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THE $RO(G)$ COHOMOLOGY OF EQUIVARIANT CLASSIFYING SPACES

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ABSTRACT

We compute the $RO(G)$-graded cohomology of various G-equivariant classifying spaces, where G is a cyclic 2-group. We then relate these descriptions to "genuine" G-equivariant characteristic class and power operations. Depending on the context, we take coefficients $\underline{R}$ in the constant Green functors $\underline{\mathbb{Z}}$, $\underline{\mathbb{F}}_2$ or the rational Burnside Green functor $\underline{A}_{\mathbb{Q}}$. The classifying spaces we study are $B_G L$ for L a compact connected Lie group such as $U(m), SO(m), Sp(m)$. For certain combinations of G, L and $\underline{R}$, we compute $\underline{H}_G^{\bigstar}(B_G L; \underline{R})$ as an $RO(G)$-graded Green functor algebra over the cohomology of a point $\underline{H}_G^{\bigstar}(*; \underline{R})$. We also develop a computer program to partially verify and automate these computations. The results in this dissertation first appeared in the author's preprints [3, 4, 5, 6, 7].

CHAPTER 1

INTRODUCTION

Equivariant algebraic topology has classical origins, going back to the work of P.A. Smith in [25]. Classically, one considers a group G acting on a space X and the first algebraic invariant of interest is the Borel equivariant cohomology given by:

$$H^*_{G,Borel}(X;R) = H^*(EG \times_G X; R)$$

where EG is a contractible free G-space and R is our ring of coefficients. In the modern language of spectra, this corresponds to the homotopy fixed points of the associated function spectrum:

$$H^*_{G,Borel}(X;R) = \pi_* F(X_+, HR)^{hG}$$

In many cases however, the Borel equivariant theory is inadequate and does not provide additional information compared to the nonequivariant theory ([20]). There is a more general and powerful theory, called Bredon (or $RO(G)$-graded) equivariant cohomology that does not suffer from this defect, and is given by the genuine fixed points of the function spectrum:

$$\underline{H}^{\bigstar}_G(X;\underline{R}) = \underline{\pi}_{\bigstar} F(X_+, H\underline{R})^G$$

The notation is somewhat loaded: First, the underlines indicate that instead of rings we use Green functors: the data of a Green functor $\underline{R}$ consist of a ring $\underline{R}(G/H)$ for each orbit G/H and maps between those rings satisfying certain axioms (see [24] for an introduction to Green functors). Second, the $\bigstar$ indicates that we are grading over the real representation ring $RO(G)$; in other words, $\bigstar$ is a real virtual G-representation.

Bredon cohomology is not only a more complicated algebraic gadget compared to Borel cohomology, but turns out to be notoriously difficult to compute even when the space X is a point and the group G is cyclic (see [16, 29, 3] for sample computations when G is a cyclic p-group).

On the other hand, there are significant upsides to using Bredon cohomology, even when the group G is as simple as possible, say $G = C_2$. First, the $RO(C_2)$ graded cohomology of C_2-equivariant classifying spaces contains nontrivial characteristic classes for C_2-bundles that cannot be seen through lenses of the nonequivariant or Borel theory ([6]). Moreover, the associated C_2-equivariant Steenrod algebra features new power operations ([14, 28]) and there are deep connections between $RO(C_2)$-graded cohomology and motivic cohomology ([10]). One can also not overstate the importance of the seminal work [11] which uses $RO(C_8)$-graded cohomology in an essential way to solve the Kervaire invariant problem in all but one dimension. Even outside the strict confines of algebraic topology, Bredon cohomology has been used in mathematical physics and in particular, string theory ([26]).

In this dissertation, we restrict ourselves to cyclic 2-groups $G = C_{2^n}$. The goal is to compute

$$\underline{H}_G^{\bigstar}(B_G L; \underline{R})$$

for classical Lie groups L such as $U(m), SO(m)$ and $Sp(m)$. Here, L does not have a G-action, but the equivariant classifying space $B_G L$ does (see section 1.2 for an example).

The coefficients $\underline{R}$ vary from chapter to chapter. In chapter 4 we take rational Burnside Green functor coefficients $\underline{R} = \underline{A}_{\mathbb{Q}}$ and describe the associated theory of C_2 Chern, Pontryagin and symplectic characteristic classes. In chapter 3 we take $\underline{R} = \underline{\mathbb{F}}_2$ and compute:

$$\underline{H}_{C_4}^{\bigstar}(B_{C_4} O(1); \underline{\mathbb{F}}_2)$$

generalizing the computation of ([14]) from $G = C_2$ to $G = C_4$. This specific computation has particular significance due to its connection to Steenrod operations and the dual Steenrod algebra.

By “computation", we mean the determination of $\underline{H}_G^{\bigstar}(X;\underline{R})$ as an $RO(G)$-graded Green functor algebra over the cohomology of a point $\underline{H}_G^{\bigstar}(*;\underline{R})$, in terms of generators and relations. Therefore, one first needs to compute:

$$\underline{H}_G^{\bigstar}(*;\underline{R})$$

which is a nontrivial undertaking in its own right. In the rational case, namely when $\underline{R} = \underline{A}_{\mathbb{Q}}$, a result of [9] implies that all modules over $\underline{R}$ are projective and injective, which makes it possible to give universal descriptions for all groups $G = C_{2^n}$ (which we do in chapter 4). That is not the case when using $\underline{R} = \underline{\mathbb{Z}}$ or $\underline{\mathbb{F}}_2$ coefficients, which is why we limit ourselves to $G = C_4$ computations in chapters 2 and 3.

With the exception of the rational case, all computations are done using spectral sequences. These spectral sequences generally have nontrivial differentials and lead to nontrivial extensions. There is an alternative approach to spectral sequences that involves Tate diagrams ([29]) but that also presents its own set of challenges. A seemingly elementary approach to all these problems is to set up a co-chain complex $\underline{C}^*(X;\underline{R})$ from the equivariant CW-decomposition of our space X and compute its homology directly. Unfortunately, due to the $RO(G)$ grading, we must instead to use the larger co-chain complex:

$$\underline{C}^*(X;\underline{R}) \boxtimes_{\underline{R}} \underline{C}^*(S^V;\underline{R})$$

where V ranges over the real virtual G-representations. Owing to the lack of a Kunneth formula, the homology of this box (tensor) product of co-chain complexes does not split into the box (tensor) product of homologies. The option now is to either use spectral sequences (which we do in section 2.5) or compute the box product and its homology directly. Direct computation is infeasible for pen-paper calculations as we are presented with the problem of diagonalizing matrices with no obvious patterns and dimensions that increase exponentially.

This is not a problem for computers however, and we have created a computer program that largely automates and partially verifies these computations (see section 2.6).

All the author's work present in this thesis was first made publicly available on the arXiv, in the form of 5 preprints: [3, 4, 5, 6, 7]. Due to the technical nature of the work, we have opted not to include all the details from the 5 papers here. Instead, we are aiming for a more readable presentation, citing the 5 papers when needed; the reader is advised to peruse the references for more detailed statements and complete descriptions and proofs.

1.1 Organization

This dissertation is organized in three chapters:

- Chapter 2 is concerned with computing the cohomology of a point

$$\underline{H}^{\bigstar}_{C_4}(*;\underline{\mathbb{Z}})$$

and partially covers the content of [3]. It also includes a section on the computer program and a conjecture regarding

$$\underline{H}^{\bigstar}_{C_{2^n}}(*;\underline{\mathbb{Z}})$$

for all $n \geq 1$.

- Chapter 3 is concerned with computing:

$$\underline{H}^{\bigstar}_{C_4}(B_{C_4}O(1);\underline{\mathbb{F}}_2)$$

We also discuss its relevance to the genuine C_4-equivariant dual Steenrod algebra, as well as the computation of the Borel C_{2^n}-equivariant dual Steenrod algebra. This chapter partially covers the contents of [4] and [5].

- Chapter 4 is concerned with computing:

$$\underline{H}^{\bigstar}_{C_2}(B_{C_2}L;\underline{A}_{\mathbb{Q}})$$

where $L = U(m), SO(m), Sp(m)$, and developing the associated theory of Chern, Pontryagin and symplectic classes. We further investigate generalizations to groups $G = C_{2^n}$. This chapter partially covers the contents of [6] and [7].

All chapters use the conventions and notations of section 1.2 which should be read first.

1.2 Background, Conventions and Notations

We will assume that the reader is familiar with the fundamentals of Mackey functors, Green functors and $RO(G)$-graded Mackey functor-valued homology theories. Introductory references for all that include [21], [24].

Henceforth $G = C_{2^n}$ is a cyclic 2-group and X is an unbased G-space.

1.2.1 $RO(G)$-graded homology and cohomology

We consider the *unreduced* $RO(G)$-graded homology and cohomology of X in $\underline{R}$ coefficients:

$$\underline{H}^G_{\bigstar}(X;\underline{R}) \ , \ \underline{H}^{\bigstar}_G(X;\underline{R})$$

The reduced versions feature an extra tilde:

$$\underline{H}^G_{\bigstar}(X;\underline{R}) = \underline{\widetilde{H}}^G_{\bigstar}(X_+;\underline{R}) \ , \ \underline{H}^{\bigstar}_G(X;\underline{R}) = \underline{\widetilde{H}}^{\bigstar}_G(X_+;\underline{R})$$

We shall generally suppress $\underline{R}$ from the notation when either $\underline{R}$ is understood from the context, or when the statement in question is true irrespective of $\underline{R}$.

When $X = *$ is a point, homology and cohomology are the same up to a regrading:

$$\underline{H}^G_\bigstar := \underline{\pi}^G_\bigstar(H\underline{R}) = \underline{H}^G_\bigstar(*;\underline{R}) = \underline{H}^{-\bigstar}_G(*;\underline{R})$$

This is an $RO(G)$-graded Green functor algebra over $\underline{R}$. For general X, both

$$\underline{H}^G_\bigstar(X) \ , \ \underline{H}^\bigstar_G(X)$$

are Mackey functor modules over $\underline{H}^G_\bigstar$. It is the cup-product structure that renders $\underline{H}^\bigstar_G(X)$ into a Green functor algebra over the homology of a point, which is why we prefer to use cohomology to homology for general spaces.

1.2.2 Mackey functors and Green functors

We write $\mathrm{Res}^{2^i}_{2^{i-1}}, \mathrm{Tr}^{2^i}_{2^{i-1}}$ for the restriction and transfer map along the subgroup inclusion $C_{2^{i-1}} \subseteq C_{2^i}$ in a C_{2^n}-Mackey functor.

We shall use Lewis diagrams to represent Mackey functors. For example, a C_4-Mackey functor $\underline{M}$ has Lewis diagram:

$$\begin{array}{c}
\underline{M}(C_4/C_4) \\
\mathrm{Res}^4_2 \downarrow \ \uparrow \mathrm{Tr}^4_2 \\
\underline{M}(C_4/C_2) \circlearrowleft C_4/C_2 \\
\mathrm{Res}^2_1 \downarrow \ \uparrow \mathrm{Tr}^2_1 \\
\underline{M}(C_4/e) \circlearrowleft C_4
\end{array}$$

When $G = C_{2^n}$ the Lewis diagram is always a tower with top level $\underline{M}(G/G)$ and bottom level $\underline{M}(G/e)$. When $G = C_4$ we shall refer to $\underline{M}(C_4/C_2)$ as the middle level.

The initial Green functor is the Burnside Green functor $\underline{A}_{\mathbb{Z}}$. There are three associated

Green functors that we shall use as our coefficients throughout this dissertation:

- The rational Burnside Green functor $\underline{A}_{\mathbb{Q}}$, obtained by rationalizing $\underline{A}_{\mathbb{Z}}$.
- The constant Green functor $\underline{\mathbb{Z}}$, obtained by enforcing the relations

$$\mathrm{Tr}_H^K(1) = |K : H|$$

  in $\underline{A}_{\mathbb{Z}}$, for all subgroups $H \subseteq K$.

- The constant Green functor $\underline{\mathbb{F}}_2$, obtained from $\underline{\mathbb{Z}}$ by reducing modulo 2.

In every Green functor, we have the Frobenius relation:

$$\mathrm{Tr}_H^K(x \operatorname{Res}_H^K y) = \mathrm{Tr}_H^K(x)y \tag{1.1}$$

1.2.3 The top level

We shall generally only be interested in the top level of $\underline{H}_\bigstar^G(X)$. This is because all lower levels can be computed by induction on the group G:

$$\underline{H}_\bigstar^G(X)(G/H) = \underline{H}_\bigstar^H(X)(H/H)\ ,\ \bigstar \in RO(G)$$

for any subgroup $H \subseteq G$. So the computation of $\underline{H}_\bigstar^{C_{2^n}}(X)$ by induction on n reduces to the computation of the top level and the determination of the restriction/transfer maps between the $2^n, 2^{n-1}$ levels.

We denote the top level by:

$$H_\bigstar^G(X) := \underline{H}_\bigstar^G(X)(G/G) = \underline{H}_\bigstar^G(X; \underline{R})(G/G)$$

and when $X = *$ is a point:

$$H^G_\bigstar := H^G_\bigstar(*) = \underline{H}^G_\bigstar(G/G)$$

1.2.4 Rational Mackey functors

In this subsection, we use $\underline{A}_{\mathbb{Q}}$ coefficients. [9] prove that

- All rational Mackey functors (i.e. $\underline{A}_{\mathbb{Q}}$ modules) are projective and injective, so we have the Kunneth formula:

$$\underline{H}^G_\bigstar(X \times Y) = \underline{H}^G_\bigstar(X) \boxtimes_{\underline{H}^G_\bigstar} \underline{H}^G_\bigstar(Y)$$

  and duality formula:

$$\underline{H}^\bigstar_G(X) = \underline{Hom}_{\underline{H}^G_\bigstar}(\underline{H}^G_\bigstar(X), \underline{H}^G_\bigstar)$$

- When $G = C_2$ we have the isomorphism of graded Green functors:

$$\underline{H}^*_{C_2}(X) = \begin{array}{c} H^*(X)^{C_2} \\ \downarrow\uparrow \\ H^*(X) \end{array} \oplus \begin{array}{c} H^*(X^{C_2}) \\ \downarrow\uparrow \\ 0 \end{array}$$

  (see section 4.9 for a generalization to all $G = C_{2^n}$).

The second bullet allows us to reduce equivariant computations to nonequivariant ones, as long as we use integer grading $* \in \mathbb{Z}$. Using the first bullet, integer graded cohomology together with the homology of a point recovers the $RO(G)$-graded cohomology:

$$\underline{H}^\bigstar_G(X) = \underline{H}^*_G(X) \boxtimes_{\underline{A}_{\mathbb{Q}}} \underline{H}^\bigstar_G$$

As such, once $\underline{H}^\bigstar_G$ is computed, we need only worry about integer grading.

The homology of a point in $\underline{A}_{\mathbb{Q}}$ itself has special significance, as it agrees with the G-equivariant rational stable stems:

$$\underline{\pi}^G_\bigstar(S)\otimes\mathbb{Q}=\underline{\pi}^G_\bigstar(H\underline{A}_{\mathbb{Q}})=\underline{H}^G_\bigstar$$

1.2.5 The real representation ring of C_{2^n}

The real representation ring $RO(C_{2^n})$ is spanned by the irreducible representations $1,\sigma,\lambda_{s,k}$ where σ is the 1-dimensional sign representation and $\lambda_{s,k}$ is the 2-dimensional representation given by rotation by $2\pi s(k/2^n)$ degrees for $1\leq k$ dividing 2^{n-2} and odd $1\leq s<2^n/k$. Note that 2-locally, $S^{\lambda_{s,k}}\simeq S^{\lambda_{1,k}}$ as C_{2^n}-equivariant spaces, by the s-power map. Therefore, to compute $\underline{H}^{C_{2^n}}_\bigstar(X)$, it suffices to only consider $\bigstar$ in the span of $1,\sigma,\lambda_k:=\lambda_{1,2^k}$ for $0\leq k\leq n-2$ ([11]). Note that $\lambda_{n-1}=2\sigma$ and $\lambda_n=2$. When $G=C_4$ we write λ for λ_0. We shall use ρ to denote the real regular representation of G.

