THE UNIVERSITY OF CHICAGO

$T\overline{T}$ DEFORMATIONS AND GRAVITY

A DISSERTATION SUBMITTED TO

THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES

IN CANDIDACY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

BY

CHIH-KAI CHANG

CHICAGO, ILLINOIS

MARCH 2022

To my parents

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to thank my advisor Sav Sethi, my parents, my brother, DB, and Wei for constant support and never blame me for keeping doing things that are either stupid or irrelevant. I want to thank Christian, David, Gabriele, Asrat for enjoyable academic discussion. Thank D&D team Brendan, Kyle, Mark, and Matt for inspiration. Thank lunch buddy Larry for countless lunch talks.

ABSTRACT

To understand the space of quantum field theories, we often begin with a tractable theory like a CFT and deform it by a local operator. When the operator is relevant, it generates an RG flow, which modifies the behavior of the theory in infrared. When the operator is exactly marginal, it parametrizes the moduli space of the CFT. However, the theory deformed by an irrelevant operator changes the UV behavior as well as the definition of the theory, so the resulting theory is usually difficult to understand. In this thesis, I study a very special example of an irrelevant deformation triggered by the $T\overline{T}$ operator in two dimensions and its relation to gravity, supersymmetry, and supergravity. I propose a manifestly supersymmetric version of the $T\overline{T}$ operator in superspace. In addition, I show that the deformed theory is classically equivalent to the undeformed theory coupled to two-dimensional gravity, which signals a deep connection to gravity. A closely related deformation appears in the holographic definition of supergravity solutions that interpolate between BTZ black holes and asymptotically linear dilaton backgrounds. This opens up the possibility of defining holography between non-AdS spacetimes and theories deformed by irrelevant deformations.

CHAPTER 1

INTRODUCTION

Renormalizable quantum field theory has been extraordinarily successful since infinitely many predictions can be made with just a handful measurement that determines a finite set of relevant and marginal couplings. Despite of this, its predictive power requires computation to higher and higher loops, which is not always feasible because of factorial growth of Feynman diagrams and IR divergence. On the other hand, an effective theory with irrelevant coupling is often used to make prediction and compare to experiment quickly although they are not reliable to extract high energy and short distance phenomenon, because new physics may emerge at higher energy scale and change the theory significantly. This thesis focuses on a surprising irrelevant operator in 2 dimension that allows the deformed theory to remains analytic in some sense. This operator was first proposed by Zamolodchikov[1][2] to study 2 dimensional quantum field theory. It can be obtained by taking the determinant of energy momentum tensor over its spacetime indices,

$$\det\big(T_{\mu\nu}\big) = T_{00}T_{11} - T_{10}T_{01}. \tag{1.0.1}$$

Although being quadratic in the energy momentum tensor, which can cause divergence when 2 operators are brought close to each other, it's shown that all divergence are cancelled, leaving it a well-defined operator. In Euclidean complex coordinate, this operator is written as $T\overline{T} - \theta^2$, where T is the holomorphic energy momentum tensor while $\overline{T}$ is the anti-holomorphic one and θ is the trace of $T_{\mu\nu}$, so this operator is usually called "$T\overline{T}$ operator". When a CFT is deformed by $T\overline{T}$ operator, it doesn't mean we add this operator to the original theory, instead, the deformed theory's lagrangian $\mathcal{L}_\lambda$ is required to satisfy the flow equation:

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = \det\big(T_{\mu\nu}\big), \tag{1.0.2}$$

where the theory is put on the flat metric and λ is the coupling constant. This is equivalent to adding the $\det\big(T_{\mu\nu}\big)$ coupling to the original theory but the energy momentum tensor is calculated from the "deformed" theory, so it's expressed as a differential equation. One may worry that this operator is not invariant under Lorentz transformation because the indices are not fully contracted. To resolve this issue, we can combine the determinant of metric with it to form

$$\frac{\det\big(T_{\mu\nu}\big)}{\det\big(g_{\mu\nu}\big)}, \tag{1.0.3}$$

which is Lorentz invariant. We always put the theory on a flat Euclidean spacetime and follow the convention that $g = 1$, so $\det\big(g_{\mu\nu}\big)$ is usually omitted. The flow equation defines a curve in the space of quantum field theory parametrized by λ with a few remarkable properties.

1. The deformation preserves integrability: if one begins with a theory $\mathcal{L}_0$ which is integrable, in the sense that the theory has infinitely many local integrals of motion, then the deformed theory $\mathcal{L}_\lambda$ at finite λ is also integrable[2].

2. The deformation is "solvable", in the sense that one can make precise statements about properties of the deformed theory $\mathcal{L}_\lambda$ in terms of the un-deformed theory $\mathcal{L}_0$. If the theory is put on a cylinder with spatial direction compactified, the energy levels of the deformed theory are related to those of the un-deformed one by a differential equation of inviscid Burgers type. The partition function of the deformed theory can also be obtained even if there are strong indication that it is not a conventional local quantum field theory[3]. This property doesn't require the un-deformed theory to be integrable or conformal.

3. It preserves supersymmetry. If the un-deformed theory is invariant under some number of supersymmetry, then the deformed theory must also be invariant under those supersymmetry[4][5][6].

4. The deformation is "unique" in the sense of the following: If we want to deform a 2 dimensional CFT $\mathcal{L}_0$ by a coupling λ with conformal dimension $(-1,-1)$ and we want the energy of a state in the deformed theory to depend only on the un-deformed energy of the same state, then $T\overline{T}$ deformation is the unique deformation that satisfies these requirements[7]. Besides, the modular invariant is preserved if λ is assigned with modular dimension $(-1,-1)$.

Thus $T\overline{T}$ deformation is a unique deformation as an example of irrelevant coupling. In chapter 2, we will review basic properties that characterize $T\overline{T}$ operator.

A few exact $T\overline{T}$ deformed theory with explicit Lagrangian is provided in [8][9]. It was noted that the $T\overline{T}$ deformation of a pure free boson is classically equivalent to the Nambu-Goto action with 3 dimensional target space. There is then intense discussion about the connection between $T\overline{T}$ deformation and string theory[3][10][11][12]. Furthermore, this deformation can be interpreted as gravitational coupling to 2 dimensional gravity. In [13], Cardy expresses $T\overline{T}$ into $\det(T_{\mu\nu})$ so it bacomes natural to couple it to metric. He shows that an infinitesimal $T\overline{T}$ transformation can be made with Hubbard–Stratonovich transformation on metric. In [14][15], it's proposed that $T\overline{T}$ deformation can be obtained by coupling the original theory to topological gravity which serves as a mapping from world sheet to target space. Several methods of determining the exact deformed Lagragian thorugh integrating out vielbeins or metric are also discovered by [16][17][18]. Then [19] and [20] propose generalizations of this idea to curved space. In chapter 3, we will review different approaches to understand $T\overline{T}$ from this point of view.

Some theories with supersymmetry can be deformed by $T\overline{T}$ in superspace which makes supersymmetry manifest. In these theories, the energy momentum tensor is part of super current multiplet which provides more structure to the irrelevant deformation[4][5]. Among them, $(2,2)$ supersymmetry is the most interesting one because it's closely related to dimensional reduction of theories in 4 dimensions, which may pave a way to understand irrelevant deformation in higher dimension[6][21][22]. In chapter 4, we will review the deformation with $(0,1)$ and $(1,1)$ supersymmetry.

There are also intense development in understanding the connection between $T\overline{T}$ deformation and string theory or non AdS geometry holograph[23][24][25][26]. Especially, it's proposed that the string theory with pure NS flux under certain background is dual to a CFT deformed by $T\overline{T}$-like operator. This background interpolates between AdS geometry or BTZ black hole in the bulk and linear dilaton on the boundary. in chapter 5, we will explore this idea from general relativity prospective.

CHAPTER 2

THE $T\overline{T}$ DEFORMATION

Often in quantum field theory we encounter short-distance divergence when two operator are brought close to each other and take the coincident point limit, so the definition of composite operator is ambiguous unless further investigation of divergence is examined. In this section we will discuss the precise definition of $T\overline{T}$ following Zamolodchikov's proof[1]. We will work in Euclidean signature and complex coordinate $z = x + iy$. A few basic assumptions need to be made in advance.

1. Local translation and rotation symmetry are assumed so there exist local energy momentum tensor that is symmetric and satisfies continuity equation. The components of energy momentum tensor are usually written as

$$\begin{aligned} &T = -T_{zz}, \quad \overline{T} = -T_{\overline{z}\overline{z}}, \\ &\theta = T_{z\overline{z}} = T_{\overline{z}z}, \end{aligned} \tag{2.0.1}$$

so the continuity equation takes a simple form:

$$\begin{aligned} \overline{\partial} T(z) &= \partial \theta(z) \\ \partial \overline{T}(z) &= \overline{\partial} \theta(z). \end{aligned} \tag{2.0.2}$$

Here $\partial = \partial_z = \frac{\partial}{\partial z}$ and $\overline{\partial} = \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}}$ are abbreviations which we will follow in this thesis.

2. Global translational symmetry is also assumed so the expectation value of some local operator $\langle \mathcal{O}(z) \rangle$ is a constant and two point function depends on the distance only,

$$\left\langle \mathcal{O}_i(z) \mathcal{O}_j(z') \right\rangle = G_{ij}(z - z'), \tag{2.0.3}$$

where $\mathcal{O}_i(z)$ are some local operators in the theory.

3. We also assume that there exists at least one direction, say along vector $\tilde{e}$, for every operators $\mathcal{O}_i(z)$ such that the correlation function tends to

$$\lim_{t\to\infty} \left\langle \mathcal{O}_i(z+\tilde{e}t)\mathcal{O}_j(z')\right\rangle = \left\langle \mathcal{O}_i(z)\right\rangle \left\langle \mathcal{O}_j(z')\right\rangle . \tag{2.0.4}$$

   At large distance along $\tilde{e}$, correlation vanishes and the correlation function is decoupled to the product of the expectations from two operators.

4. The theory is assumed to be controlled by a UV CFT.

2.1 Point-Splitting Argument

We will need to prove $T\overline{T} = T(z)\overline{T}(z) - \theta^2(z)$ is a well-defined operator by first considering the point-splitting version $T(z)\overline{T}(z') - \theta(z)\theta(z')$ and then taking the coincident point limit $z \to z'$ later. The derivative of it is

$$\begin{aligned} \partial\left(T(z)\overline{T}(z') - \theta(z)\theta(z')\right) &= \partial T(z)\overline{T}(z') - \partial\theta(z)\theta(z') \\ &= \partial T(z)\overline{T}(z') - \overline{\partial}T(z)\theta(z') \\ &= \partial T(z)\overline{T}(z') - \overline{\partial}T(z)\theta(z') + T(z)\left(\partial'\overline{T}(z') - \overline{\partial}'\theta(z')\right) \\ &= \left(\partial + \partial'\right)\left(T(z)\overline{T}(z')\right) - \left(\overline{\partial} + \overline{\partial}'\right)\left(T(z)\theta(z')\right). \end{aligned} \tag{2.1.1}$$

where ∂' is derivative with respect to z' and we have used both equations of (2.0.2) in the second and the third lines. Similarly, we can take derivative with respect to $\overline{z}$ and get

$$\overline{\partial}\left(T(z)\overline{T}(z') - \theta(z)\theta(z')\right) = \left(\partial + \partial'\right)\left(\theta(z)\overline{T}(z')\right) - \left(\overline{\partial} + \overline{\partial}'\right)\left(\theta(z)\theta(z')\right). \tag{2.1.2}$$

The meaning of these equations can be seen by substituting the operator product expansion into them,

$$\begin{aligned} T(z)\theta(z') &= \sum_i A_i(z-z')\mathcal{O}_i(z'), \\ \theta(z)\overline{T}(z') &= \sum_i B_i(z-z')\mathcal{O}_i(z'), \\ \theta(z)\theta(z') &= \sum_i C_i(z-z')\mathcal{O}_i(z'), \\ T(z)\overline{T}(z') &= \sum_i D_i(z-z')\mathcal{O}_i(z'). \end{aligned} \tag{2.1.3}$$

In the operator product expansion above we have let $\mathcal{O}_i$ runs through all operators in the theory and clearly we have

$$T(z)\overline{T}(z') - \theta(z)\theta(z') = \sum_i \left(D_i(z-z') - C(z-z')\right)\mathcal{O}_i(z') = \sum_i F_i(z-z')\mathcal{O}_i(z'). \tag{2.1.4}$$

where $F_i = D_i - C_i$. Inserting those expansion into (2.1.1) and (2.1.2) gives

$$\begin{aligned} &\sum_i \overline{\partial} F_i(z-z')\mathcal{O}_i(z') = \\ &\sum_i \left(B_i(z-z')\ \partial'\mathcal{O}_i(z') - C_i(z-z')\ \overline{\partial}'\mathcal{O}_i(z')\right), \\ &\sum_i \partial F_i(z-z')\mathcal{O}_i(z') = \\ &\sum_i \left(D_i(z-z')\ \partial'\mathcal{O}_i(z') - A_i(z-z')\ \overline{\partial}'\mathcal{O}_i(z')\right). \end{aligned} \tag{2.1.5}$$

On the right side of equality, all operators appear to have coordinate derivative, while those on the left side of the equality doesn't. Imagine now we take expectation value on both side, so, according to our assumption of translation symmetry, $\langle \partial\mathcal{O}\rangle = \left\langle \overline{\partial}\mathcal{O}\right\rangle = 0$ on the right. In order for the left side to have consistent result, operator $\mathcal{O}_i$ must either be the coordinate derivative of some other operator or have constant coefficient $F_i(z-z')$. As a result, the operator product expansion in (2.1.4) consist either linear combination of local operators or derivative terms. We can rewrite it as

$$T(z)\overline{T}(z') - \theta(z)\theta(z') = \mathcal{O}_{T\overline{T}}(z') + \text{derivative terms.} \tag{2.1.6}$$

When we evaluate the expectation value of the above equation, derivative terms have no contribution, leaving only $\mathcal{O}_{T\overline{T}}$, which is the linear combination of local operators. It also implies that the expectation value of $T\overline{T}$ is $\langle \mathcal{O}_{T\overline{T}}(z') \rangle$, a constant independent of coordinate. We now define the $T\overline{T}$ operator as $\mathcal{O}_{T\overline{T}}$. This definition may be too formal, but it shows that $T\overline{T}$ can be defined locally and unambiguously.

