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TOPICS IN RAMIFICATION THEORY

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ABSTRACT

This thesis treats several topics in ramification theory. Let K be a complete discrete valuation field whose residue field is perfect and of positive characteristic.

The first topic treated is ramification of étale cohomology. More precisely, let X be a connected, proper scheme over $\mathcal{O}_K$, and U be the complement in X of a divisor with simple normal crossings. Assume that the pair (X, U) is strictly semi-stable over $\mathcal{O}_K$ of relative dimension one and K is of equal characteristic. We prove that, for any smooth ℓ-adic sheaf $\mathscr{G}$ on U of rank one, at most tamely ramified on the generic fiber, if the ramification of $\mathscr{G}$ is bounded by $t+$ for the logarithmic upper ramification groups of Abbes-Saito at points of codimension one of X, then the ramification of the étale cohomology groups with compact support of $\mathscr{G}$ is bounded by $t+$ in the same sense.

The second topic treated is ramification in transcendental extensions of local fields. Let L/K be a separable extension of complete discrete valuation fields. The residue field of L is *not assumed to be perfect*. We prove a formula for the Swan conductor of the image of a character $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$ in $H^1(L, \mathbb{Q}/\mathbb{Z})$ for χ sufficiently ramified.

Finally, we treat generalized Hasse-Herbrand ψ-functions. We define generalizations $\psi^{\mathrm{ab}}_{L/K}$ and $\psi^{\mathrm{AS}}_{L/K}$ of the classical Hasse-Herbrand ψ-function and study their properties. In particular, we prove a formula for $\psi^{\mathrm{ab}}_{L/K}(t)$ for sufficiently large $t \in \mathbb{R}$.

CHAPTER 1

INTRODUCTION

Ramification theory flourished as an area of number theory in the 19th century, when Hilbert first introduced the concept of higher ramification groups. Subsequent developments were achieved by several authors, such as Hasse and Herbrand, who defined the classical ψ-function, and Artin, who defined the conductor of a character of the Galois group of a local or global field. The classical theory is strongly related to the study of local fields, nomenclature here used to describe complete discrete valuation fields with *perfect residue fields*. Modern ramification theory seeks to generalize the classical theory in settings that often involve the study of complete discrete valuation fields with *imperfect residue fields*. Examples are Kato's Swan conductor of an abelian character of the Galois group $G(L/K)$, where L/K is a finite extension of complete discrete valuation fields with imperfect residue fields ([8]), Abbes and Saito's upper filtrations of this Galois group ([1, 2, 23]), and Kedlaya's ([13, 14]) and Xiao's ([26, 27]) works on ramification for p-adic differential modules.

From classical ramification theory, if L/K is a finite Galois extension of local fields and $G = G(L/K)$ is the Galois group, we know that there are lower and upper ramification filtrations G_t and G^t of G, where $t \in [0, \infty)$, related to each other by the Hasse-Herbrand ψ-function:

$$G^t = G_{\psi_{L/K}(t)}.$$

This function is a continuous, piecewise linear, increasing and convex function on $[0, \infty)$.

In parallel, there is the notion of ramification of characters: one can measure the ramification of a character χ of G by its Swan conductor $\operatorname{Sw}\chi$. This invariant plays an important role in the Grothendieck–Ogg–Shafarevich formula, which allows us to compute the Euler-Poincaré characteristic of an ℓ-adic sheaf on a curve under certain assumptions.

Modern advances include Kato's definition of the Swan conductor and refined Swan conductor of an abelian character of G when the residue field of K is no longer assumed to be perfect ([8]), as well as the notion of Swan divisor ([9]). In the 2000's, Abbes and Saito used rigid geometry to construct a generalization of the upper ramification filtration of G without assuming that the residue field of K is perfect ([1, 2, 23]). The connection between Kato's invariants and the filtration of Abbes and Saito has been established in characteristic $p > 0$ ([3]), but remains open in mixed characteristic.

In this work, we explore these modern concepts to provide further advances to ramification theory. In chapter 2, we investigate the ramification of étale cohomology groups. Roughly, we use a conductor formula established by Kato and Saito in [11] and a twisting argument to obtain the following result, which will be stated with more precision later. Let K be local field of positive characteristic, and X a family of curves over $\mathcal{O}_K$. Let U be the complement in X of a divisor with simple normal crossings. Under some hypotheses, if the ramification of a smooth ℓ-adic sheaf $\mathscr{G}$ on U of rank one is bounded by $t+$ in the sense of Abbes and Saito, then the ramification of the étale cohomology groups with compact support of $\mathscr{G}$ is also bounded by $t+$.

In chapter 3, we investigate ramification in transcendental extensions of local fields. More precisely, we consider a separable extension L/K of complete discrete valuation fields, where the residue field of K is perfect and of positive characteristic, but the residue field of L may be imperfect. Then we prove a formula for the Swan conductor of the image of a character $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$ in $H^1(L, \mathbb{Q}/\mathbb{Z})$ for χ sufficiently ramified. This formula has an interpretation in terms of generalized Hasse-Herbrand ψ-functions, which are defined and studied in chapter 4. In this chapter, we consider such (possibly transcendental) extension L/K and define generalizations $\psi^{\mathrm{ab}}_{L/K}$ and $\psi^{\mathrm{AS}}_{L/K}$ of the classical Hasse-Herbrand ψ-function. Further, we obtain a formula for $\psi^{\mathrm{ab}}_{L/K}(t)$ for sufficiently large $t \in \mathbb{R}$.

Part of this work has been published by the author in [17, 18].

1.1 A taste of classical ramification theory

In this section, we give a very brief overview of classical ramification theory, which we hope will provide the reader with feelings and intuitions that will help grasp the ideas behind our non-classical definitions and results. For a more complete exposition, we refer to [25].

Let L/K be a finite Galois extension of local fields, i.e., complete discrete valuation fields with *perfect residue fields*. Let G be the Galois group of L/K and x be an $\mathcal{O}_K$-generator of $\mathcal{O}_L$. We can define, for $i \in \mathbb{R}_{\geq 0}$, the lower ramification subgroup

$$G_i = \{\sigma \in G : v_L(\sigma x - x) \geq i + 1\}.$$

We remark that the subgroup G_0 is the inertia subgroup of G. Lower ramification groups behave well under taking subgroups. More precisely, let H be a subgroup of G and recall that, by Galois theory, H is the Galois group of the extension L/L^H. We have, for every $i \in \mathbb{R}_{\geq 0}$,

$$H_i = G_i \cap H.$$

These lower ramification groups are used to define the classical Hasse-Herbrand ψ-function. More precisely, one first defines

$$\varphi_{L/K}(t) := \int_0^t \frac{ds}{(G_0 : G_s)}.$$

This function $\varphi : [0, \infty) \to [0, \infty)$ is continuous, piecewise linear, increasing and concave. The Hasse-Herbrand function $\psi_{L/K} : [0, \infty) \to [0, \infty)$ is then defined to be $\psi_{L/K} := \varphi_{L/K}^{-1}$.

We make a few remarks about $\psi_{L/K}$. First, it satisfies a transitivity formula. More precisely, for Galois extensions $L/F/K$, we have $\psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}$. Second, when the extension L/K is tamely ramified, the Hasse-Herbrand function takes a very simple form: $\psi_{L/K}(t) = e(L/K)t$.

We can now introduce the upper ramification groups. For $i \in \mathbb{R}_{\geq 0}$, one defines

$$G^i := G_{\psi_{L/K}(i)}.$$

One of the most fundamental properties of this upper ramification filtration is that it is compatible with quotients. More precisely, let H be a normal subgroup of G. Then G/H is the Galois group of the extension L^H/K, and we have

$$(G/H)^i = G^i H/H.$$

Because of this property, we have an induced filtration on the absolute Galois group of K:

$$G_K^i = \varprojlim_{L/K} G(L/K)^i.$$

This upper ramification groups may be used to introduce the notion of ramification of a character. More precisely, for an abelian character $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(G_K^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$, we can define a Swan conductor as follows. If $\chi(G_K^0) \neq 0$, put

$$\mathrm{Sw}\,\chi = \max\left\{n \in \mathbb{Z}_{\geq 0} : \chi(G_K^n) \neq 0\right\},$$

and, if $\chi(G_K^0) = 0$ put $\mathrm{Sw}\,\chi = 0$. The Swan conductor measures the wild ramification of χ; when χ is unramified or tamely ramified, the Swan conductor is zero.

Remark 1.1.1. Our choice of domain for $\psi_{L/K}$ is not usual. In most of the literature, $\psi_{L/K}$ is defined on $[-1, \infty)$.

1.2 Background

In this section we go over some concepts that will be necessary through the rest of this work.

1.2.1 Higher dimensional local fields

In this section we give a quick overview of q-dimensional local fields (for more on this subject, see [29, 20]). Subsequently, we shall use q-dimensional local fields to construct some residue maps.

Let K be a complete discrete valuation field with valuation v_K and residue field k. The field $K\{\{T\}\}$ is defined as the set

$$K\{\{T\}\} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in K,\ \inf v_K(a_i) > -\infty,\ \text{and } v_K(a_i) \to \infty \text{ as } i \to -\infty \right\}$$

with addition and multiplication as follows:

$$\sum_{i=-\infty}^{\infty} a_i T^i + \sum_{i=-\infty}^{\infty} b_i T^i = \sum_{i=-\infty}^{\infty} (a_i + b_i) T^i$$

and

$$\sum_{i=-\infty}^{\infty} a_i T^i \sum_{i=-\infty}^{\infty} b_i T^i = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_j b_{i-j} T^i.$$

We can define a discrete valuation on $K\{\{T\}\}$ by setting

$$v_{K\{\{T\}\}} \left(\sum_{i=-\infty}^{\infty} a_i T^i \right) = \min v_K(a_i).$$

Endowed with this valuation, $K\{\{T\}\}$ becomes a complete discrete valuation field with residue field $k((T))$.

When K is a local field, the field

$$K\{\{T_1\}\} \cdots \{\{T_m\}\}((T_{m+1})) \cdots ((T_{q-1})),$$

where $1 \leq m \leq q-1$, is a q-dimensional local field. Fields of this form are called standard q-dimensional local fields.

1.2.2 The Abbes-Saito upper ramification

For a complete discrete valuation field K, with possibly imperfect residue field, A. Abbes and T. Saito constructed (logarithmic) upper ramification groups $(G^t_{K,\log})_{t\in\mathbb{Q}_{>0}} \subset G_K$. When the residue field of K is perfect, $(G^t_{K,\log})_{t\in\mathbb{Q}_{>0}}$ coincides with the classical upper ramification. Furthermore, $(G^t_{K,\log})_{t\in\mathbb{Q}_{>0}}$ is stable under tame base change; more precisely, if L is a finite separable extension of K of ramification index e that is tamely ramified, we have $G^{et}_{L,\log} = G^t_{K,\log}$. In general, for a finite separable extension L/K of ramification index e, not necessarily tamely ramified, we have $G^{et}_{L,\log} \subset G^t_{K,\log}$. For a real number $s \geq 0$, the authors also defined

$$G^{s+}_{K,\log} = \overline{\bigcup_{t\in\mathbb{Q}, t>s} G^t_{K,\log}}.$$

These groups satisfy the following property:

Lemma 1.2.1 ([2], Lemma 5.2). *Let K be a complete discrete valuation field with residue field k of characteristic p. Assume that there is a map of complete discrete valuation fields $K \to L$ inducing a local homomorphism $\mathcal{O}_K \to \mathcal{O}_L$, that the ramification index is prime to p, and that the induced extension of residue fields is separable. Then, for $t \in \mathbb{Q}_{>0}$, the map $G_L \to G_K$ induces a surjection $G^{et}_{L,\log} \to G^t_{K,\log}$.*

As a consequence, we also have surjections $G^{et+}_{L,\log} \to G^{t+}_{K,\log}$.

1.2.3 Ramification of characters

In this subsection, assume that the residue field k of K has characteristic $p > 0$ and is not necessarily perfect.

We recall the definition of the k-vector space $\Omega_k(\log)$. There exists a canonical map $d\log : K^\times \to \Omega_k$, and $\Omega_k(\log)$ is the amalgamate sum of the differential module Ω_k with

$k \otimes_{\mathbb{Z}} K^\times$ over $k \otimes_{\mathbb{Z}} \mathcal{O}_K^\times$ with respect to $d\log : \mathcal{O}_K^\times \to \Omega_k$ and $\mathcal{O}_K^\times \hookrightarrow K^\times$. There is a residue map $\mathrm{Res} : \Omega_k(\log) \to k$ induced by the valuation map of K and an exact sequence

$$0 \longrightarrow \Omega_k \longrightarrow \Omega_k(\log) \xrightarrow{\mathrm{Res}} k \longrightarrow 0.$$

In [8], Kato constructs an increasing filtration $(F_r H^1(K, \mathbb{Q}/\mathbb{Z}))_{r\in\mathbb{N}}$ and defines, putting $\mathrm{Gr}_r H^1(K, \mathbb{Q}/\mathbb{Z}) = F_r H^1(K, \mathbb{Q}/\mathbb{Z})/F_{r-1} H^1(K, \mathbb{Q}/\mathbb{Z})$ for $r \geq 1$, an injection

$$\mathrm{rsw}_{r,K} : \mathrm{Gr}_r H^1(K, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_k(\mathfrak{m}_K^r/\mathfrak{m}_K^{r+1}, \Omega_k(\log)),$$

where $\mathfrak{m}_K$ denotes the maximal ideal of $\mathcal{O}_K$. For $\chi \in F_r H^1(K, \mathbb{Q}/\mathbb{Z}) \backslash F_{r-1} H^1(K, \mathbb{Q}/\mathbb{Z})$, the injection

$$\mathrm{rsw}_{r,K}(\chi) : \mathfrak{m}_K^r/\mathfrak{m}_K^{r+1} \to \Omega_k(\log)$$

is denoted by $\mathrm{rsw}_K(\chi)$ and called the refined Swan conductor of χ.

In [3], Corollary 9.12, Abbes and Saito relate Kato's construction to the upper ramification groups defined in [1]. More specifically, they prove that, when K is of equal characteristic, $\chi \in F_r H^1(K, \mathbb{Q}/\mathbb{Z})$ if and only if χ kills $G_{K,\log}^{r+}$.

Remark 1.2.2. The comparison between Kato's filtration and the Abbes-Saito logarithmic upper ramification groups remains open in the mixed characteristic case, but is expected by experts.

Consider now the following case. Let $S = \mathrm{Spec}\, \mathcal{O}_K$ and X be a regular flat separated scheme over S. Let $D = \bigcup_{i=1}^{n} D_i$ be a divisor with simple normal crossings, where D_i denotes the irreducible components of D. For each i let ξ_i be a generic point for D_i, $\mathcal{O}_{M_i} = \mathcal{O}_{X,\xi_i}^h$ the henselization of the local ring at ξ_i, M_i its field of fractions, and k_i the residue field of M_i. Let $U = X - D$ and $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$. For each i, denote by $\chi_i \in H^1(M_i, \mathbb{Q}/\mathbb{Z})$ the restriction of χ, and by r_i the Swan conductor $\mathrm{Sw}_{M_i}\chi_i$. Define the Swan divisor

$$D_\chi = \sum_i r_i D_i$$

and let

$$E = \sum_{r_i > 0} D_i$$

be the support of D_χ. It's shown by [8, (7.3)], that there exists an injection

$$\operatorname{rsw}\chi : \mathcal{O}_X(-D_\chi) \otimes_{\mathcal{O}_X} \mathcal{O}_E \to \Omega^1_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E$$

inducing $\operatorname{rsw}_{M_i}(\chi_i)$ at ξ_i. We say that χ is clean if $\operatorname{rsw}\chi$ is a locally splitting injection.

1.2.4 Semi-stable pairs

In this subsection, we let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$, X a proper scheme of finite presentation over $\mathcal{O}_K$, and U an open and dense subscheme of X. We recall the definition of a semi-stable pair ([22, Definition 1.6]):

Definition 1.2.3. The pair (X, U) is said to be semi-stable over $\mathcal{O}_K$ of relative dimension d if, étale locally on X, X is étale over $\operatorname{Spec}\mathcal{O}_K[T_0, \ldots, T_d]/(T_0 \cdots T_r - \pi)$ and U is the inverse image of $\operatorname{Spec}\mathcal{O}_K[T_0, \ldots, T_d, T_0^{-1}, \ldots, T_m^{-1}]/(T_0 \cdots T_r - \pi)$ for some $0 \le r \le m \le d$ and prime π of K.

When (X, X_K) is semi-stable over $\mathcal{O}_K$, we say that X is semi-stable over $\mathcal{O}_K$.

If we substitute the condition "étale locally" by "Zariski locally", the pair (X, U) is then said to be strictly semi-stable.

