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ABSTRACT

Let G be a finite, discrete group. This thesis studies equivariant symmetric monoidal G-categories and the operads that parametrize them. We devise explicit tools for working with these objects, and then we use them to tackle two conjectures of Blumberg and Hill and a presentation problem of Guillou-May-Merling-Osorno, with varying degrees of success.

The first half of this thesis introduces normed symmetric monoidal categories, and develops their basic theory. These are direct generalizations of the classical structures, and they are presented by generators and isomorphism relations. We explain how to construct an operad action from these generators via an equivariant version of the Kelly-Mac Lane coherence theorem, and then we study the resulting operads in their own right. We show that the operads for normed symmetric monoidal categories are precisely the cell complexes in a certain model structure, and that they are cofibrant replacements for the commutativity operad in a family of other model structures. Our work resolves a conjecture of Blumberg and Hill on the classification of N_∞ operads in the affirmative. Finally, we prove a number of homotopy invariance results for the structures under consideration. We show that weak equivalences between certain categorical N_∞ operads induce equivalences on the level of algebras, and that pseudoalgebras over such operads are strict algebras over larger, equivalent operads. We deduce that the symmetric monoidal G-categories of Guillou-May-Merling-Osorno are equivalent to E_∞ normed symmetric monoidal categories.

The second half of this thesis studies a number of examples. We explain how to construct normed symmetric monoidal structures by twisting a given operation over a diagram, and we examine a shared link between the symmetric monoidal G-categories of Guillou-May-Merling-Osorno and the G-symmetric monoidal categories of Hill and Hopkins. We give functorial constructions of N_∞ operads, and we examine how the lattice of indexing systems is reflected on the level of operads. We prove a combinatorial analogue to a conjecture of Blumberg and Hill on the Boardman-Vogt tensor product of N_∞ operads, and while our work does not solve their original problem, it does imply a space-level interchange result.

CHAPTER 1

INTRODUCTION

Suppose that G is a finite, discrete group and that X is a G-set. For every subgroup $H \subset G$, we have a subset $X^H \subset X$ of H-fixed elements, and inclusions of subgroups $K \subset H$ induce reverse inclusions $X^K \supset X^H$ on fixed points. In algebraic situations, further structure appears in the form of "wrong way" transfer maps. For example, if F/k is a finite Galois extension and $G = \mathrm{Gal}(F/k)$, then there are norm and trace maps $F \rightrightarrows k$, defined by multiplying and summing Galois conjugates together.

The transfer also plays an important role in equivariant homotopy theory. Additive transfers are the basis of a recognition principle for equivariant loop spaces (cf. [12], [18], [33], and [39]), while multiplicative transfers were instrumental in Hill-Hopkins-Ravenel's solution to the Kervaire invariant one problem [24]. An important observation of Hill and Hopkins [22] is that localization can destroy the multiplicative transfers, or "norms" on commutative ring G-spectra, and Blumberg and Hill [5] subsequently introduced N_∞ operads and algebras to axiomatize these partial systems of norms. This is the starting point for this thesis.

Our work was motivated by a number of problems, which we briefly explain. One line of inquiry stems from questions in pure operad theory. In [5], Blumberg and Hill introduce N_∞ operads and develop much of the surrounding theory, but some questions were left unanswered. In particular, they gave a classification of N_∞ operads in terms of an invariant called an *indexing system*, but it was only conjectured that every indexing system was realized [5, p. 4]. While they had precise candidates to do this, later work of Bonventre [7] revealed that these candidates generally fail. We prove their conjecture by giving an explicit construction (theorem 3.19), and other, independent solutions have also been found by Bonventre-Pereira [8] and Gutiérrez-White [21]. Blumberg and Hill also observed that in many cases, N_∞ operads interchange with themselves. Accordingly, they conjectured [5, conjecture 6.27] that under suitable cofibrancy conditions, the tensor product of N_∞ operads should again be N_∞, with norms generated by the factors. This should be compared to the additivity of

E_n-structures, as studied by Boardman [3], Dunn [14], and Fiedorowicz and Vogt [17].

Another line of inquiry stems from considerations in equivariant category theory and equivariant homotopical algebra. As indicated above, a key feature of equivariant spectra are their homotopy-coherent systems of transfer maps. A number of structures have been devised to formalize this situation. One line of thought leads to the symmetric monoidal G-categories of Guillou-May-Merling-Osorno [20]. Another leads to the G-commutative monoids of Hill and Hopkins [23], which themselves reside in G-symmetric monoidal categories. While these two notions of equivariant symmetric monoidal structure are decidedly different, there are notable similarities between them. As explained by Hill and Hopkins [23, §3.2], there is even a general, non-invertible procedure for converting a symmetric monoidal G-category into a G-symmetric monoidal category. We sought to clarify the relationship between these objects further, and to find a means of presenting them (cf. [20, problem 1.36]). Indeed, no description by generators and relations was known for either structure, which was surprising, given their resemblance to the classical objects.

While N_∞ operads and algebras were invented to account for space and spectrum-level phenomena, they can profitably be studied in other contexts. This thesis examines N_∞ structures on the level of categories. This setting is a common focal point for the preceding questions, and it facilitates an analysis of strict and coherent structure from a 2-categorical perspective. We recover topological results by taking classifying spaces.

In what follows, we develop the foundations of a theory of N_∞ symmetric monoidal G-categories, which we hope will have applications in equivariant homotopy theory. We introduce a new kind of N_∞-G-category, which we call a *normed symmetric monoidal category.* These are direct generalizations of the classical structures, and they are presented by generators and isomorphism relations. We prove an equivariant, operadic version of the Kelly-Mac Lane coherence theorem for them (theorem 2.10), and we prove a number of homotopy invariance results with respect to their parametrizing operads (theorems 4.21 and 4.33). We use our framework to address the problems above, with mixed results. As mentioned earlier, we resolve Blumberg and Hill's conjecture [5, p. 4] on indexing systems in the affirmative (theorem 3.19). We also prove a combinatorial analogue to their conjecture [5, conjecture 6.27] on the tensor product of N_∞ operads (theorem 6.27). Our work does not solve the original problem, but it does imply a space-level interchange result of the desired sort. Along similar lines, we do not find a presentation for the symmetric monoidal and permutative G-categories of [20], but our invariance theorems imply that they are equivalent to E_∞ normed symmetric monoidal categories. Thus, the presentation that defines normed symmetric monoidal structure can also be used to produce the objects of interest in [20] (theorems 4.27 and 4.34).

In summary, this thesis is organized as follows. Chapters 2 – 4 develop the theory of normed symmetric monoidal categories and a closely related class of categorical operads, while chapters 5 – 6 outline how the theory looks in specific examples.

In chapter 2, we introduce normed symmetric monoidal categories and the operads that parametrize them. Normed symmetric monoidal categories are defined without reference to operads, and the coherence theorem (theorem 2.10) roughly states that their presenting data generate actions by categorical N_∞ operads. These operads are obtained by freely generating an operad from a G-fixed constant e, a G-equivariant binary product $\otimes$, and a set $\mathcal{N}$ of norms $\bigotimes_T$, and then inserting a unique isomorphism between every pair of operations of the same arity. We denote these operads $\mathcal{SM}_\mathcal{N}$, and we reiterate that this is the first equivariant coherence theorem of its sort.

In chapter 3, we study a special class of categorical operads that includes the operads $\mathcal{SM}_\mathcal{N}$ and the equivariant Barratt-Eccles operad $\mathscr{P}_G$ of [18]. We call such operads *homogeneous.*[1] We prove that the operads $\mathcal{SM}_\mathcal{N}$ realize all indexing systems (theorem 3.19), and we develop a well-behaved homotopy theory of homogeneous operads that frames our results (theorems 3.24 and 3.40). In particular, the operads $\mathcal{SM}_\mathcal{N}$ are cell complexes in a certain model structure, and they are cofibrant replacements of the commutativity operad **Com** in

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[1] In [19] and [20], such operads are called *chaotic.*

a related family of indexing system model structures.

In chapter 4, we prove a few homotopy invariance results for the algebras and pseudoalgebras over homogeneous operads. Our results are in the spirit of Boardman and Vogt's theory of homotopy invariant algebraic structures [4]. We show that a weak equivalence between N_∞ homogeneous operads induces a biequivalence between the associated 2-categories of algebras (theorem 4.21), and that the 2-category of pseudoalgebras over a homogeneous N_∞ operad $\mathscr{O}$ is isomorphic to the 2-category of strict algebras over a cellular approximation $W\mathscr{O}$ of the operad $\mathscr{O}$ (theorem 4.33). The conceptual point is that N_∞ homogeneous operads play the role of Σ-cofibrant operads in Berger and Moerdijk's homotopy theory of operads [2]. One upshot is that the symmetric monoidal and permutative G-categories of [18] and [20] are equivalent to E_∞ normed symmetric monoidal categories.

In chapter 5, we give explicit examples of normed symmetric monoidal categories. Our work focuses on diagram categories, and we explain how to twist a given operation over a diagram to construct a full normed symmetric monoidal structure. In section 5.3, we follow Hill and Hopkins' prescription [23, §3.2] to identify a striking thread between our work, and the work in [20] and [23]. In short, the canonical example of a symmetric monoidal G-category in the sense of [20] is an E_∞ normed symmetric monoidal category, and its transfers give rise to the canonical example of G-symmetric monoidal structure defined by monoidal induction [23]. See theorem 5.12 for a more precise statement.

In chapter 6, we study a handful of homogeneous N_∞ operads, and we refine a few of the results from chapter 3. We give uniform, functorial constructions of N_∞ operads (theorem 6.3 and proposition 6.19), we construct N_∞ permutativity operads for all indexing systems (theorem 6.18), and we analyze the ways that the lattice structure on the poset of indexing systems is mirrored on the level of homogeneous operads (theorem 6.24). Our work proves a combinatorial analogue to Blumberg and Hill's conjecture on the Boardman-Vogt tensor product of N_∞ operads (theorem 6.27), but we do not resolve the original space-level conjecture.

One theme that emerges from our work is the canonical nature of the operads $\mathcal{SM}_{\mathcal{N}}$, and by extension, of normed symmetric monoidal categories. We stumbled on these structures entirely by accident, but our subsequent investigations suggest that they are fundamental to this area of mathematics. We hope to convince the reader of these facts.

CHAPTER 2

NORMED SYMMETRIC MONOIDAL CATEGORIES

2.1 Introduction and summary of results

A *symmetric monoidal category* is a category $\mathscr{C}$, equipped with a bifunctor $\otimes : \mathscr{C}^{\times 2} \to \mathscr{C}$ and a distinguished object $e \in \mathscr{C}$, such that the product $\otimes$ is associative and commutative, and the object e is a two-sided unit for $\otimes$, up to isomorphism. The isomorphisms witnessing the associativity, commutativity, and unitality of these data are regarded as additional structure on $\mathscr{C}$, and thus we also specify natural isomorphisms $\alpha : (C \otimes D) \otimes E \to C \otimes (D \otimes E)$, and $\lambda : e \otimes C \to C$, and $\rho : C \otimes e \to C$, and $\beta : C \otimes D \to D \otimes C$, which are required to make an associativity pentagon, a braid hexagon, and three triangle diagrams commute.

Remarkably, these five commutativity conditions are enough to ensure that all sensible diagrams built purely from the maps α, λ, ρ, and β will also commute. This is the Kelly-Mac Lane coherence theorem (cf. [26] and [30]), and it has at least two distinct roles. If we regard $\mathscr{C}$ as an ambient setting for doing mathematics, then the coherence theorem assures us that the familiar canonical isomorphisms are available. If we regard $\mathscr{C}$ as a model for a space, then the coherence theorem implies that the product $\otimes$ is an E_∞ operation on $\mathscr{C}$.

Fix a finite group G. The most immediate equivariantization of symmetric monoidal structure arises by placing a G-action on all data in sight. More precisely, we assume that $\mathscr{C}$ is equipped with a G-action through functors, that $\otimes$ preserves the G-action on objects and morphisms, that e is G-fixed, and that the isomorphisms α, λ, ρ, and β are preserved by the G-action, e.g. $g\alpha_{C,D,E} = \alpha_{gC,gD,gE}$. This generalization is not logically incorrect, but it leaves much to be desired because it does not equip $\mathscr{C}$ with transfers.

Typical examples of equivariant transfer arise by summing or multiplying the translates of an element over an orbit. For instance, if M is a G-module and $K \subsetneq H \subset G$ are subgroups, then the transfer $\mathrm{tr}_K^H : M^K \to M^H$ is given by the formula $\mathrm{tr}_K^H(x) = \sum_{H/K} h_i x$, where the elements h_i are chosen H/K coset representatives. This map factors as a composite of a twisted diagonal map $\Delta^{\text{tw}}(x) = (h_1 x, \dots, h_{|H:K|} x)$ and a $|H : K|$-fold sum, which we consider in turn. First, note that the twisted diagonal gives an isomorphism $\Delta^{\text{tw}} : M^K \to (M^{\times H/K})^H$, where $M^{\times H/K}$ is the H-module with action $h(x_1, \dots, x_{|H:K|}) = (hx_{\sigma^{-1}1}, \dots, hx_{\sigma^{-1}|H:K|})$, and the permutation σ is determined by the equation $hh_iK = h_{\sigma i}K$. Next, observe that the $|H : K|$-fold sum $\sum_{H/K} : M^{\times H/K} \to M$ is H-equivariant, because addition in M is G-equivariant and *strictly commutative.* Thus, $\sum_{H/K}$ descends to H-fixed points, and the transfer is the composite

$$\text{tr}_K^H = \Sigma_{H/K} \circ \Delta^{\text{tw}} : M^K \to (M^{\times H/K})^H \to M^H.$$

While the sum $\sum_{H/K} : M^{\times H/K} \to M$ is reducible to binary addition for G-modules M, the same cannot be said in homotopical situations, because one almost never has a strictly commutative operation. To get equivariant transfers for a G-category $\mathscr{C}$, we must specify H-functors $\bigotimes_{H/K} : \mathscr{C}^{\times H/K} \to \mathscr{C}$ explicitly, in addition to any other product $\otimes : \mathscr{C}^{\times 2} \to \mathscr{C}$ that might be present. We call such H-functors *norms*, and in this chapter, we shall study *normed symmetric monoidal categories.* These are G-categories, equipped with an ordinary symmetric monoidal structure $(\mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \beta)$ for which all data is G-equivariant, and a set of additional norms $\bigotimes_T : \mathscr{C}^{\times T} \to \mathscr{C}$ that are coherently, but nonequivariantly, isomorphic to ordinary $|T|$-fold tensor products. The untwisting isomorphisms $\upsilon_T : \bigotimes_T(C_1, \dots, C_{|T|}) \to (\cdots (C_1 \otimes C_2) \otimes \cdots) \otimes C_{|T|}$ are regarded as part of the structure on $\mathscr{C}$, and they must be compatible with the rest of the structure through a "twisted equivariance" diagram. We index the H-sets T over some specified set $\mathcal{N}$.

In what follows, we give the basic definitions (section 2.2) and prove a coherence theorem for normed symmetric monoidal categories (sections 2.3 – 2.8). The coherence theorem is the main result of this chapter, and it boils down to the construction of an operad action from the generating data in an $\mathcal{N}$-normed symmetric monoidal category. The operad in question parametrizes all diagrams that commute for formal reasons. It is obtained by freely generating an operad out of a G-fixed constant e, a G-equivariant product $\otimes$, and a norm map $\bigotimes_T$ for every $T \in \mathcal{N}$, and then inserting a unique isomorphism between every pair of operations of the same arity (cf. section 2.4). We denote it $\mathcal{SM}_{\mathcal{N}}$. In these terms, the coherence theorem (theorem 2.10) reads as follows.

Theorem. *The* 2*-category of* $\mathcal{N}$*-normed symmetric monoidal categories, lax (resp. strong, strict) monoidal functors, and monoidal transformations is isomorphic to the* 2*-category of* $\mathcal{SM}_{\mathcal{N}}$*-algebras in* G*-categories, lax (resp. pseudo, strict)* $\mathcal{SM}_{\mathcal{N}}$*-morphisms, and* $\mathcal{SM}_{\mathcal{N}}$*-transformations. Moreover, this isomorphism does not affect underlying* G*-categories,* G*-functors, or* G*-natural transformations.*

The coherence theorem may be alternately regarded as a presentation theorem for $\mathcal{SM}_{\mathcal{N}}$-algebras, and this will be a more useful perspective going forward. When combined with the homotopy invariance theorems in chapter 4, it implies that the presentation defining normed symmetric monoidal categories can also be used to produce algebras over a reasonably large class of categorical operads, including the equivariant Barratt-Eccles operad $\mathscr{P}_G$ of [18].

2.2 Basic definitions

Fix a finite group G throughout, and let $G\underline{\mathbf{Cat}}$ denote the 2-category of all small G-categories, G-functors, and G-natural transformations.

Definition 2.1. A *symmetric monoidal object in* $G\underline{\mathbf{Cat}}$ is a tuple $(\mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \beta)$ such that $\mathscr{C}$ is a small G-category, $\otimes : \mathscr{C}^{\times 2} \to \mathscr{C}$ is a G-bifunctor, $e \in \mathscr{C}$ is a G-fixed object, and

$$(C \otimes D) \otimes E \xrightarrow{\alpha} C \otimes (D \otimes E), \quad e \otimes C \xrightarrow{\lambda} C, \quad C \otimes e \xrightarrow{\rho} C, \quad C \otimes D \xrightarrow{\beta} D \otimes C$$

are G-natural isomorphisms that make the usual associativity pentagon, braid hexagon, and triangle diagrams commute (cf. [27, Ch. 1]). Define the *standard n-fold tensor products* on $\mathscr{C}$ by $\bigotimes_0() := e$, $\bigotimes_1(C) = C$, and $\bigotimes_{n+1}(C_1, \dots, C_{n+1}) = \bigotimes_n(C_1, \dots, C_n) \otimes C_{n+1}$.

Definition 2.2. Suppose that $H \subset G$ is a subgroup and that T is a finite H-set equipped with a linear order $T \cong \{1, \ldots, |T|\}$. Let $\sigma : H \to \Sigma_{|T|}$ denote the corresponding permutation representation on $\{1, \ldots, |T|\}$. For any G-category $\mathscr{C}$, we define $\mathscr{C}^{\times T}$ to be the $|T|$-fold cartesian power $\mathscr{C}^{\times |T|}$ equipped with the H-action $h(C_1, \ldots, C_{|T|}) = (hC_{\sigma^{-1}1}, \ldots, hC_{\sigma^{-1}|T|})$, and similarly for morphisms. We define a *T-norm* on $\mathscr{C}$ to be an H-functor $\mathscr{C}^{\times T} \to \mathscr{C}$.

Definition 2.3. Suppose that $\mathcal{N} = (\mathcal{N}(H))_{H \subset G}$ is a graded set of finite ordered H-sets T. We call $\mathcal{N}$ a *set of exponents.* A *$\mathcal{N}$-normed symmetric monoidal category* is a symmetric monoidal object $(\mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \beta)$ in $G\underline{\mathbf{Cat}}$, together with

1. a T-norm $\bigotimes_T : \mathscr{C}^{\times T} \to \mathscr{C}$ for every $T \in \mathcal{N}$, and

2. (*untwistors*) a nonequivariant natural isomorphism

$$\upsilon_T : \bigotimes_T(C_1, \ldots, C_{|T|}) \to \bigotimes_{|T|}(C_1, \ldots, C_{|T|})$$

for every H-set $T \in \mathcal{N}$, such that for every $h \in H$, the "twisted equivariance" diagram

$$\begin{tikzcd}
h\bigotimes_T(C_1, \ldots, C_{|T|}) \arrow[r, "\mathrm{id}"] \arrow[dd, "h\upsilon_T"'] & \bigotimes_T(hC_{\sigma^{-1}1}, \ldots, hC_{\sigma^{-1}|T|}) \arrow[d, "\upsilon_T"] \\
 & \bigotimes_{|T|}(hC_{\sigma^{-1}1}, \ldots, hC_{\sigma^{-1}|T|}) \arrow[d, "\sigma^{-1}"] \\
h\bigotimes_{|T|}(C_1, \ldots, C_{|T|}) \arrow[r, "\mathrm{id}"'] & \bigotimes_{|T|}(hC_1, \ldots, hC_{|T|})
\end{tikzcd}$$

commutes. Here σ^{-1} denotes the canonical isomorphism for the symmetric monoidal category $\mathscr{C}$, which permutes the factors of $\bigotimes_{|T|}$ by σ^{-1}.

Notation 2.4. We shall write $\otimes^{\mathscr{C}}$, $\alpha^{\mathscr{C}}$, $\lambda^{\mathscr{C}}$, etc. when we wish to emphasize that these data are associated to a particular normed symmetric monoidal category $\mathscr{C}$.

Remark 2.5. The coherence theorem for $\mathcal{N}$-normed symmetric monoidal categories roughly states the following. Consider all composite operations on $\mathscr{C}$ generated by e, $\otimes$, and $\bigotimes_T$ for $T \in \mathcal{N}$. We say that a natural isomorphism between two such operations is *basic* if it is the identity transformation, or if it is obtained by applying a single instance of $\alpha^{\pm 1}$, $\lambda^{\pm 1}$, etc. to a sub-operation. We say that a natural isomorphism is *canonical* if it is a componentwise (vertical) composite of basic natural isomorphisms. Then:

1. there is a unique canonical map between any two composites of the same arity,
2. canonical maps are closed under conjugation by elements of G,
3. canonical maps are closed under permutations of inputs,
4. canonical maps are closed under componentwise (vertical) composition, and
5. canonical maps are closed under operadic (horizontal) composition.

As usual, these statements are not literally correct because generically distinct operations might accidentally become equal in some particular $\mathscr{C}$, and the resulting diagrams need not all commute. One must restrict attention to certain formally definable diagrams in order to get commutativity in general. This statement is made precise using the operad $\mathcal{SM}_{\mathcal{N}}$, which we construct in section 2.4.

We now generalize the usual notions of (lax, strong, strict) monoidal functors and monoidal natural transformations to the normed symmetric monoidal setting.

Definition 2.6. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are $\mathcal{N}$-normed symmetric monoidal categories. A *lax* $\mathcal{N}$*-normed functor* $(F, F_\bullet) : \mathscr{C} \to \mathscr{D}$ consists of the following data:

1. a G-functor $F : \mathscr{C} \to \mathscr{D}$,
2. a G-fixed morphism $F_e : e^{\mathscr{D}} \to F e^{\mathscr{C}}$,
3. a G-natural transformation $F_\otimes : FC \otimes^{\mathscr{D}} FC' \to F(C \otimes^{\mathscr{C}} C')$, and

4. for every subgroup $H \subset G$ and $T \in \mathcal{N}(H)$, an H-natural transformation[1] $F_{\bigotimes_T} : \bigotimes_T^{\mathscr{D}}(FC_1, \dots, FC_{|T|}) \to F\big(\bigotimes_T^{\mathscr{C}}(C_1, \dots, C_{|T|})\big)$,

such that the usual lax symmetric monoidal diagrams relating α, λ, ρ, and β to the comparison maps F_e and $F_\otimes$ commute (cf. [31, Ch. XI.2]), and the square

$$\begin{tikzcd}
\bigotimes_T^{\mathscr{D}}(FC_1, \dots, FC_{|T|}) \arrow[r, "\upsilon_T^{\mathscr{D}}"] \arrow[d, "F_{\bigotimes_T}"'] & \bigotimes_{|T|}^{\mathscr{D}}(FC_1, \dots, FC_{|T|}) \arrow[d, "\Big(\text{iterated } F_\otimes\text{'s and id's}\Big) =: F_{\bigotimes_{|T|}}"] \\
F\Big(\bigotimes_T^{\mathscr{C}}(C_1, \dots, C_{|T|})\Big) \arrow[r, "F\upsilon_T^{\mathscr{C}}"'] & F\Big(\bigotimes_{|T|}^{\mathscr{C}}(C_1, \dots, C_{|T|})\Big)
\end{tikzcd}$$

commutes for every $T \in \mathcal{N}$. More precisely, the right hand map $F_{\bigotimes_{|T|}}$ is given by $F_{\bigotimes_0} := F_e$ and $F_{\bigotimes_1} := \mathrm{id}_F$ for $n = 0, 1$, and $F_{\bigotimes_{n+1}} := F_\otimes \circ (F_{\bigotimes_n} \otimes^{\mathscr{D}} \mathrm{id})$ for $n > 1$. We say that a lax $\mathcal{N}$-normed morphism is *strong* (resp. *strict*) if the natural transformations F_e, $F_\otimes$, and $F_{\bigotimes_T}$ are all isomorphisms (resp. identities).

Definition 2.7. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are $\mathcal{N}$-normed symmetric monoidal, and that $(F, F_\bullet)$, $(F', F'_\bullet) : \mathscr{C} \rightrightarrows \mathscr{D}$ is a pair of lax $\mathcal{N}$-normed functors between them. An *$\mathcal{N}$-normed monoidal transformation* $\omega : (F, F_\bullet) \Rightarrow (F', F'_\bullet)$ is a G-natural transformation $\omega : F \Rightarrow F'$ such that the usual monoidal transformation squares relating F_e, F'_e, $F_\otimes$, and $F'_\otimes$ to ω commute (cf. [31, Ch. XI.2]), and the square

$$\begin{tikzcd}
\bigotimes_T^{\mathscr{D}}(FC_1, \dots, FC_{|T|}) \arrow[r, "\bigotimes_T^{\mathscr{D}}(\omega, \dots, \omega)"] \arrow[d, "F_{\bigotimes_T}"'] & \bigotimes_T^{\mathscr{D}}(F'C_1, \dots, F'C_{|T|}) \arrow[d, "F'_{\bigotimes_T}"] \\
F\Big(\bigotimes_T^{\mathscr{C}}(C_1, \dots, C_{|T|})\Big) \arrow[r, "\omega"'] & F'\Big(\bigotimes_T^{\mathscr{C}}(C_1, \dots, C_{|T|})\Big)
\end{tikzcd}$$

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[1] We regard both sides as H-functors $\mathscr{C}^{\times T} \to \mathscr{D}$.

commutes for every $T \in \mathcal{N}$.

Notation 2.8. The 2-category structure on $G\underline{\mathbf{Cat}}$ lifts to normed symmetric monoidal categories. The composite of lax maps $(G, G_\bullet) \circ (F, F_\bullet)$ is obtained by composing underlying functors and comparison data, e.g.

$$GF_\otimes \circ G_\otimes : GFC \otimes GFC' \to G(FC \otimes FC') \to GF(C \otimes C').$$

Vertical and horizontal composites of transformations are computed in $G\underline{\mathbf{Cat}}$, and identities are also inherited from $G\underline{\mathbf{Cat}}$. Let $\mathcal{N}\mathbf{SMLax}$ be the 2-category of all small $\mathcal{N}$-normed symmetric monoidal categories, lax $\mathcal{N}$-normed monoidal functors, and $\mathcal{N}$-normed monoidal transformations. There are sub-2-categories $\mathcal{N}\mathbf{SMSt} \subset \mathcal{N}\mathbf{SMStg} \subset \mathcal{N}\mathbf{SMLax}$ of strong and strict maps, and there is a forgetful 2-functor $\mathcal{N}\mathbf{SMLax} \to G\underline{\mathbf{Cat}}$.

Remark 2.9. There are also operadic coherence theorems for normed symmetric monoidal functors and transformations. All things said and done, there is an isomorphism between the 2-category $\mathcal{N}\mathbf{SMLax}$ (or $\mathcal{N}\mathbf{SMStg}$ or $\mathcal{N}\mathbf{SMSt}$) and the corresponding 2-category of algebras over the operad $\mathcal{SM}_\mathcal{N}$ (cf. theorem 2.10).

2.3 An overview of the coherence theorem

The coherence theorem for normed symmetric monoidal categories is an explicit family of isomorphisms between the various 2-categories of $\mathcal{N}$-normed symmetric monoidal categories, and the corresponding 2-categories of algebras over an operad called $\mathcal{SM}_\mathcal{N}$. There is a coherence theorem for every set of exponents $\mathcal{N}$ and for every flavor of monoidal functor, but we focus on the situation for lax morphisms. The other cases are similar.

Theorem 2.10. *There is a commutative triangle*

$$\begin{tikzcd}
\mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgLax} \arrow[rr, "\mathrm{ev}"] \arrow[dr, "\text{forget}"'] & & \mathcal{N}\mathbf{SMLax} \arrow[dl, "\text{forget}"] \\
& G\underline{\mathbf{Cat}} &
\end{tikzcd}$$

of 2*-categories and* 2*-functors, and the evaluation* 2*-functor* ev *is an isomorphism. Similarly in the strong and strict cases.*

In the remainder of this chapter, we shall explain what the terms in the triangle above mean, and then prove the theorem. The most technical details are treated in section 2.8. We recommend skimming or skipping sections 2.4 – 2.8 on a first reading.

We begin by recalling the notion of an *operad*. Let $(\mathscr{V}, \otimes, e)$ be a closed symmetric monoidal category. An operad $\mathscr{O}$ in $\mathscr{V}$ is a sequence of objects $\mathscr{O}(0)$, $\mathscr{O}(1)$, $\mathscr{O}(2)$, … of $\mathscr{V}$, together with certain additional structure. We think of the object $\mathscr{O}(n)$ as a parameter space for n-ary operations, and thus we require each object $\mathscr{O}(n)$ to have a right Σ_n-action that corresponds to permutation of inputs. We also require composition operations

$$\gamma : \mathscr{O}(k) \otimes \mathscr{O}(j_1) \otimes \cdots \otimes \mathscr{O}(j_k) \to \mathscr{O}(j_1 + \cdots + j_k)$$

that formalize the composition operation $g \circ (f_1 \otimes \cdots \otimes f_k)$, and a distinguished point $\mathrm{id} : e \to \mathscr{O}(1)$ that plays the role of an identity operation. These data must satisfy natural associativity, unit, and equivariance axioms, which are most easily discerned by examining the properties of the endomorphism operad $\mathbf{End}(X)(n) = \hom(X^{\otimes n}, X)$. An $\mathscr{O}$-algebra structure on an object $X \in \mathscr{V}$ is simply a map $\mathscr{O} \to \mathbf{End}(X)$ that preserves all structure. More generally, one can consider $\mathscr{O}$-algebras in any $\mathscr{V}$-enriched category.

Our present work concerns operads and their algebras in G-categories. Note that the 1-category of all small G-categories and G-functors is cartesian closed. The product $\mathscr{C} \times \mathscr{D}$

of two G-categories is given the diagonal action and the terminal G-category $*$ consists of a single object and its identity morphism. The internal hom $\underline{\mathbf{Cat}}_G(\mathscr{C}, \mathscr{D})$ is the category of all *nonequivariant* functors $\mathscr{C} \to \mathscr{D}$ and natural transformations between them, equipped with the conjugation G-action. The nth component of the endomorphism operad of a G-category $\mathscr{C}$ is defined by $\mathbf{End}(\mathscr{C})(n) := \underline{\mathbf{Cat}}_G(\mathscr{C}^{\times n}, \mathscr{C})$.

It follows that the G-fixed subcategory of $\underline{\mathbf{Cat}}_G(\mathscr{C}, \mathscr{D})$ consists precisely of the G-functors $\mathscr{C} \to \mathscr{D}$ and the G-natural transformations between them. In particular, the G-fixed points of $\mathbf{End}(\mathscr{C})(0)$ and $\mathbf{End}(\mathscr{C})(2)$ correspond to G-fixed constants in $\mathscr{C}$ and G-bifunctors on $\mathscr{C}$, respectively. We can understand norms in similar terms. Suppose that T is a finite, ordered H-set, write $\sigma : H \to \Sigma_{|T|}$ for the permutation representation of the corresponding action on $\{1, \dots, |T|\}$, and let $\Gamma_T \subset G \times \Sigma_{|T|}$ be the subgroup

$$\Gamma_T := \{(h, \sigma(h)) \mid h \in H\}.$$

Then the Γ_T-fixed subcategory of $\mathbf{End}(\mathscr{C})(|T|)$ consists of the T-norms $\mathscr{C}^{\times T} \to \mathscr{C}$ and the H-equivariant natural transformations between them.

Consider the structure on a normed symmetric monoidal category once more. The underlying constant e and operations $\otimes$ and $\bigotimes_T$ satisfy no strict relations, but all composite operations of a fixed arity should be coherently isomorphic. Thus, the operad $\mathcal{SM}_{\mathcal{N}}$ that parametrizes $\mathcal{N}$-normed symmetric monoidal structures should have the corresponding two properties:

1. its objects should be free on a G-fixed constant, a G-equivariant binary product, and a T-norm for every $T \in \mathcal{N}$, and
2. there should be a unique isomorphism between every pair of objects of the same arity.

We shall give an explicit description of such an operad in section 2.4, and then prove that its strict algebras in G-categories are precisely the same thing as $\mathcal{N}$-normed symmetric monoidal categories (sections 2.6 – 2.8).

A priori, the operad $\mathcal{SM}_{\mathcal{N}}$ parametrizes far more structure than is needed to define an $\mathcal{N}$-normed symmetric monoidal category. Thus, it is fairly straightforward to write down an evaluation map $\mathrm{ev} : \mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgLax} \to \mathcal{N}\mathbf{SMLax}$ that picks out the relevant generating data. The work is in showing how to construct a full $\mathcal{SM}_{\mathcal{N}}$-action from the structure on a given normed symmetric monoidal category. The issue is that all diagrams in $\mathcal{SM}_{\mathcal{N}}$ commute, and thus all diagrams in the image of an operad map $|\cdot|_{\mathscr{C}} : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ must also commute. This is the sense in which theorem 2.10 is a coherence theorem.

Remark 2.11. In the language of Hill and Hopkins [23], we are constructing an N_∞ operad action out of the structure on a (pseudo) $\mathscr{O}$-commutative monoid.

There is another perspective on categorical coherence, due to Kelly. It is a standard observation that the coherence theorem for a (non-symmetric) monoidal category $\mathscr{C}$ is tantamount to an equivalence between $\mathscr{C}$ and a strictly associative monoid. Kelly's school of thought takes this as the starting point for coherence theory in general, but there are difficulties implementing these ideas equivariantly. The basic problem is that it is not possible to fully strictify a normed symmetric monoidal category, and there is no obvious "maximally strict" version of $\mathcal{N}$-normed symmetric monoidal structure for general sets of exponents $\mathcal{N}$.

2.4 The operad $\mathcal{SM}_{\mathcal{N}}$

The construction of the operad $\mathcal{SM}_{\mathcal{N}}$ splits into two steps. First, we generate a suitable free operad in G-sets, and then we formally insert isomorphisms.

2.4.1 The free G-operad on a symmetric sequence

Recall that a symmetric sequence of G-sets is a sequence $S = (S_n)_{n \geq 0}$ of left $G \times \Sigma_n$-sets, and that S is *Σ-free* if each set S_n is Σ_n-free. We have the following standard lemma.

Lemma 2.12. *Suppose that S_n is a $G \times \Sigma_n$-set and that the group Σ_n acts freely on it. If Λ is any subgroup of $G \times \Sigma_n$ for which $S_n^{\Lambda} \neq \varnothing$, then there is a unique subgroup $H \subset G$ and* *group homomorphism* $\sigma : H \to \Sigma_n$ *such that* $\Lambda = \{(h, \sigma(h)) \,|\, h \in H\}$.

Proof. The subgroup H is the image of Λ under the projection $\pi : G \times \Sigma_n \to G$. The Σ_n-freeness of S_n guarantees that for every $h \in \pi(\Lambda)$, there is exactly one $\sigma(h) \in \Sigma_n$ for which $(h, \sigma(h)) \in \Lambda$, and the closure of Λ under multiplication implies that $\sigma : H \to \Sigma_n$ is a group homomorphism. □

It follows that if S is a Σ-free symmetric sequence of G-sets, then after choosing an orbit decomposition, we have

$$S_n \cong \coprod_{T \in I_n} (G \times \Sigma_n)/\Gamma_T,$$

where I_n is a set of G-subgroup actions on n letters. We shall write $\bigotimes_T$ for the coset $e\Gamma_T$, because it represents a T-norm on algebras. From here, the free G-operad $\mathbb{F}(S)$ on S may be described as follows.

For all subgroups $H \subset G$, choose a set $\{e = g_1^H, \dots, g_{|G:H|}^H\}$ of G/H coset representatives once and for all. Consider the formal symbols below.

$$\begin{array}{rl} x_n & (n = 1, 2, 3, \dots) \\ r\bigotimes_T & (T \in \coprod_{n \geq 0} I_n \text{ an } H\text{-set, and } r \text{ a } G/H \text{ coset representative}) \\ (\ \ )\ \ , & (\text{punctuation}) \end{array}$$

By convention, $x_m = x_n$ if the numbers m and n are equal, and $r\bigotimes_T = r'\bigotimes_{T'}$ if $r = r'$ and $T = T'$. The elements of the free operad $\mathbb{F}(S)$ will be suitable finite sequences of these symbols. We start by defining *terms*:

1. every variable x_n is a term, and
2. if $t_1, \dots, t_{|T|}$ are terms, then so is $r\bigotimes_T(t_1, \dots, t_{|T|})$.

The *complexity* of a term t is the length of the longest chain of nested pairs of left and right parentheses in t. The *arity* of a term t is the number of distinct variable symbols x_i that occur in t. We say that an n-ary term t is *operadic* if each of the variables x_1, ..., x_n occur in t exactly once. Define $\mathbb{F}(S)(n)$ to be the set of all n-ary operadic terms.

We require a result to ensure that we can parse a term into its constituents. The following lemma follows by induction on complexity.

Lemma 2.13. *Suppose that t is a term. Then either*

1. *t is a variable x_n for some $n \geq 1$, or*
2. *t starts with a letter of the form $r\bigotimes_T$, has the same number of left and right parentheses, and every strict initial segment of t contains fewer right parentheses than left parentheses.*

Proposition 2.14. *If $r \bigotimes_T(t_1, \ldots, t_{|T|}) = r' \bigotimes_{T'}(t'_1, \ldots, t'_{|T'|})$, for terms t_i and t'_i, then $r = r'$, $T = T'$, and $t_i = t'_i$ for every i.*

Proof. The first letters must be the same. Now compare t_i to t'_i in succession. ☐

We deduce that every term can be expanded into a syntax tree, and conversely, all suitable syntax trees can be understood as terms. This provides a useful device for visualizing the elements of the operad $\mathbb{F}(S)$. The reader should consult [8] and [36] for a systematic discussion of these matters.

We return to the construction of $\mathbb{F}(S)$. For any permutation σ of $\{1, \ldots, n\}$ and n-ary operadic term t, we write $t\sigma$ for the n-ary operadic term obtained by replacing each variable x_i in t with $x_{\sigma^{-1}i}$. This defines a right Σ_n action on $\mathbb{F}(S)(n)$, and a left action is given by $\sigma t := t\sigma^{-1}$. Note that this is a free action.

There is also a left G-action on each of the sets $\mathbb{F}(S)(n)$. First, we construct a G-action on all terms. Fix an element $g \in G$. We define

1. $g \cdot x_n := x_n$ for all natural numbers n,
2. $g \cdot r \bigotimes_T(t_1, \ldots, t_{|T|}) := r' \bigotimes_T(g \cdot t_{\sigma(h)^{-1}1}, \ldots, g \cdot t_{\sigma(h)^{-1}|T|})$, where $gr = r'h$ for a unique G/H coset representative r' and $h \in H$, and $(h, \sigma(h)) \in \Gamma_T$.

It follows inductively that this is a G-action. Note that this action shuffles symbols around, but since G acts trivially on the x_n, it does not ultimately affect which variables appear in a given term. Therefore this action restricts to a G-action on operadic terms. Moreover, the Σ_n and G-actions interchange, because the manner in which multiplication by $g \in G$ permutes the symbols in a term t depends only on the positions of the symbols $r \bigotimes_T$ and not on the particular variables.

Given a k-ary operadic term t and j_i-ary operadic terms s_i for $i = 1, \dots, k$, the operadic term $\gamma(t; s_1, \dots, s_k) \in \mathbb{F}(S)(j_1 + \cdots + j_k)$ is obtained by

1. adding $j_1 + \cdots + j_{i-1}$ to the subscript of every variable appearing in every s_i, and then
2. substituting the terms $s_1, \dots, s_k$, with indices on variables shifted, in for the variables $x_1, \dots, x_k$ in t.

It is straightforward to check that γ is associative and that the variable x_1 is the G-fixed identity element for the operation γ. The G-equivariance of γ follows the recursive definition of the G-action. Thus $\mathbb{F}(S)$ is an operad in G-sets.

The unit map $\eta : S \to \mathbb{F}(S)$ sends the coset $e\Gamma_T \equiv \bigotimes_T$ to the term $\bigotimes_T(x_1, \dots, x_{|T|})$, and the rest is determined by $G \times \Sigma_n$-equivariance. If $f : S \to \mathscr{O}$ is any map of S into a G-operad $\mathscr{O}$, then the unique extension of f to an operad map $f : \mathbb{F}(S) \to \mathscr{O}$ is defined as follows. Every term t can be written uniquely as $t = \bar{t}\sigma$, where the variables in $\bar{t}$ appear in ascending order as we read from left to right. The term $\bar{t}$ is an operadic composite of terms of the form $r \bigotimes_T(x_1, \dots, x_{|T|}) = \eta(r \bigotimes_T)$ alone. For the sake of definiteness, we start with the leftmost operation, then use partial composition products $\circ_i$ to insert the first layer of operations from left to right, then do the same for the second layer, and so on. We define $f(t)$ to be the corresponding sequence of partial composites of $f(r \bigotimes_T)$'s, followed by $(-)\sigma$.

We conclude this section with an alternative description of the free operad $\mathbb{F}(S)$. It is more canonical than the above, and it works for non Σ-free symmetric sequences S, but one must contend with equivalence classes of terms.

Construction 2.15. Suppose that S is a symmetric sequence of G-sets. To construct $\mathbb{F}(S)$, consider the symbols

$$\begin{array}{rl} x_n & n = 1, 2, 3, \ldots \\ a & a \in S \\ (\ \ )\ \ , & \text{(punctuation)} \end{array}$$

and define terms, operadic terms, and the nonequivariant operad structure just as before. Define the G-action by $g \cdot x_n = x_n$ and $g \cdot a(t_1, \ldots, t_n) = ga(g \cdot t_1, \ldots, g \cdot t_n)$. We obtain a G-operad, but it is larger than $\mathbb{F}(S)$, and the natural candidate for the unit map $\eta : S \to \mathbb{F}(S)$ is not Σ-equivariant.

Next, write tEt' if the term t' can be obtained from t by replacing a subterm of the form

$$a\sigma(s_1, \ldots, s_n) \quad \text{with} \quad a(s_{\sigma^{-1}1}, \ldots, s_{\sigma^{-1}n}),$$

and write $t \equiv t'$ if t' can be obtained from t through a finite sequence of such transformations. This relation repsects the G-operad structure, and the resulting quotient is the free operad on S (cf. section 4.3 for a more detailed discussion of quotient operads). The unit η sends an n-ary element a to the congruence class $[a(x_1, \ldots, x_n)]$. The previous description of the free operad is obtained by restricting the elements a to a set of Σ-orbit representatives. The key observation is that when S is Σ-free, restricting the symbols a in this manner completely determines the representing term.

2.4.2 Specialization to $\mathcal{SM}_{\mathcal{N}}$

For any set of exponents $\mathcal{N}$, let $S_{\mathcal{N}}$ be the symmetric sequence given by the graded coproduct

$$S_{\mathcal{N}} := (G \times \Sigma_0)/G \ \sqcup \ (G \times \Sigma_2)/G \ \sqcup \ \coprod_{T \in \mathcal{N}} (G \times \Sigma_{|T|})/\Gamma_T.$$

As before, we write e for the coset $eG \in (G \times \Sigma_0)/G$, we write $\otimes$ for the coset $eG \in (G \times \Sigma_2)/G$, and we write $\bigotimes_T$ for the coset $e\Gamma_T \in (G \times \Sigma_{|T|})/\Gamma_T$.

We define the object operad of $\mathcal{SM}_{\mathcal{N}}$ to be the free operad $\mathbb{F}(S_{\mathcal{N}})$. Its elements are operadic terms built from the formal symbols

$$\begin{array}{rl} x_n & (n = 1, 2, \dots) \\ e & \\ \otimes & \\ r\bigotimes_T & (T \in \mathcal{N} \text{ an } H\text{-set and } r \text{ a } G/H \text{ coset representative}) \\ (\quad)\quad , & (\text{punctuation}) \end{array}$$

and its G-operad structure is just as before, provided that we understand e and $\otimes$ to be nullary and binary G-trivial norms, respectively. Concretely, this means that in the recursive definition of the G-action, we set

3. $g \cdot e() := e()$, and
4. $g \cdot \otimes(t_1, t_2) := \otimes(g \cdot t_1, g \cdot t_2)$.

