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TAYLOR–WILES–KISIN PATCHING AND THE MOD ℓ LANGLANDS CORRESPONDENCE

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BY
JEFFREY MANNING

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TABLE OF CONTENTS

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ABSTRACT

We use the Taylor–Wiles–Kisin patching method to investigate the multiplicities with which Galois representations occur in the mod ℓ cohomology of Shimura curves over totally real number fields. Our method relies on explicit computations of local deformation rings done by Shotton, which we use to compute the Weil class group of various deformation rings. Exploiting the natural self-duality of the cohomology groups, we use these class group computations to precisely determine the structure of a patched module in many new cases in which the patched module is *not* free (and so multiplicity one fails).

Our main result is a "multiplicity 2^k" theorem in the minimal level case (which we prove under some mild technical hypotheses), where k is a number that depends only on local Galois theoretic information at the primes dividing the discriminant of the Shumira curve. Our result generalizes Ribet's classical multiplicity 2 result and the results of Cheng, and provides progress towards the Buzzard-Diamond-Jarvis local-global compatibility conjecture. We also prove a statement about the endomorphism rings of certain modules over the Hecke algebra, which may have applications to the integral Eichler basis problem.

CHAPTER 1

INTRODUCTION

We use the Taylor–Wiles–Kisin patching method to investigate the multiplicities with which Galois representations occur in the mod ℓ cohomology of Shimura curves over totally real number fields. Our method relies on explicit computations of local deformation rings done in [Sho16], which we use to compute the Weil class group of various deformation rings. Exploiting the natural self-duality of the cohomology groups, we use these class group computations to precisely determine the structure of a patched module in many new cases in which the patched module is *not* free (and so multiplicity one fails).

Our main result is a "multiplicity 2^k" theorem in the minimal level case (which we prove under some mild technical hypotheses), where k is a number that depends only on local Galois theoretic information at the primes dividing the discriminant of the Shumira curve. Our result generalizes Ribet's classical multiplicity 2 result [Rib90] and the results of Cheng [Che], and provides progress towards the Buzzard-Diamond-Jarvis local-global compatibility conjecture of [BDJ10]. We also prove a statement about the endomorphism rings of certain modules over the Hecke algebra, which may have applications to the integral Eichler basis problem.

1.1 Overview

One of the most powerful tools in the study of the Langlands program is the *Taylor–Wiles–Kisin patching method* which, famously, was originally introduced by Taylor and Wiles [Wil95, TW95] to prove Fermat's Last Theorem, via proving a special case of Langlands reciprocity for GL_2.

In its modern formulation (due to Kisin [Kis09] and others) this method glues together various cohomology groups to construct a maximal Cohen–Macaulay module M_∞, over a ring R_∞ which can be determined explicitly from local Galois theoretic data. Due to its construction, M_∞ is closely related to certain automorphic representations, and so determining its structure has many applications in the Langlands program beyond simply proving reciprocity.

A few years after Wiles' proof, Diamond [Dia97] discovered that patching can also be used to prove mod ℓ multiplicity one statements, in cases where the q-expansion principle does not apply. In his argument, he considers a case when the ring R_∞ is formally smooth, and so the Auslander-Buchsbaum formula allows him to show that M_∞ is free over R_∞, a fact which easily implies multiplicity one.

There are however, many situations arising in practice in which R_∞ is not formally smooth, and so Diamond's method cannot be used to determine multiplicity one statements. In fact, Ribet [Rib90] has constructed examples arising in the cohomology of Shimura curves over $\mathbb{Q}$ where multiplicity one fails. Buzzard, Diamond and Jarvis [BDJ10] have given a conjectural generalization of Ribet's result, which predicts the multiplicity with which a given Galois representation appears in the cohomology of a Shimura curve at at any level, in terms of a mod ℓ local global compatibility conjecture.

In this paper, we introduce a new method for determining the structure of a patched module M_∞, which applies in cases when R_∞ is not formally smooth. Our method relies on an explicit calculation of the ring R_∞, together with its Weil class group. While these computations may be quite difficult in higher dimensions, all of the relevant local deformation rings have been computed by Shotton [Sho16] in the GL_2 case, and moreover his computations show that the ring R_∞/λ is (the completion of) the ring of functions on a toric variety. This observation makes it fairly straightforward to apply our method in the GL_2 case. Using this, we are able to compute the multiplicities for Shimura curves over totally real number fields in the minimal level case, under some technical hypotheses. We obtain a multiplicity 2^k result (where k is a number depending on Galois theoretic data) generalizing Ribet's result in many new directions, and providing progress towards the BDJ conjecture.

Additionally, our explicit description of the patched module M_∞ allows us to extract more refined data about the Hecke module structure of the cohomology groups, beyond just the multiplicity statements. This has potential applications to the integral Eichler basis problem.

1.2 Definitions and Notation

Let F be a totally real number field, with ring of integers $\mathcal{O}_F$. We will always use v to denote a finite place $v \subseteq \mathcal{O}_F$. For any such v, let F_v be the completion of F and let $\mathcal{O}_{F,v}$ be its ring of integers. Let ϖ_v be a uniformizer in $\mathcal{O}_{F,\mathfrak{p}}$ and let $\mathbb{k}_v = \mathcal{O}_{F,v}/\varpi_v = \mathcal{O}_F/v$ be the residue field. Let $N(v) = \#\mathbb{k}_v$ be the *norm* of v.

Let D be a quaternion algebra over F with discriminant $\mathfrak{D}$ (i.e. $\mathfrak{D}$ is the product of all finite primes of F at which D is ramified). Assume that D is either ramified at all infinite places of F (the *totally definite* case), or split at exactly one infinite place (the *indefinite* case).

Now fix a prime $\ell \geq 3$ which is relatively prime to $\mathfrak{D}$ and does not ramify in F. For the rest of this paper we will fix a finite extension $E/\mathbb{Q}_\ell$. Let $\mathcal{O}$ be the ring of integers of E, $\lambda \in \mathcal{O}$ be a uniformizer and $\mathbb{F} = \mathcal{O}/\lambda$ be its residue field.

We define a *level* to be a compact open subgroup

$$K = \prod_{v \subseteq \mathcal{O}_F} K_v \subseteq \prod_{v \subseteq \mathcal{O}_F} D^\times(\mathcal{O}_{F,v}) \subseteq D^\times(\mathbb{A}_{F,f})$$

where we have $K_v = D^\times(\mathcal{O}_{F,v})$ for each $v|\mathfrak{D}$, and for each $v \nmid \mathfrak{D}$ we have

$$K_v \supseteq \Gamma_1(v^{e_v}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{v^{e_v}} \right\}$$

for some $e_v \geq 0$ depending on v, taken to be minimal (here we must have $e_v = 0$ for all but finitely many v). We say that K is *unramified* at some $v \nmid \mathfrak{D}$ if $K_v = \mathrm{GL}_2(\mathcal{O}_{F,v})$.

If D is totally definite, let

$$S^D(K) = \left\{ f : D^\times(F)\backslash D^\times(\mathbb{A}_{F,f})/K \to \mathcal{O} \right\}.$$

If D is indefinite, let $X^D(K)$ be the Riemann surface $D^\times(F)\backslash \left(D^\times(\mathbb{A}_{F,f}) \times \mathcal{H}\right)/K$ (where $\mathcal{H}$ is the complex upper half plane). Give $X^D(K)$ its canonical structure as an algebraic curve over F, and let $S^D(K) = H^1(X^D(K), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$.

For any finite prime ideal v of F with $v \nmid \mathfrak{D}$, consider the double-coset operators $T_v, S_v, \langle\delta\rangle_v : S^D(K) \to S^D(K)$ (for any $\delta \in (\mathcal{O}_F/v^{e_v})^\times$) given by

$$T_v = \left[K \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} K \right], \quad S_v = \left[K \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} K \right], \quad \langle\delta\rangle_v = \left[K \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} K \right].$$

where $d \in \mathcal{O}_F^\times$ is a lift of δ.

Let

$$\mathbb{T}^D(K) = \mathcal{O}\left[T_v, S_v, \langle\delta\rangle_v \middle| v \subseteq \mathcal{O}_F, v \nmid \mathfrak{D}, \delta \in (\mathcal{O}_F/v^{e_v})^\times \right] \subseteq \mathrm{End}_{\mathcal{O}}(S^D(K))$$

be the full Hecke algebra.

Now let $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ be the absolute Galois group of F. For any v, let $G_v = \mathrm{Gal}(\overline{F_v}/F_v)$ be the absolute Galois group of F_v, and let $I_v \trianglelefteq G_v$ be the inertia group. Fix an embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{F_v}$ for all v, and hence embeddings $G_v \hookrightarrow G$.

Let $\varepsilon_\ell : G_F \to \mathcal{O}^\times$ be the cyclotomic charachter (given by $\sigma(\zeta) = \zeta^{\varepsilon_\ell(\sigma)}$ for any $\sigma \in G_F$ and $\zeta \in \mu_{\ell^\infty}$), and let $\overline{\varepsilon}_\ell : G_F \to \mathbb{F}^\times$ be its mod ℓ reduction.

Now take a maximal ideal $m \subseteq \mathbb{T}^D(K)$, and note that $T^D(K)/m$ is a finite extension of $\mathbb{F}$.

It is well known (see [Car86]) that the ideal m corresponds to a two-dimensional Galois representation $\overline{\rho}_m : G_F \to \mathrm{GL}_2(\mathbb{T}^D(K)/m) \subseteq \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ satisfying:

1. $\overline{\rho}_m$ is *odd.*

2. If $v \nmid \mathfrak{D}, \ell$ and K is unramified at v, then $\overline{\rho}_m$ is unramified at v and we have

$$\mathrm{tr}(\overline{\rho}_m(\mathrm{Frob}_v)) \equiv T_v \pmod{m}$$
$$\det(\overline{\rho}_m(\mathrm{Frob}_v)) \equiv N(v)S_v \pmod{m}.$$

3. If $v|\ell$ and K is unramified at v, then $\overline{\rho}_m$ is finite flat at v.

4. If $v|\mathfrak{D}$ then

$$\overline{\rho}_m|_{G_v} \sim \begin{pmatrix} \overline{\chi}\,\overline{\varepsilon}_\ell & * \\ 0 & \overline{\chi} \end{pmatrix}.$$

   where $\overline{\chi} : G_v \to \overline{\mathbb{F}}_\ell^\times$ is an unramified character.

In keeping with property (4) above, for any $\mathcal{O}$-algebra A we will say that a local representation $r : G_v \to \mathrm{GL}_2(A)$ is *Steinberg* if it can be written (in some basis) as

$$r = \begin{pmatrix} \chi\varepsilon_\ell & * \\ 0 & \chi \end{pmatrix}$$

for some character $\chi : G_v \to A^\times$. We say that a global representation $r : G_F \to \mathrm{GL}_2(A)$ is *Steinberg at* v if $r|_{G_v}$ is Steinberg.

Now for any continuous *absolutely irreducible* representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, define:

$$\mathcal{K}^D(\overline{\rho}) = \left\{ K \subseteq D^\times(\mathbb{A}_{F,f}) \middle| \overline{\rho} \sim \overline{\rho}_m \text{ for some } m \subseteq \mathbb{T}^D(K) \right\}$$

(that is, $\mathcal{K}^D(\overline{\rho})$ is the set of levels K at which the representation $\overline{\rho}$ can occur.)

Note that if $\mathcal{K}^D(\overline{\rho})$ is nonempty, then it has the form $\{K | K \subseteq K^{\min}\}$ for some level $K^{\min} = \prod_{v \subseteq \mathcal{O}_F} K_v^{\min}$, called the *minimal level* of $\overline{\rho}$. Given $K = \prod_{v \subseteq \mathcal{O}_F} K_v \in \mathcal{K}^D(\overline{\rho})$, we say that K is of *minimal level* at some $v \subseteq \mathcal{O}_F$ if $K_v = K_v^{\min}$.

1.3 Main Results

From now on, fix an absolutely irreducible Galois representation $\overline{\rho}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ for which $\mathcal{K}^D(\overline{\rho}) \neq \varnothing$ (i.e. $\overline{\rho}$ is "automorphic for D"). In particular, this implies that $\overline{\rho}$ is *odd*, and satisfies the numbered conditions in Section 1.2.

Now given $K \in \mathcal{K}^D(\overline{\rho})$ and $m \subseteq \mathbb{T}^D(K)$ for which $\overline{\rho} \sim \overline{\rho}_m$ we define the number:

$$\nu_{\overline{\rho}}(K) := \begin{cases} \dim_{\mathbb{T}^D(K)/m} S^D(K)[m] & \text{if } D \text{ is totally definite} \\ \frac{1}{2}\dim_{\mathbb{T}^D(K)/m} S^D(K)[m] & \text{if } D \text{ is indefinite} \end{cases}$$

called the *multiplicity* of $\overline{\rho}$ at level K. This number is closely related to the mod ℓ local-global compatibility conjectures given in [BDJ10].

Our main result is the following:

Theorem 1.3.1. *Take some $\overline{\rho}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ for which $\mathcal{K}^D(\overline{\rho}) \neq \varnothing$. Assume that:*

1. *$\overline{\rho}|_{G_v}$ is finite flat for any $v|\ell$.*
2. *If $v|\mathfrak{D}$, then $N(v) \not\equiv -1 \pmod{\ell}$.*
3. *The restriction $\overline{\rho}|_{G_{F(\zeta_\ell)}}$ is absolutely irreducible.*

Let

$$k = \#\left\{v|\mathfrak{D} \,\middle|\, \overline{\rho} \text{ is unramified at } v \text{ and } \overline{\rho}(\mathrm{Frob}_v) \text{ is a scalar}\right\}.$$

Then we have $\nu_{\overline{\rho}}(K^{\min}) = 2^k$.

Remark. Condition (1) above is essentially an assumption that the minimal level of $\overline{\rho}$ is prime to ℓ. It, together with the earlier assumption that ℓ does not ramify in F, is included to ensure that the local deformation rings $R_v^{\square,\mathrm{fl},\psi}(\overline{\rho}|_{G_v})$ considered in Section 2 are formally smooth. As the local deformation rings at $v|\ell$ are known to be formally smooth in more general situations, this condition can likely be relaxed somewhat with only minimal modifications to our method. Even more generally, it is likely that our techniques can be extended to certain other situations in which the local deformation rings at $v|\ell$ are *not* formally smooth, provided we can still explicitly compute these rings.

Condition (2) ensures that the Steinberg deformation ring, $R^{\square,\mathrm{st},\psi}(\overline{\rho}|_{G_v})$ from Section 2 is a domain. As the ring $R^{\mathrm{st},\square,\psi}(\overline{\rho}|_{G_v})$ has been computed by Shotton [Sho16] in all cases, it is likely that a more careful analysis of the excluded case will allow us to remove this condition as well.

Lastly, condition (3) is the classical "Taylor–Wiles condition"[1], which is a technical condition necessary for our construction in Section 4. It is unlikely that this condition can be removed without a significant breakthrough.

In addition, our computations allow us to explicitly compute the endomorphism ring of $S^D(K^{\min})_m$.

Theorem 1.3.2. *Let $\overline{\rho}$ satisfy the conditions of Theorem 1.3.1. If D to totally definite then trace map $S^D(K^{\min})_m \otimes_{\mathbb{T}^D(K^{\min})_m} S^D(K^{\min})_m \to \omega_{\mathbb{T}^D(K^{\min})_m}$, induced by the self-duality of $S^D(K^{\min})_m$ is surjective (where $\omega_{\mathbb{T}^D(K^{\min})_m}$ is the dualizing sheaf of $\mathbb{T}^D(K^{\min})_m$), and moreover the natural map $\mathbb{T}^D(K^{\min})_m \to \mathrm{End}_{\mathbb{T}^D(K^{\min})_m}(S^D(K^{\min})_m)$ is an isomorphism. If D is indefinite, then the natural map $\mathbb{T}^D(K^{\min})_m \to \mathrm{End}_{\mathbb{T}^D(K^{\min})_m[G_F]}(S^D(K^{\min})_m)$ is an isomorphism.*

As explained in [Eme02], this statement has applications towards the integral Eichler basis problem, so can likely be used to strengthen the results of Emerton [Eme02].

1. Experts will note that there is also another Taylor–Wiles one must assume in the case when $\ell = 5$ and $\sqrt{5} \in F$. In our case however, this situation is already ruled out by the assumption that ℓ is unramified in F, and so we do not need to explicitly rule it out.

CHAPTER 2

GALOIS DEFORMATION RINGS

In this chapter we will define the various Galois deformation rings which we will consider in the rest of the paper, and review their relevant properties.

2.1 Local Deformation Rings

Fix a finite place v of F and a representation $\overline{r} : G_v \to \mathrm{GL}_2(\mathbb{F})$.

Let $\mathcal{C}_{\mathcal{O}}$ (resp. $\mathcal{C}_{\mathcal{O}}^{\wedge}$) be the category of Artinian (resp. complete Noetherian) local $\mathcal{O}$-algebras with residue field $\mathbb{F}$. Consider the (framed) deformation functor $D^{\square}(\overline{r}) : \mathcal{C}_{\mathcal{O}} \to \mathbf{Set}$ defined by

$$
\begin{aligned}
D^{\square}(\overline{r})(A) &= \{r : G_v \to \mathrm{GL}_2(A),\ \text{continouos lift of } \overline{r}\} \\
&= \left\{ (M, r, e_1, e_2) \,\middle|\, \begin{array}{l} M \text{ is a free rank 2 } A\text{-module with a basis } (e_1, e_2) \text{ and } r : G_v \to \\ \mathrm{Aut}_A(M) \text{ such that the induced map } G \to \mathrm{Aut}_A(M) = \mathrm{GL}_2(A) \to \\ \mathrm{GL}_2(\mathbb{F}) \text{ is } \overline{r} \end{array} \right\}_{/\sim}
\end{aligned}
$$

It is well-known that this functor is *pro-representable* by some $R^{\square}(\overline{r}) \in \mathcal{C}_{\mathcal{O}}^{\wedge}$, in the sense that $D^{\square}(\overline{r}) \equiv \mathrm{Hom}_{\mathcal{O}}(R^{\square}(\overline{r}), -)$. Furthermore, $\overline{r}$ admits a universal lift $r^{\square} : G_v \to \mathrm{GL}_2(R^{\square}(\overline{r}))$.

For any continuous homomorphism, $x : R^{\square}(\overline{r}) \to \overline{E}$, we obtain a Galois representation $r_x : G_v \to \mathrm{GL}_2(\overline{E})$ lifting $\overline{r}$, from the composition $G_v \xrightarrow{r^{\square}} \mathrm{GL}_2(R^{\square}(\overline{r})) \xrightarrow{x} \mathrm{GL}_2(\overline{E})$.

Now for any character $\psi : G_v \to \mathcal{O}^{\times}$ with $\det \overline{r} \cong \psi \pmod{\lambda}$ define $R^{\square,\psi}(\overline{r})$ to be the quotient of $R^{\square}(\overline{r})$ on which $\det r^{\square}(g) = \psi(g)$ for all $g \in G_v$. Equivalently, $R^{\square,\psi}(\overline{r})$ is the ring pro-representing the functor of deformations of $\overline{r}$ with determinant ψ.

Given any two characters $\psi_1, \psi_2 : G_v \to \mathcal{O}^{\times}$ with $\det \overline{r} \cong \psi_1 \cong \psi_2 \pmod{\lambda}$ we have $\psi_1\psi_2^{-1} \equiv 1 \pmod{\lambda}$, and so (as $1 + \lambda\mathcal{O}$ is pro-ℓ and $\ell \neq 2$) there is a unique $\chi : G_v \to \mathcal{O}^{\times}$ with $\psi_1 = \psi_2\chi^2$. But now the map $r \mapsto r \otimes \chi$ is an automorphism of the functor $D^{\square}(\overline{r})$ which can be shown to induce a natural isomorphism $R^{\square,\psi_1}(\overline{r}) \cong R^{\square,\psi_2}(\overline{r})$. Thus, up to isomorphism, the ring $R^{\square,\psi}(\overline{r})$ does not depend choice of ψ.

We call $R^{\square}(\overline{r})$ (respectively $R^{\square,\psi}(\overline{r})$) the *deformation ring* (respectively the *fixed determinant deformation ring*) of $\overline{r}$.

In order to prove our main results, we will also need to consider various deformation rings with *fixed type*. Instead of defining these in general, we will consider only the specific examples which will appear in our arguments.

If $v|\ell$ and $\overline{r}$ and ψ are both *flat*, define $R^{\square,\mathrm{fl},\psi}(\overline{r})$ to be the ring pro-representing the functor of (framed) flat deformations of $\overline{r}$ with determinant ψ. We will refrain from giving a precise definition of this, as it is not relevant to our discussion. We will refer the reader to [Kis09], [FL82], [Ram93] and [CHT08] for more details, and use only the following result from [CHT08, Section 2.4]:

Proposition 2.1.1. *If $F_v/\mathbb{Q}_\ell$ is unramified, then* $R^{\square,\mathrm{fl},\psi}(\overline{r}) \cong \mathcal{O}[[X_1, \ldots, X_{3+[F_v:\mathbb{Q}_\ell]}]]$.

Also if $v \nmid \ell$, let $R^{\square,\min,\psi}(\overline{r})$ be the maximal reduced λ-torsion free quotient of $R^{\square,\psi}(\overline{r})$ with the following property: If $x : R^{\square,\min,\psi}(\overline{r}) \to \overline{E}$ is a continuous homomorphism, then the corresponding lift $r_x : G_v \to \mathrm{GL}_2(\overline{E})$ of $\overline{r}$ has minimal level among all lifts of $\overline{r}$ with determinant ψ. Again, we will refrain from giving a more detailed description of this, and instead we will use only the following well-known result (cf [Sho16, CHT08]):

Proposition 2.1.2. $R^{\square,\min,\psi}(\overline{r}) \cong \mathcal{O}[[X_1, X_2, X_3]]$.

Now assume that $v \nmid \ell$ and $\overline{r}$ is *Steinberg* (in the sense of Section 1.2). We define $R^{\square,\mathrm{st}}(\overline{r})$ (called the *Steinberg deformation ring*) to be the maximal reduced λ-torsion free quotient of $R^{\square}(\overline{r})$ for which $r_x : G_v \to \mathrm{GL}_2(\overline{E})$ is Steinberg for every continuous homomorphism $x : R^{\square,\mathrm{st}}(\overline{r}) \to \overline{E}$.

Similarly if $\psi : G_v \to \mathcal{O}^\times$ is an *unramified* character with $\psi \equiv \det \overline{r} \pmod{\lambda}$ (by assumption, $\overline{r}$ is Steinberg, and hence $\det \overline{r}$ is unramified), we define $R^{\square,\mathrm{st},\psi}(\overline{r})$ (called the *fixed determinant Steinberg deformation ring*) to be the maximal reduced λ-torsion free quotient of $R^{\square,\psi}(\overline{r})$ for which $r_x : G_v \to \mathrm{GL}_2(\overline{E})$ is Steinberg for every continuous homomorphism $x : R^{\square,\mathrm{st},\psi}(\overline{r}) \to \overline{E}$.

It follows from our definitions that $R^{\square,\mathrm{st},\psi}(\overline{r})$ is the maximal reduced λ-torsion free quotient of $R^{\square,\mathrm{st}}(\overline{r})$ on which $\det \rho^{\square}(g) = \psi(g)$ for all $g \in G_v$.

2.2 Global Deformation Rings

Now take a representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ satisfying:

1. $\overline{\rho}$ is absolutely irreducible.
2. $\overline{\rho}$ is odd.
3. For each $v|\ell$, $\overline{\rho}|_{G_v}$ is finite flat.
4. For each $v|\mathfrak{D}$, $\overline{\rho}$ is Steinberg at v.
5. $\mathcal{K}^D(\overline{\rho}) \neq \varnothing$.

Let Σ_ℓ^D be a set of finite places of F containing:

- All places v at which $\overline{\rho}$ is ramified
- All places $v|\mathfrak{D}$ (i.e. places at which D is ramified)
- All places $v|\ell$

(we allow Σ_ℓ^D to contain some other places in addition to these), and let $\Sigma \subseteq \Sigma_\ell^D$ consist of those $v \in \Sigma_\ell^D$ with $v \nmid \ell, \mathfrak{D}$.