2.2 Energy Level

Zamolodchikov then further elaborates on how $T\overline{T}$ can affect the energy level. Consider a theory with one Cartesian coordinate compactified on a circle of circumference R, (x, y) $(x+R, y)$. We then deform the theory according to the flow equation (3.2.2). For nth energy level in the deformed theory, we denote its eigenstate as $|n\rangle$, energy as E_n, and momentum as P_n. If the initial theory is a CFT, then both its energy and momentum must be scaled as R^{-1}. The expectation value of $T\overline{T}$ on nth excitation state is

$$\mathcal{C}_n = \langle n|\ T(z)\overline{T}(z')\ |n\rangle - \langle n|\ \theta(z)\theta(z')\ |n\rangle. \tag{2.2.1}$$

According to our result in (2.1.6), we know this must be a constant. Consider the first term only, it can be expanded as

$$\begin{aligned}\langle n|\ T(z)\overline{T}(z')\ |n\rangle &= \sum_{n'}\langle n|T(z)|n'\rangle\langle n'|\overline{T}(z')|n\rangle \\ &= \sum_{n'}\langle n|T(z)|n'\rangle\langle n'|\overline{T}(z)|n\rangle e^{(E_n-E_{n'})|y-y'|+i(P_n-P_{n'})|x-x'|}.\end{aligned} \tag{2.2.2}$$

In the second line, spectral decomposition is used. A similar expansion can be made for $\theta(z)\theta(z')$. For their combination to be independent of coordinate, term with $n' \neq n$ must cancel each other and we are left with

$$\langle n|\ T(z)\overline{T}(z')\ |n\rangle = \langle n|T(z)|n\rangle\langle n|\overline{T}(z')|n\rangle - \langle n|\theta(z)|n\rangle\langle n|\theta(z')|n\rangle. \tag{2.2.3}$$

By taking the $z \to z'$ limit, we obtain

$$\langle n|\ T\overline{T}\ |n\rangle = \langle n|T|n\rangle\langle n|\overline{T}|n\rangle - \langle n|\theta|n\rangle\langle n|\theta|n\rangle. \tag{2.2.4}$$

In Cartesian coordinate, we can rewrite the above equation by using

$$T\overline{T} - \theta^2 = \left|\frac{1}{2}(T_{xx} - T_{yy} - 2iT_{xy})\right|^2 - \frac{1}{4}(T_{xx} + T_{yy})^2 = -T_{xx}T_{yy} + T_{xy}^2 \tag{2.2.5}$$

and get

$$\langle n|\ T\overline{T}\ |n\rangle = -\langle n|T_{xx}|n\rangle\langle n|T_{yy}|n\rangle + \langle n|T_{xy}|n\rangle\langle n|T_{xy}|n\rangle. \tag{2.2.6}$$

The right hand side can be computed directly from how energy momentum tensor is defined:

$$\begin{aligned}\langle n| \, T_{yy} \, |n\rangle &= -\frac{1}{R}E_n(R), \\ \langle n| \, T_{xy} \, |n\rangle &= \frac{i}{R}P_n(R), \\ \langle n| \, T_{xx} \, |n\rangle &= -\frac{\partial}{\partial R}E_R.\end{aligned} \tag{2.2.7}$$

Thus the expectation value is

$$\langle n| \, T\overline{T} \, |n\rangle = -\frac{1}{R}\left(E_n(R)\frac{\partial}{\partial R}E_n(R) + \frac{1}{R}P_n^2(R)\right). \tag{2.2.8}$$

Since the theory is deformed according to the flow equation (3.2.2), the expectation value is

$$\langle n| \, T\overline{T} \, |n\rangle = \langle n| \, -\frac{\partial \mathcal{L}}{\partial \lambda} \, |n\rangle = \frac{1}{R}\frac{\partial E_n}{\partial \lambda}. \tag{2.2.9}$$

Combining this with (2.2.6) leads to

$$\frac{\partial}{\partial \lambda}E_n(R,\lambda) = E_n(R,\lambda)\frac{\partial}{\partial R}E_n(R,\lambda) + \frac{1}{R}P_n^2(R). \tag{2.2.10}$$

The momentum is fixed by the quantization condition on cylinder so it is not modified by $T\overline{T}$ deformation. This is inviscid Burgers' equation, which describes how energy is shift as we deform the theory. If we begin with a CFT, in which the energy and momentum are proportional to R^{-1}, we can solve the above equation:

$$E_n(R,\lambda) = \frac{R}{2\lambda}\left(-1 + \sqrt{1 + \frac{4\lambda E_n(R,0)}{R} + \frac{4\lambda^2 P_n^2(R)}{R^2}}\right). \tag{2.2.11}$$

Assume $E_n(R,0)$ is positive. For positive λ, the deformed energy level is always well-defined. However, for negative λ, the argument in the square root may go negative, making the energy imaginary. The argument in the square root can be written as

$$\frac{4P_n^2(R)}{R^2}\left(\lambda+\frac{RE_n(R,0)}{2P_n^2(R)}\right)^2+1-\frac{E_n^2(R,0)}{P_n^2(R)}. \tag{2.2.12}$$

Thus if $E_n^2(R,0) > P_n^2(R)$ the energy level goes imaginary if λ is negatively large enough. It's believed that this king of theory has holographic interpretation[25] but theories with negative λ is still mysterious now. In this dissertation, we focus on positive λ only.

CHAPTER 3

GRAVITY

$T\overline{T}$ deformation has deep connection to the energy momentum tensor, so it has long been suspected that the deformed theory can be regarded as a gravitational action[13][27]. The undeformed theory is coupled to gravitational field, which will then been integrated out as it has no dynamics in two dimensions. In this section we review several different approaches that attempts to interpret $T\overline{T}$ deformation as gravity-coupling.

3.1 Review of Previous Gravitational Proposal

3.1.1 $T\overline{T}$ as Random Geometry

First consider how to add $T\overline{T}$ deformation to the first order in λ. The undeformed theory is a 2D CFT on flat space with static metric $g_{\mu\nu}$. We now imagine a small perturbation to the metric so, to first order, it's shifted by $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$. In the weak field limit, this theory depends on $h_{\mu\nu}$ through a coupling to energy momentum tensor, so we can add $\int d^2x\, h_{\mu\nu}T^{\mu\nu}$ to the action. On the other hand, we assume the gravitational action takes the following form in the weak field approximation[13]

$$S_h = -\frac{1}{2\lambda}\int d^2x\, \epsilon^{\mu\alpha}\epsilon^{\nu\beta}h_{\mu\nu}h_{\alpha\beta}. \tag{3.1.1}$$

Combining both terms together, we have

$$-\frac{1}{2\lambda}\int d^2x\, \epsilon^{\mu\alpha}\epsilon^{\nu\beta}h_{\mu\nu}h_{\alpha\beta} + \int d^2x\, h_{\mu\nu}T^{\mu\nu}. \tag{3.1.2}$$

It's interesting to note that the coupling constant is inverse proportional to λ. This is because, as we integrate out $h_{\mu\nu}$, its equation of motion forced it to be proportional to λ.

The equation of motion is

$$\frac{1}{\lambda}\epsilon^{\mu\alpha}\epsilon^{\nu\beta}h_{\alpha\beta} = T_{\mu\nu}. \tag{3.1.3}$$

Then substitute this equation into the action (3.1.2) and get

$$\frac{\lambda}{2}\int d^2x\ \epsilon^{\mu\alpha}\epsilon^{\nu\beta}T_{\mu\nu}T_{\alpha\beta} = \lambda\int d^2x\ \det\big(T_{\mu\nu}\big). \tag{3.1.4}$$

This is exactly the first order perturbation of $T\overline{T}$ operator. As we impose the equation of motion in (3.1.3), we are effectively integrating out $h_{\mu\nu}$, which is the variation of metric around the flat space. The spirit here is similar to the saddle point approximation, in which we integrate out the field configuration around the mean field. It's amusing to ask why flat space is a special solution here and what metric the deformed theory lives on, after the gravitational degree of freedom is integrated out. Besides, why is the action proportional to $\det\big(h_{\mu\nu}\big)$ in the weak field limit in (3.1.1). We will postpone answering these question later.

3.1.2 Topological Gravity

We next consider the following action[14][17],

$$S_\lambda = S_0[e] - \frac{1}{2\lambda}\int d^2x\ \epsilon^{\mu\nu}\epsilon_{ab}\ \left(\partial_\mu X^a - e^a_\mu\right)\left(\partial_\nu X^b - e^b_\nu\right). \tag{3.1.5}$$

Here $S_0[e]$ is the un-deformed theory put on a space with vielbein denoted by e^a_μ. $X^a(x)$ are auxiliary fields transformed under local Lorentz transformation on upper index a. It is claimed that this defines a $T\overline{T}$ deformed theory upon integrating our both e^a_μ and X^a. Although there are derivative on X^a, it is not a dynamical field due to the topological nature of this action. To see this, we can first compute its equation of motion,

$$\epsilon^{\mu\nu}\partial_\mu e^a_\nu = 0. \tag{3.1.6}$$

Since X^a is not involved in its equation of motion, it is a Lagrange multiplier. Furthermore, this equation forces e^a_μ to parametrize a flat space, so we can fix a gauge and set $e^a_\mu = \delta^a_\mu$. The equation of motion of e^a_μ is

$$\epsilon^{\mu\nu}\epsilon_{ab}\left(\partial_\nu X^b - e^b_\nu\right) = \lambda\frac{\delta S_0}{\delta e^a_\mu} = -\lambda\sqrt{-g}T^\mu_a. \tag{3.1.7}$$

We can rewrite this relation to

$$\partial_\mu X^a = e^a_\mu - \lambda\sqrt{-g}\ \epsilon_{\mu\nu}\epsilon^{ab}T^\nu_b. \tag{3.1.8}$$

and substitute it into (3.1.5).

$$\begin{aligned} S_\lambda &= S_0[e] - \frac{\lambda}{2}\int d^2x\ \epsilon^{\mu\nu}\epsilon_{ab}\ \left(\partial_\mu X^a - e^a_\mu\right)\left(\partial_\nu X^b - e^b_\nu\right) \\ &= S_0[\delta] - \frac{\lambda}{2}\int d^2x\ \det(g)\ \epsilon_{\mu\nu}\epsilon^{ab}T^\mu_a T^\nu_b. \end{aligned} \tag{3.1.9}$$

Here we have set $e^a_\mu = \delta^a_\mu$. The above equation can be further reduced by using $\sqrt{-g}\epsilon^{ab} = \epsilon^{\alpha\beta}e^a_\alpha e^b_\beta$,

$$S_\lambda = S_0[\delta] + \frac{\lambda}{2}\int d^2x\ \sqrt{-g}\ \epsilon_{\mu\nu}\epsilon^{\alpha\beta}T^\mu_\alpha T^\nu_\beta. \tag{3.1.10}$$

We obtained, again, the infinitesimal deformation. The advantage of this model is that the deformed theory is put on a flat space naturally as required by the Lagrange multiplier, which then serves as a map parametrizes from the 2D space x^μ to the target space X^a. Here we are not careful enough and use the energy momentum tensor of the un-deformed theory, so the above result is only valid to the first order. To get the finite deformation, one must be more careful and compute the integration with the energy momentum tensor of the deformed theory. This is done in [14] to compute the partition function. In [17], the deformation of free boson is computed to all order using this action.

3.2 Gravity Interpretation

We now propose a different model to interpret $T\overline{T}$ deformation. Assume there are two different rank 2 tensors, $g_{\mu\nu}$ and $h_{\mu\nu}$ and work with Euclidean signature in this section. $g_{\mu\nu}$ is a rank 2 tensor coupled to the undeformed CFT, while $h_{\mu\nu}$ is a non-dynamical background metric of the deformed theory. It's important to notice the difference between tensor with lower indices and upper indices, since we have two rank 2 tensors now. We will use $g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$ and $h^{\mu\nu}$ as the inverse of $h_{\mu\nu}$. Aside from them, we adopt the convention that all other tensors with upper indices are raised by $h^{\mu\nu}$. The following action is equivalent to $T\overline{T}$ deformation at classical level,

$$S_\lambda[h_{\mu\nu}] = S_0[g_{\mu\nu}] + \frac{1}{4\lambda}\int d^2x\ \sqrt{g}g^{\mu\nu}h_{\mu\nu} - \frac{1}{2\lambda}\int d^2x\ \sqrt{h}. \tag{3.2.1}$$

Here we assume $S_0[g_{\mu\nu}]$ to be a 2d CFT coupled to $g_{\mu\nu}$. At classical level, its $T\overline{T}$ deformation is equivalent to $S_\lambda[h_{\mu\nu}]$ on metric $h_{\mu\nu}$. Although we keep $h_{\mu\nu}$ arbitrary here, we have to set $h_{\mu\nu}$ to be a flat metric in the end. To show this action is indeed a $T\bar{T}$ deformation, the first step is calculate and confirm the flow equation,

$$\frac{\partial}{\partial\lambda}\mathcal{L}_\lambda = \det\big(T_{\mu\nu}\big). \tag{3.2.2}$$

We must emphasize that $T_{\mu\nu}$ is the energy momentum tensor calculated from the deformed theory $\mathcal{L}_\lambda$. It can be computed from the action (3.2.1),

$$T^{\mu\nu} = \frac{2}{\sqrt{h}}\frac{\partial S_\lambda}{\partial h_{\mu\nu}} = \frac{1}{2\lambda\sqrt{h}}\left(\sqrt{g}g^{\mu\nu} - \sqrt{h}h^{\mu\nu}\right). \tag{3.2.3}$$

where, we emphasize again, $h^{\mu\nu}$ is defined as the inverse of $h_{\mu\nu}$, instead of raised by $g^{\mu\nu}$, since $h_{\mu\nu}$ is regarded as the metric. Similarly $T^{\mu\nu}$ is raised by $h^{\mu\nu}$. We can calculate the left hand side of (3.2.2) directly through taking the derivative of the action (3.2.1) and observe that

$$T^{\mu\nu}h_{\mu\nu} = \frac{1}{2\lambda\sqrt{h}}\left(\sqrt{g}g^{\mu\nu}h_{\mu\nu} - 2\sqrt{h}\right) = -\lambda\frac{2}{\sqrt{h}}\frac{\partial\mathcal{L}_\lambda}{\partial\lambda}. \tag{3.2.4}$$

To calculate $\det T_{\mu\nu}$, we can use the fact that, in Euclidean 2D flat space,

$$2h^{-1}\det T_{\mu\nu} = \left(T^{\mu\nu}h_{\mu\nu}\right)^2 - T_{\mu\nu}T^{\mu\nu}. \tag{3.2.5}$$

It's easy to see

$$\begin{aligned} T_{\mu\nu}T^{\mu\nu} &= T^{\mu\nu}T^{\alpha\beta}h_{\mu\alpha}h_{\nu\beta} \\ &= \frac{1}{4\lambda^2 h}\left(\sqrt{g}g^{\mu\nu} - \sqrt{h}h^{\mu\nu}\right)\left(\sqrt{g}g^{\alpha\beta} - \sqrt{h}h^{\alpha\beta}\right)h_{\mu\alpha}h_{\nu\beta}. \\ &= \frac{1}{4\lambda^2 h}\left(g\cdot g^{\mu\nu}g^{\alpha\beta}h_{\mu\alpha}h_{\nu\beta} + 2h - 2\sqrt{g}\sqrt{h}g^{\mu\nu}h_{\mu\nu}\right) \end{aligned} \tag{3.2.6}$$

On the other hand, we can use (3.2.4),

$$\left(T^{\mu\nu}h_{\mu\nu}\right)^2 = \frac{1}{4\lambda^2 h}\left(g\cdot\left(g^{\mu\nu}h_{\mu\nu}\right)^2 + 4h - 4\sqrt{g}\sqrt{h}g^{\mu\nu}h_{\mu\nu}\right), \tag{3.2.7}$$

So the flow equation can be verified by

$$\begin{aligned} T_{\mu\nu}T^{\mu\nu} - \left(T^{\mu\nu}h_{\mu\nu}\right)^2 &= -2h^{-1}\det T_{\mu\nu} \\ &= \frac{1}{4\lambda^2 h}\left(g\left(g^{\mu\nu}g^{\alpha\beta}h_{\mu\alpha}h_{\nu\beta} - \left(g^{\mu\nu}h_{\mu\nu}\right)^2\right) - 2h + 2\sqrt{g}\sqrt{h}g^{\mu\nu}h_{\mu\nu}\right) \\ &= \frac{1}{4\lambda^2 h}\left(-4h + 2\sqrt{g}\sqrt{h}g^{\mu\nu}h_{\mu\nu}\right) \\ &= \frac{1}{4\lambda^2 h}\left(-4h + 4\lambda\cdot h\cdot T^{\mu\nu}h_{\mu\nu} + 4h\right) \\ &= \frac{1}{\lambda}T^{\mu\nu}h_{\mu\nu} = -2\frac{1}{\sqrt{h}}\frac{\partial\mathcal{L}_\lambda}{\partial\lambda}. \end{aligned} \tag{3.2.8}$$

We see,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{1}{\sqrt{h}} \det\big(T_{\mu\nu}\big). \tag{3.2.9}$$

Here we have set $h_{\mu\nu}$ to be flat and $h = 1$, so the deformed theory lives on a flat space. We also use the fact that $g^{\mu\nu} g^{\alpha\beta} h_{\mu\alpha} h_{\nu\beta} - \big(g^{\mu\nu} h_{\mu\nu}\big)^2 = -2hg^{-1}$ in the third line and (3.2.4) in the fourth line. Let's use a free scalar as an example to illustrate how to integrate out the field $g_{\mu\nu}$. Consider an un-deformed CFT with a single real scalar field ϕ,

$$S_0 = \int d^2x \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{3.2.10}$$

According to our formalism, the $T\overline{T}$ deformed action is equivalent to

$$S_\lambda = \int d^2x \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4\lambda} \int d^2x \sqrt{g} g^{\mu\nu} h_{\mu\nu} - \frac{1}{2\lambda} \int d^2x \sqrt{h}. \tag{3.2.11}$$

