-groups $K_r^M(k)$. This description will give rise to generalizations, namely the Somekawa K-groups. For $r \geq 0$ we define a new K-group $K'_r(k)$ as follows.

Definition 1.2.7. We define $K'_r(k) := (\overbrace{\mathbb{G}_m \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathbb{G}_m}^{r})(k)/R$ where R is generated by the following family of elements. If C is a smooth complete curve over k and we have functions $g_0, g_1, \cdots, g_r \in k(C)^\times$ with disjoint supports, then we require

$$\sum_{x \in C} N_{k(x)/k}(\partial'_x(g_0 \otimes \cdots \otimes g_r)) \in R,$$

where

$$\partial'_x(g_0 \otimes \cdots \otimes g_r) := (g_1(x), \cdots, \partial_x(g_0, g_i), \cdots, g_r(x))_{k(x)/k(x)},$$

if x is in the support of g_i for some $i = 1, \cdots, r$, and

$$\partial'_x(g_0 \otimes \cdots \otimes g_r) := \mathrm{ord}_x(g_0)(g_1(x), \cdots, g_i(x), \cdots, g_r(x))_{k(x)/k(x)}$$

otherwise.

In the above definition, ∂_x is the tame symbol defined in remark 1.2.2.

Somekawa in [37] proved an isomorphism $K'_r(k) \simeq K^M_r(k)$. The advantage of looking at the Milnor K-groups in this way is that the definition 1.2.7 can be generalized to more general algebraic groups, as long as we can describe a Weil reciprocity. Somekawa gave a definition of a K-group $K(k; G_1, \cdots, G_r)$ attached to semi-abelian varieties over k. A semi-abelian variety G over k is an extension of an abelian variety by a torus,

$$0 \to T \to G \to A \to 0.$$

His definition has been generalized to include also some additive variants by Hiranouchi [18]. However, as it turned out Somekawa's original definition was not very suitable for geometric applications. Some more suitable variants have been introduced by Kahn and Yamazaki in [22] and by Ivorra and Rülling in [20]. We will discuss about these variants in a later section of this thesis. In the case G_i is an abelian variety for every $i \in \{1, \cdots, r\}$, all the different definitions agree. This is the case we will be interested in in the next chapter, so we review this definition here.

Definition 1.2.8. Let $A_1, \cdots, A_r$ be abelian varieties over a field k. The Somekawa K-group $K(k; A_1, \cdots, A_r)$ attached to $A_1, \cdots, A_r$ is defined as

$$K(k; A_1, ..., A_n) = [\bigoplus_{k'/k} A_1(k') \otimes ... \otimes A_n(k')]/R,$$

where the sum extends over all finite extensions $k' \supset k$ and R is the subgroup generated by the following two families of elements:

1. If $L \supset E \supset k$ are two finite extensions of k and we have points $a_i \in A_i(L)$, for some $i \in \{1, 2, ..., n\}$, and $a_j \in A_j(E)$, for all $j \neq i$, then

$$a_1 \otimes ... \otimes \mathrm{Tr}_{L/E}(a_i) \otimes ... \otimes a_n - \mathrm{res}_{L/E}(a_1) \otimes ... \otimes a_i \otimes ... \otimes \mathrm{res}_{L/E}(a_n) \in R$$

2. Let K/k be a function field in one variable over k. Let $f \in K^\times$ and $x_i \in A_i(K)$, $i = 1, ..., n$. Then we define

$$\sum_{v \text{ place of K/k}} \mathrm{ord}_v(f)(s_v(x_1) \otimes ... \otimes s_v(x_n)) \in R,$$

where the sum extends over all places v of K over k. If v is such a place, the morphism s_v is the specialization map $A_i(K) \to A_i(k_v)$, where k_v is the residue field of the place v, and is defined as follows: Let K_v be the completion of K with respect to the valuation v and $\mathcal{O}_v$ be its ring of integers. The properness of A_i over k yields isomorphisms $A_i(K_v) \simeq A_i(\mathcal{O}_v)$, for all $i = 1, ..., n$. Further, we have a natural map

$A_i(\mathcal{O}_v) \to A_i(k_v)$ induced by $\mathcal{O}_v \twoheadrightarrow k_v$. This gives,

$$\begin{array}{ccc} A_i(K) & \longrightarrow & A_i(\mathcal{O}_v) \\ & \searrow^{s_v} & \downarrow \\ & & A_i(k_v), \end{array}$$

where the horizontal map is the composition $A_i(K) \xrightarrow{\text{res}} A_i(K_v) \xrightarrow{\simeq} A_i(\mathcal{O}_v)$.

The above definition is precisely the definition introduced by Somekawa (after a suggestion of Kato). Let us rewrite the definition of $K(k; A_1, \cdots, A_r)$ in a more geometric way. We will see in a later chapter that this more geometric version will give rise to variants of the Somekawa K-group.

Definition 1.2.9. Let $A_1, \cdots, A_r$ be abelian varieties over a field k. The Somekawa K-group $K(k; A_1, \cdots, A_r)$ attached to $A_1, \cdots, A_r$ is defined as

$$K(k; A_1, \cdots, A_r) := [(A_1 \bigotimes^{M} \cdots \bigotimes^{M} A_r)(k)]/R,$$

where R is generated by the following family of elements. Let C be a smooth complete curve over k such that we have elements $g_i \in A_i(k(C))$, for every $i \in \{1, \cdots, r\}$. Since A_i is proper, each inclusion $g_i : \mathrm{Spec}(k(C)) \hookrightarrow A_i$ extends to a morphism $g_i : C \to A_i$. Let $f \in k(C)^\times$ be a function. We require

$$\sum_{x \in C} \mathrm{ord}_x(f)(g_1(x), \cdots, g_r(x))_{k(x)/k} \in R.$$

Notation 1.2.10. 1. The elements of $K(k; A_1, ..., A_n)$ will be from now on denoted as symbols. In particular if k'/k is a finite extension and $a_i \in A_i(k')$ for every $i = 1, \cdots, r$, then the class of $a_1 \otimes \cdots \otimes a_r$ in $K(k; A_1, \cdots, A_r)$ will be denoted by $\{a_1, ..., a_n\}_{k'/k}$.

2. If $A_1 = A_2 = \cdots = A_r$, then we introduce the notation $K_r(k; A) = K(k; \overbrace{A, \ldots, A}^{r})$.

Functoriality.

Let L/k be a finite extension of k. Then the trace map of Mackey functors descends to a trace map

$$\begin{aligned} \mathrm{Tr}_{L/k}: \quad & K(L; A_1 \times_k L, ..., A_n \times_k L) \to K(k; A_1, ..., A_n) \\ & \{a_1, \dots, a_n\}_{E/L} \to \{a_1, \dots, a_n\}_{E/k}. \end{aligned}$$

Similarly, if $j : k \hookrightarrow L$ is any field extension, then the restriction map defined in section 1.2.2 descends to a map

$$\mathrm{res}_{L/k} : K(k; A_1, ..., A_n) \to K(L; A_1 \times_k L, ..., A_n \times_k L).$$

1.2.4 The Galois symbol

Let $G_1, \cdots, G_r$ be semi-abelian varieties over a field k. Let $n \geq 1$ be an integer invertible in k. For every $i \in \{1, \cdots, r\}$ we consider the Kummer sequence for G_i,

$$0 \longrightarrow G_i[n] \longrightarrow G_i \xrightarrow{n} G_i \longrightarrow 0,$$

which is a short exact sequence of étale sheaves on $\mathrm{Spec}(k)_{et}$. This induces a long exact sequence in Galois cohomology, and in particular we obtain a connecting homomorphism $\delta_i : G_i(k) \to H^1(k, G_i[n])$. Let L/k be a finite extension. The cup product and the Corestriction map of Galois cohomology induce a map

$$\begin{array}{ccc} G_1(L) \otimes \cdots \otimes G_r(L) & \xrightarrow{\cup} & H^r(L, G_1[n] \otimes \cdots \otimes G_r[n]) \\ & \searrow^{s_n} & \downarrow \mathrm{Cor}_{L/k} \\ & & H^r(k, G_1[n] \otimes \cdots \otimes G_r[n]). \end{array}$$

Somekawa proved (prop. 1.5 in [37]) that the composition s_n factors through the Somekawa K-group $K(k; G_1, \cdots, G_r)/n$. To give an explicit description of s_n, if $\{a_1, \cdots, a_r\}_{k'/k}$ is a generator of $K(k; G_1, \cdots, G_r)$, then

$$s_n : \quad \frac{K(k; G_1, \cdots, G_r)}{n} \longrightarrow H^r(k, G_1[n] \otimes \cdots \otimes G_r[n]).$$
$$\{a_1, \cdots, a_r\}_{k'/k} \longrightarrow \mathrm{Cor}_{k'/k}(\delta_1(a_1) \cup \cdots \cup \delta_r(a_r)).$$

Remark 1.2.11. The above map is traditionally called the Galois symbol and it is a generalization of the Galois symbol map for Milnor K-groups, $K_r^M(k) \longrightarrow H^r(k, \mu_n^{\otimes r})$. The famous motivic Bloch-Kato conjecture [6] predicted that the latter map is an isomorphism. The case $r = 2$ was first conjectured by Milnor in [27] and it was proved by A. Merkurjev and A. Suslin [24], who later prove also the $r = 3$ case [25]. The general case was proved by V. Voevodsky in a sequence of papers (see for example [41], [42]) and nowadays this isomorphism is called the the norm-residue isomorphism Theorem.

The generalized Galois symbol s_n was conjectured by Somekawa to always be injective. However this has been disproved by M.Spiess and T.Yamazaki, who in [39] give a counterexample of a non-split torus T that has the property that the Galois symbol map $K(k; T, T)/n \xrightarrow{s_n} H^2(k, T[n]^{\otimes 2})$ is not injective. It is an interesting question whether the only counterexamples are given by non-split tori. Examples where the injectivity holds include [30], [44], [17].

1.2.5 Notation

1. If k is a field, we denote by $\overline{k}$ its algebraic closure.

2. If $L \supset k$ is any field extension and X is a variety over k, we denote by $X_L = X \times_k \mathrm{Spec}\, L$ its base change to L.

3. If x is any closed point of X, we denote by $k(x)$ its residue field. Moreover, if X is

irreducible, we denote by $k(X)$ the function field of X.

4. Let C be a smooth complete curve over a field k and $P \in C$ a closed point. We write ord_P for the normalized discrete valuation on $k(C)$ defined by the point P.

5. If B is a discrete abelian group, we denote by $B^\star$ the group $\mathrm{Hom}(\mathrm{B}, \mathbb{Q}/\mathbb{Z})$.

6. If X is a smooth variety, and $\mathcal{F}$ is an abelian sheaf on the étale site of X, we denote $H^r(X, \mathcal{F}),\ r \geq 0$, the étale cohomology groups of X with coefficients in $\mathcal{F}$. In the special case when $X = \operatorname{Spec} k$, we denote $H^r(k, \mathcal{F})$ the Galois cohomology of k (or equivalently the étale cohomology on $\operatorname{Spec} k$).

7. For an abelian variety A over k and n any integer, we denote by $A[n] = \ker(A \xrightarrow{n} A)$ the n-torsion points of A. Further, we denote by $\widehat{A}$, the dual abelian variety of A.

8. For abelian groups $A,\ B$ we will write for simplicity $A \otimes B$ instead of $A \otimes_{\mathbb{Z}} B$.

CHAPTER 2

A FILTRATION OF CH_0 FOR AN ABELIAN VARIETY

2.1 Introduction

Let A be an abelian variety over a field k. We consider the Somekawa K-group $K_r(k;A)$ attached to r copies of A. In this chapter we study the relation between this group and the group $CH_0(A)$ of zero cycles modulo rational equivalence on A. Both those groups are highly incomputable, so our effort is focusing on obtaining some information for $CH_0(A)$ by looking at $K_r(k;A)$ and vice versa.

We introduce the group $S_r(k;A)$, which is the quotient of $K_r(k;A)$ by the subgroup generated by elements of the form $\{x_1,\cdots,x_r\}_{k'/k}-\{x_{\sigma(1)},\cdots,x_{\sigma(r)}\}_{k'/k}$, where σ is any permutation of the set $\{1,\cdots,r\}$. Moreover, for $r=0$ we set $S_0(k;A)=\mathbb{Z}$. In section 2.2 we construct a decreasing filtration F^r of the group $CH_0(A)$, such that the successive quotients are "almost" isomorphic to $S_r(k;A)$. In particular, the main theorem of this chapter is the following.

Theorem 2.1.1. *Let k be a field and A an abelian variety over k. There exists a decreasing filtration $\{F^r\}_{r\geq 0}$ of $CH_0(A)$ such that there are canonical isomorphisms of abelian groups:*

$$\Phi_r:\mathbb{Z}[\frac{1}{r!}]\otimes\frac{F^r}{F^{r+1}}\xrightarrow{\simeq}\mathbb{Z}[\frac{1}{r!}]\otimes S_r(k;A),\ r\geq 0.$$

The first pieces of the filtration are $F^0=CH_0(A)$, $F^1=A_0(A)$ and $F^2=\ker(\mathrm{alb}_A)$.

The advantage of our result is that it holds over any base field k, allowing us to obtain different properties of $CH_0(A)$ by changing the base field. In the case of an algebraically closed field $k=\overline{k}$, the filtration F^r coincides, after $\otimes\mathbb{Q}$, with the filtration defined by S. Bloch in [4] and independently by A. Beauville [3]. Recall that the filtration of Beauville-Bloch, which we denote by G^r, was defined (in the case $k = \overline{k}$) as follows:

$$G^0CH_0(A) = CH_0(A),$$

$$G^1CH_0(A) =< [a] - [0] : a \in A >,$$

$$G^2CH_0(A) =< [a+b] - [a] - [b] + [0] : a,\ b \in A >,$$

$$G^3CH_0(A) =< [a+b+c] - [a+b] - [a+c] - [b+c] + [a] + [b] + [c] - [0] : a,\ b,\ c \in A >,$$

$$G^rCH_0(A) =< \sum_{j=0}^{r}(-1)^{r-j} \sum_{1\leq\nu_1<\cdots<\nu_j\leq r} [a_{\nu_1} + \cdots + a_{\nu_j}], a_i \in A > .$$

Beauville in fact showed that $\{G^r \otimes \mathbb{Q}\}$ is the motivic filtration of $CH_0(A)$ arising from the decomposition of the diagonal. Moreover, he conjectured that the successive quotients $(G^r/G^{r+1}) \otimes \mathbb{Q}$ should have an expression as symmetric products of A, but he couldn't find the relations.

In section 2.3 we prove that our filtration F^r and the Beauville-Bloch filtration agree rationally, and therefore the symmetric expressions Beauville was expecting are provided by our theorem 2.1.1, namely $(G^r/G^{r+1}) \otimes \mathbb{Q} \simeq S_r(k; A) \otimes \mathbb{Q}$.

One other case of particular interest is when k is a finite extension of $\mathbb{Q}_p$. A famous conjecture of Colliot-Thélène predicts that for a smooth projective variety X over a p-adic field k, the kernel of the Albanese map is the direct sum of a finite group and a divisible group.

Using a computation of Raskind and Spiess ([31]), we prove that for an abelian variety A over a p-adic field k with split semi-ordinary reduction, the quotients F^r/F^{r+1} are divisible for every $r \geq 3$ and $(F^2/F^3) \otimes \mathbb{Z}[\frac{1}{2}]$ is the direct sum of a finite group and a divisible group. If we therefore knew that the filtration vanishes for some big enough $r > 0$, then we would be able to establish the conjecture for odd primes p and for abelian varieties A with split semi-ordinary reduction.

2.1.1 Main Results

Theorem (2.1.1) is build up from two main propositions. The first one provides the key point in order to construct the filtration $\{F^r\}_{r\geq 0}$, while the second proves the isomorphisms.

Proposition 2.1.2. *Let k be a field and A an abelian variety over k. For any $r \geq 0$ there is a well defined abelian group homomorphism*

$$\Phi_r : CH_0(A) \longrightarrow S_r(k; A)$$
$$[a] \longrightarrow \{a, a, \ldots, a\}_{k(a)/k},$$

where a is any closed point of A. For $r = 0$, we define Φ_0 to be the degree map.

Our next step is to define the filtration F^r of $CH_0(A)$. We define $F^0CH_0(A) = CH_0(A)$ and for $r \geq 1$, $F^rCH_0(A) = \bigcap_{j=0}^{r-1} \ker \Phi_j$. In particular, $F^1CH_0(A)$ is the subgroup of degree zero cycles.

The next key step is to prove an inclusion $F^r \supset G^r$, for every $r \geq 0$. Using this inclusion, we prove the following proposition.

Proposition 2.1.3. *Let $r \geq 0$ be an integer. There is a well defined abelian group homomorphism*

$$\Psi_r : S_r(k; A) \longrightarrow \frac{F^r}{F^{r+1}}$$
$$\{a_1, \ldots, a_r\}_{k'/k} \longrightarrow \sum_{j=0}^{r} (-1)^{r-j} \operatorname{Tr}_{k'/k}\Big(\sum_{1\leq \nu_1 < \cdots < \nu_j \leq r} [a_{\nu_1} + \cdots + a_{\nu_j}]_{k'} \Big),$$

where the summand corresponding to $j = 0$ is $(-1)^r \operatorname{Tr}_{k'/k}([0]_{k'})$. Moreover, the homomorphisms Ψ_r satisfy the property, $\Phi_r \circ \Psi_r = \cdot r!$ on $S_r(k; A)$.

Note that the element $\sum_{j=0}^{r} (-1)^{r-j} \operatorname{Tr}_{k'/k}\Big(\sum_{1\leq \nu_1 < \cdots < \nu_j \leq r} [a_{\nu_1} + \cdots + a_{\nu_j}]_{k'} \Big)$ is the distinguished generator of G^r corresponding to the r-tuple $(x_1, \cdots, x_r) \in A(k') \times \cdots \times A(k')$.

Theorem (2.1.1) follows easily by the previous proposition, since for every $r \geq 0$, the inclusion $\Phi_r : F^r/F^{r+1} \hookrightarrow S_r(k; A)$ becomes an isomorphism after $\otimes\mathbb{Z}[\frac{1}{r!}]$ with inverse $\frac{1}{r!}\Psi_r$.

2.1.2 Corollaries

In section 2.3 we obtain various corollaries and properties of the filtration F^r. We first treat the case of an algebraically closed field k. In this case we prove that theorem (2.1.1) holds integrally and as a consequence, the filtrations F^r and G^r coincide rationally over an arbitrary base field k. We continue by describing a recursive algorithm to compute generators of the group $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$, for $r \geq 1$. Finally, we use the Galois symbol s_n (see section 1.2.4 for a definition), to obtain a cycle map to Galois cohomology

$$\frac{F^r/F^{r+1}}{n} \longrightarrow H^r(k, \bigwedge^r A[n]),$$

where n is any integer invertible in k.

2.1.3 The p-adic Case

In the last section of this chapter, we focus on the case when the base field k is a finite extension of $\mathbb{Q}_p$. Using a result of W. Raskind and M. Spiess, [31], we obtain the following corollary.

Corollary 2.1.4. *Let A be an abelian variety over a p-adic field k having split semi-ordinary reduction. Then for the filtration defined above, it holds:*

1. *For $r \geq 3$, the groups F^r/F^{r+1} are divisible.*
2. *The group $F^2/F^3 \otimes \mathbb{Z}[\frac{1}{2}]$ is the direct sum of a divisible group and a finite group.*

Using these divisibility results, we obtain important results about the kernel of the cycle map to étale cohomology, $\rho_n : CH_0(A)/n \to H^2(A, \mu_n^{\otimes d})$ (here $d = \dim A$), and the kernel of the map

$$CH_0(A) \otimes \mathbb{Z}[\frac{1}{2}] \to \mathrm{Hom}(\mathrm{Br(A)}, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}],$$

induced by the Brauer-Manin pairing $<,>_A$: $CH_0(A) \times Br(A) \to \mathbb{Q}/\mathbb{Z}$. We obtain the following results.

Proposition 2.1.5. *Let A be an abelian variety over k. Let $n \geq 1$ be an integer. Then the cycle map ρ_n when restricted to F^3/n is the zero map. In the special case when A has split multiplicative reduction, then* $\ker(\rho_n) = \mathrm{Im}(F^3/n \to CH_0(A)/n)$.

Theorem 2.1.6. *Let A be an abelian variety over k. The subgroup F^3 is contained in the kernel of the map*

$$j : CH_0(A) \to \mathrm{Hom}(\mathrm{Br(A)}, \mathbb{Q}/\mathbb{Z}).$$

If moreover A has split multiplicative reduction, then the kernel of the map

$$CH_0(A) \otimes \mathbb{Z}[\frac{1}{2}] \overset{j\otimes\mathbb{Z}[\frac{1}{2}]}{\longrightarrow} \mathrm{Hom}(\mathrm{Br(A)}, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}]$$

is the subgroup D of $F^2 \otimes \mathbb{Z}[\frac{1}{2}]$, which contains $F^3 \otimes \mathbb{Z}[\frac{1}{2}]$ and is such that $D/(F^3 \otimes \mathbb{Z}[\frac{1}{2}])$ is the maximal divisible subgroup of $F^2/F^3 \otimes \mathbb{Z}[\frac{1}{2}]$.

We point out that our results concerning the cycle map and the pairing of $CH_0(A)$ with $Br(A)$ were motivated by a result of T. Yamazaki, who in [44] computes the kernel of the map $j : CH_0(X) \to Br(X)^\star$, when $X = C_1 \times \cdots \times C_d$ is a product of Mumford curves.

Convention-Notation 1. Let k be any field and A a variety over k. If a is a closed point of A, then a induces a unique $k(a)$-rational point $\widetilde{a}$ of $A_{k(a)}$, and for the push-forward map $\mathrm{Tr}_{k(a)/k} : CH_0(A_{k(a)}) \to CH_0(A)$, we have the equality $\mathrm{Tr}_{k(a)/k}([\widetilde{a}]) = [a]$.

If now $k' \supset k(a) \supset k$ is a finite extension, then a can be considered by restriction as a k'-rational point of A. We will denote by $[a]_{k'}$ the class of $[\mathrm{res}_{k'/k(a)}(a)]$ in $CH_0(A_{k'})$. Note that for the push-forward map $\mathrm{Tr}_{k'/k} : CH_0(A_{k'}) \to CH_0(A)$, the equality $\mathrm{Tr}_{k'/k}([a]_{k'}) =$

$[k' : k(a)] \cdot [a]$ holds. (See [13], section 1.4). The necessity of this remark will become apparent in proposition 2.2.3.

2.2 The canonical Isomorphisms

In this section we define a filtration $F^r CH_0(A)$ of $CH_0(A)$ and prove the existence of canonical morphisms $\Phi_r : F^r/F^{r+1} \to S_r(k; A)$, and $\Psi_r : S_r(k; A) \to F^r/F^{r+1}$, for all $r \geq 0$, so that Φ_r and Ψ_r become "almost" each other inverses.

