

THE UNIVERSITY OF CHICAGO

POINTS AND LINES ON CUBIC SURFACES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS

JUNE 2021

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Acknowledgements

First and foremost, I would like to express my sincerest gratitude towards my advisor Benson Farb for his absolute and incessant support. His infinite enthusiasm for mathematics has been both a driving force and a constant source of inspiration. I also thank Benson for suggesting the problems that led to contents of this thesis.

I would like to thank Maxime Bergeron, Weiyan Chen, Nir Gadish, Nate Harman, Sean Howe, Eduard Looijenga, Nathaniel Mayer, Ben O'Connor, Jesse Wolfson and others in and around the UChicago mathematics department for helpful conversations and comments. I am grateful towards Priyavrat Deshpande, without whose mathematical and moral support I would probably not have been in a PhD program at all.

I am thankful to my friends in Chicago, including Claudio Gomez-Gonzales, Dylan Quintana, Karl Schaefer, Noah Taylor and the rest of the “crossword crew”, for making my time here much more enjoyable. Thanks to Isabella Scott for their love and support over the last four years. Finally, I want to thank my mother Swati Ghosh and the rest of my family for believing in me and helping me believe in myself.

Abstract

This thesis is a compilation of three papers.

The Cayley–Salmon theorem implies the existence of a 27-sheeted covering space of the parameter space of smooth cubic surfaces, marking each of the 27 lines on each surface. In Chapter 2 we compute the rational cohomology of the total space of this cover, using the spectral sequence in the method of simplicial resolution developed by Vassiliev. The covering map is an isomorphism in cohomology (in fact of mixed Hodge structures) and the cohomology ring is isomorphic to that of $\mathrm{PGL}(4, \mathbb{C})$. We derive as a consequence that over the finite field \mathbb{F}_q the average number of lines on a smooth cubic surface equals 1 (away from finitely many characteristics); this average is $1 + O(q^{-1/2})$ by a standard application of the Weil conjectures.

In Chapter 3 we compute the rational cohomology of the universal family of smooth cubic surfaces using the same method of simplicial resolution. Modulo embedding, the universal family has cohomology isomorphic to that of \mathbb{P}^2 . It again follows that over the finite field \mathbb{F}_q , away from finitely many characteristics, the average number of points on a smooth cubic surface is $q^2 + q + 1$.

In Chapter 4 we compute the distributions of various other markings on smooth cubic surfaces defined over the finite field \mathbb{F}_q , for example the distribution of pairs of points, ‘tritangents’ or ‘double sixes’. We also compute the (rational) cohomology of some of the associated bundles and covers over complex numbers.

Chapter 1

Introduction

The three following chapters are essentially independent; there is considerable overlap in the background sections, but we consider this to be helpful in reading them in isolation, rather than wasted space. They originally appeared as individual papers: [Das20c], [Das20b] and [Das20a] respectively. Each chapter is about spaces of smooth cubic surfaces with certain markings, with results on their cohomology over \mathbb{C} and point counts over the finite field \mathbb{F}_q . A brief summary follows.

Let $X_{3,3}$ be the parameter space of smooth degree 3 hypersurfaces in $\mathbb{C}\mathbb{P}^3$. Vassiliev, in [Vas99], showed that $H^*(X_{3,3}; \mathbb{Q}) \cong H^*(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q})$ (see Theorem 2.4 or Theorem 3.10 below). The proof works by first reducing the problem to studying the cohomology of the complement Σ , that consists of *singular* cubic surfaces. A classification of the possible singular loci of each cubic surface provides a stratification of this *discriminant locus* Σ . Vassiliev then replaces Σ by its simplicial resolution, a simplicial space whose spaces of k -simplices for various k are simpler moduli spaces of each class of singular loci. For instance, since a cubic surface can be singular at a pair of points in \mathbb{P}^3 , there is a contribution from the space $\mathrm{UConf}^2 \mathbb{P}^3$ parametrizing such pairs.

Extending these methods, we look at spaces over $X_{3,3}$ that correspond to additional geometric markings on each cubic surface. In Chapter 2, we look at the 27-sheeted cover $X_{3,3}(1)$ of $X_{3,3}$ corresponding to marking one of the 27 lines on each cubic surface, as given by the Cayley–Salmon theorem. We prove (see Theorem 2.1, though the notation is different) that this cover

has isomorphic rational cohomology to $X_{3,3}$. The isomorphism in Theorem 2.1 preserves enough structure that we can apply comparison theorems between $H_{\text{sing}}^*(X(\mathbb{C}))$ and $H_{\text{ét}}^*(X/\mathbb{F}_q)$ for a finite field \mathbb{F}_q and then the Grothendieck–Lefschetz trace formula to either side and obtain equivalent answers. Thus we prove Corollary 2.7 establishing that the average number of lines over a smooth cubic surface over \mathbb{F}_q is 1 (possibly away from finitely many characteristics, though this exception is eliminated in Theorem 4.1).

Similarly, let $U_{3,3}$ denote the incidence variety of points and smooth cubic surfaces; this is a fiber bundle over $X_{3,3}$ whose fiber over the cubic surface $S \in X_{3,3}$ is the set of points $S \subset \mathbb{P}^3$. In Chapter 3, specifically Theorem 3.3, we compute the cohomology of this universal family.

A more straightforward application of the trace formula to individual cubic surfaces S shows that the number of its \mathbb{F}_q points $\#S(\mathbb{F}_q) = q^2 + tq + 1$ for some $t \in \{-2, -1, 0, 1, 2, 3, 4, 5, 7\}$. As a corollary of Theorem 3.3, we obtain in Theorem 3.2 that the average of $\#S(\mathbb{F}_q)$ over $S \in X_{3,3}(\mathbb{F}_q)$ is $q^2 + q + 1$, i.e. the average of t is 1. However, not every t appears for every q ; see Theorem 3.1. Theorem 4.9 below establishes a much more refined version of Theorem 3.1 by determining the exact distribution of t for each q . The same Theorem 4.9 can be used to also recover results of Loughran–Trepalin [LT18] that explore when some of these distributions are 0 (i.e. for which q do there not exist smooth cubic surfaces with certain properties).

It is worth noting that analogues to Corollary 2.7, Theorem 3.2 or Theorem 4.9 that are only asymptotic in q are often much easier to obtain, using standard arithmetic tools like the Chebotarev density theorem or a more naïve application of the trace formula.

In Chapter 4 we look at more general markings. The monodromy group of the cover $X_{3,3}(1) \rightarrow X_{3,3}$ is isomorphic to $W(E_6)$, the Weyl group of type E_6 . In other words, the associated Galois cover, which we denote by $X_{3,3}(27)$, has deck group $W(E_6)$, which acts on the 27 lines on $S \in X_{3,3}$ (i.e. the fiber over S of the map $X_{3,3}(1) \rightarrow X_{3,3}$). The intermediate cover $X_{3,3}(1)$ corresponds to the subgroup stabilizing a given line. By the Galois correspondence, other subgroups of $W(E_6)$ correspond to other intermediate covers, many of which are the parameter spaces of cubic surfaces with markings of classically studied patterns of lines (e.g. “tritangents”

or “double sixes”). For each of these there is an enumerative question over \mathbb{F}_q , analogous to Corollary 2.7. Refining this question, we can also ask for the exact distribution instead of just the average. Using the computation of $H^*(X_{3,3}(27); \mathbb{Q})$ by Bergvall–Gounelas [BG19], we determine in Theorem 4.9 all of these distributions and in Theorem 4.6 the Betti numbers of these covers.

Further, we can choose to mark a pattern of lines as above in addition to some number of points on each cubic surface to produce more bundles over $X_{3,3}$ by combining the constructions of $U_{3,3}$ (and its fiberwise products) along with $X_{3,3}(27)$ (see Section 4.2 for the details). Theorem 4.9 computes the associated distribution for each of these bundles and Theorem 4.6 computes the rational cohomology in many of the cases.

Chapter 2

The space of cubic surfaces equipped with a line

2.1 Introduction

One of the first theorems of modern algebraic geometry and specifically enumerative geometry is the Cayley–Salmon theorem [Cay49]. This classical theorem states that every smooth cubic surface (over an algebraically closed field, in particular \mathbb{C}) contains exactly 27 lines. A cubic (hyper)surface in $\mathbb{P}^3 = \mathbb{C}P^3$ is the zero set $S = \mathcal{V}(F)$ of a homogeneous polynomial F of degree 3 in 4 variables. The surface S is singular (i.e. not smooth) if and only if the 20 coefficients of F are a zero of a discriminant polynomial $\Delta : \mathbb{C}^{20} \rightarrow \mathbb{C}$. Thus the space of smooth cubic surfaces is an open locus $M = M_{3,3} := \mathbb{P}^{19} \setminus \mathcal{V}(\Delta)$. The Cayley–Salmon theorem can be reinterpreted as a covering map $\pi : \tilde{M} \rightarrow M$, where \tilde{M} is the incidence variety of lines and smooth cubic surfaces (see (2.1) and the preceding discussion for precise definitions). The fiber $\pi^{-1}(S)$ over $S \in M$ is the set of 27 lines on S .

The automorphism group of \mathbb{P}^3 is $\mathrm{PGL}(4, \mathbb{C})$ and this group acts on lines and cubic surfaces, preserving smoothness. In particular the covering map $\pi : \tilde{M} \rightarrow M$ is $\mathrm{PGL}(4, \mathbb{C})$ -equivariant. It was shown by Vassiliev (in [Vas99]) that the space M has the same rational cohomology as $\mathrm{PGL}(4, \mathbb{C})$, and it follows from the results of Peters–Steenbrink ([PS03]) that the orbit map given by $g \mapsto g(S_0)$ induces an isomorphism for any choice of $S_0 \in M$ (see Theorem 2.4). See

also [Tom14].

The main result of this chapter is that the covering space \tilde{M} also has the same rational cohomology.

Theorem 2.1. *For a choice $(S_0, L_0) \in \tilde{M}$, the orbit map $\mathrm{PGL}(4, \mathbb{C}) \rightarrow \tilde{M}$ given by $g \mapsto g(S_0, L_0)$ induces an isomorphism*

$$H^*(\tilde{M}; \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}[a_3, a_5, a_7]/(a_3^2, a_5^2, a_7^2),$$

where $a_i \in H^i(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q})$. Since the composition $\mathrm{PGL}(4, \mathbb{C}) \rightarrow \tilde{M} \xrightarrow{\pi} M$ also induces an isomorphism on $H^*(_; \mathbb{Q})$, the map

$$\pi^* : H^*(M; \mathbb{Q}) \rightarrow H^*(\tilde{M}; \mathbb{Q})$$

is an isomorphism. Since the orbit map and π are algebraic, the isomorphisms respect mixed Hodge structures.

Remark 2.2. In particular, $H^k(\tilde{M}; \mathbb{Q})$ is pure of Tate type; the generator a_{2k-1} is of Hodge bidegree (k, k) .

The main tool in our proof of Theorem 2.1 is simplicial resolution à la Vassiliev. However the introduction of a line significantly increases the combinatorics of the casework. We devote all of Section 2.3 to this computation, while Section 2.2.2 contains the rest of the proof. The approach is similar to the one developed in [BT07].

2.1.1 Applications: moduli space, representations of $W(E_6)$ and point counts

Before presenting a proof of Theorem 2.1, which we postpone to Section 2.2.2 and the particularly tedious details further to Section 2.3, we describe a few applications. All of the corollaries in this section are corollaries to Theorem 2.1.

Cohomology of moduli spaces

The map $\pi : \tilde{M} \rightarrow M$ is $\mathrm{PGL}(4, \mathbb{C})$ equivariant and each orbit (in either M or \tilde{M}) is closed (see e.g. [ACT02]). Thus passing to the geometric quotient we get a covering map

$$\mathcal{H}_{3,3}(1) \rightarrow \mathcal{H}_{3,3},$$

where

$$\mathcal{H}_{3,3} = M / \mathrm{PGL}(4, \mathbb{C})$$

is the moduli space of smooth cubic surfaces and

$$\mathcal{H}_{3,3}(1) = \tilde{M} / \mathrm{PGL}(4, \mathbb{C})$$

is the moduli space of cubic surfaces equipped with a line. Note that both $\mathcal{H}_{3,3}$ and $\mathcal{H}_{3,3}(1)$ are coarse moduli spaces. For example the Fermat cubic defined by $x^3 + y^3 + z^3 + w^3$ equipped with the line $\{x = y, z = w\}$ has non-trivial (but finite) stabilizer in $\mathrm{PGL}(4, \mathbb{C})$.

Peters and Steenbrink show that a generalization of the Leray–Hirsch theorem ([PS03, Theorem 2]) applies to the action of $\mathrm{PGL}(4)$ on M . It automatically follows (see [BT07, Theorem 10] for the argument applied to a similar case) that it also applies to the action on \tilde{M} and hence

$$H^*(\tilde{M}; \mathbb{Q}) \cong H^*(\mathrm{PGL}(4); \mathbb{Q}) \otimes H^*(\mathcal{H}_{3,3}(1); \mathbb{Q}).$$

Thus we have the following corollary of Theorem 2.1.

Corollary 2.3. *The space $\mathcal{H}_{3,3}(1)$ is \mathbb{Q} -acyclic: $H^i(\mathcal{H}_{3,3}(1); \mathbb{Q}) = 0$ for $i > 1$.*

For comparison, Vassiliev’s results imply that $\mathcal{H}_{3,3}$ is \mathbb{Q} -acyclic.

Theorem 2.4 (Vassiliev [Vas99], Peters–Steenbrink [PS03]). *The map $\mathrm{PGL}(4) \rightarrow M$ given by $g \mapsto g(S_0)$ induces an isomorphism*

$$H^*(M; \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{PGL}(4); \mathbb{Q}).$$

Equivalently, $\mathcal{H}_{3,3}$ is \mathbb{Q} -acyclic, i.e. $H^i(\mathcal{H}_{3,3}; \mathbb{Q}) = 0$ for $i > 1$.

Various compactifications of $\mathcal{H}_{3,3}$, $\mathcal{H}_{3,3}(1)$ and other covers can be found in [DvGK05], in particular the two moduli spaces mentioned here are rational. Also relevant are the computation of $\pi_1(\mathcal{H}_{3,3})$ (as an orbifold) by Looijenga [Loo08], the identification of a compactification of $\mathcal{H}_{3,3}$ as a quotient of complex hyperbolic 4-space by Allcock, Carlson and Toledo [ACT02] and explicit descriptions of the moduli space of cubic surfaces given by Brundu and Logar [BL98].

The cohomology of the normal cover as a representation of $W(E_6)$

The combinatorics of how the 27 lines intersect is extremely well-studied. Let L be the graph with vertices the 27 lines and edges corresponding to intersecting pairs for the generic cubic surface [Cay49]. It was classically known that the automorphism group of L is realized as the Galois group of the extension given by adjoining the coefficients defining the lines over the field containing the coefficients of a cubic form. Camille Jordan proved [Jor89] that this group is the Weyl group $W(E_6)$ of the root system E_6 (see also [Man86, Remark 23.8.2]). The Galois group can also be realized as the monodromy of the covering space $\tilde{M} \rightarrow M$ and hence the deck group of its normal closure; see [Har79].

The cover $\tilde{M} \rightarrow M$ is in fact not normal (Galois): its normal closure is the space \tilde{M}_{nor} consisting of pairs (S, α) , where α is an identification of the intersection graph of the 27 lines on S with L . The deck group of \tilde{M}_{nor} is $W(E_6)$, as mentioned, and so $H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})$ is a $W(E_6)$ representation. We can restrict this representation to the index-27 subgroup that stabilizes a line, which can be identified with $W(D_5)$ (see [Nar82]). The intermediate cover corresponding to this $W(D_5)$ is exactly \tilde{M} . We can now deduce the following corollary about $H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})$ from Theorem 2.1.

Corollary 2.5. *For any non-trivial irreducible representation V of $W(E_6)$ appearing in $H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})$, the restriction of V to $W(D_5)$ cannot have a trivial summand. Equivalently, the non-trivial irreducible representations of $W(E_6)$ that occur in the 27-dimensional permutation representation given by the action on left cosets of $W(D_5)$ in $W(E_6)$ cannot occur in $H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})$.*

Proof. By Theorem 2.1 and transfer,

$$H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})^{W(E_6)} = H^*(M; \mathbb{Q}) = H^*(\tilde{M}; \mathbb{Q}) = H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})^{W(D_5)}.$$

The second statement is equivalent to the first by Frobenius reciprocity. □

Computing the cohomology $H^*(\tilde{M}_{\text{nor}}; \mathbb{Q})$ (as a $W(E_6)$ representation) would be an obvious and major generalization of Theorem 2.1. While the above corollary provides a restriction towards which irreducible representations can occur, it only rules out a small fraction: the order of $W(E_6)$ is 51840, and it has 24 non-trivial irreducible representations (see [Car93, pp. 428–429] for a character table).

There are other intermediate covers of M , by marking different configurations of the 27 lines. For instance, taking unordered triples of lines that intersect pairwise, we get a 45-sheeted cover marking the ‘tritangents’ of a cubic surface. See [Nar82] and the appendix by Looijenga for more on this cover and its quotient under $\text{PGL}(4, \mathbb{C})$.

Lines over \mathbb{F}_q

The spaces \tilde{M} and M as defined above are (the complex points of) quasiprojective varieties defined by polynomials with integer coefficients. To be more explicit, the discriminant Δ is a polynomial with integer coefficients, as are the polynomials defining the incidence of a line and a cubic surface. For a finite field \mathbb{F}_q of characteristic p , we can base change to \mathbb{F}_q . That is, reducing the defining polynomials mod p defines spaces

$$M(\mathbb{F}_q) \subset \mathbb{P}^{19}(\mathbb{F}_q),$$

$$\tilde{M}(\mathbb{F}_q) \subset \mathbb{P}^{19}(\mathbb{F}_q) \times \text{Gr}(2, 4)(\mathbb{F}_q),$$

and a projection map

$$\pi : \tilde{M}(\mathbb{F}_q) \rightarrow M(\mathbb{F}_q).$$

For $p \neq 3$, the discriminant Δ continues to characterize singular polynomials, so $M(\mathbb{F}_q)$ is the space of smooth cubic surfaces defined over \mathbb{F}_q (where a homogeneous cubic polynomial is

smooth if it is smooth at all $\overline{\mathbb{F}_q}$ points). Similarly, $\tilde{M}(\mathbb{F}_q)$ is the space of pairs (S, L) of smooth cubic surfaces S and lines L defined over \mathbb{F}_q such that $L \subset S$. Thus, $\frac{\#\tilde{M}(\mathbb{F}_q)}{\#M(\mathbb{F}_q)}$ is the average number of \mathbb{F}_q -lines on a cubic surface defined over \mathbb{F}_q . The Grothendieck–Lefschetz fixed point formula (see e.g. [Mil13]) lets us use our results to deduce consequences about the cardinality of $\#\tilde{M}(\mathbb{F}_q)$.