To integrate out $g_{\mu\nu}$, we first compute its equation of motion,

$$\frac{\delta S_\lambda}{\delta g^{\mu\nu}} = \sqrt{g} \left(\partial_\mu \phi \partial_\nu \phi + \frac{1}{4\lambda} h_{\mu\nu} \right) - \frac{1}{2} \sqrt{g} g_{\mu\nu} \left(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{4\lambda} g^{\alpha\beta} h_{\alpha\beta} \right) = 0. \tag{3.2.12}$$

To simplify the expression, we introduce $G_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \frac{1}{4\lambda} h_{\mu\nu}$ so the equation of motion becomes

$$G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} G_{\alpha\beta} \right), \tag{3.2.13}$$

while the action becomes

$$S_\lambda = \int d^2x \sqrt{g} g^{\mu\nu} G_{\mu\nu} - \frac{1}{2\lambda} \int d^2x \sqrt{h}. \tag{3.2.14}$$

Compute the determinant of equation of motion (3.2.13) on both side and take its square root,

$$\sqrt{G} = \frac{1}{2}\sqrt{g}\left(g^{\alpha\beta}G_{\alpha\beta}\right), \tag{3.2.15}$$

where $G = \det\left(G_{\mu\nu}\right)$. Substitute this back into the action (3.2.14), we get

$$S_\lambda = \int d^2x \, \left(2\sqrt{G} - \frac{1}{2\lambda}\sqrt{h}\right). \tag{3.2.16}$$

The determinant of $G_{\mu\nu}$ is

$$\det\left(G_{\mu\nu}\right) = \frac{1}{16\lambda^2}\det\left(h_{\mu\nu} + 4\lambda\partial_\mu\phi\partial_\nu\phi\right). \tag{3.2.17}$$

We can use the fact that $h_{\mu\nu}$ is a flat metric in 2 dimension to get

$$\begin{aligned}\det\left(G_{\mu\nu}\right) &= \frac{1}{16\lambda^2}\det\left(h_{\mu\nu} + 4\lambda\partial_\mu\phi\partial_\nu\phi\right) \\ &= \frac{1}{16\lambda^2}h\left(1 + 4\lambda h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right),\end{aligned} \tag{3.2.18}$$

so the action becomes

$$\begin{aligned}S_\lambda &= \int d^2x \, \left(2\sqrt{G} - \frac{1}{2\lambda}\sqrt{h}\right) \\ &= \int d^2x \, \sqrt{h}\frac{1}{2\lambda}\left(\sqrt{1 + 4\lambda h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} - 1\right).\end{aligned} \tag{3.2.19}$$

3.3 Partition Function

The equivalence between (3.2.1) and $T\overline{T}$ deformation is only a classical result. Quantum effects, Weyl anomaly for example, can destroy the equivalence. In this section, I attempt to study the partition function of $T\overline{T}$ deformed theory based on the action given in (3.2.1). We need to compute a path integral over metric $g_{\mu\nu}$ and integrate out all genus 1 geometry. First the undeformed theory is put on a genus 1 torus characterized by complex moduli $\tau = \tau_1 + i\tau_2$. A metric on torus can be brought to the following form by coordinate transformation,

$$d^2 s = |dx + \tau dy|^2 \tag{3.3.1}$$

or

$$g_{\mu\nu} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{pmatrix}, \qquad g^{\mu\nu} = \frac{1}{\tau_2^2} \begin{pmatrix} \tau_1^2 + \tau_2^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}. \tag{3.3.2}$$

Here we neglect a Weyl factor since the theory is assumed to be conformal invariant. Similarly, we have the same components for $h_{\mu\nu}$ which parametrizes a torus with moduli $\zeta = \zeta_1 + i\zeta_2$,

$$h_{\mu\nu} = \begin{pmatrix} 1 & \zeta_1 \\ \zeta_1 & \zeta_1^2 + \zeta_2^2 \end{pmatrix}. \tag{3.3.3}$$

The partition function is given by

$$Z_\lambda[\zeta] = \int D[g_{\mu\nu}] \; e^{-\frac{1}{4\lambda} \int d^2x \left(\sqrt{g} g^{\mu\nu} h_{\mu\nu} - 2\sqrt{h}\right)} Z_0[\tau]. \tag{3.3.4}$$

It's now straightforward to calculate the second and the third terms in (3.2.1),

$$\frac{1}{4\lambda} \int d^2x \left(\sqrt{g} g^{\mu\nu} h_{\mu\nu} - 2\sqrt{h}\right) = \frac{R^2 |\tau - \zeta|^2}{4\lambda\tau_2}, \tag{3.3.5}$$

where R is the length scale. On the other hand, the integral measure $D[g_{\mu\nu}]$ are consists of three contributions from Weyl transformation, coordinate transformation, and τ transformation. It can be written as

$$D[g_{\mu\nu}] = Z_X \frac{d^2\tau}{\tau_2^2}, \tag{3.3.6}$$

where Z_X is a Jacobian. Combining everything together gives

$$Z_\lambda[\zeta] = \int_{H^+} \frac{d^2\tau}{\tau_2^2} Z_X \; e^{-\frac{R^2|\tau-\zeta|^2}{4\lambda\tau_2}} \; Z_0[\tau], \tag{3.3.7}$$

where H^+ is upper half plane. We can't move on without knowing the exact structure of Z_X. I do not know the exact expression of Z_X. However, we now postulate that the deformed partition function is a trace over the spectrum of the deformed theory with integer coefficients[28]. As we will see later, this requires $Z_X = \frac{\zeta_2 R^2}{4\pi\lambda}$, so

$$Z_\lambda[\zeta] = \int_{H^+} \frac{d^2\tau}{\tau_2^2} \left(\frac{\zeta_2 R^2}{4\pi\lambda}\right) \; e^{-\frac{R^2|\tau-\zeta|^2}{4\lambda\tau_2}} \; Z_0[\tau]. \tag{3.3.8}$$

Given the partition function, we can compute the deformed energy level. Assume the undeformed theory has energy E_n and momentum P_n for the n-th excited state. It's partition function $Z_0[\tau]$ is

$$Z_0[\tau] = \sum_n e^{-\tau_2 R E_n + i\tau_1 R P_n}. \tag{3.3.9}$$

We now focus on how the partition function is deformed on the n-th excitation state. Following (3.3.8), we need to compute

$$\begin{aligned}
&\frac{\zeta_2 R^2}{4\lambda} \int_{H^+} \frac{d\tau_1 d\tau_2}{\tau_2^2} \exp\left(-\frac{R^2}{4\lambda\tau_2}\left[(\tau_2-\zeta_2)^2 + (\tau_1-\zeta_1)^2\right]\right) \exp\left(-\tau_2 R E_n + i\tau_1 R P_n\right) \\
&\quad = \exp\left(-\zeta_2 R \cdot \frac{R}{2\lambda}\left(-1 + \sqrt{1 + \frac{4\lambda E_n}{R} + \frac{4\lambda^2 P_n^2}{R^2}}\right) + i\zeta_1 R P_n\right).
\end{aligned} \tag{3.3.10}$$

We conclude that the deformed energy is

$$E_n(\lambda) = \frac{R}{2\lambda}\left(-1 + \sqrt{1 + \frac{4\lambda E_n}{R} + \frac{4\lambda^2 P_n^2}{R^2}}\right). \tag{3.3.11}$$

This is consistent with (2.2.11). However, I have made a strong assumption that there is a one-to-one correspondence between the undeformed energy level and the deformed energy level. So far, I am still looking for a path integral treatment of Z_X.

CHAPTER 4

SUPERSYMMETRY

$T\overline{T}$ deformation preserves supersymmetry because of it's a "solvable" deformation. Consider we begin from a supersymmetric theory in which every bosonic excited state is paired with a fermionic state of equal energy and deform the theory with $T\overline{T}$ flow equation. Recall that the energy level of the deformed theory can be determined by the energy level of the same state before deformation. There is a one-to-one correspondence between the deformed energy and the undeformed energy. We conclude that the supersymmetry is preserved since there are equal number of bosonic and fermionic states at each energy level even after deformation.[1] However, the Lagrangian of the deformed theory may not respect the supersymmetry made manifest by superspace construction even if we begin with one in superspace. In this chapter, we discuss how the action is deformed under $T\overline{T}$ deformation with manifest supersymmetry in superspace.

4.1 Bi-Spinor Conventions

In this section, we review the bi-spinor convention which is useful for the analysis with supersymmety involved. Consider two-dimensional field theories in Lorentzian signature with coordinates (x^0, x^1). It will be convenient to change coordinates to light-cone variables using bi-spinor notation. That is, we define

$$x^{\pm\pm} = \frac{1}{\sqrt{2}} \left(x^0 \pm x^1 \right) , \tag{4.1.1}$$

and write the corresponding derivatives as $\partial_{\pm\pm} = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$. In these conventions, we have $\partial_{\pm\pm} x^{\pm\pm} = 1$ and $\partial_{\pm\pm} x^{\mp\mp} = 0$. Spinors in two dimensions carry a single index which

1. This argument is kindly explained to me by David Kutasov.

is raised or lowered as follows:

$$\psi^{+} = -\psi_{-}, \qquad \psi^{-} = \psi_{+}. \tag{4.1.2}$$

The advantage of writing all vector indices as pairs of spinor indices is that it allows us to more easily compare terms in equations which involve a combination of spinor, vector, spinor-vector, and tensor quantities. For instance, in this notation the supercurrent has components $S_{+++}, S_{---}, S_{+--}$, and S_{-++}, which we can immediately identify as a spinor-vector because it has three indices. Likewise, the stress-energy tensor carries two vector indices so its components will have four bispinor indices; they are written as $T_{++++}, T_{----}, T_{++--} = T_{--++}$. When we consider $(0,1)$ and $(1,1)$ supersymmetric theories, we will have to introduce anti-commuting coordinates $\theta^{\pm}$. A derivative in superspace involves not only the coordinate derivative but also a linear combination with derivative on $\theta^{\pm}$. They are called supercovariant derivative defined, in our conventions, as

$$D_{\pm} = -i\frac{\partial}{\partial\theta^{\pm}} + \theta^{\pm}\partial_{\pm\pm}, \tag{4.1.3}$$

which satisfy $D_{\pm}D_{\pm} = -i\partial_{\pm\pm}$ and $\{D_{+}, D_{-}\} = 0$. There are also two supercharges $Q_{\pm}$ given by

$$Q_{\pm} = -i\frac{\partial}{\partial\theta^{\pm}} - \theta^{\pm}\partial_{\pm\pm}, \tag{4.1.4}$$

which satisfy $Q_{\pm}Q_{\pm} = i\partial_{\pm\pm}$.

4.1.1 Review of $T\overline{T}$

Let us review some basics of the $T\overline{T}$ deformation again with bi-spinor convention. An integrable field theory contains an infinite set of local integrals of motion generated by conserved currents. Among those currents, we are mostly interested in the stress energy tensor, $(T_{\pm\pm\pm\pm}, T_{\pm\pm\mp\mp})$, which satisfies the continuity equations

$$\begin{aligned}
\partial_{--}T_{++++} + \partial_{++}T_{++--} &= 0, \\
\partial_{++}T_{----} + \partial_{--}T_{--++} &= 0.
\end{aligned} \tag{4.1.5}$$

Given this set of continuity equations, the authors of [2] suggest considering spinless composite operators by first constructing bilinear operators, $T_{++++}(x)T_{----}(x')$ and $T_{++--}(x)T_{--++}(x')$, and taking the limit $x \to x'$. Although such limit is usually singular, the combination

$$T_{++++}(x)T_{----}(x') - T_{++--}(x)T_{--++}(x') \tag{4.1.6}$$

can be shown to contain no non-derivative divergences in its OPE. It is then natural to define a local operator $T\bar{T}(x)$ by

$$T\bar{T}(x) + \ldots = T_{++++}(x)T_{----}(x') - T_{++--}(x)T_{--++}(x'), \tag{4.1.7}$$

where $\ldots$ contains derivative terms that are not necessarily regular. A $T\bar{T}$ deformed theory on flat metric with action S_λ is a two dimensional integrable field theory deformed by the local operator $T\bar{T}(x)$ that satisfies the flow equation

$$\frac{\partial}{\partial\lambda}S_\lambda = -\int d^2x\ T\bar{T}(x). \tag{4.1.8}$$

The undeformed theory is the initial condition at $\lambda = 0$. It should be emphasized that $T\bar{T}$ in (4.1.8) is constructed from the stress energy tensor of the deformed action S_λ, thus it depends on the coupling λ implicitly and the flow equation becomes a non-linear differential equation. As a simple example, consider a free massless scalar ϕ in two dimensions with action given by

$$S_0 = \int d^2x \ \partial_{++}\phi\partial_{--}\phi. \tag{4.1.9}$$

Here d^2x means the measure dx^1dx^2. Solving the differential equation (4.1.8), the $T\bar{T}$ deformed action turns out to be

$$S_\lambda = \int d^2x \ \frac{1}{2\lambda}\left(-1+\sqrt{1+4\lambda\partial_{++}\phi\partial_{--}\phi}\right). \tag{4.1.10}$$

4.2 (0, 1) Supersymmetry

To begin, we first consider one single supercharge with $(0,1)$ supersymmetry. A $(1,1)$ superspace contains a pair of Grassmanian coordinate θ^+ and θ^-, while $(0,1)$ is the subspace when θ^- is set to zero. The supercharge and super-derivative in $(1,1)$ space can be expressed in terms of $\theta^\pm$,

$$Q_\pm = -i\frac{\partial}{\partial\theta^\pm} - \theta^\pm\partial_{\pm\pm}, \qquad D_\pm = -i\frac{\partial}{\partial\theta^\pm} + \theta^\pm\partial_{\pm\pm}. \tag{4.2.1}$$

When θ^- is set to zero, we are left with

$$Q_+ = -i\frac{\partial}{\partial\theta^+} - \theta^+\partial_{++}, \qquad D_+ = -i\frac{\partial}{\partial\theta^+} + \theta^+\partial_{++}. \tag{4.2.2}$$

A $(0,1)$ superfield $\Phi(x,\theta^+)$, can be expanded as

$$\Phi(x,\theta^+) = \phi(x) + \theta^+\psi_+(x), \tag{4.2.3}$$

where $\phi(x)$ and $\psi_+(x)$, depending only on spacetime coordinate x, are its bosonic and fermionic components respectively. Lagrangian in real space $\mathcal{L}$ is the d term of Lagrangian in superspace L^+,

$$\mathcal{L} = \int d\theta^+ L^+(x, \theta^+). \tag{4.2.4}$$

It's important to notice here that, while $\mathcal{L}$ is bosonic, L^+ must be Grassmanian. For example, a massless free theory with a scalar boson and a Weyl fermion is

$$\int d\theta^+ \; D_+\Phi\partial_{--}\Phi = \partial_{++}\phi\partial_{--}\phi + i\psi_+\partial_{--}\psi_+, \tag{4.2.5}$$

where $\Phi = \phi + \theta^+\psi_+$ is a scalar superfield. To define a $T\overline{T}$ deformation in superspace, we can begin with finding a counterpart of energy momentum tensor in superspace. Since energy momentum tensor is the conserved Noether current of translational symmetry, we can consider similar definition in superspace. Under an infinitesimal shift of coordinate $x^{++} \to x^{++} + a^{++}$, the field Φ is shifted by $\delta_{++}\Phi = \partial_{++}\Phi$, while the Lagrangian is shifted by

$$\begin{aligned}\delta_{++}L &= \partial_{++}L \\ &= \frac{\delta L}{\delta D_+\Phi}\delta_{++}D_+\Phi + \frac{\delta L}{\delta\partial_{++}\Phi}\delta_{++}\partial_{++}\Phi + \frac{\delta L}{\delta\partial_{--}\Phi}\delta_{++}\partial_{--}\Phi \\ &= D_+\left(\frac{\delta L}{\delta D_+\Phi}\delta_{++}\Phi + iD_+\left(\frac{\delta L}{\delta\partial_{++}\Phi}\delta_{++}\Phi\right)\right) + \partial_{--}\left(\frac{\delta L}{\delta\partial_{--}\Phi}\delta_{++}\Phi\right) + \cdots ,\end{aligned} \tag{4.2.6}$$

where we have used integration by parts and $\cdots$ denotes terms proportional to the equation of motion. We can summarize the above equation and get

$$D_+\left(\frac{\delta L}{\delta D_+\Phi}\partial_{++}\Phi + iD_+\left(\frac{\delta L}{\delta\partial_{++}\Phi}\partial_{++}\Phi\right) - iD_+L\right) + \partial_{--}\left(\frac{\delta L}{\delta\partial_{--}\Phi}\partial_{++}\Phi\right) = 0. \tag{4.2.7}$$

