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# A MARKOV MODEL OF DYNAMIC NETWORK FORMATION: A PRODUCTION NETWORK PERSPECTIVE

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# Contents

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# Abstract

This paper develops a tractable model of dynamic network formation with heterogeneous forward-looking agents. The model bridges the gap between recent Macroeconomic models with exogenous production networks and static econometric models of networks formation games. I model network formation as a sequence of Bayesian incomplete information games in which the dynamic state dependencies of agents' strategies are Markov. This feature of aggregate state dependence allows me to investigate the role of network externality through a global interaction channel. I characterize the Bayesian Markov Perfect symmetric equilibrium by a set of fixed-point equations in conditional choice probabilities. I motivate this approach by developing a second-stage general equilibrium model of production networks in an open economy. This second stage model provides the payoff structure for the network formation and relevant Markov sufficient statistics. I propose a simple two-step maximum likelihood estimator and develop its asymptotic properties for a single large network. I apply this model to US input-output data. In counterfactual experiments, I find that network externality is quantitatively important for endogenous network formation. Furthermore, negative network externality provides an alternative explanation for network persistence. In an extension, I show how endogenous entry and exit of nodes be can jointly be formulated with endogenous network formation.

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# Chapter 1

# Introduction

## 1.1 Motivation and Literature Review

This paper studies dynamic network formation in the propagation of macroeconomic shocks in both theoretical and empirical context. There is a growing literature on the interplay between macroeconomics and production networks since the seminal works of Gabaix (2011) and Acemoglu et al. (2012). However, this literature has largely ignored the endogenous network adjustments from forward-looking agents' optimal network formation problems. This paper addresses this insufficiency, and I show how uncertainty can affect the macroeconomic through the network adjustment channel. This feature allows researcher to answer questions such as how a future China shock could impact on the US economy through the production network.

On the other hand, recent development on the estimation of network formation games has generally overlooked network dynamics and aggregate state dependence of links (Leung, 2017; Graham, 2017; Mele, 2017). Moreover, the payoff structures of these models and their dependence on network sufficient statistics are not economically informed. In this paper, I address network dynamics, network state dependence, and network externalities by exploiting the Markov structure and the continuum of agents. In addition, the payoff structure of my network game is micro-founded by a simplified version of Eaton and Kortum (2002) general equilibrium model of an open economy. I propose a likelihood-based approach for estimating the underlying structural parameters that governs the adjustment cost of network formation.

Therefore, this framework bridges the gap between the economic and econometric aspects of dynamic network formation by combining techniques in conditional independent network formation models, stochastic games with incomplete information, and estimations of dynamic discrete choice models. In simulation exercises, I find that network externality has a sizable effect on the macroeconomic impact, accounting for up to 50% of the total network effect on welfare in response to an aggregate productivity shock.

To understand the framework's core idea, consider the following decomposition of the network effect on aggregate welfare to productivity shocks to individual firms or sectors

$$\textit{Network effect} = \textit{Granularity} + \textit{Higher order fixed network effect} + \boldsymbol{\textit{Endogenous network effect}}$$

Each of the terms on the right-hand side corresponds to a strand of recent literature investigating the role of network structure in macroeconomics. I refer to the first two terms as exogeneous network effect. First, granularity refers to the relevance of network structure regarding aggregating idiosyncratic shocks. Lucas (1977) argues that sectorial shocks get averaged out in the aggregate in the absence of input-output linkages[1]. Acemoglu et al. (2012) explore how intersectoral networks could create "cascade" effects that deviate from Lucas' network irrelevance result. In particular, they find that the decay rate of aggregate volatility is related to sectors' degree. Gabaix (2011) micro-founded aggregate shocks using the concept of granularity: granular (large) firms' volatility can explain aggregate shocks. Gabaix (2016) argues that network structure is a particular case of granularity. Network irrelevance still holds when idiosyncratic shocks of infinitesimal firms average out in the aggregate. In this paper, firms are infinitesimal but shocks are sectorial. I define the granularity term as the effect caused by asymmetric input-output linkages of large sectors.

Second, higher-order network effects can be present even when the network is exogenous. In the classical Hulten (1978) "network irrelevance" result, the input-output network structure is

[1]The diversification argument would still work with symmetric input-output linkages

irrelevant for the aggregate impact of a sectorial productivity shock once the sector's sales share has been accounted for. Baqaee and Farhi (2019) explain why Hulten's theorem only describes the first-order effect. Higher-order network effects disappear under Cobb-Douglas technology and exogenous network because shares of sectors' sales are unresponsive to shocks. Therefore for constant-elasticity-of-substitution (CES) technology, network structure may affect how idiosyncratic shock propagates through higher-order effects.

Finally, the endogenous network adjustments in response to idiosyncratic shocks provide an additional channel for network effects. Lim (2019) proposed a dynamic network formation model in a nested CES economy and quantified the network adjustment size to be 4% of observed output fluctuations, using Compustat firm-level data. Huneeus (2018) extend Lim's framework to a small open economy and quantified the welfare impact of production network adjustment cost in response to an international trade shock. For tractability, both Lim (2019) and Huneeus (2018) assume that sellers in the intermediate goods market form links subject to exogenous probability of altering links. Tintelnot et al. (2018) consider instead the sourcing problem in which buyers in the intermediate goods market form links. For tractability, Tintelnot et al. (2018) restricts the network to be acyclical. Using Belgium firm-level data, they also find that production network adjustment to be quantitatively important for a given foreign price shock. This paper extends Lim (2019) and Huneeus (2018) to a richer network formation game setting in which sellers form links freely without restrictions on the aggregate network. In particular, this model implies a decomposition of the endogenous network adjustment effect into a isolated adjustment component and a novel network externality component.

$$\textit{Network adjustment effect} = \textit{Isolated adjustment effect} + \boldsymbol{\textit{Network externality effect}}$$

I define the isolated adjustment effect as the network adjustment effect when global interaction is shut down, i.e, when firms ignore the equilibrium impact of network formation on the aggregate network.

In this model, each sector comprises a continuum of infinitesimal firms so that sectorial shocks are relevant for the aggregate shock in the same way as in Acemoglu et al. (2012). The linkages between sectors are weighted by the matching probability between firms in each respective sector. This matching probability is also a sufficient statistic for the production network in the economy. In addition, this sufficient network statistic is Markov and follows the law of motion that depends on the equilibrium of an underlying network game. Unlike Tintelnot et al. (2018), Lim (2019), and Huneeus (2018), this model allows for game-theoretical interdependence of individual strategies. This Markov aggregate network state serves two critical functions in this model: first, it reduces the dimension of the symmetric equilibrium strategies as functions of the entire network; second, agents use this aggregate network state to make a forecast of their future payoff. Therefore, network externality emerges through dependence on this aggregate network state even though each firm is infinitesimal. In this paper, I quantify this novel externality component of the endogeneous network adjustment effect. I gain tractability by exploiting the additive separable utility function and continuum of firms.

The network formation model in this paper belongs to the class of conditional independence network formation model (Chandrasekhar, 2016) that builds upon the Erdös-Rényi random graph model (Erdos and Renyi, 1959). In essence, this paper explores ways to endogenize and micro-found the linking probability $p$ in the Erdös-Rényi model through $p$'s dependence on dyad-specific characteristic, individual network state, and aggregate network state. The conditional dyad-link formation is also related to stochastic block models studied extensively in the statistics literature (Snijders and Nowicki, 1997; Daudin et al., 2008) , statistical exponential random graph models in (Chandrasekhar and Jackson, 2014), and fixed effect econometric models of network formation (Graham, 2015, 2017). It is common in the peer-effect literature (Goldsmith-Pinkham and Imbens, 2013; Blume et al., 2015; Auerbach, 2018) to formulate a network formation model as the first stage and an economic model conditional on the fixed network as the second stage, which is also the strategy adopted in this paper. This paper's econometric techniques are closely related to the econometrics of dynamic discrete choice models (Hotz and Miller, 1993; Aguirregabiria and Mira,

2007) and the econometrics of static network formation game with incomplete information (Leung, 2017; Ridder and Sheng, 2020).

## 1.2 Outline

The outline of this paper is as follows. I first describe the general equilibrium model of open economy in section 2, which provides the payoff structure for the network formation game. Then I present the endogenous network formation model and existence results in section 3. In section 4, I describe how to estimate the key structure parameter using nested pseudo likelihood estimation. In section 5, I conduct various counterfactual analysis to study the role of network adjustment in aggregate open economy in response to aggregate shocks. In section 5, I show how to extend the baseline network formation model in order to feature endogenous entry and exit decision. Section 6 concludes the paper.

# Chapter 2

# General Equilibrium Model with Exogeneous Network

The open economy has a representative household who consumes final goods and supplies a fixed labor supply of $L$ and a continuum of heterogeneous firms $i \in I$ who are differentiated by finite discrete productivity types $\phi \in \Phi$ and choose to produce an unique variety at each discrete time $t = 0, 1, \ldots$. Let $\gamma$ denote the time invariant total measure of firms and $\Pi(\phi)$ denote the time invariant proportion of firms with type $\phi$ (or $\phi$-firms for short). Then $\gamma\Pi(\phi)$ is the total measure of $\phi$-firms.

## 2.1 Household

The representative household has the following CES (constant elasticity of substitution) period utility function

$$u_t = \left( \gamma \sum_{\phi \in \Phi} \Pi(\phi) [C_t(\phi)]^{\frac{\sigma-1}{\sigma}} + \gamma^M [C_t^M]^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

where $\sigma$ is the demand elasticity of substitution across varieties and $C_t(\phi)$ is the consumption of a final good variety produced by a $\phi$-firm at $t$. Due to symmetry, $C_t(\phi)$ is identical for all firms with type $\phi$ and all variables depend only on $\phi$ instead of $i$. $C_t^I$ is the consumption of imports. At each period $t = 1, 2, \ldots$, the representative household maximizes the expected discounted utility

$$U_t = E_t \left( \sum_{k=t}^{\infty} \beta^{k-t} u_t \right).$$

The household's period by period utility maximization gives domestic final goods demand for a variety produced by a $\phi$-firm is given by

$$C_t(\phi) = p_t^C(\phi)^{-\sigma} \mathcal{P}_t^{1-\sigma} E_t. \tag{2.1}$$

Define the aggregate demand shifter

$$\Delta_t^D = \mathcal{P}_t^{1-\sigma} E_t.$$

Let $p_t^M$ be the exogeneous price of import, then the demand for imported final good is given by

$$C_t^I = (p_t^M)^{-\sigma} \Delta_t^D$$

Then $\mathcal{P}_t$ is the ideal price index that is give by

$$\mathcal{P}_t = \left( \gamma \sum_{\phi \in \Phi} \Pi(\phi) p_t^C(\phi)^{1-\sigma} + \gamma^M (p_t^M)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{2.2}$$

With aggregate demand shifter $\Delta_t^D$ and aggregregate consumption $C_t(\phi)$ defined, I can now fully describe the environment and profit maximization problem for each firm in the economy.

## 2.2 Firms

Let $D_{ijt}$ be the indicator of whether firm $i$ sells/links to $j$ at time $t$. $\{D_{ijt}\}_{(i,j) \in I^2}$ describes the complete production network at $t$, which is exogeneously given in this general equilibrium model.

At each period $t = 1, 2, \ldots$, each $\phi$-firm $i$ hire $\ell(\phi)$ unit of labor from the labor market taking price at $w_t$ and $x_t^I(\phi', \phi)$ unit of intermediate goods at price $p_t^I(\phi', \phi)$ from each $\phi'$-firm $j$ that links to $i$, that is, $i$ buys $x_t^I(\phi_i, \phi_j)$ from each $j \in I$ with $D_{ijt} = 1$. This model assumes that firms cannot buy factors or inputs from foreign market. Then, all the intermediate goods $x_t^I$ are aggregated according to CES technology into an input bundle $x_t^B$

$$x_{i,t}^B = \left( \int_{j \in I} D_{ijt} x_t^I(\phi_i, \phi_j)^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}} .$$

Note that this can be written equivalently as

$$x_t^B(\phi) = \left( \gamma \sum_{\phi' \in \Phi} \Pi_t(\phi') m_t(\phi', \phi) x_t^I(\phi', \phi)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} . \tag{2.3}$$

where the matching probability $m_t(\phi, \phi')$ is defined as

$$m_t(\phi, \phi') = [\gamma^2 \Pi(\phi) \Pi(\phi')]^{-1} \int_{j \in I} \int_{i \in I} D_{ijt} \mathbf{1}\{\phi_i = \phi\} \mathbf{1}\{\phi_j = \phi'\} di dj.$$

A key feature of this model is that $m_t \equiv \{m_t(\phi, \phi')\}_{(\phi,\phi') \in \Phi^2}$ is the sufficient aggregate network statistic for the network $\{D_{ijt}\}_{(i,j) \in I^2}$ in this general equilibrium model under the following assumption

**Assumption 1** (Initial Condition)**.** The network $\{D_{ijt}\}_{ij}$ at $t$ satisfies the following: if for any pair of firms $i, i' \in I$, $i \neq i'$

$$\phi_i = \phi_{i'} \quad \Rightarrow \quad \begin{cases} \int_{j \in I} D_{ijt} \mathbf{1}\{\phi_j = \phi'\} dj = \int_{j \in I} D_{i'jt} \mathbf{1}\{\phi_j = \phi'\} dj \\ \int_{j \in I} D_{jit} \mathbf{1}\{\phi_j = \phi'\} dj = \int_{j \in I} D_{ji't} \mathbf{1}\{\phi_j = \phi'\} dj \end{cases} , \quad \forall \phi' \in \Phi$$

This assumption guarantees symmetry for $i$ and $i'$ with the same type $\phi$ in the general equilibrium model so that the model can be analyzed at aggregate type level. Note that the assumption does not rule out the possibility that $i$ and $i'$ have different local networks $\{D_{ijt}\}_{j \in I}$ and $\{D_{i'jt}\}_{j \in I}$.

It only imposes restrictions on the aggregate behaviors of these network that is sufficient for calculating the general equilibrium model.

In later sections, I will show that the equilibrium network generated by the endogeneous network formation model satisfies this assumption. In addition, I will show that $m_t$ is also a Markov state whose transition function is determined by the equilibrium in the network formation model. $m_t$ reduces the dimension of the aggregate state space significantly.

Having obtained factor $\ell_t(\phi)$ and input $x_t^B(\phi)$, firm produce goods at period $t$ according to the following CES production function

$$y_t(\phi) = \left(\alpha(z_t\phi\ell_t(\phi))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)x_t^B(\phi)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

$\alpha$ is the labor intensity. $\sigma$ is the constant elasticity of technical substitution which coincides with demand elasticity of substitution for the varieties.

Firms sell on three types of markets: domestic final goods market, foreign final goods market, and domestic intermediate goods market. A $\phi$-firm has $e(\phi)$ probability of selling on the foreign final goods market. Let $\Delta_t^F$ denote the exogeneous foreign aggreagte demand shifter. Then standard CES results give us the conditional factor demand functions

$$\ell_t(\phi) = \alpha^\sigma(z_t\phi)^{\sigma-1}\left(\frac{w_t}{c_t(\phi)}\right)^{-\sigma} y_t(\phi) \tag{2.4}$$

$$x_t^I(\phi,\phi') = (1-\alpha)^\sigma\left(\frac{p_t^I(\phi,\phi')}{c_t(\phi')}\right)^{-\sigma} y_t(\phi') \tag{2.5}$$

and the following final goods demand functions

$$C_t(\phi) = p_t^C(\phi)^{-\sigma}\Delta_t^D \tag{2.6}$$

$$C_t^F(\phi) = p_t^F(\phi)^{-\sigma}\Delta_t^F \tag{2.7}$$

Given these demand functions, CES standard results gives the following optimal price setting conditions

$$p_t^C(\phi) = p_t^F(\phi) = p_t^I(\phi, \phi') = \underbrace{\left(\frac{\sigma}{\sigma - 1}\right)}_{\equiv \mu} c_t(\phi) \tag{2.8}$$

where the marginal cost $c_t(\phi)$ is given by

$$c_t(\phi) = \left(\alpha^\sigma (w_t/z_t\phi)^{1-\sigma} + (1-\alpha)^\sigma p_t^B(\phi)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{2.9}$$

$$p_t^B(\phi) = \left(\gamma \sum_{\phi \in \Phi} \Pi(\phi') m_t(\phi', \phi) p_t^I(\phi', \phi)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{2.10}$$

Each $\phi$-firm set the same price across all markets because there is no heterogeneous markup in this model. The constant demand elasticity of substitution and the constant price elasticity of demand is assumed to be the same $\sigma$ across all market. However, there is price heterogeneity across different types of firms because the more productive type has lower marginal cost. In addition, high productivity types does not always have lower marginal cost than the low productivity types. It also depends on how well connected the type is in the production network. Better connected type has greater access to varieties of the intermediate inputs and hence has a lower marginal cost due to the complementarities of intermediate inputs. The precise measure of this "well-connectedness" is characterized in the following section.

Each firm receive profit form the intermediate goods market and the final goods market. Define the firm-specific demand shifters

$$\Delta_t(\phi) = y_t(\phi) c_t(\phi)^\sigma.$$

Firms' revenue is linear in revenues obtain from selling to other firm

$$r_t(\phi) = \underbrace{r_t(\phi)^C + r_t(\phi)^F}_{\text{revenue from final goods market}} + \underbrace{\gamma \sum_{\phi' \in \Phi} \Pi(\phi') m_t(\phi, \phi') r^I(\phi, \phi')}_{\text{revenue from intermediate goods market}}$$

$$r^I(\phi, \phi') = (1-\alpha)^\sigma p_t^I(\phi, \phi')^{1-\sigma} \Delta_t(\phi')$$

The total revenue function can also be written as

$$r_t(\phi) = \mu c_t(\phi) y_t(\phi) = z_t^{1-\sigma} \mu H_t^c(\phi) w_t^{1-\sigma} \Delta_t(\phi) \tag{2.11}$$

By standard CES result, the realized profit for a $\phi$-firm selling to a $\phi'$-firm is given by

$$\pi_t^I(\phi, \phi') = \sigma^{-1}(1-\alpha)^\sigma p_t^I(\phi, \phi')^{1-\sigma} \Delta_t(\phi'). \tag{2.12}$$

This profit function for $\phi$-firm selling intermediate input to $\phi'$-firm is the main component of the payoff between a $\phi$-firm and $\phi'$-firm in the first stage network formation game.

## 2.3 Market Clearing

Let $L_t$ denote the exogeneous total costs of maintaining firm-to-firm links in terms of labor at time $t$. The labor market clearing condition is

$$L - L_t = \gamma \sum_{\phi' \in \Phi} \Pi(\phi') \ell_t(\phi') \tag{2.13}$$

This implies the following expression for the real wage

$$w_t^\sigma = z_t^{\sigma-1} \left( \frac{\gamma \alpha^\sigma}{L - L_t} \right) \sum_{\phi \in \Phi} \Pi(\phi) \phi^{\sigma-1} \Delta_t(\phi) \tag{2.14}$$

Suppose that there is no trade deficit, then the current account balance condition is

$$p_t^M C_t^M = \gamma \sum_{\phi \in \Phi} \Pi(\phi) e(\phi) p_t^F(\phi) C_t^F(\phi) \tag{2.15}$$

Normalize $p^M = 1$, then the current account balance condition immediately gives us the solution for the aggregate demand shifter

$$\Delta_t^D = (w_t/z_t)^{1-\sigma} \underbrace{\gamma\mu^{1-\sigma} \sum_{\phi\in\Phi} \Pi(\phi)e(\phi)H_t^c(\phi)}_{\equiv H_t^F} \Delta_t^F \tag{2.16}$$

Combine the above equation with the wage equation (2.14) and equation for firm-specific demand shifter (2.22) yield the following equation that determines the wage

$$\begin{aligned} w_t^\sigma = z_t^{2\sigma-2}\mu^{1-\sigma}\left(\frac{\gamma^2\alpha^\sigma}{L-L_t}\right)\Delta_t^F \sum_{\phi\in\Phi}\Pi(\phi)e(\phi)H_t^c(\phi)\sum_{\phi\in\Phi}\Pi(\phi)\phi^{\sigma-1}H_t^\Delta(\phi)w_t^{1-\sigma} \\ + z_t^{\sigma-1}\left(\frac{\gamma\alpha^\sigma}{L-L_t}\right)\Delta_t^F\sum_{\phi\in\Phi}\Pi(\phi)\phi^{\sigma-1}H_t^{\Delta F}(\phi) \end{aligned} \tag{2.17}$$

This equation has a unique strictly positive for the real wage for $\sigma > 1$. The rest of the models can be solved using this wage solution.

## 2.4 Marginal Cost and Upstream Network Centrality

Combining the price for the bundle (2.10), firms' marginal cost equation (2.9) and optimal pricing equation (2.8) yields a fixed point equation in the marginal cost

$$c_t(\phi)^{1-\sigma} = \alpha^\sigma(w_t/z_t\phi)^{1-\sigma} + (1-\alpha)^\sigma\mu^{1-\sigma}\gamma\sum_{\phi'\in\Phi}\Pi(\phi')m_t(\phi',\phi)c_t(\phi')^{1-\sigma} \tag{2.18}$$

which leads to the following proposition

**Proposition 1.** *The solution for the marginal cost given wage $w_t$ is*

$$c_t(\phi)^{1-\sigma} = \underbrace{\alpha^\sigma\left(\sum_{\phi'\in\Phi}(\phi')^{\sigma-1}\sum_{d=0}^{\infty}(\kappa^c)^d(B_t^c)^d(\phi',\phi)\right)}_{\equiv H_t^c(\phi)}(w_t/z_t)^{1-\sigma} \tag{2.19}$$

*where $\kappa^c$ and $B_t^c$ are defined in the Appendix.*

The term $H_t^c(\phi)$ is a Bonacich centrality measure for the "well-connectedness" of type $\phi$-firm in the upstream network: it is the mass of all paths leading into the $\phi$-firm weighted inversely to the distance $d$ by the decay rate $\kappa^c$, summed across productivity types $\phi$ and weighted by productivities $\phi$ (Jackson 2010; Huneeus 2018). Note that the effect of productivity of type $\phi$ is captured in the terms with distance $0$. This centrality measure captures the idea that the marginal cost of one firm depends the marginal costs of all other firms in the upstream production network both directly and indirectly. The network decay rate $\kappa^c$ is governed by 1) the intensity of intermediate input bundle $1-\alpha$ (increasing); 2) the constant markup $\mu$ (decreasing); 3) total mass/size $\gamma$ of firms (increasing); 4) constant elasticity of technical substitution $\sigma$.

Let $\tilde{w} \equiv w_t/z_t$ denote the productivity adjusted wage. $H_t^c(\phi)$ characterizes completely the well-connectedness of $\phi$-firm in the upstream network as well as the network effect of a shock to the fundamental factor price $\tilde{w}_t$ on marginal cost

$$\frac{dc_t(\phi)}{d\tilde{w}_t} = H_t(\phi)^{1-\sigma}.$$

$H_t^c(\phi)$ depends on $t$ only through dependence on the sufficient network statistic $m_t$. Therefore, the type with greater $H_t(\phi)$ has greater marginal cost response to an aggregate supply. Hence, network connenctedness amplifies both positive and negative aggregate supply shocks.