Remark 2.6. The fact that \tilde{M} is a connected cover of M already implies $H^0(\tilde{M}; \mathbb{Q}) \cong \mathbb{Q}$. Given Deligne’s theorem [Del80, Théorème 3.3.1] we get that both $\#M(\mathbb{F}_q)$ and $\#\tilde{M}(\mathbb{F}_q)$ are $q^{19}(1 + O(q^{-1/2}))$, since $\dim M = \dim \tilde{M} = 19$. Hence the average number of lines on a \mathbb{F}_q -cubic surface is $1 + O(q^{-1/2})$ as $q \rightarrow \infty$. One needs much more information to compute this number exactly.

Corollary 2.7. *There is a finite set of characteristics, so that for a fixed q with p not in this set,*

$$\#M(\mathbb{F}_q) = \#\tilde{M}(\mathbb{F}_q) = q^4(\#\mathrm{PGL}(4, \mathbb{F}_q)) = q^4 \frac{(q^4 - 1)(q^4 - q)(q^4 - q^2)(q^4 - q)}{q - 1}.$$

Thus the average number of lines defined over \mathbb{F}_q on a smooth cubic surface defined over \mathbb{F}_q is exactly 1.

To the best of our knowledge, the point count for $\tilde{M}(\mathbb{F}_q)$ and the consequence about the average number of lines is new. The point count for $M(\mathbb{F}_q)$ follows from Theorem 2.4.

Proof of Corollary 2.7. The varieties M and \tilde{M} are smooth since M is open in \mathbb{P}^{19} . For a smooth quasiprojective variety Y , the \mathbb{F}_q points are exactly the fixed points of Frob_q on $Y(\overline{\mathbb{F}_q})$, and $\#Y(\mathbb{F}_q)$ is determined by the Grothendieck–Lefschetz fixed point formula (see e.g. [Mil13]):

$$\#Y(\mathbb{F}_q) = q^{\dim Y} \sum_{i \geq 0} (-1)^i \mathrm{Tr}(\mathrm{Frob}_q : H_{\mathrm{ét}}^i(Y; \mathbb{Q}_\ell)^\vee),$$

where ℓ is a prime other than p . Further, there are comparison theorems implying isomorphisms

$$H_{\mathrm{ét}}^i(Y; \mathbb{Q}_\ell) \cong H^i(Y(\mathbb{C}); \mathbb{Q}_\ell) \cong H^i(Y(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}_\ell,$$

away from a finite set of characteristics (see e.g. [Del77, Théorème 1.4.6.3, Théorème 7.1.9]).

In particular, as a corollary of Theorem 2.1 we obtain $\#\tilde{M}(\mathbb{F}_q) = \#M(\mathbb{F}_q) = q^4(\#\mathrm{PGL}(4, \mathbb{F}_q))$ and hence the corollary. \square

Remark 2.8. One can define $\mathcal{H}_{3,3}(\mathbb{F}_q)$ and $\mathcal{H}_{3,3}(1)(\mathbb{F}_q)$ as base-changes of $\mathcal{H}_{3,3}$ and $\mathcal{H}_{3,3}(1)$ from above. Using an analogue of the Grothendieck–Lefschetz fixed-point formula, it is possible to conclude that

$$\#\mathcal{H}_{3,3}(1)(\mathbb{F}_q) = \#\mathcal{H}_{3,3}(\mathbb{F}_q) = q^4,$$

again away from finitely many characteristics. However, as a point of caution, these are point counts on a stack and a deeper discussion of the arguments involved is out of the scope of this thesis.

2.2 Rational cohomology of the incidence variety

2.2.1 Definitions and setup

From now on we will work over the field \mathbb{C} of complex numbers. Let $X = X_{3,3}$ be the space of *smooth* homogeneous degree 3 (complex) polynomials over 4 variables, for concreteness a subset of $\mathbb{C}[x, y, z, w]_3 \cong \mathbb{C}^{20}$. A polynomial $F \in \mathbb{C}[x, y, z, w]_3$ is smooth precisely when $\{F_x, F_y, F_z, F_w\}$ do not have a common root, by Euler’s formula. This is equivalent to a certain ‘discriminant’ in the coefficients not vanishing; there is an irreducible polynomial $\Delta : \mathbb{C}^{20} \rightarrow \mathbb{C}$ with integer coefficients that vanishes on (the coefficients of) F if and only if F is not smooth. In other words, X is the complement of the *discriminant locus*, $\Sigma = \mathcal{V}(\Delta) \subset \mathbb{C}^{20}$.

We also have the ‘incidence variety’ of a line and a (not necessarily smooth) cubic polynomial

$$\Pi = \{(F, L) \mid F|_L \equiv 0\} \subset \mathbb{C}[x, y, z, w]_3 \times \text{Gr}(2, 4),$$

where $\text{Gr}(2, 4)$ is the Grassmannian of lines in \mathbb{P}^3 (that is, 2-planes in \mathbb{C}^4). This space comes equipped with two projections. The first, $\pi : (F, L) \mapsto F$ forgets the line, and we denote the inverse image $\pi^{-1}(X)$ of X by \tilde{X} , which by (a version of) the Cayley–Salmon theorem is a 27-sheeted cover $\pi : \tilde{X} \rightarrow X$.

The second projection is to $\text{Gr}(2, 4)$, given by $(F, L) \mapsto L$, and is a fiber bundle with fiber $\Pi_\ell \cong \mathbb{C}^{16}$ over $\ell \in \text{Gr}(2, 4)$. To be explicit, Π_ℓ is the space of (not necessarily smooth) cubic

polynomials that vanish on ℓ . The restriction of the projection to \tilde{X} is also a fiber bundle, and we will denote the fiber over ℓ by \tilde{X}_ℓ , this is the space of smooth homogeneous cubic polynomials in 4 variables that vanish on ℓ . Let

$$\Sigma_\ell = \Pi_\ell \setminus \tilde{X}_\ell = \Sigma \cap \Pi_\ell.$$

Since a polynomial and its scalar multiples define the same vanishing locus, to go from the space of polynomials to the space of cubic surfaces, we need to quotient by the action of $\lambda \in \mathbb{C}^\times$ mapping $F \mapsto \lambda F$. This action of course preserves smoothness, i.e., Δ is a homogeneous polynomial and Σ is a conical hypersurface in \mathbb{C}^{20} , so passing to the quotient by \mathbb{C}^\times produces spaces

$$\begin{aligned} M &= X_{3,3}/\mathbb{C}^\times \subset \mathbb{P}^{19}, \\ \tilde{M} &= \tilde{X}/\mathbb{C}^\times \subset M \times \text{Gr}(2,4) \end{aligned} \tag{2.1}$$

and a covering map $\tilde{M} \rightarrow M$, which we will also denote by π . By transfer we know that $\pi^* : H^*(M; \mathbb{Q}) \rightarrow H^*(\tilde{M}; \mathbb{Q})$ is an injection. In fact, there is no new cohomology that appears in this cover, as in Theorem 2.1.

The map $\tilde{M} \rightarrow \text{Gr}(2,4)$ continues to be a fiber bundle, we denote the fiber over $\ell \in \text{Gr}(2,4)$ by

$$\tilde{M}_\ell = \{(S, \ell) \mid S \in M, S \supset \ell\}.$$

All these spaces and the maps described so far fit into the following commuting diagram:

$$\begin{array}{ccccc} \tilde{X}_\ell & \xrightarrow{\mathbb{C}^\times} & \tilde{M}_\ell & & \\ & \searrow & & \searrow & \\ & & \tilde{X} & \xrightarrow{\mathbb{C}^\times} & \tilde{M} \\ & \swarrow & & \swarrow & \\ X & \xrightarrow{27} & M & & \\ & & \downarrow & & \downarrow \\ & & \text{Gr}(2,4) & \xlongequal{\quad\quad\quad} & \text{Gr}(2,4) \end{array} \tag{2.2}$$

There is one more action to consider, which is important for both our theorem and its proof. As mentioned in the introduction, $GL(4) := GL(4, \mathbb{C})$ acts on \mathbb{C}^4 and $PGL(4) = GL(4)/(\mathbb{C}^\times I)$ acts on the quotient \mathbb{P}^3 . There are induced actions on the spaces defined above: on X and \tilde{X} by $GL(4)$; on M and \tilde{M} by $PGL(4)$. The action of $GL(4)$ on $Gr(2, 4)$ also factors through $PGL(4)$. Fixing a line $\ell \in Gr(2, 4)$, the respective stabilizers in $GL(4)$ and $PGL(4)$ act on the fibers \tilde{X}_ℓ and \tilde{M}_ℓ . If we fix a basepoint $(F_0, L_0) \in \tilde{X}$, and set $S_0 = \mathcal{V}(F_0)$ so that $(S_0, L_0) \in \tilde{M}$, we get orbit maps $g \mapsto g(S_0, L_0) = (g \cdot S_0, g \cdot L_0)$, and so on. Then we also have the following commuting diagram:

$$\begin{array}{ccccc}
 \mathbb{C}^\times & \xrightarrow{z \mapsto z^3} & \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times \\
 & \searrow & & \searrow & \searrow \\
 & & GL(4) & \longrightarrow & \tilde{X} \xrightarrow{\pi} X \\
 & & \downarrow & & \downarrow \\
 & & PGL(4) & \longrightarrow & \tilde{M} \xrightarrow{\pi} M
 \end{array} \tag{2.3}$$

All the four maps in the bottom-left square are in fact maps of bundles over the same base $Gr(2, 4)$, and all the vertical maps are bundles with fiber \mathbb{C}^\times . The second and third vertical maps are elaborated in the previous diagram (2.2).

Remark 2.9. It is worth noting that the map on the fibers $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$ induced by the first horizontal map is not identity but the degree 3 map $z \mapsto z^3$. This is an isomorphism on cohomology with rational coefficients, so this does not affect our computations.

Remark 2.10. Since \tilde{M} is connected, the orbit maps for different choices of basepoint $(S_0, L_0) \in \tilde{M}$ are homotopic.

2.2.2 Proof of Theorem 2.1 and the role of simplicial resolution

Vassiliev's method of simplicial resolution works by first reducing the computation of the cohomology of the discriminant complement X to computing the Borel–Moore homology $\bar{H}_*(\Sigma) = H_*^{\text{BM}}(\Sigma)$ of the discriminant locus Σ via Alexander duality. The space Σ consisting of the singular cubic surfaces is itself highly singular, and stratifies based on the how big the

singular set of each $F \in \Sigma$ is. The space Σ is replaced with its *simplicial resolution* σ with an induced stratification and applying the spectral sequence of a filtration to this stratification of σ produces a spectral sequence converging to $\bar{H}_*(\sigma) \xrightarrow{\simeq} \bar{H}_*(\Sigma)$. For more details in our context, see Section 2.3.1.

While \tilde{M} or \tilde{X} is not an open subset of a vector space, recall that the fiber \tilde{X}_ℓ over ℓ of the map $\tilde{X} \rightarrow \text{Gr}(2, 4)$ is open in the vector space Π_ℓ of polynomials vanishing on ℓ . So we can apply the Vassiliev spectral sequence to each \tilde{X}_ℓ to find $H^*(\tilde{X}_\ell; \mathbb{Q})$. For this, we need to stratify $\Sigma_\ell = \Sigma \cap \Pi_\ell$ by not just how big the singular sets are, but how they are configured with respect to the line ℓ . These are the types and subtypes described in Section 2.3.1. For now we will assume that we can perform this computation (which takes up all of Section 2.3), and when needed we refer to the answer described in Proposition 2.19.

Lemma 2.11. *For a fixed $\ell \in \text{Gr}(2, 4)$, let $\text{Stab}_{\text{GL}(4)}(\ell)$ be the stabilizer of ℓ in $\text{GL}(4)$. Then for a choice of basepoint $F_0 \in \tilde{X}_\ell$, the orbit map $\text{Stab}(\ell) \rightarrow \tilde{X}_\ell$ given by $g \mapsto g(F_0) = F_0 \circ g$ induces a surjection*

$$H^*(\tilde{X}_\ell; \mathbb{Q}) \twoheadrightarrow H^*(\text{Stab}_{\text{GL}(4)}(\ell); \mathbb{Q}) \cong H^*(\text{GL}(2) \times \text{GL}(2); \mathbb{Q}).$$

Proof. First, fix a complement ℓ^\perp of ℓ . Then $\text{Stab}_{\text{GL}(4)}(\ell)$ deformation retracts to

$$G = \text{Stab}_{\text{GL}(4)}(\ell, \ell^\perp) = \text{GL}(\ell) \times \text{GL}(\ell^\perp),$$

the subgroup of elements that fix both ℓ and ℓ^\perp .

As in the computation of $H^*(\tilde{X}_\ell; \mathbb{Q})$ in Section 2.3, it is important to identify via Alexander duality $H^*(\tilde{X}_\ell; \mathbb{Q})$ with $\bar{H}_*(\Sigma_\ell)$, and similarly $H^*(\text{GL}(2); \mathbb{Q})$ with $\bar{H}_*(\text{Mat}(2) \setminus \text{GL}(2))$, where $\text{Mat}(2)$ is the space of all 2×2 matrices. The generators of $H^*(\text{GL}(2); \mathbb{Q})$ (as a ring) are represented by the locus of matrices whose first i columns are linearly dependent¹, for $i = 1, 2$.

¹For $i = 1$ this means the first column is 0. This description of the generators generalizes to $\text{GL}(n) \subset M(n)$.

Fix $P \in \ell$ and $P' \in \ell^\perp$ non-zero and extend to bases of ℓ and ℓ^\perp respectively. This identifies $\mathrm{GL}(\ell) \times \mathrm{GL}(\ell^\perp) \cong \mathrm{GL}(2) \times \mathrm{GL}(2)$. The orbit map extends to a map

$$\mathrm{Mat}(2) \times \mathrm{Mat}(2) \rightarrow \Pi_\ell = \tilde{X}_\ell \cup \Sigma_\ell.$$

It is enough to identify subspaces of Σ_ℓ that pull-back to (a rational multiple of) the corresponding subspaces of $\mathrm{Mat}(2) \times \mathrm{Mat}(2)$. Then directly from arguments in [PS03, section 6], it is enough to pick the following four subspaces of polynomials that are: (i) singular at P , (ii) singular at some (non-zero) point of ℓ , (iii) singular at P' , (iv) singular at some (non-zero) point of ℓ^\perp . \square

Now we can prove Theorem 2.1, restated here for convenience.

Theorem 2.1. *For a choice $(S_0, L_0) \in \tilde{M}$, the orbit map $\mathrm{PGL}(4, \mathbb{C}) \rightarrow \tilde{M}$ given by $g \mapsto g(S_0, L_0)$ induces an isomorphism*

$$H^*(\tilde{M}; \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}[a_3, a_5, a_7]/(a_3^2, a_5^2, a_7^2),$$

where $a_i \in H^i(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q})$. Since the composition $\mathrm{PGL}(4, \mathbb{C}) \rightarrow \tilde{M} \xrightarrow{\pi} M$ also induces an isomorphism on $H^*(_; \mathbb{Q})$, the map

$$\pi^* : H^*(M; \mathbb{Q}) \rightarrow H^*(\tilde{M}; \mathbb{Q})$$

is an isomorphism. Since the orbit map and π are algebraic, the isomorphisms respect mixed Hodge structures.

Proof of Theorem 2.1. First we prove the analogous statement for \tilde{X} . Note that the surjection induced on $H^*(_; \mathbb{Q})$ by the orbit map $\mathrm{Stab}_{\mathrm{GL}(4)}(\ell) \rightarrow \tilde{X}_\ell$, as in Lemma 2.11, must actually be an isomorphism since from the spectral sequence in Proposition 2.19 we get that

$$\dim H^*(\tilde{X}_\ell; \mathbb{Q}) \leq 16 = \dim H^*(\mathrm{GL}(2) \times \mathrm{GL}(2); \mathbb{Q}) = \dim H^*(\mathrm{Stab}_{\mathrm{GL}(4)}(\ell); \mathbb{Q}).$$

In particular that spectral sequence degenerates at the E^1 page.

Thus we have a map of bundles (as in (2.2))

$$\begin{array}{ccc}
 \text{Stab}_{\text{GL}(4)}(\ell) & \longrightarrow & \tilde{X}_\ell \\
 & \searrow & \downarrow \\
 & & \text{GL}(4) \longrightarrow \tilde{X} \\
 & & \downarrow \\
 & & \text{Gr}(2, 4) = \text{Gr}(2, 4)
 \end{array}$$

that fiberwise induces an isomorphism

$$H^*(\tilde{X}_\ell; \mathbb{Q}) \xrightarrow{\sim} H^*(\text{Stab}_{\text{GL}(4)}(\ell); \mathbb{Q}).$$

There is no monodromy in either bundle since $\text{Gr}(2, 4)$ is simply connected. Therefore from naturality of the Serre spectral sequence, the map $\text{GL}(4) \rightarrow \tilde{X}$ must also be an isomorphism on cohomology.

