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CHARACTERIZING SYMMETRIC SPACES BY THEIR LYAPUNOV SPECTRA

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BY
CLARK BUTLER

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Abstract

This thesis uses methods from hyperbolic dynamics, Riemannian geometry, and analysis on metric spaces to obtain new rigidity results for negatively curved Riemannian manifolds.

We prove that closed, negatively curved locally symmetric spaces are locally characterized up to isometry by the Lyapunov spectra of the periodic orbits of their geodesic flows. This is done by constructing a new invariant measure for the geodesic flow that we refer to as the horizontal measure. We show that the Lyapunov spectrum of the horizontal measure alone suffices to locally characterize these locally symmetric spaces up to isometry.

Our methods extend to give rigidity theorems for smooth flows obtained as perturbations of the geodesic flows of these locally symmetric spaces. The techniques developed in this paper are focused on the symmetric spaces of nonconstant negative curvature and extend many methods used to prove rigidity theorems for uniformly quasiconformal Anosov diffeomorphisms and flows.

1 Introduction

This thesis is based on the paper [14]. Proofs of some of the propositions in Section 3 3 appeared previously in [13]. Section 4 is based on results from [13].

The primary objective of this thesis is to show how geometric properties of an Anosov flow may be derived from properties of the Lyapunov spectra of its invariant measures. Our focus will be on the geodesic flows of negatively curved Riemannian manifolds that are obtained as perturbations of negatively curved locally symmetric spaces. The connection between the Lyapunov spectra of invariant measures of the geodesic flow and the geometry of the manifold itself is mediated through the quasiconformal geometry of the visual boundary of the universal cover. This connection will feature prominently in our proofs. While we cover primarily geometric applications in this paper, many of the techniques developed here are more widely applicable, particularly to the study of the geometry of perturbations of hyperbolic toral automorphisms.

All Riemannian manifolds we consider in this work are assumed to be closed. Given a Riemannian manifold Y, we consider the geodesic flow $g_Y^t : T^1Y \to T^1Y$ on its unit tangent bundle T^1Y. Set $n = \dim Y$. By applying the multiplicative ergodic theorem [48] to the derivative cocycle Dg_Y^t of the geodesic flow, we may associate to each g_Y^t-invariant ergodic probability measure ν a list of real numbers

$$\lambda_1(g_Y^t, \nu) \leq \lambda_2(g_Y^t, \nu) \leq \cdots \leq \lambda_{2n-1}(g_Y^t, \nu),$$

such that for ν-a.e. unit tangent vector $v \in T^1Y$ and every unit vector $\xi \in T_vT^1Y$, we have

$$\lim_{t\to\infty} \frac{\|Dg_Y^t(\xi)\|}{t} = \lambda_i(g_Y^t, \nu),$$

for some $1 \leq i \leq 2n-1$. These are known as the *Lyapunov exponents* of g_Y^t with respect to ν; they describe all possible asymptotic exponential growth rates of vectors $\xi \in T_vT^1Y$ for ν-

a.e. $v \in T^1Y$. Equivalently, they describe the possible asymptotic exponential growth rates of Jacobi fields along a randomly chosen geodesic in Y, where this random choice is made according to the distribution ν. We write $\vec{\lambda}(g_Y^t, \nu)$ for the vector in $\mathbb{R}^n$ whose components are the Lyapunov exponents $\lambda_i(g_Y^t, \nu)$, written in increasing order. For any choice of ergodic invariant measure ν we always have $\lambda_n(g_Y^t, \nu) = 0$; this corresponds to the fact that g_Y^t acts isometrically on the direction of its flow. The smooth involution $v \to -v$ on T^1Y conjugates g_Y^t to g_Y^{-t} and thus we also have $\lambda_i(g_Y^t, \nu) = -\lambda_{2n-i}(g_Y^t, \nu)$ for $1 \leq i \leq 2n-1$.

When X is a negatively curved locally symmetric space, the Lyapunov spectrum does not depend on the choice of invariant measure ν, i.e., there is a fixed vector $\vec{\lambda}(g_X^t) \in \mathbb{R}^n$ such that $\vec{\lambda}(g_X^t, \nu) = \vec{\lambda}(g_X^t)$ for all g_X^t-invariant ergodic probability measures ν. Hence we will omit the choice of invariant measure when we refer to the Lyapunov spectrum of g_X^t. The vector $\vec{\lambda}(g_X^t)$ depends only on the isometry type of the universal cover of X; for example, any negatively curved locally symmetric space with universal cover the real hyperbolic space $\mathbf{H}^3_{\mathbb{R}}$ has Lyapunov spectrum $(-1, -1, 0, 1, 1)$, while any locally symmetric space with universal cover the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ has Lyapunov spectrum $(-2, -1, -1, 0, 1, 1, 2)$, provided we normalize $\mathbf{H}^2_{\mathbb{C}}$ to have sectional curvatures $-4 \leq K \leq -1$. The Lyapunov spectra of negatively curved locally symmetric spaces in general are described in Section 2.1 below.

For our rigidity theorems, we let X be a negatively curved locally symmetric space with $\dim X \geq 3$. We consider a neighborhood of X in the space of Riemannian manifolds in the following sense: let S denote the underlying smooth manifold obtained by forgetting the Riemannian metric on X. We then think of X as (S, η_X), where η_X is the inner product on TS defining the Riemannian metric on X. We say that a Riemannian manifold Y is C^r close to X if $Y = (S, \eta_Y)$, where the inner product η_Y on TS is C^r close to η_X.

Throughout this paper we fix a particular open neighborhood $\mathcal{U}_X$ of X in the space of Riemannian manifolds that is described in Section 3.5. The neighborhood $\mathcal{U}_X$ is C^2 open when X is complex hyperbolic, and C^3 open when X is quaternionic or Cayley hyperbolic. When X is real hyperbolic one may take $\mathcal{U}_X$ to be the space of all closed Riemannian manifolds of strictly 1/4-pinched negative curvature.

We say that Y is *homothetic* to X if there is a constant $c > 0$ such that Y is isometric to $(S, c \cdot \eta_X)$. For each periodic point p of the geodesic flow g^t_Y we let $\nu^{(p)}$ denote the unique g^t_Y-invariant probability measure supported on the orbit of p. Our theorem below shows that the Lyapunov spectra of g^t_Y with respect to the invariant measures $\nu^{(p)}$, ranging over all periodic orbits p of g^t_Y, suffices to determine whether Y is homothetic to X.

Theorem 1.1. *Let X be a closed negatively curved locally symmetric space with* $\dim X \geq 3$. *Let $Y \in \mathcal{U}_X$. Then Y is homothetic to X if and only if, for each periodic point p of g^t_Y, there exists a constant $\xi(p) > 0$ such that*

$$\vec{\lambda}(g^t_Y, \nu^{(p)}) = \xi(p)\vec{\lambda}(g^t_X).$$

Note that we only need to assume a priori that the Lyapunov spectrum of g^t_Y at a given periodic point p is some multiple $\xi(p)$ of the Lyapunov spectrum of g^t_X, where $\xi(p)$ is allowed to depend in an arbitrary fashion on p. As part of the conclusion of the theorem one obtains that $\xi(p)$ is actually constant in p.

Theorem 1.1 will be deduced as a corollary of Theorem 1.2 below, which characterizes the locally symmetric space X up to isometry by the Lyapunov spectrum of a *single* invariant measure of the geodesic flow. We associate to each $Y \in \mathcal{U}_X$ a certain canonically defined g^t_Y-invariant ergodic probability measure μ_Y that we refer to as the *horizontal measure* for g^t_Y. We give a brief description of the horizontal measure here, using the thermodynamic formalism. For the formal construction of the horizontal measure see Section 2.1.

Given $Y \in \mathcal{U}_X$, we will construct a Hölder continuous function

$$\zeta_Y : T^1Y \to (-\infty, 0),$$

in a natural way out of the action of Dg^t_Y on a certain Dg^t_Y-invariant subbundle of $T(T^1Y)$. We then solve the Bowen equation $P(s\zeta_Y) = 0$, $s > 0$, where $P(s\zeta_Y)$ denotes the topological pressure of the function $s\zeta_Y$ with respect to the flow g_Y^t. We obtain a unique number $Q_Y > 0$ such that $P(Q_Y\zeta_Y) = 0$; we refer to Q_Y as the *horizontal dimension* of g_Y^t. The horizontal measure μ_Y is then defined to be the unique equilibrium state of the potential $Q_Y\zeta_Y$. For the locally symmetric space X itself, ζ_X is a constant function, Q_X is easily described in terms of the Lyapunov spectrum $\vec{\lambda}(g_X^t)$, and μ_X coincides with the Liouville measure m_X, which is the invariant volume for g_X^t.

Theorem 1.2. *Let X be a closed negatively curved locally symmetric space with $\dim X \geq 3$. Let $Y \in \mathcal{U}_X$. Then Y is isometric to X if and only if*

$$\vec{\lambda}(g_Y^t, \mu_Y) = \vec{\lambda}(g_X^t).$$

The horizontal measure μ_Y is a g_Y^t-invariant measure that is specifically adapted to the nonconstant negative curvature case; in general it does not coincide with any well-known previously considered invariant measures, such as the Liouville measure or the measure of maximal entropy for g_Y^t. One of the objectives of this paper is to show that, for rigidity problems involving Lyapunov exponents, the horizontal measure is often the natural invariant measure to consider.

Remark 1.3. Like the Liouville measure and the Bowen-Margulis measure of maximal entropy, for $Y, Z \in \mathcal{U}_X$ the horizontal measure is a homothety invariant in the following sense: a homothety $F : Y \to Z$ gives a derivative map $DF : T^1Y \to T^cZ$ for some constant $c > 0$. Let $\Pi : T^cZ \to T^1Z$ denote the natural projection. Then $(\Pi \circ DF)_*(\mu_Y) = \mu_Z$. The horizontal dimension is a homothety invariant as well, i.e., $Q_Y = Q_Z$ if Y is homothetic to Z. See Proposition 6.7 below.

The author [13] previously obtained the conclusions of Theorems 2.2 and 1.2 in the case in which X has constant negative curvature. In that setting one may replace μ_Y in Theorem 1.2 with an equilibrium state for g_Y^t with respect to any given Hölder potential. The novelty of Theorems 1.1 and 1.2 is that they treat the case of locally symmetric spaces of nonconstant negative curvature, which is significantly more delicate and difficult than the constant negative curvature case, and which requires many fundamentally new ideas.

Hernandez [34] and independently Yau and Zheng [62] proved that any 1/4-pinched negatively curved metric on a closed complex hyperbolic manifold is isometric to the standard symmetric metric. Gromov [24] extended these theorems to obtain 1/4-pinching rigidity for closed quaternionic hyperbolic manifolds as well. Our final geometric rigidity theorem addresses possible ways to generalize these rigidity theorems by weakening hypotheses on curvature pinching to hypotheses on pinching inequalities among the Lyapunov exponents of the geodesic flow.

Recall that we set $n = \dim Y$. We say that the Lyapunov spectrum of g_Y^t with respect to a g_Y^t-invariant ergodic probability measure ν is 1/2-*pinched* if there is a constant $a > 0$ such that

$$a \leq |\lambda_i(g_Y^t, \nu)| \leq 2a, \text{ for } 1 \leq i \leq 2n-1,\ i \neq n,$$

i.e., excluding the exponent corresponding to the flow direction of g_Y^t, the Lyapunov exponents of g_Y^t are pinched in absolute value between a and $2a$.

Curvature pinching estimates on a negatively curved Riemannian manifold Y give rise to pinching estimates on the Lyapunov exponents of g_Y^t: if the sectional curvatures of Y satisfy $-b^2 \leq K \leq -a^2$ for constants $b > a > 0$, then for any given g_Y^t-invariant ergodic probability measure ν we have $a \leq |\lambda_i(g_Y^t, \nu)| \leq b$ for all $i \neq n$. In particular, if Y has 1/4-pinched negative curvature then the Lyapunov spectrum of g_Y^t with respect to any ergodic invariantmeasure ν is 1/2-pinched. Theorem 1.4 below gives a partial result toward understanding whether the curvature 1/4-pinching hypothesis in the rigidity theorems of Hernandez, Yau and Zheng, and Gromov above can be weakened to a 1/2-pinching hypothesis on the Lyapunov spectrum of a special choice of invariant measure for the geodesic flow.

Recall that, for $Y \in \mathcal{U}_X$, we denote the horizontal measure for g_Y^t by μ_Y and denote the horizontal dimension of g_Y^t by Q_Y. Our theorem shows that, under the additional hypothesis of a lower bound on the horizontal dimension Q_Y, one can obtain a Lyapunov spectrum

1/2-pinching rigidity theorem for g_Y^t with respect to the measure μ_Y.

Theorem 1.4. *Let X be a closed negatively curved locally symmetric space of nonconstant negative curvature. Let $Y \in \mathcal{U}_X$. Suppose that $Q_Y \geq Q_X$ and that the Lyapunov spectrum of g_Y^t with respect to μ_Y is 1/2-pinched. Then Y is homothetic to X.*

Establishing the lower bound $Q_Y \geq Q_X$ under the hypothesis that $\vec{\lambda}(g_Y^t, \mu_Y) = \vec{\lambda}(g_X^t)$ is a critical step in the proof of Theorem 1.2. This, together with the role of this lower bound in the hypotheses of Theorem 1.4, prompts the following question.

Question 1.5. *Let X be a closed negatively curved locally symmetric space of nonconstant negative curvature. Let $Y \in \mathcal{U}_X$. Do we always have $Q_Y \geq Q_X$?*

An affirmative answer to this question would give a full generalization of the 1/4-curvature pinching rigidity theorems for nonconstant negative curvature locally symmetric spaces to 1/2-pinching rigidity theorems for their Lyapunov spectra with respect to their horizontal measures. It would also lead to a positive answer to a question of Boland and Katok [8] that is presented in Section 2.2.

Our rigidity results should be viewed in conjuction with other dynamical rigidity results that characterize negatively curved locally symmetric spaces using dynamical invariants. Perhaps the most famous of these dynamical invariants is the topological entropy $h_{\text{top}}(g_Y^t)$ of the geodesic flow of Y. For nonpositively curved Riemannian manifolds the topological entropy measures the exponential growth rate of the volumes of metric balls in the universal cover. For X a hyperbolic surface of genus $g \geq 2$ and Y another negatively curved surface of the same area and genus, Katok [42] proved that $h_{\text{top}}(g_Y^t) \geq h_{\text{top}}(g_X^t)$ with equality if and only if Y has constant negative curvature. The minimal entropy rigidity theorem of Besson, Courtois, and Gallot [6] gives a remarkable generalization of Katok's theorem to higher dimensions: on a negatively curved locally symmetric space X, the topological entropy among all negatively curved metrics on X of the same volume is uniquely minimized at the symmetric metric on X. Hence the symmetric metrics are extremal for topological entropy, and are actually characterized by the topological entropy of their geodesic flows.

A previous result in the spirit of Theorem 1.2 is Hamenstädt's hyperbolic rank rigidity theorem [28]. In our context her theorem is best viewed through the perspective of an extension due to Connell: if Y is a closed negatively curved Riemannian manifold with sectional curvatures satisfying $K \leq -a^2$ for a given $a > 0$ then Y is isometric to a locally symmetric space X if and only if the minimal positive Lyapunov exponent of g_Y^t with respect to m_Y is a. These hyperbolic rank rigidity theorems have been extended further by Constantine [18] and Connell, Nguyen, and Spatzier [17]. These theorems all have a fundamentally geometric character; hypotheses on hyperbolic rank impose very strong restrictions on the geodesic flow of the manifold, e.g. the existence of proper smooth invariant subbundles for the flow if the manifold does not have constant negative curvature. In contrast, as we explain in Section 2.2, our theorems extend beyond the geometric setting to cover arbitrary perturbations of the geodesic flows of negatively curved locally symmetric spaces among smooth flows.

There is a natural question that arises from the statement of Theorem 1.2 and the methods that are used in its proof. Recall that we let m_Y denote the Liouville measure for a Riemannian manifold Y.

Question 1.6. *Let X be a closed negatively curved locally symmetric space of nonconstant negative curvature and let $Y \in \mathcal{U}_X$. Suppose that $\vec{\lambda}(g_Y^t, m_Y) = \vec{\lambda}(g_X^t)$. Is Y isometric to X?*

As remarked above, the author [13] obtained a positive answer to the analogous question in the case in which X has constant negative curvature. There is a specific technical obstruction to obtaining a positive answer to Question 1.6 using the techniques from the proof of Theorem 1.2; this is discussed in Remark 7.3.

One can also ask whether our theorems hold globally, that is, only under the hypothesis that Y is a negatively curved manifold quasi-isometric to X, instead of Y being in the open neighborhood $\mathcal{U}_X$ of X. Our techniques make strong use of the persistence of certain g_X^t-invariant structures to g_Y^t-invariant structures for Y close to X; it's unclear whether these techniques would continue to be effective globally. In particular, it is less clear how to define the horizontal measure and how to use it effectively when Y is far from X.

2 The dynamical rigidity theorems

2.1 The horizontal measure

Let X be a locally symmetric space of negative curvature. In this section we construct the horizontal measure μ_Y for Riemannian manifolds Y that are C^2 close to X. To do this we recall some basic concepts in smooth dynamics. We will also need a description of the dynamics of the geodesic flow on negatively curved symmetric spaces. We take this opportunity to establish conventions for notation and terminology, as these vary in the literature. A standard reference for the claims made below about Anosov flows is [44].

Let M be a closed Riemannian manifold and let $f^t : M \to M$ be a C^r flow on M, $r \geq 1$. We say that f^t is an *Anosov flow* if there is a Df^t-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ such that E^c is tangent to the flow direction of f^t, and there are constants $C \geq 1$ and $a > 0$ such that for every $v \in E^s$ and $t \geq 0$,

$$\|Df^t(v)\| \leq e^{-at}\|v\|,$$

and for every $v \in E^u$ and $t \geq 0$,

$$\|Df^{-t}(v)\| \leq e^{-at}\|v\|.$$

In other words, E^s is exponentially contracted by f^t, and E^u is exponentially contracted by f^{-t}. For all of the Anosov flows considered in this paper, the distributions E^u and E^s will have the same dimension; we set $l = \dim E^u = \dim E^s$. Each of the distributions E^u, E^c, E^s, $E^c \oplus E^u$, and $E^c \oplus E^s$ is uniquely integrable and tangent to a foliation W^u, W^c, W^s, W^{cu}, and W^{cs} respectively. The geodesic flow g_Y^t of a negatively curved manifold Y is an

Anosov flow on T^1Y.

We define two continuous flows f^t and $g^t : M \to M$ to be *orbit equivalent* if there is a homeomorphism $\varphi : M \to M$ and a continuous map $\alpha : \mathbb{R} \times M \to \mathbb{R}$ such that $\varphi(g^t(x)) = f^{\alpha(t,x)}(\varphi(x))$ for all $t \in \mathbb{R}$ and $x \in M$. We say that φ is a *conjugacy* if $\alpha(t,x) \equiv t$.

A key fact that we will use about Anosov flows is that they are *structurally stable*: if f^t is a C^r Anosov flow and g^t is a C^r flow that is C^1 close to f^t, then g^t is also an Anosov flow, and furthermore there is a Hölder continuous orbit equivalence φ from f^t to g^t. The orbit equivalence φ is close to the identity and is unique up to time changes in the flow direction. The invariant subbundles E^u, E^c, and E^s all depend continuously on f^t in the C^1 topology.

The negatively curved Riemannian symmetric spaces fit into four families: the real hyperbolic spaces $\mathbf{H}^n_{\mathbb{R}}$, the complex hyperbolic spaces $\mathbf{H}^n_{\mathbb{C}}$, the quaternionic hyperbolic spaces $\mathbf{H}^n_{\mathbb{H}}$, and the Cayley hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$. We normalize $\mathbf{H}^n_{\mathbb{R}}$ to have sectional curvatures $K \equiv -1$, and we normalize the other nonconstant negative curvature hyperbolic spaces to have sectional curvatures $-4 \leq K \leq -1$. A reference for the discussion below on the structure of these spaces is [49].

The structure of the unstable manifolds of the geodesic flows of each of these symmetric spaces is given by a 2-step Carnot group G, where for $\mathbf{H}^n_{\mathbb{R}}$ this group is simply $\mathbb{R}^{n-1}$, and for $\mathbf{H}^n_{\mathbb{C}}$, $\mathbf{H}^n_{\mathbb{H}}$, and $\mathbf{H}^2_{\mathbb{O}}$ the groups G are the complex, quaternionic, and octonionic Heisenberg groups respectively. We recall from [49] that a 2-step Carnot group is a 2-step nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$, where $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{g}$. We will consider G as equipped with a left-invariant Riemannian metric for which $\mathfrak{h}$ and $\mathfrak{v}$ are orthogonal.

In the identification of the unstable manifolds with G, the geodesic flow acts by an expanding automorphism on G. More precisely, for the left-invariant inner product $\langle \, , \rangle$ on TG, we have

$$\langle Dg^t_X(v), Dg^t_X(w) \rangle = e^t \langle v, w, \rangle; v, w \in \mathfrak{h},$$

and

$$\langle Dg^t_X(v), Dg^t_X(w) \rangle = e^{4t} \langle v, w, \rangle; v, w \in \mathfrak{v}.$$

Hence Dg_X^t expands $\mathfrak{h}$ by a factor of e^t and $\mathfrak{v}$ by a factor of e^{2t}.

Letting $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ we have for the $\mathbb{K}$-hyperbolic spaces that $\dim \mathfrak{v} = \dim_{\mathbb{R}} \mathbb{K} - 1$, where $\dim_{\mathbb{R}} \mathbb{K}$ is the dimension of the division algebra $\mathbb{K}$ as a vector space over $\mathbb{R}$. For a negatively curved locally symmetric space X we then set

$$k(X) := \dim \mathfrak{h} = \dim X - \dim_{\mathbb{R}} \mathbb{K}$$

Then for a $\mathbb{K}$-hyperbolic space X the geodesic flow g_X^t has $k(X)$ positive Lyapunov exponents of value 1 and has $\dim_{\mathbb{R}} \mathbb{K} - 1$ positive Lyapunov exponents of value 2. Similarly the geodesic flow has $k(X)$ negative Lyapunov exponents of value -1 and has $\dim_{\mathbb{R}} \mathbb{K} - 1$ negative Lyapunov exponents of value -2. Lastly we set

$$h(X) := h_{\text{top}}(g_X^t) = k(X) + 2(\dim_{\mathbb{R}} \mathbb{K} - 1)$$

The splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ gives rise to a global Dg_X^t-invariant splitting $E^{u,g} = H^{u,g} \oplus V^{u,g}$, where, in the G-coordinates, $H^{u,g}$ and $V^{u,g}$ are the left-invariant distributions tangent to $\mathfrak{h}$ and $\mathfrak{v}$ respectively. This splitting is *dominated* and therefore persists under C^1 small perturbations of the flow g_X^t. For the definition and properties of dominated splittings used below, see Section 3.2.

We conclude from the above that, for a C^r Anosov flow $f^t : T^1X \to T^1X$ that is C^1 close to g_X^t, there is also a dominated splitting $E^{u,f} = H^{u,f} \oplus V^{u,f}$ that is close to the corresponding splitting for g_X^t. The subbundles $H^{u,f}$ and $V^{u,f}$ are Hölder continuous and the restriction of $V^{u,f}$ to $W^{cu,f}$ leaves is a C^{r-1} subbundle; see Proposition 3.5 below. We set $\mathcal{B}^{u,f} = E^{u,f}/V^{u,f}$, to be the quotient bundle of $E^{u,f}$ by $V^{u,f}$. The bundle $\mathcal{B}^{u,f}$ is Hölder continuous and is C^{r-1} along $W^{cu,f}$ leaves as well.

We define a potential function $\zeta_f : M \to \mathbb{R}$ on $x \in T^1X$ by,

$$\zeta_f(x) = - \left.\frac{d}{dt}\right|_{t=0} \log \operatorname{Jac}\left(Df_x^t | \mathcal{B}_x^{u,f}\right).$$

The potential ζ_f is Hölder continuous and strictly negative. We show at the beginning of Section 6.3 that there is a unique number $Q(f) > 0$ such that the topological pressure of $Q(f)\zeta_f$ with respect to f^t satisfies $P(Q(f)\zeta_f) = 0$. The *unstable horizontal measure* μ_f for f^t is defined to be the unique equilibrium state for $Q(f)\zeta_f$. We define $Q(f)$ to be the *unstable horizontal dimension* of f^t.

Considering $X = (S, \eta_X)$ and $Y = (S, \eta_Y)$ as Riemannian manifolds with the same underlying smooth manifold S and with η_Y being C^2 close to X, we may consider g_X^t and g_Y^t as smooth flows on TS. The codimension-one submanifolds T^1X and T^1Y of TS are C^2 close to one another and thus we have a natural smooth diffeomorphism $\Psi : T^1X \to T^1Y$ obtained by projecting T^1X to T^1Y. We then consider $f^t = \Psi^{-1} \circ g_Y^t \circ \Psi$ as a smooth flow on T^1X. The flow f^t is C^1 close to g_X^t and so the above discussion applies; letting μ_f be the horizontal measure for f^t and $Q(f)$ the horizontal dimension, we set $\mu_Y = \Psi_*^{-1}(\mu_f)$ and $Q_Y = Q(f)$. Both μ_Y and Q_Y are independent of the choice of diffeomorphism Ψ by Proposition 6.7.

For the locally symmetric space X we have $\zeta_{g_X^t} \equiv -k(X)$, and therefore μ_X is the measure of maximal entropy for g_X^t, which is just the Liouville measure m_X. The horizontal dimension Q_X is given by the formula $Q_X = \frac{h(X)}{k(X)}$.

Remark 2.1. When we vary f^t in the C^2 topology, the potential ζ_f varies continuously in the β-Hölder topology for a small enough exponent $\beta > 0$. The pressure function $(s, f) \to P(s\zeta_f)$ is then jointly continuous in f^t in the C^2 topology and in $s \in \mathbb{R}$ [55], [10]. We conclude that $Q(f)$ and μ_f vary continuously with f^t.

2.2 The dynamical rigidity theorems

Our main dynamical rigidity theorems extend beyond the geometric setting to cover *all* C^r small perturbations of the geodesic flow g_X^t on T^1X, with $r = 1$ or 2 depending on X.

We say that a continuous map $\psi : M \to N$ between two C^r-manifolds M and N is $C^{r+\alpha}$ for some $r \geq 0$, $0 \leq \alpha < 1$, if ψ is C^r and the rth-order derivatives of ψ are α-Hölder continuous. For a C^r Anosov flow f^t that is C^1-close to g^t_X, our standard for rigidity in this section is the existence of an orbit equivalence $\varphi : T^1X \to T^1X$ from g^t_X to f^t such that, for each $x \in T^1X$, the restriction $\varphi : W^{cu,g}(x) \to W^{cu,f}(\varphi(x))$ is $C^{1+\alpha}$ for some $\alpha > 0$ independent of x. We will refer to this as the orbit equivalence φ being $C^{1+\alpha}$ on center-unstable leaves. Since f^t is C^1 close to g^t_X, an orbit equivalence between these flows always exists by structural stability. The questions the theorems below address is: under what conditions can the regularity of this orbit equivalence be improved?

To state our theorems, we first recall the formal definition of the Lyapunov exponents of an f^t-invariant ergodic probability measure ν. For a linear transformation $T : V \to W$ between two l-dimensional inner product vector spaces V and W, we let $\sigma_1(T) \leq \cdots \leq \sigma_l(T)$ denote the singular values of T with respect to these inner products, listed in increasing order. As a consequence of the multiplicative ergodic theorem [48], for each f^t-invariant probability measure ν there are positive constants $0 < \lambda^u_1(f^t, \nu) \leq \cdots \leq \lambda^u_l(f^t, \nu)$ – referred to as the unstable Lyapunov exponents of f^t with respect to ν – such that for each $1 \leq i \leq l$ we have

$$\lambda^u_i(f^t, \nu) = \lim_{t\to\infty} \frac{\log \sigma_i(Df^t_x|E^u_x)}{t} \text{ for } \nu\text{-a.e. } \; x \in M.$$

We write $\vec{\lambda}^u(f^t, \nu)$ for the vector whose components are the unstable Lyapunov exponents written in increasing order. Likewise we have negative constants $0 > \lambda^s_1(f^t, \nu) \geq \cdots \geq \lambda^s_l(f^t, \nu)$ – referred to as the stable Lyapunov exponents of f^t with respect to ν – such that for each $1 \leq i \leq l^s$ we have

$$\lambda^s_i(f^t, \nu) = \lim_{t\to\infty} \frac{\log \sigma_{l-i+1}(Df^t_x|E^s_x)}{t} \text{ for } \nu\text{-a.e. } \; x \in M.$$

We write $\vec{\lambda}^s(f^t, \nu)$ for the vector whose components are the stable Lyapunov exponents written in decreasing order. We have chosen the differing orders on the stable and unstable Lyapunov exponents in order for the equation $\vec{\lambda}^s(f^t, \nu) = -\vec{\lambda}^u(f^{-t}, \nu)$ to hold. We note that what we called $\vec{\lambda}(g_Y^t, \nu)$ before is described in this notation as the vector

$$\vec{\lambda}(g_Y^t, \nu) = (\underline{\vec{\lambda}}^s(g_Y^t, m_Y), 0, \vec{\lambda}^u(g_Y^t, m_Y)).$$

where $\underline{\vec{\lambda}}^s(g_Y^t, m_Y)$ denotes the vector $\vec{\lambda}^s(g_Y^t, m_Y)$ with its entries written in reverse order.

Below we write $\mathcal{V}_X$ for a certain C^r open neighborhood of g_X^t in the space of C^3 Anosov flows on T^1X. This neighborhood is described in Section 3.5. When X is complex hyperbolic we may take $r = 1$, and when X is quaternionic or Cayley hyperbolic we may take $r = 2$. Note that if two Riemannian manifolds X and Y are C^r close then their geodesic flows are C^{r-1} close, once we smoothly identify T^1X and T^1Y as at the end of Section 2.1.

We first state the counterpart of Theorem 1.1 for flows.

Theorem 2.2. *Let X be a closed negatively curved locally symmetric space with* $\dim X \geq 3$. *Let $f^t \in \mathcal{V}_X$. Suppose that, for each periodic point p of f^t, there exists a constant $\xi(p) > 0$ such that*

$$\vec{\lambda}^u(f^t, \nu^{(p)}) = \xi(p)\vec{\lambda}^u(g_X^t).$$

Then there is an orbit equivalence φ from g_X^t to f^t that is $C^{1+\alpha}$ on center-unstable leaves.

We next state the counterpart of Theorem 1.2 for flows.

Theorem 2.3. *Let X be a closed negatively curved locally symmetric space with* $\dim X \geq 3$. *Let $f^t \in \mathcal{V}_X$. Suppose there exists a constant $\xi > 0$ such that*

$$\vec{\lambda}^u(f^t, \mu_f) = \xi\vec{\lambda}^u(g_X^t).$$

Then there is an orbit equivalence φ from g_X^t to f^t that is $C^{1+\alpha}$ on center-unstable leaves.

Lastly we state the counterpart of Theorem 2.5 for flows. Theorem 2.5 below is motivated by the following question of Boland and Katok [8].