Now as in [Kis09] define $R_{F,S}^{\square}(\overline{\rho})$ (where $\Sigma_\ell^D \subseteq S$) to be the $\mathcal{O}$-algebra pro-representing the functor $\mathcal{C}_{\mathcal{O}} \to \mathbf{Set}$ which sends A to the set tuples $\left(\rho : G_{F,S} \to \mathrm{End}_A(M), \{(e_1^v, e_2^v)\}_{v\in\Sigma_\ell^D}\right)$, where M is a free rank 2 A-module with an identification $M/m_A = \mathbb{F}^2$ and for each $v \in \Sigma_\ell^D$, (e_1^v, e_2^v) is a basis for M, lifting the standard basis for $M/m_A = \mathbb{F}^2$, up to equivalence.

Also define the *unframed* deformation ring $R_{F,S}(\overline{\rho})$ to be the $\mathcal{O}$-algebra pro-representing the functor $\mathcal{C}_{\mathcal{O}} \to \mathbf{Set}$ which sends A to the set of free rank 2 A modules M with action $\rho : G_{F,S} \to \mathrm{End}_A(M)$, up to equivalence. This exists because $\overline{\rho}$ is absolutely irreducible.

Now take any character $\psi : G_F \to \mathcal{O}^\times$ for which:

1. $\psi \equiv \det \overline{\rho} \pmod{\lambda}$.

2. ψ is unramified at all places outside of Σ_ℓ^D, *and* all places dividing $\mathfrak{D}$.

3. ψ is flat at all places dividing ℓ.

4. $\psi\varepsilon_\ell^{-1}$ has finite image.

Define $R^{\square,\psi}_{F,S}(\overline{\rho})$ and $R^{\psi}_{F,S}(\overline{\rho})$ to be the quotients of $R^{\square}_{F,S}(\overline{\rho})$ and $R_{F,S}(\overline{\rho})$, respectively, on which $\det\rho = \psi$.

Now note that the morphism of functors

$$\left(\rho, \{(e_1^v, e_2^v)\}_{v\in\Sigma_\ell^D}\right) \mapsto \left(\rho|_{G_v} : G_v \to \mathrm{End}_A(M), (e_1^v, e_2^v)\right)_{v\in\Sigma_\ell^D}$$

induces a map:

$$\pi : \hat{\bigotimes_{v\in\Sigma_\ell^D}} R^{\square,\psi}(\overline{\rho}|_{G_v}) \to R^{\square,\psi}_{F,S}.$$

Define $R^{\square,D,\psi}_{F,S}(\overline{\rho})$ (respectively $R^{D,\psi}_{F,S}(\overline{\rho})$) to be the quotient of $R^{\square,\psi}_{F,S}(\overline{\rho})$ (respectively $R^{\psi}_{F,S}(\overline{\rho})$) parameterizing lifts of $\overline{\rho}$ for which:

- $\overline{\rho}|_{G_v}$ is flat at all $v|\ell$
- $\overline{\rho}|_{G_v}$ is Steinberg at all $v|\mathfrak{D}$
- $\overline{\rho}|_{G_v}$ has minimal level at all $v \in \Sigma_\ell^D$, $v \nmid \ell, \mathfrak{D}$.

Also define

$$R^{\square,\psi}_{\Sigma,\mathfrak{D},\ell} := \left[\hat{\bigotimes_{v|\ell}} R^{\square,\mathrm{fl},\psi}(\overline{\rho}|_{G_v})\right] \widehat{\otimes} \left[\hat{\bigotimes_{v\in\Sigma}} R^{\square,\min,\psi}(\overline{\rho}|_{G_v})\right] \widehat{\otimes} \left[\hat{\bigotimes_{v|\mathfrak{D}}} R^{\square,\mathrm{st},\psi}(\overline{\rho}|_{G_v})\right].$$

Then the map π from above induces a map $\pi : R^{\square,\psi}_{\Sigma,\mathfrak{D},\ell} \to R^{\square,\psi}_{F,S}$.

Also note that the morphism of functors $\left(\rho, \{(e_1^v, e_2^v)\}_{v\in\Sigma_\ell^D}\right) \mapsto \rho$ induces a maps $R^{\psi}_{F,S} \to R^{\square,\psi}_{F,S}(\overline{\rho})$ and $R^{D,\psi}_{F,S}(\overline{\rho}) \to R^{\square,D,\psi}_{F,S}(\overline{\rho})$. By [Kis09] these maps are formally smooth of dimension $j := 3|\Sigma_\ell^D| - 1$, and so we may identify $R^{\square,\psi}_{F,S} = R^{\psi}_{F,S}[[w_1,\ldots,w_j]]$ and $R^{\square,D,\psi}_{F,S}(\overline{\rho}) = R^{D,\psi}_{F,S}(\overline{\rho})[[w_1,\ldots,w_j]]$.

2.3 Two Lemmas about Deformation Rings

We finish this section by stating two standard results (cf. [Kis09]) which will be essential for our discussion of Taylor–Wiles–Kisin patching in Chapter 4.

The first concerns the existence of an "$R \to \mathbb{T}$" map:

Lemma 2.3.1. *Assume that $\overline{\rho}$ satisfies all of the numbered conditions listed in Section 2.2. Take $K \in \mathcal{K}^D(\overline{\rho})$ and let S be a set of finite places of F containing Σ_ℓ^D such that K is* unramified *outside of S (that is, for $v \notin S$, the composition*

$$K \hookrightarrow \prod_{\mathfrak{p} \subseteq \mathcal{O}_F} D^\times(\mathcal{O}_{F,\mathfrak{p}}) \twoheadrightarrow D^\times(\mathcal{O}_{F,v}) \cong \mathrm{GL}_2(\mathcal{O}_{F,v})$$

is surjective). Then there is surjective map $R^{D,\psi}_{F,S}(\overline{\rho}) \twoheadrightarrow \mathbb{T}^D(K)$.

The second concerns the existence of "Taylor–Wiles" primes:

Lemma 2.3.2. *Assume that $\overline{\rho}$ satisfies all of the numbered conditions listed in Section 2.2* and *condition (3) of Theorem 1.3.1. There exist integers $r, g \geq 0$ such that for any $n \geq 1$, there is a finite set Q_n of primes of F for which:*

- *$\#Q_n = r$.*
- *For any $v \in Q_n$, $N(v) \equiv 1 \pmod{\ell^n}$*
- *For any $v \in Q_n$, $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues.*
- *There is a surjection $R^{\square,\psi}_{\Sigma,\mathfrak{D},\ell}[[x_1,\dots,x_g]] \twoheadrightarrow R^{\square,D,\psi}_{F,S\cup Q_n}(\overline{\rho})$*

Moreover, we have $\dim R^{\square,\psi}_{\Sigma,\mathfrak{D},\ell} = r + j - g + 1$.

From now on we will write R_∞ to denote $R^{\square,\psi}_{\Sigma,\mathfrak{D},\ell}[[x_1,\dots,x_g]]$ so that $\dim R_\infty = r+j+1$. By the results of Section 2.1 we have

$$R_\infty = \left[\hat{\bigotimes_{v|\mathfrak{D}}} R^{\square,\mathrm{st},\psi}(\overline{\rho}|_{G_v})\right][[x_1,\dots,x_{g'}]]$$

for some integer g'. In Section 3 below, we will use the results of [Sho16] to explicitly compute the ring R_∞, and then use the theory of toric varieties to study modules over R_∞.

In Chapter 4, we will use Lemma 2.3.1 and 2.3.2 to construct a particular module M_∞ over R_∞ out of a system of modules over the rings $\mathbb{T}^D(K)$, and then use the results of Chapter 3 to deduce the structure of M_∞. This will allow us to prove Theorems 1.3.1 and 1.3.2.

CHAPTER 3

CLASS GROUPS OF LOCAL DEFORMATION RINGS

In our situation, all of the local deformation rings which will be relevant to us were computed in [Sho16]. In this chapter, we will use this description to explicitly describe the ring R_∞, and to study its class group.

We first introduce some notation which we will use for the rest of this paper. If R is any Noetherian local ring, we will always use m_R to denote its maximal ideal. Also if R is Cohen–Macaulay, we will use ω_R to denote the dualizing sheaf of R.

For any finitely generated R-module M, we will let $M^* = \mathrm{Hom}_R(M, \omega_R)$. We say that M is *reflexive* if the natural map[1] $M \to M^{**}$ is an isomorphism.

If R is a domain we will write $K(R)$ for its fraction field. If M is a finitely generated R-module, then we we will say that the *rank* of M, denoted $\mathrm{rank}_R M$ is the $K(R)$-dimension of $M \otimes_R K(R)$ (that is, the rank of M at the generic point of R).

Lastly, given any reflexive module M, the natural perfect pairing $M^* \times M \to \omega_R$ gives rise to a natural map $\tau_M : M^* \otimes_R M \to \omega_R$ (defined by $\tau_M(\varphi \otimes x) = \varphi(x)$) called the *trace map*.

Also we will let

$$k = \#\left\{ v|\mathfrak{D} \,\middle|\, \overline{\rho}(\mathrm{Frob}_v) = c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

as in Theorem 1.3.1.

Our main result of this chapter is the following:

Theorem 3.0.3. *If M_∞ is a finitely-generated module over R_∞ satisfying:*

1. *M_∞ is maximal Cohen–Macaulay over R_∞.*
2. *We have $M_\infty^* \cong M_\infty$ (and hence M_∞ is reflexive).*

---

1. As is it fairly easy to show that the dual of a finitely generated R-module is reflexive (cf [Sta17, Tag 0AV2]) this definition is equivalent to simply requiring that there is *some* isomorphism $M \xrightarrow{\sim} M^{**}$. In particular, if $M \cong M^*$ then M is automatically reflexive.

3. $\operatorname{rank}_{R_\infty} M_\infty = 1$.

then $\dim_{\mathbb{F}} M_\infty/m_{R_\infty} = 2^k$. *Moreover, the trace map* $\tau_{M_\infty} : M_\infty \otimes_{R_\infty} M_\infty \to \omega_{R_\infty}$ *is surjective.*

Thus, to prove Theorem 1.3.1, it will suffice to construct a module M_∞ over R_∞ satisfying the conditions of Theorem 3.0.3 with $\dim_{\mathbb{F}} M_\infty/m_{R_\infty} = \nu_{\overline{\rho}}(K^{\min})$. The last statement, that τ_{M_∞} is surjective, will be used to prove Theorem 1.3.2 (see the end of Section 4).

Our primary strategy for proving Theorem 3.0.3 will be to compute the Weil class group, $\mathrm{Cl}(R_\infty)$ (defined below), of the ring R_∞ (or more precisely, the ring $\overline{R}_\infty := R_\infty/\lambda$), using the fact that this ring corresponds to a *toric* variety. Note that the conditions of the theorem imply that $2[M_\infty] = [\omega_{R_\infty}]$ in $\mathrm{Cl}(R_\infty)$, and so a sufficiently explicit computation of $\mathrm{Cl}(R_\infty)$ will allow us to precisely determine M_∞.

In Section 3.1 we summarize the computations in [Sho16] and reduce Theorem 3.0.3 to a "mod λ version," which will be more convenient to work with. In Section 3.2 we recall the standard theory of toric varieties, and compute an affine toric variety corresponding to $\overline{R}_\infty$. Finally in Section 3.3 we use the theory of toric varieties to compute $\mathrm{Cl}(\overline{R}_\infty)$, and complete the proof of Theorem 3.0.3.

3.1 Explicit Calculations of Local Deformation Rings

In order to prove Theorem 3.0.3, it will be necessary to first compute the ring R_∞, or equivalently to compute $R^{\square,\mathrm{st},\psi}(\overline{\rho}|_{G_v})$ for all $v|\mathfrak{D}$.

These computations were essentially done by Shotton [Sho16], except that he considers the non fixed determinant version, $R^{\mathrm{st},\square}(\overline{\rho}|_{G_v})$ instead of $R^{\mathrm{st},\square,\psi}(\overline{\rho}|_{G_v})$. Fortunately, it is fairly straightforward to recover $R^{\mathrm{st},\square,\psi}(\overline{\rho}|_{G_v})$ from $R^{\mathrm{st},\square}(\overline{\rho}|_{G_v})$. Specifically, we get:

Theorem 3.1.1. *Take any place* $v|\mathfrak{D}$. *Recall that we have assumed that* $N(v) \not\equiv -1 \pmod{\ell}$. *If the residual representation* $\overline{\rho}|_{G_v} : G_v \to \mathrm{GL}_2(\mathbb{F})$ *is not scalar, then* $R^{\square,\mathrm{st},\psi}(\overline{\rho}|_{G_v}) \cong \mathcal{O}[[X_1, X_2, X_3]]$.

If $\overline{\rho}|_{G_v} : G_v \to \mathrm{GL}_2(\mathbb{F})$ is scalar (which can only happen when $N(v) \equiv 1 \pmod{\ell}$) then

$$R^{\square,\mathrm{st},\psi}(\overline{\rho}|_{G_v}) \cong \mathcal{S}_v := \mathcal{O}[[A,B,C,X,Y,Z]]/\mathcal{I}_v$$

where $\mathcal{I}_v$ is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} A & B & X & Y \\ C & A & Z & X + 2\frac{N(v)-1}{N(v)+1} \end{pmatrix}.$$

The ring $\mathcal{S}_v$ is a Cohen–Macaulay and non-Gorenstein domain of relative dimension 3 over $\mathcal{O}$. Moreover, $\mathcal{S}_v[1/\lambda]$ is formally smooth of dimension 3 over E.

Proof. For convenience, let $R_{\mathrm{st}} = R^{\mathrm{st},\square}(\overline{\rho}|_{G_v})$ and $R_{\mathrm{st}}^{\psi} = R^{\mathrm{st},\square,\psi}(\overline{\rho}|_{G_v})$. By definition, R_{st}^{ψ} is the maximal reduced λ-torsion free quotient of R_{st} on which $\det\rho^{\square}(g) = \psi(g)$ for all $g \in G_v$.

Now let $I_v/\tilde{P}_v \cong \mathbb{Z}_\ell$ be the maximal pro-ℓ quotient of I_v, so that $\tilde{P}_v \trianglelefteq G_v$ and $T_v := G_v/\tilde{P}_v \cong \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$. Now let $\sigma, \phi \in T_v$ be topological generators for $\mathbb{Z}_\ell$ and $\widehat{\mathbb{Z}}$, respectively (chosen so that ϕ is a lift of arithmetic Frobenius, so that $\phi\sigma\phi^{-1} = \sigma^{N(v)}$).

Now as in [Sho16], we may assume that the universal representation $\rho^{\square} : G_v \to \mathrm{GL}_2(R_{\mathrm{st}})$ factors through T_v. As we already have $\det\rho^{\square}(\sigma) = 1 = \psi(\sigma)$, it follows that R_{st}^{ψ} is the maximal reduced λ-torsion free quotient of R on which $\det\rho^{\square}(\phi) = \psi(\phi)$.

As explained in Section 2, up to isomorphism the ring R_{st}^{ψ} is unaffected by the choice of ψ, so it will suffices to prove the claim for a particular choice of ψ. Thus from now on we will assume that ψ is unramified and $\psi(\phi) = \dfrac{N(v)}{(N(v)+1)}t^2$ where

$$t = \begin{cases} N(v)+1 & N(v) \not\equiv \pm 1 \pmod{\ell} \\ 2 & N(v) \equiv 1 \pmod{\ell} \end{cases}$$

so that $t \equiv N(v)+1 \equiv \mathrm{tr}\,\overline{\rho}(\phi) \pmod{\ell}$ (this particular choice of t is made to agree with the computations of [Sho16]).

But now by the definition of $R_{\mathrm{st}} = R^{\mathrm{st},\square}(\overline{\rho}|_{G_v})$ we have that $N(v)\left(\operatorname{tr}\rho^{\square}(\phi)\right)^2 = (N(v)+1)^2 \det\rho^{\square}(\phi)$ and so

$$\det\rho^{\square}(\phi) = \frac{N(v)}{(N(v)+1)^2}\left(\operatorname{tr}\rho^{\square}(\phi)\right)^2$$

(where we have used the fact that $N(v) \not\equiv -1 \pmod{\ell}$, and so $N(v)+1$ is a unit in $\mathcal{O}$).

It follows that

$$\begin{aligned}\det\rho^{\square}(\phi) - \psi(\phi) &= \frac{N(v)}{(N(v)+1)^2}\left(\operatorname{tr}\rho^{\square}(\phi)\right) - \frac{N(v)}{(N(v)+1)}t^2\\ &= \frac{N(v)}{(N(v)+1)^2}(\operatorname{tr}\rho^{\square}(\phi)+t)(\operatorname{tr}\rho^{\square}(\phi)-t).\end{aligned}$$

But now

$$\frac{N(v)}{(N(v)+1)^2}\left(\operatorname{tr}\rho^{\square}(\phi)+t\right) \equiv \frac{2N(v)}{(N(v)+1)^2}\operatorname{tr}\overline{\rho}(\phi) \equiv \frac{2N(v)}{N(v)+1} \pmod{m_R}$$

and so as $\ell \nmid 2, N(v), N(v)+1$ we get that $\frac{N(v)}{(N(v)+1)^2}\left(\operatorname{tr}\rho^{\square}(\phi)+t\right)$ is a unit in R_{st}. It follows that R_{st}^{ψ} is the maximal reduced λ-torsion free quotient of

$$R_{\mathrm{st}}^{\psi,\circ} := \frac{R_{\mathrm{st}}}{\left(\det\rho^{\square}(\phi)-\psi(\phi)\right)} = \frac{R_{\mathrm{st}}}{\operatorname{tr}\rho^{\square}(\phi)-t}$$

It now follows immediately from Shotton's computations that in each case $R_{\mathrm{st}}^{\psi,\circ}$ is already reduced and λ-torsion free (and so $R_{\mathrm{st}}^{\psi} = R_{\mathrm{st}}^{\psi,\circ}$) and has the form described in the statement of Theorem 3.1.1 above.

Indeed, first assume that $N(v) \not\equiv \pm 1 \pmod{\ell}$. By [Sho16, Proposition 5.5] we may write

$R_{\mathrm{st}} = \mathcal{O}[[B,P,X,Y]]$ with

$$\rho^{\square}(\sigma) = \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x+B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix}$$
$$\rho^{\square}(\phi) = \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix}^{-1} \begin{pmatrix} N(v)(1+P) & 0 \\ 0 & 1+P \end{pmatrix} \begin{pmatrix} 1 & X \\ y & 1 \end{pmatrix},$$

for some $x \in \mathcal{O}$. Thus we have

$$\operatorname{tr} \rho^{\square}(\phi) = (N(v)+1)(1+P) = t + tP$$

and so (as $t = N(v)+1 \in \mathcal{O}$ is a unit), $R_{\mathrm{st}}^{\psi,\circ} = R_{\mathrm{st}}/(tP) \cong \mathcal{O}[[B,P,X,Y]]/(P) \cong \mathcal{O}[[B,X,Y]]$, as desired.

Now assume that $N(v) \equiv 1 \pmod{\ell}$. Again, following the computations of [Sho16, Proposition 5.8] we can write

$$\rho^{\square}(\sigma) = \begin{pmatrix} 1+A & x+B \\ C & 1-A \end{pmatrix}$$
$$\rho^{\square}(\phi) = \begin{pmatrix} 1+P & y+R \\ S & 1+Q \end{pmatrix}$$

(for $x, y \in \mathcal{O}$) where A, B, C, P, Q, R and S topologically generate R_{st}. Now following Shotton's notation, let $T = P+Q$, so that $\operatorname{tr} \rho^{\square}(\phi) = 2+T = t+T$ and thus $R_{\mathrm{st}}^{\psi,\circ} = R_{\mathrm{st}}/(T)$. In both cases ($\overline{\rho}|_{G_v}$ non-scalar and scalar) Shotton's computations immediately give our desired result[2] $\square$

2. In Shotton's notation, when $\overline{\rho}|_{G_v}$ is scalar R_{st}^{ψ} would be cut out by the 2×2 minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & -X_1 & Y_3 & X_3 + 2\frac{N(v)-1}{N(v)+1} \end{pmatrix}$. This is equivalent to the form stated in Theorem 3.1.1 via the variable substitutions $A = X_1$, $B = -X_2$, $C = Y_1$, $X = X_3$, $Y = X_4$ and $Z = Y_3$.

Thus letting $\mathfrak{D}_1|\mathfrak{D}$ be the product of the places $v|\mathfrak{D}$ at which $\overline{\rho}|_{G_v} : G_v \to \mathrm{GL}_2(\mathbb{F})$ is scalar, we have

$$R_\infty = \left[\hat{\bigotimes_{v|\mathfrak{D}_1}} S_v\right][[x_1, \dots, x_s]]$$

for some integer s.

Now note that the description of S_v in Theorem 3.1.1 becomes much simpler if we work in characteristic ℓ. Indeed, if $\overline{\rho}|_{G_v}$ is scalar then $N(v) \equiv 1 \pmod{\ell}$ and so, $\overline{S} := S_v/\lambda$ is an explicit graded ring not depending on v. Specifically, we have $\overline{S} = \mathbb{F}[[A, B, C, X, Y, Z]]/\overline{\mathcal{I}}$ where $\overline{\mathcal{I}}$ is the (homogeneous) ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} A & B & X & Y \\ C & A & Z & X \end{pmatrix}$.

It thus follows that

$$\overline{R}_\infty := R_\infty/\lambda \cong \overline{S}^{\hat{\otimes} k}[[x_1, \dots, x_s]],$$

which will be much easier to work with than R_∞. It will thus be useful to reduce Theorem 3.0.3 the following "mod λ" version:

Theorem 3.1.2. *If $\overline{M}_\infty$ is a finitely-generated module over $\overline{R}_\infty$ satisfying:*

1. *$\overline{M}_\infty$ is maximal Cohen–Macaulay over $\overline{R}_\infty$*
2. *We have $\overline{M}_\infty^* \cong \overline{M}_\infty$.*
3. $\operatorname{rank}_{\overline{R}_\infty} \overline{M}_\infty = 1$.

then $\dim_{\mathbb{F}} \overline{M}_\infty/m_{\overline{R}_\infty} = 2^k$*. Moreover, the trace map* $\tau_{\overline{M}_\infty} : \overline{M}_\infty \otimes_{\overline{R}_\infty} \overline{M}_\infty \xrightarrow{\sim} \omega_{\overline{R}_\infty}$ *is surjective.*

Proof that Theorem 3.1.2 implies 3.0.3. Assume that Theorem 3.1.2 holds, and that M_∞ satisfies the hypotheses of Theorem 3.0.3. As R_∞ is flat over $\mathcal{O}$, it is λ-torsion free and thus λ is not a zero divisor on M_∞ (by condition (1)). It follows that $\overline{M}_\infty := M_\infty/\lambda$ is maximal Cohen–Macaulay over $\overline{R}_\infty$ and $\operatorname{rank}_{\overline{R}_\infty} \overline{M}_\infty = \operatorname{rank}_{R_\infty} M_\infty = 1$. (In general, if R is Cohen–Macaulay and M is maximal Cohen–Macaulay over R, then for any regular element $x \in R$, $\operatorname{rank}_{R/x}(M/x) = \operatorname{rank}_R M$.)

Moreover, as M_∞ and ω_{R_∞} are both flat over $\mathcal{O}$, we have that

$$\overline{M}_\infty = M_\infty/\lambda = \mathrm{Hom}_{R_\infty}(M_\infty, \omega_{R_\infty})/\lambda = \mathrm{Hom}_{R_\infty/\lambda}(M_\infty/\lambda, \omega_{R_\infty}/\lambda) = \mathrm{Hom}_{\overline{R}_\infty}(\overline{M}_\infty, \omega_{\overline{R}_\infty}),$$

where we have used the fact that $\omega_{R_\infty}/\lambda \cong \omega_{\overline{R}_\infty}$, by [Eis95]. Thus $\overline{M}_\infty$ is self-dual. Thus $\overline{M}_\infty$ satisfies all of the hypotheses of Theorem 3.1.2, and so $\dim_{\mathbb{F}} \overline{M}_\infty/m_{\overline{R}_\infty} = 2^k$ and $\tau_{\overline{M}_\infty}$ is surjective.

Now we obviously have that $\overline{M}_\infty/_{\overline{R}_\infty} \cong M_\infty/m_{R_\infty}$, so the first conclusion of Theorem 3.0.3 follows.