We shall need the following property of strictly semi-stable pairs, which is a consequence of [22, Theorem 2.9]:

Theorem 1.2.4. *Let (X, U) be a strictly semi-stable pair over $\mathcal{O}_K$ and L be a finite separable extension of K. Then there exists a proper birational morphism $X' \to X_{\mathcal{O}_L}$ inducing an isomorphism $U' \to U_{\mathcal{O}_L}$, where U' is the inverse image of $U_{\mathcal{O}_L}$, and such that (X', U') form a strictly semi-stable pair over $\mathcal{O}_L$.*

CHAPTER 2

RAMIFICATION OF ÉTALE COHOMOLOGY GROUPS

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$. Let X be a connected, proper scheme over $\mathcal{O}_K$, D a divisor with simple normal crossings on X, and $U = X - D$. Assume that the pair (X, U) is strictly semi-stable over $\mathcal{O}_K$ of relative dimension d (see Definition 1.2.3).

Let ℓ be a prime number different from p and $\mathscr{G}$ be a smooth ℓ-adic sheaf on U, by which we mean a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on U. Assume that $\mathscr{G}$ is at most tamely ramified on the generic fiber X_K. Write $D = \bigcup_{i=1}^{n} D_i$, where D_i are the irreducible components of D. Let ξ_i be the generic point of D_i, $\mathcal{O}_{M_i} = \mathcal{O}^h_{X,\xi_i}$ the henselization of the local ring at ξ_i, M_i its field of fractions, and $\eta_i = \operatorname{Spec} M_i$.

Let G_{M_i} and G_K denote the absolute Galois groups of M_i and K, respectively, and $(G^t_{M_i,\log})_{t\in\mathbb{Q}_{\geq 0}}$, $(G^t_{K,\log})_{t\in\mathbb{Q}_{\geq 0}}$ the corresponding Abbes-Saito logarithmic upper ramification filtrations (see [1]). Put, for a real number $s \geq 0$, $G^{s+}_{M_i,\log} = \overline{\bigcup_{t\in\mathbb{Q},t>s} G^t_{M_i,\log}}$ and $G^{s+}_{K,\log} = \overline{\bigcup_{t\in\mathbb{Q},t>s} G^t_{K,\log}}$. We are interested in exploring to what extent the following conjecture holds:

Conjecture 1. *Under the assumptions above, if $G^{t+}_{M_i,\log}$ acts trivially on $\mathscr{G}_{\overline{\eta_i}}$ for every i, then $G^{t+}_{K,\log}$ acts trivially on $H^j_c(U_{\overline{K}}, \mathscr{G})$ for every j.*

Our main result in from this chapter is Theorem 2.3.1, in which we prove the conjecture in the special case where $\mathscr{G}$ is of rank 1, K has characteristic p, and the relative dimension is $d = 1$.

The structure of this chapter is as follows: In the first section, we give a criterion for $G^{t+}_{K,\log}$ to act trivially on an ℓ-adic sheaf. In the second section, we provide an application of the Kato-Saito conductor formula. In the third section, we present and prove the main result of this chapter.

2.1 The action of $G^{t+}_{K,\log}$

In this section, we let K be a complete discrete valuation field of equal characteristic with perfect residue field k of characteristic $p > 0$, ℓ be a prime different than p, and M, N be finite-dimensional representations of G_K over $\overline{\mathbb{Q}_\ell}$ which come from finite-dimensional continuous representations of G_K over a finite extension of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}_\ell}$. We shall provide a criterion for $G^{t+}_{K,\log}$ to act trivially on M.

There is a canonical slope decomposition (see [12, Proposition 1.1], or [4, Lemma 6.4])

$$M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)}$$

characterized by the following properties: if P is the wild inertia subgroup of G_K, then $M^P = M^{(0)}$. Further, for all $r > 0$,

$$(M^{(r)})^{G^{r}_{K,\log}} = 0$$

and

$$(M^{(r)})^{G^{r+}_{K,\log}} = M^{(r)}.$$

We have $M^{(r)} = 0$ except for finitely many r. The values of r for which $M^{(r)} \neq 0$ are called slopes of M.

Definition 2.1.1. We say that M is isoclinic if it has only one slope.

The following proposition gives our criterion:

Proposition 2.1.2. *Let t be a nonnegative real number. Assume that, for any totally tamely ramified extension L/K of degree e prime to p, we have the following: if M_L denotes the representation of G_L induced by M, then, for any character $\chi : G_L \to \overline{\mathbb{Q}_\ell}^{\times}$ for which* $\mathrm{Sw}_L(\chi) > et$*, we have*

$$\mathrm{Sw}_L(M_L \otimes \chi) = \mathrm{rk}(M_L)\mathrm{Sw}_L(\chi).$$

Then G_K^{t+} acts trivially on M.

The proof will be presented shortly. The general strategy is the following:

- We first show that the behavior of the tensor product of isoclinic M and N is similar to that of the tensor product of characters;

- Next, we use the previous result to understand the slope decomposition of the tensor product $M \otimes \chi$ and prove the proposition.

We start with the lemma:

Lemma 2.1.3. *If M is isoclinic of slope r and N is isoclinic of slope s, where $r > s$, then $M \otimes N$ is isoclinic of slope r.*

Proof. We have

$$M^{G_K^r} = 0,$$

$$M^{G_K^{r+}} = M,$$

$$N^{G_K^s} = 0,$$

and

$$N^{G_K^{s+}} = N.$$

Since $r > s$, $(M \otimes N)^{G_K^{r+}} = M \otimes N$. On the other hand, G_K^r acts trivially on N and $M^{G_K^r} = 0$, so $(M \otimes N)^{G_K^r} = 0$. Hence $M \otimes N$ is isoclinic of slope r. □

Proof of Proposition 2.1.2. We need to show that, if $r > t$, then $M^{(r)} = 0$. Let R be the maximum slope of M. Assume, by contradiction, that $R > t$. Let m, e be positive integers such that:

(i) e is prime to p,

(ii) $\frac{m}{e} < R$,

(iii) $\frac{m}{e}$ is strictly greater than any other slope of M,

(iv) $\frac{m}{e} > t$.

Let L be a totally tamely ramified extension of degree e of K. By [1, Proposition 3.15], $G^s_{K,\log} = G^{es}_{L,\log}$ for any $s \in \mathbb{Q}_{\geq 0}$, so the slopes of M_L are of the form er, where r is a slope of M. Take χ with $\mathrm{Sw}_L(\chi) = m$. Then, by assumption,

$$\mathrm{Sw}_L(M_L \otimes \chi) = \mathrm{rk}(M_L)\mathrm{Sw}_L(\chi) = \mathrm{rk}(M_L)m.$$

By Lemma 2.1.3, for all $r < m$ we have that $M_L^{(r)} \otimes \chi$ is isoclinic of slope m, while $M_L^{(eR)} \otimes \chi$ is isoclinic of slope eR. It follows that

$$\mathrm{Sw}_L(M_L \otimes \chi) = \sum_{r \in \mathbb{Q}_{\geq 0}} \mathrm{Sw}_L(M_L^{(r)} \otimes \chi) = \sum_{r \in \mathbb{Q}_{\geq 0}, r < m} \mathrm{rk}(M_L^{(r)})m + \mathrm{rk}(M_L^{(eR)})eR.$$

Combining the two expressions we get

$$\mathrm{rk}(M_L^{(eR)})eR = \mathrm{rk}(M_L^{(eR)})m,$$

which is a contradiction, since, by assumption, $m < eR$ and $M_L^{(eR)} \neq 0$. □

2.2 The Kato-Saito conductor formula

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$. Let ℓ be a prime number different from p, U be a smooth separated scheme of finite type over K, and $\mathscr{F}$ be a smooth ℓ-adic sheaf of constant rank on U. In [11], Kato and Saito defined the Swan class $\mathrm{Sw}_U \mathscr{F}$, a 0-cycle class with coefficients in $\mathbb{Q}$ supported on the special fiber of a compactification of U over $\mathcal{O}_K$, and proved the conductor formula

$$\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{F}) = \deg \mathrm{Sw}_U \mathscr{F} + \mathrm{rk}(\mathscr{F})\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}),$$

where $\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{F})$ denotes the alternating sum $\sum_j (-1)^j \mathrm{Sw}_K H_c^j(U_{\overline{K}}, \mathscr{F})$.

In this section, assume that X is a regular flat separated scheme of finite type over $S = \operatorname{Spec} \mathcal{O}_K$. Let $D \subset X$ be a divisor with simple normal crossings and write $D = \bigcup_{i=1}^{n} D_i$, where D_i are the irreducible components of D. Put $U = X - D$ and consider a smooth ℓ-adic sheaf $\mathscr{F}$ of rank 1 on U, at most tamely ramified on X_K and with clean ramification with respect to X.

The Swan 0-cycle class $c_{\mathscr{F}}$ of $\mathscr{F}$ is defined as follows. Let E be the support of the Swan divisor $D_{\mathscr{F}} = \sum r_i D_i$. Then define $c_{\mathscr{F}} \in CH_0(E)$ as

$$c_{\mathscr{F}} = \{c(\Omega^1_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E)^* \cap (1 + D_{\mathscr{F}})^{-1} \cap D_{\mathscr{F}}\}_{\dim 0}.$$

Under the assumption that $\dim U_K \leq 1$, by Corollary 8.3.8 of [11], the Kato-Saito conductor formula becomes simply

$$\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{F}) = \deg c_{\mathscr{F}} + \mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}).$$

The following proposition is an application of this formula that will be useful in the next section:

Proposition 2.2.1. *Let X, S and $U = X - D$ be as above. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two smooth ℓ-adic sheaves on U of rank one, $\mathscr{F}_2$ having clean ramification with respect to X. Write $D_{\mathscr{F}_1} = \sum r_i D_i$ and $D_{\mathscr{F}_2} = \sum s_i D_i$. Assume that $r_i < s_i$ for every i. Then $\mathscr{F}_1 \otimes \mathscr{F}_2$ has clean ramification and*

$$c_{\mathscr{F}_1 \otimes \mathscr{F}_2} = c_{\mathscr{F}_2}.$$

Proof. Since $r_i < s_i$ for every i, we have $D_{\mathscr{F}_1 \otimes \mathscr{F}_2} = D_{\mathscr{F}_2}$ and the refined Swan conductors of $\mathscr{F}_1 \otimes \mathscr{F}_2$ and $\mathscr{F}_2$ coincide. Denote by E_i the support of $D_{\mathscr{F}_i}$ and by E be the support of

$D_{\mathscr{F}_1\otimes\mathscr{F}_2}$. We have $E = E_2$, so

$$\begin{aligned} c_{\mathscr{F}_1\otimes\mathscr{F}_2} &= \{c(\Omega^1_{X/S}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_E)^*\cap(1+D_{\mathscr{F}_1\otimes\mathscr{F}_2})^{-1}\cap D_{\mathscr{F}_1\otimes\mathscr{F}_2}\}_{\dim 0} \\ &= \{c(\Omega^1_{X/S}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{E_2})^*\cap(1+D_{\mathscr{F}_2})^{-1}\cap D_{\mathscr{F}_2}\}_{\dim 0} \\ &= c_{\mathscr{F}_2}. \end{aligned}$$

□

2.3 Ramification and étale cohomology

In this section, we let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$ and of equal characteristic, $S = \operatorname{Spec}\mathcal{O}_K$, and $s = \operatorname{Spec} k$. We will denote by X a proper, connected scheme of finite presentation over $\mathcal{O}_K$, and U an open and dense subscheme of X. We assume that $D = X - U$ is a divisor with simple normal crossings and write $\bigcup_{i=1}^{n} D_i$, where D_i are the irreducible components. We also assume that the pair (X, U) is strictly semi-stable over $\mathcal{O}_K$ of relative dimension 1, and that $\mathscr{G}$ is a smooth ℓ-adic sheaf on U, where ℓ is a prime number different from p. Further, we assume that $\mathscr{G}$ is of rank 1 and at most tamely ramified on the generic fiber X_K. Denote by ξ_i the generic point of D_i, $\mathcal{O}_{M_i} = \mathcal{O}^h_{X,\xi_i}$ the henselization of the local ring at ξ_i, M_i its field of fractions, k_i the residue field of M_i, and $\eta_i = \operatorname{Spec} M_i$.

We shall prove the following theorem:

Theorem 2.3.1. *Conjecture 1 is true when $\mathscr{G}$ is of rank 1, the relative dimension is 1, and K is of equal characteristic.*

Remark 2.3.2. When the relative dimension is greater than 1, one should still be able to prove Conjecture 1 using the same methods, as long as it is true that

$$\operatorname{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{F}) = \deg c_{\mathscr{F}} + \operatorname{Sw}_K R\Gamma_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell})$$

for smooth ℓ-adic sheaves $\mathscr{F}$ of rank 1 on U, at most tamely ramified on X_K and with clean ramification with respect to X.

The proof is divided in two cases. First observe that, since the total constant field of X_K is a finite unramified extension of K, we may assume that K is the total constant field of X_K. Then there is an exact sequence of fundamental groups

$$1 \longrightarrow \pi_1(U_{\overline{K}}) \longrightarrow \pi_1(U) \longrightarrow G_K \longrightarrow 1.$$

Let M be the function field of X and $\eta = \operatorname{Spec} M$. We first consider the case in which the action of $\pi_1(U_{\overline{K}})$ is trivial on $\mathscr{G}_{\bar{\eta}}$, and then the case in which it is non-trivial.

To prove the first case, we shall need the following lemma:

Lemma 2.3.3. *In addition to the assumptions of Theorem 2.3.1, assume that $\mathscr{G}_{\bar{\eta}}$ is the pullback of some ℓ-adic representation $\mathscr{H}$ of G_K. If $G^{t+}_{M_i,\log}$ acts trivially on $\mathscr{G}_{\overline{\eta_i}}$, then G^{t+}_K acts trivially on $\mathscr{H}$.*

Proof. This follows from Lemma 1.2.1. □

Proposition 2.3.4. *Theorem 2.3.1 holds if $\pi_1(U_{\overline{K}})$ acts trivially on $\mathscr{G}_{\bar{\eta}}$.*

Proof. In this case, by the homotopy exact sequence of étale fundamental groups, we have that $\mathscr{G}_{\bar{\eta}}$ is the pullback of some ℓ-adic representation $\mathscr{H}$ of G_K. Then

$$H^j_c(U_{\overline{K}}, \mathscr{G}) = H^j_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}) \otimes \mathscr{H}.$$

By Lemma 2.1.3, and the fact that $H^j_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell})$ is at most tamely ramified ([21, Corollary 2]), we have that the slope decomposition of $H^j_c(U_{\overline{K}}, \mathscr{G})$ coincides with that of $\mathscr{H}$, in the following sense:

$$(H^j_c(U_{\overline{K}}, \mathscr{G}))^{(r)} = H^j_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}) \otimes \mathscr{H}^{(r)}.$$

It follows that $G^t_{K,\log}$ acts trivially on $H^j_c(U_{\overline{K}}, \mathscr{G})$ if and only if it acts trivially on $\mathscr{H}$. By Lemma 2.3.3, the result follows. □

We shall now prove Theorem 2.3.1 for the case in which $\pi_1(U_{\overline{K}})$ does not act trivially on $\mathscr{G}_{\overline{\eta}}$. The core of strategy is the following: using the Kato-Saito conductor formula and the fact that $H^0_c(U_{\overline{K}}, \mathscr{G}) = H^2_c(U_{\overline{K}}, \mathscr{G}) = 0$, we show that $H^1_c(U_{\overline{K}}, \mathscr{G})$ satisfies the hypotheses of Proposition 2.1.2.

Lemma 2.3.5. *Keep the assumptions of Theorem 2.3.1. Let e be a natural number prime to p and L be a totally tamely ramified extension of K of degree e. If $\chi : G_L \to \overline{\mathbb{Q}_\ell}^\times$ is a character such that $\mathrm{Sw}_L(\chi) > et$, then*

$$\mathrm{Sw}_L(R\Gamma_c(U_{\overline{L}}, \mathscr{G}) \otimes \chi) = \mathrm{rk}(R\Gamma_c(U_{\overline{L}}, \mathscr{G}))\mathrm{Sw}_L(\chi).$$

Proof. First consider the following. By Theorem 1.2.4, there exists a proper birational morphism $X' \to X_{\mathcal{O}_L}$ inducing an isomorphism $U' \to U_{\mathcal{O}_L}$, where U' is the inverse image of $U_{\mathcal{O}_L}$, and such that (X', U') is strictly semi-stable over $\mathcal{O}_L$.

Let $D' = X' - U'$ and write $D' = \bigcup_{i=1}^{n'} D'_i$, where D'_i are the irreducible components of D'. For each $1 \leq i \leq n'$ let ξ'_i be the generic point of D'_i, $\mathcal{O}_{M'_i} = \mathcal{O}^h_{X', \xi'_i}$ the henselization of the local ring at ξ'_i, M'_i its field of fractions, and $\eta'_i = \mathrm{Spec}\, M'_i$.

There is a composition of blowups of closed points $\tilde{X} \to X$ and a point $\tilde{\xi}_i$ such that $\mathcal{O}_{\tilde{X}, \tilde{\xi}_i} = \mathcal{O}_{X', \xi'_i} \cap M$. Let $\tilde{M}_i$ be the field of fractions of $\mathcal{O}^h_{\tilde{X}, \tilde{\xi}_i}$. Put $\tilde{\eta}_i = \mathrm{Spec}\, \tilde{M}_i$. Denote by e'_i and $\tilde{e}_i$ the ramification indices of $M'_i/\tilde{M}_i$ and $\tilde{M}_i/K$, respectively. We have $e = e'_i \tilde{e}_i$.

