THE UNIVERSITY OF CHICAGO

# $G$-TAMBARA FUNCTORS ARE $G$-COMMUTATIVE MONOIDS

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Dedication Text

To my parents

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## ACKNOWLEDGMENTS

Firstly, I would like to express my gratitude to my advisor Peter May for the continuous support of my research, for his patience, and willingness to share knowledge. I would also like to thank my fellow graduate students for their companionship that made the hard Chicago winters bearable. Lastly, I would like to thank my husband for his encouragement and unwavering faith in me.

# ABSTRACT

The interplay between trace, norm and restriction maps connecting the Burnside rings of subgroups of a finite group G has been studied extensively by Tambara. It became apparent that certain relations between those maps are satisfied by the trace, norm and restriction maps between cohomology groups as well as between representation rings of subgroups of G. A collection of abelian groups with trace, norm and restriction maps satisfying such relations is called a TNR-functor in [4] and is now referred to as a Tambara functor. A major motivation for studying Tambara functors is their connection with equivariant stable homotopy theory, i.e. for any $G$-spectrum $R$, with an equivariant $E_\infty$ structure, the Mackey functor $\pi_0^G R$ turns out to be a Tambara functor [5]. In [2], Hill and Hopkins introduce the notions of a $G$-symmetric monoidal structure on a symmetric monoidal coefficient system $\underline{\mathcal{C}}$ and of a $G$-commutative monoid in $\underline{\mathcal{C}}(G/G)$. If we think of the unbiased definition of a commutative monoid as an object with a collection of operations indexed by finite sets satisfying certain coherence relations, then a $G$-commutative monoid is an object with a collection of operations indexed by finite $G$-sets subject to appropriate coherence relations. In this thesis we construct a $G$-symmetric monoidal structure on the symmetric monoidal coefficient system formed by the collection of categories $Mackey_H$ of Mackey functors over $H$, with $H \subset G$, together with restriction functors. We then show that Tambara functors are precisely the $G$-commutative monoids in $Mackey_G$.

# CHAPTER 1

# INTRODUCTION

Classically, a Mackey functor is defined as a pair of functors $Orb_G \to Ab$ one contravariant, the other covariant, subject to certain compatibility conditions. Strickland proposes a way to view them as the additive completion of a product-preserving functor from the Burnside category $\mathcal{A}_G$ into $Set$. The domain category, $\mathcal{A}_G$, is constructed in such a way as to encapsulate the information that is otherwise explicit in the first definition. In this setting, it is straightforward to lift the symmetric monoidal structure on $\mathcal{A}_G$ given by the cartesian product of $G$-sets to the level of Mackey functors via a left Kan extension. This technique gives rise to the box product and is reviewed in Section 2.1.

In [2], Hill and Hopkins introduce the notions of a symmetric monoidal coefficient system and a symmetric monoidal Mackey functor. Furthermore, they define the concept of a $G$-symmetric monoidal structure on a symmetric monoidal coefficient system. Interestingly, it turns out that given a symmetric monoidal Mackey functor, there is an induced $G$-symmetric monoidal structure on the underlying coefficient system.

It is easy to see that the collection of categories $Mackey_H$, of Mackey functors over $H$, for $H \subset G$, together with restriction functors between them, form a symmetric monoidal coefficient system. One feels tempted to try and mimic the technique used to obtain the box product in order to construct a $G$-symmetric monoidal structure on the symmetric monoidal coefficient system of Mackey functors. To do this, two questions need to be answered:

(i) Is it possible to define a symmetric monoidal Mackey functor taking as values the categories $\mathcal{A}_G/(G/H)$ of $G$-sets over $G/H$?

(ii) Can we lift it to a symmetric monoidal Mackey functor with values $Mackey_H$, for $H \subset G$, with the aid of left Kan extensions?

The answer is yes on both fronts and this is carried out in Chapter 3.

Given a $G$-symmetric monoidal structure on a symmetric monoidal coefficient system, one has the concept of a $G$-commutative monoid as defined in [2]. Now that we have constructed a $G$-symmetric monoidal structure on the coefficient system of Mackey functors, it is natural to ask what are the $G$-commutative monoids in $Mackey_G$.

As before, we seek inspiration from the non-equivariant setting. There is an equivalence of categories between the category of monoids in $Mackey_G$ and the category of Green functors over $G$, see for instance [1]. A careful analysis of this equivalence is performed in Chapter 2 and it highlights the way in which the construction of the box product and the monoidal structure map come together and generate the extra structure necessary for the monoid to be a Green functor. This serves as a starting point for the proof of the main result of this thesis, namely that there is an equivalence of categories between $G$-commutative monoids in $Mackey_G$ and Tambara functors over $G$.

# CHAPTER 2

# EXPOSITORY REVIEW

## 2.1 A symmetric monoidal structure on Mackey functors

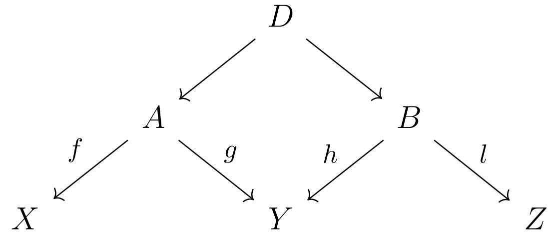
Fix a finite group $G$.

In this section we review the symmetric monoidal structure on the category of Mackey functors over $G$. The role of the symmetric monoidal product is played by the box product and the identity is the Burnside Mackey functor. We open the discussion with remarks on two equivalent definitions of Mackey functors, the second of which is the one most prevalent in this work. The construction of the box product is an application of a general technique ([1]) of lifting a symmetric monoidal structure from the level of a small semiadditive category, say $\mathcal{A}$, to the level of the category of product-preserving functors : $\mathcal{A} \to Set$ via a left Kan extension. Finally, the requirement for a Mackey functor to be a monoid with respect to this symmetric monoidal category is equivalent to it being a Green functor.

To begin, let us recall the Burnside category of $G$, denoted $\mathcal{A}_G$. This is the category with objects given by finite $G$-sets and morphisms given by isomorphism classes of spans. A span between two $G$-sets $X$ and $Y$ is a diagram of the form $X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y$, with $A$ another finite $G$-set. We say that two spans $X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y$ and $X \overset{f'}{\leftarrow} A' \overset{g'}{\rightarrow} Y$ are isomorphic if there is a $G$-equivariant isomorphism $h : A \to A'$ making the following diagram commute.

$$\begin{array}{ccccc} X & \xleftarrow{f} & A & \xrightarrow{g} & Y \\ \| & & \downarrow{h} & & \| \\ X & \xleftarrow[f']{} & A' & \xrightarrow[g']{} & Y \end{array} \tag{2.1.1}$$

The law of composition is as follows. Given two spans $X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y$ and $Y \overset{h}{\leftarrow} B \overset{l}{\rightarrow} Z$, their composition is given by the outer sides of the diagram below, where the middle square is a pullback square.



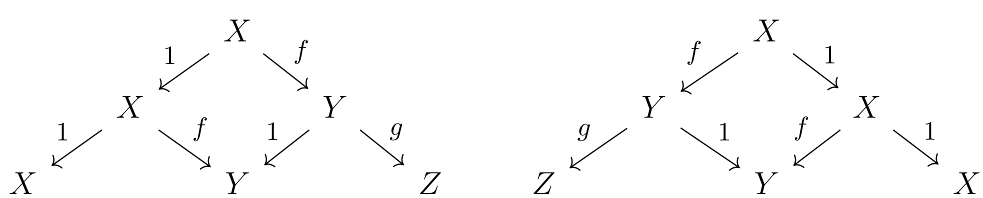
(2.1.2)

Note that composing respectively isomorphic spans gives rise to isomorphic spans, and thus the construction above is well defined.

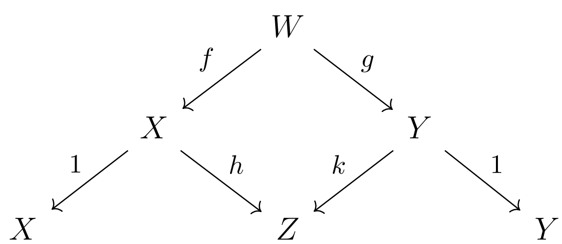
Let us look a bit more closely at morphisms in $\mathcal{A}_G$. Associated to any $G$-equivariant map $f : X \to Y$, we have two distinguished spans: $X \xleftarrow{1} X \xrightarrow{f} Y$ and $Y \xleftarrow{f} X \xrightarrow{1} X$, that we will refer to as $T_f$ and $R_f$ respectively. Any morphism in $\mathcal{A}_G$ can be expressed as a composition of a restriction followed by a transfer.

The commutative diagrams below exhibit two functoriality properties.

$$T_gT_f \quad = \quad T_{gf} \qquad\qquad R_fR_g \quad = \quad R_{gf}$$



The diagram below shows that, given any pullback square, we have $R_kT_h = T_gR_f$.