Now to obtain the result for \tilde{M} , we use another map of bundles (as in (2.3)):

$$\begin{array}{ccc}
 \mathbb{C}^\times & \xrightarrow{z \mapsto z^3} & \mathbb{C}^\times \\
 & \searrow & \downarrow \\
 & & \text{GL}(4) \longrightarrow \tilde{X} \\
 & & \downarrow \\
 & & \text{PGL}(4) \longrightarrow \tilde{M}
 \end{array}$$

Since both total spaces are complements of conical hypersurfaces, these bundles satisfy the Leray–Hirsch theorem and the fiberwise map $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$ is degree 3, so induces an isomorphism on $H^*(\mathbb{C}^\times; \mathbb{Q})$. Thus the map of bases $\text{PGL}(4) \rightarrow \tilde{M}$ must also induce an isomorphism

$$H^*(\tilde{M}; \mathbb{Q}) \xrightarrow{\sim} H^*(\text{PGL}(4); \mathbb{Q}). \quad \square$$

2.3 Rational cohomology of \tilde{X}_ℓ

2.3.1 Definitions and plan of attack

We will suppress constant rational coefficients throughout this section, and use \bar{H} to denote Borel–Moore homology (for a definition see e.g. [Bre97, Chapter V]). Note that for an orientable but not necessarily compact $2n$ -manifold M , Poincaré duality takes the form

$$\bar{H}_i(M) \cong H^{2n-i}(M) \cong (H_{2n-i}(M))^\vee \cong (H_c^i(M))^\vee.$$

We use the spectral sequence developed by Vassiliev in [Vas92] and closely follow the approach in [Vas99]. We refer the reader to Vassiliev’s works for the theory, but summarize how the computation works in practice. Recall that $\tilde{X}_\ell \subset \Pi_\ell \cong \mathbb{C}^{16}$, and set $\Sigma_\ell = \Pi_\ell \setminus \tilde{X}_\ell = \Pi_\ell \cap \Sigma$, the set of singular cubic polynomials that vanish on the line ℓ . Then via Alexander duality,

$$\tilde{H}^i(\tilde{X}_\ell) = \bar{H}_{31-i}(\Sigma_\ell). \quad (2.4)$$

Note that Σ_ℓ is a hypersurface in Π_ℓ , being the vanishing locus of $\Delta_\ell = \Delta|_{\Pi_\ell}$.

Remark 2.12. The complex variety \tilde{X}_ℓ , being the complement of a hypersurface, is affine and hence a 16-dimensional Stein manifold. Thus by the Andreotti–Frankel theorem, $H^i(\tilde{X}_\ell) = 0$ for $i > 16$. This along with Eq. (2.4) imply that $\bar{H}_i(\Sigma_\ell)$ can only be non-zero for $15 \leq i \leq 31$.

Let $F \in \Sigma_\ell$ be a singular cubic polynomial and let K be its singular locus. Then K , as a subset of \mathbb{P}^3 , can be one of the following 11 types (see [Vas99, Proposition 8]):

- (I) a point;
- (II) two distinct points;
- (III) a line;
- (IV) three points, not on a line;
- (V) a smooth conic contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$;
- (VI) a pair of intersecting lines;
- (VII) four points, not on a plane;
- (VIII) a plane;

- (IX) three lines through a point, not all on the same plane;
- (X) a smooth conic contained in a plane along with another point not on that plane;
- (XI) all of \mathbb{P}^3 .

These further break up as *subtypes* depending on their configuration with respect to ℓ . For most of the types, how they break up will not be relevant to us; we list those that will. We list names for the points for convenience, they are still to be thought of as a priori *unordered* sets of points: $\{P, Q\} = \{Q, P\}$ and so on.

- (I) a point P
 - (a) $P \in \ell$
 - (b) $P \notin \ell$
- (II) two points P, Q
 - (a) $P, Q \in \ell$
 - (b) $P \in \ell, Q \notin \ell$
 - (c) $P, Q \notin \ell, P$ and Q coplanar with ℓ
 - (d) $P, Q \notin \ell, P$ and Q not coplanar with ℓ
- (IV) three points P, Q, R , not collinear
 - (a) $P, Q \in \ell, R \notin \ell$
 - (b) $P \in \ell, Q, R \notin \ell, Q, R$ coplanar with ℓ
 - (c) $P \in \ell, Q, R \notin \ell, Q, R$ not coplanar with ℓ
 - (d) $P, Q, R \notin \ell, P, Q, R$ and ℓ all coplanar
 - (e) $P, Q, R \notin \ell, P, Q$ and ℓ coplanar, R not on that plane
 - (f) $P, Q, R \notin \ell$, no two coplanar with ℓ
- (VII) four points P, Q, R, S , not coplanar
 - (a) $P, Q \in \ell, R, S \notin \ell$
 - (b) $P \in \ell, Q, R, S \notin \ell, Q, R, \ell$ coplanar

- (c) $P \in \ell$, no two of Q, R, S coplanar with ℓ
- (d) $P, Q, R, S \notin \ell$, P, Q, R coplanar with ℓ , but S not on that plane
- (e) $P, Q, R, S \notin \ell$, P, Q and ℓ coplanar, R, S and ℓ coplanar
- (f) $P, Q, R, S \notin \ell$, P, Q and ℓ coplanar, no other pair coplanar with ℓ
- (g) $P, Q, R, S \notin \ell$, no two coplanar with ℓ

Remark 2.13. The types correspond to orbits of the singular loci under the $\mathrm{PGL}(4)$ action on \mathbb{P}^3 and the subtypes correspond to orbits under $\mathrm{Stab}(\ell) \subset \mathrm{PGL}(4)$, but this will not be explicitly important for us.

Definition 2.14. For a manifold M and natural number n , the *ordered configuration space* of n points on M is given by

$$\mathrm{PConf}_n(M) := \{(a_1, \dots, a_n) \in M^n \mid a_i \neq a_j \text{ for } i \neq j\}.$$

This space comes with a natural action of the symmetric group \mathfrak{S}_n by permuting the coordinates and the quotient is the *unordered configuration space* $\mathrm{UConf}_n(M)$ of n points on M .

Definition 2.15. For any $A \subseteq \mathrm{UConf}_n(M)$, the *sign local coefficients* on A , denoted by $\pm\mathbb{Q}$, is given by the composition

$$\pi_1(A) \rightarrow \pi_1(\mathrm{UConf}_n(M)) \rightarrow \mathfrak{S}_n \rightarrow \{\pm 1\} \subset \mathbb{Q}^\times$$

thought of as a representation on \mathbb{Q} . Explicitly, a loop in A acts on \mathbb{Q} by the sign of the induced permutation on the n points.

The method of simplicial resolution produces for us a space σ with a map $f : \sigma \rightarrow \Sigma_\ell$ with the following properties:

- (1) The map $f_* : \bar{H}_*(\sigma) \rightarrow \bar{H}_*(\Sigma_\ell)$ is an isomorphism.
- (2) The space σ has a stratification

$$\sigma = \bigcup_i F_i,$$

where i varies over all the *subtypes* (not just the ones listed, but all of them). That is, F_i is a stratum corresponding to the subtype i . The strata are (partially) ordered by degeneracy: F_i intersects $\overline{F_j}$ only if polynomials with singularity of subtype i can degenerate to a polynomial with singularity of subtype j .

(3) Let

$$A_i = \{\text{singular sets } K \text{ of subtype } i\}$$

and $K \in A_i$. Let $L(K)$ be the linear subspace of Π_ℓ consisting of polynomials that are singular on K and possibly elsewhere. Then there are spaces Φ_i and $\Lambda(K)$ along with fiber bundles:

$$\begin{array}{ccc} L(K) & \hookrightarrow & F_i \\ & & \downarrow \\ & & \Downarrow \\ \Lambda(K) & \hookrightarrow & \Phi_i \\ & & \downarrow \\ & & \Downarrow \\ & & K \in A_i \end{array}$$

(4) The space $\Lambda(K)$ is an *open cone* with vertex representing K and captures the combinatorics and topology of the subsets of K that can appear as singular sets of other polynomials in Σ_ℓ . The homeomorphism type of $\Lambda(K)$ depends only on the *type* of K and not its subtype. Further, $\overline{H}_*(\Lambda(K)) = 0$ unless K is of type **I**, **II**, **IV**, **VII** or **XI**. For K of type **I**, **II**, **IV** and **VII** respectively, i.e. when K is a finite set of points, $\Lambda(K)$ can be identified with the open simplex with vertex set K . In particular, setting $n = \#K$,

$$\overline{H}_*(\Lambda(K)) = 0 \text{ for } * \neq n - 1,$$

and

$$\overline{H}_{n-1}(\Lambda(K)) \cong \mathbb{Q},$$

generated by the *fundamental class*² representing an orientation on the simplex $\Lambda(K) \cong B^{n-1}$. Further, A_i is a subset of $\text{UConf}_n(\mathbb{P}^3)$, and the monodromy on $\overline{H}_*(\Lambda(K))$ is given by

²Recall that the fundamental class of an orientable but not necessarily compact n -manifold M without boundary is a generator of $\overline{H}_n(M)$, and the choice of the generator corresponds to the choice of an orientation on M .

$\pm\mathbb{Q}$ (permuting the points of K changes the orientation of the simplex by the sign of the permutation).

- (5) For the type **XI** (note that **XI** has only one subtype, itself), A_{XI} is singleton, the only element being $K = \mathbb{P}^3$. The only polynomial singular on K is 0, so $L(K) = \{0\}$. Thus $F_{\text{XI}} = \Phi_{\text{XI}} = \Lambda(\mathbb{P}^3)$. Further, the space $\Phi_{\text{XI}} = \Lambda(\mathbb{P}^3)$ is the open cone over $\bigcup_{j \neq \text{XI}} \Phi_j$ for certain gluings.

Example 2.16. For the subtype **IIB**, a point on ℓ and a point not on ℓ , we have $A_{\text{IIB}} = \ell \times \mathbb{P}^3 \setminus \ell$. For the subtype **IId**, two points not coplanar with ℓ , the space A_{IId} is an open set in $\text{UConf}_2(\mathbb{P}^3 \setminus \ell)$.

We refer the reader to [Vas99] for details of the construction and proofs for (1)–(5). Everything we use for our computation has been summarized in these properties. We now go through the steps of the computation before digging into the details.

By the isomorphism given by Alexander duality (Eq. (2.4)), we are reduced to computing $\bar{H}_*(\Sigma_\ell)$. By (1), this is the same as $\bar{H}_*(\sigma)$. Let

$$\deg(i) = 14 - \dim L(K)$$

for any $K \in A_i$ (these numbers, along with $\dim A_i$, can be found in Table 2.1). This is monotonic on the poset described in (2), in the sense that if F_i intersects \bar{F}_j , then $\deg(i) \leq \deg(j)$. Using the filtration of σ given by $\bigcup_{\deg(i) \leq p} F_i$ there is a spectral sequence $E_{p,q}^r \implies \bar{H}_{p+q}\sigma$, with the E^1 page given by

$$E_{p,q}^1 = \bigoplus_{\deg(i)=p} \bar{H}_{p+q}(F_i). \quad (2.5)$$

To compute each term, since $L(K) \rightarrow F_i \rightarrow \Phi_i$ is a complex vector bundle, we have the Thom isomorphism

$$\bar{H}_*(F_i) = \bar{H}_{*-2\dim_{\mathbb{C}} L(K)}(\Phi_i). \quad (2.6)$$

For the right-hand side, if $\Lambda(K)$ is acyclic then so must be Φ_i , so this automatically vanishes unless i is a subtype of **I**, **II**, **IV**, **VII**, or **XI**.

i	Ia	Ib	IIa	IIb	IIc	IId	IVa	IVb	IVc	IVd	IVe	IVf
$\dim A_i$	1	3	2	4	5	6	5	6	7	7	8	9
$\dim L(K)$	14	12	12	10	9	8	8	8	6	7	5	4

i	VIIa	VIIb	VIIc	VIIId	VIIe	VIIIf	VIIg	XI
$\dim A_i$	8	9	10	10	10	11	12	0
$\dim L(K)$	4	3	2	3	2	1	0	0

Table 2.1: $\dim A_i$ and $\dim L(K)$ for $K \in A_i$ for each subtype i .

For the (sub)type **XI**, from (5) we have that $\Phi_{\text{XI}} = CZ$, the open cone on Z , where

$$Z = \bigcup_{i \neq \text{XI}} \Phi_i.$$

So we get a spectral sequence $e_{p,q}^r \implies H_{p+q}(Z)$ with

$$e_{p,q}^1 = \bigoplus_{\substack{\deg(j)=p, \\ j \neq \text{XI}}} \bar{H}_{p+q}(\Phi_j).$$

But then we also have

$$\bar{H}_*(CZ) = H_*(CZ, Z) = \tilde{H}_{*-1}(Z).$$

For all the other i , the set K is finite, of say n points ($1 \leq n \leq 4$). Then as described in (4), $\bar{H}_*(\Lambda(K))$ is concentrated in degree $n-1$, so

$$\bar{H}_*(\Phi_i) = \bar{H}_{*-n+1}(A_i; \pm\mathbb{Q}) = H^{2\dim_{\mathbb{C}} A_i + n - 1 - *}(A_i; \pm\mathbb{Q}), \quad (2.7)$$

where the latter isomorphism is by (twisted) Poincaré duality, since the A_i are complex manifolds. So the computation eventually boils down to computing $H^*(A_i; \pm\mathbb{Q})$ for these i (see Propositions 2.18 and 2.19) and bookkeeping.

2.3.2 Case work

We now state the results of some general computations that we will use in the case work. We would like to point out that these are well known, can be proved in many ways and collected here only for the reader's convenience.

Lemma 2.17.

$$H^*(\mathrm{UConf}_2(\mathbb{C}); \pm\mathbb{Q}) \cong H^*(\mathrm{UConf}_2(\mathbb{C}^2), \pm\mathbb{Q}) \cong H^*(\mathrm{UConf}_4(\mathbb{C}^2); \pm\mathbb{Q}) = 0.$$

$$H^*(\mathrm{UConf}_4(\mathbb{P}^2 \setminus \{\bullet\}); \pm\mathbb{Q}) \cong H^*(\mathrm{UConf}_4(\mathbb{P}^3 \setminus \mathbb{P}^1); \pm\mathbb{Q}) = 0$$

$$H^*(\mathrm{UConf}_2(\mathbb{P}^1); \pm\mathbb{Q}) \cong H^*(\mathrm{UConf}_2(\mathbb{P}^3 \setminus \mathbb{P}^1); \pm\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $\mathrm{UConf}_2(\mathbb{C})$ or $\mathrm{UConf}_2(\mathbb{C}^2)$, we can use that $\mathrm{PConf}_2(\mathbb{R}^{2n}) \simeq S^{2n-1}$, and the \mathfrak{S}_2 action is by the antipodal map, which is degree 1 and hence by transfer $H^*(\mathrm{UConf}_2(\mathbb{R}^{2n}); \pm\mathbb{Q}) = 0$.

For all the other spaces of the form $\mathrm{UConf}_n(Z)$, [Tot96] provides spectral sequences that converge to $\mathrm{PConf}_n(Z)$ as an \mathfrak{S}_n representation. The computation of each of these is straightforward from [Tot96, Theorem 1]. The conclusion again follows from transfer, since $H^*(\mathrm{UConf}_n(Z); \pm\mathbb{Q})$ is the $\pm\mathbb{Q}$ summand of $H^*(\mathrm{PConf}_n(Z); \mathbb{Q})$ as a \mathfrak{S}_n representation.

For $H^*(\mathrm{UConf}_2(\mathbb{P}^1); \pm\mathbb{Q})$ we can also use [Vas99, Lemma 2B] and the argument therein applies to the other cases as well. \square

The rest of this section contains the details of the arguments to compute the various $H^*(A_i; \pm\mathbb{Q})$. The main idea is decomposing these spaces as fiber bundles, where both the fiber and base are ‘simpler’. In many instances the bases are A_j for some lower j , and the computation is ‘inductive’ or recursive. We separate out the cases where the answer is 0 in Proposition 2.18, the recursive nature of the argument makes some of the cases relatively easy and the details are postponed to the end of the section. The remaining cases are treated in Proposition 2.19.

Proposition 2.18. *Suppose that i is a subtype of I, II, IV or VII. Then $H^*(A_i; \pm\mathbb{Q}) = 0$ unless i is one of Ia, Ib, IIa, IIb, IId, IVa, IVc, VIIa.*

Recall that by (2) we have spectral sequences $E_{p,q}^r \implies \bar{H}_{p+q}(\sigma)$ and $e_{p,q}^r$ that let us compute $\bar{H}_*(F_{\mathrm{XI}}) = \tilde{H}_{*-1}(Z)$, where

$$Z = \bigcup_{i \neq \mathrm{XI}} \Phi_i.$$

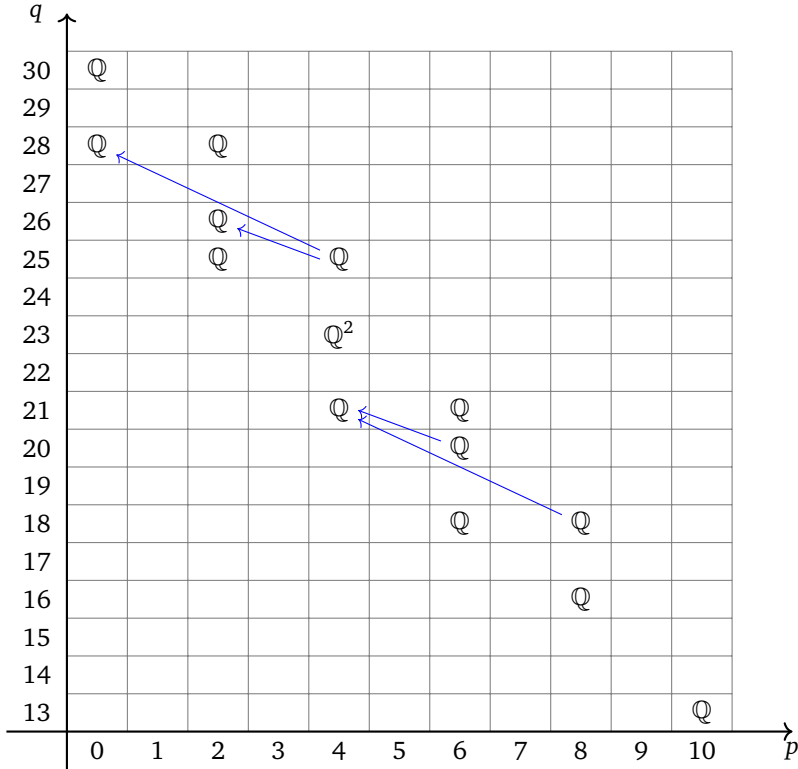


Figure 2.1: Spectral sequence page $E_{p,q}^1$ for $\bar{H}_{p+q}(\sigma)$ (with 0s omitted) and all potentially non-zero differentials in subsequent pages

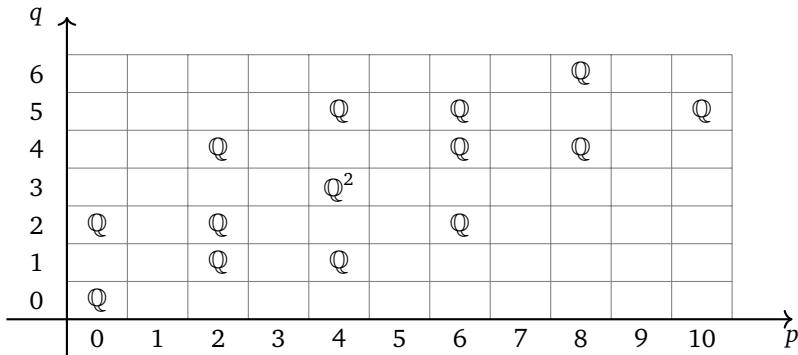


Figure 2.2: Spectral sequence page $e_{p,q}^1$ for $H_{p+q}(Z)$ (with 0s omitted)

Proposition 2.19. *The spectral sequence $E_{p,q}^r \implies \bar{H}_{p+q}(\sigma)$ has the page $E_{p,q}^1$ as in Fig. 3.2. The spectral sequence $e_{p,q}^r \implies H_{p+q}(Z)$ has the page $e_{p,q}^1$ as in Fig. 3.3.*

Proof. Recall that by construction, the terms of E^1 and e^1 are related by Thom isomorphisms:

$$E_{p,q+2(14-p)}^1 \cong e_{p,q}^1$$

except for $p = 14$, where $e_{14,*}^1 \equiv 0$. So we first go through case work to establish columns $p \neq 14$.