Also, the trace map $\tau_{\overline{M}_\infty} : \overline{M}_\infty \otimes_{\overline{R}_\infty} \overline{M}_\infty \to \omega_{\overline{R}_\infty}$ is just the mod-λ reduction of the map $\tau_{M_\infty} : M_\infty \otimes M_\infty \to \omega_{R_\infty}$, so it follows that $\tau_{\overline{M}_\infty}$ is surjective if and only if τ_{M_∞} is. Thus the second conclusion of Theorem 3.0.3 follows. □

As hinted above, we will prove Theorem 3.1.2 by computing the class group of $\overline{R}_\infty$.

For any local Cohen–Macaulay ring R, we will let $\mathrm{Cl}(R)$ denote the *Weil divisor class group* of R, which is isomorphic the group of rank 1 reflexive modules over R. For any rank 1 reflexive sheaf M, let $[M] \in \mathrm{Cl}(R)$ denote the corresponding element of the class group. The group operation is then defined by $[M]+[N] := [(M \otimes_R N)^{**}]$. Note that $[\omega_R] \in \mathrm{Cl}(R)$ and we have $[M^*] = [\omega_R] - [M]$ for any $[M] \in \mathrm{Cl}(R)$.

Now conditions (2) and (3) of Theorem 3.1.2 imply that $\overline{M}_\infty$ corresponds to an element of $\mathrm{Cl}(\overline{R}_\infty)$, and the self-duality implies that $2[\overline{M}_\infty] = [\omega_{\overline{R}_\infty}]$. Thus (provided that $\mathrm{Cl}(\overline{R}_\infty)$ is 2-torsion free, which will turn out to be the case), the conditions in Theorem 3.1.2 uniquely characterize the module $\overline{M}_\infty$. Proving the theorem will thus simply be a matter of computing the unique module $\overline{M}_\infty$ explicitly enough.

We finish this section by proving the following lemma, which will make the second conclusion of Theorem 3.1.2 easier to prove (and will also be useful in the proof of Theorem 1.3.2):

Lemma 3.1.3. *If R is a local Cohen–Macaulay ring and M is a reflexive R-module, then* *the trace map* $\tau_M : M^* \otimes_R M \to \omega_R$ *is surjective if and only if there* exists *an* R*-module surjection* $M^* \otimes_R M \twoheadrightarrow \omega_R$.

Proof. Assume that $f : M^* \otimes_R M \to \omega_R$ is a surjection. Take any $\alpha \in \omega_R$. Then we can write

$$\alpha = f\left(\sum_{i\in I} b_i \otimes c_i\right) = \sum_{i\in I} f(b_i \otimes c_i)$$

for some finite index set I and some $b_i \in M^*$ and $c_i \in M$. For each $i \in I$, consider the R-linear map $\varphi_i : M \to \omega_R$ defined by $\varphi_i(c) = f(b_i \otimes c)$. Then we have $\varphi_i \in M^*$ for all i and so

$$\alpha = \sum_{i\in I} f(b_i \otimes c_i) = \sum_{i\in I} \varphi_i(c_i) = \sum_{i\in I} \tau_M(\varphi_i \otimes c_i) = \tau_M\left(\sum_{i\in I} \varphi_i \otimes c_i\right).$$

Thus τ_M is surjective. □

3.2 Toric Varieties

The key insight that allows us to easily make computations with $\overline{R}_\infty$ is that the ring $\overline{S}$ is (the completion of) the ring of functions on a *toric* variety.

We shall primarily follow the presentation of toric varieties from [CLS11]. Unfortunately [CLS11] works exclusively with toric varieties over $\mathbb{C}$, whereas we are working in positive characteristic. All of the results we will rely on work over arbitrary base field, usually with identical proofs, so we will freely cite the results of [CLS11] as if they were stated over arbitrary fields.

We recall the following definitions. For any integer $d \geq 1$, let $T_d = \mathbb{G}_m^d = (\mathbb{F}^\times)^d$, thought of as a group variety. Define the two lattices

$$M := \operatorname{Hom}(T_d, \mathbb{G}_m) \qquad\qquad N := \operatorname{Hom}(\mathbb{G}_m, T_d),$$

called the *character lattice* and the *lattice of one-parameter subgroups*, respectively. Note that $M \cong N \cong \mathbb{Z}^d$. We shall write M and N additively. For $m \in M$ and $u \in N$, we will write $\chi^m : T_d \to \mathbb{G}_m$ and $\lambda^u : \mathbb{G}_m \to T_d$ to denote the corresponding morphisms.

First note that there is a perfect pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ given by $t^{\langle m,u\rangle} = \chi^m(\lambda^u(t))$. We shall write $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ for $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$, which are each d-dimensional real vector spaces. We will extend the pairing $\langle\ ,\ \rangle$ to a perfect pairing $\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

For the rest of this chapter, we will (arbitrarily) fix a choice of basis $e_1, \ldots, e_d$ for M, and so identify M with $\mathbb{Z}^d$. We will also identify N with $\mathbb{Z}^d$ via the dual basis to $e_1, \ldots, e_d$. Under these identifications, $\langle\ ,\ \rangle$ is simply the usual (Euclidean) inner product on $\mathbb{Z}^d$.

We can now define:

Definition 3.2.1. An *(affine) toric variety* of dimension d is a pair (X, ι), where X is an affine variety $X/\mathbb{F}$ of dimension d and ι is an open embedding $\iota : T_d \hookrightarrow X$ such that the natural action of T_d on itself extends to a group variety action of T_d on X. We will usually write X instead of the pair (X, ι).

For such an X, we define the *semigroup* of X to be

$$\mathsf{S}_X := \{m \in M | \chi^m : T_d \to \mathbb{G}_m \text{ extends to a morphism } X \to \mathbb{A}^1\} \subseteq M$$

For convenience, we will also say that a finitely generated $\mathbb{F}$-algebra R (together with an inclusion $R \hookrightarrow \mathbb{F}[M]$) is *toric* if $\operatorname{Spec} R$ is toric.

The primary significance of affine toric varieties is that they are classified by their semigroups. Specifically:

Proposition 3.2.2. *If X is an affine toric variety of dimension d, then $X = \operatorname{Spec} \mathbb{F}[\mathsf{S}_X]$, and the embedding $\iota : T_d \hookrightarrow X$ is induced by $\mathbb{F}[\mathsf{S}_X] \hookrightarrow \mathbb{F}[M]$ (using the fact that $T_d = \operatorname{Spec} \mathbb{F}[M]$). Moreover we have*

1. *The semigroup S_X spans M (that is, $\mathbb{Z}\mathsf{S}_X$ has rank d).*
2. *If S_X is* saturated *in M (in the sense that $km \in \mathsf{S}_X$ implies that $m \in \mathsf{S}_X$ for all $k > 0$ and $m \in M$) then X is a* normal *variety.*

Conversely, if $\mathsf{S} \subseteq M$ *is a finitely generated semigroup spanning* M *then the inclusion* $\mathbb{F}[\mathsf{S}] \hookrightarrow \mathbb{F}[M]$ *gives* $\operatorname{Spec}\mathbb{F}[\mathsf{S}]$ *the structure of a d-dimensional affine toric variety.*

Proof. cf. [CLS11] Proposition 1.1.14 and Theorems 1.1.17 and 1.3.5. ☐

If R is a toric $\mathbb{F}$-algebra, we will write S_R to mean $\mathsf{S}_{\operatorname{Spec} R}$.

While it can be difficult to recognize toric varieties directly from Definition 3.2.1, the following Proposition makes it fairly easy to identify toric varieties in $\mathbb{A}^s$.

Proposition 3.2.3. *Fix an integer* $h \geq 1$ *and let* $\Phi : \mathbb{Z}^h \to M$ *be any homomorphism with finite cokernel, and let* $L = \ker\Phi$. *Let* $\mathsf{S} \subseteq M$ *be the semigroup generated by* $\Phi(e_1), \ldots, \Phi(e_h) \in M$. *Then we have an isomorphism* $\mathbb{F}[z_1, \ldots, z_h]/I_L \cong \mathbb{F}[\mathsf{S}]$ *given by* $z_i \mapsto \Phi(e_i)$, *where*

$$I_L := \left(z^\alpha - z^\beta \middle| \alpha, \beta \in \mathbb{Z}^h_{\geq 0} \text{ such that } \alpha - \beta \in L\right) \subseteq \mathbb{F}[z_1, \ldots, z_h].$$

(Where, for any $\alpha = (\alpha_1, \ldots, \alpha_h) \in \mathbb{Z}^h_{\geq 0}$, *we write* $z^\alpha := z_1^{\alpha_1} \cdots z_h^{\alpha_h} \in \mathbb{F}[z_1, \ldots, z_h]$.*)*

Moreover I_L *can be explicitly computed as follows: Assume that* $\mathbb{L} = (\ell^1, ..., \ell^r)$ *is a* $\mathbb{Z}$*-basis for* L, *with* $\ell^i = (\ell^i_1, \ldots, \ell^i_h) \in \mathbb{Z}^h$. *Write each* ℓ^i *as* $\ell^i = \ell^i_+ - \ell^i_i$ *where*

$$\ell^i_+ = (\max\{\ell^i_1, 0\}, \ldots, \max\{\ell^i_h, 0\}) \in \mathbb{Z}^h_{\geq 0}$$
$$\ell^i_- = (\max\{-\ell^i_1, 0\}, \ldots, \max\{-\ell^i_h, 0\}) \in \mathbb{Z}^h_{\geq 0}.$$

Then if $I_{\mathbb{L}} := \left(z^{\ell^1_+} - z^{\ell^1_-}, \ldots, z^{\ell^h_+} - z^{\ell^h_-}\right) \subseteq \mathbb{F}[z_1, \ldots, z_h]$, I_L *is the* saturation *of* $I_{\mathbb{L}}$ *with respect to* $z_1 \cdots z_h$, *that is:*

$$\begin{aligned} I_L = (I_{\mathbb{L}} : (z_1 \cdots z_h)^\infty) &:= \left\{ f \in \mathbb{F}[z_1, \ldots, z_h] \middle| (z_1 \cdots z_h)^m f \in I_{\mathbb{L}} \text{ for some } m \geq 0 \right\} \\ &= \left(z^\alpha - z^\beta \middle| \alpha, \beta \in \mathbb{Z}^h_{\geq 0} \text{ such that } (z_1 \cdots z_h)^m \left(z^\alpha - z^\beta\right) \in I_{\mathbb{L}} \text{ for some } m \geq 0\right). \end{aligned}$$

Conversely, if $I \subseteq \mathbb{F}[z_1, \ldots, z_h]$ *is any* prime *ideal which can be written in the form* $I = \left(z^{\alpha_i} - z^{\beta_i} \middle| i \in \mathscr{A}\right)$ *for a finite index set* $\mathscr{A}$ *and* $\alpha_i, \beta_i \in \mathbb{Z}^h_{\geq 0}$, *then* $I = I_L$ *for some* L *and so*

$\mathbb{F}[z_1, \ldots, z_h]/I$ *can be given the structure of a toric* $\mathbb{F}$*-algebra.*

Proof. This mostly follows from [CLS11, Propositions 1.1.8, 1.1.9 and 1.1.11]. The statement that $I_L = (I_{\mathbb{L}} : (z_1 \cdots z_h)^\infty)$ is [CLS11, Exercise 1.1.3] or [MS05, Lemma 7.6] □

The definition of a toric $\mathbb{F}$-algebra given in Definition 3.2.1 applies to finitely generated $\mathbb{F}$-algebras, whereas $\overline{R}_\infty$ and $\overline{S}$ are merely *topologically* finitely generated. Therefore in order to apply the results of this chapter to our situation, we will need to consider the completions of toric $\mathbb{F}$-algebras.

Define $M^+ = \{c_1 e_1 + \cdots + c_d e_d | c_1, \ldots, c_d \geq 0\} \subseteq M$ (note that this is a non-canonical choice, depending on our choice of basis $e_1, e_2, ..., e_n$) so that we have $\mathbb{F}[M^+] = \mathbb{F}[x_1, \ldots, x_d]$. Now let R be a toric $\mathbb{F}$-algebra, and assume $\mathsf{S}_R \subseteq M^+$, so that $R \subseteq \mathbb{F}[x_1, \ldots, x_d]$ (which will be the case for all toric $\mathbb{F}$-algebras we will consider). Consider the maximal ideal $\mathfrak{m}_R \subseteq R$ defined by

$$\mathfrak{m}_R := R \cap (x_1, \ldots, x_d) = \left(\chi^m \middle| m \in \mathsf{S}_R \smallsetminus \{0\}\right),$$

and define the *completion* of R to be $\widehat{R} := \varprojlim R/\mathfrak{m}_R^n$.

So now define $\mathcal{S} = \mathbb{F}[A, B, C, X, Y, Z]/\overline{\mathcal{I}}$, where $\overline{\mathcal{I}}$ is (as above) the ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} A & B & X & Y \\ C & A & Z & X \end{pmatrix}$, and define $\mathcal{R} = \mathcal{S}^{\otimes k}[x_1, \ldots, x_s]$. We will see below that $\mathcal{S}$ and $\mathcal{R}$ are toric with $\widehat{\mathcal{S}} = \overline{S}$ and $\widehat{\mathcal{R}} = \overline{R}_\infty$.

Now applying the above results to the $\mathbb{F}$-algebra $\mathcal{S}$, we get:

Proposition 3.2.4. $\mathcal{S}$ *may be given the structure of a* 3*-dimensional toric* $\mathbb{F}$*-algebra, with semigroup*

$$\mathsf{S}_{\mathcal{S}} := \{(a, b, c) \in \mathbb{Z}^3 | a, b, c \geq 0, 2a + 2b \geq c\} \subseteq M^+ \subseteq M$$

under some choice of basis e_1, e_2, e_3 *for* M*. Moreover:*

1. $\operatorname{Spec} \mathcal{S}$ *is the affine cone over a surface* $\mathcal{V} \subseteq \mathbb{P}^5$ *isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$*.*
2. $\mathfrak{m}_{\mathcal{S}} = (A, B, C, X, Y, Z) \subseteq \mathcal{S}$ *and* $\widehat{\mathcal{S}} = \overline{S}$*.*

3. *$\mathcal{S}$ is isomorphic to the ring $\mathbb{F}[x, xz, xz^2, y, yz, yz^2] \subseteq \mathbb{F}[x, y, z]$ and*

   $\overline{S} \cong \mathbb{F}[[x, xz, xz^2, y, yz, yz^2]]$.

4. *$\operatorname{Spec} \mathcal{S}$ is a* normal *variety.*

5. *$\mathcal{R}$ is toric of dimension $3k + s$, and $\widehat{\mathcal{R}} = \overline{R}_\infty$.*

Proof. Take $d = 3$ in the above discussion, and fix an isomorphism $M \cong \mathbb{Z}^3$ (which in particular, determines $M^+ = \mathbb{Z}^3_{\geq 0} \subseteq \mathbb{Z}^3 = M$).

Write $\mathsf{S} := \{(a, b, c) \in \mathbb{Z}^3 | a, b, c \geq 0, 2a + 2b \geq c\}$. We will first show that $\mathcal{S} \cong \mathbb{F}[\mathsf{S}]$. Note that S is generated by the (transposes of) the columns of the matrix

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 \end{pmatrix},$$

which in particular gives an isomorphism $\mathbb{F}[\mathsf{S}] \cong \mathbb{F}[x, xz, xz^2, y, yz, yz^2] \subseteq \mathbb{F}[x, y, z]$.

Let $L = \ker \Phi$. By Proposition 3.2.3 it follows that $\mathbb{F}[\mathsf{S}] \cong \mathbb{F}[A, B, C, X, Y, Z]/I_L$ (where we have identified the ring $\mathbb{F}[z_1, z_2, z_3, z_4, z_5, z_6]$ with $\mathbb{F}[A, B, C, X, Y, Z]$ in the obvious way, in order to keep are notation consistent).

But now note that L is a rank 3 lattice with basis $\mathbb{L} = (\ell^1, \ell^2, \ell^3)$ given by the vectors:

$$\ell^1 := \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \ell^2 := \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \ell^3 := \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

It follows that

$$I_{\mathbb{L}} = (AZ - CX, AX - CY, AX - BZ)$$

from whence it is straightforward to compute that

$$\begin{aligned} I_L &= (I_{\mathbb{L}} : (ABCXYZ)^{\infty}) \\ &= \left(A^2 - BC, AZ - CX, AX - CY, AX - BZ, AY - BX, X^2 - YZ\right) = \overline{\mathcal{I}}. \end{aligned}$$

Thus $\mathcal{S} = \mathbb{F}[A, B, C, X, Y, Z]/\overline{\mathcal{I}}$ is indeed toric with $\mathsf{S}_{\mathcal{S}} = \mathsf{S}$, and we have

$$\mathcal{S} \cong \mathbb{F}[\mathsf{S}_{\mathcal{S}}] \cong \mathbb{F}[x, xz, xz^2, y, yz, yz^2].$$

Moreover, this isomorphism sends the ideal $(A, B, C, X, Y, Z) \subseteq \mathcal{S}$ to the ideal $(x, xz, xz^2, y, yz, yz^2)$, from which (2) and (3) easily follow.

Now the semigroup $\mathsf{S}_{\mathcal{S}} = \{(a, b, c) \in \mathbb{Z}^3 | a, b, c \geq 0, 2a + 2b \geq c \geq 0\}$ is clearly saturated in M, and so $\operatorname{Spec} \mathcal{S}$ is indeed normal, proving (4).

Also, the isomorphism $\mathcal{S} \cong \mathbb{F}[x, xz, xz^2, y, yz, yz^2]$ easily implies that $\operatorname{Spec} \mathcal{S} \subseteq \mathbb{A}^6$ is the cone over the image $\mathcal{V}$ of the (injective) morphism $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$ defined by

$$f([s : t], [u : v]) = [stu : s^2u : t^2u : stv : s^2v : t^2v],$$

which proves (1).

Lastly, recalling that $M = \mathbb{Z}^3$, let $M' = M = M^{\oplus k} \oplus \mathbb{Z}^s = \mathbb{Z}^{3k+s}$ and define:

$$\mathsf{S}' = (\mathsf{S})^{\otimes k} \otimes \mathbb{Z}^s_{\geq 0} \subseteq (M')^+$$

so that

$$\mathbb{F}[\mathsf{S}'] = \mathbb{F}[\mathsf{S}]^{\otimes k} \otimes \mathbb{F}[x]^{\otimes s} \cong \mathcal{S}^{\otimes k}[x_1, \ldots, x_s] = \mathcal{R}$$

so that $\mathcal{R}$ is indeed toric of dimension $3k + s$, with an embedding $\mathcal{R} \hookrightarrow \mathbb{F}[(M')^+]$. Moreover, the isomorphism $\mathcal{R} = \mathcal{S}^{\otimes k}[x_1, \ldots, x_s] \cong \mathbb{F}[\mathsf{S}']$ clearly identifies $\mathfrak{m}_{\mathbb{F}[\mathsf{S}']} \subseteq \mathbb{F}[\mathsf{S}']$ with $\mathfrak{m}_{\mathcal{S}}^{\boxtimes k} \boxtimes (x_1, \ldots, x_s) \subseteq \mathcal{S}^{\otimes k}[x_1, \ldots, x_s]$ and so $\widehat{\mathcal{R}} \cong \overline{S}^{\hat{\otimes} k}[[x_1, \ldots, x_s]] = \overline{R}_{\infty}$, proving (5). $\square$

We will now restrict our attention to *normal* affine toric varieties. The advantage to doing this is that Proposition 3.2.2 has a refinement (see Proposition 3.2.6 below) that allows us to characterize normal toric varieties much more simply, using cones instead of semigroups.

We now make the following definitions:

Definition 3.2.5. A *convex rational polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form:

$$\sigma = \mathrm{Cone}(S) := \left\{ \sum_{\lambda \in S} x_\lambda \lambda \middle| x_\lambda \geq 0 \text{ for all } \lambda \in S \right\} \subseteq N_{\mathbb{R}}$$

for some finite subset $S \subseteq N$.

A *face* of σ is a subset $\tau \subseteq \sigma$ which can be written as $\tau = \sigma \cap H$ for some hyperplane $H \subseteq N_{\mathbb{R}}$ which does not intersect the interior of σ. We write $\tau \preceq \sigma$ to say that τ is a face of σ. It is clear that any face of σ is also a convex rational polyhedral cone. We say that σ is *strongly convex* if $\{0\}$ is a face of σ.

We write $\mathbb{R}\sigma$ for the subspace of $N_{\mathbb{R}}$ spanned by σ, and we will let the dimension of σ be $\dim \sigma := \dim_{\mathbb{R}} \mathbb{R}\sigma$.

We make analogous definitions for cones in $M_{\mathbb{R}}$.

For a convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ (or similarly for $\sigma \subseteq M_{\mathbb{R}}$), we define its *dual* cone to be:

$$\sigma^\vee := \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

It is easy to see that $\sigma^\vee$ is also convex rational polyhedral cone. If σ is strongly convex and $\dim \sigma = d$ then the same is true of $\sigma^\vee$. Moreover, for any σ we have $\sigma^{\vee\vee} = \sigma$.

We now have the following:

Proposition 3.2.6. *If X is a normal affine toric variety of dimension d, then there is a (uniquely determined) strongly convex rational polyhedral cone $\sigma_X \subseteq N_{\mathbb{R}}$ for which $\sigma_X^\vee \cap M =$*

S_X *and*

$$\begin{aligned}\sigma_X \cap N &= \{u \in N | \lambda^u : \mathbb{G}_m \to T_d \textit{ extends to a morphism } \mathbb{A}^1 \to X\} \\ &= \left\{u \in N \middle| \lim_{t\to 0} \lambda^u(t) \textit{ exists in } X\right\}.\end{aligned}$$

We call σ_X *the* cone associated to X. *Again, if* R *is a toric* $\mathbb{F}$*-algebra, then we write* σ_R *for* $\sigma_{\operatorname{Spec} R}$.

Proof. This follows from [CLS11] Theorem 1.3.5 (for $\sigma_X^\vee \cap M$) and Proposition 3.2.2 (for $\sigma_X \cap N$). Note that it is clear from our definitions that a convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is uniquely determined by $\sigma \cap N$. □

Remark. Based on the statement of Proposition 3.2.6, it would seem more natural to simply define the cone associated to $\mathcal{X}$ to be $\sigma_{\mathcal{X}}^\vee$, and not mention the lattice N at all. The primary reason for making this choice in the literature is to simplify the description of non-affine toric varieties, which is not relevant to our applications. Nevertheless we shall use the convention established in Proposition 3.2.6 to keep our treatment compatible with existing literature, and specifically to avoid having to reformulate Theorem 3.3.1, below.

Rephrasing the statement of Proposition 3.2.4 in terms of cones, we get:

Corollary 3.2.7. *We have* $\sigma_{\mathcal{S}} = \operatorname{Cone}(e_1, e_2, e_3, 2e_1 + 2e_2 - e_3)$.

Proof. The description of $\mathsf{S}_{\mathcal{X}}$ in Proposition 3.2.4 immediately implies that

$$\sigma_{\mathcal{X}}^\vee = \{(x,y,z) \in \mathbb{R}^3 | x, y, x \geq 0, 2x + 2y \geq z\} = \operatorname{Cone}(e_1, e_2, e_1 + 2e_3, e_2 + 2e_3).$$

Thus we get

$$\begin{aligned}\sigma_{\mathcal{X}} = \sigma_{\mathcal{X}}^{\vee\vee} &= \{u \in N | \langle m, u\rangle \geq 0 \text{ for all } m \in \sigma_{\mathcal{X}}^{\vee}\} \\ &= \{u \in N | \langle m, e_1\rangle, \langle m, e_2\rangle, \langle m, e_1 + 2e_3\rangle, \langle m, e_2 + 2e_3\rangle \geq 0\} \\ &= \{(x, y, z) \in \mathbb{R}^3 | x, y \geq 0, x + 2z \geq 0, y + 2z \geq 0\} \\ &= \text{Cone}(e_1, e_2, e_1 + 2e_3, e_2 + 2e_3).\end{aligned}$$

□

3.3 Class Groups of Toric Varieties

The benefit of this entire discussion is that Weil divisors on toric varieties are much easier work with than they are for general varieties. In order to explain this, we first introduce a few more definitions.

For any variety X, we will let $\text{Div}(X)$ denote the group of Weil divisors of X. If $X = \operatorname{Spec} R$ is normal, affine and toric of dimension d, then the torus T_d acts on X, and hence acts on $\text{Div}(X)$. We say that a divisor $D \in \text{Div}(X)$ is *torus-invariant* if it is preserved by this action. We will write $\text{Div}_{T_d}(X) \subseteq \text{Div}(X)$ for the group of torus invariant divisors.