Everything else is exactly the same.

From here, we introduce a unique isomorphism between every pair of elements of the same arity in $\mathbb{F}(S_{\mathcal{N}})$. This is accomplished as follows.

Definition 2.16. For any set X, let $\widetilde{X}$ denote the category whose object set is X, and which has a unique morphism $(x, y) : x \to y$ for every pair of objects $x, y \in X$. We shall sometimes call $\widetilde{X}$ the *homogenization* of X. In general, we shall say that a category $\mathscr{C}$ is *homogeneous* if there is a unique morphism $! : x \to y$ for every pair of objects $x, y \in \mathscr{C}$.

The morphisms $x \to y$ and $y \to x$ are inverse, and therefore all pairs of objects in $\widetilde{X}$ are uniquely isomorphic. There is an adjunction $\mathrm{Ob} : \mathbf{Cat} \rightleftarrows \mathbf{Set} : \widetilde{(-)}$, and hence an induced adjunction $\mathrm{Ob} : G\mathbf{Cat} \rightleftarrows G\mathbf{Set} : \widetilde{(-)}$ on categories of G-objects. It follows that $\widetilde{(-)}$ preserves products, and thus it takes operads in G-sets to operads in G-categories.

Definition 2.17. The operad $\mathcal{SM}_{\mathcal{N}}$ is defined by $\mathcal{SM}_{\mathcal{N}}(n) := \mathbb{F}(\widetilde{S_{\mathcal{N}})}(n)$.

Remark 2.18. Note that in [19, definition 1.4], $\widetilde{X}$ is called the *chaotic category* on X, the intuition being that every object of $\widetilde{X}$ is the same as every other object. We have kept their notation, but we shall usually avoid that nomenclature because when X is a G-set, different objects generally have different isotropy groups. While the objects of $\widetilde{X}$ are nonequivariantly isomorphic, we find that the structure on $\widetilde{X}$ loosely resembles the tangent bundle of a homogeneous space, hence our terminology.

2.5 2-categories of algebras over operads

There are 2-categories of $\mathcal{SM}_{\mathcal{N}}$-algebras that are precisely analogous to the 2-categories of $\mathcal{N}$-normed symmetric monoidal categories considered earlier. We review the definitions here. We find it easier to work in adjoint form to [10] and [20], and there are some minor differences between our definitions. Hence we shall give complete details.

Notation 2.19. We shall use $\bullet$ to denote vertical composition of natural transformations, and $\circ$ to denote horizontal composition of functors and natural transformations.

Definition 2.20. Suppose that $\mathscr{O}$ is an operad in $G\mathbf{Cat}$. A *strict $\mathscr{O}$-algebra* in $G\mathbf{Cat}$ is a G-category $\mathscr{C}$, equipped with an operad map $|\cdot|_{\mathscr{C}} : \mathscr{O} \to \mathbf{End}(\mathscr{C})$.

We think of the map $|\cdot|_{\mathscr{C}}$ as realizing an abstract symbol as an operation on $\mathscr{C}$.

Definition 2.21. Suppose that $\mathscr{O}$ is an operad in $G\mathbf{Cat}$, and that $\mathscr{C}$ and $\mathscr{D}$ are strict $\mathscr{O}$-algebras in $G\mathbf{Cat}$. A *lax $\mathscr{O}$-algebra morphism* $(F, \partial_\bullet) : \mathscr{C} \to \mathscr{D}$ consists of:

1. a G-functor $F : \mathscr{C} \to \mathscr{D}$, and
2. for each $n \geq 0$ and $x \in \mathscr{O}(n)$, a natural transformation $(\partial_n)_x : |x|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F \circ |x|_{\mathscr{C}}$,

which are required to satisfy the following conditions.

(i) for each $n \geq 0$, the maps $(\partial_n)_x$ vary naturally in $x \in \mathscr{O}(n)$,

(ii) for each $n \geq 0$, $x \in \mathscr{O}(n)$, and $(g,\sigma) \in G \times \Sigma_n$, the equation $(g,\sigma) \cdot (\partial_n)_x = (\partial_n)_{(g,\sigma)\cdot x}$ holds, i.e. $(\partial_n)_x$ is $(G \times \Sigma_n)$-equivariant in x,

(iii) $(\partial_1)_{\mathrm{id}} = \mathrm{id}_F : |\mathrm{id}|_{\mathscr{D}} \circ F^{\times 1} \Rightarrow F \circ |\mathrm{id}|_{\mathscr{C}}$, and

(iv) for any $y \in \mathscr{O}(m)$ and $x_i \in \mathscr{O}(k_i)$, the transformations $(\partial_{k_1+\cdots+k_m})_{\gamma(y;x_1,\dots,x_m)}$ and $\Big[(\partial_m)_y \circ \Big(\mathrm{id}_{|x_1|_{\mathscr{C}}} \times \cdots \times \mathrm{id}_{|x_m|_{\mathscr{C}}}\Big)\Big] \bullet \Big[\mathrm{id}_{|y|_{\mathscr{D}}} \circ \Big((\partial_{k_1})_{x_1} \times \cdots \times (\partial_{k_m})_{x_m}\Big)\Big]$ are equal natural transformations $|\gamma(y;x_\bullet)|_{\mathscr{D}} \circ F^{\times \Sigma k_\bullet} \Rightarrow F \circ |\gamma(y;x_\bullet)|_{\mathscr{C}}$.

A *pseudomorphism* (resp. *strict morphism*) is a lax morphism such that $(\partial_n)_x$ is an isomorphism (resp. identity) for every $n \geq 0$ and $x \in \mathscr{O}(n)$.

Remark 2.22. Our pseudomorphisms are closely related, but not identical, to the pseudomorphisms considered in [10] and [20]. Conditions (iii) and (iv) correspond to the pasting diagrams in [10, definition 2.4], but we have enforced additional equivariance in (ii). On the other hand, [20] only considers pseudomorphims between algebras over a reduced operad, and they require their morphisms to preserve basepoints strictly.

As we now explain, conditions (ii) – (iv) essentially state that the assignment $x \mapsto (\partial_n)_x$ is an operad map.

Definition 2.23. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are strict $\mathscr{O}$-algebras and that $F : \mathscr{C} \to \mathscr{D}$ is a G-functor. We define an operad $\mathcal{L}ax = \mathcal{L}ax(\mathscr{O},\mathscr{C},\mathscr{D},F)$ in $G\mathbf{Set}$ as follows.

1. For each integer $n \geq 0$, let $\mathcal{L}ax(n)$ be the set of pairs (x,ξ), where $x \in \mathscr{O}(n)$ and $\xi : |x|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F \circ |x|_{\mathscr{C}}$ is a nonequivariant natural transformation in $\underline{\mathbf{Cat}}_G(\mathscr{C}^{\times n},\mathscr{D})$. The $G \times \Sigma_n$-action is $(g,\sigma)\cdot(x,\xi) = \big((g,\sigma)\cdot x, (g,\sigma)\cdot\xi\big)$.
2. Define the identity for $\mathcal{L}ax$ to be the pair $(\mathrm{id}, \mathrm{id}_F)$.

3. Define composition maps

$$\gamma_{\mathcal{L}} : \mathcal{L}ax(k) \times \mathcal{L}ax(j_1) \times \cdots \times \mathcal{L}ax(j_k) \to \mathcal{L}ax(j_1 + \cdots + j_k)$$

   by setting $\gamma_{\mathcal{L}}((z,\zeta);(x_1,\xi_1),\dots,(x_k,\xi_k))$ equal to

$$\Big(\gamma_{\mathscr{O}}(z;x_1,\dots,x_k)\ ,\ [\zeta \circ (\mathrm{id}_{|x_1|_{\mathscr{C}}} \times \cdots \times \mathrm{id}_{|x_k|_{\mathscr{C}}})] \bullet [\mathrm{id}_{|z|_{\mathscr{D}}} \circ (\xi_1 \times \cdots \times \xi_k)]\Big).$$

The first coordinate projection defines a map $\pi_1 : \mathcal{L}ax \to \mathrm{Ob}(\mathscr{O})$ of G-operads.

Remark 2.24. There are suboperads $\mathrm{Ob}\mathscr{O} \cong \mathcal{S}t \subset \mathcal{P}s \subset \mathcal{L}ax$ obtained by restricting all ξ's to be identity transformations and natural isomorphisms, respectively.

The following is a quick check of definitions.

Lemma 2.25. *Suppose that $\mathscr{C}$ and $\mathscr{D}$ are strict $\mathscr{O}$-algebras, that $F : \mathscr{C} \to \mathscr{D}$ is a G-functor, and that for each $n \geq 0$ and $x \in \mathscr{O}(n)$, we are given a natural transformation $(\partial_n)_x : |x|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F \circ |x|_{\mathscr{C}}$. Then conditions (ii) – (iv) of definition 2.21 hold if and only if the section $s : \mathrm{Ob}(\mathscr{O}) \to \mathcal{L}ax$ of $\pi_1 : \mathcal{L}ax \to \mathrm{Ob}(\mathscr{O})$ defined by $s_n(x) := (x, (\partial_n)_x)$ is a map of G-operads.*

Remark 2.26. If $\mathscr{O} = \mathcal{SM}_{\mathcal{N}}$, then the object operad $\mathrm{Ob}(\mathcal{SM}_{\mathcal{N}}) = \mathbb{F}(S_{\mathcal{N}})$ is free, and conditions (ii) – (iv) are easily satisfied. In this case, the nontrivial coherence condition in definition 2.21 is the naturality of $(\partial_n)_x$ in all x.

Definition 2.27. Suppose that $\mathscr{O}$ is an operad in G**Cat**, that $\mathscr{C}$ and $\mathscr{D}$ are strict $\mathscr{O}$-algebras, and that $(F,\partial_\bullet),(F',\partial'_\bullet) : \mathscr{C} \rightrightarrows \mathscr{D}$ is a pair of lax $\mathscr{O}$-algebra morphisms. An *$\mathscr{O}$-transformation* $\omega : (F,\partial_\bullet) \Rightarrow (F',\partial'_\bullet)$ is a G-natural transformation $\omega : F \Rightarrow F'$ such that for every $n \geq 0$ and $x \in \mathscr{O}(n)$, the maps $(\omega \circ \mathrm{id}_{|x|_{\mathscr{C}}}) \bullet (\partial_n)_x$ and $(\partial'_n)_x \bullet (\mathrm{id}_{|x|_{\mathscr{D}}} \circ \omega^{\times n})$ are equal natural transformations $|x|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F' \circ |x|_{\mathscr{C}}$.

As with normed symmetric monoidal categories, the 2-category structure on $G\underline{\mathbf{Cat}}$ lifts to $\mathscr{O}$-algebras.

Notation 2.28. Let $\mathscr{O}$-**AlgLax** be the 2-category of all strict $\mathscr{O}$-algebras in G**Cat**, lax $\mathscr{O}$-morphisms, and $\mathscr{O}$-transformations between them. The composite of lax $\mathscr{O}$-morphisms $(G, \partial_\bullet) \circ (F, \varepsilon_\bullet)$ is obtained by composing underlying functors and comparison data, e.g.

$$[\mathrm{id}_G \circ (\partial_n)_x] \bullet [(\varepsilon_n)_x \circ \mathrm{id}_{F^{\times n}}] : |x| \circ G^{\times n} \circ F^{\times n} \Rightarrow G \circ |x| \circ F^{\times n} \Rightarrow G \circ F \circ |x|,$$

and the vertical and horizontal composites of transformations are computed in $G\underline{\mathbf{Cat}}$. Identities of both sorts are also inherited from $G\underline{\mathbf{Cat}}$. There are sub-2-categories $\mathscr{O}$-**AlgSt** $\subset$ $\mathscr{O}$-**AlgPs** $\subset$ $\mathscr{O}$-**AlgLax** of pseudo and strict morphisms, and there is a forgetful 2-functor $\mathscr{O}$-**AlgLax** $\to G\underline{\mathbf{Cat}}$.

2.6 The evaluation 2-functor

In this section, we define the evaluation 2-functor ev : $\mathcal{SM}_{\mathcal{N}}$-**AlgLax** $\to \mathcal{N}$**SMLax**. This part is straightforward. The more difficult task is constructing its inverse, and we explain how to do that in the next section.

ev on categories

Suppose that $\mathscr{C}$ is a strict $\mathcal{SM}_{\mathcal{N}}$-algebra and let $|\cdot| : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ be the corresponding structure map. We can extract a $\mathcal{N}$-normed symmetric monoidal structure on $\mathscr{C}$ as follows. First, consider the values of $|\cdot|$ on the generators of $\mathcal{SM}_{\mathcal{N}}$. Define

1. $\otimes^{\mathscr{C}} := |\otimes(x_1, x_2)| : \mathscr{C}^{\times 2} \to \mathscr{C}$,
2. $e^{\mathscr{C}} := |e()| : \mathscr{C}^{\times 0} \to \mathscr{C}$, and
3. $\bigotimes_T^{\mathscr{C}} := \left| \bigotimes_T(x_1, \dots, x_{|T|}) \right| : \mathscr{C}^{\times T} \to \mathscr{C}$ for all $T \in \mathcal{N}$.

To get coherence isomorphisms, we evaluate $|\cdot|$ on the relevant morphisms in $\mathcal{SM}_{\mathcal{N}}$.

4. $\alpha^{\mathscr{C}} := |\otimes(\otimes(x_1, x_2), x_3) \to \otimes(x_1, \otimes(x_2, x_3))|$,

5. $\lambda^{\mathscr{C}} := |\otimes(e(), x_1) \to x_1|$,

6. $\rho^{\mathscr{C}} := |\otimes(x_1, e()) \to x_1|$,

7. $\beta^{\mathscr{C}} := |\otimes(x_1, x_2) \to \otimes(x_2, x_1)|$, and

8. $\upsilon_T^{\mathscr{C}} := \left| \bigotimes_T(x_1, \ldots, x_{|T|}) \to \otimes(\otimes(\ldots \otimes (\otimes(x_1, x_2), x_3) \ldots, x_{|T|-1}), x_{|T|}) \right|$ for $T \in \mathcal{N}$.

We let ev$\mathscr{C}$ denote the G-category $\mathscr{C}$, together with the functors and natural transformations specified above. These data satisfy the commutativity conditions for a normed symmetric monoidal category.

Proof. Every diagram in $\mathcal{SM}_{\mathcal{N}}$ commutes, and $|\cdot|$ is a map of operads in G-categories. For example, the pentagon axiom for $\otimes$ comes from a pentagon in $\mathcal{SM}_{\mathcal{N}}(4)$ whose vertices are $\otimes(\otimes(\otimes(x_1, x_2), x_3), x_4)$, $\otimes(\otimes(x_1, x_2), \otimes(x_3, x_4))$, etc. The other ordinary symmetric monoidal axioms are visible in $\mathcal{SM}_{\mathcal{N}}(1) - \mathcal{SM}_{\mathcal{N}}(3)$.

Twisted equivariance for υ_T can be deduced from a diagram in $\mathcal{SM}_{\mathcal{N}}(|T|)$ as follows. Write $\bigotimes_n(x_1, \ldots, x_n)$ as shorthand for the n-ary term

$$\otimes(\otimes(\ldots \otimes (\otimes(x_1, x_2), x_3) \ldots, x_{n-1}), x_n).$$

Given any H-set $T \in \mathcal{N}$ and group element $h \in H$, there is a commutative diagram

$$\begin{tikzcd}
h \cdot \bigotimes_T(x_1, \ldots, x_{|T|}) \arrow[r, "\mathrm{id}"] \arrow[dd] & \bigotimes_T(x_{\sigma(h)^{-1}1}, \ldots, x_{\sigma(h)^{-1}|T|}) \arrow[d] \\
 & \bigotimes_{|T|}(x_{\sigma(h)^{-1}1}, \ldots, x_{\sigma(h)^{-1}|T|}) \arrow[d] \\
h \cdot \bigotimes_{|T|}(x_1, \ldots, x_{|T|}) \arrow[r, "\mathrm{id}"'] & \bigotimes_{|T|}(x_1, \ldots, x_{|T|})
\end{tikzcd}$$

in $\mathcal{SM}_{\mathcal{N}}(|T|)$, where $(h, \sigma(h)) \in \Gamma_T$. The left vertical arrow maps to $h \cdot \upsilon_T^{\mathscr{C}}$, which is the untwistor $\upsilon_T^{\mathscr{C}}$ conjugated by h. The upper right vertical arrow maps to $\upsilon_T^{\mathscr{C}} \cdot \sigma(h)$, which is the untwistor $\upsilon_T^{\mathscr{C}}$ with its inputs permuted by $\sigma(h)$. The lower right vertical arrow can be factored as a sequence of associativity and commutativity arrows in $\mathcal{SM}_{\mathcal{N}}$, and thus it maps to the symmetric monoidal coherence isomorphism for $\mathscr{C}$ that permutes the factors of $\bigotimes_{|T|}^{\mathscr{C}}$ by $\sigma(h)^{-1}$. Thus, the image of this square under $|\cdot|$ produces the twisted equivariance diagrams for $\upsilon_T^{\mathscr{C}}$, once we evaluate at tuples of the form $(hC_1, \dots, hC_{|T|})$. $\square$

ev on functors

Given a lax $\mathcal{SM}_{\mathcal{N}}$-morphism $(F, \partial_\bullet) : (\mathscr{C}, |\cdot|_{\mathscr{C}}) \to (\mathscr{D}, |\cdot|_{\mathscr{D}})$ between strict $\mathcal{SM}_{\mathcal{N}}$-algebras, we obtain a lax monoidal functor $\mathrm{ev}F : \mathrm{ev}\mathscr{C} \to \mathrm{ev}\mathscr{D}$ by taking

1. the G-functor $F : \mathscr{C} \to \mathscr{D}$,
2. the G-fixed morphism $F_e := (\partial_0)_{e()}$,
3. the G-transformation $F_\otimes := (\partial_2)_{\otimes(x_1,x_2)}$, and
4. the transformation $F_{\bigotimes_T} := (\partial_{|T|})_{\bigotimes_T(x_1,\dots,x_{|T|})}$ for every H-set $T \in \mathcal{N}$.

Note that the natural transformations above have the required equivariance because $(\partial_n)_x$ is $G \times \Sigma_n$-equivariant in x. Naturality of $\partial_\bullet$ with respect to the morphisms $\otimes(\otimes(x_1,x_2),x_3) \to \otimes(x_1, \otimes(x_2,x_3))$, $\otimes(e(), x_1) \to x_1$, etc., together with the compatibility of $\partial_\bullet$ with composition imply that these data are a lax monoidal functor $(F, F_\bullet) : \mathrm{ev}\mathscr{C} \to \mathrm{ev}\mathscr{D}$.

ev on transformations

Suppose that $(\mathscr{C}, |\cdot|_{\mathscr{C}})$ and $(\mathscr{D}, |\cdot|_{\mathscr{D}})$ are strict $\mathcal{SM}_{\mathcal{N}}$-algebras, that $(F, \partial_\bullet), (F', \partial'_\bullet) : \mathscr{C} \rightrightarrows \mathscr{D}$ is a pair of lax maps, and that $\omega : (F, \partial_\bullet) \Rightarrow (F', \partial'_\bullet)$ is a $\mathcal{SM}_{\mathcal{N}}$-transformation between them. Then ω is just a G-natural transformation $F \Rightarrow F'$ that is compatible with $\partial_\bullet$ and $\partial'_\bullet$. By specializing to $(\partial_0)_{e()}$, $(\partial_2)_{\otimes(x_1,x_2)}$, and $(\partial_{|T|})_{\bigotimes_T(x_1,\dots,x_{|T|})}$, we deduce that ω also is a $\mathcal{N}$-normed monoidal transformation $\omega : \mathrm{ev}F \Rightarrow \mathrm{ev}F'$.

2.7 The inverse to the evaluation 2-functor

In this section, we describe the inverse to the 2-functor $\mathrm{ev} : \mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgLax} \to \mathcal{N}\mathbf{SMLax}$.

The inverse for categories

Suppose that

$$\left(\mathscr{C}, \otimes^{\mathscr{C}}, e^{\mathscr{C}}, (\bigotimes_T^{\mathscr{C}})_{T\in\mathcal{N}}, \alpha^{\mathscr{C}}, \lambda^{\mathscr{C}}, \rho^{\mathscr{C}}, \beta^{\mathscr{C}}, (\upsilon_T^{\mathscr{C}})_{T\in\mathcal{N}}\right)$$

is a $\mathcal{N}$-normed symmetric monoidal category, and consider the endomorphism operad $\mathbf{End}(\mathscr{C})$ of $\mathscr{C}$. We begin with some formalities. There is a map of symmetric sequences $|\cdot| : S_{\mathcal{N}} \to \mathrm{Ob}(\mathbf{End}(\mathscr{C}))$ that sends the generating cosets of $S_{\mathcal{N}}$ to the corresponding operations on $\mathscr{C}$. By adjunction, we obtain an operad map $|\cdot| : \mathbb{F}(S_{\mathcal{N}}) \to \mathrm{Ob}(\mathbf{End}(\mathscr{C}))$ such that $|e()| = e^{\mathscr{C}}$, $|\otimes(x_1, x_2)| = \otimes^{\mathscr{C}}$, and $\left|\bigotimes_T(x_1, \ldots, x_{|T|})\right| = \bigotimes_T^{\mathscr{C}}$ for every $T \in \mathcal{N}$. The remainder of the construction consists of the following two steps:

(i) defining a functorial extension of $|\cdot| : \mathbb{F}(S_{\mathcal{N}})(n) \to \mathbf{End}(\mathscr{C})(n)$ to the category $\mathcal{SM}_{\mathcal{N}}(n)$ for every $n \geq 0$, and

(ii) proving that this levelwise functorial extension of $|\cdot|$ to $\mathcal{SM}_{\mathcal{N}}$ is a map of operads in G-categories.

We outline both of these steps below, and the details are treated in section 2.8.

For the first task, the idea is to embed $\mathbb{F}(S)(n)$ as the vertex set of a directed graph $\mathbf{Bas}(n)$, whose edges represent "basic maps" between the n-ary operations in $\mathbb{F}(S_{\mathcal{N}})(n)$. Roughly speaking, a basic map $|t| \Rightarrow |t'|$ is a natural isomorphism that changes a subterm of t using a single instance of $\alpha^{\pm 1}$, $\lambda^{\pm 1}$, etc. For every $n \geq 0$, there is an extension of $|\cdot| : \mathbb{F}(S)(n) \to \mathbf{End}(\mathscr{C})(n)$ to a map

$$|\cdot| : \mathbf{Bas}(n) \to \mathbf{End}(\mathscr{C})(n)$$

of directed graphs, and by adjunction, we obtain a functor

$$|\cdot| : \mathbf{Fr}(\mathbf{Bas}(n)) \to \mathbf{End}(\mathscr{C})(n)$$

out of the free category on $\mathbf{Bas}(n)$. The free category $\mathbf{Fr}(\mathbf{Bas}(n))$ is too large, but we can adapt Mac Lane's techniques [31, Ch. VII.2] to prove that for any prescribed source and target vertices $t, t' \in \mathbf{Fr}(\mathbf{Bas}(n))$, all paths of basic edges starting at t and ending at t' have the same image under $|\cdot|$. We call this common value a "canonical isomorphism" of $\mathscr{C}$. Thus, the functor $|\cdot| : \mathbf{Fr}(\mathbf{Bas}(n)) \to \mathbf{End}(\mathscr{C})(n)$ descends to the quotient of $\mathbf{Fr}(\mathbf{Bas}(n))$ obtained by identifying all pairs of parallel morphisms, and this quotient is $\mathcal{SM}_{\mathcal{N}}(n)$. The map $|\cdot| : \mathcal{SM}_{\mathcal{N}}(n) \to \mathbf{End}(\mathscr{C})(n)$ sends the unique morphism $t \to t'$ in $\mathcal{SM}_{\mathcal{N}}$ to the canonical isomorphism can : $|t| \Rightarrow |t'|$.

For the second task, the essential point is that the canonical isomorphisms of $\mathscr{C}$ are closed under the G-operad structure on $\mathbf{End}(\mathscr{C})$. Closure under the nonequivariant operad structure follows easily from the definition of a basic map, and closure under the G-action follows from the G-equivariance and the twisted equivariance of the coherence isomorphisms for $\mathscr{C}$. From here, one uses the uniqueness of canonical maps in $\mathscr{C}$ to deduce that $|\cdot| : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ preserves the G-operad structure.

The inverse for functors

Suppose that $\mathscr{C}$ and $\mathscr{D}$ are $\mathcal{N}$-normed symmetric monoidal categories and let $|\cdot|_{\mathscr{C}} : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ and $|\cdot|_{\mathscr{D}} : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{D})$ be the corresponding $\mathcal{SM}_{\mathcal{N}}$-algebra structures described above. Given any lax monoidal morphism $(F, F_\bullet) : \mathscr{C} \to \mathscr{D}$, we construct a lax $\mathcal{SM}_{\mathcal{N}}$-morphism $(F, \partial_\bullet) : (\mathscr{C}, |\cdot|_{\mathscr{C}}) \to (\mathscr{D}, |\cdot|_{\mathscr{D}})$ as follows.

First, note that the isotropy conditions on the maps F_e, $F_\otimes$, and $F_{\bigotimes_T}$ imply that there is a G-operad map $\Phi : \mathbb{F}(S_{\mathcal{N}}) \to \mathcal{L}ax(\mathcal{SM}_{\mathcal{N}}, \mathscr{C}, \mathscr{D}, F)$ such that $\Phi_0(e()) = (e(), F_e)$, $\Phi_2(\otimes(x_1, x_2)) = (\otimes(x_1, x_2), F_\otimes)$, and $\Phi_{|T|}(\bigotimes_T(x_1, \ldots, x_{|T|})) = (\bigotimes_T(x_1, \ldots, x_{|T|}), F_{\bigotimes_T})$

for every $T \in \mathcal{N}$. We define $(\partial_n)_x := \pi_2(\Phi_n(x))$. Noting that the map $\pi_1 \circ \Phi : \mathbb{F}(S_{\mathcal{N}}) \to \mathbb{F}(S_{\mathcal{N}})$ fixes the generators of $\mathbb{F}(S_{\mathcal{N}})$, it follows $\Phi_n(x) = (x, (\partial_n)_x)$ for all $x \in \mathcal{SM}_{\mathcal{N}}$, and hence $(\partial_n)_x$ is a map $|x|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F \circ |x|_{\mathscr{C}}$ for all $n \geq 0$ and $x \in \mathcal{SM}_{\mathcal{N}}(n)$. Applying lemma 2.25 shows that $(F, \partial_\bullet)$ satisfies (ii) – (iv) of definition 2.21.

It remains to check that $(\partial_n)_x$ is natural in $x \in \mathcal{SM}_{\mathcal{N}}(n)$ for every $n \geq 0$. However, it is enough to check that every $(\partial_n)_\bullet$ is natural with respect to residue classes of basic edges, since such morphisms generate the components of $\mathcal{SM}_{\mathcal{N}}$ as G-categories. So, suppose $e : x \to x'$ is such an edge. We argue inductively on the complexity of the domain. The edge e modifies a subterm of x, and that subterm either contains the first letter of x, or it does not. In the latter case, the recursive definition of $\partial_\bullet$ allows us to reduce to a subterm, and the conclusion follows by induction. In the former case, the result follows from the compatibility of F_e, $F_\otimes$, and $F_{\bigotimes_T}$ with the coherence data for $\mathscr{C}$ and $\mathscr{D}$.

The inverse for transformations

Suppose that $(F, F_\bullet), (F', F'_\bullet) : \mathscr{C} \rightrightarrows \mathscr{D}$ are a pair of lax $\mathcal{N}$-normed functors, and let $(F, \partial_\bullet), (F', \partial'_\bullet) : (\mathscr{C}, |\cdot|_{\mathscr{C}}) \rightrightarrows (\mathscr{D}, |\cdot|_{\mathscr{D}})$ be the corresponding pair of lax $\mathcal{SM}_{\mathcal{N}}$-morphisms constructed above. If $\omega : (F, F_\bullet) \Rightarrow (F', F'_\bullet)$ is any $\mathcal{N}$-normed monoidal transformation, then ω also is a $\mathcal{SM}_{\mathcal{N}}$-transformation $\omega : (F, \partial_\bullet) \Rightarrow (F', \partial'_\bullet)$. Indeed, the set of all $x \in \mathcal{SM}_{\mathcal{N}}$ for which the equation $(\partial'_n)_x \bullet (\mathrm{id}_{|x|_{\mathscr{D}}} \circ \omega^{\times n}) = (\omega \circ \mathrm{id}_{|x|_{\mathscr{C}}}) \bullet (\partial_n)_x$ holds is closed under the $(G \times \Sigma_\bullet)$-action and the operad structure on $\mathcal{SM}_{\mathcal{N}}$. Here n is the arity of x. Thus, if $\omega : (F, F_\bullet) \Rightarrow (F', F'_\bullet)$ is monoidal, then the preceding equality holds for $x = e()$, $\otimes(x_1, x_2)$, and $\bigotimes_T(x_1, \dots, x_{|T|})$, and these terms generate the operad $\mathrm{Ob}(\mathcal{SM}_{\mathcal{N}}) = \mathbb{F}(S_{\mathcal{N}})$.

2.8 Appendix: the construction of $\mathcal{SM}_{\mathcal{N}}$-actions

In this section, we explain the formal construction of an operad map $|\cdot| : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ from an $\mathcal{N}$-normed symmetric monoidal structure on $\mathscr{C}$, thus completing the proof of theorem 2.10. We take the operad map $|\cdot| : \mathbb{F}(S_{\mathcal{N}}) \to \mathrm{Ob}(\mathbf{End}(\mathscr{C})$ defined by $|e()| = e^{\mathscr{C}}$, $|\otimes(x_1, x_2)| = \otimes^{\mathscr{C}}$, and $\left|\bigotimes_T(x_1, \ldots, x_{|T|})\right| = \bigotimes_T^{\mathscr{C}}$ as our starting point.

The morphisms in $\mathcal{SM}_{\mathcal{N}}$ parametrize coherence isomorphisms, which are generated by the coherence data for $\mathscr{C}$. We say that a coherence isomorphism is a "basic map" if it modifies an operation using a single instance of $(\alpha^{\mathscr{C}})^{\pm 1}$, $(\lambda^{\mathscr{C}})^{\pm 1}$, etc. General coherence isomorphisms are composites of basic maps. Thus, to define $\mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$, we start by introducing formal lifts of basic maps to $\mathbb{F}(S_{\mathcal{N}})$.

Definition 2.29. A *basic edge* is a pair of terms $t, t' \in \mathbb{F}(S_{\mathcal{N}})$ and a chosen subterm $s \subset t$, such that t' is obtained from t by modifying s in one of the following ways.

$$\begin{aligned}
\text{id-basic:}\quad & s \rightsquigarrow s \quad \text{(no change)} \\
\alpha\text{-basic:}\quad & s = \otimes(\otimes(s_1, s_2), s_3) \rightsquigarrow \otimes(s_1, \otimes(s_2, s_3)) \\
\lambda\text{-basic:}\quad & s = \otimes(e(), s_1) \rightsquigarrow s_1 \\
\rho\text{-basic:}\quad & s = \otimes(s_1, e()) \rightsquigarrow s_1 \\
\beta\text{-basic:}\quad & s = \otimes(s_1, s_2) \rightsquigarrow \otimes(s_2, s_1) \\
\upsilon\text{-basic:}\quad & s = g_i^H \bigotimes_T(s_1, \ldots, s_{|T|}) \rightsquigarrow \otimes(\cdots \otimes (\otimes(s_1, s_2), s_3) \cdots s_{|T|})
\end{aligned}$$

where, in the last line, T is an H-set in $\mathcal{N}$ and g_i^H is one of the chosen G/H coset representatives. One defines α^{-1}, λ^{-1}, ρ^{-1}, and υ^{-1}-basic edges similarly. Note that we regard the location of s in t, and the type of modification (id, $\alpha^{\pm 1}$, $\lambda^{\pm 1}$, etc.) as part of the data of a basic edge.

The directed graph $\mathbf{Bas}(n)$ consists of the terms in $\mathbb{F}(S_{\mathcal{N}})(n)$ and the basic edges between them. We extend the set map $|\cdot| : \mathbb{F}(S_{\mathcal{N}})(n) \to \mathrm{Ob}(\mathbf{End}(\mathscr{C})(n))$ to a morphism of directed graphs using the coherence isomorphisms for $\mathscr{C}$. By adjunction, we obtain a functor $|\cdot| : \mathbf{Fr}(\mathbf{Bas}(n)) \to \mathbf{End}(\mathscr{C})(n))$ out of the free category.

Notation 2.30. As before, we shall use $\bullet$ to denote vertical composition in $\mathbf{End}(\mathscr{C})(n)$, in contrast to the horizontal composition $\circ$ that makes $\mathbf{End}(\mathscr{C})$ into an operad. For consistency, we shall also use $\bullet$ for composition in $\mathbf{Fr}(\mathbf{Bas}(n))$.

We now prove that the functor $|\cdot| : \mathbf{Fr}(\mathbf{Bas}(n)) \to \mathbf{End}(\mathscr{C})(n)$ factors through the quotient category $\mathbf{Fr}(\mathbf{Bas}(n))/\langle p \sim q \,|\, p,\, q \text{ parallel}\rangle \cong \mathcal{SM}_{\mathcal{N}}(n)$, borrowing techniques and results from Mac Lane. The strategy is to reduce the problem to the nonequivariant case by separating υ-basic maps out from the rest of a composite of basic edges. We begin with the following interchange lemma.

Lemma 2.31. *Suppose that* $r \xrightarrow{e} s \xrightarrow{u} t$ *is a composable pair of basic edges in* $\mathbf{Bas}(n)$*, and*

(a) the edge e *is* ε*-basic, where* ε *is one of* $\alpha^{\pm 1}$*,* $\lambda^{\pm 1}$*,* $\rho^{\pm 1}$*, or* β*, and*

(b) the edge u *is* υ*-basic.*

Then there is a composable pair of basic edges $r \xrightarrow{u'} s' \xrightarrow{e'} t$ *such that*

(i) the edge e' *is* ε*-basic,*

(ii) the edge u' *is* υ*-basic, and*

(iii) $|e'| \bullet |u'| = |u| \bullet |e|$.

Proof. The edge u' modifies r according u, and the edge e' modifies s' according to e. If e and u modify disjoint subterms of t, then the equation $|e'| \bullet |u'| = |u| \bullet |e|$ follows from the functoriality of the operations $\otimes^{\mathscr{C}}$ and $\bigotimes_{T}^{\mathscr{C}}$. If the modified subterms are not disjoint, then the identity $|e'| \bullet |u'| = |u| \bullet |e|$ follows from the naturality of the isomorphisms $(\alpha^{\mathscr{C}})^{\pm 1}$, $(\lambda^{\mathscr{C}})^{\pm 1}$, etc. $\square$

Next, we follow Mac Lane. We show that the intepretations of certain parallel "υ-directed morphisms" always coincide, and then we prove the general case. Suppose that p is a morphism in $\mathbf{Fr}(\mathbf{Bas}(n))$. We say that p is *υ-directed* if p uniquely decomposes into a (possibly empty) composite $p = b_k \bullet \cdots \bullet b_1$ of basic edges, none of which are υ^{-1}-basic. In what follows, we write $\bigotimes_n(x_1, \ldots, x_n)$ as shorthand for the term $\otimes(\cdots \otimes (\otimes(x_1, x_2), x_3) \cdots, x_n)$.

Lemma 2.32. *Suppose that* $t \in \mathbf{Fr}(\mathbf{Bas}(n))$, *and that* $p, q : t \rightrightarrows \bigotimes_n(x_1, \ldots, x_n)$ *are* υ-*directed. Then the natural transformations* $|p|, |q| : |t| \rightrightarrows |\bigotimes_n(x_1, \ldots, x_n)|$ *are equal.*

Proof. Consider p alone first. We may write $p = b_k \bullet \cdots \bullet b_1$ for unique basic edges, and then $|p| = |b_k| \bullet \cdots \bullet |b_1|$. By applying lemma 2.31 repeatedly, we may move all images of υ-basic edges to the right in this composite, and thus we obtain morphisms $p_\upsilon : t \to t^{\mathrm{red}}$ and $p_s : t^{\mathrm{red}} \to \bigotimes_n(x_1, \ldots, x_n)$ in $\mathbf{Fr}(\mathbf{Bas}(n))$ such that

(a) $|p| = |p_s| \bullet |p_\upsilon|$,

(b) p_υ is a composite of υ-basic edges only, and

(c) p_s is a composite of ordinary symmetric monoidal basic edges.

Observe that the term t^{red} is uniquely determined by the above. It is the term obtained by replacing every subterm $s \subset t$ of the form $s = g_i^H \bigotimes_T(s_1, \ldots, s_{|T|})$ with the term $\otimes(\cdots \otimes (\otimes(s_1, s_2), s_3) \cdots s_{|T|})$. Now do the same thing for q. We obtain parallel pairs of maps $p_\upsilon, q_\upsilon : t \rightrightarrows t^{\mathrm{red}}$ and $p_s, q_s : t^{\mathrm{red}} \rightrightarrows \bigotimes_n(x_1, \ldots, x_n)$ such that $|p| = |p_s| \bullet |p_\upsilon|$ and $|q| = |q_s| \bullet |q_\upsilon|$.

The ordinary Kelly-Mac Lane coherence theorem implies that $|p_s| = |q_s|$, since $|p_s|$ and $|q_s|$ come from the underlying symmetric monoidal structure on $\mathscr{C}$. The equality $|p_\upsilon| = |q_\upsilon|$ holds by the same argument given in the previous lemma, because the υ-basic factors of p_υ and q_υ modify distinct letters in the term t. The naturality of υ_T implies that we may perform such changes in any order. □

Now for the general case. The following is taken nearly verbatim from [31].

Proposition 2.33. *Suppose that* $t, t' \in \mathbf{Fr}(\mathbf{Bas}(n))$ *and that* $p : t \to t'$ *is arbitrary. Choose* υ-*directed morphisms* $d : t \to \bigotimes_n(x_1, \ldots, x_n)$ *and* $d' : t' \to \bigotimes_n(x_1, \ldots, x_n)$. *Then* $|p| = \left|d'\right|^{-1} \bullet |d| : |t| \to |\bigotimes_n(x_1, \ldots, x_n)| \to \left|t'\right|$.

Mac Lane's proof. Consider $p = b_k \bullet \cdots \bullet b_1 : t \to t'$, where the b_i are basic edges. By regrouping the factors, we may write

$$p = \Big[t = t_0 \xrightarrow{s_0} t_1 \xrightarrow{u_1} t_2 \xrightarrow{s_1} t_3 \xrightarrow{u_2} t_4 \to \cdots \to t_{2j-1} \xrightarrow{u_j} t_{2j} \xrightarrow{s_j} t_{2j+1} = t'\Big]$$

where each u_i is $\upsilon^{\pm 1}$-basic, and no instances of $\upsilon^{\pm 1}$-basic maps occur in the s_i. Applying $|\cdot|$ and replacing each υ^{-1}-basic map edge with its reverse, we see

$$|p| = \Big[|t_0| \xrightarrow{|s_0|} |t_1| \xrightarrow{|u_1|^{\pm 1}} |t_2| \xrightarrow{|s_1|} |t_3| \xrightarrow{|u_2|^{\pm 1}} |t_4| \to \cdots \to \left|t_{2j-1}\right| \xrightarrow{|u_j|^{\pm 1}} \left|t_{2j}\right| \xrightarrow{|s_j|} \left|t_{2j+1}\right|\Big]$$

where each of the u_i are now υ-basic edges, possibly pointing backwards. Now, for each t_i, choose a υ-directed morphism $d_i : t_i \to \bigotimes_n(x_1, \ldots, x_n)$, taking $d_0 = d$ and $d_{2j+1} = d'$ from the theorem statement. We obtain a diagram

$$\begin{tikzcd}
{|t_0|} \arrow[r, "|s_0|"] \arrow[ddrrrr] & {|t_1|} \arrow[r, "|u_1|^{\pm 1}"] \arrow[ddrrr] & {|t_2|} \arrow[r, "|s_1|"] \arrow[ddrr] & {|t_3|} \arrow[r, "|u_2|^{\pm 1}"] \arrow[ddr] & {|t_4|} \arrow[r] \arrow[dd] & \cdots \arrow[r] & {\left|t_{2j-1}\right|} \arrow[r, "|u_j|^{\pm 1}"] \arrow[ddll] & {\left|t_{2j}\right|} \arrow[r, "|s_j|"] \arrow[ddlll] & {\left|t_{2j+1}\right|} \arrow[ddllll] \\
 & & & & & \cdots & & & \\
 & & & & {\left|\bigotimes_n\right|} & & & &
\end{tikzcd}$$

in $\mathbf{End}(\mathscr{C})(n)$, where the diagonal map $|t_i| \to |\bigotimes_n(x_1, \ldots, x_n)|$ is $|d_i|$. Every triangle commutes by lemma 2.32, hence $|p| = \left|d_{2j+1}\right|^{-1} \bullet |d_0| = \left|d'\right|^{-1} \bullet |d|$. □

Corollary 2.34. *The functor* $|\cdot| : \mathbf{Fr}(\mathbf{Bas}(n)) \to \mathbf{End}(\mathscr{C})(n)$ *factors through the quotient* $\pi : \mathbf{Fr}(\mathbf{Bas}(n)) \to \mathbf{Fr}(\mathbf{Bas}(n))/\langle p \sim q \,|\, p,\, q \text{ parallel}\rangle$*, and this quotient is isomorphic to the category* $\mathcal{SM}_{\mathcal{N}}(n)$*.*

Proof. Every term t maps down to $\bigotimes_n(x_1, \ldots, x_n)$, and vice versa. Thus, for any terms t and t', there is a map $t \to \bigotimes_n(x_1, \ldots, x_n) \to t'$, which implies that the quotient has a unique map between any pair of objects. □

We conclude by showing that the functors $|\cdot| : \mathcal{SM}_{\mathcal{N}}(n) \to \mathbf{End}(\mathscr{C})(n)$ assemble into a G-operad map $\mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$. The basic points are that the image of $|\cdot|$ is closed under the G-operad structure on $\mathbf{End}(\mathscr{C})$, and that the canonical coherence maps for $\mathscr{C}$ are unique. Thus, the map $|\cdot| : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ essentially has "no choice" but to preserve the G-operad structure.

Lemma 2.35. *The map* $|\cdot| : \mathcal{SM}_{\mathcal{N}}(n) \to \mathbf{End}(\mathscr{C})(n)$ *preserves operadic composition and identities.*

Proof. The map $|\cdot| : \mathbb{F}(S_{\mathcal{N}}) \to \mathrm{Ob}(\mathbf{End}(\mathscr{C}))$ is an operad map, so its extension must also preserve the identity. Now suppose that $\overline{p} : s \to s' \in \mathcal{SM}_{\mathcal{N}}(k)$ and $\overline{q_i} : t_i \to t_i' \in \mathcal{SM}_{\mathcal{N}}(j_i)$ for $j = 1, \ldots, k$. The morphisms p and q_i factor as composites of basic edges, and the natural isomorphisms $|\overline{p}|$ and $|\overline{q_i}|$ factor as the corresponding composites of basic maps. By functoriality, the composite $\gamma(|\overline{p}|; |\overline{q_1}|, \ldots, |\overline{q_k}|)$ also factors as a composite of basic maps, which we may lift to a chain of basic edges $c : \gamma(s; t_1, \ldots, t_k) \to \gamma(s'; t_1', \ldots, t_k')$. The residue classes $\overline{c}, \gamma(\overline{p}; \overline{q_1}, \ldots, \overline{q_k}) : \gamma(s; t_1, \ldots, t_k) \rightrightarrows \gamma(s'; t_1', \ldots, t_k')$ are parallel, and hence equal. We conclude that $|\gamma(\overline{p}; \overline{q_1}, \ldots, \overline{q_k})| = |\overline{c}| = \gamma(|\overline{p}|; |\overline{q_1}|, \ldots, |\overline{q_k}|)$. $\square$

Lemma 2.36. *The map* $|\cdot| : \mathcal{SM}_{\mathcal{N}}(n) \to \mathbf{End}(\mathscr{C})(n)$ *preserves the* $G \times \Sigma_n$ *action.*

Proof. It is enough to check when $\overline{b} : t \to t'$ is the residue class of a basic edge b. We begin with the Σ_n-action. Consider the natural isomorphism $\left|\overline{b}\right| \cdot \sigma$. It is obtained by permuting the inputs to $\left|\overline{b}\right|$, so it is a basic map of the same sort. Thus, there is a basic edge $c : t \cdot \sigma \to t' \cdot \sigma$ that lifts $\left|\overline{b}\right| \cdot \sigma$. Since $\overline{b} \cdot \sigma$ and $\overline{c}$ are parallel, and hence equal, we deduce $\left|\overline{b} \cdot \sigma\right| = |\overline{c}| = \left|\overline{b}\right| \cdot \sigma$.