## 2.5 Firm specific Demand Shifter and Network Connectivity

For each firm, the following total production identity must be satisfied

$$y_t(\phi) = C_t(\phi) + e_t(\phi)C_t^F(\phi) + \gamma \sum_{\phi' \in \Phi} \Pi(\phi')m_t(\phi,\phi')x_t^I(\phi,\phi') \tag{2.20}$$

Combining the total production identity (2.20), final goods demand function (2.1), intermediate goods demand function (2.5), the input bundle demand function (2.3), and the price setting conditions yields the following recursion in $y_t(\phi)$

$$y_t(\phi) = c_t(\phi)^{-\sigma}\mu^{-\sigma}\Delta_t^D + e(\phi)c_t(\phi)^{-\sigma}\mu^{-\sigma}\Delta_t^F + \gamma(1-\alpha)^\sigma\mu^{-\sigma}\sum_{\phi'\in\Phi}\Pi(\phi')m_t(\phi,\phi')c_t(\phi)^{-\sigma}y_t(\phi')c_t(\phi')^{-\sigma} \quad (2.21)$$

which leads to the following proposition

**Proposition 2.** *The solution for firm's demand shifter for a given aggregate demand shifter $\Delta_t^D$ is*

$$\Delta_t(\phi) = \underbrace{\mu^{-\sigma}\left(\sum_{\phi'\in\Phi}\sum_{d=0}^{\infty}(\kappa^y)^d(B_t^\Delta)^{(d)}(\phi,\phi')\right)}_{\equiv H_t^\Delta(\phi)}\Delta_t^D + \underbrace{\mu^{-\sigma}\left(\sum_{\phi'\in\Phi}e_t(\phi')\sum_{d=0}^{\infty}(\kappa^y)^d(B_t^\Delta)^{(d)}(\phi,\phi')\right)}_{\equiv H_t^{\Delta F}(\phi)}\Delta_t^F \quad (2.22)$$

*where $\kappa^y$ and $B_t^\Delta$, and $B_t^{\Delta F}$ are defined in the Appendix.*

Similar to the marginal cost, the Bonacich centralities $H_t^\Delta(\phi)$ and $H_t^{\Delta F}$ measures $\phi$-firm's demand shifter's exposures to aggregregate domestic demand $\Delta_t^D$ and aggregate foreign demand $\Delta_t^F$, respectively. The network decay rate $\kappa^y$ is governed by 1) the intensity of intermediate input bundle $1-\alpha$ (increasing); 2) the constant markup $\mu$ (decreasing); 3) total mass/size $\gamma$ of firms (increasing); 4) constant elasticity of technical substitution $\sigma$.

## 2.6 General Equilibrium

I now define the dynamic general equilibrium of the model described in this section for exogeneously given $\{m_t\}$, $L_t$, and $z_t$.

**Definition 1** (Exogeneous Network Equilibrium)**.** The dynamic general equilibrium of the economy described in this section for a given sequence of exogeneous network $\{m_t\}$, exogeneous period relationship costs $L_t$, aggregate productivity $z_t$, and aggregate foreign demand shifters $\Delta_t^F$, and parameters $\sigma$, $\alpha$, $\gamma$, $L$, $\{\phi\}_{\phi\in\Phi}$, $\{e(\phi)\}_{\phi\in\Phi}$, $\{\Pi(\phi)\}_{\phi\in\Phi}$ are the set of prices

$$\{p_t^I(\phi,\phi'), p_t^B(\phi), p_t^C(\phi), p_t^F(\phi), \mathcal{P}_t, w_t\}_{t,\phi\in\Phi,\phi'\in\Phi}$$

and allocations

$$\{x_t^I(\phi,\phi'), x_t^B(\phi), C_t(\phi), C_t^F(\phi), \ell_t(\phi), y_t(\phi), \Delta_t^D\}_{t,\phi\in\Phi,\phi'\in\Phi}$$

such that for every $t = 0, 1, \ldots$, representative households maximize expected utility so that final goods demand equations (2.1) and (2.2) are satisfied; each firm sets prices in both intermediate goods and final goods market in each region to maximize total profit so that demand functions for immediate goods (2.5), demand functions for input bundles (2.3), labor demand functions (2.4), constant markup pricing equations (2.8) are satisfied. Finally, the labor market clearing condition (2.13) and total production identity (2.20) must satisfy. In addition, intermediate goods and final goods market for all varieties must clear. We call such a equilibrium *exogeneous network Markov equilibrium* when each of the pricing and allocation function $g_t(\cdot)$ can be written as $g(\cdot, s_t)$ where $s_t$ is a Markov state.

**Assumption 2** (Parameter Restrictions)**.** The parameters $\sigma$, $\alpha$, $\gamma$, $L$, $\{\phi\}_{\phi\in\Phi}$, $\{e(\phi)\}_{\phi\in\Phi}$, $\{\Pi(\phi)\}_{\phi\in\Phi}$ satisfies the following restrictions $\forall\phi\in\Phi$

$$\sigma > 1, \quad \gamma > 0, \quad \phi > 0, \quad L > \max_t(L_t), \quad \alpha \in (0,1), \quad e(\phi) \in [0,1], \quad \Pi(\phi) \in [0,1].$$

The static equilibrium can be solved using equation (1)-(2.16) once equilibrium wage is solved. Equation (2.17) yield a unique and strictly positive solution for the equilibrium wage. Since firms and the representative household optimize period by period, the exogeneous network (dynamic) Markov equilbrium exist when the state variables $s_t = (m_t, L_t, z_t, \Delta_t^F)$ are Markov.

**Proposition 3.** *Suppose Assumption 2 hold and $s_t = (m_t, L_t, z_t, \Delta_t^F)$ is Markov, then the exogeneous network Markov equilibrium defined by equation (1)-(2.22) exists and is unique.*

Given the equilibrium profit function, I can now fully describe the network formation problem that governs evolution of the aggregate state $s_t$.

# Chapter 3

# Endogeneous Network Formation Model

**Environment and Timing** For every $t = 1, 2, \ldots$, there are two stages of decision making for each agent. In the first stage, each firm choose optimally buyers in the intermediate inputs market given the aggregate state of the previous period $s_{t-1}$ and their idiosyncratic draw of relationship cost $\epsilon_{it}$ in a dynamic Bayesian incomplete information game. At the end of the first stage, aggregate productivity $z_t$ evolves independently according to an exogeneous process. The optimal equilibrium linking strategies give rise to the matching probability $m_t$ and the relationship cost $L_t$ given the draw of $z_t$. The foreign aggregate demand shifter $\Delta_t^F$ is assumed to be constant at $\Delta^F$ over time. In the second stage, firms behave in the same way as described in the previous section, taking the Markov state $s_t = (m_t, L_t, z_t)$ as given. The equilibrium in the first stage network formation game is defined in a way such that $m_t$ is Markov.

Each firm $i$ draws an iid idiosyncratic asymmetric linking cost $\epsilon_{ij} \sim G_\epsilon$ for each other firms $j$ in the network. $\epsilon_{it} = \{\epsilon_{ijt}\}_{j \in I}$ is the private information in this incomplete information game. This is the key assumption that guarantees that optimal linking decision $D_{ijt}$ are independent conditional on the Markov state $s_{t-1}$ at time $t$.

**Assumption 3** (Conditional Independence)**.** The dyad specific relationship cost $\epsilon_{ijt} \sim G_\epsilon$ for some absolute continuous distribution $G_\epsilon$ is independent across all dyads and across time

$$\epsilon_{ijt} \perp\!\!\!\perp \epsilon_{khs}, \quad \forall (i,j) \in I^2, (k,h) \in I^2, t \neq s$$
$$\epsilon_{ijt} \perp\!\!\!\perp \epsilon_{kht}, \quad \forall (i,j) \neq (k,h) \in I^2, \quad \forall t = 1, 2, \ldots$$

By imposing Assumption (3), this model belongs to the class of conditional independence network formation models that have been widely studied since the Erdos-Renyi model. There are two important consequences of this assumption: 1) when agents are forming expectations of other's actions in equilibrium, with the absence of network externality, it suffice to use the identical marginal distribution $G_\epsilon$. This feature improves the tractability of the model; 2) the likelihood function can be easily using conditional independence assumption, which naturally leads to likelihood-based efficient estimation procedure.

Given this setup, I focus on symmetric equilibrium Markov pure strategy of the following form

$$\Sigma^*(\phi, \phi', d, s, \epsilon) \in \{0, 1\}, \quad P^*(\phi, \phi', d, s) \equiv E_\epsilon[\Sigma^*(\phi, \phi', d, s, \epsilon)]$$

where $d$ is an indicator of whether $\phi$-firm linked to the $\phi'$-firm in the previous period; $m$ is a sufficient statistic for past network peer of the firm. $\Sigma$ and $P$ are the strategy and the corresponding conditional choice probability (CCP) for linking decisions between a specific pair of firms with the type dyad $\phi\phi'$, conditional on the whether this $\phi$-firm linked to the $\phi'$-firm in the previously period or not and the event that the aggregate state in the previous period is $s$. For example, $P^*(\phi, \phi', 1, s)$ is the equilibrium probability of a $\phi$-firm links or sells to a $\phi'$-firm given that $\phi$-firm has previously linked to the $\phi'$-firm when the previous aggregate state is $s$. In this section, I describe the firms' network formation problem and define the equilibrium concept that determines the equilibrium CCPs $P^*$ then I show their existence and construction.

## 3.1 Firms' Network Formation Problem

Let $P$ denote the conditional choice function $P(\phi, \phi', d, s)$. At each $t = 1, 2, \ldots$. Firm $i$ choose $D_{ih} \equiv \{D_{ijh} | j \in I \ldots\}$ in order to maximize its expected present value discounted profit given believes $P^i$

$$\max_{D_{ih} \in \{0,1\}^I, \forall h \geq t} E^{P^i}_{(s,\epsilon),t} \left[ \sum_{h=t}^{\infty} \beta^{h-t} \pi(\phi_i, s_h, D_{ih}, D_{ih-1}, \epsilon_{ih}) \right]. \tag{3.1}$$

$E_{(s,\epsilon),t}$ denote expectation with respect to $s_h$ and $\epsilon_{ih}$ given the information at time $t$ and $i$'s belief for the conditional choice probability $P$. In equilibrium, the sufficient statistic for information at time $t$ is the aggregate state $s_t = (m_{t-1}, z_{t-1})$. Period profit function $\pi$ are determined by optimal price setting in the second stage as described by the previous section, taking aggregate states $m_t$, $z_t$, and $L_t$ as given,

$$\pi(\phi_i, s_t, D_{it}, D_{it-1}, \epsilon_{it}) = \int_{j \in I} \underbrace{D_{ijt}}_{\text{network}} \left( \underbrace{E^{P^i}_s [\pi^I(\phi_i, \phi_j, s_{t+1}) | s_t]}_{\text{intermediate good}} - \underbrace{\rho(1 - D_{ijt-1})}_{\text{sunk cost}} + \epsilon_{ijt} \right) dj$$

where $\rho$ is the measure of "stickiness" of network or the measure of sunk cost of establishing a link. Operator $E^{P^i}_s[\cdot|s_t]$ means taking expectation with respect to some Markov transition probability $T^{P^i}(s_{t+1}|s_t)$, which depends on the (belief of) conditional choice probabilities $P$.

The tractability of this model comes from the following property for the payoff function $\pi$

**Definition 2** (Network Additive Separability)**.** The Markov period payoff function $\pi(D_{it}, D_{it-1}, s_t, P^i)$ is *network additive separable* if there exists functions $\pi^{ij}(D_{it}, D_{it-1}, s_t, P^i)$ such that

$$\pi(D_{it}, D_{it-1}, s_t, P^i) = \int_{j \in I} D_{ijt} \pi^{ij}(D_{it}, D_{it-1}, s_t, P^i) dj$$

and

$$\pi^{ij}(D_{it}, D_{it-1}, s_t, P^i) \perp\!\!\!\perp D_{ikt}, \quad \pi^{ij}(D_{it}, D_{it-1}, s_t, P^i) \perp\!\!\!\perp D_{ikt-1}, \quad k \neq i.$$

Network additive separability rules out direct network externalities among $i$'s local out-network $D_{it} = \{D_{ijt} | j \neq i\}$. The $ij$ specific payoff does not depend on any link other than $ij$. However, $ij$ specific payoff depend on the realization of aggregate state $s_t$ and the belief $P^i$ over the strategies of all dyads $ij \in I^2$. This generates interdependence of the CCP for all dyad $ij \in I^2$ in equilibrium.

The intermediate profit function $\pi^I$ is given by equation (2.12). Although the period profit function depends on decision $D_{it}$ and $D_{it-1}$, the associated Bellman equation can be written in a way that does not depend on $D_{it-1}$, $D_{it-1}$. The payoff relevant information in the past network is encoded in the network sufficient statistic $m_{t-1}$.

## 3.2 Markov Transition Probability

A key feature of this model is that the transition probability $Q^P(s'|s)$ for the aggregate Markov state depends on the conditional choice probabilities $P$. Each agent use her belief $P^i$ in order to forecast the evolution of aggregate Markov state and to calculate expected payoffs. Let $s_t = (m_{t-1}, z_{t-1})$. Given a exogeneous transition probability $Q_z$ for $z'|z$

$$Q^P(z'|z, m) = Q_z(z'|z)$$

The transition function $Q_m$ for $m'|m$ can be derived from the following law of motion $F^m$

$$m_t(\phi, \phi') = m_{t-1}(\phi, \phi')P(\phi, \phi', 1, s_t) + m_{t-1}(\phi, \phi')P(\phi, \phi', 0, s_t), \quad \forall(\phi, \phi') \in \Phi^2 \tag{3.2}$$

$$m' = F_m^P(m, z) \tag{3.3}$$

It follows that

$$Q^P(s'|s) = Q^P(m'|z', z, m)Q^P(z'|z, m) = \begin{cases} Q_z(z'|z) & \text{if } m' = F_m^P(m, z) \\ 0 & \text{if } m' \neq F_m^P(m, z) \end{cases} \tag{3.4}$$

The conditional choice probabilities $P$ governs the deterministic transition of the network sufficient statistic $m$. The aggregate productivity state $z$ controls the random switching between different deterministic paths of transition dynamics.

## 3.3 Bellman Equation

The Bellman equation for firm $i$ with belief $i$ and network formation problem described in equation (6.1) is defined as follows

$$V^{P^i}(\phi_i, D_{it-1}, s_t) = E\Bigg[\max_{D_{it}\in\{0,1\}^I}\Bigg(\int_{j\in I} D_{ijt}\Big[E_s^{P^i}[\pi^I(\phi_i, \phi_j, s_{t+1})|s_t] - \rho(1-D_{ijt-1}) + \epsilon_{ijt}\Big]dj\Bigg) + \beta E_s^{P^i}[V^{P^i}(\phi_i, D_{it}, s_{t+1})|s_t]\Bigg] \quad (3.5)$$

It is simply to verify that the Blackwell condition is satisfied. Now I introduce the solution concept which determines the equilibrium CCP $P^*$.

**Definition 3.** A symmetric *Bayesian Markov perfect equilibrium* in the open economy with endogenous network described in this section and the previous section is the set of condition choice probabilities $\{P^*(\phi, \phi', d, s)\}$ that satisfies

1. *Rationality*: the associated Markov perfect strategies $\Sigma^*$ of $P^*$ satisfies

$$D^*_{ijt} = \Sigma^*(\phi_i, \phi_j, D_{ijt-1}, s_t, \epsilon_{ijt}), \quad \forall i \in I, t = 1, 2, \ldots$$

   which solves the Bellman equation (3.5).

2. *Consistency*: $P^i = P^*$ for all $i \in I$.

Note that if the distribution $G_\epsilon$ is the common prior, then Bayesian Markov perfect equilibrium coincide with Bayes Nash equilibrium for each static subgame at $t$ defined by a given continuation value $V^{P^*}(\phi_i, D_{it}, s_{t+1})$. Consistency condition on the prior beliefs of other agents' distribution of idiosyncratic relationship cost is weaker than the common prior assumption. In addition, Bayesian Markov perfect equilibrium nests pairwise stability commonly used in the literature of strategic network formation, when the a undirected link between $i$ and $j$ is defined as $D_{ij}D_{ji}$.

The Bellman equation (3.5) is equivalent to a set of "mini" Bellman equations that are easy to solve individually.

**Proposition 4.** *Suppose that Assumption (2) is satisfied and the value function $V^P$ satisfies equation (3.5).. Then there exists $v^P_{\phi\phi'}$ such that $V^P$ can expressed as*

$$V^P(\phi, s) = \gamma \sum_{\phi' \in \Phi} \Pi(\phi')[m(\phi, \phi')v^P_{\phi\phi'}(1, s) + (1 - m(\phi, \phi'))v^P_{\phi\phi'}(0, s)].$$

*where $v^P_{\phi\phi'}(0, s)$ is given by*

$$v^P_{\phi\phi'}(d, s) = E_\epsilon \left[ \max_{d' \in \{0,1\}} d' \left( E^P_{s'|s}[\pi^I(\phi, \phi', s')] - \rho(1 - d) + \epsilon \right) + \beta E^P_{s'|s}[v^P_{\phi\phi'}(d', s')] \right] \tag{3.6}$$

## 3.4 Existence of Bayesian Markov Perfect Equilibrium

Solving each "mini" Bellman in Proposition 2 yields fixed point equations that determines the equilibrium CCP $P$.

**Proposition 5.** *Suppose that profit functions $\pi^I$ is given by equation (2.12). Then an associated Bayesian Markov perfect equilibrium $P^*$ is characterized by the following system of fixed point equation*

$$P^*(\phi, \phi', d, s) = G_\epsilon \left( E^{P^*}_{s'|s}[\pi^I(\phi, \phi', s')] - \rho(1 - d) + \beta E^{P^*}_{s'|s}[v^{P^*}_{\phi\phi'}(1, s') - v^{P^*}_{\phi\phi'}(0, s')] \right) \tag{3.7}$$

The following proposition shows that solution to equation (3.7) exists.

**Proposition 6** (Existence of BMPE). *When $s$ is discrete, a Bayesian Markov Perfect equilibrium exists. When $s$ has continuous support, suppose that the profit functions $\pi^I$ is Lipschitz continuous in $s$ some constant $M^\pi > 0$ and bounded from above by $\bar{\pi}$. Suppose also that $\epsilon \sim$ Logistic$(0, \nu)$ and $Q_z(z'|z)$ is Lipschitz continuous in $z$ with constant $M^q > 0$. Then there exists (componentwise) Lipschitz function $P^*$ in $s$ for all $(\phi, \phi', d) \in \Phi^2 \times \{0, 1\}$ with sufficiently large Lipschitz constant $M > 0$ that satisfies the fixed point equation (3.7).*

When $s$ is discrete, existence can be shown by applying Brouwer's fixed point theorem to the fixed point equation (3.7) on the compact product space of $[0, 1]$. When $s$ has continuous support, the key to showing existence is to make sure that the right hand side functional preserve Lipschitz continuity with a common constant. Then one can apply Schauder's fixed point theorem to show existence of a Lipschitz continuous $P^*$. Uniqueness requires contraction mapping which imposes stronger condition on the right hand side continuous functional of $P^*$.

# Chapter 4

# Estimation

In this section, I discuss how the model can be estimated using US input-output table in a two-step likelihood estimation. The main difficulty for estimation is to estimate the equilibrium CCP $\hat{P}$ using just input-output account. I show how this can be done by imputing the aggregate network sufficient statistic using the first-stage general equilibrium model.

## 4.1 Specification

### 4.1.1 Productivity Process

Recall that $z_t\phi$ is the labor augmenting productivity at $t$. I consider two different processes for aggregate productivity. In the simple case, $z_t$ is binary with values $z_H$ and $z_L$ and transition matrix

$$Q^z = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

In the second case, the aggregate productivity shock $z_t$ evolves according to the following autoregressive process

$$\ln z_t = \rho^z \ln z_{t-1} + \varepsilon^z, \quad E(\varepsilon^z) = 0, \quad \text{Var}(\varepsilon^z) = \nu_z^2$$

Note that in both cases, the productivity type $\phi$ determines the long run expected productivity level.

### 4.1.2 The Likelihood Function

When the sequence of firm-to-firm production network $(D_0, D_1, \dots, D_T)$ and firm-to-firm profits $(\pi_1^I, \pi_2^I, \dots, \pi_T^I)$ are observed[1], by the fixed point equation (3.7)

$$
\begin{aligned}
Q_{N,T}(D|D_0,\hat{P};\nu,\rho) = \frac{1}{TN^2}\sum_{i=1}^{N}\sum_{j\neq i}\Bigg\{\sum_{t=1}^{T} D_{ijt}\ln G\bigg(&\nu^{-1}\rho[\beta\Gamma_2^{\hat{P}}(\phi_i,\phi_j,f^d(s_t)) + (D_{ijt-1}-1)] \\
&+ \nu^{-1}(\beta\Gamma_1^{\hat{P}}(\phi_i,\phi_j,f^d(s_t)) + \pi_{ij,t}^I)\bigg) \\
&+ (1-D_{ijt})\ln\bigg[1 - G\bigg(\nu^{-1}\rho[\beta\Gamma_2^{\hat{P}}(\phi_i,\phi_j,f^d(s_t)) + (D_{ijt-1}-1)] \\
&+ \nu^{-1}(\beta\Gamma_1^{\hat{P}}(\phi_i,\phi_j,f^d(s_t)) + \pi_{ij,t}^I)\bigg)\bigg]\Bigg\}
\end{aligned}
$$

where $\nu$ is the standard deviation of $\epsilon$, $f^d$ is a discretization function defined in Appendix A.2, $G$ is the cdf of standard logistic distributed random variable, coefficient matrices $\Gamma_1^{\hat{P}}$ and $\Gamma_2^{\hat{P}}$ defines the solution for the difference of mini value functions[2]

$$
v^{\hat{P}}(\phi,\phi',1,s^j) - v^{\hat{P}}(\phi,\phi',0,s^j) = \Gamma_1^{\hat{P}}(\phi,\phi',s^j) + \rho\Gamma_2^{\hat{P}}(\phi,\phi',s^j)
$$

[1] $\pi_t^I \equiv \{\pi_{ij,t}^I\}_{ij}$

[2] The formula for $\Gamma_1^{\hat{P}}(\phi,\phi',s^j)$ and $\Gamma_2^{\hat{P}}(\phi,\phi',s^j)$ are given in Appendix

In the case of two types $\phi \in \{\phi_L, \phi_H\}$, the above likelihood function can be rewritten in terms of sufficient network statistic $m_t$

$$
\begin{aligned}
Q_T(m|m_0, \hat{P}; \nu, \rho) = \frac{1}{T} \sum_{\phi \in \{\phi_L, \phi_H\}} \sum_{\phi' \in \{\phi_L, \phi_H\}} \Pi(\phi)\Pi(\phi') \Bigg( & \sum_{t=1}^{T} m_{\phi\phi',t} m_{\phi\phi',t-1} \ln \hat{P}(\phi, \phi', 1, f^d(s_t); \nu, \rho) \\
& + \sum_{t=1}^{T} m_{\phi\phi',t}(1 - m_{\phi\phi',t-1}) \ln \hat{P}(\phi, \phi', 0, f^d(s_t); \nu, \rho) \\
& + \sum_{t=1}^{T} (1 - m_{\phi\phi',t}) m_{\phi\phi',t-1} \ln(1 - \hat{P}(\phi, \phi', 1, f^d(s_t); \nu, \rho)) \\
& + \sum_{t=1}^{T} (1 - m_{\phi\phi',t})(1 - m_{\phi\phi',t-1}) \ln(1 - \hat{P}(\phi, \phi', 0, f^d(s_t); \nu, \rho)) \Bigg)
\end{aligned}
\tag{4.1}
$$

where

$$
\hat{P}(\phi, \phi', d, f^d(s_t); \nu, \rho) = G(\nu^{-1}\rho[\beta\Gamma_2^{\hat{P}}(\phi, \phi', f^d(s_t)) + (d-1)] + \nu^{-1}(\beta\Gamma_1^{\hat{P}}(\phi, \phi', f^d(s_t)) + \pi^I_{\phi\phi',t}))
\tag{4.2}
$$

$\nu, \rho$ are identified and therefore can estimated using conditional MLE. I show how to discrete the state $s_t$ in Appendix A.2 and how to estimate $\hat{P}$, $\Gamma_1^{\hat{P}}$, and $\Gamma_1^{\hat{P}}$ in the following implementation section.