1.2.6 Euler and orientation classes

We shall now define some very useful classes in the $RO(G)$-graded homology of a point. We first have Euler classes $a_\sigma:S^0\hookrightarrow S^\sigma$ and $a_{\lambda_k}:S^0\hookrightarrow S^{\lambda_k}$ given by the inclusion of the north and south poles; under the Hurewicz map these classes become elements in the homology of a point irrespective of coefficients:

$$a_\sigma\in H^G_{-\sigma}\ ,\ a_{\lambda_k}\in H^G_{-\lambda_k}$$

The orientation classes are slightly trickier to define as they depend on the coefficients. If R is a ring and $\underline{R}$ is the corresponding constant Mackey functor, then to any R-orientable real representation V we have by [11]:

$$H^G_{|V|-V}(*;\underline{R}) = \underline{R}$$

We denote a generator of the top level of this Mackey functor by u_V (this is uniquely determined upon choosing an R-orientation for V). By [6] this result also holds for $\underline{R} = \underline{A}_{\mathbb{Q}}$. For $G = C_{2^n}$, we have the following orientation classes:

- When $\underline{R} = \underline{\mathbb{Z}}$ or $\underline{A}_{\mathbb{Q}}$ there are orientation classes:

$$u_\sigma \in \underline{H}^G_{1-\sigma}(*;\underline{R})(G/C_{2^{n-1}}) \ , \ u_{2\sigma} \in H^G_{2-2\sigma}(*;\underline{R}) \ , \ u_{\lambda_k} \in H^G_{2-\lambda_k}(*;\underline{R})$$

- When $\underline{R} = \underline{\mathbb{F}}_2$ there are orientation classes:

$$u_\sigma \in H^G_{1-\sigma}(*;\underline{R}) \ , \ u_{\lambda_k} \in H^G_{2-\lambda_k}(*;\underline{R})$$

Essentially, the difference between integral (or rational Burnside) and modulo 2 coefficients is that the representation σ is only orientable over $\mathbb{F}_2$ while 2σ is orientable over any choice of coefficients. Thus, u_σ does not exist as an element in the top level of the homology of a point for $\underline{R} = \underline{\mathbb{Z}}$ or $\underline{A}_{\mathbb{Q}}$ coefficients but does for $\underline{R} = \underline{\mathbb{F}}_2$. In the latter case, $u_{2\sigma} = u_\sigma^2$.

1.2.7 Quotients

Suppose we have elements $x \in H^G_V$ and $y \in H^G_W$ that are not both 0. We will say that y/x exists if H^G_{W-V} has a cyclic subgroup C such that multiplication by x maps $C \subseteq H^G_{W-V}$ isomorphically onto the cyclic subgroup $\langle y \rangle \subseteq H^G_W$ generated by y:

$$\begin{array}{ccc} H^G_{W-V} & \xrightarrow{x} & H^G_W \\ \uparrow & & \uparrow \\ C & \xrightarrow[\approx]{x} & \langle y \rangle \end{array}$$

If C is unique with this property, then the preimage of y under multiplication by x is a single element in H^G_{W-V} denoted by y/x or by:

$$\frac{y}{x}$$

For example, $1/x$ exists iff x is invertible, and in that case $1/x = x^{-1}$ (and we will continue to use the x^{-1} notation for inverses).

However in general, y/x is less ambiguous than $y^{-1}x$ as the latter notation might suggest that y^{-1} exists by itself and is multiplied with x. For instance, when using $G = C_2$ and $\underline{R} = \underline{\mathbb{Z}}$ coefficients, $2/u_{2\sigma}$ exists because $H^{C_2}_{2\sigma-2} \xrightarrow{u_{2\sigma}} H^{C_2}_0 = \mathbb{Z}$ is an isomorphism onto $2\,\mathbb{Z} \subseteq \mathbb{Z}$. On the other hand, $1/u_{2\sigma}$ does not exist.

Let us note here that if the subgroup C in the definition above is not unique, then there are multiple candidates for y/x. This raises a subtle and technical point further expounded upon in [3, 4].

1.2.8 Equivariant classifying spaces

If K is any group, the G-classifying space of K is defined by

$$B_G K = E_G K/K$$

where $E_G K$ is a $G \times K$ space that's K-free and $(E_G K)^\Gamma$ is contractible for any subgroup $\Gamma \subseteq G \times K$ with $\Gamma \cap (\{1\} \times K) = \{1\}$ (a graph subgroup). We will usually take $K = L$ to be a classical Lie group such as $U(m), O(m), SO(m), Sp(m)$.

We do not assume any action of G on K; instead $B_G K$ derives a G action from the action on $E_G K$. The underlying space of $B_G K$ is a model for the usual classifying space BK; in other words, $B_G K$ is BK with an appropriate G-action. For example, if $K = S^1$ then $BS^1 = \mathbb{C}P^\infty$ and $B_{C_4} S^1 = \mathbb{C}P^{\infty\rho}$ is the complex projective plane in $\mathbb{C}^{\infty\rho} = \mathbb{C}^{\infty(1+\sigma+\lambda)}$, namely a generator $g \in C_4$ acts on the homogeneous coordinates by:

$$g(z_0 : z_1 : z_2 : z_3 : z_4 : \cdots) = (z_0 : -z_1 : -z_3 : z_2 : z_4 : \cdots)$$

1.2.9 Borel vs Bredon cohomology

Borel cohomology is given by the homotopy fixed points of a function spectrum:

$$H^*_{G,Borel}(X; \mathbb{Z}) = \pi_* F(X_+, H\,\mathbb{Z})^{hG} = \pi_0 F(\Sigma^{-*} X_+, H\,\mathbb{Z})^{hG}$$

This definition is equivalent to $H^*_{G,Borel}(X; \mathbb{Z}) = H^*(EG \times_G X; \mathbb{Z})$ because

$$F(EG_+ \wedge_G Y, Z) \simeq F(EG_+ \wedge Y, Z)^G \simeq F(EG_+, F(Y, Z))^G = F(Y, Z)^{hG}$$

for any G-spectra Y, Z with Z split ([21, pg. 198]). In our case, we take $Y = \Sigma^\infty X_+$ and $Z = H\underline{\mathbb{Z}}$.

We can extend the integer grading to $RO(G)$ grading by:

$$H^{\bigstar}_{G,Borel}(X; \mathbb{Z}) := \pi_0 F(\Sigma^{-\bigstar} X_+, H\,\mathbb{Z})^{hG}$$

By comparison, Bredon cohomology is given by the genuine fixed points of the same function spectrum:

$$H^{\bigstar}_G(X; \underline{\mathbb{Z}}) = \pi_0 F(\Sigma^{-\bigstar} X_+, H\underline{\mathbb{Z}})^G$$

For a general G-spectrum Y, there is always a map $Y^G \to Y^{hG}$ which in our case, induces a map from Bredon to Borel cohomology. This map is localization on all orientation classes:

$$H^{\bigstar}_{G,Borel}(X; \mathbb{Z}) = S^{-1} H^{\bigstar}_G(X; \underline{\mathbb{Z}})$$

where $S = \{u_{2\sigma}, u_{\lambda_k}, 0 \leq k \leq n-2\}$. This is also true when using $\underline{\mathbb{F}}_2$ coefficients (replacing $\mathbb{Z}$ by $\mathbb{F}_2$ and $u_{2\sigma}$ by u_σ).

1.2.10 The Tate diagram

We can better understand the difference between Y^G and Y^{hG} for a G-spectrum Y using the Tate diagram.

Let $\widetilde{EG}$ be the cofiber of the collapse map $EG_+ \to S^0$. We use the notation $Y_h = EG_+ \wedge Y$, $\widetilde{Y} = \widetilde{EG} \wedge Y$, $Y^h = F(EG_+, Y)$ and $Y^t = \widetilde{Y^h}$.

The Tate diagram ([9]) then takes the form:

$$\begin{array}{ccccc} Y_h & \longrightarrow & Y & \longrightarrow & \widetilde{Y} \\ \downarrow \simeq & & \downarrow & & \downarrow \\ Y_h & \longrightarrow & Y^h & \longrightarrow & Y^t \end{array}$$

The square on the right is a homotopy pullback diagram and is called the Tate square.

Applying $\pi^G_\bigstar$ on the Tate diagram gives:

$$\begin{array}{ccccc} Y_{hG\bigstar} & \longrightarrow & Y^G_\bigstar & \longrightarrow & \widetilde{Y}^G_\bigstar \\ \downarrow \approx & & \downarrow & & \downarrow \\ Y_{hG\bigstar} & \longrightarrow & Y^{hG}_\bigstar & \longrightarrow & Y^{tG}_\bigstar \end{array}$$

The rows give long exact sequences and there is another long exact sequence coming from the square on the right (homotopy pullback).

CHAPTER 2

THE $RO(C_4)$ INTEGRAL HOMOLOGY OF A POINT

2.1 Introduction

This chapter partially covers the results of [3]; the reader is referred to [3] for more details and complete proofs.

The computation of the $RO(G)$-graded homology of a point

$$\underline{H}^G_\bigstar := \underline{\pi}^G_\bigstar(H\underline{R}) = \underline{H}^G_\bigstar(*;\underline{R}) = \underline{H}_G^{-\bigstar}(*;\underline{R})$$

has been historically a very difficult problem. Stong and Lewis completely determined it for $G = C_p$, the cyclic group of prime order p, using coefficients $\underline{R} = \underline{A}_{\mathbb{Z}}$ ([16]).

In this chapter we take $G = C_4$ and $\underline{R} = \underline{\mathbb{Z}}$. Since $RO(C_4)$ is spanned by $1, \sigma, \lambda$ this boils down to computing

$$\underline{\widetilde{H}}^{C_4}_k(S^{n\sigma+m\lambda})$$

for $k, n, m \in \mathbb{Z}$.

When $n, m \geq 0$, we have an explicit and simple equivariant cellular decomposition for the space $S^{n\sigma+m\lambda}$ and we can compute the homology using the cellular chain complex $\underline{C}_* S^{n\sigma+m\lambda}$. When $n, m \leq 0$, we can appeal to Spanier Whitehead Duality:

$$\underline{\widetilde{H}}^{C_4}_k(S^{n\sigma+m\lambda}) = \underline{\widetilde{H}}^{-k}_{C_4}(S^{-n\sigma-m\lambda})$$

This is the homology of the cochain complex $\underline{C}^{-*}(S^{|n|\sigma+|m|\lambda})$ dual to the chain complex $\underline{C}_{-*}(S^{|n|\sigma+|m|\lambda})$ we had before.

The more difficult part of the computation is when $nm < 0$, for example when $n >$

$0, m < 0$. In this case, we could in principle work with the box product of chain complexes

$$\underline{C}_* S^{n\sigma} \boxtimes \underline{C}^{-*} S^{|m|\lambda}$$

but these complexes get intractably large for calculations by hand as $n, |m|$ get large. In place of that, we instead make use of three algebraic spectral sequences associated to these complexes: Two Atiyah-Hirzebruch spectral sequences and a Kunneth spectral sequence. Comparison of these three allows us to get the answer through fairly intuitive (if lengthy) arguments ([3]). A complication is that everything needs to be performed on the Mackey functor level: for example, the Tor terms in the Kunneth spectral sequence are computed in the symmetric monoidal category of $\underline{\mathbb{Z}}$-modules.

The main result is that $\underline{H}_{\bigstar}^{C_4}$ is generated, in a generalized sense, by the Euler and orientation classes $a_\sigma, a_\lambda, u_{2\sigma}, u_\lambda$. These classes, under the operations of multiplication, division, restriction and transfer, don't quite generate the entire $\underline{H}_{\bigstar}^{C_4}$, missing the generator s of

$$\widetilde{H}_{-3}^{C_4}(S^{-2\lambda}) = \mathbb{Z}/4$$

However, it turns out that this $\mathbb{Z}/4$ fits in a short exact sequence of abelian groups

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

where the generators of the $\mathbb{Z}/2$'s are obtainable using just the Euler and orientation classes. So if we include this group extension into our list of "operations", then the closure of the Euler and orientation classes under said operations is the entire $\underline{H}_{\bigstar}^{C_4}$. To be more precise, since we are interested in the homology as a Mackey functor, we shouldn't adjoin a group extension but rather the Mackey functor extensions that induce it.

At this point, we should mention earlier work by Zeng on this topic. [29] calculates the integer coefficient $RO(C_{p^2})$-graded homology of a point for all primes p, using the associated

Tate-square diagram as opposed to the cellular chains approach we use here. His description for the multiplicative structure is in terms of the connecting homomorphism of certain cofiber sequences, while our description is solely in terms of the Euler and orientation classes. Modulo this difference, our results agree with his for the case $p = 2$.

Another novelty in this work is the computerization of this computation, not just for $G = C_4$ but indeed for any $G = C_{p^n}$. We have devised a computer program that automatically produces the answer for both the additive and multiplicative structures of $\underline{\pi}^G_\bigstar(H\underline{\mathbb{Z}})$ or more generally $\underline{\pi}^G_\bigstar(H\underline{R})$ where $\underline{R}$ is constant coefficients in a user specified ring such as $\mathbb{F}_p$ or $\mathbb{Q}$. It can also compute the Massey products present in $\underline{\pi}^G_\bigstar(H\underline{R})$ together with their indeterminacy. Of course the program can only work in a finite range, i.e. it can produce the answer for S^V where the dimension of V is bounded.

We have used the program to verify our results in a finite range, predict/verify a d^2 differential in the spectral sequence of section 3.5 and formulate a conjecture for $\underline{\pi}^{C_{2^n}}_\bigstar(H\underline{\mathbb{Z}})$ and all $n = 1, 2, ...$ based on computational data gathered for groups $G = C_2, C_4, C_8, C_{16}$ and C_{32} (see section 2.4).

The source code is publicly available on `github.com/NickG-Math/Mackey`, where the interested reader can not only inspect it, but also contribute to its improvement and expansion, which we highly encourage.

In this chapter, we always use $\underline{\mathbb{Z}}$ coefficients.

2.2 Organization

This chapter is organized as follows:

- Section 2.3 gives a brief description of $\underline{H}^{C_4}_\bigstar$ (see [3] for the complete answer and some subtleties we do not go into here).

- Section 2.4 contains a conjecture for $\underline{H}^{C_{2^n}}_\bigstar$, $n = 1, 2, ...$

- Section 2.5 gives an overview of the method used in [3] to compute $\underline{H}^{C_4}_{\bigstar}$.
- Section 2.6 describes how the computer program works on a high level and gives an example of typical usage.

2.3 Generators and Relations

The Euler and orientation classes generate the following homology groups:

- a_σ generates $H^{C_4}_{-\sigma} = \mathbb{Z}/2$
- a_λ generates $H^{C_4}_{-\lambda} = \mathbb{Z}/4$
- $u_{2\sigma}$ generates $H^{C_4}_{2-2\sigma} = \mathbb{Z}$
- u_λ generates $H^{C_4}_{2-\lambda} = \mathbb{Z}$
- u_σ generates $\underline{H}^{C_4}_{1-\sigma}(C_4/C_2) = \mathbb{Z}_-$; the minus indicates that the Weyl group action of $C_4/C_2 = \langle g \rangle$ is given by $g \cdot u_\sigma = -u_\sigma$.