We denote the first bracket by $\mathcal{T}_{++--}$ and the second bracket by $\mathcal{S}_{+++}$, so

$$D_+\mathcal{T}_{++--} + \partial_{--}\mathcal{S}_{+++} = 0. \tag{4.2.8}$$

This is the analogy of the continuity equation (2.0.2) in superspace. Similarly, we can get another equation by considering a shift in $x^{--} \to x^{--} + a^{--}$ and get

$$D_+\left(\frac{\delta L}{\delta D_+\Phi}\partial_{--}\Phi + iD_+\left(\frac{\delta L}{\delta\partial_{++}\Phi}\partial_{--}\Phi\right)\right) + \partial_{--}\left(\frac{\delta L}{\delta\partial_{--}\Phi}\partial_{--}\Phi - L\right) = 0. \tag{4.2.9}$$

We denote the first bracket by $\mathcal{T}_{----}$ and the second one by $\mathcal{S}_{+--}$ so the continuity equation can be cast into

$$\begin{aligned} &D_+\mathcal{T}_{++--} + \partial_{--}\mathcal{S}_{+++} = 0, \\ &D_+\mathcal{T}_{----} + \partial_{--}\mathcal{S}_{+--} = 0. \end{aligned} \tag{4.2.10}$$

Each superfield has two components so there are eight components in $\mathcal{T}$ and $\mathcal{S}$. It's easy to see four of them are conventional energy momentum tensor $T_{\pm\pm\pm\pm}$ and $T_{\pm\pm\mp\mp}$. Acutually,

$$\begin{aligned} &\mathcal{S}_{+++} = \cdots + \theta^+ T_{++++} \\ &\mathcal{S}_{+--} = \cdots + \theta^+ T_{--++} \\ &\mathcal{T}_{++--} = T_{++--} + \theta^+ \cdots \\ &\mathcal{T}_{----} = T_{----} + \theta^+ \cdots , \end{aligned} \tag{4.2.11}$$

where $\cdots$ denotes other terms that are not part of energy momentum tensor. We will denote them by P and R for later convenience,

$$\begin{aligned}
\mathcal{S}_{+++} &= P_{+++} + \theta^+ T_{++++}, \\
\mathcal{S}_{+--} &= P_{+--} + \theta^+ T_{--++}, \\
\mathcal{T}_{++--} &= T_{++--} + \theta^+ R_{+++--}, \\
\mathcal{T}_{----} &= T_{----} + \theta^+ R_{+----}.
\end{aligned} \tag{4.2.12}$$

P and R are both fermionic and connected through (4.2.10),

$$\begin{aligned}
\partial_{--} P_{+++} + R_{+++--} &= 0, \\
\partial_{--} P_{+--} + R_{+----} &= 0.
\end{aligned} \tag{4.2.13}$$

We can now propose that the $T\overline{T}$ deformation in superspace is

$$-\int d\theta^+ \left(\mathcal{T}_{----} \mathcal{S}_{+++} - \mathcal{T}_{++--} \mathcal{S}_{+--} \right). \tag{4.2.14}$$

To see why this is a good candidate, we expand its components and find

$$\begin{aligned}
&\int d\theta^+ \left(\mathcal{T}_{----} \mathcal{S}_{+++} - \mathcal{T}_{++--} \mathcal{S}_{+--} \right) \\
&\qquad = T_{++++} T_{----} - T_{++--} T_{--++} - P_{+++} R_{+----} + P_{+--} R_{+++--}.
\end{aligned} \tag{4.2.15}$$

In addition to regular $T\overline{T}$ operator in the first two terms we get extra contributions from the last two terms. However, the extra terms can be combined into a total derivative by using (4.2.13), so they don't change the energy spectrum and the deformed theory remain solvable. To see how supersymmetry is made manifest in superspace, we begin with a free theory

$$\mathcal{L} = \int d\theta^+ L_0(x, \theta^+) = \int d\theta^+ D_+ \Phi \partial_{--} \Phi \tag{4.2.16}$$

and deform it according to the operator in (4.2.14). In superspace, the flow equation is

$$\frac{\partial}{\partial\lambda}L_\lambda = -\left(\frac{\delta L_\lambda}{\delta D_+\Phi}\partial_{--}\Phi + iD_+\left(\frac{\delta L_\lambda}{\delta\partial_{++}\Phi}\partial_{--}\Phi\right)\right)\left(\frac{\delta L_\lambda}{\delta\partial_{--}\Phi}\partial_{++}\Phi\right) + \left(\frac{\delta L_\lambda}{\delta D_+\Phi}\partial_{++}\Phi + iD_+\left(\frac{\delta L_\lambda}{\delta\partial_{++}\Phi}\partial_{++}\Phi\right) - iD_+L_\lambda\right)\left(\frac{\delta L_\lambda}{\delta\partial_{--}\Phi}\partial_{--}\Phi - L_\lambda\right). \tag{4.2.17}$$

To solve it, we substitute a general ansatz,

$$L_\lambda = \mathcal{F}(\lambda\partial_{++}\Phi\partial_{--}\Phi)D_+\Phi\partial_{--}\Phi \tag{4.2.18}$$

with some differentiable function $\mathcal{F}(x)$ into the flow equation. The flow equation then reduces to a differential equation of $\mathcal{F}(x)$,

$$\mathcal{F}' = -(x\mathcal{F}' + \mathcal{F})^2 + (x\mathcal{F}')^2 \tag{4.2.19}$$

bounded to the initial condition $\mathcal{F}(0) = 1$. The solution is

$$\mathcal{F}(x) = \frac{1}{2x}\left(-1 + \sqrt{1+4x}\right), \tag{4.2.20}$$

making the deformed Lagrangian

$$L_\lambda = \frac{1}{2\lambda\partial_{++}\Phi\partial_{--}\Phi}\left(-1 + \sqrt{1+4\lambda\partial_{++}\Phi\partial_{--}\Phi}\right)D_+\Phi\partial_{--}\Phi. \tag{4.2.21}$$

One can confirm that the bosonic part of this Lagrangian is identical to (4.1.10). For an interacting theory with $(0,1)$ supersymmetry, its Lagrangian in superspace is $g(\Phi)D_+\Phi\partial_{--}\Phi$ with $g(\Phi)$ an arbitrary differentiable function. One can follow the above procedure and make the same ansatz to find the deformed theory defined by

$$L_\lambda = \frac{1}{2\lambda\partial_{++}\Phi\partial_{--}\Phi}\left(-1+\sqrt{1+4\lambda g(\Phi)\partial_{++}\Phi\partial_{--}\Phi}\right)D_+\Phi\partial_{--}\Phi. \tag{4.2.22}$$

The deformation made by the operator in 4.2.14 is called supercurrent-squared deformation since it's quadratic in supercurrent. We have seen it preserves both supersymmetry and solvability of $T\overline{T}$.

4.3 $(1,1)$ Supersymmetry

We would like to generalize our strategy in $(0,1)$ supersymmetric theory to $(1,1)$ theory. Consider a supersymmetric Lagrangian which is written as an integral over $(1,1)$ superspace as $\mathcal{L} = \int d^2\theta\,\mathcal{A}$, where $d^2\theta = d\theta^+ d\theta^-$. We allow $\mathcal{A}$ to depend on a superfield Φ and a particular set of Φ derivatives listed below:

$$\mathcal{A} = \mathcal{A}\left(\Phi, D_+\Phi, D_-\Phi, \partial_{++}\Phi, \partial_{--}\Phi, D_+D_-\Phi\right). \tag{4.3.1}$$

The supercovariant derivatives $D_\pm$ are defined in (4.1.3). To define the supercurrent-squared operator, we first construct Noether current of translation. As in the derivation of the usual stress tensor T, we now consider a spatial translation of the form $\delta x^{\pm\pm} = a^{\pm\pm}$ for some constant $a^{\pm\pm}$. The variation $\delta\mathcal{A}$ of the superspace Lagrangian is given by

$$\begin{aligned}
\delta\mathcal{A} = {} & D_+\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi}\right) + D_-\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi}\right) + \partial_{++}\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) \\
& + \partial_{--}\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) + \frac{1}{2}\left(D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\delta\Phi\right) + D_-\left(\delta\Phi D_+\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\right) \\
& - \frac{1}{2}\left(D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\delta\Phi\right) + D_+\left(\delta\Phi D_-\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\right) \\
& - \delta\Phi\left(-\frac{\delta\mathcal{A}}{\delta\Phi} + D_+\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_-\frac{\delta\mathcal{A}}{\delta D_-\Phi} + \partial_{++}\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi} + \partial_{--}\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right. \\
& \left. -D_+D_-\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right).
\end{aligned} \tag{4.3.2}$$

Here we have chosen to symmetrize the term involving $D_+D_-\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}$ using $\{D_+, D_-\} = 0$ and imposed integration by parts to move derivatives around. The last two lines of (4.3.2) are the superspace equation of motion; we now specialize to the case of on-shell variations, for which this last term vanishes. Further, the left side of (4.3.2) is $\delta\mathcal{A} = a^{++}\partial_{++}\mathcal{A}+a^{--}\partial_{--}\mathcal{A}$, which is a total derivative. We use $\partial_{\pm\pm} = iD_\pm D_\pm$ to express (4.3.2) in the form

$$\begin{aligned}
0 = {} & a^{++}D_+\Big[\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + iD_+\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{++}\Phi\right) \\
& \qquad - \frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_+\mathcal{A}\Big] \\
& + a^{++}D_-\Big[\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + iD_-\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{++}\Phi\right) \\
& \qquad + \frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\Big] \\
& + a^{--}D_+\Big[\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + iD_+\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{--}\Phi\right) \\
& \qquad - \frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\Big] \\
& + a^{--}D_-\Big[\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + iD_-\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{--}\Phi\right) \\
& \qquad + \frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_-\mathcal{A}\Big].
\end{aligned} \tag{4.3.3}$$

This equation gives a conservation law for a superfield $\mathcal{T}$ which we define by

$$
\begin{aligned}
\mathcal{T}_{++-} &= \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + iD_+\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{++}\Phi\right) \\
&\quad - \frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_+\mathcal{A}, \\
\mathcal{T}_{+++} &= \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + iD_-\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{++}\Phi\right) \\
&\quad + \frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right), \\
\mathcal{T}_{---} &= \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + iD_+\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{--}\Phi\right) \\
&\quad - \frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right), \\
\mathcal{T}_{--+} &= \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + iD_-\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{--}\Phi\right) \\
&\quad + \frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_-\mathcal{A}.
\end{aligned}
\tag{4.3.4}
$$

In terms of $\mathcal{T}$, then, equation (4.3.3) implies the superspace conservation laws:

$$
D_+\mathcal{T}_{++-} + D_-\mathcal{T}_{+++} = 0, \qquad D_+\mathcal{T}_{---} + D_-\mathcal{T}_{--+} = 0. \tag{4.3.5}
$$

We are now in a position to propose the supercurrent-squared deformation of $(1,1)$ supersymmetry. Consider again an one-parameter family of superspace Lagrangians labeled by λ, which satisfy the ordinary differential equation

$$
\frac{\partial}{\partial\lambda}\mathcal{A}_\lambda = \mathcal{T}_{+++}\mathcal{T}_{---} - \mathcal{T}_{--+}\mathcal{T}_{++-}, \tag{4.3.6}
$$

where $\mathcal{T}$ is the supercurrent superfield (4.3.4) computed from the superspace Lagrangian $\mathcal{A}_\lambda$. This uniquely defines the supercurrent-squared deformation of an initial Lagrangian $\mathcal{A}_0$ at finite deformation parameter λ.

4.3.1 Reduction to Components for a Free Theory

To illustrate the relationship between the flow equation (4.3.6) and the usual $T\overline{T}$ operator, let us explicitly compute the components of the supercurrent-squared deformation for a free $(1,1)$ superspace Lagrangian

$$\mathcal{A} = D_+\Phi D_-\Phi, \tag{4.3.7}$$

where Φ is a superfield with component expansion

$$\Phi = \phi + i\theta^+\psi_+ + i\theta^-\psi_- + \theta^+\theta^- f. \tag{4.3.8}$$

The entries of $\mathcal{T}$, defined by (4.3.4), for the free theory are

$$\begin{aligned}
\mathcal{T}_{++-} &= \partial_{++}\Phi D_-\Phi - D_+\left(D_+\Phi D_-\Phi\right), \\
\mathcal{T}_{+++} &= -\partial_{++}\Phi D_+\Phi, \\
\mathcal{T}_{---} &= \partial_{--}\Phi D_-\Phi, \\
\mathcal{T}_{--+} &= -\partial_{--}\Phi D_+\Phi - D_-\left(D_+\Phi D_-\Phi\right).
\end{aligned} \tag{4.3.9}$$

In components, (4.3.9) is

$$\begin{aligned}
\mathcal{T}_{++-} =& - i\psi_+ f + \theta^+ \left(-f\partial_{++}\phi + \psi_+\partial_{++}\psi_-\right) + \theta^- \left(-f^2 - \psi_+\partial_{--}\psi_+\right) \\
&+ i\theta^+\theta^- \left(-\partial_{++}\phi\partial_{--}\psi_+ - \partial_{++}\psi_+\partial_{--}\phi - f\partial_{++}\psi_- + \psi_-\partial_{++}f\right) \\
&+ i\theta^+\theta^-\partial_{++}\left(\psi_+\partial_{--}\phi - \psi_- f\right), \\
\mathcal{T}_{+++} =& - i\psi_+\partial_{++}\phi - \theta^+ \left(\psi_+\partial_{++}\psi_+ + (\partial_{++}\phi)^2\right) - \theta^- \left(f\partial_{++}\phi + \psi_+\partial_{++}\psi_-\right) \\
&- i\theta^+\theta^- \left(2\partial_{++}\phi\partial_{++}\psi_- + \psi_+\partial_{++}f - f\partial_{++}\psi_+\right), \\
\mathcal{T}_{---} =& i\psi_-\partial_{--}\phi + \theta^+ \left(\psi_-\partial_{--}\psi_+ - f\partial_{--}\phi\right) + \theta^- \left(\psi_-\partial_{--}\psi_- + (\partial_{--}\phi)^2\right) \\
&+ i\theta^+\theta^- \left(\psi_-\partial_{--}f - f\partial_{--}\psi_- - 2\partial_{--}\phi\partial_{--}\psi_+\right), \\
\mathcal{T}_{--+} =& - i\psi_- f + \theta^+ \left(f^2 + \psi_-\partial_{++}\psi_-\right) + \theta^- \left(-f\partial_{--}\phi - \psi_-\partial_{--}\psi_+\right) \\
&+ i\theta^+\theta^- \left(-\partial_{--}\phi\partial_{++}\psi_- + f\partial_{--}\psi_+ - \partial_{--}\psi_-\partial_{++}\phi - \psi_+\partial_{--}f\right) \\
&+ i\theta^+\theta^-\partial_{--}\left(\psi_+ f + \psi_-\partial_{++}\phi\right).
\end{aligned} \tag{4.3.10}$$

To compare with the bosonic $T\overline{T}$ deformation, we identify the components of the usual stress tensor T for the theory of a free boson ϕ and fermions $\psi_\pm$ which one obtains by performing the integrals over $\theta^\pm$. In our conventions, these take the form:

$$\begin{aligned}
T_{++++} &= (\partial_{++}\phi)^2 + \psi_+\partial_{++}\psi_+, \\
T_{----} &= (\partial_{--}\phi)^2 + \psi_-\partial_{--}\psi_-.
\end{aligned} \tag{4.3.11}$$