For the G-action, there are two cases. Either b is basic for one of the ordinary symmetric monoidal coherence maps, or it is υ-basic. The first case is similar to the above. The isomorphism $g \cdot \left|\overline{b}\right|$ is obtained by conjugating everything by g, and the isomorphisms $(\alpha^{\mathscr{C}})^{\pm 1}$, $(\lambda^{\mathscr{C}})^{\pm 1}$, etc. are all G-equivariant. Therefore $g \cdot \left|\overline{b}\right|$ is a basic map of the same sort as $\left|\overline{b}\right|$, and there is a basic edge $c : g \cdot t \to g \cdot t'$ that lifts it. We find $\left|g \cdot \overline{b}\right| = |\overline{c}| = g \cdot \left|\overline{b}\right|$.

If b is υ-basic, then $g \cdot \left|\overline{b}\right|$ is not υ-basic. However, twisted equivariance lets us write $g \cdot \left|\overline{b}\right|$ as a composite of an υ-basic map with a chain of ordinary symmetric monoidal basic maps, which lifts to a chain of basic edges $c : g \cdot t \to g \cdot t'$. Therefore the equation $\left|g \cdot \bar{b}\right| = |\bar{c}| = g \cdot \left|\bar{b}\right|$ holds here as well. $\square$

To summarize, we have constructed an operad map $|\cdot| : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ from any given normed symmetric monoidal structure $(\mathscr{C}, \otimes, e, (\bigotimes_T), \alpha, \lambda, \rho, \beta, (\upsilon_T))$ on $\mathscr{C}$, and direct inspection of the definitions reveals that $\mathrm{ev}(\mathscr{C}, |\cdot|) = \mathscr{C}$. On the other hand, every operad map $|\cdot| : \mathcal{SM}_{\mathcal{N}} \to \mathbf{End}(\mathscr{C})$ is determined by its values on the terms $e()$, $\otimes(x_1, x_2)$, $\bigotimes_T(x_1, \ldots, x_{|T|})$ and the morphisms $\otimes(\otimes(x_1, x_2), x_3) \to \otimes(x_1, \otimes(x_2, x_3))$, $\otimes(e(), x_1) \to x_1$, etc. It follows that the construction given in this section is also left inverse to ev.

This completes the proof of theorem 2.10.

CHAPTER 3

HOMOGENEOUS CATEGORICAL OPERADS

3.1 Introduction and summary of results

As usual, fix a finite group G.

Definition 3.1. An N_∞ *operad* is an operad $\mathscr{O}$ in G-spaces such that

1. the operad $\mathscr{O}$ is Σ-free, i.e. for every integer $n \geq 0$, the space $\mathscr{O}(n)$ is Σ_n-free,
2. for every integer $n \geq 0$ and subgroup $\Gamma \subset G \times \Sigma_n$, the space $\mathscr{O}(n)^\Gamma$ is either empty or contractible, and
3. the spaces $\mathscr{O}(0)^G$ and $\mathscr{O}(2)^G$ are contractible.

An N_∞ operad is an E_∞ *operad* if there are fixed points for every subgroup $\Gamma \subset G \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = *$. Let N_∞-**Op** be the category of all N_∞ operads in G-spaces.

In [5], Blumberg and Hill introduce N_∞ operads to parametrize the multiplicative structure present on localizations of equivariant commutative ring spectra. From this standpoint, the Σ-freeness assumption is natural because one almost never has homotopical operations that satisfy strict commutativity relations. It follows that the only subgroups $\Gamma \subset G \times \Sigma_n$ for which $\mathscr{O}(n)^\Gamma$ can be nonempty must be of the form $\Gamma = \{(h, \sigma(h)) \mid h \in H\}$, for some subgroup $H \subset G$ and homomorphism $\sigma : H \to \Sigma_n$ (cf. lemma 2.12). The elements $c \in \mathscr{O}(n)^\Gamma$ parametrize *norms* on $\mathscr{O}$-algebras, which give rise to transfers. The second condition above ensures that we get at most one norm of each type, up to coherent homotopy. These two observations justify the label "N_∞." The third condition guarantees that every $\mathscr{O}$-algebra has an underlying nonequivariant E_∞ structure. It is motivated by the fact that localizations of equivariant commutative ring spectra always retain this minimum of structure. The key point, however, is that equivariant E_∞ ring spectra possess all possible norms, but their localizations need not.

In this chapter, we study combinatorial precursors to N_∞ operads in G-spaces.

Definition 3.2. A *categorical N_∞ operad* is an operad $\mathscr{O}$ in G-categories, whose classifying space $B\mathscr{O}$ is an N_∞ operad in G-spaces. A *homogeneous* categorical N_∞ operad is an operad $\mathscr{O}$ in G-categories such that

1. the operad $\mathscr{O}$ is Σ-free,
2. for every integer $n \geq 0$ and subgroup $\Gamma \subset G \times \Sigma_n$, the category $\mathscr{O}(n)^\Gamma$ is either empty or (categorically) equivalent to the terminal category, and
3. the categories $\mathscr{O}(0)^G$ and $\mathscr{O}(2)^G$ are equivalent to the terminal category.

Let N_∞-$\mathbf{Op}_{cat}$ be the category of all categorical N_∞ operads, and let N_∞-$\mathbf{Op}_h$ be the category of all homogeneous categorical N_∞ operads.

The operad $\mathcal{SM}_{\mathcal{N}}$ considered in chapter 2 is a typical example of a homogeneous N_∞ operad, as is the equivariant Barratt-Eccles operad $\mathscr{P}_G$ of [18].

Proposition 3.3. *Every homogeneous categorical N_∞ operad is a categorical N_∞ operad.*

Proof. The properties that define N_∞ operads can be encoded using equalizer diagrams of the form below.

$$\begin{tikzcd}
\varnothing \arrow[r] & X \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\sigma"'] & X
\end{tikzcd} \qquad (\sigma \neq \mathrm{id}),$$

$$\begin{tikzcd}
X^\Gamma \arrow[r] & X \arrow[r, shift left, "\Delta"] \arrow[r, shift right, "\text{all } \gamma \cdot (-)\text{'s}"'] & \prod_{\gamma \in \Gamma} X
\end{tikzcd} \quad (\Gamma \subset G \times \Sigma_n)$$

Now observe that taking classifying spaces preserves finite limits and initial objects, and that it takes equivalences of categories to homotopy equivalences of spaces. □

One virtue of homogeneous N_∞ operads is their simplicity. Indeed, if $\mathscr{O}$ is homogeneous, then $\mathscr{O}$ is determined by its object operad alone. Recalling the adjunction $\mathrm{Ob} : \mathbf{Cat} \rightleftarrows \mathbf{Set} : \widetilde{(-)}$ (cf. definition 2.16), we have that $\mathscr{O} \cong \widetilde{\mathrm{Ob}(\mathscr{O})}$. This motivates the following definition.

Definition 3.4. An N *operad* is an operad $\mathscr{O}$ in G-sets such that

1. the operad $\mathscr{O}$ is Σ-free, and
2. the sets $\mathscr{O}(0)^G$ and $\mathscr{O}(2)^G$ are nonempty.

Let N-**Op** be the category of all N operads and G-operad maps between them.

Proposition 3.5. *If $\mathscr{O}$ is a homogeneous categorical N_∞ operad, then* $\mathrm{Ob}(\mathscr{O})$ *is an N operad. If $\mathscr{O}$ is an N operad, then $\widetilde{\mathscr{O}}$ is a homogeneous categorical N_∞ operad. Thus* $\mathrm{Ob} : \mathbf{Cat} \rightleftarrows \mathbf{Set} : \widetilde{(-)}$ *induces an equivalence* N-$\mathbf{Op} \simeq N_\infty$-$\mathbf{Op}_h$.

Proof. The functors Ob and $\widetilde{(-)}$ preserve (finite) limits and initial objects, and for every set X, the category $\widetilde{X}$ is either empty or equivalent to the terminal category. □

The preceding two results allow us to study N_∞ operads in terms of operads in G-sets. We shall use this connection to produce explicit, combinatorial models for N_∞ operads.

The remainder of this chapter is organized as follows. We begin with a review of the classification of N_∞ operads (sections 3.2 and 3.3). This is due to Blumberg and Hill [5], and it is based on the notion of an *indexing system* (cf. definition 3.15). These are combinatorial invariants that track the norms parametrized by an N_∞ operad, and we say that a finite G-subgroup action T is an *admissible set* of an operad $\mathscr{O}$ if $\mathscr{O}$ parametrizes a T-norm (cf. definition 3.8). Blumberg and Hill prove that the homotopy category of N_∞ operads maps fully and faithfully into the poset of indexing systems, but the realizability of every indexing system was left as a conjecture [5, p.4]. Our main contribution to this discussion is a calculation of the admissible sets of a free homogeneous N_∞ operad (theorem 3.19).

Theorem. *For any set of exponents $\mathcal{N}$, the class of admissible sets of the operad $\mathcal{SM}_\mathcal{N}$ is the indexing system generated by $\mathcal{N}$.*

This immediately implies that every indexing system can be realized by one of the operads $\mathcal{SM}_\mathcal{N}$, which proves Blumberg and Hill's conjecture. Two independent proofs have also been given by Bonventre-Pereira [8] and Gutiérrez-White [21]. In some sense, the computation of the admissible sets of $\mathbb{F}(S)$ in theorem 3.19 was inevitable, because this is the universal case. That indexing systems appear naturally in free operads is a further indication of their intrinsic nature.

We continue by developing a homotopy theory of operads that puts our constructions in perspective (sections 3.4 and 3.5). For technical reasons, it is easier to study homogeneous operads with a *marked* constant and binary operation, and we denote the category of such operads by $\mathbf{Op}_{h,m}$. We equip $\mathbf{Op}_{h,m}$ with an elementary, but illuminating model category structure (theorem 3.40).

Theorem. *There is a cofibrantly generated, simplicial model category structure on $\mathbf{Op}_{h,m}$, whose weak equivalences are the maps $f : \mathscr{O}_1 \to \mathscr{O}_2$ that induce equivalences between all subspaces of norm maps, and for which every object is fibrant. The cell complexes are the operads $\mathcal{SM}_{\mathcal{N}}$, and the cofibrant objects are deformation retracts thereof. Every mapping space in $\mathbf{Op}_{h,m}$ is either empty or contractible.*

It follows that Blumberg and Hill's classification of N_∞ operads in G-spaces applies equally well to $\mathbf{Op}_{h,m}$, and that taking classifying spaces induces an equivalence between the homotopy category of homogeneous N_∞ operads and the homotopy category of N_∞ operads in G-spaces (theorem 3.41). Note that there are also more general fixed-point model category structures, for which the operads $\mathcal{SM}_{\mathcal{N}}$ become cofibrant replacements of the commutativity operad **Com** (cf. corollary 3.25).

3.2 Admissible sets

In this section and the next, we review the classification of N_∞ operads. Almost all of the following was developed by Blumberg and Hill in [5]. Our contribution is theorem 3.19, which implies that all possible values of their homotopy invariant are taken.

Definition 3.6. A map $f : \mathscr{O}_1 \to \mathscr{O}_2$ between N_∞ operads in G-spaces is a *weak equivalence* if $f_n : \mathscr{O}_1(n)^\Gamma \to \mathscr{O}_2(n)^\Gamma$ is a weak homotopy equivalence of spaces for every integer $n \geq 0$ and subgroup $\Gamma \subset G \times \Sigma_n$. A map f between N_∞ operads in G-categories is a weak equivalence if Bf is a weak equivalence. Two N_∞ operads $\mathscr{O}_1$ and $\mathscr{O}_2$ are *weakly equivalent* if they are isomorphic after we invert the weak equivalences.

These conditions ensure that weakly equivalent operads $\mathscr{O}_1$ and $\mathscr{O}_2$ parametrize norms in an equivalent fashion, and a weak equivalence $f : \mathscr{O}_1 \to \mathscr{O}_2$ must witnesses that fact. Note that a map between homogeneous N_∞ operads is a weak equivalence if and only if the maps on Γ-fixed points are all equivalences of categories.

We have the following standard observation.

Proposition 3.7. *A map $f : \mathscr{O}_1 \to \mathscr{O}_2$ between N_∞ operads in G-spaces is a weak equivalence if and only if for every integer $n \geq 0$, subgroup $H \subset G$, and homomorphism $\sigma : H \to \Sigma_n$, the space $\mathscr{O}_1(n)^\Gamma$ is contractible whenever the space $\mathscr{O}_2(n)^\Gamma$ is contractible, where $\Gamma = \{(h, \sigma(h)) \,|\, h \in H\} \subset G \times \Sigma_n$.*

Proof. For every integer $n \geq 0$ and subgroup Γ, the spaces $\mathscr{O}_1(n)^\Gamma$ and $\mathscr{O}_2(n)^\Gamma$ are either empty or contractible. By Σ-freeness, it is enough to consider the subgroups $\Gamma = \{(h, \sigma(h)) \,|\, h \in H\}$, and for such Γ, it is enough to exclude the case that $\mathscr{O}_1(n)^\Gamma = \varnothing$ and $\mathscr{O}_2(n)^\Gamma \simeq *$, because we have a map $f_n : \mathscr{O}_1(n)^\Gamma \to \mathscr{O}_2(n)^\Gamma$. □

We can push this a bit further. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are N_∞ operads such that $\mathscr{O}_1(n)^\Gamma \simeq *$ precisely when $\mathscr{O}_2(n)^\Gamma \simeq *$, then both projections in the diagram $\mathscr{O}_1 \leftarrow \mathscr{O}_1 \times \mathscr{O}_2 \to \mathscr{O}_2$ are weak equivalences, even if there is no single operad map between $\mathscr{O}_1$ and $\mathscr{O}_2$. Thus, the homotopy type of an N_∞ operad is completely determined by which of the spaces $\mathscr{O}(n)^\Gamma$ are contractible, or equivalently, nonempty. Said differently, an N_∞ operad is determined by the norms that it parametrizes.

Thus, it is reasonable to ask which systems of norms could possibly arise from an N_∞ operad $\mathscr{O}$. They cannot be arbitrary because $\mathscr{O}(0)^G$ and $\mathscr{O}(2)^G$ are nonempty, and the equivariance of the maps $\gamma : \mathscr{O}(k) \times \mathscr{O}(j_1) \times \cdots \times \mathscr{O}(j_k) \to \mathscr{O}(j_1 + \cdots + j_k)$ implies inclusions between the various fixed point subspaces of $\mathscr{O}$. In [5], Blumberg and Hill introduce *indexing systems* to codify these properties. We give a modified treatment that is better suited for our combinatorial setting.

The notion of an indexing system is most easily formulated in coordinate-free language. Instead of considering subgroups $\Gamma \subset G \times \Sigma_n$, one considers finite G-subgroup actions. We make the following definition.

Definition 3.8. Suppose that $H \subset G$ is a subgroup, and that T is a finite H-set.

1. Given a symmetric sequence of G-sets S, we say that T is an *admissible set* for S if the set $S_{|T|}^{\Gamma_T}$ is nonempty.
2. Given an N_∞ operad $\mathscr{O}$ in G-spaces, we say that T is an *admissible set* for $\mathscr{O}$ if $\mathscr{O}(|T|)^{\Gamma_T}$ is contractible (or equivalently, nonempty). We say that T is an admissible set for a categorical N_∞ operad $\mathscr{O}$ if it is admissible for $B\mathscr{O}$.

The case for operads in G-spaces is the notion of admissibility considered in [5]. Thus T is admissible for a symmetric sequence of G-sets S if S parametrizes a T-norm, and it is admissible for an N_∞ operad $\mathscr{O}$ if $\mathscr{O}$ parametrizes a homotopically unique norm. Note that we must choose an order on T to define the subgroup $\Gamma_T \subset G \times \Sigma_{|T|}$, but the definition of admissibility is independent of this choice because different orders on T give rise to subgroups that are conjugate to Γ_T.

Notation 3.9. We write $\mathbb{A}(S)$ for the class of all admissible finite G-subgroup actions for S. This class is graded over the set of all subgroups $H \subset G$.

There is a noncommutative diagram

$$\begin{tikzcd} N\text{-}\mathbf{Op} \arrow[r, shift left, "\widetilde{(-)}"] & N_\infty\text{-}\mathbf{Op}_{cat} \arrow[l, shift left, "\mathrm{Ob}"] \arrow[r, "B"] & N_\infty\text{-}\mathbf{Op} \arrow[ll, bend left, "U"] \end{tikzcd}$$

and we have the following consistency statements.

Proposition 3.10. *Admissibility is preserved by the preceding functors, i.e.*

1. *if* $\mathscr{O} \in N$-$\mathbf{Op}$*, then* $\mathbb{A}(\mathscr{O}) = \mathbb{A}(\widetilde{\mathscr{O}})$*,*
2. *if* $\mathscr{O} \in N_\infty$-$\mathbf{Op}_{cat}$*, then* $\mathbb{A}(\mathrm{Ob}(\mathscr{O})) = \mathbb{A}(\mathscr{O}) = \mathbb{A}(B\mathscr{O})$*, and*
3. *if* $\mathscr{O} \in N_\infty$-$\mathbf{Op}$*, then* $\mathbb{A}(U\mathscr{O}) = \mathbb{A}(\mathscr{O})$*.*

Proof. Working levelwise, all of the functors under consideration preserve finite limits, empty objects, and nonempty objects. □

Thus, it is enough to study the admissible sets of N operads in G-sets. We actually go one level lower.

Definition 3.11. An N *symmetric sequence* is a symmetric sequence of G-sets S such that

1. the symmetric sequence S is Σ-free, and
2. the sets S_0^G and S_2^G are nonempty.

Let N-$\mathbf{Sym}$ be the category of all N symmetric sequences and all maps of G-symmetric sequences between them.

Every N symmetric sequence is the generating data for one of the operads $\mathrm{Ob}(\mathcal{SM}_{\mathcal{N}})$ (cf. section 2.4). In general, N symmetric sequences give rise to N operads.

Proposition 3.12. *The free-forgetful adjunction between symmetric sequences of G-sets and operads in G-sets restricts to an adjunction* $\mathbb{F} : N\text{-}\mathbf{Sym} \rightleftarrows N\text{-}\mathbf{Op} : \mathbb{U}$*.*

Proof. Suppose that S is an N symmetric sequence. The operad $\mathbb{F}(S)$ must be Σ-free by universality, and the nontriviality condition on fixed points is transferred along the unit map $\eta : S \to \mathbb{F}(S)$. Alternatively, this follows by direct inspection of the construction of $\mathbb{F}(S)$ given in section 2.4. □

Suppose that S is an N symmetric sequence. This already implies certain closure conditions on $\mathbb{A}(S)$, which are axiomatized in the next definition.

Definition 3.13. Consider graded classes $(\underline{\mathcal{C}}(H))_{H \subset G}$ such that each $\underline{\mathcal{C}}(H)$ is a proper class of finite H-sets, and H ranges over all subgroups of G. Let N-**Coef** be the poset of such tuples which satisfy the following four closure conditions:

(i) for every subgroup $H \subset G$, the class $\underline{\mathcal{C}}(H)$ contains every trivial H-action with cardinality 0 or 2,

(ii) for every subgroup $H \subset G$ and pair of finite H-sets S and T, if $S \in \underline{\mathcal{C}}(H)$ and $S \cong T$, then $T \in \underline{\mathcal{C}}(H)$,

(iii) for every pair of subgroups $K \subset H \subset G$ and finite H-set T, if $T \in \underline{\mathcal{C}}(H)$, then $\mathrm{res}^H_K T \in \underline{\mathcal{C}}(K)$, and

(iv) for every subgroup $H \subset G$, finite H-set T, and element $a \in G$, if $T \in \underline{\mathcal{C}}(H)$, then $aT \in \underline{\mathcal{C}}(aHa^{-1})$.

We order N-**Coef** under levelwise inclusion. Observe that N-**Coef** has a maximum element $\underline{\mathbf{Set}}$, which contains all finite G-subgroup actions. In the language of [5] and [23], the elements of N-**Coef** are the object classes of full, isomorphism-closed subcoefficient systems of $\underline{\mathbf{Set}}$ that contain all trivial actions on $\varnothing$ and on 2-element sets. Thus, we shall abusively refer to the elements of N-**Coef** as *coefficient systems*.

Proposition 3.14. *Taking admissible sets defines a functor* $\mathbb{A} : N\text{-}\mathbf{Sym} \to N\text{-}\mathbf{Coef}$.

Proof. Suppose that S is an N symmetric sequence. Condition (i) on $\mathbb{A}(S)$ follows from the fact that S_0^G and S_2^G are nonempty. Conditions (ii) – (iv) on $\mathbb{A}(S)$ hold because the set of subgroups $\Gamma \subset G \times \Sigma_n$ for which a $G \times \Sigma_n$-set X has Γ-fixed points is closed under subconjugacy. Thus $\mathbb{A}(S) \in N$-**Coef**. Morphisms in N-**Sym** are taken to inclusions because every map $f : S \to S'$ of N symmetric sequences restricts to a map on Γ_T-fixed points. ☐

3.3 Indexing systems and the classification of N_∞ operads

Now we describe what an operad structure buys.

Definition 3.15. A coefficient system $\underline{\mathcal{F}} \in N$-**Coef** is an *indexing system* if it satisfies the following three additional conditions:

(v) for every subgroup $H \subset G$, $\underline{\mathcal{F}}(H)$ is closed under passage to subobjects,

(vi) for every subgroup $H \subset G$, $\underline{\mathcal{F}}(H)$ is closed under finite coproducts, and

(vii) (closure under *self-induction*) for all subgroups $K \subset H \subset G$, if $T \in \underline{\mathcal{F}}(K)$ and $H/K \in \underline{\mathcal{F}}(H)$, then $\mathrm{ind}_K^H T = H \times_K T \in \underline{\mathcal{F}}(H)$.

It follows that indexing systems are completely determined by the orbits that they contain. Let **Ind** be the subposet of N-**Coef** spanned by the indexing systems.[1]

Remark 3.16. The definition in [5] is equivalent to this. To start, conditions (i), (ii), (v), and (vi) imply that every indexing system in our sense contains all trivial G-subgroup actions. We can also deduce closure under cartesian products from our conditions. Indeed, if $S, T \in \underline{\mathcal{F}}(H)$ and we write $S \cong \coprod_i H/K_i$ and $T \cong \coprod_j H/L_j$, then $S \times T \cong \coprod_{i,j}(H/K_i \times H/L_j)$ and thus it will be enough to show $H/K \times H/L \in \underline{\mathcal{F}}(H)$ for any subgroups $K, L \subset H \subset G$ and orbits $H/K, H/L \in \underline{\mathcal{F}}(H)$. However, the K-map $\mathrm{res}_K^H H/L \to H/K \times H/L$ sending hL to (K, hL) induces a surjective, and hence bijective, H-map $\mathrm{ind}_K^H \mathrm{res}_K^H H/L \to H/K \times H/L$.

Crucially, Blumberg and Hill show that the class of admissible sets of any N_∞ operad is an indexing system [5, theorem 4.17]. Their proof work equally well for N operads, because they only ever use the nonemptiness of fixed point subspaces. We give a slightly modified argument, just for variety.

Theorem 3.17 (Blumberg and Hill)**.** *The admissible sets functor* $\mathbb{A} : N$-**Sym** $\to N$-**Coef** *restricts to a functor* $\mathbb{A} : N$-**Op** $\to$ **Ind**.

---

[1] The poset **Ind** is denoted $\mathcal{I}$ in [5].

Proof. Suppose that $\mathscr{O}$ is an N operad. Then $\mathbb{A}(\mathscr{O}) \in N$-**Coef** by proposition 3.14. To verify the remaining conditions, note first that the element $(g, \sigma) \in G \times \Sigma_n$ stabilizes $c \in \mathscr{O}(n)$ if and only if $g \cdot c = c \cdot \sigma$. We treat conditions (v) – (vii) in turn.

For (v), suppose that T is an admissible H-set for $\mathscr{O}$ and that $S \subset T$ is a subobject of T. Without loss of generality, we may assume that T is ordered so that S occurs as its first $|S|$ elements. Write $\Gamma_T = \{(h, \alpha(h)) \,|\, h \in H\}$, choose elements $c \in \mathscr{O}(|T|)^{\Gamma_T}$ and $u \in \mathscr{O}(0)^G$, and consider the element

$$x = \gamma(c; \mathrm{id}, \ldots, \mathrm{id}, u, \ldots, u) \in \mathscr{O}(|S|). \qquad (|S| \text{ copies of id})$$

Then for any $h \in H$, the equivariance of γ and the equation $h \cdot c = c \cdot \alpha(h)$ imply that $h \cdot x = x \cdot \alpha(h)|_{\{1,\ldots,|S|\}}$. Since $\Gamma_S = \{(h, \alpha(h)|_{\{1,\ldots,|S|\}}) \,|\, h \in H\}$, we deduce that $x \in \mathscr{O}(|S|)^{\Gamma_S}$, and hence S is also an admissible H-set for $\mathscr{O}$.

For (vi), suppose that S and T are admissible H-sets for $\mathscr{O}$ and choose $c \in \mathscr{O}(|S|)^{\Gamma_S}$ and $d \in \mathscr{O}(|T|)^{\Gamma_T}$. Let $p \in \mathscr{O}(2)^G$. Then the element

$$x = \gamma(p; c, d) \in \mathscr{O}(|S| + |T|)$$

is $\Gamma_{S \sqcup T}$-fixed, and hence $S \sqcup T$ is admissible for $\mathscr{O}$.

For (vii), suppose that T is an admissible K-set for $\mathscr{O}$ and that the orbit H/K is also admissible. Choose H/K coset representatives $e = h_1, h_2, \ldots, h_{|H:K|}$ and for each $h \in H$, define the permutation $\sigma(h) \in \Sigma_{|H:K|}$ by $h \cdot h_i K = h_{\sigma(h)i} K$. If we order H/K as $\{K < h_2 K < \cdots < h_{|H:K|} K\}$, then $\Gamma_{H/K} = \{(h, \sigma(h)) \,|\, h \in H\}$. Now choose $c \in \mathscr{O}(|H : K|)^{\Gamma_{H/K}}$ and $d \in \mathscr{O}(|T|)^{\Gamma_T}$, and consider

$$x = \gamma(c; d, h_2 d, \ldots, h_{|H:K|} d) \in \mathscr{O}(|H : K| \times |T|).$$

A quick check shows that for every $h \in H$, there is some $\rho(h) \in \Sigma_{|H:K| \times |T|}$ such that

$h \cdot x = x \cdot \rho(h)$. Let S be the H-set structure on $\{1, \ldots, |H : K| \times |T|\}$ corresponding to the subgroup $\{(h, \rho(h)) \,|\, h \in H\}$. It is admissible for $\mathscr{O}$. Moreover, the first $|T|$ elements of $\mathrm{res}^H_K S$ are isomorphic to T, and multiplying by the coset representative h_i sends these elements bijectively to the $[(i-1)|T|+1]$-st through $i|T|$-th elements of S. It follows that the inclusion $T \to \mathrm{res}^H_K S$ induces a surjection $\mathrm{ind}^H_K T \to S$, and hence $\mathrm{ind}^H_K T \cong S$ because both sides have the same finite cardinality. ☐

We now consider how the free-forgetful adjunction $\mathbb{F} : N\text{-}\mathbf{Sym} \rightleftarrows N\text{-}\mathbf{Op} : \mathbb{U}$ manifests on the level of admissible sets.

Proposition 3.18. *There is an adjunction* $\mathbb{I} : N\text{-}\mathbf{Coef} \rightleftarrows \mathbf{Ind} : \iota$*. For any* $\underline{\mathcal{C}} \in N\text{-}\mathbf{Coef}$*, the indexing system* $\mathbb{I}(\underline{\mathcal{C}})$ *is the smallest indexing system that contains* $\underline{\mathcal{C}}$*. The functor* ι *is the inclusion.*

Proof. The intersection of indexing systems is an indexing system, and every coefficient system is contained in $\underline{\mathbf{Set}}$, the indexing system that contains all finite G-subgroup actions. Thus, we can define $\mathbb{I}(\underline{\mathcal{C}})$ to be the intersection of all indexing systems that contain $\underline{\mathcal{C}}$. ☐

We are led to consider the following two squares.

$$\begin{tikzcd}
N\text{-}\mathbf{Sym} \arrow[r, "\mathbb{F}"] \arrow[d, "\mathbb{A}"'] & N\text{-}\mathbf{Op} \arrow[d, "\mathbb{A}"] \\
N\text{-}\mathbf{Coef} \arrow[r, "\mathbb{I}"'] & \mathbf{Ind}
\end{tikzcd}
\qquad
\begin{tikzcd}
N\text{-}\mathbf{Sym} \arrow[d, "\mathbb{A}"'] & N\text{-}\mathbf{Op} \arrow[l, "\mathbb{U}"'] \arrow[d, "\mathbb{A}"] \\
N\text{-}\mathbf{Coef} & \mathbf{Ind} \arrow[l, "\iota"]
\end{tikzcd}$$

The right hand square commutes by definition. The left hand square also commutes, but this requires proof.

Theorem 3.19. *Suppose that* S *is a* Σ*-free symmetric sequence of* G*-sets. Then* $\mathbb{A}(\mathbb{F}(S)) \subset \mathbb{I}(\mathbb{A}(S))$*. If* S *is an* N *symmetric sequence, then* $\mathbb{A}(\mathbb{F}(S)) = \mathbb{I}(\mathbb{A}(S))$*. In particular, if* $\mathcal{N}$

is a set of exponents, then the class of admissible sets of the operad $\mathcal{SM}_{\mathcal{N}}$ is the indexing system generated by $\mathcal{N}$.

A few preliminary remarks are in order. We prove this theorem using the first model of $\mathbb{F}(S)$ described in section 2.4. For any finite K-set U and operadic term $t \in \mathbb{F}(S)$, we say that U is *t-admissible* if t is Γ_U-fixed for some choice of order $U \cong \{1, \ldots, |U|\}$. Every admissible set of $\mathbb{F}(S)$ arises in this way for some t, and we shall show $\mathbb{A}(\mathbb{F}(S)) \subset \mathbb{I}(\mathbb{A}(S))$ by induction on the complexity of t.

The general strategy is to decompose a t-admissible set using the G-action on $\mathbb{F}(S)$. Observe that if the K-set U is t-admissible, then for every $(k, \sigma(k)) \in \Gamma_U$, we have $k \cdot t = t \cdot \sigma(k)$. The term $t \cdot \sigma(k)$ is obtained by permuting the variables $x_1, \ldots, x_{|U|}$ in t isomorphically to $k \cdot (-) : U \to U$, while the term $k \cdot t$ is obtained by modifying individual letters in t and shuffling subterms around. That $k \cdot t$ and $t \cdot \sigma(k)$ are equal places strong symmetry conditions on t, and as we shall see, it also provides a description of U that is parallel to the recursive definition of the G-action on $\mathbb{F}(S)$.

Proof. We begin with the result for general Σ-free symmetric sequences. Suppose that $t = x_1$ or $t = e()$, and that U is a t-admissible K-set. Then U is a trivial K-action, and it must be contained in $\mathbb{I}(\mathbb{A}(S))$.

Now suppose that $t = r \bigotimes_T (t_1, \ldots, t_{|T|})$ and that U is a t-admissible K-set. Here T is a finite H-action on $\{1, \ldots, |T|\}$ for some subgroup $H \subset G$ and r is a G/H coset representative, and the set T corresponds to a coset $e\Gamma_T$ in an orbit decomposition $S_n \cong \coprod (G \times \Sigma_n)/\Gamma_T$. Assume inductively that for every operadic term s of lower complexity than t, if V is an s-admissible set, then $V \in \mathbb{I}(\mathbb{A}(S))$. We must show that $U \in \mathbb{I}(\mathbb{A}(S))$, and we do this by working from the outside of t in.

Suppose that $(k, \sigma(k)) \in \Gamma_U$. Then $k \cdot t = t \cdot \sigma(k)$, and thus the first letter of t and the first letter of $k \cdot t$ must be the same. Therefore $kr = rh$ for a unique $h \in H$, and if $(h, \tau(h)) \in \Gamma_T$, then $k \cdot t = r \bigotimes_T (k \cdot t_{\tau(h)^{-1}1}, \ldots, k \cdot t_{\tau(h)^{-1}|T|})$. We conclude that the action of K on the variables of $r \bigotimes_T (x_1, \ldots, x_{|T|})$ is isomorphic to the K-set $\mathrm{res}_K^{rHr^{-1}}(rT)$, so we identify $\{1, \ldots, |T|\} \cong \mathrm{res}_K^{rHr^{-1}}(rT)$. Since $(G \times \Sigma_n)/\Gamma_T \subset S_n$, we have $T \in \mathbb{A}(S)$, and hence $\mathrm{res}_K^{rHr^{-1}}(rT) \in \mathbb{A}(S)$ as well. Since indexing systems are closed under subobjects, we conclude that every orbit $K/L \subset \mathrm{res}_K^{rHr^{-1}}(rT)$ is also contained in $\mathbb{I}(\mathbb{A}(S))$.

Now write V_i for the set of variables in t_i. Then there are set bijections

$$U \cong \coprod_{\mathrm{res}_K^{rHr^{-1}}(rT)} V_i \cong \coprod_{\substack{\text{orbits } K/L \text{ of} \\ \mathrm{res}_K^{rHr^{-1}}(rT)}} \coprod_{K/L} V_i$$

and the preceding observations imply that $k \cdot V_i \subset V_{k \cdot i}$. Therefore $\coprod_{K/L} V_i$ is closed under the K-action. Since indexing systems are closed under coproducts, we have reduced the problem to showing that $\coprod_{K/L} V_i \in \mathbb{I}(\mathbb{A}(S))$ for every orbit K/L.

Consider the K-set $\coprod_{K/L} V_i$ and write V_L for the summand corresponding to $eL \in K/L$. The inclusions $k \cdot V_i \subset V_{k \cdot i}$ and $k^{-1} \cdot V_{k \cdot i} \subset V_i$ imply that all of the sets V_i have the same cardinality, and therefore $|\coprod_{K/L} V_i| = |K : L| \cdot |V_L|$. Moreover, the K-action on $\coprod_{K/L} V_i$ restricts to an L-action on V_L because $l \cdot V_L \subset V_{l \cdot L} = V_L$. The inclusion $V_L \to \mathrm{res}_L^K \coprod_{K/L} V_i$ induces a surjective K-map $\mathrm{ind}_L^K V_L \to \coprod_{K/L} V_i$, and therefore $\coprod_{K/L} V_i \cong \mathrm{ind}_L^K V_L$, because both sides have the same finite cardinality. We know that $K/L \in \mathbb{I}(\mathbb{A}(S))$ from above, and since indexing systems are closed under self-induction, we have reduced the problem to showing that $V_L \in \mathbb{I}(\mathbb{A}(S))$.

The L-set V_L is isomorphic to the action of L on the variables of one of the terms t_i appearing in $t = r \bigotimes_T (t_1, \ldots, t_{|T|})$. By renumbering the variables of t_i, we obtain an operadic term $\overline{t_i}$, and V_L is also isomorphic the action of L on the new variables. Therefore V_L is $\overline{t_i}$-admissible, and since $\overline{t_i}$ has lower complexity than t, we conclude inductively that $V_L \in \mathbb{I}(\mathbb{A}(S))$. This proves that $\mathbb{A}(\mathbb{F}(S)) \subset \mathbb{I}(\mathbb{A}(S))$ for all Σ-free symmetric sequences S.

Now suppose that S is an N symmetric sequence. The inclusion $\mathbb{A}(S) \subset \mathbb{A}(\mathbb{F}(S))$ holds because we have a unit map $\eta : S \to \mathbb{F}(S)$. Since $\mathbb{F}(S)$ is an N operad, theorem 3.17 guarantees that $\mathbb{A}(\mathbb{F}(S))$ is an indexing system. Therefore $\mathbb{I}(\mathbb{A}(S)) \subset \mathbb{A}(\mathbb{F}(S))$. $\square$

We obtain the following classification.

Theorem 3.20. *Indexing systems are a complete weak homotopy invariant for N_∞ operads, and every indexing system is realized.*

Proof. Suppose that $\mathscr{O}_1$ and $\mathscr{O}_2$ are N_∞ operads in G-spaces or G-categories. As explained earlier, if $\mathbb{A}(\mathscr{O}_1) = \mathbb{A}(\mathscr{O}_2)$, then the product diagram $\mathscr{O}_1 \leftarrow \mathscr{O}_1 \times \mathscr{O}_2 \rightarrow \mathscr{O}_2$ proves that $\mathscr{O}_1$ and $\mathscr{O}_2$ are weakly equivalent. For the converse, note that $\mathbb{A} : N_\infty\text{-}\mathbf{Op} \rightarrow \mathbf{Ind}$ sends weak equivalences to equalities in $\mathbf{Ind}$, and therefore it induces a functor $\mathbb{A} : \mathrm{Ho}(N_\infty\text{-}\mathbf{Op}) \rightarrow \mathbf{Ind}$.

Now suppose that $\underline{\mathcal{F}}$ is an indexing system. If S is any N symmetric sequence such that the admissible sets of S generate $\underline{\mathcal{F}}$, then theorem 3.19 implies that $\mathbb{A}(\mathbb{F}(S)) = \underline{\mathcal{F}}$. We deduce that $\widetilde{\mathbb{F}(S)}$ and $B\widetilde{\mathbb{F}(S)}$ are N_∞ operads that realize $\underline{\mathcal{F}}$. To produce such a sequence S, one can take $S = \coprod (G \times \Sigma_{|T|})/\Gamma_T$, where the sets T are representatives for all isoclasses of finite G-subgroup actions in $\underline{\mathcal{F}}$, or even just the nontrivial orbits $H/K \in \underline{\mathcal{F}}$ for subgroups $K \subsetneq H \subset G$ (cf. section 6.2). $\square$

This classification can be refined to account for higher data. In [5, proposition 5.5], Blumberg and Hill prove the following result.

Theorem 3.21 (Blumberg and Hill)**.** *The derived mapping space between any two N_∞ operads in G-spaces is either empty or contractible, and hence the admissible sets functor* $\mathbb{A} : \mathrm{Ho}(N_\infty\text{-}\mathbf{Op}) \rightarrow \mathbf{Ind}$ *is full and faithful.*

Thus, we deduce the following.

Theorem 3.22. *Taking the admissible sets of N_∞ operads in G-spaces determines an equivalence of categories* $\mathbb{A} : \mathrm{Ho}(N_\infty\text{-}\mathbf{Op}) \rightarrow \mathbf{Ind}$.

This resolves Blumberg and Hill's conjecture [5, p. 4] in the affirmative. Again, two independent solutions have also been given in [8] and [21]. In the next section, we shall prove the analogous results for homogeneous N_∞ operads (cf. theorem 3.41).

3.4 The homotopy theory of homogeneous categorical operads

In this section, we develop an elementary homotopy theory of operads that gives context for our results. We allow for non Σ-free operads to ensure the existence of enough colimits.

Definition 3.23. Suppose that $\mathscr{O}$ is an operad in G-categories. We say that the operad $\mathscr{O}$ is *homogeneous* if every $G \times \Sigma_n$-category $\mathscr{O}(n)$ is either empty or (nonequivariantly) equivalent to the terminal category, i.e. $\widetilde{\mathrm{Ob}(\mathscr{O})} \cong \mathscr{O}$. Write $\mathbf{Op}_h$ for the category of all homogeneous categorical operads.

We begin by summarizing the properties of a class of fixed point model structures available on the category $\mathbf{Op}_h$. The reader should compare to the work in [8] and [21].

Theorem 3.24. *Fix a subclass $\underline{\mathcal{T}} \subset \underline{\mathbf{Set}}$ of finite G-subgroup actions. The category $\mathbf{Op}_h$ is locally finitely presentable, and there is a cofibrantly generated, right proper, simplicial $\underline{\mathcal{T}}$-model category structure on $\mathbf{Op}_h$ with the following properties.*

(a) The weak equivalences are the maps $f : \mathscr{O}_1 \to \mathscr{O}_2$ with the following property: for every $T \in \underline{\mathcal{T}}$, if $\mathscr{O}_2$ has a Γ_T-fixed point, then some Γ_T-fixed point of $\mathscr{O}_2$ lifts up f to a Γ_T-fixed point of $\mathscr{O}_1$. Equivalently, f is a weak equivalence if $\mathbb{A}(\mathscr{O}_1) \cap \underline{\mathcal{T}} = \mathbb{A}(\mathscr{O}_2) \cap \underline{\mathcal{T}}$.

(b) The fibrations are the maps $f : \mathscr{O}_1 \to \mathscr{O}_2$ with the following property: for any $T \in \underline{\mathcal{T}}$, if some Γ_T-fixed point of $\mathscr{O}_2$ lifts up f to a Γ_T-fixed point of $\mathscr{O}_1$, then every Γ_T-fixed point in $\mathscr{O}_2$ lifts up f to a Γ_T-fixed point of $\mathscr{O}_1$.

(c) The cofibrations are the retracts of the maps $i_1 : \mathscr{O} \to \mathscr{O} * \mathscr{F}$, where $*$ is the coproduct in $\mathbf{Op}_h$, and $\mathscr{F}$ is freely generated by orbits of the form $G \times \Sigma_{|T|}/\Gamma_T$, with $T \in \underline{\mathcal{T}}$.

(d) The generating cofibrations are the maps $\{\mathrm{id}\} \to \widetilde{\mathbb{F}}(G \times \Sigma_{|T|}/\Gamma_T)$, where $T \in \underline{\mathcal{T}}$, $\{\mathrm{id}\}$ is the initial operad, and $\widetilde{\mathbb{F}} = \widetilde{(-)} \circ \mathbb{F}$. The generating acyclic cofibrations are the maps $\widetilde{\mathbb{F}}(G \times \Sigma_{|T|}/\Gamma_T) \to \widetilde{\mathbb{F}}(G \times \Sigma_{|T|}/\Gamma_T) * \widetilde{\mathbb{F}}(G \times \Sigma_{|T|}/\Gamma_T)$, where $T \in \underline{\mathcal{T}}$.

(e) Every object is fibrant.

(f) The cell complexes are the operads that are free on orbits of the form $G \times \Sigma_{|T|}/\Gamma_T$*, with* $T \in \underline{\mathcal{T}}$*. General cofibrant operads are retracts*[2] *of these free operads.*

(g) The hom space between any $\mathscr{O}_1$ *and* $\mathscr{O}_2$ *is either empty or contractible.*

We call this the $\underline{\mathcal{T}}$*-model structure on* $\mathbf{Op}_h$*.*

We do not require the class $\underline{\mathcal{T}}$ to satisfy any closure conditions, but one can specialize to coefficient systems or to indexing systems. The proofs are not simplified in any way, though.

Corollary 3.25. *The operad* $\mathcal{SM}_{\mathcal{N}}$ *is a cofibrant replacement for* **Com** *in the* $\mathbb{I}(\mathcal{N})$*-model structure on* $\mathbf{Op}_h$*.*

We now construct the advertised $\underline{\mathcal{T}}$-model structures. They are ultimately based on $\underline{\mathcal{T}}$-model structures on the category of symmetric sequences of G-sets, henceforth denoted **Sym**. Fix a class $\underline{\mathcal{T}} \subset \underline{\mathbf{Set}}$.