Proposition 2.2.1. *Let k be a field and A an abelian variety over k. For any $r \geq 0$ there is a well defined abelian group homomorphism*

$$\Phi_r : CH_0(A) \longrightarrow S_r(k; A)$$
$$[a] \longrightarrow \{a, a, \ldots, a\}_{k(a)/k}.$$

Proof. For $r = 0$ we define $\Phi_0 : CH_0(A) \longrightarrow \mathbb{Z}$ to be the degree map. Let now $r > 0$ be a fixed integer. We define a map $Z_0(A) \xrightarrow{\phi_r} S_r(k; A)$ first at the level of cycles as follows. Let a be any closed point of A with residue field $k(a)$. Then we define $\phi_r(a) = \{a, a, \ldots, a\}_{k(a)/k}$. To check that ϕ_r factors through rational equivalence, let $C \subset A$ be a closed irreducible curve with function field $K = k(C)$ and let $f \in K^\times$. Let $\widetilde{C}$ be the normalization of C and let p

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{p} & A \\ \downarrow & \nearrow & \\ C & & \end{array}$$

be the canonical map. We need to show $\phi_r(p_\star(\mathrm{div}(f)) = 0$. By the definition of ϕ_r we obtain:

$$
\begin{aligned}
&\phi_r(p_\star(\operatorname{div}(f)) = \phi_r(\sum_{x\in\widetilde{C}} \operatorname{ord}_x(f)[k(x):k(p(x))][p(x)]) = \\
&\sum_{x\in\widetilde{C}} \operatorname{ord}_x(f)[k(x):k(p(x))]\{p(x),\dots,p(x)\}_{k(p(x))/k} = \\
&\sum_{x\in\widetilde{C}} \operatorname{ord}_x(f)\{[k(x):k(p(x))]p(x),p(x),\dots,p(x)\}_{k(p(x))/k} = \\
&\sum_{x\in\widetilde{C}} \operatorname{ord}_x(f)\{\operatorname{Tr}_{k(x)/k(p(x))}(\operatorname{res}_{k(x)/k(p(x))}(p(x)),p(x),\dots,p(x)\}_{k(p(x))/k} = \\
&\sum_{x\in\widetilde{C}} \operatorname{ord}_x(f)\{\operatorname{res}_{k(x)/k(p(x))}(p(x)),\dots,\operatorname{res}_{k(x)/k(p(x))}(p(x))\}_{k(x)/k}.
\end{aligned}
$$

Let $\operatorname{Spec} K \overset{\eta}{\hookrightarrow} \widetilde{C}$ be the generic point inclusion and let x be a closed point of $\widetilde{C}$. Let K_x be the completion of K at the place x and $\mathcal{O}_{K_x}$ its ring of integers. Then the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K_x & \longrightarrow & \operatorname{Spec} K \\
\downarrow{\scriptstyle \eta_x} & \swarrow & \\
\widetilde{C} & &
\end{array}
$$

yields a K_x-rational point $p\eta_x$ of A. The valuative criterion for properness gives a unique $\mathcal{O}_{K_x}$-valued point of A,

$$
\begin{array}{ccc}
\operatorname{Spec} K_x & \xrightarrow{p\eta_x} & A \\
\downarrow & \overset{\exists! p_x}{\nearrow} & \downarrow \\
\operatorname{Spec} \mathcal{O}_{K_x} & \longrightarrow & \operatorname{Spec} k.
\end{array}
$$

Then, we claim that for the specialization map s_x corresponding to the valuation x, it holds $s_x(p\eta) = \operatorname{res}_{k(x)/k(p(x))}(p(x))$. To see this, we follow the composition

$$
\begin{array}{ccccccc}
A(K) & \xrightarrow{\operatorname{res}} & A(K_x) & \xrightarrow{\simeq} & A(\mathcal{O}_{K_x}) & \xrightarrow{\operatorname{res}} & A(k_x) \\
p\eta & \longrightarrow & p\eta_x & \longrightarrow & p_x & \longrightarrow & \operatorname{res}_{k(x)/k(p(x))}(p(x)).
\end{array}
$$

This in turn yields:

$$\phi_r(p_\star(\mathrm{div}(f)) = \sum_{x\in\widetilde{C}} \mathrm{ord}_x(f)\{\mathrm{res}_{k(x)/k(p(x))}(p(x)),\ldots,\mathrm{res}_{k(x)/k(p(x))}(p(x))\}_{k(x)/k} =$$
$$\sum_{x\in\widetilde{C}} \mathrm{ord}_x(f)\{s_x(p\eta),\ldots,s_x(p\eta)\}_{k(x)/k} = 0,$$

where the last equality comes from the defining relation (2) of the K-group $K_r(k;A)$. We thus obtain a homomorphism $CH_0(A) \xrightarrow{\Phi_r} S_r(k;A)$ as desired.

□

Definition 2.2.2. We define a descending filtration F^r of $CH_0(A)$ by $F^0CH_0(A) = CH_0(A)$ and for $r \geq 1$, $F^r = \bigcap_{j=0}^{r-1} \ker \Phi_j$. In particular $F^1CH_0(A) = A_0(A)$ is the subgroup of degree zero elements.

Proposition 2.2.3. *The filtration $F^rCH_0(A)$ just defined contains the filtration $G^rCH_0(A)$ defined as follows:*

$$G^0CH_0(A) = CH_0(A),$$
$$G^1CH_0(A) =< \mathrm{Tr}_{k'/k}([a]_{k'} - [0]_{k'}) : a \in A(k') >,$$
$$G^2CH_0(A) =< \mathrm{Tr}_{k'/k}([a+b]_{k'} - [a]_{k'} - [b]_{k'} + [0]_{k'}) : a,\ b \in A(k') >,$$
$$\ldots$$
$$G^rCH_0(A) =< \sum_{j=0}^{r}(-1)^{r-j}\,\mathrm{Tr}_{k'/k}(\sum_{1\leq\nu_1<\cdots<\nu_j\leq r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{k'}) : a_1,\ldots,a_r \in A(k') >,$$

where the summand corresponding to $j = 0$ is $(-1)^r\,\mathrm{Tr}_{k'/k}([0]_{k'})$, and k' runs through all finite extensions of k.

Proof. The claim is clear for $r = 0$. Let $r \geq 1$ and let $a_1,\ldots,a_r \in A(k')$. We denote by

$\Phi^{k'}_{r-1}$ the map $CH_0(A_{k'}) \to S_{r-1}(k'; A\times_k k')$ defined as in proposition 2.2.1. We claim that:

$$\Phi_{r-1}(\sum_{j=0}^{r}(-1)^{r-j}\operatorname{Tr}_{k'/k}(\sum_{1\leq\nu_1<\cdots<\nu_j\leq r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{k'})) =$$
$$\sum_{j=0}^{r}(-1)^{r-j}\operatorname{Tr}_{k'/k}(\Phi^{k'}_{r-1}(\sum_{1\leq\nu_1<\cdots<\nu_j\leq r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{k'})) = 0.$$

The last equality is deduced by the multilinearity of the symbol $\{x_1,\ldots,x_{r-1}\}_{k'/k'}$ and the fact that Φ_{r-1} is a group homomorphism.

To justify the first equality, we need to verify the following commutativity

$$\operatorname{Tr}_{k'/k}(\Phi^{k'}_{r-1}([a]_{k'})) = \Phi_{r-1}(\operatorname{Tr}_{k'/k}([a]_{k'})),$$

where $a\in A(k')$ is a k'-rational point of A. Notice that in general the residue field $k(a)$ might be strictly smaller than k'. (See Convention-Notation 1). We have,

$$\begin{aligned}
\operatorname{Tr}_{k'/k}(\Phi^{k'}_{r}([a]_{k'}) &= \operatorname{Tr}_{k'/k}(\{\operatorname{res}_{k'/k(a)}(a),\ldots,\operatorname{res}_{k'/k(a)}(a)\}_{k'/k'}\\
&= \{\operatorname{res}_{k'/k(a)}(a),\ldots,\operatorname{res}_{k'/k(a)}(a)\}_{k'/k}.\\
\Phi_r(\operatorname{Tr}_{k'/k}([a]_{k'})) &= \Phi_r([k':k(a)]\cdot[a])) = [k':k(a)]\{a,\ldots,a\}_{k(a)/k}\\
&= \{[k':k(a)]a,\ldots,a\}_{k(a)/k} = \{\operatorname{Tr}_{k'/k(a)}(\operatorname{res}_{k'/k(a)}(a)),a,\ldots,a\}_{k(a)/k}\\
&= \{\operatorname{res}_{k'/k(a)}(a),\ldots,\operatorname{res}_{k'/k(a)}(a)\}_{k'/k}.
\end{aligned}$$

□

Proposition 2.2.4. *Let $r\geq 0$ be an integer. There is a well defined abelian group homo-*

morphism

$$\begin{aligned}\Psi_r : S_r(k;A) &\longrightarrow \frac{F^r CH_0(A)}{F^{r+1}CH_0(A)}\\ \{a_1,\ldots,a_r\}_{k'/k} &\longrightarrow \sum_{j=0}^{r}(-1)^{r-j}\operatorname{Tr}_{k'/k}(\sum_{1\le\nu_1<\cdots<\nu_j\le r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{k'}),\end{aligned}$$

where the summand corresponding to $j=0$ is $(-1)^r\operatorname{Tr}_{k'/k}([0]_{k'})$. Moreover, the homomorphisms Ψ_r satisfy the property, $\Phi_r\circ\Psi_r=\cdot r!$ on $S_r(k;A)$.

Proof. Step 1: We define a map

$$\begin{aligned}\Psi_r : \bigoplus_{k'/k}(A(k')\times A(k')\times\cdots\times A(k')) &\longrightarrow \frac{F^r CH_0(A)}{F^{r+1}CH_0(A)}\\ (a_1,\ldots,a_r) &\longrightarrow \sum_{j=0}^{r}(-1)^{r-j}\operatorname{Tr}_{k'/k}(\sum_{1\le\nu_1<\cdots<\nu_j\le r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{k'}),\end{aligned}$$

where the direct sum extends over all finite extensions of k. Notice that the inclusion $G^{r+1}\subset F^{r+1}$ proved in proposition 2.2.3, forces the map Ψ_r to be multilinear, and thus we obtain a well defined map

$$\Psi_r : \bigoplus_{k'/k}(A(k')\otimes A(k')\otimes\cdots\otimes A(k')) \longrightarrow \frac{F^r CH_0(A)}{F^{r+1}CH_0(A)}.$$

Step 2: We claim that the composition

$$\bigoplus_{k'/k}(A(k')\otimes A(k')\otimes\cdots\otimes A(k')) \xrightarrow{\Psi_r} \frac{F^r CH_0(A)}{F^{r+1}CH_0(A)} \xrightarrow{\Phi_r} S_r(k;A)$$

sends $a_1\otimes\cdots\otimes a_r$ to $r!\{a_1,\ldots,a_r\}_{k'/k}$. For, we observe that,

$$\Phi_r([\sum_{i=1}^{r}a_i])=\{\sum_{i=1}^{r}a_i,\ldots,\sum_{i=1}^{r}a_i\}_{k'/k}=\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_r=1}^{r}\{a_{i_1},\ldots,a_{i_r}\}_{k'/k}$$

and by a combinatorial counting we can see that the only terms of this sum that do not get canceled by $\Phi_r(\sum_{j=0}^{r-1}(-1)^{r-j}(\sum_{1\leq\nu_1<\cdots<\nu_j\leq r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{k'}))$ are those where all the a_{i_l} are distinct. Thus, using the symmetry of the symbol in $S_r(k;A)$, we get all the possible combinations of the set $\{a_1,\ldots,a_r\}$ without repetition, which are exactly $r!$.

Notice that the above property forces the elements of the form $(a_1\otimes\cdots\otimes\operatorname{Tr}_{E/L}(a_i)\otimes\cdots\otimes a_r)$ and $\operatorname{res}_{E/L}(a_1)\otimes\cdots\otimes a_i\otimes\cdots\otimes\operatorname{res}_{E/L}(a_r)$ to have the same image under Ψ_r, where $E\supset L\supset k$ is a tower of finite extensions, $a_i\in A(E)$ and $a_j\in A(L)$, for all $j\neq i$. For,

$$\begin{aligned}
&\Phi_r\circ\Psi_r((a_1\otimes\cdots\otimes\operatorname{Tr}_{E/L}(a_i)\otimes\cdots\otimes a_r))=r!\{a_1,\ldots,\operatorname{Tr}_{E/L}(a_i),\ldots,a_r\}_{L/k}\\
&=r!\{\operatorname{res}_{E/L}(a_1),\ldots,a_i,\ldots,\operatorname{res}_{E/L}(a_r)\}_{E/k}\\
&=\Phi_r\circ\Psi_r(\operatorname{res}_{E/L}(a_1)\otimes\cdots\otimes a_i\otimes\cdots\otimes\operatorname{res}_{E/L}(a_r)).
\end{aligned}$$

Step 3: Let $K\supset k$ be a function field in one variable over k and assume we are given $f\in K^\times$ and $x_1,\ldots,x_r\in A(K)$. We need to show that

$$\sum_{v\text{ place of K/k}}\operatorname{ord}_v(f)(\sum_{j=0}^{r}(-1)^{r-j}\operatorname{Tr}_{k_v/k}(\sum_{1\leq\nu_1<\cdots<\nu_j\leq r}[s_v(x_{\nu_1})+\cdots+s_v(x_{\nu_j})]_{k(v)}))=0.$$

This will follow by the fact that for every place v of K over k, the map s_v is a group homomorphism and by the following lemma.

Lemma 2.2.5. *For every $x\in A(K)$ it holds $\sum_v\operatorname{ord}_v(f)\operatorname{Tr}_{k_v/k}([s_v(x)]_{k_v})=0$, where the sum runs through all the places of K over k.*

Proof. Let C be the unique smooth projective curve that corresponds to the extension K/k. By the valuative criterion of properness, we obtain that the map $x:\operatorname{Spec}K\to A$ factors through the generic point inclusion $\eta : \operatorname{Spec} K \hookrightarrow C$ as follows:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{x} & A \\ \downarrow{\scriptstyle \eta} & \nearrow{\scriptstyle \widetilde{x}} & \\ C. & & \end{array}$$

Here the map $\widetilde{x} : C \to A$ is given by $\widetilde{x}(v) = s_v(x)$. Since $\widetilde{x}$ is proper, it induces a push-forward map

$$\widetilde{x}_\star : CH_0(C) \to CH_0(A).$$

Since $CH_0(C) = Pic(C)$ and $\operatorname{div}(f) = 0$ in $Pic(C)$, this yields

$$\widetilde{x}_\star(\operatorname{div}(f)) = \sum_v \operatorname{ord}_v(f) \operatorname{Tr}_{k_v/k}([s_v(x)]_{k_v}) = 0.$$

□

The last fact completes the argument that the map Ψ_r factors through $K_r(k; A)$. Finally, it is clear that if σ is any permutation of the set $\{1, \ldots, r\}$, it holds $\Psi_r(\{a_1, \ldots, a_r\}) = \Psi_r(\{a_{\sigma(1)}, \ldots, a_{\sigma(r)}\})$. Therefore, we obtain a morphism $S_r(k; A) \xrightarrow{\Psi_r} \frac{F^r}{F^{r+1}}$ as stated in the proposition.

□

Remark 2.2.6. We observe that for every $r \geq 0$ the image of the map Ψ_r is contained in $(G^r + F^{r+1})/F^{r+1}$. Furthermore, the composition

$$\Psi_r \circ \Phi_r : (G^r + F^{r+1})/F^{r+1} \to S_r(k; A) \to (G^r + F^{r+1})/F^{r+1}$$

is multiplication by $r!$.

Corollary 2.2.7. *The canonical map $A(k) \xrightarrow{\iota} K_1(k; G)$ sending $a \in A(k)$ to the symbol $\{a\}_{k/k}$ is an isomorphism.*

Proof. It follows by lemma (2.2.5) that the inverse map

$$
\begin{aligned}
K_1(k;G) &\to A(k) \\
\{a\}_{k'/k} &\to \mathrm{Tr}_{k'/k}(a)
\end{aligned}
$$

is well defined. $\square$

Our main Theorem now follows easily by the two previous propositions.

Theorem 2.2.8. *Let k be a field and A an abelian variety over k. For the filtration $F^r CH_0(A)$ defined above, there are canonical isomorphisms of abelian groups,*

$$\Phi_r : \mathbb{Z}[\frac{1}{r!}] \otimes \frac{F^r}{F^{r+1}} \xrightarrow{\simeq} \mathbb{Z}[\frac{1}{r!}] \otimes S_r(k;A),\ r \geq 1,$$

with $\Phi_r^{-1} = \dfrac{1}{r!}\Psi_r$. Moreover, the group $F^2 CH_0(A)$ is precisely the Albanese kernel of A.

Proof. Definition (2.2.2) gives that $F^{r+1} = \ker \Phi_r|_{F^r}$. Thus, for every $r \geq 1$, we get an exact sequence

$$0 \longrightarrow \frac{F^r}{F^{r+1}} \xrightarrow{\Phi_r} S_r(k;A) \longrightarrow \frac{S_r(k;A)}{Im(\Phi_r)} \longrightarrow 0.$$

Now notice that step 2 of proposition (2.2.4) yields an inclusion $Im(\Phi_r) \supset r!S_r(k;A)$. Thus the group $S_r(k;A)/Im(\Phi_r)$ is $r!$-torsion, which forces $(S_r(k;A)/Im(\Phi_r)) \bigotimes \mathbb{Z}[\frac{1}{r!}] = 0$. We conclude that after $\bigotimes \mathbb{Z}[\frac{1}{r!}]$, the map Φ_r becomes an isomorphism with inverse $\widetilde{\Psi}_r = \frac{1}{r!}\Psi_r$.

Our next claim is that $F^2 CH_0(A) = \ker \mathrm{alb}_A$. Using the isomorphism $A(k) \xrightarrow{\simeq} K_1(k;A)$ (corollary 2.2.7), the claim follows immediately from the commutative diagram

$$
\begin{array}{ccc}
CH_0(A)/F^2 & \xrightarrow{\Phi_0 \oplus \Phi_1} & \mathbb{Z} \oplus K_1(k;A) \\
{\scriptstyle \deg \oplus \mathrm{alb}_A} \downarrow & \swarrow {\scriptstyle \simeq} & \\
\mathbb{Z} \oplus A(k), & &
\end{array}
$$

and the fact that F^2 is precisely the kernel of $\Phi_0 \oplus \Phi_1$.

$\square$

2.3 Properties of the Filtration

2.3.1 The case $k = \overline{k}$

Proposition 2.3.1. *If A is an abelian variety over an algebraically closed field k, then for every $r \geq 1$ the groups $F^r CH_0(A) \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$ and $G^r CH_0(A) \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$ coincide.*

Proof. Notice that the statement holds trivially, if $r = 1$. Let $r \geq 1$. Since the base field k is algebraically closed, the group $S_r(k;A)$ is divisible, and we therefore have an equality $r!S_r(k;A) = S_r(k;A)$. Thus, for every $r \geq 1$, we obtain an isomorphism, $\Phi_r : F^r/F^{r+1} \xrightarrow{\simeq} S_r(k;A)$. We will show by induction on r that $F^r CH_0(A) \otimes \mathbb{Z}[\frac{1}{(r-1)!}] = G^r CH_0(A) \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$. Assume $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}] = G^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$ for some $r \geq 1$. Call $\overline{\Phi}_r : G^r/G^{r+1} \to S_r(k;A)$ the map induced by Φ_r. Notice that the map

$$\begin{aligned}
&\overline{\Psi}_r : S_r(k;A) \to G^r/G^{r+1} \\
&\{a_1, \ldots, a_r\} \longrightarrow \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq r} [a_{\nu_1} + \cdots + a_{\nu_j}]
\end{aligned}$$

is well defined. The proof is essentially the same as the one of the well definedness of Ψ_r (proposition 2.2.4). Namely, steps 1 and 3 of the proof apply directly in this setting, while step 2 is a tautology, since there are no finite extensions of k, and hence no non-trivial Trace-Restriction maps. We therefore obtain a commutative diagram as follows:

$$\begin{array}{ccc}
\frac{F^r}{F^{r+1}} \otimes \mathbb{Z}[\frac{1}{r!}] & \xrightarrow{\simeq} & S_r(k;A) \otimes \mathbb{Z}[\frac{1}{r!}] \\
\uparrow p & \nearrow \overline{\Phi}_r & \\
\frac{G^r}{G^{r+1}} \otimes \mathbb{Z}[\frac{1}{r!}], & &
\end{array}$$

where the map p is the natural projection. The induction hypothesis clearly implies that $F^r \otimes \mathbb{Z}[\frac{1}{r!}] = G^r \otimes \mathbb{Z}[\frac{1}{r!}]$. Moreover, the composition

$$G^r/G^{r+1} \xrightarrow{\overline{\Phi}_r} S_r(k;A) \xrightarrow{\overline{\Psi}_r} G^r/G^{r+1}$$

is the multiplication by $r!$ (see remark 2.2.6). In particular, after $\otimes\mathbb{Z}[\frac{1}{r!}]$, the map $\overline{\Phi}_r$ admits a section $\frac{1}{r!}\overline{\Psi}_r$. We thus obtain an equality $F^{r+1} \otimes \mathbb{Z}[\frac{1}{r!}] = G^{r+1} \otimes \mathbb{Z}[\frac{1}{r!}]$.

□

Remark 2.3.2. We note that the filtration $G^r CH_0(A)$ has been studied before by S. Bloch, A. Beauville and others. We refer to [4], [3] and [2] for some results concerning this filtration. This filtration is sometimes referred in the literature as the Pontryagin filtration. The reason for this name is that it arises from the Pontryagin product on A. To become more specific, the addition law on A, endows $CH_0(A)$ with a ring structure, by defining the Pontryagin product $[a] \star [b] = [a+b]$, for closed points a, b of A. We can then easily see that the group $G^r CH_0(A)$ is the r-th power of the augmentation ideal G^1 of $(CH_0(A), \star)$ generated by elements of the form $\{[a]-[0],\ a \in A\}$. As it was already mentioned in the introduction, A. Beauville in [2] showed that the filtration $\{G^r \otimes \mathbb{Q}\}$ is the motivic filtration of $CH_0(A)$ arising from the decomposition of the diagonal.

Remark 2.3.3. We now come back to the case of a non-algebraically closed field k. If $L \supset k$ is any field extension, the flat map $\pi_L : A_L \to A$ induces a pull back map $\text{res}_{L/k} : CH_0(A) \to CH_0(A_L)$, with $\text{res}_{L/k}([a]) = \sum_{\pi_L(\widetilde{a})=a} e_{\widetilde{a}}[\widetilde{a}]$, where a is any closed point of A and $e_{\widetilde{a}}$ is the length of the Artin local ring $A_L \times_A k(a)$ at $\widetilde{a}$. Notice that for $a \in A$ we have

$$\text{Tr}_{L/k}(\text{res}_{L/k}([a])) = (\sum_{\pi_L(\widetilde{a})=a} e_{\widetilde{a}})[a] = [L:k][a].$$

It is an immediate consequence of the definition of the restriction map between the K-groups (see functoriality in the subsection 2.2) that the following diagram commutes, for every $r \geq 0$.

$$\begin{array}{ccc} CH_0(A) & \xrightarrow{\mathrm{res}_{L/k}} & CH_0(A_L) \\ \downarrow{\Phi_r} & & \downarrow{\Phi_r^L} \\ S_r(k;A) & \xrightarrow{\mathrm{res}_{L/k}} & S_r(L;A_L). \end{array}$$

This in particular implies that the filtration $\{F^r\}_{r\geq 0}$ is preserved under restriction maps. For, if $x \in F^r CH_0(A)$, then by definition $\Phi_r(x) = 0$. Thus,

$$\Phi_r^L(\mathrm{res}_{L/k}(x)) = \mathrm{res}_{L/k}(\Phi_r(x)) = 0,$$

and therefore $\mathrm{res}_{L/k}(x) \in F^r CH_0(A_L)$.

Moreover, if L/k is finite, then filtration $\{G^r\}_{r\geq 0}$ is preserved under the Trace map $\mathrm{Tr}_{L/k}$. To see this, we observe that the generators of the group $G^r CH_0(A_L)$ are of the form

$$\sum_{j=0}^{r}(-1)^{r-j}\,\mathrm{Tr}_{L'/L}(\sum_{1\leq \nu_1<\cdots<\nu_j\leq r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{L'}),$$

where L'/L is a finite extension and $a_i \in A_L(L')$. Then

$$\mathrm{Tr}_{L/k}(\sum_{j=0}^{r}(-1)^{r-j}\,\mathrm{Tr}_{L'/L}(\sum_{1\leq \nu_1<\cdots<\nu_j\leq r}[a_{\nu_1}+\cdots+a_{\nu_j}]_{L'})) =$$
$$\sum_{j=0}^{r}(-1)^{r-j}\,\mathrm{Tr}_{L'/k}(\sum_{1\leq \nu_1<\cdots<\nu_j\leq r}[\pi_L(a_{\nu_1})+\cdots+\pi_L(a_{\nu_j})]_{L'}),$$

which is a generator of $G^r CH_0(A)$.