We have $\mathrm{Res}^4_2(u_{2\sigma}) = u_\sigma^2$.

Apart from the Frobenius relation (1.1) which holds for every Green functor, we have the Gold relation ([11]):

$$a_\sigma^2 u_\lambda = 2u_{2\sigma} a_\lambda \tag{2.1}$$

The Euler and orientation classes generate multiplicatively all of $\widetilde{H}^{C_4}_*(S^{n\sigma+m\lambda})$ for $n, m \geq 0$ under the Gold relation.

To generate $\widetilde{H}_*^{C_4}(S^{-n\sigma-m\lambda})$ we must also include the elements:

$$\frac{2}{u_{2\sigma}^i}, \quad \frac{4}{u_\lambda^j}\ , \quad \frac{4}{u_{2\sigma}^i u_\lambda^j}$$
$$w_n := \mathrm{Tr}_2^4(u_\sigma^{-n})\ ,\ n \geq 3 \text{ and } n \text{ odd}$$
$$x_{n,m} := \mathrm{Tr}_1^4\left(\mathrm{Res}_1^2(u_\sigma)^{-n}\,\mathrm{Res}_1^4(u_\lambda)^{-m}\right)\ ,\ n, m \geq 1 \text{ and } n \text{ odd}$$
$$\frac{w_n}{a_\sigma^i a_\lambda^j u_\lambda^k}, \quad \frac{x_{n,m}}{a_\sigma^i}$$

Here, $i, j, k \geq 0$. We don't consider w_1 because $\mathrm{Tr}_2^4(u_\sigma^{-1}) = 0$. The $w_n, x_{n,m}$ are all 2-torsion elements.

The first element in $\widetilde{H}_*^{C_4}(S^{-n\sigma-m\lambda})$ not obtained by Euler and orientation classes through the operations of multiplication, division (wherever possible), transfers and restrictions is the generator

$$s \in H_{2\lambda-3}^{C_4} = \widetilde{H}_{-3}^{C_4} S^{-2\lambda} = \mathbb{Z}/4$$

The remaining elements of $H_\bigstar^{C_4}$ are:

$$\frac{s}{u_{2\sigma}^i a_\lambda^j u_\lambda^k}, \quad \frac{u_\lambda}{u_{2\sigma}^i}, \quad \frac{2a_\lambda}{a_\sigma u_{2\sigma}^i}, \quad \frac{2u_{2\sigma}}{u_\lambda}, \quad \frac{4u_{2\sigma}}{u_\lambda^i}, \quad \frac{a_\sigma^2}{a_\lambda}, \quad \frac{a_\sigma^3}{a_\lambda^m}$$

We also have the relations:

$$2s = w_3 \frac{a_\sigma^3}{a_\lambda^2}$$
$$a_\sigma s = \mathrm{Tr}_2^4\left(\frac{2u_\sigma}{\mathrm{Res}_2^4(u_\lambda^2)}\right)$$

In the second equation, multiplication by a_σ is the projection $\mathbb{Z}/4 \to \mathbb{Z}/2$ so by reducing modulo 2 we can write:

$$s \equiv \frac{\mathrm{Tr}_2^4\left(\frac{2u_\sigma}{\mathrm{Res}_2^4(u_\lambda^2)}\right)}{a_\sigma} \mod 2$$

We have expressed $2s$ and s mod 2 in terms of Euler and orientation classes which means that s is obtained from Euler and orientation classes through the extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

This actually arises from an extension of Mackey functors ([3]).

As we have observed, every element can be written in terms of the Euler and orientation classes under the operations of multiplication, division, transfer, restriction and the extension giving s. Moreover, the only relation is the Gold relation (2.1), in the sense that all other relations can be obtained from it, the Frobenius relation (1.1) and relations of the form:

$$x\frac{y}{x} = y$$

See [3] for a precise formulation of this "presentation" of $\underline{H}_{\bigstar}^{C_4}$ in terms of generators and relations.

2.4 A conjecture for $G = C_{2^n}$

For $G = C_{2^n}$, the Mackey functor $\underline{\widetilde{H}}_{-3}^{G} S^{-2\lambda_n}$ is on each orbit:

$$\underline{\widetilde{H}}_{-3}^{G} S^{-2\lambda_n}(G/C_{2^k}) = \mathbb{Z}/2^k$$

for $k > 0$ and $\underline{\widetilde{H}}_{-3}^{G} S^{-2\lambda_n}(G/e) = 0$. Transfers are the usual inclusion maps $\mathbb{Z}/2^k \hookrightarrow \mathbb{Z}/2^l$ for $k \leq l$, while restrictions are the projection maps $\mathbb{Z}/2^l \to \mathbb{Z}/2^k$ for $k \leq l$.

If s_n denotes a generator of $H^G_{2\lambda_n-3} = \mathbb{Z}/2^n$ then

$$s_n a_\sigma = \mathrm{Tr}_1^{2^n}[(\mathrm{Res}_1^{2^{n-1}} u_\sigma)(\mathrm{Res}_1^{2^n} u_{\lambda_n})^{-2}]$$

generating a $\mathbb{Z}/2$. Furthermore, $2s_n$ is the transfer of the $C_{2^{n-1}}$ generator s_{n-1}.

Therefore, by induction, s_n is generated by the Euler and orientation classes of C_{2^k} for $2 \leq k \leq n$ through the extension

$$0 \to \mathbb{Z}/2^{n-1}(2s_n) \to \mathbb{Z}/2^n(s_n) \to \mathbb{Z}/2(s_n a_\sigma) \to 0$$

Conjecture: For all $G = C_{2^n}$, the Euler classes, orientation classes and s_n together generate $\underline{H}^G_\bigstar$ under the operations of multiplication, division, transfer and restriction.

This conjecture has been verified in a finite range for $n \leq 5$. We further expect the Gold relation (2.1) to generate all relations; see [3] for a more precise formulation.

2.5 The computation of $\underline{H}^{C_4}_\bigstar$

In this section, $G = C_4$. To compute $\underline{H}^G_\bigstar$ we first need to compute the additive structure, namely the Mackey functors

$$\widetilde{\underline{H}}^G_*(S^V) = \widetilde{\underline{H}}^G_*(S^{n\sigma+m\lambda})$$

for $n, m \in \mathbb{Z}$. When $n, m \geq 0$ this is easy: $S^{n\sigma+m\lambda}$ is a G-space and has an explicit G-CW structure with skeletal filtration $X_0 \subseteq \cdots \subseteq X_{n+2m} = S^{n\sigma+m\lambda}$ where:

- If $i \leq n$ or $i = n+2k$ for $1 \leq k \leq m$, X_i consists of coordinates $(x_1, ..., x_{i+1}, 0, ..., 0) \in \mathbb{R}^{n+2m+1}$ with $\sum_j x_j^2 = 1$.
- If $i = n+2k+1$ for $1 \leq k \leq m-1$, X_i consists of coordinates $(x_1, ..., x_{i+2}, 0, ..., 0) \in \mathbb{R}^{n+2m+1}$ with $x_{i+1}x_{i+2} = 0$ and $\sum_j x_j^2 = 1$.

The cells we attach, namely the cofibers of $X_{i-1} \hookrightarrow X_i$, are $C_{4+}/C_{2+} \wedge S^i$ when $i \leq n$ and $C_{4+} \wedge S^i$ when $n < i \leq n + 2m$. This means that we have a simple and explicit description of the chains

$$\underline{C}_*(S^{n\sigma+m\lambda})$$

from which we can compute the homology.

When $n, m \leq 0$, $S^{n\sigma+m\lambda}$ is not a space, but we can appeal to Spanier-Whitehead duality:

$$\underline{\widetilde{H}}_*^G(S^{n\sigma+m\lambda}) = \underline{\widetilde{H}}_G^{-*}(S^{-n\sigma-m\lambda})$$

is the cohomology of the cochain complex

$$\underline{C}^*(S^{|n|\sigma+|m|\lambda})$$

up to a regrading. This is computed directly from the equivariant CW decomposition of $S^{|n|\sigma+|m|\lambda}$.

The problematic cases are then $\underline{\widetilde{H}}_*^G(S^{n\sigma-m\lambda})$ and $\underline{\widetilde{H}}_*^G(S^{-n\sigma+m\lambda})$ for $n, m \geq 0$. Without loss of generality, let us only deal with $\underline{\widetilde{H}}_*^G(S^{n\sigma-m\lambda})$, which is the homology of

$$\underline{C}_*(S^{n\sigma-m\lambda}) \simeq \underline{C}_*(S^{n\sigma}) \boxtimes \underline{C}^{-*}(S^{m\lambda}) \tag{2.2}$$

Such box products get intractably large to compute by hand (see section 2.6), so we instead filter them by smaller subcomplexes and get spectral sequences. In general, for any tensor product of chain complexes $C \otimes D$ in a sufficiently good symmetric monoidal abelian category (like that of Mackey functors), we have three spectral sequences converging to $H_*(C \otimes D)$. If we filter the double complex underlying the tensor product either horizontally or vertically, we get two spectral sequences with E_2 terms

$$E_2 = H_*(C; H_*D) \implies H_*(C \otimes D)$$
$$E_2 = H_*(D; H_*C) \implies H_*(C \otimes D)$$

Using Cartan-Eilenberg resolutions we obtain a Kunneth spectral sequence

$$E_2 = \mathrm{Tor}^{*,*}(H_*C, H_*D) \implies H_*(C \otimes D)$$

See [27] and [22] for more details on these spectral sequences. In our case of $\underline{C}_*(S^V) \boxtimes \underline{C}^{-*}(S^W)$ (where $V = n\sigma, W = m\lambda$) the spectral sequences take the form:

$$\begin{aligned} \underline{E}^2_{p,q} &= \underline{H}^G_p(S^V, \underline{H}^{-q}_G S^W) \implies \underline{H}^G_{p+q} S^{V-W} \\ \underline{E}^{p,q}_2 &= \underline{H}^p_G(S^W, \underline{H}^G_{-q} S^V) \implies \underline{H}^G_{p+q} S^{V-W} \\ \underline{E}^2_{p,q} &= \underline{\mathrm{Tor}}^{p,q}(\underline{H}^G_p S^V, \underline{H}^G_q S^{-W}) \implies \underline{H}^G_{p+q} S^{V-W} \end{aligned} \tag{2.3}$$

These are all spectral sequences of $\underline{\mathbb{Z}}$-modules and the final one uses the Tor in the symmetric monoidal category of $\underline{\mathbb{Z}}$-modules. These three spectral sequences can also be obtained topologically: the first two are the Atiyah-Hirzebruch spectral sequences for the homology theory H_V and the final one is the topological Kunneth spectral sequence. See also [17].

These spectral sequences are computed in [3]. The idea is to use all three and compare them as needed: vanishing in a certain degree may be trivial for one but nontrivial for the other. In this way, we can leverage partial information from each to complete the computation. This work is too technical to present here, so we refer the reader to [3] for full details.

2.5.1 The three spectral sequences: an example

We include here an example for how the method in general works. We use the abbreviations HSS, CSS, KSS for the three spectral sequences in (2.3) in that order. We also use [3]'s notation of C_4-Mackey functors explained in table 2.6.2.

We depict the E_2 page of the HSS for $S^{4\sigma-3\lambda}$ in figure 2.1 using the Serre (p,q) grading. The E_2 terms are computed directly, the arrows denote possible nontrivial differentials and the dashed lines indicate possible extensions (if these terms survive to the E_∞ page). We shall prove that the d^2's are nontrivial; this leads to figure 2.2 for the E_3 page. The indicated d^3 turns out to be nontrivial as well leading to figure 2.3 for the $E_4 = E_\infty$ page.

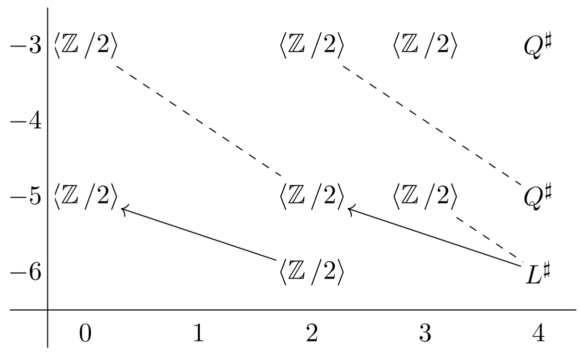


Figure 2.1: E_2 page of the HSS for $S^{4\sigma-3\lambda}$

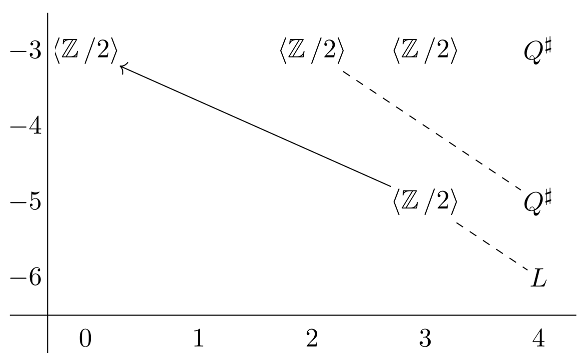


Figure 2.2: E_3 page of the HSS for $S^{4\sigma-3\lambda}$

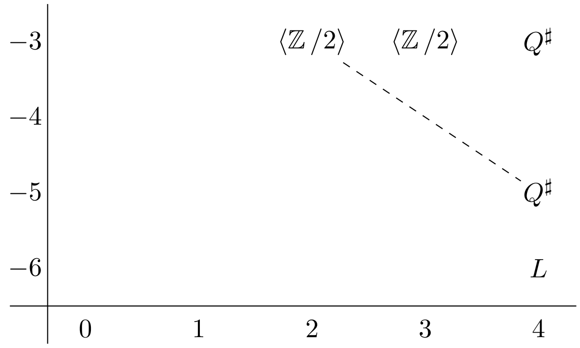


Figure 2.3: $E_\infty = E_4$ page of the HSS for $S^{4\sigma-3\lambda}$

To prove that all three differentials are nontrivial, we can assume otherwise and compare the answer given here with that given by the CSS that we've drawn in figures 2.4 and 2.5 (using again the Serre grading).

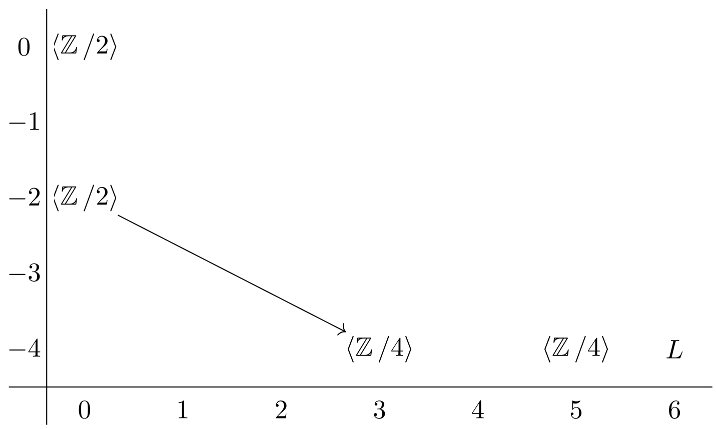


Figure 2.4: E_3 page of the CSS for $S^{4\sigma-3\lambda}$

0	$\langle \mathbb{Z}/2 \rangle$						
-1							
-2							
-3							
-4				$Q^\sharp$		$\langle \mathbb{Z}/4 \rangle$	L
	0	1	2	3	4	5	6

Figure 2.5: $E_\infty = E_4$ page of the CSS for $S^{4\sigma-3\lambda}$

In the E_2 page of the HSS, if the differential out of $L^\sharp$ were to vanish, we would obtain an extension of $L^\sharp, \langle \mathbb{Z}/2 \rangle$ at degree -2 and such an extension can never be L (which is what the CSS predicts is the answer at degree -2). So this differential has to be nontrivial. The other two differentials in the HSS are analogously shown to be nontrivial. The differential in the CSS is nontrivial as we must get $Q^\sharp$ at degree -1 and $\langle \mathbb{Z}/2 \rangle$ at degree 0 by the HSS.