Lemma 3.26. *The category* **Sym** *is locally finitely presentable.*

Proof. The orbits $G \times \Sigma_n/\Xi$ are finitely presentable because taking fixed points commutes with directed colimits in $G \times \Sigma_n$-sets. As $n \geq 0$ ranges over all nonnegative integers and $\Xi \subset G \times \Sigma_n$ ranges over all subgroups of $G \times \Sigma_n$, we obtain a strong generator for **Sym**. Now apply [1, theorem 1.11]. □

We declare a map $f : S \to S'$ in **Sym** to be a *weak equivalence* if $\mathbb{A}(S) \cap \underline{\mathcal{T}} = \mathbb{A}(S') \cap \underline{\mathcal{T}}$, where $\mathbb{A}(S)$ denotes the class of admissible sets of S. Write $\mathscr{W}$ for the class of all weak equivalences in **Sym**. We take

$$\mathscr{I} = \left\{ \varnothing \longrightarrow (G \times \Sigma_{|T|})/\Gamma_T \,\middle|\, T \in \underline{\mathcal{T}} \right\}$$
$$\mathscr{J} = \left\{ (G \times \Sigma_{|T|})/\Gamma_T \xrightarrow{i_1} (G \times \Sigma_{|T|})/\Gamma_T \sqcup (G \times \Sigma_{|T|})/\Gamma_T \,\middle|\, T \in \underline{\mathcal{T}} \right\}$$

---

[2] Note that if $\mathscr{O}_1$ and $\mathscr{O}_2$ are homogeneous, then every parallel pair of operad maps $\mathscr{O}_1 \rightrightarrows \mathscr{O}_2$ are naturally isomorphic. Therefore retracts are really deformation retracts in $\mathbf{Op}_h$.

to be sets of generating cofibrations and generating acyclic cofibrations, respectively. Note that the sets $\mathscr{I}$ and $\mathscr{J}$ are at most countably infinite, even though the index T could range over a proper class.

Lemma 3.27. *Every relative* $\mathscr{J}$*-cell complex is a split monomorphism.*

Proof. Each successor stage of the construction is a pushout square of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_1"'] & X \arrow[d, "i_1"] \\
A \sqcup A \arrow[r, "f \sqcup \mathrm{id}"'] & X \sqcup A
\end{tikzcd}$$

and the map $r = [\mathrm{id}, f] : X \sqcup A \to X$ does the job in this case. Now we proceed inductively up the cell complex. □

Lemma 3.28. *Suppose that* $f : S \to S'$ *is a map of symmetric sequences.*

1. *The map* f *is a weak equivalence if and only if for every* $T \in \underline{T}$*, if* S' *has a* Γ_T*-fixed point, then some* Γ_T*-fixed point of* S' *lifts up the map* f *to a* Γ_T*-fixed point of* S*.*
2. *The map* f *has the right lifting property against* $\mathscr{I}$ *if and only if for every* $T \in \underline{T}$*, every* Γ_T*-fixed point of* S' *lifts up* f *to a* Γ_T*-fixed point of* S*.*
3. *The map* f *has the right lifting property against* $\mathscr{J}$ *if and only if for every* $T \in \underline{T}$*, if some* Γ_T*-fixed point of* S' *lifts up* f *to a* Γ_T*-fixed point of* S*, then every* Γ_T*-fixed point of* S' *lifts up* f *to a* Γ_T*-fixed point of* S*.*

Proof. The first part is a restatement of the inclusion $\mathbb{A}(S) \cap \underline{T} \supset \mathbb{A}(S') \cap \underline{T}$. The second and third parts hold because orbits represent fixed points. □

Proposition 3.29. *There is a cofibrantly generated $\underline{\mathcal{T}}$-model category structure on* **Sym** *with weak equivalences $\mathscr{W}$, generating cofibrations $\mathscr{I}$, and generating acyclic cofibrations $\mathscr{J}$. The acyclic fibrations and fibrations are given by (2) and (3) in lemma 3.28, respectively.*

Proof. We use the usual recognition theorem (cf. [34, theorem 15.2.3]). First, observe that the sets $\mathscr{I}$ and $\mathscr{J}$ admit the small object argument. There is nothing to prove for the set $\mathscr{I}$. For $\mathscr{J}$, note that every relative $\mathscr{J}$-cell complex is an ascending union by lemma 3.27, and that taking $\Gamma_{\mathcal{T}}$-fixed points commutes with unions.

Next, observe that every relative $\mathscr{J}$-cell complex $j : S \to S'$ is a weak equivalence, because the splitting $S \to S' \to S$ implies $\mathbb{A}(S) \subset \mathbb{A}(S') \subset \mathbb{A}(S)$.

The equality $\mathscr{I}^{\boxslash} = \mathscr{J}^{\boxslash} \cap \mathscr{W}$ follows from lemma 3.28. $\square$

The next observation follows easily from lemma 3.28 and a little diagram chase.

Lemma 3.30. *The $\underline{\mathcal{T}}$-model structure on* **Sym** *is right proper, and every object of* **Sym** *is fibrant in it.*

We construct a $\underline{\mathcal{T}}$-model structure for operads using Kan transport (cf. [34, theorem 16.2.5]). Write **Op** for the category of operads in G-sets and write $\mathbb{F} : \mathbf{Sym} \rightleftarrows \mathbf{Op} : \mathbb{U}$ for the free-forgetful adjunction.

Lemma 3.31. *The adjunction* $\mathrm{Ob} : \mathbf{Cat} \rightleftarrows \mathbf{Set} : \widetilde{(-)}$ *induces an equivalence of categories* $\mathrm{Ob} : \mathbf{Op}_h \overset{\simeq}{\rightleftarrows} \mathbf{Op} : \widetilde{(-)}$. *Therefore*

1. *there is an adjunction* $\widetilde{(-)} \circ \mathbb{F} : \mathbf{Sym} \rightleftarrows \mathbf{Op}_h : \mathbb{U} \circ \mathrm{Ob}$, *and*

2. *the category* $\mathbf{Op}_h$ *is bicomplete, with limits computed levelwise, and colimits computed by applying* $\widetilde{(-)}$ *to the colimit* $\mathrm{colim}_j \mathrm{Ob}(\mathscr{O}_j)$ *in* **Op**.

Proof. We get an equivalence $\mathrm{Ob} : \mathbf{Op}_h \rightleftarrows \mathbf{Op} : \widetilde{(-)}$ directly from the definition of $\mathbf{Op}_h$. Therefore $\widetilde{(-)}$ is *left and right* adjoint to Ob when we restrict its codomain to $\mathbf{Op}_h$, and the first claim follows by composing adjunctions.

For the second claim, it is standard that **Op** is bicomplete (cf. [16]), and both limits and colimits in $\mathbf{Op}_h$ can be computed by mapping to **Op**, taking the limit or colimit there, and then mapping back to $\mathbf{Op}_h$. Since $\widetilde{(-)}$ is a right adjoint, it commutes with the levelwise limits in **Op**, but the same cannot be said for colimits. $\square$

We shall usually shorten $\widetilde{(-)} \circ \mathbb{F}$ to $\widetilde{\mathbb{F}}$ aand $\mathbb{U} \circ \mathrm{Ob}$ to Ob.

Lemma 3.32. *The category* $\mathbf{Op}_h$ *is locally finitely presentable.*

Proof. Directed colimits in **Op** are computed levelwise in **Sym**, so this follows from lemma 3.26. The operads $\widetilde{\mathbb{F}}(G \times \Sigma_n/\Xi)$ are finitely presentable strong generators. $\square$

Write $\widetilde{\mathbb{F}}(\mathscr{I})$ and $\widetilde{\mathbb{F}}(\mathscr{J})$ for the images of the sets $\mathscr{I}$ and $\mathscr{J}$ in $\mathbf{Op}_h$. The following lemma is proven exactly as before, because $\widetilde{\mathbb{F}}$ preserves coproducts.

Lemma 3.33. *Every relative* $\widetilde{\mathbb{F}}(\mathscr{J})$*-cell complex is a split monomorphism.*

Proposition 3.34. *There is a cofibrantly generated* $\underline{\mathcal{T}}$*-model category structure on* $\mathbf{Op}_h$ *with weak equivalences and fibrations created by* $\mathrm{Ob} : \mathbf{Op}_h \to \mathbf{Sym}$*, and with generating cofibrations and acyclic cofibrations* $\widetilde{\mathbb{F}}(\mathscr{I})$ *and* $\widetilde{\mathbb{F}}(\mathscr{J})$*, respectively. The adjunction* $\widetilde{\mathbb{F}} \dashv \mathrm{Ob}$ *lifts to a Quillen adjunction.*

Proof. First of all, the sets $\widetilde{\mathbb{F}}(\mathscr{I})$ and $\widetilde{\mathbb{F}}(\mathscr{J})$ admit the small object argument. There is nothing to prove for $\widetilde{\mathbb{F}}(\mathscr{I})$, because $\widetilde{\mathbb{F}}$ preserves initial objects. For $\widetilde{\mathbb{F}}(\mathscr{J})$, the previous lemma implies that every relative $\widetilde{\mathbb{F}}(\mathscr{J})$-cell complex is an ascending union of operads, and these colimits are created levelwise in **Sym**. Therefore $\widetilde{\mathbb{F}}(\mathscr{J})$ inherits smallness from $\mathscr{J}$.

As before, every relative $\widetilde{\mathbb{F}}(\mathscr{J})$-cell complex $\mathscr{O} \to \mathscr{O}'$ is a weak equivalence because it is a split monomorphism. $\square$

Thus, we have the following standard conclusions.

Proposition 3.35. *The* $\underline{\mathcal{T}}$*-model structure on* $\mathbf{Op}_h$ *is right proper, and every object of* $\mathbf{Op}_h$ *is fibrant in it. The* $\widetilde{\mathbb{F}}(\mathscr{I})$*-cell complexes are the operads that are freely generated by orbits*

$G \times \Sigma_{|T|}/\Gamma_T$, where $T \in \underline{\mathcal{T}}$. The cofibrant operads are the retracts of these cell operads. Cofibrations are retracts of coproduct structure maps $i_1 : \mathscr{O} \to \mathscr{O} * \mathscr{F}$, where $\mathscr{F}$ is freely generated by orbits $G \times \Sigma_{|T|}/\Gamma_T$ with $T \in \underline{\mathcal{T}}$, and $*$ denotes the coproduct in $\mathbf{Op}_h$.

Remark 3.36. We make no claims about left properness because we are unable to compute the admissible sets of the coproduct $\mathscr{O} * \mathscr{F}$ when $\mathscr{O}$ is not Σ-free.

We now turn to enrichment. The category $\mathbf{Op}_h$ is most naturally a 2-category, whose hom category $\underline{\mathbf{Op}}_h(\mathscr{O}_1, \mathscr{O}_2)$ consists of the operad maps $\mathscr{O}_1 \to \mathscr{O}_2$ and the natural transformations between them. However, if $f, g : \mathscr{O}_1 \rightrightarrows \mathscr{O}_2$ is any pair of operad maps, then there is a unique natural transformation $f \Rightarrow g$ because $\mathscr{O}_2$ is homogeneous. It follows that $\underline{\mathbf{Op}}_h(\mathscr{O}_1, \mathscr{O}_2) \cong \widetilde{\mathbf{Op}_h}(\mathscr{O}_1, \mathscr{O}_2)$, which is either empty or contractible.

For model categorical purposes, it is more useful to have a simplicial enrichment. Applying the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$, we see that

$$N\widetilde{\mathbf{Op}_h}(\mathscr{O}_1, \mathscr{O}_2)_q = \mathbf{Op}_h(\mathscr{O}_1, \mathscr{O}_2)^{\times q+1}.$$

Moreover, there is an adjunction $(-)_0 : \mathbf{sSet} \rightleftarrows \mathbf{Set} : N \circ \widetilde{(-)}$, and both the left and right adjoint preserve products. Thus, we can perform the standard change of enrichment and (co)tensoring [38, theorem 3.7.11].

Lemma 3.37. *Suppose that $\mathscr{C}$ is a 1-category that is tensored and cotensored over* **Set**. *Then $\mathscr{C}$ is also a* **sSet***-enriched category that is tensored and cotensored over* **sSet***, with hom objects $N\widetilde{\mathscr{C}}(X, Y)$, tensors $K \odot X := \coprod_{K_0} X$, and cotensors $K \pitchfork Y := \prod_{K_0} Y$.*

Proposition 3.38. *With the enrichment of lemma 3.37, the $\underline{\mathcal{T}}$-model structure on $\mathbf{Op}_h$ upgrades to a simplicial model structure.*

Proof. It remains to check axiom SM7. In light of the adjunction $(-)_0 \dashv N \circ \widetilde{(-)}$ and the fact that the horn Λ^n_k and n-simplex Δ^n have the same 0-simplices for $n > 1$, it will suffice to find a diagonal lift for every square

$$\begin{tikzcd}
* \arrow[rr] \arrow[d] & & \mathbf{Op}_h(\mathscr{B},\mathscr{X}) \arrow[d, "i^* \times p_*"] \\
* \sqcup * \arrow[r] \arrow[urr, dashed] & \mathbf{Op}_h(\mathscr{A},\mathscr{X}) & \underset{\mathbf{Op}_h(\mathscr{A},\mathscr{Y})}{\times} & \mathbf{Op}_h(\mathscr{B},\mathscr{Y})
\end{tikzcd}$$

provided that $i : \mathscr{A} \to \mathscr{B}$ is a cofibration and $p : \mathscr{X} \to \mathscr{Y}$ is a fibration in the $\underline{\mathcal{T}}$-model structure on $\mathbf{Op}_h$. This will ensure that the map $i^* \times p_*$ on enriched homs is a Kan fibration, and since the simplicial sets $N\widetilde{\mathbf{Op}_h}(\mathscr{O}_1, \mathscr{O}_2)$ are all either empty or contractible, and $N \circ \widetilde{(-)}$ is limit-preserving, the extra condition for acyclics is automatic.

We need to check that if one square in $\mathbf{Op}_h$ between i and p has a diagonal lift, then every such square does. This is straightforward to check when i is of the form $\mathscr{O} \to \mathscr{O} * \widetilde{\mathbb{F}}(S)$, and the general statement follows by passing to retracts. $\square$

3.5 The homotopy theory of homogeneous N_∞ operads

We would like to understand the homotopy theory of N_∞-$\mathbf{Op}_h$ in terms of a homotopy theory on $\mathbf{Op}_h$. We do this by passing to an undercategory.

Definition 3.39. A *marked* homogeneous operad is an operad $\mathscr{O} \in \mathbf{Op}_h$, equipped with chosen points $u \in \mathscr{O}(0)^G$ and $p \in \mathscr{O}(2)^G$. Let $\mathbf{Op}_{h,m}$ be the category of marked homogeneous operads. Similarly, let N_∞-$\mathbf{Op}_{h,m}$ denote the category of marked homogeneous N_∞ operads.

Write $\mathcal{SM} = \widetilde{\mathbb{F}}(G \times \Sigma_0/G \sqcup G \times \Sigma_2/G)$ for the operad that parametrizes symmetric monoidal objects. Then a marked operad $\mathscr{O} \in \mathbf{Op}_h$ is just a map $\mathcal{SM} \to \mathscr{O}$, and $\mathbf{Op}_{h,m} \cong \mathcal{SM}/\mathbf{Op}_h$. The standard machinery lets us transport model structures on $\mathbf{Op}_h$ to $\mathbf{Op}_{h,m}$.

Theorem 3.40. *The category $\mathbf{Op}_{h,m}$ is locally finitely presentable, and there is a cofibrantly generated, right proper, simplicial $\underline{\mathbf{Set}}$-model category structure on $\mathbf{Op}_{h,m}$ with analogous properties to the $\underline{\mathbf{Set}}$-model structure on $\mathbf{Op}_h$. In particular,* *(a) A map $f : \mathscr{O}_1 \to \mathscr{O}_2$ in $\mathbf{Op}_{h,m}$ is a weak equivalence, fibration, or cofibration if it is one in the $\underline{\mathbf{Set}}$-model structure on $\mathbf{Op}_h$.*

(b) The generating cofibrations and acyclic cofibrations of $\mathbf{Op}_{h,m}$ are obtained by applying the functor $\mathcal{SM} * (-)$ to the corresponding data in $\mathbf{Op}_h$.

(c) Every operad in $\mathbf{Op}_{h,m}$ is fibrant.

(d) The cell complexes are the operads $\mathcal{SM}_{\mathcal{N}}$, equipped with their distinguished constant and tensor product. General cofibrant operads are retracts of the $\mathcal{SM}_{\mathcal{N}}$'s.

(e) The hom space between any $\mathscr{O}_1$ and $\mathscr{O}_2$ is either empty or contractible.

Proof. For local finite presentability, note that directed colimits in $\mathbf{Op}_{h,m}$ are also computed levelwise in **Sym**. It follows that the operads $\mathcal{SM} * \widetilde{\mathbb{F}}(G \times \Sigma_n/\Xi)$ are finitely presentable strong generators. All of the model categorical statements except (e) follow from the usual theory [34, theorem 15.3.6], and the proof of proposition 3.38 also works for $\mathbf{Op}_{h,m}$. □

From here, we can easily identify the simplicial localization N_∞-$\mathbf{Op}_h$.

Theorem 3.41. *The functor $\mathcal{SM} \overset{\mathbb{L}}{*} (-) : N_\infty\text{-}\mathbf{Op}_h \to \mathbf{Op}_{h,m}$ induces a Dwyer-Kan equivalence on simplicial localizations. Therefore*

1. *the derived mapping space between any pair of homogeneous categorical N_∞ operads $\mathscr{O}_1$ and $\mathscr{O}_2$ is either empty or contractible,*
2. *the functor $\mathbb{A} : \mathrm{Ho}(N_\infty\text{-}\mathbf{Op}_h) \to \mathbf{Ind}$ is an equivalence of categories, and*
3. *the classifying space functor $B : N_\infty\text{-}\mathbf{Op}_h \to N_\infty\text{-}\mathbf{Op}$ induces an equivalence of homotopy categories.*

Proof. Write Q for cellular cofibrant replacement functors on $\mathbf{Op}_h$ and $\mathbf{Op}_{h,m}$. In each pair of homotopical functors below

$$\begin{tikzcd} N_\infty\text{-}\mathbf{Op}_h \arrow[r, shift left, "Q"] & N_\infty\text{-}\mathbf{Op}_{cell} \arrow[l, shift left, "\text{inc}"] \arrow[r, shift left, "\mathcal{SM} * (-)"] & N_\infty\text{-}\mathbf{Op}_{cell,m} \arrow[l, shift left, "\text{forget}"] \arrow[r, shift left, "\text{inc}"] & \mathbf{Op}_{h,m} \arrow[l, shift left, "Q"] \end{tikzcd}$$

the maps are inverse up to natural weak equivalence. Therefore they all induce Dwyer-Kan equivalences on simplicial localizations [15, propositions 3.3 and 3.5]. Since every simplicial hom set in $\mathbf{Op}_{h,m}$ is either empty or contractible, the same is true for the simplicial localization of N_∞-$\mathbf{Op}_h$.

From here, the proof that $\mathbb{A} : \text{Ho}(N_\infty\text{-}\mathbf{Op}_h) \to \mathbf{Ind}$ is an equivalence of categories proceeds exactly as in [5]. We deduce that it is faithful from the considerations above, the product diagrams $\mathscr{O}_1 \leftarrow \mathscr{O}_1 \times \mathscr{O}_2 \to \mathscr{O}_2$ prove that it is full, and theorem 3.19 implies that it is surjective. Since the functor B preserves admissible sets, we deduce that $B : \text{Ho}(N_\infty\text{-}\mathbf{Op}_h) \to \text{Ho}(N_\infty\text{-}\mathbf{Op})$ is also an equivalence of categories. $\square$

CHAPTER 4

THE UNIVERSALITY OF NORMED SYMMETRIC MONOIDAL STRUCTURE

4.1 Introduction and summary of results

There are many structures that could sensibly be called equivariant symmetric monoidal categories. The purpose of this chapter is to elucidate the situation. In what follows, we shall explain why a significant portion of these structures are equivalent to normed symmetric monoidal categories.

One expects certain features to be present in any definition of equivariant symmetric monoidal structure. For example, one would like to have a product that is associative, commutative, and unital up to isomorphism, and one would like to have transfer maps that interact well with the product. For the sake of applications, one also wants a means of presenting these structures, but we do not require this in the definition.

Fix a finite group G, and assume for the moment that a G-category $\mathscr{C}$, rather than a presheaf of categories, should underlie every equivariant symmetric monoidal category. From this perspective, transfers arise as composites $\bigotimes_{H/K} \circ \Delta^{\mathrm{tw}} : \mathscr{C}^K \to (\mathscr{C}^{\times H/K})^H \to \mathscr{C}^H$ of a twisted diagonal map with a product over an orbit. As explained in section 5.3, the familiar example of monoidal induction can be recovered by specializing to the functor category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$, where $\mathbb{T}G$ is the translation category of G and $\mathscr{C}$ is a nonequivariant symmetric monoidal category. Thus, assuming $\mathscr{C}$ to be a G-category is not unreasonable. From here, one might ask for the following:

1. a G-equivariant bifunctor $\otimes : \mathscr{C}^{\times 2} \to \mathscr{C}$ and a G-fixed unit object $e \in \mathscr{C}$, with G-equivariance ensuring that this structure descends to fixed points,
2. a collection of H-equivariant maps $\bigotimes_{H/K} : \mathscr{C}^{\times H/K} \to \mathscr{C}$ to construct transfers, and
3. that all composite operations of a given arity are coherently isomorphic.

The objects that represent this constellation of data are precisely the marked, homogeneous operads $\mathscr{O} \in \mathbf{Op}_{h,m}$ studied in chapter 3. Thus, we think of $\mathbf{Op}_{h,m}$ as a category of generalized theories of equivariant symmetric monoidal structure.

The homotopy theory of $\mathbf{Op}_{h,m}$ is particularly simple. By theorem 3.41, we know that taking admissible sets determines an equivalence between the homotopy category $\mathrm{Ho}(\mathbf{Op}_{h,m})$ and the poset **Ind** of indexing systems. Moreover, $\mathbf{Op}_{h,m}$ has a cofibrantly generated model structure, whose cell complexes are precisely the operads $\mathcal{SM}_{\mathcal{N}}$ that parametrize $\mathcal{N}$-normed symmetric monoidal categories (cf. section 2.4 and theorem 3.40). We deduce that on the level of operads, there is nothing but normed symmetric monoidal structure, up to equivalence. The purpose of this chapter is to explain how the homotopical properties of $\mathbf{Op}_{h,m}$ are reflected on the level of algebras. We prove a number of results in the spirit of Boardman and Vogt's theory of homotopy invariant algebraic structures [4].

We start by analyzing the formation of 2-categories of algebras (section 4.2). For any operad $\mathscr{O} \in \mathbf{Op}_{h,m}$, let $\mathbf{Alg}(\mathscr{O})$ denote the 2-category of $\mathscr{O}$-algebras in G-categories. The basic observation is that $\mathbf{Op}_{h,m}$ is a 2-category, and that by pulling back, the morphisms and transformations between these operads give rise to 2-functors and 2-natural transformations. Thus, **Alg** is a 2-functor (contravariant in morphisms and transformations), and it must send the internal equivalences in $\mathbf{Op}_{h,m}$ to 2-equivalences. This is quite useful, because it implies the following "change of norm" theorem (theorem 4.2) when combined with theorem 3.19.

Theorem. *If $\mathcal{M}$ and $\mathcal{N}$ are sets of exponents that generate the same indexing system, then $\mathcal{M}$**SMLax** and $\mathcal{N}$**SMLax** are 2-equivalent, and this equivalence does not change underlying G-categories, G-functors, or G-natural transformations. Similarly in the strong case.*

That said, we are ultimately concerned with the *weak* homotopy type of operads in $\mathbf{Op}_{h,m}$, because this is where Blumberg and Hill's classification applies. Unfortunately, the functor **Alg** does not preserve all weak equivalences in $\mathbf{Op}_{h,m}$ (cf. example 4.4), but it does in the cases that we care most about. Consider the right derived functor $\mathbb{R}\mathbf{Alg}$ and the comparison map $\mathbf{Alg}(\mathscr{O}) \to \mathbb{R}\mathbf{Alg}(\mathscr{O})$. We prove the following homotopy invariance theorem (theorem 4.21).

Theorem. *The comparison map* $\mathbf{Alg}(\mathscr{O}) \to \mathbb{R}\mathbf{Alg}(\mathscr{O})$ *is a biequivalence of* 2*-categories if and only if* $\mathscr{O}$ *is an* N_∞ *operad in* $\mathbf{Op}_{h,m}$*. Therefore* $\mathbf{Alg}$ *sends weak equivalences between homogeneous* N_∞ *operads to biequivalences of* 2*-categories.*

It follows that Blumberg and Hill's classification of N_∞ operads carries over to 2-categories of algebras over homogeneous operads. Concretely, $\mathbb{R}\mathbf{Alg}(\mathscr{O})$ is the 2-category of algebras over an operad $\mathcal{SM}_{\mathcal{N}}$ with the same admissible sets as $\mathscr{O}$, and the map $\mathbf{Alg}(\mathscr{O}) \to \mathbb{R}\mathbf{Alg}(\mathscr{O})$ regards an $\mathscr{O}$-algebra as an $\mathcal{N}$-normed symmetric monoidal category satisfying extra strict relations. For this comparison to be a biequivalence, every $\mathcal{N}$-normed symmetric monoidal category must be equivalent to an $\mathscr{O}$-algebra, which is a matter of categorical strictification. Accordingly, our central construction is an equivariant generalization of Isbell's strictification (construction 4.18).

By combining Isbell's construction, the change of norm theorem, and the coherence theorem for normed symmetric monoidal categories (theorem 2.10), we obtain a method of constructing algebras over any homogeneous N_∞ operad (theorem 4.27). For any set of exponents $\mathcal{N}$, let $\mathcal{N}\mathbf{SMStg}$ denote the 2-category of $\mathcal{N}$-normed symmetric monoidal categories and strong monoidal functors, and for any operad $\mathscr{O}$ in G-categories, let $\mathscr{O}$-$\mathbf{AlgSt}$ denote the 2-category of strict $\mathscr{O}$-algebra G-categories and strict $\mathscr{O}$-morphisms.

Theorem. *Suppose that* $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$ *and that* $\mathcal{N}$ *is a set of exponents that generates the class of admissible sets of* $\mathscr{O}$*. Then there is a* 2*-adjunction*

$$L : \mathcal{N}\mathbf{SMStg} \rightleftarrows \mathscr{O}\text{-}\mathbf{AlgSt} : R$$

whose unit is a levelwise internal equivalence, and whose right adjoint does not affect underlying G*-categories,* G*-functors, or* G*-natural transformations.*

If one knows explicit generators and relations for an operad $\mathscr{O}$, then a more precise identification of $\mathscr{O}$-algebras is possible. In section 4.3, we analyze quotient operads and we explain how to present their algebras (cf. theorem 4.14 and example 4.15). The point of theorem 4.27, however, is that strictification theory can sometimes make this information unnecessary. For example, by taking $\mathscr{O}$ to be the G-Barratt-Eccles operad $\mathscr{P}_G$ (cf. [18]) and $\mathcal{N}$ to be any set of exponents that generates all norms, we can bypass the presentation problem posed by Guillou-May-Merling-Osorno [20, problem 1.36].

The final portion of this chapter analyzes the relationship between operadic algebras and pseudoalgebras, with an eye towards comparing the work in this thesis to [20]. We prove the following analogue to Boardman and Vogt's classical result [4] on homotopy algebras (theorem 4.33). Let $W\mathscr{O}$ be the free homogeneous operad on the objects of $\mathscr{O}$.

Theorem. *Suppose that $\mathscr{O} \in N_\infty$-$\mathbf{Op}_h$. Then the 2-category of pseudoalgebras over $\mathscr{O}$ is isomorphic to the 2-category of strict algebras over $W\mathscr{O}$, and this isomorphism does not affect underling G-categories, G-functors, or G-natural transformations.*

It follows that the study of $\mathscr{O}$-pseudoalgebras is subsumed by the study of normed symmetric monoidal categories, because $W\mathscr{O}$ is isomorphic to $\mathcal{SM}_{\mathcal{N}(\mathscr{O})}$ for some set of exponents $\mathcal{N}(\mathscr{O})$. Note that our notion of pseudoalgebra is slightly more general than the notion considered in [20]. One recovers their objects by enforcing additional strict normality and unitality relations, which amounts to working over an E_∞ quotient of $W\mathscr{O}$ (cf. example 4.38). Therefore theorem 4.27 applies equally well to construct the normal, strictly unital pseudoalgebras of [20] from E_∞ normed symmetric monoidal categories.

Remark. Our results should also be compared to their counterparts in Berger and Moerdijk's homotopy theory of operads [2]. Indeed, N_∞ operads in $\mathbf{Op}_{h,m}$ may be regarded as the Σ-cofibrant objects in the $\underline{\mathbf{Set}}$-model structure on $\mathbf{Op}_{h,m}$ (cf. section 3.5).

4.2 The algebra 2-functors

We begin by outlining the general situation, before specializing to homogeneous categorical operads. There is a 2-category $\mathbf{Op}(G\mathbf{Cat})$ whose objects are operads $\mathscr{O}$ in G-categories, whose 1-morphisms are operad maps $\varphi : \mathscr{N} \to \mathscr{O}$, and whose 2-morphisms are transformations $\eta : \varphi \Rightarrow \psi : \mathscr{N} \rightrightarrows \mathscr{O}$ between operad maps. This means that for every $n \geq 0$, we have a $G \times \Sigma_n$-equivariant natural transformation $\eta_n : \varphi_n \Rightarrow \psi_n : \mathscr{N}(n) \rightrightarrows \mathscr{O}(n)$, that $(\eta_1)_{\mathrm{id}_{\mathscr{N}}} = \mathrm{id}_{\mathrm{id}_{\mathscr{O}}}$, and that $(\eta_{j_1+\cdots+j_k})_{\gamma(y;x_1,\ldots,x_k)} = \gamma((\eta_k)_y; (\eta_{j_1})_{x_1}, \ldots, (\eta_{j_k})_{x_k})$ for every $y \in \mathscr{N}(k)$ and $x_i \in \mathscr{N}(j_i)$. All composites and identities are taken levelwise.

For every $\mathscr{O} \in \mathbf{Op}(G\mathbf{Cat})$, we have the 2-categories $\mathscr{O}$-**AlgSt** $\subset$ $\mathscr{O}$-**AlgPs** $\subset$ $\mathscr{O}$-**AlgLax**, and there is a forgetful 2-functor $U : \mathscr{O}\text{-}\mathbf{AlgLax} \to G\underline{\mathbf{Cat}}$. Moreover, operad maps and transformations induce 2-functors and 2-natural transformations between 2-categories of algebras, but in the opposite direction. In summary:

Proposition 4.1. *Forming the* 2*-categories* $\mathscr{O}$-**AlgLax***, and pulling back along morphisms and transformations, determines a* 2*-functor*

$$\mathbf{Alg}_{lax} : \mathbf{Op}(G\mathbf{Cat})^{\mathrm{coop}} \to 2\mathbf{CAT}/G\underline{\mathbf{Cat}}$$

where $\mathbf{Op}(G\mathbf{Cat})^{\mathrm{coop}}$ *is obtained by reversing the* 1*-morphisms and* 2*-morphisms of* $\mathbf{Op}(G\mathbf{Cat})$*, and* $2\mathbf{CAT}/G\underline{\mathbf{Cat}}$ *is the (improper)* 2*-category of all large* 2*-categories over* $G\underline{\mathbf{Cat}}$*. Similarly, forming the* 2*-categories* $\mathscr{O}$-**AlgPs** *determines a* 2*-functor*

$$\mathbf{Alg}_{ps} : \mathbf{Op}(G\mathbf{Cat})^{\mathrm{coop}}_{\text{2-iso}} \to 2\mathbf{CAT}/G\underline{\mathbf{Cat}}$$

where $\mathbf{Op}(G\mathbf{Cat})^{\mathrm{coop}}_{\text{2-iso}} \subset \mathbf{Op}(G\mathbf{Cat})^{\mathrm{coop}}$ *is the sub-*2*-category that only contains invertible natural transformations.*

Proof. If $\varphi : \mathscr{N} \to \mathscr{O}$ is an operad map, then there is a pullback 2-functor $\varphi^* : \mathscr{O}\text{-}\mathbf{AlgLax} \to \mathscr{N}\text{-}\mathbf{AlgLax}$ defined by $\varphi^*(\mathscr{C}, |\cdot|_{\mathscr{C}}) = (\mathscr{C}, |\cdot|_{\mathscr{C}} \circ \varphi)$ on algebras, $\varphi^*(F, (\partial_n)) = (F, (\partial_n \circ \mathrm{id}_{\varphi_n}))$ on morphisms, and by $\varphi^*\omega = \omega$ on transformations. Thus, φ^* restricts to $\mathscr{O}$-**AlgPs** and $\mathscr{O}$-**AlgSt**, and $U \circ \varphi^* = U$.

If $\eta : \varphi \Rightarrow \psi : \mathscr{N} \rightrightarrows \mathscr{O}$ is a transformation, then for each $\mathscr{O}$-algebra $\mathscr{C}$, there is a lax

$\mathscr{N}$-algebra morphism $\eta^*_{(\mathscr{C},|\cdot|_{\mathscr{C}})} : \psi^*\mathscr{C} \to \varphi^*\mathscr{C}$ given by the identity functor $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, together with the comparison maps $|(\eta_n)_x|_{\mathscr{C}} : |x|_{\varphi^*\mathscr{C}} \circ \mathrm{id}_{\mathscr{C}}^{\times n} \Rightarrow \mathrm{id}_{\mathscr{C}} \circ |x|_{\psi^*\mathscr{C}}$. We obtain a 2-natural transformation $\eta^* : \psi^* \Rightarrow \varphi^* : \mathscr{O}\text{-}\mathbf{AlgLax} \rightrightarrows \mathscr{N}\text{-}\mathbf{AlgLax}$, and $\mathrm{id}_U \circ \eta^* = \mathrm{id}_U$. If $\eta : \varphi \Rightarrow \psi$ is a natural isomorphism, then the components of η^* are $\mathscr{N}$-pseudomorphisms, and we also obtain a 2-natural isomorphism $\eta^* : \psi^* \Rightarrow \varphi^* : \mathscr{O}\text{-}\mathbf{AlgPs} \rightrightarrows \mathscr{N}\text{-}\mathbf{AlgPs}$.

Thus, applying $(-)^*$ reverses the direction of 1-morphisms and 2-morphisms. One can check that it also reverses vertical and horizontal composition, and that it preserves vertical and horizontal identities. □

We now specialize to the sub-2-category $\mathbf{Op}_{h,m} \subset \mathbf{Op}(G\mathbf{Cat})$ of marked, homogeneous operads. As observed earlier, if $\mathscr{N}$ and $\mathscr{O}$ are homogeneous and $\varphi, \psi : \mathscr{N} \rightrightarrows \mathscr{O}$ is a parallel pair of operad maps, then there is a unique transformation $\eta : \varphi \Rightarrow \psi$, and it is an isomorphism. Therefore $\mathbf{Alg}_{lax}$ and $\mathbf{Alg}_{ps}$ restrict to 2-functors

$$\mathbf{Alg}_{lax} : \mathbf{Op}_{h,m}^{\mathrm{coop}} \to 2\mathbf{CAT}/G\underline{\mathbf{Cat}}$$
$$\mathbf{Alg}_{ps} : \mathbf{Op}_{h,m}^{\mathrm{coop}} \to 2\mathbf{CAT}/G\underline{\mathbf{Cat}}.$$

In particular, $\mathbf{Alg}_{lax}$ and $\mathbf{Alg}_{ps}$ preserve internal equivalences, and these are very easy to come by in $\mathbf{Op}_{h,m}$. This has the following simple, but useful consequence.

Theorem 4.2. *If $\mathscr{N}, \mathscr{O} \in \mathbf{Op}_{h,m}$ and there are maps $\mathscr{N} \rightleftarrows \mathscr{O}$, then $\mathscr{N}$-$\mathbf{AlgLax}$ and $\mathscr{O}$-$\mathbf{AlgLax}$ are 2-equivalent over $G\underline{\mathbf{Cat}}$. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are sets of exponents that generate the same indexing system, then $\mathcal{M}\mathbf{SMLax}$ and $\mathcal{N}\mathbf{SMLax}$ are 2-equivalent over $G\underline{\mathbf{Cat}}$. Similarly for 2-categories of pseudomorphisms and strong monoidal functors.*

Proof. Any pair of maps $\mathscr{N} \rightleftarrows \mathscr{O}$ in $\mathbf{Op}_{h,m}$ are part of an internal equivalence. If $\mathcal{M}$ and $\mathcal{N}$ generate the same indexing system, then the free operads $\mathbb{F}(S_{\mathcal{M}})$ and $\mathbb{F}(S_{\mathcal{N}})$ have the same admissible sets (cf. theorem 3.19), and therefore there are operad maps $\mathbb{F}(S_{\mathcal{M}}) \rightleftarrows \mathbb{F}(S_{\mathcal{N}})$. Applying the functor $\widetilde{(-)} : G\mathbf{Set} \to G\mathbf{Cat}$ gives maps $\mathcal{SM}_{\mathcal{M}} \rightleftarrows \mathcal{SM}_{\mathcal{N}}$. By the coherence theorem 2.10, we conclude that $\mathcal{M}\mathbf{SMLax} \cong \mathcal{SM}_{\mathcal{M}}\text{-}\mathbf{AlgLax} \simeq \mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgLax} \cong \mathcal{N}\mathbf{SMLax}$ over $G\underline{\mathbf{Cat}}$, and similarly in the strong monoidal case. □

Remark 4.3. We regard the second part of the preceding result as a "change of norm" theorem. It says that if we are willing to work with strong monoidal functors, then the only relevant feature of a set of exponents $\mathcal{N}$ is the indexing system that it generates.

The internal equivalences in $\mathbf{Op}_{h,m}$ are a form of strong homotopy equivalence, and they are preserved by $\mathbf{Alg}_{lax}$ and $\mathbf{Alg}_{ps}$. However, we are primarily interested in the weak homotopy type of the operads in $\mathbf{Op}_{h,m}$. Every weak equivalence between the cellular operads $\mathcal{SM}_{\mathcal{N}}$ is a strong equivalence, but in general, the algebra 2-functors do not preserve weak equivalences.

Example 4.4. Let G be the trivial group. Then every map in $\mathbf{Op}_{h,m}$ is a weak equivalence, and in particular, we may consider the map $\mathscr{P} \to \mathbf{Com}$. The $\mathscr{P}$-algebras in $\mathbf{Cat}$ are permutative categories, while the $\mathbf{Com}$-algebras in $\mathbf{Cat}$ are strictly commutative monoids. It is well-known that not every permutative category is equivalent to a stricly commutative monoid object, i.e. the pullback $\mathbf{Alg}(\mathbf{Com}) \to \mathbf{Alg}(\mathscr{P})$ is not surjective up to equivalence. We generalize this point in the proof of theorem 4.21.

Thus, we are obliged to take the right derived functors of $\mathbf{Alg}_{lax}$ and $\mathbf{Alg}_{ps}$. Given any operad $\mathscr{O} \in \mathbf{Op}_{h,m}$, we obtain cellular approximations of $\mathscr{O}$ by choosing an operad $\mathcal{SM}_{\mathcal{N}(\mathscr{O})}$ with the same admissible sets as $\mathscr{O}$, and a map $\varphi : \mathcal{SM}_{\mathcal{N}(\mathscr{O})} \to \mathscr{O}$. We may arrange for φ to be surjective, and it is often convenient to assume that it is. The difference between $\mathbb{R}\mathbf{Alg}(\mathscr{O})$ and $\mathbf{Alg}(\mathscr{O})$ is measured by the pullback 2-functors $\varphi^* : \mathscr{O}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$ and $\varphi^* : \mathscr{O}\text{-}\mathbf{AlgPs} \to \mathcal{N}(\mathscr{O})\mathbf{SMStg}$. In what follows, we shall analyze the image of φ^*, and we shall give conditions for when φ^* is a biequivalence.

4.3 Algebras over quotient operads

We begin this section with some basic points concerning quotients of operads in G-sets, and then we consider quotients of homogeneous categorical operads.

Definition 4.5. Let $\mathscr{O}$ be an operad in G-sets. A *congruence relation* on $\mathscr{O}$ is a graded relation $\sim = (\sim_n)_{n\geq 0}$ such that

(i) for each integer $n \geq 0$, $\sim_n$ is an equivalence relation on $\mathscr{O}(n)$,

(ii) for each integer $n \geq 0$, pair $(g,\sigma) \in G \times \Sigma_n$, and elements $x, x' \in \mathscr{O}(n)$, if $x \sim_n x'$, then $(g,\sigma)\cdot x \sim_n (g,\sigma)\cdot x'$,

(iii) for all elements $y, y' \in \mathscr{O}(k)$ and $x_i, x_i' \in \mathscr{O}(j_i)$ for $i = 1, \ldots, k$, if $y \sim_k y'$ and $x_i \sim_{j_i} x_i'$ for all i, then $\gamma(y; x_1, \ldots, x_k) \sim_{\Sigma j_i} \gamma(y'; x_1', \ldots, x_k')$.

Example 4.6. If $\varphi : \mathscr{O} \to \mathscr{O}'$ is a map of G-operads, then the relation $\sim$, defined by $x \sim x' \iff \varphi(x) = \varphi(x')$, is a congruence relation on $\mathscr{O}$. We shall usually refer to this relation as $\ker(\varphi)$.

The definition of a congruence relation is devised to make the following familiar fact true.

Proposition 4.7. *If $\sim$ is a congruence relation on $\mathscr{O}$, then there is a unique G-operad structure on the levelwise quotient $\mathscr{O}/\sim$ that makes the projection $\pi : \mathscr{O} \to \mathscr{O}/\sim$ into a G-operad map. This projection takes $\sim$-equivalent elements of $\mathscr{O}$ to equal elements of $\mathscr{O}/\sim$, and any other G-operad map $\varphi : \mathscr{O} \to \mathscr{O}'$ with this property factors uniquely through π as a G-operad map $\overline{\varphi} : \mathscr{O}/\sim \to \mathscr{O}'$.*

Proof. The conditions on $\sim$ say that the $G\times\Sigma_n$ action $(g,\sigma)\cdot[x] := [(g,\sigma)\cdot x]$ and composition operation $\gamma([y];[x_1],\ldots,[x_k]) := [\gamma(y; x_1, \ldots, x_k)]$ are well-defined. The operad axioms for $\mathscr{O}/\sim$ then follow from those for $\mathscr{O}$, and the projection map $\pi : \mathscr{O} \to \mathscr{O}/\sim$ preserves structure by design. Finally, if $\varphi : \mathscr{O} \to \mathscr{O}'$ sends equivalent elements to equal elements, then $\overline{\varphi}([x]) := \varphi(x)$ is well-defined. $\square$

Observe that the intersection of congruence relations is a congruence relation, and that there is a maximum congruence relation that identifies all elements of any given arity. Thus, we can make the following definition.

Definition 4.8. If $R = (R_n)_{n \geq 0}$ is a graded binary relation on $\mathscr{O}$, we call the intersection of all congruence relations containing R the *congruence relation generated by* R. We denote this congruence relation $\langle R \rangle$.

Note the following formal result.

Corollary 4.9. *Suppose that $\mathscr{O}$ is an operad in G-sets and that R is a graded binary relation on $\mathscr{O}$. If $\varphi : \mathscr{O} \to \mathscr{O}'$ is a map of G-operads that sends R-related elements of $\mathscr{O}$ to equal elements of $\mathscr{O}'$, then there is a unique map of G-operads $\overline{\varphi} : \mathscr{O}/\langle R \rangle \to \mathscr{O}'$ such that $\varphi = \overline{\varphi} \circ \pi : \mathscr{O} \to \mathscr{O}/\langle R \rangle \to \mathscr{O}'$.*

Thus, it is possible to introduce a set of prescribed relations R into an operad. That said, one usually needs to find an explicit description of the relation $\langle R \rangle$ to get a real handle on the quotient $\mathscr{O}/\langle R \rangle$. This is analogous to a word problem because operadic composition γ is noncommutative and noninvertible.

If $\mathscr{O} \in \mathbf{Op}_{h,m}$ is a homogeneous operad and $\sim$ is a congruence relation on $\mathrm{Ob}\mathscr{O}$, then the quotient $\mathscr{O}/\!\sim\, \in \mathbf{Op}_{h,m}$ is computed by forgetting down to G-sets, taking the quotient of $\mathrm{Ob}\mathscr{O}$ there, and then applying $\widetilde{(-)} : G\mathbf{Set} \to G\mathbf{Cat}$. This makes quotients of homogeneous operads completely elementary. To start, we have the first isomorphism theorem.