Corollary 2.3.4. *If A is an abelian variety over some field k, not necessarily algebraically closed, then the groups $F^r CH_0(A)\otimes\mathbb{Q}$ and $G^r CH_0(A)\otimes\mathbb{Q}$ coincide.*

Proof. Let $x \in F^r CH_0(A)\otimes\mathbb{Q}$, for some $r \geq 0$. Then x induces by restriction an element $\overline{x} = \mathrm{res}_{\overline{k}/k}(x)$ of $F^r CH_0(A_{\overline{k}})\otimes\mathbb{Q}$ (see remark 2.3.3). By proposition 2.3.1, we deduce that

$\overline{x} \in G^r CH_0(A_{\overline{k}}) \otimes \mathbb{Q}$ and we can therefore write it as $\overline{x} = \sum_{i=1}^{N} q_i x_i$, with $x_i \in G^r CH_0(A_{\overline{k}})$ and $q_i \in \mathbb{Q}$, for $i = 1, \ldots, N$. Let $L \supset k$ be a finite extension of k such that all the x_i are defined over L. Then we obtain:

$$\mathrm{Tr}_{L/k}(\mathrm{res}_{L/k}(x)) = \sum_{i=1}^{N} q_i \, \mathrm{Tr}_{L/k}(x_i) \in G^r CH_0(A) \otimes \mathbb{Q}.$$

The corollary then follows from the fact that $\mathrm{Tr}_{L/k}(\mathrm{res}_{L/k}(x)) = [L : k]x$.

□

2.3.2 The finiteness of the Filtration

Let A be an abelian variety of dimension d over a field k. In this section we elaborate the question if the filtration F^r defined in section 3 stabilizes for large enough $r > 0$. We have the following fact.

Fact: The filtration $\{G^r\}_{r \geq 0}$ has the property $G^{d+1} \otimes \mathbb{Q} = 0$.

S.Bloch in [4] proves the above fact for the case of an algebraically closed base field, while A. Beauville in [2] gives a different proof for an abelian variety A defined over $\mathbb{C}$. A few years later, C. Deninger and J.Murre in [11], generalize Beauville's argument for an abelian variety over an arbitrary base field k, not necessarily algebraically closed.

Corollary 2.3.5. *For every $r \geq d+1$, it holds $F^r CH_0(A) \otimes \mathbb{Q} = 0$ and $S_r(k; A) \otimes \mathbb{Q} = 0$, where F^r is the filtration defined in section 3.*

Proof. The first equality follows from corollary 2.3.4, while the second from theorem 2.2.8.

□

Remark 2.3.6. We briefly recall the argument used by A. Beauville, and later by C. Deninger,

J. Murre in their articles. A. Beauville uses the following Fourier Mukai transform:

$$F: \quad CH_\bullet(A)\otimes\mathbb{Q} \to CH_\bullet(\widehat{A})\otimes\mathbb{Q}$$
$$x \to \widehat{\pi}_\star(\exp(\mathcal{L})\cdot \pi^\star(x)),$$

where $\widehat{A}$ is the dual abelian variety of A, π, $\widehat{\pi}$ are the projections of $A\times\widehat{A}$ to A and $\widehat{A}$ respectively, $\mathcal{L}$ is the Poincaré line bundle on $A\times\widehat{A}$ and the exponential $\exp(\mathcal{L})$ is defined as $\exp(\mathcal{L}) = \sum_{n=0}^{\infty}\frac{c_1(\mathcal{L})^n}{n!}$. Here we denote by $\cdot$ the intersection product in $CH_\bullet(A\times\widehat{A})$ and by $c_1(\mathcal{L})$ the image of $\mathcal{L}$ in $CH^1(A\times\widehat{A})$.

The map F is induced by the Fourier Mukai isomorphism, $F_D : D(A)\to D(\widehat{A})$, between the derived categories of A and $\widehat{A}$, defined by S. Mukai in [28], by first passing to the K-groups and then using the chern character isomorphism, $ch : K_0(A)\otimes\mathbb{Q}\xrightarrow{\simeq} CH_\bullet(A)\otimes\mathbb{Q}$, where $CH_\bullet(A)$ is the Chow ring with operation the intersection product. The map F has further the property of interchanging the intersection product of the ring $CH_\bullet(A)\otimes\mathbb{Q}$ with the Pontryagin product of $CH_\bullet(\widehat{A})\otimes\mathbb{Q}$. This property in turn implies that $G^{d+1}CH_0(A)\otimes\mathbb{Q} = 0$, since $CH^s(\widehat{A}) = 0$ for $s > d$.

We believe that the above arguments will work after only $\otimes\mathbb{Z}[\frac{1}{(2d)!}]$. First, notice that the Fourier Mukai transform F can be considered as a map $F : CH_\bullet(A)\otimes\mathbb{Z}[\frac{1}{d!}] \to CH_\bullet(\widehat{A})\otimes\mathbb{Z}[\frac{1}{d!}]$, because the chern character isomorphism does hold after only $\otimes\mathbb{Z}[\frac{1}{d!}]$, (since $\frac{c_1(\mathcal{E})^n}{n!} = 0$ for every $n > d$ and for every line bundle $\mathcal{E}$ on A). If after $\otimes\mathbb{Z}[\frac{1}{(2d)!}]$, the relative tangent bundle of the map $\widehat{\pi} : A\times\widehat{A}\to A$ is trivial as an element of $K_0(A\times\widehat{A})\otimes\mathbb{Z}[\frac{1}{(2d)!}]$, then by the Grothendieck Riemann-Roch theorem, the map

$$F : CH_\bullet(A)\otimes\mathbb{Z}[\frac{1}{(2d)!}] \to CH_\bullet(\widehat{A})\otimes\mathbb{Z}[\frac{1}{(2d)!}]$$

will attain the above concrete description and will still interchange the two products. This would imply that $G^{d+1}\otimes\mathbb{Z}[\frac{1}{(2d)!}] = 0$ and further that $F^r\otimes\mathbb{Z}[\frac{1}{(2d)!}] = F^{d+1}\otimes\mathbb{Z}[\frac{1}{(2d)!}]$, for every $r \geq d+1$.

2.3.3 An algorithm to compute generators of F^r

It is rather complicated to give a precise description of the generators of F^r, for $r \geq 3$, but things become much more concrete after $\otimes\mathbb{Z}[\frac{1}{r!}]$, because then the map Φ_{r-1} has a very concrete inverse, namely the map $\frac{1}{(r-1)!}\Psi_{r-1}$. In this section we will describe a recursive algorithm to compute generators of $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$. As an application, we will give a complete set of generators of the Albanese kernel F^2 and of the group $F^3 \otimes \mathbb{Z}[\frac{1}{2!}]$.

Notation: If $k' \supset k$ is a finite extension and $a_1, \dots, a_r \in A(k')$, we will denote by $w_{a_1,\dots,a_r}$ the generator of G^r corresponding to the $r-$tuple $(a_1, \dots, a_r)$, namely

$$w_{a_1,\dots,a_r} := \sum_{j=0}^{r} (-1)^{r-j} \operatorname{Tr}_{k'/k}\Big(\sum_{1 \leq \nu_1 < \cdots < \nu_j \leq r} [a_{\nu_1} + \cdots + a_{\nu_j}]_{k'} \Big).$$

Definition 2.3.7. Let $r \geq 1$. We consider the subgroup $R^{r+1} \subset F^r$ generated by the following two families of elements:

1. For any finite extension $k' \supset k$ and $a_1, \dots, a_{r+1} \in A(k')$, we require

$$w_{a_1,\dots,a_{r+1}} \in R^{r+1}. \tag{2.1}$$

   (notice that this yields an inclusion $G^{r+1} \subset R^{r+1}$).

2. If $L \supset E \supset k$ is a tower of finite extensions, and we have elements $a_i \in A(L)$ for some $i \in \{1, \dots, r\}$, and $a_j \in A(E)$, for all $j \neq i$, then we require

$$w_{a_1,\dots,\operatorname{Tr}_{L/E}(a_i),\dots,a_r} - w_{\operatorname{res}_{L/E}(a_1),\dots,a_i,\dots,\operatorname{res}_{L/E}(a_r)} \in R^{r+1}. \tag{2.2}$$

Lemma 2.3.8. *For every $r \geq 1$, R^{r+1} is the smallest subgroup of F^r that makes the homomorphism*

$$\begin{aligned} \Psi_r : \bigoplus_{k'/k}(A(k') \times A(k') \times \cdots \times A(k')) &\longrightarrow F^r/R^{r+1} \\ (a_1, \ldots, a_r)_{k'/k} &\longrightarrow w_{a_1,\ldots,a_r} \end{aligned}$$

factor through $S_r(k;A)$. We therefore have an inclusion $R^{r+1} \subset F^{r+1}$, for every $r \geq 1$, and the composition

$$S_r(k;A) \otimes \mathbb{Z}[\frac{1}{r!}] \xrightarrow{\frac{1}{r!}\Psi_r} (F^r/R^{r+1}) \otimes \mathbb{Z}[\frac{1}{r!}] \xrightarrow{\Phi_r} S_r(k;A) \otimes \mathbb{Z}[\frac{1}{r!}]$$

is the identity map.

Proof. Notice that if $H \subset F^r$ is any subgroup, such that the map

$$\Psi_r : \bigoplus_{k'/k}(A(k') \times A(k') \times \cdots \times A(k')) \longrightarrow F^r/H$$

factors through $S_r(k;A)$, we definitely have $G^{r+1} \subset H$, since Ψ_r needs to be multilinear. Further, elements of the form described in (2.2) above are necessarily in H, since $\{a_1, \ldots, \mathrm{Tr}_{L/E}(a_i), \ldots, a_r\}_{E/k} = \{\mathrm{res}_{L/E}(a_1), \ldots, a_i, \ldots, \mathrm{res}_{L/E}(a_r)\}_{L/k}$ in $S_r(k;A)$. Therefore $R^{r+1} \subset H$.

We have no other restrictions, since if K is a function field in one variable over k, then the relation $\sum_v \mathrm{ord}_v(f)\,\mathrm{Tr}_{k_v/k}(x) = 0$ holds already in $CH_0(A)$, for $x \in A(K)$, and $f \in K^\times$ (see lemma 2.2.5). The other statements follow directly from proposition 2.2.4.

□

We are now ready to describe our inductive argument.

Proposition 2.3.9. *For the Albanese kernel F^2 we have an equality $F^2 = R^2$, and hence it can be generated by the following two families of elements:*

1. For any finite extension $k' \supset k$, and points $a,\ b \in A(k')$,

$$\mathrm{Tr}_{k'/k}([a+b]_{k'} - [a]_{k'} - [b]_{k'} + [0]_{k'}) \in F^2,$$

2. If $L \supset k$ is a finite extension, and $a \in A(L)$, then

$$\mathrm{Tr}_{L/k}([a]_L - [0]_L) - ([\mathrm{Tr}_{L/k}(a)] - [0]) \in F^2.$$

In general for $r \geq 2$ the group $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$ can be generated by $R^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$ and elements of the form $z - \frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1}(z)$, with $z \in F^{r-1}$.

Proof. Step 1: Compute generators of F^2: We already have an inclusion $R^2 \subset F^2$ and an isomorphism $\Phi_r : F^1/F^2 \simeq S_1(k;A)$. To show that $R^2 \supset F^2$, it suffices therefore to prove commutativity of the following diagram.

$$\begin{array}{ccc} (F^1/R^2) & \xrightarrow{\Phi_1} & S_1(k;A) \\ \downarrow 1 & \swarrow_{\Psi_1} & \\ (F^1/R^2). & & \end{array}$$

(Notice that by lemma 2.3.8 we have an equality $\Phi_1 \circ \Psi_1 = 1$). We need to verify this only for the generators of F^1/R^2, namely for $\mathrm{Tr}_{L/k}([a] - [0]_L)$ with $a \in A(L)$. We have:

$$\Psi_1 \circ \Phi_1(\mathrm{Tr}_{L/k}([a] - [0]_L)) = \Psi_1(\{a\}_{L/k}) = \mathrm{Tr}_{L/k}([a] - [0]_L).$$

Step 2: Let $r \geq 3$. Consider the group

$$B^r := R^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}] + < z - \frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1}(z) : z \in F^{r-1} > .$$

We want to show that $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}] = B^r$.

proof of ($\supset$): We already know $R^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}] \subset F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$ (lemma 2.3.8). Moreover, if z is any element of F^{r-1}, then

$$\begin{aligned} \Phi_{r-1}(z - \frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1}(z)) &= \Phi_{r-1}(z) - \Phi_{r-1}(\frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1}(z)) \\ &= \Phi_{r-1}(z) - (\Phi_{r-1} \circ \frac{1}{(r-1)!}\Psi_{r-1})(\Phi_{r-1}(z)) \\ &= \Phi_{r-1}(z) - \Phi_{r-1}(z) = 0. \end{aligned}$$

Thus, $z - \frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1}(z) \in \ker \Phi_{r-1} \cap F^{r-1} \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$, which by definition is $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$.

proof of ($\subset$):. Since the group B^r contains $R^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$, lemma (2.3.8) implies that the map

$$\Psi_{r-1} : S_{r-1}(k; A) \otimes \mathbb{Z}[\frac{1}{(r-1)!}] \to \frac{F^{r-1} \otimes \mathbb{Z}[\frac{1}{(r-1)!}]}{B^r}$$

is well defined and $\Phi_{r-1} \circ \frac{1}{(r-1)!}\Psi_{r-1}$ is the identity map. To complete the argument, it suffices to show that

$$\frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1} : \frac{F^{r-1} \otimes \mathbb{Z}[\frac{1}{(r-1)!}]}{B^r} \to \frac{F^{r-1} \otimes \mathbb{Z}[\frac{1}{(r-1)!}]}{B^r}$$

is also the identity. This follows from the definition of B^r. Namely by definition, if $z \in F^{r-1}$, then $z - \frac{1}{(r-1)!}\Psi_{r-1} \circ \Phi_{r-1}(z) \in B^r$.

$\square$

Remark 2.3.10. Notice that the proposition (2.3.9) describes a recursive algorithm to compute generators of the groups $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$, for $r \geq 3$. Namely, having computed a complete set of generators of $F^r \otimes \mathbb{Z}[\frac{1}{(r-1)!}]$, the formula $F^{r+1} \otimes \mathbb{Z}[\frac{1}{r!}] = R^{r+1} \otimes \mathbb{Z}[\frac{1}{r!}] + < z - \frac{1}{r!}\Psi_r \circ \Phi_r(z) : z \in F^r >$ allows us to compute generators of $F^{r+1} \otimes \mathbb{Z}[\frac{1}{r!}]$. As an example, we compute below a set of generators of the group $F^3 \otimes \mathbb{Z}[\frac{1}{2!}]$.

Generators of $F^3 \otimes \mathbb{Z}[\frac{1}{2!}]$: According to proposition 2.3.9, we have the following families of generators:

1. The ones that come from $R^3 \otimes \mathbb{Z}[\frac{1}{2!}]$, namely:

   (a) $\frac{1}{2^m}(\mathrm{Tr}_{k'/k}([a+b+c]_{k'} - [a+b]_{k'} - [a+c]_{k'} - [b+c]_{k'} + [a]_{k'} + [b]_{k'} + [c]_{k'} - [0]_{k'}))$, where $a,\ b,\ c \in A(k')$, and $m \geq 0$.

   (b)

   $$\frac{1}{2^m}(\mathrm{Tr}_{E/k}([a + \mathrm{Tr}_{L/E}(b)]_E - [a]_E - [\mathrm{Tr}_{L/E}(b)]_E + [0]_E) - (\mathrm{Tr}_{L/k}([\mathrm{res}_{L/E}(a) + b]_L - [\mathrm{res}_{L/E}(a)]_L - [b]_L + [0]_L))),$$

   where $L \supset E \supset k$ is a tower of finite extensions, $a \in A(E)$, $b \in A(L)$ and $m \geq 0$.

2. The ones that come from $z - \frac{1}{2}\Psi_2 \circ \Phi_2(z)$ with $z \in F^2$.
   Notice that if $z \in G^2$, then $z - \frac{1}{2}\Psi_2 \circ \Phi_2(z) = 0$, thus no new generator is obtained in this way. The only remaining generating family is of the form

   $$\frac{1}{2^m}([\mathrm{Tr}_{L/k}(a)] - [0] - \mathrm{Tr}_{L/k}([a]_L - [0]_L) - \frac{1}{2}\Psi_2\Phi_2([\mathrm{Tr}_{L/k}(a)] - [0] - \mathrm{Tr}_{L/k}([a]_L - [0]_L))),$$

   where $L \supset k$ is a finite extension, $a \in A(L)$ and $m \geq 0$.

2.4 A cycle map to Galois cohomology

In this section we construct a cycle map to Galois cohomology by using theorem 2.2.8 and the Galois symbol (see section 1.2.4 for a definition of the Galois symbol). Let n be an integer invertible in k and A an abelian variety over k.

Definition 2.4.1. Let $r \geq 1$ be a positive integer. We define the wedge product $\bigwedge^r A[n]$ as the cokernel of the map $0 \to Sym^r(A[n]) \to A[n]^{\otimes r}$, where $Sym^r(A[n])$ is the subgroup of

$A[n]^{\otimes r}$ fixed by the action of Σ_r.

Proposition 2.4.2. *Let A be an abelian variety over k and let n be an integer which is invertible in k. Then the Galois symbol map induces*

$$\frac{S_r(k;A)}{n} \xrightarrow{s_n} H^r(k, \bigwedge^r A[n]).$$

Proof. The projection $A[n]^{\otimes r} \xrightarrow{p_\wedge} \bigwedge^r A[n]$ induces a morphism

$$H^r(k, A[n]^{\otimes r}) \xrightarrow{p_\wedge} H^r(k, \bigwedge^r A[n]).$$

Let $\{a_1, ..., a_r\}_{L/k}$ be any symbol in $K_r(k;A)$ and let $\sigma \in \Sigma_r$ be any permutation of the set $\{1, ..., r\}$. We need to show that $p_\wedge \circ s_n(\{a_1, ..., a_r\}_{L/k}) = p_\wedge \circ s_n(\{a_{\sigma(1)}, ..., a_{\sigma(r)}\}_{L/k})$. Since any permutation σ can be written as a product of transpositions of the form $\tau = (i, i+1)$, it suffices to show that for all $i \in \{1, ..., r-1\}$,

$$p_\wedge \circ s_n(\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_r\}_{L/k}) = p_\wedge \circ s_n(\{\{a_1, \ldots, a_{i+1}, a_i, \ldots, a_r\}_{L/k}).$$

We consider the map

$$t : A[n] \otimes A[n] \otimes \cdots \otimes A[n] \to A[n] \otimes A[n] \otimes \cdots \otimes A[n]$$
$$a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_r \to a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_r.$$

Then for the induced map $t_\star : H^r(k, A[n]^{\otimes^r}) \to H^r(k, A[n]^{\otimes^r})$ it holds

$$\mathrm{Cor}_{k'/k}(\delta(a_1) \cup \cdots \cup \delta(a_{i+1}) \cup \delta(a_i) \cup \cdots \cup \delta(a_r)) =$$
$$-t_\star(\mathrm{Cor}_{k'/k}(\delta(a_1) \cup \cdots \cup \delta(a_i)) \cup \delta(a_{i+1}) \cup \cdots \cup \delta(a_r)).$$

(The last equality is a general fact about cup products in group cohomology. For a proof, we refer to [8], p.111). Next notice that the following diagram is commutative,

$$\begin{array}{ccc} H^r(k, \otimes^r A[n]) & \xrightarrow{t_\star} & H^r(k, \otimes^r A[n]) \\ \downarrow{p_\wedge} & & \downarrow{p_\wedge} \\ H^r(k, \bigwedge^r A[n]) & \xrightarrow{-1} & H^r(k, \bigwedge^r A[n]) \end{array}$$

To conclude, we have,

$$\begin{aligned} p_\wedge(\mathrm{Cor}_{k'/k}(\delta(a_1) \cup \cdots \cup \delta(a_{i+1}) \cup \delta(a_i) \cup \cdots \cup \delta(a_r))) &= \\ p_\wedge(-t_\star(\mathrm{Cor}_{k'/k}(\delta(a_1) \cup \cdots \cup \delta(a_i) \cup \delta(a_{i+1}) \cup \cdots \cup \delta(a_r))) &= \\ p_\wedge(\mathrm{Cor}_{k'/k}(\delta(a_1) \cup \cdots \cup \delta(a_i) \cup \delta(a_{i+1}) \cup \cdots \cup \delta(a_r))). \end{aligned}$$

The result now follows. □

Corollary 2.4.3. *For any integer n invertible in k and any $r > 0$, the Somekawa map and the map Φ_r induce a cycle map to Galois cohomology:*

$$\frac{F^r CH_0(A)/F^{r+1}CH_0(A)}{n} \longrightarrow H^r(k, \bigwedge^r A[n]).$$

2.5 The p-adic Case

In all this section we assume that the base field k is a finite extension of $\mathbb{Q}_p$, where p is a prime number. Let $\mathcal{O}_k$ be its ring of integers and κ its residue field. Using results of W. Raskind and M. Spiess [31], we obtain some divisibility results for our filtration. Furthermore, we obtain some results about the kernel of the cycle map to étale cohomology and the kernel of the Brauer-Manin pairing.

2.5.1 Divisibility Results

Let $\mathcal{A}$ be the Néron model of A, which is a smooth commutative group scheme over $\mathrm{Spec}(\mathcal{O}_k)$. Let $\mathcal{A}_s = \mathcal{A} \times_{\mathcal{O}_k} \kappa$ be its special fiber and $\mathcal{A}_s^0$ be the connected component of zero. We note that $\mathcal{A}_s^0$ is a commutative algebraic group over the finite field κ. For definitions and properties regarding the Néron model $\mathcal{A}$ of A we refer to [7] and [12].

Definition 2.5.1.

1. We say that A has semi-abelian reduction, if $\mathcal{A}_s^0$ is a semi-abelian variety over κ. This means that $\mathcal{A}_s^0$ fits into a short exact sequence

$$0 \to T \to \mathcal{A}_s^0 \to B \to 0,$$

where T is a torus and B an abelian variety over κ.

2. Further, we say that A has split semi-ordinary reduction, if it has semi-abelian reduction with T a split torus and B an ordinary abelian variety.

Raskind and Spiess obtained the following important result.

Theorem 2.5.2. *([31] theorem 4.5) Let $A_1, ..., A_n$ be abelian varieties over k with split semi-ordinary reduction. Then for $n \geq 2$, the group $K(k; A_1, ..., A_n)$ is the direct sum of a finite group F and a divisible group D. For $n \geq 3$, the group $K(k; A_1, ..., A_n)$ is in fact divisible (remark 4.4.5).*

Thus, in our set up, if we assume that the abelian variety A has split semi-ordinary reduction, then theorem 2.2.8 has the following corollary:

Corollary 2.5.3. *Let A be an abelian variety over a p-adic field k having split semi-ordinary reduction. Then for the filtration $\{F^r\}_{r\geq 0}$ defined above, it holds:*

1. *For $r \geq 3$, the groups F^r/F^{r+1} are divisible.*
2. *The group $F^2/F^3 \otimes \mathbb{Z}[\frac{1}{2}]$ is the direct sum of a divisible group and a finite group.*

Proof. Everything follows directly from theorem 2.2.8, once we notice that for $r \geq 3$ the divisibility of $S_r(k;A)$ yields an equality $S_r(k;A) = r!S_r(k;A)$. Thus, the injective map $F^r/F^{r+1} \overset{\Phi_r}{\hookrightarrow} S_r(k;A)$ is also surjective.

□

Remark 2.5.4. We note that for a smooth projective variety X over a p-adic field k, Colliot-Thélène conjectured (see for example [9]) that the Albanese kernel has a decomposition $T(X) \simeq D \oplus F$, where D is a divisible group and F is a finite group. A weaker form of this conjecture states that the degree zero subgroup has a decomposition $A_0(X) \simeq D \oplus F$, where F is a finite group and D is a group divisible by any integer coprime to p. This weaker conjecture has now been proved by S. Saito and K. Sato in [34].

We observe that if for an abelian variety A we proved finiteness of the filtration $\{F^r\}_{r \geq 0}$, then we would be able to establish Colliot-Thélène's conjecture for odd primes p and at least for abelian varieties with split semi-ordinary reduction.

2.5.2 The cycle map and the Brauer-Manin pairing

Let A be an abelian variety over k of dimension d. In this section we will prove some results about the cycle map to étale cohomology and the Brauer-Manin pairing. First, we review some necessary definitions.