Note $\emptyset$ is both an initial and a final object in $\mathcal{A}_G$. Consider two finite $G$-sets, $X$ and $Y$, and denote by $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ the canonical inclusion maps. A straightforward set theoretic argument yields that the diagrams below are a coproduct and a product diagram respectively.

$$X \xleftarrow{T_{i_X}} X \sqcup Y \xrightarrow{T_{i_Y}} Y \qquad\qquad X \xleftarrow{R_{i_Y}} X \sqcup Y \xrightarrow{R_{i_X}} Y$$

**Definition 2.1.3.** A semi-Mackey functor is a product-preserving functor $\underline{M} : \mathcal{A}_G \to Set$.

Unpacking the product-preserving requirement shows that, for finite $G$-sets $X$ and $Y$, we have two isomorphisms inverse to each other, as displayed in the diagram below.

$$\underline{M}(X \sqcup Y) \underset{\mu_{X,Y}}{\overset{(\underline{M}(R_{i_X}),\underline{M}(R_{i_Y}))}{\rightleftarrows}} \underline{M}(X) \times \underline{M}(Y).$$

To make the notation less cumbersome, we will forthwith refer to the morphisms $\underline{M}(R_f)$ and $\underline{M}(T_f)$ as $\mathbf{R}_f$ and $\mathbf{T}_f$ respectively.

**Remark 2.1.4.** For a finite $G$-set $X$, let $\nabla : X \sqcup X \to X$ be the folding map and $i : \emptyset \to X$ the canonical inclusion. The set $\underline{M}(X)$ can be endowed with a semigroup structure with addition and identity maps defined as below.

$$\begin{array}{ccc} \underline{M}(X) \times \underline{M}(X) & \xrightarrow{+} & \underline{M}(X) \\ \downarrow{\scriptstyle \mu_{X,X}} & \nearrow{\scriptstyle \mathbf{T}_\nabla} & \\ \underline{M}(X \sqcup X) & & \end{array} \qquad\qquad * \simeq \underline{M}(\emptyset) \xrightarrow{\mathbf{T}_i} \underline{M}(X)$$

Furthermore, given any $G$-equivariant map $f : X \to Y$, the maps $\mathbf{R}_f : \underline{M}(Y) \to \underline{M}(X)$ and $\mathbf{T}_f : \underline{M}(X) \to \underline{M}(Y)$ are maps of semigroups.

**Definition 2.1.5.** A Mackey functor is a semi-Mackey functor $\underline{M}$ with the property that the value at an arbitrary set $X$, $\underline{M}(X)$, together with the addition and identity maps defined above form an abelian group.

The classical definition of a Mackey functor is formulated in terms of the category $Fin_G$ with objects consisting of finite $G$-sets and morphisms given by equivariant maps.

**Definition 2.1.6.** A Mackey functor consists of a pair of functors $(M_*, M^*) : Fin_G \to \mathcal{A}b$, the first covariant and the second one contravariant, that agree on objects. We write $M(X)$ for their common value at $X$. Moreover, the following two conditions need to be fulfilled.

1. For any pullback square,

$$\begin{CD} X @>{f}>> Y \\ @V{h}VV @VV{l}V \\ Z @>>{g}> W \end{CD}$$

   in $G$-set it holds that $M^*(g)M_*(l) = M_*(h)M^*(f)$.

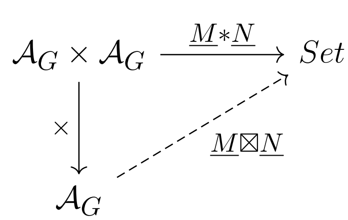
2. The map $M(X \sqcup Y) \xrightarrow{(M^*(i_X), M^*(i_Y))} M(X) \oplus M(Y)$ is an isomorphism, with $i_X$ and $i_Y$ denoting the canonical inclusion maps.

To see that the two definitions align, note that $\mathbf{T}_f$, $\mathbf{R}_f$ play the role of $M_*(f)$ and $M^*(f)$ respectively in the second definition. With this analogy in mind, observe that the first condition in 2.1.6 translates into the fact that $\underline{M}$ respects composition in $\mathcal{A}_G$, i.e. it is a functor, while the second condition is equivalent to the product-preserving requirement in 2.1.3.

**Example 2.1.7.** The Burnside Mackey functor is a representable Mackey functor, defined on objects as $\underline{A}(X) = \mathcal{A}_G(*, X)$ and on morphisms as $\underline{A}(f) = f \circ$ .

Denote the category of Mackey functors by $Mackey_G$.

**Definition 2.1.8.** Given two Mackey functors $\underline{M}, \underline{N}$, in the sense of definition 2.1.5, define $\underline{M} * \underline{N} : \mathcal{A}_G \times \mathcal{A}_G \to Set$ by $(\underline{M} * \underline{N})(X \times Y) = \underline{M}(X) \times \underline{N}(Y)$. Then their box product $\underline{M} \boxtimes \underline{N}$ is the left Kan extension of $\underline{M} * \underline{N}$ along $\times$, as shown in the diagram below.



**Remark 2.1.9.** The box product is usually defined as the left Kan extension of a functor taking values in the category *Ab* of abelian groups. It is proven in the appendix of [1] that the set-valued functor we have defined actually takes values in Ab. That is, the abelian group structures on M(X) and N(X) naturally induce an abelian group structure on $(M \boxtimes N)(X)$.

To get a better understanding of the box product construction, let us unpack its definition using the characterization of the left Kan extension as a colimit over a comma category.

We see that an element in $(\underline{M} \boxtimes \underline{N})(X)$ is an equivalence class of triples of the form

$$\Big(U \times V \xleftarrow{q} W \xrightarrow{p} X \ , \ m \in \underline{M}(U) \ , \ n \in \underline{N}(V)\Big) \tag{2.1.10}$$

under the equivalence relation $\sim$ given by

$$\begin{aligned}&( \ U \times V \xleftarrow{q} W \xrightarrow{p} X, \ \underline{M}(\omega)(m'), \ \underline{N}(\phi)(n') \ ) \sim \\ &\qquad ( \ U' \times V' \xleftarrow{\omega_q \times \phi_q} A \times B \xrightarrow{\omega_p \times \phi_q} U \times V \xleftarrow{q} \ W \xrightarrow{p} X, \ m', \ n' \ )\end{aligned} \tag{2.1.11}$$

where $\omega : U' \xleftarrow{\omega_q} A \xrightarrow{\omega_p} U$ and $\phi : V' \xleftarrow{\phi_q} B \xrightarrow{\phi_p} V$ are spans, $m' \in \underline{M}(U')$ and $n' \in \underline{N}(V')$. Note that, for the sake of transparency, the first entry of the second triple is expressed as a composition of spans, rather than in canonical form.

In this language, the transfer and restriction maps giving the Mackey functor structure on $\underline{M} \boxtimes \underline{N}$ take the form:

$$\mathbf{T}_f \Big((U \times V \xleftarrow{q} W \xrightarrow{p} X, m, n)\Big) = (U \times V \xleftarrow{q} W \xrightarrow{p} X \xrightarrow{f} Y, m, n)$$

$$\mathbf{R}_f \Big((U \times V \xleftarrow{q} W \xrightarrow{p} Y, m, n)\Big) = (U \times V \xleftarrow{q} W \xrightarrow{p} Y \xleftarrow{f} X, m, n),$$

for any equivariant map $f : X \to Y$.

**Definition 2.1.12.** Given $m \in \underline{M}(X)$ and $n \in \underline{N}(X)$ denote by $m \otimes n$ the element in $(\underline{M} \boxtimes \underline{N})(X)$ corresponding to the equivalence class of the triple $\Big(X \times X \xleftarrow{\Delta} X, m, n\Big)$. Here

$\Delta : X \to X \times X$ stands for the diagonal map.

For a $G$-map $p : W \to X$ the following squares are pullbacks.

$$\begin{array}{ccc} W & \xrightarrow{\quad p \quad} & X \\ {\scriptstyle (1\times p)\circ\Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ W \times X & \xrightarrow[p\times 1]{\quad} & X \times X \end{array} \qquad\qquad \begin{array}{ccc} W & \xrightarrow{\quad p \quad} & X \\ {\scriptstyle (p\times 1)\circ\Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times W & \xrightarrow[1\times p]{\quad} & X \times X \end{array}$$

We can use these two pullback squares to deduce two Frobenius type identities in $(\underline{M} \boxtimes \underline{N})(X)$. Indeed, for $m' \in \underline{M}(W)$ and $n \in \underline{N}(X)$,

$$\begin{aligned} \mathbf{T}_p\left(m' \otimes \mathbf{R}_p(n)\right) &= \left(W \times W \xleftarrow{\Delta} W \xrightarrow{p} X, m', \mathbf{R}_p(n)\right) \\ &= \left(W \times X \xleftarrow{1\times p} W \times W \xleftarrow{\Delta} W \xrightarrow{p} X, m', n\right) \\ &= \left(W \times X \xrightarrow{p\times 1} X \times X \xleftarrow{\Delta} X, m', n\right) \\ &= \left(X \times X \xleftarrow{\Delta} X, \mathbf{T}_p(m'), n\right) \\ &= \mathbf{T}_p(m') \otimes n. \end{aligned} \tag{2.1.13}$$

More precisely, the transitions from the first to the second line and from the third to the fourth line follow from the equivalence relation 2.1.11, while utilizing the first pullback square above yields the second equality.