Proposition 4.10. *Every operad map $\varphi : \mathscr{O}_1 \to \mathscr{O}_2$ in $\mathbf{Op}_{h,m}$ induces an isomorphism $\overline{\varphi} : \mathscr{O}_1/\mathrm{ker}(\varphi) \to \mathrm{im}(\varphi)$, where $\mathrm{ker}(\varphi)$ denotes the congruence relation on $\mathrm{Ob}\mathscr{O}_1$ that relates x and y if and only if $\varphi(x) = \varphi(y)$.*

Proof. The induced map on $\mathrm{Ob}\,\mathscr{O}_1/\mathrm{ker}(\varphi) \to \mathrm{Ob}\,\mathrm{im}(\varphi)$ is an isomorphism of operads in G-sets, and applying $\widetilde{(-)}$ preserves it. □

Corollary 4.11. *Every operad map $\varphi : \mathscr{O}_1 \to \mathscr{O}_2$ in $\mathbf{Op}_{h,m}$ that is surjective on objects is a quotient map.*

That said, the work in this chapter concerns algebras over operads, and therefore we must leave the category $\mathbf{Op}_{h,m}$. The key point is that a quotient $\mathscr{O}_1 \twoheadrightarrow \mathscr{O}_2$ in $\mathbf{Op}_{h,m}$ also has a universal property relative to all operads in G-categories.

Proposition 4.12. *Suppose that $\mathscr{O}$ is an operad in G-sets, that R is a binary relation on $\mathscr{O}$, and that $\varphi : \widetilde{\mathscr{O}} \to \mathscr{C}$ is a map of operads in G-categories. Then the map φ factors through the quotient $\pi : \widetilde{\mathscr{O}} \to \widetilde{\mathscr{O}}/\langle R\rangle$ if and only if for all $n \geq 0$ and $x, y \in \mathscr{O}(n)$, we have $\varphi(x) = \varphi(y)$ and $\varphi(x \to y) = \mathrm{id} : \varphi(x) \to \varphi(y)$ whenever xRy. In such a case, the induced map $\overline{\varphi} : \widetilde{\mathscr{O}}/\langle R\rangle \to \mathscr{C}$ is unique.*

Proof. The "only if" direction holds because π identifies R-related elements of $\mathscr{O}$. Now suppose that φ satisfies the hypotheses for the "if" direction. The universal property of quotient operads in G-sets implies that the map $\varphi : \mathscr{O} \to \mathrm{Ob}\mathscr{C}$ on objects factors as $\overline{\varphi} \circ \pi : \mathscr{O} \to \mathscr{O}/\langle R\rangle \to \mathrm{Ob}\mathscr{C}$. We must extend $\overline{\varphi}$ to a map $\overline{\varphi} : \widetilde{\mathscr{O}}/\langle R\rangle = \widetilde{\mathscr{O}/\langle R\rangle} \to \mathscr{C}$ of operads in G-categories.

Define a congruence relation $\equiv$ on $\mathscr{O}$ by $x \equiv y$ if and only if $\varphi(x) = \varphi(y)$ and $\varphi(x \to y) = \mathrm{id}$. Then $R \subset \equiv$, and hence $\langle R\rangle \subset \equiv$ as well. Given congruence classes $[x], [y] \in \widetilde{\mathscr{O}}/\langle R\rangle$, define $\overline{\varphi}([x] \to [y]) := \varphi(x \to y)$, where x and y are any representatives for $[x]$ and $[y]$. The inclusion $\langle R\rangle \subset \equiv$ implies that $\overline{\varphi}$ is well-defined, and $\overline{\varphi}$ is a map of operads in G-categories because φ is. The factorization $\varphi = \overline{\varphi} \circ \pi : \widetilde{\mathscr{O}} \to \widetilde{\mathscr{O}}/\langle R\rangle \to \mathscr{C}$ holds by design, and the equation $\varphi = \overline{\varphi} \circ \pi$ uniquely determines $\overline{\varphi}$. □

We can use the proposition above to identify the 2-category of algebras over a quotient.

Proposition 4.13. *Suppose that $\mathscr{O}$ is an operad in $\mathbf{Op}_{h,m}$, that R is a binary relation on $\mathrm{Ob}\mathscr{O}$, and let $\pi : \mathscr{O} \to \mathscr{O}/\langle R\rangle$ be the quotient map. Then the pullback 2-functor $\pi^* : (\mathscr{O}/\langle R\rangle)$-$\mathbf{AlgLax} \to \mathscr{O}$-$\mathbf{AlgLax}$ induces an isomorphism between $(\mathscr{O}/\langle R\rangle)$-$\mathbf{AlgLax}$ and the full sub-2-category of $\mathscr{O}$-$\mathbf{AlgLax}$ spanned by the $\mathscr{O}$-algebras $|\cdot| : \mathscr{O} \to \mathbf{End}(\mathscr{C})$ such that for all $n \geq 0$ and $x, y \in \mathscr{O}(n)$, we have $|x| = |y| : \mathscr{C}^{\times n} \to \mathscr{C}$ and $|x \to y| = \mathrm{id} : |x| \Rightarrow |y|$ whenever xRy. Similarly for the 2-categories of pseudomorphisms and strict morphisms.*

Proof. We construct an inverse 2-functor π_*. If $(\mathscr{C}, |\cdot|_{\mathscr{C}})$ is an $\mathscr{O}$-algebra with the property above, then its structure map $|\cdot|_{\mathscr{C}} : \mathscr{O} \to \mathbf{End}(\mathscr{C})$ induces a map $\overline{|\cdot|_{\mathscr{C}}} : \mathscr{O}/\langle R \rangle \to \mathbf{End}(\mathscr{C})$ by proposition 4.12. We define $\pi_*\mathscr{C} := (\mathscr{C}, \overline{|\cdot|_{\mathscr{C}}})$.

If $(F, \partial_\bullet) : \mathscr{C} \to \mathscr{D}$ is a lax $\mathscr{O}$-morphism between two such $\mathscr{O}$-algebras, then we let $\pi_* F : \pi_*\mathscr{C} \to \pi_*\mathscr{D}$ be the G-functor $F : \mathscr{C} \to \mathscr{D}$, together with the transformations $(\pi_*\partial_n)_{[x]} = (\partial_n)_x$. To see that this is well-defined, introduce the following congruence on $\mathscr{O}$: declare $x \equiv y$ if and only if $|x|_{\mathscr{C}} = |y|_{\mathscr{C}}$, $|x|_{\mathscr{D}} = |y|_{\mathscr{D}}$, and $(\partial_n)_x = (\partial_n)_y$. If xRy, then the naturality of $(\partial_n)_x$ in x implies that $x \equiv y$, and therefore $\langle R \rangle \subset \equiv$.

The 2-functor π_* does nothing to $\mathscr{O}$-transformations. ☐

We deduce that $\mathscr{O}$-algebras, for any operad $\mathscr{O} \in \mathbf{Op}_{h,m}$, are just normed symmetric monoidal categories that satisfy some extra strict relations.

Theorem 4.14. *Suppose that $\mathscr{O}$ is any operad in $\mathbf{Op}_{h,m}$. Then there is a set of exponents $\mathcal{N}$ and a binary relation R on $\mathcal{SM}_{\mathcal{N}}$ such that $\mathscr{O}$-$\mathbf{AlgLax}$ is isomorphic over $G\underline{\mathbf{Cat}}$ to the full sub-2-category of $\mathcal{N}\mathbf{SMLax}$ spanned by the objects satisfying the strict relations in R. Similarly in the strong and strict case.*

Proof. Given any $\mathscr{O} \in \mathbf{Op}_{h,m}$, choose a set of exponents $\mathcal{N}$ and a surjective map $\pi : \mathcal{SM}_{\mathcal{N}} \to \mathscr{O}$. If R is any binary relation that generates $\ker(\pi)$, then $\mathcal{SM}_{\mathcal{N}}/\langle R \rangle \cong \mathscr{O}$, and the rest follows from the previous proposition. ☐

If the set of exponents $\mathcal{N}$ and relations R are known, then theorem 4.14 gives a means of presenting $\mathscr{O}$-algebras.

Example 4.15. The N_∞ permutativity operad $\mathscr{P}_{\mathcal{N}}$ (cf. section 6.4) is defined to be the operad $\mathcal{SM}_{\mathcal{N}}$ modulo the relations

$$\otimes(\otimes(x_1, x_2), x_3) \sim \otimes(x_1, \otimes(x_2, x_3)) \quad \otimes(e(), x_1) \sim x_1 \quad \otimes(x_1, e()) \sim x_1$$

$$\bigotimes_T(e(), e(), \ldots, e()) \sim e() \quad (\text{all } T \in \mathcal{N}).$$

Therefore $\mathscr{P}_{\mathcal{N}}$-**AlgLax** is isomorphic to the full sub-2-category of $\mathcal{N}$**SMLax** spanned by the $\mathscr{C}$ for which all of the isomorphisms

$$(C_1 \otimes C_2) \otimes C_3 \xrightarrow{\alpha} C_1 \otimes (C_2 \otimes C_3) \qquad e \otimes C \xrightarrow{\lambda} C \qquad C \otimes e \xrightarrow{\rho} C$$

$$\bigotimes_T (e, e, \ldots, e) \xrightarrow{\upsilon_T} e \quad (\text{all } T \in \mathcal{N})$$

are identity maps. Similarly in the strong and strict cases.

That said, in the next section we shall explain why the precise set of exponents $\mathcal{N}$ and relations R can sometimes be ignored.

4.4 Isbell's construction and applications

Suppose that $\psi : \mathscr{O} \to \mathscr{N}$ is a map in $\mathbf{Op}_{h,m}$ and consider the pullback 2-functor $\psi^* : \mathscr{N}\text{-}\mathbf{AlgLax} \to \mathscr{O}\text{-}\mathbf{AlgLax}$. If there is an operad map $\varphi : \mathscr{N} \to \mathscr{O}$ in the other direction, then ψ^* is an equivalence over $G\underline{\mathbf{Cat}}$ by theorem 4.2, but this is usually too much to hope for. In the case of greatest interest, the operad $\mathscr{O}$ is free and the map $\psi : \mathscr{O} \to \mathscr{N}$ is a cellular approximation of $\mathscr{N}$. Thus, we shall study what happens when there is only a map of symmetric sequences $\varphi : \mathscr{N} \to \mathscr{O}$ in the other direction.

Proposition 4.16. *Suppose that $\mathscr{N}, \mathscr{O} \in \mathbf{Op}_{h,m}$ and that $\varphi : \mathscr{N} \to \mathscr{O}$ is a map of symmetric sequences. Then there is a pullback 2-functor*

$$\varphi^*_{\mathbb{N}} : \mathscr{O}\text{-}\mathbf{AlgPs} \to \mathscr{N}\text{-}\mathbf{AlgSt}$$

such that for every $\mathscr{O}$-algebra $\mathscr{C}$, there is an equivalence $\eta : \mathscr{C} \rightleftarrows \varphi^*_{\mathbb{N}}\mathscr{C} : \varepsilon$ in $G\underline{\mathbf{Cat}}$.

Proof. We give an explicit construction of $\varphi^*_{\mathbb{N}}$ below. □

Remark 4.17. The conceptual point is that pulling back along φ converts strict $\mathscr{O}$-algebras into $\mathscr{N}$-pseudoalgebras (cf. proposition 4.32), and as discussed in Guillou-May-Merling-

Osorno [20], these can be strictified using general 2-category theory of Power and Lack (cf. [28] and [37]).

The following construction is a direct generalization of Isbell's strictification [25].

Construction 4.18. Let $(\mathscr{C}, |\cdot|_{\mathscr{C}} : \mathscr{O} \to \mathbf{End}(\mathscr{C}))$ be an $\mathscr{O}$-algebra. We construct an $\mathscr{N}$-algebra $\varphi_{\mathbb{N}}^{*}\mathscr{C}$ from $\mathscr{C}$ as follows.

Let the object set of $\varphi_{\mathbb{N}}^{*}\mathscr{C}$ be the free $\mathrm{Ob}\mathscr{N}$-algebra on $\mathrm{Ob}\mathscr{C}$, i.e.

$$\mathrm{Ob}\,\varphi_{\mathbb{N}}^{*}\mathscr{C} := \coprod_{j\geq 0} \mathrm{Ob}\mathscr{N}(j) \times_{\Sigma_j} \mathrm{Ob}\mathscr{C}^{\times j},$$

and write $[x; C_1, \ldots, C_j]$ or $[x; C_\bullet]$ for its generic element. There is an evaluation G-map $\varepsilon : \mathrm{Ob}\,\varphi_{\mathbb{N}}^{*}\mathscr{C} \to \mathrm{Ob}\mathscr{C}$ that sends $[x; C_1, \ldots, C_k]$ to $|\varphi(x)|_{\mathscr{C}}(C_1, \ldots, C_k)$. We use the function ε to create a G-category structure on $\varphi_{\mathbb{N}}^{*}\mathscr{C}$. Define

$$\varphi_{\mathbb{N}}^{*}\mathscr{C}\Big([x; C_\bullet], [y; D_\bullet]\Big) := \mathscr{C}\Big(\varepsilon[x; C_\bullet], \varepsilon[y; D_\bullet]\Big),$$

declare the morphisms $\mathrm{id}_{\varepsilon[x;C_\bullet]} : [x; C_\bullet] \to [x; C_\bullet]$ to be identities, and equip $\varphi_{\mathbb{N}}^{*}\mathscr{C}$ with the composition and G-action on $\mathscr{C}$. This makes $\varphi_{\mathbb{N}}^{*}\mathscr{C}$ into a G-category.

We extend ε to a G-functor $\varepsilon : \varphi_{\mathbb{N}}^{*}\mathscr{C} \to \mathscr{C}$ by making it the identity map on homs. In the other direction, we define a G-functor $\eta : \mathscr{C} \to \varphi_{\mathbb{N}}^{*}\mathscr{C}$ by $\eta(C) = [\mathrm{id}_{\mathscr{N}}; C]$ on objects. Given a morphism $f : C \to D$ in $\mathscr{C}$, we let ηf be the composite

$$|\varphi(\mathrm{id}_{\mathscr{N}})|_{\mathscr{C}}(C) \xrightarrow{\kappa_C} C \xrightarrow{\quad f \quad} D \xrightarrow{\kappa_D^{-1}} |\varphi(\mathrm{id}_{\mathscr{N}})|_{\mathscr{C}}(D),$$

where κ is the G-natural isomorphism $|\varphi(\mathrm{id}_{\mathscr{N}}) \to \mathrm{id}_{\mathscr{O}}|_{\mathscr{C}} : |\varphi(\mathrm{id}_{\mathscr{N}})|_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$. The composite $\varepsilon \circ \eta$ is just $|\varphi(\mathrm{id}_{\mathscr{N}})|_{\mathscr{C}}$, and therefore κ defines a G-natural isomorphism $\varepsilon \circ \eta \cong \mathrm{id}_{\mathscr{C}}$. For the other composite, let $\lambda_{[x;C_\bullet]} = \kappa^{-1}_{\varepsilon[x;C_\bullet]} : [x; C_\bullet] \to \eta\varepsilon[x; C_\bullet]$. Then $\lambda : \mathrm{id}_{\varphi_{\mathbb{N}}^{*}\mathscr{C}} \Rightarrow \eta \circ \varepsilon$ is a G-natural isomorphism, and $\mathscr{C} \simeq \varphi_{\mathbb{N}}^{*}\mathscr{C}$.

We now make $\varphi_{\mathbb{N}}^{*}\mathscr{C}$ into an $\mathscr{N}$-algebra. Given any $k \geq 0$ and $p \in \mathscr{N}(k)$, define the functor $|p|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}} : (\varphi_{\mathbb{N}}^{*}\mathscr{C})^{\times k} \to \varphi_{\mathbb{N}}^{*}\mathscr{C}$ on objects using the free algebra structure

$$|p|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}}\Big([x^1; C^1_\bullet], \ldots, [x^k; C^k_\bullet]\Big) := [\gamma_{\mathscr{N}}(p; x^1, \ldots, x^k); C^1_\bullet, \ldots, C^k_\bullet].$$

We order the objects $C^1_\bullet, \ldots, C^k_\bullet$ lexicographically. On morphisms, proceed as follows. For any $[x^1; C^1_\bullet], \ldots, [x^k, C^k_\bullet] \in \varphi_{\mathbb{N}}^{*}\mathscr{C}$, let $\mathrm{can}_{[x^1;C^1_\bullet],\ldots,[x^k;C^k_\bullet]}$ be the $\mathscr{O}$-algebra coherence map

$$\Big|\gamma_{\mathscr{O}}(\varphi(p); \varphi(x^1), \ldots, \varphi(x^k))\Big|_{\mathscr{C}}(C^1_\bullet, \ldots, C^k_\bullet) \to \Big|\varphi(\gamma_{\mathscr{N}}(p; x^1, \ldots, x^k))\Big|_{\mathscr{C}}(C^1_\bullet, \ldots, C^k_\bullet).$$

Then, given $f^i : [x^i; C^i_\bullet] \to [y^i; D^i_\bullet]$ for $i = 1, \ldots, k$, define $|p|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}}(f^1, \ldots, f^k)$ to be

$$\mathrm{can}_{[y^1;D^1_\bullet],\ldots,[y^k;D^k_\bullet]} \;\circ\; |\varphi(p)|_{\mathscr{C}}(f^1, \ldots, f^k) \;\circ\; \mathrm{can}^{-1}_{[x^1;C^1_\bullet],\ldots,[x^k;C^k_\bullet]}.$$

Finally, given the (unique) morphism $p \to q$ in $\mathscr{N}(k)$, and any $[x^1; C^1_\bullet], \ldots, [x^k; C^k_\bullet]$, we let $(|p \to q|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}})_{[x^1;C^1_\bullet],\ldots,[x^k;C^k_\bullet]}$ be the $\mathscr{O}$-algebra coherence isomorphism

$$\Big|\varphi(\gamma_{\mathscr{N}}(p; x^1, \ldots, x^k))\Big|_{\mathscr{C}}(C^1_\bullet, \ldots, C^k_\bullet) \to \Big|\varphi(\gamma_{\mathscr{N}}(q; x^1, \ldots, x^k))\Big|_{\mathscr{C}}(C^1_\bullet, \ldots, C^k_\bullet).$$

This defines a natural isomorphism $|p \to q|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}} : |p|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}} \Rightarrow |q|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}}$, and unwinding the definitions reveals that $|\cdot|_{\varphi_{\mathbb{N}}^{*}\mathscr{C}} : \mathscr{N} \to \mathbf{End}(\varphi_{\mathbb{N}}^{*}\mathscr{C})$ is a map of G-operads.

Next, we consider 1-morphisms. Given an $\mathscr{O}$-pseudomorphism $(F, \partial_\bullet) : \mathscr{C} \to \mathscr{D}$, we define a strict $\mathscr{N}$-morphism $\varphi_{\mathbb{N}}^{*}F : \varphi_{\mathbb{N}}^{*}\mathscr{C} \to \varphi_{\mathbb{N}}^{*}\mathscr{D}$ by $\varphi_{\mathbb{N}}^{*}F[x; C_\bullet] := [x; FC_\bullet]$ on objects. Given a morphism $f : [x; C_\bullet] \to [y; D_\bullet]$ in $\varphi_{\mathbb{N}}^{*}\mathscr{C}$, we define $\varphi_{\mathbb{N}}^{*}Ff := \partial^{-1} \circ Ff \circ \partial : [x; FC_\bullet] \to [y; FD_\bullet]$. The compatibility between $\partial_\bullet$ and the coherence data for the $\mathscr{O}$-algebra structures on $\mathscr{C}$ and $\mathscr{D}$ ensures that $\varphi_{\mathbb{N}}^{*}F : \varphi_{\mathbb{N}}^{*}\mathscr{C} \to \varphi_{\mathbb{N}}^{*}\mathscr{D}$ preserves $\mathscr{N}$-algebra operations and coherence data strictly.

Finally, if $\omega : (F, \partial_\bullet) \Rightarrow (F', \partial'_\bullet) : \mathscr{C} \rightrightarrows \mathscr{D}$ is an $\mathscr{O}$-transformation, then we define the $\mathscr{N}$-

transformation $\varphi_{\mathbb{N}}^{*}\omega : \varphi_{\mathbb{N}}^{*}F \Rightarrow \varphi_{\mathbb{N}}^{*}F'$ by the formula $(\varphi_{\mathbb{N}}^{*}\omega)_{[x;C_1,\dots,C_j]} = |\varphi(x)|_{\mathscr{D}}(\omega_{C_1},\dots,\omega_{C_j}) : [x; FC_\bullet] \to [x; F'C_\bullet]$.

Lemma 4.19. *Suppose that $\mathscr{N}, \mathscr{O} \in \mathbf{Op}_{h,m}$, that $\varphi : \mathscr{N} \to \mathscr{O}$ is a map of symmetric sequences, and that $\psi : \mathscr{O} \to \mathscr{N}$ is a map of operads. Then the $\mathscr{O}$-algebras $\psi^*\varphi_{\mathbb{N}}^{*}\mathscr{C}$ and $\mathscr{C}$ are equivalent in $\mathscr{O}$-$\mathbf{AlgPs}$ for any $\mathscr{O}$-algebra $\mathscr{C}$.*

Proof. Fix an $\mathscr{O}$-algebra $\mathscr{C}$. We make η into an $\mathscr{O}$-pseudomorphism $\eta : \mathscr{C} \to \psi^*\varphi_{\mathbb{N}}^{*}\mathscr{C}$ by defining the $(C_1,\dots,C_n)$-component of $|p|_{\psi^*\varphi_{\mathbb{N}}^{*}\mathscr{C}} \circ \eta^{\times n} \Rightarrow \eta \circ |p|_{\mathscr{C}}$ to be the $\mathscr{O}$-algebra coherence isomorphism

$$|\varphi(\psi(p))|_{\mathscr{C}}(C_1,\dots,C_n) \to |\gamma_{\mathscr{O}}(\varphi(\mathrm{id}_{\mathscr{N}}); p)|_{\mathscr{C}}(C_1,\dots,C_n),$$

considered as a morphism $[\psi(p); C_\bullet] \to [\mathrm{id}_{\mathscr{N}}; |p|_{\mathscr{C}}(C_\bullet)]$. We make ε into an $\mathscr{O}$-pseudomorphism $\varepsilon : \psi^*\varphi_{\mathbb{N}}^{*}\mathscr{C} \to \mathscr{C}$ by defining the $([x^1; C_\bullet^1],\dots,[x^n; C_\bullet^n])$-component of $|p|_{\mathscr{C}} \circ \varepsilon^{\times n} \Rightarrow \varepsilon \circ |p|_{\psi^*\varphi_{\mathbb{N}}^{*}\mathscr{C}}$ to be the $\mathscr{O}$-algebra coherence isomorphism

$$\Big|\gamma_{\mathscr{O}}(p; \varphi(x^1),\dots,\varphi(x^n))\Big|_{\mathscr{C}}(C_\bullet^1,\dots,C_\bullet^n) \to \Big|\varphi(\gamma_{\mathscr{N}}(\psi(p); x^1,\dots,x^n))\Big|_{\mathscr{C}}(C_\bullet^1,\dots,C_\bullet^n).$$

With these additional data, the G-natural isomorphisms $\kappa : \varepsilon \circ \eta \Rightarrow \mathrm{id}_{\mathscr{C}}$ and $\lambda : \mathrm{id}_{\psi^*\varphi_{\mathbb{N}}^{*}\mathscr{C}} \Rightarrow \eta \circ \varepsilon$ become invertible $\mathscr{O}$-transformations. □

Recall that a *strict 2-equivalence* is a pair of 2-functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ together with specified 2-natural isomorphisms $\mathrm{id}_{\mathscr{C}} \cong GF$ and $FG \cong \mathrm{id}_{\mathscr{D}}$, while a *biequivalence* $F : \mathscr{C} \to \mathscr{D}$ is a 2-functor that is surjective on objects up to equivalence, and locally an equivalence on every hom category. If F is part of a strict 2-equivalence, then it is a biequivalence, and biequivalences satisfy the 2 out of 3 property.

Proposition 4.20. *Suppose that $\mathscr{N}, \mathscr{O} \in N_\infty$-$\mathbf{Op}_{h,m}$ and that $\pi : \mathscr{O} \to \mathscr{N}$ is an operad map that is both a quotient and a weak equivalence. Then the pullback 2-functor*

$\pi^* : \mathscr{N}$-**AlgLax** $\to \mathscr{O}$-**AlgLax** *is an embedding and a biequivalence. Similarly for* 2-*categories of pseudomorphisms.*

Proof. In this case, $\mathscr{N}$ and $\mathscr{O}$ have the same admissible sets, and since $\mathscr{N}$ is Σ-free, we can construct a map of symmetric sequences $\varphi : \mathscr{N} \to \mathscr{O}$ in the other direction. By proposition 4.13, the pullback π^* is injective on objects and an isomorphism on hom categories, and the previous lemma says that π^* is surjective on objects, up to equivalence. ☐

This proposition has several useful consequences. To start, it helps us resolve the difference between $\mathbf{Alg}_{lax}$ and $\mathbb{R}\mathbf{Alg}_{lax}$.

Theorem 4.21. *Consider the category* $\mathbf{Op}_{h,m}$.

1. *If* $\mathscr{O} \in N_\infty$-$\mathbf{Op}_{h,m}$, *then the pullback* $\mathbf{Alg}_{lax}(\mathscr{O}) \to \mathbb{R}\mathbf{Alg}_{lax}(\mathscr{O})$ *is a biequivalence and an embedding.*
2. *If* $\varphi : \mathscr{N} \to \mathscr{O}$ *is a weak equivalence between operads* $\mathscr{N}, \mathscr{O} \in N_\infty$-$\mathbf{Op}_{h,m}$, *then the pullback* $\varphi^* : \mathbf{Alg}_{lax}(\mathscr{O}) \to \mathbf{Alg}_{lax}(\mathscr{N})$ *is a biequivalence.*
3. *If* $\mathscr{O} \in \mathbf{Op}_{h,m}$ *is not* Σ*-free, then the pullback* $\mathbf{Alg}_{lax}(\mathscr{O}) \to \mathbb{R}\mathbf{Alg}_{lax}(\mathscr{O})$ *is not a biequivalence.*

Analogous statements hold for 2*-categories of pseudomorphisms.*

Proof. Define $W\mathscr{O} := \widetilde{\mathbb{F}\mathrm{Ob}\mathscr{O}}$. If $\mathscr{O} \in N_\infty$-$\mathbf{Op}_{h,m}$, then the counit $\varepsilon : W\mathscr{O} \to \mathscr{O}$ is a cellular approximation of $\mathscr{O}$ in $\mathbf{Op}_{h,m}$ because $\mathscr{O}$ is Σ-free. The map ε is a quotient and a weak equivalence because the unit $\eta : \mathscr{O} \to W\mathscr{O}$ splits it, and thus the pullback ε^* is a biequivalence and an embedding.

For the second claim, apply $\mathbf{Alg}_{lax}$ to the commutative square

$$\begin{tikzcd}
W\mathscr{N} \arrow[r, "W\varphi"] \arrow[d, "\varepsilon"'] & W\mathscr{O} \arrow[d, "\varepsilon"] \\
\mathscr{N} \arrow[r, "\varphi"'] & \mathscr{O}
\end{tikzcd}$$

and note that $W\varphi$ is an equivalence in $\mathbf{Op}_{h,m}$. Hence $(W\varphi)^*$ is a strict 2-equivalence, and φ^* is a biequivalence by the 2 out of 3 property.

For the last claim, suppose that $\mathscr{O} \in \mathbf{Op}_{h,m}$ is not Σ-free, and consider the cellular approximation

$$Q\mathscr{O} := G \times \Sigma_0/G \;\sqcup\; G \times \Sigma_2/G \;\sqcup \coprod_{\substack{n \geq 0 \\ x \in \mathscr{O}(n) \\ \Gamma \subset \mathrm{Stab}(x)}} G \times \Sigma_n/\Gamma.$$

The map $q : Q\mathscr{O} \to \mathscr{O}$ sends the first two factors to the marked operations in $\mathscr{O}$, and it sends the coset $e\Gamma$ to the point $x \in \mathscr{O}(n)$. We shall show by example that the pullback $q^* : \mathbf{Alg}_{lax}(\mathscr{O}) \to \mathbf{Alg}_{lax}(Q\mathscr{O})$ is not surjective on equivalence classes of objects.

Choose a set of exponents $\mathscr{N}$ corresponding to the subgroups Γ, so that $Q\mathscr{O} \cong \mathcal{SM}_{\mathscr{N}}$, and let $\mathscr{C}$ be $\mathbf{Fun}(\mathbb{T}G, \mathbf{Set})$ equipped with the standard $\mathscr{N}$-normed symmetric monoidal structure obtained from the coproduct on $\mathbf{Set}$ (cf. section 5.2 for a thorough discussion of this category). Suppose for contradiction that $\mathscr{C} \simeq q^*\mathscr{D}$ in $\mathbf{Alg}_{lax}(Q\mathscr{O})$ for some $\mathscr{O}$-algebra $\mathscr{D}$. Then there are pseudomorphisms $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ and $Q\mathscr{O}$-natural isomorphisms $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathscr{D}}$. Choose $x \in \mathscr{O}(n)$ and a nontrivial permutation $\sigma \in \Sigma_n$ such that $x\sigma = x$, and let $\coprod_T \in \mathcal{SM}_{\mathscr{N}}$ lift x. Then $|\coprod_T \to \coprod_T \sigma|_{q^*\mathscr{D}}$ is the identity transformation, and the ladder diagram

$$\begin{tikzcd}
G \circ |\bigsqcup_T|_{q^*\mathscr{D}} \circ F^{\times n} \arrow[r, "GF_{\bigsqcup_T}"] \arrow[d, "\mathrm{id}"'] & G \circ F \circ |\bigsqcup_T|_{\mathscr{C}} \arrow[r, "\eta^{-1}|\bigsqcup_T|_{\mathscr{C}}"] \arrow[d] & |\bigsqcup_T|_{\mathscr{C}} \arrow[r, "\upsilon_T"] \arrow[d] & |\bigsqcup_n|_{\mathscr{C}} \arrow[d, "\sigma"] \\
G \circ |\bigsqcup_T \sigma|_{q^*\mathscr{D}} \circ F^{\times n} \arrow[r, "GF_{\bigsqcup_T}\sigma"'] & G \circ F \circ |\bigsqcup_T \sigma|_{\mathscr{C}} \arrow[r, "\eta^{-1}|\bigsqcup_T|_{\mathscr{C}}\sigma"'] & |\bigsqcup_T \sigma|_{\mathscr{C}} \arrow[r, "\upsilon_T \sigma"'] & |\bigsqcup_n \sigma|_{\mathscr{C}}
\end{tikzcd}$$

of natural isomorphisms commutes. Evaluating at the tuple $(*, *, \dots, *)$ of n copies of the terminal object shows that $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is the identity permutation, contrary to our choice. □

We also obtain the following familiar result [32, proposition 3.4].

Proposition 4.22. *Suppose that $\mathscr{O}_1, \mathscr{O}_2 \in N_\infty$-$\mathbf{Op}_{h,m}$ and that the operads $\mathscr{O}_1$ and $\mathscr{O}_2$ have the same admissible sets. Then the product diagram $\mathscr{O}_1 \leftarrow \mathscr{O}_1 \times \mathscr{O}_2 \to \mathscr{O}_2$ induces a zig-zag of biequivalences and embeddings*

$$\pi_1^* : \mathscr{O}_1\text{-}\mathbf{AlgLax} \to (\mathscr{O}_1 \times \mathscr{O}_2)\text{-}\mathbf{AlgLax} \leftarrow \mathscr{O}_2\text{-}\mathbf{AlgLax} : \pi_2^*.$$

Similarly for 2-categories of pseudomorphisms.

Proof. The operads $\mathscr{O}_1$, $\mathscr{O}_2$, and $\mathscr{O}_1 \times \mathscr{O}_2$ all have the same admissible sets, and the product projections are surjective. Now apply proposition 4.20. □

Roughly speaking, proposition 4.20 allows us to introduce "sensible" strict relations in our algebras. We illustrate by way of example.

Example 4.23. Consider the quotient $\pi : \mathcal{SM}_{\mathcal{N}} \to \mathscr{P}_{\mathcal{N}}$. Both of these operads are Σ-free, and they have the same admissible sets. Therefore proposition 4.20 applies, and we deduce that $\pi^* : \mathscr{P}_{\mathcal{N}}$-**AlgLax** $\to \mathcal{SM}_{\mathcal{N}}$-**AlgLax** is a biequivalence. In light of the identification of $\mathscr{P}_{\mathcal{N}}$-**AlgLax** given in example 4.15, this says that every $\mathcal{N}$-normed symmetric monoidal category $\mathscr{C}$ is strong monoidally equivalent to some $\mathscr{D}$ for which the tensor product $\otimes$ is strictly associative and unital, and for which the norms $\bigotimes_T$ are strictly basepoint-preserving.

Remark 4.24. Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$ and that R is a binary relation on $\mathscr{O}$. We can always form the quotient $\pi : \mathscr{O} \to \mathscr{O}/\langle R \rangle$ in $\mathbf{Op}_{h,m}$, and we always have an embedding $\pi^* : (\mathscr{O}/\langle R \rangle)\text{-}\mathbf{AlgLax} \to \mathscr{O}\text{-}\mathbf{AlgLax}$. To deduce that π^* is a biequivalence using proposition 4.20, we must verify that $\mathscr{O}/\langle R \rangle$ is still Σ-free, and that it does not have any more admissible sets than $\mathscr{O}$. This means that the identifications in R must not introduce any strict commutativity relations or create new norms.

We conclude by considering one further specialization of Isbell's construction.

Theorem 4.25. *Supppose that $\mathscr{N}, \mathscr{O} \in \mathbf{Op}_{h,m}$, that $\varphi : \mathscr{N} \to \mathscr{O}$ is a map of symmetric sequences, and that $\psi : \mathscr{O} \to \mathscr{N}$ is a map of operads. Suppose further that $\psi \circ \varphi = \mathrm{id}_{\mathscr{N}}$. Then there is a 2-adjunction*

$$\varphi^*_{\mathbb{N}} : \mathscr{O}\text{-}\mathbf{AlgPs} \rightleftarrows \mathscr{N}\text{-}\mathbf{AlgSt} : \psi^*,$$

and the unit $\eta_{\mathscr{C}} : \mathscr{C} \to \psi^* \varphi^*_{\mathbb{N}} \mathscr{C}$ is an internal equivalence for all $\mathscr{C}$.

Proof. If $\psi \circ \varphi = \mathrm{id}$, then the map $\varepsilon : \varphi^*_{\mathbb{N}} \psi^* \mathscr{D} \to \mathscr{D}$ of construction 4.18 is actually a strict $\mathscr{N}$-algebra morphism for every $\mathscr{D} \in \mathscr{N}\text{-}\mathbf{AlgSt}$. The $\mathscr{O}$-pseudomorphisms $\eta_{\mathscr{C}} : \mathscr{C} \to \psi^* \varphi^*_{\mathbb{N}} \mathscr{C}$ define a 2-natural transformation $\eta : \mathrm{id} \Rightarrow \psi^* \varphi^*_{\mathbb{N}}$, the strict $\mathscr{N}$-morphisms $\varepsilon_{\mathscr{D}} : \varphi^*_{\mathbb{N}} \psi^* \mathscr{D} \to \mathscr{D}$ define a 2-natural transformation $\varepsilon : \varphi^*_{\mathbb{N}} \psi^* \Rightarrow \mathrm{id}$, and the triangle identities hold. Thus, we obtain a 2-adjunction, and lemma 4.19 says that $\eta_{\mathscr{C}}$ is an internal equivalence. □

Example 4.26. Working nonequivariantly, we have maps $\mathscr{P} \to \mathcal{SM} \to \mathscr{P}$. In this case, the constructions specialize to Isbell's original strictification of symmetric monoidal categories to permutative ones.

There are analogous equivariant statements. For any set of exponents $\mathcal{N}$, we have the operads $\mathcal{SM}_{\mathcal{N}}$ and $\mathscr{P}_{\mathcal{N}}$, and there is a quotient map $\mathcal{SM}_{\mathcal{N}} \to \mathscr{P}_{\mathcal{N}}$ by definition (cf. section

6.4). However, as indicated in the proof of lemma 6.17 (cf. section 6.6), the operad $\mathscr{P}_{\mathcal{N}}$ may also be presented as a quotient and sub-symmetric sequence of $\mathscr{O} = \widetilde{\mathbb{F}}(\mathbf{As} \sqcup \coprod G \times \Sigma_{|T|}/\Gamma_T)$, and thus the quotient $\mathscr{O} \twoheadrightarrow \mathscr{P}_{\mathcal{N}}$ has a section in symmetric sequences. Since this map factors as a pair of quotients $\mathscr{O} \twoheadrightarrow \mathcal{SM}_{\mathcal{N}} \twoheadrightarrow \mathscr{P}_{\mathcal{N}}$, the same is true for $\mathcal{SM}_{\mathcal{N}} \twoheadrightarrow \mathscr{P}_{\mathcal{N}}$, and thus we obtain a strictification 2-adjunction $\mathrm{st} : \mathcal{N}\mathbf{SMStg} \rightleftarrows \mathscr{P}_{\mathcal{N}}\text{-}\mathbf{AlgSt} : \mathrm{inc}$, whose unit maps are internal equivalences.

If $\mathscr{O}$ is any operad in $\mathbf{Op}_{h,m}$, then the unit and counit $\mathscr{O} \to \widetilde{\mathbb{F}}(\mathrm{Ob}\mathscr{O}) \to \mathscr{O}$ give an example of an operad map with a section in symmetric sequences. Therefore theorem 4.25 applies quite generally. Combining everything, we obtain the following presentation theorem.

Theorem 4.27. *Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$ and that $\mathcal{N}$ is a set of exponents that generates the class of admissible sets of $\mathscr{O}$. Then there is a 2-adjunction*

$$L : \mathcal{N}\mathbf{SMStg} \rightleftarrows \mathscr{O}\text{-}\mathbf{AlgSt} : R$$

whose unit maps are internal equivalences, and whose right adjoint does not affect underlying G-categories, G-functors, and G-natural transformations.

Proof. Let $W\mathscr{O} = \widetilde{\mathbb{F}}(\mathrm{Ob}\mathscr{O})$, and consider the following chain.

$$\mathcal{N}\mathbf{SMStg} \overset{\text{iso}}{\rightleftarrows} \mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgPs} \overset{\text{equiv}}{\rightleftarrows} W\mathscr{O}\text{-}\mathbf{AlgPs} \overset{\text{adj}}{\rightleftarrows} \mathscr{O}\text{-}\mathbf{AlgSt}$$

The operads $\mathscr{O}$, $W\mathscr{O}$, and $\mathcal{SM}_{\mathcal{N}}$ all have the same admissible sets. Since $W\mathscr{O}$ and $\mathcal{SM}_{\mathcal{N}}$ are free, there is an equivalence $\mathcal{SM}_{\mathcal{N}} \rightleftarrows W\mathscr{O}$ in $\mathbf{Op}_{h,m}$, which induces a 2-equivalence $\mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgPs} \simeq W\mathscr{O}\text{-}\mathbf{AlgPs}$. The isomorphism $\mathcal{N}\mathbf{SMStg} \cong \mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgPs}$ is theorem 2.10. Finally, the unit $\eta : \mathscr{O} \to W\mathscr{O}$ is a section of $\varepsilon : W\mathscr{O} \to \mathscr{O}$, and thus theorem 4.25 provides a 2-adjunction $\eta_{\mathbb{O}}^* : W\mathscr{O}\text{-}\mathbf{AlgPs} \rightleftarrows \mathscr{O}\text{-}\mathbf{AlgSt} : \varepsilon^*$ whose unit maps are internal equivalences. □

4.5 Pseudoalgebras and the W-construction

Thus far, we have only considered strict algebras over operads, but Guillou-May-Merling-Osorno have developed a substantial theory of equivariant symmetric monoidal structure based on operadic pseudoalgebras. In this section, we explain how the W-construction $W\mathscr{O} = \widetilde{\mathbb{F}}(\mathrm{Ob}\mathscr{O})$ can be used to rigidify pseudoactions of homogeneous N_∞ operads $\mathscr{O}$. This should be understood as an equivariant categorical analogue to Boardman and Vogt's original work [4].

Definition 4.28. Suppose that $\mathscr{O}$ is an operad in G-categories. An *$\mathscr{O}$-pseudoalgebra* in $G\mathbf{Cat}$ is a G-category $\mathscr{C}$, equipped with

1. a map of symmetric sequences $\|\cdot\| : \mathscr{O} \to \mathbf{End}(\mathscr{C})$,
2. a natural isomorphism $\phi : \mathrm{id}_{\mathscr{C}} \Rightarrow \|\mathrm{id}_{\mathscr{O}}\|$, and
3. for every $k \geq 0$, $y \in \mathscr{O}(k)$, and $x_i \in \mathscr{O}(j_i)$ for $i = 1, \ldots, k$, a natural isomorphism $\phi_{y;x_1,\ldots,x_k} : \|y\| \circ (\|x_1\| \times \cdots \times \|x_k\|) \Rightarrow \|\gamma_{\mathscr{O}}(y; x_1, \ldots, x_k)\|$,

which have the following properties.

(i) The isomorphism $\phi : \mathrm{id}_{\mathscr{C}} \Rightarrow \|\mathrm{id}_{\mathscr{O}}\|$ is G-natural.

(ii) The isomorphism $\phi_{y;x_\bullet}$ is natural in y and $x_\bullet$.

(iii) The isomorphism $\phi_{y;x_\bullet}$ is G-equivariant in $y; x_\bullet$, i.e. $\phi_{gy;gx_\bullet} = g \cdot \phi_{y;x_\bullet}$, where $g \cdot \phi_{y;x_\bullet}$ denotes conjugation.

(iv) For any $y \in \mathscr{O}(k)$, $x_i \in \mathscr{O}(j_i)$, and $\tau_i \in \Sigma_{j_i}$ for $i = 1, \ldots, k$, the equation $\phi_{y;x_\bullet \tau_\bullet} = \phi_{y;x_\bullet} \cdot (\tau_1 \oplus \cdots \oplus \tau_k)$ holds.

(v) For any $y \in \mathscr{O}(k)$, $x_i \in \mathscr{O}(j_i)$ for $i = 1, \ldots, k$, and $\sigma \in \Sigma_k$, the equation $\phi_{y\sigma;x_1,\ldots,x_k} = \phi_{y;x_{\sigma^{-1}1},\ldots,x_{\sigma^{-1}k}} \cdot \sigma(j_1, \ldots, j_k)$ holds.

(vi) For any $x \in \mathscr{O}(j)$, the composite

$$\mathrm{id}_{\mathscr{C}} \circ \|x\| \Rightarrow \|\mathrm{id}_{\mathscr{O}}\| \circ \|x\| \Rightarrow \|\gamma_{\mathscr{O}}(\mathrm{id}_{\mathscr{O}}; x)\|$$

is the identity transformation $\mathrm{id}_{\|x\|} : \|x\| \Rightarrow \|x\|$.

(vii) For any $y \in \mathscr{O}(k)$, the composite

$$\|y\| \circ (\mathrm{id}_{\mathscr{C}} \times \cdots \times \mathrm{id}_{\mathscr{C}}) \Rightarrow \|y\| \circ (\|\mathrm{id}_{\mathscr{O}}\| \times \cdots \times \|\mathrm{id}_{\mathscr{O}}\|) \Rightarrow \|\gamma_{\mathscr{O}}(y; \mathrm{id}_{\mathscr{O}}, \ldots, \mathrm{id}_{\mathscr{O}})\|$$

is the identity transformation $\mathrm{id}_{\|y\|} : \|y\| \Rightarrow \|y\|$.

(viii) For any $z \in \mathscr{O}(k)$, $y_a \in \mathscr{O}(j_a)$, and $x_{ab} \in \mathscr{O}(i_{ab})$ for $a = 1, \ldots, k$ and $b = 1, \ldots, j_a$, the composite transformations below are equal.

$$\gamma_{\mathbf{End}}(\gamma_{\mathbf{End}}(\|z\|; \|y_a\|); \|x_{ab}\|) \Rightarrow \gamma_{\mathbf{End}}(\|\gamma_{\mathscr{O}}(z; y_a)\|; \|x_{ab}\|) \Rightarrow \|\gamma_{\mathscr{O}}(\gamma_{\mathscr{O}}(z; y_a); x_{ab})\|$$
$$\gamma_{\mathbf{End}}(\|z\|; \gamma_{\mathbf{End}}(\|y_a\|; \|x_{ab}\|)) \Rightarrow \gamma_{\mathbf{End}}(\|z\|; \|\gamma_{\mathscr{O}}(y_a; x_{ab})\|) \Rightarrow \|\gamma_{\mathscr{O}}(z; \gamma_{\mathscr{O}}(y_a; x_{ab}))\|$$

Definition 4.29. Suppose that $\mathscr{O}$ is an operad in G-categories and that $(\mathscr{C}, \|\cdot\|_{\mathscr{C}}, \phi_{\mathscr{C}})$ and $(\mathscr{D}, \|\cdot\|_{\mathscr{D}}, \phi_{\mathscr{D}})$ are $\mathscr{O}$-pseudoalgebras in $G\mathbf{Cat}$. A *lax $\mathscr{O}$-pseudoalgebra morphism* $(F, \partial_\bullet) : \mathscr{C} \to \mathscr{D}$ consists of

1. a G-functor $F : \mathscr{C} \to \mathscr{D}$, and
2. for every integer $n \geq 0$ and element $x \in \mathscr{O}(n)$, a natural transformation $(\partial_n)_x : \|x\|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F \circ \|x\|_{\mathscr{C}}$,

which have the following properties.