1. Let X be a smooth, projective, geometrically connected variety over the p-adic field k. By the Brauer group of X we will always mean the group $H^2(X_{\text{et}}, \mathbb{G}_m)$ and we will denote it by $Br(X)$. There is a well defined pairing of abelian groups

$$<,>_X : CH_0(X) \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

defined as follows. If $\alpha \in Br(X)$ is an element of the Brauer group and $x \in X$ a closed point of X, then the closed immersion $\iota_x : \text{Spec}(k(x)) \to X$ induces the pullback

$\iota_x^\star : Br(X) \to Br(k(x))$. We define

$$< x, \alpha >_X = \mathrm{Cor}_{k(x)/k}(\iota_x^\star(\alpha)) \in Br(k) \simeq \mathbb{Q}/\mathbb{Z},$$

where $\mathrm{Cor}_{k(x)/k} : Br(k(x)) \to Br(k)$ is the Correstriction map and the isomorphism $Br(k) \simeq \mathbb{Q}/\mathbb{Z}$ is via the invariant map of local class field theory. To show that this definition factors through rational equivalence, we reduce to the case of curves, where the well-definedness of the pairing follows by a result of S. Lichtenbaum, [23].

2. We say that an abelian variety A of dimension d over k has split multiplicative reduction, if the connected component $\mathcal{A}_s^0$, containing the neutral element of the special fiber $\mathcal{A}_s$ of the Néron model $\mathcal{A}$ of A is a split torus. In this case, the theory of degeneration of abelian varieties ([12], Chapter III, Proposition 8.1) yields that there exists a split torus $T \simeq \mathbb{G}_m^{\oplus d}$ over k and a finitely generated free abelian group $L \subset T(k)$ of rank d, such that for any finite extension k'/k, there is an isomorphism $A(k') \simeq T(k')/L$. A special example of such an abelian variety is a Tate elliptic curve.

3. Let X be a smooth projective and geometrically connected variety over k of dimension d. The cycle map to étale cohomology

$$CH_0(X)/n \xrightarrow{\rho_{X,n}} H^{2d}(X_{\mathrm{et}}, \mu_n^{\otimes d})$$

is defined as follows. For a closed point x of X, we have the Gysin map

$$H^0(x_{et}, \mathbb{Z}/n) \longrightarrow H^{2d}(X_{et}, \mu_n^{\otimes d}).$$

Then the cycle map is defined as $\rho_{X,n}([x]) = G_x(1)$.

We start with the following lemma, which shows the relation between the cycle map and the Brauer-manin pairing.

Lemma 2.5.5. *Let X be a smooth, projective, geometrically connected variety over k. There is a commutative diagram*

$$\begin{array}{ccc} CH_0(X)/n & \xrightarrow[\rho_{X,n}]{} & H^{2d}(X_{\text{et}}, \mu_n^{\otimes d}) \\ {\scriptstyle <,>_X}\downarrow & & \downarrow \\ (Br(X)[n])^\star & \longrightarrow & (H^2(X, \mu_n))^\star. \end{array}$$

Proof. First we observe that Tate and Poincaré duality induce a non-degenerate pairing of finite abelian groups

$$H^2(X, \mu_n) \times H^{2d}(X, \mu_n^{\otimes d}) \to \mathbb{Z}/n.$$

(see [33] for a proof of this statement, due to S. Saito). Notice that this pairing induces the right vertical map of the diagram stated in the lemma, $H^{2d}(X, \mu_n^{\otimes d}) \xrightarrow{\simeq} H^2(X, \mu_n)^\star$, which is therefore an isomorphism. Let now x be a closed point of X. We obtain a commutative diagram

$$\begin{array}{ccc} H^0(x, \mathbb{Z}/n) & \xrightarrow{G_x} & H^{2d}(X_{\text{et}}, \mu_n^{\otimes d}) \\ \simeq\downarrow & & \downarrow \\ (H^2(x, \mu_n))^\star & \xrightarrow{\iota_x^\star} & (H^2(X, \mu_n))^\star, \end{array}$$

where the left vertical map is the isomorphism induced by Tate duality and $H^2(X, \mu_n) \xrightarrow{\iota_x^\star} H^2(x, \mu_n)$ is the pullback map. The result now follows from the commutative diagram

$$\begin{array}{ccc} H^2(X, \mu_n) & \xrightarrow{\iota_x^\star} & H^2(x, \mu_n) \\ \downarrow & & \simeq\downarrow \\ Br(X)[n] & \xrightarrow{\iota_x^\star} & Br(k(x))[n]. \end{array}$$

Here the two vertical maps arise from the Kummer sequence on X_{et} and x_{et} respectively. Notice that the commutativity of the last diagram follows from the functoriality properties of étale cohomology (universality of the functor $H^\star(X_{\text{et}}, \_)$).

We can thus conclude that the map $H^0(x, \mathbb{Z}/n) \to H^{2d}(X, \mu_n^{\otimes d}) \to (H^2(X, \mu_n))^\star$ factors through $H^0(x,\mathbb{Z}/n) \to (Br(X)[n])^\star$ and the lemma follows.

$\square$

Corollary 2.5.6. *The kernel of the map* $CH_0(X)/n \stackrel{<,>_X}{\longrightarrow} (Br(X)[n])^\star$ *coincides with the kernel of the cycle map* $\rho_{X,n} : CH_0(X)/n \to H^{2d}(X, \mu_n^{\otimes d})$.

Proof. This follows immediately from the commutative diagram of lemma 2.5.5, as soon as we notice that the right vertical map is an isomorphism, and the bottom horizontal map is injective. The injectivity of $(Br(X)/n)^\star \to (H^2(X,\mu_n))^\star$ follows by applying the exact functor $\mathrm{Hom}(\_,\mathbb{Q}/\mathbb{Z})$ to the short exact sequence

$$0 \to \mathrm{Pic(X)/n} \to \mathrm{H^2(X,\mu_n)} \to \mathrm{Br(X)[n]} \to 0,$$

arising from the Kummer sequence for X.

$\square$

The Hochschild-Serre spectral sequence

We now go back to the case of an abelian variety A of dimension d over the p-adic field k . We consider the Hochschild-Serre spectral sequence,

$$E_2^{pq} = H^p(k, H^q(A_{\overline{k}}, \mathcal{F})) \Rightarrow H^{p+q}(A, \mathcal{F}),$$

where $\mathcal{F}$ is any abelian sheaf on A_{et}. For any $q \geq 0$, the spectral sequence gives a descending filtration

$$H^q(A_{\mathrm{et}}, \mathcal{F}) = H_0^q \supset H_1^q \supset \dots H_{q-1}^q \supset H_q^q \supset 0,$$

with quotients $H_i^q/H_{i+1}^q \simeq E_\infty^{i,q-i}$. First we observe that $H_i^q = 0$, for $i \geq 3$. For, the p-adic field k has cohomological dimension 2, which forces $E_2^{i,q-i}$ to be zero for $i \geq 3$. We will use this filtration for the groups $H^{2d}(A, \mu_n^{\otimes d})$ and $Br(A) = H^2(A, \mathbb{G}_m)$.

Lemma 2.5.7. *After $\otimes\mathbb{Z}[\frac{1}{2}]$, the spectral sequence*

$$H_2^{pq} = H^p(k, H^q(A_{\overline{k}}, \mu_n^{\otimes d})) \Rightarrow H^{p+q}(A, \mu_n^{\otimes d})$$

degenerates at level 2.

Proof. We need to show that all the differentials d_2^{pq} become zero after $\otimes\mathbb{Z}[\frac{1}{2}]$. The statement is clear when $p \geq 1$ or $p < 0$ or $q < 1$ even before $\otimes\mathbb{Z}[\frac{1}{2}]$. We will show that for $q \geq 1$, the map

$$d_2^{0,q} : H^0(k, H^q(\overline{A}, \mu_n^{\otimes d})) \to H^2(k, H^{q-1}(\overline{A}, \mu_n^{\otimes d}))$$

has the property $2d_2^{0,q} = 0$. Let $m \in \mathbb{Z}$ be an integer. We consider the multiplication by m map $A \xrightarrow{m} A$ on A. The map m induces a pull back map on cohomology,

$$H^p(k, H^q(A_{\overline{k}}, \mu_n^{\otimes d})) \xrightarrow{m^\star} H^p(k, H^q(A_{\overline{k}}, \mu_n^{\otimes d})),$$

for every p, q. Moreover, since m is a morphism of schemes, the pullback $m^\star$ is compatible with the differentials, i.e. the following diagram commutes, for every $q \geq 1$.

$$\begin{array}{ccc} H^0(k, H^q(A_{\overline{k}}, \mu_n^{\otimes d})) & \xrightarrow{m^\star} & H^0(k, H^q(A_{\overline{k}}, \mu_n^{\otimes d})) \\ \downarrow d^{0,q} & & \downarrow d^{0,q} \\ H^2(k, H^{q-1}(A_{\overline{k}}, \mu_n^{\otimes d})) & \xrightarrow{m^\star} & H^2(k, H^{q-1}(A_{\overline{k}}, \mu_n^{\otimes d})). \end{array}$$

The action of $m^\star$ on $H^0(k, H^q(A_{\overline{k}}, \mu_n^{\otimes d}))$ is multiplication by m^q. For, the action is induced by the action of $m^\star$ on $H^1(A_{\overline{k}}, \mathbb{Z}/n) = \mathrm{Hom}(A[n], \mathbb{Z}/n)$, which is multiplication by m. Let $\alpha \in H^0(k, H^q(\overline{A}, \mu_n^{\otimes d}))$. Taking $m = -1$ and using the fact that $d^{0,q}$ is a group homomorphism, we get

$$d^{0,q}((-1)^\star(\alpha)) = d^{0,q}((-1)^q\alpha) = (-1)^q d^{0,q}(\alpha).$$

On the other hand, using the commutativity of the diagram above, we obtain

$$d^{0,q}((-1)^{\star}(\alpha)) = (-1)^{\star} d^{0,q}(\alpha) = (-1)^{q-1} d^{0,q}(\alpha).$$

We conclude that $d^{0,q}(\alpha) = -d^{0,q}(\alpha)$ and hence $2d^{0,q}(\alpha) = 0$.

□

Corollary 2.5.8. *The filtration $H_0^{2d} \supset H_1^{2d} \supset H_2^{2d} \supset 0$ of the group $H^{2d}(A, \mu_n^{\otimes d})$ induced by the Hochschild-Serre spectral sequence has successive quotients*

$$H_i^{2d}/H_{i+1}^{2d} = \begin{cases} H^0(k, H^{2d}(A, \mu_n^{\otimes d})), \ i = 0 \\ H^1(k, H^{2d-1}(A, \mu_n^{\otimes d})), \ i = 1 \\ H^2(k, H^{2d-2}(A, \mu_n^{\otimes d})), \ i = 2 \end{cases}$$

Proof. The third equality follows directly from lemma 2.5.7. We claim that for $p = 0, 1$, $E_\infty^{p,2d-p} = E_2^{p,2d-p}$ before $\otimes\mathbb{Z}[\frac{1}{2}]$. For $p = 1$ the statement follows immediately from the observation that both the differentials $d_2^{1,2d-1}$ and $d_2^{-1,2d}$ are zero.
For $p = 0$, we first observe that $E_\infty^{0,2d} = E_3^{0,2d} = \ker d_2^{0,2d}$. For, the map $d_3^{0,2d} : E_3^{0,2d} \to E_3^{3,2d-3}$ is the zero map, since $E_3^{3,2d-3} = 0$. Thus, we have an inclusion

$$H_0^{2d}/H_1^{2d} = \ker d_2^{0,2d} \overset{j}{\hookrightarrow} E_2^{0,2d} = H^0(k, H^2(A_{\overline{k}}, \mu_n^{\otimes d})).$$

Since A is projective, Poincaré duality yields an isomorphism

$$H^{2d}(A_{\overline{k}}, \mu_n^{\otimes d}) \simeq \mathrm{Hom}(\mathrm{H}^0(\mathrm{A}_{\overline{\mathrm{k}}}, \mathbb{Z}/\mathrm{n}), \mathbb{Z}/\mathrm{n}) \simeq \mathbb{Z}/\mathrm{n}.$$

We therefore obtain the following commutative diagram:

$$\begin{array}{ccccc} & & H_0^{2d} & \xleftarrow[\rho_{A,n}]{} & CH_0(A)/n \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \deg} \\ 0 & \longrightarrow & H_0^{2d}/H_1^{2d} & \xrightarrow{j} & \mathbb{Z}/n \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

Notice that since A is an abelian variety, there exists a k-rational point, and hence the degree map is surjective. Since $\deg = j \circ p \circ \rho_{A,n}$, we conclude that the map j is surjective.

□

Proposition 2.5.9. *Let A be an abelian variety over k and $n \geq 1$ a positive integer. The cycle map*

$$\rho_{A,n} : CH_0(A)/n \to H^{2d}(A, \mu_n^{\otimes d}),$$

when restricted to F^3/n is the zero map. Moreover, if A has split multiplicative reduction, then after $\otimes\mathbb{Z}[\frac{1}{2}]$, the kernel of the cycle map is precisely the group $((F^3 + nCH_0(A))/n) \otimes \mathbb{Z}[\frac{1}{2}]$.

Proof. Consider the filtration $H_0^{2d} \supset H_1^{2d} \supset H_2^{2d} \supset 0$ of the group $H^{2d}(A, \mu_n^{\otimes d})$. Then from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1/n & \longrightarrow & CH_0(A)/n & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_{A,n}} & & \downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & H_1^{2d} & \longrightarrow & H^{2d}(A, \mu_n^{\otimes d}) & \longrightarrow & H_0^{2d}/H_1^{2d} & \longrightarrow & 0 \end{array}$$

we obtain that F^1/n is mapped to H_1^{2d} via the cycle map and the kernel of $\rho_{A,n}$ is contained in F^1/n. Notice that the right vertical map is an isomorphism by lemma 2.5.8. Next we consider the commutative diagram

$$\begin{array}{ccccccccc} & & F^2/n & \longrightarrow & F^1/n & \longrightarrow & (F^1/F^2)/n & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_{A,n}} & & \downarrow & & \\ 0 & \longrightarrow & H_2^{2d} & \longrightarrow & H_1^{2d} & \longrightarrow & H_1^{2d}/H_2^{2d} & \longrightarrow & 0 \end{array}$$

from where we obtain that F^2/n maps to H_2^{2d} via the cycle map and the kernel of $\rho_{A,n}$ is contained in the image of the map $F^2/n \to F^1/n$. Notice that in this case, the right vertical map is injective. For, by Poincaré duality and the étale cohomology of abelian varieties over an algebraically closed field, we obtain isomorphisms

$$H^{2d-1}(A_{\overline{k}}, \mu_n^{\otimes d}) \simeq H^1(A_{\overline{k}}, \mathbb{Z}/n)(-1) \simeq \mathrm{Hom(A[n], \mathbb{Z}/n)(-1)} \simeq \mathrm{A[n]}.$$

and therefore the map $(F^1/F^2)/n \to H_1^{2d}/H_2^{2d}$ coincides with the map $A(k)/n \hookrightarrow H^1(k, A[n])$ arising from the Kummer sequence for A, $0 \to A[n] \to A \xrightarrow{n} A \to 0$.

Next we turn our attention to the map $\rho_{A,n} : F^2/n \to H_2^{2d}$. Again, by Poincaré duality we obtain

$$H^{2d-2}(A_{\overline{k}}, \mu_n^{\otimes d}) \simeq \mathrm{Hom(H^2(A_{\overline{k}}, \mathbb{Z}/n), \mathbb{Z}/n)} \simeq$$
$$\mathrm{Hom(\wedge^2(Hom(A[n], \mathbb{Z}/n), \mathbb{Z}/n)} \simeq \wedge^2\mathrm{A[n]}.$$

Thus, the cycle map induces $F^2/n \xrightarrow{\rho_{A,n}} H^2(k, \wedge^2 A[n])$.

Using now the map $s_n : (F^2/F^3)/n \to H^2(k, \wedge^2 A[n])$ obtained in corollary (2.4.3), we deduce that $\rho_{A,n} : F^2/n \to H_2^{2d}$ factors through $(F^2/F^3)/n$ and therefore the group F^3/n, being the kernel of $F^2/n \to (F^2/F^3)/n$, is contained in the kernel of the map $\rho_{A,n}$. This concludes the proof of the first statement of the proposition.

Assume now that A has split multiplicative reduction. We will prove that the map

$$(F^2/F^3)/n \otimes \mathbb{Z}[\frac{1}{2}] \to H^2(k, \wedge^2 A[n]) \otimes \mathbb{Z}[\frac{1}{2}]$$

is injective. By theorem (2.2.8), it suffices to prove that the Somekawa map

$$s_n : S_2(k; A)/n \otimes \mathbb{Z}[\frac{1}{2}] \to H^2(k, \wedge^2 A[n]) \otimes \mathbb{Z}[\frac{1}{2}]$$

is injective. T.Yamazaki, in [44], proved that in the split multiplicative reduction case, the map

$$s_n : K_2(k; A)/n \to H^2(k, A[n] \otimes A[n])$$

is injective. Consider the following commutative diagram

$$\begin{array}{ccc} K_2(k; A)/n & \overset{s_n}{\hookrightarrow} & H^2(k, A[n] \otimes A[n]) \\ \downarrow & & \downarrow \\ S_2(k; A)/n & \xrightarrow{s_n} & H^2(k, \wedge^2 A[n]). \end{array}$$

Notice that after $\otimes\mathbb{Z}[\frac{1}{2}]$ both vertical maps have sections. Namely, the maps

$$S_2(k; A)/n \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{i} K_2(k; A)/n \otimes \mathbb{Z}[\frac{1}{2}]$$
$$\{a, b\}_{k'/k} \to \frac{\{a, b\}_{k'/k} + \{b, a\}_{k'/k}}{2}$$

and

$$H^2(k, \wedge^2 A[n]) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{j} H^2(k, A[n] \otimes A[n]) \otimes \mathbb{Z}[\frac{1}{2}],$$

induced by the map

$$\wedge^2 A[n] \otimes \mathbb{Z}[\frac{1}{2}] \to A[n] \otimes A[n] \otimes \mathbb{Z}[\frac{1}{2}]$$
$$x \wedge y \to \frac{x \otimes y - y \otimes x}{2}.$$

The injectivity of the map

$$s_n : S_2(k; A)/n \otimes \mathbb{Z}[\frac{1}{2}] \to H^2(k, \wedge^2 A[n]) \otimes \mathbb{Z}[\frac{1}{2}]$$

hence follows from the commutative diagram

$$\begin{array}{ccc} K_2(k;A)/n \otimes \mathbb{Z}[\frac{1}{2}] & \overset{s_n}{\hookrightarrow} & H^2(k, A[n] \otimes A[n]) \otimes \mathbb{Z}[\frac{1}{2}] \\ i \uparrow & & \uparrow j \\ S_2(k;A)/n \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{s_n} & H^2(k, \wedge^2 A[n]) \otimes \mathbb{Z}[\frac{1}{2}]. \end{array}$$

□

Theorem 2.5.10. *Let A be an abelian variety over k. The subgroup F^3 is contained in the kernel of the map*

$$j : CH_0(A) \to Br(A)^\star.$$

If moreover A has split multiplicative reduction, then the kernel of the map

$$CH_0(A) \otimes \mathbb{Z}[\frac{1}{2}] \overset{j \otimes \mathbb{Z}[\frac{1}{2}]}{\longrightarrow} Br(A)^\star \otimes \mathbb{Z}[\frac{1}{2}]$$

is the subgroup D of $F^2 \otimes \mathbb{Z}[\frac{1}{2}]$, which contains $F^3 \otimes \mathbb{Z}[\frac{1}{2}]$ and is such that $D/(F^3 \otimes \mathbb{Z}[\frac{1}{2}])$ is the maximal divisible subgroup of $F^2/F^3 \otimes \mathbb{Z}[\frac{1}{2}]$.

Proof. Assume to contradiction that $F^3 \subsetneq \ker j$ and let $w \in F^3$ be such that $j(w) \neq 0$. This means that there exists some element $\alpha \in Br(A)$ such that $< w, \alpha > \neq 0$. Notice that the group $Br(A)$ is torsion, because it is a subgroup of $Br(K)$, where K is the function field of A (for a proof of the last statement see [16], II, corollary 1.10). Let m be the order of α. Then $j(w)$ gives a nonzero morphism $Br(A)[m] \to \mathbb{Q}/\mathbb{Z}$. The map $F^3 \to Br(A)[m]^\star$ factors through F^3/m and by theorem (2.5.9) , we get a commutative diagram

$$\begin{array}{ccc} F^3/m & \xrightarrow{0} & H^{2d}(X_{\text{et}}, \mu_n^{\otimes d}) \\ \downarrow & & \downarrow \\ Br(X)[m]^\star & \longrightarrow & (H^2(X, \mu_n))^\star. \end{array}$$

Since the bottom map is injective, we conclude that the map $F^3/m \to Br(A)[m]^\star$ is zero, which is the desired contradiction.

Next, proposition 2.5.3 gives us an isomorphism

$$F^2/F^3 \otimes \mathbb{Z}[\frac{1}{2}] \simeq D_0 \oplus F_0,$$

where F_0 is a finite group and D_0 is divisible. Let D be the subgroup of $F^2 \otimes \mathbb{Z}[\frac{1}{2}]$ such that $D/(F^3 \otimes \mathbb{Z}[\frac{1}{2}]) \simeq D_0$. It is clear that $D \subset \ker(CH_0(A) \otimes \mathbb{Z}[\frac{1}{2}] \to Br(A)^\star \otimes \mathbb{Z}[\frac{1}{2}])$, since $Br(A)$ is a torsion group.

Assume now that A has split multiplicative reduction. We will show that D is in fact equal to $\ker(j \otimes \frac{1}{2})$. First, we consider the filtration $H_0^2 \supset H_1^2 \supset H_2^2 \supset 0$ of $Br(A)$ arising from the Hochschild-Serre spectral sequence, $H^p(k, H^q(\overline{A}, \mathbb{G}_m)) \Rightarrow H^{p+q}(A, \mathbb{G}_m)$.
We can easily see that $E_\infty^{1,1} = E_2^{1,1}$. For, both the differentials $d_2^{1,1}$ and $d_2^{-1,2}$ are zero. This yields an isomorphism $H_1^2/H_2^2 \simeq H^1(k, H^1(A_{\overline{k}}, \mathbb{G}_m))$. Next we observe that $E_\infty^{2,0} = E_3^{2,0} = E_2^{2,0}/\operatorname{Im}(E_2^{0,1} \to E_2^{2,0})$. For, both the differentials $d_3^{2,0}$ and $d_3^{-1,2}$ are zero. In particular, we have a surjection $E_2^{2,0} \to H_2^2 \to 0$. Dualizing, we obtain an inclusion $0 \to (H_2^2)^\star \to (E_2^{2,0})^\star$. Since A is proper, we have an isomorphism

$$E_2^{2,0} \simeq H^2(k, H^2(A_{\overline{k}}, \mathbb{G}_m)) \simeq Br(k) \simeq \mathbb{Q}/\mathbb{Z}.$$

We have a commutative diagram as follows:

$$\begin{CD}
0 @>>> F^1 @>>> CH_0(A) @>{\deg}>> \mathbb{Z} @>>> 0\\
@. @. @VV{j}V @VVV @.\\
0 @>>> (H_0^2/H_2^2)^\star @>>> (H_0^2)^\star @>>> (H_2^2)^\star @>>> 0
\end{CD}$$

We claim that the right vertical map is an inclusion. To see this, we observe that the composition

$$\mathbb{Z} \to (H_2^2)^\star \hookrightarrow (E_2^{2,0})^\star$$

coincides with the inclusion $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}} = (Br(k))^\star$. (We note here that T.Yamazaki is using this exact same argument for the injectivity in the proof of his proposition 3.1 in [44]). We conclude that $\ker j \subset F^1$. Moreover, under this map, F^1 is sent to the subgroup $(H_0^2/H_2^2)^\star$. Next we consider the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow F^2 & \longrightarrow & F^1 & \xrightarrow{\mathrm{alb}_A} & A(k) & \longrightarrow & 0\\
& \swarrow^{j} & & & \Big\downarrow{\scriptstyle \lambda\,\simeq} & & \\
(H_0^2/H_2^2)^\star \longrightarrow & H^1(k, Pic(A_{\overline{k}}))^\star & & \longrightarrow & H^1(k, Pic^0(A_{\overline{k}}))^\star & \longrightarrow & 0.
\end{array}$$

The right vertical map is the isomorphism obtained by Tate duality (see for example [26] for the definition of this isomorphism). We can therefore deduce that $\ker j \subset F^2$. For, if $x \in F^1$ is such that $j(x) = 0$, then $\lambda(\mathrm{alb}_A(x)) = 0$, and since λ is injective, we conclude that $\mathrm{alb}_A(x) = 0$.