Similarly, we obtain that, for $m \in \underline{M}(X)$ and $n' \in \underline{N}(W)$, $\mathbf{T}_p(\mathbf{R}_p(m) \otimes n') = m \otimes \mathbf{T}_p(n')$.

**Definition 2.1.14.** A commutative monoid in $Mackey_G$ is a Mackey functor together with a map of Mackey functors $\mu : \underline{S} \boxtimes \underline{S} \to \underline{S}$ that is commutative, associative and unital.

**Definition 2.1.15.** A Green functor over $G$ is a Mackey functor $\underline{S}$, satisfying the following conditions:

(i) For any finite $G$-set $X$, $\underline{S}(X)$ is a commutative ring and for any $G$-map $f : X \to Y$, $\mathbf{R}_f : \underline{S}(Y) \to \underline{S}(X)$ is a ring homomorphism.

(ii) For any $f : X \to Y$, a Frobenius reciprocity relation holds:

$$a \cdot \mathbf{T}_f(b) = \mathbf{T}_f(a \cdot \mathbf{R}_f(b)),$$

where $a \in \underline{S}(Y)$ and $b \in \underline{S}(X)$.

**Proposition 2.1.16.** *The categories of Green functors and commutative monoids in $Mackey_G$ are naturally isomorphic.*

*Proof.* While this is a standard result, we include the proof here both because full details are hard to find in the literature and because the proof provides insight that will come in handy later on, during our discussion of Tambara functors.

Note that given an arbitrary representative in an equivalence class in $(\underline{M} \boxtimes \underline{N})(X)$, we have the following string of equivalences, for $m \in \underline{M}(U)$ and $n \in \underline{N}(V)$.

$$\begin{aligned} \left(U \times V \xleftarrow{q} W \xrightarrow{p} X, m, n\right) &\sim \left(U \times V \xleftarrow{(i\ ,\ j)} W \times W \xleftarrow{\Delta} W \xrightarrow{p} X,\ m, n\right) \\ &\sim \left(W \times W \xleftarrow{\Delta} W \xrightarrow{p} X,\ \mathbf{R}_i(m),\ \mathbf{R}_j(n)\right) \\ &\sim \mathbf{T}_p\left(W \times W \xleftarrow{\Delta} W,\ \mathbf{R}_i(m),\ \mathbf{R}_j(n)\right). \end{aligned}$$

Thus, any element in $(\underline{M} \boxtimes \underline{M})(X)$ has a representative of the form $\mathbf{T}_p\left(\mathbf{R}_i(m) \otimes \mathbf{R}_j(n)\right)$.

Specifying a map of Mackey functors $\mu : \underline{M} \boxtimes \underline{M} \to \underline{M}$ is the same as giving, for each finite $G$-set $X$, a map of abelian groups $\mu_X : (\underline{M} \boxtimes \underline{M})(X) \to \underline{M}(X)$ such that for an arbitrary $G$-map $p : W \to X$,

$$\mathbf{T}_p \circ \mu_W = \mu_X \circ \mathbf{T}_p \text{ and } \mathbf{R}_p \circ \mu_X = \mu_W \circ \mathbf{R}_p.$$

Since $\mu_X(\mathbf{T}_p(\mathbf{R}_i(m) \otimes \mathbf{R}_j(n))) = \mathbf{T}_p(\mu_W(\mathbf{R}_i(m) \otimes \mathbf{R}_j(n)))$, the maps $\mu_X$ are determined by their values at elements of the form $m \otimes n$, for all $m, n \in \underline{M}(X)$.

Consider the binary operation on $\underline{M}(X)$, obtained by setting $m_1 \cdot m_2 = \mu_X(m_1 \otimes m_2)$. This operation is associative and unital by virtue of the fact that $\mu$ is associative and unital.

Furthermore, multiplication, as defined above, distributes over addition. Indeed, for $m_1, m_2, n \in \underline{M}(X)$ we have:

$$
\begin{aligned}
(m_1 + m_2) \cdot n &= \mu_X \left( \left( X \times X \xleftarrow{\Delta} X, m_1 + m_2, n \right) \right) \\
&= \mu_X \left( \left( X \times X \xleftarrow{\Delta} X, \mathbf{T}_\nabla(m_1, m_2), n \right) \right) \\
&= \mu_X \left( \left( (X \sqcup X) \times X \xleftarrow{\nabla \times 1} X \times X \xleftarrow{\Delta} X, (m_1, m_2), n \right) \right) \\
&= \mu_X \left( \left( (X \sqcup X) \times X \xleftarrow{\cong} (X \times X) \sqcup (X \times X) \xleftarrow{\Delta \sqcup \Delta} X \sqcup X \xrightarrow{\nabla} X, (m_1, m_2), n \right) \right) \\
&= \mu_X \left( \mathbf{T}_\nabla \left( (X \times X) \sqcup (X \times X) \xleftarrow{\Delta \sqcup \Delta} X \sqcup X, (m_1, n), (m_2, n) \right) \right) \\
&= \mu_X \left( \left( X \times X \xleftarrow{\Delta}, m_1, n \right) \right) + \mu_X \left( \left( X \times X \xleftarrow{\Delta} X, m_2, n \right) \right) \\
&= (m_1 \cdot n) + (m_2 \cdot n).
\end{aligned}
$$

We can now conclude that the set $\underline{M}(X)$, together with the two operations $+$ and $\cdot$ , is a ring for any $G$-set $X$.

Let us check that $\mathbf{R}_p : \underline{M}(X) \to \underline{M}(W)$ is a ring homomorphism.

$$
\begin{aligned}
\mathbf{R}_p \left( \mu_X \left( (X \times X \xleftarrow{\Delta} X, m, n) \right) \right) &= \mu_W \left( (X \times X \xleftarrow{\Delta} X \xrightarrow{p} W, m, n) \right) \\
&= \mu_W \left( (X \times X \xleftarrow{(p,p)} W, m, n) \right) \\
&= \mu_W \left( (W \times W \xleftarrow{\Delta}, \mathbf{R}_p(m), \mathbf{R}_p(n)) \right) \\
&= \mathbf{R}_p(m) \cdot \mathbf{R}_p(n)
\end{aligned}
$$

The fact that the Frobenius reciprocity formula holds follows immediately from 2.1.13.

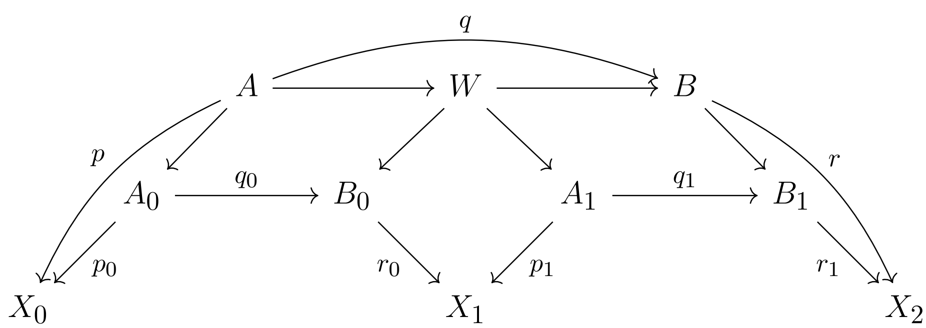
This shows that the requirement that a Mackey functor is a commutative monoid in $Mackey_G$ coincides with the requirement that it is a Green functor. □

sectionTambara functors In this section we give the definition of Tambara functors as additive completions of product preserving functors from the category of bispans of $G$-equivariant finite sets into $Set$. We'll show how norms corresponding to certain maps are constructed in the case of the Green functors, thus preparing the terrain for explaining why Tambara functors are precisely $G$-commutative monoids in the category of Mackey functors.

We start by defining the category of $G$-equivariant bispans, denoted by $\mathcal{U}_G$. The objects of this category are finite $G$-sets, while the set of morphisms between two objects, $X$ and $Y$ are isomorphism classes of equivariant bispans $X \leftarrow A \rightarrow B \rightarrow Y$. Generalizing the idea of isomorphic spans, we say that two bispans are isomorphic if there are vertical equivariant isomorphisms making the following diagram commute.

$$\begin{array}{ccccccc} X & \longleftarrow & A & \longrightarrow & B & \longrightarrow & Y \\ \| & & \downarrow \simeq & & \downarrow \simeq & & \| \\ X & \longleftarrow & A' & \longrightarrow & B' & \longrightarrow & Y \end{array}$$

The diagram below gives the composition law for two bispans.