We will also drop terms involving the auxiliary field f, since in the bosonic part of the supercurrent-squared deformation, these terms vanish after integrating out f using its equation of motion. Then the bilinears appearing in our flow equation (4.3.6) are

$$\begin{aligned}
\mathcal{T}_{+++}\mathcal{T}_{---} &= \psi_+\psi_-\partial_{++}\phi\partial_{--}\phi + i\theta^+\left(\psi_+\psi_-\partial_{++}\phi\partial_{--}\psi_+ - T_{++++}\psi_-\partial_{--}\phi\right) \\
&\quad + i\theta^-\left(\psi_+\partial_{++}\phi T_{----} + \psi_+\psi_-\partial_{++}\psi_-\partial_{--}\phi\right) - \theta^+\theta^-\big(T_{++++}T_{----} \\
&\quad + 2\partial_{++}\phi\partial_{--}\phi\left(\psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-\right) - \psi_-\partial_{++}\psi_-\psi_+\partial_{--}\psi_+\big), \\
\mathcal{T}_{++-}\mathcal{T}_{--+} &= -2\theta^+\theta^-\left(\psi_+\partial_{--}\psi_+\psi_-\partial_{++}\psi_-\right).
\end{aligned} \tag{4.3.12}$$

The superspace integral of the deformation $\mathcal{T}_{+++}\mathcal{T}_{---} + \mathcal{T}_{++-}\mathcal{T}_{--+}$ picks out the top component, which is

$$\begin{aligned}
&\int d^2\theta\, \left(\mathcal{T}_{+++}\mathcal{T}_{---} + \mathcal{T}_{++-}\mathcal{T}_{--+}\right) = \\
&\quad - T_{++++}T_{----} - 2\partial_{++}\phi\partial_{--}\phi\left(\psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-\right) - \psi_-\partial_{++}\psi_-\psi_+\partial_{--}\psi_+.
\end{aligned} \tag{4.3.13}$$

We see that (4.3.13) contains the usual $T\overline{T}$ deformation, given in our bi-spinor notation by $-T_{++++}T_{----}$, along with extra terms which are all proportional to the fermion equations of motion, $\partial_{\pm\pm}\psi_{\mp} = 0$. These added terms vanish on-shell and, as we will argue in section (4.3.3), do not spoil solvability: the energy levels of the deformed theory can still be expressed in terms of those in the undeformed theory, as in the purely bosonic $T\overline{T}$ case.

4.3.2 Relationship with the $\mathcal{S}$-multiplet

The $(1,1)$ superfield $\mathcal{T}$ contains the conserved stress-energy tensor $T_{\mu\nu}$ and the supercurrent $S_{\mu\alpha}$. Such current multiplets have received much attention in the literature; the first construction for four-dimensional theories was the FZ multiplet [29], which was later shown to be a special case of the more general $\mathcal{S}$-multiplet [30]. For the two-dimensional theories we consider here, it is known that the $\mathcal{S}$-multiplet is the most general multiplet containing the stress tensor and supercurrent, subject to assumptions that the multiplet be indecomposable and contain no other operators with spin greater than one. Since our supercurrent superfield $\mathcal{T}$ satisfies these properties, it must be equivalent to the $\mathcal{S}$-multiplet. As we will show, the four superfields contained in $\mathcal{T}$ are identical to the four superfields of the $\mathcal{S}$-multiplet, up to terms which vanish on-shell and therefore do not affect the conservation equations for the currents. The $\mathcal{S}$-multiplet is a reducible but indecomposable set of two superfields $\mathcal{S}$ and χ satisfying the constraints

$$\begin{aligned} D_{\mp}\mathcal{S}_{\pm\pm\pm} &= D_{\pm}\chi_{\pm}, \\ D_{-}\chi_{+} &= D_{+}\chi_{-}. \end{aligned} \tag{4.3.14}$$

In components, the $\mathcal{S}$-multiplet for $(1,1)$ theories contains the usual stress tensor $T_{\mu\nu}$, the supercurrent $S_{\mu\alpha}$, and a vector Z_{μ} which is associated with a scalar central charge:

$$\begin{aligned} \mathcal{S}_{+++} &= S_{+++} + \theta^{+}T_{++++} + \theta^{-}Z_{++} - \theta^{+}\theta^{-}\partial_{++}S_{-++}, \\ \mathcal{S}_{---} &= S_{---} + \theta^{+}Z_{--} + \theta^{-}T_{----} + \theta^{+}\theta^{-}\partial_{--}S_{+--}, \\ \chi_{+} &= S_{-++} + \theta^{+}Z_{++} + \theta^{-}T_{++--} - \theta^{+}\theta^{-}\partial_{++}S_{+--}, \\ \chi_{-} &= S_{+--} + \theta^{+}T_{++--} + \theta^{-}Z_{--} + \theta^{+}\theta^{-}\partial_{--}S_{-++}. \end{aligned} \tag{4.3.15}$$

In terms of these component fields, the constraints (4.3.14) give conservation equations for the currents:

$$\begin{aligned} \partial_{++}T_{----} + \partial_{--}T_{++--} &= 0 = \partial_{++}T_{--++} + \partial_{--}T_{++++}, \\ \partial_{++}S_{+--} + \partial_{--}S_{+++} &= 0 = \partial_{++}S_{---} + \partial_{--}S_{-++}, \\ \partial_{++}Z_{--} + \partial_{--}Z_{++} &= 0. \end{aligned} \tag{4.3.16}$$

We claim that the components (4.3.10) of our superspace supercurrent are the same as those in the two superfields $\mathcal{S}$ and χ appearing in the $(1,1)$ $\mathcal{S}$-multiplet (4.3.15), up to signs and terms which vanish on-shell. In particular, after discarding terms which are proportional to the equations of motion, we find the identifications:

$$\mathcal{S}_{\pm\pm\pm} = \mp\mathcal{T}_{\pm\pm\pm}, \qquad \chi_+ = \mathcal{T}_{++-}, \qquad \chi_- = \mathcal{T}_{--+}. \tag{4.3.17}$$

We will check this explicitly for the free theory, $\mathcal{A} = D_+\Phi D_-\Phi$, for which we computed the components of $\mathcal{T}$ in section (4.3.1). Writing only those terms that survive when the component equations of motion $f = 0$, $\partial_{++}\psi_- = 0 = \partial_{--}\psi_+$, and $\partial_{++}\partial_{--}\phi = 0$ are all satisfied, (4.3.10) becomes

$$\begin{aligned}
\mathcal{T}_{++-} &\overset{\text{on-shell}}{=} 0, \\
\mathcal{T}_{+++} &\overset{\text{on-shell}}{=} -i\psi_+\partial_{++}\phi - \theta^+\left(\psi_+\partial_{++}\psi_+ + (\partial_{++}\phi)^2\right), \\
\mathcal{T}_{---} &\overset{\text{on-shell}}{=} i\psi_-\partial_{--}\phi + \theta^-\left(\psi_-\partial_{--}\psi_- + (\partial_{--}\phi)^2\right), \\
\mathcal{T}_{--+} &\overset{\text{on-shell}}{=} 0.
\end{aligned} \tag{4.3.18}$$

For the free $(1,1)$ superfield considered here, the supercurrent is given in our conventions by

$$\begin{aligned}
S_{+++} &= \psi_+\partial_{++}\phi, \\
S_{---} &= \psi_-\partial_{--}\phi, \\
S_{+--} &= 0 = S_{-++},
\end{aligned} \tag{4.3.19}$$

To find expressions for the scalar central charge current $Z_{\pm\pm}$, we use the supersymmetry algebra implied by the $\mathcal{S}$-multiplet constraints, which gives

$$\begin{aligned}
\{Q_\pm, S_{\pm\pm\pm}\} &= T_{\pm\pm\pm\pm}, \\
\{Q_\pm, S_{\pm\mp\mp}\} &= T_{\pm\pm\mp\mp}, \\
\{Q_\pm, S_{\mp\pm\pm}\} &= Z_{\pm\pm}, \\
\{Q_\pm, S_{\mp\mp\mp}\} &= Z_{\mp\mp}.
\end{aligned} \tag{4.3.20}$$

Note that the $\mathcal{S}$-multiplet constraints only hold when the conservation equations for the currents hold, so the relations (4.3.20) should be viewed as an on-shell algebra. Acting with the supercharges $Q_{\pm}$ on the stress tensor and supercurrent components, one finds that $Z_{--} \sim \psi_- \partial_{--} \psi_+$ and $Z_{++} \sim \psi_+ \partial_{++} \psi_-$, both of which vanish when the fermion equations of motion are satisfied. Thus, after imposing the equations of motion, we can write our supercurrent superfield components as

$$\mathcal{T}_{++-} = \chi_+ = 0, \qquad \mathcal{T}_{--+} = 0 = \chi_-,$$
$$\mathcal{T}_{+++} = -S_{+++} - \theta^+ T_{++++} = -\mathcal{S}_{+++}, \quad \mathcal{T}_{---} = S_{---} + \theta^- T_{----} = \mathcal{S}_{---}. \tag{4.3.21}$$

Since terms which vanish on-shell do not affect conservation equations, one can view $\mathcal{T}$ as an improvement transformation of the $\mathcal{S}$-multiplet. The constraint equation $D_{\mp}\mathcal{S}_{\pm\pm\pm} - D_{\pm}\chi_{\pm} = 0$ is expressed by our conservation equations $D_+\mathcal{T}_{++-} + D_-\mathcal{T}_{+++} = 0$ and $D_+\mathcal{T}_{---} + D_-\mathcal{T}_{+--} = 0$.

4.3.3 Solvability

In this section we prove the theory deformed by (4.3.6) is solvable just like the usual $T\bar{T}$ deformation. Let's begin with the conservation law in superspace (4.3.5). It is straightforward to solve these constraints in components by using the conservation of the stress energy tensor:

$$\begin{aligned}
\mathcal{T}_{+++} &= H_{+++} - \theta^+ T_{++++} - \theta^- W_{++} + \theta^+\theta^- G_{+++}, \\
\mathcal{T}_{---} &= H_{---} + \theta^- T_{----} + \theta^+ W_{--} - \theta^+\theta^- G_{---}, \\
\mathcal{T}_{--+} &= H_{--+} - \theta^+ T_{--++} - \theta^- W_{--} + \theta^+\theta^- G_{--+}, \\
\mathcal{T}_{++-} &= H_{++-} + \theta^- T_{++--} + \theta^+ W_{++} - \theta^+\theta^- G_{++-}.
\end{aligned} \tag{4.3.22}$$

Here $(H_{\pm\pm\pm}, H_{\mp\mp\pm})$ denote the lowest components of $\mathcal{T}$ while $(G_{\pm\pm\pm}, G_{\mp\mp\pm})$ are its highest components. The conservation law in (4.3.5) implies constraints on G and H:

$$
\begin{aligned}
G_{\mp\mp\pm} &= \partial_{\pm\pm} H_{\mp\mp\mp}, \\
G_{\pm\pm\pm} &= \partial_{\pm\pm} H_{\mp\pm\pm}.
\end{aligned}
\tag{4.3.23}
$$

In terms of these components, the deformation in (4.3.6) becomes

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}\mathcal{L}_\lambda =& -\int d^2\theta \left(\mathcal{T}_{+++}\mathcal{T}_{---} + \mathcal{T}_{++-}\mathcal{T}_{--+}\right) \\
=& -\left(T_{++++}T_{----} - T_{++--}T_{--++}\right) \\
&+ \left(H_{+++}G_{---} - G_{+++}H_{---} - H_{++-}G_{--+} + G_{++-}H_{--+}\right).
\end{aligned}
\tag{4.3.24}
$$

The first bracket of the right hand side is the usual $T\bar{T}$ deformation. To understand how the second bracket changes the energy level, we consider the two-point correlation function.

$$
\begin{aligned}
\mathcal{C} = & \left\langle H_{+++}(x)G_{---}(x')\right\rangle - \left\langle G_{+++}(x)H_{---}(x')\right\rangle \\
& - \left\langle H_{++-}(x)G_{--+}(x')\right\rangle + \left\langle G_{++-}(x)H_{--+}(x')\right\rangle.
\end{aligned}
\tag{4.3.25}
$$

Up to contact terms that vanish at separated points, we can replace G by using the conservation equation (4.3.23):

$$
\begin{aligned}
\mathcal{C} = & \left\langle H_{+++}(x)\partial'_{--}H_{--+}(x')\right\rangle - \left\langle \partial_{++}H_{++-}(x)H_{---}(x')\right\rangle \\
& - \left\langle H_{++-}(x)\partial'_{++}H_{---}(x')\right\rangle + \left\langle \partial_{--}H_{+++}(x)H_{--+}(x')\right\rangle.
\end{aligned}
\tag{4.3.26}
$$

Here ∂' means the derivative with respect to the coordinate x'. Now we can use translational invariance to move the derivative from x' to x. Then the first term cancels the fourth term and the third term cancels the second one because both H and G are fermionic, hence $\mathcal{C}$ vanishes at separated points. This implies the extra term can have no effect on the energy level. The presence of the extra term is only to make the action supersymmetric. The theory remains solvable, like the usual $T\bar{T}$ deformation, with the same relation between deformed and undeformed energy levels.

4.3.4 Free $(1,1)$ *Superfield*

First consider an undeformed superspace Lagrangian $\mathcal{A}_\lambda = D_+\Phi D_-\Phi$. We make the following ansatz for the deformed Lagrangian at finite λ:

$$\mathcal{A}_\lambda = F\left(\lambda\partial_{++}\Phi\partial_{--}\Phi,\, \lambda\left(D_+D_-\Phi\right)^2\right) D_+\Phi D_-\Phi. \tag{4.3.27}$$

Here F may only depend on the two dimensionless combinations which we define by

$$x = \lambda\,\partial_{++}\Phi\partial_{--}\Phi, \qquad y = \lambda\left(D_+D_-\Phi\right)^2. \tag{4.3.28}$$

In order to reduce to the free theory as $\lambda \to 0$, we must also impose the boundary condition $F(0,0) = 1$. After computing the components of the supercurrent-squared deformation and simplifying, the flow equation (4.3.6) yields

$$\begin{aligned}\frac{\partial}{\partial\lambda}F = &\left((D_+D_-\Phi)^2 - \partial_{++}\Phi\partial_{--}\Phi\right)F^2 \\ &- 2F\left(\partial_{++}\Phi\partial_{--}\Phi\right)\left(\partial_{++}\Phi\partial_{--}\Phi + (D_+D_-\Phi)^2\right)\frac{\partial F}{\partial x}.\end{aligned} \tag{4.3.29}$$

In terms of the dimensionless variables x and y, equation (4.3.29) becomes

$$\frac{\partial F}{\partial x}x + \frac{\partial F}{\partial y}y = (y-x)F^2 - 2F\frac{\partial F}{\partial x}x(x+y). \tag{4.3.30}$$

Supplemented with the boundary condition $F(0,0) = 1$, the partial differential equation (4.3.29) uniquely determines the deformed Lagrangian at finite t. As a check, we would like to verify that the bosonic structure of the solution to (4.3.29) reduces to the known results for the $T\overline{T}$-deformed theory of a free boson. We will argue that, in fact, it suffices to set $y = 0$ in (4.3.29) and note that the result agrees with the flow equation obtained in the purely bosonic case [31]. Indeed, let us write the components of the superfield Φ as $\Phi = \phi + i\theta^+\psi_+ + i\theta^-\psi_- + \theta^+\theta^- f$. To probe the bosonic structure, it suffices to set $\psi_\pm = 0$, perform the superspace integration, and then integrate out the auxiliary field f using its equation of motion. Thus consider an arbitrary superspace integral of the form

$$\mathcal{L}_\lambda = \int d^2\theta \ F(x, y) D_+\Phi D_-\Phi. \tag{4.3.31}$$

The lowest component of the superfield $y = \lambda D_+\Phi D_-\Phi$ is $-f$, and the higher components will not contribute to the bosonic part because they come multiplying $D_+\Phi D_-\Phi$, which is already proportional to $\theta^+\theta^-$ after setting the fermions to zero. Thus the purely bosonic piece of the physical Lagrangian associated with a superspace Lagrangian $\mathcal{A}_\lambda = F(x, y)D_+\Phi D_-\Phi$ is

$$\mathcal{L}_\lambda = F\left(\lambda\partial_{++}\phi\partial_{--}\phi, \lambda f^2\right)\left(f^2 + 4\partial_{++}\phi\partial_{--}\phi\right). \tag{4.3.32}$$

The equation of motion for the auxiliary field f is

$$2\lambda f \frac{\partial F}{\partial y}\left(f^2 + 4\partial_{++}\phi\partial_{--}\phi\right) + 2fF = 0, \tag{4.3.33}$$

which admits the solution $f = 0$. The Lagrangian for the bosonic field ϕ is then

$$\mathcal{L}_\lambda = 4F\left(\lambda\partial_{++}\phi\partial_{--}\phi, 0\right)\partial_{++}\phi\partial_{--}\phi. \tag{4.3.34}$$

Therefore, to determine the terms in the Lagrangian which involve only ϕ, we may solve the simpler partial differential equation

$$\frac{\partial F}{\partial x} x = -xF^2 - 2Fx^2 \frac{\partial F}{\partial x}, \tag{4.3.35}$$

which holds upon setting $y = 0$ in (4.3.30). But this is precisely the equation discussed in section 4.1.1, whose solution is equation (4.1.10):

$$\mathcal{L}_\lambda = \frac{\sqrt{1 + 4\lambda \partial_{++}\phi \partial_{--}\phi} - 1}{2\lambda}. \tag{4.3.36}$$

We see that the supercurrent-squared deformation of the free superfield is indeed a generalization of the $T\overline{T}$ deformation of a free boson, in the sense that it yields the same modification to the purely bosonic terms in the action but also includes additional terms which affect only the fermions.