## 4.2 Implementation

### 4.2.1 Data Description and Estimation Procedure

The dataset used in this section is the merged input output accounts for the year 1964-2018 from the Bureau of Economic Analysis. The dataset contains yearly value of the use of commodities by one industry from another industry in millions of dollars. All industries are identified by NACIS code. While the dataset does not provide details information of firm level production network, it provides the most comprehensive coverage of the US economy which is important for studying aggregate economic implications of network adjustments. In addition, I make use of the GDP deflator and average earning time series from FRED for inflation adjustment and constructing real wage.

The main challenge with this dataset is that production network is not directly observed. After identifying the productivity type $\phi$, the network sufficient statistic $m_t(\phi, \phi')$ is also not directly available. To solve this data problem, I propose an imputation procedure to map the total use value between types $R_t(\phi, \phi')$ into aggregate network states $m_t(\phi, \phi')$ and the productivity state $z_t$ using the general equilibrium model. The details of the imputation procedure described in Appendix A.1.

After obtaining $\{\hat{m}_t\}_t$ and $\{\hat{z}_t\}_t$, I then proceed to estimate the implicit conditional choice probabilities that is consistent with both the imputed $\hat{m}_t(\phi, \phi')$ and the law of motion for $m_t$, that is, find $\hat{P}$ that solves the following problem

$$\min_{P} \quad \frac{1}{T}\sum_{t=1}^{T}[\hat{m}_t - F_m^P(\hat{m}_{t-1}, \hat{z}_{t-1})]^2.$$

The solution give us the nonparametric estimate $\hat{P}$ for the equilibrium conditional choice probabilities.[3]

An overview of the estimation procedure is described as follows

1. Discretize the state space of $s_t = (m_t, z_t)$ using methods in Appendix A.2.
2. Identify the productivity types and calibration standard parameters (section 4.2.2)
3. Impute $\{\hat{m}_t\}_t$ and $\{\hat{z}_t\}_t$ using observations $\{R_t\}_t$ and methods in Appendix A.1.
4. Estimate $\hat{P}$ using $\{\hat{m}_t\}_t$ and $\{\hat{z}_t\}_t$ and the method in section 4.2.3.
5. Calculate the mini value functions using value function iteration (4.3) in section 4.2.4.
6. Estimate $\nu$ using scaling method in section 4.2.5.
7. Estimate $\rho$ using NPL estimation (Definition 4) and likelihood function (4.1).

[3]With firm-level data, I can estimate $\hat{P}$ directly using standard nonparametric estimator.

### 4.2.2 Identification of types and Calibration

The number of types and the type of industries are identified directly. I consider two types: manufacturing $\phi_H$ and non-manufacturing $\phi_L$. Industry $i$ has type $\phi_i = \phi_H$ if its NACIS code has format 3XX and has $\phi_i = \phi_L$ otherwise. The observed total use value by $\phi'$ from $\phi$ is interpreted as total sales revenue from $\phi'$ to $\phi$, that is,

$$R_t(\phi, \phi') = \underbrace{\gamma^2 \Pi(\phi)\Pi(\phi')m_t(\phi, \phi')}_{\text{total measure of } \phi \to \phi'} r_t(\phi, \phi').$$

is directly observed from the data. Exporting probabilities: $e(\phi)$ are identified directly from the data by comparing the average total value of output vs total value of export for $\phi$

$$\hat{e}(\phi_H) = 0.090, \quad \hat{e}(\phi_L) = 0.025.$$

The macroeconomic parameter are identified with standard calibration $\sigma$ following Lim (2019)

- Elasticity: $\sigma = 2, 3, 4$
- Labor share: $\alpha = 2/3$
- Discount factor: $\beta = 0.95$

The estimation routine will be carried out with different values of $\sigma$. The foreign demand shifter $\Delta_t^F$ is normalized to $1$.

### 4.2.3 Identification and Estimation of Conditional Choice Probabilities

Using the input-output table, only the sufficient network statistic $m_t$ is identified instead of the actual firm-to-firm links. Suppose that $\{\hat{m}_t\}$ and $\{\hat{z}_t\}_t$ are imputed by using Appendix A.1, the conditional choice probabilities of each pair of firm types can be estimated by using the law of motion equation (3.2)

Suppose that $m_t$ is discretized using a discretization function $f^d : [0,1] \to \mathbb{D}_2$ described in section A.2. Suppose that $z_t$ is binary with values $z_H$ and $z_L$ and transition matrix

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Then the discrete Markov state can be enumerated as $s^1, s^2, \ldots, s^{32}$. Run the following LS regressions for $k = 1, 2, \ldots, 16$,

$$\hat{m}_t(\phi, \phi') = a_{\phi\phi',k} + b_{\phi\phi',k}\hat{m}_{t-1}(\phi, \phi') + \varepsilon_{\phi\phi',k,t}, \quad \forall(\hat{m}_{t-1}, z_{t-1})|(f^d(\hat{m}_{t-1}), z_{t-1}) = s^k.$$

where $\varepsilon_{\phi\phi',k}$ is the measurement error

$$\varepsilon_{\phi\phi',k,t} = [\hat{m}_t(\phi, \phi') - m_t(\phi, \phi')] + b_{\phi\phi',k}[m_{t-1}(\phi, \phi') - \hat{m}_{t-1}(\phi, \phi')]$$

with the following constraints

$$a_{\phi\phi',k} \in [0,1], \quad a_{\phi\phi',k} + b_{\phi\phi',k} \in [0,1].$$

Suppose that measurement error is independent of imputated network sufficient statistic $m_{t-1}$.

**Assumption 4** (Random Measurement Error)**.** For all $(\phi, \phi') \in \Phi^2$ and $k = 1, 2, \ldots, K$,

$$\eta^m_{\phi\phi,t} \equiv m_t(\phi, \phi') - \hat{m}_t(\phi, \phi'), \quad \text{Cov}(\eta^m_{\phi\phi',t}, \hat{m}_s(\phi, \phi')) = 0, \quad \forall t, s = 1, 2, \ldots$$

Under Assumption (4), unbiased estimators for $a_{\phi\phi',k}$ and $b_{\phi\phi',k}$ can be obtained using constrained OLS estimators and quadratic programming provided that there are at least two data points for each $k = 1, 2, \ldots, K$.

The CCPs are identified through the individual Markov state dependence: when the dependence on $d$ is large, then $P(\phi, \phi', 0, s_k) - P(\phi, \phi', 1, s_k)$ is large which means that network evolution has greater dependence on the $t-1$ network statistics $m_{t-1}$, conditional on $\phi, \phi', s^k$. Given the scarcity of yearly data when the state is large, the above regressions are mostly saturated. Then we obtain the following estimates for the CCPs for all $\phi$, $\phi'$, $s^k$

$$\hat{P}(\phi, \phi', 0, s^k) = \hat{a}_{\phi\phi',k},$$
$$\hat{P}(\phi, \phi', 1, s^k) = \hat{a}_{\phi\phi',k} + \hat{b}_{\phi\phi',k}.$$

By contrast, when firm level data $\{D_{ijt}\}_{i,j,t}$ is observed, the equilibrium CCP can be estimated nonparametrically

$$\hat{P}(\phi, \phi', d, s^k) = \frac{\sum_{t=2}^T \sum_i \sum_j D_{ijt} \mathbf{1}\{\phi_i = \phi, \phi_j = \phi', D_{ijt-1} = d, s_t^k = s_k\}}{\sum_{t=2}^T \sum_i \sum_j \mathbf{1}\{\phi_i = \phi, \phi_j = \phi', D_{ijt-1} = d, s_t^k = s_k\}}.$$

The value function can be constructed a function of structure parameters[4] using Hotz and Miller (1993) type argument once we have estimated $\hat{P}$.

### 4.2.4 The Value Function Iteration

Using the law of motion (3.2) for $m_t$ and $Q^z$, the transition matrix $Q^{\hat{P}}$ for the discretized aggregate state $s^k$ can be obtained by simulating a long Markov chain. Similarly, the payoff function can be calculated

$$\pi^I(\phi, \phi', s^k, s^j) = \sigma^{-1} \frac{\sum_{t=1}^T R_t(\phi, \phi') \mathbf{1}\{(s_t, s_{t-1}) = (s^k, s^j)\}}{\sum_{t=1}^T \mathbf{1}\{(s_t, s_{t-1}) = (s^k, s^j)\}}.$$

[4]The structural parameter in this model is $\theta = (\rho, \nu)$ where $\rho$ is the parameter of relationship sunk cost and $\nu$ is the standard deviation of the idiosyncratic dyad-specific relationship cost $\epsilon_{ijt}$.

By the mini Bellman equation (3.6)

$$
\begin{aligned}
v^{\hat{P}}(\phi, \phi', d, s^j) = \sum_{k \in \{1,2,\ldots,K\}} Q^{\hat{P}}(s^k|s^j) \Bigg( & \hat{P}(\phi, \phi', d, s^j)[\pi^I(\phi, \phi', s^k, s^j) - \rho(1-d) \\
& + E[\epsilon|\hat{\Sigma}(\phi, \phi', d, s^j, \epsilon) = 1]] \\
& + \beta \left[ \hat{P}(\phi, \phi', d, s^k) v^{\hat{P}}(\phi, \phi', 1, s^k) + (1 - \hat{P}(\phi, \phi', d, s^k)) v^{\hat{P}}(\phi, \phi', 0, s^k) \right] \Bigg)
\end{aligned} \tag{4.3}
$$

By stacking the value function $v^{\hat{P}}$ as a vector, $v^{\hat{P}}$ can be solved and we can write

$$
v^{\hat{P}}(\phi, \phi', 1, s^j) - v^{\hat{P}}(\phi, \phi', 0, s^j) = \Gamma_1^{\hat{P}}(\phi, \phi', s^j) + \rho \Gamma_2^{\hat{P}}(\phi, \phi', s^j)
$$

for some $\Gamma_1^{\hat{P}}$ and $\Gamma_2^{\hat{P}}$ that can be calculated using formulas in Appendix A.3.

### 4.2.5 Scaling

Note that $\nu^{-1}$ have to be applied when forming $\Gamma_1^{\hat{P}}$ in the step of solving for the value function, since I have to choose a scaling for $\pi^I$ from the GE model to match the expected relationship cost $E[\epsilon|\Sigma(\phi, \phi', d, m, \epsilon) = 1]$ where $\epsilon$ is standard logistic. This suggests the iterative process:

1. guess $\nu$ and construct $\Gamma_1^{\hat{P}}$
2. perform CMLE to obtain estimate for $\hat{\nu}$
3. if $\hat{\nu} > 1$ then repeat step 1 by decreasing $\nu$; if $\hat{\nu} < 1$ then repeat step 1 by increasing $\nu$
4. stop when $\hat{\nu}$ is sufficiently close to $1$

When $\hat{\nu} = 1$, this suggests that the original guess of scaling in step 1 is correct because the estimated standard error for the relationship cost is standard logistic.

### 4.2.6 Multiple Equlibria and NPL Estimation

Since Proposition (6) does not guarantee uniqueness, there could be multiple Bayesian Markov perfect equilibria and the model is incomplete (Aguirregabiria and Mira, 2019). To complete the model, I assume a selection mechanism. Let $\mathcal{E}(m^T, z^T, \theta_0)$ be the set of BMPE[5]. By Proposition (6), $\mathcal{E}(m^T, z^T, \theta_0)$ is nonempty. Define the selection mechanism $\lambda_T : (\upsilon^T, \theta_0) \mapsto P^T \in \mathcal{E}(m^T, z^T, \theta_0)$ where $\upsilon^T$ is the auxiliary random public signal from which the agents coordinate and $\upsilon^T \perp\!\!\!\perp \epsilon$.

**Assumption 5.** There exists a sequence of selection mechanisms $\{\lambda_T\}_{T=1}^{\infty}$ and public signals $\{\upsilon^T\}_{T=1}^{\infty}$ such that

$$\mathbb{P}(m^T, z^T) = \sum_{P^T \in \mathcal{E}(m^T, z^T, \theta_0)} \mathbb{P}(\lambda_T(\upsilon^T, \theta_0) = P^T)$$

Assumption 5 implies that the observed network sufficient statistics $\{\hat{m}_t\}_{t=1}^{T}$ together with the aggregate productivity state $\{\hat{z}_t\}_{t=1}^{T}$ is rationalized by a symmetric BMPE. I assume that the same equilibrium $P$ is played for a given sequence of observed Markov states $(m^1, z^1)$, $(m^2, z^2)$, .... This would guarantee point identification of $\theta_0 = (\nu, \rho)$.

I use the nested pseudo likelihood (NPL) estimator for the second step. The NPL estimation generates a sequence $\{\hat{\theta}^K, \hat{P}^K\}$ according to Step 1:

$$\hat{\theta}_K = \arg\max_{\theta \in \Theta} Q_T(m|m_0, \hat{P}_{K-1}; \theta)$$

Then the CCP estimate is updated in Step 2:

$$\hat{P}_K = \Gamma(\hat{\theta}_K, \hat{P}_{K-1})$$

with $\hat{P}_0 = \hat{P}$ obtained in section. Define the NPL estimator as the fixed points to the above procedure.

[5]Note that $m^T = \{m_t\}_{t=1}^{T}$ and $z^T = \{z_t\}_{t=1}^{T}$

**Definition 4** (NPL Estimator)**.** Let $P = \Gamma(\theta, P)$ denote the fixed point equation (3.7) for the conditional choice probabilities and $Q_T$ denote the conditional log-likelihood function defined in equation (4.1). Then the *nested pseudo likelihood estimator* $(\hat{\theta}_{\text{NPL}}, \hat{P}_{\text{NPL}})$ for $(\theta, P)$ is defined as the solution to

$$\hat{P}_{\text{NPL}} = \Gamma(\hat{\theta}_{\text{NPL}}, \hat{P}_{\text{NPL}}), \quad \hat{\theta}_{\text{NPL}} = \arg\max_{\theta \in \Theta} Q_T(m|m_0, \hat{P}_{K-1}; \theta)$$

Convergence is not guaranteed. Proposition 2 in Aguirregabiria and Mira (2007) establishes asymptotic properties NPL estimators.

# Chapter 5

# Counterfactual Experiment

## 5.1 Uncertainty Shock to Productivity Process

In this section, I decompose the network effect on aggregate welfare of the general equilibrium economy in response to an uncertainty shock to the aggregate productivity process by performing simulation exercises. The model described by section 2 and 3 are simulated for a large number of period ($T = 5000$) in order to numerically integrate future Markov state $s_t = (m_t, z_t)$ on discretized aggregate state space. Here I interpret the discretization as being exact: I assume that firms are making linking decisions based on the discretized aggregate state $m_t^d$. This implies that firms have bounded rationality.

Let the values of aggregate productivity state $z_t$ take discrete values on a uniform grid of the interval $[\underline{z}, \bar{z}]$ with $N^z$ number of states. For example, take $\underline{z} = 0.5$, $\bar{z} = 1.5$, and $N^z = 6$, then the productivity states $z$ takes the following values

$$z \in \{0.5, 0.7, 0.9, 1.1, 1.3, 1.5\}.$$

In addition, I parameterize the Markov transition matrix by persistence parameter $p^z$.

$$Q_z = [p^z - (1 - p^z)/5] \times I_{5\times5} + (1 - p^z)/5 \times J_{5\times5}$$

where $J_{5\times5}$ is a 5 matrix of 1s and $p^z$ controls the uncertainty of the productivity state $z$. A greater $p^z$ implies less aggregate macroeconomic uncertainty. Let the discretized aggregate network sufficient statistic $m_t^d \in \{L, H\}$ be defined as

$$m_t^d = \begin{cases} L & \text{if } \sum_{(\phi,\phi')\in\{\phi_L,\phi_H\}^2} m_t(\phi,\phi') \leq 2 \\ H & \text{if } \sum_{(\phi,\phi')\in\{\phi_L,\phi_H\}^2} m_t(\phi,\phi') > 2 \end{cases}$$

$m_t^d = L$ corresponds to a low network connectivity state whereas $m_t^d = H$ corresponds to a low network connectivity. Firms are only using the discretized network state $m_t^d$ in their first stage network formation problem. They know the $12 \times 12$ transition probability $Q^{d,P}$ of $(z_t, m_t^d)$ for each equilibrium CCP $P$, which in turns depends on the aggregate states $(z, m^d)$. Then they form expectation using $Q^s$ in formulating the individual link formation problem in (3.6). The actual Markov network state $m_t$ follows a law of motion $Q^P$ . For the second stage profit maximizing problem, firms know the realized $m_t \in [0,1]^4$.

The endogenous network state $m_t$ serves two important functions in the model. First, it is the network sufficient statistic for the general equilibrium model described in section 2. Most importantly, $m_t$ determines how central or granular each sector $\phi$ is in the production network. Second, $m_t$ affect equilibrium CCP $P$ through global interaction. Firms must consider the impact of other players' equilibrium strategies on the aggregate network sufficient state $m_t$ in their network formation games. This dependence of $P$ on $m_t$ creates network externality through global interaction.

Multiple equilibria do arise in this setting, but it can be addressed precisely in practice. Let $\bar{Q}^{m^d}$ denote the stationary distribution of the discretized aggregate network sufficient statistic $m_t$ induced by transition probability $Q^{d,P}$. Call an equilibrium $P$ degenerated when $\bar{Q}^{m^d}(m^d) = 0$ for some $m^d \in \{L, H\}$. In practice, I find multiplicity arises precisely due to this degeneracy. Intuitively, it is common to have multiplicity with externality. In this case, a degenerate equilibrium corresponds to a network equilibrium without externality since there is no global interaction through network state $m_t^d$ with probability 1. Hence, nondegenerate equilibrium and degenerate equilibria may exist at the same time. However, I find that nondegenerate equilibrium is always unique in practice, provided that it exists[1]. Therefore, I will focus only on counterfactual analysis in which nondegenerate equilibrium exists.

Consider an uncertainty shock $p^z$ from $0.3$ to $0.7$ so that each productivity state will become more persistent ex-post. There are three different margins to consider: 1) the counterfactual with the sequence of aggregate network sufficient statistics $m_t$ fixed; this is the exogeneous network effect; 2) the counterfactual with a sequence of $m_t$ flexible, but each firm takes $m_t$ as fixed in its network formation problem; this measures the endogenous network effect without the externality through the aggregate state; 3) the counterfactual with fully flexible network adjustment; this margin gives me the externality effect.

| Parameter Values | $\rho$ | $\sigma$ | $\nu$ | $\iota$ | $L$ | $\gamma$ | $\alpha$ | $\Pi(\phi_H)$ | $\phi_L$ | $\phi_H$ | $p^z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Simulation Ex ante | 1 | 2 | 0.0025 | -2 | 10 | 10 | 2/3 | 0.5 | 1.14 | 2.32 | 0.3 |
| Simulation Ex post | 1 | 2 | 0.0025 | -2 | 10 | 10 | 2/3 | 0.5 | 1.14 | 2.32 | 0.7 |

Table 5.1: Parameter Values for Different Simulations

The parameter values used in the counterfactual analysis is reported in Table 1. The long-run productivity type $\phi_L$ and $\phi_H$ are obtained from empirical exercise. Note that the aggregate welfare in the general equilibrium model described in section 2 is given by the simple relation

$$U_t = E_t/\mathcal{P}_t^{1-\sigma}$$

Both the aggregate earning $E_t$ and the ideal price index $\mathcal{P}_t$ depend on the Markov network state $m_t$ and aggregate productivity state $z_t$.

[1]The set of nondegenerate equilibria is either a singleton set or an empty set.

| $(z_t, m_t^d)$ | Ex-ante | Ex-post (fixed) | Ex-post (no externality) | Ex-post |
|---|---|---|---|---|
| 0.5,L | 124.12 | 72.06 | 82.12 | 79.68 |
| 0.7,L | 112.16 | 93.55 | 99.55 | 96.61 |
| 0.9,L | 116.29 | 114.95 | 116.79 | 118.62 |
| 1.1,L | 129.92 | 132.02 | 137.53 | 131.18 |
| 1.3,L | 129.92 | 151.71 | 137.53 | 155.39 |
| 1.5,L | 136.52 | 168.53 | 176.71 | 169.21 |
| 0.5,H | 113.11 | 78.68 | 83.82 | 79.99 |
| 0.7,H | 118.28 | 96.80 | 99.18 | 99.61 |
| 0.9,H | 123.40 | 113.87 | 118.61 | 120.65 |
| 1.1,H | 126.04 | 133.75 | 138.20 | 138.50 |
| 1.3,H | 132.00 | 150.73 | 158.54 | 157.30 |
| 1.5,H | 138.21 | 168.72 | 179.39 | 177.56 |

Table 5.2: Average Welfare across Aggregate States

For the first margin, I first compute the average welfare $\bar{U}^0(z, m^d)$ over each aggregate state $(z_t, m_t^d)$ using $p^z = 0.3$. This is reported in the first column under the name of "Ex-ante" in Table 2. Then I calculate the average welfare $\bar{U}^1(z, m^d)$ over each aggregate state $(z_t, m_t^d)$ using $p^z = 0.3$ and fixing the network statistics $m_t$ at ex-ante value for all $t$. This is reported in the second column under the name "Ex-post (fixed)." The difference between these two is the exogenous network effect

$$\textit{exogeneous network effect} = \bar{U}^1(z, m^d) - \bar{U}^0(z, m^d).$$

The exogenous network effect is reported in column 2 of Table 3 under the name "Fix Network." From Table 2 and 3, it is clear that the uncertainty shock increases the persistence of both good and bad states, which leads to greater dispersion of welfare across states.