Next, notice that the map $j \otimes \mathbb{Z}[\frac{1}{2}]$ induces

$$(F^2 \otimes \mathbb{Z}[\frac{1}{2}])/D \stackrel{j\otimes\mathbb{Z}[\frac{1}{2}]}{\longrightarrow} Br(A)^\star \otimes \mathbb{Z}[\frac{1}{2}].$$

Let n be the order of $(F^2 \otimes \mathbb{Z}[\frac{1}{2}])/D = F_0$. Since D is divisible, we have an equality

$$\frac{(F^2/F^3) \otimes \mathbb{Z}[\frac{1}{2}]}{n} = F_0.$$

We obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & (F^2 \otimes \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & F_0 & \longrightarrow & 0 \\ & & & & \downarrow j & & \downarrow j & & \\ & & & & (Br(A)^\star \otimes \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & (Br(A)[n]^\star \otimes \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & 0. \end{array}$$

Since the kernel of the map

$$j \otimes \mathbb{Z}[\frac{1}{2}] : \frac{(F^2/F^3) \otimes \mathbb{Z}[\frac{1}{2}]}{n} \to Br(A)[n]^\star \otimes \mathbb{Z}[\frac{1}{2}]$$

coincides with the kernel of the cycle map

$$\rho_{A,n} \otimes \mathbb{Z}[\frac{1}{2}] : \frac{(F^2/F^3) \otimes \mathbb{Z}[\frac{1}{2}]}{n} \to H^{2d}(A, \mu_n^{\otimes d})\mathbb{Z}[\frac{1}{2}]$$

(corollary (2.5.6)), the result follows by the second part of theorem (2.5.9).

□

CHAPTER 3

SEMI-ABELIAN VARIETIES AND SUSLIN'S SINGULAR HOMOLOGY

3.1 Introduction

The main goal of this chapter is to generalize theorem (2.1.1) for semi-abelian varieties. Recall that a semi-abelian variety G over a field k is an extension of an abelian variety by a torus. In particular, semi-abelian varieties are quasi-projective.

For a smooth quasi-projective variety X over a field k, the Chow group of zero cycles $CH_0(X)$ still provides a fundamental invariant of the variety. When X is not projective though, the group $CH_0(X)$ loses some of its properties. For example, the degree map $\deg : Z_0(X) \to \mathbb{Z}$ no longer factors through $CH_0(X)$. This group is too small and it needs to be replaced by a larger class group of zero cycles.

The next question that arises is if we can have any reasonable analogue of the Albanese map in this quasi-projective setting. We note that J.P. Serre proved the existence, for every smooth quasi-projective variety X over a perfect field k, of a generalized Albanese variety, Alb_X. This is a semi-abelian variety which is universal for regular maps from X to semi-abelian varieties. Therefore the question comes down to whether we can find a new geometric invariant of the variety, namely an appropriate quotient of $Z_0(X)$, such that both the degree map and the generalized Albanese map become well defined. The answer is given by a theorem of Spiess and Szamueli in [38]. It turns out that the appropriate group is Suslin's singular homology group, $H_0^{sing}(X)$.

We can now reset our goal for this chapter. Given a semi-abelian variety G over a perfect field k, we want to construct a decreasing filtration $\{F^r\}_{r\geq 0}$ of $H_0^{sing}(G)$ such that the successive quotients are "almost" isomorphic to some Somekawa K-group.

For semi-abelian varieties $G_1, \ldots, G_r$ over k, Somekawa gave a definition of a K-group $K(k; G_1, \ldots, G_r)$. His definition though turned out to be not very suitable for geometric applications. The problem is that the Weil reciprocity of $K(k;G_1,\dots,G_r)$ is very hard to describe, when some of the coordinates have a toric part. Kahn and Yamazaki in [22] suggested a more geometric variant of this group, namely the group which we call the geometric K-group $K^{geo}(k;G_1,\dots,G_r)$ attached to $G_1,\dots,G_r$. We can now state the main theorem of this chapter.

Theorem 3.1.1. *Let k be a perfect field and G a semi-abelian variety over k. There exists a decreasing filtration $\{F^r\}_{r\geq 0}$ of $H_0^{sing}(G)$ such that there are canonical isomorphisms of abelian groups:*

$$\Phi_r : \mathbb{Z}[\frac{1}{r!}]\otimes\frac{F^r}{F^{r+1}} \xrightarrow{\simeq} \mathbb{Z}[\frac{1}{r!}]\otimes S_r^{geo}(k;G),\ r\geq 0.$$

In the above theorem we use a similar notation as in the previous chapter, namely we denote by $S_r^{geo}(k;G) := \dfrac{K_r^{geo}(k;G)}{<\{x_1,\cdots,x_r\}_{k'/k}-\{x_{\sigma(1)},\cdots,x_{\sigma(r)}\}_{k'/k}>}$, where σ runs in the set of permutations of $\{1,\cdots,r\}$.

Remark 3.1.2. Theorem (3.1.1) is stated in terms of Suslin's singular homology, $H_0^{sing}(G)$. In reality though, the group we will be using is a variant of Wiesend's class group, $W(G)$, a group introduced by Yamazaki in [45]. For a quasi-projective variety X over a perfect field k the groups $W(X)$ and $H_0^{sing}(X)$ coincide. In the next section we will review the definitions of both those groups.

Remark 3.1.3. Kahn and Yamazaki in [22] define a geometric K-group $K^{geo}(k;\mathcal{F}_1,\dots,\mathcal{F}_r)$ for more general coordinates, namely for homotopy invariant Nisnevich sheaves with transfers. We will discuss more about their definition in section 3.4. When G_i is a semi-abelian variety for every $i=1,\cdots,r$, they in fact prove ([22], theorem 11.14) a canonical isomorphism

$$K(k;G_1,\dots,G_r)\simeq K^{geo}(k;G_1,\dots,G_r).$$

To prove this coincidence with the classical Somekawa K-group, they use very deep and long computations of symbols.

An important application of their theory in the study of zero cycles is that Kahn and Yamazaki generalize the result of Raskind and Spiess mentioned in example (1.1.1) for a product of smooth open curves. In particular, let $X = C_1 \times C_2$ be the product of two smooth open curves over a perfect field k. They prove an isomorphism $\ker(\mathrm{alb}_X) \simeq K(k; J_{\mathfrak{m}_1}, J_{\mathfrak{m}_2})$, where $J_{\mathfrak{m}_i}$ is the generalized Jacobian variety of C_i corresponding to the modulus $\mathfrak{m}_i = \sum_{P \in \overline{C}_i - C_i} P$, $i = 1, 2$. Here by $\overline{C}_i$ we denote the smooth completion of C_i.

Remark 3.1.4. The rational analogue of theorem (3.1.1) was obtained in a recent paper of Sugiyama [40]. In this paper, Sugiyama studies the motive of a semi-abelian variety G generalizing the results of Beauville [2]. In this chapter we give the integral version of his construction.

3.2 Suslin's singular homology and geometric K-group

In this section we review the definitions of Suslin's homology and Wiesend's class group and we introduce the geometric K-group $K^{geo}(k; G_1, \cdots, G_r)$ attached to semi-abelian varieties $G_1, \cdots, G_r$. We start with the case of curves.

3.2.1 Generalized Jacobians

Let C be a smooth curve over a perfect field k. Let $\overline{C}$ be its smooth compactification and let $\mathfrak{m} = \sum_{P \in \overline{C} \setminus C} P$ be the reduced Cartier divisor on $\overline{C}$ supported on $\overline{C} \setminus C$. The divisor $\mathfrak{m}$ is usually called a modulus condition on $\overline{C}$. Then there exists a generalized Jacobian variety $J_{\mathfrak{m}}$ of $\overline{C}$ corresponding to the modulus $\mathfrak{m}$, which is a semi-abelian variety satisfying the following universal property. There is a morphism $\varphi : C \to J_{\mathfrak{m}}$ such that if $\psi : \overline{C} \dashrightarrow G$ is a rational map to some semi-abelian variety G over k, which is regular on C, then ψ factors uniquely through φ. When $\overline{C} = C$, or alternatively $\mathfrak{m} = 0$, $J_{\mathfrak{m}}$ coincides with the usual Jacobian variety J of $\overline{C}$ and in the case $\overline{C}$ has a k-rational point, the Abel-Jacobi map gives an isomorphism

$$Pic^0(\overline{C}) \simeq \frac{Z_0(\overline{C})^{\deg=0}}{<(\operatorname{div}(f) : f \in k(\overline{C})^\times>} \xrightarrow{\simeq} J(k).$$

When $C \subsetneq \overline{C}$ and $C(k) \neq \emptyset$, the analogous expression in terms of zero cycles for the generalized Jacobian $J_{\mathfrak{m}}$, where $\mathfrak{m} = \sum_{P \in \overline{C} \setminus C} P$, is the following.

$$\frac{Z_0(C)^{\deg 0}}{< \operatorname{div}(f) : f \in k(\overline{C})^\times, f(P) = 1, P \in \overline{C} \setminus C >} \xrightarrow{\simeq} J_{\mathfrak{m}}(k).$$

This is part of the more general theory of generalized Jacobians of Rosenlicht-Serre. We will discuss some generalizations of this theory in the next chapter. As mentioned in section (3.1), Serre showed the existence of generalized Albanese varieties for every smooth quasi-projective variety X.

3.2.2 The groups $W(X)$ and $H_0^{sing}(X)$

Let X be a smooth quasi-projective variety over a field k. We denote by $X_{(1)}$ the set of all closed irreducible curves in X and by $Z_0(X)$ the free abelian group of zero cycles on X. Moreover, for $C \in X_{(1)}$, we denote $\widetilde{C} \xrightarrow{\pi} C$ its normalization and $\iota : \widetilde{C} \hookrightarrow \overline{C}$ its smooth completion.

Definition 3.2.1. We define the Wiesend tame ideal class group $W(X)$ to be the quotient of $Z_0(X)$, by the subgroup generated by cycles of the form $\{\operatorname{div}(f) : f \in k(C)^\times\}_{C \in X_{(1)}}$ with f having the property that $f(P) = 1$, for every $P \in \overline{C} - \widetilde{C}$.

Remark 3.2.2. This definition is a direct generalization of the generalized Jacobian $J_{\mathfrak{m}}$ of a smooth complete curve corresponding to a reduced modulus $\mathfrak{m}$. We note that Wiesend in [43] defined a similar class group for arithmetic schemes of finite type over $\operatorname{Spec}\mathbb{Z}$. Definition (3.2.1) is a variant due to Yamazaki [45], who in fact used this group to obtain a generalization of the Brauer-Manin pairing for quasi-projective varieties. Namely, he showed that for a smooth quasi-projective variety X over a p-adic field k there is a well defined pairing

$$Br(X) \times W(X) \to \mathbb{Q}/\mathbb{Z},$$

defined similarly as in the case of a proper variety.

Definition 3.2.3. Let X be a smooth quasi-projective variety over a field k. Suslin's singular homology group, $H_0^{sing}(X)$, is the quotient of $Z_0(X)$ modulo the subgroup generated by zero cycles of the form $\iota_0^\star(Z) - \iota_1^\star(Z)$, where $\iota_\lambda : X \to X \times \mathbb{A}^1$ is the inclusion $x \to (x, \lambda)$, for $\lambda = 0, 1$, and Z runs through all closed integral subvarieties of $X \times \mathbb{A}^1$ such that the projection $Z \to \mathbb{A}^1$ is finite and surjective.

Properties

1. When the base field k is perfect, it is a theorem of Schmidt ([35], theorem 5.1) that the groups $W(X)$ and $H_0^{sing}(X)$ are isomorphic. From now on we assume that k is perfect. In most of the statements we will be using $W(X)$, but we will be interchanging between the two definitions without further notice.

2. When X is proper over k, the groups $CH_0(X)$ and $W(X)$ coincide.

3. It is clear that the degree map $\deg : Z_0(X) \to \mathbb{Z}$, $x \to [k(x) : k]$, factors through $W(X)$. We will denote by $W^0(X)$ the subgroup of degree zero cycle classes.

4. When $\overline{C}$ is a smooth, complete, geometrically irreducible curve over k and S a finite set of points of $\overline{C}$, then for the smooth curve $C = \overline{C} - S$, the group $W(C)$ coincides with the group of classes of divisors on $\overline{C}$ prime to S modulo S-equivalence, as defined in [36](Ch.V). Notice that when C has a k-rational point, the abelian group $W^0(C)$ is isomorphic to the generalized Jacobian of $\overline{C}$ corresponding to the modulus $\mathfrak{m} = \sum_{P \in S} P$.

5. The group $W(X)$ is covariant functorial for morphisms of varieties $X \to Y$ (lemma 2.3 in [45]).

6. Generalized Albanese map: If X is a smooth variety over a perfect field k, there is a generalized albanese map $\mathrm{alb}_X : W^0(X) \to G_X(k)$, where G_X is the generalized Albanese variety of X. For a proof of the well defined of the generalized Albanese map we refer to [38]. In this article, T.Szamueli and M.Spiess use the group $H_0^{sing}(X)$ instead.

3.2.3 The geometric K-group

In this subsection we define the geometric variant of Somekawa's K-group introduced by Kahn and Yamazaki in [22].

Definition 3.2.4. Let $G_1, \ldots, G_r$ be semi-abelian varieties over a perfect field k. We define the geometric K-group attached to $G_1, \ldots, G_r$ as follows.

$$K^{geo}(k; G_1, \ldots, G_r) = (G_1 \bigotimes^M \cdots \bigotimes^M G_r)(k)/R_0,$$

where R_0 is the subgroup generated by the following family of elements:

Weil reciprocity II: Let K be a function field in one variable over k and let C be the smooth complete curve having function field K. Let $g_i \in G_i(K)$, for $i = 1, \ldots, r$. Each g_i extends to a rational map $g_i : C \dashrightarrow G_i$, which is regular outside a finite set of places S_i of C. We consider the set $S = \bigcup_{i=1}^{r} S_i$. Let $f \in K^\times$ be a function such that $f(P) = 1$, for every $P \in S$. Then we require

$$\sum_{P \notin S} \mathrm{ord}_P(f)(g_1(P) \otimes \cdots \otimes g_r(P))_{k(P)/k} \in R_0.$$

In the above definition, $\bigotimes^M$ is the product of Mackey functors defined in section 1.2.2.

Remark 3.2.5. Let K be a function field in one variable over k and C be the unique smooth complete curve having function field K. From now on we will refer to C as the corresponding smooth complete curve. To make definition (3.2.4) more precise, let K be such a function field, let $g_i \in G_i(K)$ for some $i \in \{1, \ldots, r\}$ and let P be a closed point of C. If K_P is the completion of K at P and $\mathcal{O}_{K_P}$ its ring of integers, we will denote by $\mathcal{O}_P$ the algebraic local ring $K \cap \mathcal{O}_{K_P}$. Then, the set S_i that appears in the definition (3.2.4) is precisely the set

$$S_i = \{P \in C : g_i \notin G_i(\mathcal{O}_P)\}.$$

From now on we will refer to this set as the set of bad places of g_i.

3.3 Main Theorem

Let G be a semi-abelian variety over a perfect field k of dimension d. We will write the group law in G in multiplicative notation with 1 the neutral element.

The Pontryagin Filtration

We consider the group of zero cycles $Z_0(G)$. Since the closed points of G have a multiplication law, this group becomes a group ring with multiplication given by the Pontryagin product, $(\sum_{j=1}^{s} n_j x_j) \odot (\sum_{i=1}^{t} m_i y_i) = \sum_{i,j} n_j m_i x_j y_i$, for x_j, y_i closed points of G and $n_j,\ m_i$ integers.

Lemma 3.3.1. *The subgroup* $M =< \operatorname{div}(f) : f \in k(C)^{\times}, C \in G_{(1)},\ f(x) = 1,\ \forall x \in \overline{C} - \widetilde{C} >$ *is an ideal of* $Z_0(G)$ *and therefore* $W(G)$ *becomes a ring with the Pontryagin product.*

Proof. It suffices to show that if $x \in G$ is any closed point of G and $\operatorname{div}(f)$ is a generator of M, then $x \odot \operatorname{div}(f) \in M$. We consider the translation map

$$\begin{aligned} \tau_x : \quad & G \to G \\ & y \to xy. \end{aligned}$$

Then we observe that $x \odot \operatorname{div}(f) = \tau_{x\star}(\operatorname{div}(f))$ and since $W(G)$ has covariant functoriality, we conclude that $x \odot \operatorname{div}(f) \in M$.

□

Under this ring structure, the subgroup of degree zero elements, $W^0(G)$, becomes an ideal I of $W(G)$. By taking its powers, we can define the following filtration.

Definition 3.3.2. We consider the Pontryagin filtration of $W(G)$,

$$I^0W(G) = W(G),$$

$$I^1W(G) =< Tr_{k'/k}([x] - [1]_{k'}) : x \in G(k') >= W^0(G),$$

$$I^2W(G) =< Tr_{k'/k}([xy] - [x] - [y] + [1]_{k'}) : x,\ y \in G(k') >,$$

$$I^3W(G) =< Tr_{k'/k}([xyz] - [xy] - [xz] - [yz] + [x] + [y] + [z] - [1]_{k'}) : x,\ y,\ z \in G(k') >,$$

$$\ldots$$

$$I^rW(G) =< \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \le \nu_1 < \cdots < \nu_j \le r} Tr_{k'/k}([x_{\nu_1} x_{\nu_2} \ldots x_{\nu_j}]),\ x_i \in G(k') >,$$

where the summand corresponding to $j = 0$ is $Tr_{k'/k}((-1)^r[1]_{k'})$.

Notation 3.3.3. From now on we will denote by $\omega_{x_1,\ldots,x_r}$ the generator of I^r corresponding to the points $x_1, \ldots, x_r \in G(k')$ described above.

Remark 3.3.4. We will see that after $\otimes\mathbb{Q}$ the filtration $\{I^rW(G)\}_{r\ge0}$ just defined has the property that $I^r/I^{r+1}\otimes\mathbb{Q} \simeq S_r(k;G)\otimes\mathbb{Q}$. This is exactly what was proved also by Sugiyama in [40].

3.3.1 Definition of the Filtration

We start this subsection with the observation that G is its own generalized Albanese variety. Therefore the Albanese map takes on the form $\mathrm{alb}_G : W^0(G) \to G(k)$.

Proposition 3.3.5. *For any semi-abelian variety G over k, the natural map $j : G(k) \to K_1(k; G)$ is an isomorphism.*

Proof. First we prove surjectivity. Let k'/k be a finite extension and $x \in G(k')$. Notice that in $K_1(k; G)$ we have the following equality, $\{x\}_{k'/k} = \{\mathrm{Tr}_{k'/k}(x)\}_{k/k}$. We conclude that $j(\mathrm{Tr}_{k'/k}(x)) = \{x\}_{k'/k}$ and hence surjectivity follows.

To prove injectivity, it suffices to show that Weil reciprocity II holds already in $G(k)$. Let K be a function field in one variable over k and C be the corresponding complete smooth curve. Let $g \in G(K)$ and $f \in K^\times$. According to the remark (3.2.5), we obtain a regular map $C - S \to G$, where $S = \{P \in C : g \notin G(\mathcal{O}_P)\}$. Theorem 1 in [36] tells us that the map g admits a modulus $\mathfrak{m}$. Proposition 13 in [36] shows that in the case G is an extension of an abelian variety A by $\mathbb{G}_m$, $\mathfrak{m} = \sum_{P \in S} P$ is a modulus for g. By a simple argument using the projections of $\mathbb{G}_m^{\oplus d}$ to each factor, we can conclude that $\mathfrak{m} = \sum_{P \in S} P$ is a modulus for g when G is an arbitrary semi-abelian variety. This means that for every function $f \in K^\times$ such that $f(P) = 1$ for every $P \in S$, it holds $\sum_{P \notin S} \mathrm{ord}_P(f) \, \mathrm{Tr}_{k(P)/k}(g(P)) = 0$. Notice that this implies that Weil reciprocity II holds in $G(k)$.

□

The next proposition is analogous to proposition (2.2.1).

Proposition 3.3.6. *For every $r \geq 1$, there exists a well defined morphism*

$$\begin{aligned} \Phi_r : \quad & W(G) \to S_r(k; G) \\ & [x] \to \{x, x, \ldots, x\}_{k(x)/k}. \end{aligned}$$

Furthermore for $r = 0$, we define $S_0(k; G) = \mathbb{Z}$ and Φ_0 to be the degree map.

Proof. Let $r > 0$ be a positive integer. We define the map $\Phi_r : Z_0(G) \to S_r(k; G)$ first on the level of zero cycles. Let $C \in G_{(1)}$ be a closed irreducible curve in G, let $\widetilde{C} \xrightarrow{\pi} C$ be its normalization and $\iota : \widetilde{C} \hookrightarrow \overline{C}$ the smooth completion of $\widetilde{C}$. Let $f \in k(\overline{C})^\times$ be a function such that $f(P) = 1$, for every $P \in \overline{C} - \widetilde{C}$. We need to show $\Phi_r(\pi_\star(\mathrm{div}(f))) = 0$. More precisely, we need to prove

$$\Phi_r(\sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)[k(x) : k(\pi(x))]) = \sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)[k(x) : k(\pi(x))]\{\pi(x), \cdots, \pi(x)\}_{k(\pi(x))/k} = 0.$$

First we have the following equalities.

$$\begin{aligned}
&\sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)[k(x) : k(\pi(x))]\{\pi(x), \cdots, \pi(x)\}_{k(\pi(x))/k} = \\
&\sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)\{[k(x) : k(\pi(x))]\pi(x), \cdots, \pi(x)\}_{k(\pi(x))/k} = \\
&\sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)\{Tr_{k(x)/k(\pi(x))}(\mathrm{res}_{k(x)/k(\pi(x))}(\pi(x))), \cdots, \pi(x)\}_{k(\pi(x))/k} = \\
&\sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)\{\mathrm{res}_{k(x)/k(\pi(x))}(\pi(x)), \cdots, \mathrm{res}_{k(x)/k(\pi(x))}(\pi(x))\}_{k(x)/k}.
\end{aligned}$$

Let $K = k(C)$ and consider the generic point inclusion $\eta : \mathrm{Spec}\, K \hookrightarrow \widetilde{C}$. We set $g = \pi\eta \in G(K)$ and observe that $S = \{x \in \widetilde{C} : g \notin G(\mathcal{O}_P)\} = \overline{C} - \widetilde{C}$. Then we can easily see that

$$\sum_{x \in \widetilde{C}} \mathrm{ord}_x(f)\{\mathrm{res}_{k(x)/k}(\pi(x)), \ldots, \mathrm{res}_{k(x)/k}(\pi(x))\}_{k(x)/k} = \sum_{x \notin S} \mathrm{ord}_x(f)\{g(x), \ldots, g(x)\}_{k(x)/k}.$$

The result therefore follows from the reciprocity relation of the group $S_r(k; G)$.

□

We can now proceed to the definition of the filtration. First notice that the isomorphism obtained in proposition (3.3.5) yields an equality $\Phi_1 | \ker(\Phi_0) = \mathrm{alb}_G$. This in turn implies that $\ker(\Phi_0) \cap \ker(\Phi_1) = \ker(\mathrm{alb}_G)$.

Definition 3.3.7. We define a decreasing filtration $\{F^r\}_{r \geq 0}$ of $W(G)$ with $F^0 = W(G)$ and for $r \geq 1$, $F^r = \bigcap_{j=0}^{r-1} \ker \Phi_j$. In particular, $F^1 = W^0(G)$ and $F^2 = \ker(\mathrm{alb}_G)$.

Proposition 3.3.8. *For every $r \geq 0$ we have inclusions $I^r \subset F^r$. Moreover,*

$$\Phi_r(\omega_{x_1,\ldots,x_r}) = r!\{x_1, \ldots, x_r\}_{k(x)/k}.$$

Proof. This is analogous to proposition (2.2.3) and part of proposition (2.2.4). For the inclusion $I^r \subset F^r$ we use the commutativity $\Phi_{r-1} \operatorname{Tr}_{k'k} = \operatorname{Tr}_{k'/k} \Phi^{k'}_{r-1}$, the multilinearity of the symbol in $S_r(k;G)$ and the fact that the map Φ_{r+1} is a group homomorphism. The second statement follows by a combinatorial counting.