Now consider the (strongly convex, rational polyhedral) cone $\sigma_R \subseteq N_{\mathbb{R}}$. We will let $\sigma_R(1)$ denote the set of *edges* (1 dimensional faces) of σ_R. For any $\rho \in \sigma_R(1)$, note that $\rho \cap N$ is a semigroup isomorphic to $\mathbb{Z}_{\geq 0}$, and so there is a unique choice of generator $u_\rho \in \rho \cap N$ (called a minimal generator). By Proposition 3.2.6, the limit $\gamma_\rho := \lim_{t \to 0} \lambda^{u_\rho}(t) \in X$ exists. Thus we may consider its orbit closure $D_\rho := \overline{T_d \cdot \gamma_\rho} \subseteq X$.

The following theorem allows us to characterize $\text{Cl}(R)$, and $[\omega_R] \in \text{Cl}(R)$, entirely in terms of the set $\sigma_R(1)$.

Theorem 3.3.1. *Let $X = \operatorname{Spec} R$ be a normal affine toric variety, with cone $\sigma_R \subseteq N_{\mathbb{R}}$. We have the following:*

1. *For any* $\rho \in \sigma_R(1)$, $D_\rho \subseteq X$ *is a torus-invariant prime divisor. Moreover,* $\mathrm{Div}_{T_d}(X) = \bigoplus_{\rho \in \sigma_X(1)} \mathbb{Z} D_\rho$.
2. *Any divisor* $D \in \mathrm{Div}(X)$ *is rationally equivalent to a torus-invariant divisor.*
3. *For any* $m \in M$*, the rational function* $\chi^m \in K(X)$ *has divisor* $\mathrm{div}(\chi^m) = \sum_{\rho \in \sigma_X(1)} \langle m, u_\rho \rangle D_\rho$.
4. *For any torus-invariant divisor* D,

$$\mathcal{O}(D) := \{f \in K(X) \,|\, \mathrm{div}(f) + D \geq 0\} = \bigoplus_{\chi^m \in \mathcal{O}(D)} \mathbb{F}\chi^m = \bigoplus_{\mathrm{div}(\chi^m)+D \geq 0} \mathbb{F}\chi^m \subseteq K(X)$$

5. *There is an exact sequence*

$$M \to \mathrm{Div}_{T_d}(X) \to \mathrm{Cl}(R) \to 0$$

where the first map is $m \mapsto \mathrm{div}(\chi^m)$ *and the second map is* $D \mapsto \mathcal{O}(D)$.

6. R *is Cohen–Macaulay and we have* $\omega_R \cong \mathcal{O}\left(- \sum_{\rho \in \sigma_X(1)} D_\rho \right)$

Proof. By the orbit cone correspondence ([CLS11] Theorem 3.2.6), it follows that each D_ρ is a torus-invariant prime divisor, and moreover that these are the only torus-invariant prime divisors. The rest of (1) follows easily from this (cf. [CLS11] Exercise 4.1.1).

(3) is just [CLS11] Proposition 4.1.2. (4) follows from [CLS11] Proposition 4.3.2. (5) is [CLS11] Theorem 4.1.3, and (2) is an immediate corollary of (5). Lastly (6) is [CLS11] Theorems 8.2.3 and 9.2.9. □

Next we show that the results of Theorem 3.3.1 are not affected by taking completions.

Proposition 3.3.2. *Let* R *be a normal toric* $\mathbb{F}$*-algebra of dimension* d*, with* $\mathsf{S}_R \subseteq M^+$*. The natural completion map* $\mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R})$*, sending* $[A]$ *to* $[A^\wedge_{\mathfrak{m}_R}]$*, is an isomorphism sending* $[\omega_R]$ *to* $[\omega_{\widehat{R}}]$.

Proof. The fact that $(\omega_R)^\wedge_{\mathfrak{m}_R} \cong \omega_{\widehat{R}}$ is just [Eis95] Corollaries 21.17 and 21.18, so it suffices to show that the map $\mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R})$ is an isomorphism.

First note that as $\mathsf{S}_R \subseteq M^+$ we have that $R = \mathbb{F}[\mathsf{S}]$ is a subalgebra of $\mathbb{F}[M] = \mathbb{F}[x_1, \ldots, x_d]$ generated by monomials, and $\mathfrak{m}_R$ is the ideal generated by all nonconstant monomials in R. In particular, it follows that R has the natural structure of a graded ring, and $\mathfrak{m}_R$ is simply the irrelevant ideal (i.e. the ideal generated by all homogeneous elements of positive degree).

We will call an ideal $I \subseteq R$ *torus-invariant* if it is generated by monomials of $\mathbb{F}[M]$ (and hence is fixed by the natural action of T_d on $\mathbb{F}[M]$). In this case we have that $I = \bigoplus_{\chi^m \in I} \mathbb{F}\chi^m \subseteq R$ and $I\widehat{R} = \prod_{\chi^m \in I} \mathbb{F}\chi^m \subseteq \widehat{R}$ as $\mathbb{F}$-vector spaces. We can now prove the following useful Lemma:

Lemma 3.3.3. *If $P \subseteq R$ is a prime torus-invariant ideal, then $P\widehat{R}$ is a prime ideal of $\widehat{R}$.*

Proof. We say that a *monomial ordering* on $\mathbb{F}[M]$ is a well-ordering $\preceq$ on M with the property that for any $m, m', m'' \in M$, $m \preceq m'$ implies $m + m'' \preceq m' + m''$. Fix such an ordering. (The lexicographic ordering under the identification $M = \mathbb{Z}^d_{\geq 0}$ is one such ordering.)

Now pick any $\alpha, \beta \in \widehat{R} \smallsetminus P\widehat{R}$. We must show that $\alpha\beta \notin P\widehat{R}$. We may uniquely write

$$\alpha \equiv \sum_{m_1 \in A} a_{m_1}\chi^{m_1} \pmod{P\widehat{R}} \qquad \text{and} \qquad \beta \equiv \sum_{m_2 \in B} b_{m_2}\chi^{m_2} \pmod{P\widehat{R}}$$

where $A, B \subseteq \{m \in M | \chi^m \notin P\}$ and the a_m's and b_m's are *nonzero*. For any $m_1 \in A$ and $m_2 \in B$ we get that $\chi^{m_1+m_2} \notin P$, since P is prime and $\chi^{m_1}, \chi^{m_2} \notin P$. Thus we have

$$\alpha\beta \equiv \sum_{\substack{m \in M \\ \chi^m \notin P}} \left(\sum_{\substack{m_1+m_2=m \\ m_1 \in A, m_2 \in B}} a_{m_1}b_{m_2} \right) \chi^m \pmod{P\widehat{R}}$$

and so if $\alpha\beta \in P\widehat{R}$, then $\sum_{\substack{m_1+m_2=m \\ m_1 \in A, m_2 \in B}} a_{m_1}b_{m_2} = 0$ for all m.

But now since $\alpha, \beta \notin P\widehat{R}$, the sets A and B are both nonempty. As $\prec$ is a well-ordering, we may thus pick minimal elements m_A and m_B for A and B. But then the properties of $\preceq$ imply that for any $m_1 \in A$ and $m_2 \in B$, $m_A + m_B \preceq m_1 + m_2$, with equality iff $m_1 = m_A$

and $m_2 = m_B$. Thus

$$\sum_{\substack{m_1+m_2=m_A+m_B \\ m_1\in A, m_2\in B}} a_{m_1}b_{m_2} = a_{m_A}b_{m_B} \neq 0$$

and so $\alpha\beta \notin P\widehat{R}$. $\square$

Now we show that the map is injective. Take some $[\mathcal{M}] \in \mathrm{Cl}(R)$, and assume that $\mathcal{M}^{\wedge}_{m_{\mathcal{R}}} \cong \widehat{R}$. This gives that $\mathcal{M}/\mathfrak{m}_R \cong \mathcal{M}^{\wedge}_{\mathfrak{m}_R}/\mathfrak{m}_{\widehat{R}} \cong \widehat{R}/\mathfrak{m}_{\widehat{R}} \cong \mathbb{F}$.

Moreover, by Theorem 3.3.1(4), $\mathcal{M}$ is isomorphic to a torus invariant ideal of R, and so $\dim_{\mathbb{F}} \mathcal{M}/m_{\mathcal{R}}$ is simply the minimal number of generators of $\mathcal{M}$. It follows that $\mathcal{M} \subseteq R$ is a *principal* ideal, and so as $R \subseteq \mathbb{F}[x_1, \ldots, x_d]$ is a domain, $\mathcal{M} \cong R$. Thus the map is indeed injective.

It remains to show that the map is surjective. For any $\rho \in \sigma_R(1)$, let $\mathfrak{p}_\rho \in \operatorname{Spec} R$ be the prime ideal of R corresponding to the prime divisor D_ρ. By Theorem 3.3.1(4), $\mathfrak{p}_\rho$ is a torus-invariant ideal, and thus Lemma 3.3.3 implies that $\mathfrak{p}_\rho\widehat{R}$ is prime in $\widehat{R}$. Let $\widehat{D_\rho} \subseteq \operatorname{Spec}\widehat{R}$. It follows from the properties of completions that $\widehat{D_\rho}$ is a prime divisor of $\operatorname{Spec}\widehat{R}$ and $\mathcal{O}(\widehat{D_\rho}) = \mathcal{O}(D_\rho)^{\wedge}_{\mathfrak{m}_R}$. Thus each $[\mathcal{O}(\widehat{D_\rho})]$ is in the image of $\mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R})$, and so it will suffice to show that the $[\mathcal{O}(\widehat{D_\rho})]$'s generate $\mathrm{Cl}(\widehat{R})$.

As in the proof of [CLS11], Theorem 4.1.3, consider the open embedding $T_d \subseteq \operatorname{Spec} R$, and note that

$$(\operatorname{Spec} R) \smallsetminus T_d = \bigcup_{\rho\in\sigma_R(1)} D_\rho.$$

Now consider the scheme $\widehat{T_d} := \operatorname{Spec}\mathbb{F}[[x_1, \ldots, x_d]]_{(x_1\cdots x_d)}$, so that we have an open embedding $\widehat{T_d} \subseteq \operatorname{Spec}\widehat{R}$. It follows that

$$\left(\operatorname{Spec}\widehat{R}\right) \smallsetminus \widehat{T_d} = \bigcup_{\rho\in\sigma_R(1)} \widehat{D_\rho}.$$

Now using the fact that each $\widehat{D_\rho}$ is a prime divisor (and the fact that R, and hence $\widehat{R}$, is a noetherian integral separated scheme which is regular in codimension 1) Proposition 6.5 of

[Har77] implies that there is an exact sequence

$$\bigoplus_{\rho} \mathbb{Z}\widehat{D}_\rho \to \operatorname{Cl}(\widehat{R}) \to \operatorname{Cl}(\widehat{T}_d) \to 0$$

where the first map is given by $\widehat{D}_\rho \mapsto [\mathcal{O}(\widehat{D}_\rho)]$.

But now as $\widehat{T}_d = \operatorname{Spec} \mathbb{F}[[x_1, \dots, x_d]]_{(x_1 \cdots x_d)}$ and $\mathbb{F}[[x_1, \dots, x_d]]_{(x_1 \cdots x_d)}$ is a UFD (since it is a localization of the UFD $\mathbb{F}[[x_1, \dots, x_d]]$), it follows that $\operatorname{Cl}(\widehat{T}_d) = 0$, and so indeed $\operatorname{Cl}(\widehat{R})$ is generated by the $[\mathcal{O}(\widehat{D}_\rho)]$'s, and so the map $\operatorname{Cl}(R) \to \operatorname{Cl}(\widehat{R})$ is surjective. □

It will thus suffice to compute $\operatorname{Cl}(\mathcal{R})$ and $[\omega_{\mathcal{R}}]$.

But now Corollary 3.2.7 and Theorem 3.3.1 make it straight-forward to compute $\operatorname{Cl}(\overline{S})$ and $[\omega_{\overline{S}}]$:

Proposition 3.3.4. *Let $e_0 = 2e_1 + 2e_2 - e_3$, so that $\sigma_{\mathcal{X}} = \operatorname{Cone}(e_0, e_1, e_2, e_3)$. For each i, let $\rho_i = \mathbb{R}_{\geq 0} e_i$ and $D_i = D_{\rho_i}$, so that $u_{\rho_i} = e_i$ and $\sigma_{\mathcal{X}}(1) = \{\rho_0, \rho_1, \rho_2, \rho_3\}$. Then:*

1. *We have an isomorphism $\operatorname{Cl}(\mathcal{S}) \cong \mathbb{Z}$ given by $k \mapsto \mathcal{O}(kD_0)$.*
2. *$\omega_{\mathcal{S}} \cong \mathcal{O}(2D_0)$.*
3. *If $\mathcal{M}$ is a rank 1 reflexive, self-dual module over $\mathcal{S}$, then $\mathcal{M} \cong \mathcal{O}(D_0)$.*
4. *Identifying $\mathcal{S}$ with $\mathbb{F}[x, xz, xz^2, y, yz, yz^2] \subseteq \mathbb{F}[x, y, z]$ as in Proposition 3.2.4 we get*

$$\mathcal{O}(D_0) \cong \mathcal{S} \cap xz\mathbb{F}[x, y, z] = (xz, xz^2) \subseteq \mathcal{S}$$
$$\omega_{\mathcal{S}} = \mathcal{O}(2D_0) \cong \mathcal{S} \cap x\mathbb{F}[x, y, z] = (x, xz, xz^2) \subseteq \mathcal{S},$$

   so in particular, $\dim_{\mathbb{F}} \mathcal{O}(D_0)/m_{\mathcal{S}} = 2$.
5. *There is a surjection $\mathcal{O}(D_0) \otimes_{\mathcal{S}} \mathcal{O}(D_0) \twoheadrightarrow \omega_{\mathcal{S}}$.*

Proof. Write $x = \chi^{e_1}, y = \chi^{e_2}$ and $z = \chi^{e_3}$, so that $\mathbb{F}[M] = \mathbb{F}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. By Theorem 3.3.1(3) we get that

$$\operatorname{div}(x) = 2D_0 + D_1, \qquad \operatorname{div}(y) = 2D_0 + D_2, \qquad \operatorname{div}(z) = D_3 - D_0.$$

It follows that $D_1 \sim -2D_0$, $D_2 \sim -2D_0$ and $D_3 \sim D_0$, and so from Theorem 3.3.1(5) we get that $\mathrm{Cl}(\mathcal{S})$ is generated by $\mathcal{O}(D_0)$. Moreover, the exactness in Theorem 3.3.1(5) gives that the above relations are the only ones between the D_i's, and so $\mathcal{O}(D_0)$ is non-torsion in $\mathrm{Cl}(\mathcal{S})$, indeed giving the isomorphism $\mathrm{Cl}(\mathcal{S}) \cong \mathbb{Z}$. (Alternatively, Theorem 3.3.1(5) implies that the $\mathbb{Z}$-rank of $\mathrm{Cl}(\mathcal{S})$ is at least $\operatorname{rank} \operatorname{div}_{T_3}(\mathcal{X}) - \operatorname{rank} M = 4 - 3 = 1$.) This proves (1).

For (2), we simply use Theorem 3.3.1(6):

$$\omega_{\mathcal{S}} \cong \mathcal{O}(-D_0 - D_1 - D_2 - D_3) \cong \mathcal{O}\Big(-D_0 - (-2D_0) - (-2D_0) - D_0\Big) = \mathcal{O}(2D_0).$$

Now by (1), any rank 1 reflexive $\mathcal{S}$-module is in the form $\mathcal{O}(kD_0)$ for some $k \in \mathbb{Z}$, and by (2) $\mathcal{O}(kD_0)^* \cong \mathcal{O}((2-k)D_0)$. Thus if $\mathcal{O}(kD_0)$ is self-dual then $k = 1$, giving (3).

By the above computations, we get that

$$\operatorname{div}(x^a y^b z^c) = (2a + 2b - c)D_0 + 2aD_1 + 2bD_2 + cD_3$$

for any $(a, b, c) \in \mathbb{Z}^3$. Now note that $\mathcal{O}(D_0) \cong \mathcal{O}(-D_1 - D_3)$ and $\mathcal{O}(2D_0) \cong \mathcal{O}(-D_1)$ which are both ideals of $\mathcal{S}$. But now by Theorem 3.3.1(4) as ideals of $\mathcal{S}$ we have

$$\begin{aligned}
\mathcal{O}(D_0) \cong \mathcal{O}(-D_1 - D_3) &= \Big(x^a y^b z^c \Big| 2a + 2b - c \geq 0, 2a \geq 1, 2b \geq 0, c \geq 1\Big) \\
&= \Big(x^a y^b z^c \Big| 2a + 2b - c \geq 0, a \geq 1, b \geq 0, c \geq 1\Big) \\
&= \Big(x^a y^b z^c \Big| x^a y^b z^c \in \mathcal{S}, xz | x^a y^b z^c\Big) \\
&= \mathcal{S} \cap xz\mathbb{F}[x, y, z] = (xz, xz^2) \\
\mathcal{O}(2D_0) \cong \mathcal{O}(-D_1) &= \Big(x^a y^b z^c \Big| 2a + 2b - c \geq 0, 2a \geq 1, 2b \geq 0, c \geq 0\Big) \\
&= \Big(x^a y^b z^c \Big| 2a + 2b - c \geq 0, a \geq 1, b \geq 0, c \geq 0\Big) \\
&= \Big(x^a y^b z^c \Big| x^a y^b z^c \in \mathcal{S}, x | x^a y^b z^c\Big) \\
&= \mathcal{S} \cap x\mathbb{F}[x, y, z] = (x, xz, xz^2),
\end{aligned}$$

proving (4).

Now identify $\mathcal{O}(D_0)$ with $(xz, xz^2) \subseteq \mathcal{S}$ and $\omega_{\mathcal{S}}$ with $(x, xz, xz^2) \subseteq \mathcal{S}$. Notice that

$$\mathcal{O}(D_0)\mathcal{O}(D_0) = (xz, xz^2)(xz, xz^2) = (x^2z^2, x^2z^3, x^2z^4) = xz^2(x, xz, xz^2) = xz^2\omega_{\mathcal{S}}.$$

Thus we can define a surjection $f : \mathcal{O}(D_0) \otimes_{\mathcal{S}} \mathcal{O}(D_0) \twoheadrightarrow \omega_{\mathcal{S}}$ by $f(\alpha \otimes \beta) = \dfrac{1}{xz^2}\alpha\beta$, proving (5). □

We can now compute $\mathrm{Cl}(\mathcal{R})$ and $\omega_{\mathcal{R}}$, by using the following lemma:

Lemma 3.3.5. *For any normal, affine toric varieties X and Y the natural map $\mathrm{Cl}(X) \oplus \mathrm{Cl}(Y) \to \mathrm{Cl}(X \times Y)$ given by $([A], [B]) \mapsto [A \boxtimes B]$ is an isomorphism which sends $([\omega_X], [\omega_Y])$ to $\omega_{X\times Y}$.*

Proof. By [CLS11] Proposition 3.1.14, $X \times Y$ is a toric variety with cone $\sigma_{X\times Y} = \sigma_X \times \sigma_Y$. It follows that $\sigma_{X\times Y}(1) = \sigma_X(1) \sqcup \sigma_Y(1)$. The claim now follows immediately from Theorem 3.3.1. □

Thus we have:

Corollary 3.3.6. *The map $\varphi : \mathrm{Cl}(\mathcal{S})^k \to \mathrm{Cl}(\mathcal{R})$ given by*

$$([A_1], \ldots, [A_k]) \mapsto \left[(A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_a)[x_1, \ldots, x_s]\right]$$

is an isomorphism which sends $([\omega_{\mathcal{S}}], \ldots, [\omega_{\mathcal{S}}])$ to $[\omega_{\mathcal{R}}]$.

Consequently there is a unique self-dual rank 1 reflexive module $\mathcal{M}$ over $\mathcal{R}$, which is the image of $([\mathcal{O}(D_0)], \ldots, [\mathcal{O}(D_0)])$. We have that $\dim_{\mathbb{F}} \mathcal{M}/m_{\mathcal{R}} = 2^k$ and there is a surjection $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{M} \twoheadrightarrow \omega_{\mathcal{R}}$.

Proof. The isomorphism follows immediately from Corollary 3.3.5 (noting that $\mathbb{A}^1$ is a toric variety with $\mathrm{Cl}(\mathbb{A}^1) = 0$ and $\omega_{\mathbb{A}^1} = \mathbb{A}^1$).

Now for any self-dual rank 1 reflexive module $\mathcal{M}$ over $\mathcal{R}$, it follows that $[\mathcal{M}] = \varphi([A_1], \ldots, [A_a])$ where each A_i is self-dual. Proposition 3.3.4 implies that each A_i is isomorphic to $\mathcal{O}(D_0)$, as claimed.

For this $\mathcal{M}$ we indeed have

$$\mathcal{M}/m_{\mathcal{R}} = \frac{\mathcal{O}(D_0)^{\boxtimes k}[x_1, \ldots, x_s]}{m_{\mathcal{S}}^{\boxtimes k} \boxtimes (x_1, \ldots, x_s)} \cong \left(\frac{\mathcal{O}(D_0)}{m_{\mathcal{S}}}\right)^{\boxtimes k} = \left(\mathbb{F}^2\right)^{\boxtimes k} = \mathbb{F}^{2^k}.$$

Also, the surjection $\mathcal{O}(D_0) \otimes_{\mathcal{S}} \mathcal{O}(D_0) \twoheadrightarrow \omega_{\mathcal{S}}$ from Proposition 3.3.4 indeed gives a surjection

$$\begin{aligned}\mathcal{M} \otimes_{\mathcal{R}} \mathcal{M} &= \left(\mathcal{O}(D_0)^{\boxtimes k}[x_1, \ldots, x_s]\right) \otimes_{\mathcal{R}} \left(\mathcal{O}(D_0)^{\boxtimes k}[x_1, \ldots, x_s]\right) \\ &\cong \left(\mathcal{O}(D_0) \otimes \mathcal{O}(D_0)\right)^{\boxtimes k}[x_1, \ldots, x_s] \twoheadrightarrow \omega_{\mathcal{S}}^{\boxtimes k}[x_1, \ldots, x_s] \cong \omega_{\mathcal{R}}.\end{aligned}$$

□

Theorem 3.1.2 now follows from Corollary 3.3.6 by applying Proposition 3.3.2. (For the statement about surjectivity, note that $(A \otimes_{\mathcal{R}} B)^{\wedge}_{\mathfrak{m}_{\mathcal{R}}} \cong A_{\mathfrak{m}_{\mathcal{R}}} \otimes_{\overline{R}_\infty} B_{\mathfrak{m}_{\mathcal{R}}}$ so if $\omega_{\mathcal{R}}$ is a quotient of $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}$ then $\omega_{\overline{R}_\infty}$ is a quotient of $\mathcal{M}^{\wedge}_{\mathfrak{m}_{\mathcal{R}}} \otimes_{\overline{R}_\infty} \mathcal{M}^{\wedge}_{\mathfrak{m}_{\mathcal{R}}}$.)

Remark. In our proof of Theorem 3.1.2, we never actually used the first condition, namely that $\overline{M}_\infty$ was maximal Cohen–Macaulay over $\overline{R}_\infty$. We only used the (strictly weaker) assumption that $\overline{M}_\infty$ was reflexive, which, combined with the fact that $\overline{M}_\infty$ was self-dual, was enough to uniquely determine the structure on $\overline{M}_\infty$.

In most situations, the modules M_∞ produced by the patching method will be maximal Cohen–Macaulay, but it is possible that might fail to be self-dual (e.g. if they arise from the cohomology of a non self-dual local system).