(i) For every $n \geq 0$, the map $(\partial_n)_x$ is natural in x.

(ii) For every $n \geq 0$, the map $(\partial_n)_x$ is $G \times \Sigma_n$-equivariant in x.

(iii) The composite transformations below are equal.

$$\mathrm{id}_{\mathscr{D}} \circ F \Rightarrow \|\mathrm{id}_{\mathscr{O}}\|_{\mathscr{D}} \circ F \Rightarrow F \circ \|\mathrm{id}\|_{\mathscr{C}}$$

$$\mathrm{id}_{\mathscr{D}} \circ F = F \circ \mathrm{id}_{\mathscr{C}} \Rightarrow F \circ \|\mathrm{id}_{\mathscr{O}}\|_{\mathscr{C}}$$

(iv) For any $k \geq 0$, $y \in \mathscr{O}(k)$, and $x_i \in \mathscr{O}(j_i)$ for $i = 1, \dots, k$, the composite transformations below are equal.

$$\|y\|_{\mathscr{D}} \circ (\|x_1\|_{\mathscr{D}} \times \cdots \times \|x_k\|_{\mathscr{D}}) \circ F^{\times \Sigma j_\bullet} \Rightarrow \|y\|_{\mathscr{D}} \circ F^{\times k} \circ (\|x_1\|_{\mathscr{C}} \times \cdots \times \|x_k\|_{\mathscr{C}}) \Rightarrow$$
$$F \circ \|y\|_{\mathscr{C}} \circ (\|x_1\|_{\mathscr{C}} \times \cdots \times \|x_k\|_{\mathscr{C}}) \Rightarrow F \circ \|\gamma_{\mathscr{O}}(y; x_1, \dots, x_k)\|_{\mathscr{C}}$$

$$\|y\|_{\mathscr{D}} \circ (\|x_1\|_{\mathscr{D}} \times \cdots \times \|x_k\|_{\mathscr{D}}) \circ F^{\times \Sigma j_\bullet} \Rightarrow \|\gamma_{\mathscr{O}}(y; x_1, \dots, x_k)\|_{\mathscr{D}} \circ F^{\times \Sigma j_\bullet}$$
$$\Rightarrow F \circ \|\gamma_{\mathscr{O}}(y; x_1, \dots, x_k)\|_{\mathscr{C}}$$

We say that a lax $\mathscr{O}$-morphism is a pseudomorphism or a strict morphism if the maps $(\partial_n)_x$ are isomorphisms or identity maps, respectively.

Definition 4.30. Suppose that $\mathscr{O}$ is an operad in G-categories, that $\mathscr{C}$ and $\mathscr{D}$ are $\mathscr{O}$-pseudoalgebras, and that $(F, \partial_\bullet), (F', \partial'_\bullet) : \mathscr{C} \rightrightarrows \mathscr{D}$ is a pair of lax $\mathscr{O}$-morphisms between them. An $\mathscr{O}$-transformation $\omega : (F, \partial_\bullet) \Rightarrow (F', \partial'_\bullet)$ is a G-natural transformation $\omega : F \Rightarrow F'$ such that for every $n \geq 0$ and $x \in \mathscr{O}(n)$, the composites

$$\|x\|_{\mathscr{D}} \circ F^{\times n} \Rightarrow F \circ \|x\|_{\mathscr{C}} \Rightarrow F' \circ \|x\|_{\mathscr{C}}$$

$$\|x\|_{\mathscr{D}} \circ F^{\times n} \Rightarrow \|x\|_{\mathscr{D}} \circ (F')^{\times n} \Rightarrow F' \circ \|x\|_{\mathscr{C}}$$

are equal transformations.

As usual, the 2-category structure on $G\underline{\mathbf{Cat}}$ lifts to $\mathscr{O}$-pseudoalgebras.

Notation 4.31. Let **Ps-$\mathscr{O}$-AlgLax** be the 2-category of all $\mathscr{O}$-pseudoalgebras in G**Cat**, lax $\mathscr{O}$-morphisms, and $\mathscr{O}$-transformations between them. The composite of lax $\mathscr{O}$-morphisms is obtained by composing underlying functors and comparison data, and the vertical and horizontal composites of $\mathscr{O}$-transformations are computed in $G\underline{\mathbf{Cat}}$. Identities are also inherited from $G\underline{\mathbf{Cat}}$. There are sub-2-categories **Ps-$\mathscr{O}$-AlgSt** $\subset$ **Ps-$\mathscr{O}$-AlgPs** $\subset$ **Ps-$\mathscr{O}$-AlgLax** of pseudomorphisms and strict morphisms respectively, and there is a forgetful 2-functor **Ps-$\mathscr{O}$-AlgLax** $\to G\underline{\mathbf{Cat}}$.

We shall now explain how to identify $\mathscr{O}$-pseudoalgebras with strict $W\mathscr{O}$-algebras, provided that $\mathscr{O}$ is a homogeneous N_∞ operad.

Proposition 4.32. *Suppose that $\mathscr{N}, \mathscr{O} \in \mathbf{Op}_{h,m}$ and that $\psi : \mathscr{N} \to \mathscr{O}$ is a map of symmetric sequences. Then there is a pullback 2-functor*

$$\psi^* : \mathscr{O}\text{-}\mathbf{AlgLax} \to \mathbf{Ps}\text{-}\mathscr{N}\text{-}\mathbf{AlgLax}$$

over $G\underline{\mathbf{Cat}}$, which restricts to sub-2-categories of pseudo and strict morphisms.

Proof. We give formulas for ψ^*. If $(\mathscr{C}, |\cdot|_{\mathscr{C}} : \mathscr{O} \to \mathbf{End}(\mathscr{C}))$ is a strict $\mathscr{O}$-algebra, then the pullback $\psi^*(\mathscr{C}, |\cdot|_{\mathscr{C}})$ is the G-category $\mathscr{C}$, together with the map $\|\cdot\|_{\psi^*\mathscr{C}} = |\cdot|_{\mathscr{C}} \circ \psi : \mathscr{N} \to \mathscr{O} \to \mathbf{End}(\mathscr{C})$, and the natural isomorphisms

$$\begin{aligned} \phi &:= |\mathrm{id}_{\mathscr{O}} \to \psi(\mathrm{id}_{\mathscr{N}})|_{\mathscr{C}} \\ \phi_{y;x_\bullet} &:= |\gamma_{\mathscr{O}}(\psi y; \psi x_\bullet) \to \psi\gamma_{\mathscr{N}}(y; x_\bullet)|_{\mathscr{C}}. \end{aligned}$$

If $(F, \partial_\bullet) : \mathscr{C} \to \mathscr{D}$ is a lax $\mathscr{O}$-morphism, then $\psi^* F = (F, \partial_n \circ \mathrm{id}_{\psi_n}) : \psi^*\mathscr{C} \to \psi^*\mathscr{D}$. The 2-functor ψ^* does nothing to transformations. ☐

Theorem 4.33. *Suppose that $\mathscr{O} \in N_\infty$-$\mathbf{Op}_{h,m}$. Then pulling back along the unit of the* *adjunction $\eta : \mathscr{O} \to W\mathscr{O}$ induces an isomorphism*

$$\eta^* : W\mathscr{O}\text{-}\mathbf{AlgLax} \xrightarrow{\cong} \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax}$$

of 2-categories over $G\underline{\mathbf{Cat}}$. Similarly for pseudo and strict morphisms.

Sketch of proof. The operad $W\mathscr{O}$ is free on the Σ-free symmetric sequence $\mathrm{Ob}\mathscr{O}$, and since $\mathscr{O}$ is marked, we are given $u \in \mathscr{O}(0)^G$ and $p \in \mathscr{O}(2)^G$. Therefore $W\mathscr{O} = \mathcal{SM}_{\mathcal{N}(\mathscr{O})}$ for some set of exponents $\mathcal{N}(\mathscr{O})$, to be determined. Define $p_0 := u$, $p_1 := \mathrm{id}_{\mathscr{O}}$, $p_{n+1} := \gamma_{\mathscr{O}}(p; p_n, \mathrm{id}_{\mathscr{O}})$, and extend the set $\{p_n \,|\, n \geq 0\}$ to a set R of $G \times \Sigma$-orbit representatives for $\mathrm{Ob}\mathscr{O}$. Then

$$\mathrm{Ob}\mathscr{O} \cong G \times \Sigma_0/G \;\sqcup\; G \times \Sigma_2/G \;\sqcup \coprod_{n \neq 0,2} G \times \Sigma_n/G \;\sqcup \coprod_{\substack{r \,\in\, R \\ r \,\neq\, p_n}} G \times \Sigma_{n_r}/\mathrm{Stab}(r),$$

where n_r is the arity of r. Choose finite G-subgroup actions T_r such that $\Gamma_{T_r} = \mathrm{Stab}(r)$. Then $W\mathscr{O} \cong \mathcal{SM}_{\mathcal{N}(\mathscr{O})}$ for $\mathcal{N}(\mathscr{O}) = \{\varepsilon^* n = (n, \mathrm{triv}) \,|\, n \neq 0, 2\} \sqcup \{T_r \,|\, r \in R\,,\, r \neq p_n\}$.

As explained in section 4.6, there is an evaluation 2-functor $\mathrm{ev} : \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$ such that $\mathrm{ev} \circ \eta^* : W\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$ is just the evaluation $\mathrm{ev} : \mathcal{SM}_{\mathcal{N}(\mathscr{O})}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$. The latter is an isomorphism by theorem 2.10, and therefore $\mathrm{ev} : \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$ has a section. On the other hand, in section 4.6 we also prove that this 2-functor is injective on categories, functors, and transformations. Therefore the evaluation 2-functor for $\mathscr{O}$-pseudoalgebras is an isomorphism, and so is the pullback η^*. □

We obtain the following presentation theorem for $\mathscr{O}$-pseudoalgebras.

Theorem 4.34. *Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$ and that $\mathcal{N}$ is any set of exponents that generates the admissible sets of $\mathscr{O}$. Then $\mathcal{N}\mathbf{SMLax}$ and $\mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax}$ are 2-equivalent over $G\underline{\mathbf{Cat}}$. Similarly for strong and strict morphisms.*

Proof. We have the chain

$$\mathcal{N}\mathbf{SMLax} \overset{\text{iso}}{\rightleftarrows} \mathcal{SM}_{\mathcal{N}}\text{-}\mathbf{AlgLax} \overset{\text{equiv}}{\rightleftarrows} W\mathscr{O}\text{-}\mathbf{AlgLax} \overset{\text{iso}}{\rightleftarrows} \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax}$$

of 2-categories and 2-functors over $G\underline{\mathbf{Cat}}$. □

Example 4.35. Working nonequivariantly, we can specialize to the case where $\mathcal{N} = \varnothing$, and $\mathscr{O}$ is the Barratt-Eccles operad $\mathscr{P}$. In this case, we obtain the chain

$$\mathbf{SMLax} \cong \mathcal{SM}\text{-}\mathbf{AlgLax} \simeq W\mathscr{P}\text{-}\mathbf{AlgLax} \cong \mathbf{Ps}\text{-}\mathscr{P}\text{-}\mathbf{AlgLax},$$

which corresponds to an isomorpism between $\mathscr{P}$-pseudoalgebras and unbiased symmetric monoidal categories, and a further equivalence to symmetric monoidal categories in the usual sense.

Equivariantly, one can consider the G-Barratt-Eccles operad $\mathscr{P}_G$ of [18]. This is an E_∞ operad, and thus, if $\mathcal{N}$ is any set of exponents that generates $\underline{\mathbf{Set}}$, then one has a similar chain from $\mathcal{N}\mathbf{SMLax}$ to $\mathbf{Ps}\text{-}\mathscr{P}_G\text{-}\mathbf{AlgLax}$.

By combining theorems 4.25 and 4.33, we also obtain the strictification for pseudoalgebras considered in [20].

Proposition 4.36. *Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$. Then there is a 2-adjunction*

$$\mathrm{st} : \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgPs} \rightleftarrows \mathscr{O}\text{-}\mathbf{AlgSt} : \mathrm{inc}$$

whose unit maps are internal equivalences.

Proof. Consider the chain

$$\mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgPs} \overset{\text{iso}}{\rightleftarrows} W\mathscr{O}\text{-}\mathbf{AlgPs} \overset{\text{adj}}{\rightleftarrows} \mathscr{O}\text{-}\mathbf{AlgSt}$$

of 2-categories and 2-functors over $G\underline{\mathbf{Cat}}$. □

The isomorphism $W\mathscr{O}$-**AlgLax** $\cong$ **Ps**-$\mathscr{O}$-**AlgLax** formalizes the intuition that passing to pseudoalgebras loosens up the relations parametrized by $\mathscr{O}$. We round out this discussion by explaining how to reintroduce strictness.

Proposition 4.37. *Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$, and that R is a graded binary relation contained in $\ker(\varepsilon : W\mathscr{O} \to \mathscr{O})$. Then $W\mathscr{O}/\langle R\rangle$ is a homogeneous N_∞ operad, there are biequivalences and embeddings*

$$\mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax} \leftarrow (W\mathscr{O}/\langle R\rangle)\text{-}\mathbf{AlgLax} \leftarrow \mathscr{O}\text{-}\mathbf{AlgLax},$$

and there is a 2-adjunction

$$\mathrm{st} : (W\mathscr{O}/\langle R\rangle)\text{-}\mathbf{AlgPs} \rightleftarrows \mathscr{O}\text{-}\mathbf{AlgSt} : \mathrm{inc}$$

whose unit maps are internal equivalences.

Proof. The counit factors as a pair $W\mathscr{O} \to W\mathscr{O}/\langle R\rangle \to \mathscr{O}$ of quotient maps. We deduce that $W\mathscr{O}/\langle R\rangle$ is Σ-free, because it maps into $\mathscr{O}$, and that all three operads have the same admissible sets. Applying proposition 4.20 shows that both pullbacks are biequivalences and embeddings. Next, the map $W\mathscr{O}/\langle R\rangle \to \mathscr{O}$ has a section, obtained by composing $\eta : \mathscr{O} \to W\mathscr{O}$ with the quotient map $W\mathscr{O} \to W\mathscr{O}/\langle R\rangle$. By theorem 4.25, we obtain the desired strictification 2-adjunction. $\square$

Example 4.38. Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$. An $\mathscr{O}$-pseudoalgebra $(\mathscr{C}, \|\cdot\|, \phi)$ is *normal* if $\phi : \mathrm{id} \Rightarrow \|\mathrm{id}\|$ is the identity. This transformation is represented by $\mathrm{id}_{W\mathscr{O}} \to \eta(\mathrm{id}_{\mathscr{O}})$ in $W\mathscr{O}(1)$, and $\varepsilon(\mathrm{id}_{W\mathscr{O}} \to \eta(\mathrm{id}_{\mathscr{O}})) = \mathrm{id} : \mathrm{id}_{\mathscr{O}} \to \mathrm{id}_{\mathscr{O}}$. Therefore strict $W\mathscr{O}/\langle \mathrm{id}_{W\mathscr{O}} \sim \eta(\mathrm{id}_{\mathscr{O}})\rangle$-algebras coincide with normal $\mathscr{O}$-pseudoalgebras, and proposition 4.37 applies.

Let $\mathscr{P}_G$ be the G-Barratt-Eccles operad. In [20], the authors consider normal $\mathscr{P}_G$-

pseudoalgebras $\|\cdot\| : \mathscr{P}_G \to \mathbf{End}(\mathscr{C})$, which satisfy the additional strict unitality relations

$$\|c\|(x_1, \ldots, \|0\|, \ldots, x_n) = \|c \circ_i 0\|(x_1, \ldots, \widehat{x_i}, \ldots, x_n)$$

for $n > 0$, $c \in \mathscr{P}_G(n)$, and $i = 1, \ldots, n$. Here 0 is the unique element of $\mathscr{P}_G(0)$. The comparison map between the terms above is represented by

$$\gamma(\eta(c); \mathrm{id}, \ldots, \eta(0), \ldots, \mathrm{id}) \to \eta(\gamma(c; \mathrm{id}, \ldots, 0, \ldots, \mathrm{id}))$$

in $W\mathscr{P}_G$, and ε sends this edge to an identity map in $\mathscr{P}_G$. Therefore there is an equivariant E_∞ operad $W\mathscr{P}_G/\langle R\rangle$, whose strict algebras are the normal, strictly unital $\mathscr{P}_G$-pseudoalgebras of [20], and proposition 4.37 applies. By choosing $G \times \Sigma$-orbit representatives for $\mathscr{P}_G$, we can write $W\mathscr{P}_G = \mathcal{SM}_{\mathcal{N}(\mathscr{P}_G)}$, and therefore proposition 4.13 implies that the normal, strictly unital $\mathscr{P}_G$-pseudoalgebras of [20] are precisely the same thing as $\mathcal{N}(\mathscr{P}_G)$-normed symmetric monoidal categories satisfying the strict relations in R.

That said, if $\mathcal{N}$ is any set of exponents that generates $\underline{\mathbf{Set}}$, then theorem 4.27 supplies a strictification 2-adjunction $\mathcal{N}\mathbf{SMStg} \leftrightarrows (W\mathscr{P}_G/\langle R\rangle)\text{-}\mathbf{AlgSt}$. Thus, we can rigidify any E_∞ normed symmetric monoidal category to normal, strictly unital $\mathscr{P}_G$-pseudoalgebra. Similarly, there is a strictification 2-adjunction $\mathcal{N}\mathbf{SMStg} \rightleftarrows \mathscr{P}_G\text{-}\mathbf{AlgSt}$. Thus, while we have not solved the presentation problem [20, problem 1.36], one can construct E_∞ normed symmetric monoidal categories in a biased fashion, and then use our invariance theorems to get equivalent $\mathscr{P}_G$-algebras and pseudoalgebras.

4.6 Appendix: the evaluation 2-functor for pseudoalgebras

Suppose that $\mathscr{O} \in N_\infty\text{-}\mathbf{Op}_{h,m}$ and keep notation as in the proof of theorem 4.33. In this section, we show that the evaluation 2-functor $\mathrm{ev} : W\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$ factors through an analogous 2-functor $\mathrm{ev} : \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathcal{N}(\mathscr{O})\mathbf{SMLax}$, which is injective on categories, functors, and transformations. This will complete the proof of theorem 4.33.

4.6.1 A partial coherence theorem for pseudoalgebras

We begin by sketching a coherence theorem for the maps ϕ in an $\mathscr{O}$-pseudoalgebra $(\mathscr{C}, \|\cdot\|, \phi)$. This will streamline our subsequent work. Consider the free operad $\mathbb{F}(\mathrm{Ob}\mathscr{O} \sqcup G \times \Sigma_1/G)$. Its elements can be identified with operadic terms built from the formal symbols

$$\begin{array}{rl} x_n & n = 1, 2, 3, \ldots \\ a & a \in \mathrm{Ob}\mathscr{O} \\ \iota & \text{(corresponding to the element of } G \times \Sigma_1/G) \\ (\quad) \quad , & \text{(punctuation)} \end{array}$$

modulo the congruence relation generated by $c\sigma(x_1, \ldots, x_n) \equiv c(x_{\sigma^{-1}1}, \ldots, x_{\sigma^{-1}n})$ (cf. section 2.4 and construction 2.15). There is an operad map $\varepsilon : \mathbb{F}(\mathrm{Ob}\mathscr{O} \sqcup G \times \Sigma_1/G) \to \mathrm{Ob}\mathscr{O}$, given by the identity map on $\mathrm{Ob}\mathscr{O}$, and which sends the generator $\iota(x_1)$ to $\mathrm{id}_{\mathscr{O}}$. There is also an operad map $\|\cdot\| : \mathbb{F}(\mathrm{Ob}\mathscr{O} \sqcup G \times \Sigma_1/G) \to \mathbf{End}(\mathscr{C})$ given by $\|\cdot\|$ on $\mathrm{Ob}\mathscr{O}$, and which sends the generator $\iota(x_1)$ to $\mathrm{id}_{\mathscr{C}}$.

For any congruence class $[t] \in \mathbb{F}(\mathrm{Ob}\mathscr{O} \sqcup G \times \Sigma_1/G)$, we say that $[s]$ is a *contraction* of $[t]$ if, for some representatives s and t, the term s is obtained by modifying a subterm of t in one of the ways below.

$$\begin{aligned} a(t_1, \ldots, a'(t_{i1}, \ldots, t_{ij}), \ldots, t_k) &\rightsquigarrow a \circ_i a'(t_1, \ldots, t_{i1}, \ldots, t_{ij}, \ldots, t_k) \\ \iota(a(t_1, \ldots, t_k)) &\rightsquigarrow a(t_1, \ldots, t_k) \\ a(t_1, \ldots, \iota(t_i), \ldots, t_k) &\rightsquigarrow a(t_1, \ldots, t_i, \ldots, t_k) \\ \iota(t_1) &\rightsquigarrow \mathrm{id}_{\mathscr{O}}(t_1) \end{aligned}$$

Given a congruence class $[t]$, we consider a directed graph $\mathbf{Con}[t]$. Its vertices are $[t]$ and all of its contractions. Its edges are pairs $([r],[s])$, together with a chosen contraction of $[r]$ to $[s]$. There is a map $\Phi : \mathbf{Con}[t] \to \mathbf{End}(\mathscr{C})(n)$ of directed graphs, where n is the arity of $[t]$. On objects, it is obtained by restricting the map $\|\cdot\| : \mathbb{F}(\mathrm{Ob}\mathscr{O} \sqcup G \times \Sigma_1/G) \to \mathbf{End}(\mathscr{C})$ above. It sends edges to whiskerings of the natural transformations

$$\|a\| \circ (\mathrm{id} \times \cdots \times \|a'\| \times \cdots \times \mathrm{id}) \Rightarrow \|a\| \circ (\|\mathrm{id}\| \times \cdots \times \|a'\| \times \cdots \times \|\mathrm{id}\|) \Rightarrow \|a \circ_i a'\|$$

$$\mathrm{id} \circ \|a\| = \|a\| \quad \text{(identity transformation)}$$

$$\|a\| \circ (\mathrm{id} \times \cdots \times \mathrm{id} \times \cdots \times \mathrm{id}) = \|a\| \quad \text{(identity transformation)}$$

$$\mathrm{id} \Rightarrow \|\mathrm{id}\|,$$

defined in terms of ϕ and the necessary identity transformations. By adjunction, we obtain a functor $\Phi : \mathbf{Fr}(\mathbf{Con}[t]) \to \mathbf{End}(\mathscr{C})(n)$.

Lemma 4.39. *The functor $\Phi : \mathbf{Fr}(\mathbf{Con}[t]) \to \mathbf{End}(\mathscr{C})(n)$ takes parallel morphisms to equal natural transformations.*

Proof. One argues by induction on the complexity $c[t]$ of $[t]$, where

$$c[t] = (\text{number of symbols } a \in \mathrm{Ob}\mathscr{O} \text{ in } t) + 2(\text{number of } \iota \text{ symbols in } t).$$

First, note that there is a unique $[u] \in \mathbf{Con}[t]$ of complexity 1, because the image of a term under ε is invariant under contractions. Then, since all of the maps ϕ are isomorphisms, it will be enough to show that all parallel morphisms $[r] \rightrightarrows [u]$ have the same value under Φ. When $[r] = [u]$, the only possibility is the identity map. Now suppose that two paths $[q] \rightrightarrows [u]$ are given, and that the desired result holds for classes $[r]$ of lower complexity than $[q]$. Consider the first edges $e : [q] \to [r]$ and $e' : [q] \to [r']$ in each path. From the axioms for a pseudoalgebra, we can always find a class $[s]$ and paths $p : [r] \to [s]$ and $p : [r'] \to [s]$ of length 0 or 1 such that $\Phi(p \circ e) = \Phi(p' \circ e')$, and the result for $[q]$ follows. □

4.6.2 The proof of theorem 4.33

Keep notation as in the proof of theorem 4.33.

Construction 4.40. Given any $\mathscr{O}$-pseudoalgebra $(\mathscr{C}, \|\cdot\|, \phi)$, we let $\mathrm{ev}\mathscr{C}$ be the G-category $\mathscr{C}$, together with the operations

$$e := \|p_0\|, \quad \otimes := \|p_2\|, \quad \bigotimes_{\varepsilon^* n} := \|p_n\| \quad (\text{if } n \neq 0, 2), \quad \bigotimes_{T_r} := \|r\| \quad (\text{if } r \neq p_n),$$

and the natural isomorphisms

$$\begin{aligned}
\alpha &:= \Big[\otimes \circ (\otimes \times \mathrm{id}) \Rightarrow \|\gamma(p_2; p_2, p_1)\| \Rightarrow \|\gamma(p_2; p_1, p_2)\| \Rightarrow \otimes \circ (\mathrm{id} \times \otimes) \Big], \\
\lambda &:= \Big[\otimes \circ (e \times \mathrm{id}) \Rightarrow \|\gamma(p_2; p_0, p_1)\| \Rightarrow \|p_1\| \Rightarrow \mathrm{id} \Big], \\
\rho &:= \Big[\otimes \circ (\mathrm{id} \times e) \Rightarrow \|\gamma(p_2; p_1, p_0)\| \Rightarrow \|p_1\| \Rightarrow \mathrm{id} \Big], \\
\beta &:= \Big[\otimes = \|p_2\| \Rightarrow \|p_2(12)\| = \otimes(12) \Big], \\
\upsilon_{\varepsilon^* n} &:= \Big[\bigotimes_{\varepsilon^* n} = \|p_n\| \Rightarrow \otimes \circ (\|p_{n-1}\| \times \mathrm{id}) \Rightarrow \cdots \Rightarrow \bigotimes_n \Big] \quad (\text{if } n \neq 0, 2), \\
\upsilon_{T_r} &:= \Big[\bigotimes_{T_r} = \|r\| \Rightarrow \Big\| p_{|T_r|} \Big\| = \bigotimes_{\varepsilon^* |T_r|} \overset{\upsilon}{\Rightarrow} \bigotimes_{|T_r|} \Big] \quad (\text{if } r \neq p_n).
\end{aligned}$$

Here, all maps are either composites of copies of $\phi^{\pm 1}$, or the image of a morphism in $\mathscr{O}$ under the map $\|\cdot\| : \mathscr{O} \to \mathbf{End}(\mathscr{C})$. To prove that these data form an $\mathcal{N}(\mathscr{O})$-normed symmetric monoidal category, observe that all diagrams in any component of $\mathscr{O}$ commute, and that applying ϕ repeatedly lets us reduce to this case. Lemma 4.39 ensures that the particular sequence of ϕ's used does not affect the end result.

If $(F, \partial_\bullet) : \mathscr{C} \to \mathscr{D}$ is a lax $\mathscr{O}$-morphism, we let $\mathrm{ev}F : \mathrm{ev}\mathscr{C} \to \mathrm{ev}\mathscr{D}$ be the G-functor $F : \mathscr{C} \to \mathscr{D}$, together with the comparison data

$$F_e := (\partial_0)_{p_0}, \quad F_\otimes := (\partial_2)_{p_2}, \quad F_{\bigotimes_{\varepsilon^* n}} := (\partial_n)_{p_n} \quad (n \neq 0, 2), \quad F_{\bigotimes_{T_r}} := (\partial_{|T_r|})_r \quad (r \neq p_n).$$

As before, the main point is that $(\partial_n)_x$ is natural in x, and we can reduce to this case by applying ϕ. Evaluation leaves transformations unchanged. These assignments determine a 2-functor over $G\underline{\mathbf{Cat}}$.

The equation $\mathrm{ev} \circ \eta^* = \mathrm{ev}$ holds because on both sides, we are evaluating at $G \times \Sigma$-orbit representatives for $\mathscr{O}$, and the isomorphisms ϕ for $\eta^*(\mathscr{C}, |\cdot|_{\mathscr{C}})$ are defined in terms of the $W\mathscr{O}$-algebra coherence maps for $\mathscr{C}$.

Now for the injectivity of the evaluation 2-functor.

Lemma 4.41. *The* 2*-functor* $\mathrm{ev} : \mathbf{Ps}\text{-}\mathscr{O}\text{-}\mathbf{AlgLax} \to \mathscr{N}(\mathscr{O})\mathbf{SMLax}$ *is injective on categories, functors, and transformations.*

Proof. Injectivity on transformations is immediate. Next, recall that the elements p_n and r generate $\mathrm{Ob}\mathscr{O}$ as a symmetric sequence. Thus, the injectivity of ev on functors $(F, \partial_\bullet)$ follows from the equivariance of $(\partial_n)_x$ in x. We now consider the case for categories $(\mathscr{C}, \|\cdot\|, \phi)$. The value of $\|\cdot\|$ on $\mathrm{Ob}\mathscr{O}$ is determined by its values on p_n and r by $G \times \Sigma$-equivariance. The map $\left\| r \to p_{|T_r|} \right\|$ is the composite $\upsilon^{-1}_{\varepsilon^*|T_r|} \bullet \upsilon_{T_r}$, and if $n \in \Sigma_n$, then the commutative square

$$\begin{tikzcd}
\bigotimes_n \arrow[r, "\mathrm{can}"] & \bigotimes_n \cdot \sigma \\
\|p_n\| \arrow[u, "\upsilon_{\varepsilon^* n}"] \arrow[r, "\|p_n \to p_n \cdot \sigma\|"'] & \|p_n \cdot \sigma\| \arrow[u, "\upsilon_{\varepsilon^* n}\sigma"']
\end{tikzcd}$$

shows that $\|p_n \to p_n\sigma\|$ is also determined by $\mathrm{ev}\mathscr{C}$. Here can denotes the symmetric monoidal coherence isomorphism that permutes the factors of $\bigotimes_n$ by σ. By $G \times \Sigma$-equivariance, it follows that for any $x \in \mathscr{O}(n)$, the value of $\|x \to p_n\|$ is determined, and thus $\|x \to y\| = \|y \to p_n\|^{-1} \bullet \|x \to p_n\|$ is, too. Thus the map $\|\cdot\| : \mathscr{O} \to \mathscr{C}$ is determined by $\mathrm{ev}\mathscr{C}$.

It remains to consider the maps ϕ. The map $\phi : \mathrm{id}_{\mathscr{C}} \to \|\mathrm{id}_{\mathscr{O}}\|$ is equal to $\upsilon^{-1}_{\varepsilon^* 1}$. Next, given integers $k \geq 0$, and $j_1, \dots, j_k \geq 0$, write $\gamma_{\mathscr{O}}(p_k; p_{j_\bullet}) = r\sigma$ for some orbit representative

r. Then the commutative diagram

$$\begin{tikzcd}
& \gamma_{\mathbf{End}}(\bigotimes k; \bigotimes j_\bullet) \arrow[r, "\text{can}"] \arrow[d, "\phi\text{'s}"] & \bigotimes \Sigma j_\bullet \cdot \sigma \arrow[d, "\phi\text{'s}"] \\
\gamma_{\mathbf{End}}(\|p_k\|; \|p_{j_\bullet}\|) \arrow[ur, "\gamma_{\mathbf{End}}(\upsilon_{\varepsilon^* k}; \upsilon_{\varepsilon^* j_\bullet})"] \arrow[r, "\phi_{p_k;p_{j_\bullet}}"'] & \|\gamma_{\mathscr{O}}(p_k; p_{j_\bullet})\| \arrow[ur, "\upsilon_{T_r} \cdot \sigma"] \arrow[r, "\|\gamma_{\mathscr{O}}(p_k; p_{j_\bullet}) \to p_{\Sigma j_\bullet} \sigma\|"'] & \|p_{\Sigma j_\bullet} \cdot \sigma\|
\end{tikzcd}$$

shows that $\phi_{p_k;p_{j_\bullet}}$ is determined by ev$\mathscr{C}$. From here, we recover the values of all the maps $\phi_{y;x_\bullet}$ using the naturality of $\phi_{y;x_\bullet}$ in $y; x_\bullet$, and the fact that each component of $\mathscr{O}$ is a connected groupoid. □

CHAPTER 5

EXAMPLES OF NORMED SYMMETRIC MONOIDAL CATEGORIES

5.1 Introduction and summary of results

The previous three chapters develop the basic theory of normed symmetric monoidal categories and homogeneous operads. The purpose of this chapter and the next is to give some indication of how things work in practice. In this chapter, we describe some notable examples of normed symmetric monoidal categories. We hope that they will provide a useful baseline for future work. We make no claims of completeness, but we have tried to give some variety.

Fix a finite group G. The normed symmetric monoidal categories we shall consider are primarily diagram categories. The idea is to make the group G act on the diagram shape, and possibly on the individual diagram entries, too. We show how to produce new operations on diagrams by twisting an existing product by the group action. We give a general overview in section 5.2, and in section 5.3, we focus on diagrams indexed over the translation category of G. In section 5.4, we describe a construction specific to the case $G = C_2$ and diagrams over the translation category of C_2.

Most of the work in this chapter is specialized. However, there is a general conceptual point that is clarified by these considerations. The analysis in section 5.3 identifies an important, shared link between our work, the work of Guillou-May-Merling-Osorno [20], and the work of Hill and Hopkins [23]. For any finite group G, let $\mathbb{T}G$ denote the translation category of G (cf. definition 5.8), and consider the functor category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ for some nonequivariant symmetric monoidal category $\mathscr{C}$. Up to notation, this is a standard example of input to the infinite loop space machinery of [20]. However, $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ is also an E_∞ normed symmetric monoidal category, and its transfers are noteworthy (cf. theorem 5.12).

Theorem. *For any subgroup $H \subset G$, the H-fixed subcategory $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H$ is equivalent to* *the category $H\mathscr{C}$ of H-actions in $\mathscr{C}$, and for any subgroups $K \subsetneq H \subset G$, the transfer* tr_K^H : $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^K \to \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H$ *is equivalent to the monoidal induction functor* $K\mathscr{C} \to H\mathscr{C}$.

We deduce that $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ is a precursor to the standard G-symmetric monoidal structure on the coefficient system of all $H\mathscr{C}$ (cf. [23]). That said, normed symmetric monoidal categories and G-symmetric monoidal categories were invented for very different purposes, and we do not believe that these two notions of structure are equivalent. Informally, we think of G-symmetric monoidal categories as large, ambient settings for doing mathematics, and normed symmetric monoidal categories as small, algebraic objects that model N_∞-G-spaces. These perspectives are not completely disjoint, because we have a recursion. The category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ is simultaneously a structure that encodes the actions of all subgroups of G in $\mathscr{C}$, and a kind of (pseudo) G-commutative monoid in the G-symmetric monoidal category determined by $\mathbf{Fun}(\mathbb{T}G, \mathbf{Cat})$.

5.2 Normed symmetric monoidal structures from configurations

The classical algebraic counterpart to a normed symmetric monoidal category is a G-module, or more generally, a commutative monoid, equipped with a G-action through monoid automorphisms. We have in mind the underlying additive structure on a linear representation, or the Galois action on a field extension, considered only as an additive or multiplicative monoid. Normed symmetric monoidal categories are higher analogues to these structures, in which the algebraic operation is coherently associative, commutative, and unital, but the group action is strict.

This particular combination of strict and weak structure presents difficulties. Consider a finite-dimensional linear G-representation V. In this case, group elements act on V by matrix multiplication, and the strict associativity and unitality of the G-action ultimately follows from the strict algebraic relations satisfied by addition and multiplication in the ground field. If we weaken those conditions on addition and multiplication, we should not expect to retain a strict G-action, except under very particular circumstances.

In this section, we shall primarily consider "configurations" of algebraic data, where the configuration itself is equipped with a group action. In this case, the group G essentially acts by permuting factors, which avoids the issue above.

For consistency, we begin with a trivial example.

Example 5.1. Suppose that $\mathscr{C}$ is a commutative monoid object in $G\mathbf{Cat}$. Then $\mathscr{C}$ is a symmetric monoidal object with trivial coherence data, and it can be made into a normed symmetric monoidal category by taking each T-norm $\bigotimes_T$ to be the $|T|$-fold product on $\mathscr{C}$, and each T-untwistor υ_T to be the identity map. Note that if M is a commutative monoid in $G\mathbf{Set}$, then FM is a commutative monoid in $G\mathbf{Cat}$ for any product-preserving functor $F : \mathbf{Set} \to \mathbf{Cat}$. In particular, this applies to the discrete category M^{disc}, obtained by attaching an identity morphism to each element of M and nothing more. This also applies to the category $\widetilde{M}$, obtained by inserting a unique isomorphism between each pair of elements in M (cf. definition 2.16 and [19, definition 1.4]).

The next example is a thickened version of the sign representation of C_2 on $\mathbb{Z}$. Note that a pair (X, Y) is the same thing as map out of a pair of points.

Example 5.2. Let $\mathscr{C}$ be the category whose objects are pairs of finite sets (X, Y) and which has the hom sets

$$\mathscr{C}\Big((X,Y),(X',Y')\Big) = \begin{cases} * & \text{if } |X| - |Y| = \left|X'\right| - \left|Y'\right| \\ \varnothing & \text{otherwise} \end{cases} .$$

Write $C_2 = \{e, g\}$. We make $\mathscr{C}$ into a C_2-category via the formula $g(X,Y) = (Y,X)$. More conceptually, there is a C_2-equivariant evaluation map $\varepsilon : \mathrm{Ob}(\mathscr{C}) \to \mathbb{Z}$ defined by $\varepsilon(X,Y) = |X| - |Y|$, which creates the C_2-category structure on $\mathscr{C}$.

Define operations $\sqcup$ and $\sqcup_{C_2/e}$ on $\mathscr{C}$ by

$$(X_1, Y_1) \sqcup (X_2, Y_2) = (X_1 \sqcup X_2, Y_1 \sqcup Y_2)$$
$$(X_1, Y_1) \sqcup_{C_2/e} (X_2, Y_2) = (X_1 \sqcup X_2, Y_2 \sqcup Y_1)$$

and consider the object $(\varnothing, \varnothing)$. Then $\sqcup : \mathscr{C}^{\times 2} \to \mathscr{C}$ is a C_2-bifunctor, $(\varnothing, \varnothing)$ is C_2-fixed, and $\sqcup_{C_2/e} : \mathscr{C}^{C_2/e} \to \mathscr{C}$ is a C_2/e-norm. It is easy to check that these data, together with the only possible isomorphisms, define a $\{C_2/e\}$-normed symmetric monoidal structure on $\mathscr{C}$. This normed symmetric monoidal structure is strong monoidally equivalent to the sign representation of C_2 on $\mathbb{Z}^{\text{disc}}$, given by the previous example.

We now turn our attention to categories of diagrams over a fixed G-category J. We regard the functor category $\mathbf{Fun}(J, \mathscr{C})$ as a higher analogue to the representation $k[X]$ when X is a finite G-set, or k^X in the infinite case.

Example 5.3. Suppose that $\mathscr{C}$ is a symmetric monoidal object in $G\underline{\mathbf{Cat}}$ (cf. definition 2.1) and that J is a right G-category, and consider the functor category $\mathbf{Fun}(J, \mathscr{C})$ of J-diagrams in $\mathscr{C}$. This category inherits a left G-action by composing and precomposing with the actions on $\mathscr{C}$ and J, respectively. It also has a levelwise ordinary symmetric monoidal product that is equivariant with respect to this action. The monoidal unit of $\mathbf{Fun}(J, \mathscr{C})$ is the constant functor valued at the unit of $\mathscr{C}$, and the coherence data are all taken levelwise.

The finite G-subgroup actions T for which $\mathbf{Fun}(J, \mathscr{C})$ supports a T-norm depend on the stabilizers of the objects of J. In general, the above symmetric monoidal structure on $\mathbf{Fun}(J, \mathscr{C})$ extends to include a T-norm $\bigotimes_T$ and untwistor v_T if and only if

$$\bigcup_{j \in J} \mathrm{Stab}_H(j) \subset \bigcap_{t \in T} \mathrm{Stab}_H(t) = \Big\{ h \in H \;\Big|\; h \cdot (-) = \mathrm{id} : T \to T \Big\}.$$

In words, this says that every $h \in H$ that stabilizes a single object of J must fix all of T. We shall momentarily construct norms for all such T (construction 5.6), and we shall show what goes wrong when T does not have this property (nonexample 5.7).

Remark 5.4. For any right G-category J, the class of all finite H-sets T that satisfy the condition $\bigcup_{j \in J} \mathrm{Stab}_H(j) \subset \bigcap_{t \in T} \mathrm{Stab}_H(t)$ forms an indexing system $\underline{\mathcal{F}}_J$. By theorem 2.10, one concludes that $\mathbf{Fun}(J, \mathscr{C})$ is generally an N_∞-algebra structured over $\underline{\mathcal{F}}_J$.

Note that the normed symmetric monoidal structure on $\mathbf{Fun}(J, \mathscr{C})$ has an operadic source, which we examine in section 6.3. In the special case that $J = \mathbb{T}G$ is the translation category of G (cf. definition 5.8), the category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ has norms for every finite G-subgroup action T. We treat this example in detail in section 5.3.

Example 5.5. Suppose that X is a left G-space, and consider the equivariant fundamental groupoid $\Pi_G X$ of X (cf. [11] and [13]). Its objects are pairs $(G/H, x : G/H \to X)$, where $x : G/H \to X$ denotes the G-map that sends the coset eH to $x \in X^H$. A morphism $(x : G/H \to X) \to (y : G/K \to X)$ consists of a G-map $(-)a : G/H \to G/K$, together with a homotopy class of paths $\omega : x \to ay$ in X^H.

$$\begin{tikzcd}
x \arrow[rr, "\omega"] & & ay \\
G/H \arrow[rr, "(-)a"] \arrow[dr, "x"'] & & G/K \arrow[dl, "y"] \\
& X &
\end{tikzcd}$$

Given $g \in G$, we define $g \cdot (-) : \Pi_G X \to \Pi_G X$ by $g \cdot (x : G/H \to X) = (gx : G/gHg^{-1} \to X)$ on objects. The map $g \cdot (-)$ sends the morphism above to

$$\begin{tikzcd}
gx \arrow[rr, "g\omega"] & & gay \\
G/gHg^{-1} \arrow[rr, "(-)gag^{-1}"] \arrow[dr, "gx"'] & & G/gKg^{-1} \arrow[dl, "gy"] \\
& X &
\end{tikzcd}$$

and we obtain a functor $g \cdot (-) : \Pi_G X \to \Pi_G X$. These maps make $\Pi_G X$ into a left G-category, and we obtain a right G-action by taking $(-) \cdot g := g^{-1} \cdot (-)$. If $\mathscr{C}$ is any symmetric monoidal object in $G\underline{\mathbf{Cat}}$, then $\mathbf{Fun}(\Pi_G X, \mathscr{C})$ is a normed symmetric monoidal category. Functors of this sort arise in connection with equivariant bundle theory. We suspect it will be worthwhile to study the normed symmetric monoidal category $\mathbf{Fun}(\Pi_G X, \mathbf{Fun}(\mathbb{T}G, \mathscr{C}))$ for X a G-space and $\mathscr{C} = \mathbb{R}\text{-}\mathbf{Vect}$ or $\mathbf{Set}$, but we shall not pursue this line of thought here.

Construction 5.6. Suppose that $H \subset G$ is a subgroup, and that T is an ordered, finite H-set. Let the subgroup

$$\Gamma_T = \{(h, \sigma(h)) \,|\, h \in H\} \subset G \times \Sigma_{|T|}$$

be the graph of the corresponding permutation representation on $\{1, \dots, |T|\}$, so that $h \cdot i = \sigma(h)(i)$ for all $i \in \{1, \dots, |T|\}$, and assume that $\bigcup_{j \in J} \mathrm{Stab}_H(j) \subset \ker(\sigma)$. Choose a set of H-orbit representatives $\{j_a \,|\, a \in A\}$ for $\mathrm{Ob}(J)$.