The KSS is depicted in figure 2.6.

-1			$Q^\sharp$
-2	L	$\langle \mathbb{Z}/4 \rangle$	$\langle \mathbb{Z}/2 \rangle$
	0	1	2

Figure 2.6: $E_2 = E_\infty$ page of the KSS for $S^{4\sigma-2\lambda} \wedge S^{-\lambda}$

2.6 The C++ library `mackey`

The computations in section 2.5 depend on filtering box products such as (2.2) in different ways and comparing the resulting spectral sequences. Ideally, we would be working directly with that box product, but there are two major complications that prohibit this: Firstly, the box product of Mackey functors is not the level-wise tensor product. Instead, only the bottom level (corresponding to the orbit G/e) can be obtained as the tensor product, while all the higher levels are obtained by transferring (our chains consist solely of free Mackey functors). Secondly, the bottom level tensor product itself gets arbitrarily large as we increase n, m in (2.2), making it impractical to compute with it.

The idea underlying computer computations is that our chains consist solely of free Mackey functors over $\underline{\mathbb{Z}}$, so every differential can be completely described by a matrix with integer entries. The operations of transfer, restriction and group action can all be performed algorithmically for free Mackey functors, and their effect can be described in terms of these matrices. Similarly, the tensor product can also be computed algorithmically, and then the box product is just obtained by transferring it to higher levels. At the final step, we need to take homology and that can be achieved via a Smith Normal Form algorithm over $\mathbb{Z}$.

There are a few more technicalities in this procedure that we haven't addressed here, but once these details are dealt with, this process allows us to algorithmically compute the additive structure of the $RO(G)$ homology, in any given range for our representations (for $G = C_4$, this amounts to a given range for n, m in $S^{n\sigma+m\lambda}$).

For the multiplicative structure we need to be able to compute the product of any two generators. Just like with tensor products, this can be directly performed only on the bottom level. If the generators live in a higher level, the idea is to first restrict them to the bottom level, multiply these restrictions, and then invert the restriction map. This is possible because in free Mackey functors, restrictions are injective, and our chain complexes consist exclusively of such Mackey functors.

There is a final algorithm that allows us to automatically write our generators in terms of Euler and orientation classes (like in sections 2.3 and 3.3). This "factorization" algorithm works by forming a multiplication table for the $RO(G)$ homology, and then turning this table into a colored graph, somewhat analogous to the Cayley graph of a group. There are two colors, corresponding to multiplication and division, and traversing this graph is equivalent to factorizing elements.

This chains-based approach also works remarkably well with Massey products. And indeed, our program can compute Massey products, and their indeterminacy, directly from their definition. Finally, we can replace $\underline{\mathbb{Z}}$ with other constant Green functors such as $\underline{\mathbb{F}}_2$.

From an implementation standpoint, a significant challenge is memory usage: the naive approach to storing chain complexes requires extreme amounts of memory for groups $G = C_8$ and beyond. It is thus critical to use a sparse matrix format to store the differentials: these differential matrices are extremely sparse in the sense that 99% of their entries are 0 and thus we need not waste memory storing said zeros. We also use a variant of algebraic Morse theory (see [15]) that preserves equivariance. This allows us to reduce our chain complexes to smaller ones in the same equivariant homotopy type. With this reduction, we can compress box products of chain complexes anywhere between 30% to 90%, with larger chain complexes leading to better compression ratios.

The interested reader can find these details and much more, together with the source code, documentation and demo binaries on `github.com/NickG-Math/Mackey`. Subsection 2.6.1 includes a short tutorial for the curious reader to get an idea of the input/output of the program. A much more detailed tutorial is present on `nickg-math.github.io/Mackey/html/index.html`.

2.6.1 A typical use case

First we set up our C++ working environment as explained in `nickg-math.github.io/Mackey/html/tuto.html`.

Next, we can compute the additive structure of $\widetilde{\underline{H}}_*^{C_4}(S^{n\sigma+m\lambda})$ in a range, say for $-1 \leq n \leq 5, -4 \leq m \leq 6$, using:

```
std::cout « mackey::AdditiveStructure<C4<int64_t»({-1,-4},{5,6});
```

Here the template `C4<int64_t>` indicates that we are using group C_4 and constant $\underline{\mathbb{Z}}$ coefficients (with 64bit precision).

Running the code will print the result to the console in the following format:

```
The k=-9 homology of the -1,-4 sphere is 112#01
```

which means:

$$\underline{\widetilde{H}}^{C_4}_{-9}(S^{-\sigma-4\lambda}) = 112\#01 = \begin{array}{c} \mathbb{Z}/2 \\ \big\downarrow \big\uparrow 1 \\ \mathbb{Z} \circlearrowleft -1 \\ 2 \big\downarrow \big\uparrow 1 \\ \mathbb{Z} \circlearrowleft -1 \end{array}$$

See subsection 2.6.2 for an explanation of the computer notation `112#01`.

If we want to multiply two top level generators together, say

$$\widetilde{H}^{C_4}_0(S^{2\sigma-2\lambda}) \otimes \widetilde{H}^{C_4}_{-4}(S^{-2\sigma-\lambda}) \to \widetilde{H}^{C_4}_{-2}(S^{-2\sigma})$$

we can use the code:

```
std::cout « mackey::ROGreen<C4<int64_t»(2, {-4, -2, -1}, {2, 0, 1});
```

Here the first input 2 signifies that we are using the top level C_4/C_4 ($4 = 2^2$). Running the code will print `2` to the console, which indicates

$$ab = 2c$$

where a, b, c are the generators of

$$\widetilde{H}^{C_4}_0(S^{2\sigma-2\lambda}), \widetilde{H}^{C_4}_{-4}(S^{-2\sigma-\lambda}), \widetilde{H}^{C_4}_{-2}(S^{-2\sigma})$$

respectively. And indeed, we have:

$$\frac{4}{u_{2\sigma}u_\lambda} \cdot u_\lambda = 2 \cdot \frac{2}{u_{2\sigma}}$$

See `nickg-math.github.io/Mackey/html/tuto.html` for more examples including printing factorizations of all elements (in a range) in terms of Euler and orientation classes, computing Massey products and much more.

2.6.2 Mackey functor notation

The library `mackey` uses the notation

$$a_0 \cdots a_n \sharp b_0 \cdots b_m$$

to describe the following C_{2^n}-Mackey functor $\underline{M}$:

$$\underline{M}(C_{2^n}/C_{2^i}) = \begin{cases} \mathbf{Z}/a_i & \text{if } a_i \neq 1 \\ \mathbf{Z} & \text{if } a_i = 1 \end{cases}$$

$$Tr_{2^i}^{2^{i+1}}(x) = \begin{cases} x & \text{if } i = b_j \text{ for some } j \\ 2x & \text{otherwise} \end{cases}$$

$$Res_{2^i}^{2^{i+1}}(x) = \begin{cases} 2x & \text{if } i = b_j \text{ for some } j \\ x & \text{otherwise} \end{cases}$$

where $0 \leq b_0 < \cdots < b_m < n$. The Weyl group actions are determined by the double coset formula. Every Mackey functor appearing in $\underline{H}_\bigstar^{C_{2^n}}$ is a sum of Mackey functors of the above form when $n \leq 2$; if $n > 2$ there are "exceptional" Mackey functors that cannot be written in this notation.

There are various notations for C_4 Mackey functors in the literature and the following table provides a glossary for them.

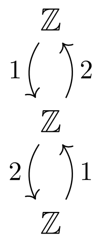
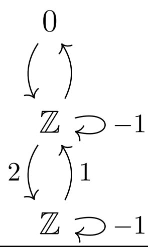
Table 2.1: Glossary of C_4-Mackey functor notations

Lewis Diagram	Notation in [12]	Notation in [3]	Computer notation
$\mathbb{Z}$ $1 \downarrow \uparrow 2$ $\mathbb{Z}$ $1 \downarrow \uparrow 2$ $\mathbb{Z}$	$\square$	$\underline{\mathbb{Z}}$	111
0 $\downarrow \uparrow$ $\mathbb{Z} \circlearrowleft -1$ $1 \downarrow \uparrow 2$ $\mathbb{Z} \circlearrowleft -1$	$\overline{\square}$	$\underline{\mathbb{Z}}_-$	110
$\mathbb{Z}/4$ $1 \downarrow \uparrow 2$ $\mathbb{Z}/2$ $\downarrow \uparrow$ 0	$\circ$	$\langle \mathbb{Z}/4 \rangle$	024
$\mathbb{Z}/2$ $\downarrow \uparrow$ 0 $\downarrow \uparrow$ 0	$\bullet$	$\langle \mathbb{Z}/2 \rangle$	002
0 $\downarrow \uparrow$ $\mathbb{Z}/2$ $\downarrow \uparrow$ 0	$\overline{\bullet}$	$\overline{\langle \mathbb{Z}/2 \rangle}$	020

Glossary of C_4-Mackey functor notations (continued)

$\mathbb{Z}$ $2 \downarrow \; \uparrow 1$ $\mathbb{Z}$ $2 \downarrow \; \uparrow 1$ $\mathbb{Z}$	◪	L	111#01
$\mathbb{Z}$ $2 \downarrow \; \uparrow 1$ $\mathbb{Z}$ $1 \downarrow \; \uparrow 2$ $\mathbb{Z}$	⬕	$p^*L = L^\flat$	111#1
$\mathbb{Z}/2$ $\downarrow \; \uparrow 1$ $\mathbb{Z} \circlearrowleft -1$ $2 \downarrow \; \uparrow 1$ $\mathbb{Z} \circlearrowleft -1$	$\dot{⬕}$	L_-	112#01
$\mathbb{Z}/2$ $\downarrow \; \uparrow 1$ $\mathbb{Z} \circlearrowleft -1$ $1 \downarrow \; \uparrow 2$ $\mathbb{Z} \circlearrowleft -1$	$\dot{□}$	$p^*L_- = L_-^\flat$	112#1
$\mathbb{Z}/2$ $0 \downarrow \; \uparrow 1$ $\mathbb{Z}/2$ $\downarrow \; \uparrow$ 0	▼	Q	022#1
$\mathbb{Z}/2$ $1 \downarrow \; \uparrow 0$ $\mathbb{Z}/2$ $\downarrow \; \uparrow$ 0	▲	$Q^\sharp$	022

Glossary of C_4-Mackey functor notations (continued)

		$L^\sharp$	111#0
		$\mathbb{Z}_-^\flat$	110#0

CHAPTER 3

THE $RO(C_4)$ COHOMOLOGY OF $B_{C_4}O(1)$

3.1 Introduction

This chapter partially covers the content of [4, 5]; the reader is referred to [4, 5] for more details and complete proofs.

The computation of the classical (nonequivariant) dual Steenrod algebra

$$(H\,\mathbb{F}_2)_*(H\,\mathbb{F}_2) = \mathbb{F}_2[\xi_i]$$

relies on the construction of the Milnor generators ξ_i. These generators can be defined through the completed coaction of the dual Steenrod algebra on the cohomology of the classifying space $BO(1) = B\Sigma_2 = \mathbb{R}\,P^\infty$: $H^*(B\Sigma_2; \mathbb{F}_2) = \mathbb{F}_2[x]$ and the completed coaction $\mathbb{F}_2[x] \to (H\,\mathbb{F}_2)_*(H\,\mathbb{F}_2)[[x]]$ is:

$$x \mapsto \sum_i x^{2^i} \otimes \xi_i$$

In the C_2-equivariant world, the space replacing $B\Sigma_2$ is the equivariant classifying space $B_{C_2}\Sigma_2$. This is still $\mathbb{R}\,P^\infty$ but now equipped with a nontrivial C_2 action, given in homogeneous coordinates by:

$$g \cdot (x_0 : x_1 : x_2 : x_3 : \cdots) = (x_0 : -x_1 : x_2 : -x_3 : \cdots)$$

Over the homology of a point, we no longer have a polynomial algebra on a single generator x, but rather a polynomial algebra on two generators c, b modulo a single relation:

$$H^{\bigstar}_{C_2}(B_{C_2}\Sigma_2; \underline{\mathbb{F}}_2) = \frac{H^{\bigstar}_{C_2}(*; \underline{\mathbb{F}}_2)[c, b]}{c^2 = a_\sigma c + u_\sigma b}$$

([14]). As a module, this is still free over the homology of a point, and the completed coaction is given by:

$$c \mapsto c \otimes 1 + \sum_i b^{2^i} \otimes \tau_i$$

$$b \mapsto \sum_i b^{2^i} \otimes \xi_i$$

The τ_i, ξ_i are the C_2-equivariant analogues of the Milnor generators, and Hu-Kriz show that they span the genuine dual Steenrod algebra under the family of relations:

$$\tau_i^2 = \tau_{i+1} a_\sigma + \xi_{i+1}(u_\sigma + \tau_0 a_\sigma)$$

Attempting to do the same for C_4 involves computing the cohomology of $B_{C_4}\Sigma_2$ which turns out to be significantly more complicated (see section 3.4) and most importantly is *not* a free module over the homology of a point. In fact, it's not even flat (Proposition 3.4.1) bringing into question whether we even have a coaction by the dual Steenrod algebra in this case.

There is another related reason to consider the space $B_{C_4}\Sigma_2$. In [28], Wilson describes a framework for equivariant total power operations over an $H\mathbb{F}_2$ module A equipped with a symmetric multiplication. The total power operation is induced from a map of spectra

$$A \to A^{t\Sigma_{[2]}}$$

where $(-)^{t\Sigma_{[2]}}$ is a variant Tate construction defined in [28].

In the nonequivariant case, $A \to A^{t\Sigma_{[2]}}$ induces a map $Q : A_* \to A_*((t))$ and the Dyer-Lashof operations Q^i can be obtained as the components of this map:

$$Q(x) = \sum_i Q^i(x) t^i$$

In the C_2 equivariant case, we have a map $Q : A_\bigstar^{C_2} \to A_\bigstar^{C_2}[c, b^{\pm}]/(c^2 = a_\sigma c + u_\sigma b)$ and we get power operations

$$Q(x) = \sum_i Q^{i\rho}(x)b^i + \sum_i Q^{i\rho+\sigma}(x)cb^i$$

When $A = H\,\mathbb{F}_2$, $A_\bigstar^{C_2}[c, b^\pm]/(c^2 = a_\sigma c + u_\sigma b)$ is the cohomology of $B_{C_2}\Sigma_2$ localized at the class b.

For C_4 we would have to use the cohomology of $B_{C_4}\Sigma_2$ (localized at a certain class) but that is no longer free, meaning that the resulting power operations would have extra relations between them and further complicating the other arguments in [28].

In this chapter, we always use $\underline{\mathbb{F}}_2$ coefficients.