By [8, Theorem 8.1], and the fact that $G^{t+}_{M_i, \log}$ acts trivially on $\mathscr{G}_{\overline{\eta_i}}$ for every $1 \leq i \leq n$, we have that $G^{\tilde{e}_i t+}_{\tilde{M}_i, \log}$ acts trivially on $\mathscr{G}_{\overline{\tilde{\eta}_i}}$ for every $1 \leq i \leq n'$. Further, since we have $G^{e'_i \tilde{e}_i t+}_{M'_i, \log} \subset G^{\tilde{e}_i t+}_{\tilde{M}_i, \log}$, we get that $G^{et+}_{M'_i, \log}$ acts trivially on $\mathscr{G}_{\overline{\eta'_i}}$ for all $1 \leq i \leq n'$. Thus it is enough to prove that

$$\mathrm{Sw}_K(R\Gamma_c(U_{\overline{K}}, \mathscr{G}) \otimes \chi) = \mathrm{rk}(R\Gamma_c(U_{\overline{K}}, \mathscr{G}))\mathrm{Sw}_K(\chi)$$

for $\chi : G_K \to \overline{\mathbb{Q}_\ell}^\times$ such that $\mathrm{Sw}_K(\chi) > t$.

Put $r = \mathrm{Sw}_K(\chi)$ and denote by $\tilde{\chi}$ the pullback of χ to U. $\tilde{\chi}$ has clean ramification because the following diagram

$$\begin{array}{ccc}
\mathfrak{m}_K^r/\mathfrak{m}_K^{r+1} & \xrightarrow{\mathrm{rsw}_K\,\chi} & \Omega_k(\log) \\
\downarrow & & \downarrow \\
\mathfrak{m}_{M_i}^r/\mathfrak{m}_{M_i}^{r+1} & \xrightarrow{\mathrm{rsw}_{M_i}\,\tilde{\chi}} & \Omega_{k_i}(\log)
\end{array}$$

is commutative. Indeed, since χ is clean and $\Omega_k(\log) \hookrightarrow \Omega_{k_i}(\log)$ is a splitting injection, $\mathrm{rsw}\,\tilde{\chi}$ is a locally splitting injection. Further, by Lemma 1.2.1, $\mathrm{Sw}_{M_i}(\tilde{\chi}) > t$ for every i. From the Kato-Saito conductor formula, Proposition 2.2.1, and the fact that (X, U) is semi-stable over $\mathcal{O}_K$, we have that $\mathscr{G} \otimes \tilde{\chi}$ is clean and

$$\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{G} \otimes \tilde{\chi}) = \deg c_{\mathscr{G} \otimes \tilde{\chi}} = \deg c_{\tilde{\chi}}.$$

Again by the Kato-Saito conductor formula,

$$\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \tilde{\chi}) = \deg c_{\tilde{\chi}}.$$

Therefore, we have

$$\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{G} \otimes \tilde{\chi}) = \mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \tilde{\chi}) = \mathrm{Sw}_K(R\Gamma_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}) \otimes \chi).$$

Since

$$\mathrm{Sw}_K R\Gamma_c(U_{\overline{K}}, \mathscr{G} \otimes \tilde{\chi}) = \mathrm{Sw}_K(R\Gamma_c(U_{\overline{K}}, \mathscr{G}) \otimes \chi)$$

and

$$\mathrm{Sw}_K(R\Gamma_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}) \otimes \chi) = \mathrm{rk}(R\Gamma_c(U_{\overline{K}}, \overline{\mathbb{Q}_\ell}))\mathrm{Sw}_K(\chi) = \mathrm{rk}(R\Gamma_c(U_{\overline{K}}, \mathscr{G}))\mathrm{Sw}_K(\chi),$$

we conclude that

$$\mathrm{Sw}_K(R\Gamma_c(U_{\overline{K}},\mathscr{G})\otimes\chi)=\mathrm{rk}(R\Gamma_c(U_{\overline{K}},\mathscr{G}))\mathrm{Sw}_K(\chi).$$ □

Lemma 2.3.6. *Let the assumptions be the same as in Lemma 2.3.5, and assume further that $\pi_1(U_{\overline{K}})$ does not act trivially on $\mathscr{G}$. Then*

$$H^j_c(U_{\overline{L}},\mathscr{G})=0$$

for every $j\neq 1$.

Proof. By Poincaré duality, it's enough to show that $H^0(U_{\overline{L}},\mathscr{G})=0$. Since $\pi_1(U_{\overline{K}})$ does not act trivially on $\mathscr{G}_{\bar{\eta}}$ and $\mathrm{rk}(\mathscr{G})=1$, we get that $H^0(U_{\overline{L}},\mathscr{G})=\mathscr{G}_{\bar{\eta}}^{\pi(U_{\overline{L}})}=0$. □

Proof of Theorem 2.3.1. The theorem has already been proved in Proposition 2.3.4 for $\mathscr{G}$ such that $\pi_1(U_{\overline{K}})$ acts trivially on it, so we assume that $\pi_1(U_{\overline{K}})$ does not act trivially. By Lemma 2.3.6, it's enough to prove that $G^{t+}_{K,\log}$ acts trivially on $H^1_c(U_{\overline{K}},\mathscr{G})$.

From Lemmas 2.3.5 and 2.3.6, it follows that

$$\mathrm{Sw}_L(H^1_c(U_{\overline{L}},\mathscr{G})\otimes\chi)=\mathrm{rk}(H^1_c(U_{\overline{L}},\mathscr{G}))\mathrm{Sw}_L(\chi)$$

for any totally tamely ramified extension L of K of degree e prime to p and arbitrary character $\chi:G_L\to\overline{\mathbb{Q}_\ell}^\times$ satisfying $\mathrm{Sw}_L(\chi)>et$.

From Proposition 2.1.2, we have that $G^{t+}_{K,\log}$ acts trivially on $H^1_c(U_{\overline{K}},\mathscr{G})$. Hence $G^{t+}_{K,\log}$ acts trivially on $H^j_c(U_{\overline{K}},\mathscr{G})$ for every j. □

CHAPTER 3

RAMIFICATION IN TRANSCENDENTAL EXTENSIONS OF LOCAL FIELDS

Let K be a complete discrete valuation field. Classical ramification theory has extensively studied finite Galois extensions L/K when the residue field of K is perfect. Much progress has also been achieved when the residue field is no longer assumed to be perfect, such as K. Kato's generalization of the classical Swan conductor $\operatorname{Sw}\chi \in \mathbb{Z}_{\geq 0}$ for abelian characters $\chi : G(L/K) \to \mathbb{Q}/\mathbb{Z}$ ([8]) and A. Abbes and T. Saito's generalization of the upper ramification filtration $G(L/K)$ ([1]). Yet there are still many open questions, both when the residue field of K is imperfect and when the extension L/K is transcendental.

Let L/K be a finite Galois extension of complete discrete valuation fields with perfect residue fields. Denote by $e(L/K)$ the ramification index of L/K and by $D^{\log}_{L/K}$ the wild different of L/K, i.e., $D^{\log}_{L/K} = D_{L/K} - e(L/K) + 1$, where $D_{L/K}$ is the different of L/K. It is classically known that, if $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$ and χ_L is its image in $H^1(L, \mathbb{Q}/\mathbb{Z})$, then, when $\operatorname{Sw}\chi \gg 0$,

$$\operatorname{Sw}\chi_L = \psi_{L/K}(\operatorname{Sw}\chi) = e(L/K)\operatorname{Sw}\chi - D^{\log}_{L/K}, \tag{3.0.1}$$

where $\psi_{L/K}$ is the classical ψ-function (see, for example, [25]).

In this chapter, we obtain a formula resembling (3.0.1) for (possibly transcendental) separable extensions L/K of complete discrete valuation fields when the residue field of K is perfect but the residue field of L is not necessarily perfect, and then define generalizations of the classical ψ-function. More precisely, we first prove the two following results, the first when L is of equal positive characteristic and the second when L is of mixed characteristic. Here $\hat{\Omega}^1_{\mathcal{O}_L/\mathcal{O}_K}(\log)$ denotes the completed $\mathcal{O}_L$-module of relative differential forms with log poles, $\delta_{\mathrm{tor}}(L/K)$ the length of its torsion part, and e_K the absolute ramification index of K. For a character $\chi \in H^1(L, \mathbb{Q}/\mathbb{Z})$, $\operatorname{Sw}\chi$ denotes Kato's Swan conductor of χ (defined in [8]).

Main Result 1 (Theorem 3.1.12). *Let L/K be a separable extension of complete discrete valuation fields of equal characteristic $p > 0$. Assume that K has perfect residue field and $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$ is such that*

$$\operatorname{Sw}\chi > \frac{p}{p-1}\frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)}.$$

Denote by χ_L its image in $H^1(L, \mathbb{Q}/\mathbb{Z})$. Then

$$\operatorname{Sw}\chi_L = e(L/K)\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K).$$

Main Result 2 (Theorem 3.3.13). *Let L/K be an extension of complete discrete valuation fields of mixed characteristic. Assume that K has perfect residue field of characteristic $p > 0$ and $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$ is such that*

$$\operatorname{Sw}\chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by χ_L its image in $H^1(L, \mathbb{Q}/\mathbb{Z})$. Then

$$\operatorname{Sw}\chi_L = e(L/K)\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K).$$

In the next chapter, we relate this discussion to the $\psi_{L/K}$ function for L/K. More precisely, we define two ψ-functions $\psi^{\mathrm{AS}}_{L/K}$ and $\psi^{\mathrm{ab}}_{L/K}$ when K has perfect residue field but L has residue field not necessarily perfect. We then show that, in the classical case of finite L/K, both these definitions coincide with the classical $\psi_{L/K}$ function. Finally, we prove that we can regard our first two main theorems as formulas for $\psi^{\mathrm{ab}}_{L/K}(t)$ for $t \gg 0$:

Main Result 3 (Theorem 4.0.4). *Let L/K be a separable extension of complete discrete valuation fields. Assume that K has perfect residue field of characteristic $p > 0$. Let $t \in \mathbb{R}_{\geq 0}$*

be such that

$$\begin{cases} t \geq \dfrac{2e_K}{p-1} + \dfrac{1}{e(L/K)} + \left\lceil \dfrac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil & \textit{if } K \textit{ is of characteristic } 0, \\ t > \dfrac{p}{p-1} \dfrac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} & \textit{if } K \textit{ is of characteristic } p. \end{cases}$$

Then

$$\psi^{\mathrm{ab}}_{L/K}(t) = e(L/K)t - \delta_{\mathrm{tor}}(L/K).$$

Our methods for the proof of Main Result 1 differ greatly from those for the proof of Main Result 2. In the equal characteristic case, we use Artin-Schreier-Witt theory. In the mixed characteristic case, we use M. Kurihara's exponential map ([16]) and a modified version of higher dimensional local class field theory.

Notation. We introduce the following notation for Chapters 3 and 4. For a complete discrete valuation field K, $\mathcal{O}_K$ denotes its ring of integers, $\mathfrak{m}_K$ the maximal ideal, π_K a prime element, and G_K the absolute Galois group. Lowercase k denotes the residue field of K, and v_K the discrete valuation. We write $U_K^n = 1 + \mathfrak{m}_K^n$.

When we say that K is a local field, we mean that K is a complete discrete valuation field with perfect (not necessarily finite) residue field. Similarly, when we say K is a q-dimensional local field, we mean that there is a chain of fields $K = K_q, K_{q-1}, \dots, K_1, K_0$ such that, for each $1 \leq i \leq q$, K_i is a complete discrete valuation field with residue field K_{i-1} and K_0 is a perfect field. When the last residue field K_0 is finite, we say that K is a q-dimensional local field with finite last residue field.

We write

$$\hat{\Omega}^1_{\mathcal{O}_K}(\log) = \varprojlim_m \Omega^1_{\mathcal{O}_K}(\log)/\mathfrak{m}_K^m \Omega^1_{\mathcal{O}_K}(\log),$$

where

$$\Omega^1_{\mathcal{O}_K}(\log) = (\Omega^1_{\mathcal{O}_K} \oplus (\mathcal{O}_K \otimes_{\mathbb{Z}} K^\times))/(da - a \otimes a,\, a \in \mathcal{O}_K,\, a \neq 0).$$

We shall denote by P_{tor} the torsion part of an abelian group P. Let L/K a separable extension of complete discrete valuation fields (of either mixed characteristic or positive characteristic $p > 0$). Throughout these chapters, $e(L/K)$ shall denote the ramification index of L/K and e_K the absolute ramification index of K. When k is perfect, $\delta_{\text{tor}}(L/K)$ shall denote the length of $\left(\dfrac{\hat{\Omega}^1_{\mathcal{O}_L}(\log)}{\mathcal{O}_L \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K}(\log)}\right)_{\text{tor}}$.

The r-th Milnor K-group of L shall be denoted by $K_r(L)$. We denote by $U^n K_r(L)$ the subgroup of $K_r(L)$ generated by elements $\{a, b_1, \ldots, b_{r-1}\}$ where $a \in U_L^n$, $b_i \in L^\times$, and we write

$$\hat{K}_r(L) = \varprojlim_n K_r(L)/U^n K_r(L)$$

and

$$U^n \hat{K}_r(L) = \varprojlim_{n'} U^n K_r(L)/U^{n'} K_r(L).$$

Following the notation in [8], we write, for A a ring over $\mathbb{Q}$ or a smooth ring over a field of characteristic $p > 0$, and $n \neq 0$,

$$H^q_n(A) = H^q((\operatorname{Spec} A)_{\text{et}}, \mathbb{Z}/n\mathbb{Z}(q-1))$$

and

$$H^q(A) = \varinjlim_n H^q_n(A).$$

3.1 Swan conductor in positive characteristic

Let L be complete discrete valuation field of equal characteristic $p > 0$. In this section, we will study separable extensions L/K where K is a local field (and therefore k is perfect). To be precise, we shall show that, if $\chi \in H^1(K)$ has Swan conductor sufficiently large, then

$$\operatorname{Sw} \chi_L = e \operatorname{Sw} \chi - \delta_{\text{tor}}(L/K),$$

where χ_L is the image of χ in $H^1(L)$ and $e = e(L/K)$. For that goal, we will use valuations on differential forms and Witt vectors, as well as the notion of a Witt vector being "best", defined later.

First of all, we review some concepts necessary for our discussion. By completed free $\mathcal{O}_L$-module with basis $\{e_\lambda\}_{\lambda\in\Lambda}$, we mean $\varprojlim_m M/\mathfrak{m}_L^m M$, where M is the free $\mathcal{O}_L$-module with basis $\{e_\lambda\}_{\lambda\in\Lambda}$. Write $L = l((\pi_L))$ for some prime $\pi_L \in L$, where l is the residue field of L. Let $\{b_\lambda\}_{\lambda\in\Lambda}$ be a lift of a p-basis of l to $\mathcal{O}_L$. Then $\hat{\Omega}^1_{\mathcal{O}_L}(\log)$ is the completed free $\mathcal{O}_L$-module with basis $\{db_\lambda, d\log\pi_L : \lambda \in \Lambda\}$. Write $\hat{\Omega}^1_L = L \otimes_{\mathcal{O}_L} \hat{\Omega}^1_{\mathcal{O}_L}(\log)$.

Recall that, when K is a local field of positive characteristic, $\hat{\Omega}^1_{\mathcal{O}_K}(\log)$ is free of rank one and, for an extension of complete discrete valuation fields L/K, $\delta_{\mathrm{tor}}(L/K)$ is the length of the torsion part of $\hat{\Omega}^1_{\mathcal{O}_L/\mathcal{O}_K}(\log)$.

Denote by $W_s(L)$ the Witt vectors of length s. There is a homomorphism $d : W_s(L) \to \hat{\Omega}^1_L$ given by

$$a = (a_{s-1}, \dots, a_0) \mapsto \sum_i a_i^{p^i-1} da_i.$$

Remark 3.1.1. In the literature, the operator $d : W_s(L) \to \hat{\Omega}^1_L(\log)$ is often denoted by $F^{s-1}d$.

We can define valuations on $\hat{\Omega}^1_L$ and $W_s(L)$ as follows. If $\omega \in \hat{\Omega}^1_L$ and $a \in W_s(L)$, let

$$v_L^{\log}\omega = \sup\left\{n : \omega \in \pi_L^n \otimes_{\mathcal{O}_L} \hat{\Omega}^1_{\mathcal{O}_L}(\log)\right\},$$

and

$$v_L(a) = -\max_i\{-p^i v_L(a_i)\} = \min_i\{p^i v_L(a_i)\}.$$

These valuations define increasing filtrations of $\hat{\Omega}^1_L$ and $W_s(L)$ by the subgroups

$$F_n\hat{\Omega}^1_L = \{\omega \in \hat{\Omega}^1_L : v_L^{\log}\omega \geq -n\}$$

and

$$F_n W_s(L) = \{a \in W_s(L) : v_L(a) \geq -n\},$$

respectively, where $n \in \mathbb{Z}_{\geq 0}$. The latter filtration was defined by Brylinski in [5].