For the second margin, I compute the average welfare $\bar{U}^2(z, m^d)$ by using $p^z = 0.7$ and by solving for the Bayesian Markov perfect equilibrium $P_2$ using the fixed point equation in CCP (3.7) while holding fix the aggregate network state $m_t$ at the ex-ante level. By doing this, the equilibrium $P_2$ is computed without considering how $P_2$ could affect the distribution of $m_t$. The actual Markov network state $m_t$ is allowed to adjust according to $P_2$ and to equation (3.2). Hence $\bar{U}^2$ measures the average welfare of the ex-post economy with the Markov network states adjusting according to the new equilibrium $P_2$, but without network externality through $m_t$. $\bar{U}^2(z, m^d)$ is reported in the third column of Table 2. The increase in the aggregate productivity state $z_t$'s persistence leads to greater dependence on the average welfare on $z_t$.

For the third margin, I compute the average welfare $\bar{U}^3(z, m^d)$ by using $p^z = 0.7$ and letting both the equilibrium CCP $P_3$ and the Markov network state $m_t$ adjust freely. This is reported in the fourth column of table 3.

| $(z_t, m_t^d)$ | Total Effect | | Endogenous Network Adjustment | |
|---|---|---|---|---|
| | Fixed Network | Endogenous Network | Network Externality | Without Externality |
| 0.5,L | -52.06 | 7.62 | -2.44 | 10.05 |
| 0.7,L | -18.61 | 3.06 | -2.95 | 6.00 |
| 0.9,L | -1.34 | 3.67 | 1.82 | 1.84 |
| 1.1,L | 2.10 | -0.84 | -6.35 | 5.51 |
| 1.3,L | 13.50 | 3.57 | 0.02 | 3.55 |
| 1.5,L | 31.90 | 0.79 | -7.49 | 8.28 |
| 0.5,H | -34.43 | 1.31 | -3.83 | 5.14 |
| 0.7,H | -21.49 | 2.81 | 0.42 | 2.39 |
| 0.9,H | -9.54 | 6.70 | 1.96 | 4.74 |
| 1.1,H | 7.71 | 4.75 | 0.30 | 4.45 |
| 1.3,H | 18.72 | 6.58 | -1.24 | 7.82 |
| 1.5,H | 30.51 | 8.96 | -1.71 | 10.68 |

Table 5.3: Decomposition of Impact on Welfare

Therefore, the total effect, reported in column 1 of table 3, is given by

$$\textit{total effect} = \bar{U}^3(z, m^d) - \bar{U}^0(z, m^d).$$

The total effect is negative for low productivity states and positive for high productivity states due to a higher state persistence. The total effect is also slightly higher for high discretized network state $H$ than for $L$, since higher persistence of productivity state implies higher persistence of the discretized network state $m_t^d$.

The endogenous network adjustment effect, reported in column 2 of Table 3, is given by

$$\textit{endogeneous network adjustment effect} = \bar{U}^3(z, m^d) - \bar{U}^1(z, m^d).$$

The endogeneous network adjustment leads to higher welfare in general. In the case of $(z, m^d) = (0.9, H)$ for example, network adjustment offsets more than 70% of the negative exogeneous network effect. Network adjustment effect also appear to be greater at the extreme low state $(z_t, m_t^d) = (0.5, L)$ and extreme high state $(z_t, m_t^d) = (1.5, H)$.

The network externality effect, reported in column 3, is given by

$$\textit{network externality effect} = \bar{U}^3(z, m^d) - \bar{U}^2(z, m^d).$$

The network externality effect is negative in general. In the state $(z, m^d) = (1.1, L)$, the network externality effect is substantial and offsets the isolated network adjustment effect completely. The network externality effect is reported in column 4 of Table 3.

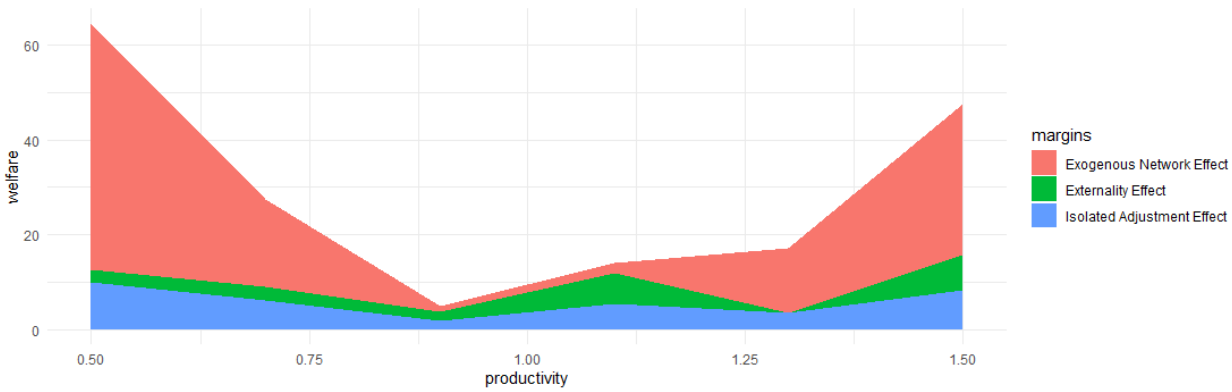
I define the network adjustment effect in the absence of network externality as

$$\textit{isolated adjustment effect} = \bar{U}^2(z, m^d) - \bar{U}^1(z, m^d).$$

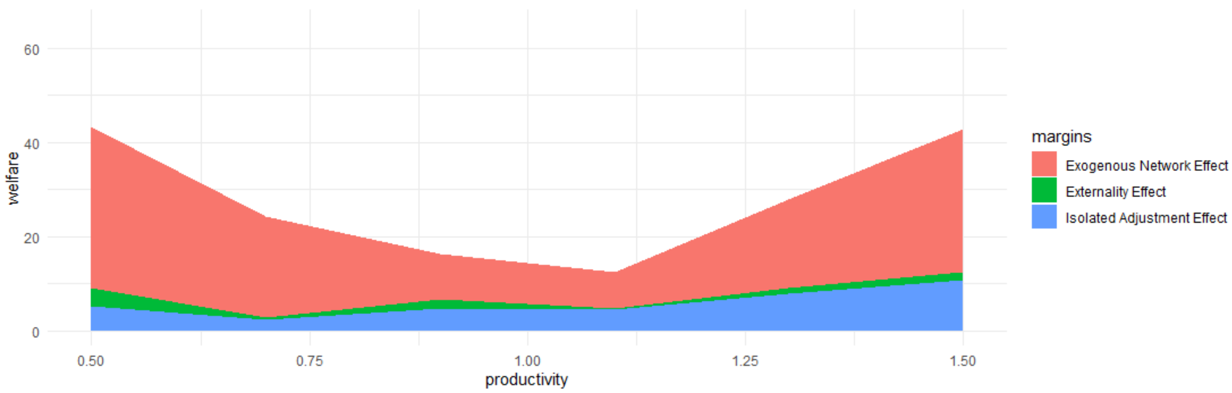
The isolated adjustment effect is reported in column 5 of Table 3. The isolated adjustment effect is all positive since firms cannot do worse than keeping their linking probability fixed when they can optimize their conditional choice probabilities.

To get a sense of the relative magnitudes of different network effects, I graph the decomposition of an uncertainty shock's total effect in absolutely in Figure 1. The decomposition for low $L$ and high $H$ network states are graphed separately in figure 1(a) and figure 1(b). The y-axis corresponds to the aggregate welfare level, and the x-axis corresponds to different aggregate productivity states. The region represents the exogenous network effect; the green region represents the magnitude of the network externality effect through global interaction; the blue region represents the magnitude of the endogenous network effect without global interaction.

By comparing figure 1(a) and 1(b), I find that the network externality effect appears to be greater in the low aggregate network state $L$. In the ex-post economy with a higher productivity persistence level, the equilibrium CCP varies more with the productivity state $z_t$. When the




(a) Decomposition of Network Effects for $m_t^d = L$




(b) Decomposition of Network Effects for $m_t^d = H$

Figure 5.1: Decomposition of Network Effects in Absolute Values

equilibrium linking probability is high ex-ante, then the uncertainty shock would lower the linking probability for low productivity states than it raises the linking probability for the high productivity states. This means a greater impact on the network sufficient statistic $m_t$ in the low connectivity region $L$ than on the $m_t$ in the high connectivity region $H$. Therefore, this impact raises the stationary probability of staying in the network state[2] $L$. Taking into account the lower network connectivity in the ex post economy, each firm's incentive to link is dampened due to such global interaction. This explains why the network externality effect is negative and appears to be greater in the low aggregate network state $L$.

From the results obtained from this counterfactual exercise, I find that the endogenous network adjustment could have a sizable effect on the aggregate welfare, accounting for up to 100%

[2]In the simulation exercise, the uncertainty shock raises the stationary probability of being in $L$ from $0.12$ to $0.14$.

of the total welfare effect of an uncertainty shock, counteracting the exogeneous network effect in certain states. Additionally, network externality through global interaction is also crucial for understanding the endogenous effect. When an uncertainty shock raises the aggregate productivity state's persistence, network externality dampens the endogenous network adjustment effect. Knowing that their re-optimization would lead to less aggregate network connectivity, firms adjust their linking probability less. In some instances, this network externality effect completely offsets any positive welfare gain from re-adjusting production networks.

# Chapter 6

# Extension: Network Formation with Endogenous Node Entry and Exit

In this section, I present an extension of the endogenous network formation model described in section 2 and 3. Specifically, this extension endogenizes both the entry and exit problem and export decisions all in an unified framework with the network formation game. By allowing for entry and exit decision, this model becomes the first strategic network formation model to feature endogeneous birth and death of nodes as well as growth of local out-networks. In addition, the endogeneous entry and exit decision introduces another source of interdependence among network links, since each link between firm $i$ and $j$ depends on the fact that both $i$ and $j$ decides to be present in the market, which in turn, depend on whether the sum of payoffs from $i$ and $j$'s respective links justify the entry cost. Another consequence of this extension is that firms are distinguishable within the same type by the amount of period they have stayed in the market. In this case, the relevant network sufficient statistic is the distribution of out-degrees across individual agents.

**Environment and Timing** For every $t = 1, 2, \ldots$, there are two stages of decision making for each agent. In the first stage, each firm choose optimally buyers in the intermediate inputs market given the aggregate state and their idiosyncratic draw of relationship cost in a dynamic Bayesian incomplete information game. Firms also decide 1) whether to stay or to enter the market; 2) whether to export or not given their idiosyncratic draw of fixed costs. Firms are forward looking and fully rational. At the end of the first stage, the optimal linking decisions and exporting decisions then yield the matching probability $m_t$ as well as other Markov states at $t$ according to law of motions. In the second stage, firms behave in the same way as described in the previous section, taking $m_t$, $e_t$, $\gamma_t$, and $\Pi_t$ as given.

For every $t = 1, 2, \ldots$, there are $\bar{\gamma} - \gamma_{t-1}$ potential entrants and $\gamma_{t-1}$ incumbent firms. Each firm $i$ draws an iid idiosyncratic asymmetric linking cost $\epsilon_{ij} \sim G_\epsilon$ for each other firms $j$ in the network, an iid idiosyncratic fixed sell-off value or entry cost $f_i^S \sim G_{f^S}$ depending on whether it is a incumbent firm or a potential entrant[1], an iid idiosyncratic fixed exporting cost $f_i^E \sim G_{f^E}$. I focus on symmetric equilibrium Markov strategy of the following form

$$\begin{aligned}
\text{Link:} \quad & \Sigma^*(\phi, \phi', d, s, \epsilon) \in \{0, 1\}, \quad P^*(\phi, \phi', d, s) \equiv E_\epsilon[\Sigma^*(\phi, \phi', d, s, \epsilon)] \\
\text{Export:} \quad & \tilde{\Sigma}^*(\phi, m, s, f^E) \in \{0, 1\}, \quad \tilde{P}^*(\phi, m, s) \equiv E_{f^E}[\tilde{\Sigma}^*(\phi, m, s, f^E)] \\
\text{Entry:} \quad & \dot{\Sigma}^*(\phi, m, s, f^S) \in \{0, 1\}, \quad \dot{P}^*(\phi, m, s) \equiv E_{f^S}[\dot{\Sigma}^*(\phi, m, s, f^S)]
\end{aligned}$$

where $d$ is an indicator of whether $\phi$-firm linked to the $\phi'$-firm in the previous period; $m$ is a sufficient statistic for past network peer of the firm. $\Sigma$ and $P$ are the strategy and the corresponding conditional choice probability (CCP) for linking decisions, conditional on the event that the both firms of the link decide to be present in the market and that the $\phi$-firm decides to export if the link is between firms in different regions; $\tilde{\Sigma}$ and $\tilde{P}$ are the strategy and corresponding CCP for exporting decisions conditional on the event that the $\phi$-firm decides to be present in the market; finally, $\dot{\Sigma}$ and $\dot{P}$ are the strategy and corresponding CCP for the entry and exit decision of the $\phi$-firm. It suffices to characterize the equilibrium entirely using $P$, $\dot{P}$ $\tilde{P}$. To illustrate, $P^*(\phi, \phi', 1, s)$ is the equilibrium probability of a $\phi$-firm links or sells to a $\phi'$-firm given that $\phi$ type firm has previously linked to the $\phi'$ type firm for a given aggregate state $s$. In this section, I describe the firms' network formation problem and define the equilibrium concept that determines the equilibrium CCPs $P^*$,

[1]In this framework, I assume that both the sell-off value and the entry cost are fully independent and are drawn from identical distributions. Thus there is no reason to distinguish the two or to keep track of the incumbent state

$\tilde{P}^*$, and $\dot{P}$, then I show their existence and construction.

## 6.1 Firms' Network Formation Problem

For every $t = 1, 2, \ldots$, in a first stage, each firm $i$ draws independently idiosyncratic private costs $\{\epsilon_{ijt}|j \in J_t\}$, $f^S_{it}$, and $f^E_{it}$. $\epsilon_{ijt}$, $f^S_{it}$, and $f^E_{it}$ are fully independent from one another across $i$, $j$, and $t$. Let $D_{ijt}$ be the indicator of whether firm $i$ links to or sells to $j$ at time $t$ and $\tilde{D}_{it}$ be the indicator of whether firm $i$ exports at $t$. Since firm $i$ do not observe $\epsilon_{jkt}$ for $j \neq i$, firm $i$ do not observe $D_{jkt}$ so that $i$ has believes $P^i$ where

$$E^i_\epsilon[D_{jkt}|\phi_j, \phi_k, D_{jkt-1}, s_t] = P^i(\phi_j, \phi_k, D_{jkt-1}, s_t), \quad \forall j \in J_t.$$

Similarly, $i$ has believes $\tilde{P}^i$ and $\dot{P}^i$ where

$$\begin{aligned} E^i_{f^S}[\dot{D}_{jt}|\phi_j, \phi_k, D_{jt-1}, s_t] &= \dot{P}^i(\phi_j, m_{jt}, s_t), \\ E^i_{f^E}[\tilde{D}_{jt}|\phi_j, \phi_k, D_{jt-1}, s_t] &= \tilde{P}^i(\phi_j, m_{jt}, s_t), \quad \forall j \in J_t \end{aligned}$$

In order to mitigate the curse of dimension, I formulate the network formation problem so that out of the three CCPs, only the CCP for linking decisions $P$ depends on linking decision $D_{jkt-1}$ of the previous period. At each $t = 1, 2, \ldots$. Firm $i$ choose $\{D_{ijh}|j \in J_h, h = t, t+1, \ldots\}$, $\{\dot{D}_{ih}|h = t, t+1, \ldots\}$, and $\{\tilde{D}_{ih}|h = t, t+1, \ldots\}$ to maximize its expected present value discounted profit given believes $P^i$, $\dot{P}^i$, and $\tilde{P}^i$

$$\max_{\dot{D}_{ih} \in \{0,1\}, \forall h \geq t} \max_{\tilde{D}_{ih} \in \{0,1\}, \forall h \geq t} \max_{D_{ih} \in \{0,1\}^{J_h}, \forall h \geq t} E_{(s, f^S, f^E, \epsilon), t}\left[\sum_{h=t}^{\infty} \beta^{h-t} \pi(\phi_i, s_h, \mathbb{D}_{ih}, P^i, \dot{P}^i, \tilde{P}^i, f^S_{ih}, f^E_{ih}, \epsilon_{ih})\right]. \tag{6.1}$$

The period profit function $\pi$ are determined by optimal price setting in the second stage as described by the previous section, taking matching probability $m_t$, exporting probability $e_t$, and type-specific measures of firms $\Pi_t$ as given. The ex-post (realization of private costs) period profit

$\pi$ is given by[2]

$$\pi(\phi_i, s_t, D_{it}, \dot{D}_t, \tilde{D}_{it}, D_{it-1}, \dot{D}_{it-1}, \tilde{D}_{it-1}, P^i, \dot{P}^i, \tilde{P}^i, f^S_{it}, f^E_{it}, \epsilon_{it}) =$$

$$\underbrace{\dot{D}_{it}}_{\text{incumbent}} \Bigg[ \underbrace{\pi^C_D(\phi_i, m_{it}, s_t, P^i, \dot{P}^i, \tilde{P}^i)}_{\text{domestic final good}} +$$

$$\int_{j \in J_t, \mathcal{R}_j = \mathcal{R}_i} \dot{D}_{jt} \underbrace{D_{ijt}}_{\text{network}} \Bigg( \underbrace{\pi^I(\phi_i, \phi_j, m_{it}, s_t, P^i, \dot{P}^i, \tilde{P}^i)}_{\text{intermediate good}} - \underbrace{\rho_{\phi_i}(1 - \dot{D}_{jt-1}\dot{D}_{it-1}D_{ijt-1})}_{\text{sunk cost}} + \epsilon_{ijt} \Bigg) dj - f^S_{it}$$

$$+ \underbrace{\tilde{D}_{it}}_{\text{export}} \Bigg\{ \underbrace{\pi^C_E(\phi_i, s_t, P^i, \dot{P}^i, \tilde{P}^i)}_{\text{exported final good}}$$

$$+ \textstyle\int_{j \in J_t, \mathcal{R}_j \neq \mathcal{R}_i} \dot{D}_{jt} \underbrace{D_{ijt}}_{\text{network}} \Bigg( \underbrace{\pi^I(\phi_i, \phi_j, m_{it}, s_t, P^i, \dot{P}^i, \tilde{P}^i)}_{\text{intermediate good}} - \underbrace{\rho_{\phi_i}(1 - \dot{D}_{jt-1}\dot{D}_{it-1}\tilde{D}_{it-1}D_{ijt-1})}_{\text{sunk cost}} + \epsilon_{ijt} \Bigg) dj - f^E_{it} \Bigg\} \Bigg].$$

where $\rho_{\phi_i}$ is the measure of "stickiness" of network and exporting. $\pi^C$ and $\pi^I$ are given by equation (2.11) and (2.12). The network stickiness is heterogeneous across type $\phi$. This allows for heterogeneous network response to aggregate shocks, since types with low $\rho$ can adjust their network more freely than types with high $\rho$. Although the period profit function depends on decision $D$, $\dot{D}$, $\tilde{D}$ made in the present and previous period, the associated Bellman equation can be written in a way that does not depend on $D_{it-1}$, $\dot{D}_{it-1}$, $\tilde{D}_{it-1}$, and $\dot{D}_{jt-1}$ by introducing a firm specific sufficient statistic.

## 6.2 Law of Motions

First, I introduce two new aggregate state $\dot{m}_t(\phi)$ and $\tilde{m}_t(\phi)$ that keeps track of the proportion of the population of firms that 1) made the entry/staying decision $\dot{D}_{it}$; 2) made the exporting decision

[2] Note that the total ex-post profit function $\pi$ depend on both the measure of the incumbent firms $\gamma_{t-1}$ and the measure of all firms $\bar{\gamma}_{t-1}$ but profit functions for specific markets $\pi^C$ and $\pi^I$ do not depend on $\gamma_{t-1}$ and $\bar{\gamma}_{t-1}$ given the optimal decisions in the first stage. This is due to the scale-invariant nature of the second stage model. Therefore, the long run growth of the global economy (growth rate of $\gamma_t$ and $\bar{\gamma}_t$) affects the short run equilibrium only through the endogeneous network channel.

$\tilde{D}_{it}$:

$$\dot{m}_t(\phi) = \frac{\int_{j\in J} \dot{D}_{jt}\mathbf{1}\{\phi_j = \phi\}dj}{\int_{j\in J} \mathbf{1}\{\phi_j = \phi\}dj} = \frac{\gamma_t \Pi_t(\phi)}{\bar{\gamma}_t(\phi)}$$
$$\tilde{m}_t(\phi) = \frac{\int_{j\in J} \tilde{D}_{jt}\dot{D}_{jt}\mathbf{1}\{\phi_j = \phi\}dj}{\int_{j\in J} \dot{D}_{jt}\mathbf{1}\{\phi_j = \phi\}dj}$$

where $\bar{\gamma}_t(\phi)$ is the measure of all incumbent and potential firms for a given type $\phi$ at $t$ whereas $\gamma_t\Pi_t(\phi)$ is the measure of all incumbant firms for a given type $\phi$ at $t$ (entering the second stage of decision making). Next, I introduces a private state $m_{it-1}(\phi')$ that acts as a sufficient statistic for each firm $i$'s realized past network peers $D_{it-1}$

$$m_{it-1}(\phi') = \begin{cases} \frac{\int_{j\in J} \dot{D}_{jt-1}D_{ijt-1}\dot{D}_{it-1}\mathbf{1}\{\phi_j=\phi'\}dj}{\gamma_{t-1}\Pi_{t-1}(\phi')}, & \mathcal{R}' = \mathcal{R} \\ \frac{\int_{j\in J} \dot{D}_{jt-1}D_{ijt-1}\dot{D}_{it-1}\tilde{D}_{it-1}\mathbf{1}\{\phi_j=\phi'\}dj}{\gamma_{t-1}\Pi_{t-1}(\phi')}, & \mathcal{R}' = \mathcal{R} \end{cases}$$

$m_{it-1}(\phi')$ is the average out-degree of $i$ to $\phi'$-firms at $t-1$. Let $f^m_{\phi,\phi',t-1}(m_{it-1})$ denote the out-degree distribution of the private state $m_{it-1}(\phi')$ at time $t-1$ for $\phi_i = \phi$. $\{f^m_{\phi,\phi',t-1}(m_{it-1})\}_{(\phi,\phi'),t}$ is both common knowledge and an aggregate state that is sufficient statistic for the entire production network. Since firms are infinitesimal, each firm can use $f^m_{\phi,\phi',t-1}(m_{it-1})$ to integrate out the private states of other firms when forming believes about their actions. For instance, define the entry probability $\dot{P}$ average over $\dot{D}_{it-1}$, $\tilde{D}_{it-1}$, and $m_{it}$ across all firms

$$\bar{\dot{P}}(\phi, s_t) = \int_0^1 \dot{P}(\phi, m, s_t) f^m_{\phi,\phi',t-1}(m)dm$$

$\bar{\tilde{P}}$ and $\bar{P}$ can be defined analogously. Then the equilibrium CCPs $P^*$, $\dot{P}^*$, $\tilde{P}^*$, $m_{it}$, and the aggregate state $s_t$[3] are sufficient statistics for the production network $D$. The aggregate measure of firms

[3] $s_t = (\gamma_{t-1}, \Pi_{t-1}, \tilde{m}_{t-1}, f^m_{t-1})$. Note $m_{t-1}$ is also redundant once both $m_{it-1}$ $f^m_{t-1}$ are known.

evolves according to the following equations

$$\gamma_t = \sum_{\phi \in \Phi} \bar{\dot{P}}^*(\phi, s_t)\bar{\gamma}_{t-1}(\phi) \tag{6.2}$$

$$\Pi_t(\phi) = \gamma_t^{-1} \bar{\dot{P}}^*(\phi, s_t)\bar{\gamma}_{t-1}(\phi) \tag{6.3}$$

$$\bar{\gamma}_t(\phi) = \bar{\gamma}(\phi) \tag{6.4}$$

The firm-specific private state (conditional out-degree) $m_{it}(\phi)$ is Markov and evolve according to

$$\begin{aligned} m_{it}(\phi') &= H(\phi', \dot{D}_{it}, \tilde{D}_{it}, m_{it-1}, s_t) \qquad (6.5) \\ &\equiv \dot{D}_{it}\bar{\dot{P}}(\phi', s_t)\Big[P(\phi, \phi', 1, m_{it-1}, s_t)m_{it-1}(\phi') + P(\phi, \phi', 0, m_{it-1}, s_t)[1 - m_{it-1}(\phi')]\Big] \end{aligned}$$

$$\begin{aligned} m_{it}(\phi') &= H(\phi', \dot{D}_{it}, \tilde{D}_{it}, m_{it-1}, s_t) \qquad (6.6) \\ &\equiv \dot{D}_{it}\tilde{D}_{it}\bar{\dot{P}}(\phi', s_t)\Big[P(\phi, \phi', 1, m_{it-1}, s_t)m_{it-1}(\phi') + P(\phi, \phi', 0, m_{it-1}, s_t)[1 - m_{it-1}(\phi')]\Big] \end{aligned}$$

The variation in $m_{it}$ across $i$ and $t$ comes from the variations in the initial conditional out-degree $m_{i0}$ and the variation of the realized path $\dot{D}_i^t = (\dot{D}_{i0}, \ldots, \dot{D}_{it-1})$ and $\tilde{D}_i^t = (\tilde{D}_{i0}, \ldots, \tilde{D}_{it-1})$. Note that without the binary decision $\dot{D}_{it}$ and $\tilde{D}_{it}$, $f^m$ becomes degenerate and $m_{it} = m_t$ for all $i$.