In this situation it is possible to formulate a weaker version of 3.0.3, where one drops the self-duality assumption. Specifically one can show (in the notation of Proposition 3.3.4) that the only Cohen–Macaulay rank one modules over the ring $\overline{S}$ are the 5 modules:

$$\begin{aligned}
\mathcal{O}(-D_0) &= (xz, xz^2, yz, yz^2)\\
\mathcal{O} &= \overline{S}\\
\mathcal{O}(D_0) &= (xz, xz^2)\\
\mathcal{O}(2D_0) &= (x, xz, xz^2)\\
\mathcal{O}(3D_0) &= (x^2, x^2z, x^2z^2, x^2z^3).
\end{aligned}$$

This can be done quite simply by noting that any regular sequence for $\overline{S}$, such as $(x, yz^2, y - xz^2)$, must also be a regular sequence for any maximal Cohen–Macaulay module $\overline{M}_\infty$ over $\overline{S}$, and thus if $\operatorname{rank}_{\overline{S}} \overline{M}_\infty = 1$ we must have

$$\dim_{\mathbb{F}} \overline{M}_\infty / m_{\overline{S}} \overline{M}_\infty \leq \dim_{\mathbb{F}} \overline{M}_\infty/(x, yz^2, y - xz^2) = \dim_{\mathbb{F}} \overline{S}/(x, yz^2, y - xz^2) = 4,$$

reducing the problem to simply checking that the above 5 modules are indeed all maximal Cohen–Macaulay.

This unfortunately does not allow us to uniquely deduce the structure of $\overline{M}_\infty$, but it does give us the bound $\dim_{\mathbb{F}} M_\infty/m_{R_\infty} \leq 4^k$, and could potentially lead to more refined information about M_∞, which may be of independent interest.

CHAPTER 4

THE CONSTRUCTION OF M_∞

From now on assume that $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ satisfies the final condition of Theorem 1.3.1 (i.e. the "Taylor–Wiles" condition). The goal of this chapter is to construct a module M_∞ over R_∞ satisfying the conditions of Theorem 3.0.3.

We shall construct M_∞ by applying the Taylor–Wiles–Kisin patching method [Wil95, TW95, Kis09] to a natural system of modules over the rings $T^D(K)$. For convenience we will follow the "Ultrapatching" construction introduced by Scholze in [Sch15], which we have summarized in the appendix. The primary advantage to doing this is that Scholze's construction is somewhat more "natural" than the classical construction, and thus it will be easier to show that M_∞ satisfies additional properties (in our case, that it is self-dual).

From now on, we will freely assume all of the definitions and results from the appendix. In particular, this means we are fixing a nonprincipal ultrafilter $\mathfrak{F}$ on $\mathbb{N}$.

4.1 A Patching System Producing M_∞

Recall that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ is assumed to be a Galois representation satisfying all of the conditions of Theorem 1.3.1.

In this section, we will construct the desired module M_∞ over R_∞ by applying the results of the appendix, and using Lemmas 2.3.1 and 2.3.2.

From now on, we will take the ring S_∞ from the appendix to be $\mathcal{O}[[y_1, \ldots, y_r, w_1, \ldots, w_j]]$, where r is an in Lemma 2.3.2 and $j = 3|\Sigma_\ell^D| - 1$ is as in Section 2.2, and let $\mathfrak{n} = (y_1, \ldots, y_r, w_1, \ldots, w_j)$ as in the appendix. Note that $\dim S_\infty = r + j + 1 = \dim R_\infty$ by Lemma 2.3.2.

We will construct a weak patching algebra $\mathscr{R}^\square$ covered by R_∞ together with a free patching $\mathscr{R}^\square$-module $\mathscr{M}^\square$, and let $M_\infty := \mathscr{P}(\mathscr{M}^\square)$. By Theorem A.3.4 and Lemma A.3.2 it will then follow that M_∞ is maximal Cohen–Macaulay over R_∞. In Section 4.2, we will show that M_∞ satisfies the remaining conditions of Theorem 3.0.3.

We start by defining a natural system of modules over the Hecke algebras $\mathbb{T}^D(K)_m$. For any $K \in \mathcal{K}^D(\overline{\rho})$, define $M(K) := S^D(K)_m^\vee$ if D is totally definite and

$$M(K) := \mathrm{Hom}_{\mathbb{T}^D(K)_m[G_F]}(\rho^{\mathrm{univ}}, S^D(K)_m^\vee)$$

if D is indefinite. Give $M(K)$ its natural $\mathbb{T}^D(K)_m$-module structure.

Remark. The purpose of the definition of $M(K)$ in the indefinite case is to "factor out" the Galois action on $S^D(K)^\vee$. This construction was described by Carayol in [Car94]. If we did not do this, and only worked with $S^D(K)_m^\vee$, then the module M_∞ we will construct would have rank 2 instead of rank 1, and so we would not be able to directly apply Theorem 3.0.3.

Note that it follows from the definitions that $\dim_{\mathbb{T}^D(K)/m} M(k)/m = \nu_{\overline{\rho}}(K)$ for all K.

From now on, fix a collection of *sets* of primes $\mathcal{Q} = \{Q_n\}_{n\geq 1}$ satisfying the conclusion of Lemma 2.3.2. For any n, let Δ_n be the maximal ℓ-power quotient of $\prod_{v\in Q_n} (\mathcal{O}_F/v)^\times$. Consider the ring $\Lambda_n := \mathcal{O}[\Delta_n]$, and note that:

$$\Lambda_n \cong \frac{\mathcal{O}[[y_1, \dots, y_r]]}{\left((1+y_1)^{\ell^{e(n,1)}}, \dots, (1+y_r)^{\ell^{e(n,r)}}\right)}$$

where $\ell^{e(n,i)}$ is the ℓ-part of $N(v) - 1 = \#(\mathcal{O}_F/v)^\times$, so that $e(n,i) \geq n$ by assumption. Let $\mathfrak{a}_n = (y_1, \dots, y_r) \subseteq \Lambda_n$ be the augmentation ideal.

Also let $H_n = \ker\left(\prod_{v\in Q_n} (\mathcal{O}_F/v)^\times \twoheadrightarrow \Delta_n\right)$. For any finite place v of F, note that $\Gamma_1(v)$ is a normal subgroup of

$$\Gamma_0(v) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{v} \right\}$$

and there is a group isomorphism $\Gamma_0(v)/\Gamma_1(v) \to (\mathcal{O}/v)^\times$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{v}$.

Now let $\Gamma_H(Q_n) \subseteq \prod_{v\in Q_n} \Gamma_0(v)$ be the preimage of $H_n \subseteq \prod_{v\in Q_n} (\mathcal{O}/v)^\times$ under the map

$$\prod_{v\in Q_n} \Gamma_0(v) \twoheadrightarrow \prod_{v\in Q_n} \Gamma_0(v)/\Gamma_1(v) \cong \prod_{v\in Q_n} (\mathcal{O}/v)^\times$$

finally let $K_n \subseteq K^{\min}$ be the preimage of $\Gamma_H(Q_n)$ under

$$K^{\min} \hookrightarrow \prod_{\mathfrak{p}\subseteq\mathcal{O}_F} D^\times(\mathcal{O}_{F,\mathfrak{p}}) \twoheadrightarrow \prod_{v\in Q_n} D^\times(\mathcal{O}_{F,v}).$$

Now let $M_n = M(K_n)$, $\mathbb{T}_n = \mathbb{T}^D(K_n)$ and $R_n = R^{D,\psi}_{F,S\cup Q_n}(\overline{\rho})$. Also let $M_0 = M(K^{\min})$, $\mathbb{T}_0 = \mathbb{T}^D(K^{\min})$ and $R_0 = R_{F,S}(\overline{\rho})$. By Lemma 2.3.1 there is a surjection $R_n \twoheadrightarrow \mathbb{T}_n$ and so we may view M_n as a R_n-module for all $n \geq 0$. Now we have:

We now have the following standard result (cf [Kis09])

Proposition 4.1.1. *There exists an embedding $\Lambda_n \hookrightarrow R_n$ under which M_n is a finite rank free Λ_n-module. Moreover, we have $R_n/\mathfrak{a}_n \cong R_0$ and $M_n/\mathfrak{a}_n \cong M_0$ (so in particular, $\operatorname{rank}_{\Lambda_n} M_n = \operatorname{rank}_{\mathcal{O}} M_0$)*

Now let $R_n^\square = R^{\square,D,\psi}_{F,S\cup Q_n}(\overline{\rho})$, and recall from Section 2.2 that $R_n^\square = R_n[[w_1,\dots,w_j]]$ for some integer j. Using this, we may define framed versions of $\mathbb{T}_n$ and M_n. Namely

$$\mathbb{T}_n^\square := R_n^\square \otimes_{R_n} \mathbb{T}_n \cong \mathbb{T}_n[[w_1,\dots,w_j]]$$
$$M_n^\square := R_n^\square \otimes_{R_n} M_n \cong M_n[[w_1,\dots,w_j]]$$

so that $M_n^\square$ inherits a natural $\mathbb{T}_n^\square$-module structure, and we still have a surjective map $R_n^\square \twoheadrightarrow \mathbb{T}_n^\square$ (and so $M_n^\square$ inherits a $R_n^\square$-module structure). Note that the ring structure of $\mathbb{T}_n^\square$ and the $\mathbb{T}_n^\square$-module structure of $M_n^\square$ do not depend on the choice of the set S, and so we may define this without reference to a specific S.

Also for any n, consider the ring $\Lambda_n^\square := \Lambda_n[[w_1, \dots, w_j]] = \mathcal{O}[\Delta_n][[w_1, \dots, w_j]]$, which we will view as a quotient of the ring $S_\infty = \mathcal{O}[[y_1, \dots, y_r, w_1, \dots, w_j]]$ from above.

Rewriting Proposition 4.1.1 in terms of the framed versions of R_n and M_n, we get:

Proposition 4.1.2. *There exists an embedding $\Lambda_n^\square \hookrightarrow R_n^\square$ under which $M_n^\square$ is a finite rank free $\Lambda_n^\square$-module. Moreover, we have $R_n^\square/\mathfrak{n} \cong R_0$ and $M_n^\square/\mathfrak{n} \cong M_0$ (so in particular, $\operatorname{rank}_{S_\infty} M_n^\square = \operatorname{rank}_{\Lambda_n^\square} M_n^\square = \operatorname{rank}_{\mathcal{O}} M_0$).*

So in particular, the rings $R_n^\square$ are S_∞-algebras and the modules $M_n^\square$ are S_∞-modules. But now the collections $\mathscr{R}^\square := \{R_n^\square\}_{n\geq 1}$ and $\mathscr{M}^\square := \{M_n^\square\}_{n\geq 1}$ satisfy the conditions of Lemma A.4.2 and so we get that $\mathscr{R}^\square$ is a weak patching algebra, $\mathscr{M}^\square$ is a free patching $\mathscr{R}^\square$-algebra.

The final statement follows from Lemma A.3.2 after noting that R_∞ is a domain (by Theorem 3.1.1 and the discussion following it) and $\dim R_\infty = \dim S_\infty$.

Thus we may define $M_\infty := \mathscr{P}(\mathscr{M}^\square)$. By Theorem A.3.4 we get that M_∞ is maximal Cohen–Macaulay over $\mathscr{P}(\mathscr{R}^\square) \cong R_\infty$ and

$$\dim_{\mathbb{F}} M_\infty/m_{R_\infty} = \dim_{\mathbb{F}}(M_\infty/\mathfrak{n})/m_{R_\infty} = \dim_{\mathbb{F}} M_0/m_{R_0} = \nu_{\overline{\rho}}(K^{\min}).$$

4.2 The Properties of M_∞

We shall now show that M_∞ satisfies the remaining conditions of Theorem 3.0.3.

We start by showing that it is self-dual. This will follow from the fact that the modules $M(K)$ were naturally self-dual:

Lemma 4.2.1. *There is a $\mathbb{T}^D(K)_m$-equivariant perfect pairing $M(K) \times M(K) \to \mathcal{O}$.*

Proof. First note that there is a $\mathbb{T}^D(K)$-equivariant perfect pairing $S^D(K) \times S^D(K) \to \mathbb{Z}$. In the totally definite case, this is the monodromy pairing, in the indefinite case it is Poincare duality (although this must be modified slightly in order to make the pairing $\mathbb{T}^D(K)$-

equivariant, see [Car94, 3.1.4]). Completing and dualizing gives a $\mathbb{T}^D(K)_m$-equivariant perfect pairing $S^D(K)_m^\vee \times S^D(K)_m^\vee \to \mathcal{O}$.

In the totally definite case, this is already the desired pairing. In the indefinite case, it follows from [Car94, 3.2.3] that the self-duality on $S^D(K)_m^\vee$ implies that $M(K) := \mathrm{Hom}_{\mathbb{T}^D(K)_m[G_F]}(\rho^{\mathrm{univ}}, S^D(K)_m^\vee)$ is also $\mathbb{T}^D(K)_m$-equivariantly self-dual. ☐

Thus by Lemma A.4.2 and the construction from Section 4.1, it follows that $\mathscr{M}^\square$ is dualizable and $(\mathscr{M}^\square)^* \cong \mathscr{M}^\square$. The fact that $M_\infty^* \cong M_\infty$ now follows from Theorem A.4.4.

It remains to show that $\mathrm{rank}_{R_\infty} M_\infty = 1$. First, the fact that $R_\infty[1/\lambda]$ is formally smooth implies that:

Lemma 4.2.2. *$M_0[1/\lambda]$ is free of rank 1 over $R_0[1/\lambda]$. In particular, the natural map $R_0[1/\lambda] \twoheadrightarrow \mathbb{T}_0[1/\lambda]$ is an isomorphism.*

Proof. As M_∞ is maximal Cohen–Macaulay over R_∞, $M_\infty[1/\lambda]$ is also maximal Cohen–Macaulay over $R_\infty[1/\lambda]$. Since $R_\infty[1/\lambda]$ is a formally smooth domain, this implies that $M_\infty[1/\lambda]$ is *locally* free over $R_\infty[1/\lambda]$ of some rank, d.

Now quotienting by $\mathfrak{n}$ we get that $M_\infty[1/\lambda]/\mathfrak{n} \cong M_0[1/\lambda]$ is locally free over $R_\infty[1/\lambda]/\mathfrak{n} \cong R_0[1/\lambda]$ of constant rank d. But now $R_0[1/\lambda]$ is a finite dimensional commutative E-algebra, and hence is a product of local rings. Thus as $M_0[1/\lambda]$ is locally free of rank d, it must actually be free of rank d.

But now by classical generic multiplicity 1 results we get that $M_0[1/\lambda]$ is free of rank 1 over $\mathbb{T}_0[1/\lambda]$, which is a quotient of $R_0[1/\lambda]$. Thus $d = 1$ and hence $M_0[1/\lambda] \cong R_0[1/\lambda]$.

Lastly, as the action of $R_0[1/\lambda]$ on $M_0[1/\lambda]$ is free *and* factors through $R_0[1/\lambda] \twoheadrightarrow \mathbb{T}_0[1/\lambda]$, we get that $R_0[1/\lambda] \twoheadrightarrow \mathbb{T}_0[1/\lambda]$ is an isomorphism. ☐

It is now straightforward to compute $\mathrm{rank}_{R_\infty} M_\infty$. Let $\kappa(R_\infty)$ and $\kappa(S_\infty)$ be the fraction fields of R_∞ and S_∞, respectively. As R_∞ is a finite type S_∞-algebra, $\kappa(R_\infty)$ is a finite extension of $\kappa(S_\infty)$. It follows that

$$M_\infty \otimes_{R_\infty} \kappa(R_\infty) \cong M_\infty \otimes_{S_\infty} \kappa(S_\infty).$$

Since R_∞ and M_∞ are both finite free S_∞-modules, we thus get

$$\begin{aligned}\operatorname{rank}_{R_\infty} M_\infty = \dim_{\kappa(R_\infty)} \left[M_\infty \otimes_{R_\infty} \kappa(R_\infty)\right] &= \dim_{\kappa(R_\infty)} \left[M_\infty \otimes_{S_\infty} \kappa(S_\infty)\right] \\ = \frac{\dim_{\kappa(S_\infty)} \left[M_\infty \otimes_{S_\infty} \kappa(S_\infty)\right]}{\dim_{\kappa(S_\infty)} K(R_\infty)} &= \frac{\operatorname{rank}_{S_\infty} M_\infty}{\operatorname{rank}_{S_\infty} R_\infty}\end{aligned}$$

But now for any finite free S_∞ module A we have

$$\operatorname{rank}_{S_\infty} A = \operatorname{rank}_{S_\infty/\mathfrak{n}} A/\mathfrak{n} = \operatorname{rank}_{\mathcal{O}} A/\mathfrak{n} = \dim_E (A/\mathfrak{n})[1/\lambda]$$

and so the fact that $M_\infty[1/\lambda]/\mathfrak{n} \cong M_0[1/\lambda] \cong R_0[1/\lambda] \cong R_\infty[1/\lambda]/\mathfrak{n}$ implies that $\operatorname{rank}_{S_\infty} M_\infty = \operatorname{rank}_{S_\infty} R_\infty$, giving $\operatorname{rank}_{R_\infty} M_\infty = 1$.

This shows that M_∞ indeed satisfies the conditions of Theorem 3.0.3, and so completes the proof of Theorem 1.3.1.

Remark. It is worth mentioning here that Shotton's computations of local deformation rings (particularly the fact that R_∞ is Cohen–Macaulay, by Theorem 3.1.1) actually imply an integral "$R = \mathbb{T}$" theorem. This result is likely known to experts, but we include it here for the sake of completeness.

Specifically one considers the surjection $f : R_0 \twoheadrightarrow \mathbb{T}_0$. As shown in Lemma 4.2.2 (see also, [Kis09]), f is an isomorphism after inverting λ (i.e. $R_0[1/\lambda] \cong \mathbb{T}_0[1/\lambda]$). This means that $\ker f \subseteq R_0$ is a torsion $\mathcal{O}$-module.

But now R_∞ is Cohen–Macaulay, and M_∞ is a maximal Cohen–Macaulay module over R_∞. Since $(\lambda, y_1, \dots, y_r, w_1, \dots, w_j)$ is a regular sequence for M_∞ (by Theorem A.3.4(2)) it follows that it is also a regular sequence for R_∞. Thus $R_0 \cong R_\infty/\mathfrak{n} = R_\infty/(y_1, \dots, y_r, w_1, \dots, w_j)$ is Cohen–Macaulay and λ is a regular element on R_0 (i.e. a non zero divisor).

But this implies that R_0 is λ-torsion free, giving that $\ker f = 0$, so indeed $f : R_0 \twoheadrightarrow \mathbb{T}_0$ is an isomorphism.

It remains to show Theorem 1.3.2:

Proof of Theorem 1.3.2. We first rephrase the statement in terms of the module M_0. Recall that we've defined $\mathbb{T}_0 = \mathbb{T}^D(K^{\min})_m$. In the definite case we get that

$$\operatorname{End}_{\mathbb{T}_0}(S^D(K^{\min})_m) = \operatorname{End}_{\mathbb{T}_0}(S^D(K^{\min})_m^\vee) = \operatorname{End}_{\mathbb{T}_0}(M(K^{\min})) = \operatorname{End}_{\mathbb{T}_0}(M_0)$$

as $\mathbb{T}_0$-algebras.

In the indefinite case, we have an isomorphism

$$M_0 \otimes_{\mathbb{T}_0} \rho^{\text{univ}} = \operatorname{Hom}_{\mathbb{T}_0[G_F]}(\rho^{\text{univ}}, S^D(K^{\min})_m) \otimes_{\mathbb{T}_0} \rho^{\text{univ}} \xrightarrow{\sim} S^D(K^{\min})_m$$

by [Car94]. It now follows that

$$\begin{aligned}
\operatorname{End}_{\mathbb{T}_0[G_F]}(S^D(K^{\min})_m) &\cong \operatorname{End}_{\mathbb{T}_0[G_F]}(S^D(K^{\min})_m^\vee) \\
&\cong \operatorname{Hom}_{\mathbb{T}_0[G_F]}(M_0 \otimes_{\mathbb{T}_0} \rho^{\text{univ}}, S^D(K^{\min})_m^\vee) \\
&\cong \operatorname{Hom}_{\mathbb{T}_0[G_F]}((M_0 \otimes_{\mathbb{T}_0} \mathbb{T}_0[G_F]) \otimes_{\mathbb{T}_0[G_F]} \rho^{\text{univ}}, S^D(K^{\min})_m^\vee) \\
&\cong \operatorname{Hom}_{\mathbb{T}_0[G_F]}(M_0 \otimes_{\mathbb{T}_0} \mathbb{T}_0[G_F], \operatorname{Hom}_{\mathbb{T}_0[G_F]}(\rho^{\text{univ}}, S^D(K^{\min})_m^\vee)) \\
&\cong \operatorname{Hom}_{\mathbb{T}_0}(M_0, \operatorname{Hom}_{\mathbb{T}_0[G_F]}(\rho^{\text{univ}}, S^D(K^{\min})_m^\vee)) \\
&\cong \operatorname{Hom}_{\mathbb{T}_0}(M_0, M_0) = \operatorname{End}_{\mathbb{T}_0}(M_0)
\end{aligned}$$

as $\mathbb{T}_0$-algebras (where we used [Sta17, Tag 00DE, Tag 05DQ]).

Thus in either case, it suffices to show that the structure map $\mathbb{T}_0 \to \operatorname{End}_{\mathbb{T}_0}(M_0)$ is an isomorphism. Alternatively, using the surjection $R_0 \twoheadrightarrow \mathbb{T}_0$, and viewing M_0 as an R_0-module, it suffices to prove that the structure map $R_0 \to \operatorname{End}_{R_0}(M_0)$ is an isomorphism[1].

1. This will also imply that the surjection $R_0 \twoheadrightarrow \mathbb{T}_0$ is an isomorphism. However as noted in the remark

Now by the last conclusion of Theorem 3.0.3, we indeed get that the trace map $\tau_{M_\infty} : M_\infty \otimes_{R_\infty} M_\infty \to \omega_{R_\infty}$ is surjective.

As noted above, $(y_1, \ldots, y_r, w_1, \ldots, w_j)$ is a regular sequence for M_∞, and hence for R_∞. It follows that $R_0 \cong R_\infty/\mathfrak{n}$ is Cohen–Macaulay and $M_0 \cong M_\infty/\mathfrak{n}$ is maximal Cohen–Macaulay over R_0. Moreover, we get that the dualizing module of R_0 is just $\omega_{R_0} \cong \omega_{R_\infty}/\mathfrak{n}$.

But now quotienting out by $\mathfrak{n}$, we thus get a surjective map $M_0 \otimes_{R_0} M_0 \twoheadrightarrow \omega_{R_0}$, which (by Lemma 3.1.3) implies that the trace map $\tau_{M_0} : M_0 \otimes_{R_0} M_0 \to \omega_{R_0}$ is also surjective.

But now, as in [Eme02, Lemmas 2.4 and 2.6], we have the following commutative algebra result:

Lemma 4.2.3. *Let B be an $\mathcal{O}$-algebra and let U and V be B-modules. Assume that B, U and V are all finite rank free $\mathcal{O}$-modules, and we have a B-bilinear perfect pairing $\langle\ ,\ \rangle : V \times U \to \mathcal{O}$. Moreover, assume that $U[1/\lambda]$ is free over $B[1/\lambda]$. Define $\phi : U \otimes_B V \to \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O})$ by $\phi(u \otimes v)(b) = \langle bu, v\rangle = \langle u, bv\rangle$. Then ϕ is surjective if and only if the natural map from B to the center of $\mathrm{End}_B(U)$ is an isomorphism.*

Applying this with $B = R_0$, $U = M_0$, $V = M_0^*$ and $\langle\ ,\ \rangle : M_0 \times M_0 \to \mathcal{O}$ being the natural perfect pairing implies that the natural map $R_0 \to Z(\mathrm{End}_{R_0}(M_0))$ is an isomorphism. (Here we have used the fact that $\omega_{R_0} \cong \mathrm{Hom}_{\mathcal{O}}(R_0, \mathcal{O})$ as in the proof of Lemma A.4.3, and $M_0[1/\lambda] \cong R_0[1/\lambda]$ by Lemma 4.2.2.)

But now as M_0 is free over $\mathcal{O}$, we get that

$$\mathrm{End}_{R_0}(M_0) \hookrightarrow \mathrm{End}_{R_0[1/\lambda]}(M_0[1/\lambda]) \cong \mathrm{End}_{R_0[1/\lambda]}(R_0[1/\lambda]) = R_0[1/\lambda]$$

and so $\mathrm{End}_{R_0}(M_0)$ is commutative. Hence the natural map $R_0 \to \mathrm{End}_{R_0}(M_0)$ is an isomorphism, as desired. □

following Lemma 4.2.2 this follows more simply from the fact that R_∞ is a Cohen-Macauly domain, flat over $\mathcal{O}$ (as shown in [Sho16]) and the fact that $R_0[1/\ell] \cong \mathbb{T}_0[1/\ell]$ (shown in [Kis09]).