Here,

$$B = \left\{ (b_1, s) \mid s : q_1^{-1}\{b_1\} \rightarrow B_0, r_0 \circ s = p_1 \right\},$$
$$W = \left\{ (a_1, s) \mid s : q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0 \circ s = p_1 \right\},$$
$$A = \left\{ (a_0, a_1, s) \mid s : q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0 \circ s = p_1, a_0 \in q_0^{-1}\{a_1\} \right\}.$$

While the two rhombi are pullback squares, the middle square is in general not a pullback.

**Remark 2.1.17.** The following observation might shed some light on the somewhat opaque way of composing bispans described above. A bispan $\phi \in \mathcal{U}_G(X_0, X_1)$, with a representative given by $X_0 \xleftarrow{p_0} A_0 \xrightarrow{q_0} B_0 \xrightarrow{r_0} X_1$ can be interpreted as a collection of polynomials in elements of $X_0$, indexed by elements in $X_1$ . Indeed, for every $x_1 \in X_1$, we have $\phi_{x_1} = \sum_{b_0 \in r_0^{-1}\{x_1\}} \prod_{a_0 \in q_0^{-1}\{b_0\}} p_0(a_0)$. In this context, composition of bispans boils down to composition of polynomials and the maps $s : q_0^{-1}(b_0) \to B_0$ encode the distributivity of multiplication over addition.

In this vein, $\mathcal{U}_G(X_0, X_1)$ can be made into a groupoid by defining addition of bispans $X_0 \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} X_1$ and $X_0 \xleftarrow{p'} A' \xrightarrow{q'} B' \xrightarrow{r'} X_1$ to be

$$X_0 \xleftarrow{\nabla\circ(p \sqcup p')} A \sqcup A' \xrightarrow{q \sqcup q'} B \sqcup B' \xrightarrow{\nabla\circ(r \sqcup r')} X_1.$$

Mirroring the discussion of morphisms in $\mathcal{A}_G$, we have that associated to any equivariant map $f : X \to Y$, there are three types of distinguished morphisms in $\mathcal{U}_G\,(X, Y)$, namely

$$R_f : Y \xleftarrow{f} X \xrightarrow{1} X \xrightarrow{1} X$$
$$T_f : X \xleftarrow{1} X \xrightarrow{1} X \xrightarrow{f} Y$$
$$N_f : X \xleftarrow{1} X \xrightarrow{f} Y \xrightarrow{1} Y$$

Every morphism $\phi$ in $\mathcal{U}_G$ can be expressed as a composition of the form $T_f \circ N_g \circ R_h$, for some equivariant maps $f$, $g$ and $h$.

There are two natural ways of viewing spans as bispans. Indeed, for a span, $X \xleftarrow{f} A \xrightarrow{g} Y$, we have two corresponding bispans $X \xleftarrow{f} A \xrightarrow{g} Y \xrightarrow{1} Y$ and $X \xleftarrow{f} A \xrightarrow{1} A \xrightarrow{g} Y$. A routine check yields that both ways of associating bispans respect composition. In other words, the Burnside category $\mathcal{A}_G$ embeds into $\mathcal{U}_G$ in two ways. We can now utilize the work we've done in the beginning of section 2.1 to obtain the following result.

**Proposition 2.1.18.** *(i) For every two equivariant maps $f : X \to Y$ and $g : Y \to Z$, we have $R_{g\circ f} = R_f \circ R_g$, $T_{g\circ f} = T_g \circ T_f$ and $N_{g\circ f} = N_g \circ N_f$.*

*(ii) For any pullback square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle l} \\ Z & \xrightarrow[g]{} & W \end{array}$$

*we have that* $R_l \circ T_g = T_f \circ R_h$ *and* $R_l \circ N_g = N_f \circ R_h$.

We look more closely at the interaction between norms and transfers. To this end, let us introduce the notion of a distributor.

**Definition 2.1.19.** Given the diagram of equivariant maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we define its distributor, $\Delta(f,g)$, to be the bispan $X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z$, where

$$B = \left\{(z,s) \mid z \in Z,\ s : g^{-1}\{z\} \to X,\ f \circ s = 1\right\}$$

$$A = Y \times_Z B$$

$$p((y,s)) = s(y)\ ,\ q((y,s)) = (g(y),s) \text{ and } r((z,s)) = z.$$

From the definition of the law of composition of bispans, we easily deduce that $N_f \circ T_g$ is precisely $\Delta(f,g)$.

Finally, note that products in this newly defined category $\mathcal{U}_G$ are the same as the products in $\mathcal{A}_G$, i.e. given by taking the disjoint union of the two finite $G$-sets and letting the restriction maps associated to the canonical inclusions play the role of the projection maps.

**Definition 2.1.20.** A semi-Tambara functor is a product-preserving functor $\underline{R} : \mathcal{U}_G \to Set$.

In section 2.1 we explained how the requirement that the Mackey functor $\underline{M}$ be product-preserving gives rise to an associative and unital operation on each of the sets $\underline{M}(X)$, for $X$ a finite $G$-set. Each of the two embeddings of $\mathcal{A}_G$ in $\mathcal{U}_G$ induces an associative and unital operation on $\underline{R}(X)$. Addition is constructed via the transfer associated to the folding map as we've already shown in detail and analogously, while multiplication is constructed via the norm associated to the folding map.

To see that multiplication distributes over addition, consider the composite

$$\mathbf{N}_\nabla \circ \mathbf{T}_{1\sqcup\nabla} : \underline{R}(X \sqcup (X \sqcup X)) \to \underline{R}(X),$$

representing the correspondence $(a, b, c) \mapsto a(b+c)$. The distributor $\Delta(1\sqcup\nabla, \nabla)$ is given by

$$X \sqcup (X \sqcup X) \xleftarrow{\nabla} (X \sqcup X) \sqcup (X \sqcup X) \xrightarrow{\nabla\sqcup\nabla} X \sqcup X \xrightarrow{\nabla} X,$$

which is the precisely the way to formalize the correspondence $(a, b, c) \mapsto ab + ac$. Thus, $\underline{R}$ admits the structure of a semi-ring. From the way addition and multiplication were defined, it follows that transfer maps are additive, norm maps are multiplicative and restriction maps are semi-ring homomorphisms.

**Definition 2.1.21.** A Tambara functor is a product-preserving functor $\underline{R} : \mathcal{U}_G \to Set$ such that each set $\underline{R}(X)$ is a ring with addition and multiplication defined as above.

**Remark 2.1.22.** Tambara functors are automatically Green functors. The only fact that needs to be verified is the Frobenius identity. To this end observe that the following sequence of equalities holds true for $X \xrightarrow{p} Y$, $m \in \underline{R}(X)$ and $n \in \underline{R}(Y)$:

$$\mathbf{T}_p(m \cdot \mathbf{R}_p(n)) = (\mathbf{T}_p \circ \mathbf{N}_\nabla \circ \mathbf{R}_{1\sqcup p})(m, n) = (\mathbf{N}_\nabla \circ \mathbf{T}_{p\sqcup 1})(n, m) = \mathbf{T}_p(m) \cdot n.$$

The middle equality comes from the fact that $\Delta(p\sqcup 1, \nabla) = X \sqcup Y \xleftarrow{1\sqcup p} X \sqcup X \xrightarrow{\nabla} X \xrightarrow{p} Y$.

Tambara functors are trivially Mackey functors and so we can consider their box product. In [1] it is shown that, given two Tambara functors $\underline{R}$ and $\underline{S}$, there is a unique way of making $\underline{R} \boxtimes \underline{S}$ into a Tambara functor such that $\mathbf{N}_f(m \otimes n) = \mathbf{N}_f(m) \otimes \mathbf{N}_f(n)$, for $f : X \to Y$ and all $(m, n) \in \underline{R}(X) \times \underline{S}(X)$. Moreover, the box product is the coproduct in the category of Tambara functors.

# CHAPTER 3

# A G-SYMMETRIC MONOIDAL STRUCTURE ON MACKEY FUNCTORS

## 3.1 Constructing a G-symmetric monoidal structure on Mackey functors

Let $Sym$ be the category of symmetric monoidal categories and strong monoidal functors.

The concept of a genuine $G$-symmetric monoidal structure on a symmetric monoidal coefficient system was introduced in [2]. For the sake of completeness we quote the relevant definitions below.

**Definition 3.1.1.** [2] A symmetric monoidal coefficient system is a contravariant pseudo-functor $Orb_G \to Sym$, where $Orb_G$ is the orbit category of the group $G$.

Equivalently, a symmetric monoidal coefficient system can be viewed as a product-preserving pseudo-functor $Fin_G^{op} \to Sym$.