Next, the following law of motions for the Markov states $m_t$, $\tilde{m}_t$ and must satisfy[4]

$$
\begin{aligned}
f^m_{\phi,t}(m) &= \bar{\gamma}(\phi)^{-1}\int_{i\in J|\phi_i=\phi}\mathbf{1}\{m_{it}=m\}di \\
&= \bar{\gamma}(\phi)^{-1}\sum_{(\dot{D}_t,\tilde{D}_t)\in\{0,1\}^2}\tilde{\Pi}^{\phi}(\dot{D}_t,\tilde{D}_t,s_t)\int_{i\in J|\phi_i=\phi}\mathbf{1}\{H(\dot{D}_t,\tilde{D}_t,m_{it-1},s_t)=m\}di \\
&= \bar{\gamma}(\phi)^{-1}\sum_{(\dot{D}_t,\tilde{D}_t)\in\{0,1\}^2}\tilde{\Pi}^{\phi}(\dot{D}_t,\tilde{D}_t,s_t)\int_{m^-\in H^{-1}(\dot{D}_t,\tilde{D}_t,m,s_t)}\int\mathbf{1}\{m_{it-1}=m^-\}didm^- \\
&= \sum_{(\dot{D}_t,\tilde{D}_t)\in\{0,1\}^2}\tilde{\Pi}^{\phi}(\dot{D}_t,\tilde{D}_t,s_t)\int_{m^-\in H^{-1}(\dot{D}_t,\tilde{D}_t,m,s_t)} f^m_{\phi,t-1}(m^-)dm^- &\quad (6.7)\\
m_t(\phi,\phi') &= [\gamma_t\Pi_t(\phi)]^{-1}\int_0^1 f^m_{\phi,t}(m)mdm, \quad \mathcal{R}=\mathcal{R}' &\quad (6.8)\\
\dot{m}_t(\phi) &= \int f^m_{\phi,t-1}(m)\dot{P}(\phi,m,s_t)dm = \bar{\dot{P}}(\phi,s_t) &\quad (6.9)\\
\tilde{m}_t(\phi) &= \int f^m_{\phi,t-1}(m)\tilde{P}(\phi,m,s_t)dm = \bar{\tilde{P}}(\phi,s_t) &\quad (6.10)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Pi}^{\phi}(\dot{D}_t,\tilde{D}_t,s_t) &= \dot{D}_t\tilde{D}_t\dot{m}_t\tilde{m}_t + [1-\dot{D}_t]\tilde{D}_t(1-\dot{m}_t)\tilde{m}_t \\
&\quad + \dot{D}_t[1-\tilde{D}_t]\dot{m}_t[1-\tilde{m}_t] + [1-\dot{D}_t][1-\tilde{D}_t](1-\dot{m}_t)(1-\tilde{m}_t) \quad (6.11)
\end{aligned}
$$

The probability of exporting for a $\phi$-firm is redundant once $\tilde{m}_t(\phi)$ is included in the aggregate state $s_t$

$$
\begin{aligned}
e_{\mathcal{R}',t}(\phi) &= \tilde{m}_t(\phi), \quad \mathcal{R}'\neq\mathcal{R} \\
e_{\mathcal{R}',t}(\phi) &= 1, \quad \mathcal{R}'=\mathcal{R} \quad (6.12)
\end{aligned}
$$

[4]Note that $m_t$ is now effectively the matching probability taking into account both the entry decisions of both parties

$$m_t(\phi,\phi') = [\gamma_t^2\Pi_t(\phi)\Pi_t(\phi')]^{-1}\int_{i\in J,\phi_i=\phi}\int_{j\in J,\phi_j=\phi'} D^*_{ijt}\dot{D}^*_{it}\dot{D}^*_{jt}djdi$$

for $\mathcal{R}_i$=$\mathcal{R}_j$ and the exporting decision $m_t(\phi,\phi') = [\gamma_t^2\Pi_t(\phi)\Pi_t(\phi')]^{-1}\int_{i\in J,\phi_i=\phi}\int_{j\in J,\phi_j=\phi'} D^*_{ijt}\dot{D}^*_{it}\tilde{D}_{it}\dot{D}^*_{jt}djdi$ for $\mathcal{R}_i\neq\mathcal{R}_j$, since I can freely set $D^*_{ijt}=0$ when either one of $\dot{D}^*_{it}$, $\tilde{D}_{it}$, $\dot{D}^*_{jt}$ is zero to avoid ambiguity. This convention is embedded in the laws of motions.

To understand these laws of motions intuitively, let $P_{\phi\phi',t}$ denote the probability measure of events on the measure of firm dyads with a given pair of types $\phi$-$\phi$' at $t$ and let $P_{\phi,t}$ denote the probability measure of events on the measure of firms with types $\phi$ conditional on $s_t$, then

$$P_{\phi\phi',t}\{D_{ijt} = 1|\dot{D}_{it} = 1, \dot{D}_{jt} = 1\} = m_t(\phi, \phi')$$
$$P_{\phi,t}\{\tilde{D}_{it} = 1|\dot{D}_{it} = 1\} \equiv \tilde{m}_t(\phi)$$
$$P_{\phi,t}\{\dot{D}_{it} = 1\} \equiv \dot{m}_t(\phi)$$

Given the public network statistic $m_t$, private network statistic $m_{it}$, export probability $e_t$ or $\tilde{m}_t$, and aggregate states $\gamma_t, \Pi_t$, equilibrium profits $\pi^C$ and $\pi^I$ at $t$ are determined by profit functions without the knowledge of $f^m_{\phi,t}$. Recall that the firm-specific demand shifter satisfies the following modified version of recursion (2.21)

$$\begin{aligned} y_{it}(\phi)c_{it}(\phi)^\sigma = \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} \sum_{\mathcal{R}'\in\{\mathcal{F},\mathcal{W},\mathcal{H}\}} \tau(\mathcal{R},\mathcal{R}')^{-\sigma}\alpha_{\mathcal{R}'}^{\sigma-1}(P_{\mathcal{R}',t})^\sigma u_{\mathcal{R}',t} + \\ \gamma_t \sum_{\phi'\in\Phi} \Pi_t(\phi')m_{it}(\phi,\phi')(1-\alpha^L)^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}\tau(\mathcal{R},\mathcal{R}')^{-\sigma}y_t(\phi')c_t(\phi')^\sigma \end{aligned} \quad (6.13)$$

The firm-specific term $y_{it}(\phi)c_{it}(\phi)^\sigma$ is now $i$-specific instead of $\phi$-specific due to the heterogeneity in $i$'s local network $m_{it}(\phi')$ instead of the aggregate matching probability $m_t(\phi, \phi')$. To obtain the original recursion, simply integrate out $i$ using the condition out-degree distribution $f^m_{\phi,t}$ for type $\phi$ at time $t$. Therefore, the second stage equilibrium can be solved in the same way to derive the original profit functions $\pi^I$ and $\pi^C$ (does no depend on $m_{it}$), which are now interpreted as profit functions aggregated at the type-level with local network $m_{it}$ averaged out. Firm $i$-specific profit functions $\pi^I$ and $\pi^C$ (depends on $m_{it}$) can therefore be derived using equation (6.13).

## 6.3 Bellman Equation

Let $\mathbb{P}^i$ denote firm $i$'s belief functions of other firms' strategies $(P^i, \tilde{P}^i, \dot{P}^i)$. The Bellman equation for the $\phi$-firm's linking problem described in equation (6.1) is as follows

$$
\begin{aligned}
V(\phi_i, D_{it-1}, \dot{D}_{it-1}, \tilde{D}_{it-1}, m_{it-1}, s_t) = E_{(\epsilon, f^S, f^E)}\Bigg[ \max_{\dot{D}_{it}\in\{0,1\}} \max_{D_{it}\in\{0,1\}^J} \max_{\tilde{D}_{it}\in\{0,1\}} \dot{D}_{it}\Bigg\{ \\
\Bigg(\int_{j\in J^D} D_{ijt}\dot{D}_{jt}\Big[\pi^I(\phi_i, \phi_j, s_t, m_{it-1}, \mathbb{P}^i) - \rho_i(1 - D_{ijt-1}\dot{D}_{it-1}\dot{D}_{jt-1}) - \epsilon_{ijt}\Big] \\
+ \tilde{D}_{it}\Bigg(\pi_E^C(\phi_i, s, \mathbb{P}^i) + \Bigg(\int_{j\in J^E} D_{ijt}\dot{D}_{jt}[\pi^I(\phi_i, \phi_j, m_{it-1}, s_t, \mathbb{P}^i) - \rho_i(1 - D_{ijt-1}\dot{D}_{it-1}\tilde{D}_{it-1}\dot{D}_{jt-1}) \\
+ \beta E[V(\phi_i, D_{it}, \dot{D}_{it}, \tilde{D}_{it}, m', s', \mathbb{P}^i)|\dot{D}_{it} = 1, \tilde{D}_{it} = 1] - f_{it}^E\Bigg)\Bigg) \\
+ \pi_E^C(\phi_i, s, \mathbb{P}^i) + (1 - \tilde{D}_{it})\beta E[V(\phi_i, D_{it}, \dot{D}_{it}, \tilde{D}_{it}, m', s', \mathbb{P}^i)|\dot{D}_{it} = 1, \tilde{D}_{it} = 0] - f_{it}^S\Bigg\} \\
+ (1 - \dot{D}_{it})\beta E[V(\phi_i, D_{it}, \dot{D}_{it}, \tilde{D}_{it}, m', s', \mathbb{P}^i)|\dot{D}_{it} = 0, \tilde{D}_{it} = 0]\Bigg]
\end{aligned} \tag{6.14}
$$

where

$$
\begin{aligned}
J^D &= J \cap \{j : \mathcal{R}_j = \mathcal{R}_i\} \\
J^E &= J \cap \{j : \mathcal{R}_j \neq \mathcal{R}_i\} \\
D_{it}(j) &= D_{ijt} \\
\dot{D}_t(j) &= \dot{D}_{jt}
\end{aligned}
$$

The states $\dot{D}_{it-1}$, $\tilde{D}_{it-1}$ and $D_{jt-1}$ are encoded in $m_{it-1}$. Now I introduce the solution concept which determines the equilibrium CCP $P^*$, $\tilde{P}^*$, and $\dot{P}^*$.

**Definition 5.** A symmetric *Bayesian Markov perfect equilibrium* in the open economy with endogenous network described in this section and the previous section is the set of condition choice probabilities $\{P^*(\phi, \phi', d, s)\}$, $\{\dot{P}^*(\phi, m, s)\}$, and $\{\tilde{P}^*(\phi, m, s)\}$ that satisfies

1. *Rationality*: the associated Markov strategies $\Sigma^*$, $\tilde{\Sigma}^*$, and $\dot{\Sigma}^*$ of $P^*$, $\dot{P}^*$, and $\tilde{P}^*$

$$
\begin{aligned}
D^*_{ijt} &= \Sigma^*(\phi_i, \phi_j, D_{ijt-1}, m_{it-1}, s_t, \epsilon_{ijt}) \\
\tilde{D}^*_{it} &= \tilde{\Sigma}^*(\phi_i, m_{it-1}, s_t, f^E_{it}) \\
\dot{D}^*_{it} &= \dot{\Sigma}^*(\phi_i, m_{it-1}, s_t, f^S_{it})
\end{aligned}
$$

   solves the Bellman equation (6.14) for every $i \in J$ and for every $t$.

2. *Consistency*: $P^i = P^*$, $\dot{P}^i = \dot{P}^*$, and $\tilde{P}^i = \tilde{P}^*$ for all $i \in J$ and for all $t = 1, 2, \ldots$.

Note that the equilibrium strategy functions $\Sigma^*$, $\tilde{\Sigma}^*$, $\dot{\Sigma}^*$ are all Markovian and only depend on sufficient statistics $f^m_{\phi,t-1}\}_\phi$ of the past network included in the aggregate state $s_t = (\gamma_{t-1}, \Pi_{t-1}, \{f^m_{\phi,t-1}\}_\phi)$ and private sufficient statistic $m_{it-1}$ of the local network. In particular, the information in $D_{ijt-1}$ is completely encoded in private sufficient statistic $m_{it-1}$. This feature mitigates the curse of dimensionality of dependence on past networks when estimating $P^*$, $\dot{P}^*$ and $\tilde{P}^*$ nonparametrically.

The Bellman equation (6.14) is equivalent to a set of "mini" Bellman equations that are easy to solve individually.

**Proposition 7.** *If the value function $V$ satisfies equation (6.14) then there exists $V^D_{\phi,\phi'}$ and $V^E_{\phi,\phi'}$ such that $V$ can expressed as*

$$
V(\phi, m, s, \mathbb{P}) = \underbrace{\sum_{(\phi',d)|\mathcal{R}'=\mathcal{R}} \tilde{\Pi}^D(\phi', d, m) V^D_{\phi,\phi'}(d, m, s, \mathbb{P})}_{\equiv V^D_\phi(m,s,\mathbb{P})} + \underbrace{\sum_{(\phi',d)|\mathcal{R}'=\mathcal{R}} \tilde{\Pi}^E(\phi', d, m) V^E_{\phi,\phi'}(d, m, s, \mathbb{P})}_{\equiv V^E_\phi(m,s,\mathbb{P})}
$$

*where $V^D_{\phi,\phi}$ and $V^E_{\phi,\phi'}$ take the following forms*

$$
\begin{aligned}
V^D_{\phi,\phi'}(d,m,s,\mathbb{P}) = E_\epsilon\Bigg[\max_{d'\in\{0,1\}}\Bigg\{&\dot{P}(\phi,m,s)\bar{\dot{P}}(\phi',s)\\
&\times\Big(d'[\pi^I(\phi,\phi',m',s,\mathbb{P})-\rho(1-d)-\epsilon]+\zeta^D(\phi,\phi',d,m,s)E[f^S|\dot{D}^*_{it}=1]+\beta E[V^D_{\phi,\phi'}(d',m',s',\mathbb{P})|\dot{D}^*_{it}=1]\Big)\\
&+\dot{P}(\phi,m,s)[1-\bar{\dot{P}}(\phi',s)]\beta E[V^D_{\phi,\phi'}(0,m',s',\mathbb{P})|\dot{D}^*_{it}=1]+[1-\dot{P}(\phi,m,s)]\beta E[V^D_{\phi,\phi'}(0,m',s',\mathbb{P})|\dot{D}^*_{it}=0]\Bigg\}\Bigg] \quad (6.15)
\end{aligned}
$$

*and*

$$
\begin{aligned}
V^E_{\phi,\phi'}(d,m,s,\mathbb{P}) = E_\epsilon\Bigg[\max_{d'\in\{0,1\}}\Bigg\{&\dot{P}(\phi,m,s)\bar{\dot{P}}(\phi',s)\tilde{P}(\phi,m,s)\\
\times\Bigg(&d'[\pi^I(\phi,\phi',m',s,\mathbb{P})-\rho(1-d)-\epsilon]+\zeta^E(\phi,\phi',d,m,s)(\pi^C_E(\phi,s,\mathbb{P})-E[f^E|\tilde{D}^*_{it}=1])\\
&+\beta E[V^E_{\phi,\phi'}(d',m',s',\mathbb{P})|\dot{D}^*_{it}=1,\tilde{D}^*_{it}=1]\Bigg)\\
&+\dot{P}(\phi,m,s)\tilde{P}(\phi,m,s)[1-\bar{\dot{P}}(\phi',s)]\beta E[V^E_{\phi,\phi'}(0,m',s',\mathbb{P})|\dot{D}^*_{it}=1,\tilde{D}^*_{it}=1]\\
&+[1-\dot{P}(\phi,m,s)\tilde{P}(\phi,m,s)]\beta E[V^E_{\phi,\phi'}(0,m',s',\mathbb{P})|\dot{D}^*_{it}\tilde{D}^*_{it}=0]\Bigg\}\Bigg] \quad (6.16)
\end{aligned}
$$

*where $\zeta^D$ and $\zeta^E$ satisfies*

$$
\sum_{(\phi',d)\in\Phi\times\{0,1\}|\mathcal{R}'=\mathcal{R}}\tilde{\Pi}^D(\phi',m,s)\zeta^D(\phi,\phi,d,m,s)=1,\quad \forall\phi_i\in\Phi,s\in\mathbb{S},m\in[0,1]^{|\Phi|}
$$
$$
\sum_{(\phi',d)\in\Phi\times\{0,1\}|\mathcal{R}'\neq\mathcal{R}}\tilde{\Pi}^E(\phi',m,s)\zeta^E(\phi,\phi,d,m,s)=1,\quad \forall\phi_i\in\Phi,s\in\mathbb{S},m\in[0,1]^{|\Phi|}
$$

*and $\tilde{\Pi}^D$ and $\tilde{\Pi}^E$ are defined as*

$$
\tilde{\Pi}^D(\phi',d,m)\equiv\bar{\gamma}(\phi')[dm_{it}(\phi')+(1-d)(1-m_{it}(\phi'))],\quad \mathcal{R}'=\mathcal{R},
$$
$$
\tilde{\Pi}^E(\phi',d,m)\equiv\bar{\gamma}(\phi')[dm_{it}(\phi')+(1-d)(1-m_{it}(\phi'))],\quad \mathcal{R}'\neq\mathcal{R}.
$$

Although $V^D$/$V^E$ are unique only up to some functions $\zeta^E$/$\zeta^D$ that can be freely chosen, the linking decision problems do not depend on the form of $\zeta^E/\zeta^D$. In addition, note that agents are infinitesimal so that the probability of $s'|s$ does not depend on each individual firm's choice $D_{ijt}$, $\dot{D}_{it}$, and $\tilde{D}_{it}$. This is the key assumption that give rise to additive separability of the value function. However, the continuation value still depends on the choice $D_{ijt}$ through the term that captures the sunk relationship cost and the private state $m_{it}$ also depend on $\dot{D}_{it}$, and $\tilde{D}_{it}$ in a simple way.

## 6.4 Fixed Point Equations for the Equilibrium CCPs

Solving each “mini" Bellman in Proposition 2 yields fixed point equations that determines the equilibrium CCP $P$, $\dot{P}$, and $\tilde{P}$.