By the theory of Artin-Schreier-Witt, there are isomorphisms

$$W_s(L)/(F-1)W_s(L) \simeq H^1(L, \mathbb{Z}/p^s\mathbb{Z}),$$

where F is the endomorphism of Frobenius. Kato defined in [8] the filtration $F_n H^1(L, \mathbb{Z}/p^s\mathbb{Z})$ as the image of $F_n W_s(L)$ under this map. We recall that, for $\chi \in H^1(L, \mathbb{Z}/p^s\mathbb{Z})$, the Swan conductor $\mathrm{Sw}\,\chi$ is the smallest n such that $\chi \in F_n H^1(L, \mathbb{Z}/p^s\mathbb{Z})$.

We shall now define what it means for a Witt vector $a \in W_s(L)$ to be "best", as well as the notion of relevance length.

Definition 3.1.2. Let $a \in W_s(L)$, and n be the smallest non-negative integer such that $a \in F_n W_s(L)$. We say that a is best if there is no $a' \in W_s(L)$ mapping to the same element as a in $H^1(L, \mathbb{Z}/p^s\mathbb{Z})$ such that $a' \in F_{n'} W_s(L)$ for some non-negative integer $n' < n$.

When $v_L(a) \geq 0$, a is clearly best. When $v_L(a) < 0$, a is best if and only if there are no $a', b \in W_s(L)$ satisfying

$$a = a' + (F-1)b$$

and $v_L(a) < v_L(a')$.

Observe that $a \in F_n W_s(L) \backslash F_{n-1} W_s(L)$ is best if and only if $n = \mathrm{Sw}\,\chi$, where χ is the image of a under $F_n W_s(L) \to H^1(L, \mathbb{Z}/p^s\mathbb{Z})$. We remark that "best a" is not unique.

We shall start by deducing a simple criterion for determining when a is best. When $s = 1$ the characterization of "best a" is well-known: every $a \in \mathcal{O}_L$ is best, and $a \in L \setminus \mathcal{O}_L$ is best if and only if either $p \nmid v_L(a)$ or $p \mid v_L(a)$ but $\bar{a} \notin l^p$, where $\bar{a}$ denotes the residue class of $a/\pi_L^{v_L(a)}$ for a prime element $\pi_L \in L$. In this section we will characterize best a for arbitrary s. We shall prove that a is best if and only if a_i is best for some relevant position i, in the sense of the following definition.

Definition 3.1.3. We shall say that the i-th position of a is relevant if $v_L(a) = p^i v_L(a_i)$. Let $j = \max\{i : v_L(a) = p^i v_L(a_i)\}$. Then $j+1$ shall be called the relevance length of a.

Lemma 3.1.4. *Let $a \in W_s(L)$ be of negative valuation. We have $v_L(a) = v_L^{\log}(da)$ if and only if there is some relevant position k such that $v_L(a_k) = v_L^{\log}(da_k)$.*

Proof. Let I denote the subset of $\{0, \ldots, s-1\}$ consisting of i such that the i-th position is relevant and $v(a_i) = v_L^{\log}(da_i)$. Let $j+1$ denote the relevance length of a. We have

$$da = \sum_{i \in I} a_i^{p^i - 1} da_i + \sum_{i \notin I} a_i^{p^i - 1} da_i.$$

Clearly

$$v_L^{\log}\left(\sum_{i \notin I} a_i^{p^i - 1} da_i\right) > v_L(a),$$

so it is enough to prove that

$$v_L^{\log}\left(\sum_{i \in I} a_i^{p^i - 1} da_i\right) = v_L(a)$$

if I is nonempty.

Assume I nonempty. Since the relevance length of a is $j+1$, we get that $p^j \mid v_L(a)$. We have $v_L(a) = -np^j$ for some $n \in \mathbb{N}$. For each $i \in I$, we have $v_L(a_i) = -np^{j-i}$. Write $a_i = \pi_L^{-np^{j-i}} u_i$, where $u_i \in \mathcal{O}_L$ is a unit.

Then

$$\sum_{i \in I} a_i^{p^i - 1} da_i = \pi_L^{-np^j} \left(\sum_{i \in I} u_i^{p^i - 1} du_i - nu_j^{p^j} \frac{d\pi_L}{\pi_L}\right).$$

If $p \nmid n$, then

$$v_L^{\log}\left(\sum_{i \in I} a_i^{p^i - 1} da_i\right) = v_L(a).$$

On the other hand, if $p \mid n$,

$$\sum_{i\in I} a_i^{p^i-1} da_i = \pi_L^{-np^j} \sum_{i\in I} u_i^{p^i-1} du_i.$$

Let $\bar{u}_i$ denote the image of u_i in the residue field l. Then

$$v_L^{\log}\left(\pi_L^{-np^j} \sum_{i\in I} u_i^{p^i-1} du_i\right) > v_L(a)$$

if and only if

$$\sum_{i\in I} \bar{u}_i^{p^i-1} d\bar{u}_i = 0.$$

If

$$\sum_{i\in I} \bar{u}_i^{p^i-1} d\bar{u}_i = 0,$$

then, by repeatedly applying the Cartier operator, we see that $\bar{u}_i \in l^p$ for every $i \in I$. This implies $v_L(a_i) < v_L^{\log}(da_i)$ for every $i \in I$, a contradiction. Hence we must have

$$v_L^{\log}(da) = v_L(a).$$

□

Lemma 3.1.5. *Let $a \in W_s(L)$ be of negative valuation. Assume that $v_L(a) < v_L^{\log}(da)$ and the relevance length of a is 1. Then a is not best.*

Proof. Since the relevance length is 1, we have $v_L^{\log}(a_i^{p^i-1} da_i) \geq p^i v_L(a_i) > v_L(a_0)$ for $i > 0$. Therefore we must have $v_L(a_0) < v_L^{\log}(da_0)$, which implies that there exist $a_0', b_0 \in L$ such that $a_0 = a_0' + b_0^p - b_0$ and $v_L(a_0) < v_L(a_0')$. Let $a' = (0, \ldots, 0, a_0')$ and $b = (0, \ldots, 0, b_0)$. We have

$$a = a' + (F-1)b,$$

and $v_L(a) = v_L(a_0) < v_L(a')$, so a is not best. □

Lemma 3.1.6. *Let $a \in W_s(L)$ be an element of negative valuation. Assume that $v_L(a) <$*

$v_L^{\log}(da)$. *Then a is not best.*

Proof. We shall prove by induction on the relevance length. The case in which a has relevance length 1 has been proven in Lemma 3.1.5. Assume now that a has relevance length $j+1$.

From Lemma 3.1.4, $v(a_j) < v_L^{\log}(da_j)$, so there exist $a'_j, b_j \in L$ such that $a_j = a'_j + b_j^p - b_j$ and $v_L(a_j) < v_L(a'_j)$. Observe that $v_L(a_j) = pv_L(b_j)$. Let $b = (0, \dots, 0, b_j, 0, \dots, 0)$ and $a' = a - (F-1)b$. Then

$$a' = a - Fb + b = (a_{s-1}, \dots, a_{j+1}, a'_j, \tilde{a}_{j-1}, \dots \tilde{a}_0),$$

where $p^i v_L(\tilde{a}_i) \geq v_L(a)$ for every $0 \leq i \leq j-1$.

We have two cases. If $p^i v_L(\tilde{a}_i) > v_L(a)$ for all $0 \leq i \leq j-1$, then $v_L(a') > v_L(a)$, so a is not best.

On the other hand, if $v_L(\tilde{a}_i) = v_L(a)$ for some $0 \leq i \leq j-1$, then a' has relevance length at most j and $v_L(a') = v_L(a)$. Further, $da' = da + db$. Since $v_L(a) < v_L^{\log}(da)$ and $v_L(a) = pv_L(b) \leq pv_L^{\log}(db)$, we have $v_L(a) < v_L^{\log}(da')$. Thus $v_L(a') < v_L^{\log}(da')$ and a' is of relevance length at most j. By induction, a' is not best, i.e., there are $a'', c \in W_s(L)$ such that

$$a' = a'' + (F-1)c,$$

with $v(a') < v(a'')$. Then

$$a = a' + (F-1)b = a'' + (F-1)(b+c),$$

with $v_L(a) < v_L(a'')$. Thus a is not best. □

Theorem 3.1.7. *Let $a \in W_s(L)$. The following conditions are equivalent:*

(i) a is best.

(ii) There exists some relevant position i such that a_i is best in the sense of length one.

(iii) $v_L(a) = v_L^{\log}(da)$.

Proof. Observe that, when a has non-negative valuation, $(i), (ii)$ and (iii) are all simultaneously satisfied, so in the following we assume $v_L(a) < 0$.

$(ii) \Leftrightarrow (iii)$ by Lemma 3.1.4.

Lemma 3.1.6 proves $(i) \Rightarrow (iii)$.

To prove $(iii) \Rightarrow (i)$, assume that a is not best. Then there are $a', b \in W_s(L)$ such that $a = a' + (F-1)b$ and $v_L(a) < v_L(a')$. We have $pv_L^{\log}(db) \geq pv_L(b) = v_L(a)$, so both $v_L^{\log}(db) > v_L(a)$ and $v_L^{\log}(da') \geq v_L(a') > v_L(a)$. Since $da = da' - db$, we get that $v_L^{\log}(da) > v_L(a)$. ☐

The notion of "best a" allows us to construct a homomorphism $F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \to F_n\hat{\Omega}_L^1/F_{\lfloor n/p \rfloor}\hat{\Omega}_L^1$ satisfying some useful properties. Given an element of $H^1(L, \mathbb{Z}/p^s\mathbb{Z})$, it is easy to show the existence of a best $a \in W_s(L)$ in its preimage. We then have the following proposition:

Proposition 3.1.8.

(i) There is a unique homomorphism

$$\mathrm{rsw} : F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \to F_n\hat{\Omega}_L^1/F_{\lfloor n/p \rfloor}\hat{\Omega}_L^1,$$

called refined Swan conductor, such that the composition

$$F_nW_s(L) \longrightarrow F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \longrightarrow F_n\hat{\Omega}_L^1/F_{\lfloor n/p \rfloor}\hat{\Omega}_L^1$$

coincides with

$$d : F_nW_s(L) \to F_n\hat{\Omega}_L^1/F_{\lfloor n/p \rfloor}\hat{\Omega}_L^1.$$

(ii) For $\lfloor n/p \rfloor \leq m \leq n$, the induced map

$$\mathrm{rsw} : F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z})/F_mH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \to F_n\hat{\Omega}_L^1/F_m\hat{\Omega}_L^1$$

is injective.

Proof. To prove assertion *(i)*, define rsw as follows. Given an element $\chi \in F_n H^1(L, \mathbb{Z}/p^s\mathbb{Z})$, take $a \in F_n W_s(L)$ such that a is best and the image of a is χ. Then put $\operatorname{rsw}\chi = da$.

We must show that this map is well-defined. Let $a' \in F_n W_s(L)$ be another element that is best and maps to χ. Then

$$a = a' + (F-1)b$$

for some $b \in W_s(L)$. We get that $pv_L^{\log}(db) \geq pv_L(b) \geq -n$, so $db \in F_{\lfloor n/p \rfloor}\hat{\Omega}^1_L$. Since $da = da' - db$, da and da' define the same class in $F_n\hat{\Omega}^1_L / F_{\lfloor n/p \rfloor}\hat{\Omega}^1_L$. Uniqueness of the map is clear.

We shall now prove *(ii)*. Let $\chi \in F_n H^1(L, \mathbb{Z}/p^s\mathbb{Z})$ such that $\operatorname{rsw}\chi \in F_m\hat{\Omega}^1_L$. Take $a \in F_n W_s(L)$ that is best and such that $da = \operatorname{rsw}\chi$. Since a is best, we have

$$v_L^{\log}(\operatorname{rsw}\chi) = v_L^{\log}(da) = v_L(a) \geq -m,$$

so $a \in F_m W_s(L)$. It follows that $\chi \in F_m H^1(L, \mathbb{Z}/p^s\mathbb{Z})$. □

Remark 3.1.9. Related results were obtained by Y. Yatagawa in [28], where the author compares the non-logarithmic filtrations of Matsuda ([19]) and Abbes-Saito ([1]) in positive characteristic.

Remark 3.1.10. Our refined Swan conductor rsw is a refinement of the refined Swan conductor defined by K. Kato in [8, §5].

Let L/K be a separable extension of complete discrete valuation fields of positive characteristic $p > 0$, and assume that K has perfect residue field k. Let $\chi \in H^1(K)$ and χ_L its image in $H^1(L)$. We shall now use Proposition 3.1.8 to compute the Swan conductor of χ_L. We will need the following lemma:

Lemma 3.1.11. *Let L/K be a separable extension of complete discrete valuation fields of equal characteristic $p > 0$. Write $e = e(L/K)$ and assume that k is perfect.*

Let $\omega \in \hat{\Omega}^1_K$, and ω_L be the image of ω in $\hat{\Omega}^1_L$. Then

$$v_L^{\log}(\omega_L) = ev_K^{\log}(\omega) + \delta_{\mathrm{tor}}(L/K).$$

Proof. Since the residue field k of K is perfect, $\hat{\Omega}^1_{\mathcal{O}_K}(\log) = \mathcal{O}_K \frac{d\pi_K}{\pi_K}$. Let $\{b_\lambda\}_{\lambda\in\Lambda}$ be a lift of a p-basis of l to $\mathcal{O}_L$, so that $\hat{\Omega}^1_{\mathcal{O}_L}(\log)$ is the completed free module with basis $\{db_\lambda, d\log\pi_L : \lambda \in \Lambda\}$. Write $\frac{d\pi_K}{\pi_K} = \sum \alpha_\lambda db_\lambda + \alpha d\log\pi_L$, where $\alpha_\lambda, \alpha \in \mathcal{O}_L$. Then

$$\delta_{\mathrm{tor}}(L/K) = \min\{\{v_L(\alpha)\} \cup \{v_L(\alpha_\lambda) : \lambda \in \Lambda\}\} = v_L^{\log}\left(\frac{d\pi_K}{\pi_K}\right).$$

Writing $\omega = \gamma\frac{d\pi_K}{\pi_K}$ for some $\gamma \in K$, we see that

$$v_L^{\log}(\omega) = v_L(\gamma) + v_L^{\log}\left(\frac{d\pi_K}{\pi_K}\right) = ev_K(\gamma) + \delta_{\mathrm{tor}}(L/K) = ev_K^{\log}(\omega) + \delta_{\mathrm{tor}}(L/K). \qquad \square$$

Theorem 3.1.12. *Let L/K be a separable extension of complete discrete valuation fields of equal characteristic $p > 0$. Assume that K has perfect residue field.*

Denote by $e(L/K)$ the ramification index of L/K. Assume that $\chi \in H^1(K)$ is such that

$$\mathrm{Sw}\,\chi > \frac{p}{p-1}\frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)}.$$

Let χ_L be its image in $H^1(L)$. Then

$$\mathrm{Sw}\,\chi_L = e(L/K)\,\mathrm{Sw}\,\chi - \delta_{\mathrm{tor}}(L/K).$$

Proof. Write $e = e(L/K)$. It is enough to show that, for a character $\chi \in H^1(K, \mathbb{Z}/p^s\mathbb{Z})$ corresponding to the Artin-Schreier-Witt equation $(F-1)X = a$, we have that, if $\mathrm{Sw}\,\chi > p(p-1)^{-1}e^{-1}\delta_{\mathrm{tor}}(L/K)$, then

$$\mathrm{Sw}\,\chi_L = e\,\mathrm{Sw}\,\chi - \delta_{\mathrm{tor}}(L/K).$$

To simplify notation, write $n = \operatorname{Sw}\chi$, $\delta_{\mathrm{tor}} = \delta_{\mathrm{tor}}(L/K)$. The case $e = 1$ is simple, so we assume $e > 1$. Since $\operatorname{Sw}\chi > p(p-1)^{-1}e^{-1}\delta_{\mathrm{tor}}(L/K)$, we have that $\frac{en}{p} < en - \delta_{\mathrm{tor}}$, so $\lfloor \frac{en}{p} \rfloor \leq en - \delta_{\mathrm{tor}} - 1$. From that, Theorem 3.1.7, and Lemma 3.1.11, we get that the diagram

$$\begin{array}{ccc} F_nH^1(K,\mathbb{Z}/p^s\mathbb{Z})/F_{n-1}H^1(K,\mathbb{Z}/p^s\mathbb{Z}) & \longrightarrow & F_n\hat{\Omega}^1_K/F_{n-1}\hat{\Omega}^1_K \\ \downarrow & & \downarrow \\ F_{en}H^1(L,\mathbb{Z}/p^s\mathbb{Z})/F_{en-\delta_{\mathrm{tor}}-1}H^1(L,\mathbb{Z}/p^s\mathbb{Z}) & \longrightarrow & F_{en}\hat{\Omega}^1_L/F_{en-\delta_{\mathrm{tor}}-1}\hat{\Omega}^1_L \end{array}$$

commutes, and the horizontal arrows are injective. Thus

$$\operatorname{Sw}\chi_L = e(L/K)\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K). \qquad \square$$

3.2 The example of a two-dimensional local field of mixed characteristic with finite last residue field

In Section 3.1, we proved Main Result 1. We shall now focus on proving Main Result 2. Let L/K be an extension of complete discrete valuation fields of mixed characteristic, and assume that K has perfect residue field. We will show that, if $\chi \in H^1(K)$ has Swan conductor sufficiently large, then

$$\operatorname{Sw}\chi_L = e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K),$$

where χ_L is the image of χ in $H^1(L)$ and $e = e(L/K)$ is the ramification index of L/K.