4.3.5 Interacting $(1,1)$ *Superfield*

Next, we consider the case with a superpotential: that is, we begin from the undeformed superspace Lagrangian

$$\mathcal{A}_\lambda = D_+ \Phi D_- \Phi + h(\Phi), \tag{4.3.37}$$

where $h(\Phi)$ is an arbitrary function (it need not give rise to a theory with infinitely many integrals of motion). After performing the superspace integral, the physical Lagrangian is

$$\mathcal{L}_0 = \int d^2\theta \; \mathcal{A}^{(0)} = \partial_{++}\phi \partial_{--}\phi + \psi_+ \partial_{--}\psi_+ + \psi_- \partial_{++}\psi_- + f^2 + h'(\phi) f. \tag{4.3.38}$$

Integrating out the auxiliary field using its equation of motion $f = -\frac{1}{2}h'(\phi)$, we see that the physical potential V is given by $V = -\frac{1}{4}h'(\phi)^2$. We might expect that both the kinetic and potential terms are modified by a finite supercurrent-squared deformation, which would lead us to make the ansatz

$$\mathcal{A}_\lambda = F(x,y) D_+\Phi D_-\Phi + G(\lambda, \Phi), \tag{4.3.39}$$

where G is a new function to be determined, and $x = \lambda \partial_{++}\Phi\partial_{--}\Phi$, $y = \lambda \left(D_+D_-\Phi\right)^2$ as above. However, the deformation does not induce any change in the potential h, so in fact we may put $G = h$ for all λ. To see this, we can write down the supercurrent-squared deformation associated with the ansatz (4.3.39), which gives

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} F(x,y) D_+\Phi D_-\Phi + \frac{\partial}{\partial\lambda} G(\lambda,\Phi) = \\
&\frac{1}{\lambda}\left((y-x) F^2 - 2Fx(x+y)\frac{\partial F}{\partial x} + \left(G'\right)^2 + 2G'\sqrt{y}\left(x\frac{\partial F}{\partial x} - F\right) - 2\sqrt{y}xG'\frac{\partial F}{\partial y}\right) D_+\Phi D_-\Phi.
\end{aligned} \tag{4.3.40}$$

The details of the calculation leading to (4.3.40) are discussed in Appendix A. We see that deformation is proportional to $D_+\Phi D_-\Phi$, so it does not source any change in the potential $h(\Phi)$; thus we may take $G(h,\Phi) = h(\Phi)$ in our ansatz. This leaves us with a single partial differential equation for F, namely

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = (y-x) F^2 - 2Fx(x+y)\frac{\partial F}{\partial x} + \left(h'\right)^2 + 2h'\sqrt{y}\left(x\frac{\partial F}{\partial x} - F\right) - 2\sqrt{y}xh'\frac{\partial F}{\partial y}. \tag{4.3.41}$$

In the second line, we have used the constraint that F can depend only on the dimensionless combinations $x = \lambda\partial_{++}\Phi\partial_{--}\Phi$ and $y = \lambda\left(D_+D_-\Phi\right)^2$. As in the free case, we would like to study the purely bosonic terms in the physical Lagrangian resulting from (4.3.41) and compare them to known results. Here the auxiliary will play a more important role since $f = 0$ is no longer a solution. We can expand both the Lagrangian

$\mathcal{L} = \int d^2\theta \, (F(x,y) D_+ \Phi D_- \Phi + h(\Phi))$ and the auxiliary field f as power series in λ:

$$\mathcal{L} = \sum_{j=0}^{\infty} \lambda^j \mathcal{L}^{(j)}, \qquad f = \sum_{j=0}^{\infty} \lambda^j f^{(j)}, \tag{4.3.42}$$

and then integrate out the auxiliary order-by-order in λ. Doing so to order λ^3, we arrive at

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4} h'(\phi)^2 + \frac{x}{\lambda} + \lambda \left(\frac{1}{16} h'(\phi)^4 - \left(\frac{x}{\lambda}\right)^2 \right) + \lambda^2 \left(-\frac{1}{4} \left(\frac{x}{\lambda}\right)^2 h'(\phi)^2 - \frac{1}{64} h'(\phi)^6 + 2 \left(\frac{x}{\lambda}\right)^3 \right) \\ &+ \lambda^3 \left(\left(\frac{x}{\lambda}\right)^3 h'(\phi)^2 + \frac{1}{256} h'(\phi)^8 - 5 \left(\frac{x}{\lambda}\right)^4 \right) + \mathcal{O}(\lambda^4),\end{aligned} \tag{4.3.43}$$

after setting the fermions to zero. Up to conventions, this matches the Taylor expansion of the known result [31][32] for the $T\overline{T}$ deformation of a boson with a generic potential V, which is given in our conventions as

$$\mathcal{L}_\lambda = -\frac{1}{2\lambda} \frac{1 - 2\lambda V}{1 - \lambda V} + \frac{1}{2\lambda} \sqrt{\frac{\lambda \left(4V + \partial_{++}\phi \partial_{--}\phi\right)}{1 - \lambda V} + \frac{(1 - 2\lambda V)^2}{(1 - \lambda V)^2}}. \tag{4.3.44}$$

Again the physical potential V is related to h via $V = -\frac{1}{4} h'(\phi)^2$. We have checked explicitly that the bosonic part of the series solution to the PDE (4.3.41) matches the Taylor expansion of (4.3.44) up to $\mathcal{O}(\lambda^7)$.

CHAPTER 5

SUPERGRAVITY SOLUTION

5.1 Review of Basic Ideas

In this section, we consider a non AdS solution of supergravity that is dual to a CFT deformed by $T\overline{T}$-like operator. The analysis in [3] is made in worldsheet string theory. I explore the same idea in general relativity.

5.1.1 Review of Type IIB Supergravity

Let's begin with the action of low energy Type IIB superstring in ten dimensions. The bosonic sector is approximately

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\hat{g}} \left\{ e^{-2\Phi} \left(\hat{R} + 4(\partial\Phi)^2 - \frac{1}{12}|H_3|^2 \right) - \frac{1}{2}|F_1|^2 \right.$$
$$\left. - \frac{1}{12}|F_3 - C_0 H_3|^2 - \frac{1}{4 \cdot 5!}|\widetilde{F}_5|^2 \right\}, \tag{5.1.1}$$

where $\widetilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$ and we ignore the problem of dC_4 being self-dual. We will focus on cases with pure NS fluxes so it's consistent to set $C_0 = C_2 = C_4 = 0$. Furthermore, we will restrict the geometry to solutions of the form $\mathcal{M}_3 \times S^3 \times T^4$. Through the S^3, the flux quantization condition requires that

$$\frac{1}{4\pi^2\alpha'} \int_{S^3} H_3 = m_5 \,, \tag{5.1.2}$$

where m_5 is an integer corresponding to the NS 5-brane charge. There is another quantization condition on the dual field strength. If we define

$$H_7 = e^{-2\Phi} * H_3 \,, \tag{5.1.3}$$

then this form satisfies the following quantization condition for any 7-cycle Σ_7:

$$\frac{1}{(2\pi)^6(\alpha')^3}\int_{\Sigma_7} H_7 = m_1 \in \mathbb{Z}\,. \tag{5.1.4}$$

There is a well-known class of solutions in type IIB supergravity which satisfy the flux quantization conditions outlined in the previous subsection. These much-studied solutions are interpreted as bound states of m_1 fundamental strings and m_5 NS-5 branes, where m_1, m_5 are the integers appearing in the quantization conditions (5.1.2) and (5.1.4), respectively. One way of presenting these solutions [33], which is nicely reviewed in [34], is

$$\begin{aligned} e^{-2\Phi} = \frac{1}{g_s^2}\frac{f_1}{f_5}\,, \qquad B_{05} = \frac{1}{f_1} - 1\,, \qquad H_{mnp} = \epsilon_{mnpq}\partial_q f_5 \text{ for } m,n,p,q = 1,2,3,4\,, \\ ds^2 = \frac{-dt^2 + dx_5^2}{f_1} + f_5\left(dx_1^2 + \cdots + dx_4^2\right) + \left(dx_6^2 + \cdots dx_9^2\right)\,, \end{aligned} \tag{5.1.5}$$

where

$$f_1 = 1 + \frac{r_1^2}{r^2} = 1 + \frac{16\pi^4 g_s^2 \alpha'^3 m_1}{V_4 r^2}\,, \qquad f_5 = 1 + \frac{r_5^2}{r^2} = 1 + \frac{\alpha' m_5}{r^2}\,. \tag{5.1.6}$$

In these expressions, r represents a radial coordinate in the four-dimensional space transverse to the NS5-branes (parameterized by x_1, x_2, x_3, x_4), while the coordinates x^6, x^7, x^8, x^9 are periodically identified so that these four directions span a torus with volume V_4. Note that the flux H_3 has two separate contributions, one arising from the explicit H_{mnp} components and one arising from $H = dB$ with $B = B_{05}\,dx^0 \wedge dx^5$ and B_{05} as indicated in (5.1.1). This solution (5.1.1) connects two different asymptotic regions. For large $r \gg r_1 \& r_5$, the geometry tends to a flat Minkowski space. For small $r \ll r_1 \& r_5$, the sphere (S^3 parameterized by x_1, x_2, x_3, x_4 at fixed r) decouples from the geometry, so the space is effectively a three dimensional gravity that is diffeomorphic to a Poincaré patch of AdS$_3$. As a result, we call this solution interpolates between the AdS$_3$ and the flat space. We will call the asymptotically flat region as the undecoupled solution while the near horizon limit at small

r as the decoupled solution. In the intermediate region between undecoupled and decoupled solution, we can take $r \gg r_1$ and restrict to $r_1 \gg r_5$. In this limit, the dilaton is approximately $\Phi \sim 2\log(r)$, which is usually refer to linear dilaton background. We will call this solution the partially decoupled solution. It has been argued [3, 24, 35] to have field theory duals which are obtained by deforming a conformal field theory by an irrelevant operator related to $T\overline{T}$. These solutions have also been shown [26] to arise from TsT transformations, a relationship which generalizes to the case of $J\overline{T}$ type deformations [36].

5.1.2 BTZ Black Hole

Following [37], we introduce some basic ideas of BTZ black hole. The authors of [37] attempt to find a black hole solution in 2+1 dimension in Einstein theory with negative cosmological constant. The action in consideration is

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2}\right). \tag{5.1.7}$$

where the length scale ℓ is related to the cosmological constant by $-\Lambda = \ell^{-2}$. The BTZ black hole solution with mass M and angular momentum J is given by

$$d^2s = -N^2dt^2 + N^{-2}dr^2 + r^2\left(N^{\phi}dt + d\phi\right)^2, \tag{5.1.8}$$

where

$$\begin{aligned} N^2(r) &= -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}, \\ N^{\phi}(r) &= -\frac{4GJ}{r^2}, \end{aligned} \tag{5.1.9}$$

with $t \in (-\infty, \infty)$, $r \in [0, \infty)$, and $\phi \in [0, 2\pi]$. We emphasize here that, although the parameter M is called mass here, it's not clearly here how it's related to mass. We will discuss on this topic later. We can assume M is mass for now. The vacuum state, which is the empty space without the black hole, can be obtained by making the horizon disappear in the $M \to 0$ and $J \to 0$ limit. This reaches the Poincaré patch of AdS_3, thus AdS_3 is a bound state in the BTZ spectrum. However, this is not the "ground state". One can go further to negative M and encounter the conical deficit[38]. Those conical deficit are not protected by the horizon so they are naked singularity and must be excluded from the spectrum. However, one can show that $M = -\frac{1}{8G}$ is the unique point that is free of conical deficit and it parametrizes the global coordinate of AdS_3. It's regarded as the ground state of BTZ black hole spectrum and there is a mass gap from $M = -\frac{1}{8G}$ to $M = 0$. In $3+1$ dimension, black hole mass is usually determined by probing the gravitational potential from the asymptotically flat region. However, BTZ black hole resides in AdS_3 space with constant negative cosmological constant so the definition of mass can be ambiguous. We will return to this point in the later section.

5.1.3 Covariant Phase Space Formalism of Mass

As previuosly stated, the BTZ black hole lives in a space with constant negative curvature, so the usual definition of mass in $3+1$ dimension fails to apply on it. Here we review the covariant phase space formalism[39]. which defines the conserved charge in covariant theories. In the next section we will apply this formalism to compute the mass of supergravity solution. Consider a theory with Lagrangian $\boldsymbol{L}(\Phi)$, where Φ is the field content in the theory. We will now use boldface symbols to denote differential form. Lagrangian is an n-form so we use $\boldsymbol{L} = L\, d^n x$ to denote it. Consider a variation δ of the field Φ. The corresponding change in Lagrangian is

$$\delta \boldsymbol{L} = \frac{\delta \boldsymbol{L}}{\delta \Phi}\, \delta\Phi - d\boldsymbol{\Theta}[\delta\Phi; \Phi]\,. \tag{5.1.10}$$

δ is a one-form anti-commuting with dx^μ. The object $\boldsymbol{\Theta}[\delta\Phi; \Phi]$ is called presymplectic potential. It depends on how we vary Φ but not on the coordinate explicitly. It's also convinient to define presymplectic form $\boldsymbol{\omega}$

$$\boldsymbol{\omega}[\delta_2\Phi, \delta_1\Phi; \Phi] = \delta_2\boldsymbol{\Theta}[\delta_1\Phi; \Phi] \,. \tag{5.1.11}$$

Here we use δ_1 and δ_2 to denote two different variations. $\boldsymbol{\Theta}[\delta_1\Phi; \Phi]$ depends only on δ_1. On the other hand, $\boldsymbol{\omega}[\delta_2\Phi, \delta_1\Phi; \Phi]$ depends on both δ_1 and δ_2. It's constructed by taking the variation d_2 on $\boldsymbol{\Theta}[\delta_1\Phi; \Phi]$. Among all the variations, we will be most interested in the infinitesimal coordinate transformation, $x^\mu \to x^\mu + \xi^\mu$. We will denote such transformation as δ_ξ for some vector ξ^μ. If one of the δ in (5.1.11) is coordinate transformation, it can be shown that $\boldsymbol{\omega}[\delta_\xi\Phi, \delta\Phi; \Phi]$ is exact,

$$\boldsymbol{\omega}[\delta_\xi\Phi, \delta\Phi; \Phi] = d\boldsymbol{k}_\xi[\delta\Phi; \Phi] \,, \tag{5.1.12}$$

under the requirement that Φ solves the equation of motion and $\delta\Phi$ solves the linearized equation of motion around the configuration Φ. The vector $\boldsymbol{k}_\xi[\delta\Phi; \Phi]$ is unique up to total derivative, but it's not a problem here because the physical charge is the integration of some closed surface. $\boldsymbol{k}_\xi[\delta\Phi; \Phi]$ is given by the Noether surface charge and the presymplectic potential,

$$\boldsymbol{k}_\xi[\delta\Phi; \Phi] = -\delta\boldsymbol{Q}_\xi[\delta\Phi; \Phi] + i_\xi\boldsymbol{\Theta}[\delta\Phi; \Phi] \,. \tag{5.1.13}$$

The surface charge variation between the solution Φ and $\Phi + \delta\Phi$ is

$$\delta Q_\xi = \oint_S \boldsymbol{k}_\xi[\delta\Phi; \Phi], \tag{5.1.14}$$

where S is chosen to be a sphere at fixed radius and time. We also restrict to the case when ξ^μ is killing vector of the solution. To calculate the mass, we can first choose δ to be the variation with respect to some parameter, say α, in the solution Φ and ξ^μ to be time-like killing vector. Then we can use (5.1.14) to compute δQ_ξ. This is the variation of mass with respect to α. We can now integrate it to get mass. Lets' use the BTZ black hole as an example. The Lagrangian is given by (5.1.7). The solution Φ is BTZ metric (5.1.8). We choose δ as the variation with respect to parameter M. The time-like killing vector is $\xi = \partial_t$. Thus, we can repeat the above analysis and get

$$\delta Q_\xi = \oint_S \boldsymbol{k}_\xi[\delta\Phi;\Phi] = \delta M. \tag{5.1.15}$$

This implies that the mass is M up to some constant.