Question 2.4. *Let X be a closed complex hyperbolic manifold, $\dim X \geq 4$. Let f^t be a contact Anosov flow that is a C^1 small perturbation of g_X^t. Suppose that $\lambda_l^u(f^t,\nu) \leq 2\lambda_1^u(f^t,\nu)$, $l = \dim X - 1$, for all f^t-invariant ergodic probability measures ν. Is f^t smoothly orbit equivalent to g_X^t?*

Our theorem uses the 1/2-pinching inequality of Question 2.4 for the horizontal measure, together with a lower bound on the horizontal dimension of f^t, to build an orbit equivalence from f^t to g_X^t that is $C^{1+\alpha}$ on center-unstable leaves.

Theorem 2.5. *Let X be a closed negatively curved locally symmetric space of nonconstant negative curvature. Let $l = \dim X - 1$. Let $f^t \in \mathcal{V}_X$. Suppose that*

$$Q(f) \geq Q_X, \text{ and } \lambda_l^u(f^t,\mu_f) \leq 2\lambda_1^u(f^t,\mu_f).$$

Then there is an orbit equivalence φ from g_X^t to f^t that is $C^{1+\alpha}$ on center-unstable leaves.

Observe that in the theorems above we do not make any assumptions on the specific *values* of the Lyapunov exponents of f^t. We instead only make assumptions on the *structure* of the Lyapunov exponents of the flow. We are only imposing the condition that the Lyapunov spectrum of f^t should have the same multiplicities and ratios between exponents as the flow g_X^t. We assume no structure on f^t beyond this.

We are able to deal with such general flows f^t because our techniques draw on the classification theorems for uniformly quasiconformal Anosov diffeomorphisms and Anosov flows. The cumulative results of Kanai [41], Sadovskaya [56], Kalinin and Sadovskaya [38], [37], and Fang [19], [20] show that all such Anosov diffeomorphisms are linear toral automorphisms and that all such Anosov flows are smoothly orbit equivalent to the geodesic flow of a real hyperbolic manifold. The uniform quasiconformality condition – defined at the beginning of Section 3 – is strong enough on its own to obtain these rigidity theorems without additional hypotheses. In the proof of Theorem 2.3, we will exploit a limited form of uniform quasiconformality on a certain proper invariant subbundle for Df^t inside of E^u.

Remark 2.6. By replacing g_X^t and f^t with g_X^{-t} and f^{-t} in the above theorems, one obtains analogues of these rigidity theorems for the stable Lyapunov exponents of f^t, which gives a similar conclusion that there is an orbit equivalence from g_X^t to f^t that is $C^{1+\alpha}$ on center-stable leaves. When f^t is a contact Anosov flow, the hypotheses of the above dynamical rigidity theorems hold for the unstable Lyapunov exponents if and only if they hold for the stable Lyapunov exponents. In this case one can show that f^t is actually $C^{1+\alpha}$ orbit equivalent to g_X^t.

2.3 The real hyperbolic case

In the case of real hyperbolic manifolds we do not need to use perturbative techniques. This allows us to obtain global rigidity results in terms of the Lyapunov spectrum for Anosov flows that are only orbit equivalent to the geodesic flow of some negatively curved Riemannian manifold. In this case we do not need to use perturbative techniques.

In the hypotheses of Theorem 2.7 we impose two conditions, one on the flow f^t and the other on the f^t-invariant ergodic probability measures ν^u and ν^s under consideration, which are defined later in the paper. The 1-bunching condition on f^t should be thought of as the analogue of the strict 1/4-pinching hypothesis for Anosov flows. 1-bunching is defined at the beginning of Section 3.3. The local product structure condition on the measures ν^u and ν^s should be thought of as both a nontriviality and a regularity hypothesis on the measures, which requires them to respect the structure of the invariant foliations of f^t. Local product structure for f^t-invariant measures is defined at the beginning of Section 3. All equilibrium states associated to Hölder continuous potentials for a transitive Anosov flow have local product structure [10], [9].

Theorem 2.7. *Let f^t be a C^∞ Anosov flow which is orbit equivalent to the geodesic flow of a closed negatively curved manifold of dimension $l+1$, with $l \geq 2$. Suppose that f^t is 1-bunched and that there exist fully supported f^t-invariant ergodic probability measures ν^u and ν^s for f^t with local product structure such that $\lambda_1^u(f^t,\nu^u) = \lambda_l^u(f^t,\nu^u)$ and $\lambda_1^s(f^t,\nu^s) = \lambda_l^s(f^t,\nu^s)$.*

Then there exists a closed real hyperbolic manifold X such that f^t is C^∞ orbit equivalent to g_X^t.

Note that we do not assume any relationship between the measures ν^u and ν^s. The hypotheses imply that all positive Lyapunov exponents of ν^u with respect to f^t are equal and all negative Lyapunov exponents of f^t with respect to ν^s are equal. Although we do not pursue this direction here, the condition that f^t is orbit equivalent to the geodesic flow of a negatively curved manifold can likely be weakened to the hypothesis that f^t is orbit equivalent to the geodesic flow of a closed Finsler manifold of negative flag curvature or the geodesic flow of a Riemannian manifold whose geodesic flow is Anosov.

As with the above theorems, we have a theorem that characterizes by periodic orbits as well. This theorem does not require the bunching condition of Theorem 2.7.

Theorem 2.8. *Let f^t be a C^∞ Anosov flow which is orbit equivalent to the geodesic flow of a closed negatively curved manifold of dimension $l+1$, with $l \geq 2$. Suppose that f^t is 1-bunched and that for each periodic point p of f^t we have*

$$\lambda_1^*(f^t, \nu^{(p)}) = \lambda_l^*(f^t, \nu^{(p)}), * \in \{s, u\}.$$

Then there exists a closed real hyperbolic manifold X such that f^t is C^∞ orbit equivalent to g_X^t.

We prove Theorems 2.7 and 2.8 at the end of Section 5.1.

3 Uniform quasiconformality and Anosov flows

In this section we recall in detail the setup of our previous work [13], closely following the presentations of Sections 2 and 4 from that paper. This requires us to introduce linear cocycles over Anosov flows and several concepts associated to them, e.g. dominated splittings, holonomies, and uniform quasiconformality. We caution that we have made some changes in notation from our previous paper.

3.1 Holonomies for invariant foliations

Throughout the paper we will use a convention that we also used in [15]: let $f^t : M \to M$ be an Anosov flow on a Riemannian manifold M and let W^* denote any one of the dynamically defined invariant foliations for f^t. We fix constants $R > r > 0$ depending only on f^t and, for each $x \in M$, we write $W^*_{loc}(x)$ for any connected open neighborhood of x inside $W^*(x)$ that satisfies $B^*(x, r) \subseteq W^*_{loc}(x) \subseteq B^*(x, R)$, where $B^*(x, r)$ is the ball of radius r inside $W^*(x)$ of the restriction of the metric d on M to $W^*(x)$. The constants r, R are chosen such that all of the neighborhoods $W^*_{loc}(x)$ lie inside foliation charts for the foliation W^*.

Given two nearby transversals $T(x)$ and $T(y)$ to an invariant foliation W^* of f^t passing through x and y respectively, for each $z \in T(x)$ we define $h^*_{xy}(z)$ to be the unique point in the intersection $W^*_{loc}(z) \cap T(y)$. We refer to these as the *holonomy maps* for the foliation W^*. The choice of transversals we are taking will always be clear from context; we will often be taking local leaves of another invariant foliation as transversals. In this case, for example, for $x \in M$, $y \in W^{cs}_{loc}(x)$ when we write the center-stable holonomy

$$h^{cs}_{xy} : W^u_{loc}(x) \to W^u_{loc}(y),$$

this should be understood as saying that the neighborhoods $W^u_{loc}(x)$ and $W^u_{loc}(y)$ have been chosen such that h^{cs}_{xy} is a homeomorphism onto its image, and such that they satisfy the hypotheses imposed in the previous paragraph. Similar interpretations should be used for all other holonomy maps.

We will further choose the local neighborhoods W^*_{loc} to be compatible with the *local product structure* of the invariant foliations for f^t: for any $x \in M$ there is an open neighborhood U_x of x such that, if we consider the transversal $W^{cs}_{loc}(x)$ to the local unstable leaf $W^u_{loc}(x)$

of x, then we have $W^u_{loc}(x) \times W^{cs}_{loc}(x) \cong U_x$ via the map $(y, z) \to h^{cs}_{xz}(y)$, where we are considering cs-holonomy from $W^u_{loc}(x)$ to $W^u_{loc}(z)$. We also require that we have a similar homeomorphism $W^{cu}_{loc}(x) \times W^s_{loc}(x) \cong U_x$.

We say that a fully supported f^t-invariant ergodic probability measure μ has *local product structure* if for every $x \in M$ there is a neighborhood U_x of x and a uniformly continuous function $\xi_x : U_x \to (0, \infty)$ such that in the holonomy chart $U_x \cong W^u_{loc}(x) \times W^{cs}_{loc}(x)$ the measure μ splits as a product

$$\mu = \xi_x \cdot (\mu_x \times \mu'_x),$$

of conditional measures μ_x and μ'_x on $W^u_{loc}(x)$ and $W^{cs}_{loc}(x)$ respectively. We also require that the analogous statement is true when we consider the product splitting $U_x \cong W^{cu}_{loc}(x) \times W^s_{loc}(x)$ instead. We require the functions ξ_x to also be uniformly continuous in the variable $x \in M$. Equilibrium states of Hölder continuous potentials always have local product structure [10].

3.2 Dominated splittings

Let M be a compact metric space. We define an l-dimensional vector bundle $\pi : \mathcal{E} \to M$ over M to be α-Hölder continuous if there is an open cover of M by open sets U_i admitting linear trivializations $T_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^l$ such that the transition maps $T_i \circ T_j^{-1}$ are α-Hölder continuous with respect to the Riemannian metric on M and the Euclidean metric on $\mathbb{R}^l$. We equip $\mathcal{E}$ with an α-Hölder continuous inner product $\langle\ ,\ \rangle$ coming from these trivializations which gives rise to a norm $\|\cdot\|$.

Let $f^t : M \to M$ be an α-Hölder continuous flow. We define an α-Hölder *linear cocycle* on $\mathcal{E}$ over f^t to be an α-Hölder flow $A^t : \mathcal{E} \to \mathcal{E}$, such that $f^t \circ \pi = \pi \circ A^t$, and such that for each $x \in M$ and $t \in \mathbb{R}$ the map $A^t_x : \mathcal{E}_x \to \mathcal{E}_{f^t(x)}$ is a linear isomorphism. Observe that the linear maps A^t_x satisfy the *cocycle condition*: for every $s, t \in \mathbb{R}$ and $x \in M$,

$$A^{t+s}_x = A^s_{f^t x} \circ A^t_x$$

When $\dim \mathcal{E} = 1$ we refer to $\psi^t := A^t$ as a *multiplicative cocycle*. When we consider multiplicative cocycles in this paper we will always consider the case where $\mathcal{E}$ is trivial and ψ^t is positive, so that we obtain a map $\psi^t : M \to (0, \infty)$ for each $t \in \mathbb{R}$ which satisfies the cocycle condition above. We define a continuous map $(t, x) \to \tau(t, x)$ for $t \in \mathbb{R}$, $x \in M$ to be an *additive cocycle* over f^t if it satisfies the additive cocycle identity,

$$\tau(t + s, x) = \tau(s, f^t(x)) + \tau(t, x).$$

Additive cocycles correspond to multiplicative cocycles through exponentiation $\tau \to \exp(\tau)$.

A *dominated splitting* of index k for the linear cocycle A^t over f^t is an A^t-invariant splitting $\mathcal{E} = H \oplus V$ of $\mathcal{E}$ into two continuous subbundles H and V, with $\dim H = k$, such that there are constants $C \geq 1$, $\chi > 0$ for which we have for all $x \in M$ and $t \geq 0$,

$$\frac{\sigma_k(A^t_x | H_x)}{\sigma_1(A^t_x | V_x)} \leq C e^{-\chi t},$$

where we recall that σ_i denotes the ith singular value of a linear transformation between two normed vector spaces, listed in increasing order. Here we equip H and V with the norm induced from our norm on $\mathcal{E}$. The domination condition implies that there is a uniform exponential gap between the largest singular value of A^t on H and the smallest singular value of A^t on V. The subbundles of a dominated splitting for an α-Hölder linear cocycle A^t are always uniformly β-Hölder continuous for some exponent β that only depends on α and the exponential expansion rates of A^t on H and V respectively. In particular the restrictions $A^t|H$ and $A^t|V$ are both Hölder continuous linear cocycles as well.

Remark 3.1. It is always possible to choose a new continuous Riemannian metric on $\mathcal{E}$ such that the domination estimate for the splitting $\mathcal{E} = H \oplus V$ above holds for all $t \geq 0$ with constant $C = 1$ and the same exponent χ [23]. In particular this holds for the trivial splitting $\mathcal{E}$, for which we conclude that if we have bounds for $x \in M$, $t \geq 0$, $a \leq b \in \mathbb{R}$, and some constant $C \geq 1$,

$$C^{-1}e^{at} \leq \sigma_1(A_x^t) \leq \sigma_l(A_x^t) \leq Ce^{bt},$$

then a new Riemannian metric on $\mathcal{E}$ can be chosen such that these bounds hold with the same exponents a and b but with constant $C = 1$. We will require this fact at a few points in the paper.

Dominated splittings are stable under C^0-small perturbations of the linear cocycle A^t [58]. Thus if B^t is a linear cocycle which is C^0-close to A^t and A^t admits a dominated splitting $\mathcal{E} = H \oplus V$ of index k then B^t also admits a dominated splitting $\mathcal{E} = H' \oplus V'$ of index k with H' and V' uniformly close to H and V respectively.

We define a linear cocycle A^t to be *uniformly quasiconformal* if there is a constant $C \geq 1$ such that for all $x \in M$ and $t \in \mathbb{R}$ we have

$$C^{-1} \leq \frac{\sigma_1(A_x^t)}{\sigma_l(A_x^t)} \leq C,$$

where we recall that $l = \dim \mathcal{E}$. We say that A^t is *conformal* if this holds with $C = 1$. Note that the uniform quasiconformality condition is trivial if $l = 1$ but highly nontrivial if $l \geq 2$. We will see in Sections 5.1 and 5.2 below that one can often derive uniform quasiconformality of a cocycle A^t from information about the Lyapunov exponents of A^t, with respect to those f^t-invariant ergodic probability measures which carry a certain amount of structure. Lastly, by Remark 3.1, given a uniformly quasiconformal linearly cocycle A^t it is always possible to choose a Riemannian metric on $\mathcal{E}$ for which A^t is conformal in that metric.

3.3 Holonomies for linear cocycles

We now assume that A^t is an α-Hölder linear cocycle on an α-Hölder l-dimensional vector bundle $\mathcal{E}$ over an Anosov flow f^t on a closed Riemannian manifold M. The standard example one should keep in mind in this section is the restriction of the derivative cocycle Df^t to a Df^t-invariant α-Hölder continuous subbundle of TM.

We say that A^t is *fiber bunched* if there is some constant $C \geq 1$ such that for all $x \in M$ and $t \geq 0$ we have,

$$\frac{\sigma_l(A_x^t)}{\sigma_1(A_x^t)} < C \min\{\sigma_l(Df_x^t|E^s), \sigma_1(Df_x^t|E^u)^{-1}\}^{-\alpha}. \tag{1}$$

Fiber bunching is a technical condition which guarantees the existence of A^t-equivariant identifications of the fibers of $\mathcal{E}$ along stable and unstable manifolds of f^t which we refer to as *holonomies*. These identifications are essential for our work in this paper. If A^t is uniformly quasiconformal then it is fiber bunched. Fiber bunching should be thought of as a bound – in terms of the hyperbolicity of the base system – on the defect of A^t from being uniformly quasiconformal.

Sometimes it will be necessary to specify the choice of exponent in the fiber bunching inequality (1). We define A^t to be *fiber bunched with exponent* α if A^t is an α-Hölder continuous linear cocycle which satisfies the inequality (1), with exponent α on the right-hand side. This is stronger than the fiber bunching condition alone: if A^t is β-Hölder continuous for some $\beta > \alpha$ then it can be fiber bunched – meaning it satisfies (1) with exponent β – without being fiber bunched with exponent α.

We next define unstable holonomies for A^t. The definition and basic properties of these maps are given in Proposition 3.2 below; we refer to [39] for the proof of this proposition and additional discussion. As a consequence of α-Hölder continuity of the vector bundle $\mathcal{E}$, there is an open neighborhood $\mathcal{O}$ of the diagonal $\{(x,x) : x \in M\}$ in $M \times M$ such that, for each $(x,y) \in \mathcal{O}$, there is a linear isomorphism $I_{xy} : \mathcal{E}_x \to \mathcal{E}_y$ for which I_{xy} depends uniformly α-Hölder continuously on the points $(x,y) \in \mathcal{O}$. We state Hölder continuity properties of the unstable holonomies L^u_{xy} below through comparison to the identifications I_{xy}.

Proposition 3.2. *Let A^t be an α-Hölder linear cocycle over an Anosov flow f^t and suppose that A^t is fiber bunched. Then for each $x \in M$, $y \in W^u_{loc}(x)$ there is a linear isomorphism $L^u_{xy} : \mathcal{E}_x \to \mathcal{E}_y$ with the following properties,* *1.* $L^u_{xx} = Id$ *and* $L^u_{yz} \circ L^u_{xy} = L^u_{xz}$*;*

2. $L^u_{xy} = (A^t_y)^{-1} \circ L^u_{f^t x f^t y} \circ A^t_x$ *for every* $t \in \mathbb{R}$*.*

3. There is a constant $C \geq 1$ *such that* $\|L^u_{xy} - I_{xy}\| \leq Cd(x,y)^\alpha$ *for all* $y \in W^u_{loc}(x)$*.*

Furthermore the family of linear maps L^u *satisfying the above properties is unique.*

Note that from the third property and the Hölder continuity of the identifications I_{xy} we obtain that the maps L^u_{xy} are jointly α-Hölder in x and y.

In the proof of Proposition 3.8 below, we will need a more explicit statement of the uniqueness of u-holonomies. The following may be extracted from the proofs of [39] (or alternatively derived from the statements of Proposition 3.2 above),

Proposition 3.3. *Let* A^t *be an* α*-Hölder linear cocycle over an Anosov flow* f^t*. Suppose that* A^t *is* β*-fiber bunched for some* $\beta \leq \alpha$*. Let* $x \in M$*,* $y \in W^u_{loc}(x)$ *be given. Let* $t_n \to \infty$ *be a sequence of real numbers and let* J_n *be a sequence of linear maps* $J_n : \mathcal{E}_{f^{-t_n}x} \to \mathcal{E}_{f^{-t_n}y}$ *satisfying*

$$\|J_n - I_{f^{-t_n}x f^{-t_n}y}\| \leq Cd(f^{-t_n}x, f^{-t_n}y)^\beta,$$

for some constant $C \geq 1$*. Then* $Df^{t_n}_{f^{-t_n}y} \circ J_n \circ Df^{-t_n}_x \to L^u_{xy}$ *as* $t_n \to \infty$*.*

We refer to the family of maps L^u as the *unstable holonomies* for A^t. Likewise, by considering the cocycle A^{-t} over f^{-t} and applying Proposition 3.2, we obtain an A^t-equivariant family of linear isomorphisms $L^s_{xy} : \mathcal{E}_x \to \mathcal{E}_y$ for $y \in W^s_{loc}(x)$ which we refer to as the *stable holonomies* of A^t. We also define *center holonomies* for A^t along the flow direction of f^t: if $x \in M$ and $y = f^s x \in W^c_{loc}(x)$ then we set $L^c_{xy} := A^s(x)$. When $x \in M$ and $y \in W^{cs}_{loc}(x)$ we define *center-stable holonomies* L^{cs}_{xy} of A^t by

$$L^{cs}_{xy} = L^c_{h^s_{xy}(x)y} \circ L^s_{xh^s_{xy}(x)} = L^s_{h^c_{xy}(x)y} \circ L^c_{xh^c_{xy}(x)}.$$

In the second expression we are considering stable holonomy $h^s : W^c_{loc}(x) \to W^c_{loc}(y)$ and in the third expression we are considering center holonomy $h^c : W^s_{loc}(x) \to W^s_{loc}(y)$. The maps

L^c and L^s commute by the A^t-equivariance properties of the stable holonomies. For $x \in M$ and $y \in W^{cu}_{loc}(x)$ we define the *center-unstable holonomies* L^{cu}_{xy} similarly, replacing L^s by L^u in the above. As a shorthand we will often refer to unstable holonomies as u-holonomies, center-stable holonomies as cs-holonomies, etc.

Lastly we state a proposition that we will use in Section 7 to upgrade measurable conjugacies between linear cocycles to Hölder continuous conjugacies. We let $f^t : M \to M$ be a *volume-preserving* C^2 transitive Anosov flow on a closed Riemannian manifold M, and let $\mathcal{E}$ be an α-Hölder vector bundle over M. We write m for the invariant volume for f^t. Given two α-Hölder linear cocycles A^t and B^t over f^t, we say that a fiber-preserving continuous map $\Phi : \mathcal{E} \to \mathcal{E}$ is an *α-Hölder conjugacy* if $\Phi : \mathcal{E}_x \to \mathcal{E}_x$ is a linear isomorphism for each $x \in M$, and $\Phi \circ A^t = B^t \circ \Phi$ for every $t \in \mathbb{R}$. We say that Φ is a *measurable conjugacy* between A^t and B^t if instead Φ is only defined m-a.e. , and the equation $\Phi \circ A^t = B^t \circ \Phi$ only holds m-a.e for each $t \in \mathbb{R}$. We assume that both A^t and B^t are fiber bunched; we write $L^{*,A}$ and $L^{*,B}$ for the holonomies of A^t and B^t respectively.

Proposition 3.4. *Let $A^t, B^t : \mathcal{E} \to \mathcal{E}$ be uniformly quasiconformal α-Hölder linear cocycles over f^t. Suppose that there is a measurable conjugacy Φ between A^t and B^t. Then Φ coincides m-a.e. with an α-Hölder conjugacy $\widetilde{\Phi}$ between A^t and B^t such that for $x \in M$, $y \in W^*_{loc}(x)$, $* \in \{u, c, s, cu, cs\}$,*

$$\widetilde{\Phi}_y \circ L^{*,A}_{xy} = L^{*,B}_{xy} \circ \widetilde{\Phi}_x.$$

Proof. First suppose that the foliations W^u and W^s for f^t are jointly integrable. Then f^t is a special flow constructed over an Anosov diffeomorphism [53]. This easily reduces to the case in which A and B are instead cocycles over an Anosov diffeomorphism, for which the above proposition was proved by Sadovskaya in [57].

More precisely, there is some $\tau > 0$ such that we have $M = N \times [0, \tau]/(x, \tau) \sim (F(x), 0)$ for a closed Riemannian manifold N, where $F : N \to N$ is a C^2 volume-preserving Anosov diffeomorphism. We consider A^τ and B^τ as uniformly quasiconformal linear cocycles over

F and apply [57, Theorem 2.7] to obtain that $\Phi|_{N\times\{0\}}$ coincides m-a.e. with an α-Hölder continuous (on N) function $\widetilde{\Phi}$, for which we have for $x \in N$, $y \in W^*_{loc}(x)$, $* \in \{u,s\}$, that $\widetilde{\Phi}_y \circ L^{*,A}_{xy} = L^{*,B}_{xy} \circ \widetilde{\Phi}_x$. We then extend $\widetilde{\Phi}$ to $M = N \times [0,\tau]/\sim$ by setting, for $x \in N$, $s \in \mathbb{R}$, $\widetilde{\Phi}_{(x,s)} = A^s_x \circ \Phi_{(x,0)} \circ (A^s_x)^{-1}$. It's then easy to check that $\widetilde{\Phi}$ is well-defined, α-Hölder continuous, and equivariant with respect to all holonomies, so we obtain the desired conclusion.

Now consider the case where the invariant foliations W^u and W^s for f^t are not jointly integrable. Then, for each $s \in \mathbb{R}\backslash\{0\}$, the map f^s is a volume-preserving C^2 center-bunched accessible partially hyperbolic diffeomorphism [12]. Consequently, by recent work of Kalinin and Sadovskaya [40], for each $s \neq 0$ the measurable conjugacy Φ coincides m-a.e. with a continuous conjugacy $\widetilde{\Phi}^s : \mathcal{E} \to \mathcal{E}$ such that for $x \in M$, $y \in W^*_{loc}(x)$, $* \in \{u,s\}$ we have $\widetilde{\Phi}^s_y \circ L^{*,A}_{xy} = L^{*,B}_{xy} \circ \widetilde{\Phi}^s_x$. Since the conjugacies $\widetilde{\Phi}^s$ all coincide m-a.e. with Φ, we conclude that $\widetilde{\Phi}^s = \widetilde{\Phi}^1 := \widetilde{\Phi}$ for all $s \neq 0$. Consequently $\widetilde{\Phi}$ is equivariant under c-holonomy as well as s- and u-holonomy since $\widetilde{\Phi} \circ A^t = B^t \circ \widetilde{\Phi}$ for all $t \in \mathbb{R}$. Since the holonomies $L^{*,A}$ and $L^{*,B}$ for A^t and B^t are all locally uniformly α-Hölder, we conclude from the equivariance of $\widetilde{\Phi}$ with respect to these holonomies that $\widetilde{\Phi}$ is an α-Hölder conjugacy from A^t to B^t. □

3.4 Anosov flows with dominated splittings

Let $f^t : M \to M$ be a C^r Anosov flow, $r \geq 2$. We say that f^t has a *u-splitting* of index k if there is a dominated splitting $E^u = H^u \oplus V^u$ for the linear cocycle $Df^t|E^u$, such that $\dim H^u = k$. Here H^u denotes the directions of weaker expansion for Df^t, and V^u denotes the directions of stronger expansion in the splitting. We will sometimes refer to H^u as the *horizontal bundle* and V^u as the *vertical bundle*. We do not preclude the possibility that the splitting is trivial; in this case our convention will always be that $H^u = E^u$ and $V^u = \{0\}$. In particular we will take $H^u = E^u$ in the proofs of Theorems 2.7 and 2.8. We define an *s-splitting* $E^s = H^s \oplus V^s$ of index k similarly, with V^s being the direction of stronger contraction under Df^t in the splitting.

Implicit in all of the statements of regularity results below is that the regularity is *uniform*. For example, when we say that the leaves of a foliation W^* are C^r, we mean that the leaves are r-times differentiable with uniformly continuous derivatives of order r.

We have the following proposition which guarantees the existence of a foliation tangent to V^u,

Proposition 3.5. *Suppose that* $\dim V^u \geq 1$. *Then the subbundle* V^u *is uniquely integrable. The foliation* W^{uu} *tangent to* V^u *is Hölder continuous with* C^r *leaves. The subbundle* V^u *is* C^{r-1} *when restricted to* W^u *leaves and thus the foliation* W^{uu} *gives a* C^r *subfoliation of each* W^u *leaf of* f.

Proof. We take $F := f^1$ to be the time one map of f^t, $E^{u,F} = V^u$, $E^{c,F} = H^u \oplus E^c$, and $E^{s,F} = E^s$. Then F is a C^r partially hyperbolic diffeomorphism of M with V^u as its unstable bundle. From standard results on partially hyperbolic diffeomorphisms, the bundle $E^{u,F}$ is uniquely integrable and tangent to a Hölder continuous foliation $W^{u,F}$ with C^r leaves. Furthermore, since there is a foliation $W^{cu,F} = W^{cu}$ with C^r leaves tangent to $E^{u,F} \oplus E^{c,F}$ given by the center-unstable foliation for f, we obtain that the unstable foliation of F is a C^r subfoliation of the center-unstable leaves $W^{cu,F}$ of F, again from standard results. We refer to [35] for the proofs of these claims.

Since the bundle V^u is f^t-invariant, the foliation $W^{uu} := W^{u,F}$ is f^t-invariant. The desired conclusions of the proposition then follow from the claims of the previous paragraph.

□

Since W^{uu} subfoliates $W^u(x)$ for each $x \in M$, we may define a quotient space $\mathcal{Q}^u(x)$ by the following equivalence relation: $y \sim z$ if and only if $z \in W^{uu}(y)$, for $y, z \in W^u(x)$. We establish below that this quotient space is a C^r manifold diffeomorphic to $\mathbb{R}^k$.

Proposition 3.6. *For each* $x \in M$, *there is a proper* C^r *embedding* $\iota_x : \mathbb{R}^k \to W^u(x)$ *with* $\iota_x(0) = p$ *such that* $\iota_x(\mathbb{R}^k)$ *meets each* W^{uu} *leaf inside of* $W^u(x)$ *in exactly one point.*

Proof. Let $F = f^1$ and consider this as a partially hyperbolic map as in Proposition 3.5 with $E^{u,F} = V^u$ and $E^{c,F} = H^u \oplus E^c$. The theory of partially hyperbolic diffeomorphisms then tells us that there is some $R > 0$ such that on any ball of radius R in M, the foliation tangent to $E^{u,F} = V^u$ is trivial [35]. Furthermore, since there is a foliation tangent to $E^{u,F} \oplus E^{c,F}$, and the unstable foliation $W^{u,F}$ tangent to $E^{u,F}$ always C^r subfoliates $W^{cu,F}$, we can choose this trivialization to be C^r along W^u leaves. Choose a sequence of times $t_n \to \infty$ such that $f^{-t_n}(x) \to x$ in M. For each $n \in \mathbb{N}$, let $D_{n,R}$ be the disk of radius R centered at $f^{-t_n}(x)$ in $W^u(f^{-t_n(x)})$.

By shrinking R if necessary, we can assume that f^{-t} is a contracting map on $D_{n,R}$ for each n in the induced Riemannian metric on W^u, which implies that $f^{t_n - t_s}(D_{n,R}) \subset D_{s,R}$ for $s > n$. For each n, choose a compact transversal submanifold $K_n \subset D_{n,R}$ to the W^{uu} foliation which contains $f^{-t_n}(p)$ and is tangent to $H^u_{f^{-t_n}(x)}$ at $f^{-t_n}(x)$. K_n meets each leaf of the induced foliation of $D_{n,R}$ by W^{uu} in exactly one point.

Consider the collection of k-dimensional submanifolds $f^{t_n}(K_n)$ of $W^u(x)$. We make three claims. First we claim that if a W^{uu} leaf intersects $f^{t_n}(K_n)$, then it intersects $f^{t_s}(K_s)$ for any $s > n$. Second, we claim that each W^{uu} leaf meets each submanifold $f^{t_n}(K_n)$ in at most one point. Lastly, we claim that for each W^{uu} leaf in $W^u(x)$, there is an $n \in \mathbb{N}$ such that $f^{t_n}(K_n)$ intersects this leaf.

For the first claim, if $s > n$, then $f^{-t_s}(f^{t_n}(K_n)) \subset D_{s,R}$ by construction. Since K_s is a full transversal inside of $D_{s,R}$, any W^{uu} leaf intersecting $f^{-t_s}(f^{t_n}(K_n))$ also intersects K_s. By f^t-invariance of the W^{uu} foliation, any W^{uu} leaf intersecting $f^{t_n}(K_n)$ thus also intersects $f^{t_s}(K_s)$.