The proof of this result is based on two key ideas: the commutativity of a diagram of the form

$$\begin{array}{ccc} \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\ \downarrow{\scriptstyle \operatorname{Res}_{L/K}} & & \downarrow{\scriptstyle \operatorname{Res}_{L/K}} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

and a modified version of higher dimensional local class field theory. In order to facilitate comprehension and illustrate the main ideas, in the present section we will consider, in a brief and expository way, the special case in which L is a two-dimensional local field with finite last residue field. In this special case, the second key idea is simpler, since we can use two-dimensional local class field theory without any modification. In Section 3.3 we consider the general case in which L is a complete discrete valuation field of mixed characteristic.

Through this section, we let L be a two-dimensional local field of mixed characteristic with residue field l of characteristic $p > 0$, and $K \subset L$ a one-dimensional local field with finite residue field k.

As a consequence of [20], there is a residue homomorphism

$$\mathrm{Res}_{L/K} : \hat{\Omega}^1_{\mathcal{O}_L} \to \mathcal{O}_K$$

which induces

$$\mathrm{Res}_{L/K} : \hat{\Omega}^1_{\mathcal{O}_L}(\log) \otimes_{\mathcal{O}_L} L \to K.$$

Example 3.2.1. When $L = K\{\{T\}\}$ (see page 5),

$$\mathrm{Res}_{L/K}\left(\sum_{i=-\infty}^{\infty} a_i T^i \frac{dT}{T}\right) = a_0.$$

From [16], if $\eta \in \mathcal{O}_L$ is such that $v_L(\eta) \geq \frac{2e_L}{p-1} + 1$, there exists an exponential map

$$\exp_\eta : \hat{\Omega}^1_{\mathcal{O}_L}(\log) \to \hat{K}_2(L).$$

This map is used in the following theorem, which is the first key step in the proof of the main result for the special case of a two-dimensional local field with finite last residue field. Its proof is omitted due to similarity with that of Theorem 3.3.11.

Theorem 3.2.2. *Let L be a two-dimensional local field of mixed characteristic and with finite last residue field, and $K \subset L$ a local field. Write $e = e(L/K)$. Let $\eta \in \mathcal{O}_K$ be such* *that*

$$n = v_K(\eta) \geq \frac{2e_K}{p-1} + \frac{1}{e}.$$

Then, if $n' \in \mathbb{N}$ *satisfies*

$$n' \geq \frac{\delta_{\mathrm{tor}}(L/K)}{e},$$

we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^1_{\mathcal{O}_L}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_2(L) \\ \downarrow{\scriptstyle \mathrm{Res}_{L/K}} & & \downarrow{\scriptstyle \mathrm{Res}_{L/K}} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

where the right vertical arrow is the residue homomorphism from K-theory defined in [7] and the top and bottom horizontal maps are, respectively, the exponential maps $\exp_{\eta,2}$ *and* $\exp_{\eta,1}$ *defined in [16].*

We observe that $\mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^1_{\mathcal{O}_L}(\log) \to \mathfrak{m}_K^{n'}$ in the diagram above is surjective (see Proposition 3.3.10) and the images of

$$\exp_\eta : \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^1_{\mathcal{O}_L}(\log) \to \hat{K}_2(L)$$

and

$$\exp_\eta : \mathfrak{m}_K^{n'} \to K^\times$$

are, respectively, $U^{e(n+n')-\delta_{\mathrm{tor}}(L/K)}\hat{K}_2(L)$ and $U_K^{n+n'}$ (see Lemma 3.3.2).

Theorem 3.2.2 is then combined with two-dimensional local class field theory to prove the main result in the particular case of a two-dimensional local field of mixed characteristic with finite last residue field:

Theorem 3.2.3. *Let L be a two-dimensional local field of mixed characteristic with finite last residue field, and* $K \subset L$ *be a local field. Assume that* $\chi \in H^1(K)$ *is such that*

$$\mathrm{Sw}\,\chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by χ_L its image in $H^1(L)$. Then

$$\operatorname{Sw}\chi_L = e(L/K)\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K).$$

Proof. Write $e = e(L/K)$. Let $n' = \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e} \right\rceil$ and $n = \operatorname{Sw}\chi - n'$. Pick $\eta \in \mathcal{O}_K$ with $v_K(\eta) = n$. By two-dimensional local class field theory, the diagram

$$\begin{array}{ccc} \hat{K}_2(L) & \longrightarrow & G_L^{\mathrm{ab}} \\ \downarrow{\scriptstyle \mathrm{Res}_{L/K}} & & \downarrow \\ K^\times & \longrightarrow & G_K^{\mathrm{ab}} \end{array}$$

commutes. Together with Theorem 3.2.2, this gives us a commutative diagram

$$\begin{array}{ccccc} \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^1_{\mathcal{O}_L}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_2(L) & \longrightarrow & G_L^{\mathrm{ab}} \\ \downarrow{\scriptstyle \mathrm{Res}_{L/K}} & & \downarrow{\scriptstyle \mathrm{Res}_{L/K}} & & \downarrow \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times & \longrightarrow & G_K^{\mathrm{ab}} \end{array}$$

From Proposition 3.3.10, the left vertical arrow is surjective. We know that $\operatorname{Sw}\chi = m$ if and only if χ kills U_K^{m+1} but not U_K^m, and $\operatorname{Sw}\chi_L = m$ if and only if χ_L kills $U^{m+1}\hat{K}_2(L)$ but not $U^m\hat{K}_2(L)$ (see the proof of Proposition 3.3.12 for details). Then it follows from the commutative diagram above and Lemma 3.3.2 that

$$\operatorname{Sw}\chi_L = e(n'+n) - \delta_{\mathrm{tor}}(L/K) = e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K). \qquad \square$$

As a guide for Section 3.3, we will use Theorem 3.2.3 to get the same result for a complete discrete valuation field of mixed characteristic L which has residue field that is a function field in one variable over a finite field. In Section 3.3, Proposition 3.3.12 will be used to obtain Theorem 3.3.13 in an analogous way.

Corollary 3.2.4. *Let L be a complete discrete valuation field of mixed characteristic, and $K \subset L$ be a local field. Assume that the residue field l of L is a function field in one variable over the finite residue field k of K.*

Assume that $\chi \in H^1(K)$ *is such that*

$$\mathrm{Sw}\,\chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by χ_L *its image in* $H^1(L)$*. Then*

$$\mathrm{Sw}\,\chi_L = e(L/K)\,\mathrm{Sw}\,\chi - \delta_{\mathrm{tor}}(L/K).$$

Proof. It is sufficient to prove that this case can be reduced to that of a two-dimensional local field with finite last residue field.

Since l is a function field in one variable over k, l is a finite separable extension of $k(T)$ for some transcendental element T. Then there is an embedding of l into a finite separable extension E of $k((T))$. Note that $\{T\}$ is a p-basis for both l and E. Then there is a complete discrete valuation field $L(E)$ which is an extension of L satisfying $\mathcal{O}_L \subset \mathcal{O}_{L(E)}$, $\mathcal{O}_{L(E)}\mathfrak{m}_L = \mathfrak{m}_{L(E)}$, and the residue field of $L(E)$ is isomorphic to E over l.

From [8, Lemma 6.2], we get that $\mathrm{Sw}\,\chi_{L(E)} = \mathrm{Sw}\,\chi_L$. Further, since E is a one-dimensional local field, $L(E)$ is a two-dimensional local field. Finally, since E and l have the same p-basis $\{T\}$, and π_L is a prime for both L and $L(E)$, the map $\mathcal{O}_{L(E)} \otimes_{\mathcal{O}_L} \hat{\Omega}^1_{\mathcal{O}_L}(\log) \to \hat{\Omega}^1_{\mathcal{O}_{L(E)}}(\log)$ is an isomorphism and we get $\mathcal{O}_{L(E)} \otimes_{\mathcal{O}_L} \hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}} \simeq \hat{\Omega}^1_{\mathcal{O}_{L(E)}}(\log)_{\mathrm{tor}}$. Therefore, by definition, $\delta_{\mathrm{tor}}(L(E)/K) = \delta_{\mathrm{tor}}(L/K)$.

Thus it is sufficient to prove that

$$\mathrm{Sw}\,\chi_{L(E)} = e(L(E)/K)\,\mathrm{Sw}\,\chi - \delta_{\mathrm{tor}}(L(E)/K),$$

which follows from Theorem 3.2.3. □

3.3 Swan conductor in the general mixed characteristic case

In this section, we shall generalize the results of the previous section to the more general case in which L is any complete discrete valuation field of mixed characteristic. We start by briefly reviewing some necessary background and proving some preliminary results.

Let L be a complete discrete valuation field of mixed characteristic. Let B be a lift of a p-basis of the residue field l to $\mathcal{O}_L$. Write $\{e_\lambda\}_{\lambda\in\Lambda} = \{db : b \in B\} \cup \{d\log \pi_L\}$. The $\mathcal{O}_L$-module $\hat{\Omega}^1_{\mathcal{O}_L}(\log)$ has the structure

$$\hat{M} \oplus \mathcal{O}_L/\mathfrak{m}_L^a \mathcal{O}_L$$

for some $a \in \mathbb{Z}_{\geq 0}$ (see [15, Lemma 1.1] and [10, 4.3]). Here $\hat{M}$ is the completed free $\mathcal{O}_L$-module with basis $\{e_\lambda\}_{\lambda\in\Lambda-\{\mu\}}$, i.e., $\hat{M} = \varprojlim_m M/\mathfrak{m}_L^m M$ where M is the free $\mathcal{O}_L$-module with basis $\{e_\lambda\}_{\lambda\in\Lambda-\{\mu\}}$ for some $\mu \in \Lambda$.

We have, from [16, Theorem 0.1], the existence of an exponential map

$$\exp_{\eta,r+1} : \hat{\Omega}^r_{\mathcal{O}_L}(\log) \to \hat{K}_{r+1}(L)$$

when $\eta \in \mathcal{O}_L$ satisfies

$$v_L(\eta) \geq \frac{2e_L}{p-1} + 1.$$

This exponential map satisfies

$$a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_r}{b_r} \mapsto \{\exp(\eta a), b_1, \ldots, b_r\}$$

for $a \in \mathcal{O}_L$, $b_i \in \mathcal{O}_L^\times$. We shall denote $\exp_{\eta,r+1}$ simply by $\exp_\eta$ through this chapter.

Remark 3.3.1. More precisely, in [16], M. Kurihara proved the existence of an exponential map

$$\exp_{\eta,r+1} : \hat{\Omega}^r_{\mathcal{O}_L} \to \hat{K}_{r+1}(L)$$

when $\eta \in \mathcal{O}_L$ satisfies

$$v_L(\eta) \geq \frac{2e_L}{p-1}.$$

Considering the existence of a map $\hat{\Omega}^r_{\mathcal{O}_L}(\log) \to \hat{\Omega}^r_{\mathcal{O}_L}$ satisfying the commutative diagram

$$\begin{array}{ccc} \hat{\Omega}^r_{\mathcal{O}_L}(\log) & \xrightarrow{\pi_L} & \hat{\Omega}^r_{\mathcal{O}_L} \\ \uparrow & \overset{\pi_L}{\nearrow} & \\ \hat{\Omega}^r_{\mathcal{O}_L} & & \end{array}$$

we can define, for

$$v_L(\eta) \geq \frac{2e_L}{p-1} + 1,$$

an exponential map

$$\exp^{\log}_{\eta,r+1} : \hat{\Omega}^r_{\mathcal{O}_L}(\log) \to \hat{K}_{r+1}(L)$$

by taking the composite

$$\exp^{\log}_{\eta,r+1} = \exp_{\frac{\eta}{\pi_L},r+1} \circ \pi_L.$$

To simplify the notation, we omit the superscript log when we write this exponential map.

Lemma 3.3.2. *Let L be a complete discrete valuation field of mixed characteristic, with residue field l of characteristic $p > 0$. Assume that $\eta \in \mathcal{O}_L$ satisfies*

$$n = v_L(\eta) \geq \frac{2e_L}{p-1} + 1.$$

Then the image of the exponential map

$$\exp_\eta : \mathfrak{m}_L^{n'} \hat{\Omega}^r_{\mathcal{O}_L}(\log) \to \hat{K}_{r+1}(L)$$

is $U^{n+n'}\hat{K}_{r+1}(L)$.

Proof. Let $a \in \mathfrak{m}_L^{n'}$, $b_i \in \mathcal{O}_L^\times$. Observe that, from the definition of the exponential map and

[16, Proposition 3.2],

$$a\frac{d\pi_L}{\pi_L}\wedge\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_{r-1}}{b_{r-1}}\mapsto\{\exp(pa\eta),\pi_L,b_1,\ldots,b_{r-1}\}$$

and

$$a\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_r}{b_r}\mapsto\{\exp(a\eta),b_1,\ldots,b_r\}.$$

Then the image is contained in $U^{n+n'}\hat{K}_{r+1}(L)$. Let $\tilde{n}\geq n+n'$. Observe that the maps

$$\frac{\mathfrak{m}_L^{\tilde{n}}}{\mathfrak{m}_L^{\tilde{n}+1}}\otimes\Omega^r_{\mathcal{O}_L}(\log)\to U^{\tilde{n}}K_{r+1}(L)/U^{\tilde{n}+1}K_{r+1}(L)$$

given by

$$\alpha\otimes\beta\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_r}{b_r}\mapsto\{1+\alpha\beta,b_1,\ldots,b_r\},$$

where $\alpha\in\mathfrak{m}_L^{\tilde{n}}$, $\beta\in\mathcal{O}_L, b_i\in L^\times$, are surjective. Passing to the limit, we get that $\exp_\eta$: $\mathfrak{m}_L^{n'}\hat{\Omega}^r_{\mathcal{O}_L}(\log)\to U^{n+n'}\hat{K}_{r+1}(L)$ is surjective. □

We shall now construct some tools and intermediate steps necessary for the obtainment of the main result. For an extension of complete discrete valuation fields of mixed characteristic L/K, where k is not necessarily perfect, denote by $\delta_{\mathrm{tor}}(L/K)$ the length of

$$\frac{\hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}{\mathcal{O}_L\otimes_{\mathcal{O}_K}\hat{\Omega}^1_{\mathcal{O}_K}(\log)_{\mathrm{tor}}}.$$

Remark 3.3.3. When k is perfect, the $\mathcal{O}_K$-module $\hat{\Omega}^1_{\mathcal{O}_K}(\log)$ is a torsion module, and therefore $\delta_{\mathrm{tor}}(L/K)$ is simply the length of

$$\left(\hat{\Omega}^1_{\mathcal{O}_L/\mathcal{O}_K}(\log)\right)_{\mathrm{tor}},$$

which coincides with the definition of $\delta_{\mathrm{tor}}(L/K)$ introduced previously.

We have the following property:

Lemma 3.3.4. *Let L/M be a finite extension of complete discrete valuation fields of characteristic zero. Assume that the residue field l of L has characteristic $p>0$ and $[l:l^p]=p^r$. Write $e=e(L/M)$. Then*

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta_{\mathrm{tor}}(L/M)}\frac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right)=\mathfrak{m}_M^n\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}$$

and

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta_{\mathrm{tor}}(L/M)+1}\frac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right)=\mathfrak{m}_M^{n+1}\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}$$

for every integer n.