3.2 Organization

This chapter is organized as follows:

- Section 3.3 gives a brief description of $\underline{H}_\bigstar^{C_4}$ (see [4] for the complete answer and some subtleties we do not go into here).

- Section 3.4 summarizes the result of the computation of $\underline{H}_{C_4}^\bigstar(B_{C_4}\Sigma_2)$ and explains the problem with flatness.

- Section 3.5 gives an overview of the proof of the computation of $\underline{H}_{C_4}^\bigstar(B_{C_4}\Sigma_2)$ in [4].

- Section 3.6 computes the C_{2^n} equivariant *Borel* dual Steenrod algebra in characteristic 2. We also compare with the dual description in [8]. This section is independent of the rest of the chapter and partially covers the content of [5].

3.3 The $RO(C_4)$ homology of a point in $\underline{\mathbb{F}}_2$ coefficients

The reader should compare the description in this section with the integral case of section 2.3.

The classes $a_\sigma, a_\lambda, u_\sigma, u_\lambda$ live in degrees $\bigstar = -\sigma, -\lambda, 1-\sigma, 2-\lambda$ of $H_\bigstar^{C_4}$ respectively. The Gold Relation (2.1) takes the form:

$$a_\sigma^2 u_\lambda = 0$$

To generate $H_\bigstar^{C_4}$ we use, in addition to the Euler and orientation classes, the following quotients:

$$\frac{u_\lambda}{u_\sigma^i}, \frac{a_\sigma^2}{a_\lambda^i}, \frac{\theta}{a_\sigma^i u_\sigma^j a_\lambda^r}, \frac{\frac{\theta}{a_\lambda} a_\sigma^{1+\epsilon}}{u_\sigma^i a_\lambda^j}, \frac{\frac{\theta}{a_\lambda} a_\sigma^{1+\epsilon}}{u_\sigma^r a_\lambda^j u_\lambda^{1+m}} \tag{3.1}$$

where the indices i, j, m range in $0, 1, 2, ...$, r ranges in $\mathbb{Z}$ and ϵ ranges in $0, 1$.

The class θ of (3.1) is defined as

$$\theta = \mathrm{Tr}_2^4 \left(\mathrm{Res}_2^4(u_\sigma)^{-2} \right)$$

We further introduce the elements:

$$x_{n,m} = \mathrm{Tr}_2^4 \left(\mathrm{Res}_2^4(u_\sigma)^{-n} \mathrm{Res}_2^4(u_\lambda)^{-m} \right) = \frac{x_{0,1}}{u_\sigma^n u_\lambda^{m-1}} \ , \ m \geq 1$$

where

$$x_{0,1} = a_\sigma^2 \frac{\theta}{a_\lambda} = \theta \frac{a_\sigma^2}{a_\lambda}$$

With this notation, the final two families of elements in (3.1) take the form:

$$\frac{x_{n,1}}{a_\sigma^\epsilon a_\lambda^i} \ , \ \frac{x_{n,m}}{a_\lambda^i} \ , \ \frac{x_{n,m+1}}{a_\sigma a_\lambda^i}$$

The mod 2 reduction of the element s from section 2.3 is:

$$s := \frac{\frac{\theta}{a_\lambda} a_\sigma}{u_\lambda} u_\sigma = \frac{x_{0,2} u_\sigma}{a_\sigma}$$

3.4 The cohomology of $B_{C_4}\Sigma_2$

Proposition 3.4.1. *There exist elements $e^a, e^u, e^\lambda, e^\rho$ in degrees $\sigma + \lambda, \sigma + \lambda - 2, \lambda, \rho$ of $H^\bigstar_{C_4}(B_{C_4}\Sigma_2)$ respectively, such that*

$$H^\bigstar_{C_4}(B_{C_4}\Sigma_2) = \frac{H^\bigstar_{C_4}\left[e^a, \frac{e^u}{u_\sigma^i}, \frac{e^\lambda}{u_\sigma^i}, e^\rho\right]_{i \geq 0}}{S}$$

See [4] for an explicit description of the relation set S (which contains both module and multiplicative relations), as well as a description of the Mackey functor $\underline{H}^\bigstar_{C_4}(B_{C_4}\Sigma_2)$. As a module over $\underline{H}^\bigstar_{C_4}$, $\underline{H}^\bigstar_{C_4}(B_{C_4}\Sigma_2)$ is not flat.

Proof. See [4]. □

We can describe the restrictions of the classes $e^a, e^u, e^\lambda, e^\rho$ in terms of the Hu-Kriz generators c, b of $H^\bigstar_{C_2}(B_{C_2}\Sigma_2)$ (see (3.1)):

$$\begin{aligned}
\mathrm{Res}^4_2(e^a) &= \mathrm{Res}^4_2(u_\sigma)(a_{\sigma_{C_2}} b + bc) \\
\mathrm{Res}^4_2(e^u) &= \mathrm{Res}^4_2(u_\sigma) u_{\sigma_{C_2}} c \\
\mathrm{Res}^4_2(e^\lambda) &= c^2 \\
\mathrm{Res}^4_2(e^\rho) &= \mathrm{Res}^4_2(u_\sigma) b^2
\end{aligned}$$

Here, we write σ_{C_2} for the C_2 sign representation to distinguish it from the C_4 sign representation σ.

We can also express the map to Borel cohomology terms of our generators:

Proposition 3.4.2. *There is a choice of the degree* 1 *element* w *in*

$$H^{\bigstar}_{C_4,Borel}(B_{C_4}\Sigma_2) = H^{\bigstar}_{C_4,Borel}(*)[w] = \mathbb{F}_2[a_\sigma, a_\lambda, u_\sigma^{\pm}, u_\lambda^{\pm}, w]/a_\sigma^2$$

so that the localization map from Bredon to Borel cohomology is:

$$e^u \mapsto u_\sigma u_\lambda w$$
$$e^\lambda \mapsto u_\lambda w^2$$
$$e^a \mapsto u_\sigma u_\lambda w^3 + u_\sigma a_\lambda w$$
$$e^\rho \mapsto u_\sigma u_\lambda w^4 + a_\sigma u_\lambda w^3 + u_\sigma a_\lambda w^2 + a_\sigma a_\lambda w$$

Proof. See [4]. □

In other words, the one dimensional Stiefel-Whitney class $w_1 = \mathrm{Res}^4_1(w)$ splits into 4 C_4-equivariant Stiefel-Whitney classes $e^u, e^\lambda, e^a, e^\rho$, whose restrictions are w_1, w_1^2, w_1^3, w_1^4 (up to multiplication with $\mathrm{Res}^4_1(u_\sigma), \mathrm{Res}^4_1(u_\lambda)$).

3.5 An overview of the computation of $\underline{H}^{\bigstar}_{C_4}(B_{C_4}\Sigma_2)$

3.5.1 A G-CW decomposition of $B_{C_4}\Sigma_2$

The space $B_{C_4}\Sigma_2 = \mathbb{R}P^{\infty\rho}$ is $\mathbb{R}P^\infty$ with nontrivial C_4 action:

$$g(x_0, x_1, x_2, x_3, x_4, ...) = (x_0, -x_1, -x_3, x_2, x_4, ...)$$

in homogeneous coordinates. It is an infinite dimensional C_4-equivariant CW complex with skeletal filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq \cup X_i = B_{C_4}\Sigma_2$ where:

- If $i \not\equiv 2 \bmod 4$ then X_i consists of coordinates $(x_0, ..., x_i, 0, ...)$.

- If $i \equiv 2 \bmod 4$ then X_i consists of coordinates $(x_0, ..., x_i, x_{i+1}, 0, ...)$ with $x_i x_{i+1} = 0$.

The equivariant cells, namely the cofibers of $X_{i-1} \hookrightarrow X_i$, are:

- $S^{j\rho}$ when $i = 4j$
- $S^{j\rho+\sigma}$ when $i = 4j+1$
- $C_{4+}/C_{2+} \wedge S^{j\rho+\lambda}$ when $i = 4j+2$
- $C_{4+}/C_{2+} \wedge S^{j\rho+\lambda+1}$ when $i = 4j+3$

Applying the homology theory $\underline{H}_{C_4}^{\bigstar}(-)$ on this filtration gives an Atiyah-Hirzebruch spectral sequence of $\underline{H}_{C_4}^{\bigstar}$-modules converging to $\underline{H}_{C_4}^{\bigstar}(B_{C_4}\Sigma_2)$. This spectral sequence has nontrivial differentials and extensions ([4]).

This is in contrast to the C_2 case, where there are no differentials and all extensions are trivial [28]. The reason for this discrepancy is as follows: A CW decomposition similar to the one above for $B_{C_2}\Sigma_2$ has cells of the form $S^{j\rho_{C_2}}, S^{j\rho_{C_2}+\sigma_{C_2}}$ which means that the resulting E_1 page is free as a module over the homology of a point. There are no differentials for purely degree reasons so $E_1 = E_\infty$; by freeness, all extensions are trivial. By comparison, the CW decomposition of $B_{C_4}\Sigma_2$ uses *induced* cells $C_{4+}/C_{2+} \wedge S^V$ where $V = j\rho+\lambda, j\rho+\lambda+1$. This means that the middle level of the E_1 page features 2 dimensional vector spaces $\mathbb{F}_2^2$; since that middle level computes $H_{C_2}^{\bigstar}(B_{C_2}\Sigma_2)$, we must get only 1 dimensional vector spaces in E_∞ by the C_2 computation. This means that there are nontrivial differentials in the middle level, which transfer to give nontrivial differentials in the top level.

The differentials obtained by the above argument are all d^1's (the middle level spectral sequence collapses in $E_2 = E_\infty$). We shall now discuss the existence of nontrivial d^2's which is much more subtle (see also [4]).

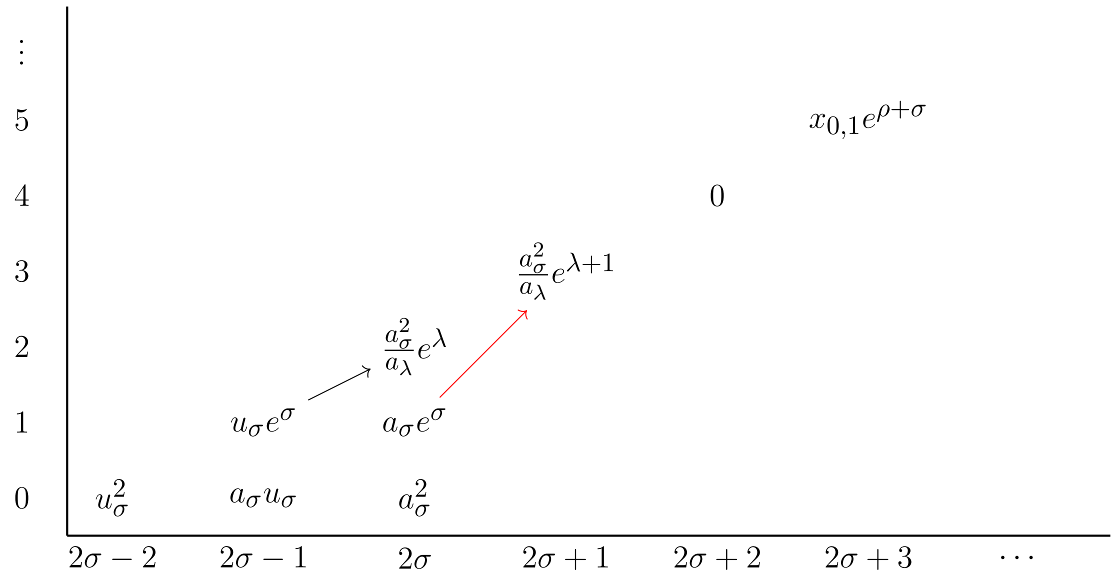


Figure 3.1: The E_1 page of the spectral sequence computing $H_{C_4}^{\bigstar}(B_{C_4}\Sigma_2)$

In figure 3.1 the nontrivial d^2 appears as a red arrow. The vertical axis is the filtration degree, while the horizontal axis is the total degree (dimension $\bigstar$ in $H_{C_4}^{\bigstar}(B_{C_4}\Sigma_2)$). The elements $e^\sigma, e^\lambda, e^{\lambda+1}, e^{\rho+\sigma}$ correspond to the $1, 2, 3, 5$ dimensional cells respectively in the C_4-CW structure of $B_{C_4}\Sigma_2$.

As we can see from the figure, the existence of the dashed d^2 is equivalent to the vanishing of $H_{C_4}^{2\sigma+1}(B_{C_4}\Sigma_2)$ which can be verified using the computer program of chapter 2 (see also subsection 3.5.2). There is a different argument given in [4] that does not use computer calculations.

The existence of differentials implies that the resulting E_∞ page is not free, and indeed there are many extension problems one needs to deal with. The work is too technical to present here, so we refer the reader to [4] for details.

3.5.2 A decomposition using trivial spheres

The cellular decomposition of $B_{C_4}\Sigma_2$ established in subsection 3.5.1, consists of one cell in every dimension, whereby “cell" we mean a space of the form $(C_4/H)_+ \wedge S^V$ for H a subgroup of C_4 and V a real non-virtual C_4-representation; let us call this a “type I" decomposition. It is also possible to obtain a decomposition using only “trivial spheres", namely with cells of the form $(C_4/H)_+ \wedge S^n$; we shall refer to this as a “type II" decomposition. A type I decomposition can be used to produce a type II decomposition by replacing each type I cell $(C_4/H)_+ \wedge S^V$ with its type II decomposition. This is useful for computer-based calculations, since type II decompositions lead to chain complexes as opposed to spectral sequences ($H_*((C_4/H)_+ \wedge S^V)$ is concentrated in a single degree if and only if V is trivial). Equipped with a type II decomposition, the computer program of chapter 2 can calculate the additive structure of $H^{\bigstar}_{C_4}(B_{C_4}\Sigma_2)$ in a finite range.

We note however that a minimal type I decomposition may expand to a non-minimal type II decomposition; this is the case for $B_{C_4}\Sigma_2$, where the minimal type II decomposition uses $2d+3$ cells in each dimension $d \geq 1$, while the one obtained by expanding the type I decomposition uses $3d+3$ cells in each dimension $d \geq 1$. It is the minimal decomposition that we have used as input for the computer program of chapter 2.

3.6 The C_{2^n} Borel equivariant dual Steenrod algebra

In this section let $k = H\mathbb{F}_2$ be the naive $G = C_{2^n}$ spectrum with trivial action. We compute the Borel dual Steenrod algebra $(k \wedge k)^{hG}_{\bigstar}$ as an $RO(G)$-graded Hopf algebroid over the Borel homology of a point $k^{hG}_{\bigstar}$.

3.6.1 The Borel homology of a point

Proposition 3.6.1. *When $n > 1$,*

$$k_\bigstar^{hG} = \mathbb{F}_2[a_\sigma, a_{\lambda_0}, u_\sigma^\pm, u_{\lambda_0}^\pm, ..., u_{\lambda_{n-2}}^\pm]/a_\sigma^2$$

while for $n = 1$:

$$k_\bigstar^{hC_2} = \mathbb{F}_2[a_\sigma, u_\sigma^\pm]$$

Proof. The homotopy fixed point spectral sequence becomes:

$$H^*(G; \mathbb{F}_2)[u_\sigma^\pm, u_{\lambda_0}^\pm, ..., u_{\lambda_{n-2}}^\pm] \implies k_\bigstar^{hG}$$

We have $H^*(G; \mathbb{F}_2) = k^*BG = \mathbb{F}_2[a]/a^2 \otimes \mathbb{F}_2[b]$ where $|a| = 1$ and $|b| = 2$. The spectral sequence collapses with no extensions and we can identify $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$.