For the second claim, suppose that $W^{uu}(y)$ intersects $f^{t_n}(K_n)$ in the points y and y', for $y \neq y'$. $W^u(x)$ is exponentially contracted under f^{-t}, so for large enough s, there will be a curve contained entirely in $f^{-t_s}(W^{uu}(y)) \cap D_{s,R}$ which joins $f^{-t_s}(y)$ to $f^{-t_s}(y')$. On the other hand, since the splitting $E^u = V^u \oplus H^u$ is dominated, as $s \to \infty$, the tangent spaces to $f^{t_n - t_s}(K_n)$ are uniformly asymptotic to the sequence of planes $H^u_{f^{t_s}(p)}$. Thus for large enough s, $f^{t_n-t_s}(K_n)$ will be a small disk that is almost parallel to $H^u_{f^{t_s}(p)}$; in particular it will meet each leaf of $W^{uu} \cap D_{s,R}$ in at most one point. But this contradicts the existence of the segment joining $f^{-t_s}(y)$ to $f^{-t_s}(y')$ inside of $f^{-t_s}(W^{uu}(y)) \cap D_{s,R}$.

For the last claim, recall that $W^u(x)$ is defined as the set of points in M asymptotic to the orbit of x under f^{-t}. Since $f^{-t_n}(x) \to x$, it follows that for any $y \in W^u(x)$, there is some $n > 0$ such that $f^{-t_n}(y) \in D_{n,R}$; the last claim follows.

Having proven those three claims, we now construct the desired embedding inductively. Set $U_1 := f^{t_1}(K_1)$. To construct U_n from U_{n-1}, take the submanifold $f^{t_n}(K_n)$ of $W^u(x)$ and use the smoothness of the W^{uu} foliation of W^u to map $f^{t_n}(K_n)$ smoothly onto a submanifold of $W^u(x)$ which contains $y \in U_{n-1}$ for each y such that $W^{uu}(y) \cap f^{t_n}(K_n)$ is nonempty. By the first claim $U_n \subset U_s$ for $s \geq n$. By the second and third claim, the submanifold $U := \bigcup_{n=1}^{\infty} U_n$ meets each W^{uu} leaf in exactly one point. Properness of the embedding follows from the fact that the W^{uu} foliation is locally trivial and that U meets each W^{uu} leaf in only one point. $\square$

Since the foliation W^{uu} is f^t-invariant, for each $t \in \mathbb{R}$ the map f^t descends to a C^r diffeomorphism $\bar{f}^t : \mathcal{Q}^u(x) \to \mathcal{Q}^u(f^t(x))$. When $V^u = \{0\}$, we set $\mathcal{Q}^u(x) := W^u(x)$ in our arguments. We let $\pi : W^u(x) \to \mathcal{Q}^u(x)$ denote the projection for each $x \in M$. Note that the projection map π is C^r-smooth, uniformly as we vary x.

We define $\mathcal{B}^u = E^u/V^u$ to be the quotient of E^u by the subbundle V^u. Note that $\mathcal{B}^u$ is a Hölder continuous vector bundle over M which is C^{r-1} when restricted to any W^u leaf. The derivative of the projection map gives a linear isomorphism $D\pi_x : \mathcal{B}^u_x \to T_x\mathcal{Q}^u(x)$ for each $x \in M$.

In order to proceed further, we need to impose additional hypotheses on the Anosov flow f^t as well as the linear cocycle $Df^t|H^u$. For $\beta > 0$ we say that an Anosov flow f^t is *β-u-bunched* if there is a constant $C \geq 1$ such that for all $x \in M$ and $t \geq 0$,

$$\frac{\sigma_l(Df^t|E^u_x)}{\sigma_1(Df^t|E^u_x)} < C \min\{\sigma_l(Df^t_x|E^s), \sigma_1(Df^t_x|E^u)^{-1}\}^{-\beta}.$$

and we say that f^t is β-*s-bunched* if f^{-t} is β-u-bunched. We say that f^t is β-*bunched* if it is both β-u-bunched and β-s-bunched. Note that the β-u-bunching condition is equivalent to requiring that the linear cocycle $Df^t|E^u$ over f^t be fiber bunched.

When f^t is β-u-bunched, the center-unstable foliation W^{cu} is β-Hölder continuous and likewise, when f^t is β-s-bunched, the center-stable foliation W^{cs} is β-Hölder continuous. When $\beta \geq 1$ the bunching conditions imply that the corresponding foliations are $C^{1+\delta}$ foliations for $\delta = \beta - 1$ [31]. Furthermore in the case $\beta = 1$ one actually obtains that the foliations are $C^{1+\varepsilon}$ for some small $\varepsilon > 0$.

In what follows the β-bunching conditions will not be used explicitly. We will only use the regularity of the foliations W^{cu} and W^{cs}, which one obtains as a consequence of the bunching inequalities. The reason we state the bunching inequalities here is that these inequalities are essentially the only way one can prove regularity of the foliations W^{cu} and W^{cs} in general. In fact, in many settings these inequalities predict the optimal regularity of these bundles [32].

For the rest of this section, given a u-splitting $E^u = H^u \oplus V^u$ for f^t we will always assume that $Df^t|H^u$ is fiber bunched as a linear cocycle. We then have holonomy maps L^* for the linear cocycle $Df^t|H^u$. The key proposition regarding these holonomy maps, the u-splitting of f^t, and the fiber bunching conditions is that if $Df^t|H^u$ is fiber bunched with a small enough exponent then we can still have differentiability of the center-stable holonomy maps when restricted to the horizontal bundle H^u. This is despite the fact that the holonomies of the foliation W^{cs} may not necessarily be C^1 under our hypotheses.

Proposition 3.7. *Let $E^u = H^u \oplus V^u$ be a u-splitting of f^t. Suppose that E^u is α-Hölder continuous and that $Df^t|H^u$ is fiber bunched with exponent α. Then for each $x \in M$, $y \in W^{cs}_{loc}(x)$ the local cs-holonomy map $h^{cs}_{xy} : W^u_{loc}(x) \to W^u_{loc}(y)$ is differentiable along H^u. For $z \in W^u_{loc}(x)$ and $w = h^{cs}_{xy}(z)$ the derivative is given by*

$$D_z h^{cs}_{xy}|H^u = L^{cs}_{zw} : H^u_z \to H^u_w.$$

In particular, h^{cs}_{xy} *maps* C^1 *curves tangent to* H^u *to* C^1 *curves tangent to* H^u.

Proof. Set $d = \dim M - 1$. Let x, y be two points in M such that $x \in W^{cs}_{loc}(y)$. Set $x_n = f^n x$ and $y_n = f^n y$. For each $n \geq 0$, choose a hypersurface S_n of uniform size and biLipschitz to an open subset of $\mathbb{R}^d$ with Lipschitz constants independent of n that is transverse to the direction of the flow E^c, and contains $W^u_{loc}(x_n)$ and $W^u_{loc}(y_n)$. Let $g_n : S_{n-1} \to S_n$ be the smooth map defined by $g_n(z) = f^{t(z)}(z)$ for $z \in S_{n-1}$, where $t(z)$ is the unique time, smoothly depending on z, with $t(x_{n-1}) = 1$ and such that $f^{t(z)}(z) \in S_n$. g_n is defined on a neighborhood of x_{n-1} of uniform size, independent of n. Further, g_n is uniformly hyperbolic on the interior of this neighborhood with the same contraction and expansion estimates (up to multiplicative constants) as f^1 on the stable and unstable bundles E^u and E^s. Set $F^n = g_n \circ g_{n-1} \circ \cdots \circ g_1$. Note that F^n is defined on increasingly small neighborhoods of x as $n \to \infty$; the only points for which F^n is defined for all $n \geq 1$ are the points on the intersection of $W^{cs}_r(x)$ with $S := S_0$.

Let β be the minimum of the Hölder exponents of H^u and E^u viewed as subbundles of TM. As remarked in earlier in this section, there is a β-Hölder system of linear identifications $I_{pq} : E^u_p \to E^u_q$ defined for p near q with I_{pp} being the identity on E^u_p. We can choose these identifications so that $I_{pq}(H^u_p) = H^u_q$. For each n, let $A_n : W^u_r(x_n) \to W^u_r(y_n)$ be a diffeomorphism with $A_n(x_n) = y_n$. Since the unstable foliation is Hölder continuous in the C^1 topology with Hölder exponent β, we can choose A_n such that

$$\|I_{qp} \circ DA_n - Id\| \leq Cd(p, q)^\beta$$

$$\|DA_n \circ I_{pq} - Id\| \leq Cd(p, q)^\beta$$

for some constant $C > 0$ and $p \in W^u_r(x_n)$, $q \in W^u_r(y_n)$. For $z \in S_n$, let $\widehat{W}^s(z)$ denote the smooth projection of $W^s_r(z)$ onto S_n along the orbit foliation E^c, given by using g^t to flow these leaves onto S_n. Let $\widehat{H}^u$, $\widehat{E}^u$, and $\widehat{E}^s$ denote the projection of these subbundles onto TS_n by flowing along the orbit foliation.

Let φ be the holonomy map between $W^u_r(x)$ and $W^u_r(y)$ induced by the projected stable foliation $\widehat{W}^s$. Let $\varphi_n = F^{-n} \circ A_n \circ F^n$, which is defined on a neighborhood of x (dependent on n) inside of $W^u_r(x)$. Let $\gamma : [-1, 1]$ be a C^1 curve tangent to H^u inside of $W^u_r(x)$ with $\gamma(0) = x$. Our first goal is to prove that $\varphi \circ \gamma$ is differentiable at 0, i.e., that the image of the curve γ under $\widehat{W}^s$-holonomy along the transversal S is differentiable at p.

We first claim that the sequence of linear maps

$$\left\{(DF_y^{-n} \circ DA_n \circ DF_x^n)|H_x^u : n \in \mathbb{N}\right\}$$

is Cauchy (note that we have restricted the domain of these maps to H^u_x). We closely follow the proof of Proposition 3.2 given in [39]. We begin with the formula

$$(DF_y^n)^{-1} \circ DA_n \circ DF_x^n = DA_0 + \sum_{i=0}^{n-1} (DF_y^i)^{-1} \circ R_i \circ DF_x^i$$

where $R_i = (D_{y_i} g_{i+1})^{-1} \circ DA_{i+1} \circ D_{x_i} g_{i+1} - DA_i$. For the rest of the proof we will consider all linear maps as restricted to $\widehat{E}^u$ for the purpose of calculating norms. We want to estimate the product

$$\begin{aligned}
\|(DF_y^n)^{-1}\| \cdot \|DF_x^n|\widehat{H}^u\| &\leq \prod_{i=0}^{n-1} \|(D_{y_i} g_i)^{-1}\| \cdot \|D_{x_i} g_i|\widehat{H}^u\| \\
&= \left(\prod_{i=0}^{n-1} \|(D_{y_i} g_i)^{-1}\| \cdot \|(D_{x_i} g_i)^{-1}|\widehat{H}^u\|^{-1}\right) \\
&\cdot \left(\prod_{i=0}^{n-1} \|(D_{x_i} g_i)^{-1}|\widehat{H}^u\| \cdot \|D_{x_i} g_i|\widehat{H}^u\|\right)
\end{aligned}$$

To bound the first factor, we observe that $\|(D_{x_i} g_i)^{-1}|\widehat{H}^u\| = \|(D_{x_i} g_i)^{-1}\|$ since $\widehat{H}^u$ is the less expanded term of the dominated splitting $\widehat{E}^u = \widehat{V}^u \oplus \widehat{H}^u$. We then use the estimate

$$\begin{aligned}\frac{\|(D_{y_i}g_i)^{-1}\|}{\|(D_{x_i}g_i)^{-1}\|} &\leq \frac{\|(D_{y_i}g_i)^{-1} - I_{x_iy_i} \circ (D_{x_i}g_i)^{-1} \circ I^{-1}_{x_{i+1}y_{i+1}}\|}{\|(D_{x_i}g_i)^{-1}\|} \\ &+ \frac{\|I_{x_iy_i} \circ (D_{x_i}g_i)^{-1} \circ I^{-1}_{x_{i+1}y_{i+1}}\|}{\|(D_{x_i}g_i)^{-1}\|} \\ &\leq C'd(x_i, y_i)^\beta + 1\end{aligned}$$

for some constant C'. Here we use the fact that $\|I_{pq}\|$ is uniformly bounded when p and q are close (say $d(p, q) \leq r$), and that the derivative D_pf_i is smooth as a function of p, hence when we use the identifications I_{pq}, it becomes Hölder with Hölder exponent β.

To bound the second factor, we note that $Dg_i|\widehat{H}^u$ is fiber bunched since the cocycle $Df^t|H^u$ we derived it from was fiber bunched. Hence there is a constant $\delta < 1$ such that

$$\|(D_pg_i)^{-1}|\widehat{H}^u\| \cdot \|D_pg_i|\widehat{H}^u\| \leq \|D_pg_i|\widehat{E}^s\|^{-\beta}\delta$$

for all $p \in S_i$, where δ is independent of i.

Putting these two bounds together, we obtain

$$\|(DF^n_y)^{-1}\| \cdot \|DF^n_x|\widehat{H}^u\| \leq \prod_{i=0}^{n-1}(C'd(x_i, y_i)^\beta + 1)\prod_{i=0}^{n-1}\delta\|D_{x_i}f_i|\widehat{E}^s\|^{-\beta}$$

The first product is uniformly bounded since $d(x_i, y_i) \to 0$ exponentially in i, so we get a constant C'' such that

$$\|(DF^n_y)^{-1}\| \cdot \|DF^n_x|\widehat{H}^u\| \leq C''\delta^n \prod_{i=0}^{n-1}\|D_{x_i}g_i|\widehat{E}^s\|^{-\beta}$$

Now we can also estimate

$$\begin{aligned}\|R_i\| &\le \|(D_{y_i}g_{i+1})^{-1}\circ DA_{i+1}\|\cdot\|D_{x_i}g_{i+1} - DA_{i+1}^{-1}\circ D_{y_i}g_{i+1}\circ DA_i\| \\ &\le Cd(x_i,y_i)^\beta \\ &\le Cd(x,y)^\beta \prod_{i=0}^{n-1}\|D_{x_i}g_i|\widehat{E}^s\|^\beta\end{aligned}$$

for some constant C. In the first inequality we used the Hölder closeness of DA_i to the identity, together with uniform bounds on the norms of all of the linear maps involved. In the second inequality we use the fact that x and y lie on the same stable manifold in S. We have the basic bound

$$\begin{aligned}\|(DF_y^n)^{-1}\circ DA_n\circ DF_x^n - DA_0|\widehat{H}^u\| &\le \sum_{i=0}^{n-1}\|(DF_y^i)^{-1}\circ R_i\circ DF_x^i|\widehat{H}^u\| \\ &\le \sum_{i=0}^{n-1}\|(DF_y^i)^{-1}\|\cdot\|DF_x^i|\widehat{H}^u\|\cdot\|R_i\|\end{aligned}$$

We replace the right side with the previously obtained bounds on the factors $\|(DF_y^i)^{-1}\|\cdot\|DF_x^i|\widehat{H}^u\|$ and $\|R_i\|$. This gives an upper bound of

$$\sum_{i=0}^{n-1}\left(C''\delta^i\prod_{j=0}^{i-1}\|D_{x_i}g_i|\widehat{E}^s\|^{-\beta}\cdot Cd(x,y)^\beta\prod_{j=0}^{i-1}\|D_{x_i}g_i|\widehat{E}^s\|^\beta\right)\le C^*d(x,y)^\beta$$

for some constant C^*. Also note that

$$\begin{aligned}\|(DF_y^{n+1})^{-1}\circ DA_{n+1}\circ DF_x^{n+1} &- (DF_y^n)^{-1}\circ DA_n\circ DF_x^n|\widehat{H}^u\| \\ &= \|(DF_y^n)^{-1}\circ R_n\circ DF_x^n|\widehat{H}^u\| \\ &\le C^*\delta^n d(x,y)^\beta\end{aligned}$$

This second inequality immediately implies that the sequence of linear maps

$$\left\{(DF_y^n)^{-1} \circ DA_n \circ DF_x^n | H_x^u : n \in \mathbb{N}\right\}$$

is Cauchy. Hence this sequence converges to a linear map $T_{xy} : \widehat{H}_x^u \to \widehat{E}_y^u$. However, for any given vector $v \in \widehat{H}_x^u$, $DA_n \circ DF_x^n(v)$ is a vector which makes an angle θ_n with $\widehat{H}_y^u$, where θ_n is uniformly bounded away from $\pi/2$, independent of n. Applying DF_y^{-n} exponentially contracts this angle since the splitting $\widehat{E}^u = \widehat{V}^u \oplus \widehat{H}^u$ is dominated, so letting $n \to \infty$, we conclude that T_{xy} must have image in $\widehat{H}_y^u$.

For each $j \geq 0$, we can also consider the sequence of linear maps

$$\left\{(DF_y^{n+j} \circ (DF_y^j)^{-1})^{-1} \circ DA_{n+j} \circ DF_x^{n+j} \circ (DF_x^j)^{-1} | H_x^u : n \in \mathbb{N}\right\}$$

For the same reasons as for the original sequence, this sequence is Cauchy and converges to a limit that we denote $T_{x_j y_j}$ which is a linear map from $\widehat{H}_{x_j}^u$ to $\widehat{H}_{y_j}^u$. It is straightforward to check that for each n we have $(DF_y^n)^{-1} \circ T_{x_n y_n} \circ DF_x^n = T_{xy}$ by writing out the limiting expression for $T_{x_n y_n}$. Since we chose the transversal S to contain $W_r^u(x)$ and $W_r^u(y)$, we have $\widehat{H}_x^u = H_x^u$ and the same for y. We now consider the center stable holonomy map $L_{xy}^{cs} : H_x^u \to H_y^u$. This is equivariant with respect to DF^n as well and also depends in a β-Hölder manner on the points x and y. Then

$$\begin{aligned}
\|L_{xy}^{cs} - P_{xy}\| &= \|(DF_y^n)^{-1} \circ (L_{x_n y_n}^{cs} - T_{x_n y_n}) \circ DF_x^n | H_x^u\| \\
&\leq \|(DF_y^n)^{-1}\| \cdot \|DF_x^n | H_x^u\| \cdot \|L_{x_n y_n}^{cs} - T_{x_n y_n}\| \\
&\leq C\delta^n \prod_{i=0}^{n-1} \|D_{x_i} g_i | \widehat{E}^s\|^{-\beta} d(x_n, y_n)^\beta \\
&\leq C^* \delta^n
\end{aligned}$$

for some constant C^*. As $n \to \infty$, $\delta^n \to 0$, so $L_{xy}^{cs} = T_{xy}$.

To prove differentiability of $\varphi \circ \gamma$, take a coordinate chart on S (as well as each of the transversals S_n) so that we can work with the linear structure on $\mathbb{R}^d$. Let y correspond to the origin. We will not change the notation for the maps, so they should be understood in this chart. Let $v = \gamma'(0)$. We need to show that $\varphi(\gamma(s))$ agrees with its claimed linearization $s \cdot L^{cs}_{xy}(v)$ to first order at the origin. First observe that the calculations above are valid if we replace x and y by any two points x', y' in S such that $y' \in \widehat{W}^s_{loc}(x')$, whenever n is small enough (relative to x' and y') that the iterates $F, F^2, \ldots, F^n$ are all defined on a neighborhood of x' and y'. This implies that

$$\|(DF^n)^{-1} \circ DA_n \circ DF^n_{\gamma(s)}(\gamma'(s)) - DA_0(\gamma'(s))\| \leq C|s|^\beta$$

whenever s is small enough that F^n is defined on a neighborhood of $\gamma(s)$ and $A_n(F^n(\gamma(s)))$ lies in the image of F^n. The constant C is independent of n, so $(DF^n)^{-1} \circ DA_n \circ DF^n_{\gamma(s)}(\gamma'(s))$ is a Hölder continuous function of s with Hölder exponent and constant independent of n for $|s|$ small. Note that $A_n(F^n(\gamma(s)))$ will not necessarily lie on $\widehat{W}^s(F^n(\gamma(s)))$, but it will be β-Hölder close to the intersection of $\widehat{W}^s(F^n(\gamma(s)))$ with $W^u_r(y_n)$, so our estimates remain valid. By the mean value inequality, we thus obtain

$$\|\varphi_n(\gamma(s)) - s \cdot D\varphi_n(\gamma'(0))\| \leq C|s|^{1+\beta}$$

for a constant C.

We next estimate the difference between φ and φ_n near $\gamma(0)$. Observe that $\varphi = (F^n)^{-1} \circ \psi_n \circ F^n$, where ψ_n is the $\widehat{W}^s$-holonomy map from $\widehat{W}^u_r(x_n)$ to $W^u_r(y_n)$. Hence for s small enough that $\gamma(s)$ is in the domain of definition of the expressions below,

$$\begin{aligned}\|\varphi_n(\gamma(s)) - \varphi(\gamma(s))\| &= \|((F^n)^{-1} \circ A_n \circ F^n - (F^n)^{-1} \circ \psi_n \circ F^n)(\gamma(s))\| \\ &\leq C\|(DF^n)^{-1}|E^u\| \cdot \|A_n \circ F^n(\gamma(s)) - \psi_n \circ F^n(\gamma(s))\|\end{aligned}$$

since F^{-n} exponentially contracts distances on unstable leaves. Next we note that ψ_n and A_n are β-Hölder close in the C^0 topology. As a consequence, since they both map x to y,

$$\begin{aligned}
C\|(DF^n)^{-1}|E^u\| \cdot \|A_n \circ F^n(\gamma(s)) - \psi_n \circ F^n(\gamma(s))\| \\
&\leq C\|(DF^n)^{-1}|E^u\| d(F^n(x), F^n(y))^\beta \|F^n(\gamma(s))\| \\
&\leq C\|(DF^n)^{-1}|E^u\| \cdot \|DF^n|E^s\|^\beta \cdot \|F^n(\gamma(s))\| \\
&\leq C\|(DF^n)^{-1}|E^u\| \cdot \|DF^n|E^s\|^\beta \cdot \|DF^n|\widehat{H}^u\| \cdot |s| \\
&\leq C\delta^n |s|
\end{aligned}$$

where we have not paid much attention to the constant C in front (which will change from line to line). In the third line we use the exponential contraction of stable leaves by F^n, and in the fourth line we use the fiber bunching property on H^u transferred to the induced bundle $\widehat{H}^u$, noting that $\|(DF^n)^{-1}|E^u\| = \|(DF^n)^{-1}|\widehat{H}^u\|$.

We now compare $\varphi \circ \gamma$ to the linearization $L_{xy}^{cs}(v) \cdot s$ at 0. Fix $n \in \mathbb{N}$. For $|s|$ small enough that all of the expressions above are defined for this n, we obtain

$$\begin{aligned}
\|\varphi(\gamma(s)) - L_{xy}^{cs}(v) \cdot s\| &\leq \|\varphi(\gamma(s)) - \varphi_n(\gamma(s))\| + \|\varphi_n(\gamma(s)) - s \cdot D\varphi_n(v)\| \\
&\quad + |s| \cdot \|D\varphi_n(v) - L_{xy}^{cs}(v)\| \\
&\leq C(\delta^n |s| + |s|^{1+\beta} + |s| \cdot \|D\varphi_n(\gamma'(0)) - L_{xy}^{cs}(v)\|)
\end{aligned}$$

Dividing through by $|s|$, we obtain

$$\frac{\|\varphi(\gamma(s)) - L_{xy}^{cs}(v) \cdot s\|}{|s|} \leq C(\delta^n + |s|^\beta + \|D\varphi_n(v) - L_{xy}^{cs}(v)\|)$$

We can consider $n := n(s)$ as an integer function of s such that $n(s) \to \infty$ as $s \to 0$. Then as $s \to 0$, the right side converges to 0. We thus obtain that $\varphi \circ \gamma$ agrees to first order with its linearization at 0, i.e., $\varphi \circ \gamma$ is differentiable at 0, and furthermore, $(\varphi \circ \gamma)'(0) =$

$L^{cs}_{\gamma(0)\varphi(\gamma(0))}(\gamma'(0))$.

Now observe that holonomy from $W^u_r(x)$ to $W^u_r(y)$ along the projected stable foliation $\widehat{W}^s$ corresponds precisely to W^{cs}-holonomy in M. Hence the curve $\varphi \circ \gamma$ is also the image of γ under the W^{cs}-holonomy h^{cs}_{xy}. We can apply our calculations to the other points of γ by recentering at each pair of points x', y' lying on γ and $\varphi \circ \gamma$ respectively with $y' \in W^{cs}_r(x')$. This proves that $\varphi \circ \gamma$ is differentiable for every $t \in [-1, 1]$, and furthermore we have the derivative formula

$$(\varphi \circ \gamma)'(t) = L^{cs}_{\gamma(t)\varphi(\gamma(t))}(\gamma'(t))$$

which completes the proof. □

The holonomy maps L^* for $Df^t|H^u$ also serve as holonomy maps for the quotient cocycle $Df^t|\mathcal{B}^u$, by the canonical Hölder identification $H^u \to \mathcal{B}^u$, which we will still denote by L^*. It will always be clear from context which bundle we are considering. We extend the definition of L^u to all of a given unstable leaf by setting, for $x \in M$ and $y \in W^u(x)$,

$$L^u_{xy} = Df^t_{f^{-t}y} \circ L^u_{f^{-t}xf^{-t}y} \circ Df^{-t}_x,$$

where we choose t such that $f^{-t}y \in W^u_{loc}(x)$.

Lemma 3.8. *Let $x \in M$ and let $S_1, S_2 \subseteq W^u(x)$ be any two local transversals to the W^{uu} foliation such that $\pi_x(S_1) = \pi_x(S_2) \subseteq \mathcal{Q}^u(x)$. Identify the tangent bundles TS_1 and TS_2 with the restrictions of $\mathcal{B}^u$ to S_1 and S_2 respectively. Then the derivative of the W^{uu}-holonomy map h^{uu} from S_1 to S_2 is given by*

$$Dh^{uu} = L^u : \mathcal{B}^u|_{S_1} \to \mathcal{B}^u|_{S_2}.$$

Proof. Fix a Hölder continuous family of identifications $I_{xy} : \mathcal{B}^u_x \to \mathcal{B}^u_y$, $y \in W^u_{loc}(x)$, that is uniformly C^1 on W^u leaves (this can be done since $\mathcal{B}^u$ is C^1 on W^u leaves). Let S_1, S_2 be two given local transversals to the W^{uu} foliation inside of $W^u(x)$ with $\pi_x(S_1) = \pi_x(S_2)$.

Let $h^{uu} : S_1 \to S_2$ denote the W^{uu}-holonomy map and for each $x \in M$ let $D_x h^{uu}$ denote the derivative considered as a map $\mathcal{B}^u_x \to \mathcal{B}^u_{h^{uu}x}$. Our goal is to show that $D_x h^{uu} = L^u_{xh^{uu}x}$.

For each $t \geq 0$ we consider the holonomy map $h^{uu,t} : f^{-t}(S_1) \to f^{-t}(S_2)$ that satisfies $h^{uu} = f^t \circ h^{uu,t} \circ f^{-t}$. Since $E^u = H^u \oplus V^u$ is a dominated splitting, it is straightforward to show that, for $i = 1, 2$, all tangent spaces of the transversal $f^{-t}(S_i)$ make a uniformly small angle with H^u. Since W^{uu} is a uniformly C^1 subfoliation of W^u, this implies that the holonomy maps $h^{uu,t}$ are uniformly C^1 in t. In particular, there is a constant $C \geq 1$ such that, for each $y \in S_1$,

$$\left\| D_{f_y^{-t}} h^{uu,t} - I_{f_y^{-t} h^{uu,t}(f^{-t}y)} \right\| \leq C d(f^{-t}y, h^{uu,t}(f^{-t}y)).$$

Then, by Proposition 3.3, for each $x \in S_1$ we have

$$D_x h^{uu} = Df^t_{h^{uu,t}(f^{-t}x)} \circ D_{f^{-t}x} h^{uu,t} \circ Df^{-t}_x \to L^u_{xh^{uu}x},$$

as $t \to \infty$, which completes the proof. □

Let $\bar{f}^t$ be the induced action of f^t on the quotient spaces $\mathcal{Q}^u$. We define u-holonomies $\bar{L}^u$ for the derivative action $D\bar{f}^t$ on $T\mathcal{Q}^u(x)$ by, for each y, $z \in \mathcal{Q}^u(x)$,

$$\bar{L}^u_{\pi(y)\pi(z)} = D\pi_z \circ L^u_{yz} \circ (D\pi_y)^{-1}.$$

Using the relation $L^u_{xz} = L^u_{xy} \circ L^u_{yz}$, for x, y, z in the same unstable leaf, together with the fact that L^u gives the derivative of W^{uu}-holonomy maps between transversals by Proposition 3.8, it is straightforward to show that $\bar{L}^u$ does not depend on the choice of preimage of $\pi(y)$ or $\pi(z)$ in $W^u(x)$ under π. For any $x \in M$, the u-holonomies $\bar{L}^u_{yz}$ on $T\mathcal{Q}^u(x)$ are jointly uniformly locally Hölder continuous in y, z, and x.

We use the holonomies $\bar{L}^u$ to define a connection on $T\mathcal{Q}^u(x)$ whose parallel transport is given by $\bar{L}^u$.

Proposition 3.9. *For each $x \in M$ there is a complete, flat, torsion-free $\bar{f}^t$-invariant C^1 connection ∇ on $\mathcal{Q}^u(x)$ such that the parallel transport of ∇ on $T\mathcal{Q}^u(x)$ is given by the u-holonomies $\bar{L}^u$.*

Proof. Since V^u is a smooth subbundle of E^u when restricted to W^u, the quotient bundle $\mathcal{B}^u = E^u/V^u$ over W^u is smooth. The projection $E^u \to \mathcal{B}^u$ induces a bundle isomorphism $H^u \to \mathcal{B}^u$ which is equivariant with respect to the action of Df^t on H^u and the induced action of Df^t on $\mathcal{B}^u$. We push forward the Riemannian metric on H^u to a Riemannian metric on $\mathcal{B}^u$, with respect to which the induced action of Df^t is fiber bunched. The isomorphism $H^u \to \mathcal{B}^u$ also induces an unstable holonomy $\bar{L}^u$ for the action of Df^t on $\mathcal{B}^u$. Since $\mathcal{B}^u$ has a smooth structure along W^u leaves with respect to which Df^t is smooth and the action of Df^t on $\mathcal{B}^u$ is fiber bunched, the unstable holonomy $\bar{L}^u$ is C^1 along W^u leaves.

By Lemma 3.8 above, we have the following alternative construction of $\bar{L}^u$. Take two compact transversals K_1 and K_2 to the W^{uu} foliation which meet the same collection of W^{uu} leaves (or equivalently, they have the same projection to $Q^u(p)$). The projection $E^u \to \mathcal{B}^u$ induces natural bundle isomorphisms $TK_i \to \mathcal{B}^u$ over each of these transversals. Then the derivative of the chart transition map $(\Pi|_{K_2})^{-1} \circ \Pi|_{K_1}$ is the unstable holonomy $\bar{L}^u$ when we make the identifications $TK_i \cong \mathcal{B}^u$.