Proof. We shall prove the first equality. Let $\delta(L/M)$ be the length of the $\mathcal{O}_L$-module $\hat{\Omega}^1_{\mathcal{O}_L/\mathcal{O}_M}(\log)$. Observe that $\dfrac{\hat{\Omega}^1_{\mathcal{O}_L}(\log)}{\hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}$ and $\dfrac{\hat{\Omega}^1_{\mathcal{O}_M}(\log)}{\hat{\Omega}^1_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}$ are free of rank r. We have an exact sequence

$$0\to\mathcal{O}_L\underset{\mathcal{O}_M}{\otimes}\frac{\hat{\Omega}^1_{\mathcal{O}_M}(\log)}{\hat{\Omega}^1_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}\to\frac{\hat{\Omega}^1_{\mathcal{O}_L}(\log)}{\hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\to\frac{\dfrac{\hat{\Omega}^1_{\mathcal{O}_L}(\log)}{\hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}}{\mathcal{O}_L\underset{\mathcal{O}_M}{\otimes}\dfrac{\hat{\Omega}^1_{\mathcal{O}_M}(\log)}{\hat{\Omega}^1_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}}\to 0.$$

Since the length of

$$\frac{\hat{\Omega}^1_{\mathcal{O}_L}(\log)}{\hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\Bigg/\left(\mathcal{O}_L\otimes_{\mathcal{O}_M}\frac{\hat{\Omega}^1_{\mathcal{O}_M}(\log)}{\hat{\Omega}^1_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}\right)$$

is $\delta(L/M)-\delta_{\mathrm{tor}}(L/M)$, we have that the length of

$$\frac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\Bigg/\left(\mathcal{O}_L\otimes_{\mathcal{O}_M}\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}\right)$$

is also $\delta(L/M)-\delta_{\mathrm{tor}}(L/M)$. Since $\dfrac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}$ and $\dfrac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}$ are both free of rank one, we have

$$\frac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}=\mathfrak{m}_L^{\delta_{\mathrm{tor}}(L/M)-\delta(L/M)}\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}.$$

Therefore

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta_{\mathrm{tor}}(L/M)}\frac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right)=$$

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta_{\mathrm{tor}}(L/M)}\mathfrak{m}_L^{\delta_{\mathrm{tor}}(L/M)-\delta(L/M)}\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}\right)=$$

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta(L/M)}\right)\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}.$$

Let $\tilde{\delta}(L/M)$ be the length of the $\mathcal{O}_L$-module $\hat{\Omega}^1_{\mathcal{O}_L/\mathcal{O}_M}$. Since

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{e(n+1)-\tilde{\delta}(L/M)-1}\right)=\mathfrak{m}_M^n$$

and $\tilde{\delta}(L/M)=\delta(L/M)+e-1$, we get

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta(L/M)}\right)=\mathfrak{m}_M^n.$$

Hence

$$\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{en-\delta_{\mathrm{tor}}(L/M)}\frac{\hat{\Omega}^r_{\mathcal{O}_L}(\log)}{\hat{\Omega}^r_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right)=\mathfrak{m}_M^n\frac{\hat{\Omega}^r_{\mathcal{O}_M}(\log)}{\hat{\Omega}^r_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}.$$

The second equality is obtained similarly. □

We shall now make the constructions necessary for defining a residue map

$$\mathrm{Res}_{L/K}:\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)\to\mathcal{O}_K$$

for a finite extension L of $K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$, where K is a local field of mixed characteristic.

Definition 3.3.5. Let K be a complete discrete valuation field. Write $L_0=K, L_1=$

$K\{\{T_1\}\}, \ldots, L = L_{q-1} = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$. Define

$$c_{L_i/L_{i-1}} : L_i \to L_{i-1}$$

by

$$c_{L_i/L_{i-1}} \left(\sum_{k \in \mathbb{Z}} a_k T_i^k \right) = a_0.$$

Then define $c_{L/K} = c_{L_1/L_0} \circ \cdots \circ c_{L_{q-1}/L_{q-2}}$.

Definition 3.3.6. Let K be a local field of mixed characteristic and $L_0 = K, L_1 = K\{\{T_1\}\}, \ldots, L = L_{q-1} = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$. Define the residue map $\mathrm{Res}_{L_i/L_{i-1}}$ as the composition

$$\hat{\Omega}^i_{\mathcal{O}_{L_i}}(\log) \to \hat{\Omega}^i_{\mathcal{O}_{L_i}/\mathcal{O}_{L_{i-1}}}(\log) \to \hat{\Omega}^{i-1}_{\mathcal{O}_{L_{i-1}}}(\log),$$

where $\hat{\Omega}^i_{\mathcal{O}_{L_i}/\mathcal{O}_{L_{i-1}}}(\log) \to \hat{\Omega}^{i-1}_{\mathcal{O}_{L_{i-1}}}(\log)$ is the homomorphism that satisfies

$$a d\log T_1 \wedge \cdots \wedge d\log T_i \mapsto c_{L_i/L_{i-1}}(a) d\log T_1 \wedge \cdots \wedge d\log T_{i-1}$$

for $a \in \mathcal{O}_{L_i}$. Then define

$$\mathrm{Res}_{L/K} : \hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \to \mathcal{O}_K$$

as the composition

$$\mathrm{Res}_{L/K} = \mathrm{Res}_{L_1/L_0} \circ \cdots \circ \mathrm{Res}_{L_{q-1}/L_{q-2}}.$$

It induces

$$\mathrm{Res}_{L/K} : \hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \otimes_{\mathcal{O}_L} L \to K.$$

Definition 3.3.7. Let L be a finite extension of $M = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$, where K is a local field of mixed characteristic. Define the residue map

$$\mathrm{Res}_{L/K} : \hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \otimes_{\mathcal{O}_L} L \to K$$

by

$$\mathrm{Res}_{L/K} = \mathrm{Res}_{M/K} \circ \mathrm{Tr}_{L/M}.$$

Remark 3.3.8. In Definition 3.3.7, $\mathrm{Res}_{L/K}$ is expected to be independent of M. Independence has been proven when L is a two-dimensional local field ([20, 2.3.3]), but appears to remain open in the general case. This property shall not be necessary for us.

We will now start to obtain some properties of the trace and residue maps that will be necessary for the proof of the main theorem of this section.

Proposition 3.3.9. *Let L be a complete discrete valuation field that is a finite extension of $M = K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$, where K is a local field of mixed characteristic. Write $e = e(L/K)$. Then, for any integer n,*

$$\mathrm{Res}_{L/K}\left(\mathfrak{m}_L^{ne-\delta_{\mathrm{tor}}(L/K)}\frac{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)}{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right) = \mathfrak{m}_K^n$$

and

$$\mathrm{Res}_{L/K}\left(\mathfrak{m}_L^{ne-\delta_{\mathrm{tor}}(L/K)+1}\frac{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)}{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right) = \mathfrak{m}_K^{n+1}.$$

Proof. We shall prove the first equality; the second is obtained in a similar way.

Observe that $\mathrm{Res}_{L/K} = \mathrm{Res}_{M/K} \circ \mathrm{Tr}_{L/M}$. Further, $\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log)$ is generated by dT_i and $d\log\pi_K$, and its torsion part is generated by $d\log\pi_K$. Thus we have an isomorphism $\mathcal{O}_M \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K}(\log) \simeq \hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log)_{\mathrm{tor}}$. We get, by definition,

$$\delta_{\mathrm{tor}}(L/K) = \delta_{\mathrm{tor}}(L/M).$$

Then, using Lemma 3.3.4, we get

$$\mathrm{Res}_{L/K}\left(\mathfrak{m}_L^{ne-\delta_{\mathrm{tor}}(L/K)}\frac{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)}{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right)=$$

$$\mathrm{Res}_{M/K}\left(\mathrm{Tr}_{L/M}\left(\mathfrak{m}_L^{ne-\delta_{\mathrm{tor}}(L/K)}\frac{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)}{\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)_{\mathrm{tor}}}\right)\right)=$$

$$\mathrm{Res}_{M/K}\left(\mathfrak{m}_M^{n}\frac{\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log)}{\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log)_{\mathrm{tor}}}\right)=\mathfrak{m}_K^n.$$

□

Proposition 3.3.10. *Let L, K, and e be as in Proposition 3.3.9. Then, if $n\in\mathbb{N}$ satisfies*

$$n\geq\frac{\delta_{\mathrm{tor}}(L/K)}{e},$$

we have

$$\mathrm{Res}_{L/K}\left(\mathfrak{m}_L^{ne-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)\right)=\mathfrak{m}_K^n$$

and

$$\mathrm{Res}_{L/K}\left(\mathfrak{m}_L^{ne-\delta_{\mathrm{tor}}(L/K)+1}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log)\right)=\mathfrak{m}_K^{n+1}.$$

Proof. In this case $en-\delta_{\mathrm{tor}}(L/K)\geq 0$, so this follows from Proposition 3.3.9. □

We will now use the previous properties of residue and trace maps, the exponential map defined by M. Kurihara ([16]), and a modification of higher dimensional class field theory to prove that, when L is a q-dimensional local field that is a finite extension of $K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$, Main Result 2 holds. This will then be used to prove the general result. We start with the following theorem:

Theorem 3.3.11. *Let L be a q-dimensional local field that is a finite extension of $M=K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$, where K is a local field of mixed characteristic with residue field k*

of characteristic $p > 0$. Write $e = e(L/K)$. Assume that $n \in \mathbb{N}$ satisfies

$$n \geq \frac{2e_K}{p-1} + \frac{1}{e}$$

and let $n' \in \mathbb{N}$ be such that $n' \geq \frac{\delta_{\mathrm{tor}}(L/K)}{e}$. Take $\eta \in \mathcal{O}_K$ such that $v_K(\eta) = n$.

Then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\ \downarrow{\scriptstyle \mathrm{Res}_{L/K}} & & \downarrow{\scriptstyle \mathrm{Res}_{L/K}} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

where the right vertical arrow is the residue homomorphism from K-theory defined in [7] and the top and bottom horizontal maps are, respectively, the exponential maps $\exp_{\eta,q}$ and $\exp_{\eta,1}$ defined in [16]. Further, the left vertical arrow is surjective.

Proof. First, observe that the condition

$$n \geq \frac{2e_K}{p-1} + \frac{1}{e}$$

implies

$$en \geq \frac{2e_K e}{p-1} + 1 = \frac{2e_L}{p-1} + 1.$$

Therefore this condition guarantees the convergence of both the top and the bottom exponential maps (by Theorem 0.1 in [16]). Furthermore, the condition

$$n' \geq \frac{\delta_{\mathrm{tor}}(L/K)}{e}$$

guarantees that we can apply Proposition 3.3.10.

We need to prove that the diagram

$$\begin{array}{ccc}
\mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\
\downarrow \mathrm{Tr}_{L/M} & & \downarrow N_{L/M} \\
\mathfrak{m}_M^{n'}\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(M) \\
\downarrow \mathrm{Res}_{M/K} & & \downarrow \mathrm{Res}_{M/K} \\
\mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times
\end{array}$$

commutes.

By Proposition 3.3.10, the map $\mathrm{Res}_{L/K} : \hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \to \mathcal{O}_K$ induces a surjection

$$\mathrm{Res}_{L/K} : \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \twoheadrightarrow \mathfrak{m}_K^{n'},$$

and the map $\mathrm{Res}_{M/K} : \hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log) \to \mathcal{O}_K$ induces a surjection

$$\mathrm{Res}_{M/K} : \mathfrak{m}_M^{n'}\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log) \twoheadrightarrow \mathfrak{m}_K^{n'}.$$

A similar argument shows that $\mathrm{Tr}_{L/M} : \hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \to \hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log)$ induces a surjection

$$\mathrm{Tr}_{L/M} : \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) \twoheadrightarrow \mathfrak{m}_M^{n'}\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log).$$

The commutativity of the top square is shown in [16]. The commutativity of the bottom square can be checked explicitly as follows. Let $M_0 = K, M_1 = K\{\{T_1\}\}, \ldots, M_{q-1} = M = K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$. It is enough to show that each one of the squares in the diagram

$$\begin{array}{ccc}
\hat{\Omega}^{q-1}_{\mathcal{O}_M}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(M) \\
\Big\downarrow{\scriptstyle \mathrm{Res}_{M/M_{q-2}}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{M/M_{q-2}}} \\
\hat{\Omega}^{q-2}_{\mathcal{O}_{M_{q-2}}}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_{q-1}(M_{q-2}) \\
\Big\downarrow{\scriptstyle \mathrm{Res}_{M_{q-2}/M_{q-3}}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{M_{q-2}/M_{q-3}}} \\
\vdots & & \vdots \\
\Big\downarrow{\scriptstyle \mathrm{Res}_{M_2/M_1}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{M_2/M_1}} \\
\hat{\Omega}^{1}_{\mathcal{O}_{M_1}}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_2(M_1) \\
\Big\downarrow{\scriptstyle \mathrm{Res}_{M_1/K}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{M_1/K}} \\
\mathcal{O}_K & \xrightarrow{\exp_\eta} & K^\times
\end{array}$$

commutes.

Let $a \in \mathcal{O}_{M_i}$ and write

$$a = \sum_{k<0} a_k T_i^k + a_0 + \sum_{k>0} a_k T_i^k,$$

where $a_k \in \mathcal{O}_{M_{i-1}}$ for every $k \in \mathbb{Z}$. Put $a_- = \sum\limits_{k<0} a_k T_i^k$ and $a_+ = \sum\limits_{k>0} a_k T_i^k$. Observe first that, since

$$\hat{K}_i(M_{i-1}) \xrightarrow{\{\,,T_i\}} \hat{K}_{i+1}(M_i) \xrightarrow{\mathrm{Res}_{M_i/M_{i-1}}} \hat{K}_i(M_{i-1})$$

is the identity map ([7, Theorem 1]), we get

$$\mathrm{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left(a_0 \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) =$$
$$\mathrm{Res}_{M_i/M_{i-1}} \{\exp(\eta a_0), T_1, \ldots, T_i\} = \{\exp(\eta a_0), T_1, \ldots, T_{i-1}\} =$$
$$\exp_\eta \circ \mathrm{Res}_{M_i/M_{i-1}} \left(a_0 \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right).$$

Further, the same theorem gives

$$\mathrm{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left(a_+ \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) =$$
$$\exp_\eta \circ \mathrm{Res}_{M_i/M_{i-1}} \left(a_+ \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) = 0.$$

We will now show that we also have

$$\mathrm{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left(a_- \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) =$$
$$\exp_\eta \circ \mathrm{Res}_{M_i/M_{i-1}} \left(a_- \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) = 0.$$

From Theorem 1 in [7], we have, for $k \in \mathbb{Z}_{<0}$ and $m \in \mathbb{N}$,

$$\mathrm{Res}_{M_i/M_{i-1}} \left\{ 1 + \eta a_k T_i^k + \cdots + \frac{(\eta a_k)^m T_i^{mk}}{m!}, T_1, \ldots, T_i \right\} = 0.$$

Since $v_{M_{i-1}}((\eta a_k)^m/m!\,) \to \infty$ and the residue map is continuous, we have

$$\mathrm{Res}_{M_i/M_{i-1}} \left\{ \exp(\eta a_k T_i^k), T_1, \ldots, T_i \right\} = 0. \qquad (*)$$

Given $k \in \mathbb{Z}_{<0}$, write $s_k = \sum_{k \leq k' < 0} a_{k'} T_i^{k'}$. From $(*)$ we have that

$$\mathrm{Res}_{M_i/M_{i-1}} \left\{ \exp(\eta s_k), T_1, \ldots, T_i \right\} = 0.$$

By continuity and $s_k \to a_-$, we get

$$\mathrm{Res}_{M_i/M_{i-1}} \left\{ \exp(\eta a_-), T_1, \ldots, T_i \right\} = 0.$$

Hence we conclude that

$$\operatorname{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left(a\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) =$$
$$\exp_\eta \circ \operatorname{Res}_{M_i/M_{i-1}} \left(a\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right).$$

A similar argument shows that

$$\operatorname{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left(a\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_{i-1}}{T_{i-1}} \wedge \frac{d\pi_K}{\pi_K} \right) =$$
$$\exp_\eta \circ \operatorname{Res}_{M_i/M_{i-1}} \left(a\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_{i-1}}{T_{i-1}} \wedge \frac{d\pi_K}{\pi_K} \right) = 0,$$

so we conclude that each square in the diagram is commutative. □

We have now developed all the necessary tools in order to prove Proposition 3.3.12, which states that Main Result 2 holds when L is a q-dimensional local field that is a finite extension of $K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$. We will then use Proposition 3.3.12 to prove Theorem 3.3.13, which gives Main Result 2 in full generality.

Proposition 3.3.12. *Let L be a q-dimensional local field that is a finite extension of $M = K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$, where K is a local field of mixed characteristic with residue field k of characteristic $p > 0$. Assume that $\chi \in H^1(K)$ is such that*

$$\operatorname{Sw}\chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil .$$

Denote by χ_L its image in $H^1(L)$. Then

$$\operatorname{Sw}\chi_L = e(L/K)\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K).$$

Proof. Using the same argument as in [8, (7.6)], we can assume $H^1_p(k) \neq 0$. Let $L = L_q, l = L_{q-1}, \ldots, L_1, L_0$ be the chain of residue fields of the q-dimensional local field L. Since there are isomorphisms ([6, Theorem 3])

$$H^{q+1}(L)\{p\} \simeq H^{q}(L_{q-1})\{p\} \simeq H^{q-1}(L_{q-2})\{p\} \simeq \cdots \simeq H^{1}(L_0)\{p\}$$

and

$$H^{2}(K)\{p\} \simeq H^{1}(k)\{p\},$$

we have a commutative diagram

$$\begin{array}{ccccccc}
H^1(L) & \times & \hat{K}_q(L) & \xrightarrow{\{\ ,\ \}_L} & H^{q+1}(L)\{p\} & \xrightarrow{\ \simeq\ } & H^1(L_0)\{p\} \\
\uparrow & & \downarrow{\scriptstyle \mathrm{Res}_{L/K}} & & & & \downarrow \\
H^1(K) & \times & K^{\times} & \xrightarrow{\{\ ,\ \}_K} & H^{2}(K)\{p\} & \xrightarrow{\ \simeq\ } & H^1(k)\{p\}
\end{array}$$

Here, the pairing $H^1(L) \times \hat{K}_q(L) \to H^{q+1}(L)\{p\}$ is the one constructed in [8]. Denote the composition $H^1(L) \times \hat{K}_q(L) \to H^{q+1}(L)\{p\} \to H^1(k)\{p\}$ by $\{\ ,\ \}_k$. Similarly, denote the composition $H^1(L) \times \hat{K}_q(L) \to H^{q+1}(L)\{p\} \to H^q(l)\{p\}$ by $\{\ ,\ \}_l$. Since the last arrow is an isomorphism, $\{A, B\}_L = 0$ if and only if $\{A, B\}_l = 0$, where $A \in H^1(L)$ and $B \in \hat{K}_q(L)$.