□

Theorem 3.3.9. *Let $r \geq 0$ be an integer. There is a well defined abelian group homomorphism*

$$\begin{aligned} \Psi_r : S_r(k;G) &\longrightarrow \frac{F^r W(G)}{F^{r+1} W(G)} \\ \{x_1, \ldots, x_r\}_{k'/k} &\longrightarrow \omega_{x_1,\ldots,x_r}. \end{aligned}$$

Moreover, the homomorphism Ψ_r satisfies the property, $\Phi_r \circ \Psi_r = \cdot r!$ on $S_r(k;G)$. As a conclusion, after $\otimes\mathbb{Z}[\frac{1}{r!}]$ the map Φ_r becomes an isomorphism with inverse $\dfrac{1}{r!}\Psi_r$.

Proof. The first step is to obtain a well defined map, for every $r \geq 0$,

$$\Psi_r : \frac{(\overbrace{G \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} G}^{r})(k)}{(x_1, \cdots, x_r)_{k'/k} - (x_{\sigma(1)}, \cdots, x_{\sigma(r)})_{k'/k}} \longrightarrow \frac{F^r W(G)}{F^{r+1} W(G)}.$$

The argument is exactly the same as the proof of the first two steps of proposition (2.2.4).

We will now show that this map factors through $S_r(k;G)$. Let C be a smooth complete curve over k having function field K. Let $g \in G(K)$ and $S = \{P \in C : g \notin G(\mathcal{O}_P)\}$ be the set of bad places of g. Let $f \in K^\times$ be a function such that $f(P) = 1$, for every $P \in S$. We need to show that

$$\sum_{P \notin S} \operatorname{ord}_P(f) \operatorname{Tr}_{k(P)/k}([g(P)]) = 0$$

in $W(G)$. Here by $\operatorname{Tr}_{k(P)/k}$ we denote the push-forward map $W(G \times_k k(P)) \to W(G)$. Set $C_0 = C - S$. Then g induces a morphism $g : C_0 \to G$. Since Wiesend's ideal class group is covariant with respect to morphisms, we obtain a push forward $g_\star : W(C_0) \to W(G)$. By property (3) of Wiesend's class group, we have that the group $W(C_0)$ is equal to the group of divisors on C prime to S modulo S-equivalence, and therefore $\operatorname{div}(f) = 0$ in $W(C_0)$. This forces

$$g_\star(\operatorname{div}(f)) = \sum_{P \notin S} \operatorname{ord}_P(f) \operatorname{Tr}_{k(P)/k}([g(P)]) = 0 \in W(G).$$

□

3.4 Motivation towards Reciprocity Functors

The purpose of this section is to provide a short introduction to the next chapter. The main result of chapter 4 concerns a newly developed theory of reciprocity functors introduced by Ivorra and Rülling in [20]. Before we introduce this new category RF of reciprocity functors, we discuss what appears to have motivated this definition.

3.4.1 Presheaves with transfers

Let k be a field. Let Sm/k be the category of smooth schemes over k. This category is far from being abelian, because it does not have enough morphisms. Voevodsky defined a larger abelian category, namely the category Cor_k. The objects of this category are precisely the objects of Sm/k but the morphisms are given by finite correspondences. A finite correspondence $Y \to X$ is given by a closed subvariety $Z \hookrightarrow Y \times X$ such that the projection to the first factor is finite and surjective. Note that if $Y \xrightarrow{f} X$ is a morphism of smooth schemes, then f induces a finite correspondence given by the graph Γ_f of f. Hence we have an inclusion of categories $(Sm/k) \subset \mathrm{Cor}_k$.

A presheaf with transfers $\mathcal{F}$ is a presheaf on Cor_k. Alternatively, a presheaf with transfers $\mathcal{F}$ is a presheaf on Sm/k with the additional property that for any finite correspondence $Y \to X$ we have a pull-back map $\mathcal{F}(X) \to \mathcal{F}(Y)$.

Remark 3.4.1. A presheaf with transfers is in particular a Mackey functor. For, if L/k is a finite field extension, then we have the usual morphism $\operatorname{Spec} L \to \operatorname{Spec} k$ of schemes, but also we have a finite correspondence $\operatorname{Spec} k \to \operatorname{Spec} L$.

In section 1.2.2 we mentioned that every commutative algebraic group G over k induces a Mackey functor. In fact, every such group induces a presheaf with transfers.

3.4.2 The category $\mathrm{HI}_{\mathrm{Nis}}$

Let k be a perfect field. We consider the category PST of presheaves with transfers. Kahn and Yamazaki in [22] introduced a new category, $\mathrm{HI}_{\mathrm{Nis}}$, with objects homotopy invariant Nisnevich sheaves with transfers. An object $\mathcal{F}$ of $\mathrm{HI}_{\mathrm{Nis}}$ is a presheaf with transfers satisfying the following two additional properties.

1. $\mathcal{F}$ is a sheaf for the Nisnevich topology.

2. $\mathcal{F}$ is homotopy invariant. This means that for every smooth connected scheme U over k the pull-back map $\mathcal{F}(U) \to \mathcal{F}(U \times \mathbb{A}^1)$ is an isomorphism.

Example 3.4.2. Let G be a semi-abelian variety over k. Then G induces a Nisnevich sheaf with transfers by assigning to every smooth scheme U over k the group $\mathrm{Hom}_k(U, G)$. Furthermore, G is homotopy invariant. For, every morphism $\mathbb{A}^1 \to G$ must be constant and therefore the restriction $G(U) \to G(U \times \mathbb{A}^1)$ is an isomorphism.

As we already discussed in chapter 1, for a smooth variety X over a field k, the group $CH_0(X)$ is a birational invariant of the variety. This property does not pass to Suslin's singular homology group, $H_0^{sing}(X)$, unless X is proper. The property that these two geometric invariants have in common though is that they both induce homotopy invariant sheaves with transfers.

Example 3.4.3. Let X be a smooth projective variety over a field k. It is a theorem of Huber and Kahn (2.2 in [19]) that the group $CH_0(X)$ induces a homotopy invariant Nisnevich sheaf with transfers $\underline{CH}_0(X)$ by assigning to a smooth connected scheme U over k the group

$$\underline{CH}_0(X)(U) = CH_0(X \times_k k(U)).$$

Kahn and Yamazaki showed that the category $\mathrm{HI}_{\mathrm{Nis}}$ is abelian with a tensor product $\mathcal{F}_1 \bigotimes \cdots \bigotimes \mathcal{F}_r$. This tensor product when evaluated at $\operatorname{Spec} k$ is given by a Somekawa type K-group,

$$\mathcal{F}_1 \bigotimes \cdots \bigotimes \mathcal{F}_r(\operatorname{Spec} k) = K(k; \mathcal{F}_1, \cdots, \mathcal{F}_r).$$

The group $K(k; \mathcal{F}_1, \cdots, \mathcal{F}_r)$ is defined using local symbols imitating the original definition of the Somekawa K-group. They prove that this group is isomorphic to its geometric variant $K^{geo}(k; \mathcal{F}_1, \cdots, \mathcal{F}_r)$ which is defined similarly to definition 3.2.4. We note that the main relation in both those groups is given by Weil reciprocity.

3.4.3 First Examples of more general reciprocity: Generalized Jacobians and the modulus condition

The category $\mathrm{HI}_{\mathrm{Nis}}$ captures all the homotopy invariant phenomena. The question that arises is whether one can define a larger triangulated category that will encode also unipotent information.

We note that a commutative algebraic group G with a non-trivial unipotent part is a presheaf with transfers that is no longer homotopy invariant. It might still contain significant geometric information though as the next motivating example indicates.

Example 3.4.4. Let C be a smooth complete curve over a field k. Let $f : C \dashrightarrow G$ be a rational map to a commutative algebraic group G over k. The theory of Rosenlicht-Serre described in [36] tells us that f admits a modulus $\mathfrak{m} = \sum_{P \in C} n_P P$. This means that f is regular outside the support S of $\mathfrak{m}$ and moreover for every function $g \in k(C)^\times$ such that $\mathrm{ord}_P(1-g) \geq n_P$ for every $P \in S$ the following reciprocity holds

$$\sum_{P \notin S} \mathrm{ord}_P(g) \,\mathrm{Tr}_{k(P)/k}(f(P)) = 0 \in G(k).$$

This example from the theory of curves was the main motivation for the definition of a larger category of sheaves, containing the homotopy invariant ones, satisfying more general reciprocity relations. We give a quick overview of some aspects of this theory in the next chapter.

CHAPTER 4

LOCAL SYMBOLS AND RECIPROCITY FUNCTORS

4.1 Introduction

Let F be a perfect field. We consider the category $\mathcal{E}_F$ of finitely generated field extensions of F. In [20], F. Ivorra and K. Rülling created a theory of reciprocity functors. A reciprocity functor is a presheaf with transfers in the category $\mathrm{Reg}^{\leq 1}$ of regular schemes of dimension at most one over some field $k \in \mathcal{E}_F$ that satisfies various properties.

Some examples of reciprocity functors include commutative algebraic groups, homotopy invariant Nisnevich sheaves with transfers, Kähler differentials. Moreover, if $\mathcal{M}_1, \cdots, \mathcal{M}_r$ are reciprocity functors, Ivorra and Rülling construct a K-group $T(\mathcal{M}_1, \cdots, \mathcal{M}_r)$ which is itself a reciprocity functor.

One of the crucial properties of a reciprocity functor $\mathcal{M}$ is that it has local symbols. Namely, if C is a smooth, complete and geometrically connected curve over some field $k \in \mathcal{E}_F$ with generic point η, then at each closed point $P \in C$ there is a local symbol assignment

$$(.;.)_P : \mathcal{M}(\eta) \times \mathbb{G}_m(\eta) \to \mathcal{M}(k),$$

satisfying three characterizing properties, one of which is a reciprocity relation $\sum_{P \in C} (g; f)_P = 0$, for every $g \in \mathcal{M}(\eta)$ and $f \in \mathbb{G}_m(\eta)$. We note here that if G is a commutative algebraic group over an algebraically closed field k, then the local symbol of G coincides with the local symbol constructed by Rosenlicht-Serre in [36]. The reciprocity relation induces a local symbol complex ($\underline{\underline{C}}$)

$$(\mathcal{M} \bigotimes^{M} \mathbb{G}_m)(\eta) \xrightarrow{((.;.)_P)_{P \in C}} \bigoplus_{P \in C} \mathcal{M}(k) \xrightarrow{\sum_P} \mathcal{M}(k),$$

where by $\otimes^M$ we denote the product of Mackey functors (see def. 1.2.3). The main goal of this chapter is to give a description of the homology $H(\underline{\underline{C}})$ of the above complex in terms of K-groups of reciprocity functors. Our computations work well for curves C over an algebraically closed field k. In the last section we describe some special cases where the method could be refined to include non-algebraically closed base fields. To obtain a concrete result, we need to impose two conditions on the reciprocity functor $T(\mathcal{M}, \underline{CH}_0(C)^0)$ (see assumptions 4.3.2, 4.3.9). In section 4.3 we prove the following theorem.

Theorem 4.1.1. *Let C be a smooth, complete curve over an algebraically closed field k. Let $\mathcal{M}$ be a reciprocity functor such that the K-group of reciprocity functors $T(\mathcal{M}, \underline{CH}_0(C)^0)$ satisfies the assumptions 4.3.2 and 4.3.9. Then the homology of the local symbol complex $(\underline{\underline{C}})$ is canonically isomorphic to the K-group $T(\mathcal{M}, \underline{CH}_0(C)^0)(\operatorname{Spec} k)$.*

Here $\underline{CH}_0(C)^0$ is a reciprocity functor that is identified with the Jacobian variety J of C.

In section 4.4 we give some examples of reciprocity functors that satisfy the two assumptions. In particular, we prove the following theorem.

Theorem 4.1.2. *Let $\mathcal{F}_1, \cdots, \mathcal{F}_r$ be homotopy invariant Nisnevich sheaves with transfers, and consider the reciprocity functor $\mathcal{M} = T(\mathcal{F}_1, \cdots, \mathcal{F}_r)$. Let C be a smooth, complete curve over an algebraically closed field k. Then there is an isomorphism*

$$H(\underline{\underline{C}}) \simeq T(\mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)(\operatorname{Spec} k).$$

In particular, if $G_1, \cdots, G_r$ are semi-abelian varieties over k, then we obtain an isomorphism

$$H(\underline{\underline{C}}) \simeq T(G_1, \cdots, G_r, \underline{CH}_0(C)^0)(\operatorname{Spec} k) \simeq K(k; G_1, \cdots, G_r, \underline{CH}_0(C)^0),$$

where $K(k; G_1, \cdots, G_r, \underline{CH}_0(C)^0)$ is the Somekawa K-group attached to $G_1, \cdots, G_r$.

Another case where the assumptions of theorem 4.1.1 are satisfied is when $\mathcal{M}$ is of the form $\mathcal{M} = T(\mathcal{M}_1, \cdots, \mathcal{M}_r)$ such that $\mathcal{M}_i = \mathbb{G}_a$ for some $i \in \{1, \cdots, r\}$. Using the main result of [32] together with theorem 5.4.7. in [20], we obtain the following corollary.

Corollary 4.1.3. *Let $\mathcal{M}_1, \cdots, \mathcal{M}_r$ be reciprocity functors. Let $\mathcal{M} = T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)$. Then for any smooth complete curve C over k, $H(\underline{\underline{C}}) = 0$. In particular, if $\operatorname{char} k = 0$, the complex $\Omega^{n+1}_{k(C)} \xrightarrow{\mathrm{Res}_P} \bigoplus_{P \in C} \Omega^n_k \xrightarrow{\sum_P} \Omega^n_k$ is exact.*

The idea for theorem 4.1.1 stems from the special case when $\mathcal{M} = \mathbb{G}_m$. In this case the local symbol $k(C)^\times \otimes^M k(C)^\times \xrightarrow{(.,.)_P} k^\times$ at a closed point $P \in C$ factors through the group $T(\mathbb{G}_m, \mathbb{G}_m)(\eta_C)$. By a theorem in [20] this group is isomorphic to the usual Milnor K-group $K_2^M(k(C))$ and we recover the Milnor complex

$$K_2^M(k(C)) \to \bigoplus_{P \in C} k^\times \xrightarrow{\sum_P} k^\times.$$

This complex was studied by M. Somekawa in [37] and R. Akhtar in [1]. Using different methods, they both prove that the homology of the above complex is isomorphic to the Somekawa K-group $K(k; \mathbb{G}_m, \underline{CH}_0(C)^0)$. This group turns out to be isomorphic to the group $T(\mathbb{G}_m, \underline{CH}_0(C)^0)(\operatorname{Spec} k)$. (by [20], theorem 5.1.8. and [22], theorem 11.14). A similar result was proved by T. Hiranouchi in [18] for his Somekawa-type additive K-groups. Our method to prove theorem 4.1.1 is similar to the method used by R. Akhtar and T. Hiranouchi.

Notation 4.1.4. Let C be a smooth complete curve over $k \in \mathcal{E}_F$ and $P \in C$ a closed point. For an integer $n \geq 1$, we set $U^{(n)}_{C,P} = \{f \in k(C)^\times : \operatorname{ord}_P(1-f) \geq n\}$.

4.2 Review of Definitions

4.2.1 Reciprocity Functors

Let $\mathrm{Reg}^{\leq 1}$ be the category with objects regular F-schemes of dimension at most one which are separated and of finite type over some $k \in \mathcal{E}_F$. Let $\mathrm{Reg}^{\leq 1}\mathrm{Cor}$ be the category with the same objects as $\mathrm{Reg}^{\leq 1}$ and with morphisms finite correspondences. A reciprocity functor $\mathcal{M}$ is a presheaf of abelian groups on $\mathrm{Reg}^{\leq 1}\,\mathrm{Cor}$ which satisfies various properties. Here we only recall those properties that we will need later in the paper.

Notation 4.2.1. Let $\mathcal{M}$ be a reciprocity functor. For $k \in \mathcal{E}_F$ we will write $\mathcal{M}(k) := \mathcal{M}(\operatorname{Spec} k)$.

Let E/k be a finite extension of fields in $\mathcal{E}_F$. The morphism $\operatorname{Spec} E \to \operatorname{Spec} k$ induces a pull-back map $\mathcal{M}(k) \to \mathcal{M}(E)$, which we call restriction and will denote by $\mathrm{res}_{E/k}$. Moreover, there is a finite correspondence $\operatorname{Spec} k \to \operatorname{Spec} E$ which induces a push-forward $\mathcal{M}(E) \to \mathcal{M}(k)$, which we will call the trace and denote it by $\mathrm{Tr}_{E/k}$.

Injectivity: Let C be a smooth, complete curve over $k \in \mathcal{E}_F$. Each open set $U \subset C$ induces a pull-back map $\mathcal{M}(C) \to \mathcal{M}(U)$ that is required to be injective. Additionally, if η_C is the generic point of C, we have an isomorphism

$$\varinjlim \mathcal{M}(U) \xrightarrow{\simeq} \mathcal{M}(\eta_C),$$

where the limit extends over all open subsets $U \subset C$.

Specialization and Trace maps: Let $P \in C$ be a closed point. For each open $U \subset C$ with $P \in U$, the closed immersion $P \hookrightarrow U$ induces $\mathcal{M}(U) \to \mathcal{M}(P)$. We consider the stalk $\mathcal{M}_{C,P} = \varinjlim \mathcal{M}(U)$, where the limit extends over all open $U \subset C$ with $P \in U$. The above morphisms induce a specialization map

$$s_P : \mathcal{M}_{C,P} \to \mathcal{M}(P).$$

Moreover, for every closed point $P \in C$ we obtain a Trace map, which we will denote by

$$\mathrm{Tr}_{P/k} : \mathcal{M}(P) \to \mathcal{M}(k).$$

4.2.2 The modulus condition and local symbols

Let $\mathcal{M}$ be a reciprocity functor. Let C be a smooth, projective and geometrically connected curve over $k \in \mathcal{E}_F$. The definition of a reciprocity functor imposes the existence for each section $g \in \mathcal{M}(\eta_C)$ of a modulus $\mathfrak{m}$ corresponding to g. The modulus $\mathfrak{m}$ is an effective divisor $\mathfrak{m} = \sum_{P \in S} n_P P$ on C, where S is a closed subset of C, such that $g \in \mathcal{M}_{C,P}$, for every $P \notin S$ and for every function $f \in k(C)^\times$ with $f \in \bigcap_{P \in S} U_{C,P}^{(n_P)}$, it holds

$$\sum_{P \in C \setminus S} \operatorname{ord}_P(f) \operatorname{Tr}_{P/k}(s_P(g)) = 0.$$

Notation 4.2.2. Let $f \in k(C)^\times$ be such that $f \in \bigcap_{P \in S} U_{C,P}^{(n_P)}$. Then we will write $f \equiv 1 \mod \mathfrak{m}$.

The modulus condition on $\mathcal{M}$ is equivalent to the existence, for each closed point $P \in C$, of a bi-additive pairing called the local symbol at P

$$(.;.)_P : \mathcal{M}(\eta_C) \times \mathbb{G}_m(\eta_C) \to \mathcal{M}(k)$$

which satisfies the following three characterizing properties:

1. $(g; f)_P = 0$, for $f \in U_{C,P}^{(n_P)}$, where $\mathfrak{m} = \sum_{P \in S} n_P P$ is a modulus corresponding to g.
2. $(g; f)_P = \operatorname{ord}_P(f) \operatorname{Tr}_{P/k}(s_P(g))$, for all $g \in \mathcal{M}_{C,P}$ and $f \in k(C)^\times$.
3. $\sum_{P \in C} (g; f)_P = 0$, for every $g \in \mathcal{M}(\eta_C)$ and $f \in k(C)^\times$.

The proof of existence and uniqueness of this local symbol is along the lines of Prop.1 Chapter III in [36]. In this paper we will use the precise definition of $(g; f)_P$, for $g \in \mathcal{M}(\eta_C)$ and $f \in k(C)^\times$, so we review it here.

Case 1: If $g \in \mathcal{M}_{C,P}$, property (2) forces us to define $(g; f)_P = \operatorname{ord}_P(f) \operatorname{Tr}_{P/k}(s_P(g))$.

Case 2: Let $P \in S$. Using the weak approximation theorem for valuations, we consider an auxiliary function f_P for f at P, i.e. a function $f_P \in k(C)^\times$ such that $f_P \in U_{C,P'}^{(n_{P'})}$ at every $P' \in S$, $P' \neq P$ and $f/f_P \in U_{C,P}^{(n_P)}$. Then we define

$$(g; f)_P = -\sum_{Q \notin S} \operatorname{ord}_Q(f_P) \operatorname{Tr}_{Q/k}(s_Q(g)).$$

Using the local symbol, one can define for each closed point $P \in C$, $Fil_P^0 \mathcal{M}(\eta_C) := \mathcal{M}_{C,P}$ and for $r \geq 1$

$$Fil_P^r \mathcal{M}(\eta_C) := \{g \in \mathcal{M}(\eta_C) : (g; f)_P = 0, \text{ for all } f \in U_{C,P}^{(r)}\}.$$

Then $\{Fil_P^r\}_{r \geq 0}$ form an increasing and exhaustive filtration of $\mathcal{M}(\eta_C)$.

The reciprocity functors $\mathcal{M}$ for which there exists an integer $n \geq 0$ such that it holds $\mathcal{M}(\eta_C) = Fil_P^n \mathcal{M}(\eta_C)$, for every smooth complete and geometrically connected curve C and every closed point $P \in C$, form a full subcategory of RF, which is denoted by RF_n. (see def. 1.5.7. in [20]).

4.2.3 K-group of Reciprocity Functors

Let $\mathcal{M}_1, \cdots, \mathcal{M}_n$ be reciprocity functors. The K-group of reciprocity functors $T(\mathcal{M}_1, \cdots, \mathcal{M}_n)$ is itself a reciprocity functor that satisfies various properties. (4.2.4. in [20]). We will not need the precise definition of $T(\mathcal{M}_1, \cdots, \mathcal{M}_n)$, but only the following properties.

1. For $k \in \mathcal{E}_F$, the group $T(\mathcal{M}_1, \cdots, \mathcal{M}_n)(k)$ is a quotient of $(\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_n)(k)$, where by $\overset{M}{\bigotimes}$ we denote the product of Mackey functors. The group $T(\mathcal{M}_1, \cdots, \mathcal{M}_n)(k)$ is generated by elements of the form $\operatorname{Tr}_{k'/k}(x_1 \otimes \cdots \otimes x_n)$, with $x_i \in \mathcal{M}_i(k')$, where k'/k is any finite extension.

2. Let C be a smooth, complete and geometrically connected curve over $L \in \mathcal{E}_k$ and let

$P \in C$ be a closed point. Let $g_i \in \mathcal{M}_i(\eta_C)$. Then

(a) If for some $r \geq 0$ we have elements $g_i \in Fil_P^r \mathcal{M}_i(\eta_C)$ for every $i = 1, \cdots, n$, then $g_1 \otimes \cdots \otimes g_n \in Fil_P^r T(\mathcal{M}_1, \cdots, \mathcal{M}_n)(\eta_C)$. Moreover, if the element g_i has modulus $\mathfrak{m}_i = \sum_{P \in S_i} n_P^i P$, for $i = 1, \cdots, n$, then $\mathfrak{m} = \sum_{P \in \cup S_i} \max_{1 \leq i \leq n} \{n_P^i\} P$ is a modulus for $g_1 \otimes \cdots \otimes g_n$.

(b) If $g_i \in Fil_P^0 \mathcal{M}_i(\eta_C)$, for $i = 1, \cdots, n$, then we have an equality

$$s_P(g_1 \otimes \cdots \otimes g_n) = s_P(g_1) \otimes \cdots \otimes s_P(g_n).$$

4.2.4 Examples

Some examples of reciprocity functors include constant reciprocity functors, commutative algebraic groups, homotopy invariant Nisnevich sheaves with transfers. For an explicit description of each of these examples we refer to section 2 in [20]. The following example is of particular interest to us.

Let X be a smooth projective variety over $k \in \mathcal{E}_F$. Then there is a reciprocity functor $\underline{CH}_0(X)$ such that for any scheme $U \in \mathrm{Reg}^{\leq 1}$ over k we have

$$\underline{CH}_0(X)(U) = CH_0(X \times_k k(U)).$$

Since we assumed X is projective, the degree map $CH_0(X) \to \mathbb{Z}$ induces a map of reciprocity functors $\underline{CH}_0(X) \to \mathbb{Z}$ whose kernel will be denoted by $\underline{CH}_0(X)^0$. Both $\underline{CH}_0(X)$ and $\underline{CH}_0(X)^0$ are in RF_0.