1. The norm map $\bigotimes_T : \mathbf{Fun}(J, \mathscr{C})^{\times T} \to \mathbf{Fun}(J, \mathscr{C})$ is defined as follows.

   (a) For an object $(C^1_\bullet, \dots, C^{|T|}_\bullet) \in \mathbf{Fun}(J, \mathscr{C})^{\times T}$, and $j = j_a h \in J$,

   $$\Big[\bigotimes\nolimits_T (C^1, \dots, C^{|T|})\Big]_j := \bigotimes\nolimits_{|T|} (C_j^{\sigma(h)^{-1}1}, \dots, C_j^{\sigma(h)^{-1}|T|}).$$

   For $f : j \to j'$, where $j = j_a h$ and $j' = j_b h'$,

   $$\Big[\bigotimes\nolimits_T (C^1, \dots, C^{|T|})\Big]_f := \sigma(h') \circ \sigma(h)^{-1} \circ \bigotimes\nolimits_{|T|} (C_f^{\sigma(h)^{-1}1}, \dots, C_f^{\sigma(h)^{-1}|T|}),$$

   where $\sigma(-)$ is the symmetric monoidal coherence map for $\mathscr{C}$ that permutes the factors of the tensor product by $\sigma(-)$. Note that the permutation $\sigma(h)$ is independent of the expression $j = j_a h$ because of our assumption on the stabilizers of objects in J.

(b) For a morphism $(f^1_\bullet, \dots f^{|T|}_\bullet) : (C^1_\bullet, \dots, C^{|T|}_\bullet) \to (D^1_\bullet, \dots, D^{|T|}_\bullet)$ and an object $j = j_a h \in J$,

$$\Big[\bigotimes\nolimits_T(f^1, \dots, f^{|T|})\Big]_j := \bigotimes\nolimits_{|T|}(f_j^{\sigma(h)^{-1}1}, \dots, f_j^{\sigma(h)^{-1}|T|}).$$

2. The untwistor $\upsilon = \upsilon_T : \bigotimes_T \Rightarrow \bigotimes_{|T|}$ has $j = j_a h$ component

$$(\upsilon_{C^1, \dots, C^{|T|}})_j := \sigma(h)^{-1} : \bigotimes\nolimits_{|T|}(C_j^{\sigma(h)^{-1}1}, \dots, C_j^{\sigma(h)^{-1}|T|}) \to \bigotimes\nolimits_{|T|}(C^1_j, \dots, C^{|T|}_j).$$

Again, $\sigma(h)^{-1}$ denotes the symmetric monoidal coherence map for $\mathscr{C}$ that permutes the factors of $\bigotimes_{|T|}$ by $\sigma(h)^{-1}$.

One uses the classical coherence theorem for symmetric monoidal categories and naturality to check that $\bigotimes_T(C^1, \dots, C^{|T|}) : J \to \mathscr{C}$ is a functor, and that the map υ_T is twisted H-equivariant.

Nonexample 5.7. Let J be a right G-category, let $H \subset G$ be a subgroup of G, and let T be a finite H-action. Suppose that there is some $h_0 \in H$ and $j_0 \in J$ such that $j_0 \cdot h_0 = j_0$, but $h_0 \cdot (-) : T \to T$ is not the identity map. We shall show that the levelwise symmetric monoidal structure on $\mathbf{Fun}(J, \mathscr{C})$ generally does not extend to include a compatible T-norm and untwistor.

Let $\mathscr{C} = (\mathbf{Set}, \sqcup, \varnothing)$, and give $\mathscr{C}$ a trivial G-action. Suppose for contradiction that we had a T-norm $\bigsqcup_T : \mathbf{Fun}(J, \mathbf{Set})^{\times T} \to \mathbf{Fun}(J, \mathbf{Set})$ and untwistor $\upsilon = \upsilon_T : \bigsqcup_T \Rightarrow \bigsqcup_{|T|}$. We consider T-fold coproducts of the terminal object $* : J \to \mathbf{Set}$. The twisted equivariance diagram for h_0 is

$$\begin{tikzcd}
h_0 \bigsqcup_T (*, \cdots, *) \arrow[r, "\mathrm{id}"] \arrow[dd, "h_0 \upsilon_{*,\cdots,*}"'] & \bigsqcup_T (*, \cdots, *) \arrow[d, "\upsilon_{*,\cdots,*}"] \\
 & \bigsqcup_{|T|} (*, \cdots, *) \arrow[d, "\sigma(h_0)^{-1}_{*,\cdots,*}"] \\
h_0 \bigsqcup_{|T|} (*, \cdots, *) \arrow[r, "\mathrm{id}"'] & \bigsqcup_{|T|} (*, \cdots, *)
\end{tikzcd}$$

and evaluating at j_0 yields the equation

$$(\upsilon_{*,\cdots,*})_{j_0} = (\upsilon_{*,\cdots,*})_{j_0 \cdot h_0} = (h_0 \upsilon_{*,\cdots,*})_{j_0} = (\sigma(h_0)^{-1}_{*,\cdots,*})_{j_0} \circ (\upsilon_{*,\cdots,*})_{j_0}.$$

Since υ is an isomorphism, we must have that $(\sigma(h_0)^{-1}_{*,\cdots,*})_{j_0} = \mathrm{id}$, but this is false because $(\sigma(h_0)^{-1}_{*,\cdots,*})_{j_0}$ is isomorphic to the permutation $h_0^{-1} \cdot (-) : T \to T$.

5.3 The category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$

Let G be a finite group and write $\mathbb{T}G$ for the translation category on $\mathscr{C}$ (cf. definition 5.8). In this section, we describe some notable features of the category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$. This object has been studied by many authors. Thomason [40] considers its G-fixed points in connection to the homotopy limit problem, Murayama-Shimakawa [35], [39] introduced the construction $\mathbf{Fun}(\mathbb{T}G, -)$ for use in equivariant bundle theory, and Guillou-May-Merling-Osorno [18], [19], [20] take the categories $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ as prototypical input to their infinite loop space machinery.

Our goal, however, is to explain the connection between $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ and the G-symmetric monoidal structure on the coefficient system of G-subgroup actions in $\mathscr{C}$ (cf. [23]). The entire point of equipping G-categories with norms is to get transfer maps, and the transfers of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ are particularly striking. To start, if $\mathscr{C}$ is G-trivial, then there is an isomorphism $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H \cong \mathbf{Fun}(\mathbb{T}(G/H), \mathscr{C})$, and as observed in [24], the latter is equivalent to the category $H\mathscr{C}$ of H-actions in $\mathscr{C}$. The punchline, however, is that for any subgroups $K \subsetneq H \subset G$, the transfer $\mathrm{tr}_K^H = \bigotimes_{H/K} \circ \Delta^{\mathrm{tw}} : \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^K \to \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H$ can be identified with the monoidal pushforward $p_*^{\otimes} : \mathbf{Fun}(\mathbb{T}(G/K), \mathscr{C}) \to \mathbf{Fun}(\mathbb{T}(G/H), \mathscr{C})$ of [24]. It follows that the transfer $\mathrm{tr}_K^H : \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^K \to \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H$ is equivalent to the monoidal induction functor $N_K^H : K\mathscr{C} \to H\mathscr{C}$, which specializes to induction for group actions on sets, induction for group representations over vector spaces, and the Hill-Hopkins-Ravenel norm for equivariant spectra. In this section, we explain how all of this this goes.

5.3.1 The transfers of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$

We begin by describing our conventions for the category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ and its G-action.

Definition 5.8. The *translation category* of a (left) G-set X is the groupoid $\mathbb{T}X$ whose object set is X, and whose hom sets are $\mathbb{T}X(x, y) = \{g \in G \,|\, gx = y\}$.[1] Composition is by group multiplication, and the unit $e \in G$ gives the identities. There is a functor $\mathbb{T} : G\mathbf{Set} \to \mathbf{Cat}$ that sends a G-set X to $\mathbb{T}X$, and sends a G-map $f : X \to Y$ to the functor $\mathbb{T}f : \mathbb{T}X \to \mathbb{T}Y$ defined by the formula $\mathbb{T}f(x) = f(x)$ on objects and $\mathbb{T}f(g : x \to y) = g : f(x) \to f(y)$ on morphisms. We shall sometimes write $\mathbb{T}_G$ to emphasize that we are taking the translation category of a G-set.

Example 5.9. The group G acts on itself by left and right multiplication, and these actions interchange. Thus, we may regard G asymmetrically as a left G-set equipped with a right G-action. Applying $\mathbb{T}$ makes $\mathbb{T}G$ into a right G-category.

Since G is a transitive, free left G-set, it follows that for every $x, y \in \mathbb{T}G$, there is a unique morphism $! = yx^{-1} : x \to y$, and it is an isomorphism. For any $g \in G$, the functor $(-)g : \mathbb{T}G \to \mathbb{T}G$ sends the object $x \in \mathbb{T}G$ to xg and the morphism $yx^{-1} : x \to y$ to $yx^{-1} : xg \to yg$.

---

[1] Translation categories are denoted $T(G, X)$ in [19] and $\mathcal{B}_X G$ in [24].

Note that the translation category $\mathbb{T}G$ is isomorphic to the category $\widetilde{G}$ (cf. definition 2.16), but this is just a coincidence. One can make analogous constructions for monoids M, and in that case, $\mathbb{T}M$ and $\widetilde{M}$ are usually different. See [19, §1.4] for a further discussion.

Definition 5.10. Let $\mathscr{C}$ be a nonequivariant category. The left G-category $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ is the category whose objects are the functors $C_\bullet : \mathbb{T}G \to \mathscr{C}$, and whose morphisms are the natural transformations $f_\bullet : C_\bullet \to D_\bullet$. Composition is componentwise, i.e. $(f \circ g)_x = f_x \circ g_x$, and identities are, too: $(\mathrm{id}_C)_x = \mathrm{id}_{C_x}$. The right action of G on $\mathbb{T}G$ induces a left G-action on $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ by precomposition. Explicitly, $(gC)_\bullet = C_{\bullet g}$ and $(g\eta)_\bullet = \eta_{\bullet g}$.

Remark 5.11. One might also suppose that $\mathscr{C}$ has a nontrivial G-action, and that it is a symmetric monoidal object in $G\underline{\mathbf{Cat}}$ (cf. definition 2.1). In this case, the G-action on the objects of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ becomes $(g \cdot C)_x = g(C_{xg})$, and similarly for morphisms. We have a normed symmetric monoidal structure exactly as in section 5.2, but the following identification of the fixed points and transfers breaks down.

Next, we identify the fixed points of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$. As observed by Thomason [40], if $\mathscr{C}$ is G-trivial, then the G-fixed subcategory of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ is isomorphic to the category of G-actions in $\mathscr{C}$. We generalize this point to arbitrary subgroups $H \subset G$. First, observe that the projection map $\pi : G \to G/H$ sending x to xH determines a functor $\mathbb{T}\pi : \mathbb{T}G \to \mathbb{T}(G/H)$, and pulling back defines another functor $\mathbb{T}\pi^* : \mathbf{Fun}(\mathbb{T}(G/H), \mathscr{C}) \to \mathbf{Fun}(\mathbb{T}G, \mathscr{C})$. Since $\pi \circ (-)h = \pi$ for every $h \in H$, the functor $\mathbb{T}\pi^*$ lands in $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H$. On the other hand, if the diagram $C_\bullet : \mathbb{T}G \to \mathscr{C}$ is H-fixed, then it factors uniquely through $\mathbb{T}\pi$, and thus we obtain inverse functors

$$\mathbb{T}\pi^* : \mathbf{Fun}(\mathbb{T}(G/H), \mathscr{C}) \overset{\cong}{\rightleftarrows} \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H : q.$$

Next, we recall an observation of Hill-Hopkins-Ravenel. Let the functor $s : \mathbb{T}_H(H/H) \to \mathbb{T}_G(G/H)$ be the inclusion of $\mathbb{T}_H(H/H)$ as the automorphisms of the coset $eH \in \mathbb{T}_G(G/H)$. The functor s is an equivalence of categories, and we can construct an explicit deformation retraction $r : \mathbb{T}_G(G/H) \to \mathbb{T}_H(H/H)$ by choosing a set of G/H coset representatives $\{e = g_1, \dots, g_{|G:H|}\}$, and then setting $r(g_iH) = H$ and $r(g : g_iH \to g_jH) = g_j^{-1}gg_i : H \to H$. Since $g_1 = e$, it follows that $r \circ s = \mathrm{id}$. The equivalence $\mathbb{T}_H(H/H) \simeq \mathbb{T}_G(G/H)$ induces an equivalence

$$r^* : \mathbf{Fun}(\mathbb{T}_H(H/H), \mathscr{C}) \overset{\simeq}{\rightleftarrows} \mathbf{Fun}(\mathbb{T}_G(G/H), \mathscr{C}) : s^*,$$

and the functor category $\mathbf{Fun}(\mathbb{T}_H(H/H), \mathscr{C})$ is isomorphic to the category $H\mathscr{C}$ of H-actions in $\mathscr{C}$. Thus $H\mathscr{C} \simeq \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H$. One can lift this to an equivalence between the H-category $\mathscr{C}_H$ of all H-objects in $\mathscr{C}$ and *nonequivariant* maps between them, and the full subcategory of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ spanned by the H-fixed diagrams, but we shall not need to.

Finally, we describe the transfers of $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})$.

Theorem 5.12. *Suppose that G is a finite group, $K \subsetneq H \subset G$ are subgroups, and that $\mathscr{C}$ is a nonequivariant symmetric monoidal category. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^K \arrow[r, "\Delta^{\mathrm{tw}}"] \arrow[rr, bend left, "\mathrm{tr}_K^H"] \arrow[d, shift left, "q"] & \left(\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^{\times H/K}\right)^H \arrow[r, "\bigotimes_{H/K}"] & \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^H \arrow[d, shift left, "q"] \\
\mathbf{Fun}(\mathbb{T}(G/K), \mathscr{C}) \arrow[u, shift left, "\mathbb{T}\pi^*"] \arrow[rr, "p_*^{\otimes}"'] & & \mathbf{Fun}(\mathbb{T}(G/H), \mathscr{C}) \arrow[u, shift left, "\mathbb{T}\pi^*"] \arrow[d, "s^*"] \\
K\mathscr{C} \arrow[u, "r^*"] \arrow[rr, "N_K^H"'] & & H\mathscr{C}
\end{tikzcd}$$

where $p_*^{\otimes}$ is the monoidal pushforward for the functor $p : \mathbb{T}(G/K) \to \mathbb{T}(G/H)$ that sends gK to gH, and N_K^H is the norm functor.

The proof is straightforward once the correct definitions have been made, but one must be a bit careful because some of these functors are constructed using noncanonical choices.

In general, this diagram only commutes up to natural isomorphism, but we shall explain how to achieve strict commutativity in the next section.

5.3.2 Monoidal pushforwards and the proof of theorem 5.12

Suppose that $\mathscr{C}$ is a nonequivariant symmetric monoidal category. In [6], [24], the authors explain how to construct a monoidal pushforward $p_*^{\otimes} : \mathbf{Fun}(I, \mathscr{C}) \to \mathbf{Fun}(J, \mathscr{C})$ associated to any finite covering category $p : I \to J$. For their purposes, it was not necessary to track the orderings in tensor products too carefully, and therefore those details were rightly suppressed. Unfortunately, our present work demands attention to these matters, because twisted equivariance for υ_T is entirely about the relationship between a group action and symmetric monoidal permutation maps. Thus, we review the construction of $p_*^{\otimes}$, with a focus on orderings.

Recall the following definition [24, definition A.24].

Definition 5.13. A *finite covering category* is a functor $p : I \to J$ such that

1. For every morphism $f : j \to j'$ in J and object $i \in p^{-1}(j)$, there is a unique I-morphism $\widetilde{f}$ such that $\mathrm{dom}\widetilde{f} = i$ and $p\widetilde{f} = f$.
2. For every morphism $f : j \to j'$ in J and object $i' \in p^{-1}(j')$, there is a unique I-morphism $\widetilde{f}$ such that $\mathrm{cod}\widetilde{f} = i'$ and $p\widetilde{f} = f$.
3. For every object $j \in J$, the fiber $p^{-1}(j) \subset \mathrm{Ob}(I)$ is finite.

Convention 5.14. We shall assume that every finite covering category $p : I \to J$ comes equipped with a chosen linear ordering on every fiber $p^{-1}(j)$.

We write $\widetilde{f_i}$ for the unique lift of f starting at i, and we define $f \cdot i := \mathrm{cod}\widetilde{f_i}$. For any $f : j \to j'$ in J, we obtain a set bijection $f \cdot (-) : p^{-1}(j) \to p^{-1}(j')$, but it will not generally respect the orderings of $p^{-1}(j)$ and $p^{-1}(j')$. Suppose that the fibers $p^{-1}(j)$ and $p^{-1}(j')$ both have cardinality n, and write

$$p^{-1}(j) = \{i_1 < \cdots < i_n\} \quad \text{and} \quad p^{-1}(j') = \{i'_1 < \cdots < i'_n\}.$$

We define the permutation $\sigma(f) \in \Sigma_n$ by the formula $f \cdot i_k = i'_{\sigma(f)k}$. Equivalently, there is a lift $\widetilde{f}_{i_k} : i_k \to i'_{\sigma(f)k}$ of f.

Recall (definition 2.1) that the *standard tensor products* on $\mathscr{C}$ are $\bigotimes_0 := e$, $\bigotimes_1 := \mathrm{id}$, $\bigotimes_2 := \otimes$, and $\bigotimes_{n+1} := \otimes \circ (\bigotimes_n \times \mathrm{id})$ for $n \geq 2$.

Definition 5.15. Keep notation as above. For any symmetric monoidal category $\mathscr{C}$ and finite covering category $p : I \to J$, the *monoidal pushforward* functor $p_*^{\otimes} : \mathbf{Fun}(I, \mathscr{C}) \to \mathbf{Fun}(J, \mathscr{C})$ is defined as follows.

(a) Given a functor $X : I \to \mathscr{C}$, we define $p_*^{\otimes} X : J \to \mathscr{C}$ on objects $j \in J$ by

$$(p_*^{\otimes} X)(j) := \bigotimes\nolimits_n \Big(X(i_1), \ldots, X(i_n) \Big),$$

where $p^{-1}(j) = \{i_1 < \cdots < i_n\}$ and $\bigotimes_n$ is the standard n-fold tensor product. Then, given $f : j \to j'$ in J, we define $p_*^{\otimes}(f) : p_*^{\otimes}(j) \to p_*^{\otimes}(j')$ to be the composite

$$\bigotimes\nolimits_n \Big(X(i_k) \Big) \xrightarrow{\bigotimes_n \left(X(\widetilde{f}_{i_k}) \right)} \bigotimes\nolimits_n \Big(X(i'_{\sigma(f)k}) \Big) \xrightarrow{\sigma(f)} \bigotimes\nolimits_n \Big(X(i'_k) \Big)$$

where $\sigma(f)$ is the symmetric monoidal coherence map for $\mathscr{C}$ that permutes the factors of the tensor product by $\sigma(f)$.

(b) Given a natural transformation $\eta_\bullet : X_\bullet \Rightarrow Y_\bullet$, we define

$$(p_*^{\otimes} \eta)_j := \bigotimes\nolimits_n \Big(\eta_{i_1}, \ldots, \eta_{i_n} \Big),$$

where $p^{-1}(j) = \{i_1 < \cdots < i_n\}$.

For completeness, we also recall the definition of the norm [24, definition A.52].

Definition 5.16. For subgroups $K \subset H \subset G$, the *norm functor* $N_K^H : K\mathscr{C} \to H\mathscr{C}$ is the composite $p_*^{\otimes} \circ r^* : K\mathscr{C} \to \mathbf{Fun}(\mathbb{T}_H(H/K), \mathscr{C}) \to H\mathscr{C}$, where $p_*^{\otimes}$ is the monoidal pushforward for $p : \mathbb{T}_H(H/K \to H/H)$.

Proof of theorem 5.12. The point is to make compatible choices of coset representatives and orderings. We describe one possible route.

First, choose sets $\{e = g_1, \dots, g_{|G:H|}\}$ and $\{e = h_1, \dots, h_{|H:K|}\}$ of G/H and H/K coset representatives, and give the orbits G/H and H/K the corresponding orders. We obtain a set $\{g_i h_j \,|\, 1 \le i \le |G:H|\,,\, 1 \le j \le |H:K|\}$ of G/K coset representatives, and we order G/K lexicographically as follows:

$$K < h_2K < \cdots < h_{|H:K|}K < g_2K < g_2h_2K < \cdots < g_2h_{|H:K|}K < \cdots .$$

From here, we

(a) use the relation $h \cdot h_iK = h_{\sigma(h)i}K$ to define $\Gamma_{H/K} = \{(h, \sigma(h)) \,|\, h \in H\}$, and give $\mathbf{Fun}(\mathbb{T}G, \mathscr{C})^{\times H/K}$ the diagonal H-action twisted by σ,

(b) construct the norm map $\bigotimes_{H/K} : \mathbf{Fun}(\mathbb{T}G, \mathscr{C})^{\times H/K} \to \mathbf{Fun}(\mathbb{T}G, \mathscr{C})$ as in construction 5.6, using the G/H coset representatives g_i,

(c) define $p_*^{\otimes} : \mathbf{Fun}(\mathbb{T}(G/K), \mathscr{C}) \to \mathbf{Fun}(\mathbb{T}(G/H), \mathscr{C})$ using the order on the fibers of $p : \mathbb{T}(G/K) \to \mathbb{T}(G/H)$ induced by the order on G/K,

(d) define $r : \mathbb{T}_G(G/K) \to \mathbb{T}_K(K/K)$ using the coset representatives g_ih_j, and

(e) for the norm $N_K^H : K\mathscr{C} \to H\mathscr{C}$, we define $r : \mathbb{T}_H(H/K) \to \mathbb{T}_K(K/K)$ using the coset representatives h_j, and use the order on H/K to construct the monoidal pushforward $p_*^{\otimes} : \mathbf{Fun}(\mathbb{T}_H(H/K), \mathscr{C}) \to \mathbf{Fun}(\mathbb{T}_H(H/H), \mathscr{C})$.

The remaining verifications are left to the interested reader. For the upper square in theorem 5.12, we find it easiest to compare $p_*^{\otimes}$ to the composite $q \circ \bigotimes_{H/K} \circ \Delta^{\mathrm{tw}} \circ \mathbb{T}\pi^*$. $\square$

5.4 Normed symmetric monoidal structures from twisted products

In this section, suppose that $G = C_2$ and write $g \in C_2$ for the generator. The examples considered in the previous two sections build norms out of operations $\otimes$ that satisfy the usual equivariance equation $g(C \otimes D) = gC \otimes gD$. The constructions of this section build untwisted products out an operation $\boxtimes$ satisfying the equation $g(C \boxtimes D) = gD \boxtimes gC$. As before, we focus on diagram categories, but despite this formal similarity, the examples in this section are actually quite different. The operations on $\mathbf{Fun}(J, \mathscr{C})$ considered previously are defined purely on the level of operads (cf. section 6.3), while the operations in this section mix in data from the diagrams.

Example 5.17. The relevant operation $\boxtimes$ will be obtained by passing to a small model of concatenation on the category of finite, linearly ordered sets. Write $X = (x_1, \ldots, x_m)$. The group C_2 acts by reversing orders, and we write $\overline{(x_1, \ldots, x_m)} = (x_m, \ldots, x_1)$. Given two ordered sets $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$, we define $X \sqcup Y = (x_1, \ldots, x_m, y_1, \ldots, y_n)$, and therefore $\overline{X \sqcup Y} \cong \overline{Y} \sqcup \overline{X}$.

Now consider $\mathbb{T}C_2$ diagrams of such objects. These amount to a pair of finite, ordered sets with a chosen set bijection between them, and the basic idea is to use the formulas

$$(X \cong X') \sqcup (Y \cong Y') := X \sqcup Y' \cong Y \sqcup X'$$
$$(X \cong X') \sqcup_{C_2/e} (Y \cong Y') := X \sqcup Y \cong X' \sqcup Y'$$

to equip this diagram category with a normed symmetric monoidal structure.

Formally, we proceed as follows. Let $\mathcal{S}$ be the category whose objects are the natural numbers, and whose morphisms $m \to n$ are arbitrary *partial functions* $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$. We do not require f to be order-preserving. We obtain a C_2-action on $\mathcal{S}$ from the involution $\overline{(-)} : \mathcal{S} \to \mathcal{S}$ defined by $\overline{n} = n$ on objects, and for each $f : m \to n$, we define the map $\overline{f} : \overline{m} \to \overline{n}$ by

$$\overline{f}(x) = n + 1 - f(m + 1 - x).$$

The operation $\sqcup$ is defined by $m \sqcup n = m + n$ on objects, and given $f_1 : m_1 \to n_1$ and $f_2 : m_2 \to n_2$, we define $f_1 \sqcup f_2 : m_1 + m_2 \to n_1 + n_2$ by

$$(f_1 \sqcup f_2)(x) = \begin{cases} f_1(x) & \text{if } 1 \leq x \leq m_1 \\ f_2(x - m_1) + n_1 & \text{if } m_1 + 1 \leq x \leq m_1 + m_2 \end{cases}.$$

Ignoring the C_2-action, it is easy to see that we obtain a strictly associative and unital symmetric monoidal structure on $\mathcal{S}$. The braiding $\beta_{m,n} : m + n \to n + m$ is given by the block transposition $\tau(m, n)$. Remembering the C_2-action, we have $\overline{f_1 \sqcup f_2} = \overline{f_2} \sqcup \overline{f_1}$, so that $\sqcup$ is a C_2-equivariant functor $\sqcup : \mathcal{S}^{\times C_2/e} \to \mathcal{S}$, and $\overline{\beta_{m,n}} = \beta_{\overline{n},\overline{m}}$.

Now consider the diagram category $\mathbf{Fun}(\mathbb{T}C_2, \mathcal{S})$ and write $C_2 = \{e, g\}$. Such diagrams $C_\bullet : \mathbb{T}C_2 \to \mathcal{S}$ can be identified with isomorphisms $C_{e \to g} : C_e \to C_g$. We define a C_2-equivariant sum on $\mathbf{Fun}(\mathbb{T}C_2, \mathcal{S})$ using the formula

$$(C \oplus D)_{e \to g} : C_e \sqcup D_g \to C_g \sqcup D_e \to D_e \sqcup C_g.$$

Here, the first map is the disjoint union of the structural maps for C and D, while the second map is the braiding. We define a C_2/e-norm by

$$(C \boxplus D)_{e \to g} : C_e \sqcup D_e \to C_g \sqcup D_g.$$

These data, together with the evident coherence isomorphisms, make $\mathbf{Fun}(\mathbb{T}C_2, \mathcal{S})$ into a $\{C_2/e\}$-normed symmetric monoidal category.

Informally, the C_2/e norm on $\mathcal{S}$ arises because line segments have C_2-symmetry, and gluing two line segments end-to-end yields another line segment. One can generalize the preceding example by replacing line segments with other similarly behaved shapes, such as rectangles, prisms, and so on.

We conclude by describing a labeled variant of the preceding example.

Example 5.18. Let $\mathscr{C}$ be a fixed symmetric monoidal object in $C_2\underline{\mathbf{Cat}}$. We define the C_2-category $\mathscr{C}^{\otimes}$ as follows (compare to [29, construction 2.0.0.1]): an object of $\mathscr{C}^{\otimes}$ is a finite (possibly empty) sequence $(C_1, \ldots, C_m)$ of objects of $\mathscr{C}$, and a morphism $\mathbf{f} : (C_1, \ldots, C_m) \to (D_1, \ldots, D_n)$ in $\mathscr{C}^{\otimes}$ consists of a partial function $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, together with morphisms

$$\varphi_i : \bigotimes_{f(j)=i} C_j \to D_i$$

in $\mathscr{C}$ for every $i = 1, \ldots, n$. For definiteness, if $f^{-1}(i) = \{j_1 < \cdots < j_k\}$, then we understand $\bigotimes_{f(j)=i} C_j$ to be $(\cdots((C_{j_1} \otimes C_{j_2}) \otimes C_{j_3}) \cdots \otimes C_{j_{k-1}}) \otimes C_{j_k}$. Write $C_2 = \{e, g\}$. The category $\mathscr{C}^{\otimes}$ gets a C_2-action via the involution $\overline{(-)} : \mathscr{C}^{\otimes} \to \mathscr{C}^{\otimes}$ that sends a tuple $(C_1, \ldots, C_m)$ to $(gC_m, \ldots, gC_1)$ and a morphism $\mathbf{f} : (C_1, \ldots, C_m) \to (D_1, \ldots, D_n)$ to the morphism $\overline{\mathbf{f}} : (gC_m, \ldots, gC_1) \to (gD_n, \ldots, gD_1)$, whose underlying partial function is the mirror image $\overline{f} : \{1, \ldots, m\} \to \{1, \ldots, n\}$, and whose maps $\overline{\varphi}_{n+1-i}$ are given by the composites

$$(\cdots(gC_{j_k} \otimes gC_{j_{k-1}}) \cdots) \otimes gC_{j_1} \xrightarrow{\cong} (\cdots(gC_{j_1} \otimes gC_{j_2}) \cdots) \otimes gC_{j_k} \xrightarrow{g\varphi_i} gD_i.$$

We define a concatenation operation $\sqcup$ on the objects of $\mathscr{C}^{\otimes}$ by

$$(C_1, \ldots, C_m) \sqcup (C'_1, \ldots, C'_{m'}) := (C_1, \ldots, C_m, C'_1, \ldots, C'_{m'}).$$

To concatenate a pair of morphisms $\mathbf{f} : (C_1, \ldots, C_m) \to (D_1, \ldots, D_n)$ and $\mathbf{f}' : (C'_1, \ldots, C'_{m'}) \to (D'_1, \ldots, D'_{n'})$, we concatenate the underlying partial functions $f \sqcup f' : m + m' \to n + n'$ as before, and we track the maps φ_i and φ'_i for both $\mathbf{f}$ and $\mathbf{f}'$ simultaneously. Then $\sqcup$ is a C_2-equivariant functor $\sqcup : (\mathscr{C}^{\otimes})^{\times C_2/e} \to \mathscr{C}^{\otimes}$, and ignoring all equivariance, it makes $\mathscr{C}^{\otimes}$ into a strictly associative and unital symmetric monoidal category. The braiding satisfies $\overline{\beta_{C_\bullet, D_\bullet}} = \beta_{\overline{D_\bullet}, \overline{C_\bullet}}$, and the same construction as before makes the diagram category $\mathbf{Fun}(\mathbb{T}C_2, \mathscr{C}^{\otimes})$ into a $\{C_2/e\}$-normed symmetric monoidal category.

CHAPTER 6

EXAMPLES OF HOMOGENEOUS OPERADS

6.1 Introduction and summary of results

In this chapter, we study an eclectic selection of homogeneous operads, and a few constructions that can be performed on them. The operadic considerations in chapters 2 – 4 primarily concern the properties of the operads $\mathcal{SM}_{\mathcal{N}}$, for general sets of exponents $\mathcal{N}$. In this chapter, we make more specific choices of $\mathcal{N}$, and we discuss a few other homogeneous operads, which we hope will be useful in applications. From a more conceptual standpoint, the work in chapter 3 establishes that every indexing system can be realized by a homogeneous operad of the form $\mathcal{SM}_{\mathcal{N}}$, and that taking admissible sets defines an equivalence $\mathbb{A} : \mathrm{Ho}(N_\infty\text{-}\mathbf{Op}) \to \mathbf{Ind}$ between the homotopy category of N_∞ operads and the poset of indexing systems (theorems 3.19 and 3.22). The examples in this chapter refine those results.

In section 6.2, we describe some choices of $\mathcal{N}$ that lead to uniform realizations of every indexing system, and efficient realizations of the maximum indexing system, $\underline{\mathbf{Set}}$. In section 6.3, we analyze the natural operads $\mathbf{Fun}(J, \mathscr{O})$ that act on the functor categories $\mathbf{Fun}(J, \mathscr{C})$ considered in section 5.2. The remainder of this chapter focuses on quotient operads. In section 6.4, we construct N_∞ permutativity operads for all indexing systems, and in section 6.5, we analyze the coproducts and Boardman-Vogt tensor products of homogeneous operads. Section 6.6 describes a method for controlling the combinatorics in quotient operads.

Let $N_\infty\text{-}\mathbf{Op}_h$ denote the category of homogeneous categorical N_∞ operads. In a bit more detail, we prove the following theorem (theorem 6.3 and proposition 6.19).

Theorem. *The admissible sets functor* $\mathbb{A} : N_\infty\text{-}\mathbf{Op}_h \to \mathbf{Ind}$ *has a strictly functorial section* $s : \mathbf{Ind} \to N_\infty\text{-}\mathbf{Op}_h$ *that sends inclusions of indexing systems to inclusions of operads.*

The point is that we can choose generating sets of exponents $\mathcal{N}(\underline{\mathcal{F}})$ uniformly in the indexing system $\underline{\mathcal{F}} \in \mathbf{Ind}$, and this results in the desired inclusions between the operads

$\mathcal{SM}_{\mathcal{N}(\underline{\mathcal{F}})}$. A variant of this construction can be obtained by passing to the universal reduced, strictly associative and unital quotients $\mathscr{P}_{\mathcal{N}(\underline{\mathcal{F}})}$ of the operads $\mathcal{SM}_{\mathcal{N}(\underline{\mathcal{F}})}$.

Remark. The operad $\mathscr{P}_{\mathcal{N}}$ is an N_∞ permutativity operad in a sense defined in section 6.4. Other examples are given by the Barratt-Eccles operad $\mathscr{P}$ equipped with a trivial G-action, the G-Barratt-Eccles operad $\mathscr{P}_G = \mathbf{Fun}(\widetilde{G}, \mathscr{P})$ and its N_∞ suboperads, and the operads $\mathbf{Fun}(J, \mathscr{P})$ for general right G-categories J. Bonventre showed that not every indexing system is realized as a suboperad of $\mathscr{P}_G$ (cf. [7, appendix B] and example 6.15), and we give an analogous result for the operads $\mathbf{Fun}(J, \mathscr{P})$ (example 6.12).

Our work also investigates the extent to which the lattice structure on the poset of indexing systems is reflected on the level of operads. It is straightforward to check that the class of admissible sets of a product of N_∞ operads is the intersection of the admissible sets of the factors [5, proposition 5.1]. We give similar calculations for certain coproducts and Boardman-Vogt tensor products of homogeneous operads (theorems 6.24 and 6.27).

Theorem. *Suppose that $\mathscr{O}_1$ and $\mathscr{O}_2$ are homogeneous categorical N_∞ operads. Then the class of admissible sets of the coproduct $\mathscr{O}_1 * \mathscr{O}_2$ is the join of the admissible sets of $\mathscr{O}_1$ and the admissible sets of $\mathscr{O}_2$. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are retracts of $\mathcal{SM}_{\mathcal{N}_1}$ and $\mathcal{SM}_{\mathcal{N}_2}$ respectively, then the same is true for the class of admissible sets of the tensor product $\mathscr{O}_1 \otimes \mathscr{O}_2$.*

Note that the coproduct $\mathscr{O}_1 * \mathscr{O}_2$ and the tensor product $\mathscr{O}_1 \otimes \mathscr{O}_2$ above are taken in the category of homogeneous categorical operads, and that the classifying space functor does not preserve these colimits. The second half of this theorem is the combinatorial analogue to Blumberg and Hill's conjecture on the Boardman-Vogt tensor product of N_∞ operads [5, conjecture 6.27]. Our work does not verify their conjecture as originally stated, but we do conclude that there are N_∞ operads in G-spaces that interchange with themselves, for every indexing system (corollary 6.28).

6.2 Choosing an operad $\mathcal{SM}_{\mathcal{N}}$

Suppose that we are trying realize the indexing system $\underline{\mathcal{F}}$ as an operad $\mathcal{SM}_{\mathcal{N}}$. There many different ways to generate $\underline{\mathcal{F}}$, and thus we have a considerable amount of freedom. We shall explain some choices of $\mathcal{N}$ that lead to a uniform construction of operads for every indexing system, and some choices specific to the maximum indexing system $\underline{\mathbf{Set}}$.

Definition 6.1. Let **Ind** denote the poset of all indexing systems. We say that a tuple of sets of exponents $(\mathcal{N}(\underline{\mathcal{F}}))_{\underline{\mathcal{F}}\in\mathbf{Ind}}$ is *convenient* if

1. for every indexing system $\underline{\mathcal{F}}$, the indexing system generated by $\mathcal{N}(\underline{\mathcal{F}})$ is $\underline{\mathcal{F}}$, and
2. for any indexing systems $\underline{\mathcal{F}} \subset \underline{\mathcal{G}}$, the inclusion $\mathcal{N}(\underline{\mathcal{F}}) \subset \mathcal{N}(\underline{\mathcal{G}})$ holds.

Example 6.2. Let

$$\mathbf{O}(\underline{\mathcal{F}}) = \{H/K \in \underline{\mathcal{F}} \,|\, K \subsetneq H \subset G\},$$

and choose an order on each orbit H/K. Then $(\mathbf{O}(\underline{\mathcal{F}}))_{\underline{\mathcal{F}}\in\mathbf{Ind}}$ is convenient.

For any integer $n \geq 0$, subgroup $H \subset G$, and group homomorphism $\sigma : H \to \Sigma_n$, write (n, σ) for the action on $\{1, \ldots, n\}$ determined by σ. Let

$$\mathbf{S}(\underline{\mathcal{F}}) = \{(n, \sigma) \in \underline{\mathcal{F}} \,|\, n \geq 0,\ H \subset G, \text{ and } \sigma : H \to \Sigma_n\}.$$

Then $(\mathbf{S}(\underline{\mathcal{F}}))_{\underline{\mathcal{F}}\in\mathbf{Ind}}$ is convenient.

Theorem 6.3. *If $(\mathcal{N}(\underline{\mathcal{F}}))_{\underline{\mathcal{F}}\in\mathbf{Ind}}$ is convenient, then there is a functorial section*

$$\mathcal{SM}_{\mathcal{N}(-)} : \mathbf{Ind} \to N_\infty\text{-}\mathbf{Op}_h$$

of the admissible sets functor $\mathbb{A} : N_\infty\text{-}\mathbf{Op}_h \to \mathbf{Ind}$, which takes inclusions to inclusions.

Proof. The map $\mathcal{SM}_{\mathcal{N}(-)}$ is right inverse to $\mathbb{A}$ on indexing systems by theorem 3.19, so we just need to consider the inclusions. Direct inspection of the construction of the free operad

$\mathbb{F}(S)$ in section 2.4 reveals that if $\mathcal{N}_1 \subset \mathcal{N}_2$, then we can construct $\mathcal{SM}_{\mathcal{N}_1}$ as a suboperad of $\mathcal{SM}_{\mathcal{N}_2}$ by restricting the norm symbols $\bigotimes_T$ that we use. It follows that if $(\mathcal{N}(\underline{\mathcal{F}}))_{\underline{\mathcal{F}} \in \mathbf{Ind}}$ is convenient, then we can construct the operad $\mathcal{SM}_{\mathcal{N}(\underline{\mathbf{Set}})}$ first, and then take every other operad $\mathcal{SM}_{\mathcal{N}(\underline{\mathcal{F}})}$ to be the relevant suboperad. □

The sets $\mathbf{O}(\underline{\mathcal{F}})$ and $\mathbf{S}(\underline{\mathcal{F}})$ have different virtues. By choosing orders, we see that every $T \in \underline{\mathcal{F}}$ is isomorphic to a set in $\mathbf{S}(\underline{\mathcal{F}})$, and thus the operad $\mathcal{SM}_{\mathbf{S}(\underline{\mathcal{F}})}$ parametrizes every norm in $\underline{\mathcal{F}}$ explicitly. We conclude that $\mathbf{S}(\underline{\mathcal{F}})$ is, in some sense, canonical or maximal. On the other hand, indexing systems are completely determined by their orbits, so the operad $\mathcal{SM}_{\mathbf{O}(\underline{\mathcal{F}})}$ gives a natural and *finite* biased presentation of an N_∞-structure based on $\underline{\mathcal{F}}$.

Proposition 6.4. *Fix an indexing system $\underline{\mathcal{F}}$. Every $\mathbf{O}(\underline{\mathcal{F}})$-normed symmetric monoidal category $\mathscr{C}$ has a finite presentation by generators and isomorphism relations, and the classifying space of $\mathscr{C}$ is an algebra over an N_∞ operad in G-spaces structured by $\underline{\mathcal{F}}$.*

Proof. That $\mathscr{C}$ has a finite presentation follows from the fact that G, and hence $\mathbf{O}(\underline{\mathcal{F}})$, is finite. By theorem 2.10, the category $\mathscr{C}$ is a $\mathcal{SM}_{\mathbf{O}(\underline{\mathcal{F}})}$-algebra, and the rest follows because B preserves products and admissible sets. □

We conclude with the following observation.

Example 6.5. Each of the sets

- $\mathbf{S}(\underline{\mathbf{Set}}) = \{(n, \sigma) \,|\, n \geq 0,\ H \subset G, \text{ and } \sigma : H \to \Sigma_n\}$,
- $\mathbf{O}(\underline{\mathbf{Set}}) = \{H/K \,|\, K \subsetneq H \subset G\}$,
- $\{G/H \,|\, H \subsetneq G\}$, and
- $\{\coprod_{H \subsetneq G} G/H\}$

generates the indexing system $\underline{\mathbf{Set}}$. The top set is countably infinite, the bottom three sets are finite, and the last set is a singleton. Thus, if $\mathcal{N}$ is any one of the sets above, then $\mathcal{SM}_{\mathcal{N}}$

is an equivariant E_∞ operad, and $\mathcal{N}$-normed symmetric monoidal categories are equivariant E_∞ algebras. Thus, we only need a G-fixed constant e, a G-equivariant product $\otimes$, and a single $(\coprod_{H \subsetneq G} G/H)$-norm to generate an entire E_∞ structure. In fact, just a G-fixed constant e and a $(G/G \sqcup G/G \sqcup \coprod_{H \subsetneq G} G/H)$-norm will suffice.

6.3 The operads $\mathbf{Fun}(J, \mathcal{SM})$ and $\mathbf{Fun}(J, \mathscr{P})$

As explained in section 5.2, good examples of normed symmetric monoidal categories arise as $\mathbf{Fun}(J, \mathscr{C})$, where J is a right G-category and $\mathscr{C}$ is a nonequivariant symmetric monoidal or permutative category. Such categories $\mathscr{C}$ start life as $\mathcal{SM}$-algebras or $\mathscr{P}$-algebras, where $\mathscr{P}$ is the Barratt-Eccles operad. Since the functor $\mathbf{Fun}(J, -) : \mathbf{Cat} \to G\mathbf{Cat}$ preserves products, the natural operads that act on $\mathbf{Fun}(J, \mathscr{C})$ are $\mathbf{Fun}(J, \mathcal{SM})$ or $\mathbf{Fun}(J, \mathscr{P})$, respectively. These are both homogeneous N_∞ operads, and we shall compute their admissible sets.

Throughout this section, suppose that $\mathscr{O}$ is a nonequivariant, homogeneous E_∞ operad. We have $\mathcal{SM}$ and $\mathscr{P}$ in mind, but the precise operad is not important for this discussion.