**Proposition 8.** *Suppose that profit functions* $\pi^I$ *and* $\pi^C_E$ *are given by the equilibrium define in economy described in section 3.1 with the law of motions defined in section 3.2. Then the associated Bayesian Markov perfect equilibrium* $(P^*, \dot{P}^*, \tilde{P}^*)$ *exists and is characterized by the following system of fixed point equations*

$$P^*(\phi,\phi',d,m,s) = G_\epsilon\left(\pi^I(\phi,\phi',m',s,\mathbb{P}^*) - \rho(1-d) + \beta E[V^D_{\phi,\phi'}(1,m',s',\mathbb{P}) - V^D_{\phi,\phi'}(0,m',s',\mathbb{P})|\dot{D}^*_{it}=1]\right) \quad (6.17)$$

*for* $\mathcal{R} = \mathcal{R}'$ *and*

$$P^*(\phi,\phi',d,m,s) = G_\epsilon\left(\pi^I(\phi,\phi',m',s,\mathbb{P}^*) - \rho(1-d) + \beta E[V^E_{\phi,\phi'}(1,m',s',\mathbb{P}) - V^E_{\phi,\phi'}(0,m',s',\mathbb{P})|\dot{D}^*_{it}=1,\tilde{D}_{it}=1]\right) \quad (6.18)$$

*for $\mathcal{R} \neq \mathcal{R}'$ and*

$$\tilde{P}^*(\phi, m, s) = G_{f^E}\Bigg(\pi_E^C(\phi, s, \mathbb{P}^*) + \sum_{(\phi', d)|\mathcal{R}' \neq \mathcal{R}} \tilde{\Pi}^E(\phi', m, s)\Big[\bar{\dot{P}}(\phi', s)P^*(\phi, \phi', d, m', s)$$

$$[\pi^I(\phi, \phi', m, s, \mathbb{P}^*) - \rho(1-d) - E_\epsilon[\epsilon|\Sigma^*(\phi, \phi', d, m, s, \epsilon) = 1]]\Big]$$

$$+\beta E[V_\phi^E(m', s', \mathbb{P}^*)|\dot{D}_{it}^* = 1, \tilde{D}_{it} = 1] - \beta E[V_\phi^E(m', s', \mathbb{P}^*)|\dot{D}_{it}^* = 1, \tilde{D}_{it} = 0]\Bigg) \quad (6.19)$$

*and*

$$\dot{P}^*(\phi, m, s) = G_{f^S}\Bigg(\pi_D^C(\phi, s, \mathbb{P}^*) + \sum_{(\phi', d) \in \Phi \times \{0,1\}|\mathcal{R}' = \mathcal{R}} \tilde{\Pi}^D(\phi', m, s)\Big[\bar{\dot{P}}(\phi', s)P^*(\phi, \phi', d, s)$$

$$[\pi^I(\phi, \phi', m', s, \mathbb{P}^*) - \rho(1-d) - E_\epsilon[\epsilon|\Sigma^*(\phi, \phi', d, m, s, \epsilon) = 1]$$

$$+ \beta E[V_{\phi,\phi'}^D(1, m', s', \mathbb{P}) - V_{\phi,\phi'}^D(0, m', s', \mathbb{P})|\dot{D}_{it}^* = 1]]\Big]$$

$$+ \tilde{P}^*(\phi, m, s)\Big\{\pi_E^C(\phi, s, \mathbb{P}^*) + \sum_{(\phi', d)|\mathcal{R}' \neq \mathcal{R}} \tilde{\Pi}^E(\phi', m, s)\Big[\bar{\dot{P}}(\phi', s)P^*(\phi, \phi', d, m, s)$$

$$[\pi^I(\phi, \phi', m', s, \mathbb{P}^*) - \rho(1-d) - E_\epsilon[\epsilon|\Sigma^*(\phi, \phi', d, m, s, \epsilon) = 1]]\Big]$$

$$+\beta E[V_\phi^E(m', s', \mathbb{P}^*)|\dot{D}_{it}^* = 1, \tilde{D}_{it} = 1] - \beta E[V_\phi^E(m', s', \mathbb{P}^*)|\dot{D}_{it}^* = 1, \tilde{D}_{it} = 0] + E[f^E|\tilde{D}_{it}^* = 1]\Big\}\Bigg) \quad (6.20)$$

For nonparametric estimation of $P^*$, $\dot{P}^*$, and $\tilde{P}^*$, the state space $\mathbb{S}$ is discretized so that both $V_{\phi\phi'}^D$ and $V_{\phi\phi'}^E$ can be solved directly by inverting the Markov transition probability matrix for state $s$. Link decisions to foreign firms are interdependent through the export decision captured by the equilibrium CCP $\tilde{P}^*$. This means that international trade shocks to one region of the world affect firms' link decisions to other foreign regions by shifting the overall values of exporting.

The continuation value of production link is captured in the value differential term

$$E[V_{\phi,\phi'}^{D/E}(1, m', s', P^*, \dot{P}, \tilde{P}^*) - V_{\phi,\phi'}^{D/E}(0, m', s', P^*, \dot{P}, \tilde{P}^*)|\dot{d} = 1, \tilde{d} = 1].$$

For a given pair of types $(\phi, \phi')$, this term is large when there is a high probability that the economy will transition into a favorable aggregate state $s'$ in which the sunk relationship cost is more relevant for linking decision, i.e., when the long run profit from selling in the intermediate market is high. When the expected profit from intermediate market is low, the probability of establishing a link in the next period is also low, which implies that the sunk cost saved from pre-establishing a link in the current period is also low. Therefore, the model suggests that the production network reacts to anticipated shocks. In addition, the persistence of favorable or unfavorable state also affects for the size of this value differential. As a result, the production network responds to permanent shocks more than to transitory shocks.

The value differential term also differ across the sufficient statistic for the local network $m$. This implies that incentive for new entrants to form link is lower than the incentive for firms which have stayed in the market for a long time. To see this, note that it is harder to the new firms to stay in the market next period because they are less connected.

## 6.5 Existence of Bayesian Markov Perfect Equilibrium

**Proposition 9** (Existence of BMPE)**.** *Suppose that the profit functions $\pi^I$, $\pi_E^C$, $\pi_D^C$ are all Lipschitz continuous in $s$ and $\mathbb{P}$ with some constant $M^\pi > 0$ and bounded from above by $\bar{\pi}$. Suppose also that $\epsilon \sim$ Logistic$(0, \sigma_\epsilon)$, $f^E \sim$ Logistic$(0, \sigma_{f^E})$, $f^S \sim$ Logistic$(0, \sigma_{f^S})$, and the state transition function $q^{\mathbb{P}}(s'|s)$ is Lipschitz continuous in $s$ for all $s' \in \mathbb{S}$ with some constant $M^q > 0$. In addition, the following condition on the parameter bound satisfies*

1. $\sigma_{f^E} \geq (4 - \bar{\gamma})^{-1} \left[\bar{\pi} + \bar{\rho} + \ln[G^{-1})\epsilon(-\bar{\rho})]^{-1} + 2\bar{q}\beta(1-\beta)^{-1}\bar{\pi}\right]$

2. $\sigma_{f^S} \geq 3(4 - 3\bar{\gamma})^{-1} \left[\bar{\pi} + \bar{\rho} + \ln[G_\epsilon^{-1}(-\bar{\rho})]^{-1} + 2\bar{q}\beta(1-\beta)^{-1}\bar{\pi}\right]$

*Then there exists (componentwise) Lipschitz function $\mathbb{P}^*$ in $s$ for all $(\phi, \phi', s) \in \Phi^2 \times \{0, 1\}$ with sufficiently large Lipschitz constant $M > 0$ that satisfies the fixed point equation defined by equations (6.17)-(6.20).*

Proposition 9 with normally distributed costs $\epsilon$, $f^E$, $f^S$ can be proven analogously following the proof strategy described in the appendix. The law of motion described by equations (6.2)-(6.12) also satisfies the conditions on the state transition function $q^{\mathbb{P}}$ (the transition probability for most of the state are degenerate). Moreover, for sufficiently large $\sigma_{f^E}$ and $\sigma_{f^E}$, the pair of conditions on the parameter bounds always satisfies.

## 6.6 Conclusion

In this paper, I developed a tractable network formation model that extends the existing static Bayesian network formation model with incomplete information to a dynamic Markov setting. I apply this model to study endogenous production networks' role in terms of shock propagation in an open economy. This approach's significant new feature is that the equilibrium conditional choice probabilities depend on both individual and aggregate network state. This feature allows me to investigate the importance of network externality through global interaction. I show how this model can be estimated using a two-steps likelihood approach with US input-output data. In a counterfactual experiment, I find both endogenous network adjustment and network externality to be quantitatively relevant in the context of the aggregate welfare effect of an uncertainty shock to the aggregate productivity process. Most importantly, network externality could potentially offset the welfare gain from endogenous network adjustment. Therefore, network externality could offer an alternative explanation for observed network persistence. Moreover, I show how the model can be extended to account for both endogenous entry and exit of nodes. This novel framework paves the way to study the endogenous growth of network jointly with endogeneous network formation.

# Appendix

## A.1 Imputation of Aggregate States

In this appendix, I show how to impute aggregate network state $\hat{m}_t(\phi, \phi')$ and aggregate productivity state $\hat{z}_t$ using observations $\{R_t(\phi, \phi')\}_{t,\phi\in\Phi,\phi'\in\Phi}$.

### A.1.1 Identification and Estimation of proportion of types

Define $\tilde{m}_t(\phi, \phi') \equiv \gamma m_t(\phi, \phi')$ as the matching measure. In the GE model, the equilibrium only depend on the matching measures $\tilde{m}_t(\phi, \phi')$ instead of $\gamma$ and $m_t(\phi, \phi')$ separately.

Let $R_t(\phi)$ denote the total revenue of all $\phi$-firms, then

$$R_t(\phi) = \sum_{\phi'\in\Phi} R_t(\phi, \phi').$$

By equation (2.11)

$$\Pi(\phi) = \frac{z_t^{1-\sigma} R_t(\phi)}{\gamma\mu H_t^c(\phi) w_t^{1-\sigma}\Delta_t(\phi)}, \quad \Pi(\phi') = \frac{z_t^{1-\sigma} R_t(\phi')}{\gamma\mu H_t^c(\phi') w_t^{1-\sigma}\Delta_t(\phi')}$$

By imposing $\Pi(\phi) + \Pi(\phi') = 1$, I can solve for $\gamma$

$$\gamma = z_t^{1-\sigma}\frac{R_t(\phi')/[H_t^c(\phi')\Delta_t(\phi')] + R_t(\phi)/[H_t^c(\phi)\Delta_t(\phi)]}{\mu w_t^{1-\sigma}}.$$

It follows immediately that

$$\Pi(\phi) = \frac{R_t(\phi)/[H_t^c(\phi)\Delta_t(\phi)]}{R_t(\phi')/[H_t^c(\phi')\Delta_t(\phi')] + R_t(\phi)/[H_t^c(\phi)\Delta_t(\phi)]}.$$

In the GE model, both $H_t^c(\phi')$ and $\Delta_t(\phi)$ depends on $\Pi(\phi)$, $\tilde{m}_t$ and $z_t$ for a given set of parameters, so $\Pi$ can be expressed in terms of a fixed point equation of $\tilde{m}_t$ and data $R_t(\phi)$

$$\hat{\Pi}(\phi, \tilde{m}_t, w_t, z_t) = \frac{R_t(\phi)/[H^c(\phi,\tilde{m}_t,\hat{\Pi})\Delta(\phi,\tilde{m}_t,w_t,z_t,\hat{\Pi})]}{R_t(\phi')/[H^c(\phi',\tilde{m}_t,\hat{\Pi})\Delta(\phi',\tilde{m}_t,w_t,z_t,\hat{\Pi})]+R_t(\phi)/[H^c(\phi,\tilde{m}_t,\hat{\Pi})\Delta(\phi,\tilde{m}_t,w_t,z_t,\hat{\Pi})]}. \quad \text{(A.1)}$$

Note that by using the real wage data $w_t$, the equilibrium calculation in equation (2.17) and $L_t$ can be avoided. $\Delta_t(\phi)$ is proportional to $z_t^{\sigma-1}$ which cancels out. Solution $\hat{\Pi}$ to equation (A.1) only depends $\tilde{m}_t$, $z_t$, and $w_t$. In practice, the solution is unique and converges fast for a given $\tilde{m}_t$ and $z_t$.

### A.1.2 Identification and Estimation of aggregate productivity shock

The challenge in solving fixed point equation (2.17) is that we do not observe $z_t$. The goal is to express $z_t$ in terms of $\tilde{m}_t$, $w_t$, and $\hat{\Pi}$. This can be done by imposing the constancy of $\Pi(\phi)$

$$\hat{\Pi}(\phi, \tilde{m}_t, w_t, z_t) = \Pi(\phi), \quad \forall t = 1, 2, \ldots$$

and set the initial condition $z_1 = 1$ to be the steady state. Then the solution to equation (A.1) at $t = 1$,

$$\tilde{\Pi}(\phi, \tilde{m}_1) = \hat{\Pi}(\phi, \tilde{m}_1, w_1, 1)$$

can be solved for a given $\tilde{m}_1$. For $t > 1$, $\hat{z}_t = z(\tilde{m}_t, w_t, \tilde{\Pi})$ is obtained by solving

$$\tilde{\Pi}(\phi, \tilde{m}_1) = \frac{R_t(\phi)/[H^c(\phi,\tilde{m}_t,\tilde{\Pi})\Delta(\phi,\tilde{m}_t,w_t,z_t,\tilde{\Pi})]}{R_t(\phi')/[H^c(\phi',\tilde{m}_t,\tilde{\Pi})\Delta(\phi',\tilde{m}_t,w_t,z_t,\tilde{\Pi})]+R_t(\phi)/[H^c(\phi,\tilde{m}_t,\hat{\Pi})\Delta(\phi,\tilde{m}_t,w_t,z_t,\tilde{\Pi})]}. \quad \text{(A.2)}$$

### A.1.3 Identification and Estimation of matching measures

Form the normalized total revenue shares

$$\tilde{R}_t(\phi,\phi') = \frac{R_t(\phi,\phi')}{\sum_{\phi'} R_t(\phi,\phi')} = \frac{\tilde{m}_t(\phi,\phi')r(\phi,\phi',\tilde{m}_t,w_t,\tilde{\Pi})}{\sum_{\phi'}\tilde{\Pi}(\phi',m_1)\tilde{m}_t(\phi,\phi')r(\phi,\phi',\tilde{m}_t,w_t,\tilde{\Pi})}. \tag{A.3}$$

where the GE revenue function is given by equation (2.12), (2.22), and (2.16)

$$r(\phi,\phi',\tilde{m}_t,w_t) \propto H^c(\phi,\tilde{m}_t,\tilde{\Pi}))[H^{\Delta}(\phi',\tilde{m}_t,\tilde{\Pi})]H^F(w_t/z(\tilde{m}_t,w_t,\tilde{\Pi})) + H^{\Delta F}(\phi',\tilde{m}_t,\tilde{\Pi})]$$

with $z(\tilde{m}_1,w_1,\tilde{\Pi}) = 1$. The network centrality terms $H_t^c$, $H_t^{\Delta}$, $H_t^{\Delta F}$ and $H_t^F$ depends (continuously) on $\tilde{\Pi}(\phi,\tilde{m}_1)$ and $\tilde{m}_t$ for a given set of parameters. Moreover, the centrality terms are strictly increasing in $\tilde{m}(\phi,\phi')$. Therefore, $\tilde{m}_t$ can be solved from following system of equations (A.3) along with matching the following two ratios

$$\frac{R_t(\phi)}{R_t(\phi')} = \frac{\hat{R}_t(\phi)}{\hat{R}_t(\phi')} \tag{A.4}$$

$$\frac{\sum_{\phi} R_t(\phi)}{\sum_{\phi,\phi'} R_t(\phi,\phi')} = \frac{\sum_{\phi}\hat{R}_t(\phi)}{\sum_{\phi,\phi'}\hat{R}_t(\phi,\phi')} \tag{A.5}$$

for all $(\phi,\phi') \in \{\phi_L,\phi_H\}^2$.

Provided that the solution to this nonlinear system is well behaved, this gives us the imputed $\tilde{m}_t(\phi,\phi')$. Note that to solve for $\tilde{m}_1$, $\tilde{\Pi}(\phi,\tilde{m}_1)$ must be solved for each guess of $\tilde{m}_1$ using the fixed point equation (A.1). In practice, this fixed point problem of $\Pi$ converges fast. For $t > 1$, this nested fixed point problem becomes unnecessary by exploiting the constancy of $\tilde{\Pi}(\phi,\tilde{m}_1)$.

### A.1.4 Identification and Estimation of Network Sufficient Statistic

Having imputed $\tilde{m}_t$, the following quantities can be calculated

$$\hat{H}_t^c(\phi) = H^c(\phi, \tilde{m}_t, \tilde{\Pi}(\phi, \tilde{m}_1))$$
$$\hat{\Delta}_t(\phi) = \Delta(\phi, \tilde{m}_t, w_t, \hat{z}_t, \tilde{\Pi}(\phi, \tilde{m}_1))$$
$$\hat{z}_t = z(\tilde{m}_t, w_t, \tilde{\Pi}(\phi, \tilde{m}_1))$$

The total measure of firms $\gamma$ is over identified by

$$\hat{\gamma}_t = \hat{z}_t^{1-\sigma} \frac{R_t(\phi')/[\hat{H}_t^c(\phi')\hat{\Delta}_t(\phi')] + R_t(\phi)/[\hat{H}_t^c(\phi)\hat{\Delta}_t(\phi)]}{\mu w_t^{1-\sigma}}.$$

And the estimator for the total measure of firms $\gamma$ is

$$\hat{\gamma} = \frac{1}{T}\sum_{t=1}^{T} \hat{\gamma}_t.$$

Finally, by definition of matching measures $\tilde{m}_t(\phi, \phi')$, the network sufficient statistics can be recovered using

$$\hat{m}_t(\phi, \phi') = \tilde{m}_t(\phi, \phi')/\hat{\gamma}.$$

### A.1.5 Identification and Estimation of long run productivity levels

Using the marginal cost equation (2.9) and the total revenue function (2.11)

$$\begin{bmatrix} \phi_L^{\sigma-1} \\ \phi_H^{\sigma-1} \end{bmatrix} = (\alpha^L)^{-\sigma}(w_t/z_t)^{\sigma-1}[\mu^{-1}I - (1-\alpha^L)^{\sigma}\mu^{-\sigma}\gamma M] \begin{bmatrix} R_t(\phi_L)/\Delta_t(\phi_L) \\ R_t(\phi_H)/\Delta_t(\phi_H) \end{bmatrix} \tag{A.6}$$

where

$$M_t = \begin{pmatrix} \Pi(\phi_L)m_t(\phi_L, \phi_L) & \Pi(\phi_H)m_t(\phi_H, \phi_L) \\ \Pi(\phi_L)m_t(\phi_L, \phi_H) & \Pi(\phi_H)m_t(\phi_H, \phi_H) \end{pmatrix}.$$

Intuitively, marginal cost can be estimated using $R_t(\phi)/\Delta_t(\phi)$ and $(w_t/z_t)$ on the right hand side. The empirical counterpart of equation (A.6) is

$$\begin{bmatrix} \phi_L^{\sigma-1} \\ \phi_H^{\sigma-1} \end{bmatrix} = (\alpha^L)^{-\sigma}(w_t/\hat{z}_t)^{\sigma-1}[\mu^{-1}I - (1-\alpha^L)^{\sigma}\mu^{-\sigma}\hat{\gamma}\hat{M}_t] \begin{bmatrix} R_t(\phi_L)/\hat{\Delta}_t(\phi_L) \\ R_t(\phi_H)/\hat{\Delta}_t(\phi_H) \end{bmatrix} \tag{A.7}$$

where $\hat{\gamma}$, $\hat{z}$, $\hat{\Delta}$ and $\hat{M}$ are given by the previous steps for given values of $(\phi_L, \phi_H)$. Denote the solution to equation (A.7) $\{(\hat{\phi}_{L,t}, \hat{\phi}_{H,t})\}$, then

$$\hat{\phi}_L = \frac{1}{T}\sum_{t=1}^{T}\hat{\phi}_{L,t}, \quad \hat{\phi}_H = \frac{1}{T}\sum_{t=1}^{T}\hat{\phi}_{H,t}.$$

Note that each iteration for equation (A.7) calls for evaluation of $\hat{M}_t$, which can be computationally heavy.

## A.2 Discretization of Continuous State

For estimation and simulation, the aggregate continuous state is infeasible. There are two ways to deal with this challenge: 1) divide the continuous support of $s$ and replace each region of the support with discrete states; 2) use partial sums of basis function in $s$ expansion to approximate each function continuous in $s$.

First, I show how to divide the continuous support. For exposition purposes, let $s_t$ include the aggregate network sufficient statistic $m_{t-1}$. I show existence for a sequence of approximate models indexed with $N$ in which a discretization function $f^N(\cdot) : [0,1] \to \mathbb{D}_N$ is applied to each of the continuous states in $s$, where $\mathbb{D}_N$ is a finite set contained in $[0,1]$. The discretization function $f^N$

satisfies the following condition

$$f^N(s) \to s, \quad N \to \infty.$$

For instance, I set $\mathbb{D}_N$ to be set of $2^N$ points obtained from the uniform partition of $[0,1]$. The CCPs associated with the $N$th approximate model are denoted $P^{*(N)}$. Fix a discrete state $(\phi, \phi', d)$ and consider $P$ to be a function of $s$ indexed by $(\phi, \phi', d)$. Let $D(P)$ be the domain of $P$. Then $P^{*(N)}$ can be either represented as vectors in Euclidean space $[0,1]^{|D(P^{*(N)})|}$ or piecewise constant functions in the space of real regulated functions defined on $D(P)$, denoted by Reg$(D(P))$. Reg$(D(P))$ equipped with sup norm is a complete metric space by standard analysis. The following theorem justifies this approximation.

**Proposition 10** (Convergence of Approximating Models)**.** *Let $P^*$ be a fixed point that solves the fixed point equations defined by (3.7)*

$$P^*(s) = T(P^*)(s).$$

*Let $f_N : [0,1] \to \mathbb{D}_N$ with $|\mathbb{D}_N| = 2^N$ be a sequence of discretization functions such that*

$$f^N(s) \to s, \quad N \to \infty$$

*and define the $N$th approximation of the fixed point equations (3.7) by applying $f^N(\cdot)$ to $s$ so that*

$$P^{(N)}(s) \equiv P(f^N(s)) = T(P)(f^N(s)) = T(P^{(N)})(s)$$

*Then $P^{*(N)}$ that satisfies the above fixed point equation exists for every $N \in \mathbb{N}$. In addition, if $T$ is a contraction mapping, i.e., $T$ is Lipschitz continuous in $P$ with constant $K \in (0,1)$, then $P^*$ is* *unique and there is a sequence of* $\{P^{*(N)}\}_{N=1}^{\infty}$ *such that*

$$P^{*(N)} \to P^*, \quad N \to \infty$$

*in* ***Reg***$(D(P))$.

Note that in Proposition (10), the Markov transition probability $Q^{P^{(N)}}(s'|s)$ is not discretized even if $P^{(N)}$ has discrete support. In practice, I replace $Q^{P^{(N)}}(s'|s)$ with the induced transition matrix $Q^{P^{(N)}}(s^{d'}|s^d)$ where $s^d \in \mathbb{D}_N$. The main difficulty of applying Proposition 5 is to show that $T$ is a contraction in $P$.