□

3.6.2 The Borel dual Steenrod algebra

We will implicitly be completing Borel dual Steenrod algebra

$$(k \wedge k)_\bigstar^{hG}$$

at the ideal generated by a_σ for $G = C_2$, and at the ideal generated by a_{λ_0} for $G = C_{2^n}$, $n > 1$ (see [14] pg. 373 for more details in the case of $G = C_2$).

Proposition 3.6.2 (Hu-Kriz)**.** *The C_2-Borel dual Steenrod algebra is:*

$$(k \wedge k)_\bigstar^{hC_2} = k_\bigstar^{hC_2}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ *(*$\xi_0 = 1$*). The generators* ξ_i *restrict to the Milnor generators in the nonequivariant dual Steenrod algebra and*

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k$$

$$\epsilon(\xi_i) = 0 \ , \ i \geq 1$$

$$\eta_R(a_\sigma) = a_\sigma$$

$$\eta_R(u_\sigma)^{-1} = \sum_{i=0}^{\infty} a_\sigma^{2^i-1} u_\sigma^{-2^i} \xi_i$$

Proposition 3.6.3. *For* $G = C_{2^n}$*,* $n > 1$*,*

$$(k \wedge k)^{hG}_{\bigstar} = k^{hG}_{\bigstar}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ *(*$\xi_0 = 1$*). The generators* ξ_i *restrict to the* $C_{2^{n-1}}$ *generators* ξ_i*, with*

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k$$

$$\epsilon(\xi_i) = 0 \ , \ i \geq 1$$

$$\eta_R(a_\sigma) = a_\sigma \ , \ \eta_R(a_{\lambda_0}) = a_{\lambda_0}$$

$$\eta_R(u_\sigma) = u_\sigma + a_\sigma \xi_1$$

$$\eta_R(u_{\lambda_m}) = u_{\lambda_m} \ , \ m > 0$$

$$\eta_R(u_{\lambda_0})^{-1} = \sum_{i} a_{\lambda_0}^{2^i-1} u_{\lambda_0}^{-2^i} \xi_i^2$$

Proof. The computation of $(k \wedge k)^{hG}_* = (k \wedge k)^*(BG)$ follows from the computation of $k^{hG}_* = k^*(BG) = \mathbb{F}_2[a]/a^2 \otimes \mathbb{F}_2[b]$ and the fact that nonequivariantly, $k \wedge k$ is a free k-module. To see that the homotopy fixed point spectral sequence for $k \wedge k$ converges strongly, let $F^i BG$ be the skeletal filtration on the Lens space $BG = S^\infty / C_{2^n}$; we can then compute directly that $\lim_i^1 (k \wedge k)^*(F^i BG) = \lim_i^1 \mathbb{F}_2[a]/a^2 \otimes \mathbb{F}_2[b]/b^i) = 0$.

Thus we get $(k \wedge k)_\bigstar^{hG} = k_\bigstar^{hG}[\xi_i]$ and the diagonal Δ and augmentation ϵ are the same as in the nonequivariant case. The Euler classes a_σ, a_{λ_0} are maps of spheres so they are preserved under η_R. The action of η_R on u_σ, u_{λ_0} can be computed through the right coaction on $k_\bigstar^{hG}$: The (completed) coaction of the nonequivariant dual Steenrod algebra on $k^*(BG) = \mathbb{F}_2[a]/a^2 \otimes \mathbb{F}_2[b]$ is

$$a \mapsto a \otimes 1$$
$$b \mapsto \sum_i b^{2^i} \otimes \xi_i^2$$

To verify the formula for the coaction on b we need to check that $Sq^1(b) = 0$ (the alternative is $Sq^1(b) = ab$). From the long exact sequence associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, we can see that the vanishing of the Bockstein on b follows from $H^2(C_{2^n}; \mathbb{Z}/4) = \mathbb{Z}/4$ $(n > 1)$.

After identifying $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$ we get the formula for $\eta_R(u_{\lambda_0})$ and also that

$$\eta_R(u_\sigma) = u_\sigma + \epsilon a_\sigma \xi_1$$

where ϵ is either 0 or 1. This is equivalent to

$$\eta_R(u_\sigma^{-1}) = u_\sigma^{-1} + \epsilon a_\sigma u_\sigma^{-2} \xi_1$$

and to see that $\epsilon = 1$ we use the map $k^{hC_2} = k^{h(C_{2^n}/C_{2^{n-1}})} \to k^{hC_{2^n}}$ that sends a_σ, u_σ to a_σ, u_σ respectively. Finally, to compute $\eta_R(u_{\lambda_m})$ for $m > 0$ note that

$$k^{hC_{2^{n-m}}} = k^{hC_{2^n}/C_{2^m}} \to k^{hC_{2^n}}$$

sends $a_{\lambda_0}, u_{\lambda_0}$ to $a_{\lambda_m} = 0, u_{\lambda_m}$ respectively. □

3.6.3 Comparison with Greenlees's description

We now compare with the dual description given in [8].

In our notation, the G-spectrum b of [8] is $b = k^h = F(EG_+, k)$ and $b^V(X)$ corresponds to $(k^h)^{|V|}_G(X)$; to get $(k^h)^V_G(X)$ we need to multiply with the invertible element $u_V \in k^{hG}_{|V|-V}$. The Borel Steenrod algebra is $b^\bigstar_G b = (k^h)^\bigstar_G(k^h)$ and the Borel dual Steenrod algebra is $b^G_\bigstar b = (k^h)^G_\bigstar(k^h) = (k \wedge k)^{hG}_\bigstar$.

Greenlees proves that the Borel Steenrod algebra is given by the Massey-Peterson twisted tensor product ([19]) of the nonequivariant Steenrod algebra k^*k and the Borel cohomology of a point $(k^h)^\bigstar_G = k^{hG}_{-\bigstar}$. The twisting has to do with the fact that the action of the Borel Steenrod algebra on $x \in (k^h)^\bigstar_G(X)$ is given by:

$$(\theta \otimes a)(x) = \theta(ax)$$

where $\theta \in k^*k$ and $a \in k^{hG}_\bigstar$. The product of elements $\theta \otimes a$ and $\theta' \otimes a'$ in the Borel Steenrod algebra is not $\theta\theta' \otimes aa'$, since θ does not commute with cup-products, but rather satisfies the Cartan formula:

$$\theta(ab) = \sum_i \theta'_i(a)\theta''_i(b) \ , \ \Delta\theta = \sum_i \theta'_i \otimes \theta''_i$$

Therefore:

$$(\theta \otimes a)(\theta' \otimes a')(x) = \theta(a\theta'(a'x)) = \sum_i \theta'_i(a)(\theta''_i\theta')(a'x)$$

so

$$(\theta \otimes a)(\theta' \otimes a') = \sum_i \theta'_i(a)(\theta''_i\theta' \otimes a') \tag{3.2}$$

(we have ignored signs as we are working in characteristic 2).

So the Borel Steenrod algebra is $k^*k \otimes k^{hG}_\bigstar$ with twisted algebra structured defined by (3.2).

Moreover, Greenlees expresses the action of k^*k on $(k^h)^\bigstar_G(X)$ in terms of the action of k^*k on the orientation classes u_V and the usual (nonequivariant) action of k^*k on $(k^h)^*_G(X) =$

$k^*(X \wedge_G EG_+)$. This is done through the Cartan formula: If $x \in (k^h)^V_G(X)$ then $u_V^{-1}x \in (k^h)^{|V|}_G(X)$ and

$$\theta(x) = \theta(u_V u_V^{-1} x) = \sum_i \theta'_i(u_V)\theta''_i(u_V^{-1}x)$$

What remains to compute is $\theta'_i(u_V)$, namely the action of k^*k on orientation classes.

In our case, for $G = C_{2^n}$, we can see that:

Proposition 3.6.4. *The action of k^*k on orientation classes is determined by:*

$$Sq^i(u_\sigma) = \begin{cases} u_\sigma & i = 0 \\ a_\sigma & i = 1 \\ 0 & otherwise \end{cases}$$

$$Sq^i(u_{\lambda_m}) = \begin{cases} u_{\lambda_m} & i = 0 \\ a_{\lambda_0} & i = 2, m = 0 \\ 0 & otherwise \end{cases}$$

Proof. Compare with the proof of Proposition 3.6.3. □

The twisting in the case of the Borel dual Steenrod algebra corresponds to the fact that $(k \wedge k)^{hG}_\bigstar$ is a Hopf algebroid and not a Hopf algebra; computationally this amounts to the formula for η_R of Proposition 3.6.3.

CHAPTER 4

RATIONAL EQUIVARIANT CHARACTERISTIC CLASSES

4.1 Introduction

This chapter partially covers the content of [6, 7]; the reader is referred to [6, 7] for more details and complete proofs.

Characteristic classes are classical and invaluable tools for understanding and distinguishing bundles over spaces. If we have a compact Lie group G acting on a space X, there is a corresponding theory of G-equivariant bundles and G-equivariant characteristic classes.

May proves in [20] that when Borel cohomology is used, the theory of Borel equivariant characteristic classes reduces to the nonequivariant one, in the sense that

$$H^*_{G,Borel}(B_G L) = H^*(BG) \otimes H^*(BL)$$

for any compact Lie group L (using field coefficients).

Equivariant characteristic classes in $RO(G)$-graded equivariant cohomology are much less understood, owing to the significant complexity involved in computing it. A way to simplify the algebra involved is to use coefficients in the rational Burnside Green functor $\underline{A}_{\mathbb{Q}}$. Indeed, a result by Greenlees-May reduces the computation of the $RO(G)$-graded cohomology of a space X in $\underline{A}_{\mathbb{Q}}$ coefficients to nonequivariant rational cohomology of the fixed points X^H where H ranges over the subgroups of G ([9]). This allows us to compute explicit descriptions of the Green functors $\underline{H}^{\bigstar}_G(B_G U(n); \underline{A}_{\mathbb{Q}})$, $\underline{H}^{\bigstar}_G(B_G SO(n); \underline{A}_{\mathbb{Q}})$, $\underline{H}^{\bigstar}_G(B_G Sp(n); \underline{A}_{\mathbb{Q}})$ and so on.

However, those explicit descriptions are rather inefficient: For $G = C_2$, the ring

$$H^{\bigstar}_G(B_G U(n); \underline{A}_{\mathbb{Q}})$$

according to the Greenlees-May decomposition, has n^2+2n many algebra generators over the homology of a point, which is just under double the minimal amount $\frac{n^2+2n}{2}+1$ of generators that we can obtain (see the remarks after Proposition 4.4.2).

Our method for obtaining minimal generating characteristic classes rests on equivariant generalizations of the following nonequivariant arguments: By a classical Theorem of Borel ([1]), if L is a connected compact Lie group, $T \subseteq L$ a maximal torus and $W_L T = N_L T/T$ is the Weyl group then, at least rationally,

$$H^*(BL) = H^*(BT)^{W_L T}$$

Through this result, the characteristic classes in $H^*(BL)$ can be computed from $H^*(BS^1)$, as long as the Weyl group action is understood. For example, if we take $L = U(n)$ then $T = (S^1)^n$ and $W_L T = \Sigma_n$ acts on $H^*(BT;\mathbb{Q}) = \mathbb{Q}[a_1, ..., a_n]$ by permuting the generators a_i. The fixed points under this permutation action are minimally generated by the elementary symmetric polynomials on the a_i, which are by definition the Chern classes c_i. In this way, $H^*(BU(n);\mathbb{Q}) = \mathbb{Q}[c_1, ..., c_n]$.

The same method can be performed equivariantly for $G = C_2$ and coefficients in $\underline{A}_{\mathbb{Q}}$. There is an extra degree of complexity owing to the fact that $H_G^{\bigstar}(B_G S^1; \underline{A}_{\mathbb{Q}})$ is not polynomial on one generator over $H_G^{\bigstar}(*; \underline{A}_{\mathbb{Q}})$, but rather on two generators, one of which is idempotent (see (4.1)). As such, in the $L = U(n)$ example, the elementary symmetric polynomials c_i must be replaced by a family of more complicated polynomials $\alpha, c_i, \gamma_{s,j}$ (Proposition 4.4.3). Moreover, while this family of generators is minimal, it is not algebraically independent i.e. there are relations within this family. It is true however that $H_G^{\bigstar}(B_G U(n); \underline{A}_{\mathbb{Q}})$ is a finite module over $H_G^{\bigstar}(*; \underline{A}_{\mathbb{Q}})[c_1, ..., c_n]$ where the c_i are C_2-equivariant refinements of the classical Chern classes.

We use this method to obtain explicit minimal descriptions of $\underline{H}_{C_2}^{\bigstar}(B_{C_2} L; \underline{A}_{\mathbb{Q}})$ where $L = U(n), SO(n), Sp(n)$. We also examine the cases of $L = O(n), SU(n)$ and of the noncompact Lie groups $L = U, SO, Sp, O, SU$. The resulting equivariant Chern, Pontryagin and symplectic classes are compared using the complexification, quaternionization and forgetful maps between the aforementioned Lie groups. We also compute the effect of these characteristic classes on the direct sum of bundles and on the tensor product of line bundles. Finally, we give an explicit formula for the C_2-equivariant Chern character that is an isomorphism between rational C_2-equivariant complex K-theory and C_2-equivariant Bredon cohomology in $\underline{A}_{\mathbb{Q}}$ coefficients.

In this chapter we always use $\underline{A}_{\mathbb{Q}}$ coefficients.

4.2 Organization

- Section 4.3 contains facts about rational Mackey functors and the description of $\underline{H}^{\bigstar}_{C_2}$.
- Sections 4.4-4.8 contain a summary of all our results on C_2 characteristic classes. See [6] for more details and proofs.
- Section 4.9 attempts to generalize these results from group C_2 to all groups C_{2^n}, $n = 1, 2, \ldots$. See [7] for more details.

4.3 The $RO(C_2)$ homology of a point in $\underline{A}_{\mathbb{Q}}$ coefficients

The rational C_2-Burnside Green functor $\underline{A}_{\mathbb{Q}}$ has Lewis diagram:

$$\underline{A}_{\mathbb{Q}} = \begin{array}{c} \frac{\mathbb{Q}[x]}{x^2=2x} \\ {\scriptstyle x\mapsto 2}\big\downarrow\ \big\uparrow{\scriptstyle 1\mapsto x} \\ \mathbb{Q} \end{array} = \begin{array}{c} \mathbb{Q}\,x \\ {\scriptstyle x\mapsto 2}\big\downarrow\ \big\uparrow{\scriptstyle 1\mapsto x} \\ \mathbb{Q} \end{array} \oplus \begin{array}{c} \mathbb{Q}\,y \\ \big\downarrow\ \big\uparrow \\ 0 \end{array}$$

where $x = \mathrm{Tr}(1)$ and $y = 1 - x/2$. We shall use $A_{\mathbb{Q}}$ with no underline to denote the top level of $\underline{A}_{\mathbb{Q}}$.