The projection $\Pi : W^u(p) \to Q^u(x)$ is smooth and hence induces a derivative map $D\Pi : E^u \to TQ^u$ with $V^u = \ker D\Pi$. Hence for each $x \in W^u(p)$ the induced map $\overline{D\Pi} : \mathcal{B}^u_x \to TQ^u_{\Pi(x)}$ is an isomorphism. For w, $z \in Q^u(p)$ which are the image of x and $y \in W^u(p)$ respectively, we define $P_{wz} : TQ^u_w \to TQ^u_z$ by $P_{wz} = \overline{D\Pi}_y \circ \bar{h}_{xy} \circ \overline{D\Pi}_x^{-1}$. We claim that P_{wz} does not depend on the preimages x and y of w and z which were chosen. Suppose that x' and y' are two other points projecting to w and z respectively. Then

$$\begin{aligned} \overline{D\Pi}_{y'} \circ \bar{L}_{x'y'} \circ \overline{D\Pi}_{x'}^{-1} &= \overline{D\Pi}_y \circ \bar{L}_{y'y} \circ \bar{L}_{x'y'} \circ \bar{L}_{xx'} \circ \overline{D\Pi}_x^{-1} \\ &= \overline{D\Pi}_y \circ \bar{L}_{xy} \circ \overline{D\Pi}_x^{-1} \end{aligned}$$

where we have used the observation that the derivatives of the transition maps for Π are given by the unstable holonomy $\bar{L}^u$, and also the properties of the unstable holonomy $\bar{L}^u$ itself.

It's straightforward to check that P_{wz} is equivariant with respect to the induced derivative action $\overline{Df^t} : TQ^u(p) \to TQ^u(f^t(p))$, using the equivariance property of $\bar{L}^u$. P_{wz} is also C^1 in the variables w and z and has the property that for $x, y, z \in Q^u(p)$, $P_{yz} \circ P_{xy} = P_{xz}$. This implies that for each $X \in TQ^u(p)$,

$$\mathcal{P}(X) = \{Y \in TQ^u : \ P_{xy}(X) = Y \text{ for some } x, y \in Q^u(p)\}$$

is a C^1 submanifold of TQ^u which is transverse to the tangent spaces TQ^u_x. The tangent spaces to the foliation of TQ^u by these subfoliations define an Ehresmann connection on $Q^u(p)$ which we can then use to define a connection ∇ on $Q^u(p)$. The parallel transport of a vector by ∇ is given by the linear maps P_{wz}. Thus ∇ is a C^1 flat affine connection on $Q^u(p)$. Since the maps P_{wz} are equivariant with respect to $\overline{Df^t}$, ∇ is also f^t invariant.

We next show that ∇ is torsion-free. Let T be the torsion tensor of ∇. T is a mixed tensor of type $(2,1)$ on TQ^u which is invariant under f^t. But the fact that $Df^t|H^u$ is fiber bunched implies that Df^t acts by exponential contraction on tensors of type $(2,1)$ on TQ^u. This forces $T \equiv 0$ so that ∇ is torsion-free.

Completeness of ∇ follows from the fact that ∇ is locally complete – due to the uniform continuity properties of $\bar{L}^u$ – and the fact that ∇ is $\bar{f}^{-t}$-invariant, and $\bar{f}^{-t}$ contracts any given bounded open set to lie inside a small open ball for t large enough. □

3.5 The neighborhoods in the theorems

In this section we describe the neighborhoods $\mathcal{U}_X$ and $\mathcal{V}_X$ in our main theorems. For X a complex hyperbolic manifold, we define $\mathcal{V}_X$ to be the C^1 open neighborhood of g^t_X in the space of C^3 Anosov flows on T^1X such that all of the above discussion applies to any $f^t \in \mathcal{V}_X$, i.e., there is a u-splitting $E^{u,f} = H^{u,f} \oplus V^{u,f}$ and an s-splitting $E^{s,f} = H^{s,f} \oplus V^{s,f}$ for f^t such that $Df^t|H^{u,f}$ and $Df^t|H^{s,f}$ are fiber bunched, and the hypotheses of Proposition 3.7 hold.

When X is quaternionic or Cayley hyperbolic, the C^1 open neighborhood described above is sufficient for most of the propositions in this paper. The only exception to this is Proposition 5.5. Hence for these spaces we take $\mathcal{V}_X$ to be the C^2 open neighborhood of g^t_X on which all of the assertions of the previous paragraph hold, and in addition the hypotheses of Proposition 5.5 are satisfied.

We define the neighborhood $\mathcal{U}_X$ of X in the space of Riemannian manifolds in terms of the neighborhood $\mathcal{V}_X$. For Y a Riemannian manifold C^2 close to X, we let $\Psi : T^1X \to T^1Y$ be the projection diffeomorphism as defined at the end of Section 2.1. We then define $\mathcal{U}_X$ by saying that $Y \in \mathcal{U}_X$ if and only if $\Psi^{-1} \circ g^t_Y \circ \Psi \in \mathcal{V}_X$. Since $\Psi^{-1} \circ g^t_Y \circ \Psi$ is C^r close to g^t_X if Y is C^{r+1} close to X, we see that we may take $\mathcal{U}_X$ to be C^{r+1} open if $\mathcal{V}_X$ is C^r open.

4 Continuous Amenable Reduction

We now adapt the main results of [39] to our setting. Let $\mathcal{E}$ be a d-dimensional Hölder continuous vector bundle over a Riemannian manifold X with an Anosov flow g^t. We let μ be a fully supported ergodic g^t-invariant measure with local product structure. For the results in this section we will assume that the stable and unstable distributions E^s and E^u for g^t are not jointly integrable; this is true for the geodesic flow because the geodesic flow is a contact Anosov flow.

Two Riemannian metrics τ and σ on $\mathcal{E}$ are *conformally equivalent* if there is a function $a : X \to \mathbb{R}$ such that $\tau_p = a(p)\sigma_p$. A *conformal structure* on $\mathcal{E}$ is a conformal equivalence class of Riemannian metrics on $\mathcal{E}$. A^t transforms a conformal structure by pulling back the associated Riemannian metric. A conformal structure represented by a Riemannian metric

τ is *invariant* under $\mathcal{A}$ if for each $t \in \mathbb{R}$ there is a map $\psi^t : X \to \mathbb{R}$ satisfying

$$(A^t)^* \tau = \psi^t \tau$$

In this case we say that ψ^t is the multiplicative cocycle associated to the invariant conformal structure τ. ψ^t satisfies the cocycle property

$$\psi^{t+s}(p) = \psi^t(p)\psi^s(g^t(p))$$

for any $t, s \in \mathbb{R}$.

Two multiplicative cocycles ψ^t and φ^t are *cohomologous* if there is a map $\zeta : X \to \mathbb{R}$ such that

$$\frac{\psi^t}{\varphi^t} = \frac{\zeta \circ g^t}{\zeta}$$

for every $t \in \mathbb{R}$.

If a cocycle $\mathcal{A}$ over X admits stable and unstable holonomies, we say that a subbundle $\mathcal{V} \subset \mathcal{E}$ is *holonomy invariant* if for $y \in W^*(x)$ we have $h^*_{xy}(\mathcal{V}_x) = \mathcal{V}_y$ for $* = u$ or s. Similarly we say that a conformal structure is holonomy invariant if it is invariant under pulling back by stable and unstable holonomies.

Lemma 4.1. *Let $\mathcal{A}$ be a fiber bunched cocycle over an Anosov flow g^t for which E^u and E^s are not jointly integrable. Suppose that*

$$\lambda_+(\mathcal{A}, \mu) = \lambda_-(\mathcal{A}, \mu)$$

Then any measurable $\mathcal{A}$-invariant subbundle $V \subseteq \mathcal{E}$ coincides μ-a.e. with a $\mathcal{A}$-invariant holonomy invariant continuous subbundle. Under the same hypotheses, any $\mathcal{A}$-invariant measurable conformal structure τ on $\mathcal{E}$ coincides μ-a.e. with a $\mathcal{A}$-invariant holonomy invariant continuous conformal structure.

Proof. The cocycle generated by A^1 is a fiber bunched cocycle over the partially hyperbolic diffeomorphism g^1. Since E^u and E^s are not jointly integrable, g^1 is accessible as a partially hyperbolic diffeomorphism [12]. In the first case V is a measurable invariant subbundle for A^1; in the second case, τ is an invariant measurable conformal structure for A^1. Theorem 3.3 and Theorem 3.1 respectively from [39] then apply to give the desired result. □

Lemma 4.2. *Let $\mathcal{A}$ be a fiber bunched cocycle over an Anosov flow g^t such that E^u and E^s are not jointly integrable. Suppose that*

$$\lambda_+(\mathcal{A},\mu) = \lambda_-(\mathcal{A},\mu).$$

Then there is a finite cover $\mathcal{X}$ of X and a flag

$$0 \subsetneq \mathcal{E}^1 \subsetneq \mathcal{E}^2 \subsetneq \cdots \subsetneq \mathcal{E}^k = \widetilde{\mathcal{E}}$$

of continuous holonomy-invariant subbundles $\mathcal{E}^i$ which are invariant under the action of the lifted cocycle $\widetilde{\mathcal{A}}$ on the lifted bundle $\widetilde{E}$ over $\mathcal{X}$. Furthermore the induced action of the cocycle $\widetilde{\mathcal{A}}_i$ on $\mathcal{E}^i/\mathcal{E}^{i-1}$ preserves a continuous holonomy invariant conformal structure.

Proof. The vector bundle $\mathcal{E}$ admits a measurable trivialization on a set of full μ-measure by Proposition 2.12 in [5]. Since μ is fully supported on X, this implies that there is a measurable map $P : \mathcal{E} \to X \times \mathbb{R}^d$ commuting with the projections onto X and which is linear on the fibers. $\mathcal{B} = P\mathcal{A}P^{-1}$ is a measurable linear cocycle over g^t on the trivial vector bundle $X \times \mathbb{R}^d$. We can apply Zimmer's amenable reduction theorem [63] for $\mathbb{R}$-cocycles to conclude that there is a measurable map $C : X \to GL(d,\mathbb{R})$ such that the cocycle $\mathcal{F} = C\mathcal{B}C^{-1}$ takes values in an amenable subgroup G of $GL(d,\mathbb{R})$.

The maximal amenable subgroups of $GL(d,\mathbb{R})$ are classified in [47]. Any such group G

contains a finite index subgroup K which is conjugate to a subgroup of a group of the form

$$H(d_1, \dots, d_k) = \begin{bmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & A_k \end{bmatrix}$$

where $\sum_{i=1}^k d_i = d$ and $A_i \in \mathbb{R} \cdot SO(d_i, \mathbb{R})$. Thus, by conjugating the cocycle $\mathcal{F}$ if necessary, we may assume that $\mathcal{F}$ takes values in a group G which contains a finite index subgroup K that is contained in one of the groups $H(d_1, \dots, d_k)$. Let G_* be the stabilizer in G of the flag $V^1 \subset V^2 \subset \dots \subset V^k = \mathbb{R}^d$ corresponding to the group $H(d_1, \dots, d_k)$ containing K. Thus V^j is the span of the first $\sum_{i=1}^j d_i$ coordinate axes in $\mathbb{R}^d$. Let ℓ be the index of G_* in G, which is finite since K has finite index in G and $K \subset G_*$.

Let $V^{i,j}$, $j = 1, \dots, \ell$ be the at most ℓ distinct images of the subspace V^i under the action of G. Let $U^i = \bigcup_{j=1}^{\ell} V^{i,j}$. Then let $\widehat{\mathcal{E}}_x^{i,j} = (C \circ P)^{-1}(x) \cdot V^{i,j}$, $\widehat{\mathcal{U}}^i = (C \circ P)^{-1}(x) \cdot U^i$. The proof of Theorem 3.4 in [39] shows that if the union of measurable subbundles $\widehat{\mathcal{U}}^i$ is invariant under a fiber bunched cocycle with equal extremal exponents over an accessible partially hyperbolic system (which we can take to be the time 1 map A^1 of the cocycle $\mathcal{A}$ over g^1), then there is a finite cover $\mathcal{X}$ of X such that the individual subbundles $\widehat{\mathcal{E}}_x^{i,j}$ lift to subbundles $\mathcal{E}^{i,j}$ of the lifted bundle $\widetilde{\mathcal{E}}$ over $\mathcal{X}$ which agree μ-a.e. with continuous subbundles which we will also denote $\mathcal{E}^{i,j}$. By construction the lifts $\mathcal{U}^i$ are invariant μ-a.e. under the action of the lift $\widetilde{\mathcal{A}}$ of the cocycle $\mathcal{A}$. This is because we constructed these unions of subbundles using amenable reduction over the $\mathbb{R}$ action given by $\mathcal{A}$, and under our measurable trivialization $\mathcal{A}$ takes values in the group G. Since $\mathcal{A}$ is continuous and the lifts $\mathcal{U}^i$ are continuous after modification on a μ-null set, we conclude that each $\mathcal{U}^i$ is everywhere invariant under $\mathcal{A}$.

For each $i \in \{1, \dots, k\}$, $x \in \mathcal{X}$, $t \in \mathbb{R}$, and $j \in \{1, \dots, \ell\}$, there is thus an integer $S_i(x, t, j)$ such that $A^t(\mathcal{E}_x^{i,j}) = \mathcal{E}_{g^t x}^{i, S(x,t,j)}$. For a fixed i and j, $S_i(x, t, j)$ depends continuously on x and t since both $\widetilde{A}^t$ and all of the subbundles $\mathcal{E}^{i,j}$ are continuous. Since for a fixed i and

j we have that $S_i(x,t,j)$ is continuous, integer valued, and has connected domain $\mathcal{X} \times \mathbb{R}$, we conclude that $S_i(x,t,j) := S_i(j)$ is constant in x and t. Furthermore, since $S_i(x,0,j) = j$, we conclude that $S_i(j) = j$. Hence all of the subbundles $\mathcal{E}^{i,j}$ are invariant under $\widetilde{\mathcal{A}}$ as well. In particular $\mathcal{A}$ preserves the flag $\mathcal{E}^1 \subset \cdots \subset \mathcal{E}^k$ which arises as the continuous extension of the lift of the flag coming from the standard flag $V^1 \subset V^2 \subset \cdots \subset V^k$.

To prove the second claim, note that for any $r \geq 1$, the induced action of the cocycle $\mathcal{F}$ on $V^{\sum_{i=1}^{r} d_i}/V^{\sum_{i=1}^{r-1} d_i} = \mathbb{R}^{d_r}$ preserves the standard Euclidean conformal structure on $\mathbb{R}^{d_r}$. This immediately implies that $\widetilde{\mathcal{A}}$ preserves a measurable conformal structure on the corresponding quotient bundle $\mathcal{E}^j/\mathcal{E}^{j-1}$. By Lemma 4.1, this measurable conformal structure coincides μ-a.e. with a holonomy invariant continuous conformal structure. □

Lemma 4.3. *Suppose that there is a finite cover $\mathcal{X}$ of X such that the lifted cocycle $\widetilde{\mathcal{A}}$ on the lifted bundle $\widetilde{\mathcal{E}}$ preserves a continuous holonomy-invariant conformal structure. Then $\mathcal{A}$ also preserves a continuous holonomy-invariant conformal structure.*

Proof. Let $\widetilde{\mathcal{C}}_x$ be the space of conformal structures on the vector space $\widetilde{\mathcal{E}}_x$. $\widetilde{\mathcal{C}}_x$ can be identified with the Riemannian symmetric space $SL(d,\mathbb{R})/SO(d,\mathbb{R})$ and in fact carries a canonical Riemannian metric of nonpositive curvature for which the induced map $\widetilde{\mathcal{C}}_x \to \widetilde{\mathcal{C}}_{g^t x}$ over the cocycle $\widetilde{\mathcal{A}}$ is an isometry [39]. In particular, for compact subsets $K \subset \widetilde{\mathcal{C}}_x$ there is a natural barycenter map $K \to \mathrm{bar}(K)$ mapping K to its center of mass.

Let τ be the continuous holonomy-invariant conformal structure preserved by $\widetilde{\mathcal{A}}$. Let H be the group of covering transformations for $\mathcal{X}$ over X, which also acts as the group of covering transformations for $\widetilde{\mathcal{E}}$ over $\mathcal{E}$. Let $K_x = \bigcup_{\rho \in H} \{\rho \cdot \tau_{\rho^{-1}(x)}\} \subset \widetilde{\mathcal{C}}_x$. The collection of compact subsets K_x depends continuously on x, is holonomy-invariant, and is invariant under $\mathcal{A}$. Hence all of the same is true of the family of barycenters $\sigma_x := \mathrm{bar}(K_x)$. We thus get a conformal structure σ that is continuous, holonomy-invariant, invariant under $\widetilde{\mathcal{A}}$, and also invariant under the action of the deck group H. σ then descends to the desired conformal structure on $\mathcal{E}$. □

In subsequent sections we will use Lemmas 4.2 and 4.3 together to construct invariant conformal structures for our cocycles of interest. We will first use Lemma 4.2 to construct an invariant flag on a finite cover, then we will show this flag must be trivial, then lastly we will use Lemma 4.3 to push the invariant conformal structure back down to our original bundle.

Remark 4.4. The assumption that the stable and unstable distributions E^u and E^s of g^t are not jointly integrable is likely unnecessary in Lemmas 4.1 and 4.2. Different arguments are needed in the case that E^u and E^s are jointly integrable however, as one cannot use accessibility of the time one map g^1 in this case.

5 From Lyapunov exponents to quasiconformality

5.1 From horizontal exponents to horizontal quasiconformality

In this section we will consider Anosov flows f^t that are orbit equivalent to the geodesic flow of a closed negatively curved Riemannian manifold Y, $\dim Y \geq 3$. We will consider u-splittings $E^u = H^u \oplus V^u$ for f^t and show how to derive uniform quasiconformality of $Df^t|H^u$ from hypotheses on the unstable Lyapunov exponents of f^t. We devote this section to the proof of the following proposition.

Proposition 5.1. *Let f^t be a C^2 Anosov flow which is orbit equivalent to the geodesic flow of a closed negatively curved manifold Y with $\dim Y \geq 3$. Let $E^u = H^u \oplus V^u$ be a u-splitting of index k for f^t. Suppose that there is some $\alpha > 0$ such that E^u is α-Hölder continuous and $Df^t|H^u$ is fiber bunched with exponent α. Suppose further that there exists a fully supported f^t-invariant ergodic probability measure μ with local product structure such that we have $\lambda_1^u(f^t,\mu) = \lambda_k^u(f^t,\mu)$. Then $Df^t|H^u$ is uniformly quasiconformal.*

The hypothesis that $\lambda_1^u(f,\mu) = \lambda_k^u(f,\mu)$ is equivalent to all Lyapunov exponents of the linear cocycle $Df^t|H^u$ with respect to μ being equal. We use the local product structure of

μ only to apply the combination of Lemmas 4.2 and 4.3 above.

Proof. Suppose first that f^t admits a u-splitting $E^u = H^u \oplus V^u$ of index k such that f^t is β-bunched and $Df^t|H^u$ is β-fiber bunched. From the combination of Lemmas 4.2 and 4.3, if there exists a fully supported f^t-invariant ergodic probability measure μ with local product structure such that we have $\lambda_1^u(f^t,\mu) = \lambda_k^u(f^t,\mu)$ – or equivalently, in the terminology of that paper, the extremal Lyapunov exponents of $Df^t|H^u$ with respect to μ are equal – then either $Df^t|H^u$ is uniformly quasiconformal, or there is a proper nontrivial Df^t-invariant subbundle $\mathcal{E} \subset H^u$ such that $\mathcal{E}$ is both s- and u-holonomy invariant. It thus suffices to prove that, under the additional hypothesis that f^t is orbit equivalent to the geodesic flow of a closed negatively curved manifold, H^u admits no such subbundle $\mathcal{E}$.

We first establish that u-holonomy invariant vector fields tangent to H^u are (not necessarily uniquely) integrable inside of W^u leaves. For this first lemma we assume only that $Df^t|H^u$ is fiber bunched and thus admits unstable holonomies L^u.

Lemma 5.2. *Let $x \in M$ and let $X : W^u(x) \to H^u$ be a nonzero L^u-invariant vector field. Then there exists a continuous foliation $\mathcal{X}$ of $W^u(x)$ by C^1 curves tangent to X. Furthermore every curve of the foliation $\mathcal{X}$ is properly embedded in $W^u(x)$.*

Proof. Let $x \in M$ be given and let $X : W^u(x) \to H^u$ be an L^u-invariant vector field. For $t \geq 0$ let $X^t = Df^t(X) : W^u(f^t(x)) \to H^u$. Now smooth X^t to obtain a C^1 vector field Z^t satisfying $\|Z^t - X^t\| \leq 1$ (one can perform the smoothings locally and then glue using a partition of unity). Since Z^t is a C^1 vector field it is uniquely integrable; let $\mathcal{Z}^t$ be the C^1 foliation of $W^u(f^t(x))$ by C^2 curves tangent to Z^t.

Observe that $Df^{-t}(Z^t) \to X$ as $t \to \infty$, as we have

$$\|Df^{-t}(Z^t) - X\| = \|Df^{-t}(Z^t - X^t)\| \leq e^{-at}\|Z^t - X^t\| \leq e^{-at}.$$

for some constant $a > 0$. This implies that the curves of the foliation $\mathcal{Z}^t$ converge to a continuous foliation $\mathcal{X}$ of $W^u(x)$ by curves tangent to X.

To prove the last claim, let γ be any curve in the foliation $\mathcal{X}$. The vector field X projects to an $\bar{L}^u$-invariant vector field $\bar{X}$ on $\mathcal{Q}^u(x)$. This implies that $\bar{X}$ is ∇-parallel, where ∇ is the invariant connection on the quotient spaces constructed in Proposition 3.9. Thus $\bar{\gamma} = \pi \circ \gamma$ is a ∇-geodesic.

Since ∇ is complete, flat, and torsion-free, it induces a proper affine chart $\mathcal{Q}^u(x) \to \mathbb{R}^k$ in which the ∇-geodesics are straight lines in $\mathbb{R}^k$. It follows that all ∇-geodesics are properly embedded in $\mathcal{Q}^u(x)$, so in particular this is true for $\bar{\gamma}$. Thus $\bar{\gamma}(t) \to \infty$ as $t \to \pm\infty$ in $\mathcal{Q}^u(x)$. This immediately implies that $\gamma(t) \to \infty$ as $t \to \pm\infty$ in $W^u(x)$, and therefore γ is properly embedded in $W^u(x)$. □

Now let $\mathcal{E}$ be a Df^t-invariant subbundle of H^u which is invariant under u-, s-, and c-holonomies. We will show that, given any two points y, z in an unstable leaf $W^u(x)$ of f^t, we can join y and z by a well-behaved C^1 curve which is tangent to $\mathcal{E}$. Our argument, applied to the case $\mathcal{E} = H^u$, gives interesting results on accessibility properties of paths tangent to H^u even when we do not make any assumptions on the Lyapunov exponents of $Df^t|H^u$. We will use some standard properties of the visual boundary $\partial\widetilde{Y}$ of the universal cover $\widetilde{Y}$ of a negatively curved manifold Y; we refer to [3] for an exposition of these properties.

Proposition 5.3. *Suppose that $f^t : M \to M$ is a C^2 Anosov flow which is orbit equivalent to the geodesic flow g_Y^t of a closed negatively curved manifold Y. Let $E^u = H^u \oplus V^u$ be a u-splitting for f^t and assume there is an $\alpha > 0$ such that E^u is α-Hölder continuous and $Df^t|H^u$ is fiber bunched with exponent α. Let $\mathcal{E}$ be a nonzero Df^t-invariant subbundle of H^u which is invariant under s- and u-holonomies.*

Then for each $x \in M$ and $y \in W^u(x)$ there exists a continuous curve $\gamma : [0,1] \to W^u(x)$ such that $\gamma(0) = x$, $\gamma(1) = y$ and γ is C^1 on $[0,1)$ with $\gamma'(t) \in \mathcal{E}$ for each $t \in [0,1)$. Furthermore $\gamma'(t) \neq 0$ for all $t \in [0,1)$.

Proof. We write $W^{*,f}$ for the invariant foliations of f^t and $W^{*,g}$ for the invariant foliations of g_Y^t. Let $x \in M$ and $y \in W^{u,f}(x)$ be given. We assume that $x \neq y$, as otherwise the proposition is trivial. We also assume that $\dim Y \geq 3$, as for $\dim Y = 2$ the proposition is also trivial. We let $\varphi : \widetilde{M} \to T^1\widetilde{Y}$ denote the lift to the universal cover of the orbit equivalence $\bar{\varphi} : M \to T^1 Y$ from f^t to g_Y^t. For a unit tangent vector $v \in T^1\widetilde{Y}$, we let $\xi_+(v)$ denote the forward projection (as $t \to \infty$) of v to $\partial\widetilde{Y}$ along the geodesic flow g_Y^t, and let ξ_- denote the backward projection (as $t \to -\infty$) of this vector to $\partial\widetilde{Y}$.

Choose a point $z \in M$ such that its image $\varphi(z) \in T^1\widetilde{Y}$ satisfies $\xi_-(\varphi(z)) = \xi_+(\varphi(y))$ and $\xi_+(\varphi(z)) = \xi_+(\varphi(x))$. By construction, we then have $\varphi(x) \in W^{cs,g}(\varphi(z))$ and therefore $x \in W^{cs,f}(z)$. Choose a nontrivial u-holonomy invariant vector field $X : W^{u,f}(z) \to H^u$ which is tangent to $\mathcal{E}$. By Lemma 5.2, there exists a continuous foliation $\mathcal{X}$ of $W^{u,f}(z)$ by C^1 curves tangent to X. Consider the curve η in this foliation $\mathcal{X}$ for which we have $\eta(0) = z$. By Lemma 5.2, η is properly embedded in $W^{u,f}(z)$, and therefore $\eta(t) \to \infty$ as $t \to \infty$.

Since the invariant foliations $W^{*,g}$ for g_Y^t have global product structure on $T^1\widetilde{Y}$, the same is true for the invariant foliations $W^{*,f}$ for f^t on $\widetilde{M}$. Let σ be the cs-holonomy image of η in $W^{u,f}(x)$. By cs-holonomy invariance of $\mathcal{E}$ and Proposition 3.7, we conclude that σ is a C^1 curve tangent to $\mathcal{E}$ with $\sigma(0) = x$ and $\sigma'(t) \neq 0$ for all $t \in [0, \infty)$. We claim that $\sigma(t) \to y$ as $t \to \infty$.

Consider the curve $\varphi \circ \eta \subset W^{cu,g}(\varphi(z))$. By construction of η, we have $\varphi(\eta(t)) \to \infty$ as $t \to \infty$. Projecting to the boundary $\partial\widetilde{Y}$, this implies that

$$\xi_+(\varphi(\eta(t)) \to \xi_-(\varphi(z)) = \xi_+(\varphi(y)),$$

as $t \to \infty$. Let $\hat{y} \in W^{u,g}(\varphi(x))$ be the c-holonomy image of $\varphi(y)$ inside of $W^{u,g}(\varphi(x))$. The above implies that if we take the cs-holonomy image $\hat{\eta}$ of $\varphi \circ \eta$ inside of $W^{u,g}(\varphi(x))$, then $\hat{\eta}(0) = \varphi(x)$ and $\hat{\eta}(t) \to \hat{y}$ as $t \to \infty$.

The curve σ is the c-holonomy image (projection by f^t) of $\varphi^{-1} \circ \hat{\eta}$ inside of $W^{u,f}(x)$, and y is the c-holonomy image of $\varphi^{-1}(\hat{y})$. We conclude that $\sigma(t) \to y$ as $t \to \infty$. The curve $\gamma(t) = \sigma((2/\pi)\arctan(t))$, $t \in [0, 1]$, then has the desired properties of the proposition. $\square$

We now complete the proof of Proposition 5.1. Fix a point $x \in M$. Since $\mathcal{E}$ is u-holonomy invariant, it projects to a ∇-parallel C^1 subbundle $\bar{\mathcal{E}}$ of $T\mathcal{Q}^u(x)$. Since ∇ is a flat connection, by the C^1 Frobenius theorem [54] there is a C^2 foliation $\mathcal{F}$ of $\mathcal{Q}^u(x)$ which is tangent to $\bar{\mathcal{E}}$.

If $\mathcal{E}$ is a proper subbundle of H^u, then $\mathcal{F}$ is a nontrivial foliation of $\mathcal{Q}^u(x)$. Hence there exists a point $y \in W^u(x)$ such that $\pi(y) \notin \mathcal{F}(x)$. On the other hand, since we assumed that f^t is orbit equivalent to the geodesic flow g^t_Y of a closed negatively curved Riemannian manifold Y, we have by Proposition 5.3 that there is a curve $\gamma : [0,1] \to W^u(x)$ tangent to $\mathcal{E}$ with $\gamma(0) = x$, $\gamma(1) = y$, and such that γ is C^1 on $[0,1)$. Then $\pi \circ \gamma$ is a C^1 curve tangent to $\bar{\mathcal{E}}$ that joins $\pi(x)$ to $\pi(y)$ in $\mathcal{Q}^u(x)$. However, this is impossible since the curve γ must be contained inside of $\mathcal{F}(x)$ and we have assumed that $y \notin \mathcal{F}(x)$.

Thus $Df^t|H^u$ has no invariant subbundles which are both s- and u-holonomy invariant. Therefore by the discussion at the beginning of the proof we conclude that $Df^t|H^u$ is uniformly quasiconformal. □

We now prove Theorem 2.7.

Proof of Theorem 2.7. We apply Proposition 5.1 to the case $H^u = E^u$ and $V^u = \{0\}$, under the hypothesis that $\lambda^u_1(f^t, \nu^u) = \lambda^u_l(f^t, \nu^u)$ for an f^t-invariant fully supported ergodic probability measure ν^u with local product structure. We conclude that $Df^t|E^u$ is uniformly quasiconformal. Using the hypothesis $\lambda^s_1(f^t, \nu^s) = \lambda^s_l(f^t, \nu^s)$ for another f^t-invariant fully supported ergodic probability measure ν^s with local product structure, we conclude from applying Proposition 5.1 to f^{-t} (which has horizontal unstable bundle $H^s = E^s$) that $Df^t|E^s$ is uniformly quasiconformal as well. From Fang's theorem [19, 20], we conclude that f^t is smoothly orbit equivalent to the geodesic flow of a closed real hyperbolic manifold. □

We can then deduce Theorem 2.8 from Theorem 2.7.

Proof of Theorem 2.8. By Kalinin's periodic approximation theorem for Lyapunov exponents [36], the hypothesis $\lambda^u_1(f^t, \nu^{(p)}) = \lambda^u_l(f^t, \nu^{(p)})$ for all periodic points p of f^t implies that $\lambda^u_1(f^t, \nu) = \lambda^u_l(f^t, \nu)$ for all f^t-invariant measures ν. Likewise the hypothesis

$\lambda_1^s(f^t, \nu^{(p)}) = \lambda_l^s(f^t, \nu^{(p)})$ for all periodic points p of f^t implies that $\lambda_1^s(f^t, \nu) = \lambda_l^s(f^t, \nu)$ for all f^t-invariant measures ν.