Alternatively, consider the following mixed approach for approximation. Since the evolution of $m_{t-1}$ is deterministic and high dimensional compared to $z_{t-1}$, I discretize the support $[0,1]^{|\Phi|^2}$ for $m_{t-1}$ and use series approximation for dependence for $z_{t-1}$. Then consider the following approximations

$$P^{(\phi,\phi',d,m^d)}(z) \approx \sum_{i=1}^{N} \alpha_i^P(\phi,\phi',d,m^d) T_i(z)$$

where $T_i(z)$ is the $i$th Chebyshev Polynomial of the first type. Chebyshev Polynomials has several good properties when approximating a Lipschitz function. Fix a set of coefficients $\{\alpha_i^P(\phi,\phi',d,m)\}_{i=1}^N$ that determines an approximation $P^N$ of $P$. There is an induced transition function for discretized sufficient network statistics $m^j, m^k \in \mathbb{D}_N^{|\Phi|^2}$

$$Q^{P^N}(m^k|m^j,z) = \frac{Q^P(z,m^k|m^j)}{f(z)} \approx \sum_{i=1}^{N} \alpha_i^T(m^j,m^k) T_i(z)$$

Similarly, consider the following approximations

$$
\begin{aligned}
E_{z'}[\pi^I(m^j, m^k, z')|z] &\approx \sum_{i=1}^{N} \alpha_i^{\pi}(m^j, m^k) T_i(z) \\
E[\epsilon; \Sigma^*(\phi, \phi, d, m^d, z, \epsilon) = 1] &\approx \sum_{i=1}^{N} \alpha_i^{\epsilon} \pi(\phi, \phi', d, m^d) T_i(z) \\
v_{\phi\phi'}^{PN}(d, m^k, z) &\approx \sum_{i=1}^{N^v} \alpha_i^{\epsilon} \pi(\phi, \phi', d, m^k) T_i(z)
\end{aligned}
$$

Then suppressing the type dyad $\phi\phi$, the product of terms such as

$$
Q^{PN}(m^k|m^j, z) P^N(d, m^j, z) E_{z'}[\pi^I(m^j, m^k, z')|z]
$$

can be written as

$$
\sum_{i,j,k} \alpha_i^T \alpha_j^P \alpha_k^{\pi} T_i(z) T_j(z) T_k(z)
$$

Denote $\alpha_{ijk}^{TP\pi}(m^j, m^k, d, m) \equiv \alpha_i^T \alpha_j^P \alpha_k^{\pi}$, then apply the product rule

$$
T_i(z) * T_j(z) = \frac{1}{2}(T_{i+j}(z) + T_{|i-j|}(z)).
$$

The coefficient on the linear term $T_1(z)$ can be shown to be

$$
\begin{aligned}
\tilde{\alpha}_1^{TP\pi} = {} & \frac{3}{4}\alpha_{111}^{TP\pi} + \frac{1}{4}\alpha_{001}^{TP\pi} + \frac{1}{4}\sum_{k\ odd}\sum_{\tau} \alpha_{\tau(\frac{k+1}{2}\frac{k+1}{2}k)}^{TP\pi} \\
& + \frac{1}{4}\sum_{k\ odd}\sum_{\tau} \alpha_{\tau(\frac{k-1}{2}\frac{k-1}{2}k)}^{TP\pi} + + \frac{1}{2}\sum_{k\neq 1}\sum_{\tau} \alpha_{\tau(kk1)}^{TP\pi} \\
& + \frac{1}{4}\sum_{k=1}^{N-1}\sum_{\tau} \alpha_{\tau(k(k+1)0)}^{TP\pi} + \frac{1}{4}\sum_{k=2}^{N}\sum_{i\neq 1,k}^{N-k+1}\sum_{\tau} \alpha_{\tau(i(i+k-1)k)}^{TP\pi} \\
& + \frac{1}{4}\sum_{k=0}^{N-2}\sum_{i\neq k}^{N-k-1}\sum_{\tau} \alpha_{\tau(i(i+k-1)k)}^{TP\pi}
\end{aligned}
$$

Apply analogous argument to all terms to the right hand side the Bellman equation (3.6) and let $\alpha_k^v(d, m^j)$ denote the coefficient on $T_k(z)$ of the series approximation of the value function

$v^{PN}(d, m^j, z)$. The Bellman equation can be expressed in terms of coefficients on the basis functions

$$\alpha^v(\phi, \phi') = M^{PN}(\phi, \phi')\alpha^v(\phi, \phi')$$

where $\alpha^v(\phi, \phi')$ is the stacked vector of $\alpha_1^v(\phi, \phi', d, m^j)$ and for example

$$\alpha_1^v(d, m^j) = \sum_k \tilde{\alpha}_1^{TP\pi}(m^j, m^k) + \tilde{\alpha}_1^{TP\epsilon}(m^j, m^k) + d\rho\tilde{\alpha}^{TP}(m^j, m^k) \\ + \beta(\tilde{\alpha}_1^{T(1-P)V}(\alpha^v(0, m^k)) + \tilde{\alpha}_1^{TPV}(\alpha^v(1, m^k))).$$

If a Lipschitz solution exists for the Bellman equation $v = \mathbb{T}v$ in equation (3.6), then

$$v = \underbrace{\text{Proj}_{\text{span}\{T_1,\ldots,T_{N^v}\}}\mathbb{T}v}_{\equiv\hat{v}} + \text{error}$$

which implies

$$\hat{v} = \text{Proj}_{\text{span}\{T_1,\ldots,T_{N^V}\}}\mathbb{T}\hat{v} + \text{Proj}_{\text{span}\{T_1,\ldots,T_{N^v}\}}\mathbb{T}(\text{error})$$

The product rule of the Chebyshev polynomial implies that the part of $\mathbb{T}(\text{error})$ in the approximating space is of minimal order $N^V - N + 1$. Hence, $\hat{v} \to v$ as $(N^v - N) \to \infty$.

## A.3 Expression for the Difference of Mini Value Functions

Equation (4.3) can be written in the following stacked form for a given pair $(\phi, \phi')$

$$\underbrace{\begin{bmatrix} v^{\hat{P}}(\phi,\phi',0,s^1) \\ v^{\hat{P}}(\phi,\phi',1,s^1) \\ v^{\hat{P}}(\phi,\phi',0,s^2) \\ v^{\hat{P}}(\phi,\phi',1,s^2) \\ \vdots \end{bmatrix}}_{V^{\hat{P}}_{\phi\phi'}} = \underbrace{\begin{bmatrix} \sum_k Q^{\hat{P}}(s^k|s^1)\hat{P}_{\phi\phi'}(0,s^1)[\pi^I_{\phi\phi'}(s^k,s^1) + E[\epsilon|\hat{\Sigma}_{\phi\phi'}(0,s^1,\epsilon)=1]] \\ \sum_k Q^{\hat{P}}(s^k|s^1)\hat{P}_{\phi\phi'}(1,s^1)[\pi^I_{\phi\phi'}(s^k,s^1) + E[\epsilon|\hat{\Sigma}_{\phi\phi'}(1,s^1,\epsilon)=1]] \\ \sum_k Q^{\hat{P}}(s^k|s^2)\hat{P}_{\phi\phi'}(0,s^2)[\pi^I_{\phi\phi'}(s^k,s^2) + E[\epsilon|\hat{\Sigma}_{\phi\phi'}(0,s^2,\epsilon)=1]] \\ \sum_k Q^{\hat{P}}(s^k|s^2)\hat{P}_{\phi\phi'}(1,s^2)[\pi^I_{\phi\phi'}(s^k,s^2) + E[\epsilon|\hat{\Sigma}_{\phi\phi'}(1,s^2,\epsilon)=1]] \\ \vdots \end{bmatrix}}_{T^{\hat{P}}_{\phi\phi',1}}$$

$$- \rho \underbrace{\begin{bmatrix} 0 \\ \sum_k Q^{\hat{P}}(s^k|s^1)\hat{P}_{\phi\phi'}(1,s^1) \\ 0 \\ \sum_k Q^{\hat{P}}(s^k|s^2)\hat{P}_{\phi\phi'}(1,s^2) \\ \vdots \end{bmatrix}}_{T^{\hat{P}}_{\phi\phi',2}}$$

$$+ \underbrace{\begin{bmatrix} (1-\hat{P}_{\phi\phi'}(0,s^1))Q^{\hat{P}}(s^1|s^1) & \hat{P}_{\phi\phi'}(0,s^1)Q^{\hat{P}}(s^1|s^1) & (1-\hat{P}_{\phi\phi'}(0,s^1))Q^{\hat{P}}(s^2|s^1) & \cdots \\ (1-\hat{P}_{\phi\phi'}(1,s^1))Q^{\hat{P}}(s^1|s^1) & \hat{P}_{\phi\phi'}(1,s^1)Q^{\hat{P}}(s^1|s^1) & (1-\hat{P}_{\phi\phi'}(1,s^1))Q^{\hat{P}}(s^2|s^1) & \cdots \\ (1-\hat{P}_{\phi\phi'}(0,s^2))Q^{\hat{P}}(s^1|s^2) & \hat{P}_{\phi\phi'}(0,s^2)Q^{\hat{P}}(s^1|s^2) & (1-\hat{P}_{\phi\phi'}(0,s^2))Q^{\hat{P}}(s^2|s^2) & \cdots \\ (1-\hat{P}_{\phi\phi'}(1,s^2))Q^{\hat{P}}(s^1|s^2) & \hat{P}_{\phi\phi'}(1,s^2)Q^{\hat{P}}(s^1|s^2) & (1-\hat{P}_{\phi\phi'}(1,s^2))Q^{\hat{P}}(s^2|s^2) & \cdots \\ \vdots & & & \end{bmatrix}}_{\Gamma^{\hat{P}}_{\phi\phi',0}}$$

$$\times \underbrace{\begin{bmatrix} v^{\hat{P}}(\phi,\phi',0,s^1) \\ v^{\hat{P}}(\phi,\phi',1,s^1) \\ v^{\hat{P}}(\phi,\phi',0,s^2) \\ v^{\hat{P}}(\phi,\phi',1,s^2) \\ \vdots \end{bmatrix}}_{V^{\hat{P}}_{\phi\phi'}}$$

Provided that $\Gamma^{\hat{P}}_{\phi\phi',0}$ is invertible[1], the above can be solved as

$$V^{\hat{P}}_{\phi\phi'} = (\Gamma^{\hat{P}}_{\phi\phi',0})^{-1}T^{\hat{P}}_{\phi\phi',1} + \rho(\Gamma^{\hat{P}}_{\phi\phi',0})^{-1}T^{\hat{P}}_{\phi\phi',1}.$$

Let $S^i$ denote the vector for $i = 1, 2, \ldots, 16$.

$$S^i \equiv \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1 \quad -1}_{\text{ith pair}} & 0 & \cdots & 0 \end{bmatrix}$$

Then the value difference can be written as

$$v^{\hat{P}}(\phi, \phi', 1, m^i) - v^{\hat{P}}(\phi, \phi', 0, m^i) = \underbrace{S^i(\Gamma^{\hat{P}}_{\phi\phi',0})^{-1}T^{\hat{P}}_{\phi\phi',1}}_{\Gamma^{\hat{P}}_1(\phi,\phi',m^i)} + \rho \underbrace{S^i(\Gamma^{\hat{P}}_{\phi\phi',0})^{-1}T^{\hat{P}}_{\phi\phi',1}}_{\Gamma^{\hat{P}}_2(\phi,\phi',m^i)}.$$

## A.4 Decomposition of Continuation Value Difference

The continuation value of forming a link is captured in the value difference term

$$E^P_{s'|s}[v^P_{\phi\phi'}(1, s') - v^P_{\phi\phi'}(0, s')].$$

Fix a given pair of types $(\phi, \phi')$ and suppress $\phi\phi'$ in the notation. Denote both the CCP and the value difference term with respective to individual network state $d$

$$\delta_d P(s) = P(1, s) - P(0, s), \quad \delta_d v^P(s) = v^P(1, s) - v^P(0, s)$$

[1] $\Gamma^{\hat{P}}_{\phi\phi',0}$ is singular if there are degenerate state $s^k$, i.e., state with zero stationary distribution. In such cases, the degenerated states should be merged to reduce the state space.

The value difference term satisfies the following Bellman equation

$$\delta_d v^P(s) = \underbrace{\delta_d P(s) E^P_{s'|s}[\pi^I(s')]}_{\text{network access benefit}} + \underbrace{\rho P(0,s)}_{\text{sunk cost saving}} + \underbrace{[P(1,s)\xi^P(1,s) - P(0,s)\xi^P(0,s)]}_{\text{idiosyncratic cost adjustment}} + \underbrace{\beta\delta_d P(s)}_{\text{effective discount factor}} E^P_{s'|s}[\delta_d v^P(s')]$$

This decomposition shows that the continuation net benefit from forming a link today consist of (discounted sum of) three terms. The first term is the benefit from having better access to the payoff provided by the network. The term $\delta_d P(s)$ is expected to be positive since having established link in the previous period lower the relationship cost and raises uniformly in $s$ the probability of accessing the payoff provided by the link in this period. $\delta_d P(s)$ is increasing in $\rho$ and tends to zero when $\rho \to 0$. It is also governed by the size of expected payoff $E^P_{s'|s}[\pi^I(s')]$. When the aggregate state is expected to be favorable in the next period, the network access benefit term will increase.

The second term is the saving from not having to pay the sunk cost for establishing a link in the current period. Holding $P$ fixed, this term is increasing in the sunk cost or network stickiness parameter $\rho$. However, in equilibrium, $P(0,s)$ is smaller when the network stickiness $\rho$ is large to avoid paying the sunk cost frequently. Moreover, when an expected favorable aggregate state $s$ creates strong incentive for establishing a new link, that is, when $P(0,s)$ is large, the sunk cost saving term also create incentive in the previous period to establish a link through the continuation net benefit term $\delta_d v^P(s)$. Therefore, network formation is more active in periods that anticipates favorable aggregate states instead of being in periods with favorable aggregate states.

The third term is the expected difference in idiosyncratic cost. The term $\xi^P(d,s)$ is the expected idiosyncratic relationship cost conditional on the state of the previous link $d$

$$\xi^P(d,s) = E[\epsilon; \Sigma(d,s,\epsilon) = 1] = \int_{-G_\epsilon^{-1}(P(d,s))}^{\infty} \epsilon dG_\epsilon(\epsilon)$$

Note that $\xi^P(1,s) < \xi^P(0,s)$ when $P(1,s) > P(0,s)$ because agents are able to pay more idiosyncratic relationship cost when they don't have to pay the sunk cost due to having established a link in the previous period $d = 1$. For standardized symmetric distribution $G_\epsilon$, the size of this term depends on $\delta_d P(s)$ and the sign of this term is ambiguous. The term $\xi^P(d, s)P(d, s)$ is bounded within $[0, \bar{P}^2]$ where $\bar{P}$ satisfies

$$\bar{P} = \int_{-G_\epsilon^{-1}(\bar{P})}^{\infty} \epsilon dG_\epsilon(\epsilon)$$

For example, when $G_\epsilon$ is standard logistic, $\bar{P}^2 \approx 0.42$. The marginal agent pays the same relationship cost regardless of their individual network state $d$ but it is less likely to be on the marginal when $d = 0$. This implies that the expect total cost saving for the agent with the previously established link $d = 1$ is still higher than the agent without the established link $d = 0$.

Finally, $\delta_d P(s)$ determines the effective discount factor for the continuation value difference term. When $\delta_d P(s)$ is small, there is only a small difference in chances of continuing with a link established versus continuing without a link established. Forming a link today has small impact on the continuation value for all states and therefore $\delta_d P(s)$ effectively "discount" the difference of continuation values with respect to the individual network state $d$.

From the analysis above, one can see that main incentives for the marginal agent to take a temporary loss in order to establish a link today is 1) the agent anticipate that the payoff from network access will be great in the future with favorable aggregate state $E^P_{s'|s}[\pi^I(s')]$ ; 2) establish link today leads to greater probability of forming links $\delta_d P(s)$ and receive benefits in the future.

## A.5 Proofs

### A.5.1 Proof of Proposition 1

This proposition follows immediately by recursive substitution with $k^c$ and $B_t^c$ defined as belows.

$$\kappa^c \equiv (1-\alpha)^\sigma \mu^{1-\sigma}\gamma$$

$$B_t^c \equiv \begin{pmatrix} \Pi(\phi_1)m_t(\phi_1,\phi_1) & \Pi(\phi_2)m_t(\phi_2,\phi_1) & \cdots & \Pi(\phi_K)m_t(\phi_K,\phi_1) \\ \Pi(\phi_1)m_t(\phi_1,\phi_2) & \Pi(\phi_2)m_t(\phi_2,\phi_2) & \cdots & \Pi(\phi_K)m_t(\phi_K,\phi_2) \\ \vdots & \ddots & \ddots & \vdots \\ \Pi(\phi_1)m_t(\phi_1,\phi_K) & \Pi(\phi_2)m_t(\phi_2,\phi_K) & \cdots & \Pi(\phi_K)m_t(\phi_K,\phi_K) \end{pmatrix}$$

$$(B_t^c)^0 \equiv \begin{pmatrix} \mathbf{1}\{\phi_1=\phi\} & 0 & \cdots & 0 \\ 0 & \mathbf{1}\{\phi_2=\phi\} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}\{\phi_K=\phi\} \end{pmatrix}$$

$(B_t^c)^d(\phi_i,\phi_j)$ is $ij$th element of $(B_t^c)^d$, which is proportion to the measure of $\phi_i \to \phi_j$ path of distance $d > 0$. □

### A.5.2 Proof of Proposition 2

This proposition follows immediately by recursive substitution with $k^c$ and $B_t^{\Delta}$, and $B_t^{\Delta F}$ defined as belows.

$$\kappa^y \equiv (1-\alpha^L)^{\sigma-1}\mu^{-\sigma}\gamma \tag{A.8}$$

$$B_t^{\Delta} \equiv \begin{pmatrix} \Pi(\phi_1)m_t(\phi_1,\phi_1) & \Pi(\phi_2)m_t(\phi_1,\phi_2') & \cdots & \Pi(\phi_K)m_t(\phi_1,\phi_K) \\ \Pi(\phi_1)m_t(\phi_2,\phi_1) & \Pi(\phi_2)m_t(\phi_2,\phi_2') & \cdots & \Pi(\phi_K)m_t(\phi_2,\phi_K) \\ \vdots & \ddots & \ddots & \vdots \\ \Pi(\phi_1)m_t(\phi_K,\phi_1) & \Pi(\phi_2)m_t(\phi_K,\phi_2) & \cdots & \Pi(\phi_K)m_t(\phi_K,\phi_K) \end{pmatrix} \tag{A.9}$$

$$(B_t^{\Delta})^{(0)} \equiv \begin{pmatrix} \mathbf{1}\{\phi_1=\phi_1\} & \mathbf{1}\{\phi_2=\phi_1\} & \cdots & \mathbf{1}\{\phi_K=\phi_1\} \\ \mathbf{1}\{\phi_1=\phi_2\} & \mathbf{1}\{\phi_2=\phi_2\} & \cdots & \mathbf{1}\{\phi_K=\phi_2\} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{1}\{\phi_1=\phi_K\} & \mathbf{1}\{\phi_2=\phi_K\} & \cdots & \mathbf{1}\{\phi_K=\phi_K\} \end{pmatrix} \tag{A.10}$$

where $(B_t^c)^{\Delta}(\phi_i,\phi_j)$ is $ij$th element of $(B_t^{\Delta})^d$, which is proportion to the measure of $\phi_i \to \phi_j$ path of distance $d > 0$. □

### A.5.3 Proof of Proposition 3

Follows immediately by applying contraction mapping theorem to equation (2.14). The rest of the model can be solved using the unique wage solution. □

### A.5.4 Proof of Proposition 4 and 7

The proof for proposition 4 and proposition 7 are analogous. It suffice to prove the general case (Proposition 7). First, I establishes identities to simply the original Bellman equation

$$\int_{j\in J_t^D} \max_{D_{ijt}\in\{0,1\}} \left(D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-D_{ijt-1})-\epsilon_{ijt}]\right) dj \geq \int_{j\in J_t^D} D^\dagger_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-D_{ijt-1})-\epsilon_{ijt}]dj$$

for all $D^\dagger_{it}\in\{0,1\}^{J_t^D}$ so that

$$\int_{j\in J_t^D} \max_{D_{ijt}\in\{0,1\}} \left(D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-D_{ijt-1})-\epsilon_{ijt}]\right) dj = \max_{D_{it}\in\{0,1\}^{J_t^D}} \int_{j\in J_t^D} D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-D_{ijt-1})-\epsilon_{ijt}]dj$$

and

$$\int_{j\in J_t^E} \max_{D_{ijt}\in\{0,1\}} \left(D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-D_{ijt-1})-\epsilon_{ijt}]\right) dj = \max_{D_{it}\in\{0,1\}^{J_t^E}} \int_{j\in J_t^E} D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-D_{ijt-1})-\epsilon_{ijt}]dj.$$

Next, note that by the independence of $\epsilon_{ijt}$ across $j$ and by the law of large number

$$\int_{j\in J_t^D} \max_{D_{ijt}\in\{0,1\}} \left(D_{ijt}[\pi^I(\phi_i,\phi,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-d)-\epsilon_{ijt}]\right) \mathbf{1}\{D_{ijt}=d,\phi_j=\phi\}dj = \int_{j\in J_t^D} E_\epsilon\left[\max_{D_{ijt}\in\{0,1\}} \left(D_{ijt}[\pi^I(\phi_i,\phi,s,P^i,\dot{P}^i,\tilde{P}^i)-\rho_i(1-d)-\epsilon_{ijt}]\right) dj\right] \mathbf{1}\{D_{ijt}=d,\phi_j=\phi\}dj.$$

Moreover, note that by the full independence of $f_{it}^E$, we have

$$E_{f^E}\left[\max_{\tilde{D}_{it}\in\{0,1\}} \tilde{D}_{it}\left(\pi_E^C(\phi_i, s, P^i, \dot{P}^i, \tilde{P}^i)\right.\right.$$

$$\left.\left. + \int_{j\in J_t^D} E_\epsilon\left[\max_{D_{ijt}\in\{0,1\}}\left(D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i) - \rho_i(1-D_{ijt-1}) - \epsilon_{ijt}]\right)\right]dj - f_{it}^E\right)\right]$$

$$= \tilde{P}(\phi, s)\left(\pi_E^C(\phi_i, s, P^i, \dot{P}^i, \tilde{P}^i)\right.$$

$$\left. + \int_{j\in J_t^D} E_\epsilon\left[\max_{D_{ijt}\in\{0,1\}}\left(D_{ijt}[\pi^I(\phi_i,\phi_j,s,P^i,\dot{P}^i,\tilde{P}^i) - \rho_i(1-D_{ijt-1}) - \epsilon_{ijt}]\right)\right]dj - E[f_{it}^E|\tilde{D}_{it} = 1]\right).$$

Next, I establish the law of motions for $\tilde{\Pi}^{\phi,D/E}$ for a given $\phi' \in \Phi$ and a given $(d, \dot{d}, \tilde{d})$. By the law of total probability, we have

$$\Pi^{\phi,D/E}(\phi', d', \dot{d}', \tilde{d}', s_{t+1})$$

$$= \bar{\dot{P}}(\phi', s_t) \times \sum_{(d,\dot{d},\tilde{d})\in\{0,1\}^3} \left\{[d'P(\phi,\phi',d,\dot{d},\tilde{d},s_t) + (1-d')[1-P(\phi,\phi',d,\dot{d},\tilde{d},s_t)]]\right.$$

$$\times [\dot{d}'\dot{P}(\phi,\dot{d},\tilde{d},s_t) + (1-\dot{d}')[1-\dot{P}(\phi,\dot{d},\tilde{d},s_t)]]$$

$$\left.\times [\tilde{d}'\tilde{P}(\phi,\dot{d},\tilde{d},s_t) + (1-\tilde{d}')[1-\tilde{P}(\phi,\dot{d},\tilde{d},s_t)]]\Pi^{\phi,D/E}(\phi',d',\dot{d}',\tilde{d}',s_t)\right\} d'[1-\bar{\dot{P}}(\phi',s_t)]$$

$$\times \sum_{(d',d,\dot{d},\tilde{d})\in\{0,1\}^4} \left\{[d'P(\phi,\phi',d,\dot{d},\tilde{d},s_t) + (1-d')[1-P(\phi,\phi',d,\dot{d},\tilde{d},s_t)]]\right.$$

$$\times [\dot{d}'\dot{P}(\phi,\dot{d},\tilde{d},s_t) + (1-\dot{d}')[1-\dot{P}(\phi,\dot{d},\tilde{d},s_t)]]$$

$$\left.\times [\tilde{d}'\tilde{P}(\phi,\dot{d},\tilde{d},s_t) + (1-\tilde{d}')[1-\tilde{P}(\phi,\dot{d},\tilde{d},s_t)]]\Pi^{\phi,D/E}(\phi',d',\dot{d}',\tilde{d}',s_t)\right\}$$