Lemma 6.6. *The operad* $\mathbf{Fun}(J, \mathscr{O})$ *is homogeneous, and for every integer* $n \geq 0$*, the map* $\mathrm{Ob} : \mathbf{Fun}(J, \mathscr{O}(n)) \to \widetilde{\mathbf{Set}}(\mathrm{Ob}J, \mathrm{Ob}\mathscr{O}(n))$ *is an isomorphism of* $G \times \Sigma_n$*-categories.*

Proof. For every $n \geq 0$, the category $\mathbf{Fun}(J, \mathscr{O}(n))$ is homogeneous because $\mathscr{O}(n)$ is. Consider $\mathbf{Cat}(J, \mathscr{O}(n)) = \mathrm{Ob}\mathbf{Fun}(J, \mathscr{O}(n))$. It is isomorphic to $\mathbf{Set}(\mathrm{Ob}J, \mathrm{Ob}\mathscr{O}(n))$ by adjunction, and therefore $\mathbf{Fun}(J, \mathscr{O}(n)) \cong \widetilde{\mathbf{Set}}(\mathrm{Ob}J, \mathrm{Ob}\mathscr{O}(n))$. The $G \times \Sigma_n$ actions on both sides are defined by $(g, \sigma) \cdot F = \sigma \cdot (-) \circ F \circ (-) \cdot g$, and thus the isomorphism respects the action. $\square$

Lemma 6.7. *If* $J \neq \varnothing$*, then* $\mathbf{Fun}(J, \mathscr{O})$ *is* Σ*-free.*

Proof. If $\sigma \cdot (-) \circ F = F$, then evaluating at some $j \in J$ proves that $\sigma = \mathrm{id}$. $\square$

Lemma 6.8. *Suppose that* X *is a right* G*-set, that* Y *is a nonempty, free left* Σ_n*-set, and consider the set of functions* $\mathbf{Set}(X, Y)$*, equipped with the left* $G \times \Sigma_n$*-action* $[(g, \tau) \cdot f](x) =$

$\tau f(xg)$. Let $H \subset G$ be a subgroup, $\sigma : H \to \Sigma_n$ be a group homomorphism, and write $\Gamma = \{(h, \sigma(h)) \mid h \in H\}$. Then

$$\mathbf{Set}(X,Y)^{\Gamma} \neq \varnothing \quad \text{if and only if} \quad \bigcup_{x \in X} \mathrm{Stab}_H(x) \subset \ker(\sigma).$$

Proof. Suppose that $f \in \mathbf{Set}(X,Y)^{\Gamma}$. If $h \in H$ fixes x, then

$$f(x) = [(h, \sigma(h)) \cdot f](x) = \sigma(h) f(xh) = \sigma(h) f(x),$$

and therefore $\sigma(h) = \mathrm{id}$ by Σ-freeness. Conversely, if $\bigcup_{x \in X} \mathrm{Stab}_H(x) \subset \ker(\sigma)$, then we can define a Γ-fixed function $f : X \to Y$ by choosing H-orbit representatives x_i for X and a point $y \in Y$, and then setting $f(x_i h) := \sigma(h)^{-1} y$. □

Combining the lemmas above, we obtain the following.

Theorem 6.9. *Suppose that $\mathscr{O}$ is a nonequivariant homogeneous E_∞ operad, and let $J \neq \varnothing$ be a right G-category. Then $\mathbf{Fun}(J, \mathscr{O})$ is a homogeneous N_∞ operad, and for any subgroup $H \subset G$ and finite H-set T, the set T is admissible for $\mathbf{Fun}(J, \mathscr{O})$ if and only if*

$$\bigcup_{j \in \mathrm{Ob} J} \mathrm{Stab}_H(j) \subset \bigcap_{t \in T} \mathrm{Stab}_H(t) = \left\{ h \in H \,\middle|\, h \cdot (-) = \mathrm{id} : T \to T \right\},$$

i.e. every element of H that fixes an element of $\mathrm{Ob}J$ acts as the identity on all of T.

Corollary 6.10. *If $\mathrm{Ob}J$ is a free right G-category, then all finite G-subgroup actions are admissible for $\mathbf{Fun}(J, \mathscr{O})$, i.e. $\mathbf{Fun}(J, \mathscr{O})$ is an equivariant E_∞ operad.*

Remark 6.11. In particular, the preceding corollary applies when $J = \widetilde{G}$ and $\mathscr{O} = \mathscr{P}$, in which case $\mathbf{Fun}(\widetilde{G}, \mathscr{P}) = \mathscr{P}_G$ is the equivariant Barratt-Eccles operad.

For any right G-category J, write $\underline{\mathcal{F}}_J$ for the class of admissible sets of $\mathbf{Fun}(J, \mathscr{O})$. In light of theorem 3.17, this class is an indexing system, but one can also check this directly.

The defining condition of $\underline{\mathcal{F}}_J$ is fairly stringent. We now show by example that not every G-indexing system arises as $\underline{\mathcal{F}}_J$ for some category J.

Example 6.12. Suppose that $G = C_4$, choose a generator $g \in C_4$, and let $H = \{e, g^2\}$. Let $\underline{\mathcal{F}}$ be the C_4-indexing system that contains all finite H-sets, but only trivial actions otherwise. If J is a C_4-category for which $\underline{\mathcal{F}} \subset \mathbb{A}(\mathbf{Fun}(J, \mathscr{O}))$, then g^2 cannot fix any object $j \in J$, because g^2 acts nontrivially on H/e. It follows that C_4 acts freely on $\mathrm{Ob}J$, and hence $\mathbb{A}(\mathbf{Fun}(J, \mathscr{O})) = \underline{\mathbf{Set}}$. Thus the admissible sets of $\mathbf{Fun}(J, \mathscr{O})$ can never be precisely $\underline{\mathcal{F}}$.

6.4 Equivariant permutativity operads

For applications in infinite loop space theory, it is often preferable to work with symmetric monoidal categories that satisfy additional strict relations. Recall that a *permutative category* is a symmetric monoidal category $\mathscr{C}$ such that the isomorphisms

$$\alpha : (C \otimes D) \otimes E \to C \otimes (D \otimes E) \quad , \quad \lambda : e \otimes C \to C \quad , \quad \rho : C \otimes e \to C$$

are all identity maps. Permutative categories are the same thing as algebras over the categorical Barratt-Eccles operad $\mathscr{P}$, which is obtained by applying the functor $\widetilde{(-)} : \mathbf{Set} \to \mathbf{Cat}$ to the associativity operad. In this section, we consider the N_∞ analogues to $\mathscr{P}$. Let $\varepsilon^* \mathscr{P}$ denote the Barratt-Eccles operad, equipped with a trivial G-action.

Definition 6.13. An N_∞ *permutativity operad* is a reduced operad $\mathscr{O} \in N_\infty$-$\mathbf{Op}_h$, equipped with a map $\varepsilon^* \mathscr{P} \to \mathscr{O}$. This map is necessarily an embedding, and we require morphisms of N_∞ permutativity operads to respect the maps from $\varepsilon^* \mathscr{P}$.

Example 6.14. The G-Barratt-Eccles operad $\mathscr{P}_G = \mathbf{Fun}(\widetilde{G}, \mathscr{P}) \cong \widetilde{\mathbf{Set}}(G, \Sigma_\bullet)$ is an E_∞ operad by theorem 6.9, and $\varepsilon^* \mathscr{P}$ embeds diagonally as the G-fixed suboperad of $\mathscr{P}_G$. In general, if J is any right G-category, then the operad $\mathscr{P}_J = \mathbf{Fun}(J, \mathscr{P})$ is an N_∞ permutativity operad with admissible sets $\underline{\mathcal{F}}_J$.

One would like to have N_∞ permutativity operads for every indexing system $\underline{\mathcal{F}}$. As shown in example 6.12, varying J alone will not suffice, and even though $\mathscr{P}_G$ is homotopically terminal, not every indexing system can be realized as one of its suboperads. The following example is due to Bonventre [7, appendix B].

Example 6.15. Let $G = C_2 \times C_2$, let $H = C_2 \times \{e\}$, and let $\underline{\mathcal{F}}$ be the indexing system that contains all finite H-sets, but only trivial actions otherwise. Write $\Gamma \subset G \times \Sigma_2$ for the subgroup corresponding to $H/\{(e,e)\}$. Then $\left|\mathbf{Set}(G, \Sigma_2)^\Gamma\right| = 4$, and the stabilizer of every Γ-fixed $f : C_2 \times C_2 \to \Sigma_2$ is strictly larger than Γ. Thus, if $\mathscr{O} \subset \mathscr{P}_G$ is a suboperad and $H/\{(e,e)\}$ is admissible for $\mathscr{O}$, then some 2-element G-set must also be admissible for $\mathscr{O}$. Therefore $\mathbb{A}(\mathscr{O})$ can never be precisely $\underline{\mathcal{F}}$.

That said, there are universal N_∞ permutativity operads for all indexing systems $\underline{\mathcal{F}}$.

Definition 6.16. Let $\mathcal{N}$ be a set of exponents. We define the operad $\mathscr{P}_\mathcal{N}$ to be the quotient of $\mathcal{SM}_\mathcal{N}$ by the relations below.

$$\begin{aligned} &\otimes(\otimes(x_1, x_2), x_3) \sim \otimes(x_1, \otimes(x_2, x_3)) \\ &\otimes(e(), x_1) \sim x_1 \quad \text{and} \quad \otimes(x_1, e()) \sim x_1 \\ &\bigotimes_T(e(), e(), \ldots, e()) \sim e() \end{aligned}$$

When $T = \varnothing$, the last relation reads $\bigotimes_T() \sim e()$.

The following is proven in section 6.6.

Lemma 6.17. *The operad* $\operatorname{Ob} \mathscr{P}_\mathcal{N}$ *can be identified with a sub-symmetric sequence of the operad* $\mathbb{F}(\Sigma_\bullet \sqcup \coprod_{T \in \mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T)$*, equipped with a modified composition operation.*

We deduce the following result. Recall that $\mathbb{A}(\mathscr{O})$ denotes the class of admissible sets of the operad $\mathscr{O}$, while $\mathbb{I}(\mathcal{N})$ denotes the indexing system generated by the set $\mathcal{N}$. Theorem 3.19 gives the equation $\mathbb{A}(\mathcal{SM}_\mathcal{N}) = \mathbb{I}(\mathcal{N})$.

Theorem 6.18. *The operad $\mathscr{P}_{\mathcal{N}}$ is an N_∞ permutativity operad whose class of admissible sets is the indexing system generated by $\mathcal{N}$. If $\mathscr{O}$ is any other N_∞ permutativity operad such that $\mathcal{N} \subset \mathbb{A}(\mathscr{O})$, then there is a (nonunique) map $\mathscr{P}_{\mathcal{N}} \to \mathscr{O}$ of N_∞ permutativity operads.*

Proof. Write $\mathscr{F} = \mathbb{F}(\Sigma_\bullet \sqcup \coprod_{T \in \mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T)$. The embedding $\mathrm{Ob}\mathscr{P}_{\mathcal{N}} \subset \mathscr{F}$ of symmetric sequences given by lemma 6.17 implies that $\mathscr{P}_{\mathcal{N}}$ is Σ-free and that $\mathbb{A}(\mathscr{P}_{\mathcal{N}}) \subset \mathbb{A}(\mathscr{F}) = \mathbb{I}(\mathcal{N})$. The quotient map $\pi : \mathcal{SM}_{\mathcal{N}} \to \mathscr{P}_{\mathcal{N}}$ gives $\mathbb{I}(\mathcal{N}) = \mathbb{A}(\mathcal{SM}_{\mathcal{N}}) \subset \mathbb{A}(\mathscr{P}_{\mathcal{N}})$, and hence $\mathscr{P}_{\mathcal{N}}$ is an N_∞ operad with admissible sets $\mathbb{I}(\mathcal{N})$. The relation $\bigotimes_T(e(), \dots, e()) \sim e()$ inductively implies that $\mathscr{P}_{\mathcal{N}}$ is reduced, and the canonical inclusion $\mathcal{SM} \hookrightarrow \mathcal{SM}_{\mathcal{N}}$ induces a canonical embedding $\varepsilon^*\mathscr{P} \hookrightarrow \mathscr{P}_{\mathcal{N}}$ on quotients.

If $\varepsilon^*\mathscr{P} \to \mathscr{O}$ is an N_∞ permutativity operad with $\mathcal{N} \subset \mathbb{A}(\mathscr{O})$, then pulling back along the quotient $\pi : \mathcal{SM} \to \varepsilon^*\mathscr{P}$ gives a morphism $\mathcal{SM} \to \mathscr{O}$. We can map the operad $\widetilde{\mathbb{F}}(\coprod_{T \in \mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T)$ into $\mathscr{O}$ freely, and taking coproducts gives a morphism $\mathcal{SM}_{\mathcal{N}} \to \mathscr{O}$ that descends to a map $\mathscr{P}_{\mathcal{N}} \to \mathscr{O}$. □

Note that the map $\mathscr{P}_{\mathcal{N}} \to \mathscr{O}$ is not unique on the point set level, but it is homotopically unique. For every set of exponents $\mathcal{N}$, we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{SM} \arrow[r, "\pi"] \arrow[d, "\mathrm{inc}"'] & \varepsilon^*\mathscr{P} \arrow[d, "\mathrm{inc}"'] \arrow[rdd, bend left, "\Delta"] & \\
\mathcal{SM}_{\mathcal{N}} \arrow[r, "\pi"'] & \mathscr{P}_{\mathcal{N}} \arrow[rd, dashed] & \\
& & \mathscr{P}_G
\end{tikzcd}$$

Both of the quotient maps labeled π are equivalences, and the dashed map is an equivalence if $\mathbb{I}(\mathcal{N}) = \underline{\mathbf{Set}}$. If we take $\mathcal{N}$ to be large enough, we can also arrange for the map $\mathscr{P}_{\mathcal{N}} \to \mathscr{P}_G$ to be a quotient map.

Direct inspection of the construction of $\mathscr{P}_{\mathcal{N}}$ given in section 6.6 also proves the following.

Proposition 6.19. *If $(\mathcal{N}(\underline{\mathcal{F}}))_{\underline{\mathcal{F}}\in\mathbf{Ind}}$ is convenient, then there is a functorial section*

$$\mathscr{P}_{\mathcal{N}(-)} : \mathbf{Ind} \to N_\infty\text{-}\mathbf{Op}_h$$

of the admissible sets functor that takes inclusions to inclusions.

We conclude with a brief account of the homotopical properties of the operads $\mathscr{P}_{\mathcal{N}}$. Consider the adjunction $r : \mathbf{Op} \rightleftarrows \mathbf{Op}_0 : i$ between operads in G-sets and reduced operads in G-sets. If $\mathscr{O} \in \mathbf{Op}$, then $r\mathscr{O}$ is constructed by forming the coproduct $\widetilde{\mathbb{F}}(G \times \Sigma_0/G) * \mathscr{O}$, and then identifying $\gamma(c; 0, 0, \ldots, 0) \sim 0$, where 0 denotes the new G-fixed constant in $\widetilde{\mathbb{F}}(G \times \Sigma_0/G) * \mathscr{O}$, and $c \in \mathscr{O}$. There is an induced adjunction $r : \mathbf{Op}_h \rightleftarrows \mathbf{Op}_{h,0} : i$ for homogeneous categorical operads.

Directed colimits in $\mathbf{Op}_{h,0}$ are computed in $\mathbf{Op}_h$. Therefore $\mathbf{Op}_{h,0}$ is locally finitely presentable, with finitely presented strong generators $r\widetilde{\mathbb{F}}(G \times \Sigma_n/\Xi)$. Equip $\mathbf{Op}_h$ with its $\underline{\mathbf{Set}}$-model structure (cf. section 3.4). By Kan transport, we obtain a cofibrantly generated model structure on $\mathbf{Op}_{h,0}$. The functor $i : \mathbf{Op}_{h,0} \to \mathbf{Op}_h$ creates fibrations and weak equivalences, and the generating (acyclic) cofibrations are obtained by applying r to the corresponding data for $\mathbf{Op}_h$. Local finite presentability ensures that the small object argument applies, and the acyclicity condition follows because every relative acyclic cell complex is a split monomorphism. Finally, pass to the slice category $\varepsilon^*\mathscr{P}/\mathbf{Op}_{h,0}$ of reduced homogeneous operads under $\varepsilon^*\mathscr{P}$.

Proposition 6.20. *There is a cofibrantly generated* $\underline{\mathbf{Set}}$*-model structure on* $\varepsilon^*\mathscr{P}/\mathbf{Op}_{h,0}$*, whose cell complexes are the operads* $\mathscr{P}_{\mathcal{N}}$*.*

Proof. The generating cofibrations are the maps $\varepsilon^*\mathscr{P} \to \varepsilon^*\mathscr{P} *_r r\widetilde{\mathbb{F}}(G \times \Sigma_{|T|}/\Gamma_T)$, where $*_r$ is the coproduct in $\mathbf{Op}_{h,0}$, the functor r preserves colimits, and the operad $\mathscr{P}_{\mathcal{N}}$ is isomorphic to the operad $r(\varepsilon^*\mathscr{P} * \widetilde{\mathbb{F}}(\coprod_{T\in\mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T))$. □

More generally, if $\underline{\mathcal{T}} \subset \underline{\mathbf{Set}}$ is any class of finite G-subgroup actions, then we can also transport the $\underline{\mathcal{T}}$-model structure on $\mathbf{Op}_h$ over to $\varepsilon^*\mathscr{P}/\mathbf{Op}_{h,0}$.

Proposition 6.21. *The operad $\mathscr{P}_{\mathcal{N}}$ is a cofibrant replacement for the commutativity operad* **Com** *in the $\mathbb{I}(\mathcal{N})$-model structure on $\varepsilon^* \mathscr{P}/\mathbf{Op}_{h,0}$.*

6.5 Coproducts and interchange

We now study the coproduct and Boardman-Vogt tensor product of homogeneous N_∞ operads $\mathscr{O}$. As with all colimits in $\mathbf{Op}_h$, the quotient $\mathscr{O}/\sim$ is obtained by forgetting down to G-sets, taking the quotient there, and then applying the functor $\widetilde{(-)}$.

Example 6.22. Suppose that $\mathscr{O}_1$ and $\mathscr{O}_2$ are operads in G-sets. The coproduct of $\mathscr{O}_1$ and $\mathscr{O}_2$ can be constructed by forming the disjoint union $\mathscr{O}_1 \sqcup \mathscr{O}_2$ in symmetric sequences, freely generating a new operad $\mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$, and then taking the following quotient. For $i = 1, 2$, let $\alpha_i : \mathscr{O}_i \to \mathscr{O}_1 \sqcup \mathscr{O}_2 \to \mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$ be the map of symmetric sequences obtained by composing the inclusion with the unit of the adjunction. Let $\sim$ be the congruence relation on $\mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$ generated by the relations

$$\alpha_i(\mathrm{id}) \sim \mathrm{id} \quad \text{and} \quad \gamma(\alpha_i(y); \alpha_i(x_1), \dots, \alpha_i(x_k)) \sim \alpha_i(\gamma(y; x_1, \dots, x_k)),$$

and let $\beta_i = \pi \circ \alpha_i : \mathscr{O}_i \to \mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2) \to \mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)/\sim$. Then β_1 and β_2 are operad maps, and the diagram $\beta_1 : \mathscr{O}_1 \to \mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)/\sim \leftarrow \mathscr{O}_2 : \beta_2$ is a coproduct of operads in G-sets.

Note the following standard observation. We prove it in section 6.6 to illustrate a more general method.

Lemma 6.23. *If $\mathscr{O}_1$ and $\mathscr{O}_2$ are operads in G-sets, then the coproduct $\mathscr{O}_1 * \mathscr{O}_2$ can be identified with a sub-symmetric sequence of $\mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$, equipped with a modified composition operation.*

Recall that the poset **Ind** of all indexing systems is a lattice. Given any indexing systems $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$, the meet $\underline{\mathcal{F}} \wedge \underline{\mathcal{G}}$ is the intersection $\underline{\mathcal{F}} \cap \underline{\mathcal{G}}$, and the join $\underline{\mathcal{F}} \vee \underline{\mathcal{G}}$ is the smallest indexing system that contains the union $\underline{\mathcal{F}} \cup \underline{\mathcal{G}}$.

Theorem 6.24. *Suppose that* $\mathscr{O}_1$ *and* $\mathscr{O}_2$ *are homogeneous* N_∞ *operads. Then*

1. *the operads* $\mathscr{O}_1 \times \mathscr{O}_2$ *and* $\mathscr{O}_1 * \mathscr{O}_2$ *are also homogeneous* N_∞ *operads, and*
2. *the equations* $\mathbb{A}(\mathscr{O}_1 \times \mathscr{O}_2) = \mathbb{A}(\mathscr{O}_1) \wedge \mathbb{A}(\mathscr{O}_2)$ *and* $\mathbb{A}(\mathscr{O}_1 * \mathscr{O}_2) = \mathbb{A}(\mathscr{O}_1) \vee \mathbb{A}(\mathscr{O}_2)$ *hold, i.e. taking admissible sets preserves products and coproducts.*

Proof. The statements concerning products are formal, because products are preserved when we pass to fixed points [5, proposition 5.1].

Now consider the coproduct $\mathscr{O}_1 * \mathscr{O}_2$. The symmetric sequence $\mathscr{O}_1 \sqcup \mathscr{O}_2$ is Σ-free, and therefore the free operad $\mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$ also is. By lemma 6.23, we deduce that $\mathscr{O}_1 * \mathscr{O}_2$ is Σ-free. The operad $\mathscr{O}_1 * \mathscr{O}_2$ is homogeneous by definition, and since we have maps $\mathscr{O}_i \to \mathscr{O}_1 * \mathscr{O}_2$, it follows that $(\mathscr{O}_1 * \mathscr{O}_2)(n)^G \simeq *$ for all $n \geq 0$. Therefore $\mathscr{O}_1 * \mathscr{O}_2 \in N_\infty\text{-}\mathbf{Op}_h$.

To determine $\mathbb{A}(\mathscr{O}_1 * \mathscr{O}_2)$, note that the maps $\mathscr{O}_1 \to \mathscr{O}_1 * \mathscr{O}_2 \leftarrow \mathscr{O}_2$ imply the inclusion $\mathbb{A}(\mathscr{O}_1) \vee \mathbb{A}(\mathscr{O}_2) \subset \mathbb{A}(\mathscr{O}_1 * \mathscr{O}_2)$. On the other hand, the indexing system $\mathbb{A}(\mathscr{O}_1 * \mathscr{O}_2)$ only depends on the underlying symmetric sequence of $\mathscr{O}_1 * \mathscr{O}_2$, and we have an embedding $\mathscr{O}_1 * \mathscr{O}_2 \hookrightarrow \mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$ by lemma 6.23. Therefore the inclusion

$$\mathbb{A}(\mathscr{O}_1 * \mathscr{O}_2) \subset \mathbb{A}(\mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)) = \mathbb{I}(\mathbb{A}(\mathscr{O}_1 \sqcup \mathscr{O}_2)) = \mathbb{I}(\mathbb{A}(\mathscr{O}_1) \cup \mathbb{A}(\mathscr{O}_2)) = \mathbb{A}(\mathscr{O}_1) \vee \mathbb{A}(\mathscr{O}_2)$$

holds as well. □

Example 6.25. The Boardman-Vogt tensor product $\mathscr{O}_1 \otimes \mathscr{O}_2$ is obtained as follows. First, form the coproduct $\iota_1 : \mathscr{O}_1 \to \mathscr{O}_1 * \mathscr{O}_2 \leftarrow \mathscr{O}_2 : \iota_2$, and then introduce relations corresponding to the horizontal-vertical interchange formula (cf. [17])

$$\begin{aligned} &g(f(x_{11}, \dots, x_{1k}), f(x_{21}, \dots, x_{2k}), \dots, f(x_{j1}, \dots, x_{jk})) = \\ &f(g(x_{11}, \dots, x_{j1}), g(x_{12}, \dots, x_{j2}), \dots, g(x_{1k}, \dots, x_{jk})). \end{aligned}$$

Formally, given $x \in \mathscr{O}_1(j)$ and $y \in \mathscr{O}_2(k)$, we identify

$$\gamma(\iota_1(x); \iota_2(y), \ldots, \iota_2(y)) \sim \gamma(\iota_2(y); \iota_1(x), \ldots, \iota_1(x))\sigma,$$

where σ is the permutation that reorders the jk elements

$$(1,1) < (1,2) < \cdots < (1,k) < (2,1) < \cdots < (2,k) < \cdots < (j,1) < \cdots < (j,k)$$

in reverse lexicographic order

$$(1,1) < (2,1) < \cdots < (j,1) < (1,2) < \cdots < (j,2) < \cdots < (1,k) < \cdots < (j,k).$$

Note that there are nullary interchange relations. If $u \in \mathscr{O}_1(0)$, then we identify $\iota_1(u) \sim \iota_2(v)$ for $v \in \mathscr{O}_2(0)$, and we identify $\iota_1(u) \sim \gamma(\iota_2(v); \iota_1(u), \ldots, \iota_1(u))$ for $v \in \mathscr{O}_2(j)$ and $j > 0$. It follows that if both $\mathscr{O}_1(0)$ and $\mathscr{O}_2(0)$ are nonempty, then $\mathscr{O}_1 \otimes \mathscr{O}_2$ is reduced.

The following is proven in section 6.6.

Lemma 6.26. *If S_1 and S_2 are Σ-free symmetric sequences of G-sets such that $S_1(0)^G$ and $S_2(0)^G$ are nonempty, then $\mathbb{F}(S_1) \otimes \mathbb{F}(S_2)$ can be identified with a sub-symmetric sequence of $\mathbb{F}(S_1 \sqcup S_2)$, equipped with a modified composition operation.*

By theorem 3.40, we know that the cofibrant objects in $\mathbf{Op}_{h,m}$ are retracts of the operads $\mathcal{SM}_{\mathcal{N}}$. We have the following result.

Theorem 6.27. *Suppose that $\mathscr{O}_1$ and $\mathscr{O}_2$ are homogeneous categorical N_∞ operads. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are retracts of $\mathcal{SM}_{\mathcal{N}_1}$ and $\mathcal{SM}_{\mathcal{N}_2}$ respectively, then $\mathscr{O}_1 \otimes \mathscr{O}_2$ is a homogeneous N_∞ operad, and $\mathbb{A}(\mathscr{O}_1 \otimes \mathscr{O}_2) = \mathbb{A}(\mathscr{O}_1) \vee \mathbb{A}(\mathscr{O}_2)$.*

Proof. If $\mathscr{O}_i$ is equal to $\mathcal{SM}_{\mathcal{N}_i}$ for $i = 1$ and 2, then this can be proven using the same argument that established $\mathbb{A}(\mathscr{O}_1 * \mathscr{O}_2) = \mathbb{A}(\mathscr{O}_1) \vee \mathbb{A}(\mathscr{O}_2)$ in theorem 6.24. The full statement now follows from the functoriality of the Boardman-Vogt tensor product, and the fact that admissible sets are preserved under retracts of operads. ☐

This establishes a combinatorial analogue to [5, conjecture 6.27]. We deduce the following interchange results on the level of G-categories and G-spaces.

Corollary 6.28. *For any indexing system $\underline{\mathcal{F}}$, there are N_∞ operads $\mathscr{O}_1$ and $\mathscr{O}_2$ in G-categories or G-spaces, and a pair of operad maps $\varphi_1, \varphi_2 : \mathscr{O}_1 \rightrightarrows \mathscr{O}_2$ such that*

1. *the classes of admissible sets of $\mathscr{O}_1$ and $\mathscr{O}_2$ are both $\underline{\mathcal{F}}$, and*
2. *the operad maps φ_1 and φ_2 interchange.*

Proof. Let $\mathcal{N}$ be any set of exponents that generates the indexing system $\underline{\mathcal{F}}$. The desired interchange condition holds for the two natural maps $\mathbb{F}(S_{\mathcal{N}}) \rightrightarrows \mathbb{F}(S_{\mathcal{N}}) \otimes \mathbb{F}(S_{\mathcal{N}})$, and it is encoded by diagrams that only involve operad morphisms, operad structure maps, and the cartesian monoidal structure on G-sets. The functors $\widetilde{(-)}$ and B preserve such structure, and therefore the maps for $\mathcal{SM}_{\mathcal{N}} \otimes \mathcal{SM}_{\mathcal{N}}$ and $B(\mathcal{SM}_{\mathcal{N}} \otimes \mathcal{SM}_{\mathcal{N}})$ also interchange. ☐

Using this corollary, we can arrange for N_∞ actions to interchange with themselves, which is a useful technical condition (cf. [5, §7]).

Blumberg and Hill's conjecture for the space-level tensor product is considerably more delicate, and we do not know how to prove it as originally stated. Work in progress of Bonventre and Pereira [9] promises to resolve the space-level problem.

6.6 Appendix: identifying quotients of free operads

We use the same strategy to prove lemmas 6.17, 6.23, and 6.26, so we begin with some general observations.

Suppose that R is a binary relation on the set X, and let E be the equivalence relation generated by R. Suppose further that there is a complexity function $c : X \to \mathbb{N}$, and that R is complexity-reducing in the sense that $c(x) > c(y)$ whenever xRy. In this case, we can try to identify E as follows. Say that $x \in X$ is *reduced* if there is no $y \in X$ such that xRy. Say that $\overline{x}$ is a *reduced form* of x if $\overline{x}$ is reduced, and there is an integer $n \geq 0$ and a sequence of elements $x_0, \ldots, x_n \in X$ such that $x = x_0 R x_1 R x_2 R \ldots R x_n = \overline{x}$.

Lemma 6.29. *Retain the setup above. Then:*

(1) Every $x \in X$ has at least one reduced form.

(2) Suppose that whenever xRy and xRy', there is some $z \in X$ and integers $m, n \geq 0$ such that $y = y_0 R y_1 R \cdots R y_m = z$ and $y' = y'_0 R y'_1 R \cdots R y'_n = z$. Then every $x \in X$ has a unique reduced form.

(3) Suppose that every $x \in X$ has a unique reduced form $\overline{x}$, and write $x \sim y$ if $\overline{x} = \overline{y}$. Then $\sim$ is the equivalence relation generated by R, and the reduced elements in X are a set of representatives for $\sim$.

Proof. For the first part, note that complexity is measured by a nonnegative integer. The second part follows by induction. For the third part, note that if $x \sim y$, then there is a zig-zag of R-related elements connecting x and y, and that the uniqueness assumption ensures that $\overline{(-)} : X \to X$ is an idempotent function. □

Now suppose that we have a G-operad $\mathscr{O}$, a graded complexity-reducing relation R on $\mathscr{O}$, and that every element of $\mathscr{O}$ has a unique reduced form. Then $\sim$ from (3) above is the graded equivalence relation generated by R, and if $\sim$ happens to be a congruence relation, then it is actually the congruence relation generated by R. We have criteria for this, too.

Lemma 6.30. *Assume that $\mathscr{O}$ is a G-operad, that R is a graded complexity-reducing relation on $\mathscr{O}$, and that every element of $\mathscr{O}$ has a unique reduced form. Define the relation $\sim$ as in (3) above. Then:*

(4) Suppose that xRy implies $gx\sigma R gy\sigma$. Then $x \sim y$ implies $gx\sigma \sim gy\sigma$.

(5) Suppose that yRy' *implies* $\gamma(y;x_1,\ldots,x_k)R\gamma(y';x_1,\ldots,x_k)$ *and that* x_iRx_i' *implies* $\gamma(y;x_1,\ldots,x_i,\ldots,x_k)R\gamma(y;x_1,\ldots,x_i',\ldots,x_k)$. *Then* $y\sim y'$ *and* $x_i\sim x_i'$ *for* $i=1,\ldots,k$ *implies* $\gamma(y;x_1,\ldots,x_k)\sim\gamma(y';x_1',\ldots,x_k')$.

If the hypotheses of (4) and (5) both hold, then $\sim$ *is the congruence relation generated by* R*, and the inclusion* $\mathscr{O}/{\sim}\cong\{\text{reduced elements of }\mathscr{O}\}\hookrightarrow\mathscr{O}$ *is a map of symmetric sequences. The identity of* $\mathscr{O}/{\sim}$ *is the reduced form of* $\mathrm{id}\in\mathscr{O}$*, and composites in* $\mathscr{O}/{\sim}$ *are calculated by composing in* $\mathscr{O}$ *and then reducing.*

Proof. If R respects the $G\times\Sigma_n$ action, then the action must preserve reduced elements, and multiplication by (g,σ) must send any chain from x to $\overline{x}$ to a chain from $gx\sigma$ to $g\overline{x}\sigma$. It follows that $\overline{gx\sigma}=g\overline{x}\sigma$, and (4) follows.

If R respects composition in each factor, then there is a chain of R-relations from $\gamma(y;x_1,\ldots,x_k)$ to $\gamma(\overline{y};\overline{x_1},\ldots,\overline{x_k})$. It follows that we may reduce $\gamma(y;x_1,\ldots,x_k)$ by first reducing componentwise to $\gamma(\overline{y};\overline{x_1},\ldots,\overline{x_k})$, and then reducing further to $\overline{\gamma(\overline{y};\overline{x_1},\ldots,\overline{x_k})}$. Therefore $\overline{\gamma(y;x_1,\ldots,x_k)}=\overline{\gamma(\overline{y};\overline{x_1},\ldots,\overline{x_k})}$, and (5) follows.

Suppose that the hypotheses of (4) and (5) hold, and identify the congruence class $[x]$ with $\overline{x}$. Then the G-operad structure on $\mathscr{O}/{\sim}$ is obtained by reducing the structure on $\mathscr{O}$. Since the $G\times\Sigma_n$-action on $\mathscr{O}$ preserves reduced elements, it follows that $\mathscr{O}/{\sim}\hookrightarrow\mathscr{O}$ is a map of symmetric sequences. □

We return to the problem of identifying a quotient of a free operad. The basic idea is to define a family of "reduction operations" on the terms in $\mathbb{F}(S)$, and then to declare sRt if t can be obtained from s via a single reduction. If we can verify the hypotheses in parts (2), (4), and (5), then we can identify the quotient $\mathscr{O}/\langle R\rangle$ with a sub-symmetric sequence of $\mathscr{O}$, equipped with a modified operad structure. In what follows, we shall work with the model for the free operad $\mathbb{F}(S)$ described in construction 2.15.

Proof of lemma 6.23. Suppose that $\mathscr{O}_1$ and $\mathscr{O}_2$ are operads, and form $\mathbb{F}(\mathscr{O}_1\sqcup\mathscr{O}_2)$. Given an equivalence class $[t]\in\mathbb{F}(\mathscr{O}_1\sqcup\mathscr{O}_2)$, define $c([t])$ to be the number of operation symbols

$a \in \mathscr{O}_1 \sqcup \mathscr{O}_2$ in t. Next, write $[s]R[t]$ if, for some representing terms s and t, the term t is obtained by modifying a subterm of s in one of the following ways:

$$\mathrm{id}(s_1) \rightsquigarrow s_1$$
$$a(s_1, \ldots, a'(s_{i1}, \ldots, s_{ik}), \ldots, s_j) \rightsquigarrow a \circ_i a'(s_1, \ldots, s_{i1}, \ldots, s_{ik}, \ldots, s_j)$$

where id is the identity element for $\mathscr{O}_1$ or $\mathscr{O}_2$, and the symbols a and a' must come from the same operad. Then $\mathscr{O}_1 * \mathscr{O}_2$ is the quotient of $\mathbb{F}(\mathscr{O}_1 \sqcup \mathscr{O}_2)$ by the congruence relation generated by R. Note that the relation R is complexity-reducing.

Condition (2) follows from the associativity and unitality of composition. For condition (4), observe that R commutes with the Σ-action, because reduction operations do not affect variables, and R commutes with the G action because id is G-fixed and $\circ_i$ is G-equivariant. Condition (5) is clear, because the relation R is defined only with reference to subterms. □

Proof of lemma 6.17. Suppose that $\mathcal{N}$ is a set of exponents. We shall present $\mathrm{Ob}\mathscr{P}_{\mathcal{N}}$ as a quotient of $\mathbb{F}(\Sigma_\bullet \sqcup \coprod_{T \in \mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T)$, where the symmetric sequence $\Sigma_\bullet$ is equipped with a trivial G-action. For each integer $n \geq 0$, write Π_n for the identity permutation in Σ_n. Then the generic element of Σ_n is $\Pi_n \sigma$. Extend $\{\Pi_n \,|\, n \geq 0\}$ to a choice of Σ-orbit representatives for $\Sigma_\bullet \sqcup \coprod G \times \Sigma_{|T|}/\Gamma_T$, and identify each congruence class $[t]$ with the unique term t written in terms of those representatives. We declare sRt if the term t is obtained by modifying a subterm of s in one of the following ways:

$$\Pi_n(s_1, \ldots, \Pi_m(s_{i1}, \ldots, s_{im}), \ldots, s_n) \rightsquigarrow \Pi_{n+m-1}(s_1, \ldots, s_{i1}, \ldots, s_{im}, \ldots, s_n)$$
$$\Pi_1(s_1) \rightsquigarrow s_1$$
$$a(\Pi_0(), \ldots, \Pi_0()) \rightsquigarrow \Pi_0()$$

where $a \in \coprod_{T \in \mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T$ in the last case. When a is a nullary operation, this reads

$a() \rightsquigarrow \Pi_0()$. For each t, we define

$$c(t) = (\text{number of operation symbols in } t) + (\text{number of nullary } a \neq \Pi_0 \text{ in } t)$$

Then $\mathrm{Ob}\mathscr{P}_{\mathcal{N}} \cong \mathbb{F}(\Sigma_\bullet \sqcup \coprod_{T \in \mathcal{N}} G \times \Sigma_{|T|}/\Gamma_T)/\langle R\rangle$, and the relation R is complexity-reducing. Conditions (2), (4), and (5) are verified as before. □

Proof of lemma 6.26. Suppose that S_1 and S_2 are Σ-free symmetric sequences such that $S_1(0)^G, S_2(0)^G \neq \varnothing$, and choose $z \in S_2(0)^G$. The coproduct $\mathbb{F}(S_1) * \mathbb{F}(S_2)$ is just $\mathbb{F}(S_1 \sqcup S_2)$. Extend $\{z\}$ to a choice of Σ-orbit representatives for $S_1 \sqcup S_2$, and identify each congruence class $[t]$ with the unique term t written in terms of those representatives. For any $s, t \in \mathbb{F}(S_1 \sqcup S_2)$ we write sRt if the term t is obtained by modifying a subterm of s in one of the following ways:

$$a(b(s_{11}, \ldots, s_{1k}), \ldots, b(s_{j1}, \ldots, s_{jk})) \rightsquigarrow b(a(s_{11}, \ldots, s_{j1}), \ldots, a(s_{1k}, \ldots, s_{jk}))$$

$$d(z(), z(), \ldots, z()) \rightsquigarrow z()$$

where $a \in S_1(j)$ and $b \in S_2(k)$ for $j, k > 0$, while $d \in (S_1 \sqcup S_2)(j)$ for some $j \geq 0$ and $d \neq z$. If d is nullary, then the second line is $d() \rightsquigarrow z()$. Then the quotient $\mathbb{F}(S_1 \sqcup S_2)/\langle R\rangle$ is isomorphic to $\mathbb{F}(S_1) \otimes \mathbb{F}(S_2)$, because it is enough to make the generators of $\mathbb{F}(S_1)$ and $\mathbb{F}(S_2)$ interchange. The second line implies all nullary interchanges because it reduces the operad, and the first line handles the rest. Note that $\mathbb{F}(S_1) \otimes \mathbb{F}(S_2)$ is reduced.

Let $t \in \mathbb{F}(S_1 \sqcup S_2)$, and define the *height* $h(a)$ of an operation symbol a in t to be the number of nested pairs of left and right parentheses that contain a. Define the complexity of the term t to be

$$c(t) = \sum_{\substack{S_2\text{-operation} \\ \text{symbols } a \text{ in } t}} h(a) + \sum_{\substack{\text{nullary } S_1 \\ \text{symbols } d \text{ in } t}} \Big(h(d) + 1\Big) + \left(\begin{array}{c} \text{number of nullary } S_2 \\ \text{symbols } d \neq z \text{ in } t \end{array}\right)$$

Then R is complexity-reducing, and it satisfies conditions (2), (4), and (5). □

REFERENCES

[1] J. Adámek and J. Rosický. Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, 189. Cambridge University Press, Cambridge, 1994. xiv+316 pp.

[2] C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. Comment. Math. Helv. 78 (2003), no. 4, 805–831.

[3] J. M. Boardman. Homotopy structures and the language of trees. Proceedings of the Summer Institute on Algebraic Topology, University of Wisconsin (1970), A.M.S., 37–58.

[4] J. M. Boardman and R. M. Vogt. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973. x+257 pp.

[5] A. J. Blumberg and M. A. Hill. Operadic multiplications in equivariant spectra, norms, and transfers. Adv. Math. 285 (2015), 658–708.

[6] A. M. Bohmann. A comparison of norm maps. With an appendix by Bohmann and Emily Riehl. Proc. Amer. Math. Soc. 142 (2014), no. 4, 1413–1423.

[7] P. Bonventre. Comparison of models for equivariant operads. Ph.D. thesis. https://doi.org/10.18130/V39M09.

[8] P. Bonventre and L. A. Pereira. Genuine equivariant operads. Preprint. arXiv:1707.02226.

[9] P. Bonventre and L. A. Pereira. An additivity theorem for N_∞ operads. In preparation.

[10] A. S. Corner and N. Gurski. Operads with general groups of equivariance, and some 2-categorical aspects of operads in Cat. Preprint. arXiv:1312.5910.

[11] S. R. Costenoble, J. P. May, and S. Waner. Equivariant orientation theory. Homology Homotopy Appl. 3 (2001), no. 2, 265–339.

[12] S. R. Costenoble and S. Waner. Fixed set systems of equivariant infinite loop spaces. Trans. Amer. Math. Soc. 326 (1991), no. 2, 485–505.

[13] T. tom Dieck. Transformation groups. De Gruyter Studies in Mathematics, 8. Walter de Gruyter & Co., Berlin, 1987. x+312 pp.

[14] G. Dunn. Uniqueness of n-fold delooping machines. J. Pure Appl. Algebra 113 (1996), 159–193.

[15] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. J. Pure Appl. Algebra 18 (1980), no. 1, 17–35.

[16] B. Fresse. Homotopy of operads and Grothendieck-Teichmüller groups. Part 1. The algebraic theory and its topological background. Mathematical Surveys and Monographs, 217. American Mathematical Society, Providence, RI, 2017. xlvi+532 pp.

[17] Z. Fiedorowicz and R. M. Vogt. An additivity theorem for the interchange of E_n structures. Adv. Math. 273 (2015), 421–484.

[18] B. Guillou and J. P. May. Equivariant iterated loop space theory and permutative G-categories. Algebr. Geom. Topol. 17 (2017), no. 6, 3259–3339.

[19] B. Guillou, J. P. May, and M. Merling. Categorical models for equivariant classifying spaces. Algebr. Geom. Topol. 17 (2017), no. 5, 2565–2602.

[20] B. Guillou, J. P. May, M. Merling, and A. M. Osorno. Operadic and monadic coherence theory. In preparation.

[21] J. J. Gutiérrez and D. White. Encoding equivariant commutativity via operads. Preprint. arXiv:1707.02130.

[22] M. A. Hill and M. J. Hopkins. Equivariant multiplicative closure. Algebraic topology: applications and new directions, 183–199, Contemp. Math., 620, Amer. Math. Soc., Providence, RI, 2014.

[23] M. A. Hill and M. J. Hopkins. Equivariant symmetric monoidal structures. Preprint. arXiv:1610.03114.

[24] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. Ann. of Math. (2) 184 (2016), no. 1, 1–262.

[25] J. R. Isbell. On coherent algebras and strict algebras. J. Algebra 13 1969 299–307.

[26] G. M. Kelly. On MacLane's conditions for coherence of natural associativities, commutativities, etc. J. Algebra 1 1964 397–402.

[27] G. M. Kelly. Basic concepts of enriched category theory. London Mathematical Society Lecture Note Series, 64. Cambridge University Press, Cambridge-New York, 1982. 245 pp.

[28] S. Lack. Codescent objects and coherence. J. Pure and Applied Alg 175(2002), 223–241.

[29] J. Lurie. Higher Algebra. Preprint from the web page of the author, May 2018.

[30] S. Mac Lane. Natural associativity and commutativity. Rice Univ. Studies 49 1963 no. 4, 28–46.

[31] S. Mac Lane. Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

[32] J. P. May. The geometry of iterated loop spaces. Lecture Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[33] J. P. May, M. Merling, and A. M. Osorno. Equivariant infinite loop space theory, I. The space level story. Preprint. arXiv:1704.03413.

[34] J. P. May and K. Ponto. More concise algebraic topology. Localization, completion, and model categories. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. xxviii+514 pp.

[35] M. Murayama and K. Shimakawa. Universal equivariant bundles. Proc. Amer. Math. Soc. 123 (1995), 1289–1295.

[36] L. A. Pereira. Equivariant dendroidal sets. Algebr. Geom. Topol. 18 (2018), no. 4, 2179–2244.

[37] A. J. Power. A general coherence result. J. Pure and Applied Alg. 57(1989), 165–173.

[38] E. Riehl. Categorical homotopy theory. New Mathematical Monographs, 24. Cambridge University Press, Cambridge, 2014. xviii+352 pp.

[39] K. Shimakawa. Infinite loop G-spaces associated to monoidal G-graded categories. Publ. Res. Inst. Math. Sci. 25(1989), 239–262.

[40] R. W. Thomason. The homotopy limit problem. Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982), 407–419, Contemp. Math., 19, Amer. Math. Soc., Providence, R.I., 1983.