By Eqs. (2.5) to (2.7) and careful bookkeeping, it is enough to find $H^*(A_i; \pm\mathbb{Q})$ along with the numbers $\dim(A_i) = \dim_{\mathbb{C}}(A_i)$ and $\dim(L(K)) = \dim_{\mathbb{C}}(L(K))$ for $K \in A_i$, for the subtypes i of I, II, IV VII (see Table 2.1 for the relevant numerics). Further, there are only eight subtypes that need to be dealt with — the exceptions from Proposition 2.18.

Ia, $P \in \ell$ $A_{Ia} = \ell \cong \mathbb{P}^1$, since there is only one point, the coefficients $\pm\mathbb{Q}$ are trivial, so

$$H^*(A_{Ia}; \pm\mathbb{Q}) = H^*(\mathbb{P}^1) = \begin{cases} \mathbb{Q} & * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{0,28}^1 \cong e_{0,0}^1$ and $E_{0,30}^1 \cong e_{0,2}^1$ since $\dim(A_{Ia}) = 1$ and $\dim(L(K)) = 14$.

Ib, $P \notin \ell$ $A_{Ib} = \mathbb{P}^3 - \ell \simeq \mathbb{P}^1$. Again, the coefficients are trivial, so

$$H^*(A_{Ib}; \pm\mathbb{Q}) = H^*(\mathbb{P}^1) = \begin{cases} \mathbb{Q} & * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{2,26}^1 \cong e_{2,2}^1$ and $E_{2,28}^1 \cong e_{2,4}^1$ since $\dim(A_{Ib}) = 3$ and $\dim(L(K)) = 12$.

IIa, $P, Q \in \ell$ $A_{IIa} = \text{UConf}_2(\ell) \cong \text{UConf}_2(\mathbb{P}^1)$. By Lemma 2.17,

$$H^*(A_{IIa}; \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{2,25}^1 \cong e_{2,1}^1$ since $\dim(A_{IIa}) = 2$ and $\dim(L(K)) = 12$.

IIb, $P \in \ell$, $Q \notin \ell$ $A_{IIb} \cong \ell \times (\mathbb{P}^3 \setminus \ell) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and the coefficients are trivial. Hence,

$$H^*(A_{IIb}; \pm\mathbb{Q}) \cong H^*(\mathbb{P}^1 \times \mathbb{P}^1) = \begin{cases} \mathbb{Q} & * = 0, 4 \\ \mathbb{Q}^2 & * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{4,21}^1 \cong e_{4,1}^1$, $E_{4,23}^1 \cong e_{4,3}^1$ and $E_{4,25}^1 \cong e_{4,5}^1$ since $\dim(A_{\text{IIa}}) = 4$ and $\dim(L(K)) = 10$.

II d, $P, Q \notin \ell$, P and Q not coplanar with ℓ $A_{\text{II d}} = \text{UConf}_2(\mathbb{P}^3 \setminus \ell) \setminus A_{\text{II c}}$. Proposition 2.18 shows that $H^*(A_{\text{II c}}; \pm\mathbb{Q}) = 0$, so from the Gysin sequence, and by Lemma 2.17,

$$H^*(A_{\text{II d}}, \pm\mathbb{Q}) \cong H^*(\text{UConf}_2(\mathbb{P}^3 \setminus \mathbb{P}^1), \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{6,21}^1 \cong e_{6,5}^1$ since $\dim(A_{\text{IIa}}) = 6$ and $\dim(L(K)) = 8$.

IV a, $P, Q \in \ell$, $R \notin \ell$ $A_{\text{IV a}} \cong \text{UConf}_2(\ell) \times (\mathbb{P}^3 \setminus \ell)$, and the local coefficients restrict to trivial coefficients on the second factor $\mathbb{P}^3 \setminus \ell \simeq \mathbb{P}^1$. Thus,

$$H^*(A_{\text{IV a}}; \pm\mathbb{Q}) \cong \bigoplus_{a+b=*} H^a(\text{UConf}_2(\mathbb{P}^1); \pm\mathbb{Q}) \otimes H^b(\mathbb{P}^1) = \begin{cases} \mathbb{Q} & * = 2, 4 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{6,18}^1 \cong e_{6,2}^1$ and $E_{6,20}^1 \cong e_{6,4}^1$ since $\dim(A_{\text{IV a}}) = 5$ and $\dim(L(K)) = 8$.

IV c, $P \in \ell$, $Q, R \notin \ell$, Q and R not coplanar with ℓ Since the line $\langle Q, R \rangle$ doesn't intersect ℓ , P can be any point on ℓ for any choice of Q and R . Thus $A_{\text{IV c}} = \ell \times A_{\text{II d}}$ and the local coefficients are trivial on the first factor (ℓ is anyway simply connected). Hence

$$H^*(A_{\text{IV c}}; \pm\mathbb{Q}) \cong \bigoplus_{a+b=*} H^a(\mathbb{P}^1) \otimes H^b(A_{\text{II d}}; \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 2, 4 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{8,16}^1 \cong e_{8,4}^1$ and $E_{8,18}^1 \cong e_{8,6}^1$ since $\dim(A_{\text{IV a}}) = 7$ and $\dim(L(K)) = 6$.

VII a, $P, Q \in \ell$, $R, S \notin \ell$ By definition of VII, the four points cannot be coplanar. This is equivalent to R and S not being coplanar with ℓ . If $\rho : \mathbb{P}^3 \setminus \ell \rightarrow \ell^\perp$ is the projection, then this is further equivalent to $\rho(R) \neq \rho(S)$. Note that $\rho^{-1}(T) = \langle T, \ell \rangle \setminus \ell \cong \mathbb{C}^2$. Thus, mapping

$\{P, Q, R, S\} \mapsto (\{P, Q\}, \{\rho(R), \rho(S)\})$, we get a bundle:

$$\begin{array}{ccc} \mathbb{C}^4 & \hookrightarrow & A_{\text{VIIa}} \\ & & \downarrow \\ & & \text{UConf}_2(\ell) \times \text{UConf}_2(\ell^\perp) \end{array}$$

This implies, using Lemma 2.17,

$$H^*(A_{\text{VIIa}}; \pm\mathbb{Q}) \cong \bigoplus_{a+b=*} H^a(\text{UConf}_2(\mathbb{P}^1); \pm\mathbb{Q}) \otimes H^a(\text{UConf}_2(\mathbb{P}^1); \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 4 \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{10,13}^1 \cong e_{10,5}^1$ since $\dim(A_{\text{VIIa}}) = 8$ and $\dim(L(K)) = 4$.

Thus we've computed the pages $e_{p,q}^1$ and $E_{p,q}^1$ except the $p = 14$ column of the latter. For **XI**, $L(K) = 0$, so $E_{14,q}^1 \cong \bar{H}_{14+q}(\Phi_{\text{XI}})$. Now, if any term with $1 \leq d = p + q \leq 14$ remains non-zero in $e_{p,q}^\infty$, then it would appear as $\bar{H}_{d+1}(\Phi_{\text{XI}})$ and hence as a term $E_{14,d-13}^1$, which cannot interact with any of the other terms, by the shapes of the other columns, which we have already determined. That means $0 \neq \bar{H}_{d+1}(\sigma) \cong \tilde{H}^{31-d}(\tilde{X}_\ell)$, which is a contradiction with \tilde{X}_ℓ being a 16-dimensional Stein manifold, as in Remark 2.12. This implies, given the shape of $e_{p,q}^1$, that $\bar{H}_*(\Phi_{\text{XI}}) \equiv 0$, so we have also verified $E_{14,*}^1$. \square

Proof of Proposition 2.18. We need to show that $H^*(A_i; \pm\mathbb{Q}) = 0$ when i is one of **IIc**, **IVb**, **IVd**, **IVe**, **IVf**, **VIIb**, **VIIc**, **VIIId**, **VIIe**, **VIIIf**, **VIIg**. Let's deal with each in turn.

IIc, $P, Q \notin \ell$, but P, Q and ℓ coplanar Mapping $\{P, Q\} \mapsto H = \langle P, Q, \ell \rangle$, the projective span of P, Q, ℓ , i.e. the plane containing P, Q and ℓ , we get a map from A_{IIc} to the space of planes in \mathbb{P}^3 containing ℓ , which is a $\mathbb{P}^1 \cong \ell^\vee \subset (\mathbb{P}^3)^\vee$. This is a fiber bundle

$$\begin{array}{ccc} \text{UConf}_2(H \setminus \ell) & \hookrightarrow & A_{\text{IIc}} \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

and the local coefficients $\pm\mathbb{Q}$ restrict to the fiber to the sign local coefficient on $\text{UConf}_2(H \setminus \ell) \cong \text{UConf}_2(\mathbb{C}^2)$. But $H^*(\text{UConf}_2(\mathbb{C}^2), \pm\mathbb{Q}) = 0$ from Lemma 2.17, so we are done.

IVd, $P, Q, R \notin \ell$, but P, Q, R and ℓ coplanar Mapping $\{P, Q, R\} \mapsto H = \langle P, Q, R, \ell \rangle$, we get a fiber bundle:

$$\begin{array}{ccc} F & \hookrightarrow & A_{\text{IVd}} \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

The fiber is the space of three (unordered points) non-collinear points on $H \setminus \ell \cong \mathbb{C}^2$, and the local coefficients $\pm\mathbb{Q}$ restrict to the local coefficients $\pm\mathbb{Q}$ on $F \subset \text{UConf}_3(\mathbb{C}^2)$. Hence $\text{UConf}_3(\mathbb{C}^2) \setminus F$ fibers over the space of lines in \mathbb{C}^2 with fiber $\text{UConf}_3(\mathbb{C})$. Hence we are done by [Vas99, Lemma 2].

IVf, $P, Q, R \notin \ell$, no two coplanar with ℓ In this case, we go to the \mathfrak{S}_3 cover \tilde{A} of A_{IVf} , so that by transfer $H^*(A_{\text{IVf}}; \pm\mathbb{Q})$ is the sign-representation summand of $H^*(\tilde{A}; \mathbb{Q})$. Then \tilde{A} can be broken up by fiber bundles:

$$\begin{array}{ccc} (\mathbb{C} \setminus 0) \times (\mathbb{C}^2 \setminus 0) \cong \mathbb{P}^3 \setminus (\langle P, \ell \rangle \cup \langle Q, \ell \rangle \cup \langle P, Q \rangle) & \hookrightarrow & \tilde{A} \\ & & \downarrow \\ \mathbb{C}^3 \cong \mathbb{P}^3 \setminus \langle P, \ell \rangle & \hookrightarrow & \{(P, Q)\} \\ & & \downarrow \\ & & \{P\} \quad \text{=====} \quad \mathbb{P}^3 \setminus \ell \simeq \mathbb{P}^1 \end{array}$$

The transposition (PQ) acts by -1 on the fiber $(\mathbb{C} \setminus 0) \times (\mathbb{C}^2 \setminus 0)$ and trivially on $\mathbb{P}^3 \setminus \ell$. Therefore the action of \mathfrak{S}_3 on $H^*(\tilde{A})$ is trivial and we are done.

VIIe, $P, Q, R, S \notin \ell$; P, Q and ℓ coplanar; R, S and ℓ coplanar We can map $\{P, Q, R, S\}$ to $\{\langle P, Q \rangle, \langle R, S \rangle\}$, the pair of lines through PQ and RS and get a map $A_{\text{VIIe}} \rightarrow B$, where B is the set of unordered pairs of lines in \mathbb{P}^3 that both intersect ℓ , but so that the three lines are not coplanar (in particular the pair of lines do not themselves intersect). This is a fiber bundle:

$$\begin{array}{ccc} \text{UConf}_2(L_1 \setminus \ell) \times \text{UConf}_2(L_2 \setminus \ell) & \hookrightarrow & A_{\text{VIIe}} \\ & & \downarrow \\ & & B \end{array}$$

Since $L_i \setminus \ell \cong \mathbb{C}$, and $H^*(\text{UConf}_2(\mathbb{C}), \pm\mathbb{Q}) = 0$, $H^*(A_{\text{VIIe}}; \pm\mathbb{Q}) = 0$.

VIIg, $P, Q, R, S \notin \ell$, **no two coplanar with ℓ** By an argument analogous to the case of IVf, A_{VIIg} has an \mathfrak{S}_4 cover by ordering the four points that breaks up as a fiber bundle over the \mathfrak{S}_3 cover of A_{IVf} . The sign representation doesn't occur in the cohomology of this cover, so we are done.

The recursive cases The rest of the cases each fiber over one of the previous cases. For example, consider the case IVb, with $P \in \ell$, $Q, R \notin \ell$, but Q, R and ℓ coplanar. Even though P, Q and R are a priori unordered, we cannot (continuously) swap R with one of P and Q . So there is a well-defined map $\{P, Q, R\} \mapsto \{Q, R\}$, and we get a fiber bundle:

$$\begin{array}{ccc} \mathbb{C} \cong \ell \setminus \langle Q, R \rangle & \hookrightarrow & A_{IVb} \\ & & \downarrow \\ & & A_{IIc} \end{array}$$

The local coefficients $\pm\mathbb{Q}$ on the total space pull-back from $\pm\mathbb{Q}$ on base (that is, the map $\pi_1(A_{IVb}) \rightarrow \{\pm 1\}$ factors through $\pi_1(A_{IIc})$). But as we just showed, $H^*(A_{IIc}; \pm\mathbb{Q}) = 0$, so we are done.

Similarly, A_{IVe} , A_{VIIb} and A_{VIIf} fiber over A_{IIc} , A_{VIId} fibers over A_{IVd} and A_{VIIf} fibers over A_{IVf} . We leave the explicit maps in each case to the reader. □

Chapter 3

The universal smooth cubic surface

3.1 Introduction

A cubic surface $S \subset \mathbb{P}^3 = \mathbb{C}\mathbb{P}^3$ is the zero set $S = \mathcal{V}(F)$ of a homogeneous complex polynomial F of degree 3 in 4 variables. The surface S is singular (i.e. not smooth) if and only if the 20 coefficients of F are a zero of a discriminant polynomial $\Delta : \mathbb{C}^{20} \rightarrow \mathbb{C}$. Thus the space of smooth cubic surfaces is an open locus $M = M_{3,3} := \mathbb{P}^{19} \setminus \mathcal{V}(\Delta)$.

Similarly one can define a smooth cubic surface over the finite field \mathbb{F}_q as the smooth zero set of a homogeneous \mathbb{F}_q polynomials of degree 3 in 4 variables. It is a fact (see Section 3.1.1) that every smooth cubic surface over \mathbb{F}_q has $q^2 + (t + 1)q + 1$ points for some $t \in \{-3, -2, -1, 0, 1, 2, 3, 4, 6\}$. Serre asked (e.g. in [Ser12, Section 2.3.3]) which t can occur over all surfaces defined over each q .

Theorem 3.1 (Swinerton-Dyer ([Swi10]), Banwait–Fité–Loughran ([BFL18])). *For $q = 2, 3$ or 5 , the value $t = 6$ is impossible. These are the only exceptions. That is, for every other possible value of t and q , there is some cubic surface over \mathbb{F}_q with that value of t .*

Using the topology of the space M and of the ‘universal family’ of smooth cubic surfaces, we can obtain the average value of t for fixed q .

Theorem 3.2. *There is a finite set of characteristics, so that for a fixed $q = p^d$ with p not in this set, the average number of \mathbb{F}_q points on a smooth cubic surface defined over \mathbb{F}_q is exactly $q^2 + q + 1$.*

To the best of our knowledge, this result about the average number of points is new. The average number of points on *irreducible* (but not necessarily smooth) cubic surfaces was known to also be $q^2 + q + 1$ by N. Elkies (see [Kap13, Section 2.4]) using different methods.

To prove this theorem (in Section 3.1.1), we will need the *incidence variety*

$$U = \{(S, p) \mid p \in S\} \subset M \times \mathbb{P}^3$$

of points and cubic surfaces (see Eq. (3.2)); a subvariety $U \subset M \times \mathbb{P}^3$. The canonical projection map $U \rightarrow M$ is a fiber bundle, whose fiber over $S \in M$ is exactly $S \subset \mathbb{P}^3$. This bundle is the aforementioned *universal family* of cubic surfaces with embeddings in \mathbb{P}^3 , in the sense that a family of embedded smooth cubic surfaces corresponds to a pullback of this bundle by a map to M .