It can be easily verified that the Bellman equation (6.14) satisfies the Blackwell condition, thus $V$ exists and is unique. Proposition 2 follows immediately from direct substitutions for $V$ and the expressions of $V_{\phi,\phi}^D$ and $V_{\phi,\phi}^E$ and making use of the identities above. In particular, the total weighted sum of coefficients on continuation value term $E_{s'|s}V_{\phi,\phi'}^{D/E}(d', \dot{d}', \tilde{d}', s', \mathbb{P})$ for each

$(d', \dot{d}', \tilde{d}')$ matches $\Pi^{\phi, D/E}(\phi', d', \dot{d}', \tilde{d}', s')$ according to the law of motion. $\square$

### A.5.5 Proof of Proposition 5 and 8

The proof for proposition 4 and proposition 7 are analogous. It suffice to prove the general case (Proposition 8). The expressions for $P^*(\phi, \phi', d, \dot{d}, \tilde{d}, s)$ are derived directly from solving the 'mini Bellman equations' of $V^D_{\phi,\phi'}$ for $\mathcal{R}' = \mathcal{R}$ and $V^E(\phi, \phi')$ for $\mathcal{R}' \neq \mathcal{R}$. The expressions for $\tilde{P}$ and $\dot{P}$ follows immediate by applying the law of large numbers to replace $D^*_{ijt}\mathbf{1}\{\phi_i = \phi, \phi_j = \phi', D_{ijt-1} = d, s_t = s\}$ with $P^*(\phi, \phi', d, \dot{d}, \tilde{d}, s)$ in the inner maximization problem s $\tilde{D}_{it}$ and $\dot{D}_{it}$. $\square$

### A.5.6 Proof of Proposition 6 and 9

The proof for proposition 4 and proposition 7 are analogous. It suffice to prove the general case (Proposition 9). I show that the fixed point equation

$$\mathbb{P}^* = T(\mathbb{P}^*)$$

described by (6.17)-(6.20) has a fixed point $\mathbb{P}^*$ in a suitably defined space. Note that the state space is now $\mathbb{S} = [0,1]^{1+2|\Phi|+|\Phi|^2}$ where $|\Phi|$ is the cardinality of the type sapce $\Phi$. It suffices to show the existence of fixed point $\mathbb{P}^*$ for a given $(\phi, \phi', d) \in \Phi^2 \times \{0, 1\}$.Consider the space of continuous functions that maps from $\mathbb{S}$ to $[0,1]^3$ equipped with the sup norm, denoted by $C_\infty(\mathbb{S}; [0,1]^3)$, I show that the functional $T$ maps the subset of Lipschitz functions that are uniformly bounded by some Lipschitz constant $M > 0$, denoted $C^M_1(\mathbb{S}; [0,1]^3)$, into $C^M_1(\mathbb{S}; [0,1]^3)$. Note that $C^M_1(\mathbb{S}; [0,1]^3)$ is a closed, convex, and compact subset of complete metric space $C_\infty(\mathbb{S}; [0,1]^3)$. The goal is to use Schauder fixed point theorem to show existence of a fixed point $\mathbb{P}^*$. The proof consists of the following steps: 1) $T$ maps $C^M_1(\mathbb{S}; [0,1]^3)$ into $C^M_1(\mathbb{S}; [0,1]^3)$; 2) $T$ is continuous in $\mathbb{P}$ with the sup norm; 3) a fixed point $\mathbb{P}^* \in C_1^M(\mathbb{S};[0,1]^3)$ exists.

*Step I: $T$ maps $C_1^M(\mathbb{S};[0,1]^3)$ into $C_1^M(\mathbb{S};[0,1]^3)$*. For $P(\phi,\phi',d,s)$, denote the inner term of $G_\epsilon$ in equation (6.17) by $X^1_{\phi,\phi',d}(s,\mathbb{P})$

$$|X^1_{\phi,\phi',d}(s_2,\mathbb{P}) - X^1_{\phi,\phi',d}(s_1,\mathbb{P})| \leq \underbrace{|\pi^I(\phi,\phi',s_2,\mathbb{P}) - \pi^I(\phi,\phi',s_1,\mathbb{P})|}_{\text{term I}} + \beta \underbrace{\int |V^D_{\phi,\phi'}(1,s',\mathbb{P}) - V^D_{\phi,\phi'}(0,s',\mathbb{P})||q(s'|s_2) - q(s'|s_1)|ds'}_{\text{term II}}$$

*Term I*: by the assumption of $\pi^I$, we have

$$\text{term I} \leq M^\pi \cdot |s_2 - s_1|.$$

*Term II*: note that $V^D_{\phi,\phi'}(1,s',\mathbb{P})$ is bounded by $(1-\beta)^{-1}$ times the upper bound on profit $\bar{\pi}$. By the assumption of $q$, we have

$$\text{term II} \leq \beta(1-\beta)^{-1}\bar{\pi}M^q \cdot |s_2 - s_1|.$$

Adding the terms together and apply $G_\epsilon$, we see that $P = T_1(\mathbb{P})$ is Lipschitz in $s$ with constant $0.25\sigma_\epsilon^{-1}[M^\pi + \beta(1-\beta)^{-1}\bar{\pi}M^q]$ where $T_1(\mathbb{P})$ is the right hand side of equation (6.17). I have used the fact that the derivative of a $G_\epsilon$ with $\epsilon \sim \text{Logistic}(0,1)$ is at most $0.25$ (at $0$). Analogously, $P = T_2(\mathbb{P})$ is Lipschitz in $s$ with constant $0.25\sigma_\epsilon^{-1}[M^\pi + 2\beta(1-\beta)^{-1}\bar{\pi}M^q]$ where $T_2(\mathbb{P})$ is the right hand side of equation (6.18). For $M$ sufficiently large, $T_1(\mathbb{P})$ and $T_1(\mathbb{P})$ always lies in $C_1^M(\mathbb{S};[0,1]^3)$. Next, for $\tilde{P}(\phi,s) = T_3(\mathbb{P})$ from equation (6.19), denote the inner term of $G_{f^E}$ in equation (6.19) by $X^3_\phi(s,\mathbb{P})$, then we have

$$
\begin{aligned}
|X^3_\phi(s_2,\mathbb{P}) - X^3_\phi(s_1,\mathbb{P})| \leq & \underbrace{|\pi^C_D(\phi,\phi',s_2,\mathbb{P}) - \pi^C_D(\phi,\phi',s_1,\mathbb{P})|}_{\text{term I}} \\
& + \textstyle\sum_{(\phi',d)|\mathcal{R}'\neq\mathcal{R}} \Big\{ \underbrace{|\tilde{\Pi}^\phi(\phi',d,s_2)P^*(\phi,\phi,d,s_2)\pi^I(\phi,\phi',s_2,\mathbb{P}) - \tilde{\Pi}^\phi(\phi',d,s_1)P^*(\phi,\phi,d,s_1)\pi^I(\phi,\phi',s_1,\mathbb{P})|}_{\text{term II}} \\
& + \underbrace{|\tilde{\Pi}^\phi(\phi',d,s_2)P^*(\phi,\phi,d,s_1) - \tilde{\Pi}^\phi(\phi',d,s_1)P^*(\phi,\phi,d,s_2)|\rho(1-d)}_{\text{term III}} \\
& + \underbrace{|\tilde{\Pi}^\phi(\phi',d,s_2)P^*(\phi,\phi,d,s_2)E_\epsilon[\epsilon|\Sigma(\phi,\phi',d,s_2,\epsilon)=1]}_{\text{term IV}} \\
& \underbrace{-\tilde{\Pi}^\phi(\phi',d,s_1)P^*(\phi,\phi,d,s_1)E_\epsilon[\epsilon|\Sigma(\phi,\phi',d,s_1,\epsilon)=1]|}_{\text{term IV}} \\
& + \underbrace{\beta\int|V^E_{\phi,\phi'}(1,s',\mathbb{P})||\tilde{\Pi}^\phi(\phi',d,s_2)P^*(\phi,\phi,d,s_2)q(s'|s_2)}_{\text{term V}} \\
& \underbrace{-\tilde{\Pi}^\phi(\phi',d,s_1)P^*(\phi,\phi,d,s_1)q(s'|s_1)|ds'}_{\text{term V}} \\
& + \underbrace{\beta\int|V^E_{\phi,\phi'}(0,s',\mathbb{P})||\tilde{\Pi}^\phi(\phi',d,s_2)[1-P^*(\phi,\phi,d,s_2)]q(s'|s_2)}_{\text{term VI}} \\
& \underbrace{-\tilde{\Pi}^\phi(\phi',d,s_1)[1-P^*(\phi,\phi,d,s_1)]q(s'|s_1)|ds' \Big\}}_{\text{term VI}}
\end{aligned}
$$

Note that $\tilde{\Pi}^\phi(\phi',d,s)$ is Lipschitz with constant 1 since all the terms in $\tilde{\Pi}^\phi(\phi',d,s)$ are bounded and also

$$\sum_{(\phi',d)|\mathcal{R}'\neq\mathcal{R}} \tilde{\Pi}^\phi(\phi',d,s) \leq 1.$$

*Term I*: by the assumption of $\pi^C$, we have

$$\text{term I} \leq M^\pi \cdot |s_2 - s_1|.$$

*Term II-V*: for $\epsilon \sim \text{Logistic}(0, \sigma)$, we have

$$E[\epsilon|\Sigma(\phi, \phi', d, s, \epsilon) = 1] = \gamma\sigma - \ln P(\phi, \phi', d, s)$$

where $\gamma$ is the Euler-Mascheroni constant. We can write

$$|\ln P(\phi, \phi', d, s_2) - \ln P(\phi, \phi', d, s_1)| = |\ln P(\phi, \phi', d, s_2)^{-1} - \ln P(\phi, \phi', d, s_1)^{-1}| \leq |s_2 - s_1|$$

Term II-V is the difference of the products of Lipschitz function $\tilde{\Pi}^{\phi}$, $P^*$, and ex ante period payoff function. It follows that

$$\begin{aligned}&\text{term II} + \cdots + \text{term V} \leq \\ &\left(\left[\bar{\pi} + \gamma\sigma + \rho(1-d) + \ln \underline{P}^{-1} + \bar{q}\beta(1-\beta)^{-1}\bar{\pi}\right]\left[\textstyle\sum_{(\phi,d)|\mathcal{R}' \neq \mathcal{R}} 1 + M\right] + [M^{\pi} + 1 + \beta(1-\beta)^{-1}\bar{\pi}M^q]\right) \cdot |s_2 - s_1|.\end{aligned}$$

*Term VI*: Term VI is the difference of the products of Lipschitz function $\tilde{\Pi}^{\phi}$, $1 - P^*$ and $q$. Therefore

$$\text{term VI} \leq \beta(1-\beta)\bar{\pi}\left[M^q + \bar{q}M + \bar{q}\sum_{(\phi,d)|\mathcal{R}' \neq \mathcal{R}} 1\right] \cdot |s_2 - s_1|.$$

Adding the terms together and apply $G_\epsilon$, we see that $\tilde{P} = T_3(\mathbb{P})$ is Lipschitz in $s$ with constant less than $M$ (sufficiently large) if

$$0.25\sigma_{f^E}^{-1}\left[\bar{\pi} + \bar{\gamma}\sigma_{f^E} + \bar{\rho}(1-d) + \ln \underline{P}^{-1} + 2\bar{q}\beta(1-\beta)^{-1}\bar{\pi}\right] < 1$$

Now by comparing equation (6.20) and (6.19), $\dot{P} = T_4(\mathbb{P})$ is Lipschitz in $s$ with constant less than $M$ for $M$ sufficiently large if

$$3 \times 0.25\sigma_{f^S}^{-1}\left[\bar{\pi} + \bar{\gamma}\sigma_{f^S} + \bar{\rho}(1-d) + \ln \underline{P}^{-1} + 2\bar{q}\beta(1-\beta)^{-1}\bar{\pi}\right] < 1$$

with $\underline{P} \geq G_\epsilon^{-1}(-\bar{\rho})$.

*Step II: $T$ is continuous in $\mathbb{P}$.* It suffices to show that $V^D_{\phi,\phi'}(d,s)(\cdot) : C_1^M(\mathbb{S};[0,\bar{P}]^3) \to C_1^{M^D}(\mathbb{S};[0,\bar{P}]^3)$ and $V^E_{\phi,\phi'}(d,s)(\cdot) : C_1^M(\mathbb{S};[0,\bar{P}]^3) \to C_1^{M^E}(\mathbb{S};[0,\bar{P}]^3)$ are both continuous in $\mathbb{P}$. Consider the following distance

$$
\begin{aligned}
\sup_{s\in\mathbb{S}}|V^D_{\phi,\phi'}(d,s,\mathbb{P}_2) - V^D_{\phi,\phi'}(d,s,\mathbb{P}_1)| \leq\ & \underbrace{\sup_{s\in\mathbb{S}}[\dot{P}_2(\phi,s)P_2(\phi,\phi',d,s)\pi^I(\phi,\phi',d,s,\mathbb{P}_2) - \dot{P}_1(\phi,s)P_1(\phi,\phi',d,s)\pi^I(\phi,\phi',d,s,\mathbb{P}_1)]}_{\text{term I}} \\
& + \underbrace{\sup_{s\in\mathbb{S}}[\dot{P}_2(\phi,s) - \dot{P}_1(\phi,s)][\rho(1-d)]}_{\text{term II}} \\
& + \underbrace{\sup_{s\in\mathbb{S}}[\dot{P}_2(\phi,s)E[\epsilon|\Sigma_2(\phi,\phi',d,s,\epsilon)=1] - \dot{P}_1(\phi,s_1)E[\epsilon|\Sigma_1(\phi,\phi',d,s,\epsilon)=1]]}_{\text{term III}} \\
& + \underbrace{\beta\,\bar{P}\sup_{s\in\mathbb{S}}|V^D_{\phi,\phi'}(d,s',\mathbb{P}_2) - V^D_{\phi,\phi'}(d,s',\mathbb{P}_1)|}_{\text{term IV}}
\end{aligned}
$$

which implies that

$$
\sup_{s\in\mathbb{S}}|V^D_{\phi,\phi'}(d,s,\mathbb{P}_2) - V^D_{\phi,\phi'}(d,s,\mathbb{P}_1)| \leq (1-\beta\bar{P})^{-1}[\text{term I} + \text{term II} + \text{term III}].
$$

Note that term I and III are the norms of the differences of products of continuous functional in $\mathbb{P}$. Then let $\|\mathbb{P}_2 - \mathbb{P}_1\|_\infty \to 0$, term I-III all vanishes, which establishes the continuity of $V^D_{\phi,\phi'}(d,s)(\cdot)$ and $V^E_{\phi,\phi'}(d,s)(\cdot)$ analogously.

*Step III: a fixed point $\mathbb{P}^* \in C_1^M(\mathbb{S};[0,1]^3)$ exists.* I have shown thus far that under certain parameter conditions, the continuous map $T$ maps a closed convex set $C_1^M(\mathbb{S};[0,1]^3)$ into a compact set $C_1^M(\mathbb{S};[0,1]^3)$ embedded in the complete metric space $C_\infty(\mathbb{S};[0,1]^3)$. A fixed point $\mathbb{P}^* \in C_1^M(\mathbb{S};[0,1]^3)$ exists by Schauder fixed point theorem, which states the following

**Theorem 1** (Schauder Fixed Point Theorem)**.** *If $K$ is a nonempty convex closed subset of a Hausdorff topological vector space $V$ and $T$ is a continuous mapping of $K$ into itself such that $T(K)$ is contained in a compact subset of $K$, then $T$ has a fixed point.*

Note that compactness of $C_1^M(\mathbb{S};[0,1]^3)$ in $C_\infty(\mathbb{S};[0,1]^3)$ follows directly from Arzela Ascoli theorem. This completes the proof.

### A.5.7 Proof of Proposition 10

First consider the $N$th approximate model in which all continuous states are discretized by $f^N$, that is, $\mathbb{P}^N(\phi,\phi,d,s)=\mathbb{P}(\phi,\phi,d,f^N(s))$ where $f^N:[0,1]\to\mathbb{D}^N$ with finite $\mathbb{D}^N\subset[0,1]$ is the discretization function. I first establish that $V^D_{\phi,\phi'}(\mathbb{P}^{*(N)})$ and $V^E_{\phi,\phi'}(\mathbb{P}^{*(N)})$ are uniformly continuous in $\mathbb{P}^{*(N)}$ for every $N\in\mathbb{N}$.

Note that in the discretized version of 'mini Bellman' equations (6.15) and (6.16), the expectation $E_{s'|s}$ operator can be equivalently replaced by its associated finite state Markov transition matrix $Q_{f^N(s)'|f^N(s)}$. This implies that both $V^D_{\phi,\phi'}$ and $V^E_{\phi,\phi'}$ can be directly solved from equations (6.15) and (6.16). Let $[V^D_{\phi,\phi'}(\mathbb{P})]_{d,f^N(s)}$ denote the stack vector of $V^D_{\phi,\phi'}(d,f^N(s),\mathbb{P})$ over $\mathbb{D}^N$ for a given $d\in\{0,1\}$. Then in the Bayesian Markov perfect equilibrium the following must hold

$$
\begin{aligned}
[V^D_{\phi,\phi'}(\mathbb{P}^{*(N)})]_{d,f^N(s)} = {} & (I-\beta[\dot{P}(\phi,f^N(s)')/\dot{P}(\phi,f^N(s))Q_{f^N(s)'|f^N(s)}]_{(f^N(s),f^N(s)')})^{-1} \\
& \times [P(\phi,\phi',d,f^N(s),\mathbb{P}^{*(N)}) \\
& \{\pi^I(\phi,\phi',d,f^N(s),\mathbb{P}^{*(N)})-\rho(1-d)-E[\epsilon|\Sigma(\phi,\phi',d,f^N(s),\mathbb{P}^{*(N)},\epsilon)=1]\}]_{d,f^N(s)}
\end{aligned}
$$

for $d\in\{0,1\}$. When $\epsilon$ has either Gaussian or logistic distribution, the term

$$E[\epsilon|\Sigma^*(\phi,\phi',d,f^N(s),\mathbb{P}^{*(N)},\epsilon)=1]$$

is continuous in $\mathbb{P}^*$. For instance, when $\epsilon\sim\text{Logistic}(0,\sigma)$,

$$E[\epsilon|\Sigma^*(\phi,\phi',d,f^N(s),\mathbb{P}^{*(N)},\epsilon)=1]=\gamma-\ln P^*(\phi,\phi',d,f^N(s),\mathbb{P}^{*(N)})$$

where $\gamma$ is the Euler-Mascheroni constant. Thus, $V^D_{\phi,\phi'}$ is continuous in $\mathbb{P}^{*(N)}$. By analogous argument, we have established that both $V^D_{\phi,\phi'}$ and $V^E_{\phi,\phi'}$ are uniformly continuous in $\mathbb{P}^{*(N)}$ for every $N \in \mathbb{N}$.

Next, I show that equilibrium exists for the $N$th approximate model using discretization function $f^N$ for all $N \in \mathbb{N}$. Note that equations (6.17)-(6.20) defines the following fixed point equation in $\mathbb{P}^{*(N)}$

$$\mathbb{P}^{*(N)} = T^{(N)}(\mathbb{P}^{*(N)})$$

where $T^{(N)}$ is uniformly continuous in $\mathbb{P}^{*(N)}$ by what we have established before. Here I represent $\mathbb{P}^{*(N)}$ as a vector in $[0,1]^{|D(\mathbb{P}^{*(N)})|}$ where $|D(\mathbb{P}^{*(N)})|$ is the cardinality of the compact domain of $\mathbb{P}^{*(N)}$. It is also clear that $T^{(N)}$ maps $[0,1]^{|D(\mathbb{P}^{*(N)})|}$ into $[0,1]^{|D(\mathbb{P}^{*(N)})|}$ by construction. For $\mathbb{P}^N$ that contains zero, $T^{(N)}(\mathbb{P}^{*(N)})$ maybe undefined due to division by zero or taking log of zero. I instead can define $T^{(N)}(\mathbb{P}^{*(N)})$ by the limit of $T^{(N)}(\mathbb{P}^{*(N)}_n)$ as $n \to \infty$ for some sequence $\mathbb{P}^{*(N)}_n$ in $(0,1]^{|D(\mathbb{P}^{*(N)})|}$ that converges to $\mathbb{P}^{*(N)}$. Therefore, by Bouwer's fix point theorem, the equilibrium solution $P^{*(N)}$ exists for every choice of discretization function $f^N$ and $N \in \mathbb{N}$.

Finally, to show that $\mathbb{P}^{*(N)} \to \mathbb{P}^*$ in the space of regulated functions equipped with the sup norm, it suffices to show that

$$\|\mathbb{P}^{*(N)} - \mathbb{P}^* f^N\|_\infty \to 0, \quad N \to \infty,$$

since by Proposition 4, $\mathbb{P}^*$ is Lipschitz continuous in the state and $f^N(s) \to s$ by assumption. We have the following inequalities

$$\begin{aligned}\|\mathbb{P}^{*(N)} - \mathbb{P}^* f^N\|_\infty &\leq \|T^N(\mathbb{P}^{*(N)}) - T(\mathbb{P}^*)(f^N)\|_\infty \\ &\leq \|T^N(\mathbb{P}^{*(N)}) - T(\mathbb{P}^{*(N)})(f^N)\|_\infty + \|T(\mathbb{P}^{*(N)})(f^N) - T(\mathbb{P}^*)(f^N)\|_\infty\end{aligned}$$

The first term is governed by the following difference

$$\|E_{s'|f^N(s)}[V_{\phi,\phi'}^{D/E}(d,\dot{d},\tilde{d},s',\mathbb{P})] - E_{f^N(s')|f^N(s)}[V_{\phi,\phi'}^{D/E}(d,\dot{d},\tilde{d},f^N(s'),\mathbb{P})]\|_\infty, \quad \forall d,\dot{d},\tilde{d} \in \{0,1\}^3.$$

It is easy to verify that the above difference vanishes $N \to \infty$ since 1) the law of motions for most states are deterministic except for $w^0$; 2) I have established in the proof of proposition 4 that the "mini" value functions $V_{\phi,\phi'}^{D/E}$ are Lipschitz continuous in $s$. By assumption, $T$ is a contraction mapping so that

$$\|T(\mathbb{P}^{*(N)})(f^N) - T(\mathbb{P}^*)(f^N)\|_\infty \leq K\|\mathbb{P}^{*(N)} - \mathbb{P}^*\|_\infty, \quad K \in (0,1).$$

Therefore as $N \to \infty$,

$$\|\mathbb{P}^{*(N)} - \mathbb{P}^* f^N\|_\infty \leq (1-K)^{-1}\left\{\|T^N(\mathbb{P}^{*(N)}) - T(\mathbb{P}^{*(N)})(f^N)\|_\infty + \|\mathbb{P}^* f^N - \mathbb{P}^*\|_\infty\right\} \to 0.$$

This completes the proof. $\square$

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