Observe that $H^1_p(L_0) \neq 0$. Indeed, $H^1_p(L_0) \simeq L_0/(x^p - x, x \in L_0)$ and $H^1_p(k) \simeq k/(x^p - x, x \in k)$, so $H^1_p(L_0) \twoheadrightarrow H^1_p(k)$ follows from the compatibility between the corestriction map and the trace map. Since $H^1_p(k) \neq 0$, we also have $H^1_p(L_0) \neq 0$.

From [8, Proposition 6.5], we have that

$$\mathrm{Sw}\,\chi_L = m \geq 1$$

if and only if

$$\{\chi_L, U^{m+1}\hat{K}_q(L)\}_L = 0$$

but

$$\{\chi_L, U^{m}\hat{K}_q(L)\}_L \neq 0.$$

To simplify notation, put $e = e(L/K)$, $n' = \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e} \right\rceil$ and $n = \operatorname{Sw}\chi - n'$. Pick $\eta \in \mathcal{O}_K$ such that $v_K(\eta) = n$. From Lemma 3.3.2, the commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_L^{en'-\delta_{\mathrm{tor}}(L/K)}\hat{\Omega}^{q-1}_{\mathcal{O}_L}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\ \downarrow \mathrm{Res}_{L/K} & & \downarrow \mathrm{Res}_{L/K} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

given by Theorem 3.3.11, and the surjectivity of the left vertical arrow, we have that

$$\{\chi_L, U^{e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K)+1}\hat{K}_q(L)\}_k =$$
$$\{\chi_L, U^{en'-\delta_{\mathrm{tor}}(L/K)+en+1}\hat{K}_q(L)\}_k = \{\chi, U_K^{\operatorname{Sw}\chi+1}\}_k = 0$$

but

$$\{\chi_L, U^{e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K)}\hat{K}_q(L)\}_k =$$
$$\{\chi_L, U^{en'-\delta_{\mathrm{tor}}(L/K)+en}\hat{K}_q(L)\}_k = \{\chi, U_K^{\operatorname{Sw}\chi}\}_k \neq 0.$$

This clearly yields $\{\chi_L, U^{e\operatorname{Sw}\chi-\delta_{\mathrm{tor}}(L/K)}\hat{K}_q(L)\}_L \neq 0$, so $\operatorname{Sw}\chi_L \geq e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K)$. It remains to show that $\operatorname{Sw}\chi_L \leq e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K)$.

Assume that $s = \operatorname{Sw}\chi_L > e\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K)$. The key point is to show that

$$\{\chi_L, U^s\hat{K}_q(L)\}_l \supset H_p^q(l).$$

Indeed, if $\{\chi_L, U^s\hat{K}_q(L)\}_l \supset H_p^q(l)$, then, from the isomorphisms

$$H_p^q(l) \simeq \cdots \simeq H_p^1(L_0)$$

obtained in [6] and the surjectivity of $H_p^1(L_0) \twoheadrightarrow H_p^1(k)$, we get that

$$\{\chi_L, U^s\hat{K}_q(L)\}_k \neq 0.$$

This is a contradiction because

$$\{\chi_L, U^s\hat{K}_q(L)\}_k \subset \{\chi_L, U^{e\,\mathrm{Sw}\,\chi-\delta_{\mathrm{tor}}(L/K)+1}\hat{K}_q(L)\}_k = \{\chi, U_K^{\mathrm{Sw}\,\chi+1}\}_k = 0.$$

We will now show that $\{\chi_L, U^s\hat{K}_q(L)\}_l \supset H_p^q(l)$. Since l is of characteristic $p > 0$, there is an isomorphism

$$H_p^q(l) \simeq \mathrm{Coker}\left(F - 1 : \Omega_l^{q-1} \longrightarrow \Omega_l^{q-1}/d\Omega_l^{q-2}\right).$$

Denote by $\delta_1(\omega)$ the class of $\omega \in \Omega_l^{q-1}$ in $H_p^q(l)$. Let $[\pi_L^s]^{-1}\left(\alpha + \beta\frac{d[\pi_L]}{[\pi_L]}\right)$ be Kato's refined Swan conductor ([8, Definition 5.3]) of χ_L, where $\alpha \in \Omega_l^1$ and $\beta \in l$, and $(\alpha, \beta) \neq (0, 0)$.

If $\beta \neq 0$, take $u_1, \ldots, u_{q-1} \in l^\times$ and $a \in l$ such that

$$\delta_1\left(a\beta\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-1}}{u_{q-1}}\right) \neq 0.$$

Let $\tilde{a} \in \mathcal{O}_L$, $\tilde{u_i}$ be lifts of a, u_i to $\mathcal{O}_L$. Then

$$\{\chi_L, 1 + \tilde{a}\pi_L^s, \tilde{u}_1, \ldots, \tilde{u}_{q-1}\}_l = \delta_1\left(a\beta\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-1}}{u_{q-1}}\right) \neq 0.$$

Since Ω_l^{q-1} is a one-dimensional vector space over l, we know that $\beta\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-1}}{u_{q-1}}$ is a generator for Ω_l^{q-1} over l. Then

$$\begin{aligned} H_p^q(l) &= \left\{\delta_1\left(b\beta\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-1}}{u_{q-1}}\right) : b \in l\right\} \\ &= \left\{\{\chi_L, 1 + \tilde{b}\pi_L^s, \tilde{u}_1, \ldots, \tilde{u}_{q-1}\}_l : \tilde{b} \in \mathcal{O}_L\right\} \subset \{\chi_L, U^s\hat{K}_q(L)\}_l. \end{aligned}$$

Similarly, if $\beta = 0$ and $\alpha \neq 0$, take $u_1, \ldots, u_{q-2} \in l^\times$ and $a \in l$ such that

$$\delta_1\left(a\alpha \wedge \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-2}}{u_{q-2}}\right) \neq 0.$$

We have that

$$\{\chi_L, 1+\tilde{a}\pi_L^s, \tilde{u}_1, \ldots, \tilde{u}_{q-2}, \pi_L\}_l = \delta_1\left(a\alpha \wedge \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-1}}{u_{q-2}}\right) \neq 0,$$

and $\alpha \wedge \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{q-1}}{u_{q-2}}$ is a generator for Ω_l^{q-1} over l. Then, using the same reasoning as before, we get $H_p^q(l) \subset \{\chi_L, U^s\hat{K}_q(L)\}_l$. □

Theorem 3.3.13. *Let L/K be an extension of complete discrete valuation fields of mixed characteristic. Assume that K has perfect residue field of characteristic $p > 0$.*

Denote by $e(L/K)$ the ramification index of L/K. Assume that $\chi \in H^1(K)$ is such that

$$\operatorname{Sw}\chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by χ_L its image in $H^1(L)$. Then

$$\operatorname{Sw}\chi_L = e(L/K)\operatorname{Sw}\chi - \delta_{\mathrm{tor}}(L/K).$$

Proof. Following the same argument as [8, §10], we can assume that the residue field l of L is finitely generated over the residue field k of K. Since we have proven Proposition 3.3.12, it is enough to show that this case can be reduced to that of a q-dimensional local field that is a finite extension of $K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$.

Since l is finitely generated over k, there are $T_1, \ldots, T_{q-1} \in l$ such that l is a finite, separable extension of $k(T_1, \ldots, T_{q-1})$. Since there is an embedding $k(T_1, \ldots, T_{q-1}) \hookrightarrow k((T_1))\cdots((T_{q-1}))$, there is also an embedding $l \hookrightarrow E$ of l into a finite, separable extension E of $k((T_1))\cdots((T_{q-1}))$. Since $\{T_1, \ldots, T_{q-1}\}$ is a p-basis for both l and E, there is a complete, discrete valuation field $L(E)$ that is an extension of L satisfying $\mathcal{O}_L \subset \mathcal{O}_{L(E)}$, $\mathfrak{m}_L \subset \mathfrak{m}_{L(E)}$, π_L is still prime in $L(E)$, and the residue field of $L(E)$ is isomorphic to E over l.

$L(E)$ is a finite extension of $K\{\{T_1\}\}\cdots\{\{T_{q-1}\}\}$. Since $e(L(E)/L) = 1$, we get

$e(L(E)/K) = e(L/K)$. Further, since E and l have the same p-basis and π_L is a prime for both L and $L(E)$, the map $\mathcal{O}_{L(E)} \otimes_{\mathcal{O}_L} \hat{\Omega}^1_{\mathcal{O}_L}(\log) \to \hat{\Omega}^1_{\mathcal{O}_{L(E)}}(\log)$ sends generators to generators satisfying the same relations, so it is an isomorphism. In particular, $\mathcal{O}_{L(E)} \otimes_{\mathcal{O}_L} \hat{\Omega}^1_{\mathcal{O}_L}(\log)_{\mathrm{tor}} \simeq \hat{\Omega}^1_{\mathcal{O}_{L(E)}}(\log)_{\mathrm{tor}}$. Therefore, by definition, $\delta_{\mathrm{tor}}(L(E)/K) = \delta_{\mathrm{tor}}(L/K)$. From [8, Lemma 6.2], since $\mathcal{O}_L \subset \mathcal{O}_{L(E)}$, $\mathfrak{m}_{L(E)} = \mathcal{O}_{L(E)}\mathfrak{m}_L$, and the extension of residue fields is separable, we have $\mathrm{Sw}\,\chi_{L(E)} = \mathrm{Sw}\,\chi_L$. Thus it is sufficient to prove that

$$\mathrm{Sw}\,\chi_{L(E)} = e(L/K)\,\mathrm{Sw}\,\chi - \delta_{\mathrm{tor}}(L(E)/K),$$

which follows from Proposition 3.3.12. □

CHAPTER 4

GENERALIZED HASSE-HERBRAND ψ-FUNCTIONS

Through this chapter, let L/K be an extension of complete discrete valuation fields such that the residue field of K is perfect and of characteristic $p > 0$. We define generalizations of the classical Hasse-Herbrand ψ-function for this case. More precisely, we will define functions $\psi^{\mathrm{AS}}_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\psi^{\mathrm{ab}}_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and show that, in the classical case of L/K finite, they both coincide with the classical $\psi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ (see Theorem 4.0.5). The superscripts AS and ab refer, respectively, to Abbes-Saito and abelian. In the definition of $\psi^{\mathrm{AS}}_{L/K}$ we use the Abbes-Saito upper ramification filtrations of absolute Galois groups, while in the definition of $\psi^{\mathrm{ab}}_{L/K}$ we use Kato's ramification filtration of $H^1(L)$.

We also define functions $\varphi^{\mathrm{AS}}_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\varphi^{\mathrm{ab}}_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and show that, when $\varphi^{\mathrm{AS}}_{L/K}$ and $\varphi^{\mathrm{ab}}_{L/K}$ are injective, $\psi^{\mathrm{AS}}_{L/K}$ and $\psi^{\mathrm{ab}}_{L/K}$ are their respective left inverses (and vice-versa).

Assume first that the residue field k of K is algebraically closed. Recall that the residue field l of L may be imperfect.

For $t \in \mathbb{Z}_{(p)}$, $t \geq 0$, define $\psi^{\mathrm{ab}}_{L/K}(t) \in \mathbb{R}_{\geq 0}$ as

$$\psi^{\mathrm{ab}}_{L/K}(t) = \inf \left\{ s \in \mathbb{Z}_{(p)} \;\middle|\; \begin{array}{l} \mathrm{Im}(F_{e(K'/K)t}H^1(K') \to H^1(LK')) \subset F_{e(LK'/L)s}H^1(LK') \\ \text{for all finite, tame extensions } K'/K \text{ of complete discrete} \\ \text{valuation fields such that } e(LK'/L)s, e(K'/K)t \in \mathbb{Z} \end{array} \right\},$$

and then extend $\psi^{\mathrm{ab}}_{L/K}$ to $\mathbb{R}_{\geq 0}$ by putting

$$\psi^{\mathrm{ab}}_{L/K}(t) = \sup\{\psi^{\mathrm{ab}}_{L/K}(s) : s \leq t, s \in \mathbb{Z}_{(p)}\}.$$

Similarly, for $t \in \mathbb{Z}_{(p)}$, $t \geq 0$, define $\varphi^{\mathrm{ab}}_{L/K}(t) \in \mathbb{R}_{\geq 0}$ as

$$\varphi^{\mathrm{ab}}_{L/K}(t) =$$
$$\sup \left\{ s \in \mathbb{Z}_{(p)} \;\middle|\; \begin{array}{l} \mathrm{Im}(F_{e(K'/K)s}H^1(K') \to H^1(LK')) \subset F_{e(LK'/L)t}H^1(LK') \\ \text{for all finite, tame extensions } K'/K \text{ of complete discrete} \\ \text{valuation fields such that } e(LK'/L)t, e(K'/K)s \in \mathbb{Z} \end{array} \right\},$$

and then extend $\varphi^{\mathrm{ab}}_{L/K}$ to $\mathbb{R}_{\geq 0}$ by putting

$$\varphi^{\mathrm{ab}}_{L/K}(t) = \sup\{\varphi^{\mathrm{ab}}_{L/K}(s) : s \leq t, s \in \mathbb{Z}_{(p)}\}.$$

Let $G^{t+}_{K,\log}$ denote the Abbes-Saito logarithmic upper ramification filtration defined in [1]. We now define $\psi^{\mathrm{AS}}_{L/K}$ and $\varphi^{\mathrm{AS}}_{L/K}$ by putting, for $t \in \mathbb{R}_{\geq 0}$,

$$\psi^{\mathrm{AS}}_{L/K}(t) = \inf \left\{ s \in \mathbb{R} : \mathrm{Im}(G^{s+}_{L,\log} \to G_K) \subset G^{t+}_{K,\log} \right\}$$

and

$$\varphi^{\mathrm{AS}}_{L/K}(t) = \sup \left\{ s \in \mathbb{R} : \mathrm{Im}(G^{t+}_{L,\log} \to G_K) \subset G^{s+}_{K,\log} \right\}.$$

When k is not necessarily algebraically closed, we define $\psi^{\mathrm{ab}}_{L/K}$, $\varphi^{\mathrm{ab}}_{L/K}$, $\psi^{\mathrm{AS}}_{L/K}$ and $\varphi^{\mathrm{ab}}_{L/K}$ as follows. Let $\tilde{K} = \widehat{K_{\mathrm{ur}}}$ and $\tilde{L} = \widehat{LK_{ur}}$. Then define $\psi^{\mathrm{ab}}_{L/K} = \psi^{\mathrm{ab}}_{\tilde{L}/\tilde{K}}$, $\varphi^{\mathrm{ab}}_{L/K} = \varphi^{\mathrm{ab}}_{\tilde{L}/\tilde{K}}$, $\psi^{\mathrm{AS}}_{L/K} = \psi^{\mathrm{AS}}_{\tilde{L}/\tilde{K}}$ and $\varphi^{\mathrm{AS}}_{L/K} = \varphi^{\mathrm{AS}}_{\tilde{L}/\tilde{K}}$.

The above defined functions have properties similar to those of their classical counterparts. We will now prove some of these properties.

Proposition 4.0.1. *If $\varphi^{\mathrm{ab}}_{L/K}(t)$ is injective, then $\psi^{\mathrm{ab}}_{L/K}(t)$ is its left inverse. Similarly, if $\psi^{\mathrm{ab}}_{L/K}(t)$ is injective, then $\varphi^{\mathrm{ab}}_{L/K}(t)$ is its left inverse.*

Proof. From the definitions of $\varphi^{\mathrm{ab}}_{L/K}(t)$ and $\psi^{\mathrm{ab}}_{L/K}(t)$, we can assume that k is algebraically closed. We shall prove that if $\varphi^{\mathrm{ab}}_{L/K}(t)$ is injective, then $\psi^{\mathrm{ab}}_{L/K}(t)$ is its left inverse. The other statement is proved in an analogous way.