5.2 Type IIB Solution

We are interesting in a solution that depends on a parameter λ with dimension length^2. The solution parametrizes a BTZ black hole when some parameter λ is set to zero and it also has to be a solution of supergravity (5.1.1) at finite λ. Let's define a few physical parameters first. In the solution, we will include mass M and spin J of the deep-bulk BTZ black hole; the parameter α' controlling the string length; the effective AdS_3 length scale ℓ; the asymptotic string coupling g_s; and the integers m_1 and m_5 counting the number of fundamental strings and NS5-branes. The solution is expressed as

$$\begin{aligned} ds^2 = -\frac{f_e(r)}{f_1(r)}\,dt^2 + \frac{1}{f_1(r)}\,\big(dx_5 + f_j(r)\,dt\big)^2 + \frac{f_5(r)}{f_e(r)}\,dr^2 + r^2 f_5(r)\,d\Omega_3^2 + dx_6^2 + \cdots + dx_9^2\,, \\ e^{-2\Phi} = e^{-2\phi_0}\frac{f_1(r)}{f_5(r)}\,, \qquad H_3 = f_{\mathcal{M}_3}(r)e^{2\Phi}\epsilon_3^{\mathcal{M}_3} + f_{S^3}(r)\epsilon_3^{S^3}\,, \end{aligned} \tag{5.2.1}$$

where the various functions and parameters are defined by

$$f_{\mathcal{M}_3}(r) = \frac{c_1}{r^3 f_5(r)^{3/2}}\,, \quad f_{S^3}(r) = \frac{c_5}{r^3 f_5(r)^{3/2}}\,, \quad c_1 = \frac{32 m_1 \pi^4 \alpha'^3}{V_4} \quad c_5 = 2\alpha' m_5\,,$$
$$f_1(r) = \frac{\lambda}{\alpha'} + \frac{r_1^2}{r^2}\,, \quad f_e(r) = 1 - \frac{8MGr_1^2}{r^2} + \frac{16G^2J^4r_1^2}{r^4}\,, \quad f_j(r) = -\frac{4GJr_1}{r^2}\,,$$
$$f_5(r) = \frac{\lambda}{\alpha'} g_s^2 e^{-2\phi_0} + \frac{r_5^2}{r^2}\,. \tag{5.2.2}$$

Here r_1 is constant related to the AdS$_3$ length scale ℓ by the equation

$$r_1^2 = \ell^2 \left(\frac{8MG\lambda}{\alpha'} + 1 \right)\,, \tag{5.2.3}$$

and $e^{2\phi_0}$, r_5 are two constants determined by the algebraic constraints

$$e^{2\phi_0} = \frac{2r_1^2}{c_1} \cdot \sqrt{1 + \frac{8MG\lambda}{\alpha'} + \frac{16G^2J^2\lambda^2}{\alpha'^2}}\,, \tag{5.2.4}$$
$$r_5^2 = \frac{4MGe^{-2\phi_0} g_s^2 \lambda r_1^2}{\alpha'} + \frac{1}{2}\sqrt{c_5^2 + \frac{64 r_1^4 g_s^4 e^{-4\phi_0}\lambda^2}{\alpha'^2}(M^2G^2 - G^2J^2)}\,. \tag{5.2.5}$$

In the limit $\lambda \to 0$, this reduces to the BTZ black hole solution, so λ plays the role of deforming the BTZ solution. These solution are discussed in [12]. Here we emphasize that the solution (5.2.1) is in the string frame. In general, we can take advantage of the Weyl transformation and redefine the metric to go to Einstein frame.

$$g_{\mu\nu} = e^{-4\Phi} \hat{g}_{\mu\nu}. \tag{5.2.6}$$

This is a spacetime Weyl transformation that depends on the dilaton. In terms of the Einstein-frame metric g, the action now becomes

$$S_{\mathrm{D}=3} = \frac{1}{2\kappa_3^2} \int d^3x \sqrt{-g} \left(\mathcal{R} - 4(\partial\Phi)^2 - \frac{1}{12} e^{-8\Phi} |H|^2 \right) + \ldots, \tag{5.2.7}$$

where $\mathcal{R}$ is the Ricci scalar of the new, un-hatted metric g. This takes us to Einstein frame where the graviton kinetic term is canonical. We will next calculate the mass in Einstein frame.

5.3 Charge of Type IIB Supergravity

We now repeat the formalism in section (5.1.3) to the Lagrangian in (5.1.1). We now restrict the action to have pure NS flux, so, in Einstein frame, it becomes

$$S = \frac{1}{16\pi G}\int d^{10}x\,\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}e^{-\Phi}|H|^2\right)\,. \tag{5.3.1}$$

Here $\boldsymbol{H} = d\boldsymbol{B}$ is the field strength of 2-form. In the action there are three fields $g_{\mu\nu}$, $B_{\mu\nu}$, Φ so we have to calculate their contribution one by one. Here we will use δ as the variation with respect to any parameter in the solution while δ_ξ is the variation with respect to killing vector ξ. Under δ_ξ, a tensor field $T_{\mu_1\cdots\mu_p}$ transformation can be expressed as the Lie derivative $\mathcal{L}_\xi$ on it.

$$\delta_\xi T_{\mu_1\cdots\mu_p} = \mathcal{L}_\xi T_{\mu_1\cdots\mu_p}\,. \tag{5.3.2}$$

Following[40], the scalar $\boldsymbol{k}_\xi^\Phi[\delta_\xi g, \delta\Phi; g; \Phi]$ is

$$\boldsymbol{k}_\xi^{\mu\nu,\Phi}[\delta g, \delta\Phi; g; \Phi] = (\delta\Phi)\cdot\xi^{[\nu}\partial^{\mu]}\Phi. \tag{5.3.3}$$

The contribution from the 2-form B is[40]

$$\boldsymbol{k}_\xi^B[\delta g, \delta B; g, B, \phi] = -\delta\boldsymbol{Q}_\xi^B + i_\xi\boldsymbol{\Theta}_B - \boldsymbol{E}_{\mathcal{L}}^B[\mathcal{L}_\xi B, \delta B]\,, \tag{5.3.4}$$

where

$$
\begin{aligned}
\boldsymbol{Q}^B_\xi &= e^{-\Phi}\left(i_\xi B\right)\wedge *H\,,\\
\boldsymbol{\Theta}^B &= e^{-\Phi}\left(\delta B\right)\wedge *H\,,\\
\boldsymbol{E}^B_{\mathcal{L}}[\mathcal{L}_\xi B,\delta B] &= e^{-\Phi} * \left(\frac{1}{2!}\delta B_{\mu\alpha}\left(\mathcal{L}_\xi B\right)_\nu{}^\alpha\, dx^\mu\wedge dx^\nu\right)\,.
\end{aligned}
\tag{5.3.5}
$$

The contribution from metric $g_{\mu\nu}$ is[39]

$$
\boldsymbol{k}^g_\xi[\delta g;g] = -\delta\boldsymbol{Q}_\xi[g] - i_\xi\boldsymbol{\Theta}[\delta g;g]\,,
\tag{5.3.6}
$$

where

$$
\begin{aligned}
\Theta^\mu &= \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\nu\,\delta g^{\mu\nu} - \nabla^\mu\delta g\right)\,,\\
\boldsymbol{Q}_\xi &= \frac{\sqrt{-g}}{8\pi G}\nabla^\mu\xi^\nu\left(d^8x\right)_{\mu\nu}\,.
\end{aligned}
\tag{5.3.7}
$$

In components, (5.3.6) can be written as

$$
k^{g,\mu\nu}_\xi = \xi^{[\nu}\nabla^{\mu]}\delta g - \xi^{[\nu}\nabla_\alpha\delta g^{\mu]\alpha} + \xi_\alpha\nabla^{[\nu}\delta g^{\mu]\alpha} + \frac{1}{2}\delta g\nabla^{[\nu}\xi^{\mu]} - \frac{1}{2}\delta g^{\alpha[\nu}\nabla_\alpha\xi^{\mu]} + \frac{1}{2}\delta g^{\alpha[\nu}\nabla^{\mu]}\xi_\alpha\,.
\tag{5.3.8}
$$

The total contribution to the charge is

$$
\delta Q_\xi = \oint_S\left(\boldsymbol{k}^g_\xi + \boldsymbol{k}^B_\xi + \boldsymbol{k}^\Phi_\xi\right)
\tag{5.3.9}
$$

for co-dimension 2 surface S at fixed time and radius.

5.3.1 Mass

For this part we will restrict the solution to the spinless black hole with $J = 0$ in (5.2.1) and calculate the mass associated to killing vector ∂_t. We again choose δ as the variation with respect to parameter M in (5.2.1). Thus we are considering the variation of Q_ξ as we vary the parameter M. It's important to note that, although M is the mass of BTZ black hole when $\lambda = 0$, it may not the the mass of the "deformed" theory. Since now the theory contains three fields, $g_{\mu\nu}$, $B_{\mu\nu}$, and Φ, we consider their variation respectively.

$$\delta_M g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial M}\,, \qquad \delta_M B_{\mu\nu} = \frac{\partial B_{\mu\nu}}{\partial M}\,, \qquad \delta_M \Phi = \frac{\partial \Phi}{\partial M}\,. \tag{5.3.10}$$

The combined contribution from k_ξ^g, k_ξ^B, and k_ξ^Φ to Q_ξ is

$$\delta_M Q_\xi = \oint_S \left(\boldsymbol{k}_\xi^g + \boldsymbol{k}_\xi^B + \boldsymbol{k}_\xi^\Phi\right) = \frac{r_1^2\,\delta M}{\sqrt{r_1^4 + 8GMr_1^4\lambda/\alpha'}}\,. \tag{5.3.11}$$

We can integrate both side and get

$$Q_\xi = \frac{\alpha'}{4G\lambda}\left(-1 + \sqrt{1 + \frac{8GM\lambda}{\alpha'}}\right). \tag{5.3.12}$$

Here we have chosen the integration constant to match the BTZ mass in the $\lambda \to 0$ limit. We emphasize again the the mass is Q_ξ, instead of M, which is just a parameter that coincides to mass in the BTZ limit. If we now identify $R = \frac{\alpha'}{2G}$. We can rewrite the mass as

$$Q_\xi = \frac{R}{2\lambda}\left(-1 + \sqrt{1 + \frac{4M\lambda}{R}}\right). \tag{5.3.13}$$

This is consistent to (2.2.11) when M is identified as the initial energy and momentum is set to zero. When momentum is not zero, we can follow the above calculation to compute mass. However, this is a problem due to integrability. With momentum involved, we can choose δ to be the variation with respect to J, instead of M, so we have two choices of variation, δ_M

and δ_J. We could compute two separate derivatives

$$\begin{aligned}\frac{\partial Q_\xi}{\partial M} &= \oint_S \boldsymbol{k}_\xi[\partial_M g; g]\,,\\ \frac{\partial Q_\xi}{\partial J} &= \oint_S \boldsymbol{k}_\xi[\partial_J g; g]\,.\end{aligned} \tag{5.3.14}$$

If computing $\partial_J \partial_M Q_\xi$ using the first line of (5.3.14) yields the same result as computing $\partial_M \partial_J Q_\xi$ using the second line of (5.3.14), then we can unambiguously define the charge $Q_\xi(M, J)$ in a way which does not involve any choice of how to perform the integration. In this case, the charge $Q_\xi(M, J)$ is said to be integrable. More generally, the integrability condition will hold so long as

$$\delta_1 \oint_S \boldsymbol{k}_\xi[\delta_2\Phi; \Phi] = \delta_2 \oint_S \boldsymbol{k}_\xi[\delta_1\Phi; \Phi]\,, \tag{5.3.15}$$

for any pair of variations $\delta_1\Phi$, $\delta_2\Phi$. However, the integrability fails in our supergravity solution, making the definition of charge in the presence of angular momentum ambiguous. It's not yet clear why integrability fails in this case, but the agreement between black hole mass and (2.2.11) is already striking. It follows that, a non-AdS solution that reduces to a non-spinning BTZ black hole has mass as if it's deformed by $T\overline{T}$ deformation. It seems reasonable then to suspect that a field theory dual to a non-AdS spacetime, with a potentially rich spectrum of black holes, might be controlled by irrelevant deformation.