Remark 4.2.3. We note here that if X has a k-rational point, then we have a decomposition of reciprocity functors $\underline{CH}_0(X) \simeq \underline{CH}_0(X)^0 \oplus \mathbb{Z}$, where $\mathbb{Z}$ is the constant reciprocity functor. Moreover, if $\mathcal{M}_1, \cdots, \mathcal{M}_r$ are reciprocity functors, then by corollary 4.2.5. (2) in [20] we have a decomposition

$$T(\underline{CH}_0(X), \mathcal{M}_1, \cdots, \mathcal{M}_r) \simeq T(\underline{CH}_0(X)^0, \mathcal{M}_1, \cdots, \mathcal{M}_r) \oplus T(\mathbb{Z}, \mathcal{M}_1, \cdots, \mathcal{M}_r).$$

Relation to Milnor K-theory and Kähler differentials:

If we consider the reciprocity functor $T(\mathbb{G}_m^{\times n}) := T(\mathbb{G}_m, \cdots, \mathbb{G}_m)$ attached to n copies of $\mathbb{G}_m$, then for every $k \in \mathcal{E}_F$ the group $T(\mathbb{G}_m^{\times n})(k)$ is isomorphic to the usual Milnor K-group $K_n^M(k)$ (theorem 5.3.3. in [20]).

Moreover, if k is of characteristic zero, then the group $T(\mathbb{G}_a, \mathbb{G}_m^{\times n-1})(k)$, $n \geq 1$, is isomorphic to the group of Kähler differentials $\Omega^{n-1}_{k/\mathbb{Z}}$ (theorem 5.4.7 in [20]).

4.3 The homology of the complex

Convention 4.3.1. From now on, unless otherwise mentioned, we will be working over an algebraically closed base field $k \in \mathcal{E}_F$.

Let $\mathcal{M}$ be a reciprocity functor. Let C be a smooth complete curve over k with generic point η_C. At each closed point $P \in C$ we have a local symbol $(.;.)_P$. We will denote by $(.;.)_C$ the collection of all symbols $\{(.;.)_P\}_{P\in C}$, namely

$$(.;.)_C : \mathcal{M}(\eta_C) \otimes \mathbb{G}_m(\eta_C) \to \bigoplus_{P\in C} \mathcal{M}(k).$$

We note here that a reciprocity functor $\mathcal{M}$ is also a Mackey functor. Using the functoriality properties of the local symbol at each closed point $P \in C$ (prop. 1.5.5. in [20]), we obtain a complex

$$(\mathcal{M} \overset{M}{\bigotimes} \mathbb{G}_m)(\eta_C) \xrightarrow{(.;.)_C} \bigoplus_{P\in C} \mathcal{M}(k) \xrightarrow{\sum_P} \mathcal{M}(k).$$

Namely, if C' is a smooth complete curve over k with function field $k(C') \supset k(C)$ and we have a section $g \in \mathcal{M}(\eta_{C'})$ and a function $f \in k(C')^\times$, then we define

$$(g;f)_C = (\sum_{\lambda(P')=P} (g;f)_{P'})_P \in \bigoplus_{P\in C} \mathcal{M}(k),$$

where $\lambda : C' \to C$ is the finite covering induced by the inclusion $k(C) \subset k(C')$.

We will denote this complex by $(\underline{\underline{C}})$ and its homology by $H(\underline{\underline{C}})$. We consider the reciprocity functor $\underline{CH}_0(C)$. Notice that the existence of a k-rational point $P_0 \in C(k)$ yields a decomposition of reciprocity functors $\underline{CH}_0(C) \simeq \underline{CH}_0(C)^0 \oplus \mathbb{Z}$. We make the following assumption on the K-group $T(\mathcal{M}, \underline{CH}_0(C))$.

Assumption 4.3.2. Let $\mathcal{M}$ be a reciprocity functor. Let $g \in \mathcal{M}(\eta_C)$, $h \in \underline{CH}_0(C)(\eta_C)$ and $f \in k(C)^\times$. Let $P \in C$ be a closed point of C. Assume that the local symbol $(g \otimes h; f)_P \in T(\mathcal{M}, \underline{CH}_0(C))(k)$ vanishes at every point P such that $s_P(h) = 0$.

In the next section we will give examples where the assumption 4.3.2 is satisfied.

Proposition 4.3.3. *Let $\mathcal{M}$ be a reciprocity functor over k satisfying 4.3.2. Then there is a well defined map*

$$\begin{aligned} \Phi : \quad & (\bigoplus_{P\in C} \mathcal{M}(k))/\operatorname{Im}((.;.)_C) \to T(\mathcal{M}, \underline{CH}_0(C))(k) \\ & (a_P)_{P\in C} \to \sum_{P\in C} a_P \otimes [P]. \end{aligned}$$

Proof. First, we immediately observe that if $P \in C$ is any closed point of C, then the map $\phi_P : \mathcal{M}(k) \to T(\mathcal{M}, \underline{CH}_0(C))(k)$ given by $a \to a \otimes [P]$ is well defined. In particular, the map

$$\Phi = \sum_P \phi_P : \bigoplus_{P\in C} \mathcal{M}(k) \to T(\mathcal{M}, \underline{CH}_0(C))(k)$$

is well defined. Let D be a smooth complete curve over k with generic point η_D and assume there is a finite covering $\lambda : D \to C$. Let $g \in \mathcal{M}(\eta_D)$ and $f \in k(D)^\times$ be a function. For every closed point $P \in C$ we consider the element $(a_P)_P \in \bigoplus_{P \in C} \mathcal{M}(k)$ such that $a_P = (g;f)_P$. We are going to show that $\Phi(\sum_{P \in C}(g;f)_P) = 0$.

First, we treat the case $D = C$ and $\lambda = 1_C$. The element $g \in \mathcal{M}(\eta_C)$ admits a modulus $\mathfrak{m}$ with support S. We consider the zero-cycle $h = [\eta_C] \in \underline{CH}_0(C)(\eta_C)$. Notice that for a closed point $P \in C$, the specialization map $s_P : \underline{CH}_0(C)(\eta_C) \to \underline{CH}_0(C)(k)$ has the property $s_P(h) = [P]$. We are going to show that for every $P \in C$, it holds

$$\Phi((g;f)_P) = (g \otimes h; f)_P$$

and the required property will follow from the reciprocity law of the local symbol. We consider the following cases.

1. Let $P \notin S$. Then,

$$\begin{aligned}\Phi((g;f)_P) = \quad & \phi_P(\mathrm{ord}_P(f)s_P(g)) = \mathrm{ord}_P(f)s_P(g) \otimes [P] = \mathrm{ord}_P(f)s_P(g) \otimes s_P(h) = \\ & \mathrm{ord}_P(f)s_P(g \otimes h) = (g \otimes h; f)_P.\end{aligned}$$

2. Let $P \in S$ and $f \equiv 1 \bmod \mathfrak{m}$ at P. Since $\underline{CH}_0(C) \in RF_0$, h does not contribute to the modulus, and hence, by 2.3 (b)(ii) we get:

$$\Phi((g;f)_P) = \Phi(0) = 0 = (g \otimes h; f)_P.$$

3. Let now $P \in S$ and $f \in K^\times$ be any function. We consider an auxiliary function f_P for f at P. By the definition of the local symbol, we have:

$$\begin{aligned}\Phi((g;f)_P) = \quad & \phi_P(-\sum_{Q \notin S} \mathrm{ord}_Q(f_P)s_Q(g)) = -\sum_{Q \notin S} \mathrm{ord}_Q(f_P)s_Q(g) \otimes [P] = \\ & -\sum_{Q \notin S} \mathrm{ord}_Q(f_P)s_Q(g) \otimes [Q] + \sum_{Q \notin S} \mathrm{ord}_Q(f_P)s_Q(g) \otimes ([Q]-[P]).\end{aligned}$$

We observe that we have an equality

$$(g \otimes h; f)_P = -\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes [Q].$$

Next, notice that the flat embedding $k \hookrightarrow k(C)$ induces a restriction map $\mathrm{res}_{\eta/k} : CH_0(C) \to CH_0(C \times \eta_C)$. Let $h_0 = \mathrm{res}_{\eta/k}([P])$. Then we clearly have

$$\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes ([P] - [Q]) = (g \otimes (h_0 - h); f)_P.$$

Since we assumed that the assumption 4.3.2 is satisfied, we get that this last symbol vanishes. For, $s_P(h - h_0) = 0$.

The general case is treated in a similar way. Namely, if $\lambda : D \to C$ is a finite covering of smooth complete curves over k and $g \in \mathcal{M}(\eta_D)$, then the local symbol at a closed point $P \in C$ is defined to be $(g; f)_P = \sum_{\lambda(Q)=P} (g; f)_Q$. Considering the zero cycle $h = [\eta_D] \in \underline{CH}_0(C)(\eta_D)$, we can show that

$$\Phi_P((g; f)_P) = (g \otimes h; f)_P.$$

$\square$

From now on we fix a k-rational point P_0 of C. We obtain the following corollary.

Corollary 4.3.4. *The map Φ of proposition 4.3.3 induces a map*

$$\begin{aligned} \Phi : \quad & H(\underline{\underline{C}}) \to T(\mathcal{M}, \underline{CH}_0(C)^0)(k) \\ & (a_P)_{P \in C} \to \sum_{P \in C} a \otimes ([P] - [P_0]). \end{aligned}$$

which does not depend on the k-rational point P_0.

Proof. If $(a_P)_{P\in C} \in H(\underline{\underline{C}})$, then $\sum_P a_P = 0 \in \mathcal{M}(k)$ and hence $\sum_P a_P \otimes [P_0] = 0 \in T(\mathcal{M}, \underline{CH}_0(C))(k)$. We conclude that if $(a_P)_{P\in C} \in H(\underline{\underline{C}})$, then

$$\Phi((a_P)_{P\in C}) \in T(\mathcal{M}, \underline{CH}_0(C)^0)(k)$$

and clearly the map does not depend on the k-rational point P_0.

□

Definition 4.3.5. Let $\mathcal{M}_1, \cdots, \mathcal{M}_r$ be reciprocity functors over k. We consider the geometric K-group attached to $\mathcal{M}_1, \cdots, \mathcal{M}_r$,

$$K^{geo}(k; \mathcal{M}_1, \cdots, \mathcal{M}_r) = (\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_r)(k)/R,$$

where the subgroup R is generated by the following family of elements:

Let D be a smooth complete curve over k with generic point η_D. Let $g_i \in \mathcal{M}_i(\eta_D)$. Then each g_i admits a modulus $\mathfrak{m}_i$. Let $\mathfrak{m} = \sup_{1\le i\le r} \mathfrak{m}_i$ and S be the support of $\mathfrak{m}$. Let $f \in k(D)^\times$ be a function such that $f \equiv 1 \bmod \mathfrak{m}$. Then

$$\sum_{P\notin S} \mathrm{ord}_P(f) s_P(g_1) \otimes \cdots \otimes s_P(g_r) \in R.$$

Notation 4.3.6. The elements of the geometric K-group $K^{geo}(k; \mathcal{M}_1, \cdots, \mathcal{M}_r)$ will be denoted as $\{x_1 \otimes \cdots \otimes x_r\}^{geo}$.

Remark 4.3.7. In the notation of [20] the group $K^{geo}(k; \mathcal{M}_1, \cdots, \mathcal{M}_r)$ is the same as the Lax Mackey functor $LT(\mathcal{M}_1, \cdots, \mathcal{M}_r)$ evaluated at $\operatorname{Spec} k$. (def. 3.1.2. in [20]). In general the group $T(\mathcal{M}_1, \cdots, \mathcal{M}_r)(k)$ is a quotient of $K^{geo}(k; \mathcal{M}_1, \cdots, \mathcal{M}_r)$. In the next section we give some examples where these two groups coincide.

Proposition 4.3.8. *Let P_0 be a fixed k-rational point of C. The map*

$$\begin{aligned}\Psi: \qquad & K^{geo}(k;\mathcal{M},\underline{CH}_0(C)^0) \longrightarrow H(\underline{\underline{C}}) \\ & \{x\otimes([P]-[P_0])\}^{geo} \longrightarrow (x_{P'})_{P'\in C},\end{aligned}$$

with $x_{P'} = \begin{cases} x, \ P' = P \\ -x, \ P' = P_0 \\ 0, \ \text{otherwise,} \end{cases}$ *for $P \neq P_0$, is well defined and does not depend on the choice of the k-rational point P_0.*

Proof. We start by defining the map $\Psi_{P_0} : \mathcal{M}(k)\otimes\underline{CH}_0(C)^0(k) \to H(\underline{\underline{C}})$ as in the statement of the proposition. To see that Ψ_{P_0} is well defined, let $f \in k(C)^\times$. We need to verify that for every $x \in \mathcal{M}(k)$ it holds $\Psi_{P_0}(x\otimes\operatorname{div}(f)) = 0$. Let $\pi : C \to \operatorname{Spec} k$ be the structure map. Consider the pull back $g = \pi^\star(x) \in \mathcal{M}(C)$. Then $g \in \mathcal{M}(\eta_C)$ has modulus $\mathfrak{m} = 0$ and hence for a closed point $P \in C$ we have $(g,f)_P = \operatorname{ord}_P(f)s_P(\pi^\star(x)) = \operatorname{ord}_P(f)x$. Since $\Psi_{P_0}(x\otimes\operatorname{div}(f)) = (\operatorname{ord}_P(f)x)_{P\in C}$, we conclude that $\Psi_{P_0}(x\otimes\operatorname{div}(f)) \in \operatorname{Im}(.,.)_C$.

Next, notice that Ψ_{P_0} does not depend on the base point P_0. For, if Q_0 is another base point, then

$$\begin{aligned}&\Psi_{Q_0}(\{x\otimes([P]-[P_0])\}^{geo}) = \\ &\Psi_{Q_0}(\{x\otimes([P]-[Q_0])\}^{geo}) - \Psi_{Q_0}(\{x\otimes([P_0]-[Q_0])\}^{geo}).\end{aligned}$$

$\Psi_{Q_0}(\{x\otimes([P]-[Q_0])\}^{geo})$ gives the element x at the coordinate P and $-x$ at the coordinate Q_0, while $-\Psi_{Q_0}(\{x\otimes([P_0]-[Q_0])\}^{geo})$ gives $-x$ at coordinate P_0 and x at Q_0. From now on we will denote this map by Ψ. In order to show that Ψ factors through $K^{geo}(k;\mathcal{M},\underline{CH}_0(C)^0)$, we consider a smooth complete complete curve D with generic point η_D. Let $g_1 \in \mathcal{M}(\eta_D)$ admitting a modulus $\mathfrak{m}$ with support S_D and $g_2 \in \underline{CH}_0(C)^0(\eta_D)$ having modulus $\mathfrak{m}_2 = 0$. Let moreover $f \in k(D)^\times$ be a function such that $f \equiv 1 \bmod \mathfrak{m}$.

We need to show that

$$\Psi(\sum_{R\notin S_D} \mathrm{ord}_R(f)\{s_R(g_1)\otimes s_R(g_2)\}^{geo}) = 0 \in H(\underline{C}).$$

Since we assumed the existence of a k-rational point P_0, the group $\underline{CH}_0(C)^0(\eta_D)$ is generated by elements of the form $[h]-m[\mathrm{res}_{k(D)/k}(P_0)]$, where h is a closed point of $C\times k(D)$ having residue field of degree m over $k(D)$. Using the linearity of the symbol on the last coordinate, we may reduce to the case when g_2 is of the above form. Notice that $h = \mathrm{Spec}\, k(E) \hookrightarrow C\times \mathrm{Spec}\, k(D)$, where E is a smooth complete curve over k, and hence h induces two coverings

$$\begin{array}{ccc} E & \xrightarrow{\lambda} & D \\ {\scriptstyle\mu}\downarrow & & \\ C. & & \end{array}$$

Let $S_E = \lambda^{-1}(S_D)$. For a closed point $R\in D$, we obtain an equality:

$$s_R([h]) = \sum_{\lambda(Q)=R} e(Q/R)[\mu(Q)],$$

where $e(Q/R)$ is the ramification index at the point $Q \in E$ lying over $R \in D$. Since $m = [k(E):k(D)] = \sum_{\lambda(Q)=R} e(Q/R)$, we get

$$\begin{aligned}
&\Psi(\sum_{R\notin S_D} \mathrm{ord}_R(f)\{s_R(g_1)\otimes s_R(g_2)\}^{geo}) = \\
&\Psi(\sum_{R\notin S_D} \mathrm{ord}_R(f)\{s_R(g_1)\otimes (\sum_{\lambda(Q)=R} e(Q/R)[\mu(Q)] - m[P_0])\}^{geo}) = \\
&\Psi(\sum_{R\notin S_D}\sum_{\lambda(Q)=R} e(Q/R)\,\mathrm{ord}_R(f)\{s_R(g_1)\otimes ([\mu(Q)]-[P_0])\}^{geo}) = \\
&\Psi(\sum_{Q\notin S_E} \mathrm{ord}_Q(\lambda^\star(f))\{s_Q(\lambda^\star(g_1))\otimes([\mu(Q)]-[P_0])\}^{geo}).
\end{aligned}$$

In the above computation we used the fact that for a closed point $Q\in E$ lying over $R\in D$, it holds $s_R(g_1) = s_Q(\lambda^\star(g_1))$ (this equality follows from prop. 1.3.7.($\mathcal{S}_2$) in [20] and the assumption that the base field k is algebraically closed).

We conclude that

$$\Psi(\sum_{Q\notin S_E} \mathrm{ord}_Q(\lambda^\star(f))\{s_Q(\lambda^\star(g_1)) \otimes ([\mu(Q)] - [P_0])\}_{Q/k}) =$$
$$\begin{cases} \sum_{\mu(Q)=P} \mathrm{ord}_Q(\lambda^\star(f)) s_Q(\lambda^\star(g_1)), \text{at } P \neq P_0 \\ -\sum_{P\neq P_0} \sum_{\mu(Q)=P} \mathrm{ord}_Q(\lambda^\star(f)) s_Q(\lambda^\star(g_1)), \text{at } P_0 \end{cases}$$

This last computation completes the argument, after we notice that the reciprocity of the local symbol yields an equality

$$-\sum_{P\neq P_0} \sum_{\mu(Q)=P} \mathrm{ord}_Q(\lambda^\star(f)) s_Q(\lambda^\star(g_1)) = (\lambda^\star(g_1); \lambda^\star(f))_{P_0}.$$

□

We make the following assumption on $T(\mathcal{M}, \underline{CH}_0(C)^0)$.

Assumption 4.3.9. Let $\mathcal{M}$ be a reciprocity functor. Assume that the geometric K-group $K^{geo}(k; \mathcal{M}, \underline{CH}_0(C)^0)$ coincides with the group $T(\mathcal{M}, \underline{CH}_0(C)^0)(k)$.

Theorem 4.3.10. *Let $\mathcal{M}$ be a reciprocity functor such that the group $T(\mathcal{M}, \underline{CH}_0(C)^0)(k)$ satisfies both assumptions 4.3.2 and 4.3.9. Then we have an isomorphism*

$$H(\underline{\underline{C}}) \simeq T(\mathcal{M}, \underline{CH}_0(C)^0)(k).$$

Proof. By proposition 4.3.8 we obtain a homomorphism

$$\Psi : T(\mathcal{M}, \underline{CH}_0(C)^0)(k) \to H(\underline{\underline{C}}).$$

It is almost a tautology to check that Ψ is the inverse of Φ. Namely,

$$\Phi\Psi(x \otimes ([P] - [P_0])) = \Phi((x_{P'})_{P'}) = \sum_{P'} x_{P'} \otimes [P'] = x \otimes [P] - x \otimes [P_0],$$

and

$$\Psi\Phi((x_P)_P) = \Psi(\sum_{P \in C} x_P \otimes ([P] - [P_0])) = (x_P)_P.$$

Notice that for the last equality, we used the fact that $(x_P)_{P\in C} \in \ker(\sum_{P\in C})$, and hence at coordinate P_0 we have $x_{P_0} = -\sum_{P \neq P_0} x_P$.

□

4.4 Examples

In this section we give some examples of reciprocity functors $\mathcal{M}$ such that the K-group of reciprocity functors $T(\mathcal{M}, \underline{CH}_0(C)^0)$ satisfies the assumptions 4.3.2 and 4.3.9.

4.4.1 Homotopy Invariant Nisnevich sheaves with Transfers

We consider the category $\mathrm{HI}_{\mathrm{Nis}}$ of homotopy invariant Nisnevich sheaves with transfers over a perfect field F. We already introduced this category in section 3.4.2. Let $\mathcal{F}_1, \cdots, \mathcal{F}_r \in \mathrm{HI}_{\mathrm{Nis}}$. Then each $\mathcal{F}_i$ induces a reciprocity functor $\hat{\mathcal{F}}_i \in RF_1$ (see example 2.3 in [20]). Moreover, the associated K-group of reciprocity functors $T(\hat{\mathcal{F}}_1, \cdots, \hat{\mathcal{F}}_r)$ is also in RF_1. We claim that $T(T(\hat{\mathcal{F}}_1, \cdots, \hat{\mathcal{F}}_r), \underline{CH}_0(C)^0)$ satisfies both assumptions of theorem 4.3.10. The claim follows by the comparison of the K-group $T(T(\hat{\mathcal{F}}_1, \cdots, \hat{\mathcal{F}}_r), \underline{CH}_0(C)^0)(k)$ with the Somekawa type K-group $K(k; \mathcal{F}_1 \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)$ defined by B.Kahn and T.Yamazaki in [22] (def.5.1).

Remark 4.4.1. If $\mathcal{M}_1, \cdots, \mathcal{M}_r$ are reciprocity functors with $r \geq 3$, then F. Ivorra and K.

Rülling in corollary 4.2.5. of [20] prove that there is a functorial map

$$T(\mathcal{M}_1, \cdots, \mathcal{M}_r) \to T(T(\mathcal{M}_1, \cdots, \mathcal{M}_{r-1}), \mathcal{M}_r)$$

which is surjective as a map of Nisnevich sheaves. It is not clear whether this map is always an isomorphism which would imply that T is associative and we would call it a product. In the case $\mathcal{F}_i \in \mathrm{HI}_{\mathrm{Nis}}$, for every $i \in \{1, \cdots, r\}$, associativity holds. In fact, in this case there is an isomorphism of reciprocity functors

$$T(\hat{\mathcal{F}}_1, \cdots, \hat{\mathcal{F}}_r) \simeq (\mathcal{F}_1 \widehat{\bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \cdots \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}}} \cdots \mathcal{F}_r),$$

where $\mathcal{F}_1 \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \cdots \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \mathcal{F}_r$ is the product of homotopy invariant Nisnevich sheaves with transfers. (see 2.10 in [22] for the definition of the product and theorem 5.1.8 in [20] for the isomorphism).

Notation 4.4.2. By abuse of notation from now on we will write $T(\mathcal{F}_1, \cdots, \mathcal{F}_r)$ for the K-group of reciprocity functors associated to $\hat{\mathcal{F}}_1, \cdots, \hat{\mathcal{F}}_r$.

Remark 4.4.3. Let NST be the category of Nisnevich sheaves with transfers. We note here that there is a left adjoint to the inclusion functor $\mathrm{NST} \to \mathrm{HI}_{\mathrm{Nis}}$ which is denoted by h_0^{Nis} (see section 2 in [22]). If U is a smooth curve over F, then there is a Nisnevich sheaf with transfers $L(U)$, where $L(U)(V) = \mathrm{Cor}(V, U)$ is the group of finite correspondences for V smooth over F, i.e. the free abelian group on the set of closed integral subschemes of $V \times U$ which are finite and surjective over some irreducible component of V. Then the corresponding homotopy invariant Nisnevich sheaf with transfers $h_0^{\mathrm{Nis}}(U) := h_0^{\mathrm{Nis}}(L(U))$ is the sheaf associated to the presheaf of relative Picard groups

$$V \to \mathrm{Pic}(\overline{U} \times V, D \times V),$$

where $\overline{U}$ is the smooth compactification of U, $D = \overline{U} \setminus U$ and V runs through smooth F-schemes. When U is projective we have an isomorphism $h_0^{\text{Nis}}(U) \simeq \underline{CH}_0(U)$ (see lemma 11.2 in [22]). In particular, $\underline{CH}_0(C)$ is homotopy invariant Nisnevich sheaf with trasnfers.