The automorphism group of \mathbb{P}^3 is $\mathrm{PGL}(4, \mathbb{C})$ and this group takes cubic surfaces to cubic surfaces, preserving smoothness. In particular the projection map $\pi : U \rightarrow M$ is $\mathrm{PGL}(4, \mathbb{C})$ -equivariant. Vassiliev showed (in [Vas99]) that the space M has the same rational cohomology as $\mathrm{PGL}(4, \mathbb{C})$ and it following results of Peters–Steenbrink ([PS03]) this isomorphism is induced by the orbit map given by $g \mapsto g(S_0)$, for any choice of $S_0 \in M$ (see Theorem 3.10). See also Tommasi ([Tom14]).

The main result of this chapter is that the rational cohomology of U is isomorphic to that of $\mathrm{PGL}(4, \mathbb{C}) \times \mathbb{C}\mathbb{P}^2$.

Theorem 3.3 (Cohomology of the universal smooth cubic). *Let $\eta \in H^2(\mathbb{C}\mathbb{P}^3; \mathbb{Q})$ be the hyperplane class. Let $\iota : U \rightarrow M \times \mathbb{C}\mathbb{P}^3$ be the inclusion map. Then $\iota^*(1 \otimes \eta^3) = 0$ and the induced map*

$$H^*(M \times \mathbb{C}\mathbb{P}^3; \mathbb{Q})/(\eta^3) \rightarrow H^*(U; \mathbb{Q})$$

is an isomorphism. In particular, with rational coefficients,

$$H^*(U) \cong H^*(M \times \mathbb{C}\mathbb{P}^2) \cong H^*(\mathrm{PGL}(4, \mathbb{C})) \otimes H^*(\mathbb{C}\mathbb{P}^2) \cong \mathbb{Q}[\alpha_3, \alpha_5, \alpha_7, \eta]/(\alpha_3^2, \alpha_5^2, \alpha_7^2, \eta^3)$$

where $\alpha_i \in H^i(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q})$. Since the inclusion map is algebraic, each isomorphism is an isomorphism of mixed Hodge structures. In particular, $H^k(U; \mathbb{Q})$ is pure of Tate type; each generator α_{2k-1} is of bidegree (k, k) and η is of bidegree $(1, 1)$.

The key tool in our proof of Theorem 3.3 is simplicial resolution à la Vassiliev. Considering the combinatorics of how the marked point is situated with respect to possible singularities on the surfaces makes the casework fairly complicated. We devote all of Section 3.3 to this computation, while Section 3.2.2 contains the rest of the proof.

3.1.1 Applications: moduli space, representations of $W(E_6)$ and point counts

We now give a few applications of Theorem 3.3.

Cohomology of moduli spaces

The map $\pi : U \rightarrow M$ is $\mathrm{PGL}(4, \mathbb{C})$ -equivariant and each orbit (in either M or U) is closed (see e.g. [ACT02]). Further, two cubic surfaces are isomorphic exactly when they are in the same $\mathrm{PGL}(4, \mathbb{C})$ -orbit. Thus, passing to the geometric quotient gives a bundle

$$\mathcal{U}_{3,3} \rightarrow \mathcal{H}_{3,3},$$

where

$$\mathcal{H}_{3,3} := M / \mathrm{PGL}(4, \mathbb{C})$$

is the moduli space of smooth cubic surfaces and

$$\mathcal{U}_{3,3} := U / \mathrm{PGL}(4, \mathbb{C})$$

is the moduli space of cubic surfaces equipped with a point. The induced map $\mathcal{U}_{3,3} \rightarrow \mathcal{H}_{3,3}$ is the universal family of cubic surface up to isomorphism.

Note that both $\mathcal{H}_{3,3}$ and $\mathcal{U}_{3,3}$ are coarse moduli spaces. For example the Fermat cubic defined by $x^3 + y^3 + z^3 + w^3$ equipped with the point $[1 : -1 : 0 : 0]$ has non-trivial (but finite) stabilizer in $\mathrm{PGL}(4, \mathbb{C})$. Using a theorem of Peters and Steenbrink ([PS03, Theorem 2]), which is a generalization of the Leray–Hirsch theorem, we have the following corollary.

Corollary 3.4. *The space $\mathcal{U}_{3,3}$ has the rational cohomology of \mathbb{P}^2 :*

$$H^i(\mathcal{U}_{3,3}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } i = 0, 2 \text{ or } 4; \\ 0 & \text{otherwise.} \end{cases}$$

For comparison, it was known previously that $\mathcal{H}_{3,3}$ is \mathbb{Q} -acyclic; see Theorem 3.10 below.

Monodromy and the normal cover with deck group $W(E_6)$

One way of trying to compute $H^*(U; \mathbb{Q})$ would be to use the fiber bundle $U \rightarrow M$. Since the fiber over a surface $S \in M$ is exactly $S \subset \mathbb{C}\mathbb{P}^3$, this provides a spectral sequence

$$H^p(M; H^q(S)) \implies H^{p+q}(U),$$

where the coefficients are twisted by the monodromy action of $\pi_1(M)$ on

$$H^*(S; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 0, 4; \\ \mathbb{Q}^7 & \text{if } * = 2; \\ 0 & \text{otherwise.} \end{cases}$$

The monodromy action on H^0 and H^4 are of course trivial but the action on H^2 is quite interesting; to explore this we need a better description of $H^2(S)$.

There are different elements of $H^2(S)$ that can be described as the hyperplane class: the pullback η of a generic hyperplane in \mathbb{P}^3 , which also equals the anticanonical class; or the strict transform λ of a line when S is identified with \mathbb{P}^2 blown up at 6 points. Every cubic surface S famously contains 27 lines and a choice of any 6 *disjoint* lines out of the 27 when blown down produces \mathbb{P}^2 (see for instance [Har77, Section V.4, specifically Proposition V.4.10]). It is then

straightforward to see that the classes of 6 such (disjoint) lines, along with either η or λ is a basis of $H^2(S)$.

The monodromy action keeps η invariant since it preserves the embedding $S \subset \mathbb{P}^3$, but it does not preserve the choice of lines—in fact it must be transitive on the choices of 6 disjoint lines. It does act by a finite group, the automorphism group of the intersection pairing of the 27 lines, which can be identified as the Weyl group $W(E_6)$ of the root system E_6 (see [Man86, Remark 23.8.2], also [Jor89; Har79]). As a representation of $W(E_6)$, we get a decomposition of $H^2(S)$ into a one-dimensional trivial representation spanned by η and a copy of the irreducible *fundamental representation* of $W(E_6)$, denoted V_{fund} , spanned by the projections of any 6 disjoint lines. Thus,

$$H^p(M; H^2(S)) \cong H^p(M; \mathbb{Q}\langle\eta\rangle) \oplus H^p(M; V_{\text{fund}}).$$

So to use the Serre spectral sequence, for $U \rightarrow M$, we would need to compute $H^p(M; V_{\text{fund}})$. The finite quotient $\pi_1(M) \rightarrow W(E_6)$ corresponds to a normal cover $M(27)$ of M , whose points are given by decorating each $S \in M$ with a choice of ordering of the 27 lines, consistent with some chosen intersection pattern. Thus by transfer, we would need the multiplicity of V_{fund} in $H^*(M(27); \mathbb{Q})$. As the following corollary shows, it is in fact possible to turn this argument backwards and use Theorem 3.3 to compute this multiplicity.

Corollary 3.5. *The fundamental representation V_{fund} of $W(E_6)$ does not appear in $H^*(M(27); \mathbb{Q})$.*

Remark 3.6. Some of the other irreducible representations of $W(E_6)$ are also precluded from occurring in $H^*(M(27))$, see Corollary 2.5, but this is not sufficient to determine $H^*(M(27))$ entirely.

Proof of Corollary 3.5. By Bezout's theorem, $\eta^2 = \eta \cup \eta \in H^4(S)$ is 3 times the fundamental cohomology class of S and of course $\eta^3 = 0$. Moreover the pullback of a generic hyperplane to U under the map $U \rightarrow \mathbb{P}^3$ further pulls back to η for every inclusion $S \subset U$, so we also denote this class by $\eta \in H^2(U)$. By Theorem 3.3, $H^*(U) = H^*(M) \otimes \mathbb{Q}[\eta]$.

But in the Serre spectral sequence for the bundle $U \rightarrow M$ from above, the E_2 page has three rows ($q = 0, 2, 4$), which consist of $H^p(M) \otimes \mathbb{Q}\langle \eta^{q/2} \rangle$ along with $H^p(M; V_{\text{fund}})$ on the $q = 2$ row. There cannot be a non-zero differential mapping either into or out of $H^p(M) \otimes \mathbb{Q}\langle \eta \rangle$, since these terms must survive till the E_∞ page and thus all the differentials vanish. But $H^*(M) \otimes \mathbb{Q}\langle \eta \rangle$ already accounts for all of $H^*(U)$, so we must have

$$H^p(M; V_{\text{fund}}) = 0$$

for each p . But by transfer,

$$H^p(M; V_{\text{fund}}) \cong H^p(M(27)) \otimes_{W(E_6)} V_{\text{fund}},$$

so this irreducible representation cannot occur in any $H^p(M(27))$. □

Remark 3.7. The vanishing of the differentials is consistent with the bundle $U \rightarrow M$ having a (continuous) section. In fact, the existence of such a section, along with the result that $H^*(M; V_{\text{fund}}) = 0$ would be sufficient to recover Theorem 3.3.

Point counts over \mathbb{F}_q

The spaces U and M as defined are (the complex points of) quasiprojective varieties defined by integer polynomials. To be more explicit, the discriminant Δ is an integer polynomial, as are the polynomials defining the incidence of a point and a cubic surface. For a finite field \mathbb{F}_q of characteristic p , we can base change to \mathbb{F}_q . That is, reducing the defining polynomials mod p defines spaces

$$M(\mathbb{F}_q) \subset \mathbb{P}^{19}(\mathbb{F}_q), \quad \text{and} \quad U(\mathbb{F}_q) \subset \mathbb{P}^{19}(\mathbb{F}_q) \times \mathbb{P}^3(\mathbb{F}_q),$$

and a projection map

$$\pi : U(\mathbb{F}_q) \rightarrow M(\mathbb{F}_q).$$

For $p \neq 3$, the discriminant Δ continues to characterize singular polynomials, so $M(\mathbb{F}_q)$ is the space of smooth cubic surfaces defined over \mathbb{F}_q (where a homogeneous cubic polynomial is

smooth if it is smooth at all $\overline{\mathbb{F}}_q$ points). Similarly, $U(\mathbb{F}_q)$ is the space of pairs (S, p) of smooth cubic surfaces S and points p defined over \mathbb{F}_q such that $p \in S$. Thus, $\frac{\#U(\mathbb{F}_q)}{\#M(\mathbb{F}_q)}$ is the average number of \mathbb{F}_q points on a cubic surface defined over \mathbb{F}_q .

For a smooth quasiprojective variety Y , the \mathbb{F}_q points are exactly the fixed points of Frob_q on $Y(\overline{\mathbb{F}}_q)$ and $\#Y(\mathbb{F}_q)$ is determined by the Grothendieck–Lefschetz fixed point formula (see e.g. [Mil13]):

$$\#Y(\mathbb{F}_q) = q^{\dim Y} \sum_{i \geq 0} (-1)^i \text{Tr}(\text{Frob}_q : H_{\text{ét}}^i(Y; \mathbb{Q}_\ell)^\vee), \quad (3.1)$$

where ℓ is a prime other than p . Further, there are comparison theorems implying isomorphisms

$$H_{\text{ét}}^i(Y; \mathbb{Q}_\ell) \cong H^i(Y(\mathbb{C}); \mathbb{Q}_\ell) \cong H^i(Y(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}_\ell,$$

away from a finite set of characteristics (see e.g. [Del77, Théorème 1.4.6.3, Théorème 7.1.9]).

This formula lets us use our results to deduce consequences about $\#U(\mathbb{F}_q)$.

Applying the fixed-point formula to each S we get $\#S(\mathbb{F}_q) = q^2 + (t + 1)q + 1$, where t is the trace of Frobenius on the complement of $\eta \in H^2(S)$ described above. Frobenius must act on $H^2(S)$ by some element of $W(E_6)$, so the possible values of t are given by the character of $W(E_6)$ on the fundamental representation, namely the set $\{-3, -2, -1, 0, 1, 2, 3, 4, 6\}$.

Fixing a q , the average number of points over all S has to be $q^2 + q + 1 + q(t_{\text{average}})$. Theorem 3.2 shows that $t_{\text{average}} = 0$.

Proof of Theorem 3.2. By the Grothendieck–Lefschetz fixed point formula (Eq. (3.1)) and Theorem 3.3 we obtain

$$\#U(\mathbb{F}_q) = \#(M \times \mathbb{P}^2)(\mathbb{F}_q) = q^4 \cdot (\#\text{PGL}(4, \mathbb{F}_q)) \cdot (\#\mathbb{P}^2(\mathbb{F}_q)).$$

Similarly from Theorem 3.10,

$$\#M(\mathbb{F}_q) = q^4 \cdot (\#\text{PGL}(4, \mathbb{F}_q)).$$

Dividing produces the average number $\#\mathbb{P}^2(\mathbb{F}_q) = q^2 + q + 1$. □

Remark 3.8. The q^4 factor in these formulas arises from the difference in dimensions of M and $\mathrm{PGL}(4)$.

3.2 Rational cohomology of the incidence variety

3.2.1 Definitions and setup

Much of the following is analogous to Section 2.2, although here we are looking at the incidence variety of smooth cubic surfaces and points instead of *lines*. From now on we will work over the field \mathbb{C} of complex numbers.

Let $X = X_{3,3}$ be the space of *smooth* homogeneous degree 3 (complex) polynomials in 4 variables, seen as a subset of $\mathbb{C}[x, y, z, w]_3 \cong \mathbb{C}^{20}$. It will be important for us to note that smoothness of such a polynomial is defined by a ‘discriminant’ Δ : there is a homogeneous polynomial $\Delta : \mathbb{C}^{20} \rightarrow \mathbb{C}$ with integer coefficients so that a polynomial $F \in \mathbb{C}[x, y, z, w]_3$ is *not smooth* if and only if $\Delta(F) = 0$. Denoting the *discriminant locus* by $\Sigma = \mathcal{V}(\Delta) \subset \mathbb{C}^{20}$,

$$X = \mathbb{C}^{20} \setminus \Sigma \subset \mathbb{C}^{20} \setminus \{0\}.$$

Two polynomials F and F' in $\mathbb{C}[x, y, z, w]_3$ define the same cubic surface ($\mathcal{V}(F) = \mathcal{V}(F')$) exactly when they are scalar multiples, that is, $F' = \lambda F$ for some $\lambda \in \mathbb{C}^\times$. Further, F is smooth if and only if λF is smooth. Thus we can quotient by \mathbb{C}^\times and get the space

$$M = X/\mathbb{C}^\times \subset \mathbb{P}^{19}$$

of smooth cubic surfaces.

Next we have the ‘incidence variety’ of cubic polynomials and points

$$\Pi = \{(F, p) \mid F(p) = 0\} \subset \mathbb{C}[x, y, z, w]_3 \times \mathbb{P}^3.$$

The preimage of X under the projection $\pi : (F, p) \mapsto F$ is the incidence variety of *smooth* polynomials and points and will be denoted by \tilde{X} . Again taking the quotient by \mathbb{C}^\times , we get the

incidence variety of smooth cubic surfaces and points:

$$U = \tilde{X}/\mathbb{C}^\times = \{(S, p) \in M \times \mathbb{P}^3 \mid p \in S\} \subset M \times \mathbb{P}^3. \quad (3.2)$$

The projection $U \rightarrow M$ is a fiber bundle, which we will also denote by π .

Each of the incidence varieties also comes equipped with another projection, to \mathbb{P}^3 ; each of these maps is in fact a fiber bundle ($\Pi \rightarrow \mathbb{P}^3$ happens to be a vector bundle). We will denote the fiber over $p \in \mathbb{P}^3$ in Π , \tilde{X} and U by $\Pi_p \cong \mathbb{C}^{19}$, X_p and U_p respectively. To be explicit, Π_p is the space of (not necessarily smooth) cubic polynomials that vanish at p , X_p is the subset of *smooth* cubic polynomials that vanish at p and U_p is the space of smooth cubic *surfaces* that contain p .

All the spaces and maps above far fit into the following commuting diagram:

$$\begin{array}{ccccc}
 X_p & \xrightarrow{\mathbb{C}^\times} & U_p & & \\
 \searrow & & \searrow & & \\
 & & \tilde{X} & \xrightarrow{\mathbb{C}^\times} & U \\
 \swarrow & & \downarrow & & \downarrow \\
 X & \xrightarrow{\mathbb{C}^\times} & M & & \mathbb{P}^3 \\
 \swarrow & & \swarrow & & \swarrow \\
 & & \mathbb{P}^3 & \xlongequal{\quad} & \mathbb{P}^3
 \end{array} \quad (3.3)$$

The actions of $GL(4) := GL(4, \mathbb{C})$ on \mathbb{C}^4 and $PGL(4) = GL(4)/(\mathbb{C}^\times I)$ on \mathbb{P}^3 induce actions on the spaces defined above: on Π , X and \tilde{X} by $GL(4)$; on M and U by $PGL(4)$. Fixing a point $p \in \mathbb{P}^3$, the respective stabilizers in $GL(4)$ and $PGL(4)$ act on the fibers X_p and U_p . Choose a basepoint $(F_0, p_0) \in \tilde{X}$ and set $S_0 = \mathcal{V}(F_0)$ so that $(S_0, p_0) \in U$. Then the actions produce orbit maps $g \mapsto g(S_0, p_0) = (g \cdot S_0, g \cdot p_0)$ and so on. Since all the actions are compatible by construction, we also have the following commuting ‘cube’:

$$\begin{array}{ccccccc}
 \text{Stab}_{GL(4)}(p) & \longrightarrow & X_p & \xrightarrow{\quad} & & & \\
 \downarrow & & \downarrow & \searrow & & & \\
 \text{Stab}_{PGL(4)}(p) & \longrightarrow & U_p & \xrightarrow{\quad} & GL(4) & \longrightarrow & \tilde{X} \\
 & & \searrow & & \downarrow & & \downarrow \\
 & & & & PGL(4) & \longrightarrow & U
 \end{array} \quad (3.4)$$

All the horizontal maps are orbit maps, all the vertical maps are quotients by \mathbb{C}^\times and the diagonal dotted maps are inclusions of fibers over $p \in \mathbb{P}^3$.