The generating classes for $\underline{H}^{C_2}_{\bigstar}$ are the Euler and orientation classes. The Euler class a_σ generates a Mackey functor that we denote by $\underline{M}_1$:

$$\underline{M}_1\{a_\sigma\} = \begin{array}{c} \mathbb{Q}\,a_\sigma \\ \big\downarrow\big\uparrow \\ 0 \end{array}$$

The orientation class u_σ generates a Mackey functor that we denote by $\underline{M}_0^-$:

$$\underline{M}_0^-\{u_\sigma\} = \begin{array}{c} 0 \\ \big\downarrow\big\uparrow \\ \mathbb{Q}\,u_\sigma \circlearrowleft C_2 \end{array}$$

The orientation class $u_{2\sigma}$ generates Mackey functor that we denote by $\underline{M}_0$:

$$\underline{M}_0\{u_{2\sigma}\} = \begin{array}{c} \mathbb{Q}\,u_{2\sigma} \\ {\scriptstyle 1}\big\downarrow\big\uparrow{\scriptstyle 2} \\ \mathbb{Q}\,u_\sigma^2 \end{array}$$

The fact that $\mathrm{Res}(u_{2\sigma}) = u_\sigma^2$ follows from the fact $\underline{M}_0^- \boxtimes_{\underline{A}_{\mathbb{Q}}} \underline{M}_0^- = \underline{M}_0$ and by the Kunneth formula for $S^{2\sigma} = S^\sigma \wedge S^\sigma$. Note that $a_\sigma u_{2\sigma} = 0$ since $\underline{M}_1 \boxtimes_{\underline{A}_{\mathbb{Q}}} \underline{M}_0 = 0$.

We also have classes $y/a_\sigma, u_\sigma^{-1}$ and $x/u_{2\sigma}$ spanning $\underline{M}_1, \underline{M}_0^-$ and $\underline{M}_0$ respectively. In summary:

Proposition 4.3.1. *The C_2 equivariant rational stable stems are:*

$$\underline{H}^{C_2}_{k+n\sigma} = \begin{cases} \underline{M}_0 & \text{if } k=-n : \textit{ even and} \neq 0 \\ \underline{M}_0^- & \text{if } k=-n : \textit{ odd} \\ \underline{M}_1 & \text{if } k=0 \ , \ n\neq 0 \\ \underline{A}_{\mathbb{Q}} & \text{if } k=n=0 \\ 0 & \text{otherwise} \end{cases}$$

and:

- $u^j_{2\sigma}, x/u^j_{2\sigma}$ *generate a copy* $\underline{M}_0$ *for each* $j=1,2,....$
- u^{2j+1}_σ *generate a copy* $\underline{M}_0^-$ *for each* $j\in\mathbb{Z}$.
- $a^j_\sigma, y/a^j_\sigma$ *generate a copy of* $\underline{M}_1$ *for each* $j=1,2,....$
- 1 *generates* $\underline{A}_{\mathbb{Q}}$.

4.4 Rational C_2 Chern classes

As explained in section 1.2, it suffices to compute $H^\bigstar_G(X)$ over integer grading $\bigstar = *$.

We view $H^*_G(B_GU(n))$ as an augmented algebra over $H^*(BU(n))$ with the augmentation being restriction.

Proposition 4.4.1. *The augmentation*

$$\text{Res} : H^*_G(B_GU(n)) \to H^*(BU(n))$$

is a split surjection, so the nonequivariant Chern classes have C_2 equivariant refinements.

We fix a section of the augmentation, i.e. equivariant refinements $c_1, ..., c_n$ of the Chern classes, according to Proposition 4.4.4.

Proposition 4.4.2. *There exist elements* $\alpha \in H^0_G(B_GU(n))$ *and* $\gamma_{s,j} \in H^{2s}_G(B_GU(n))$ *for* $1 \leq s < n$ *and* $1 \leq j \leq n-s$*, generating* $H^*_G(B_GU(n))$ *as an augmented algebra over* $H^*(BU(n)) \otimes A_{\mathbb{Q}}$*:*

$$H^*_G(B_GU(n)) = \frac{(H^*(BU(n)) \otimes A_{\mathbb{Q}})[\alpha, \gamma_{s,j}]}{\mathrm{Res}(\alpha), \mathrm{Res}(\gamma_{s,j}), S}$$

where the finite set of relations $S \subseteq \mathbb{Q}[\alpha, c_i, \gamma_{s,j}]$ *is described in [6]*

The c_i *are algebraically independent and* $H^*_G(B_GU(n))$ *is a finitely generated module over* $\mathbb{Q}[c_1, ..., c_n]$*.*

The generating family $\{\alpha, c_i, \gamma_{s,j}\}$ *has cardinality* $\frac{n^2+2n}{2} + 1$ *and is a minimal generating set of* $H^*_G(B_GU(n))$ *as an* $A_{\mathbb{Q}}$ *algebra, in the sense that any other generating set has at least* $\frac{n^2+2n}{2} + 1$ *many elements.*

Substituting $H^*(BU(n)) = \mathbb{Q}[c_1, ..., c_n]$ in the formula for $H^*_G(B_GU(n))$ gives:

Proposition 4.4.3. *As an algebra over* $A_{\mathbb{Q}}$*,*

$$H^*_G(B_GU(n)) = \frac{A_{\mathbb{Q}}[\alpha, c_i, \gamma_{s,j}]}{x\alpha, x\gamma_{s,j}, S}$$

Two observations:

- The relations $x\alpha = 0, x\gamma_{s,j} = 0$ are equivalent to $\alpha, \gamma_{s,j}$ having trivial restrictions (i.e. augmentations) respectively. This completes the description of the Mackey functor structure of $H^*_G(B_GU(n))$.
- The $\frac{n^2+2n}{2} + 1$ many generators of the generating set $\{\alpha, c_i, \gamma_{s,j}\}$ are just over half of the $n^2 + 2n$ many generators given by the idempotent decomposition ([9]) of the Mackey functor $H^*_G(B_GU(n))$.

For $n = 1$ the computation takes a simpler form:

$$H^*_G(B_GU(1)) = A_{\mathbb{Q}}[\alpha, c_1]/(\alpha^2 = \alpha, x\alpha) \tag{4.1}$$

To simplify the notation in the next Proposition, we set $u = c_1 \in H^2_G(B_G U(1))$.

Proposition 4.4.4. *The maximal torus inclusion $U(1)^n \hookrightarrow U(n)$ induces an isomorphism*

$$H^*_G(B_G U(n)) = (H^*_G(B_G U(1))^{\otimes n})^{\Sigma_n}$$

Explicitly:

$$A_{\mathbb{Q}}[\alpha, c_i, \gamma_{s,j}]/(x\alpha, x\gamma_{s,j}, S) = (A_{\mathbb{Q}}[\alpha_i, u_i]/(\alpha_i^2 = \alpha_i, x\alpha_i))^{\Sigma_n}$$

under the identifications:

$$\alpha = \sigma_1(\alpha_1, ..., \alpha_n) = \sum_{1 \leq m \leq n} \alpha_m$$

$$c_i = \sigma_i(u_1, ..., u_n) = \sum_{m_* \in K_i} u_{m_1} \cdots u_{m_i}$$

$$\gamma_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1} \cdots u_{m_s} \alpha_{l_1} \cdots \alpha_{l_j}$$

where K_i consists of all partitions $1 \leq m_1 < \cdots < m_i \leq n$ and $K_{s,j} \subseteq K_s \times K_j$ consists of all pairs of disjoint partitions. The polynomial σ_i is the i-th elementary symmetric polynomial.

The family of generators $\alpha, c_i, \gamma_{s,j}$ is determined upon choosing $\alpha, u = c_1$ in the vector spaces $H^0_G(B_G U(1)), H^2_G(B_G U(1))$ respectively, with:

$$H^*_G(B_G U(1)) = A_{\mathbb{Q}}[\alpha, u]/(\alpha^2 = \alpha, x\alpha)$$

The choice of u is unique under the additional requirement that its restriction is the nonequivariant Chern class c_1 (in this way, the equivariant c_i are all canonically determined). There are two equally good candidates for α however: α and $y - \alpha$. They can only be distinguished upon fixing a model for $B_G U(1)$. As such, there is no canonical choice of $\alpha \in H^0_G(B_G U(1))$.

Proposition 4.4.5. *The map $B_GU(n) \to B_GU(n+1)$ given by direct sum with a trivial complex representation induces on cohomology:*

$$\alpha \mapsto y + \alpha$$

$$c_i \mapsto c_i$$

$$\gamma_{s,j} \mapsto \gamma_{s,j} + \gamma_{s,j-1}$$

using the convention $\gamma_{s,0} = yc_s$.

The map $B_GU(n) \to B_GU(n+1)$ given by direct sum with a σ representation induces on cohomology:

$$\alpha \mapsto \alpha$$

$$c_i \mapsto c_i$$

$$\gamma_{s,j} \mapsto \gamma_{s,j}$$

For both maps we use the conventions that $c_{n+1} = 0$ and $\gamma_{s,n+1-s} = 0$.

Proposition 4.4.6. *The direct sum of bundles map $B_GU(n) \times B_GU(m) \to B_GU(n+m)$ induces on cohomology:*

$$\alpha \mapsto \alpha \otimes 1 + 1 \otimes \alpha$$

$$c_i \mapsto \sum_{j+k=i} c_j \otimes c_k$$

$$\gamma_{s,j} \mapsto \sum_{s' + s'' = s \ , \ j' + j'' = j} \gamma_{s',j'} \otimes \gamma_{s'',j''}$$

using the conventions $c_0 = 1, \gamma_{s,0} = yc_s, \gamma_{0,j} = (j!)^{-1}\alpha(\alpha-1)\cdots(\alpha-j+1)$ in every RHS.

Proposition 4.4.7. *The tensor product of line bundles map $B_GU(1) \times B_GU(1) \to B_GU(1)$*

induces on cohomology:

$$\alpha \mapsto y - \alpha \otimes 1 - 1 \otimes \alpha + 2\alpha \otimes \alpha$$

$$c_1 \mapsto c_1 \otimes 1 + 1 \otimes c_1$$

The C_2-equivariant Chern character is induced by the Hurewicz map:

$$KU_{C_2} \to KU_{C_2} \wedge HA_{\mathbb{Q}}$$

Let v be the Bott element in $\pi_2^{C_2}(KU_{C_2})$.

Proposition 4.4.8. *Under the equivalence $KU_{C_2} \wedge HA_{\mathbb{Q}} = HA_{\mathbb{Q}}[v^{\pm}]$ the Chern character*

$$KU_{C_2}(X) \otimes \mathbb{Q} \to \prod_n H^{2n}_{C_2}(X; A_{\mathbb{Q}})$$

is the isomorphism determined on line bundles by:

$$L \mapsto \frac{x}{2} e^{c_1(L)} + (2\alpha(L) - y) e^{c_1(L)}$$

4.5 Rational C_2 symplectic classes

The theory of C_2 symplectic characteristic classes is entirely analogous to Chern classes, by replacing $B_G U(n)$ with $B_G Sp(n)$ and the generators $c_i, \gamma_{s,j}$ with generators $k_i, \kappa_{s,j}$ of double degree. Propositions 4.4.1-4.4.3 become:

Proposition 4.5.1. *There exist classes $\alpha, k_i, \kappa_{s,j} \in H^*_G(B_G Sp(n))$ of degrees $0, 4i, 4s$ respectively, where $1 \leq i, s \leq n$ and $1 \leq j \leq n - s$, such that*

$$H^*_G(B_G Sp(n)) = \frac{A_{\mathbb{Q}}[\alpha, k_i, \kappa_{s,j}]}{x\alpha, x\kappa_{s,j}, S}$$

where the relation set S is the same as that for $H^*_G(B_GU(n))$ with $c_i, \gamma_{s,i}$ replaced by $k_i, \kappa_{s,i}$.

The generators k_i restrict to the nonequivariant symplectic classes k_i, so the restriction map $H^*_G(B_GSp(n)) \to H^*(BSp(n))$ is a split surjection.

The maximal torus inclusion $U(1)^n \hookrightarrow Sp(n)$ induces an isomorphism

$$H^*_G(B_GSp(n)) = (H^*_G(B_GU(1))^{\otimes n})^{C_2 \wr \Sigma_n}$$

Explicitly:

$$A_{\mathbb{Q}}[\alpha, k_i, \kappa_{s,j}]/(x\alpha, x\kappa_{s,j}, S) = (A_{\mathbb{Q}}[\alpha_i, u_i]/(x\alpha_i))^{C_2 \wr \Sigma_n}$$

under the identifications:

$$\alpha = \sum_{1 \le m \le n} \alpha_m$$
$$k_i = \sum_{m_* \in K_i} u^2_{m_1} \cdots u^2_{m_i}$$
$$\kappa_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u^2_{m_1} \cdots u^2_{m_s} \alpha_{l_1} \cdots \alpha_{l_j}$$

where K_i and $K_{s,j}$ are as in Proposition 4.4.4.

Propositions 4.4.5-4.4.7 have analogous statements in the symplectic case, replacing $B_GU(n)$ by $B_GSp(n)$ and $c_i, \gamma_{s,j}$ with $k_i, \kappa_{s,j}$ respectively.

Proposition 4.5.2. *The forgetful map $B_G Sp(n) \to B_G U(2n)$ induces on cohomology:*

$$\alpha \mapsto \alpha$$

$$c_{2i+1}, \gamma_{2s+1,j} \mapsto 0$$

$$c_{2i} \mapsto (-1)^i k_i$$

$$\gamma_{2s,j} \mapsto (-1)^s \kappa_{s,j}$$

The quaternionization map $B_G U(n) \to B_G Sp(n)$ induces:

$$\alpha \mapsto \alpha$$

$$k_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$

The effect of quaternionization on the $\kappa_{s,j}$ is explained [6].

4.6 Rational C_2 Pontryagin and Euler classes

The results are analogous to the symplectic classes, but we need to distinguish between $B_G SO(2n)$ and $B_G SO(2n+1)$. The following Proposition contains the shared aspects of both cases:

Proposition 4.6.1. *The restriction map $H^*_G(B_G SO(n)) \to H^*(BSO(n))$ is a split surjection. The maximal torus inclusion $T \hookrightarrow SO(n)$ induces an isomorphism*

$$H^*_G(B_G SO(n)) = (H^*_G(B_G T))^W$$

where W is the corresponding Weyl group.

This gives us C_2 equivariant refinements p_i, χ of the Pontryagin and Euler classes respectively. Recall that for $BSO(2n)$ the characteristic classes are $p_1, ..., p_{n-1}$, χ (and $p_n = \chi^2$)

while for $BSO(2n+1)$ they are $p_1, ..., p_n$.

Proposition 4.6.2. *There exist classes $\alpha, \pi_{s,j} \in H_G^*(B_G SO(2n))$ of degrees $0, 4s$ respectively for $1 \le s < n$ and $1 \le j \le n-s$, such that*

$$H_G^*(B_G SO(2n)) = \frac{A_{\mathbb{Q}}[\alpha, p_i, \pi_{s,j}, \chi]}{x\alpha, x\pi_{s,j}, S}$$

where the relation set S is the same as that for $H_G^*(B_G U(n))$ with $c_i, \gamma_{s,i}$ replaced by $p_i, \pi_{s,i}$ and using that $p_n = \chi^2$.

Under the maximal torus isomorphism:

$$\alpha = \sum_{1 \le m \le n} \alpha_m$$

$$p_i = \sum_{m_* \in K_i} u_{m_1}^2 \cdots u_{m_i}^2$$

$$\pi_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1}^2 \cdots u_{m_s}^2 \alpha_{l_1} \cdots \alpha_{l_j}$$

$$\chi = u_1 \cdots u_n$$

where K_i and $K_{s,j}$ are as in Proposition 4.4.4.

Proposition 4.6.3. *The map $B_G SO(2n) \to B_G SO(2n+1)$ induces an injection in cohomology and:*

$$H_G^*(B_G SO(2n+1)) = \frac{A_{\mathbb{Q}}[\alpha, p_i, \pi_{s,j}]}{x\alpha, x\pi_{s,j}, S}$$

where $i = 1, ..., n$.