By [36, Theorem 1.3], we conclude that for every $\varepsilon > 0$ there is a constant c_ε such that for all $t \geq 0$ we have

$$\frac{\sigma_l(Df^t|E^u)}{\sigma_1(Df^t|E^u)} \leq c_\varepsilon e^{\varepsilon t},$$

and

$$\frac{\sigma_l(Df^t|E^s)}{\sigma_1(Df^t|E^s)} \leq c_\varepsilon e^{\varepsilon t}.$$

We can then take ε small enough that these estimates imply that f^t is 1-bunched. Now take μ to be the measure of maximal entropy for f^t, which is fully supported and has local product structure. From the above we have that $\lambda_1^u(f^t, \mu) = \lambda_l^u(f^t, \mu)$ and $\lambda_1^s(f^t, \mu) = \lambda_l^s(f^t, \mu)$. Hence, by Theorem 2.7, f^t is smoothly orbit equivalent to the geodesic flow of a closed real hyperbolic manifold. □

We end this section with an interesting corollary of our arguments. If f^t has a u-splitting $E^u = H^u \oplus V^u$ of index 1, then the linear cocycle $Df^t|H^u$ is 1-dimensional. Thus it will always be the case that there is some $\alpha > 0$ such that E^u is α-Hölder continuous and $Df^t|H^u$ is fiber bunched with exponent α. Hence we may run the arguments in the proof of Proposition 5.1 with $\mathcal{E} = H^u$ to show that Anosov flows orbit equivalent to geodesic flows of negatively curved manifolds never admit u-splittings of index 1,

Corollary 5.4. *Let f^t be a C^2 Anosov flow which is orbit equivalent to the geodesic flow of a closed negatively curved Riemannian manifold Y with $\dim Y \geq 3$. Then f^t does not admit a u-splitting or an s-splitting of index 1.*

In particular, if $f^t = g_Y^t$ is the geodesic flow of a closed negatively curved 3-manifold, then f^t does not admit a nontrivial dominated splitting of either its unstable or stable bundle.

Proof. It suffices to prove that f^t does not admit a u-splitting of index 1, as the claim regarding s-splittings of index 1 then follows by considering f^{-t}. Assume that f^t admits a

u-splitting of index 1, and choose a point $x \in M$ and a point $y \in W^{uu}_{loc}(x)$. By Proposition 5.3, there is a curve $\gamma : [0,1] \to W^u(x)$ which is C^1 on $(0,1)$, tangent to H^u with $\gamma(0) = x$, $\gamma(1) = y$, and such that $\gamma'(t) \neq 0$ for all $t \in (0,1)$.

Consider the projection $\bar{\gamma} = \pi \circ \gamma$ of γ to $\mathcal{Q}^u(x)$. Note that $\bar{\gamma}'(t) \neq 0$ for all $t \in (0,1)$. Using the invariant connection ∇ from Proposition 3.9 we identify $\mathcal{Q}^u(x)$ with $\mathbb{R}$, mapping $\pi(x) = \pi(y)$ to 0. Then $\bar{\gamma} : [0,1] \to \mathbb{R}$ is a curve which is C^1 on $(0,1)$ such that $\bar{\gamma}(0) = \bar{\gamma}(1) = 0$ and $\bar{\gamma}'(t) \neq 0$ for all $t \in (0,1)$, which is absurd. We conclude that f^t cannot admit a u-splitting of index 1. $\square$

The conclusion of Corollary 5.4 for geodesic flows on closed negatively curved 3-manifolds M should not be surprising: if M has strictly 1/4-pinched negative sectional curvature then the boundary $\partial\tilde{M}$ has a C^1 structure. Using this structure one can show that if f^t admits a u-splitting $E^u = H^u \oplus V^u$ of index 1 then H^u extends to a nonvanishing line bundle on $\partial\widetilde{M} \cong S^2$. This is impossible because S^2 has no nonvanishing line bundles.

5.2 From vertical exponents to vertical quasiconformality

In this Section we restrict specifically to the case where we obtain our flow f^t by perturbing the geodesic flow g^t_X of a closed quaternionic or Cayley hyperbolic manifold X. We assume $\dim X \geq 8$ so that X is not also real or complex hyperbolic. In this case, for a C^1 small enough perturbation f^t of g^t_X, the flow f^t will have a u-splitting $E^{u,f} = H^{u,f} \oplus V^{u,f}$ with $\dim V^{u,f} = 3$ if X is quaternionic hyperbolic, or $\dim V^{u,f} = 7$ if X is Cayley hyperbolic.

In the course of the proof of the theorems of Section 2, we must show that the hypotheses on the Lyapunov exponents of f^t imply that Df^t is uniformly quasiconformal on both $H^{u,f}$ and $V^{u,f}$. To show uniform quasiconformality on $H^{u,f}$, we use Proposition 5.1. However, the argument we use to derive quasiconformality of Df^t on $H^{u,f}$ from equality of all Lyapunov exponents on this subbundle breaks down when we consider $V^{u,f}$ instead. In particular, the cs-holonomy maps between $W^{u,f}$ leaves are not necessarily differentiable along $V^{u,f}$. Furthermore, they do not necessarily preserve the vertical unstable foliation $W^{uu,f}$ of f^t.

Note that this is not an issue when X is complex hyperbolic, as in this case $\dim V^{u,f} = 1$ and therefore uniform quasiconformality on $V^{u,f}$ is trivial.

We will instead use a strategy adapted from our work with D. Xu [15]. This strategy requires that f^t be C^2 close to g^t_X, not just C^1 close. This is why we lose C^1 openness of the neighborhood $\mathcal{V}_X$ in our dynamical theorems in the case where X is quaternionic hyperbolic or Cayley hyperbolic. Below we set $k = k(X)$.

Proposition 5.5. *Let X be a closed quaternionic hyperbolic or Cayley hyperbolic manifold. There is a C^2 open neighborhood $\mathcal{V}_X$ of g^t_X in the space of C^2 flows on T^1X with the following property: if $f^t \in \mathcal{V}_X$ and there exists a fully supported f^t-invariant ergodic probability measure μ with local product structure such that $\lambda^u_{k+1}(f^t,\mu) = \lambda^u_l(f^t,\mu)$, then $Df^t|V^{u,f}$ is uniformly quasiconformal.*

Proof. Our arguments follow the methods of [15, Section 7] very closely. Hence we only sketch the modifications to the argument given there that are necessary. We write $g^t := g^t_X$.

For f^t C^1-close enough to g^t, we have a u-splitting $E^{u,f} = H^{u,f} \oplus V^{u,f}$ which is uniformly close to the u-splitting $E^{u,g} = H^{u,g} \oplus V^{u,g}$ for g^t. Furthermore, since Dg^t is conformal on $V^{u,g}$, it's easy to see that, for f^t C^1-close enough to g^t, we have that $Df^t|V^{u,f}$ is fiber bunched. We conclude that the linear cocycle $Df^t|V^{u,f}$ admits u-holonomies $J^{u,f}$ and s-holonomies $J^{s,f}$. We let $J^{u,g}$ and $J^{s,g}$ denote the u- and s-holonomies for $Dg^t|V^{u,g}$. Let $PV^{u,*}$ denote the projectivization of the bundle $V^{u,*}$ over T^1X. Below we will think of T^1X as being equipped with its Sasaki metric corresponding to the symmetric metric on X.

An *su-path* for g^t is a piecewise C^1 path γ consisting of finitely many segments γ_i, for which each γ_i is tangent to either $W^{s,g}$ or $W^{u,g}$. We refer to each γ_i as a *leg* of the path. An *su-loop* for g^t based at $v \in T^1X$ is an su-path which starts and ends at v. We define su-loops and su-paths similarly for f^t, replacing $W^{*,g}$ with $W^{*,f}$.

There is a positive integer d such that each unit tangent vector $z \in T^1\widetilde{X}$ is tangent to a unique totally geodesic submanifold $\mathcal{S}(z)$ of $\widetilde{X}$, which is an isometrically embedded copy of a real hyperbolic space of constant negative curvature $K \equiv -4$, recalling that we normalized

X to have sectional curvatures $-4 \leq K \leq -1$. When X is quaternionic hyperbolic we have $d = 4$, and when X is Cayley hyperbolic we have $d = 8$. The tangent bundle $T^1\mathcal{S}(z) \subset T^1X$ is a g^t-invariant submanifold of T^1X that is subfoliated by the $W^{uu,g}$ and $W^{ss,g}$ foliations, which in turn are the stable and unstable foliations for g^t restricted to $T^1\mathcal{S}(z)$.

We now think of the real hyperbolic space $\mathcal{S}(z)$ in isolation for a given $z \in T^1X$. Note that we have $T(T^1\mathcal{S}(z)) = V^{u,g} \oplus E^{c,g} \oplus V^{s,g}$, and also $\dim V^{u,g} = d-1$. An su-loop γ based at z inside of $T^1\mathcal{S}(v)$ induces an isometry $T_g(\gamma) : PV^{u,g}_z \rightarrow PV^{u,g}_z$ by composing s- and u-holonomies of $Dg^t|PV^{u,g}$, since the s- and u-holonomies on both the stable and unstable bundles for real hyperbolic manifolds are conformal, hence isometric after projectivization. As a consequence of work of Brin and Karcher [11] on the frame flow for real hyperbolic manifolds, there are finitely many su-loops $\gamma_1, \dots, \gamma_m$ such that, identifying $PV^{u,g}_z$ isometrically with the projective space $\mathbb{RP}^{d-2}$, the isometries $T_g(\gamma_1), \dots, T_g(\gamma_m)$ generate the Lie group $PO(d-1)$ of isometries of $\mathbb{RP}^{d-2}$. Furthermore the total number m of loops used, and the total lengths of these loops may be chosen to only depend on the curvature and the dimension of the real hyperbolic space $\mathcal{S}(z)$ in question. In particular, they are independent of the chosen point z.

Given the above, the proposition below is a straightforward exercise with proof identical to [15, Proposition 34],

Proposition 5.6. *For any $\delta > 0$ there is a constant $L > 0$ and an integer $\ell > 0$ such that given any $z \in T^1X$ there is a finite collection $\gamma_1, \dots, \gamma_\ell$ of su-loops based at z of total length at most L for which the collection of points $\{T_g(\gamma_i)(v)\}_{i=1}^{\ell}$ is δ-dense in $PV^{u,g}_z$ for any $v \in PV^{u,g}_z$.*

Since $\mathcal{S}(z)$ is totally geodesic inside of $\widetilde{X}$, the s- and u-holonomies on $V^{s,g}$ and $V^{u,g}$ of the restriction of g^t to $T^1\mathcal{S}(v)$ coincide with the s- and u-holonomies of $Dg^t|V^{u,g}$,when g^t is considered as a flow on all of $T^1\widetilde{X}$. Each of the su-loops γ_i of Proposition 5.6 is also an su-loop for g^t in T^1X, with each leg tangent to either $W^{uu,g}$ or $W^{ss,g}$. The map $T_g(\gamma_i)$ is thus given as a composition of the holonomy maps $J^{s,g}$ and $J^{u,g}$ of $Dg^t|V^{u,g}$ along each leg of the loop.

From this point on the arguments are identical to those given in [15, Section 7], only with our bundles $V^{u,*}$ replacing the bundles $E^{u,*}$ of those arguments. We sketch the concluding arguments and refer the reader to that paper for more details.

We now pass to our C^2-small perturbation f^t of g^t. We recall that the holonomies of a fiber bunched linear cocycle vary uniformly continuously with the cocycle in the Hölder topology [2]. Hence, after perturbing g^t to f^t, we obtain the following lemma for the corresponding holonomy maps for $Df^t|V^{u,f}$ around su-loops for f^t.

Lemma 5.7. *Given any $\delta > 0$, there is a C^2-open neighborhood $\mathcal{V}_X$ of g^t such that, if $f^t \in \mathcal{V}_X$, we have that for any $z \in T^1X$ there is a finite collection $\gamma_1, \dots, \gamma_\ell$ of su-loops for f^t based at z such that the collection of points $\{T_f(\gamma_i)(v)\}_{i=1}^{\ell}$ is δ-dense in $PV_z^{u,f}$ for any $v \in PV_z^{u,f}$.*

There is a technical issue in the proof of Lemma 5.7, since an su-loop based at z for g^t is not necessarily an su-loop for f^t based at z. However there will be an su-path $\hat{\gamma}$ for f^t which is uniformly close to γ, and whose endpoint is close to z. By using a proposition of Katok and Kononenko [43] on accessibility properties of partially hyperbolic diffeomorphisms obtained as C^2-small perturbations of the time-1 map g^1 of a contact Anosov flow, we can close $\hat{\gamma}$ by a short su-path for f^t to obtain an su-loop for f^t based at z, which is uniformly close to the original su-loop γ for g^t.

Let δ be given and let $f^t \in \mathcal{V}_X$. Suppose that there exists a fully supported f^t-invariant ergodic probability measure μ with local product structure such that $\lambda^u_{k+1}(f^t, \mu) = \lambda^u_l(f^t, \mu)$. By the work of Avila, Santamaria and Viana [2], there is a Df^t-invariant probability measure ν on PV^u, which projects down to the f^t-invariant measure μ on T^1X, such that ν has a disintegration $\{\nu_z\}_{z \in T^1X}$ into probability measures ν_z on the projective fibers PV^u_z which depend continuously on the basepoint z. Furthermore this disintegration is equivariant under s- and u-holonomy.

If $Df^t|V^{u,f}$ is not uniformly quasiconformal, then using the equivariance of the conditional measures of ν on PV fibers under s- and u-holonomy we can construct a point $z \in T^1X$ and a sequence of projective linear maps $A_n : PV_z^{u,f} \to PV_z^{u,f}$ with $(A_n)_*\nu_z = \nu_z$ for all n. If we identify $PV_z^{u,f}$ with $\mathbb{RP}^{d-2}$ and realize A_n as a sequence of elements of the projective linear group $PSL(d-1, \mathbb{R})$ acting on $\mathbb{RP}^{d-2}$, then $A_n \to \infty$ in $PSL(d-1, \mathbb{R})$. See the end of the argument of [15, Section 7] for details. This implies that A_n converges to a quasi-projective transformation Q of $\mathbb{RP}^{d-2}$ [22]. From this and the equations $(A_n)_*\nu_z = \nu_z$, we conclude that ν_z is supported on the projectivization of the union of two proper linear subspaces $\ker Q$ and $\operatorname{Im} Q$ of $V_z^{u,f}$.

The support of ν_z is invariant under holonomies around su-loops based at z. By Lemma 5.7 we conclude that the support of ν_z must be δ-dense in $PV_z^{u,f}$. But we can choose δ small enough that the projectivization of the union of *any* two proper linear subspaces of $V_z^{u,f}$ is not δ-dense. This gives the contradiction that implies that $Df^t|V^{u,f}$ is uniformly quasiconformal. □

6 Hamenstädt metrics and synchronization

Throughout the rest of the paper we will write $\asymp$ for equality of two quantities up to a multiplicative constant that is independent of the parameters involved. For example, for two functions $\zeta : \mathbb{R} \times M \to \mathbb{R}$ and $\xi : \mathbb{R} \times M \to \mathbb{R}$ we write $\zeta \asymp \xi$ if there is a constant $C \geq 1$ such that $C^{-1}\xi(t, x) \leq \zeta(t, x) \leq C\xi(t, x)$ for all $t \in \mathbb{R}$ and $x \in M$.

6.1 Hamenstädt metrics

Let M be a smooth manifold and let W be a continuous foliation of M with C^1 leaves, such that TW is equipped with a continuous inner product with norm $\|\cdot\|$ that induces a Riemannian metric d_x on each leaf $W(x)$. Note that $d_x = d_y$ for $y \in W(x)$. We consider a continuous flow $f^t : M \to M$ that preserves the W foliation and is C^1 when restricted to each leaf $W(x)$. We assume that f^t uniformly expands the leaves of W; we fix $a > 0$

to be a constant such that $\sigma_1(Df^t|TW) \geq e^{at}$ for every $t \geq 0$. This implies that we have $d_{f^t x}(f^t x, f^t y) \geq e^{at} d_x(x,y)$ for every $x \in M$, $y \in W(x)$, and $t \geq 0$.

We define for $x \in M$ and $y \in W(x)$,

$$\beta(x,y) = \sup\{t \in \mathbb{R} : d_{f^t x}(f^t(x), f^t(y)) \leq 1\},$$

Note that β is finite because f^t uniformly expands W leaves by hypothesis. We then define the *Hamenstädt* metric ρ_x on $W(x)$ by, for $y, z \in W(x)$,

$$\rho_x(y,z) = e^{-a\beta(y,z)}.$$

We clearly have $\rho_y = \rho_x$ for $y \in W(x)$. We denote the ball of radius r centered at x in the metric ρ_x by $B_\rho(x,r)$. Similarly, we denote the ball of radius r centered at x in the metric d_x by $B_d(x,r)$.

Remark 6.1. It is not obvious that ρ_x satisfies the triangle inequality on $W(x)$. A proof of this fact may be found in [30]. The constant a in the definition of the Hamenstädt metric is not canonical; for definiteness we will choose the maximal a such that the inequality $\sigma_1(Df^t|TW) \geq e^{at}$ holds for every $t \geq 0$.

These metrics were introduced by Hamenstädt [27] in the context where f^t is a geodesic flow and W is the unstable foliation. Hasselblatt [30] showed that her formulation of this metric extends to the setting of Anosov flows. The metric ρ_x satisfies $\rho_{f^t x}(f^t y, f^t z) = e^{at}\rho_x(y,z)$ for every $t \in \mathbb{R}$, i.e., f^t is conformal on the W foliation in this family of metrics.

It is easy to check that the metrics ρ_x vary continuously with x, in the sense that the function $(x,y,z) \to \rho_x(y,z)$ is jointly uniformly continuous in $x \in M$ and $y, z \in W(x)$. This implies that the Hamenstädt metrics induce the Euclidean topology on the leaves of W. As a consequence of this continuity, the Hamenstädt metrics and the Riemannian metrics are uniformly comparable at any fixed scale: given any $r > 0$, there is a constant $C = C(r) \geq 1$

such that for all $x \in M$,

$$B_d(x, C^{-1}r) \subseteq B_\rho(x, r) \subseteq B_d(x, Cr),$$

and

$$B_\rho(x, C^{-1}r) \subseteq B_d(x, r) \subseteq B_\rho(x, Cr).$$

6.2 Thermodynamic formalism

In this section we let f^t be a transitive C^2 Anosov flow on a closed Riemannian manifold M. We briefly discuss here the thermodynamic formalism for Anosov flows. We refer to [10] for details and proofs of the claims made in this discussion. Let $\zeta' : M \to \mathbb{R}$ be a Hölder continuous function. Define $\zeta : \mathbb{R} \times M \to \mathbb{R}$ by $\zeta(t, x) = \int_0^t \zeta'(f^s x)\, ds$. The map ζ is an additive cocycle over f^t as defined at the beginning of Section 3.2. For our purposes the additive cocycle ζ and the function ζ' contain the same information, as we can recover ζ' through the equation $\zeta'(x) = \frac{d}{dt}\big|_{t=0} \zeta(t, x)$. We will use the additive cocycle ζ in what follows.

We let $P(\zeta)$ denote the *topological pressure* of ζ with respect to f^t, given by the formula

$$P(\zeta) = \lim_{T \to \infty} \frac{1}{T} \log \left(\sum_{\substack{(x,t): f^t(x) = x, \\ t \in [0,T]}} \exp\left(\zeta(t, x)\right) \right).$$

The *variational principle for pressure* states that $P(\zeta)$ may alternatively be described as

$$P(\zeta) = \sup_{\nu \in \mathcal{M}_{\mathrm{erg}}(f^t)} h_\nu(f^t) + \int_M \zeta(1, x)\, d\nu(x),$$

where the supremum is taken over all f^t-invariant ergodic probability measures ν, and $h_\nu(f^t)$ denotes the entropy with respect to the measure ν There is a unique f^t-invariant ergodic probability measure μ_ζ – referred to as the equilibrium state for ζ with respect to f^t –

which achieves the supremum in the variational principle. We note that one may replace the integrand $\zeta(1, x)$ in the variational principle with $\zeta'(x)$; that is the usual formulation of the variational principle.

Two additive cocycles ζ and ψ over f^t are *cohomologous* if there is a continuous function $\xi : M \to \mathbb{R}$ such that

$$\zeta(t, x) - \psi(t, x) = \xi(f^t x) - \xi(x).$$

We refer to ξ as the *transfer function* from ζ to ψ. The transfer function has the same regularity as ζ and ψ: if both ζ and ψ are $C^{r+\alpha}$ for some integer $r \geq 0$ and some $0 < \alpha < 1$ then ξ is $C^{r+\alpha}$ as well [45]. Two additive cocycles ζ and ψ have the same equilibrium state μ if and only if there is a constant $c \in \mathbb{R}$ such that ζ is cohomologous to $\psi_c(t, x) = \psi(t, x) + ct$.

The Bowen-Margulis measure of maximal entropy for f^t is the equilibrium state associated to any additive cocycle of the form $\zeta(t, x) \equiv ct$ for some constant $c \in \mathbb{R}$. Another equilibrium state of interest to us is the *SRB measure* m_f for f^t; this is the equilibrium state associated to the additive cocycle $(t, x) \to -\log \mathrm{Jac}(Df^t|E^u)$. The SRB measure is characterized by absolute continuity of the conditional measures $m_{x,f}$ on unstable leaves $W^u(x)$; when f^t is volume-preserving, m_f is the invariant volume for f^t[10].

We recall that, as noted at the beginning of Section 3, equilibrium states for Hölder potentials (i.e., equilibrium states associated to Hölder continuous additive cocycles) have local product structure. We take the measures μ_x on local unstable leaves $W^u_{loc}(x)$ defined by the local product structure to be the conditional measures of μ on unstable leaves. We caution that, in general, even if $y \in W^u_{loc}(x)$ we only have that the measures μ_y and μ_x are equivalent up to some positive continuous density; they are not necessarily equal. However we will usually be considering the case where μ is the measure of maximal entropy for f^t, for which we *do* have $\mu^u_y = \mu^u_x$ for all $y \in W^u(x)$. This can be seen, for example, from the Hamenstädt [26] and Hasselblatt [30] constructions of the unstable conditionals for μ as Hausdorff measures of the Hamenstädt metric ρ defined at the beginning of the section. It can also be seen from the holonomy invariance of the conditionals μ_x in Margulis' construction of the measure of maximal entropy for f^t [46].

Fix a Riemannian metric on E^u that gives metrics d_x on unstable leaves $W^u(x)$. The equilibrium state μ of a Hölder continuous additive cocycle ζ has the *Gibbs property*: for any $x \in M$, $t \geq 0$ and $r \leq 1$,

$$\mu_{f^{-t}x}(f^{-t}(B_d(x,r))) \asymp \exp\left(-P(\zeta) + \zeta(t,x)\right)\mu_x(B(x,r)).$$

We are particularly interested in the Gibbs property for potentials ζ for which we have $P(\zeta) = 0$.

We now let μ denote the measure of maximal entropy for f^t and let $h := h_{\text{top}}(f^t)$. Consider the additive cocycle $\zeta(t,x) \equiv -ht$ over f^t, for which we have $P(\zeta) = 0$. From the Gibbs property, we have for any $x \in M$ and $t \geq 0$,

$$\mu_{f^{-t}x}(f^{-t}(B_d(x,1))) \asymp e^{-ht}\mu_x(B_d(x,1)).$$

Let $a > 0$ be given such that $\sigma_1(Df^t|E^u) \geq e^{at}$ for $t \geq 0$, and let ρ_x be the associated Hamenstädt metric on $W^u(x)$ with exponent a. Using the uniform comparability of the Hamenstädt metric to the Riemannian metric with $r = 1$ as at the end of Section 6.1, we conclude that

$$\mu_x(B_d(x,1)) \asymp \mu_x(B_\rho(x,1)).$$

Since f^t acts conformally with respect to the Hamenstädt metrics, the previous two proportionality statements imply that we have, for all $x \in M$ and $t \in \mathbb{R}$,

$$\mu_x(B_\rho(x,e^{at})) \asymp e^{ht}.$$

Setting $r = e^{at}$, and recalling that $\mu_y = \mu_x$ and $\rho_y = \rho_x$ for $y \in W^u(x)$, this implies that for all $x \in M$, $y \in W^u(x)$, and $r \geq 0$,

$$\mu_x(B_\rho(y,r)) \asymp r^{\frac{h}{a}}.$$

This property has an important formalization in analysis on metric spaces. A *metric measure space* is a triple (W, ρ, μ), where W is a metric space with metric ρ and μ is a Borel probability measure on W. Given $Q > 0$, the metric measure space (W, ρ, μ) is *Ahlfors Q-regular* if for all $r \geq 0$ we have $\mu(B_\rho(x,r)) \asymp r^Q$, where the implied multiplicative constant is independent of r. The Gibbs property estimate for the Hamenstädt metric ρ_x implies that the metric measure space $(W^u(x), \rho_x, \mu_x)$ is Ahlfors $\frac{h}{a}$-regular. Furthermore, the multiplicative constant can be taken to be independent of x.

6.3 Synchronization

In this section we carefully review a process known as *synchronization* for transitive Anosov flows. This process goes back to Parry [52] and Ghys [21] and has proved useful in the study of rigidity problems for Anosov flows. Hamenstädt [29] also used it to describe the unstable conditional measures of an equilibrium state as Hausdorff measures with respect to modifications of her metrics constructed in Section 6.1. We will adapt the synchronization procedure to our specific setting.

Throughout this section we will assume that there is some $0 < \alpha < 1$ and an integer $r \geq 1$ such that our transitive Anosov flow f^t is $C^{r+1+\alpha}$. Note that this hypothesis is satisfied if f^t is a C^3 Anosov flow. Given a Hölder continuous additive cocycle $\zeta : M \to \mathbb{R}$ with $\zeta(t,x) < 0$ for $t > 0$, we consider the family $q\zeta$, $q > 0$ of additive cocycles. The topological pressure $P(q\zeta)$ is analytic and strictly decreasing in q (see [55] for the discrete time case, [10] for the adaptation of this method to flows). Since $P(0) = h_{\text{top}}(f) > 0$ and $P(q\zeta) \to -\infty$ as $q \to \infty$, there is a unique $Q > 0$ such that $P(Q\zeta) = 0$. We let μ denote the equilibrium state for $Q\zeta$. We will construct a Hölder time change $\hat{f}^t$ of f^t such that there is an $\hat{f}^t$-invariant measure

$\hat{\mu}$ equivalent to μ that is the measure of maximal entropy for $\hat{f}^t$.

We will assume that ζ is $C^{r+\alpha}$ along W^{cu}-leaves as a function on $\mathbb{R} \times M$. This is a natural hypothesis for the potentials we will consider in this paper, e.g., the additive cocycle $(t, x) \to -\log \mathrm{Jac}(Df^t|E^u)$ for the SRB measure is $C^{r+\alpha}$ on W^{cu} leaves if f^t is $C^{r+1+\alpha}$.

We define an additive cocycle $\tau : \mathbb{R} \times M \to \mathbb{R}$ by setting, for $t \geq 0$ and $x \in M$, $\tau(t, x)$ to be the unique positive number such that

$$\zeta(\tau(t, x), x) = -t.$$

and setting $\tau(-t, x) = -\tau(t, f^{-t}x)$ for $t < 0$. The solution to the above equation is unique because we assumed $\zeta(t, x) < 0$ for $t > 0$. By the implicit function theorem it's easy to see that τ is $C^{r+\alpha}$ along W^{cu} leaves, since ζ is $C^{r+\alpha}$ along W^{cu} leaves. Furthermore τ is Hölder continuous on M. We define a new flow $\hat{f}^t$ by $\hat{f}^t(x) = f^{\tau(t,x)}(x)$ for $x \in M$ and $t \in \mathbb{R}$. This flow is $C^{r+\alpha}$ on W^{cu} leaves but may only be Hölder continuous on M.

We summarize now the discussion from [52] of how time changes transform invariant measures and entropy. Let Z be the generating vector field for f^t, and let $\hat{Z}(x) = \tau'(x)Z(x)$ be the vector field generating our time change $\hat{f}^t(x)$, where we recall that $\tau'(x) = \frac{d}{dt}\big|_{t=0} \tau(t, x)$ is the generator for τ. Let $\omega(x) = (\tau'(x))^{-1}$ and let $\omega(t, x) = \int_0^t \omega(f^s x)\, ds$. The cocycle $\tau(t, x)$ is the inverse self-homeomorphism of $\mathbb{R}$ to $\omega(t, x)$, i.e., we have $t \equiv \omega(x, \tau(t, x)) \equiv \tau(x, \omega(x, t))$.

If ν is an ergodic f^t-invariant probability measure, then $\hat{\nu} = \omega(x)\nu / \int_M \omega\, d\nu$ is an $\hat{f}^t$-invariant ergodic probability measure equivalent to ν. By Abramov's formula for the entropy of time changes of a flow [1], we have $h_{\hat{\nu}}(\hat{f}^t) = \frac{h_\nu(f^t)}{\int_M \omega\, d\nu}$.

For the equilibrium state μ for $Q\zeta$ with respect to f^t defined above we let $\hat{\mu}$ be the corresponding $\hat{f}^t = f^{\tau(t,x)}$-invariant probability measure constructed as above.

Lemma 6.2. *The following claims hold for the time change $\hat{f}^t$ of f^t,*

1. *The foliations W^{cu}, W^{cs}, and W^c are invariant under $\hat{f}^t$. The flow $\hat{f}^t$ is $C^{r+\alpha}$ when restricted to the leaves of the W^{cu} foliation.*

2. *Define for each* $x \in M$,

$$\hat{W}^u(x) = \{y \in M : d(\hat{f}^{-t}(x), \hat{f}^{-t}(y)) \to 0 \text{ as } t \to \infty\}.$$

Then $\hat{W}^u(x)$ *is a* $C^{r+\alpha}$ *embedded submanifold of* $W^{cu}(x)$ *which is the graph of a* $C^{r+\alpha}$ *function over* $W^u(x)$. $\hat{W}^u$ *is tangent to a* $C^{r-1+\alpha}$ *subbundle* $\hat{E}^u$ *of* $E^u \oplus E^c$.

3. *For each* $x \in M$ *define*

$$\hat{W}^s(x) = \{y \in M : d(\hat{f}^t(x), \hat{f}^t(y)) \to 0 \text{ as } t \to \infty\}.$$

Then $\hat{W}^s(x)$ *is a Hölder graph over* $W^s(x)$.

4. *The measure* $\hat{\mu}$ *is the measure of maximal entropy for* $\hat{f}^t$, *and we have* $h_{\hat{\mu}}(\hat{f}) = h_{top}(\hat{f}) = Q$.

Proof. Any time-change of f^t will preserve the foliations W^{cu}, W^c, and W^{cs}. The $C^{r+\alpha}$-smoothness part of claim (1) is obvious from the discussion regarding the regularity of τ on W^{cu} leaves.

To prove (2), we give an alternative description of $\hat{W}^u(x)$. For $y \in W^u(x)$, we define

$$\xi(x, y) = \lim_{t \to \infty} \tau(-t, x) - \tau(-t, y). \tag{2}$$

Standard results on additive cocycles over hyperbolic systems imply that this limit exists and converges uniformly in $x \in M$, $y \in W^u_{loc}(x)$; one can see this for instance by applying the results of 3.2 to the multiplicative cocycle $\exp(\tau)$. Applying this to τ, as well as its derivatives, we conclude that $\xi(x, y)$ is $C^{r+\alpha}$ in x, y. We note that ξ satisfies the equation,

$$\xi(f^t(x), f^t(y)) = \xi(x, y) + \tau(t, y) - \tau(t, x), \tag{3}$$

for all $t \in \mathbb{R}$, $x \in M$, $y \in W^u(x)$.