Remark 3.9. Since \tilde{X} and X_p are connected, a different choice of basepoint $(F_0, p_0) \in \tilde{X}$ does not change the orbit maps up to homotopy.

Theorem 3.10 (Vassiliev [Vas99], Peters–Steenbrink [PS03]). *The map $\mathrm{PGL}(4) \rightarrow M$ given by $g \mapsto g(S_0)$ induces an isomorphism*

$$H^*(M; \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{PGL}(4); \mathbb{Q}).$$

3.2.2 Proof of Theorem 3.3 and the role of simplicial resolution

Vassiliev’s computation of $H^*(M; \mathbb{Q})$ and $H^*(X; \mathbb{Q})$ (as in [Vas99]) starts with a reduction, via Alexander duality, to computing the (Borel–Moore) homology of the discriminant locus Σ . The space Σ , the set of singular cubic surfaces, is itself highly singular and stratifies based on the singular locus of an $F \in \Sigma$. This stratification then produces a spectral sequence converging to $\bar{H}_*(\Sigma) = H_*^{\mathrm{BM}}(\Sigma)$ (Borel–Moore or compactly supported homology).

Similar to the proof of Theorem 2.1, we apply the same methods to each fiber X_p , over p , of the map $\tilde{X} \rightarrow \mathbb{P}^3$, since each is a ‘discriminant complement’ in the vector space Π_p of polynomials vanishing at p . We then need to stratify $\Sigma_p = \Sigma \cap \Pi_p$ not just by what the singular loci are as subsets of \mathbb{P}^3 , but also how they are configured with respect to the point p . These are the types and subtypes described in Section 3.3.1. For now we will assume that we can perform this computation (which takes up all of Section 3.3) and when needed we refer to the answer described in Proposition 3.27.

The following Lemma 3.11 is relatively elementary but included for the reader’s convenience:

Lemma 3.11. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a non-constant homogeneous polynomial of degree d , so that $\mathcal{V}(f)$ is a conical hypersurface; denote its complement by $Y = \mathbb{C}^n \setminus \mathcal{V}(f)$. Let $\mathbb{P}Y = Y/\mathbb{C}^\times =$*

$\mathbb{P}^{n-1} \setminus \mathcal{V}_{\mathbb{P}}(f)$ be the complement of the projective hypersurface given by the same polynomial f . Then $H^*(Y; \mathbb{Q}) \cong H^*(\mathbb{C}^\times) \otimes H^*(\mathbb{P}Y)$.

Lemma 3.12. For a fixed $p \in \mathbb{P}^3$, choose a complement hyperplane $H = \mathbb{P}V$, with $V \subset \mathbb{C}^4$. Then $\mathrm{GL}(V) \subset \mathrm{Stab}_{\mathrm{GL}(4)}(p)$ acts on X_p . For a choice of basepoint $F_0 \in X_p$, the orbit map $\mathrm{GL}(V) \rightarrow X_p$ given by $g \mapsto g(F_0) = F_0 \circ g$ induces a surjection

$$H^*(X_p; \mathbb{Q}) \twoheadrightarrow H^*(\mathrm{GL}(V); \mathbb{Q}) \cong H^*(\mathrm{GL}(3); \mathbb{Q}).$$

Proof. Choose a basis of V and denote the corresponding projective flag by $P \in L \subset H$. This identifies $\mathrm{GL}(V)$ with $\mathrm{GL}(3, \mathbb{C})$.

As in the computation of $H^*(X_p; \mathbb{Q})$ in Section 3.3, it is important to identify, via Alexander duality, $H^*(X_p; \mathbb{Q})$ with $\bar{H}_*(\Sigma_p)$ and similarly $H^*(\mathrm{GL}(3); \mathbb{Q})$ with $\bar{H}_*(\mathrm{Mat}(3) \setminus \mathrm{GL}(3))$, where $\mathrm{Mat}(3)$ is the space of all 3×3 matrices. The generators of $H^*(\mathrm{GL}(3); \mathbb{Q})$ (as a ring) are represented by the locus of matrices whose first i columns are linearly dependent¹, for $i = 1, 2, 3$.

The orbit map extends to a map

$$\mathrm{Mat}(3) \rightarrow \Pi_p = X_p \cup \Sigma_p.$$

It is enough to identify subspaces of Σ_p that pull back to (a rational multiple of) the corresponding subspaces of $\mathrm{Mat}(2) \times \mathrm{Mat}(2)$. Then by the proof of [PS03, Lemma 7] (which is the analogous statement for all singular polynomials, while X_p restricts to polynomials vanishing at p), appropriate choices of subspaces are the sets of polynomials that are: (i) singular at P , (ii) singular at some point of L , (iii) singular at some point of H . \square

Remark 3.13. The stabilizer of p in $\mathrm{PGL}(4)$ deformation retracts to $\mathrm{Stab}(p, H) \cong \mathrm{GL}(V)$, for any choice of complement $H = \mathbb{P}V$ above. However, there isn't a way of extending the action of $\mathrm{GL}(V)$ on X_p to an action of $\mathrm{PGL}(4)$ on \tilde{X} .

¹For $i = 1$ this means the first column is 0. This description of the generators generalizes to $\mathrm{GL}(n) \subset M(n)$.

This allows us to apply Leray–Hirsch to $X_p \rightarrow X_p/\mathrm{GL}(V)$ by [PS03, Theorem 2]. Knowing the Betti numbers of X_p from Proposition 3.27 and using Lemma 3.11 to move from X_p to U_p , we get the following.

Corollary 3.14. *As rings,*

$$H^*(X_p; \mathbb{Q}) \cong H^*(S^1 \times S^3 \times S^5 \times S^5; \mathbb{Q})$$

and

$$H^*(U_p; \mathbb{Q}) \cong H^*(S^3 \times S^5 \times S^5; \mathbb{Q}).$$

Now we can prove Theorem 3.3.

Proof of Theorem 3.3. Let us suppress rational coefficients for brevity. Setting

$$G_p = \mathrm{Stab}_{\mathrm{PGL}(4)}(p) \simeq \mathrm{GL}(3),$$

we have maps of bundles (as in (3.4)):

$$\begin{array}{ccccc} G_p & \longrightarrow & U_p & \longrightarrow & M \\ & \searrow & \searrow & & \searrow \\ & & \mathrm{PGL}(4) & \longrightarrow & U & \longrightarrow & M \times \mathbb{P}^3 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{P}^3 & \xlongequal{\quad} & \mathbb{P}^3 & \xlongequal{\quad} & \mathbb{P}^3 \end{array}$$

The pair of horizontal maps on the left are orbit maps as described above and the pair on the right are inclusions.

Since the base is simply connected, we get spectral sequences

$$H^p(\mathbb{P}^3) \otimes H^q(G_p) \implies H^{p+q}(\mathrm{PGL}(4)); \quad H^p(\mathbb{P}^3) \otimes H^q(U_p) \implies H^{p+q}(U)$$

and, since the last bundle is trivial,

$$\bigoplus_{p+q=d} H^p(\mathbb{P}^3) \otimes H^q(M) \cong H^d(M \times \mathbb{P}^3).$$

Alternatively, all the differentials in the spectral sequence for the third bundle are 0. Since the Serre spectral sequence is natural, this will help us compute the differentials in the case of U . We will also use our knowledge of the differentials in the $\mathrm{PGL}(4)$ case.

By Theorem 3.10, $H^*(M)$ is isomorphic to $H^*(\mathrm{PGL}(4))$ via the orbit map. This implies that the map $G_p \rightarrow M$ induces isomorphisms:

$$H^3(M) \cong H^3(G_p); \quad H^5(M) \cong H^5(G_p).$$

In particular, the map

$$\mathbb{Q} \cong H^3(M) \rightarrow H^3(U_p) \cong \mathbb{Q}$$

is an isomorphism and

$$\mathbb{Q} \cong H^5(M) \rightarrow H^5(U_p) \cong \mathbb{Q}^2$$

is injective. Now, by the Leibniz rule for differentials in the Serre spectral sequence and the description of $H^*(U_p)$ from Corollary 3.14, it is enough to find the ranks of the differentials (see Fig. 3.1)

$$d_4 : H^3(U_p) \rightarrow H^4(\mathbb{P}^3) \quad \text{and} \quad d_6 : H^5(U_p) \rightarrow H^6(\mathbb{P}^3).$$

By the isomorphism $H^3(M) \cong H^3(U_p)$, the differential d_4 vanishes but the injection $H^5(M) \rightarrow H^5(U_p)$ is not enough to determine if the differential d_6 has rank 1 or 0 (although the image of $H^5(M)$ must be in the kernel of d_6).

Since we are considering field coefficients, $H^*(U)$ is isomorphic to the associated graded as a vector space. Thus, the Poincaré polynomial of U is either

$$(1 + t^3)(1 + t^5)(1 + t^7)(1 + t^2 + t^4)$$

or

$$(1 + t^3)(1 + t^5)^2(1 + t^2 + t^4 + t^6).$$

But the map $H^*(U) \rightarrow H^*(\mathrm{PGL}(4))$ is surjective (since the map $H^*(M) \rightarrow H^*(\mathrm{PGL}(4))$ is), so by yet another application of [PS03, Theorem 2], the Poincaré polynomial of U must be divisible

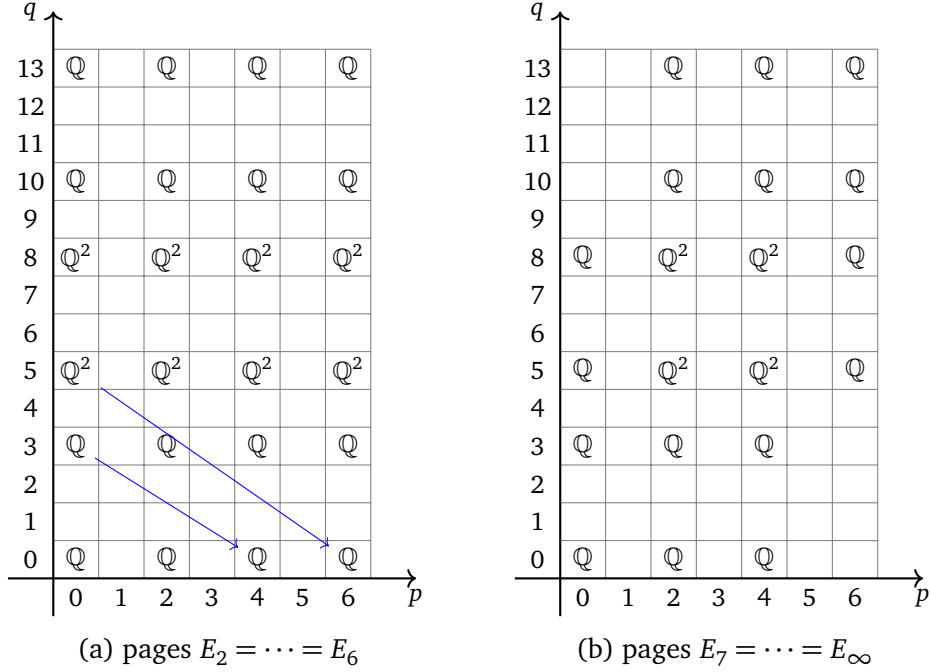


Figure 3.1: Serre spectral sequence for the bundle $U \rightarrow \mathbb{P}^3$. Only potentially non-zero differentials on *generating degrees* are shown.

by that of $\mathrm{PGL}(4)$, in particular by $(1 + t^7)$. This implies that the rank of d^6 is 1 and that $H^6(\mathbb{P}^3)$ is in the kernel of the pullback map for the fiber bundle $U \rightarrow \mathbb{P}^3$.

Finally, to establish the ring structure, it is enough to note that in addition to the version of Leray–Hirsch from [PS03], the generators in degrees 3, 5 and 7 cannot have any relations except those forced by graded commutativity since this is true for their images in $H^*(\mathrm{PGL}(4))$. \square

3.3 Rational cohomology of X_p

3.3.1 Definitions and plan of attack

We will suppress constant rational coefficients throughout this section and use \bar{H} to denote Borel–Moore homology. Recall that for an orientable but not necessarily compact $2n$ -manifold M , Poincaré duality takes the form

$$\bar{H}_i(M) \cong H^{2n-i}(M) \cong (H_{2n-i}(M))^\vee \cong (H_c^i(M))^\vee.$$

Recall that $X_p \subset \Pi_p \cong \mathbb{C}^{19}$ and set $\Sigma_p = \Pi_p \setminus X_p = \Pi_p \cap \Sigma$, the set of *singular* cubic polynomials that vanish at the point p . Then by Alexander duality,

$$\tilde{H}^i(X_p) = \bar{H}_{37-i}(\Sigma_p). \quad (3.5)$$

Remark 3.15. The ‘discriminant locus’ Σ_p is a conical hypersurface in Π_p , being the vanishing locus of $\Delta_p = \Delta|_{\Pi_p}$. The complex variety X_p , being the complement of a hypersurface, is affine and hence a 19-dimensional Stein manifold. Thus by the Andreotti–Frankel theorem, $H^i(X_p) = 0$ for $i > 19$. Hence, by Eq. (3.5), $\bar{H}_i(\Sigma_p)$ can only be non-zero for $18 \leq i \leq 37$.

Let $F \in \Sigma_p$ be a singular cubic polynomial and let K be its singular locus. Then K , as a subset of \mathbb{P}^3 , can be one of the following 11 *types* (see [Vas99, Proposition 8]):

- (I) a point
- (II) two distinct points
- (III) a line
- (IV) three points, not on a line
- (V) a smooth conic contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$
- (VI) a pair of intersecting lines
- (VII) four points, not on a plane
- (VIII) a plane
- (IX) three lines through a point, not all on the same plane
- (X) a smooth conic contained in a plane along with another point not on that plane
- (XI) all of \mathbb{P}^3

These can be further classified into *subtypes* depending on their configuration with respect to the marked point p . This will not be relevant for most of the types; we list those that are relevant. The names P, Q etc. below for the points are for convenience, the sets of points are a priori *unordered*: $\{P, Q\} = \{Q, P\}$ and so on.

- (I) a point P

- (a) $P = p$
 - (b) $P \neq p$
- (II) two points P, Q
- (a) $P = p$
 - (b) P and Q collinear with $p, P, Q \neq p$
 - (c) P and Q not collinear with p
- (IV) three points P, Q, R , not collinear
- (a) $P = p$
 - (b) P and Q collinear with $p, P, Q \neq p$
 - (c) P, Q and R coplanar with p , no two collinear with p
 - (d) P, Q and R not coplanar with p
- (VII) four points P, Q, R, S , not coplanar
- (a) $P = p$
 - (b) P and Q collinear with $p, P, Q \neq p$
 - (c) P, Q and R coplanar with p , no two collinear with p
 - (d) no three coplanar with p

Remark 3.16. The *types* correspond to singular loci that are equivalent under the $\mathrm{PGL}(4)$ -action on \mathbb{P}^3 and the *subtypes* correspond to equivalence under the action of the subgroup $\mathrm{Stab}(p) \subset \mathrm{PGL}(4)$. However, this will not be explicitly important for us.

Definition 3.17. For a singular locus K , denote by $L(K)$ the set of all polynomials in Σ_p that are singular on all of K (and perhaps elsewhere as well). This is a vector space for any $K \subset \mathbb{P}^3$.

Remark 3.18. The subtypes are (partially) ordered by degeneracy: $i \leq j$ if polynomials with singularity of subtype i can degenerate to a polynomial with singularity of subtype j . In the following we need to choose a rank function (i.e., monotonic integer-valued map) on this poset,

we use

$$\deg(i) = 16 - \dim L(K)$$

for any K of subtype i .

Definition 3.19. For a manifold M and natural number n , the *ordered configuration space* of n points on M is given by

$$\text{PConf}_n(M) := \{(a_1, \dots, a_n) \in M^n \mid a_i \neq a_j \text{ for } i \neq j\}.$$

This space comes with a natural action of the symmetric group \mathfrak{S}_n by permuting the coordinates and the quotient is the *unordered configuration space* $\text{UConf}_n(M)$ of n points on M .

Definition 3.20. For any $A \subseteq \text{UConf}_n(M)$, the *sign local coefficients* on A , denoted by $\pm\mathbb{Q}$, is given by the composition

$$\pi_1(A) \rightarrow \pi_1(\text{UConf}_n(M)) \rightarrow \mathfrak{S}_n \rightarrow \{\pm 1\} \subset \mathbb{Q}^\times$$

thought of as a representation on \mathbb{Q} . Explicitly, a loop in A acts on \mathbb{Q} by the sign of the induced permutation on the n points.

The method of simplicial resolution ultimately produces for us a spectral sequence

$$E_{*,*} \implies \bar{H}_*(\Sigma_p)$$

with the E^1 page described below. For slightly more details see an entirely analogous description in Section 2.3.1; for proofs and constructions, see [Vas99].

Let the index i vary over all the *subtypes* (not just the ones listed, but all of them). Define

$$A_i := \{\text{singular sets } K \text{ of subtype } i\}.$$

Example 3.21. For the subtype **Ila**, a point $P = p$ and a point $Q \neq p$, we have $A_{\text{Ila}} = \{p\} \times (\mathbb{P}^3 \setminus \{p\})$. For the subtype **Ilc**, two points not collinear with p , the space A_{Ilc} is an open set in $\text{UConf}_2(\mathbb{P}^3 \setminus \{p\})$.