APPENDIX A

ULTRAPATCHING

In this appendix we will develop the commutative algebra results needed for the Taylor–Wiles–Kisn patching method, as reformulated by Scholze in [Sch15].

A.1 Ultraproducts

Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the natural numbers. Recall that a *nonprincipal ultrafilter* on $\mathbb{N}$ is a collection, $\mathfrak{F}$, of subsets of $\mathbb{N}$ satisfying the following conditions:

1. $\mathfrak{F}$ does not contain any finite sets.
2. If $I, J \in \mathfrak{F}$ then $I \cap J \in \mathfrak{F}$
3. If $I \in \mathfrak{F}$ and $I \subseteq J \subseteq \mathbb{N}$, then $J \in \mathfrak{F}$ as well.
4. If $I \sqcup J = \mathbb{N}$ is a partition of $\mathbb{N}$, then either $I \in \mathfrak{F}$ or $J \in \mathfrak{F}$.

It is well known that such an $\mathfrak{F}$ must exist, if one assumes the axiom of choice.

Note that these conditions imply the following: If $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_a = \mathbb{N}$ is a partition of $\mathbb{N}$, then $I_i \in \mathfrak{F}$ for *exactly* one i.

For the remainder of this appendix, we will fix a nonprincipal ultrafilter $\mathfrak{F}$ on $\mathbb{N}$.

For convenience, we will say that a property $\mathcal{P}(i)$ holds for $\mathfrak{F}$-*many* i if there is some $I \in \mathfrak{F}$ such that $\mathcal{P}(i)$ is true for all $i \in I$. The four conditions above imply the following:

1. If $\mathcal{P}(i)$ holds for $\mathfrak{F}$-many i, then it holds for infinitely many i.
2. If $\mathcal{P}(i)$ and $\mathcal{Q}(i)$ each hold for $\mathfrak{F}$-many i, then $\mathcal{P}(i)$ and $\mathcal{Q}(i)$ are *simultaneously* true for $\mathfrak{F}$-many i.
3. $\mathcal{P}(i)$ holds for $\mathfrak{F}$-many i if and only if the set $\{i|\mathcal{P}(i) \text{ is true}\}$ is in $\mathfrak{F}$.
4. For any property $\mathcal{P}$, either $\mathcal{P}(i)$ is true for $\mathfrak{F}$-many i, or it is false for $\mathfrak{F}$-many i.

If $\mathscr{M} = \{M_n\}_{n\geq 1}$ is any sequence of sets, we define an equivalence relation $\sim$ on the set $\prod_{n\geq 1} M_n$ by $(m_1, m_2, \ldots) \sim (m_1', m_2', \ldots)$ if $m_i = m_i'$ for $\mathfrak{F}$-many i (the above properties of ultrafilters imply that this is an equivalence relation). We then define the *ultraproduct* of $\mathscr{M}$

to be

$$\mathcal{U}(\mathscr{M}) := \left(\prod_{n\geq 1} M_n\right) / \sim$$

For any $m = (m_1, m_2, \ldots) \in \prod_{n\geq 1} M_n$ we will denote the equivalence class of m in $\mathcal{U}(\mathscr{M})$ by $[m_i]_i = [m_1, m_2, \ldots]$. We will frequently define elements $m = [m_i]_i$ by only specifying m_i for $\mathfrak{F}$-many i. Doing so is unambiguous, as if m_i is specified for all $i \in I$ ($I \in \mathfrak{F}$) the choices of m_j for $j \in \mathbb{N} \smallsetminus I$ do not affect the equivalence class $[m_i]_i$.

If M is any set we will write $\underline{M} := \{M\}_{n\geq 1}$ for the constant sequence of sets, and define the *ultrapower* of M to be $M^{\mathcal{U}} := \mathcal{U}(\underline{M})$. Notice that we have a diagonal map $\Delta : M \to M^{\mathcal{U}}$ defined by $m \mapsto [m, m, \ldots]$. This map is clearly injective.

In our applications, we will generally consider the case where each M_n has a certain algebraic structure. Thus for the rest of this subsection we will fix a category, $\mathcal{C}$ of sets with algebraic structure, taken to be one of the following:

- The category of abelian groups;
- The category of (commutative) rings;
- The category of (continuous) R-modules;
- The category of (continuous) R-algebras,

for some fixed ring topological R (which we will often take to have the discrete topology, however the continuous version will be used in Lemma A.3.2). Using the language of universal algebra (or more generally, of model theory) it is possible phrase the results of this section for significantly more general categories of "sets with structure," however the specific cases we discuss here will be sufficient for our purposes.

We first show that if each M_n is in $\mathcal{C}$, then $\mathcal{U}(\mathscr{M})$ inherits a natural $\mathcal{C}$-object structure.

Proposition A.1.1. *Let $\mathscr{M} = \{M_n\}_{n\geq 1}$, and assume that each M_n is in $\mathcal{C}$. Then $\mathcal{U}(\mathscr{M})$ may be given the structure of object in $\mathcal{C}$ with the operations additions, multiplication and* *scalar multiplication (when appropriate) defined by:*

$$[a_1, a_2, \dots] + [b_1, b_2, \dots] = [a_1 + b_1, a_2 + b_2, \dots]$$

$$[a_1, a_2, \dots] \cdot [b_1, b_2, \dots] = [a_1 \cdot b_1, a_2 \cdot b_2, \dots]$$

$$r[a_1, a_2, \dots] = [ra_1, ra_2, \dots]$$

for $\alpha = [a_1, a_2, \dots], \beta = [b_1, b_2, \dots] \in \mathcal{U}(\mathscr{M})$*, the elements* $0, 1 \in \mathcal{U}(\mathscr{M})$ *(again when appropriate) defined by:*

$$0 = [0, 0, \dots] \in \mathcal{U}(\mathscr{M}), \qquad 1 = [1, 1, \dots] \in \mathcal{U}(\mathscr{M}),$$

and topology defined by the quotient map $\pi : \prod_{n \geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$*. Moreover:*

1. *The natural surjection* $\pi : \prod_{n \geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$, $(m_i)i \mapsto [m_i]_i$ *is a* $\mathcal{C}$*-morphism.*
2. *For* $M \in \mathcal{C}$*, the diagonal map* $\Delta : M \to M^{\mathcal{U}}$ *is a* $\mathcal{C}$*-morphism.*

Proof. We will prove this only in the case when $\mathcal{C}$ is taken to by the category of continuous R-algebras. The other cases are analogous.

First we check that the operations are well-defined. Take $\alpha = [a_i]_i, \alpha' = [a_i']_i, \beta = [b_i]_i, \beta' = [b_i'] \in \mathcal{U}(\mathscr{M})$ with $\alpha = \alpha'$ and $\beta = \beta'$. Then for $\mathfrak{F}$-many i we *simultaneously* have that $a_i = a_i'$ and $b_i = b_i'$. It follows that $a_i + b_i = a_i' + b_i'$, $a_i \cdot b_i = a_i' \cdot b_i'$ and $ra_i = ra_i'$ for $\mathfrak{F}$-many i, and so $\alpha + \beta = \alpha' + \beta'$, $\alpha \cdot \beta = \alpha' \cdot \beta'$ and $r\alpha = r\alpha'$.

Now as the operations are defined pointwise, they are clearly preserved by $\pi : \prod_{n \geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$.Thus as $\prod_{n \geq 1} M_n$ is a *continuous* R-algebra, and π is continuous by definition, (1) will follow if we show that the operations make $\mathcal{U}(\mathscr{M})$ into a R-algebra (the operations will automatically be continuous as $\mathcal{U}(\mathscr{M})$ has the quotient topology).

Now let

$$K = \left\{ (a_1, a_2, \ldots) \in \prod_{n \geq 1} M_n \,\middle|\, (a_1, a_2, \ldots) \sim (0, 0, \ldots) \right\}$$
$$= \left\{ (a_1, a_2, \ldots) \in \prod_{n \geq 1} M_n \,\middle|\, a_i = 0 \text{ for } \mathfrak{F}\text{-many } i \right\} \subseteq \prod_{n \geq 1} M_n$$

Now as the operations are well-defined, for any $a = (a_n)_n, b = (b_n)_n \in K$, any $m = (m_n) \in \prod_{n \geq 1} M_n$ and any $r \in R$ we get that:

$$(a_n + b_n)_n = (a_n)_n + (b_n)_n \sim (0)_n + (0)_n = (0)_n$$
$$(m_n \cdot a_n)_n = (m_n)_n \cdot (a_n)_n \sim (m_n)_n \cdot (0)_n = (0)_n$$
$$(ra_n)_n = r(a_n)_n \sim r(0)_n = (0)_n,$$

and so $a + b, ma, ra \in K$. It follows that $K \subseteq \prod_{n \geq 1} M_n$ is an *ideal*.

Also by definition, for $a = (a_n)_n, b = (b_n)_n \in \prod_{n \geq 1} M_n$, $a \sim b$ if and only if $a - b \in K$. It follows that $\pi : \prod_{n \geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$ gives an identification $\overline{\pi} : \left(\prod_{n \geq 1} M_n \right) / K \xrightarrow{\sim} \mathcal{U}(\mathscr{M})$. As π, and thus $\overline{\pi}$, preserves the operations and $\left(\prod_{n \geq 1} M_n \right) / K$ is an R-algebra, it follows that $\mathcal{U}(\mathscr{M})$ is indeed an R-algebra, and π is an R-algebra homomorphism. This proves (1).

For (2), we simply note that $\Delta : M \to M^{\mathcal{U}}$ is the composition of the $\mathcal{C}$-morphisms $M \hookrightarrow \prod_{n \geq 1} M$, $m \mapsto (m, m, \ldots)$ and $\pi : \prod_{n \geq 1} M \to \mathcal{U}(\underline{M}) = M^{\mathcal{U}}$. □

Given two sequences $\mathscr{M} = \{M_n\}_{n \geq 1}$ and $\mathscr{M}' = \{M'_n\}_{n \geq 1}$ in $\mathcal{C}$, we define an $\mathfrak{F}$-*morphism* $\varphi : \mathscr{M} \to \mathscr{M}'$ to be a collection of $\mathcal{C}$-morphisms $\varphi = \{\varphi_i : M_i \to M'_i\}_{i \in I}$ indexed by some $I \in \mathfrak{F}$. Then we have

Proposition A.1.2. *If $\varphi : \mathscr{M} \to \mathscr{M}'$ is an $\mathfrak{F}$-morphism, then the map $\varphi^{\mathcal{U}} : \mathcal{U}(\mathscr{M}) \to \mathcal{U}(\mathscr{M}')$ given by $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)]_i$ is a well-defined $\mathcal{C}$-morphism. Moreover,*

1. *If $\varphi, \psi : \mathscr{M} \to \mathscr{M}'$ are two $\mathfrak{F}$-morphisms, and $\varphi_i = \psi_i$ for $\mathfrak{F}$-many i, then $\varphi^{\mathcal{U}} = \psi^{\mathcal{U}}$. In particular, if $\varphi : \mathscr{M} \to \mathscr{M}$ satisfies $\varphi_i = \mathrm{id}_{M_i} : M_i \to M_i$ for $\mathfrak{F}$-many i, then $\varphi^{\mathcal{U}} = \mathrm{id}_{\mathcal{U}(\mathscr{M})} : \mathcal{U}(\mathscr{M}) \to \mathcal{U}(\mathscr{M})$.*
2. *For two $\mathfrak{F}$-morphisms, $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{M}' \to \mathscr{M}''$, we have $\psi^{\mathcal{U}} \circ \varphi^{\mathcal{U}} = (\psi \circ \varphi)^{\mathcal{U}}$.*

Hence $\mathcal{U}(-)$ is a functor.

Proof. As in Proposition A.1.1, we will prove this only in the case where $\mathcal{C}$ is the category of continuous R-algebras.

Let $\varphi : \mathscr{M} \to \mathscr{M}'$ be an $\mathfrak{F}$-morphism. If we have $[a_i]_i = [a_i']_i$ in $\mathcal{U}(\mathscr{M})$, then for $\mathfrak{F}$-many i we simultaneously have that φ_i exists and $a_i = a_i'$. Thus $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)] = \left[\varphi(a_i')\right] = \varphi^{\mathcal{U}}[a_i']_i$, and so $\varphi^{\mathcal{U}}$ is well-defined. As each φ_i is continuous, it follows that $\varphi^{\mathcal{U}}$ is induced by a continuous map $\prod_{n\geq 1} M_n \to \prod_{n\geq 1} M_n'$, and thus is continuous.

Now for $\alpha = [a_i]_i, \beta = [b_i]_i \in \mathcal{U}(\mathscr{M})$ and $r \in R$, as φ_i is an R-algebra homomorphism for $\mathfrak{F}$-many i, we get

$$\begin{aligned}
\varphi^{\mathcal{U}}(\alpha+\beta) &= \varphi^{\mathcal{U}}[a_i+b_i]_i = [\varphi_i(a_i+b_i)]_i = [\varphi_i(a_i)+\varphi(b_i)]_i = \varphi^{\mathcal{U}}(\alpha)+\varphi^{\mathcal{U}}(\beta)\\
\varphi^{\mathcal{U}}(\alpha\cdot\beta) &= \varphi^{\mathcal{U}}[a_i\cdot b_i]_i = [\varphi_i(a_i\cdot b_i)]_i = [\varphi_i(a_i)\cdot\varphi(b_i)]_i = \varphi^{\mathcal{U}}(\alpha)\cdot\varphi^{\mathcal{U}}(\beta)\\
\varphi^{\mathcal{U}}(r\alpha) &= \varphi^{\mathcal{U}}[ra_i]_i = [\varphi_i(ra_i)]_i = [r\varphi_i(a_i)]_i = r\varphi^{\mathcal{U}}(\alpha)\\
\varphi^{\mathcal{U}}(1) &= \varphi^{\mathcal{U}}[1]_i = [\varphi_i(1)]_i = [1]_i = 1,
\end{aligned}$$

so indeed $\varphi^{\mathcal{U}}$ is an R-algebra homomorphism.

If $\varphi_i = \psi_i$ for $\mathfrak{F}$-many i, then by definition we have $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)]_i = [\psi_i(a_i)]_i = \psi^{\mathcal{U}}[a_i]_i$, and if $\varphi_i = \mathrm{id}_{M_i}$ for $\mathfrak{F}$-many i, then $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)]_i = [a_i]_i$. So (1) holds.

For (2), simply note that for $\mathfrak{F}$-many i, φ_i and ψ_i simultaneously exist, and so

$$\left(\psi^{\mathcal{U}}\circ\varphi^{\mathcal{U}}\right)[a_i]_i = \psi^{\mathcal{U}}\left(\varphi^{\mathcal{U}}\,[a_i]_i\right) = \psi^{\mathcal{U}}\,[\varphi_i(a_i)]_i = [\psi_i\left(\varphi_i(a_i)\right)]_i = (\psi\circ\varphi)^{\mathcal{U}}[a_i]_i.$$

□

In general, $\mathcal{U}(\mathscr{M})$ can be a quite complicated object. However in our setup, the M_n's will always be taken to be finite, of uniformly bounded cardinalities. In that case, we have the following:

Proposition A.1.3. *If $M \in \mathcal{C}$ has finite cardinality, the diagonal map $\Delta : M \to M^{\mathcal{U}}$ is an isomorphism.*

Now assume that $\mathcal{C}$ is the category of abelian groups or rings, or that the ring R is topologically finitely generated (in particular, if it is finite*). If $\mathscr{M} = \{M_n\}_{n\geq 1}$ where each $M_n \in \mathcal{C}$ is a* finite *set, and the cardinalities $\#M_n$ are bounded, then $\mathcal{U}(\mathscr{M})$ is also finite and we have $\mathcal{U}(\mathscr{M}) \cong M_i$ in $\mathcal{C}$ for $\mathfrak{F}$-many i.*

Proof. As $\Delta : M \to M^{\mathcal{U}}$ is already an injective $\mathcal{C}$-morphism, it suffices to show that it is surjective. Take any $\alpha = [a_i]_i \in M^{\mathcal{U}}$. As M is finite, $\bigsqcup_{a\in M} \{i | a_i = a\}$ is a finite partition of $\mathbb{N}$, and so for some $a \in M$, $a_i = a$ for $\mathfrak{F}$-many i. But then $\alpha = [a_i]_i = [a]_i = \Delta(a)$, so indeed Δ is surjective, and hence an isomorphism.

For the second statement, the assumption on $\mathcal{C}$ implies that there are only finitely many isomorphism classes of $\mathcal{C}$-objects of any fixed cardinality d. As the $\#M_n$'s are bounded, there are only finitely many distinct cardinalities $\{\#M_n\}_{n\geq 1}$. It thus follows that there are only finitely many isomorphism classes of $\mathcal{C}$-objects in $\mathscr{M}$.

Thus we may pick some $M \in \mathcal{C}$ (which is necessarily finite) for which $M \cong M_i$ for $\mathfrak{F}$-many i. Fix isomorphisms $\varphi_i : M \xrightarrow{\sim} M_i$ for $\mathfrak{F}$-many i, and define $\mathfrak{F}$-morphisms $\varphi : \underline{M} \to \mathscr{M}$ and $\psi : \mathscr{M} \to \underline{M}$ by $\varphi = \{\varphi_i\}$ and $\psi = \{\varphi_i^{-1}\}$. It follows from Proposition A.1.2 that $\psi^{\mathcal{U}} = (\varphi^{\mathcal{U}})^{-1}$ and so $\varphi^{\mathcal{U}} : M^{\mathcal{U}} = \mathcal{U}(\underline{M}) \to \mathcal{U}(\mathscr{M})$ is an isomorphism.

Combining this with the first claim, we indeed get $\mathcal{U}(\mathscr{M}) \cong M^{\mathcal{U}} \cong M \cong M_i$ for $\mathfrak{F}$-many i. □

In the case when $\mathcal{C}$ is taken to be the category of R-modules (or R-algebras), the construction of $\mathcal{U}(\mathscr{M})$ can be reformulated as a localization of modules, and is thus quite well behaved. We finish this section by discussing this situation.

For the reminder of this section, R will denote a *local* ring with maximal ideal m_R and residue field $\mathbb{F} = R/m_R$. For convenience, we will assume that R is finite.

We will let $\mathcal{R} := \prod_{n\geq 1} R$, treated as an R-algebra via the diagonal embedding $r \mapsto (r, r, \ldots)$. Proposition A.1.1 implies that the natural map $\pi : \mathcal{R} \twoheadrightarrow R^{\mathcal{U}} = R$ is a surjective ring homomorphism.

Also for any $I \subseteq \mathbb{N}$, we will let $\mathcal{R}_I := \prod_{i\in I} R$, viewed as a quotient of $\mathcal{R}$ via the map $\pi_I : (r_n)_{n\geq 1} \mapsto (r_i)_{i\in I}$. Note that $\pi : \mathcal{R} \to R$ factors through π_I for each $I \in \mathfrak{F}$.

The key observation is that π may be viewed as a localization map:

Proposition A.1.4. *There is a unique prime ideal* $\mathfrak{z}_R \in \operatorname{Spec} \mathcal{R}$ *for which the* $\mathcal{R}$*-algebra localization map* $R \to R_{\mathfrak{z}_R}$ *is an isomorphism. For this* $\mathfrak{z}_R$ *we have:*

- *The map* $\pi_{\mathfrak{z}_R} : \mathcal{R}_{\mathfrak{z}_R} \to R$ *is an isomorphism.*
- *For all* $I \in \mathfrak{F}$ *the map* $\pi_{I,\mathfrak{z}_R} : \mathcal{R}_x \twoheadrightarrow \mathcal{R}_{I,\mathfrak{z}_R}$*, induced by* $\pi_I : \mathcal{R} \twoheadrightarrow \mathcal{R}_I$ *is an isomorphism.*

We will call $\mathfrak{z}_R$ *the* prime (of $\mathcal{R}$) associated to $\mathfrak{F}$.

Finally, if $\psi : R \to R'$ *is a surjection of local rings, inducing the surjection* $\Psi : \mathcal{R} \twoheadrightarrow \mathcal{R}' := \prod_{n\geq 1} R'$*, and* $\mathfrak{z}_{R'} \in \operatorname{Spec} \mathcal{R}'$ *is the prime associated to* $\mathfrak{F}$*, then* $\mathfrak{z}_R = \Psi^{-1}(\mathfrak{z}_{A'})$.

Proof. Assume that there is some $\mathfrak{z}_R \in \operatorname{Spec} \mathcal{R}$ which makes $R \to R_{\mathfrak{z}_R}$ into an isomorphism. Clearly we must have $\ker(\pi : \mathcal{R} \to R) \subseteq \mathfrak{z}_R$, or we would have $R_{\mathfrak{z}_R} = 0$. Thus $\mathfrak{z}_R = \Psi^{-1}(P)$ for some $P \in \operatorname{Spec} R$ and $R_P \cong R_{\mathfrak{z}_R}$. But now as R is a local ring, $R \to R_P$ is an isomorphism if and only if $P = m_R^{\mathcal{U}}$. Thus the unique prime $\mathfrak{z}_R$ satisfying the condition is $\mathfrak{z}_R = \Psi^{-1}(m_A^{\mathcal{U}})$.

We now show that the map $\pi_{\mathfrak{z}_R} : \mathcal{R}_{\mathfrak{z}_R} \to R$ is an isomorphism. As localization is exact, it is surjective.

Take any $\frac{r}{s} \in \ker(\pi_{\mathfrak{z}_R})$ where $r = (r_1, r_2, \ldots) \in \mathcal{R}$. Then $r \in \ker(\pi)$ so that $[r_i]_i = 0$ in R, and hence $r_i = 0$ for $\mathfrak{F}$-many i. Define $e = (\varepsilon_1, \varepsilon_2, \ldots) \in \mathcal{R}$ by $\varepsilon_i = 1$ if $r_i = 0$ and $e_i = 0$ if $r_i \neq 0$, and note that $er = 0$. But by definition $e_i = 1$ for $\mathfrak{F}$-many i, and so $\pi(e) = 1 \notin m_R^{\mathcal{U}}$.

Hence $e \notin \mathfrak{z}_R$, and so $\frac{e}{1}$ is a unit in $\mathcal{R}_{\mathfrak{z}_R}$. As $\frac{e}{1}\frac{a}{s} = 0$, this implies that $\frac{a}{s} = 0$. Therefore $\ker(\pi_{\mathfrak{z}_R}) = 0$ and so indeed, $\pi_{\mathfrak{z}_R}$ is an isomorphism.

The last statement follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\pi} & R \\ \Psi \big\downarrow & & \big\downarrow \psi^{\mathcal{U}} \\ \mathcal{R}' & \xrightarrow[\pi']{} & R' \end{array}$$

□

From now on we will always use $\mathfrak{z}_R$ to denote the prime of $\mathcal{R}$ associated to $\mathfrak{F}$, or just $\mathfrak{z}$ if R is clear from context.

We will now investigate ultraproducts of R-modules (and R-algebras). Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be any sequence of R-modules, and write $\mathcal{M} = \prod_{n\geq 1} M_n$ with its natural $\mathcal{R}$-module structure. We claim that the natural surjection $\pi^{\mathscr{M}} : \mathcal{M} \twoheadrightarrow \mathcal{U}(\mathscr{M})$ is an $\mathcal{R}$-module homomorphism, where the $\mathcal{R}$-action on $\mathcal{U}(\mathscr{M})$ is given by $\pi : \mathcal{R} \twoheadrightarrow R$.

Indeed for any $r = (r_1, r_2, \ldots) \in \mathcal{R}$ and $m = (m_1, m_2, \ldots) \in \mathcal{M}$ we have $r_i = \pi(r)$ for $\mathfrak{F}$-many i, and so

$$\pi^{\mathscr{M}}(rm) = [r_i m_i]_i = [\pi(r) m_i]_i = \pi(r)[m_i]_i = \pi(r)\pi^{\mathscr{M}}(m).$$

If additionally the M_n's are A-*algebras*, then $\mathcal{U}(\mathscr{M})$ is an $\mathcal{R}$-algebra, and the above morphism is of $\mathcal{R}$-algebras.