**Example 3.1.2.** Let $\underline{Fin}$ be the pseudo-functor with the property that $\underline{Fin}(G/H)$ is the category of finite $G$-sets over $G/H$. The disjoint union of finite $G$-sets makes this into a symmetric monoidal category, and the pullback along a map $G/K \to G/H$ defines a strong symmetric monoidal restriction functor $\underline{Fin}(G/H) \to \underline{Fin}(G/K)$, while the transfer $\underline{Fin}(G/K) \to \underline{Fin}(G/H)$ is defined by composition with the map $G/K \to G/H$. Define $\underline{Fin}^{\cong}(G/H)$ analogously to be the category of finite $G$-sets over $G/H$ together with isomorphisms between them. This yields a symmetric monoidal system as well.

**Example 3.1.3.** The psudo-functor $\underline{Mackey_G}$ given by

$$\underline{Mackey_G}(G/H) = (Mackey_H, \boxtimes, \underline{A})\,, \text{ for } H \subset G,$$

$$\left(\underline{Mackey_G}\,(f)\,(\underline{M})\right)(X) = \underline{M}(K \times_H X), \text{ for } f : G/H \to G/K, \ \underline{M} \in Mackey_K,$$

is a symmetric monoidal coefficient system.

**Definition 3.1.4.** [2] A symmetric monoidal Mackey functor is a pair of functors $(\underline{M}_*, \underline{M}^*)$, one covariant and one contravariant, from $Orb_G \to Sym$, which agree on objects and for which we have the standard double-coset formula (up to isomorphism). The contravariant maps are restrictions, the covariant maps are transfers.

This definition can be rephrased as: a symmetric monoidal Mackey functor is a product-preserving pseudo-functor $\mathcal{A}_G \to Sym$.

**Definition 3.1.5.** [2] Let $\underline{\mathcal{C}} : Orb_G \to Sym$ be a symmetric coefficient system. Then, a genuine $G$-symmetric monoidal structure on $\underline{\mathcal{C}}$ is a bilinear map

$$\_\,\square\_ : \underline{Fin}^{\cong} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}},$$

making $\underline{\mathcal{C}}$ into a module over $\underline{Fin}^{\cong}$ in the sense that

(i) for any $H \subset G$ and trivial $G$-set $X$, we have that $(X \times G/H)\square A = \overbrace{A \otimes A \cdots \otimes A}^{|X| \text{ times}}$, i.e. the canonical exponential map on the symmetric monoidal category $\underline{\mathcal{C}}(G/H)$,

(ii) the following diagram of symmetric monoidal coefficient systems commutes up to natural isomorphism

$$\begin{array}{ccc} \underline{Fin}^{\cong} \times \underline{Fin}^{\cong} \times \underline{\mathcal{C}} & \xrightarrow{1\times\square} & \underline{Fin}^{\cong} \times \underline{\mathcal{C}} \\ \downarrow{\scriptstyle (\_\times\_)\times 1} & & \downarrow{\scriptstyle \square} \\ \underline{Fin}^{\cong} \times \underline{\mathcal{C}} & \xrightarrow[\square]{} & \underline{\mathcal{C}} \end{array}$$

**Proposition 3.1.6.** *[2] If $\underline{\mathcal{C}}$ is a symmetric monoidal Mackey functor with transfer maps $Tr_H^K : \underline{\mathcal{C}}(G/K) \to \underline{\mathcal{C}}(G/H)$ and restriction maps $i_K^H : \underline{\mathcal{C}}(G/H) \to \underline{\mathcal{C}}(G/K)$ then the assignment*

$$K/H\square M := Tr_K^H i_H^K M \text{ for } \underline{M} \in \underline{\mathcal{C}}(G/H)$$

*gives $\underline{\mathcal{C}}$ a genuine $G$-symmetric monoidal structure.*

For each finite $G$-set $X$, there exists a category $\mathcal{A}_G/X$ with objects given by $G$-sets over $X$ and morphisms consisting of equivalence classes of spans over $X$. Note that if we take $X$ to be the $G$-set $G/H$, then $\mathcal{A}_G/(G/H)$ is equivalent to $\mathcal{A}_H$, the regular Burnside category of $H$-sets, via the pair of functors:

$$\mathcal{A}_G/(G/H) \underset{\left(T\xrightarrow{p}X\right)\mapsto p^{-1}(eH)}{\overset{G\times_H -}{\leftrightarrows}} \mathcal{A}_H \tag{3.1.7}$$

Furthermore, the category $\mathcal{A}_G/X$ admits a symmetric monoidal structure with product $- \times_X -$ and unit $(X, 1_X)$. Let us denote the category of product-preserving functors from $\mathcal{A}_G/X$ into $Set$ by $Fun^\times(\mathcal{A}_G/X, Set)$. As before, we can lift the symmetric monoidal structure on $\mathcal{A}_G/X$ to a symmetric monoidal structure on $Fun^\times(\mathcal{A}_G/X, Set)$, by means of a left Kan extension. To illustrate the close analogy with the case of the Burnside category, we refer to this new symmetric monoidal product as $- \boxtimes_X -$ and to the corresponding unit as $\underline{A}_X$.

For every $G$-set $X$, let $Fin_G/X$ denote the category of $G$-sets over $X$. To a $G$-map $f : X \to Y$ we can associate a pair of functors as shown in the diagram below.

$$Fin_G/X \underset{f_*}{\overset{f^*}{\leftrightarrows}} Fin_G/Y$$

Given $T \in Fin_G/Y$, set $f^*(Y) = X \times_Y T$, i.e. the fiber product of $X$ and $T$ over $Y$. The set $f^*(Y)$ can be realized as a set over $X$ but taking the structure map to be the projection onto the first factor.

Here, for an arbitrary $T \in Fin_G/X$, with structure map p, we have

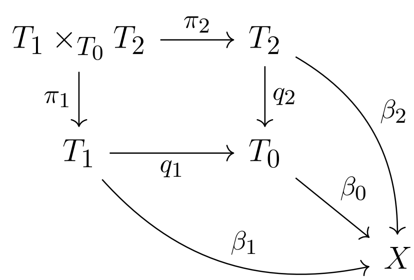
$$f_*(T) = \left\{ (y, s) \mid y \in Y \ , \ s : f^{-1}\{y\} \to T, p \circ s = 1_{f^{-1}\{y\}} \right\}, \tag{3.1.8}$$

with $G$ acting according to the rule $g \cdot (y, s) = (g \cdot y, g \cdot s \cdot g^{-1})$. The projection onto the first factor turns $f_*(T)$ into a $Y$-set.

**Lemma 3.1.9.** *For every $G$-equivariant map $f : X \xrightarrow{p} Y$, the functors $f^*$ and $f_*$, extend to functors between the categories $\mathcal{A}_G/X$ and $\mathcal{A}_G/Y$.*

*Proof.* To prove this, we need only verify that $f^*$ and $f_*$ respect composition of bispans. This is turn boils down to checking that each functor preserves pullback squares up to isomorphism.

Consider the pullback square of sets over $X$.




As expected, we have an isomorphism

$$f_* \left(T_1 \times_{T_0} T_2\right) \cong (f_*(T_1)) \times_{f_*(T_0)} (f_*(T_2)),$$

given by sending an element $(y, s)$ in the domain to the pair $((y, \pi_1 \circ s), (y, \pi_2 \circ s))$. The commutativity of the diagram above implies that the proposed map is well-defined, while the defining property of a pull-back square shows that it is indeed an isomorphism.

In turn, if $T_0$, $T_1$, $T_2$ are sets over $Y$, then we have an isomorphism

$$f^*\left(T_1 \times_{T_0} T_2\right) \cong (f^*(T_1)) \times_{(f^*(T_0))} (f^*(T_2)),$$

given by $(x, t_1, t_2) \mapsto ((x, t_1), (x, t_2))$. A routine check shows that the map is well-defined and an isomorphism. $\square$

**Lemma 3.1.10.** *Given an arbitrary pull-back square,*

$$\begin{array}{ccc} W & \xrightarrow{h} & Y \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$$

*the following diagram commutes up to isomorphism.*

$$\begin{array}{ccc} \mathcal{A}_G/W & \xrightarrow{\quad h_* \quad} & \mathcal{A}_G/Y \\ {\scriptstyle l^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathcal{A}_G/Z & \xrightarrow[\quad f_* \quad]{} & \mathcal{A}_G/X \end{array}$$

*Proof.* Denote by $\pi_1 : l^*(T) \to W$ and $\pi_2 : l^*(T) \to T$ the projections onto the first and second factors respectively.

We can rephrase the conclusion of the lemma as: there exists a family of isomorphisms $g^*(f_*(T)) \xrightarrow{\phi_{T,p}} h_*(l^*(T))$ that are functorial in $T \in \mathcal{A}_G/Z$.