Let $\mathcal{F} \in \text{HI}_{\text{Nis}}$. If we are given a section $g \in \mathcal{F}(U)$ for some open dense $U \subset C$, then g induces a map of Nisnevich sheaves with transfers $\varphi : h_0^{\text{Nis}}(U) \to \mathcal{F}$ such that

$$\begin{aligned} \varphi(U): \quad & h_0^{\text{Nis}}(U)(U) \to \mathcal{F}(U) \\ & [\Delta] \to g, \end{aligned}$$

where $[\Delta] \in h_0^{\text{Nis}}(U)(U)$ is the class of the diagonal. The existence of the map φ follows by adjunction, since we have an obvious morphism $L(U) \to \mathcal{F}$ in NST.

Lemma 4.4.4. *Let $\mathcal{F}_1, \cdots, \mathcal{F}_r \in \text{HI}_{\text{Nis}}$ be homotopy invariant sheaves with transfers. Then the K-group of reciprocity functors $T(T(\mathcal{F}_1, \cdots, \mathcal{F}_r), \underline{CH}_0(C)^0)$ satisfies the assumptions of theorem 4.3.10.*

Proof. By remark 4.4.1 we get an isomorphism

$$T(T(\mathcal{F}_1, \cdots, \mathcal{F}_r), \underline{CH}_0(C)^0)(k) \simeq T(\mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)(k).$$

Moreover, by theorem 5.1.8. in [20] we get that the groups $K^{geo}(k; \mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)$ and $T(\mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)(k)$ are equal and they coincide with the Somekawa type K-group $K(k; \mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)$. We conclude that assumption 4.3.9 holds.

Regarding the assumption 4.3.2, let $g_i \in \mathcal{F}_i(\eta_C)$ and $h \in \underline{CH}_0(C)^0(\eta_C)$ such that $s_P(h) = 0$ for some closed point $P \in C$. Let moreover $f \in k(C)^\times$. We need to verify that $(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = 0$. If $g_i \in \mathcal{F}_{i,C,P}$, for every $i \in \{1, \cdots, r\}$, then

$$(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = \text{ord}_P(f) s_P(g_1) \otimes \cdots \otimes s_P(g_r) \otimes s_P(h) = 0.$$

Assume P is in the support of g_i for some $i \in \{1, \cdots, r\}$.

We first treat the case when $\mathcal{F}_i$ is curve-like (see def. 11.1 in [22])), for $i = 1, \cdots, r$. For such $\mathcal{F}_i$ it suffices to consider elements $g_i \in \mathcal{F}_i(\eta_C)$ with disjoint supports (proposition 11.11 in [22]). In this case the claim follows by the explicit computation of the local symbol (lemma 8.5 and proposition 11.6 in [22]). Namely, if $P \in \mathrm{supp}(g_i)$, then the local symbol at P is given by the formula

$$(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = s_P(g_1) \otimes \cdots \partial_P(g_i, f) \otimes \cdots \otimes s_P(g_r) \otimes s_P(h) = 0,$$

where $\partial_P(g_i, f)$ is the symbol at P defined in section 4.1 of [22].

Assume now that $\mathcal{F}_i$ is general, for $i = 1, \cdots, r$. Since $g_i \in \mathcal{F}_i(\eta_C)$ and $\mathcal{F}_i(\eta_C) \simeq \varinjlim \mathcal{F}_i(U)$, there is an open dense subset $U_i \subset C$ such that $g_i \in \mathcal{F}(U_i)$, for $i = 1, \cdots, r$. By remark 4.4.3 we get that the sections g_i induce morphisms in $\mathrm{HI}_{\mathrm{Nis}}$, $\varphi_i : h_0^{\mathrm{Nis}}(U_i) \to \mathcal{F}$. In particular, we get a homomorphism

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_r \otimes 1 : K(k; h_0^{\mathrm{Nis}}(U_1), \cdots, h_0^{\mathrm{Nis}}(U_r), \underline{CH}_0(C)^0) \to K(k; \mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)$$

with the property

$$(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = \varphi(([\Delta_1] \otimes \cdots \otimes [\Delta_r] \otimes h; f)_P).$$

Notice that the latter element vanishes, because $h_0^{\mathrm{Nis}}(U_i)$ is curve-like, for $i = 1, \cdots, r$ (lemma 11.2(c) in [22]) and hence $([\Delta_1] \otimes \cdots \otimes [\Delta_r] \otimes h; f)_P = 0$.

□

Corollary 4.4.5. *Let $\mathcal{F}_1, \cdots, \mathcal{F}_r \in \mathrm{HI}_{\mathrm{Nis}}$. Let $\mathcal{M} = T(\mathcal{F}_1, \cdots, \mathcal{F}_r)$ and let $(\underline{\underline{C}})$ be the local symbol complex associated to $\mathcal{M}$ corresponding to the curve C. Then there is a canonical isomorphism*

$$H(\underline{\underline{C}}) \simeq T(\mathcal{F}_1, \cdots, \mathcal{F}_r, \underline{CH}_0(C)^0)(k).$$

In particular, if $G_1, \cdots, G_r$ are semi-abelian varieties over k, then

$$H(\underline{\underline{C}}) \simeq T(G_1, \cdots, G_r, \underline{CH}_0(C)^0)(k) \simeq K(k; G_1, \cdots, G_r, \underline{CH}_0(C)^0),$$

where $K(k; G_1, \cdots, G_r, \underline{CH}_0(C)^0)$ is the usual Somekawa K-group attached to semi-abelian varieties. ([37], def. 1.2)

4.4.2 The $\mathbb{G}_a$-Case

In this subsection we consider reciprocity functors $\mathcal{M}_1, \cdots, \mathcal{M}_r$, $r \geq 0$ and set $\mathcal{M}_0 = \mathbb{G}_a$. We consider the K-group of reciprocity functors $T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)$.

Lemma 4.4.6. *The K-group $T(T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)$ satisfies the assumption 4.3.2.*

Proof. We have a functorial surjection

$$T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r, \underline{CH}_0(C)^0)(k) \twoheadrightarrow T(T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)(k).$$

The first group vanishes by the main result of [32] (theorem 1.1). Therefore, the second group vanishes as well. In particular, 4.3.2 is satisfied.

□

Lemma 4.4.7. *The K-group $T(T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)$ satisfies the assumption 4.3.9.*

Proof. To prove the lemma, it suffices to show that $K^{geo}(k; T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)$ vanishes.

<u>Claim:</u> There is a well defined local symbol

$$T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\eta_C) \otimes \underline{CH}_0(C)^0(\eta_C) \otimes k(C)^\times \to K^{geo}(k; T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)$$

satisfying the unique properties (1)-(3) of section 2.2.

To have a well defined local symbol following Serre ([36]), we need for every closed point $P \in C$ the natural map

$$T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes \underline{CH}_0(C)^0(\mathcal{O}_{C,P}) \xrightarrow{h} T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\eta_C) \otimes \underline{CH}_0(C)^0(\eta_C)$$

to be injective. For, if $g_1 \in T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\eta_C)$, $g_2 \in \underline{CH}_0(C)^0(\eta_C)$, then we say that $g_1 \otimes g_2$ is regular, if $g_1 \otimes g_2 = h(\tilde{g}_1 \otimes \tilde{g}_2)$, for some $\tilde{g}_1 \otimes \tilde{g}_2 \in T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes \underline{CH}_0(C)^0(\mathcal{O}_{C,P})$. For such $g_1 \otimes g_2$ we can define $(g_1 \otimes g_2; f)_P = \text{ord}_P(f) s_P(\tilde{g}_1) \otimes s_P(\tilde{g}_2)$. For non regular $g_1 \otimes g_2$ we define the local symbol using an auxiliary function f_P for f at P as usual. (see section 2.2) The symbol $(.;.)_P$ is well defined, since there is a unique lifting $\tilde{g}_1 \otimes \tilde{g}_2$ and the unique properties (1)-(3) of section 2.2 are satisfied by the very definition of the group $K^{geo}(k; T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)$. Therefore to prove the claim, it suffices to show the injectivity of h.

Note that we have an equality

$$\underline{CH}_0(C)^0(\mathcal{O}_{C,P}) := \underline{CH}_0(C \times k(\text{Spec}(\mathcal{O}_{C,P})))^0 = \underline{CH}_0(C)^0(\eta_C).$$

Moreover, the map $T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \to T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\eta_C)$ is injective by the injectivity condition of reciprocity functors. Next, notice that $T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)$ becomes a reciprocity functor of either $\mathbb{Q}$ or $\mathbb{F}_p$-vector spaces, depending on whether char F is 0 or $p > 0$. Setting $\kappa = \mathbb{Q}$ or $\mathbb{Z}/p$ depending on the case we have,

$$\begin{aligned} &T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes_{\mathbb{Z}} \underline{CH}_0(C)^0(\mathcal{O}_{C,P}) = \\ &T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes_{\kappa} (\kappa \otimes_{\mathbb{Z}} \underline{CH}_0(C)^0(\mathcal{O}_{C,P})). \end{aligned}$$

Since the κ-module $\kappa \otimes_{\mathbb{Z}} \underline{CH}_0(C)^0(\mathcal{O}_{C,P})$ is flat, the claim follows.

To prove the lemma, we imitate the proof of Rülling-Yamazaki of the vanishing of

$T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r, \underline{CH}_0(C)^0)(k)$ in [32]. Namely, let $\{(x_0, \cdots, x_r), \zeta\}^{geo}$ be a generator of $K^{geo}(k; T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r), \underline{CH}_0(C)^0)$. Since k is algebraically closed, we may assume $\zeta = [P_0] - [P_1]$, for some closed points $P_0, P_1 \in C$. Then we can show that

$$\{(x_0, \cdots, x_r), \zeta\}^{geo} = \sum_{P \in C} ((x_0 g \otimes \mathrm{res}_{k(C)/k}(x_1) \cdots \otimes \mathrm{res}_{k(C)/k}(x_r)) \otimes \eta_C; f)_P = 0,$$

where $f \in k(C)^\times$ is a function such that $\mathrm{ord}_{P_0}(f) = 1$ and $\mathrm{ord}_{P_1}(f) = -1$ and $g \in k(C)^\times$ is obtained using the exact sequence

$$\Omega^1_{k(C)/k} \longrightarrow \bigoplus_{P \in C} \frac{\Omega^1_{k(C)/k}}{\Omega^1_{C,P}} \xrightarrow{\sum \mathrm{Res}_P} k \longrightarrow 0.$$

For more details on the above local symbol computation we refer to section 3 in [32]. In particular, we refer to 3.2 and 3.4 for the choice of the functions $f, g \in k(C)^\times$.

□

Corollary 4.4.8. *Let $\mathcal{M}_1, \cdots, \mathcal{M}_r$ be reciprocity functors. Let $\mathcal{M} = T(\mathbb{G}_a, \mathcal{M}_1, \cdots, \mathcal{M}_r)$. Then for any smooth complete curve C over k, $H(\underline{\underline{C}}) = 0$. In particular, if $\mathrm{char}\, F = 0$, the complex $\Omega^{n+1}_{k(C)} \xrightarrow{\mathrm{Res}_P} \bigoplus_{P \in C} \Omega^n_k \xrightarrow{\sum_P} \Omega^n_k$ is exact.*

Proof. When $\mathrm{char}\, F = 0$, Ivorra and Rülling showed an isomorphism of reciprocity functors $\theta : \Omega^n \simeq T(\mathbb{G}_a, \mathbb{G}^{\times n})$ (Theorem 5.4.7 in [20]). Moreover, the complex $(\underline{\underline{C}})$ factors through $\Omega^{n+1}_{k(C)}$.

□

4.5 The non-algebraically closed Case

In order to prove theorem 4.3.10, we made the assumption that the curve C is over an algebraically closed field k. The reason this assumption was necessary is that for a general reciprocity functor $\mathcal{M}$ the local symbol at a closed point $P \in C$ does not have a local description, but rather depends on the other closed points. Namely, if P is in the support of the modulus $\mathfrak{m}$ corresponding to a section $g \in \mathcal{M}(\eta_C)$, then we have an equality

$$(g; f)_P = -\sum_{Q \notin S} \operatorname{ord}_Q(f_P) \operatorname{Tr}_{Q/k}(s_Q(g)),$$

where f_P is an auxiliary function for f at P. If for some reciprocity functor $\mathcal{M}$ we have a local description $(g; f)_P = \operatorname{Tr}_{P/k}(\partial_P(g; f)))$, where $\partial_P(g; f) \in \mathcal{M}(P)$, for every $P \in C$, then we can obtain a complex $(\underline{\underline{C}})'$

$$(\mathcal{M} \bigotimes^{M} \mathbb{G}_m)(\eta_C) \xrightarrow{\partial_C} \bigoplus_{P \in C} \mathcal{M}(P) \xrightarrow{\sum_P \operatorname{Tr}_{P/k}} \mathcal{M}(k).$$

For such a reciprocity functor $\mathcal{M}$, assuming the existence of a k-rational point $P_0 \in C(k)$, we can have a generalization of theorem 4.3.10 for the complex $(\underline{\underline{C}})'$ by imposing the following two stronger conditions on $\mathcal{M}$. Namely, we make the following assumptions.

Assumption 4.5.1. Let $\mathcal{M}$ be a reciprocity functor for which we have a local description of the symbol $(g; f)_P = \operatorname{Tr}_{P/k}(\partial_P(g; f))$. Let $\lambda : D \to C$ be a finite morphism. Assume that for every $h \in \underline{CH}_0(C)(\eta_D)$ and every closed point $P \in C$ we have an equality

$$(g \otimes h; f)_P = \operatorname{Tr}_{P/k}(\partial_P(g, f) \otimes s_P(h)).$$

Assumption 4.5.2. We assume that for every finite extension L/k we have an equality

$$K^{geo}(L; \mathcal{M}, \underline{CH}_0(C)^0) \simeq T(\mathcal{M}, \underline{CH}_0(C)^0)(L).$$

Notation 4.5.3. If E/L is a finite extension and $x \in \mathcal{M}(k)$, we will denote $x_E = \operatorname{res}_{E/L}(x)$.

Theorem 4.5.4. *Let $\mathcal{M}$ be a reciprocity functor that satisfies the assumptions 4.5.1 and*

4.5.2. Then we have an isomorphism

$$\begin{aligned}\Phi' : \quad & H(\underline{\underline{C'}}) \xrightarrow{\simeq} T(\mathcal{M}, \underline{CH}_0(C)^0)(k)\\ & (a_P)_{P\in C} \to \sum_{P\in C} \mathrm{Tr}_{P/k}(a_P \otimes ([P] - P_{0,k(P)})).\end{aligned}$$

Proof. We start by considering the map

$$\begin{aligned}\Phi' : \quad & (\bigoplus_{P\in C} \mathcal{M}(P))/\operatorname{Im}\partial_C \to T(\mathcal{M}, \underline{CH}_0(C))(k)\\ & (a_P)_{P\in C} \to \sum_{P\in C} \mathrm{Tr}_{P/k}(a_P \otimes [P]).\end{aligned}$$

The map Φ' is well defined because of the assumption 4.5.1. Restricting to $H(\underline{\underline{C'}})$, we obtain the map of the proposition. Moreover, we can consider the map

$$\begin{aligned}\Psi' : \quad & T(\mathcal{M}, \underline{CH}_0(C)^0)(k) \to H(\underline{\underline{C'}})\\ & Tr_{L/k}(x \otimes ([Q] - [L(Q) : L][P_{0,L}])) \to (x_{P'})_{P'\in C},\end{aligned}$$

with $x_P = \mathrm{Tr}_{L(Q)/k(P)}(x)$, $x_{P_0} = -\mathrm{Tr}_{L(Q)/k}(x)$ and $x_{P'} = 0$ otherwise. Here L/k is a finite extension, $x \in \mathcal{M}(L)$, Q is a closed point of $C \times L$ having residue field $L(Q)$ that projects to $P \in C$ under the map $C \times L \to C$ with $P \neq P_0$. We denote by $k(P)$ the residue field of P. The map Ψ' will be well defined (using the same argument as in proposition 4.3.8), as long as we check the following.

1. If $k \subset L \subset E$ is a tower of finite extensions and we have elements $x \in \mathcal{M}(L)$, $y \in \underline{CH}_0(C)^0(E)$, then $\Psi'(\mathrm{Tr}_{L/k}(x \otimes \mathrm{Tr}_{E/L}(y))) = \Psi'(\mathrm{Tr}_{E/k}(x_E \otimes y))$.
2. If $x \in \mathcal{M}(E)$, $y \in \underline{CH}_0(C)^0(L)$, then $\Psi'(\mathrm{Tr}_{L/k}(\mathrm{Tr}_{E/L}(x) \otimes y)) = \Psi'(\mathrm{Tr}_{E/k}(x \otimes y_E))$.

For (1) we can reduce to the case when $y = [Q] - [E(Q) : E]P_{0,E}$, for some closed point Q of $C \times E$ with residue field $E(Q)$. Let Q' be the projection of Q in $C \times L$ and P the projection of Q' in C. Notice that we have an equality $\mathrm{Tr}_{E/L}([Q]) = [E(Q):L(Q')][Q']$. We compute.

$$\Psi'(\mathrm{Tr}_{E/k}(x_E \otimes ([Q]-[E(Q):E][P_{0,E}]))) = \begin{cases} \mathrm{Tr}_{E(Q)/k(P)}(x), \text{ at } P \\ -\mathrm{Tr}_{E(Q)/k}(x), \text{ at } P_0 \end{cases}$$

$$\begin{aligned} &\Psi'(\mathrm{Tr}_{L/k}(x \otimes \mathrm{Tr}_{E/L}([Q]-[E(Q):E][P_{0,E}]))) = \\ &\Psi'(\mathrm{Tr}_{L/k}(x \otimes [E(Q):L(Q')]([Q']-[L(Q'):L][P_{0,L}]))) \\ &\begin{cases} [E(Q):L(Q')]\,\mathrm{Tr}_{L(Q')/k(P)}(x), \text{ at } P \\ -[E(Q):L(Q')]\,\mathrm{Tr}_{L(Q')/k}(x), \text{ at } P_0. \end{cases} \end{aligned}$$

The claim then follows by the fact that

$$\mathrm{Tr}_{E(Q)/k(P)}(x) = \mathrm{Tr}_{L(Q')/k(P)}\mathrm{Tr}_{E(Q)/L(Q')}(x) = [E(Q):L(Q')]\,\mathrm{Tr}_{L(Q')/k(P)}(x).$$

For (2) we can again reduce to the case when $y = [Q]-[L(Q):L][P_{0,L}]$ for some closed point Q of $C \times L$ with residue field $L(Q)$. Notice that we have an equality

$$[Q]_E = \sum_{Q' \to Q} e(Q'/Q)[Q'],$$

where the sum extends over all closed points Q' of $C \times E$ such that Q' projects to Q. We compute

$$\Psi'(\mathrm{Tr}_{L/k}(\mathrm{Tr}_{E/L}(x)\otimes y)) = \begin{cases} \mathrm{Tr}_{L(Q)/k(P)}(\mathrm{Tr}_{E/L}(x)_{L(Q)}), \text{ at } P \\ -\mathrm{Tr}_{L(Q)/k}(\mathrm{Tr}_{E/L}(x)_{L(Q)}), \text{ at } P_0 \end{cases}$$

$$= \begin{cases} \mathrm{Tr}_{L(Q)/k(P)}(\sum_{Q'\to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/L(Q)}(x_{E(Q')})), \text{ at } P \\ -\mathrm{Tr}_{L(Q)/k}(\sum_{Q'\to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/L(Q)}(x_{E(Q')})), \text{ at } P_0 \end{cases}$$

$$= \begin{cases} \sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E(Q')/k(P)}(x_{E(Q')}), \text{ at } P \\ -\sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E(Q')/k}(x_{E(Q')}), \text{ at } P_0. \end{cases}$$

The equality $\mathrm{Tr}_{E/L}(x)_{L(Q)} = \sum_{Q'\to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/L(Q)}(x_{E(Q')})$ follows from remark 1.3.3, property (MF1) in [20] if we set $\varphi : \mathrm{Spec}\, E \to \mathrm{Spec}\, L$ and $\psi : \mathrm{Spec}\, L(Q) \to \mathrm{Spec}\, L$. On the other hand we have

$$\Psi'(\mathrm{Tr}_{E/k}(x\otimes y_E)) = \Psi'(\sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E/k}(x\otimes([Q'] - [L(Q):L][P_{0,E}]))$$

$$= \begin{cases} \sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E(Q')/k(P)}(x_{E(Q')}), \text{ at } P \\ -\sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E(Q')/k}(x_{E(Q')}), \text{ at } P_0 \end{cases}$$

Next we need to show that the maps Φ', Ψ' are each other inverses. It is immediate that the composition $\Psi'\Phi'$ is the identity map. For the other composition, we consider an element $x\otimes([Q] - [L(Q):L][P_{0,L}]) \in T(\mathcal{M}, \underline{CH}_0(C)^0)(L)$. If $L(Q)$ is the residue field of Q, then Q induces an $L(Q)$-rational point $\widetilde{Q}$ of $C\times L(Q)$. Then we have an equality $\mathrm{Tr}_{L(Q)/L}([\widetilde{Q}]) = [Q]$. By the projection formula we get an equality

$$x\otimes([Q] - [L(Q):L][P_{0,L}]) = \mathrm{Tr}_{L(Q)/L}(x_{L(Q)}\otimes([\widetilde{Q}] - [P_{0,L(Q)}])),$$

we are therefore reduced to the case $L(Q) = L$. Then we have

$$\begin{aligned}\Phi'\Psi'(\mathrm{Tr}_{L/k}(x \otimes ([Q] - [P_{0,L}]))) = \quad & \mathrm{Tr}_{P/k}(\mathrm{Tr}_{L/P}(x) \otimes ([P] - [P_{0,k(P)}])) = \\ & \mathrm{Tr}_{P/k}\,\mathrm{Tr}_{L/P}(x \otimes \mathrm{res}_{L/P}([P] - [P_{0,k(P)}])) = \\ & \mathrm{Tr}_{L/k}(x \otimes [Q] - [P_{0,L}]).\end{aligned}$$

This completes the proof of the theorem.

□

Remark 4.5.5. We note here that for the algebraically closed field case if instead of the assumption 4.3.2, we had made the stronger assumption 4.5.1, the proof of the proposition 4.3.3 would have become simpler. The only reason we used 4.3.2 is that in general the problem of computing the symbol $(g; f)_P$ locally is rather hard and is known only in very few cases, namely for homotopy invariant Nisnevich sheaves with transfers, as the next example indicates.

Example 4.5.6. Let $k \in \mathcal{E}_F$ be any perfect field. Let $\mathcal{F}_1, \cdots, \mathcal{F}_r$ be homotopy invariant Nisnevich sheaves with transfers. Then as mentioned in the previous section, the main theorem of [22] gives an isomorphism

$$T(\mathcal{F}_1, \cdots, \mathcal{F}_r)(L) \simeq K^{geo}(L; \mathcal{F}_1, \cdots, \mathcal{F}_r) \simeq K(L; \mathcal{F}_1, \cdots, \mathcal{F}_r),$$

where $K(L; \mathcal{F}_1, \cdots, \mathcal{F}_r)$ is the Somekawa type K-group ([22],def. 5.1) and L/k is any finite extension. In particular, let C be a smooth, complete and geometrically connected curve over k and $P \in C$ be a closed point. As in the proof of lemma 4.4.4, we can reduce to the case when $\mathcal{F}_i$ is curve-like, for $i = 1, \cdots, r$. To describe the local symbol, it therefore suffices to consider sections $g_i \in \mathcal{F}_i(\eta_C)$ with disjoint supports. In this case, if P is in the support of g_i for some $i \in \{1, \cdots, r\}$ and $f \in k(C)^\times$ is a function, then we have the following explicit local description of $(g_1 \otimes \cdots \otimes g_r; f)_P$.

$$(g_1 \otimes \cdots \otimes g_r; f)_P = \mathrm{Tr}_{P/k}(s_P(g_1) \otimes \cdots \otimes \partial_P(g_i, f) \otimes \cdots \otimes s_P(g_r)).$$

Moreover, $\underline{CH}_0(C)^0$ is itself a homotopy invariant Nisnevich sheaf with transfers. Namely, $\underline{CH}_0(C)^0 \in RF_0$ and hence the above formula implies that the assumption 4.5.1 is satisfied.

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