It is enough to show that, for $t \in \mathbb{Z}_{(p)}$, $t \geq 0$, we have $t = \psi^{\mathrm{ab}}_{L/K}(\varphi^{\mathrm{ab}}_{L/K}(t))$. If $s \in \mathbb{Z}_{(p)}$ is smaller or equal to $\varphi^{\mathrm{ab}}_{L/K}(t)$, then

$$\mathrm{Im}(F_{e(K'/K)s}H^1(K') \to H^1(LK')) \subset F_{e(LK'/L)t}H^1(LK')$$

for all finite Galois extensions K'/K of complete discrete valuation fields such that K'/K is tame and $e(LK'/L)t, e(K'/K)s \in \mathbb{Z}$. Then $t \geq \psi^{\mathrm{ab}}_{L/K}(s)$, so $t \geq \psi^{\mathrm{ab}}_{L/K}(\varphi^{\mathrm{ab}}_{L/K}(t))$.

Assume that we have $t > \psi^{\mathrm{ab}}_{L/K}(\varphi^{\mathrm{ab}}_{L/K}(t))$. Take $\tilde{t} \in \mathbb{Z}_{(p)}$ that satisfies $\psi^{\mathrm{ab}}_{L/K}(\varphi^{\mathrm{ab}}_{L/K}(t)) < \tilde{t} < t$. Let K'/K be any finite Galois extension of complete discrete valuation fields that is tame and such that $e(LK'/L)\tilde{t} \in \mathbb{Z}$. Since $\tilde{t} > \psi^{\mathrm{ab}}_{L/K}(\varphi^{\mathrm{ab}}_{L/K}(t))$,

$$\mathrm{Im}(F_{e(K'/K)s}H^1(K') \to H^1(LK')) \subset F_{e(LK'/L)\tilde{t}}H^1(LK')$$

for every $s \leq \varphi^{\mathrm{ab}}_{L/K}(t)$ in $\mathbb{Z}_{(p)}$ such that $e(K'/K)s \in \mathbb{Z}$. Then

$$\varphi^{\mathrm{ab}}_{L/K}(\tilde{t}) \geq \varphi^{\mathrm{ab}}_{L/K}(t).$$

Since $\varphi^{\mathrm{ab}}_{L/K}$ is clearly increasing and $t > \tilde{t}$, we get

$$\varphi^{\mathrm{ab}}_{L/K}(\tilde{t}) = \varphi^{\mathrm{ab}}_{L/K}(t),$$

which contradicts the injectivity assumption. Therefore

$$t = \psi^{\mathrm{ab}}_{L/K}(\varphi^{\mathrm{ab}}_{L/K}(t))$$

for every $t \geq 0$ and we conclude that $\psi^{\mathrm{ab}}_{L/K}(t)$ is the left inverse of $\varphi^{\mathrm{ab}}_{L/K}(t)$. $\square$

The analogous result for $\psi^{\mathrm{AS}}_{L/K}$ and $\varphi^{\mathrm{AS}}_{L/K}$ is also true:

Proposition 4.0.2. *If $\varphi^{\mathrm{AS}}_{L/K}(t)$ is injective, then $\psi^{\mathrm{AS}}_{L/K}(t)$ is its left inverse. Similarly, if $\psi^{\mathrm{AS}}_{L/K}(t)$ is injective, then $\varphi^{\mathrm{AS}}_{L/K}(t)$ is its left inverse.*

Proof. From the definitions of $\varphi^{\mathrm{AS}}_{L/K}(t)$ and $\psi^{\mathrm{AS}}_{L/K}(t)$, we can assume that k is algebraically closed. We shall prove that if $\varphi^{\mathrm{AS}}_{L/K}(t)$ is injective, then $\psi^{\mathrm{AS}}_{L/K}(t)$ is its left inverse. The other statement is proved in an analogous way.

If $s \in \mathbb{R}$ is less than or equal to $\varphi^{\mathrm{AS}}_{L/K}(t)$, then

$$\mathrm{Im}(G^{t+}_{L,\log} \to G_K) \subset G^{s+}_{K,\log}.$$

Hence $t \geq \psi^{\mathrm{AS}}_{L/K}(s) \geq \psi^{\mathrm{AS}}_{L/K}(\varphi^{\mathrm{AS}}_{L/K}(t))$.

Assume that we have $t > \psi^{\mathrm{AS}}_{L/K}(\varphi^{\mathrm{AS}}_{L/K}(t))$. Take $\tilde{t} \in \mathbb{R}$ such that

$$\psi^{\mathrm{AS}}_{L/K}(\varphi^{\mathrm{AS}}_{L/K}(t)) < \tilde{t} < t.$$

Then

$$\mathrm{Im}(G^{\tilde{t}+}_{L,\log} \to G_K) \subset G^{s+}_{K,\log}$$

for every $s \leq \varphi^{\mathrm{AS}}_{L/K}(t)$. Thus

$$\varphi^{\mathrm{AS}}_{L/K}(\tilde{t}) \geq \varphi^{\mathrm{AS}}_{L/K}(t).$$

Since $\varphi^{\mathrm{AS}}_{L/K}$ is clearly increasing and $t > \tilde{t}$, we get

$$\varphi^{\mathrm{AS}}_{L/K}(\tilde{t}) = \varphi^{\mathrm{AS}}_{L/K}(t),$$

which contradicts the injectivity assumption. Therefore

$$t = \psi^{\mathrm{AS}}_{L/K}(\varphi^{\mathrm{AS}}_{L/K}(t))$$

for every $t \geq 0$ and we conclude that $\psi^{\mathrm{AS}}_{L/K}(t)$ is the left inverse of $\varphi^{\mathrm{AS}}_{L/K}(t)$. $\square$

These functions satisfy formulas similar to those satisfied by the classical φ and ψ-functions, as we can see from the following lemma.

Lemma 4.0.3. *Let K' be a finite Galois extension of K that is tamely ramified and $L' = LK'$. Then*

$$
\begin{aligned}
\varphi^{\mathrm{ab}}_{L'/K'}(e(L'/L)t) &= e(K'/K)\varphi^{\mathrm{ab}}_{L/K}(t),\\
\psi^{\mathrm{ab}}_{L'/K'}(e(K'/K)t) &= e(L'/L)\psi^{\mathrm{ab}}_{L/K}(t),\\
\varphi^{\mathrm{AS}}_{L'/K'}(e(L'/L)t) &= e(K'/K)\varphi^{\mathrm{AS}}_{L/K}(t),\\
\psi^{\mathrm{AS}}_{L'/K'}(e(K'/K)t) &= e(L'/L)\psi^{\mathrm{AS}}_{L/K}(t).
\end{aligned}
$$

Proof. Follows from the definitions. For example,

$$
\begin{aligned}
\varphi^{\mathrm{AS}}_{L'/K'}(e(L'/L)t) &= \sup\left\{s \in \mathbb{R} : \mathrm{Im}(G^{e(L'/L)t+}_{L,\log} \to G'_K) \subset G^{s+}_{K',\log}\right\}\\
&= \sup\left\{s \in \mathbb{R} : \mathrm{Im}(G^{t+}_{L,\log} \to G_K) \subset G^{\frac{s}{e(K'/K)}+}_{K,\log}\right\}\\
&= e(K'/K)\sup\left\{s \in \mathbb{R} : \mathrm{Im}(G^{t+}_{L,\log} \to G_K) \subset G^{s+}_{K,\log}\right\}\\
&= e(K'/K)\varphi^{\mathrm{AS}}_{L/K}(t).
\end{aligned}
$$

□

We relate this discussion with Chapter 3. The main results that we proved in the previous chapter are, in reality, results about $\psi^{\mathrm{ab}}_{L/K}$. More precisely, we have the following theorem:

Theorem 4.0.4. *Let L/K be a separable extension of complete discrete valuation fields. Assume that K has perfect residue field of characteristic $p > 0$. Let $t \in \mathbb{R}_{\geq 0}$ be such that*

$$
\begin{cases}
t \geq \dfrac{2e_K}{p-1} + \dfrac{1}{e(L/K)} + \left\lceil \dfrac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil & \text{if } K \text{ is of characteristic } 0,\\[2ex]
t > \dfrac{p}{p-1}\dfrac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} & \text{if } K \text{ is of characteristic } p.
\end{cases}
$$

Then

$$\psi^{\mathrm{ab}}_{L/K}(t) = e(L/K)t - \delta_{\mathrm{tor}}(L/K).$$

Proof. Write

$$T(L/K) = \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil$$

when K is of characteristic 0, and

$$T(L/K) = \frac{p}{p-1}\frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)}$$

when K is of characteristic p. Let $t \in \mathbb{R}_{\geq 0}$ be such that $t \geq T(L/K)$ if K is of characteristic 0 and $t > T(L/K)$ if K is of characteristic p.

If $t \in \mathbb{Z}$, it follows from Theorems 3.1.12 and 3.3.13 that

$$\psi^{\mathrm{ab}}_{L/K}(t) = e(L/K)t - \delta_{\mathrm{tor}}(L/K).$$

If $t \in \mathbb{Z}_{(p)}$, take a finite Galois extension K'/K that is tamely ramified and such that $e(K'/K)t \in \mathbb{Z}$. Observe that, if K is of characteristic 0,

$$\begin{aligned} e(K'/K)T(L/K) &= \frac{2e_{K'}}{p-1} + \frac{e(L'/L)}{e(L'/K')} + e(K'/K)\left\lceil \frac{\delta_{\mathrm{tor}}(L/K)}{e(L/K)} \right\rceil \\ &\geq \frac{2e_{K'}}{p-1} + \frac{e(L'/L)}{e(L'/K')} + \left\lceil \frac{e(L'/L)\delta_{\mathrm{tor}}(L/K)}{e(L'/K')} \right\rceil \\ &\geq \frac{2e_{K'}}{p-1} + \frac{1}{e(L'/K')} + \left\lceil \frac{\delta_{\mathrm{tor}}(L'/K')}{e(L'/K')} \right\rceil = T(L'/K'). \end{aligned}$$

Similarly, if K is of characteristic p,

$$\begin{aligned} e(K'/K)T(L/K) &= \frac{p}{p-1}\frac{e(K'/K)\delta_{\text{tor}}(L/K)}{e(L/K)} \\ &= \frac{p}{p-1}\frac{e(L'/L)\delta_{\text{tor}}(L/K)}{e(L'/K')} \\ &= \frac{p}{p-1}\frac{\delta_{\text{tor}}(L'/K')}{e(L'/K')} = T(L'/K'). \end{aligned}$$

Then we have $e(K'/K)t \geq T(L'/K')$ if K is of characteristic 0 and $e(K'/K)t > T(L'/K')$ if K is of characteristic p. It follows that

$$\begin{aligned} \psi^{\text{ab}}_{L'/K'}(e(K'/K)t) &= e(L'/K')e(K'/K)t - \delta_{\text{tor}}(L'/K') \\ &= e(L'/L)e(L/K)t - e(L'/L)\delta_{\text{tor}}(L/K). \end{aligned}$$

From Lemma 4.0.3, we conclude that

$$\psi^{\text{ab}}_{L/K}(t) = \frac{\psi^{\text{ab}}_{L'/K'}(e(K'/K)t)}{e(L'/L)} = e(L/K)t - \delta_{\text{tor}}(L/K).$$

The result then follows from the definition of $\psi^{\text{ab}}_{L/K}$. $\square$

In the classical case, the functions we defined in fact coincide with the classical φ and ψ-functions, as is shown in the following theorem.

Theorem 4.0.5. *If L/K is a finite Galois extension and k is perfect, we have*

$$\psi_{L/K} = \psi^{\text{ab}}_{L/K} = \psi^{\text{AS}}_{L/K}.$$

Proof. From the definitions of the functions, we can assume that k is algebraically closed. We shall first show that $\psi^{\text{AS}}_{L/K} = \psi_{L/K}$. To show $\varphi_{L/K}(t) \leq \varphi^{\text{AS}}_{L/K}(t)$, just observe that, if

L'/L is a finite Galois extension over K, then

$$G(L'/L)^t = G(L'/L)_{\psi_{L'/K}\circ\varphi_{L/K}(t)} \subset G(L'/K)_{\psi_{L'/K}\circ\varphi_{L/K}(t)} = G(L'/K)^{\varphi_{L/K}(t)}.$$

Since the Abbes-Saito filtration is left continuous with rational jumps, it remains to show that $\varphi_{L/K}(t) \geq \varphi^{\mathrm{AS}}_{L/K}(t)$ for $t \in \mathbb{Q}_{\geq 0}$. Let K' be a finite Galois extension of K that is tame and write $L' = LK'$. Since L'/L and K'/K are tame extensions, we have

$$\begin{aligned}\varphi_{L'/K'}(e(L'/L)t) &= e(K'/K)\varphi_{K'/K} \circ \varphi_{L'/K'} \circ \psi_{L'/L}(t) \\ &= e(K'/K)\varphi_{L'/K} \circ \psi_{L'/L}(t) \\ &= e(K'/K)\varphi_{L/K}(t).\end{aligned}$$

From Serre's local class field theory for fields with algebraically closed residue field ([24]), for every $s \in \mathbb{Z}_{\geq 0}$, the maps

$$\frac{\left(G^{\mathrm{ab}}_{L'}\right)^{\psi_{L'/K'}(s)}}{\left(G^{\mathrm{ab}}_{L'}\right)^{\psi_{L'/K'}(s)+1}} \to \frac{\left(G^{\mathrm{ab}}_{K'}\right)^{s}}{\left(G^{\mathrm{ab}}_{K'}\right)^{s+1}}$$

have images that are of finite index and nontrivial. Taking K' such that $e(K'/K)\varphi_{L/K}(t)$ is an integer and setting $s = \varphi_{L'/K'}(e(L'/L)t)$, we see that the image of $(G^{\mathrm{ab}}_{L'})^{e(L'/L)t} = (G^{\mathrm{ab}}_{L})^t$ is not contained in $(G^{\mathrm{ab}}_{K'})^{\varphi_{L'/K'}(e(L'/L)t)+1}$.

Since $(G^{\mathrm{ab}}_{K'})^{e(K'/K)\varphi_{L/K}(t)+1} = (G^{\mathrm{ab}}_{K})^{\varphi_{L/K}(t)+\frac{1}{e(K'/K)}}$, we have that the image of $(G^{\mathrm{ab}}_{L})^t$ is not contained in $(G^{\mathrm{ab}}_{K})^{\varphi_{L/K}(t)+\frac{1}{e(K'/K)}}$. We can choose tame extensions with $e(K'/K)$ arbitrarily large, so we have that $\varphi_{L/K}(t) \geq \varphi^{\mathrm{AS}}_{L/K}(t)$. Hence $\psi^{\mathrm{AS}}_{L/K} = \psi_{L/K}$.

Now we shall prove that $\psi^{\mathrm{ab}}_{L/K} = \psi_{L/K}$. Let K'/K be a finite, separable extension of complete discrete valuation fields that is tamely ramified and such that $e(LK'/L)t$ and $e(K'/K)\varphi_{L/K}(t)$ are integers. Write $L' = LK'$. Observe that, taking into account that

K'/K and L'/L are tamely ramified, we have

$$\psi_{L'/K'}(e(K'/K)\varphi_{L/K}(t)) = \psi_{L'/K'} \circ \psi_{K'/K} \circ \varphi_{L/K}(t) = \psi_{L'/K} \circ \varphi_{L/K}(t)$$
$$= \psi_{L'/L} \circ \psi_{L/K} \circ \varphi_{L/K}(t) = \psi_{L'/L}(t) = e(L'/L)t.$$

Let

$$\chi \in F_{e(K'/K)\varphi_{L/K}(t)} H^1(K').$$

Denote by $\chi_{L'}$ its image in $H^1(L')$. Using the same argument as before we see that $\chi_{L'} \in F_{e(L'/L)t} H^1(L')$, so $\varphi_{L/K}(t) \le \varphi^{\mathrm{ab}}_{L/K}(t)$. Now, if $s = \varphi_{L/K}(t) + \frac{1}{e(K'/K)}$, then

$$F_{e(K'/K)s} H^1(K') = F_{e(K'/K)\varphi_{L/K}(t)+1} H^1(K').$$

Since $F_{e(L'/L)t} H^1(L')$ does not contain the image of $F_{e(K'/K)\varphi_{L/K}(t)+1} H^1(K')$, we have that $s > \varphi^{\mathrm{ab}}_{L/K}(t)$. Since we can take extensions K'/K with arbitrarily large $e(K'/K)$, we get that $\varphi_{L/K} = \varphi^{\mathrm{ab}}_{L/K}$. Thus $\psi^{\mathrm{ab}}_{L/K} = \psi_{L/K}$. □

The properties we proved and Theorem 4.0.5 give evidence that the above defined functions $\psi^{\mathrm{ab}}_{L/K}$ and $\psi^{\mathrm{AS}}_{L/K}$ are good generalizations of the classical ψ-function. We can conjecture:

Conjecture 2. *Let L/K be an extension of complete discrete valuation fields. Assume that k is perfect of characteristic $p > 0$. Then*

$$\psi^{\mathrm{ab}}_{L/K} = \psi^{\mathrm{AS}}_{L/K}.$$

Conjecture 3. *Let L/K be an extension of complete discrete valuation fields. Assume that k is perfect of characteristic $p > 0$. Then $\psi^{\mathrm{ab}}_{L/K}$ and $\psi^{\mathrm{AS}}_{L/K}$ are continuous, piecewise linear, increasing, and convex.*

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