Proposition A.1.4 now allows us to re-interpret $\pi^{\mathscr{M}}$ as a localization map of $\mathcal{R}$-modules:

Proposition A.1.5. *Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be a collection of R-modules and let $\mathcal{M}$ and $\pi^{\mathscr{M}} : \mathcal{M} \to \mathcal{U}(\mathscr{M})$ be as above. We have the following:*

1. *The map $\pi^{\mathscr{M}}_{\mathfrak{z}} : \mathcal{M}_{\mathfrak{z}} \to \mathcal{U}(\mathscr{M})_{\mathfrak{z}} = \mathcal{U}(\mathscr{M})$ is an isomorphism of $\mathcal{R}_{\mathfrak{z}} = R$-modules. If each M_n is an R-algebra then $\pi^{\mathscr{M}}_{\mathfrak{z}}$ is an isomorphism of R-algebras.*

2. *If* $\varphi = \{\varphi_i\}_{i\in I} : \mathscr{M} \to \mathscr{M}'$ *(for* $I \in \mathfrak{F}$*) is a* $\mathfrak{F}$*-morphism of sequences of* R*-modules, then the map* $\varphi^{\mathcal{U}} : \mathcal{U}(\mathscr{M}) \to \mathcal{U}(\mathscr{M}')$ *from Proposition A.1.1 is the localization of the map*

$$\Phi_I := \prod_{i\in I} \varphi_i : \prod_{i\in I} M_i \to \prod_{i\in I} M_i'$$

at $\mathfrak{Z}$*.*

3. *The functor* $\mathscr{M} \mapsto \mathcal{U}(\mathscr{M})$ *(from the category of sequences of* R*-modules, to the category of* R*-modules) is exact.*

Proof. As localization is exact, $\pi_{\mathfrak{Z}}^{\mathscr{M}}$ is surjective. Now arguing as in Proposition A.1.4, if $\frac{m}{s} \in \ker(\pi_{\mathfrak{Z}}^{\mathscr{M}})$ where $m = (m_1, m_2, \dots) \in \mathcal{M}$, then $[m_i]_i = 0$ in $\mathcal{U}(\mathscr{M})$ and hence $m_i = 0$ for all $i \in I$ for some $I \in \mathfrak{F}$. But then $m \in \ker(\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_I)$ and so $\frac{m}{s} \in \ker(\mathcal{M}_{\mathfrak{Z}} \to \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_{I,\mathfrak{Z}} = \mathcal{M}_{\mathfrak{Z}}) = 0$. So indeed, $\ker(\pi_{\mathfrak{Z}}^{\mathscr{M}}) = 0$, and so $\pi_{\mathfrak{Z}}^{\mathscr{M}}$ is an isomorphism of R-modules. If each M_n is an R-algebra then $\pi_{\mathfrak{Z}}^{\mathscr{M}}$ is also a homomorphism of R-algebras, and thus is an isomorphism of R-algebras. This proves (1).

For (2), note that $\mathcal{M}_I := \prod_{i\in I} \varphi_i : \prod_{i\in I} M_i = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_I$, and so $\mathcal{M}_{I,\mathfrak{Z}} = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_{I,\mathfrak{Z}} = \mathcal{M}_{\mathfrak{Z}}$, and similarly for $\mathcal{M}_I' := \prod_{i\in I} M_i'$. (2) then follows from localizing the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_I & \xrightarrow{\pi^{\mathscr{M}}} & \mathcal{U}(\mathscr{M}) \\ {\scriptstyle \Phi_I}\downarrow & & \downarrow{\scriptstyle \varphi^{\mathcal{U}}} \\ \mathcal{M}_I' & \xrightarrow[\pi^{\mathscr{M}'}]{} & \mathcal{U}(\mathscr{M}') \end{array}$$

Finally, (3) follows by noting that the functors $\{M_n\}_{n\geq 1} \mapsto \prod_{n\geq 1} M_n$ and $\mathcal{M} \mapsto \mathcal{M}_{\mathfrak{Z}}$ are both exact. □

A.2 The patching construction

We are now ready to give the main patching construction. Fix a complete DVR $\mathcal{O}$ with uniformizer λ and *finite* residue field $\mathbb{F} = \mathcal{O}/\lambda$ of characteristic ℓ. Also fix some $d \geq 1$, and consider the ring:

$$S_\infty := \mathcal{O}[[t_1, \dots, t_d]].$$

And let $\mathfrak{n} = (t_1, \dots, t_d) \subseteq S_\infty$. Note that S_∞ is a compact topological ring, so that $S_\infty/\mathfrak{a}$ is finite for all open ideals $\mathfrak{a} \subseteq S_\infty$.

The patching construction will take a sequence $\mathscr{M} = \{M_n\}_{n\geq 1}$ of finite type S_∞-modules satisfying certain properties, and produce a reasonably well behaved module $\mathscr{P}(\mathscr{M})$, which can be roughly though of as a "limit" of the M_n's.

We first make a precise definition of the sequences of S_∞-modules we will consider:

Definition A.2.1. Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be a sequence of finite type S_∞-modules.

- We say that $\mathscr{M}$ is a *weak patching system* if the S_∞-ranks of the M_n's are uniformly bounded.
- We say that $\mathscr{M}$ is a *patching system* if it is a weak patching system, and for any open ideal $\mathfrak{a} \subseteq S_\infty$, we have $\operatorname{Ann}_{S_\infty}(M_i) \subseteq \mathfrak{a}$ for all but finitely many n.
- We say that $\mathscr{M}$ is *free* if M_n is free over $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ for all but finitely many n.
- If M_0 is a finite type $\mathcal{O}$-module, we say that $\mathscr{M}$ is a *patching system over* M_0 if $\mathscr{M}$ is a patching system and we have $M_n/\mathfrak{n} \cong M_0$ for all but finitely many n.

Furthermore, assume that $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a sequence of finite type S_∞-*algebras*.

- We say that $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a *(weak) patching algebra*, if it is a (weak) patching system.
- We say that $\mathscr{R}$ is a *patching algebra over* R_0 if $R_n/\mathfrak{n} \cong R_0$ as $\mathcal{O}$-algebras for all but finitely many n.
- If M_n is an R_n-module (viewed as an S_∞-module via the S_∞-algebra structure on R_n) for all n we say that $\mathscr{M} = \{M_n\}_{n\geq 1}$ is a *(weak) patching $\mathscr{R}$-module* if it is a (weak) patching system.
- If $\mathscr{R}$ is a patching algebra over R_0 and M_0 is an R_0-module, we say that $\mathscr{M}$ is a *patching $\mathscr{R}$-module over* M_0 if $M_n/\mathfrak{n} \cong M_0$ as $R_n/\mathfrak{n} \cong R_0$-modules for all but finitely

many n.

Note that the collection of weak patching systems is an abelian category, with the obvious notion of morphism, as is the category of weak patching $\mathscr{R}$-modules for any weak patching algebra $\mathscr{R}$.

For any weak patching system $\mathscr{M}$ and any ideal $J \subseteq S_\infty$, we will write $\mathscr{M}/J := \{M_n/J\}_{n\geq 1}$.

If $\mathfrak{a} \subseteq S_\infty$ is open, note that each $M_n/\mathfrak{a}$ is a finite type $S_\infty/\mathfrak{a}$-module and the ranks of the $M_n/\mathfrak{a}$'s are bounded. As $S_\infty/\mathfrak{a}$ is finite, it follows that each $M_n/\mathfrak{a}$ is finite, and the cardinalities of the $M_n/\mathfrak{a}$'s are bounded. Proposition A.1.3 then implies that $\mathcal{U}(\mathscr{M}/\mathfrak{a}) \cong M_i/\mathfrak{a}$ as $S_\infty/\mathfrak{a}$-modules (and hence as S_∞-modules) for $\mathfrak{F}$-many i.

Now for any $\mathfrak{a}' \subseteq \mathfrak{a}$, the surjections $M_n/\mathfrak{a}' \twoheadrightarrow M_n/\mathfrak{a}$ induce a surjection $\mathcal{U}\big(\mathscr{M}/\mathfrak{a}'\big) \twoheadrightarrow \mathcal{U}(\mathscr{M}/\mathfrak{a})$. In fact, by the exactness of $\mathcal{U}(-)$, this surjection induces an isomorphism $\mathcal{U}\big(\mathscr{M}/\mathfrak{a}'\big)/\mathfrak{a} \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$ of S_∞-modules (or S_∞-algebras if $\mathscr{M}$ is a weak patching algebra).

Thus the $\mathcal{U}(\mathscr{M}/\mathfrak{a})$'s form an inverse system, and so we may make the following definition:

Definition A.2.2. For any weak patching system $\mathscr{M}$ define:

$$\mathscr{P}(\mathscr{M}) := \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{M}/\mathfrak{a})\,.$$

As $\mathcal{U}(-)$ is functorial, $\mathscr{P}(-)$ is functorial as well. For a morphism $f : \mathscr{M} \to \mathscr{N}$ of weak patching systems, let $f^{\mathscr{P}} : \mathscr{P}(\mathscr{M}) \to \mathscr{P}(\mathscr{N})$ denote the induced map.

Note that as each $\mathcal{U}(\mathscr{M}/\mathfrak{a})$ is an S_∞-module, $\mathscr{P}(\mathscr{M})$ is an S_∞-module. Similarly if $\mathscr{R}$ is a weak patching algebra then $\mathscr{P}(\mathscr{R})$ is an S_∞-algebra, and if $\mathscr{M}$ is a weak patching $\mathscr{R}$-module then $\mathscr{P}(\mathscr{M})$ is a $\mathscr{P}(\mathscr{R})$-module (with its S_∞-module structure induced from the S_∞-algebra structure on $\mathscr{P}(\mathscr{R})$).

In turns out that $\mathscr{P}$ is a reasonably well behaved functor. In fact:

Proposition A.2.3. *$\mathscr{P}$ is a right-exact functor.*

Proof. Let **Ab** be the category of abelian groups. For any index set I, let $\mathbf{finAb}^I$ be the category of inverse systems *finite* abelian groups indexed by I. We claim the the functor $\varprojlim : \mathbf{finAb}^I \to \mathbf{Ab}$ is exact.

By [Sta17, Tag 0598], it suffices to show that any $(A_i, f_{ji} : A_j \to A_i) \in \mathbf{finAb}^I$ satisfies the Mittag-Leffler condition: For any $i \in I$ there is a $j \geq i$ for which $\mathrm{im}(f_{ki}) = \mathrm{im}(f_{ji})$ for all $k \geq j$.

But as A_i is finite, it has only finitely many subgroups and so the collection $\{\mathrm{im}(f_{ji})\}_{j\geq i}$ of subgroups of A_i must have some minimal member, $\mathrm{im}(f_{ji})$, under inclusion. Then for any $k \geq j$, $\mathrm{im}(f_{ki}) = \mathrm{im}(f_{ji} \circ f_{kj}) \subseteq \mathrm{im}(f_{ji})$ and hence $\mathrm{im}(f_{ki}) = \mathrm{im}(f_{ji})$. So indeed every object of $\mathbf{finAb}^I$ satisfies the Mittag-Leffler condition, and so $\varprojlim$ is exact.

Now assume $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are weak patching systems, and we have an exact sequence

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$$

Then for any $\mathfrak{a} \subseteq S_\infty$, $\mathscr{A}/\mathfrak{a} \to \mathscr{B}/\mathfrak{a} \to \mathscr{C}/\mathfrak{a} \to 0$ is right-exact, so by the exactness of $\mathcal{U}(-)$ we get the right exact sequence

$$\mathcal{U}(\mathscr{A}/\mathfrak{a}) \to \mathcal{U}(\mathscr{B}/\mathfrak{a}) \to \mathcal{U}(\mathscr{C}/\mathfrak{a}) \to 0.$$

Thus we have a right-exact sequence of complexes

$$\big(\mathcal{U}(\mathscr{A}/\mathfrak{a})\big)_{\mathfrak{a}} \to \big(\mathcal{U}(\mathscr{B}/\mathfrak{a})\big)_{\mathfrak{a}} \to \big(\mathcal{U}(\mathscr{C}/\mathfrak{a})\big)_{\mathfrak{a}} \to 0$$

But now as $\mathcal{U}(\mathscr{A}/\mathfrak{a})$, $\mathcal{U}(\mathscr{B}/\mathfrak{a})$ and $\mathcal{U}(\mathscr{C}/\mathfrak{a})$ are all finite, the above argument shows that taking inverse limits preserves exactness, and so indeed

$$\mathscr{P}(\mathscr{A}) \to \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathscr{C}) \to 0$$

is right-exact. $\square$

Note that in general, $\mathscr{P}$ is not left-exact. Indeed, letting $\mathscr{M} = \{S_\infty\}_{n\geq 1}$ and defining $\varphi : \mathscr{M} \hookrightarrow \mathscr{M}$ via $\varphi_n(s) = \lambda^n s$ we see that $\mathscr{P}(\mathscr{M}) = S_\infty$ and $\varphi^{\mathscr{P}} : S_\infty \to S_\infty$ is the zero map (since for any $\mathfrak{a}$, $\varphi_{n,\mathfrak{a}} : S_\infty/\mathfrak{a} \to S_\infty/\mathfrak{a}$ is the zero map for all but finitely many n).

Now Proposition A.2.3, and Definition A.2.1 easily imply the following basic properties:

Proposition A.2.4. *If $\mathscr{M} = \{M_n\}_{n\geq 1}$ is a patching system then:*

1. *$\mathscr{P}(\mathscr{M})$ is a finite type, faithful S_∞-module.*
2. *For any open ideal $\mathfrak{a} \subseteq S_\infty$, $\mathscr{P}(\mathscr{M})/\mathfrak{a} \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$.*
3. *If $\mathscr{M}$ is a patching system over M_0, then $\mathscr{P}(\mathscr{M})/\mathfrak{n} \cong M_0$.*
4. *If $\mathscr{M}$ is free, then $\mathscr{P}(\mathscr{M})$ is a finite type free S_∞-module.*
5. *If M is a finite type S_∞-module and $\underline{M} = \{M\}_{n\geq 1}$ is the constant weak patching system, then there is a natural isomorphism $\mathscr{P}(\underline{M}) \cong M$.*

Proof. First, for (5), $M/\mathfrak{a}$ is finite for all open $\mathfrak{a} \subseteq S_\infty$ and so Proposition A.1.3 gives a natural isomorphism $(M/\mathfrak{a})^{\mathcal{U}} \cong M/\mathfrak{a}$. Thus

$$\mathscr{P}(\underline{M}) = \varprojlim_{\mathfrak{a}} \mathcal{U}\big(\underline{M/\mathfrak{a}}\big) = \varprojlim_{\mathfrak{a}} (M/\mathfrak{a})^{\mathcal{U}} \cong \varprojlim_{\mathfrak{a}} M/\mathfrak{a} \cong M$$

as any finite type S_∞-module is complete.

Now if $J \subseteq S_\infty$ is any ideal, then Proposition A.2.3 implies that $\mathscr{P}(\mathscr{M})/J \cong \mathscr{P}(\mathscr{M}/J)$.

Letting $J = \mathfrak{a} \subseteq S_\infty$ open, we have $(\mathscr{M}/\mathfrak{a})/\mathfrak{a}' \cong \mathscr{M}/\mathfrak{a}$ for all $\mathfrak{a}' \subseteq \mathfrak{a}$ and so $\mathcal{U}\big((\mathscr{M}/\mathfrak{a})/\mathfrak{a}'\big) \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$. Thus we have

$$\mathscr{P}(\mathscr{M})/\mathfrak{a} \cong \mathscr{P}(\mathscr{M}/\mathfrak{a}) = \varprojlim_{\mathfrak{a}'} \mathcal{U}\big((\mathscr{M}/\mathfrak{a})/\mathfrak{a}'\big) = \lim_{\mathfrak{a}'\subseteq\mathfrak{a}} \mathcal{U}\big((\mathscr{M}/\mathfrak{a})/\mathfrak{a}'\big) \cong \lim_{\mathfrak{a}'\subseteq\mathfrak{a}} \mathcal{U}(\mathscr{M}/\mathfrak{a}) \cong \mathcal{U}(\mathscr{M}/\mathfrak{a}),$$

proving (2).

Assuming that $\mathscr{M}$ is a patching system over M_0 and letting $J = \mathfrak{n}$, we get

$$\mathscr{P}(\mathscr{M})/\mathfrak{n} \cong \mathscr{P}(\mathscr{M}/\mathfrak{n}) \cong \mathscr{P}\left(\underline{M_0}\right) \cong M_0,$$

proving (3).

Now as the S_∞-ranks of of the M_n's are bounded, say by some $N \geq 1$, there is a surjection of patching systems $\underline{{S_\infty}^N} \twoheadrightarrow \mathscr{M}$ and hence a surjection ${S_\infty}^N = \mathscr{P}(\underline{{S_\infty}^N}) \twoheadrightarrow \mathscr{P}(\mathscr{M})$. Thus $\mathscr{P}(\mathscr{M})$ is a finite type S_∞-module.

Now for any open $\mathfrak{a} \subseteq S_\infty$, we have $\operatorname{Ann}_{S_\infty}(M_n) \subseteq \mathfrak{a}$ for all but finitely many n, by assumption, and hence $\operatorname{Ann}_{S_\infty}(M_n/\mathfrak{a}) = \mathfrak{a}$ for all such n. Thus $\operatorname{Ann}_{S_\infty}(\mathcal{U}(\mathscr{M}/\mathfrak{a})) = \mathfrak{a}$. But now we have

$$\operatorname{Ann}_{S_\infty}(\mathscr{P}(\mathscr{M})) \subseteq \operatorname{Ann}_{S_\infty}(\mathscr{P}(\mathscr{M})/\mathfrak{a}) = \operatorname{Ann}_{S_\infty}(\mathcal{U}(\mathscr{M}/\mathfrak{a})) = \mathfrak{a}$$

for all $\mathfrak{a}$, and so $\operatorname{Ann}_{S_\infty}(\mathscr{P}(\mathscr{M})) = 0$, proving (1).

Lastly, assume that $\mathscr{M}$ is free. Then for all but finitely many n, $M_n \cong (S_\infty/\operatorname{Ann}_{S_\infty}(M_n))^{r_n}$ for some r_n. As there r_n's are bounded, there is some r such that $r_i = r$, and hence $M_i \cong (S_\infty/\operatorname{Ann}_{S_\infty}(M_i))^r$, for $\mathfrak{F}$-many i.

Define an $\mathfrak{F}$-morphism $\varphi : \underline{{S_\infty}^r} \to \mathscr{M}$ by letting $\varphi_i : {S_\infty}^r \twoheadrightarrow (S_\infty/\operatorname{Ann}_{S_\infty}(M_i))^r \cong M_i$ for $\mathfrak{F}$-many i. Then for any open $\mathfrak{a} \subseteq S_\infty$, $\overline{\varphi_{i,\mathfrak{a}}} : {S_\infty}^r/\mathfrak{a} \to M_i/\mathfrak{a}$ is an isomorphism for $\mathfrak{F}$-many i, and so φ induces an isomorphism $\mathcal{U}\big(\underline{(S_\infty/\mathfrak{a})^r}\big) \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$ for all $\mathfrak{a}$, and thus an isomorphism $\varphi^{\mathscr{P}} : {S_\infty}^r = \mathscr{P}\left(\underline{{S_\infty}^r}\right) \to \mathscr{P}(\mathscr{M})$ is an isomorphism. So indeed, $\mathscr{P}(\mathscr{M})$ is a finite type, free S_∞-module, proving (4). $\square$

The following simple consequence of Proposition A.2.4 is central to the classical applications of this theory:

Corollary A.2.5. *If $\mathscr{R}$ is a weak patching algebra and $\mathscr{M}$ is a patching $\mathscr{R}$-module, then the homomorphism $S_\infty \to \mathscr{P}(\mathscr{R})$ inducing the S_∞-algebra structure on $\mathscr{P}(\mathscr{R})$ is injective and* *the Krull dimension of $\mathscr{P}(\mathscr{R})$ is $d+1$ $(= \dim S_\infty)$.*

If, furthermore, $\mathscr{M}$ is a free *patching $\mathscr{R}$-module then $\mathscr{P}(\mathscr{M})$ is a maximal Cohen–Macaulay module over $\mathscr{P}(\mathscr{R})$.*

Finally if $\mathscr{P}(\mathscr{R})$ is a Cohen–Macaulay ring, then $(t_1, \dots, t_d, \lambda) \subseteq S_\infty \subseteq \mathscr{P}(\mathscr{R})$ is a regular sequence for $\mathscr{P}(\mathscr{R})$.

Proof. For the first statement, the map $\iota : S_\infty \to \mathscr{P}(\mathscr{R})$ induces the S_∞-module structure on $\mathscr{P}(\mathscr{M})$, which is *faithful* by Proposition A.2.4(1), and so ι must be injective.

As $\mathscr{P}(\mathscr{R})$ is of finite type over S_∞, it follows that the Krull dimension of $\mathscr{P}(\mathscr{R})$ is just $\dim \mathscr{P}(\mathscr{R}) = \dim \iota(S_\infty) = \dim S_\infty = d+1$.

Now assume that $\mathscr{M}$ is free. By Proposition A.2.4(4), $\mathscr{P}(\mathscr{M})$ is finite free over $\iota(S_\infty) \cong S_\infty$ and so $\mathscr{P}(\mathscr{M})$ is indeed Cohen–Macaulay of dimension $d+1 = \dim S_\infty$. In particular $(t_1, \dots, t_d, \lambda) \subseteq S_\infty \subseteq \mathscr{P}(\mathscr{R})$ is a regular sequence.

Now if $\mathscr{P}(\mathscr{R})$ is also Cohen–Macaulay, then a sequence is regular for $\mathscr{P}(\mathscr{M})$ if and only if it is regular for $\mathscr{P}(\mathscr{R})$, so the last statement follows. □

A.3 Covers of Patching Algebras

In the classical setup of Taylor–Wiles–Kisin patching, one considers a patching algebra $\mathscr{R} = \{R_n\}_{n\geq 1}$, where the R_n's are all taken to be quotients of a fixed ring R_∞. We thus make a the following definition:

Definition A.3.1. If $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a weak patching algebra we say that a *cover* $(R_\infty, \{\varphi_n\})$ of $\mathscr{R}$ is:

- A complete, Cohen–Macaulay ring R_∞, which is topologically finitely generated as an $\mathcal{O}$-algebra, of Krull dimension $d+1$ $(= \dim S_\infty)$ together with:
- For each n, a continuous, surjective $\mathcal{O}$-algebra homomorphism $\varphi_n : R_\infty \twoheadrightarrow R_n$.

We will often use R_∞ to denote the cover $(R_\infty, \{\varphi_n\})$.

Note that:

Lemma A.3.2. *If $(R_\infty, \{\varphi_n\})$ is a cover of a weak patching algebra $\mathscr{R}$, then the φ_n's induce a natural continuous surjection $\varphi_\infty : R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R})$.*

Proof. The φ_n's induce a continuous map $\Phi = \prod_{n\geq 1} \varphi_n : R_\infty \to \prod_{n\geq 1} R_n$, and thus induce continuous maps

$$\Phi_{\mathfrak{a}} : R_\infty \xrightarrow{\Phi} \prod_{n\geq 1} R_n \twoheadrightarrow \prod_{n\geq 1} (R_n/\mathfrak{a}) \twoheadrightarrow \mathcal{U}(\mathscr{R}/\mathfrak{a})$$

for all open $\mathfrak{a} \subseteq S_\infty$, and thus they indeed induce a continuous map

$$\varphi_\infty = (\Phi_{\mathfrak{a}})_{\mathfrak{a}} : R_\infty \to \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{R}/\mathfrak{a}) = \mathscr{P}(\mathscr{R}).$$

We now claim that each $\Phi_{\mathfrak{a}}$ is surjective. As each map $R_\infty \xrightarrow{\varphi_n} R_n \twoheadrightarrow R_n/\mathfrak{a}$ is continuous, we may give each $R_n/\mathfrak{a}$ the structure of a continuous R_∞-algebra. Then the map $\Phi_{\mathfrak{a}} : R_\infty \to \mathcal{U}(\mathscr{R}/\mathfrak{a})$ defines the continuous R_∞-algebra structure on $\mathcal{U}(\mathscr{R}/\mathfrak{a})$ from Proposition A.1.1. By Proposition A.1.3, $\mathcal{U}(\mathscr{R}/\mathfrak{a}) \cong R_i/\mathfrak{a}$ as R-algebras for $\mathfrak{F}$-many i. But for any such i, the map $R_\infty \xrightarrow{\varphi_i} R_i \twoheadrightarrow R_i/\mathfrak{a}$ defining the R_∞-algebra structure is surjective, and so $\Phi_{\mathfrak{a}} : R_\infty \to \mathcal{U}(\mathscr{R}/\mathfrak{a})$ must indeed be surjective.