We claim that, for $x \in M$, we have

$$\hat{W}^u(x) = \{f^{\xi(x,y)}(y) : \; y \in W^u(x)\}.$$

Using (3), and setting $\hat{y} = f^{\xi(x,y)}$, we have

$$\hat{f}^{-t}(\hat{y}) = f^{\tau(-t,y)+\xi(x,y)}(y) = f^{\tau(-t,x)+\xi(f^{-t}(x),f^{-t}(y))}(y).$$

Since $y \in W^u(x)$, we have

$$d(f^{\tau(-t,x)}(x), f^{\tau(-t,x)}(y)) \to 0 \text{ as } t \to \infty.$$

Since $\xi(f^{-t}(x), f^{-t}(y)) \to 0$ as $t \to \infty$, we also have

$$d(\hat{f}^{-t}(\hat{y}), f^{\tau(-t,x)}(y)) \to 0 \text{ as } t \to \infty.$$

These two equations then imply that $d(\hat{f}^{-t}(\hat{y}), \hat{f}^{-t}(x)) \to 0$ as $t \to \infty$, which implies the desired description of $\hat{W}^u(x)$.

For part (3), we carry out the calculations of part (2) with t replacing $-t$, and $W^s(x)$ replacing $W^u(x)$. Since τ is only Hölder along $W^s(x)$, we obtain that $\hat{W}^s(x)$ is a Hölder graph over $W^s(x)$.

Lastly we prove (4). Let $\hat{B}(x,r)$ denote the ball of radius r centered at x in $\hat{W}^u(x)$. We have $\hat{\mu}_x(\hat{B}(x,1)) \asymp \mu_x(B_d(x,1))$, where we recall that $B_d(x,r)$ denotes the ball of radius r centered at x in $W^u(x)$. We now apply $\hat{f}^{-t} = f^{-\tau(t,-)}$ to each side; by equation (2) and the Gibbs property for μ we have $\mu_{f^{-\tau(t,x)}(x)}(\hat{f}^{-t}(B_d(x,1))) \asymp \mu_{f^{-\tau(t,x)}(x)}(f^{-\tau(t,x)}(B_d(x,1)))$. Applying the Gibbs property for μ again gives

$$\mu_{f^{-\tau(t,x)}(x)}(f^{-\tau(t,x)}(B_d(x,1))) \asymp \exp\left(Q\zeta(\tau(t,x),x)\right)\mu_x(B_d(x,1)) = e^{-Qt}\mu_x(B_d(x,1)).$$

Running this back through our calculations gives $\hat{\mu}_{\hat{f}^{-t}x}(\hat{f}^{-t}(\hat{B}(x,1))) \asymp e^{-Qt}\hat{\mu}_x(\hat{B}(x,1))$, for all $t \geq 0$. This implies that $\hat{\mu}$ is the equilibrium state for $\hat{f}^t$ with respect to the potential $-Q$, which is equivalent to $\hat{\mu}$ being the measure of maximal entropy with $h_{\hat{\mu}}(\hat{f}) = h_{\text{top}}(\hat{f}) = Q$, by the variational principle for pressure. □

We now assume that f^t admits a u-splitting $E^u = H^u \oplus V^u$ of index k. We will show that this transforms under synchronization into a u-splitting $\hat{E}^u = \hat{H}^u \oplus \hat{V}^u$ of index k for the flow $\hat{f}^t$. For each $x \in M$ we let $\psi^x : W^u(x) \to \hat{W}^u(x)$ denote the $C^{r+\alpha}$-projection map along the flowlines of f^t. The function ψ^x is given by $\psi^x(y) = f^{\xi(x,y)}(y)$, where ξ is defined as in the proof of Lemma 6.2.

Proposition 6.3. *Suppose that f^t admits a u-splitting $E^u = H^u \oplus V^u$ of index k. Then*

1. *$\hat{f}^t$ admits a u-splitting $\hat{E}^u = \hat{H}^u \oplus \hat{V}^u$ of index k that, for each $x \in M$, is the image of the u-splitting for f^t under the projection $D\psi^x$.*
2. *The bundles $\hat{V}^u$ and $\hat{\mathcal{B}}^u = \hat{E}^u/\hat{V}^u$ are $C^{r+\alpha}$ along W^{cu}.*
3. *For each $x \in M$, the image of the W^{uu}-foliation of $W^u(x)$ under ψ^x gives a $C^{r+\alpha}$ foliation $\hat{W}^{uu}$ of $\hat{W}^u(x)$ tangent to $\hat{V}^u$.*

Proof. We begin by observing that $D\psi^x_y(E^u_y) = \hat{E}^u_{\psi^x(y)}$ for each $y \in W^u(x)$. It's also easily checked (using the representation $\psi^x(y) = f^{\xi(x,y)}(y)$) that, for $y \in W^u(x)$, we have

$$\psi^{f^{\tau(t,x)}(x)}(f^{\tau(t,x)}(y)) = f^{\tau(t,\psi^x(y))}(\psi^x(y)) = \hat{f}^t(\psi^x(y)), \tag{4}$$

and thus by differentiating (4), we obtain the equation

$$D\psi^{f^{\tau(t,x)}(x)}_{f^{\tau(t,x)}(y)} \circ Df^{\tau(t,x)}_y = D\hat{f}^t_{\psi^x(y)} \circ D\psi^x_y. \tag{5}$$

For each $x \in M$ we define $\hat{H}^u_x = D\psi^x_x(H^u_x)$ and $\hat{V}^u_x = D\psi^x_x(V^u_x)$. By applying equation (5) at $y = x$, we obtain that the splitting $\hat{E}^u = \hat{H}^u \oplus \hat{V}^u$ is $D\hat{f}^t$-invariant. Since both $D\psi^x_x$ and

V^u_x are $C^{r+\alpha}$ in the variable $x \in W^{cu}(x)$, we conclude that $\hat{V}^u$ and $\hat{\mathcal{B}}^u$ are both $C^{r+\alpha}$ along W^{cu}.

For $x \in M$ we define $\hat{W}^{uu}(x) = \psi^x(W^{uu}(x))$. We can view $\hat{W}^{uu}(x)$ as the intersection of the leaf $W^{cuu}(x)$ tangent to $V^u \oplus E^c$ with $\hat{W}^u(x)$. Then $\hat{W}^{uu}(x)$ is tangent to the subbundle of $V^u \oplus E^c$, which is tangent to $\hat{W}^u(x)$, i.e., it is tangent to $\hat{V}^u$. This proves (3).

It remains only to check that the u-splitting $\hat{E}^u = \hat{H}^u \oplus \hat{V}^u$ is a dominated splitting for $D\hat{f}^t|\hat{E}^u$. Since the splitting $E^u = H^u \oplus V^u$ is dominated for Df^t, there are constants $C > 0$ and $\chi > 0$ such that, for every $t \geq 0$ and every $x \in M$, we have $\sigma_k(Df^t_x|H^u_x\| \leq Ce^{-\chi t}\sigma_1(Df^t|V^u_x)$. Let $x \in M$ be given, and let $\hat{v} \in \hat{H}^u_x$ and $\hat{w} \in \hat{V}^u_x$ be a pair of unit vectors. We set $v = D(\psi^x_x)^{-1}(\hat{v}) \in H^u_x$, $w = D(\psi^x_x)^{-1}(\hat{w}) \in V^u_x$. The domination estimate for Df^t then implies that, for $t \geq 0$,

$$\|Df^{\tau(t,x)}_x(v)\| \leq Ce^{-\chi\tau(t,x)}\|Df^{\tau(t,x)}_x(w)\|.$$

From the compactness of M and the continuity of τ, there is a positive constant $c \geq 1$ such that, for every $x \in M$ and $t \geq 0$,

$$\tau(t,x) \geq ct.$$

We thus conclude that

$$\|D\hat{f}^t_x(v)\| \leq Ce^{-c\chi t}\|D\hat{f}^t_x(w)\|.$$

By the uniform continuity of the function $x \to D\psi^x_x$ this implies that there is a constant C' independent of x such that we have

$$\|D\hat{f}^t_x(\hat{v})\| \leq C'e^{-c\chi t}\|D\hat{f}^t_x(\hat{w})\|.$$

Taking the supremum over all unit vectors on the left side and the infimum over all s on the right then gives

$$\sigma_k(D\hat{f}^t|\hat{H}^u_x) \leq C'e^{-c\chi t}\sigma_1(D\hat{f}^t|\hat{V}^u_x),$$

which is our desired domination estimate for the splitting $\hat{E}^u = \hat{H}^u \oplus \hat{V}^u$. □

If $Df^t|H^u$ is fiber bunched, then the u-holonomies for the linear cocycle $Df^t|H^u$ over f^t give rise to u-holonomies for $D\hat{f}^t|\hat{H}^u$, and likewise if $Df^t|V^u$ is fiber bunched then the u-holonomies for this cocycle also translate into u-holonomies for $D\hat{f}^t|\hat{V}^u$. More precisely, if L^* are the u-holonomies for $Df^t|H^u$ then we define u-holonomies $\hat{L}^u$ for $D\hat{f}^t|\hat{H}^u$ by, for $x \in M$ and $y \in \hat{W}^u(x)$,

$$\hat{L}^u_{xy} = D\psi^x_y \circ L^u_{xy} \circ (D\psi^x_x)^{-1}.$$

It's easily checked that the maps $\hat{L}^u$ with respect to the linear cocycle $D\hat{f}^t|\hat{H}^u$ over $\hat{f}^t$ satisfy the properties of Proposition 3.2. This all holds in particular if we assume that $Df^t|H^u$ is uniformly quasiconformal, which we will do in Proposition 6.4 below.

We now consider the case of a particular potential ζ_f associated to a u-splitting $E^u = H^u \oplus V^u$ for f^t of index k. We set

$$\zeta_f(t,x) = -\log \mathrm{Jac}(Df^t_x|\mathcal{B}^u_x).$$

Observe that ζ_f is $C^{r+\alpha}$ along the W^{cu}-foliation if f^t is $C^{r+1+\alpha}$, and that we also have $\zeta_f < 0$. We let $Q(f)$ be the horizontal dimension of f^t, so that $P(Q(f)\zeta_f) = 0$. We let $\hat{f}^t$ be the synchronization of f^t with respect to the additive cocycle ζ_f/k. Lastly we let μ_f be the horizontal measure for f^t, which is the equilibrium state of the potential $Q(f)\zeta_f$.

Proposition 6.4. *Suppose that $Df^t|H^u$ is uniformly quasiconformal. Then the following claims hold.*

1. *There is an inner product $(\,,\,)$ on $\hat{H}^u$ such that, for all v,$w \in \hat{H}^u_x$ and all $t \in \mathbb{R}$,*

$$(D\hat{f}^t_x(v), D\hat{f}^t_x(w))_{\hat{f}^t x} = e^t (v,w)_x.$$

2. *For all* $y \in \hat{W}^u(x)$ *and* $v,\, w \in \hat{H}^u_x$,

$$(\hat{L}^u_{xy}(v), \hat{L}^u_{xy}(w))_y = (v, w)_x.$$

Likewise for all $y \in W^{cs}(x)$,

$$(\hat{L}^{cs}_{xy}(v), \hat{L}^{cs}_{xy}(w))_y = (v, w)_x.$$

3. *We have* $h_{top}(\hat{f}) = kQ(f)$. *The measure of maximal entropy* $\hat{\mu}_f$ *for* $\hat{f}^t$ *corresponds to the horizontal measure* μ_f *for* f^t.

Proof. Since $Df^t|H^u$ is uniformly quasiconformal, we can find an inner product $\langle\, ,\, \rangle$ on H^u such that there is a multiplicative cocycle ϕ^t over f^t for which we have for every $x \in M$ and $v, w \in H^u_x$,

$$\langle Df^t_x(v), Df^t_x(w)\rangle_{f^t x} = (\phi^t)^2 \langle v, w\rangle_x,$$

and furthermore for $y \in W^*_{loc}(x)$ we have

$$\langle L^*_{xy}(v), L^*_{xy}(w)\rangle_y = (\ell^*_{xy})^2 \langle v, w\rangle_x,$$

where ℓ^*_{xy} denotes the holonomies of ϕ^t. See [39] for details. Using uniform quasiconformality, we conclude that we have

$$\|Df^t_x(v)\|_{f^t x} \asymp \exp\left(-\zeta_f(t,x)\right),$$

for every $x \in M$, every $t \in \mathbb{R}$, and every $v \in H^u_x$. This implies, taking $\|v\| = 1$ in the above, that for all $x \in M$ and $t \in \mathbb{R}$,

$$\phi^t(x) \asymp \exp\left(-\zeta_f(t,x)\right).$$

This implies that ϕ^t is cohomologous to $\exp\left(-\zeta_f(t,x)\right)$. As a consequence, $\langle , \rangle$ is equivalent to another inner product $\langle\,,\,\rangle'$, for which we have instead,

$$\langle Df_x^t(v), Df_x^t(w)\,\rangle'_{f^t x} = \exp\left(-\zeta_f(t,x)\right)\langle v, w\rangle_x,$$

i.e., $-\zeta_f$ gives the rate of expansion of the inner product $\langle\,,\,\rangle'$ under Df^t.

We define $(\,,\,)$ on $\hat{H}^u$ by setting, for $v, w \in \hat{H}^u_x$,

$$(v, w)_x = \langle (D\psi_x^x)^{-1}(v), (D\psi_x^x)^{-1}(w)\rangle'_x.$$

Since

$$D\hat{f}_x^t = D\psi_{f^{\tau(t,x)}(x)}^{f^{\tau(t,x)}(x)} \circ Df_x^{\tau(t,x)} \circ (D\psi_x^x)^{-1},$$

and $\exp\left(-\zeta_f(\tau(t,x),t)\right) = e^t$, it's straightforward to check that for all $x \in M$, $t \in \mathbb{R}$, and v, $w \in \hat{H}^u_x$,

$$(D\hat{f}_x^t(v), D\hat{f}_x^t(w))_{\hat{f}^t x} = e^t (v, w)_x.$$

This completes the proof of (1).

For (2), the inner product $(\,,\,)$ is $\hat{L}^*$-invariant up to a scalar factor, i.e., $\hat{L}^*$ is conformal with respect to this inner product, since this conformality also holds for L^*. Furthermore the holonomies of $\hat{L}^*$ correspond to the holonomies of the scaling factor of the inner product with respect to $D\hat{f}^t$. Since this scaling factor e^t is independent of the point $x \in M$, we conclude the holonomies of the scaling factor are trivial. Thus equation (2) holds.

Lastly, part (3) of the lemma follows immediately from part (4) of Lemma 6.2. □

We next give a formula for the horizontal dimension that comes from the variational principle for pressure. This formula is the source of the usefulness of $Q(f)$ for controlling the behavior of the unstable Lyapunov exponents of μ_f with respect to f^t.

Proposition 6.5. *Suppose that f^t admits a u-splitting of index k. Then we have*

$$Q(f) = \sup_{\nu \in \mathcal{M}_{erg}(f^t)} \frac{h_\nu(f^t)}{\lambda_1^u(f^t,\nu) + \cdots + \lambda_k^u(f^t,\nu)},$$

with equality if and only if $\nu = \mu_f$ is the horizontal measure for f^t.

Proof. By the variational principle for pressure for the additive cocycle $Q(f)\zeta_f$ defined above,

$$\begin{aligned} 0 &= \sup_{\nu \in \mathcal{M}_{\mathrm{erg}}(f^t)} h_\nu(f^t) + Q(f) \int_M \zeta_f(1,x)\, d\nu(x) \\ &= \sup_{\nu \in \mathcal{M}_{\mathrm{erg}}(f^t)} h_\nu(f^t) - Q(f)(\lambda_1^u(f^t,\nu) + \cdots + \lambda_k^u(f^t,\nu)). \end{aligned}$$

Note that the sum $\lambda_1^u(f^t,\nu) + \cdots + \lambda_k^u(f^t,\nu)$ is always positive for any f^t-invariant measure ν. The formula for $Q(f)$ then follows by rearranging the above expression. □

We end this section by justifying Remark 1.3 of the Introduction. We first show that the horizontal measure and horizontal dimension are both invariant under a flow conjugacy that is C^1 on unstable manifolds.

Lemma 6.6. *Let g^t and f^t be two C^2 Anosov flows on M with u-splittings of index k. Suppose that there is a constant $c > 0$ and a flow conjugacy $\varphi : M \to M$ from g^t to f^{ct} that is C^1 on center-unstable manifolds. Then $\varphi_*(\mu_g) = \mu_f$ and $Q(g) = Q(f)$.*

Proof. We first claim that, for a C^2 Anosov flow f^t with a u-splitting of index k, replacing f^t with f^{ct} for any constant $c > 0$ does not change the horizontal measure and horizontal dimension, i.e., $\mu_{f^{ct}} = \mu_{f^t}$ and $Q(f^{ct}) = Q(f^t)$. Note that a probability measure ν is invariant for f^t if and only if it is invariant for f^{ct} for all $c > 0$. For all ergodic f^t-invariant probability measures ν we then have,

$$\begin{aligned} \frac{h_\nu(f^{ct})}{\lambda_1^u(f^{ct},\nu) + \cdots + \lambda_k^u(f^{ct},\nu)} &= \frac{c h_\nu(f^t)}{c\lambda_1^u(f^t,\nu) + \cdots + c\lambda_k^u(f^t,\nu)} \\ &= \frac{h_\nu(f^t)}{\lambda_1^u(f^t,\nu) + \cdots + \lambda_k^u(f^t,\nu)}. \end{aligned}$$

We conclude from Proposition 6.5 that $Q(f^{ct}) = Q(f^t)$ and $\mu_{f^{ct}} = \mu_{f^t}$.

Thus we may assume, after possibly replacing f^t with f^{ct}, that our map φ conjugates g^t to f^t. Since φ is a conjugacy, we have, for all g^t-invariant ergodic probability measures ν, that $h_{\varphi_*\nu}(f^t) = h_\nu(g^t)$. Furthermore, since φ is C^1 on unstable manifolds, we have that

$$\lambda_i^u(f^t, \varphi_*\nu) = \lambda_i^u(g^t, \nu),$$

for each $1 \leq i \leq \dim E^u$.

We may write each ergodic f^t-invariant probability measure κ as $\kappa = \varphi_*\nu$ for a g^t-invariant probability measure ν. From the above we have

$$\frac{h_\kappa(f^t)}{\lambda_1^u(f^t, \kappa) + \cdots + \lambda_k^u(f^t, \kappa)} = \frac{h_\nu(g^t)}{\lambda_1^u(g^t, \nu) + \cdots + \lambda_k^u(g^t, \nu)}.$$

We thus conclude from Proposition 6.5 that $Q(f) = Q(g)$ and $\varphi_*(\mu_g) = \mu_f$. □

We now prove the claim of the Remark 1.3. Recall that, for a homothety $F : Y \to Z$ of two Riemannian manifolds, we let $DF : T^1Y \to T^cZ$ be the derivative map ($c > 0$ a constant) and let $\Pi \circ DF : T^1Y \to T^1Z$ denote the composition given by composing F with the natural projection $\Pi : T^cZ \to T^1Z$.

Proposition 6.7. *Let X be a closed locally symmetric space of nonconstant negative curvature. Let Y and Z be two Riemannian manifolds that are C^2 close to X. Suppose that we have a homothety $F : Y \to Z$. Then $(\Pi \circ DF)_*(\mu_Y) = \mu_Z$ and $Q_Y = Q_Z$.*

Proof. For $c > 0$, let Z_c denote the Riemannian manifold given by scaling the metric on Z by the constant factor c. By the first half of the proof of Lemma 6.6, under the projection $\Pi : T^1Z_c = T^cZ \to T^1Z$, the horizontal measures of $g^t_{Z_c}$ and g^t_Z are the same and we have $Q_{Z_c} = Q_Z$.

Hence it suffices to prove the proposition in the case where $F : Y \to Z$ is an isometry. We then have $g^t_Z \circ DF = DF \circ g^t_Y$, i.e., g^t_Y are smoothly conjugate by DF. We conclude from Lemma 6.6 that $DF_*(\mu_Y) = \mu_Z$ and $Q_Y = Q_Z$. □

6.4 Quasisymmetry of orbit equivalences

We begin this section with an important remark that will be in effect for the rest of the paper.

Remark 6.8. We will be considering Anosov flows $g^t : M \to M$ and $f^t : M \to M$ on a Riemannian manifold M, together with a Hölder continuous orbit equivalence $\varphi : M \to M$ from g^t to f^t. We will always assume that g^t is at least C^2. However, we allow for the possibility that f^t was obtained from another C^r Anosov flow $\tilde{f}^t$ by the synchronization procedure of Section 6.3. This means that f^t may only be Hölder continuous. However, we will always assume that f^t is at least C^2 on $W^{cu,f} = W^{cu,\tilde{f}}$ leaves.

We will only consider φ as a map $\varphi : W^{cu,g}(x) \to W^{cu,f}(\varphi(x))$ for one choice of x at a time. In particular the transverse regularity of f^t along $W^{s,f}$ leaves will never appear in our proofs.

Let (W, ρ_W) and (Z, ρ_Z) be two metric spaces. A homeomorphism $\varphi : W \to Z$ is *quasisymmetric* if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that, for all $x, y, z \in W$,

$$\rho_W(x,y) \leq s\rho_W(x,z) \Rightarrow \rho_Z(\varphi(x),\varphi(y)) \leq \eta(s)\rho_Z(\varphi(x),\varphi(z)).$$

When the specific form of η is important, we will say that φ is η-quasisymmetric.

We will consider unstable leaves $W^u(x)$ for an Anosov flow f^t with the Hamenstädt metric ρ_x from the beginning of this section, defined as usual using a constant $a > 0$ such that $\sigma_1(Df^t_x|E^u_x) \geq e^{at}$ for all $x \in M$ and $t \geq 0$.

We now let g^t and f^t be two Anosov flows on M which are orbit equivalent by a Hölder continuous function φ. We let $a > 0$ be a common constant such that $\sigma_1(Dg^t_x|E^{u,g}_x) \geq e^{at}$ and $\sigma_1(Df^t_x|E^{u,f}_x) \geq e^{at}$ for all $x \in M$ and $t \geq 0$, and define the Hamenstädt metrics ρ_g and ρ_f accordingly.

We let α be the Hölder continuous additive cocycle over g^t satisfying $\varphi \circ g^t = f^{\alpha(t,x)} \circ \varphi$, i.e., α describes the failure of φ to be a flow conjugacy. As in the proof of Lemma 6.2, for $x \in M$, $y \in W^{u,g}(x)$, we define

$$\xi(x,y) = \lim_{t \to \infty} \alpha(-t,x) - \alpha(-t,y),$$

with this limit existing because g^{-t} exponentially contracts $W^{u,g}$ leaves. As before, we have the relation for $t \in \mathbb{R}$ and $x \in M$, $y \in W^{u,g}(x)$,

$$\xi(g^t(x), g^t(y)) = \xi(x,y) + \alpha(t,y) - \alpha(t,x).$$

For each $x \in M$ and $y \in W^{u,g}(x)$ we define $\varphi_x(y) = f^{-\xi(x,y)}(y)$. We note that $\varphi_x(y) \in W^{u,f}(\varphi(x))$, as we have

$$\begin{aligned} d(f^{\alpha(-t,x)}\varphi_x(y), f^{\alpha(-t,x)}\varphi(x)) &= d(f^{\alpha(-t,x)-\xi(x,y)}\varphi(y), f^{\alpha(-t,x)}(\varphi(x))) \\ &= d(f^{\alpha(-t,y)-\xi(g^{-t}(x),g^{-t}(y))}\varphi(y), f^{\alpha(-t,x)}(\varphi(x))) \\ &= d(f^{-\xi(g^{-t}(x),g^{-t}(y))}\varphi(g^{-t}(y)), \varphi(g^{-t}(x))), \end{aligned}$$

and the last expression converges to 0 as $t \to \infty$ since $\xi(g^{-t}(x), g^{-t}(y)) \to 0$. Finally, we note that we have the equivariance relationship for $y \in W^{u,g}(x)$, $t \in \mathbb{R}$,

$$\varphi_{g^t x}(g^t(y)) = f^{\alpha(t,x)}(\varphi_x(y)).$$

We thus have a homeomorphism $\varphi_x : W^{u,g}(x) \to W^{u,f}(\varphi(x))$. We will show below that this homeomorphism is η-quasisymmetric with respect to the Hamenstädt metrics on each of these leaves, with η being independent of the choice of x.

Proposition 6.9. *There is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that, for every $x \in M$, the map $\varphi_x : (W^{u,g}(x), \rho_{x,g}) \to (W^{u,f}(\varphi(x)), \rho_{\varphi(x),f})$ is η-quasisymmetric.*

Proof. By the continuity of φ, f^t, and g^t, and the fact that the Hamenstädt metrics $\rho_{x,g}$ and $\rho_{x,f}$ depend continuously on the choice of point $x \in M$, there is a constant $C \geq 1$ such that, for every $x \in M$ and $y, z \in W^{u,g}_{loc}(x)$, if $\rho_{x,g}(y, z) = 1$ then

$$C^{-1} \leq \rho_{\varphi(x),f}(\varphi_x(y), \varphi_x(z)) \leq C.$$

Let $x \in M$, and let $y \in W^{u,g}(x)$. Let $\beta = \beta(x, y)$ be the unique real number such that $\rho_{g^\beta x,g}(g^\beta(x), g^\beta(y)) = 1$. We have from the above that

$$\rho_{\varphi(g^\beta x),f}(\varphi_{g^\beta x}(g^\beta x), \varphi_{g^\beta x}(g^\beta y)) \asymp 1$$

By using equivariance and the scaling property of the dynamical metrics for f^t, we have

$$\rho_{\varphi(x),f}(\varphi_x(x), \varphi_x(y)) \asymp \exp(-a\alpha(\beta(x, y), x)).$$

We next take two points $y, z \in W^{u,g}(x)$ with $\beta(x, y) \geq \beta(x, z)$. The above implies that

$$\frac{\rho_{\varphi(x),f}(\varphi_x(x), \varphi_x(y))}{\rho_{\varphi(x),f}(\varphi_x(x), \varphi_x(z))} \asymp \exp(-a(\alpha(\beta(x, y), x) - \alpha(\beta(x, z), x))).$$

By the additivity of α we have

$$\alpha(\beta(x, y), x) - \alpha(\beta(x, z), x) = \alpha(\beta(x, y) - \beta(x, z), x).$$

Continuity of α and compactness of M implies that there is a constant $b > 0$ such that $\alpha(t, x) \geq bt$ for all $t \geq 0$ and $x \in M$. This implies in particular that

$$\alpha(\beta(x, y) - \beta(x, z), x) \geq b(\beta(x, y) - \beta(x, z)),$$

which implies, recalling the definition of the dynamical metric for g^t, that there is a constant

$C \geq 1$, independent of x,y, and z, such that

$$\frac{\rho_{\varphi(x),f}(\varphi_x(x), \varphi_x(y))}{\rho_{\varphi(x),f}(\varphi_x(x), \varphi_x(z))} \leq C\frac{\exp(-ab\beta(x,y))}{\exp(-ab\beta(x,z))} = C\left(\frac{\rho_{x,g}(x,y)}{\rho_{x,g}(x,z)}\right)^b.$$

The inequality $\beta(x,y) \geq \beta(x,z)$ holds if and only if $\rho_{x,g}(x,y) \leq \rho_{x,g}(x,z)$.

Now let $w, y, z \in W^{u,g}(x)$ be any given triple of points. By applying the above with $x = w$ and recalling that $\rho_{w,g} = \rho_{x,g}$, we have the inequality

$$\frac{\rho_{\varphi(w),f}(\varphi_w(w), \varphi_w(y))}{\rho_{\varphi(w),f}(\varphi_w(w), \varphi_w(z))} \leq C\left(\frac{\rho_{x,g}(w,y)}{\rho_{x,g}(w,z)}\right)^b,$$

for $\rho_{x,g}(w,y) \leq \rho_{x,g}(w,z)$. For $y \in W^{u,g}(x)$ we have the relationship

$$\varphi_w(y) = f^{-\xi(w,y)}(y) = f^{-\xi(w,x)-\xi(x,y)}(\varphi(y)) = f^{-\xi(w,x)}(\varphi_x(y)),$$

that is, we have $\varphi_w = f^{-\xi(w,x)} \circ \varphi_x$. This implies, using the conformality of f^t with respect to the Hamenstädt metrics ρ_f, that we have

$$\rho_{\varphi(w),f}(\varphi_w(w), \varphi_w(y)) = e^{-a\xi(x,w)}\rho_{\varphi(x),f}(\varphi_x(w), \varphi_x(y))$$

This implies, going back to our inequality for φ_w, that we have

$$\frac{\rho_{\varphi(x),f}(\varphi_x(w), \varphi_x(y))}{\rho_{\varphi(x),f}(\varphi_x(w), \varphi_x(z))} \leq C\left(\frac{\rho_{x,g}(w,y)}{\rho_{x,g}(w,z)}\right)^b,$$

for $\rho_{x,g}(w,y) \leq \rho_{x,g}(w,z)$, as the factors of $e^{-a\xi(x,w)}$ on the top and bottom cancel. This inequality again is valid for $\rho^a_{x,g}(w,y) \leq \rho^a_{x,g}(w,z)$. We conclude that there is a constant $C \geq 1$ such that, for each $x \in M$ and $w, y, z \in W^{u,g}(x)$,

$$\rho_{x,g}(w,y) \leq \rho_{x,g}(w,z) \Rightarrow \rho_{\varphi(x),f}(\varphi_x(w), \varphi_x(y)) \leq C\rho_{\varphi(x),f}(\varphi_x(w), \varphi_x(z)).$$

This implies that φ_x is weakly C-quasisymmetric [33, Chapter 10]. Let $\mu_{x,g}$ and $\mu_{x,f}$ be the unstable conditionals for the measures of maximal entropy of g^t and f^t respectively. Since both of the metric measure spaces $(W^{u,g}(x), \rho_{x,g}, \mu_{x,g})$ and $(W^{u,f}(\varphi(x)), \rho_{\varphi(x),f}, \mu_{x,f})$ are Ahlfors Q_g- and Q_f-regular respectively for some Q_g, $Q_f > 0$ with constants independent of x, this implies that φ_x is η-quasiymmetric and that η can be chosen independently of x [33, Theorem 10.19]. □

6.5 A criterion for conjugacy

We next give a criterion for two Anosov flows g^t and f^t to be conjugate, given that we have a Hölder continuous orbit equivalence φ from g^t to f^t. We will use this criterion in the proof of Proposition 7.5. We recall that Remark 6.8 is in effect here, so that f^t may be a flow obtained through the synchronization of a C^3 Anosov flow $\tilde{f}^t$.

Let m be the measure of maximal entropy for g^t and let μ be the measure of maximal entropy for f^t. For an unstable leaf $W^{u,g}(x)$ we let m_x denote the conditional measure of m on $W^{u,g}(x)$, which is a Hausdorff measure for the Hamenstädt metric $\rho_{x,g}$ on $W^{u,g}(x)$, by the discussion at the end of Section 6.2. Likewise for an unstable leaf $W^{u,f}(x)$ we let μ_x be the conditional measure of μ_x on $W^{u,f}(x)$, which is also a Hausdorff measure for the Hamenstädt metric $\rho_{x,f}$.