APPENDIX A

$(1,1)$ FLOW CALCULATION

In this Appendix, we show some steps of the calculation which leads to the partial differential equation (4.3.41) defining the supercurrent-squared deformation of a free theory with a potential. By setting $h = 0$, this calculation also reproduces the PDE (4.3.29) which describes deformations of the free theory. We would like to consider what happens when we deform the superspace Lagrangian $\mathcal{A}_0 = D_+\Phi D_-\Phi + h(\Phi)$, according to the flow equation (4.3.6),

$$\frac{\partial}{\partial\lambda}\mathcal{A}_\lambda = \mathcal{T}_{+++}\mathcal{T}_{---} - \mathcal{T}_{--+}\mathcal{T}_{++-}.$$

It will help to introduce some shorthand: we define $A = D_+\Phi D_-\Phi$ so that $\mathcal{A}_0 = A$, and let $x = \lambda\partial_{++}\Phi\partial_{--}\Phi$ and $y = \lambda\left(D_+D_-\Phi\right)^2$ as before. Also define the dimensionful combinations

$$X = \partial_{++}\Phi\partial_{--}\Phi = \frac{x}{\lambda}, \qquad Y = (D_+D_-\Phi)^2 = \frac{y}{\lambda}. \tag{A.0.1}$$

Our ansatz for the superspace Lagrangian at finite λ will be $\mathcal{A}_\lambda = F(x,y)A + h(\Phi)$. With this ansatz, some of the terms in (4.3.4) will not contribute to the right side of (4.3.6). For instance, the terms $\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_\pm\partial_\pm\Phi$ will be proportional to $D_+\Phi D_-\Phi = A$. However, every term in the superspace supercurrent is proportional to $D_+\Phi$, $D_-\Phi$, or $D_+\Phi D_-\Phi$. Therefore, when we construct a bilinear in $\mathcal{T}$, any term containing $D_+\Phi D_-\Phi$ will not contribute because it can only appear multiplying another term which contains at least one of $D_\pm\Phi$, which vanishes because $(D_\pm\Phi)^2 = 0$. For our special ansatz, we will re-write the components of $\mathcal{T}$

keeping only terms which contribute to bilinears,

$$\begin{aligned}
\mathcal{T}_{++-} &\sim \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + \partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) - \frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_+\mathcal{A}, \\
\mathcal{T}_{+++} &\sim \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + \partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) + \frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right), \\
\mathcal{T}_{---} &\sim \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + \partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) - \frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right), \\
\mathcal{T}_{--+} &\sim \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + \partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) + \frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_-\mathcal{A}.
\end{aligned} \tag{A.0.2}$$

The terms are

$$
\begin{aligned}
D_+\mathcal{A} &\sim FD_+A + h'(\Phi)D_+\Phi,\\
D_-\mathcal{A} &\sim FD_-A + h'(\Phi)D_-\Phi,\\
\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} &\sim F\partial_{++}\Phi D_-\Phi,\\
\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} &\sim -F\partial_{++}\Phi D_+\Phi,\\
\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} &\sim F\partial_{--}\Phi D_-\Phi,\\
\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} &\sim -F\partial_{--}\Phi D_+\Phi,\\
\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) &\sim X\frac{\partial F}{\partial X}D_+A,\\
\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) &\sim (\partial_{++}\Phi)^2\,\frac{\partial F}{\partial X}D_-A,\\
\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) &\sim (\partial_{--}\Phi)^2\,\frac{\partial F}{\partial X}D_+A,\\
\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) &\sim X\frac{\partial F}{\partial X}D_-A,\\
\frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim \sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}\cdot D_+A,\\
\frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim \sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}\cdot D_+A,\\
-\frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim -\sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}\cdot D_-A,\\
-\frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim -\sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}\cdot D_-A,
\end{aligned}
$$

where $\sim$ means "equal modulo terms which are proportional to $D_+\Phi D_-\Phi$," since any products involving these terms will contain two nilpotent factors and thus vanish. The first piece of supercurrent-squared is

$$\begin{aligned}
\mathcal{T}_{++|+}\mathcal{T}_{--|-} &= \left(-F\partial_{++}\Phi D_{+}\Phi + (\partial_{++}\Phi)^2\frac{\partial F}{\partial X}D_{-}A + \sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}\cdot D_{+}A\right)\\
&\times\left(F\partial_{--}\Phi D_{-}\Phi + (\partial_{--}\Phi)^2\frac{\partial F}{\partial X}D_{+}A - \sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}D_{-}A\right),\\
&= -F^2XA - FX\frac{\partial F}{\partial X}\partial_{--}\Phi D_{+}\Phi D_{+}A + FX\frac{\partial F}{\partial X}\partial_{++}\Phi D_{-}AD_{-}\Phi\\
&+ X^2\left(\frac{\partial F}{\partial X}\right)^2 D_{-}AD_{+}A + F\frac{\partial F}{\partial Y}\sqrt{Y}XD_{+}AD_{-}\Phi\\
&+ FX\sqrt{Y}\frac{\partial F}{\partial Y}D_{+}\Phi D_{-}A - YX\left(\frac{\partial F}{\partial Y}\right)^2 D_{+}AD_{-}A.
\end{aligned} \tag{A.0.3}$$

The second piece is

$$\begin{aligned}
\mathcal{T}_{++|-}\mathcal{T}_{--|+} &= \left(F\partial_{++}\Phi D_{-}\Phi + \left(X\frac{\partial F}{\partial X} - F\right)D_{+}A - G'D_{+}\Phi - \sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}D_{-}A\right)\\
&\times\left(-F\partial_{--}\Phi D_{+}\Phi + \left(X\frac{\partial F}{\partial X} - F\right)D_{-}A - G'D_{-}\Phi + \sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}\cdot D_{+}A\right),\\
&= F^2XA + F\left(X\frac{\partial F}{\partial X} - F\right)\partial_{++}\Phi D_{-}\Phi D_{-}A + FX\sqrt{Y}\frac{\partial F}{\partial Y}D_{-}\Phi D_{+}A\\
&+ FX\sqrt{Y}\frac{\partial F}{\partial Y}D_{-}AD_{+}\Phi - F\left(X\frac{\partial F}{\partial X} - F\right)\partial_{--}\Phi D_{+}AD_{+}\Phi\\
&+ \left(X\frac{\partial F}{\partial X} - F\right)^2 D_{+}AD_{-}A - YX\left(\frac{\partial F}{\partial Y}\right)^2 D_{-}AD_{+}A\\
&- G'\left(X\frac{\partial F}{\partial X} - F\right)D_{+}\Phi D_{-}A + \left(G'\right)^2 D_{+}\Phi D_{-}\Phi - G'\sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}D_{+}\Phi D_{+}A\\
&- G'\left(X\frac{\partial F}{\partial X} - F\right)D_{+}AD_{-}\Phi + G'\sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}D_{-}AD_{-}\Phi.
\end{aligned} \tag{A.0.4}$$

Using the definitions $A = D_+\Phi D_-\Phi$, $X = \partial_{++}\Phi\partial_{--}\Phi$, and $\sqrt{Y} = D_+D_-\Phi$, we see that the products appearing in the above bilinears can be simplified as follows:

$$\begin{aligned}
&D_+\Phi D_+A = D_+\Phi D_+\left(D_+\Phi D_-\Phi\right) = D_+\Phi D_+D_+\Phi D_-\Phi = A\partial_{++}\Phi,\\
&D_+\Phi D_-A = D_+\Phi D_-\left(D_+\Phi D_-\Phi\right) = D_+\Phi D_-D_+\Phi D_-\Phi = -A\sqrt{Y},\\
&D_-\Phi D_+A = D_-\Phi D_+\left(D_+\Phi D_-\Phi\right) = -D_-\Phi D_+\Phi D_+D_-\Phi = A\sqrt{Y},\\
&D_-\Phi D_-A = D_-\Phi D_-\left(D_+\Phi D_-\Phi\right) = -D_-\Phi D_+\Phi D_-D_-\Phi = A\partial_{--}\Phi,\\
&D_+AD_-A = \left(\partial_{++}\Phi D_-\Phi - D_+\Phi\sqrt{Y}\right)\left(-\sqrt{Y}D_-\Phi - \partial_{--}\Phi D_+\Phi\right) = (X+Y)A.
\end{aligned} \tag{A.0.5}$$

So after simplifying,

$$\begin{aligned}
\mathcal{T}_{++|+}\mathcal{T}_{--|-} &= -F^2XA - 2FX^2\frac{\partial F}{\partial X}A - X^2\left(\frac{\partial F}{\partial X}\right)^2 A(X+Y) - 2F\frac{\partial F}{\partial Y}YXA,\\
\mathcal{T}_{++|-}\mathcal{T}_{--|+} &= F^2XA + 2FX\left(X\frac{\partial F}{\partial X} - F\right)A + 2FXY\frac{\partial F}{\partial Y}A + \left(X\frac{\partial F}{\partial X} - F\right)^2(X+Y)A\\
&\quad + \left((h')^2 + 2h'\sqrt{Y}\left(X\frac{\partial F}{\partial X} - F\right) - 2\sqrt{Y}Xh'\frac{\partial F}{\partial Y}\right)A.
\end{aligned} \tag{A.0.6}$$

In particular, we see that every term appearing in (A.0.6) is proportional to $A = D_+\Phi D_-\Phi$. This means that the deformation only generates a change in the first term of our ansatz $\mathcal{A}_\lambda = FD_+\Phi D_-\Phi + h(\Phi)$, but it does not source any change in the potential. This justifies our choice of ansatz which leaves the potential as $h(\Phi)$ rather than allowing a more general function $G(t,\Phi)$ with $G(0,\Phi) = h(\Phi)$. Adding the contributions gives,

$$\begin{aligned}
\mathcal{T}_{++|+}\mathcal{T}_{--|-} + \mathcal{T}_{++|-}\mathcal{T}_{--|+} &= \Big[(Y-X)F^2 - 2FX(X+Y)\frac{\partial F}{\partial X} + 2h'\sqrt{Y}\left(X\frac{\partial F}{\partial X} - F\right)\\
&\quad - 2\sqrt{Y}Xh'\frac{\partial F}{\partial Y} + (h')^2\Big]A.
\end{aligned} \tag{A.0.7}$$

Setting this deformation equal to $\frac{\partial}{\partial\lambda}\mathcal{A}_\lambda$, and multiplying both sides by t to convert dimensionlful variables X and Y into their dimensionless counterparts x and y, gives our final result (4.3.41),

$$x\frac{\partial}{\partial x}F + y\frac{\partial}{\partial y}F = (y-x)\,F^2 - 2Fx(x+y)\frac{\partial F}{\partial x} + \left(h'\right)^2 + 2h'\sqrt{y}\left(x\frac{\partial F}{\partial x} - F\right) - 2\sqrt{y}xh'\frac{\partial F}{\partial y}. \tag{A.0.8}$$

We were unable to find a closed-form solution to (4.3.41) in the general case. However, we can find the solution in a few special cases. If $y = 0$, (A.0.8) reduces to

$$xF'(x) = -x\left(F(x)^2 + 2F(x)F'(x)x\right), \tag{A.0.9}$$

which is solved by the Dirac-type ansatz $F(x) = \frac{\sqrt{1+4x}-1}{2x}$. If $x = 0$, equation (A.0.8) is solved by $F(y) = \frac{1}{1-y}$. If $y = -x$, the second term on the right side of (A.0.8) drops out and the solution is $F(x) = \frac{1}{1+2x}$.

REFERENCES

[1] A. B. Zamolodchikov, "Expectation value of composite field T anti-T in two-dimensional quantum field theory," hep-th/0401146.

[2] F. A. Smirnov and A. B. Zamolodchikov, "On space of integrable quantum field theories," *Nucl. Phys.* **B915** (2017) 363–383, 1608.05499.

[3] A. Giveon, N. Itzhaki, and D. Kutasov, "$\mathrm{T}\overline{\mathrm{T}}$ and LST," *JHEP* **07** (2017) 122, 1701.05576.

[4] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli, and H. Walsh, "On $T\overline{T}$ deformations and supersymmetry," *JHEP* **06** (2019) 063, 1811.00533.

[5] C.-K. Chang, C. Ferko, and S. Sethi, "Supersymmetry and $T\overline{T}$ Deformations," 1811.01895.

[6] C.-K. Chang, C. Ferko, S. Sethi, A. Sfondrini, and G. Tartaglino-Mazzucchelli, "$T\bar{T}$ Flows and (2,2) Supersymmetry," 1906.00467.

[7] O. Aharony, S. Datta, A. Giveon, Y. Jiang, and D. Kutasov, "Modular invariance and uniqueness of $T\bar{T}$ deformed CFT," *JHEP* **01** (2019) 086, 1808.02492.

[8] A. Cavaglià, S. Negro, I. M. Szécsényi, and R. Tateo, "$T\bar{T}$-deformed 2D Quantum Field Theories," *JHEP* **10** (2016) 112, 1608.05534.

[9] G. Bonelli, N. Doroud, and M. Zhu, "$T\bar{T}$-deformations in closed form," *JHEP* **06** (2018) 149, 1804.10967.

[10] M. Baggio and A. Sfondrini, "Strings on NS-NS Backgrounds as Integrable Deformations," *Phys. Rev.* **D98** (2018), no. 2, 021902, 1804.01998.

[11] S. Frolov, "TTbar deformation and the light-cone gauge," 1905.07946.

[12] S. Chakraborty, A. Giveon, and D. Kutasov, "$T\bar{T}$, $J\bar{T}$, $T\bar{J}$ and String Theory," *J. Phys. A* **52** (2019), no. 38, 384003, 1905.00051.

[13] J. Cardy, "The $T\overline{T}$ deformation of quantum field theory as random geometry," *JHEP* **10** (2018) 186, 1801.06895.

[14] S. Dubovsky, V. Gorbenko, and M. Mirbabayi, "Asymptotic fragility, near AdS_2 holography and $T\overline{T}$," *JHEP* **09** (2017) 136, 1706.06604.

[15] S. Dubovsky, V. Gorbenko, and G. Hernández-Chifflet, "$T\overline{T}$ partition function from topological gravity," *JHEP* **09** (2018) 158, 1805.07386.

[16] R. Conti, S. Negro, and R. Tateo, "The $\mathrm{T}\overline{\mathrm{T}}$ perturbation and its geometric interpretation," *JHEP* **02** (2019) 085, 1809.09593.

[17] E. A. Coleman, J. Aguilera-Damia, D. Z. Freedman, and R. M. Soni, "$T\bar{T}$-Deformed Actions and (1,1) Supersymmetry," 1906.05439.

[18] A. Ireland and V. Shyam, "$T\bar{T}$ deformed YM_2 on general backgrounds from an integral transformation," 1912.04686.

[19] A. J. Tolley, "$T\overline{T}$ deformations, massive gravity and non-critical strings," *JHEP* **06** (2020) 050, 1911.06142.

[20] E. A. Mazenc, V. Shyam, and R. M. Soni, "A $T\bar{T}$ Deformation for Curved Spacetimes from 3d Gravity," 1912.09179.

[21] M. Taylor, "TT deformations in general dimensions," 1805.10287.

[22] T. D. Brennan, C. Ferko, and S. Sethi, "A Non-Abelian Analogue of DBI from $T\overline{T}$," *SciPost Phys.* **8** (2020), no. 4, 052, 1912.12389.

[23] T. D. Brennan, C. Ferko, E. Martinec, and S. Sethi, "Defining the $T\overline{T}$ Deformation on AdS_2," 2005.00431.

[24] M. Asrat, A. Giveon, N. Itzhaki, and D. Kutasov, "Holography Beyond AdS," *Nucl. Phys.* **B932** (2018) 241–253, 1711.02690.

[25] L. McGough, M. Mezei, and H. Verlinde, "Moving the CFT into the bulk with $T\overline{T}$," *JHEP* **04** (2018) 010, 1611.03470.

[26] L. Apolo, S. Detournay, and W. Song, "TsT, $T\bar{T}$ and black strings," *JHEP* **06** (2020) 109, 1911.12359.

[27] S. Dubovsky, V. Gorbenko, and M. Mirbabayi, "Asymptotic Fragility, near AdS_2 Holography and $T\overline{T}$," *Journal of High Energy Physics* **2017** (Sept., 2017) 136.

[28] A. Hashimoto and D. Kutasov, "$T\overline{T}, J\overline{T}, T\overline{J}$ partition sums from string theory," *JHEP* **02** (2020) 080, 1907.07221.

[29] S. Ferrara and B. Zumino, "Transformation Properties of the Supercurrent," *Nucl. Phys.* **B87** (1975) 207.

[30] T. T. Dumitrescu and N. Seiberg, "Supercurrents and Brane Currents in Diverse Dimensions," *Journal of High Energy Physics* **2011** (July, 2011) 1106.0031.

[31] A. Cavaglià, S. Negro, I. M. Szécsényi, and R. Tateo, "$T\overline{T}$-Deformed 2D Quantum Field Theories," *Journal of High Energy Physics* **2016** (Oct., 2016) 1608.05534.

[32] G. Bonelli, N. Doroud, and M. Zhu, "$T\overline{T}$-Deformations in Closed Form," *Journal of High Energy Physics* **2018** (June, 2018) 1804.10967.

[33] A. A. Tseytlin, "Extreme dyonic black holes in string theory," *Mod. Phys. Lett. A* **11** (1996) 689–714, hep-th/9601177.

[34] J. Klusoň, "(p, q)-five brane and (p, q)-string solutions, their bound state and its near horizon limit," *JHEP* **06** (2016) 002, 1603.05196.

[35] A. Giveon, N. Itzhaki, and D. Kutasov, "A solvable irrelevant deformation of AdS_3/CFT_2," *JHEP* **12** (2017) 155, 1707.05800.

[36] L. Apolo and W. Song, "TsT, black holes, and $T\bar{T} + J\bar{T} + T\bar{J}$," 2111.02243.

[37] M. Bañados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, hep-th/9204099.

[38] S. Deser, R. Jackiw, and G. 't Hooft, "Three-dimensional Einstein gravity: Dynamics of flat space," *Annals of Physics* **152** (1984), no. 1, 220–235.

[39] G. Compère and A. Fiorucci, "Advanced Lectures on General Relativity," 1801.07064.

[40] G. Compere, "Note on the First Law with p-form potentials," *Phys. Rev. D* **75** (2007) 124020, hep-th/0703004.