To show this, define

$$\phi_{T,p}\left((y, (x, s))\right) = \left(y, \left(\left(l\mid_{h^{-1}\{y\}}\right)^{-1} \times s\right) \circ l\mid_{h^{-1}\{y\}}\right). \tag{3.1.11}$$

Observe that, since we start with a pullback square, $l\mid_{h^{-1}\{y\}}: h^{-1}\{y\} \to f^{-1}\{g(y)\}$ is an isomorphism and so our definition makes sense. The left hand side of 3.1.11 is indeed an element of $h_*(l^*(T))$ since

$$\pi_1 \circ \left( \left( l \mid_{h^{-1}\{y\}} \right)^{-1} \times s \right) \circ l \mid_{h^{-1}\{y\}} = \left( l \mid_{h^{-1}\{y\}} \right)^{-1} \circ l \mid_{h^{-1}\{y\}} = 1_{h^{-1}\{y\}}.$$

Consider $\psi_{T,p} : h_*(l^*(T)) \to g^*(f_*(T))$, where

$$\psi_{T,p}\left((y, r)\right) = \left( y, \left( g(y), \pi_2 \circ r \circ \left( l \mid_{h^{-1}\{y\}} \right)^{-1} \right) \right).$$

The string of equalities below shows that $\psi_{T,p}$ is well-defined.

$$\begin{aligned} p \circ \pi_2 \circ r \circ \left( l \mid_{h^{-1}\{y\}} \right)^{-1} &= l \circ \pi_1 \circ r \circ \left( l \mid_{h^{-1}\{y\}} \right)^{-1} \\ &= l \circ 1_{h^{-1}\{y\}} \circ \left( l \mid_{h^{-1}\{y\}} \right)^{-1} = 1_{f^{-1}\{g(y)\}} \end{aligned}$$

A routine check yields that $\phi_{T,p}$ and $\psi_{T,p}$ are inverse to each other and functorial in $T$ so the desired diagram commutes. □

**Proposition 3.1.12.** *Let $X$ and $Y$ be arbitrary finite $G$-sets and $f : X \to Y$ an equivariant map. The assignment*

$$\begin{aligned} \underline{\mathcal{M}}(X) &= Fun^{\times}\left(\mathcal{A}_G/X, Set\right), \\ \underline{\mathcal{M}}(T_f)(\underline{M}) &= Lan_{f_*}\underline{M}, \\ \underline{\mathcal{M}}(R_f)(\underline{M}) &= Lan_{f^*}\underline{M} \end{aligned}$$

*gives a symmetric monoidal semi-Mackey functor.*

**Remark 3.1.13.** It is implicit in our definition of $\underline{\mathcal{M}}$, that given a span $\omega = \left( T_1 \stackrel{\alpha}{\leftarrow} A \stackrel{\beta}{\rightarrow} T_2 \right)$, $\underline{\mathcal{M}}(\omega) = \underline{\mathcal{M}}(T_\beta) \circ \underline{\mathcal{M}}(R_\alpha)$. This is independent of the representative span $\omega$.

*Proof.* Recall the symmetric monoidal structure on $\underline{\mathcal{M}}(X)$ introduced at the beginning of the section. We first verify that $\underline{\mathcal{M}}(T_f) : \underline{\mathcal{M}}(X) \to \underline{\mathcal{M}}(Y)$ and $\underline{\mathcal{M}}(R_f) : \underline{\mathcal{M}}(Y) \to \underline{\mathcal{M}}(X)$ are indeed strong symmetric monoidal functors.

Lemma 3.1.9 implies that, for any $G$-map $f : X \to Y$, there are natural isomorphisms between the following pairs of functors:

$$(\_ \times_Y \_) \circ (f_* \times f_*) \cong f_* \circ (\_ \times_X \_)$$
$$(\_ \times_X \_) \circ (f^* \times f^*) \cong f^* \circ (\_ \times_Y \_)$$

This yields natural isomorphisms between the corresponding left Kan extensions. More precisely, given $\underline{M_1}, \underline{M_2} \in \underline{\mathcal{M}}(X)$ and $\underline{N_1}, \underline{N_2} \in \underline{\mathcal{M}}(Y)$ we have

$$\underline{\mathcal{M}}(T_f)(\underline{M_1} \boxtimes_X \underline{M_2}) \cong \underline{\mathcal{M}}(T_f)(\underline{M_1}) \boxtimes_Y \underline{\mathcal{M}}(T_f)(\underline{M_2}),$$
$$\underline{\mathcal{M}}(R_f)(\underline{N_1} \boxtimes_Y \underline{N_2}) \cong \underline{\mathcal{M}}(R_f)(\underline{N_1}) \boxtimes_X \underline{\mathcal{M}}(R_f)(\underline{N_2}).$$

To exhibit the natural isomorphism $\underline{\mathcal{M}}(\underline{A}_X) \cong \underline{A}_Y$ we employ the coend definition of the left Kan extension. Fix $T \in \mathcal{A}_G/Y$. An arbitrary element of $\underline{\mathcal{M}}(\underline{A}_X)(T)$ is an equivalence class with representative

$$(\omega_1, \omega_2) \in \mathcal{A}_G/Y\left(f_*(T'), T\right) \times \mathcal{A}_G/X\left(X, T'\right), \text{ with } T' \in \mathcal{A}_G/X.$$

We have that $(\omega_1, \omega_2) \sim (\omega_1 \circ f_*(\omega_2), 1_X)$. Sending this equivalence class to the element $\omega_1 \circ f_*(\omega_2) \in \mathcal{A}_G/Y(Y, T) = \underline{A}_Y(T)$ yields a well-defined natural isomorphism.

A careful, yet routine check utilizing the definition of the left Kan extension shows that the necessary associativity, symmetry and unitality diagrams commute.

We need to check that $\underline{\mathcal{M}}$ is a pseudo-functor : $\mathcal{A}_G \to \mathit{Sym}$, i.e. that given a pullback square

$$\begin{array}{ccc} W & \xrightarrow{h} & Z \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

there exist a natural isomorphism

$$\phi_{f,g} : \underline{\mathcal{M}}(R_g) \circ \underline{\mathcal{M}}(T_f) \to \underline{\mathcal{M}}(T_h \circ R_l) \tag{3.1.14}$$

that satisfy the associativity condition.

By Lemma 3.1.10, the following compositions of functors : $\mathcal{A}_G/Y \to \mathcal{A}_G/Z$ are naturally isomorphic

$$g^* \circ f_* \sim h_* \circ l^*.$$

And as a result, the corresponding left Kan extensions are naturally isomorphic.

Since $(1_X)_*$ is $1_{\mathcal{A}_G/X}$ it follows trivially that $\underline{\mathcal{M}}(1_X) = 1_{\underline{\mathcal{M}}(X)}$ for every finite $G$-set $X$.

Finally, we need to verify that $\underline{\mathcal{M}}$ is product-preserving.

$$\begin{aligned} \underline{\mathcal{M}}(X \sqcup Y) = Fun^{\times}(\mathcal{A}_G/(X \sqcup Y), Set) &= Fun^{\times}(\mathcal{A}_G/X \times \mathcal{A}_G/Y, Set) \\ &= Fun^{\times}(\mathcal{A}_G/X, Set) \times Fun^{\times}(\mathcal{A}_G/Y, Set) \\ &= \underline{\mathcal{M}}(X) \times \underline{\mathcal{M}}(Y). \end{aligned}$$

□

**Remark 3.1.15.** By construction, the coefficient system $\underline{\mathcal{M}} : \mathcal{A}_G \to Sym$ restricted to $Orb_G \hookrightarrow \mathcal{A}_G$ is actually $\underline{Mackey}_G$ as defined in Example 3.1.3. By Proposition 3.1.6, the fact that $\underline{\mathcal{M}}$ is a symmetric monoidal Mackey functor implies that we have a $G$-symmetric monoidal structure on the underlying coefficient system and thus on $\underline{Mackey}_G$.

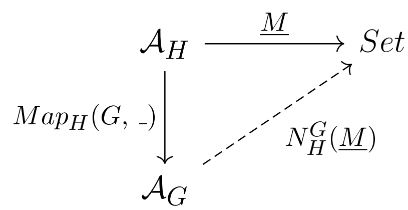
**Proposition 3.1.16.** *Fix $H \subset G$ and let $f : G/H \to G/G$. Then*

*(i) $\underline{\mathcal{M}}(T_f)$ viewed as a functor $Mackey_H \to Mackey_G$ is naturally isomorphic to $N_H^G$ as constructed in [3].*

*(ii) $\underline{\mathcal{M}}(R_f)$ viewed as a functor $Mackey_G \to Mackey_H$ is naturally isomorphic to the forgetful functor $i_H : Mackey_G \to Mackey_H$ as defined in [3].*

**Remark 3.1.17.** Note that in the formulation of this proposition, we implicitly used the equivalence between $\mathcal{A}_G/(G/H)$ and $\mathcal{A}_H$, introduced in (3.1.7).

*Proof.* (i) Recall that the map $N_H^G : Mackey_H \to Mackey_G$ is constructed via a left Kan extension. More precisely, given $\underline{M} \in Mackey_H$, we have




We mention here that the equivariance condition for maps in $Map_H(G, Y)$, for some $H$-set $Y$, relies on the left action of $H$ on $G$ and $Y$, while the action on such a function is given by precomposition with the right action of $G$ on itself, i.e. given an $H$-map $f$, we have $g \cdot f(g') = f(g'g)$.