We let φ be a Hölder continuous orbit equivalence from g^t to f^t, so that we have $\varphi(g^t(x)) = f^{\alpha(t,x)}(\varphi(x))$ for a Hölder continuous additive cocycle α. We define $\xi(x,y)$ for $y \in W^{u,g}(x)$ and we define the homeomorphism $\varphi_x : W^{u,g}(x) \to W^{u,f}(\varphi(x))$ as in Section 6.4. Since the Borel measures m_x and μ_x are σ-finite, by the Radon-Nikodym theorem there is an m_x-integrable function $J_x : W^{u,g}(x) \to [0,\infty]$, and a signed measure κ_x that is mutually singular with respect to m_x, such that we have

$$d((\varphi_x)_*^{-1}\mu_{\varphi(x)}) = J_x dm_x + d\kappa_x.$$

We thus have, for every $x \in M$, an m_x-integrable measurable function $J_x : W^{u,g}(x) \to [0, \infty]$. We give below a criterion for producing flow conjugacies in terms of the functions J_x.

Proposition 6.10. *Let g^t and f^t be two Anosov flows, with g^t being C^2, and let φ be a Hölder continuous orbit equivalence from g^t to f^t. The following are equivalent.*

1. *There is a constant $c > 0$ and a Hölder continuous function $\omega : M \to \mathbb{R}$ such that $\hat{\varphi}(x) := \varphi(g^{\omega(x)}(x))$ satisfies $\hat{\varphi} \circ g^t = f^{ct} \circ \hat{\varphi}$.*

2. *We have $m\left(\{x \in M : J_x(y) = 0 \text{ for } m_x\text{-a.e. } \ y \in W^u(x)\}\right) = 0$.*

Proof. Suppose that (1) holds. Then $\hat{\varphi}$ maps the measure of maximal entropy for g^t to the measure of maximal entropy for f^{ct} (which is the same as the measure of maximal entropy for f^t). We thus have $\hat{\varphi}_* m = \mu$. For m-a.e. $x \in M$, we then have that $\hat{\varphi}_*^{-1} \mu_{\varphi(x)}$ is absolutely continuous with respect to m_x, with a density that is positive m_x-a.e. Since g^t acts with positive Jacobian on the conditional measures m_x, this implies that for m-a.e. $x \in M$ we have that $(\varphi_x^{-1})_* \mu_{\varphi(x)}$ is absolutely continuous with respect to m_x, again with a density that is positive m_x-a.e. Thus (1) $\Rightarrow$ (2).

Now suppose that (2) holds. Let $h_g := h_{\text{top}}(f)$ and $h_f := h_{\text{top}}(f)$. For $z \in W^{u,g}(x)$, we first relate the measures $(\varphi_z)_*^{-1} \mu_{\varphi(z)}$ and $(\varphi_x)_*^{-1} \mu_{\varphi(x)}$ on $W^{u,g}(x)$. As noted in the proof of Proposition 6.9, we have $\varphi_z = f^{-\xi(z,x)} \circ \varphi_x$. We then have

$$
\begin{aligned}
(\varphi_z)_*^{-1} \mu_{\varphi(z)} &= (\varphi_x)_*^{-1} (f_*^{-\xi(z,x)} \mu_{\varphi(x)}) \\
&= e^{-h_f \xi(z,x)} (\varphi_x)_*^{-1} \mu_{\varphi(x)}.
\end{aligned}
$$

We also compute how these measures behave under iteration by g^t. For $x \in M$, we have $\varphi_{g^t x} \circ g^t = f^{\alpha(t,x)} \circ \varphi_x$. This implies that

$$
\begin{aligned}
(\varphi_{g^t x})_*^{-1} \mu_{\varphi(g^t x)} &= g_*^{-t} (\varphi_x^{-1})_* f_*^{\alpha(t,x)} \mu_{\varphi(g^t x)} \\
&= e^{h_f \alpha(t,x) - h_g t} (\varphi_x)_*^{-1} \mu_{\varphi(x)}.
\end{aligned}
$$

By the Radon-Nikodym theorem, for each $z \in W^{u,g}(x)$ there is a signed measure κ_z and a locally m_x-integrable function $J_z : W^{u,g}(x) \to [0,\infty]$ such that

$$d(\varphi_z)_*^{-1}\mu_{\varphi(z)} = J_z dm_x + d\kappa_z.$$

By our previous calculations,

$$\begin{aligned}(\varphi_z)_*^{-1}\mu_{\varphi(z)} &= e^{-h_f\xi(z,x)}(\varphi_x)_*^{-1}\mu_{\varphi(x)} \\ &= e^{-h_f\xi(z,x)}(J_x dm_x + d\kappa_x).\end{aligned}$$

We conclude that we may take $J_z = e^{-h_f\xi(z,x)}J_x$ and $d\kappa_z = e^{-h_f\xi(z,x)}\kappa_x$ in the Radon-Nikodym decomposition for $(\varphi_z)_*^{-1}\mu_{\varphi(z)}$. We now define, for m_x-a.e. $z \in W^{u,g}(x)$,

$$J(z) := J_z(z) = e^{-h_f\xi(z,x)}J_x.$$

The function J is locally m_x-integrable on $W^{u,g}(x)$, so as a consequence we have $J(z) < \infty$ for m_x-a.e. z. By the formula $J(z) = J_z(z)$ we can then extend J to a well-defined locally m-integrable function $J : M \to [0,\infty]$ that is finite m-a.e. We also have from our calculations above that

$$J(g^t x) = e^{h_f\alpha(t,x) - h_g t}J(x).$$

We conclude that $Z := \{x : J(x) \neq 0\}$ is a g^t-invariant set.

By ergodicity of g^t with respect to m, we have $m(Z) = 0$ or $m(Z) = 1$. Suppose that $m(Z) = 0$. Then for m-a.e. $x \in M$ and m_x-a.e. $z \in W^{u,g}(x)$,

$$0 = e^{h_f\xi(z,x)}J(z) = e^{h_f\xi(z,x)}J_z(z) = J_x(z).$$

Thus for m-a.e. $x \in M$ and m_x-a.e. $z \in W^{u,g}(x)$, we have $J_x(z) = 0$ This contradicts our hypothesis that (2) holds.

Thus $m(Z) = 1$. Since $J(x) < \infty$ for m-a.e. x, we have

$$\log(J(g^t x)) - \log J(x) = h_f \alpha(t,x) - h_g t.$$

Thus $\frac{h_f}{h_g}\alpha(t,x)$ and the linear additive cocycle t are measurably cohomologous over g^t. By the measurably rigidity of the Livsic equation for Anosov flows [45], these additive cocycles are continuously cohomologous. By the standard criterion for two orbit equivalent flows to be conjugate [44], this implies that g^t is conjugate to f^{ct} with $c = \frac{h_f}{h_g}$, and this conjugacy takes the form $\hat{\varphi}(x) = \varphi(g^{\omega(x)}(x))$ for some Hölder continuous function $\omega : M \to \mathbb{R}$. Thus (2) $\Rightarrow$ (1). □

7 Differentiability of the orbit equivalence

In this section we will complete the proofs of all of our major theorems. We begin by reducing all of our major theorems to Theorem 2.3. The bulk of the section is then devoted to proving Theorem 2.3.

7.1 Reductions

We first reduce Theorem 2.5 to Theorem 2.3. To do this we use the following lemma. Recall that we denote the SRB measure for an Anosov flow f^t by m_f. We also recall that $l = \dim X - 1$.

Lemma 7.1. *Let X be a closed locally symmetric space of nonconstant negative curvature. Let $f^t \in \mathcal{V}_X$. Suppose that $Q(f) \geq Q_X$ and $\lambda_l^u(f^t, \mu_f) \leq 2\lambda_1^u(f^t, \mu_f)$. Then*

1. *$\mu_f = m_f$.*
2. *$Q(f) = Q_X$.*
3. *There is a constant $\xi > 0$ such that $\vec{\lambda}^u(f^t, \mu_f) = \xi \vec{\lambda}^u(g_X^t)$.*

Proof. After a homothety of X, we may assume that X has sectional curvatures satisfying $-4 \leq K \leq -1$, as replacing f^t by f^{ct} for a constant $c > 0$ does not change the hypotheses on the flow. We then have $Q_X = h(X)/k(X)$, where $k(X)$ denotes the index of the u-splitting for g_X^t and $h(X) = k + 2(l-k)$. Set $k := k(X)$ and $h := h(X)$.

By Ruelle's inequality we have

$$Q(f) \leq \frac{\sum_{i=1}^{l} \lambda_i(f^t, \mu_f)}{\sum_{i=1}^{k} \lambda_i^u(f^t, \mu_f)}.$$

Furthermore, equality holds in the Ruelle inequality if and only if $\mu_f = m_f$ [10]. By the inequalities $Q(f) \geq Q_X$ and $\lambda_l^u(f^t, \mu_f) \leq 2\lambda_1^u(f^t, \mu_f)$, we conclude that

$$\begin{aligned} Q_X = \frac{k + 2(l-k)}{k} &\leq \frac{\sum_{i=1}^{l} \lambda_i(f^t, \mu_f)}{\sum_{i=1}^{k} \lambda_i^u(f^t, \mu_f)} \\ &\leq 1 + \frac{2(l-k)\lambda_1^u(f^t, \mu_f)}{\sum_{i=1}^{k} \lambda_i^u(f^t, \mu_f)} \\ &\leq 1 + \frac{2(l-k)\lambda_1^u(f^t, \mu_f)}{k\lambda_1^u(f^t, \mu_f)} \\ &= \frac{k + 2(l-k)}{k} = Q_X. \end{aligned}$$

Thus all of the inequalities above are actually equalities. Since the unstable Lyapunov exponents $\lambda_i^u(f^t, \mu_f)$ are positive and strictly increasing in i, equality in the second inequality above holds if and only if $\lambda_i^u(f^t, m_f^u) = 2\lambda_1^u(f^t, m_f^u)$ for all $k+1 \leq i \leq l$, and equality in the third inequality holds if and only if $\lambda_i^u(f^t, m_f^u) = \lambda_1^u(f^t, m_f^u)$ for all $1 \leq i \leq k$. This implies that there is a constant $\xi > 0$ such that $\vec{\lambda}^u(f^t, \mu_f) = \xi \vec{\lambda}^u(g_X^t)$.

The above string of equalities also immediately implies that $Q(f) = Q_X$, and also that equality holds in the Ruelle inequality. Thus we also have $\mu_f = m_f$. $\square$

From Lemma 7.1 we see that Theorem 2.3 implies Theorem 2.5. We next show that Theorem 2.3 implies Theorem 2.2.

Theorem 2.3 ⇒ Theorem 2.2. We define

$$\ell_X = \{v \in \mathbb{R}^l : v = c\vec{\lambda}^u(g_X^t),\ c \geq 0\},$$

to be the positive ray from the origin generated by $\vec{\lambda}^u(g_X^t)$. This is a closed subset of $\mathbb{R}^l$.The hypotheses of Theorem 2.2 imply that $\vec{\lambda}^u(f^t, \nu^{(p)}) \in \ell_X$ for all periodic points p of f^t.

By Kalinin's periodic exponent approximation theorem [36], any f^t-invariant ergodic probability measure ν can have its unstable Lyapunov vector $\vec{\lambda}^u(f^t, \nu)$ approximated arbitrarily well by the unstable Lyapunov vectors $\vec{\lambda}^u(f^t, \nu^{(p)})$ of periodic points. This implies that $\vec{\lambda}^u(f^t, \nu) \in \ell_X$ for all ν. In particular this holds for $\nu = \mu_f$, which implies that the hypotheses of Theorem 2.3 hold. □

We now explain how Theorem 2.3 implies Theorems 1.1, 1.2, and 1.4. First note that the hypotheses of Theorem 1.1 and Theorem 1.4 imply that the hypotheses of Theorem 2.2 and Theorem 2.5 hold respectively for $f^t = g_Y^t$. In turn this implies from the above that the hypotheses of Theorem 2.3 hold for $f^t = g_Y^t$. By a homothety of Y we may assume that the constant ξ is equal to 1.

Clearly the hypotheses of Theorem 1.2 imply that the hypotheses of Theorem 2.3 hold as well, with $\xi = 1$. Hence it suffices to show that that if we assume the hypotheses of Theorem 2.3 with $f^t = g_Y^t$ and $\xi = 1$ then Y is isometric to X. We prove this below.

For a Riemannian manifold Y we let $\sigma_Y : T^1Y \to T^1Y$ denote the involution $v \to -v$. For all $t \in \mathbb{R}$ we have $\sigma_Y \circ g_Y^t = g_Y^{-t} \circ \sigma_Y$, i.e., σ_Y smoothly conjugates g_Y^t to g_Y^{-t}.

Theorem 2.3 ⇒ Theorem 1.2. By the conclusion of Theorem 2.3 we have an orbit equivalence $\varphi : T^1X \to T^1Y$ from g_X^t to g_Y^t that is $C^{1+\alpha}$ on center-unstable leaves. By Proposition 7.12 below, we conclude that the center-stable foliation $W^{cs,Y}$ for g_Y^t is $C^{1+\alpha}$. Since g_Y^t is a contact Anosov flow, this implies that the stable foliation $W^{s,Y}$ is $C^{1+\alpha}$. Since the smooth involution σ_Y maps the $W^{s,Y}$ foliation to the $W^{u,Y}$ foliation, this implies that $W^{u,Y}$ is a $C^{1+\alpha}$ foliation of T^1Y as well. We conclude that the Anosov splitting of g_Y^t is $C^{1+\alpha}$. This implies that the boundary $\partial\widetilde{Y}$ of $\widetilde{Y}$ carries a $C^{1+\alpha}$ smooth structure.

Let $F : X \to Y$ be the diffeomorphism from X to Y given by these two Riemannian manifolds having the same underlying smooth manifold S. The map F lifts to a quasi-isometry $\tilde{F} : \widetilde{X} \to \widetilde{Y}$ that gives rise to a homeomorphism $\partial\tilde{F} : \partial\widetilde{X} \to \partial\widetilde{Y}$.

Recall that, for v a unit vector in $T^1\widetilde{X}$, $\xi_+(v)$ denotes the forward projection of v to $\partial\widetilde{X}$ along the geodesic through v, and $\xi_-(v)$ denotes the backward projection to $\partial\widetilde{X}$. We use the same notation for projections to $\widetilde{Y}$ from $T^1\widetilde{Y}$. Fix some $\zeta \in \partial\widetilde{X}$ and fix a $v \in T^1\widetilde{X}$ with $\xi_+(v) = \zeta$. Since φ is $C^{1+\alpha}$, it maps $W^{u,X}(v)$ onto a $C^{1+\alpha}$ submanifold $\varphi(W^{u,X}(v)) \subseteq W^{cu,Y}(\varphi(v))$ that is transverse to the flow direction of g_Y^t. We then have a $C^{1+\alpha}$ diffeomorphism,

$$\xi_-^Y \circ \varphi \circ (\xi_-^X)^{-1} : \partial\widetilde{X}\backslash\{\zeta\} \to \partial\widetilde{Y}\backslash\{\partial\tilde{F}(\zeta)\}.$$

But the map above is simply $\partial\tilde{F}$, as can be seen by the standard construction of this boundary homeomorphism using the Morse-Mostow lemma [3].

We conclude (after performing this construction as well for some $\zeta' \neq \zeta$) that $\partial\tilde{F}$ is a $C^{1+\alpha}$ diffeomorphism. In particular it preserves the Lebesgue measure class on these boundaries. Since the Anosov splitting of g_Y^t is C^1, by work of Hamenstädt[25, Corollary 4.6] we conclude that there is some $c > 0$ such that g_X^t is C^1-conjugate to g_Y^{ct}; the assumption that $\vec{\lambda}(g_Y^t, \mu_Y) = \vec{\lambda}(g_X^t)$ implies that $c = 1$. We can thus apply a corollary of the minimal entropy rigidity theorem [7, Theorem 1.3] to obtain that X and Y are homothetic. By again appealing to the equality $\vec{\lambda}(g_Y^t, \mu_Y) = \vec{\lambda}(g_X^t)$, this implies that X and Y are isometric. $\square$

7.2 Starting the proof

We proceed now with the proof of Theorem 2.3. We assume that $f^t \in \mathcal{V}_X$ and that there is a constant $\xi > 0$ such that $\vec{\lambda}^u(f^t, \mu_f) = \xi\vec{\lambda}^u(g_X^t)$. We will assume that X has sectional curvatures normalized to $K \equiv -1$ when X is real hyperbolic and $-4 \leq K \leq -1$ when X

has nonconstant negative curvature. Since the hypotheses and conclusion of the theorem are not affected by replacing f^t with a rescaling f^{ct}, $c > 0$, we may assume without loss of generality that $\xi = 1$.

We first observe that the equality $\vec{\lambda}^u(f^t, \mu_f) = \vec{\lambda}^u(g_X^t)$ implies that f^t satisfies the hypotheses of Propositions 5.1 and 5.5. We conclude that $Df^t|H^{u,f}$ and $Df^t|V^{u,f}$ are both uniformly quasiconformal. We next apply the synchronization of Proposition 6.4, using the fact that $Df^t|H^{u,f}$ is uniformly quasiconformal. We obtain a Hölder continuous flow $\hat{f}^t$ that is a time change of f^t, such that this flow is C^2 on $W^{cu,f}$ leaves, and we obtain an inner product $\langle\ ,\ \rangle$ on $\hat{H}^{u,f}$ such that for all $x \in M$ and $v, w \in \hat{H}_x^{u,f}$,

$$\langle D\hat{f}_x^t(v), D\hat{f}_x^t(v)\rangle_{\hat{f}^t x} = e^t\langle v, w\rangle_x.$$

Furthermore the holonomies $\hat{L}^u$ of $D\hat{f}^t|\hat{H}^{u,f}$ are isometric with respect to this inner product.

Fix a continuous inner product on $\hat{V}^{u,f}$ with norm $|\cdot|$. Since the splitting $\hat{E}^{u,f} = \hat{H}^{u,f} \oplus \hat{V}^{u,f}$ is dominated and obtained as a small perturbation of the corresponding splitting for g_X^t, there are constants $c > 0$ and $\gamma > 1$ such that for all $x \in M$,

$$\sigma_1(D\hat{f}_x^t|\hat{V}_x^{u,f}) \geq ce^{\gamma t}.$$

By Remark 3.1, we may choose a new continuous inner product on $\hat{V}^{u,f}$ with norm $\|\cdot\|$ such that the above inequality holds with the same exponent γ and with $c = 1$.

We extend the inner product $\langle\ ,\ \rangle$ defined above on $\hat{H}^{u,f}$ to an inner product on $\hat{E}^{u,f}$ by declaring $\hat{H}^{u,f}$ and $\hat{V}^{u,f}$ to be orthogonal, and using the inner product with norm $\|\cdot\|$ described above on $\hat{V}^{u,f}$. We then have, for all $x \in M$ and $t \geq 0$,

$$\sigma_1(D\hat{f}_x^t(v)|\hat{E}_x^{u,f}) \geq e^t.$$

We set $M := T^1X$ and let $\varphi : M \to M$ be the orbit equivalence from $g^t := g_X^t$ to f^t that is given by structural stability. We will use φ to create a conjugacy $\hat{\varphi} : M \to M$ satisfying $\hat{\varphi} \circ g^t = \hat{f}^t \circ \hat{\varphi}$, such that $\hat{\varphi}$ is $C^{1+\alpha}$ on $W^{cu,f}$ leaves. Since $\hat{\varphi}$ is an orbit equivalence from g^t to f^t, this will complete the proof.

To simplify notation, from now on we will write $f^t, H^{u,f}$, etc. for $\hat{f}^t, \hat{H}^{u,f}$, etc., i.e., we will proceed as if f^t already satisfies the conclusions of the synchronization. This means that Remark 6.8 will be in effect for the rest of the paper.

As shown above, in our inner product on $E^{u,f}$ we have $\sigma_1(Df^t|E^{u,f}) \geq e^t$ for $t \geq 0$. We have the analogous inequality for g^t, using the inner product on $W^{u,g}(x)$ coming from a left-invariant inner product on the Carnot group G described in Section 2.1. We let $\rho_{x,g}$ and $\rho_{x,f}$ denote the Hamenstädt metrics for g^t and f^t respectively, taking the constant $a = 1$ in the definition. From now on, we will write m for the invariant volume for g^t (which is the measure of maximal entropy) and $\mu = \mu_f$ for the horizontal measure for f^t (which is also the measure of maximal entropy for f^t).

Lemma 7.2. *We have $Q(f) \geq Q_X$.*

Proof. For each $x \in M$, the metric measure space $(W^{u,g}(x), \rho_{x,g}, m_x)$ is isometric to a Carnot group $G = G_X$ equipped with its left-invariant Carnot-Caratheodory metric and Lebesgue measure. This space is Ahlfors kQ_X-regular. Furthermore this metric measure space admits a positive kQ_X-modulus family of curves [50].

By Proposition 6.9, $\varphi_x : (W^{u,g}(x), \rho_{x,g}, m_x) \to (W^{u,f}(\varphi(x)), \rho_{\varphi(x),f}, \mu_{\varphi(x)})$ is a quasisymmetric homeomorphism between an Ahlfors kQ_X-regular space and an Ahlfors $kQ(f)$-regular space. Since, by the remarks of the previous paragraph, $(W^{u,g}(x), \rho_{x,g}, m_x)$ admits a positive kQ_X-modulus family of curves, by Tyson's theorem [59] we conclude that $kQ_X \leq kQ(f)$, i.e., $Q(f) \geq Q_X$. ☐

Given the horizontal dimension inequality of Lemma 7.2 and the fact that $\vec{\lambda}^u(f^t, \mu) = \vec{\lambda}^u(g^t_X)$, we conclude from Lemma 7.1 that we actually have $Q(f) = Q_X$. This lemma also implies that μ is the SRB measure for f^t; we will not need to use this until Section 7.5.

Remark 7.3. To provide a positive answer to Question 1.6 using our methods, one would have to replace the horizontal measure μ_f in the hypotheses of Theorem 2.3 with the SRB measure m_f. Our methods require the equality $Q(f) = Q_X$; using m_f it is possible to prove the lower bound $Q(f) \geq Q_X$ as in Lemma 7.2, but it is unclear how to obtain the upper bound $Q(f) \leq Q_X$.

7.3 From orbit equivalence to conjugacy

In this section we will assume that $Q(f) = Q_X$ and that $Df^t|H^{u,f}$ is uniformly quasiconformal. We will assume that the synchronization of f^t has been carried out as above. However, we will not make any assumptions on the Lyapunov exponents of f^t on $V^{u,f}$ for this section or the next section. Set $Q := Q_X$.

Let ∇ be the affine $\bar{f}^t$-invariant connection on $\mathcal{Q}^{u,f}(x)$, $x \in M$, from Proposition 3.9. Recall that we write $\pi : W^{u,f}(x) \rightarrow \mathcal{Q}^{u,f}(x)$ for the projection map, $\bar{f}^t$ for the induced action of f^t on $\mathcal{Q}^{u,f}(x)$, and $\bar{L}^u$ for the holonomies of $D\bar{f}^t$, which coincide with the parallel transport maps of the connection ∇. The L^u-invariant inner product $\langle \, , \, \rangle$ on $H^{u,f}$ descends by projection to an $\bar{L}^u$-invariant inner product on $\mathcal{Q}^{u,f}$, which we will also denote by $\langle \, , \, \rangle$.

We let d_x be the Riemannian metric on $W^{u,f}(x)$ coming from the continuous inner product chosen above on $E^{u,f}$, and $\bar{d}_x$ denote the Riemannian metric on $\mathcal{Q}^{u,f}(x)$ given by the projection of this inner product onto $T\mathcal{Q}^u$. Of course in our notation $d_x = d_y$ and $\bar{d}_x = \bar{d}_y$ for $y \in W^{u,f}(x)$. Then $\pi : W^{u,f}(x) \rightarrow \mathcal{Q}^{u,f}(x)$ is a Riemannian submersion; in particular we have for $y, z \in W^{u,f}(x)$,

$$\bar{d}_x(\pi(y), \pi(z)) \leq d_x(y, z),$$

and

$$\bar{d}_x(\pi(y), \pi(z)) \geq \inf_{\substack{\pi(y)=\pi(p) \\ \pi(z)=\pi(q)}} d_x(p, q).$$

Lastly, since $D\bar{f}^t$ is conformal with respect to our inner products on $T\mathcal{Q}^u$ with expansion factor e^t, we have for each $y, z \in \mathcal{Q}^{u,f}(x)$,

$$\bar{d}_{\bar{f}^t x}(\bar{f}^t(y), \bar{f}^t(z)) = e^t \bar{d}_x(y, z).$$

We show below that π is actually globally Lipschitz for the Hamenstädt metrics $\rho_{x,f}$ as well.

Lemma 7.4. *There is a constant $C \geq 1$ independent of $x \in M$ such that, for each $y, z \in W^{u,f}(x)$, we have*

$$\bar{d}_x(\pi(y), \pi(z)) \leq C \rho_{x,f}(y, z).$$

Proof. Let $x \in M$ be given. By the uniform comparability of the Hamenstädt metric $\rho_{x,f}$ to the Riemannian metric d_x, together with the fact that π is a Riemannian submersion, we conclude that there is a constant $c > 0$ independent of x such that for any $y, z \in W^{u,f}(x)$,

$$\bar{d}_x(\pi(y), \pi(z)) = 1 \Rightarrow \rho_{x,f}(y, z) \geq c.$$

Now let $y, z \in W^{u,f}(x)$ be arbitrary and set $r = \bar{d}_x(\pi(y), \pi(z))$, $s = -\log r$. Then

$$\bar{d}_{f^t x}(\pi(f^s y), \pi(f^s z)) = e^s \bar{d}_x(\pi(y), \pi(z)) = 1.$$

Thus we have

$$\rho_{f^s x, f}(f^s y, f^s z) \geq c,$$

and so

$$\begin{aligned} \rho_{x,f}(y, z) &= e^{-s} \rho_{f^s x, f}(f^s y, f^s z) \\ &\geq c e^{-s} \\ &= c \bar{d}_x(\pi(y), \pi(z)). \end{aligned}$$

$\square$

For a given $x \in M$ we let $T_x : \mathcal{Q}^{u,f}(x) \to \mathbb{R}^k$ be the affine chart given by the connection ∇. By ∇-invariance, this maps $\langle \ , \ \rangle$ to the standard Euclidean inner product on $\mathbb{R}^k$. Furthermore, the transition maps

$$T_x \circ T_y^{-1} : \mathbb{R}^k \to \mathbb{R}^k,$$

are isometries of $\mathbb{R}^k$.

Similarly, recalling that G denotes our 2-step Carnot group on which the unstable manifolds of g^t are modeled, for each $x \in M$ we have charts $S_x : W^{u,g}(x) \to G$ that map G equipped with its Carnot-Caratheodory metric ρ_G isometrically onto $(W^{u,g}(x), \rho_{x,g})$. The transition maps $S_y \circ S_x^{-1}$ are left translations of G, which are isometries for ρ_G. We write m_G for the Lebesgue measure on G, which is the kQ-dimensional Hausdorff measure for ρ_G.

We have a 1-parameter family of expanding automorphisms A^t of G such that, writing $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ for the Lie algebra of G split into horizontal and vertical subspaces, DA^t expands $\mathfrak{h}$ by e^t and $\mathfrak{v}$ by e^{2t}. Likewise we have the standard 1-parameter family of expanding linear maps B^t on $\mathbb{R}^k$ given by $B^t(x) = e^t x$. We say that a homomorphism $\psi : G \to \mathbb{R}^k$ is *homogeneous* if $\psi \circ A^t = B^t \circ \psi$, i.e., ψ commutes with these dilations.

We say that F is *Pansu differentiable* at $x \in G$ if there is a homogeneous homomorphism $DF_x : G \to \mathbb{R}^k$ such that for all $y \in G$ we have

$$DF_x(y) = \lim_{t \to \infty} B^{-t}(f(x \cdot A^t y) - f(x)).$$

We say that DF_x is the Pansu derivative of F at x. See [51] for basic properties of the Pansu derivative and its uses. The Pansu derivative may be defined more generally for continuous maps between any pair of Carnot groups, but we will only need the formulation for G and $\mathbb{R}^k$ described above.

We begin with the orbit equivalence φ from g^t to f^t coming from structural stability. We will use Proposition 6.10 below to produce a conjugacy between g^t and f^{ct} for some constant $c > 0$. Since both f^t and g^t have the same topological entropy kQ, we then have $c = 1$, and so g^t and f^t are conjugate.

Proposition 7.5. *There is a Hölder continuous homeomorphism $\hat{\varphi} : M \to M$ such that $\hat{\varphi} \circ g^t = f^t \circ \hat{\varphi}$. Furthermore there is a constant $C \geq 1$ independent of $x \in M$ such that $\hat{\varphi} : (W^{u,g}(x), \rho_{x,g}) \to (W^{u,f}(\varphi(x)), \rho_{\varphi(x),f})$ is C-Lipschitz.*

Proof. By the remarks above, to produce the conjugacy $\hat{\varphi}$ it suffices to show that assertion (2) of Proposition 6.10 holds.

Suppose that this is not the case. Then there is an $x \in M$ such that for m_x-a.e. $y \in W^{u,g}(x)$ we have $J_x(y) = 0$. From Proposition 6.9 we then have a quasisymmetric homeomorphism of Ahlfors kQ-regular metric spaces,

$$\psi := \varphi_x \circ S_x^{-1} : (G, \rho_G, m_G) \to (W^{u,f}(\varphi(x)), \rho_{\varphi(x),f}, \mu_{\varphi(x)}).$$

For $y \in G$ we define

$$\mathcal{L}(y) = \limsup_{r \to 0} \frac{\sup\{\rho_{x,f}(\psi(y), \psi(z)) : \rho_G(y,z) = r\}}{r},$$

to be the Lipschitz constant of ψ at y. For $y \in G$ we define

$$\mathcal{J}(y) = \lim_{r \to 0} \frac{\mu_{\varphi(x)}(\psi(B_{\rho_G}(y,r)))}{m_G(B_{\rho_G}(y,r))}.$$

As per the discussion in [4, Section 4], by the Radon-Nikodym theorem this limit exists and is finite for m_G-a.e. $y \in G$. Since G is a Carnot group, since ψ is quasisymmetric, and since both G and $W^{u,f}(\varphi(x))$ are Ahlfors kQ-regular, we may apply [4, Theorem 5.2] to conclude that $\mathcal{L}(y)^{kQ} \leq \mathcal{J}(y)$ for m_G-a.e. $y \in G$.

Recall that we have $d(\varphi_x^{-1})_* \mu_{\varphi(x)} = J_x dm_x + d\kappa_x$, where κ_x is mutually singular with respect to m_x. Hence we can rewrite the limit defining $\mathcal{J}$ as

$$\mathcal{J}(y) = \lim_{r\to 0} \frac{1}{m_G(B_{\rho_G}(y,r))} \int_{B_{\rho_G}(y,r)} J_x \circ S_x^{-1}\, dm_G + \lim_{r\to 0} \frac{(S_x)_*\kappa_x(B_{\rho_G}(y,r))}{m_G(B_{\rho_G}(y,r))}$$

Since G is Ahlfors kQ-regular and m_G is a σ-finite Borel measure, the Lebesgue differentiation theorem holds on G [33]. Hence, for m_G-a.e. $y \in G$, the first limit is $J_x(S_x^{-1}(y))$, and the second limit is 0. We conclude that $\mathcal{J}(y) = J_x(S_x^{-1}(y))$ for m_G-a.e. $y \in G$.