It follows that $\varphi_\infty(R_\infty) \subseteq \mathscr{P}(\mathscr{R})$ is *dense*. But now as R_∞ is topologically finitely generated over $\mathcal{O}$, it is compact, and so $\varphi_\infty(R_\infty)$ is also closed in $\mathscr{P}(\mathscr{R})$. Therefore φ_∞ is indeed surjective. □

We will say that the cover R_∞ is *minimal* if φ_∞ is an isomorphism.

Lemma A.3.2 and Corollary A.2.5 give the following useful result:

Corollary A.3.3. *If $\mathscr{R}$ is a weak patching algebra with a cover R_∞, and $\mathscr{M}$ is any* free *patching $\mathscr{R}$-module, then $\mathscr{P}(\mathscr{M})$ is maximal Cohen–Macaulay over R_∞.*

Proof. By Lemma A.3.2, $\mathscr{P}(\mathscr{R})$ may be thought of as a quotient of R_∞. From the definition of Cohen–Macaulay modules, if $f : A \twoheadrightarrow B$ is any surjective map of rings, and M is a B-module, then M is Cohen–Macaulay over B if and only if it is Cohen–Macaulay over A. Thus by Corollary A.2.5, $\mathscr{P}(\mathscr{M})$ is Cohen–Macaulay over R_∞.

Furthermore, by Corollary A.2.5 and Definition A.3.1, we have $\dim R_\infty = d+1 = \dim \mathscr{P}(\mathscr{R}) = \dim \mathscr{P}(\mathscr{M})$, so $\mathscr{P}(\mathscr{M})$ is *maximal* Cohen–Macaulay over R_∞. □

We can now prove the main result of this appendix:

Theorem A.3.4. *Let R_0 be a finite type $\mathcal{O}$-algebra and let M_0 be a nonzero R_0-module, which is finite type and free over $\mathcal{O}$. Assume that we are given:*

- *A weak patching algebra $\mathscr{R} = \{R_n\}_{n\geq 1}$ over R_0;*
- *A* free *patching $\mathscr{R}$-module $\mathscr{M} = \{M_n\}_{n\geq 1}$ over M_0;*
- *A cover R_∞ of $\mathscr{R}$, where R_∞ is a* domain.

Then we have the following:

1. *R_∞ is a minimal cover.*
2. *$R_\infty \cong \mathscr{P}(\mathscr{R})$ is Cohen–Macaulay, $(t_1, \dots, t_d, \lambda) \subseteq R_\infty$ is a regular sequence for R_∞, $R_\infty/\mathfrak{n} \cong R_0$ and R_0 is Cohen–Macaulay and λ-torsion free.*
3. *If η is any generic point of* $\operatorname{Spec} R_0$*, with function field $\kappa(\eta)$ (i.e. $\kappa(\eta)$ is the field of fractions of R_0/η), then*

$$\dim_{\kappa(\eta)} M_0 \otimes_{R_0} \kappa(\eta) \geq \dim_{K(R_\infty)} \mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(R_\infty) \geq 1,$$

where $K(R_\infty)$ is the fraction field of R_∞. In particular, $\dim_{K(\eta)} M_0 \otimes_{R_0} K(\eta)$ *is always nonzero.*

Proof. Let $\varphi_\infty : R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R})$ be the map from Lemma A.3.2. If φ_∞ is not injective, then $\dim \mathscr{P}(\mathscr{R}) = \dim R_\infty/\ker\varphi_\infty < \dim R_\infty = d+1$, contradicting Corollary A.2.5, and so φ_∞ is indeed an isomorphism, proving (1).

Now $\mathscr{P}(\mathscr{R}) \cong R_\infty$ is Cohen–Macaulay by our assumptions on R_∞, so Corollary A.2.5 implies that $(t_1, \dots, t_d, \lambda) \subseteq R_\infty$ is an R_∞-regular sequence. Also Proposition A.2.4 implies that $R_\infty/(t_1, \dots, t_d) = R_\infty/\mathfrak{n} \cong R_0$. Now by the definition of regular sequences, it follows that R_0 is Cohen–Macaulay and (λ) is an R_0-regular sequence, which implies that R_0 is λ-torsion free. This proves (2).

It remains to prove (3). Let $K(S)$ be the fraction field of S_∞. Then $R_\infty \otimes_{S_\infty} K(S_\infty)$ is a finite-dimensional $K(S_\infty)$ algebra which is still a domain, so it follows that $R_\infty \otimes_{S_\infty} K(S_\infty)$ is a field. In particular $R_\infty \otimes_{S_\infty} K(S_\infty) \cong K(R_\infty)$. It follows from this that

$$\mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(R_\infty) \cong \mathscr{P}(\mathscr{M}) \otimes_{S_\infty} K(S_\infty).$$

As $\mathscr{P}(\mathscr{M})$ is free over S_∞ (and nonzero) it follows that $\mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(R_\infty) \neq 0$ and so

$$\dim_{K(R_\infty)} \mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(R_\infty) \geq 1.$$

For any $P \in \operatorname{Spec} R_\infty$, let $\kappa(P)$ be the residue field of P (that is, the field of fractions of R_∞/P). As $\mathscr{P}(\mathscr{M})$ is a finite type R_∞ algebra, the map $P \mapsto \dim_{\kappa(P)} \mathscr{P}(M) \otimes_{R_\infty} \kappa(P)$ is upper semi-continuous on $\operatorname{Spec} R_\infty$ and so in particular

$$\dim_{\kappa(P)} \mathscr{P}(M) \otimes_{R_\infty} \kappa(P) \geq \dim_{K(R_\infty)} \mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(R_\infty) \geq 1$$

for any P, as $\operatorname{Spec} R_\infty$ is irreducible.

Now consider the quotient map $g : R_\infty \twoheadrightarrow R_\infty/\mathfrak{n} \cong R_0$ and let $P_\eta = g^{-1}(\eta) \in \operatorname{Spec} R_\infty$. As $\mathfrak{n} = \ker g$, we get that $\mathfrak{n} \subseteq P_\eta$. It follows that $R_\infty/P_\eta \cong (R_\infty/\mathfrak{n})/(P_\eta/\mathfrak{n}) \cong R_0/\eta$, so that $\kappa(P_\eta) \cong \kappa(\eta)$ and

$$\mathscr{P}(\mathscr{M}) \otimes_{R_\infty} \kappa(P_\eta) \cong (\mathscr{P}(\mathscr{M})/\mathfrak{n}) \otimes_{R_\infty/\mathfrak{n}} \kappa(\eta) \cong M_0 \otimes_{R_0} \kappa(\eta),$$

and so the above result gives (3). □

A.4 Duality

In order to prove our main result, we will also need to show that patching preserves duality. Over an arbitrary ring, duality can be tricky to define, so we will restrict ourselves to considering patching systems $\mathscr{M} = \{M_n\}_{n\geq 1}$ where the rings $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ are well behaved. Specifically we make the following definition:

Definition A.4.1. Let $\mathscr{R} = \{R_n\}_{n\geq 1}$ be a weak patching algebra, and let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be a patching $\mathscr{R}$-module. We say that $\mathscr{M}$ is *dualizable* if it is free and each $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ is a local complete intersection[1].

If $\mathscr{M}$ is dualizable, we define its dual to be the patching $\mathscr{R}$-module

$$\mathscr{M}^* = \{M_n^*\}_{n\geq 1} := \left\{\operatorname{Hom}_{S_\infty}(M_n, S_\infty/\operatorname{Ann}_{S_\infty}(M_n))\right\}_{n\geq 1},$$

where for each n, R_n acts on $M_n^* := \operatorname{Hom}_{S_\infty}(M_n, S_\infty/\operatorname{Ann}_{S_\infty}(M_n))$ by $(rf)(x) = f(rx)$.

The following lemma shows that the results of this subsection will be applicable to the specific patching systems we consider in Section 4.

Lemma A.4.2. *Let $d' \leq d$ be an integer and let*

$$S_\infty{}' := \mathcal{O}[[t_1, \ldots, t_{d'}]] \subseteq \mathcal{O}[[t_1, \ldots, t_d]] = S_\infty.$$

Assume that for each integer $n \geq 1$ we are given:

- *A finite type $S_\infty{}'$-algebra R_n, such that the $S_\infty{}'$-ranks of the R_n's are bounded.*
- *Finite type R_n-modules M_n and N_n, whose $S_\infty{}'$-ranks are again bounded.*
- *Integers $e(n,1), e(n,2), \ldots, e(n,d') \geq n$ such that M_n and N_n are free over the ring:*

$$\mathcal{O}[\Delta_n] := S_\infty{}'/I_n := \frac{\mathcal{O}[[t_1, \ldots, t_{d'}]]}{\left((1+t_1)^{\ell^{e(n,1)}} - 1, (1+t_2)^{\ell^{e(n,2)}} - 1, \ldots, (1+t_{d'})^{\ell^{e(n,d')}} - 1\right)}$$

(where the $\mathcal{O}[\Delta_n]$ action is induced by the $S_\infty{}'$ actions on M_n and N_n).

1. We could, with a little extra work, weaken this condition to only requiring that $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ be Cohen–Macaulay, but as the rings $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ are always complete intersections in practice, there is no advantage to doing this.

Define:

$$R_n^{\square} := R_n[[t_{d'+1}, \dots, t_d]] = R_n \otimes_{\mathcal{O}} \mathcal{O}[[t_{d'+1}, \dots, t_d]] = R_n \otimes_{{S_\infty}'} S_\infty$$
$$M_n^{\square} := M_n[[t_{d'+1}, \dots, t_d]] = M_n \otimes_{\mathcal{O}} \mathcal{O}[[t_{d'+1}, \dots, t_d]] = M_n \otimes_{{S_\infty}'} S_\infty$$
$$N_n^{\square} := R_n[[t_{d'+1}, \dots, t_d]] = N_n \otimes_{\mathcal{O}} \mathcal{O}[[t_{d'+1}, \dots, t_d]] = N_n \otimes_{{S_\infty}'} S_\infty$$

and let $\mathscr{R} = \{R_n^{\square}\}_{n\geq 1}$, $\mathscr{M} = \{M_n^{\square}\}_{n\geq 1}$ *and* $\mathscr{N} = \{N_n^{\square}\}_{n\geq 1}$. *Then:*

1. *$\mathscr{R}$ is a weak patching algebra, and $\mathscr{M}$ and $\mathscr{N}$ are free patching $\mathscr{R}$-algebras.*
2. *$\mathscr{M}$ and $\mathscr{N}$ are dualizable.*

Furthermore, if, for each $n \geq 1$, we are given an R_n-equivariant perfect pairing $\langle\ ,\ \rangle :$ $M_n \times N_n \to \mathcal{O}$, then we have $\mathscr{M}^* \cong \mathscr{N}$.

Proof. As the ${S_\infty}'$-ranks of the R_n's, M_n's and N_n's are bounded, so are the S_∞-ranks of the $R_n^{\square}$'s, $M_n^{\square}$'s and $N_n^{\square}$'s. Thus $\mathscr{R}$ is a weak patching algebra and $\mathscr{M}$ and $\mathscr{N}$ are weak patching $\mathscr{R}$-modules.

The show that M_n and N_n are patching $\mathscr{R}$-modules, we must show that for any open ideal $\mathfrak{a} \subseteq S_\infty$, $\operatorname{Ann}_{S_\infty}(M_n), \operatorname{Ann}_{S_\infty}(N_n) \subseteq \mathfrak{a}$ for all but finitely many n. But by assumption, we have $\operatorname{Ann}_{S_\infty}(M_n) = \operatorname{Ann}_{S_\infty}(N_n) = I_n$ for all n (where I_n is now interpreted as an ideal of S_∞). As $S_\infty/\mathfrak{a}$ is finite, and the group $1 + m_{S_\infty}$ is pro-ℓ, the group $(1 + m_{S_\infty})/\mathfrak{a} :=$ $\operatorname{im}(1+m_{S_\infty} \hookrightarrow S_\infty \twoheadrightarrow S_\infty/\mathfrak{a})$ is a finite ℓ-group. Since $1+t_i \in 1+m_{S_\infty}$ for all i, there is an integer $K \geq 0$ such that $(1 + t_i)^{\ell^K} \equiv 1 \pmod{\mathfrak{a}}$ for all $i = 1, \dots, d'$. Then for any $n \geq K$, $e(n, i) \geq n \geq K$ for all i, and so indeed $I_n \subseteq \mathfrak{a}$ by definition.

Also by assumption, M_n and N_n are free over ${S_\infty}'/I_n = \mathcal{O}[\Delta_n]$, and so $M_n^{\square}$ and $N_n^{\square}$ are free over $\mathcal{O}[\Delta_n] \otimes_{{S_\infty}'} S_\infty = \mathcal{O}[\Delta][[t_{d'+1}, \dots, t_d]] = S_\infty / \operatorname{Ann}_{S_\infty}(M_n^{\square}) = S_\infty / \operatorname{Ann}_{S_\infty}(N_n^{\square})$, so $\mathscr{M}$ and $\mathscr{N}$ are free patching $\mathscr{R}$-modules, proving (1).

For (2), simply note that $\mathcal{O}[\Delta_n]$ is a finite free $\mathcal{O}$-module, and so $\dim \mathcal{O}[\Delta] = 1 =$ $\dim {S_\infty}' - d'$. As I_n is generated by d' elements, it follows that $\mathcal{O}[\Delta_n]$ is local complete intersection. Thus $S_\infty / \operatorname{Ann}_{S_\infty}(M_n^{\square}) = S_\infty / \operatorname{Ann}_{S_\infty}(N_n^{\square}) = \mathcal{O}[\Delta][[t_{d'+1}, \dots, t_d]]$ is also a local complete intersection, and so $\mathscr{M}$ and $\mathscr{N}$ are indeed dualizable.

It remains to show that $\mathscr{M}^* \cong \mathscr{N}$, that is that

$$N_n^{\square} \cong \mathrm{Hom}_{S_\infty}(M_n, S_\infty/I_n) = \mathrm{Hom}_{S_\infty/I_n}(M_n, S_\infty/I_n)$$

as $R_n^{\square}$-modules for all n. As we are given an R_n-equivariant perfect pairing $M_n \times N_n \to \mathcal{O}$ we have that $N_n \cong \mathrm{Hom}_{\mathcal{O}}(M_n, \mathcal{O})$ as R_n-modules. As M_n is finite free over $\mathcal{O}$, it now follows that

$$N_n^{\square} = N_n \otimes_{\mathcal{O}} \mathcal{O}[[t_{d'+1}, \ldots, t_d]] \cong \mathrm{Hom}_{\mathcal{O}[[t_{d'+1},\ldots,t_d]]}\left(M_n^{\square}, \mathcal{O}[[t_{d'+1}, \ldots, t_d]]\right)$$

as $R_n^{\square} = R_n \otimes_{\mathcal{O}} \mathcal{O}[[t_{d'+1}, \ldots, t_d]]$-modules.

Now as $S_\infty/I_n = \mathcal{O}[\Delta_n][[t_{d'+1}, \ldots, t_d]]$ is a local complete intersection of the same dimension as $\mathcal{O}[[t_{d'+1}, \ldots, t_d]]$, the claim follows from the following commutative algebra lemma. $\square$

Lemma A.4.3. *If A is a local Cohen–Macaulay ring and B is an A-algebra which is also Cohen–Macaulay with* $\dim A = \dim B$*, then for and B-module M,*

$$\mathrm{Hom}_A(M, \omega_A) \cong \mathrm{Hom}_B(M, \omega_B)$$

as left $\mathrm{End}_B(M)$*-modules.*

Proof. By [Sta17, Tag 08YP] there is an isomorphism

$$\mathrm{Hom}_A(M, \omega_A) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, \omega_A))$$

sending $\alpha : M \to \omega_A$ to $\alpha' : m \mapsto (b \mapsto \alpha(bm))$, which clearly preserves the action of $\mathrm{End}_B(M)$ (as $(\alpha \circ \psi)(bm) = \alpha(b\psi(m))$ for any $\psi \in \mathrm{End}_B(M)$). It remains to show that $\mathrm{Hom}_A(B, \omega_A) \cong \omega_B$, which is just Theorem 21.15 from [Eis95] in the case $\dim A = \dim B$.

□

We are now ready to show that patching preserves duality:

Theorem A.4.4. *Let $\mathscr{R}$ be a weak patching algebra and let $\mathscr{M}$ be a dualizable patching $\mathscr{R}$-module. Then we have $\mathscr{P}(\mathscr{M}^*) \cong \operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)$ as $\mathscr{P}(\mathscr{R})$-modules.*

Furthermore, if R_∞ is a cover of $\mathscr{R}$ (which is assumed to be Cohen–Macaulay by our definition of cover) then $\mathscr{P}(\mathscr{M}^*) \cong \operatorname{Hom}_{R_\infty}(\mathscr{P}(\mathscr{M}), \omega_{R_\infty})$ as R_∞-modules.

Proof. We shall first compute $\mathcal{U}(\mathscr{M}^*/\mathfrak{a})$ for any open ideal $\mathfrak{a} \subseteq S_\infty$. For any such $\mathfrak{a}$, we have $\operatorname{Ann}_{S_\infty}(M_n) \subseteq \mathfrak{a}$ for all but finitely many n, and so $S_\infty/\mathfrak{a}$ is a $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ for all such n.

But now for all sufficiently large n, M_n is finite free over $S_\infty/\operatorname{Ann}_{S_\infty}(M_n)$ by assumption, and so it is projective. Thus the functor $\operatorname{Hom}_{S_\infty}(M_n, -) = \operatorname{Hom}_{S_\infty/\operatorname{Ann}_{S_\infty}(M_n)}(M_n, -)$ is exact and so if $\operatorname{Ann}_{S_\infty}(M_n) \subseteq \mathfrak{a}$ then

$$M_n^*/\mathfrak{a} = \operatorname{Hom}_{S_\infty}(M_n, S_\infty/\operatorname{Ann}_{S_\infty}(M_n))/\mathfrak{a} \cong \operatorname{Hom}_{S_\infty}(M_n, S_\infty/\mathfrak{a}) = \operatorname{Hom}_{S_\infty/\mathfrak{a}}(M_n/\mathfrak{a}, S_\infty/\mathfrak{a})$$

as $R_n/\mathfrak{a}$-modules.

Now by Proposition A.1.3, for $\mathfrak{F}$-many i we have that $\mathcal{U}(\mathscr{R}/\mathfrak{a}) \cong R_i/\mathfrak{a}$ and $\mathcal{U}(\mathscr{M}/\mathfrak{a}) \cong M_i/\mathfrak{a}$ and $\mathcal{U}(\mathscr{M}^*/\mathfrak{a}) \cong M_i^*/\mathfrak{a}$ as $R_i/\mathfrak{a}$-modules. Taking any such i, the above computation gives that

$$\mathcal{U}(\mathscr{M}^*/\mathfrak{a}) \cong \operatorname{Hom}_{S_\infty/\mathfrak{a}}(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a})$$

as $\mathcal{U}(\mathscr{R}/\mathfrak{a})$-modules. Taking inverse limits, it now follows that

$$\mathscr{P}(\mathscr{M}^*) \cong \varprojlim_{\mathfrak{a}} \operatorname{Hom}_{S_\infty/\mathfrak{a}}(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a})$$

as $\mathscr{P}(\mathscr{R})$-modules. It remains to show that the right hand side is just $\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)$. But using the fact that $\mathscr{P}(\mathscr{M})$, and thus $\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)$ is a finite free S_∞-module (and thus is m_{S_∞}-adically complete) we get that

$$\mathrm{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty) \cong \varprojlim_{\mathfrak{a}} \mathrm{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)/\mathfrak{a}$$

as $\mathscr{P}(\mathscr{R}) = \varprojlim_{\mathfrak{a}} \mathscr{P}(\mathscr{R})/\mathfrak{a}$-modules. But now for any $\mathfrak{a}$, as $\mathscr{P}(\mathscr{M})$ is a finite free, and hence projective, S_∞-module

$$\mathrm{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)/\mathfrak{a} \cong \mathrm{Hom}_{S_\infty/\mathfrak{a}}(\mathscr{P}(\mathscr{M})/\mathfrak{a}, S_\infty/\mathfrak{a}) \cong \mathrm{Hom}_{S_\infty/\mathfrak{a}}\left(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a}\right)$$

as $\mathscr{P}(\mathscr{R})/\mathfrak{a} = \mathcal{U}(\mathscr{R}/\mathfrak{a})$-modules. So indeed

$$\mathrm{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty) \cong \varprojlim_{\mathfrak{a}} \mathrm{Hom}_{S_\infty/\mathfrak{a}}\left(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a}\right) \cong \mathscr{P}(\mathscr{M}^*)$$

as $\mathscr{P}(\mathscr{R})$-modules.

Now assume that $(R_\infty, \{\varphi_n\}_n)$ is a cover of $\mathscr{R}$ and let $\varphi_\infty : R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R})$ be the map from Lemma A.3.2. We first note that we may pick an embedding $\iota : S_\infty \hookrightarrow R_\infty$, giving R_∞ the structure of an S_∞-algebra, which makes the map φ_∞ into an S_∞-algebra homomorphism. Indeed as φ_∞ is surjective, for each $i = 1, \dots, d$ we may simply pick a lift $\iota(t_i)$ with $\varphi_\infty(\iota(t_i)) = t_i$ (where we treat S_∞ as a subring of $\mathscr{P}(\mathscr{R})$) so that $\varphi_\infty \circ \iota = \mathrm{id}$.

It now follows that $\mathscr{P}(\mathscr{M})$ is an R_∞-module, and the S_∞-action on $\mathscr{P}(\mathscr{M})$ is induced by the S_∞-algebra structure on R_∞. But now as $\dim R_\infty = d+1 = \dim S_\infty$, Lemma A.4.3 implies that

$$\mathscr{P}(\mathscr{M}^*) \cong \mathrm{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty) \cong \mathrm{Hom}_{R_\infty}(\mathscr{P}(\mathscr{M}), \omega_{R_\infty})$$

as R_∞-modules (where we have used the fact that $\omega_{S_\infty} = S_\infty$). □

We also note that dualizable patching systems are automatically reflexive in the following sense:

Proposition A.4.5. *If $\mathscr{R}$ is a weak patching $\mathscr{R}$-algebra, and $\mathscr{M}$ is a dualizable patching $\mathscr{R}$-module, then the natural map $\mathscr{M} \to \mathscr{M}^{**}$ given by $x \mapsto (f \mapsto f(x))$ is an isomorphism.*

Consequently we have that the map $\mathscr{P}(\mathscr{M}) \to \operatorname{Hom}_{S_\infty}\left(\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty), S_\infty\right)$ is an isomorphism, and if R_∞ is a cover for $\mathscr{R}$ then $\mathscr{P}(\mathscr{M}) \to \operatorname{Hom}_{R_\infty}\left(\operatorname{Hom}_{R_\infty}(\mathscr{P}(\mathscr{M}), \omega_{R_\infty}), \omega_{R_\infty}\right)$ is an isomorphism.

Proof. As each M_n is free of finite rank over $S_\infty / \operatorname{Ann}_{S_\infty}(M_n)$, it is a reflexive $S_\infty / \operatorname{Ann}_{S_\infty}(M_n)$-module, and so the first claim follows.

It thus follows that the map $\mathscr{M} \to \mathscr{M}^{**}$ induces an isomorphism $\mathscr{P}(\mathscr{M}) \xrightarrow{\sim} \mathscr{P}(\mathscr{M}^{**})$. But now by Theorem A.4.4 we have natural isomorphisms:

$$\operatorname{Hom}_{S_\infty}\left(\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty), S_\infty\right) \cong \operatorname{Hom}_{S_\infty}\left(\mathscr{P}(\mathscr{M}^*), S_\infty\right) \cong \mathscr{P}(\mathscr{M}^{**})$$
$$\operatorname{Hom}_{R_\infty}\left(\operatorname{Hom}_{R_\infty}(\mathscr{P}(\mathscr{M}), \omega_{R_\infty}), \omega_{R_\infty}\right) \cong \operatorname{Hom}_{R_\infty}\left(\mathscr{P}(\mathscr{M}^*), \omega_{R_\infty}\right) \cong \mathscr{P}(\mathscr{M}^{**})$$

which clearly identify the maps in the the statement of the Proposition with the isomorphism $\mathscr{P}(\mathscr{M}) \xrightarrow{\sim} \mathscr{P}(\mathscr{M}^{**})$. □

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