Propositions 4.4.5-4.4.7 have analogous statements in this context. The action on the Euler class χ is the same as in the nonequivariant case; for example, under $B_G SO(n) \times$

$B_GSO(m) \to B_GSO(n+m)$ we get:

$$\chi \mapsto \chi \otimes \chi$$

Proposition 4.6.4. *The complexification map $B_GSO(2n) \to B_GU(2n)$ induces on cohomology:*

$$\alpha \mapsto \alpha$$
$$c_{2i+1}, \gamma_{2s+1,j} \mapsto 0$$
$$c_{2i} \mapsto (-1)^i p_i$$
$$\gamma_{2s,j} \mapsto (-1)^s \pi_{s,j}$$

The forgetful map $B_GU(n) \to B_GSO(2n)$ induces on cohomology:

$$\alpha \mapsto \alpha$$
$$p_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$
$$\chi \mapsto c_n$$

and the action on $\pi_{s,j}$ is explained in [6].

4.7 Stable characteristic classes

In the C_2-equivariant case, there are different notions of stability for complex bundles, represented by the following spaces:

- $B_G^+U = \text{colimit}(B_GU(1) \xrightarrow{\oplus 1} B_GU(2) \xrightarrow{\oplus 1} \cdots)$. This is the usual equivariant classifying space $B_GU = E_GU/U$ and is a G-equivariant Hopf space using the direct sum of bundles maps $B_GU(n) \times B_GU(m) \to B_GU(n+m)$.

- $B_G^-U = \text{colimit}(B_GU(1) \xrightarrow{\oplus\sigma} B_GU(2) \xrightarrow{\oplus\sigma} \cdots)$. This is equivalent to B_G^+U.
- $B_G^{\pm}U = \text{colimit}(B_G^+U \xrightarrow{\oplus\sigma} B_G^+U \xrightarrow{\oplus\sigma} \cdots) = \text{colimit}(B_G^-U \xrightarrow{\oplus 1} B_G^-U \xrightarrow{\oplus 1} \cdots)$. This becomes a G-equivariant Hopf space using the direct sum of bundles, and is the group completion of B_G^+U (and B_G^-U). Moreover, $B_G^{\pm}U \times \mathbb{Z}$ represents equivariant K-theory.

Computing $H_G^*(B_G^-U)$ in terms of the generators $\alpha, c_i, \gamma_{s,j}$ is more complicated compared to the nonequivariant case because for fixed degree $*$, the $\mathbb{Q}$-dimension of $H_G^*(B_GU(n))$ does not stabilize as $n \to +\infty$ and as a result, $H_G^*(B_G^-U)$ is infinite dimensional (dimension is $2^{\aleph_0}$). In degree $* = 0$, $H_G^0(B_G^-U)$ is linearly spanned over $A_{\mathbb{Q}}$ by series of the form

$$a_{-1} + \sum_{i \geq 0} a_i \alpha(\alpha - 1) \cdots (\alpha - i)$$

Generally, the graded algebra $H_G^*(B_G^-U)$ is generated over $H_G^0(B_G^-U)[c_1, c_2, ...]$ by series of the form

$$\sum_{j=1}^{\infty} a_j \gamma_{s,j} \in H_G^{2s}(B_G^-U)$$

for $a_j \in \mathbb{Q}$ and $s = 1, 2, ...$ See [6] for more details.

For the ring $H_G^*(B_G^{\pm}U)$ we also have to compute the effect of the $\oplus 1$ map on the series in $H_G^*(B_G^-U)$. If we restrict our attention to finite series, we are in essence dealing with characteristic classes that are stable under addition of both the $\oplus 1$ and $\oplus\sigma$ representations. Since the $\oplus 1$ map takes the form $\gamma_{s,j} \mapsto \gamma_{s,j} + \gamma_{s,j-1}$ (and $\gamma_{s,0} = yc_s$, $\gamma_{0,1} = \alpha$) we can immediately see that for $i \geq 1$, the elements

$$c_i \ , \ \gamma_i := c_i \alpha - \gamma_{i,1}$$

are stable under both $\oplus 1$ and $\oplus\sigma$. We conjecture that all classes with this property are polynomially generated by c_i, γ_i; this is equivalent to the elements $\gamma_1, \gamma_2, ...$ being algebraically independent over $\mathbb{Q}[c_1, c_2, ...]$.

In any case, the elements c_i, γ_i span sub-Hopf-algebras of $H^*_G(B^-_G U)$ and $H^*_G(B^{\pm}_G U)$ with

$$\gamma_s \mapsto \sum_{i+j=s} (c_i \otimes \gamma_j + \gamma_i \otimes c_j)$$

using the conventions $c_0 = 1$ and $\gamma_0 = 0$.

The spaces $B^+_G U, B^-_G U, B^{\pm}_G U$ are equivariant Hopf spaces, hence their equivariant homology is a Green functor dual to their equivariant cohomology. This homology can be expressed in terms of the classes $a_i, b_i, d \in H^G_*(B_G U(1))$ dual to $\alpha c_1^i, c_1^i, x/2+\alpha \in H^*_G(B_G U(1))$ respectively, where $i \geq 1$. Note that the $\gamma_1^i \in H^*_G(B_G U(2))$ map to αc_1^i under $H^*_G(B_G U(2)) \xrightarrow{\oplus 1} H^*_G(B_G U(1))$ so the a_i can be thought of as duals to the γ_1^i.

Proposition 4.7.1. *We have:*

$$H^G_*(B^-_G U) = \frac{A_{\mathbb{Q}}[d, a_i, b_i]}{xa_i, xd = x}$$
$$H^G_*(B^{\pm}_G U) = \frac{A_{\mathbb{Q}}[d^{\pm}, a_i, b_i]}{xa_i, xd = x}$$

and for the coalgebra structure:

$$d \mapsto d \otimes d$$
$$a_i \mapsto \sum_{j+k=i} a_j \otimes a_k$$
$$b_i \mapsto \sum_{j+k=i} (b_j \otimes b_k - b_j \otimes a_k - b_k \otimes a_j + 2a_j \otimes a_k)$$

using the conventions $a_0 = d - x/2$ *and* $b_0 = 1$.

The case of stable symplectic classes is entirely analogous: We can distinguish between $B^+_G Sp, B^-_G Sp$ and $B^{\pm}_G Sp$ and we have classes $k_i, \kappa_i = k_i\alpha - \kappa_{i,1}$ that are stable under both $\oplus 1, \oplus \sigma$ maps. Moreover,

Proposition 4.7.2. *The forgetful map $Sp \to U$ induces*

$$c_{2s+1}, \gamma_{2s+1} \mapsto 0$$
$$c_{2s} \mapsto (-1)^s k_s$$
$$\gamma_{2s} \mapsto (-1)^s \kappa_s$$

while quaternionization $U \to Sp$ induces

$$k_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$
$$\kappa_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a \gamma_b$$

The dual homology result can be expressed in terms of the classes

$$a_i^{sp}, b_i^{sp}, d \in H_*^G(B_G Sp(1))$$

dual to $\alpha k_1^i, k_1^i, x/2 + \alpha \in H_G^*(B_G Sp(1))$ respectively, for $i \geq 1$ (the a_i^{sp} are dual to κ_1^i). The analogue of Proposition 4.7.1 holds, and:

Proposition 4.7.3. *The forgetful map $Sp \to U$ induces*

$$d \mapsto d$$
$$a_i^{sp} \mapsto \sum_{2i=j+k} (-1)^k a_j a_k$$
$$b_i^{sp} \mapsto \sum_{2i=j+k} (-1)^k (b_j b_k - a_j b_k - a_k b_j + 2a_j a_k)$$

while quaternionization $U \to Sp$ induces

$$d \mapsto d$$
$$a_{2i+1} \mapsto 0 \ , \ a_{2i} \mapsto a_i^{sp}$$
$$b_{2i+1} \mapsto 0 \ , \ b_{2i} \mapsto b_i^{sp}$$

The case of stable Pontryagin classes is entirely analogous, replacing Sp by SO (the forgetful map $Sp \to U$ is replaced by complexification $SO \to U$ and the quaternionization map $U \to Sp$ is replaced by the forgetful map $U \to SO$). In brief, setting $\pi_i = p_i\alpha - \pi_{i,1}$ gives the analogue of 4.7.2. Moreover, we have classes $a_i^{so}, b_i^{so}, d \in H_*^G(B_G SO(2))$ dual to $\alpha p_1^i, p_1^i, x/2 + \alpha \in H_G^*(B_G SO(2))$ respectively, for $i \geq 1$, and the analogues of Propositions 4.7.1 and 4.7.3 also hold.

4.8 The cases of orthogonal and special unitary groups

4.8.1 Orthogonal groups

Unlike their nonequivariant counterparts, the C_2-equivariant classifying spaces of the orthogonal groups $O(n)$ do not generally satisfy the maximal torus isomorphism, namely the map $H_G^*(B_G O(n)) \to H_G^*(B_G T)^W$ is not generally an isomorphism, where T is the maximal torus in $O(n)$ and W the Weyl group. Moreover, $H_G^*(B_G O(2n))$ is not isomorphic to $H_G^*(B_G O(2n+1))$, but rather, the inclusion-induced map

$$H_G^*(B_G O(2n+1)) \to H_G^*(B_G O(2n))$$

is always a surjection with nontrivial kernel. The spaces $B_G O(2n+1)$ can be put into our framework using the splitting $O(2n+1) = SO(2n+1) \times O(1)$:

Proposition 4.8.1. *There is a generator* $\beta \in H^0_G(B_GO(1))$ *such that*

$$H^*_G(B_GO(2n+1)) = \frac{A_{\mathbb{Q}}[\alpha, \beta, p_i, \pi_{s,j}]}{x\alpha, x\beta, x\pi_{s,j}, S}$$

The $H^*_G(B_GO(2n))$ can then be understood as quotients of $H^*_G(B_GO(2n+1))$ (see [6]). The stable case similarly reduces to B_GSO by use of the fact that $B_GO = B_GSO \times B_GO(1)$.

4.8.2 Special unitary groups

For $SU(n)$ we have the maximal torus isomorphism equivariantly:

Proposition 4.8.2. *The maximal torus inclusion* $U(1)^{n-1} \to SU(n)$ *induces an isomorphism*

$$H^*_G(B_GSU(n)) \to H^*_G(B_GU(1)^{n-1})^{\Sigma_n}$$

We prove that for any n, the inclusion induced map

$$H^*_G(B_GU(n)) \to H^*_G(B_GSU(n))$$

is a surjection, and $c_1 = \gamma_{1,n-1} = 0$ in $H^*_G(SU(n))$. There are more relations however; for example, if $n = 2$ there is an additional relation $\alpha^2 = 2\alpha$ since $SU(2) = Sp(1)$.

In the stable case, we can distinguish between B^+_GSU, B^-_GSU and $B^{\pm}_GSU$ and we have $c_1 = \gamma_1 = 0$.

4.9 C_{2^n}-equivariant rational characteristic classes

In this section, we attempt to generalize the work summarized in sections 4.4-4.8 from $G = C_2$ to all groups $G = C_{2^n}$, $n \geq 1$. There are three fundamental components that must be generalized:

- The computation of the G-equivariant rational stable stems:

$$\underline{H}^G_\bigstar$$

- The G-equivariant maximal torus isomorphism:

$$H_G^\bigstar(B_G L) = H_G^\bigstar(B_G T)^W$$

  where T is a maximal torus of a compact connected Lie group L and W the associated Weyl group.

- The computation of minimal generating sets for $R = H_G^*(B_G T)$ and R^W.

It turns out that ([7]):

- $\underline{H}^G_\bigstar$ is generated by Euler and orientation classes (see [7] for the explicit description).

- The maximal torus isomorphism is true for $G = C_{2^n}$, $L = U(m)$ and any $n, m \geq 1$, namely:

$$H_G^\bigstar(B_G U(m)) = (\otimes^m H_G^\bigstar(B_G U(1)))^{\Sigma_m}$$

  However, it is not true for $G = C_{2^n}, L = SU(2) = Sp(1)$ and $n > 1$, namely the map:

$$H_G^\bigstar(B_G Sp(1)) \to H_G^\bigstar(B_G S^1)^{\Sigma_2}$$

  is not an isomorphism.

Let us restrict to the case $G = C_{2^n}$ and $L = U(m)$, $m \geq 1$. The issue now lies is in computing a minimal generating set for the ring of symmetric polynomials with relations R^{Σ_n}, which is more complicated than in the case $G = C_2$. The fundamental building block is:

Proposition 4.9.1. *As a Green functor algebra over the homology of a point:*

$$H_G^{\bigstar}(B_G S^1) = \frac{H_G^{\bigstar}[u, \alpha_{m,j}]_{1\le m\le n, 1\le j<2^m}}{\alpha_{m,j}\alpha_{m',j'} = \delta_{mm'}\delta_{jj'}\alpha_{m,j}\ ,\ \mathrm{Res}^{2^n}_{2^{m-1}}(\alpha_{m,j}) = 0}$$

for $|u| = 2$ *and* $|\alpha_{m,j}| = 0$.

Proof. See [7]. ☐

The α classes actually come from $B_G\Sigma_2$:

Proposition 4.9.2. *We have an isomorphism of Green functor algebras over* $H_G^{\bigstar}$*:*

$$H_G^{\bigstar}(B_G\Sigma_2) = \frac{H_G^{\bigstar}(B_G S^1)}{u}$$

where the quotient map $H_G^{\bigstar}(B_G S^1) \to H_G^{\bigstar}(B_G\Sigma_2)$ *is induced by complexification:* $B_G\Sigma_2 = B_G O(1) \to B_G U(1) = B_G S^1$.

The set of generators $\{u, \alpha_{m,j}\}$ for $H_G^*(B_G S^1)$ is not minimal. Indeed, whenever we have generators $e_1, ..., e_s$ with $e_ie_j = \delta_{ij}e_i$, we can replace them by a single generator defined by $e = e_1 + 2e_2 + \cdots + se_s$:

$$\frac{\mathbb{Q}[e_1, ..., e_s]}{e_ie_j = \delta_{ij}e_i} = \frac{\mathbb{Q}[e]}{e(e-1)\cdots(e-s)}$$

This isomorphism follows from the fact that any polynomial f on $e_1, ..., e_n$ satisfies:

$$f(e) = f(0) + (f(1) - f(0))e_1 + \cdots + (f(s) - f(0))e_s$$

and thus

$$e_i = \frac{f_i(e)}{f_i(i)} \text{ where } f_i(x) = \frac{x(x-1)\cdots(x-s)}{x-i}$$

In this way, $H^*_G(B_G S^1)$ is generated as an $A_{\mathbb{Q}}$ algebra by two elements u, α but now with α satisfying some rather complicated relations. If $n = 1$ i.e. $G = C_2$, we only have one $\alpha_{m,j}$ element, namely $\alpha = \alpha_{1,1}$, and it satisfies: $\alpha^2 = \alpha$.

For $G = C_2$, the description

$$H^*_G(B_G S^1) = A_{\mathbb{Q}}[u, \alpha]/(x\alpha = 0, \alpha^2 = \alpha)$$

is simple enough to allow an explicit computation of a minimal presentation of $H^*_G(B_G U(m))$ through the maximal torus isomorphism. Due to the greater algebraic complexity of the fundamental building block $H^*_G(B_G S^1)$ for $G = C_{2^n}$ and $n \geq 2$, we do not attempt to generalize the rest of sections 4.4-4.8 to cyclic 2-groups beyond C_2.

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