By hypothesis $J_x(S_x^{-1}(y)) = 0$ for m_G-a.e. $y \in G$. Hence by the inequality $\mathcal{L}(y)^{kQ} \leq \mathcal{J}(y)$ we conclude that $\mathcal{L}(y) = 0$ for m_G-a.e. $y \in G$. By Lemma 7.4, the projection

$$T_x \circ \pi : (W^{u,f}(x), \rho_{x,f}) \to (\mathbb{R}^k, d_{\mathbb{R}^k}),$$

is C-Lipschitz for some constant $C \geq 1$. Set

$$\bar{\psi} = T_x \circ \pi \circ \psi : (G, \rho_G) \to (\mathbb{R}^k, d_{\mathbb{R}^k}),$$

and observe that

$$\begin{aligned}\bar{\mathcal{L}}(y) &= \limsup_{r\to 0} \frac{\sup\{d_{\mathbb{R}^k}(\bar{\psi}(y), \bar{\psi}(z)) : \rho_G(y,z) = r\}}{r} \\ &\leq C\mathcal{L}(y) = 0,\end{aligned}$$

for m_G-a.e. $y \in G$.

By the Pansu-Rademacher-Stepanov theorem for Carnot groups [61], this implies that $\bar{\psi}$ is Pansu differentiable m_G-a.e., with Pansu derivative 0 at m_G-a.e. point. This implies that $\bar{\psi} : G \to \mathbb{R}^k$ is a constant function, i.e., its image is a single point. But $\bar{\psi}$ is clearly surjective, which gives a contradiction. Hence assertion (2) of Proposition 6.10 holds.

We thus have a Hölder continuous conjugacy $\hat{\varphi} : M \to M$ satisfying $f^t \circ \hat{\varphi} = \hat{\varphi} \circ g^t$. Since m is the measure of maximal entropy for g^t and μ is the measure of maximal entropy for f^t, we have $\hat{\varphi}_* m = \mu$. Since these are measures of maximal entropy, this implies that $\hat{\varphi}_* m_x = \mu_{\hat{\varphi}(x)}$, or in other words, we have $J_x \equiv 1$.

By the inequality $\mathcal{L}(y)^{kQ} \leq \mathcal{J}(y) \equiv 1$ of [4, Theorem 5.2], this implies that $\hat{\varphi}$ is Lipschitz with respect to the Hamenstädt metrics. □

7.4 The horizontal derivative

Once again for simplicity of notation, we let φ denote the homeomorphism $\hat{\varphi}$ of the previous section, so that $\varphi \circ g^t = f^t \circ \varphi$. In this section we will construct a horizontal derivative map $\Phi : H^{u,g} \to H^{u,f}$ for φ. We first give a standard lemma that characterizes rectifiable curves for the Hamenstädt metric on an unstable leaf (see [28], [16]). Below we let I be a closed subinterval of $\mathbb{R}$.

Lemma 7.6. *Let f^t be an Anosov flow with a u-splitting $E^u = H^u \oplus V^u$ such that $\|Df^t(v)\| = e^t\|v\|$ for all $v \in H^u$ and $t \in \mathbb{R}$. Then a continuous curve $\gamma : I \to W^u(x)$ is rectifiable with respect to the Hamenstädt metric $\rho_{x,f}$ if and only if γ is Lipschitz on I and for a.e. $s \in I$ we have $\gamma'(s) \in H^u_{\gamma(s)}$.*

Proof. We first suppose that γ is Lipschitz on I and that for a.e. $s \in I$ we have $\gamma'(s) \in H^u_{\gamma(s)}$. By reparametrizing γ with respect to arc length (which does not affect $\rho_{x,f}$-rectifiability), we may assume that $\|\gamma'(s)\| = 1$ for all $s \in I$. Pass to Euclidean coordinates on $W^u_{loc}(x)$, for which we have the affine structure on Euclidean space. Let $a < b \in I$. We can then write $\gamma(b) = \gamma(a) + \int_a^b \gamma'(s)\, ds$.

Since $\gamma'(s) \in H^u_{\gamma(s)}$, we have $\|Df^t(\gamma'(s))\| = e^t\|\gamma'(s)\|$ for all $t \in \mathbb{R}$. This implies that, for $t \geq 0$, we have

$$\begin{aligned}\|f^t(\gamma(b)) - f^t(\gamma(a))\| &\leq \int_a^b \|Df^t(\gamma'(s))\|\, ds \\ &= e^t \int_a^b \|\gamma'(s)\|\, ds \\ &= e^t(b-a),\end{aligned}$$

since γ is parametrized by arc length. Thus, for a fixed T for which $\|f^T(\gamma(b)) - f^T(\gamma(a))\| = 1$, we must have $e^T(b-a) \geq 1$ which implies that $T \geq -\log(b-a)$. This implies that

$$\beta(\gamma(a), \gamma(b)) \geq -\log(b-a),$$

which implies that

$$\rho(\gamma(a), \gamma(b)) = e^{-\beta(\gamma(a),\gamma(b))} \leq b - a.$$

In other words, the restriction of $\rho_{x,f}$ to γ in the parametrization of γ with respect to arc length is 1-Lipschitz. It follows immediately that γ is $\rho_{x,f}$-rectifiable.

Now assume that γ is $\rho_{x,f}$-rectifiable. By uniform comparability of the Hamenstädt metrics and Riemannian metrics at unit scale, there is a constant $c > 0$ such that, for all $x \in M$ and $y, z \in W^{u,f}(x)$,

$$d_x(y,z) = 1 \Rightarrow \rho_{x,f}(y,z) \geq c.$$

By applying f^{-t} to each side and using the estimate $\|Df^{-t}|E^u\| \leq e^{-t}$ for $t \geq 0$, this implies that

$$\begin{aligned} d_{f^{-t}x}(f^{-t}y, f^{-t}z) &\leq e^{-t} d_x(y,z) \\ &\leq c^{-1} e^{-t} \rho_{x,f}(y,z) \\ &= c^{-1} \rho_{f^{-t}x,f}(f^{-t}y, f^{-t}z). \end{aligned}$$

From this we see that $d_x(y,z) \leq c^{-1}\rho_{x,f}(y,z)$ for $x \in M$, $y, z \in W^{u,f}_{loc}(x)$. Thus rectifiability of γ with respect to $\rho_{x,f}$ implies that γ is also rectifiable with respect to d_x. Hence γ is Lipschitz, and so γ is differentiable at a.e. $s \in I$. By reparametrizing, we may assume that γ is parametrized with respect to arc length for the metric d_x. Let ℓ_d denote lengths of curves measured in the Riemannian metric d_x, and let ℓ_ρ denote lengths of curves measured in the Hamenstädt metric $\rho_{x,f}$. Let κ denote the Lipschitz constant of the curve γ in the metric $\rho_{x,f}$.

Assume to a contradiction that there is a positive measure subset $K \subseteq I$ on which $\gamma'(s) \notin H^u$. By passing to a compact subset of K, we may assume that there is a $\theta > 0$ such that $\measuredangle(\gamma'(s), H^u_{\gamma(s)}) \geq \theta$ for all $s \in K$. By our domination estimates in the norm $\|\cdot\|$, there is a constant $\chi > 1$ and a constant $\delta > 0$ such that for every $s \in K$ and $t \geq 0$,

$$\|Df^t(\gamma'(s))\| \geq \delta e^{\chi t}\|\gamma'(s)\|.$$

Let $s_0 \in K$ be a Lebesgue density point in I for the set K. Let B_r denote the interval of radius r centered at s_0. For r small enough we have

$$|B_r \cap K| \geq \frac{|B_r|}{2} = r.$$

Let $\gamma_r : B_r \to W^{u,f}(x)$ denote the restriction of γ to B_r. There is a constant $c > 0$ such that, for $t \geq 0$ such that $\ell_\rho(f^t \circ \gamma_r) \leq 1$ – so that we can still compare d_x and $\rho_{x,f}$ locally – we have

$$\begin{aligned}\ell_d(f^t \circ \gamma_r) &\leq c^{-1}\ell_\rho(f^t \circ \gamma_r)\\ &= c^{-1}e^t\ell_\rho(\gamma_r)\\ &\leq \kappa c^{-1}e^t r\end{aligned}$$

On the other hand we have, recalling that γ is parametrized with respect to arc length,

$$\begin{aligned}\ell_d(f^t \circ \gamma_r) &\geq e^t\ell_d(\gamma_r|_{B_r\setminus K}) + \delta e^{\chi t}\ell_d(\gamma_r|_{B_r\cap K})\\ &= ce^t|B_r\setminus K| + \delta e^{\chi t}|B_r \cap K|\\ &\geq r\delta e^{\chi t}.\end{aligned}$$

Choose T large enough that $\delta e^{\chi T} > \kappa c^{-1}e^T$. Then choose $r = r(T) > 0$ small enough that $\ell_\rho(f^T \circ \gamma_r) \leq 1$. We conclude from the above that we have

$$\kappa c^{-1} r e^T \geq \ell_d(f^t \circ \gamma_r) \geq r\delta e^{\chi T},$$

which, after canceling r from each side, gives the inequality $\kappa c^{-1} e^T \geq \delta e^{\chi T}$, which contradicts our choice of T. We conclude that $\gamma'(s) \in H^u_{\gamma(s)}$ for a.e. $s \in I$. □

Our next proposition gives us, for each $x \in M$, a horizontal derivative of the map $\varphi : W^{u,g}(x) \to W^{u,f}(\varphi(x))$.

Proposition 7.7. *There is a Hölder continuous homeomorphism $\Phi : H^{u,g} \to H^{u,f}$, linear on fibers, such that $\Phi \circ Dg^t = Df^t \circ \Phi$ for all $t \in \mathbb{R}$, and such that if γ is a C^1 curve tangent to $H^{u,g}$ then $\varphi \circ \gamma$ is a C^1 curve tangent to $H^{u,f}$ satisfying $(\varphi \circ \gamma)' = \Phi \circ \gamma'$.*

Proof. Fix $x \in M$. Recall that $S_x : W^{u,g}(x) \to G$ is the chart for $W^{u,g}(x)$ and the transition maps $S_x \circ S_y^{-1} : G \to G$ are isometries of G. Consider the composition of maps

$$\Psi : G \xrightarrow{S_x^{-1}} W^{u,g}(x) \xrightarrow{\varphi} W^{u,f}(x) \xrightarrow{\pi} \mathcal{Q}^{u,f}(x) \xrightarrow{T_{\pi(x)}} \mathbb{R}^k.$$

By Proposition 7.5, we have that φ is Lipschitz. By Lemma 7.4, π is also Lipschitz. Finally, $T_{\pi(x)}$ is an isometry to the Euclidean metric on $\mathbb{R}^k$. We conclude that Ψ is Lipschitz with respect to the Carnot-Caratheodory metric ρ_G on G and the Euclidean metric $d_{\mathbb{R}^k}$ on $\mathbb{R}^k$.

By the Pansu-Rademacher-Stepanov theorem for Lipschitz maps between Carnot groups [51], the map Ψ is Pansu differentiable m_G-a.e. on G. The Pansu derivative $D\Psi_y : G \to \mathbb{R}^k$ is a homogeneous homomorphism; since $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{v}$ in the Lie algebra $\mathfrak{g}$ of G, and since $\mathbb{R}^k$ is abelian, the vertical subgroup V of G satisfies $V \subseteq \ker D\Psi_y$. Hence $D\Psi_y$ descends to a homogeneous homomorphism $D\Psi_y : G/V \to \mathbb{R}^k$; since both G/V and $\mathbb{R}^k$ are abelian, $D\Psi_y$ actually gives a linear map between these two groups.

For m_x-a.e. $y \in W^{u,g}(x)$, we then define $\Phi_y : H^{u,g}_y \to H^{u,f}_{\varphi(y)}$ by the composition

$$\Phi_y : H^{u,g}_y \xrightarrow{DS_x} G/V \xrightarrow{D\Psi_y} \mathbb{R}^k \xrightarrow{(DT^{-1}_{\pi(x)})_{\Psi(\pi x)}} TQ^{u,f} \xrightarrow{(D\pi)^{-1}_y} H^{u,f}_{\varphi(y)}.$$

A straightforward exercise gives that Φ_y is independent of the chosen basepoint x above, for $y \in W^u(x)$. We can then extend Φ_y to be defined m-a.e. on all of M. Since $D\Psi_y$ is homogeneous and both G/V and $\mathbb{R}^k$ have the same scaling factor e^t, we also have the equality

$$\Phi_{g^t y} \circ Dg^t_y = Df^t_{\varphi(y)} \circ \Phi_y,$$

for all $t \in \mathbb{R}$, for m-a.e. $y \in M$.

We next show that Φ_y is invertible for m-a.e. $y \in M$. By the above measurable semiconjugacy equation, the set

$$J := \{y \in M : \Phi_y \text{ is invertible}\},$$

is g^t-invariant and thus either $m(J) = 0$ or $m(J) = 1$. Hence it suffices to show that we cannot have $m(J) = 0$.

Suppose otherwise, so that for m-a.e. $y \in M$ we have that Φ_y is not invertible at y. Then, again by the above equation, $\ker \Phi \subseteq H^{u,g}$ is a measurable Dg^t-invariant subbundle defined m-a.e. By Proposition 5.1, $\ker \Phi$ cannot be a nontrivial subbundle of $H^{u,g}$, so that we must have $\ker \Phi_y = H^{u,g}_y$ or $\ker \Phi_y = \{0\}$ for m-a.e. $y \in M$. In the first case we conclude that $D\Psi_y = 0$ for m-a.e. $y \in M$. This implies that $\Psi : G \to \mathbb{R}^k$ has Pansu derivative 0 at m_G-a.e. $y \in G$, which implies that Ψ is a constant function. This contradicts the fact that Ψ is surjective. Thus the second case must hold, so that we have $\ker \Phi_y = \{0\}$ for m-a.e. $y \in M$, i.e., Φ_y is invertible for m-a.e $y \in G$.

Since g^t and f^t are C^1 close, the Hölder continuous subbundles $H^{u,g}$ and $H^{u,f}$ of TM are uniformly close. Hence there is a Hölder continuous homeomorphism $\mathcal{I}_H : H^{u,g} \to H^{u,f}$ that is an isomorphism on fibers. From $Df^t|H^{u,f}$ we define a new Hölder cocycle A^t on

$H^{u,g}$ over g^t by $A^t_x = \mathcal{I}_H^{-1} \circ Df^t_{\varphi(x)} \circ \mathcal{I}_H$. Then $\hat{\Phi} = \mathcal{I}_H^{-1} \circ \Phi$ satisfies $\hat{\Phi} \circ Dg^t|H^{u,g} = A^t \circ \hat{\Phi}$ m-a.e. for all $t \in \mathbb{R}$. Thus, by Proposition 3.4 the map $\hat{\Phi}$ coincides m-a.e. with a Hölder continuous conjugacy between $Dg^t|H^{u,g}$ and A^t. Unwrapping the identifications, we obtain that Φ agrees m-a.e. with a Hölder continuous conjugacy – for which we use the same notation – $\Phi : H^{u,g} \to H^{u,f}$ such that $\Phi \circ Dg^t = Df^t \circ \Phi$ everywhere for all $t \in \mathbb{R}$.

As a consequence, since Φ_y is Hölder continuous in y we obtain that $D\Psi_y$ is Hölder continuous in y as well. In particular we conclude that $\Psi : G \to \mathbb{R}^k$ is continuously Pansu differentiable with Pansu derivative $D\Psi$. This implies that for any C^1 curve γ in G that is tangent to the horizontal distribution $H \subseteq TG$, we have that $\Psi \circ \gamma$ is a C^1 curve in $\mathbb{R}^k$ with $(\Psi \circ \gamma)' = D\Psi \circ \gamma'$.

Set $\bar{\Phi} = D\pi \circ \Phi$. The assertions of the previous paragraph show that if $\gamma : I \to W^{u,g}(x)$ is a C^1 curve tangent to $H^{u,g}$, then $\bar{\varphi} \circ \gamma$ is a C^1 curve in $\mathcal{Q}^{u,f}(x)$ with $(\bar{\varphi} \circ \gamma)' = \bar{\Phi} \circ \gamma'$. Since $\varphi : (W^{u,g}(x), \rho_{x,g}) \to (W^{u,f}(\varphi(x)), \rho_{\varphi(x),f})$ is Lipschitz, the curve $\sigma := \varphi \circ \gamma$ is rectifiable with respect to $\rho_{x,f}$. By Lemma 7.6, σ is a Lipschitz curve in the Riemannian metric $d_{\varphi(x)}$ on $W^{u,f}(\varphi(x))$ with $\sigma'(s) \in H^{u,f}_{\sigma(s)}$ for a.e. s. Since $\pi \circ \sigma = \bar{\varphi} \circ \gamma$, we have

$$D\pi \circ \sigma' = (\bar{\varphi} \circ \gamma)' = \bar{\Phi} \circ \gamma',$$

with this equality holding a.e. on I. Since $\sigma' \in H^{u,f}$ a.e. on I, we can invert $D\pi$ in this equation to obtain

$$\sigma' = \Phi \circ \gamma',$$

a.e. on I. This implies that $\sigma = \varphi \circ \gamma$ is a C^1 curve tangent to $H^{u,f}$ with $(\varphi \circ \gamma) = \Phi \circ \gamma'$. □

Our next lemma shows that φ maps the vertical $W^{uu,g}$-foliation to the vertical $W^{uu,f}$-foliation.

Proposition 7.8. *For each $x \in M$ we have $\varphi(W^{uu,g}(x)) = W^{uu,f}(\varphi(x))$.*

Proof. From Proposition 3.4 one obtains that the map Φ is equivariant with respect to

$L^{u,g}$ holonomy on $H^{u,g}$ and $L^{u,f}$ holonomy on $H^{u,f}$. We claim that we may assume that $\Phi : H^{u,g} \to H^{u,f}$ is an isometry on fibers. If this is not the case, then we replace the norm $\| \cdot \|_f$ on $H^{u,f}$ with the norm $| \cdot |$ given by $|v| = \|\Phi^{-1}(v)\|_g$, which in turn is given by pulling back by Φ and measuring the norm in $H^{u,g}$. By equivariance of Φ with respect to $L^{u,*}$ holonomies, the corresponding inner product $(,)$ on $H^{u,f}$ is preserved by $L^{u,f}$ holonomies, and we have $|Df^t(v)| = e^t|v|$ for all $v \in H^{u,f}$.

We thus assume that $\Phi : H^{u,g} \to H^{u,f}$ is an isometry on fibers. Let $x \in M$ and $y \in W^{uu,g}(x)$ be given. It suffices to show that $\varphi(y) \in W^{uu,f}(\varphi(x))$. We view $W^{u,g}(x)$ as a homogeneous copy of G with x at 0. Since $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{v}$, there are two orthogonal left-invariant vector fields X_1 and X_2 on G, which are tangent to $\mathfrak{h}$ at 0, such that $Y := [X_1, X_2]$ is a left-invariant vector field G for which $\exp(Y(0)) = y$, i.e., y is the image of $Y(0)$ by the exponential map $\exp : \mathfrak{g} \to G$.

Let P_g be the plane in $\mathbb{R}^k$ spanned by the projection of $X_1(0)$ and $X_2(0)$ to $\mathbb{R}^k$. Since $[X_1, X_2] = Y$, there is a clockwise oriented rectangle in P_g with sides $\bar{\gamma}_i$, $1 \leq i \leq 4$ – each of which are tangent to either the projection of X_1 or the projection of X_2 to $\mathbb{R}^k$ – such that the curve $\bar{\gamma}$ lifts to a smooth curve γ tangent to $H^{u,g}$ which has initial point x and endpoint y. We let γ_i, $1 \leq i \leq 4$, denote the lifts of the sides of the rectangle.

Thus $\sigma := \varphi \circ \gamma$ is a C^1 curve tangent to $H^{u,f}$ which starts at $\varphi(x)$ and ends at $\varphi(y)$. Set $\sigma_i := \varphi \circ \gamma_i$. Since the tangent vectors to the curve σ are given by $\Phi \circ \gamma'$, and Φ is an isometry on fibers of $H^{u,g}$ and $H^{u,g}$, we conclude that $\ell(\sigma_i) = \ell(\gamma_i)$ for $1 \leq i \leq 4$, where ℓ denotes the length of the curve. Since the projections $\pi_* : W^{u,*}(x) \to \mathcal{Q}^{u,*}$ on each side are isometries on the $H^{u,*}$ bundle (for $* \in \{f, g\}$), we have that $\bar{\sigma}_i := \pi \circ \sigma_i$ satisfies $\ell(\bar{\sigma}_i) = \ell(\bar{\gamma}_i)$.

Let P_f be the plane in $\mathbb{R}_k$ spanned by the vector fields $DT_{\varphi(x)} \circ D\pi_f \circ \Phi \circ X_i$, for $i = 1, 2$, where we recall that $T_{\varphi(x)}$ denotes the affine coordinates on $\mathcal{Q}^u(x)$ given by the f^t-invariant connection ∇. Note that these vector fields are ∇-parallel, and so in the $\mathbb{R}^k$ coordinates they are constant and tangent to a foliation of $\mathbb{R}^k$ by parallel lines. Both Φ and the projections $D\pi_f$ and $D\pi_g$ also preserve orientation and angles, so $\bar{\sigma}$ gives a clockwise oriented curve tangent to P_f such that its sides $\bar{\sigma}_1$ and $\bar{\sigma}_3$ are parallel lines of the same length in P_f and the same is true for $\bar{\sigma}_2$ and $\bar{\sigma}_4$. Furthermore, for $i = 1, 2, 3$ the lines $\bar{\sigma}_i$ and $\bar{\sigma}_{i+1}$ meet at a right angle with the clockwise orientation. This implies that $\bar{\sigma}$ is simply a rectangle in P_f. In particular the endpoint of $\bar{\sigma}_4$ coincides with the initial point of $\bar{\sigma}_1$. This implies that $\varphi(y) \in W^{uu,f}(\varphi(x))$ which completes the proof. □

7.5 Vertical differentiability

At this point we split into two cases, depending on whether $\dim V^{u,g} = 1$ or $\dim V^{u,g} > 1$, i.e., depending on whether X is complex hyperbolic or quaternionic/Cayley hyperbolic. As in the proof of Proposition 7.7, since g^t and f^t are C^1 close, the Hölder continuous subbundles $V^{u,g}$ and $V^{u,f}$ of TM are uniformly close. Thus, as above, there is a Hölder continuous homeomorphism $\mathcal{I}_V : V^{u,g} \to V^{u,f}$ that is a linear isomorphism on fibers.

We first consider the case where $\dim V^{u,g} = \dim V^{u,f} = 1$. We recall that, under our hypotheses, μ is the SRB measure for f^t, as shown in Section 7.2. This implies that μ_x is the Riemannian volume on $W^{u,f}(x)$.

Lemma 7.9. *Suppose that* $\dim V^{u,g} = \dim V^{u,f} = 1$*. Then we have that* φ *is* $C^{1+\alpha}$ *on* $W^{uu,g}$*-leaves for some* $\alpha > 0$*.*

Proof. Let $\nu_{x,g}$, $x \in M$ denote the conditional measures of m on $W^{uu,g}$-leaves, and similarly let $\nu_{x,f}$ denote the conditional measures of μ on $W^{uu,f}$-leaves. Then $\nu_{x,g}$ is the Riemannian arc length on $W^{uu,g}(x)$. Similarly, since μ is the SRB measure for f^t, μ_x is the Riemannian volume on $W^{u,f}(x)$ and so $\nu_{x,f}$ is the Riemannian arc length on $W^{uu,f}(x)$.

Since $\varphi_* m = \mu$ and $\varphi(W^{uu,g}(x)) = W^{uu,f}(\varphi(x))$ for each $x \in M$, we have $\varphi_* \nu_{x,g} = c(x)\nu_{\varphi(x),f}$ for m-a.e. $x \in M$, where $c(x) > 0$ is a measurable function of x. We conclude that for $x \in M$, $y \in W^{uu,g}(x)$,

$$\mathcal{L}(x) = \limsup_{r \to 0} \frac{\sup\{d_{x,f}(\varphi(x), \varphi(y)) : d_{x,f}(x, y) = r)\}}{r} < \infty,$$

for m-a.e. $x \in M$. This implies by the Rademacher-Stepanov theorem that φ is differentiable a.e. on $W^{uu,g}$ leaves, and thus we have an m-a.e. defined derivative map $D\varphi^V : V^{u,g} \to V^{u,f}$ that satisfies $D\varphi^V \circ Dg^t = Df^t \circ D\varphi^V$. By the same construction as at the end of the proof of Proposition 7.7, but with $\mathcal{I}_V$ instead, we conclude by applying Proposition 3.4 to this mutiplicative one-dimensional cocycle $D\varphi^V$ that $D\varphi^V$ agrees m-a.e. with a Hölder continuous homeomorphism from $V^{u,f}$ to $V^{u,g}$ that is linear on fibers. This implies that φ is $C^{1+\alpha}$ on $W^{uu,g}$-leaves for some $\alpha > 0$. □

For the case when $\dim V^{u,*} > 1$, we use the fact that our derivative cocycles are uniformly quasiconformal on the vertical subbundles as well.

Lemma 7.10. *Suppose that* $\dim V^{u,g} = \dim V^{u,f} > 1$ *and that both* $Dg^t|V^{u,g}$ *and* $Df^t|V^{u,f}$ *are uniformly quasiconformal. Then* φ *is* $C^{1+\alpha}$ *on* $W^{uu,g}$*-leaves for some* $\alpha > 0$.

Proof. We first show that there is a $K \geq 1$ independent of $x \in M$ such that $\varphi|W^{uu,g}_{loc}(x)$ is K-quasiconformal. After replacing the Riemannian metric on $V^{u,f}$ with an equivalent one (see Remark 3.1) , we may assume that there is a Hölder continuous multiplicative cocycle ψ^t over f^t such that $\|Df^t_x(v)\|_{f^t x} = \psi^t(x)\|v\|_x$. Since G is a Carnot group and we are assuming Dg^t expands $H^{u,g}$ by e^t, we then have that Dg^t expands $V^{u,g}$ by e^{2t} for each $t \in \mathbb{R}$.

Let $r > 0$ be given and let $y, z \in W^{uu,g}_{loc}(x)$ satisfy $d_{x,g}(y, z) = r$. Setting $t = -(\log r)/2$, we then have $d_{x,g}(g^t y, g^t z) = 1$. By the uniform continuity of φ, we have

$$d_f(f^t(\varphi(y)), f^t(\varphi(z)) = d_f(\varphi(g^t y), \varphi(g^t z)) \asymp 1,$$

with multiplicative constant independent of y, z, r. Since $f^t(\varphi(y)), f^t(\varphi(z)) \in W^{uu,f}(\varphi(x))$ (because φ preserves the vertical foliations by Proposition 7.8), we then have

$$d_f(\varphi(y), \varphi(z)) \asymp (\psi^{-(\log r)/2}(y))^{-1},$$

for any $z \in W^{uu,g}(x)$ that satisfies $d_{x,g}(y, z) = r$. We conclude that there is a constant $K \geq 1$

such that $\varphi : W^{uu,g}(x) \to W^{uu,f}(\varphi(x))$ is K-quasiconformal.

As a consequence of the K-quasiconformality, $\varphi|_{W^{uu,g}(x)}$ is differentiable a.e. on this leaf with respect to volume [60], and so we obtain a measurable m-a.e. defined derivative map $D\varphi^V : V^{u,g} \to V^{u,f}$ that satisfies $D\varphi^V \circ Dg^t = Df^t \circ D\varphi^V$. Once again appealing to the construction at the end of the proof of Proposition 7.7 (with $\mathcal{I}_V$ instead), we conclude by Proposition 3.4 that $D\varphi^V$ agrees m-a.e. with a Hölder continuous homeomorphism from $V^{u,f}$ to $V^{u,g}$ that is linear fibers. This implies that φ is $C^{1+\alpha}$ on $W^{uu,g}$-leaves for some $\alpha > 0$. □

We use an elementary calculus lemma to show that if f^t and g^t satisfy all of the hypotheses of the propositions of this section then φ is $C^{1+\alpha}$. Recall that, by Proposition 7.7, φ is $C^{1+\alpha}$ when restricted to each smooth curve tangent to $H^{u,g}$ for some $\alpha > 0$, i.e., φ is $C^{1+\alpha}$ along the subbundle $H^{u,g}$.

Lemma 7.11. *Let $x \in M$ and suppose $\varphi : W^{u,g}(x) \to W^{u,f}(x)$ is $C^{1+\alpha}$ along $H^{u,g}$ and $V^{u,g}$. Then φ is $C^{1+\alpha}$.*

Proof. Without loss of generality, it suffices to show that φ is differentiable at $x \in W^{u,g}(x)$. Pass to coordinates on G for which x corresponds to $0 \in G$. Let $u \in \mathfrak{g} = T_0 G$ be the direction in which we wish to show φ is differentiable. Write $u = h + v$ for $h \in \mathfrak{h}$ and $v \in \mathfrak{v}$. We may assume that $h \neq 0$ and $v \neq 0$, as otherwise φ is differentiable in the direction u by hypothesis. There are unique left-invariant vector fields X and Y on G tangent to h and v respectively. Since $[X, Y] = 0$, these two vector fields span a plane P in G and we can choose coordinates on P such that X is parallel to the x-axis (first coordinate direction) and Y is parallel to the y-axis (second coordinate direction). Then, by hypothesis, φ has $C^{1+\alpha}$ partial derivatives in these coordinates and so φ is $C^{1+\alpha}$ on P. Repeating this for all coordinate directions u, we conclude that φ has $C^{1+\alpha}$ partial derivatives in all directions at all points. We thus conclude that φ is $C^{1+\alpha}$ on G. □

This completes the proof of Theorem 2.3.

Our final proposition of this section is required to show that Theorem 2.3 implies Theorem 1.2. We will use the regularity of φ to obtain $C^{1+\alpha}$ regularity of the $W^{cs,f}$ foliation for f^t. Note that this invariant foliation is the same for all time changes of f^t, and so if we obtained f^t as the synchronization of a C^3 Anosov flow $\tilde{f}^t$, then Lemma 7.12 implies that $W^{cs,\tilde{f}}$ is a $C^{1+\alpha}$ foliation as well.

Proposition 7.12. *The $W^{cs,f}$ foliation is $C^{1+\alpha}$.*

Proof. Let $x \in M$ and let $y \in W^{cs,f}_{loc}(x)$. Consider the local center-stable holonomy map $h^{cs,f} : W^{u,f}_{loc}(x) \to W^{u,f}_{loc}(y)$ for the $W^{cs,f}$ foliation. Since φ is a conjugacy from g^t to f^t, we have $h^{cs,f} = \varphi \circ h^{cs,g} \circ \varphi^{-1}$, where

$$h^{cs,g} : W^{u,g}_{loc}(\varphi^{-1}(x)) \to W^{u,g}_{loc}(\varphi^{-1}(y)),$$

is the local center-stable holonomy map for the $W^{cs,g}$ foliation. Since the $W^{cs,g}$ foliation is smooth and φ is $C^{1+\alpha}$ on $W^{u,g}$ leaves, this implies that $h^{cs,f}$ is $C^{1+\alpha}$. We then have two continuous transverse foliations $W^{u,f}$ and $W^{cs,f}$ with C^2 leaves such that the $W^{cs,f}$ holonomy maps between $W^{u,f}$ leaves are $C^{1+\alpha}$. By [15, Lemma 31], this implies that $W^{cs,f}$ is a $C^{1+\alpha}$ foliation. □

8 References

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