To conclude that the desired functors are naturally isomorphic, it is enough to verify that the functors from $\mathcal{A}_H \to \mathcal{A}_G$ along which the the two left Kan extensions are taken are naturally isomorphic. We directly construct isomorphisms

$$\phi_{(T,p)} : f_*(T) \to Map_H\left(G, p^{-1}(eH)\right), \text{ for every } T \in \mathcal{A}_H.$$

Fix a system of representatives for the right cosets of $H$ in $G$, say $S = \{g_1, \cdots, g_k\}$ and define

$$\left(\phi_{(T,p)}(s)\right)(g) = g \cdot s(g_i^{-1}), \text{ with } g = hg_i \text{ for some } 1 \le i \le k.$$

Note that $S' = \left\{g_i^{-1} \mid g_1 \in S\right\}$ is a system of representatives for the left cosets of $H$ in $G$ and so $s \in f_*(T)$ can be viewed as a map : $\left\{g_1^{-1}, \cdots, g_k^{-1}\right\} \to T$ such that $p \circ s = 1'_S$. The latter requirement on $s$ ensures that $g \cdot s(g_i^{-1})$ belongs to $p^{-1}(eH)$. A straightforward verification shows that the above maps are equivariant isomorphism and functorial in $T$.

(ii) Observe that the functor $f^*$ featured in the construction of $\underline{\mathcal{M}}(R_f)$ is essentially the forgetful functor $i_H : \mathcal{A}_G \to \mathcal{A}_H$. Thus, the conclusion follows as soon as we manage to construct a functorial family of isomorphisms of Mackey functors over $H$

$$Lan_{i_H}\underline{M} \xrightarrow{\psi_{\underline{M}}} \underline{M} \circ (G \times_H \_)\,, \text{for every } \underline{M} \in Mackey_G.$$

Let $X \in \mathcal{A}_H$. An arbitrary element in $Lan_{i_H}\underline{M}(X)$ is an equivalence class with representative of the form

$$\left(i_H Y \xleftarrow{\alpha} A \xrightarrow{\beta} X, y \in \underline{M}(Y)\right), \text{for some } y \in \mathcal{A}_G.$$

Since $G \times_H \_$ is a left adjoint to $i_H$, we can express the above pair as

$$\left(i_H Y \xleftarrow{i_H(\alpha')} i_H\left(G \times_H A\right) \xleftarrow{\eta_A} A \xrightarrow{\beta} X, a\right)$$
$$\sim \left(i_H(G \times_H A) \xleftarrow{\eta_A} A \xrightarrow{\beta} X, R_{\alpha'}\left(a'\right)\right)$$

Using the fact that the following is a pullback square

$$\begin{array}{ccc} A & \xrightarrow{\quad\beta\quad} & X \\ {\scriptstyle\eta_A}\downarrow & & \downarrow{\scriptstyle\eta_X} \\ i_H\left(G \times_H A\right) & \xrightarrow[G\times_H\beta]{} & i_H\left(G \times_H X\right) \end{array}$$

we continue the string of equivalences:

$$\sim \left(i_H\left(G \times_H A\right) \xleftarrow{G\times_H\beta} i_H\left(G \times_H X\right) \xleftarrow{\eta_X} X, R_{\alpha'}(a)\right)$$
$$\sim \left(R_{\eta_X}, T_{G\times_H\beta}R_{\alpha'}(a)\right).$$

Defining $\psi_{\underline{M}}\left(\left(R_{\eta_X}, T_{G\times_H\beta}R_{\alpha'}(a)\right)\right) = T_{G\times_H\beta}R_{\alpha'}(a)$ yields the desired isomorphism. $\square$

## 3.2 G-Tambara functors are G-commutative monoids

**Definition 3.2.1.** [2] Given a $G$-symmetric monoidal structure $\_\Box\_ : \underline{Fin} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ on the symmetric monoidal coefficient system $\underline{\mathcal{M}}$, a $G$-commutative monoid is an object $M \in \underline{\mathcal{M}}(G/G)$ together with an extension of functors of symmetric monoidal coefficient systems.

$$\begin{array}{ccc} \underline{Fin}^{\cong} & \xrightarrow{\_\Box M} & \underline{\mathcal{M}} \\ \downarrow & \nearrow_{\mathcal{N}} & \\ \underline{Fin} & & \end{array}$$

If, in addition, $\underline{\mathcal{M}}$ is a symmetric monoidal Mackey functor, then we require $\mathcal{N}$ to be a map of symmetric monoidal Mackey functors.

**Remark 3.2.2.** This definition, while elegant, obfuscates some of the intrinsic properties of the extension functor $\mathcal{N} : \underline{Fin} \to \underline{\mathcal{M}}$.

Firstly, note that for a set $T$ over $X$ with structure map $p$, by Proposition 3.1.6, we have

$$\mathcal{N}(X)(T) = T \Box i_X \underline{M} = \underline{\mathcal{M}}(T_p \circ R_{\pi_T})(\underline{M}) \in \underline{\mathcal{M}}(X).$$

The condition that $\mathcal{N}$ is a map of symmetric monoidal Mackey functors translates into the following two diagrams being commutative.

$$\begin{array}{ccc} T_1 \Box i_Y \underline{M} & \xrightarrow{\mathcal{N}(T_f(\psi))} & T_2 \Box i_Y \underline{M} \\ \cong \downarrow & & \downarrow \cong \\ \underline{\mathcal{M}}(T_f)(T_1 \Box i_X \underline{M}) & \xrightarrow[\underline{\mathcal{M}}(T_f)(\mathcal{N}(X)(\psi))]{} & \underline{\mathcal{M}}(T_f)(T_2 \Box i_X \underline{M}) \end{array}$$

$$\begin{array}{ccc} (X \times_Y S_1) \Box i_X \underline{M} & \xrightarrow{\mathcal{N}(X)(X \times_Y \phi)} & (X \times_Y S_2) \Box i_X \underline{M} \\ \cong \downarrow & & \downarrow \cong \\ \underline{\mathcal{M}}(R_f)(S_1 \Box i_Y \underline{M}) & \xrightarrow[\underline{\mathcal{M}}(R_f)(\mathcal{N}(Y)(\phi))]{} & \underline{\mathcal{M}}(R_f)(S_2 \Box i_Y \underline{M}) \end{array}$$

Here $f : X \to Y$ is a map of $G$-sets, $\psi : T_1 \to T_2$ a map of $G$-sets over $X$ and $\phi : S_1 \to S_2$ a map of $G$-sets over $Y$. When we write $T_f(\psi)$ in the first commutative diagram, we refer to the transfer map associated to $\psi$ coming from the realization of $\underline{Fin}$ as a symmetric monoidal Mackey functor in Example 3.1.2.

**Remark 3.2.3.** In what follows we'll need a more detailed description of the $G$-symmetric monoidal structure on $\underline{Mackey}_G$ from Remark 3.1.15. Again, by Proposition 3.1.6, given a finite $G$-set $X \xrightarrow{\pi_X} G/G$ and $\underline{M} \in Mackey_G$, we have that

$$X \Box \underline{M} = \underline{\mathcal{M}}(T_{\pi_X} R_{\pi_X})(\underline{M}).$$

Equivalently, $X \Box \underline{M}$ is the left Kan extension of $\underline{M}$ along the composition $(\pi_X)_* \circ \pi_X^*$. Evaluating this composition at a $G$-set T yields

$$(\pi_X)_* \pi_X^* T = \{s : X \to X \times T \mid \pi_1 \circ S = 1_Y\},$$

where $\pi_1 : X \times T \to X$ is the projection onto the first factor. Set theoretically,

$$(\pi_X)_* \pi_X^* T = \underbrace{T \times \cdots \times T}_{|X| \text{ times}},$$

and the action of $g \in G$ is given by:

$$g \cdot \left(t_{x_1}, \cdots, t_{x_{|X|}}\right) = (g \cdot t_{g^{-1} \cdot x_1}, \cdots, g \cdot t_{g^{-1} \cdot t_{x_{|Y|}}}).$$

For the sake of notation, we refer to $(\pi_X)_* \pi_X^* T$ by $(T \times \cdots \times T)_X$. In conclusion, an element in $(X \Box \underline{M})(Z)$ is an equivalence class with representative of the form

$$((T \times \cdots \times T)_X \leftarrow A \to Z, a \in \underline{M}(T)).$$

**Theorem 3.2.4.** *There is an equivalence of categories between $Tamb_G$ and the category of $G$-commutative monoids.*

*Proof.* Start with a $G$-commutative monoid $\underline{M} \in \underline{\mathcal{M}}(G/G) = Mackey_G$. We will use the structure map $\mathcal{N} : \underline{Fin} \to \underline{\mathcal{M}}$ from Definition 3.2.1 to construct norm maps that make $\underline{M}$ into a Tambara functor.