There are spaces F_i so that the E^1 page is given by

$$E_{p,q}^1 = \bigoplus_{\deg(i)=p} \bar{H}_{p+q}(F_i). \quad (3.6)$$

There are further spaces Φ_i and $\Lambda(K)$ as well as fiber bundles:

$$\begin{array}{ccc} L(K) & \hookrightarrow & F_i \\ & & \downarrow \\ \Lambda(K) & \hookrightarrow & \Phi_i \\ & & \downarrow \\ & & K \in A_i \end{array}$$

So to compute $\bar{H}_*(F_i)$, we can use the Thom isomorphism

$$\bar{H}_*(F_i) = \bar{H}_{*-2 \dim_{\mathbb{C}} L(K)}(\Phi_i). \quad (3.7)$$

Unless i is a subtype of **I**, **II**, **IV**, **VII** or **XI**, $\bar{H}_*(\Lambda(K)) = 0$ ([Vas99, proof of Proposition 9]) and hence $\bar{H}_*(\Phi_i) = 0$. Now suppose i is a subtype of **I**, **II**, **IV** or **VII**, i.e. $K \in A_i$ is a finite set of say n points. Then A_i is a subset of $\text{UConf}_n(\mathbb{P}^3)$ and

$$\bar{H}_*(\Phi_i) = \bar{H}_{*-n+1}(A_i; \pm\mathbb{Q}) = H^{2 \dim_{\mathbb{C}} A_i + n - 1 - *}(A_i; \pm\mathbb{Q}). \quad (3.8)$$

For the type **XI** (note that **XI** has only one subtype, itself), A_{XI} is singleton, the only element being $K = \mathbb{P}^3$. The only polynomial singular on K is 0, so $L(K) = \{0\}$. Thus $F_{\text{XI}} = \Phi_{\text{XI}} = \Lambda(\mathbb{P}^3)$. Further, the space $\Phi_{\text{XI}} = \Lambda(\mathbb{P}^3)$ is the open cone $\mathring{C}Z$ over

$$Z = \bigcup_{j \neq \text{XI}} \Phi_j$$

for certain gluings. So we get a spectral sequence $e_{p,q}^r \implies H_{p+q}(Z)$ with

$$e_{p,q}^1 = \bigoplus_{\substack{\deg(j)=p, \\ j \neq \text{XI}}} \bar{H}_{p+q}(\Phi_j).$$

But then we also have

$$\bar{H}_*(\mathring{C}Z) = H_*(CZ, Z) = \tilde{H}_{*-1}(Z).$$

So the computation eventually reduces to computing $H^*(A_i; \pm\mathbb{Q})$ for the various subtypes of I, II, IV and VII (see Propositions 3.25 and 3.26) followed by bookkeeping and relatively standard arguments involving spectral sequences following [Vas99] (see Proposition 3.27).

Remark 3.22. We could keep track of the mixed Hodge structures throughout the entire computation, as in [Tom05; Tom14] (see also [Gor05]), but this ends up being unnecessary for our purposes.

3.3.2 Case work

This section contains the details of the arguments to compute the various $H^*(A_i; \pm\mathbb{Q})$. The main idea is decomposing these spaces as fiber bundles, where both the fiber and base are simpler. In many instances the bases are A_j for some lower j and the computation is ‘inductive’ or recursive.

First, a couple of general facts that we will use freely in the computation below:

Lemma 3.23. *Let $H \cong \mathbb{P}^k$ be a k -dimensional linear subspace of \mathbb{P}^n for some $0 \leq k \leq n$ and let H^\perp be the (projectivized) orthogonal complement of H . Then $\mathbb{P}^n \setminus H$ deformation retracts to $H^\perp \cong \mathbb{P}^{n-k-1}$.*

Lemma 3.24.

$$H^*(\mathrm{UConf}_2(\mathbb{C}); \pm\mathbb{Q}) = 0.$$

$$H^*(\mathrm{UConf}_2(\mathbb{P}^2); \pm\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * = 2, 4, 6; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $\mathrm{UConf}_2(\mathbb{C})$, we can use that $\mathrm{PConf}_2(\mathbb{R}^{2n}) \simeq S^{2n-1}$ and the \mathfrak{S}_2 action is by the antipodal map, which is degree 1. Hence, by transfer, $H^*(\mathrm{UConf}_2(\mathbb{R}^{2n}); \pm\mathbb{Q}) = 0$. For $H^*(\mathrm{UConf}_2(\mathbb{P}^2); \pm\mathbb{Q})$ see [Vas99, Lemma 2B]. □

Now we establish the cases where $H^*(A_i; \pm\mathbb{Q}) = 0$, the recursive nature of the argument makes some of the cases relatively easy. The remaining cases are treated in Proposition 3.26.

Proposition 3.25. *If i is **IIb**, **IVb**, **IVc**, **VIIb**, **VIIc** or **VIIId** then $H^*(A_i; \pm\mathbb{Q}) = 0$.*

Proof. Let's deal with each case in turn.

IIb. $P, Q \neq p$, **but P, Q and p collinear** Mapping $\{P, Q\} \mapsto L = \langle P, Q, p \rangle$, the projective span of P, Q and p , i.e. the line containing P, Q and p , we get a map from A_{IIb} to the space of lines in \mathbb{P}^3 containing p , which is a $\mathbb{P}^2 \subset G(1, 3)$. This is a fiber bundle

$$\begin{array}{ccc} \text{UConf}_2(L \setminus p) & \hookrightarrow & A_{\text{IIb}} \\ & & \downarrow \\ & & \mathbb{P}^2 \end{array}$$

and the local coefficients $\pm\mathbb{Q}$ restricts to the fiber to the sign local coefficient on $\text{UConf}_2(L \setminus p) \cong \text{UConf}_2(\mathbb{C})$. But $H^*(\text{UConf}_2(\mathbb{C}), \pm\mathbb{Q}) = 0$ from Lemma 3.24, so we are done.

IVb. $P, Q, R \neq p$, **P, Q and p collinear, but R not on that line** Here, even though P, Q and R are a priori unordered, we can't (continuously) interchange R with one of P and Q . So there is a well-defined map $\{P, Q, R\} \mapsto \{Q, R\}$ and we get a fiber bundle:

$$\begin{array}{ccc} \mathbb{P}^3 \setminus \mathbb{P}^1 \cong \mathbb{P}^3 \setminus \langle P, Q, p \rangle & \hookrightarrow & A_{\text{IVb}} \\ & & \downarrow \\ & & A_{\text{IIb}} \end{array}$$

The local coefficients $\pm\mathbb{Q}$ on the total space pulls back from $\pm\mathbb{Q}$ on base (that is, the map $\pi_1(A_{\text{IVb}}) \rightarrow \{\pm 1\}$ factors through $\pi_1(A_{\text{IIb}})$). But as we just showed, $H^*(A_{\text{IIb}}; \pm\mathbb{Q}) = 0$, so we are done.

IVc. $P, Q, R \neq p$, **coplanar with p and no three of P, Q, R and p collinear** Mapping

$$\{P, Q, R\} \mapsto H = \langle P, Q, R, p \rangle,$$

we get a fiber bundle:

$$\begin{array}{ccc} F & \hookrightarrow & A_{\text{IVc}} \\ & & \downarrow \\ & & \mathbb{P}^2 \end{array}$$

The fiber is the space of three (unordered) non-collinear points on $H \setminus \{p\}$ and the local coefficients $\pm\mathbb{Q}$ restricts to the local coefficients $\pm\mathbb{Q}$ on $F \subset \text{UConf}_3(\mathbb{P}^2)$. Since $\pi_1(F) \rightarrow \{\pm 1\}$ factors through \mathfrak{S}_3 , we can go to the associated \mathfrak{S}_3 cover $\tilde{F} \subset \text{PConf}_3(\mathbb{P}^2)$ and then, by transfer, $H^*(F; \pm\mathbb{Q})$ is the summand of $H^*(\tilde{F}; \mathbb{Q})$ where \mathfrak{S}_3 acts by the sign representation.

But \tilde{F} can be identified with the fiber of $(P, Q, R, S) \mapsto S$, where (P, Q, R, S) varies over all tuples in $\text{PConf}_4(\mathbb{P}^2)$ so that no three are collinear. But $\text{PGL}(3)$ acts freely and transitively on this open subset of $\text{PConf}_4(\mathbb{P}^2)$ and hence we have a fiber bundle:

$$\begin{array}{ccc} \tilde{F} & \hookrightarrow & \text{PGL}(3) \\ & & \downarrow \\ & & \mathbb{P}^2 \end{array}$$

The action of \mathfrak{S}_3 extends to $\text{PConf}_4(\mathbb{P}^2)$, permuting the first three points, so the action on the base is trivial. The action on the total space extends to the action of the entire (connected) group $\text{PGL}(3)$ by right multiplication, so is trivial on homology. As a result, the \mathfrak{S}_3 action on $H^*(\tilde{F}; \mathbb{Q})$ is trivial, which implies $H^*(F; \pm\mathbb{Q}) = 0$, as needed.

VIIb. P, Q, R and S not coplanar, P and Q collinear with p Note that if the four points are not coplanar, at most one pair can be collinear with p , so this determines the subset $\{P, Q\} \subset \{P, Q, R, S\}$. The line $L = \langle P, Q, p \rangle$ can be any line through p that is not on the plane $\langle R, S, p \rangle$ and fixing L , P and Q vary exactly in $\text{UConf}_2(L \setminus \{p\}) \cong \text{UConf}_2(\mathbb{C})$. Thus mapping $\{P, Q, R, S\} \mapsto (L, \{R, S\})$ we get a fiber bundle:

$$\begin{array}{ccc} \text{UConf}_2(L \setminus \{p\}) & \hookrightarrow & A_{\text{VIIb}} \\ & & \downarrow \\ & & \{(L, \{R, S\})\} \end{array}$$

But again $H^*(\text{UConf}_2(\mathbb{C}), \pm\mathbb{Q}) = 0$ from Lemma 3.24, so we are done.

VIIc. P, Q and R but not S coplanar with p , and no two collinear with p Mapping

$$\{P, Q, R, S\} \mapsto \{P, Q, R\}$$

we get a fiber bundle:

$$\begin{array}{ccc} \mathbb{C}^3 \cong \mathbb{P}^3 \setminus \langle P, Q, R \rangle & \hookrightarrow & A_{\text{VIIc}} \\ & & \downarrow \\ & & A_{\text{IVc}} \end{array}$$

Since $H^*(A_{\text{IVc}}, \pm\mathbb{Q}) = 0$ by previous arguments, we are done.

VIIId. $P, Q, R, S \neq p$, **no three coplanar with p** By an argument analogous to the case of **IVc**, A_{VIIId} has an \mathfrak{S}_4 cover by ordering the four points. This cover is the fiber of the bundle $\text{PGL}(3) \rightarrow \mathbb{P}^3$, where $\text{PGL}(3)$ is identified with five (ordered) points in \mathbb{P}^3 , no four of which are coplanar, by its free and transitive action. The action of \mathfrak{S}_4 is again trivial on the base and on $H^*(\text{PGL}(3))$, so we are done. \square

Recall that we have spectral sequences $E_{p,q}^r \implies \bar{H}_{p+q}(\sigma)$ and $e_{p,q}^r$ that let us compute $\bar{H}_*(F_{\text{XI}}) = \tilde{H}_{*-1}(Z)$, where

$$Z = \bigcup_{i \neq \text{XI}} \Phi_i.$$

Proposition 3.26. *The spectral sequence $E_{p,q}^r \implies \bar{H}_{p+q}(\sigma)$ has the page $E_{p,q}^1$ as in Fig. 3.2. The spectral sequence $e_{p,q}^r \implies H_{p+q}(Z)$ has the page $e_{p,q}^1$ as in Fig. 3.3.*

Proof. Recall that by construction, the terms of E^1 and e^1 are related by Thom isomorphisms

$$E_{p,q+2(16-p)}^1 \cong e_{p,q}^1,$$

except for $p = 16$, where $e_{16,*}^1 \equiv 0$. So we first go through more case work to establish columns $p \neq 16$.

By Eqs. (3.6) to (3.8) and careful bookkeeping, it is enough to find $H^*(A_i; \pm\mathbb{Q})$ along with the numbers $\dim(A_i) = \dim_{\mathbb{C}}(A_i)$ and $\dim(L(K)) = \dim_{\mathbb{C}}(L(K))$ for $K \in A_i$, for the subtypes i of **I**, **II**, **IV** and **VII** (see Table 3.1 for the relevant numerics). Further, there are only seven subtypes remaining — the ones not covered in Proposition 3.25.

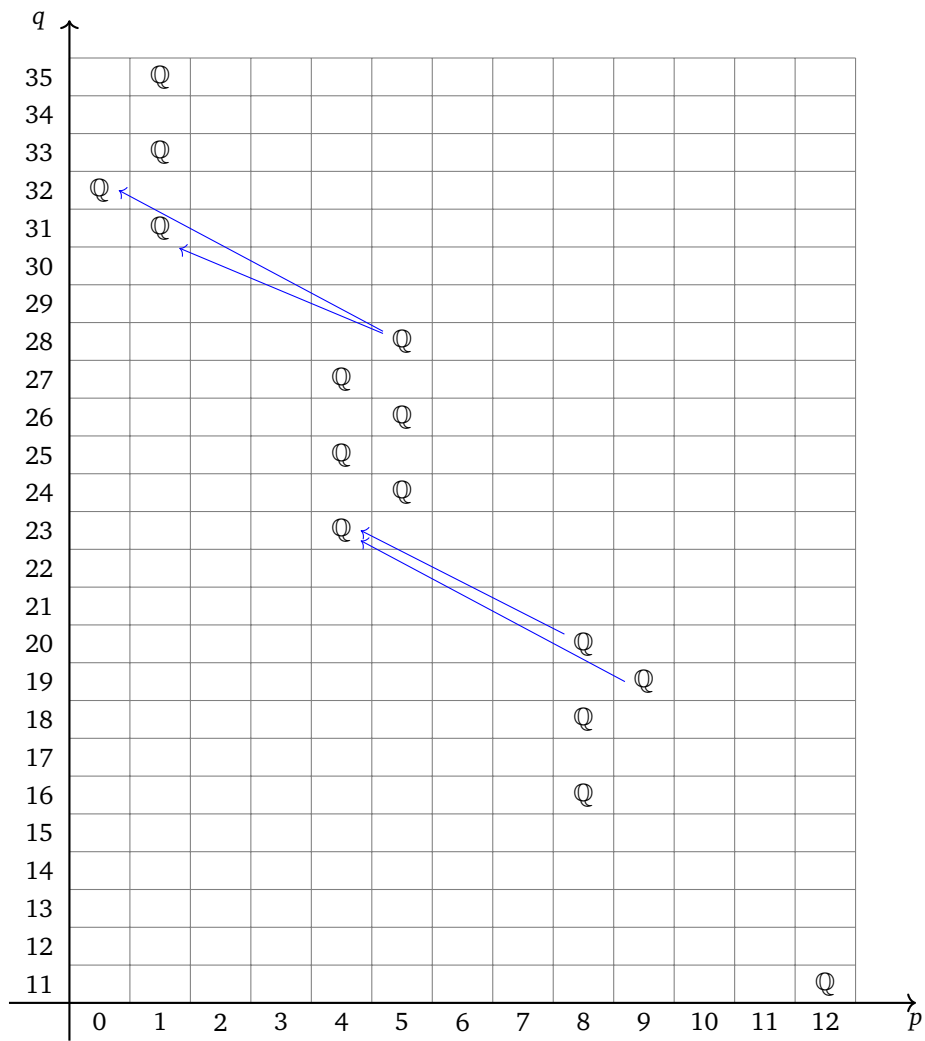


Figure 3.2: Spectral sequence page $E_{p,q}^1$ for $\overline{H}_{p+q}(\sigma)$ (with 0s omitted) and all potentially non-zero differentials in subsequent pages

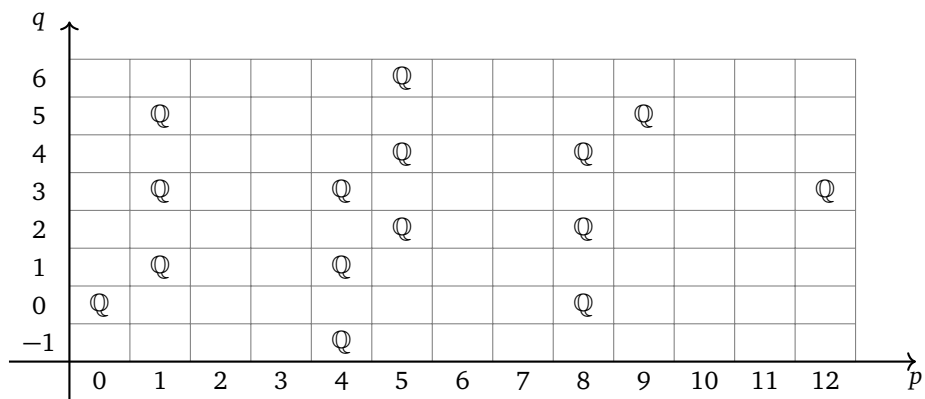


Figure 3.3: Spectral sequence page $e_{p,q}^1$ for $H_{p+q}(Z)$ (with 0s omitted)

i	Ia	Ib	IIa	IIc	IVa	IVd	VIIa	XI
$\dim A_i$	0	3	3	6	6	9	9	0
$\dim L(K)$	16	15	12	11	8	7	4	0

Table 3.1: $\dim A_i$ and $\dim L(K)$ for $K \in A_i$ for each subtype i excepted in Proposition 3.25.

Ia. $P = p$ $A_{Ia} = \{p\}$ and the coefficients $\pm \mathbb{Q}$ are trivial, so

$$H^*(A_{Ia}; \pm \mathbb{Q}) = H^*(\{p\}) = \begin{cases} \mathbb{Q} & \text{if } * = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{0,32}^1 \cong e_{0,0}^1$ since $\dim(A_{Ia}) = 0$ and $\dim(L(K)) = 16$.

Ib. $P \neq p$ $A_{Ib} = \mathbb{P}^3 - p \simeq \mathbb{P}^2$. Again, the coefficients are trivial since there is only one point, so

$$H^*(A_{Ib}; \pm \mathbb{Q}) = H^*(\mathbb{P}^2) = \begin{cases} \mathbb{Q} & \text{if } * = 0, 2, 4; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{1,31}^1 \cong e_{1,1}^1$, $E_{1,33}^1 \cong e_{1,3}^1$ and $E_{1,35}^1 \cong e_{1,5}^1$ since $\dim(A_{Ib}) = 3$ and $\dim(L(K)) = 15$.