Given $f : X \to Y$ we define the corresponding norm map $N_f : \underline{M}(X) \to \underline{M}(Y)$ to be

$$N_f(a) = \mathcal{N}(Y)(f)(Y)\left(\left(f_*\pi_X^* X \xleftarrow{\Delta} X \xrightarrow{f} Y, a\right)\right). \tag{3.2.5}$$

To make the above definition a bit more intuitive, note that

$$\mathcal{N}(Y)(f) : X\Box i_Y\underline{M} \to Y\Box i_Y\underline{M}$$

Thus, the element $\mathcal{N}(Y)(f)(Y)\left(\left(f_*\pi_X^* X \xleftarrow{\Delta} X \xrightarrow{f} Y, a\right)\right) \in Y\Box i_Y\underline{M}(Y)$.

Unpacking the definitions above and using the coend expression of the left Kan extension quickly yields that $Y\Box i_Y\underline{M}(Y) = i_Y\underline{M}(Y) \cong \underline{M}(Y)$.

Consider the pullback square below.

$$\begin{array}{ccc} W & \xrightarrow{h} & Z \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

We need to verify that $R_g \circ N_f = N_h \circ R_l$. Fix $a \in \underline{M}(Y)$, then the right hand side of the identity can be expressed as:

$$\begin{aligned} N_h(R_l(a)) &= \mathcal{N}(Z)(h)(Z)\left(\left(h_*\pi_W^* W \xleftarrow{\Delta} W \xrightarrow{h} Z, R_l(a)\right)\right) \\ &= \mathcal{N}(Z)(h)(Z)\left(\left(h_*\pi_W^* Y \xleftarrow{\tilde{l}} h_*\pi_W^* W \xleftarrow{\Delta} W \xrightarrow{h} Z, a\right)\right). \end{aligned}$$

Since $h = Z \times_X f$, we have, by Remark 3.2.2, that

$$\mathcal{N}(Z)(h)(Z) = \underline{\mathcal{M}}(R_g)\left(\mathcal{N}(X)(f)\right)(Z).$$

In order to use the above identity we need to find a way of expressing an arbitrary element of $W \Box i_Z \underline{M}(Z)$ as an element in $\underline{\mathcal{M}}(R_g)(Y \Box i_X \underline{M})(Z)$. This hinges on the existence of the isomorphism (3.1.14). Indeed, we have

$$\begin{aligned} W \Box i_Z \underline{M} &= \underline{\mathcal{M}}(T_h \circ R_l \circ R_{\pi_Y})(\underline{M}) \\ &\cong \underline{\mathcal{M}}(R_g \circ T_f \circ R_{\pi_Y})(\underline{M}) \\ &= \underline{\mathcal{M}}(R_g)(Y \Box i_X \underline{M}). \end{aligned}$$

In particular, the isomorphism (3.1.14) sends the element

$$\left( h_* \pi_W^* Y \xleftarrow{\tilde{l}} h_* \pi_W^* W \xleftarrow{\Delta} W \xrightarrow{h} Z, a \right) \in W \Box i_Z \underline{M}$$

to the element $\left( g^* Z \xleftarrow{\Delta} Z, \left( f_* \pi_Y^* Y \xleftarrow{\Delta} Y \xleftarrow{f} W \xrightarrow{g} Z, a \right) \right) \in \underline{\mathcal{M}}(R_g)(Y \Box i_X \underline{M})$.
This brings us to the desired conclusion

$$\begin{aligned} N_h(R_l(a)) &= \mathcal{N}(X)(f)(Z) \left( f_* \pi_Y^* Y \xleftarrow{\Delta} Y \xleftarrow{l} W \xrightarrow{h} Z, a \right) \\ &= \mathcal{N}(X)(f)(Z) \left( f_* \pi_Y^* Y \xleftarrow{\Delta} Y \xrightarrow{f} X \xleftarrow{g} Z, a \right) \\ &= R_g \left( \mathcal{N}(X)(f)(Z) \left( f_* \pi_Y^* Y \xleftarrow{\Delta} Y \xrightarrow{f} X, a \right) \right) \\ &= R_g(N_f(a)). \end{aligned}$$

We have left to check that, given an exponential diagram as below,

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & A & \xrightarrow{\beta} & B \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \gamma} \\ Y & & \xrightarrow[g]{} & & Z \end{array}$$

the identity $N_g \circ T_f = T_\alpha \circ N_\beta \circ R_\alpha$ holds true.

Fix $a \in \underline{M}(X)$. The right-hand side of the identity can be expressed as follows.

$$\begin{aligned} T_\gamma\left(N_\beta\left(R_\alpha(a)\right)\right) &= T_\gamma\left(\mathcal{N}(B)(\beta)(B)\left(\beta_*\pi_A^*A \xleftarrow{\Delta} A \xrightarrow{\beta} B, R_\alpha(a)\right)\right) \\ &= \mathcal{N}(B)(\beta)(Z)\left(\beta_*\pi_A^*A \xleftarrow{\Delta} A \xrightarrow{\beta} B \xrightarrow{\gamma} Z, R_\alpha(a)\right) \\ &= \mathcal{N}(B)(\beta)(Z)\left(\beta_*\pi_A^*X \xleftarrow{\tilde{\alpha}} \beta_*\pi_A^*A \xleftarrow{\Delta} A \xrightarrow{\beta} B \xrightarrow{\gamma} Z, a\right) \end{aligned}$$

Note that $\beta = B \times_Z g$, and so $\mathcal{N}(B)(\beta)(Z) = \underline{\mathcal{M}}(R_\gamma)\left(\mathcal{N}(Z)(g)\right)(Z)$.

As before, we note that the element

$$\left(\beta_*\pi_A^*X \xleftarrow{\tilde{\alpha}} \beta_*\pi_A^*A \xleftarrow{\Delta} A \xrightarrow{\beta} B \xrightarrow{\gamma} Z, a\right)$$

goes to

$$\left(B \xrightarrow{\gamma} Z, \left(g_*\pi_X^*Y \xrightarrow{\tilde{f}} g_*\pi_Y^*Y \xleftarrow{\Delta} Y \xrightarrow{g} Z, a\right)\right),$$

under the isomorphism $A\Box i_B\underline{M} \cong \underline{\mathcal{M}}(R_\gamma)(Y\Box i_Z\underline{M})$.

This implies that

$$\begin{aligned} T_\gamma\left(N_\beta\left(R_\alpha(a)\right)\right) =& \mathcal{N}(Z)(g)(Z)\left(\left(g_*\pi_X^*Y \xrightarrow{\tilde{f}} g_*\pi_Y^*Y \xleftarrow{\Delta} Y \xrightarrow{g} Z, a\right)\right) \\ =& \mathcal{N}(Z)(g)(Z)\left(g_*\pi_Y^*Y \xleftarrow{\Delta} Y \xrightarrow{g} Z, T_f(a)\right) \\ =& N_g(T_f(a)) \end{aligned}$$

We have thus succeeded in using the information of a $G$-commutative monoid structure on a Mackey functor to construct a $G$-Tambara functor structure on it.

Conversely, consider a $G$-Tambara functor $\underline{M}$. We aim to construct a map of symmetric monoidal Mackey functors $\mathcal{N} : \underline{Fin} \to \underline{\mathcal{M}}$ extending the map $\_\,\Box\underline{M} : \underline{Fin}^{\cong} \to \underline{\mathcal{M}}$. It is enough to give maps of Mackey functors

$$\mathcal{N}(G/G)(f) : X\Box i_Y\underline{M} \to Y\Box i_Y\underline{M}$$

for any $G$-map $f : X \to Y$. Indeed, requiring the second diagram in (3.2.2) to be commutative makes the map $\mathcal{N}(X)(f) : T_1\Box i_X\underline{M} \to T_2\Box i_X\underline{M}$ entirely determined by $\mathcal{N}(G/G)(f)$. Here, we first view $f$ as a map of $G$-sets over $X$ and then simply as a map of $G$-sets.

For any $G$-set $X$ and projection map $\pi_X : X \to G/G$, define

$$\mathcal{N}(G/G)(\pi_X)(Z)\left((T \times \cdots T)_X \xleftarrow{\alpha} A \xrightarrow{\beta} Z, a\right) = T_\alpha\left(R_\beta\left(N_\Delta(a)\right)\right).$$

Furthermore, given $f : X \to Y$, we set

$$\begin{aligned} &\mathcal{N}(G/G)(f)(Z)\left((T \times \cdots \times T)_X \xleftarrow{\alpha} A \xrightarrow{\beta} Y, a\right) \\ &= \left((T \times \cdots \times T)_Y \xrightarrow{\tilde{f}} (T \times \cdots \times T)_X \xleftarrow{\alpha} A \to Y, a\right). \end{aligned}$$

A routine check shows that the two maps are well-defined. Also, by construction, they are maps of Mackey functors and $N(G/G) : \underline{Fin}(G/G) \to \underline{\mathcal{M}}(G/G)$ is a well-defined functor as desired.

□

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