IIa. $P = p$, $Q \neq p$ $A_{IIa} = \{p\} \times \mathbb{P}^3 \setminus \{p\} \cong (\mathbb{P}^2)$ and the coefficients are trivial. So,

$$H^*(A_{IIa}; \pm \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 0, 2, 4; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{4,23}^1 \cong e_{4,-1}^1$, $E_{4,25}^1 \cong e_{4,1}^1$ and $E_{4,27}^1 \cong e_{4,3}^1$ since $\dim(A_{IIa}) = 3$ and $\dim(L(K)) = 12$.

IIc. P, Q and p not collinear The three points not being collinear is equivalent to the lines $\langle P, p \rangle$ and $\langle Q, p \rangle$ being distinct (lines through p). Hence mapping $\{P, Q\} \mapsto \{\langle P, p \rangle, \langle Q, p \rangle\}$, we get a fiber bundle

$$\begin{array}{ccc} \mathbb{C}^2 \cong (\langle P, p \rangle \setminus p) \times (\langle Q, p \rangle \setminus p) & \hookrightarrow & A_{IIc} \\ & & \downarrow \\ & & \text{UConf}_2(\mathbb{P}_p^2) \end{array}$$

where $\mathbb{P}_p^2 \cong \mathbb{P}^2$ is the space of lines through p . The coefficients on the total space pull back from $\pm\mathbb{Q}$ coefficients on the base, hence by Lemma 3.24,

$$H^*(A_{\text{IIC}}; \pm\mathbb{Q}) \cong H^*(\text{UConf}_2(\mathbb{P}^2); \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 2, 4, 6; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{5,24}^1 \cong e_{5,2}^1$, $E_{5,26}^1 \cong e_{5,4}^1$ and $E_{5,28}^1 \cong e_{5,6}^1$ since $\dim(A_{\text{IIC}}) = 6$ and $\dim(L(K)) = 11$.

IVa. $P = p$, Q and R not coplanar with p $A_{\text{IVa}} = \{p\} \times A_{\text{IIC}}$ and the coefficients pull back from the $\pm\mathbb{Q}$ coefficients on A_{IIC} . Hence,

$$H^*(A_{\text{IVa}}; \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 2, 4, 6; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{8,16}^1 \cong e_{8,0}^1$, $E_{8,18}^1 \cong e_{8,2}^1$ and $E_{8,20}^1 \cong e_{8,4}^1$ since $\dim(A_{\text{IVa}}) = 6$ and $\dim(L(K)) = 8$.

IVd. P , Q and R not coplanar with p Mapping $\{P, Q, R\} \mapsto \langle P, Q, R \rangle$, we get a fiber bundle whose base is $(\mathbb{P}^3)^\vee \setminus p^\perp \cong \mathbb{C}^3$ and the fiber is the space of non-collinear triples of points in \mathbb{P}^2 , whose cohomology with $\pm\mathbb{Q}$ coefficients is the same as that of $\text{UConf}_3(\mathbb{P}^2)$, by [Vas99, Lemma 4]. Thus, using Lemma 3.24,

$$H^*(A_{\text{IVd}}; \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 6; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{9,19}^1 \cong e_{9,5}^1$ since $\dim(A_{\text{IVd}}) = 9$ and $\dim(L(K)) = 7$.

VIIa. $P = p$, Q , R and S not coplanar with p $A_{\text{VIIa}} = \{p\} \times A_{\text{IVd}}$ and the coefficients pull back from the $\pm\mathbb{Q}$ coefficients on A_{IVd} . Hence

$$H^*(A_{\text{VIIa}}; \pm\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 6; \\ 0 & \text{otherwise.} \end{cases}$$

This contributes to $E_{12,11}^1 \cong e_{12,3}^1$ since $\dim(A_{\text{VIIa}}) = 9$ and $\dim(L(K)) = 4$.

Thus we've computed the pages $e_{p,q}^1$ and $E_{p,q}^1$ except the $p = 16$ column of the latter. For XI, $L(K) = 0$, so $E_{16,q}^1 \cong \bar{H}_{16+q}(\Phi_{\text{XI}})$. Now, if any term with $1 \leq d = p + q \leq 16$ remains non-zero in $e_{p,q}^\infty$, then it would appear as $\bar{H}_{d+1}(\Phi_{\text{XI}})$ and hence as a term $E_{16,d-15}^1$, which cannot interact with any of the other terms, by the shapes of the other columns, which we have already determined. That means $0 \neq \bar{H}_{d+1}(\sigma) \cong \tilde{H}^{37-d}(X_p)$, which is a contradiction with X_p being a 19-dimensional Stein manifold, as in Remark 3.15. This implies, given the shape of $e_{p,q}^1$, that $\bar{H}_*(\Phi_{\text{XI}}) \equiv 0$, so we have also verified $E_{16,*}^1$. \square

Proposition 3.27. *The spectral sequence $E_{p,q}^r$ degenerates at $r = 1$ and hence the (rational) Poincaré polynomials of X_p and U_p are given by:*

$$P(X_p; t) = (1 + t)(1 + t^3)(1 + t^5)^2$$

$$P(U_p; t) = (1 + t^3)(1 + t^5)^2$$

Proof. Recall that $E_{p,q}^r \implies \bar{H}_{p+q}(\sigma) \cong \tilde{H}^{37-p-q}(\sigma)$. The page $E_{p,q}^1$ is quite sparse to begin with, the only potentially non-zero differentials (on any page) are shown in Fig. 3.2. By Lemma 3.11, since $X_p = \Pi_l \setminus \mathcal{V}(\Delta_p)$, we must have

$$P(X_p; t) = P(\mathbb{C}^\times; t)P(U_p; t) = (1 + t)P(U_p; t).$$

This shows that $H^4(X_p) \cong \bar{H}_{33}(\sigma)$ and $H^{10}(X_p) \cong \bar{H}_{27}(\sigma)$ cannot be 0, which means all those differentials must vanish. So $E_{p,q}^\infty \cong E_{p,q}^1$ and there are no extension problems with rational coefficients. It is then straightforward to factor the polynomials in the given manner. \square

Chapter 4

Arithmetic statistics on cubic surfaces

4.1 Introduction

The classical Cayley–Salmon theorem implies that each smooth cubic surface over an algebraically closed field contains exactly 27 lines (see Section 4.2 for detailed definitions). In contrast, for a surface over a finite field \mathbb{F}_q , all 27 lines are defined over $\overline{\mathbb{F}}_q$ but not necessarily over \mathbb{F}_q itself. In other words, the action of the Frobenius Frob_q permutes the 27 lines and only fixes a (possibly empty) subset of them. It is also classical that the full monodromy group of the 27 lines, i.e. the Galois group of an appropriate extension or cover, is isomorphic to the Weyl group $W(E_6)$ of type E_6 .

The Frobenius action on the 27 lines governs much of the arithmetic of the surface S : evidently the pattern of lines defined over \mathbb{F}_q and, less obviously, the number of \mathbb{F}_q points on S (or $\text{UConf}^n S$ etc). Work of Bergvall and Gounelas [BG19] allows us to compute the number of cubic surfaces over \mathbb{F}_q where Frob_q induces a given permutation, or rather a permutation in a given conjugacy class of $W(E_6)$. The point-counting results in this chapter can be thought of as a combinatorial reinterpretation of their computation.

Theorem 4.1. *Over the finite field \mathbb{F}_q , the number of smooth cubic surfaces on whose 27 lines Frob_q acts by a given conjugacy class of $W(E_6)$ is as in Table 4.1.*

The results of Bergvall–Gounelas that we use are cohomological in nature and we use the

Table 4.1: The number of cubic surfaces over \mathbb{F}_q on whose 27 lines Frob_q acts by a given conjugacy class of $W(E_6)$. The factors in the second column are to normalize to a degree 4 monic polynomial in q (and come up naturally in the representation theoretic setup, see Section 4.2.1). The normalization factor $\frac{\#W(E_6)}{\#\text{PGL}(4, \mathbb{F}_q)}$ reappears in Tables 4.2 to 4.4 for similar reasons. The third column lists q for which the count in the second column vanishes. See Section 4.3 for the notation used for conjugacy classes.

Conjugacy class c	$\frac{\#\{S \mid \text{Frob}_{q,S} \sim c\}}{\#\text{PGL}(4, \mathbb{F}_q)} \times \frac{\#W(E_6)}{\#c}$	$\#\{S \mid \text{Frob}_{q,S} \sim c\} = 0$ for $q =$
(1^6)	$(q-2)(q-3)(q-5)^2$	2, 3, 5
$(1^2, 2^2)$	$(q+1)^2(q-2)(q-3)$	2, 3
$(1^{-2}, 2^4)$	$(q-2)(q-3)(q^2-2q-7)$	2, 3
$(1^3, 3)$	$q(q+1)(q^2-q+1)$	
$(1^{-3}, 3^3)$	$(q+1)^2(q^2+q-3)$	
(3^2)	$(q-2)(q^3-q^2-2q-6)$	2
$(1^2, 2^{-2}, 4^2)$	$(q+1)^3(q-2)$	2
$(2, 4)$	$(q+1)(q-2)(q^2+1)$	2
$(1, 5)$	$q^2(q^2+1)$	
$(1, 2, 3^{-1}, 6)$	$q(q+1)(q^2+q-1)$	
$(1^{-1}, 2^2, 3)$	$q(q+1)(q^2-q+1)$	
$(1^{-2}, 2, 6)$	$q(q-2)(q^2+q+2)$	2
$(1, 2^{-2}, 3^{-1}, 6^2)$	$(q+1)(q^3-2q^2+2q-3)$	
$(3^{-1}, 9)$	$q(q+1)(q^2-q+1)$	
$(1^{-1}, 2, 3, 4^{-1}, 6^{-1}, 12)$	$(q+1)^2(q^2-q+1)$	
$(1^4, 2)$	$q(q-1)(q^2-4q+5)$	
(2^3)	$q(q-1)(q^2-3)$	
$(1^2, 4)$	$q(q+1)^2(q-1)$	
$(1^{-2}, 2^2, 4)$	$q(q-1)^3$	
$(1, 2, 3)$	$q(q-1)(q^2-q-1)$	
$(1^{-2}, 2, 3^2)$	$q(q-1)(q^2+2q+2)$	
(6)	$q^3(q-1)$	
$(2, 4^{-1}, 8)$	$q(q+1)(q^2+1)$	
$(1^{-1}, 2, 5)$	$q^2(q+1)(q-1)$	
$(1, 2^{-1}, 3^{-1}, 4, 6)$	$q(q-1)(q^2+q+1)$	

Grothendieck–Lefschetz trace formula to convert them to point-counting; see Section 4.2.1. We also directly obtain the rational cohomology of various bundles and covers over the moduli space of smooth cubic surfaces; see Theorem 4.6 and Corollary 4.7. These spaces are the respective moduli spaces of smooth cubic surfaces with various markings of points and lines. The point-counting analogue of this is Theorem 4.9.

It is worth noting how Theorem 4.1 relates to the distribution predicted by the Chebotarev density theorem: for a fixed smooth cubic surface defined over \mathbb{Z} , the conjugacy class of Frob_p acting on the 27 lines of the mod p reduction is distributed (as $p \rightarrow \infty$) proportional to the sizes of the conjugacy classes. Theorem 4.1, on the other hand, describes for each fixed q the distribution over all smooth cubic surfaces defined over \mathbb{F}_q . The asymptotic distribution as $q \rightarrow \infty$ is still proportional to the size of the conjugacy class (with the normalization factor in Table 4.1, this corresponds to each of the polynomials listed being monic); this is a relatively easy application of the Grothendieck–Lefschetz trace formula for twisted coefficients (see Section 4.2.1).

It also follows from the trace formula that a cubic surface over \mathbb{F}_q contains $q^2 + tq + 1$ points, where t is the trace of the element of $W(E_6)$ induced by Frob_q on an appropriate representation of $W(E_6)$ (specifically, $V_1 \oplus V_6$ in the notation of Section 4.3). Adding up the counts for a given t , we get the number of surfaces containing a given number of points.

Corollary 4.2. *Over the finite field \mathbb{F}_q , the number of smooth cubic surfaces with $q^2 + tq + 1$ points is given by Table 4.2.*

In particular, a cubic surface for a pair (t, q) , for a listed t , exists unless the polynomial listed in Table 4.2 vanishes at q (i.e. unless $t = 7$ and $q = 2, 3, 5$). These pairs were known before, by Swinnerton-Dyer [Swi10] and Banwait–Fité–Loughran [BFL18], but the exact numbers of surfaces were not. We can also check when the number of cubic surfaces vanishes for each conjugacy class individually (as in the third column of Table 4.1) and recover more recent results of Loughran–Trepalin [LT18].

Table 4.2: The number of cubic surfaces over \mathbb{F}_q with a given number of points.

t	$\#\{S(\mathbb{F}_q) = q^2 + tq + 1\} \times \frac{\#W(E_6)}{\#\mathrm{PGL}(4, \mathbb{F}_q)}$
-2	$80(q^2 + q - 3)(q + 1)^2$
-1	$45(77q^4 - 43q^3 + 45q^2 - 181q - 42)$
0	$432(27q^3 - 17q^2 + 5q + 10)(q + 1)$
1	$60(347q^4 - 51q^3 + 27q^2 + 161q - 12)$
2	$144(91q^4 - 5q^3 + 36q^2 - 35q - 15)$
3	$270(9q^2 - 13q + 2)(q + 1)^2$
4	$240(q^2 - q + 1)(q + 1)q$
5	$36(q^2 - 4q + 5)(q - 1)q$
7	$(q - 2)(q - 3)(q - 5)^2$

Corollary 4.3 ([LT18, Theorem 1.1]). *Over every prime power q , every conjugacy class of $W(E_6)$ occurs as the class of Frob_q acting on the lines of some cubic surface, with the following exceptions:*

- the conjugacy class (1^6) (i.e. identity) does not appear for $q = 2, 3, 5$;
- the conjugacy classes $(1^2, 2^2)$ and $(1^{-2}, 2^4)$ do not appear for $q = 2, 3$;
- the conjugacy classes (3^2) , $(1^{-2}, 2^2, 4)$, $(2, 4)$ and $(1^{-2}, 2, 6)$ do not appear for $q = 2$.

Just like the number of lines, the intersection pattern of pairs of lines is fixed over $\overline{\mathbb{F}}_q$. Once we know how Frob_q acts on the set of lines on S , we can determine how many sets of lines with a given intersection pattern are fixed by Frob_q (and hence are defined over \mathbb{F}_q). Since we know the distribution of conjugacy classes of Frob_q , this allows us to compute the distribution of that intersection pattern over all S defined over \mathbb{F}_q . Similarly, we can also find the distribution of the number of \mathbb{F}_q points on $\mathrm{Sym}^n S$ or $\mathrm{UConf}^n S$. Two such examples are listed in Table 4.3 (tritangents), and Table 4.4 (unordered pairs of points); for a general statement see Theorem 4.9.

As mentioned above, we obtain these results on point counts by applying (an equivariant version of) the Grothendieck–Lefschetz trace formula to the cohomology of the $W(E_6)$ -cover Y of the parameter space X of smooth cubic surfaces (see Section 4.2), as computed by Bergvall–Gounelas (and restated here as Theorem 4.4). They perform their computation by first obtaining

\mathfrak{S}_6 -equivariant point counts of smooth cubic surfaces [BG19, Table 1], i.e. the analogue of Table 4.1 for $\mathfrak{S}_6 < W(E_6)$. This determines $H_{\text{ét}}^*(Y_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ as \mathfrak{S}_6 representations and leaves finitely many possibilities as $W(E_6)$ representations. They then use various constraints on the $W(E_6)$ -equivariant point counts (for example that each entry in Table 4.1 must be non-negative for each q) to reduce to a single possibility.

We also apply their computation of cohomology, along with some facts from Chapter 3, to various bundles and covers over X . More explicitly, marking n , not necessarily distinct, ordered (resp. unordered) points on each $S \in X$ corresponds to a bundle over X whose fiber over S is S^n (resp. $\text{Sym}^n S$); see Section 4.2 for details. In Theorem 4.6 and Corollary 4.7 we compute the rational cohomology of each of these bundles and their finite covers given by subgroups of $W(E_6)$; these covers correspond to marking various subsets of the 27 lines in addition to the n points on the surface S . These cohomology computations are of course related to the point-count computations in Theorem 4.1 again by the trace formula; see Theorem 4.9 and Remark 4.10.

4.2 Marking points and lines on cubic surfaces

Denote the *ordered configuration space* of n points on a space X by

$$\text{PConf}^n X := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The *unordered configuration space* of n points on X is the quotient by the action of the symmetric group \mathfrak{S}_n permuting the n points and will be denoted by

$$\text{UConf}^n X := \text{PConf}^n X / \mathfrak{S}_n.$$

A cubic surface $S \subset \mathbb{P}^3$ is defined by a homogeneous cubic polynomial F in 4 variables. Accordingly, the space of cubic surfaces is the projectivization \mathbb{P}^{19} of the affine space of homogeneous cubic polynomials in 4 variables. The cubic surface S is *smooth* if its defining polynomial F is smooth, i.e. its partials do not simultaneously vanish on S .

The subspace of non-smooth or *singular* cubic surfaces is a closed set Σ , given by a ‘discriminant’ polynomial (with integer coefficients) in the coefficients of F . We will denote the complement, the space of smooth cubic surfaces, by

$$X = \mathbb{P}^{19} \setminus \Sigma.$$

Thus X is an algebraic variety defined over \mathbb{Z} .

According to the classical theorem of Cayley and Salmon, a smooth cubic surface over \mathbb{C} contains 27 lines. This corresponds to a degree 27 cover of $X(\mathbb{C})$ given by the incidence variety of lines $\ell \in \text{Gr}(2, 4)$ (the Grassmannian of lines in \mathbb{P}^3) and cubic surfaces $S \in X$ (see Chapter 2). This cover is not Galois; its Galois group is $W(E_6)$, the Weyl group of type E_6 (see [Man86; Har79]). More precisely, this is the subgroup of \mathfrak{S}_{27} permuting the lines that preserves the set of intersecting pairs of lines. Explicitly, let $R \subset \{1, \dots, 27\}^2$ be the set of intersecting pairs of lines for some fixed $S_0 \in X$. Then the Galois cover is

$$Y = \{(S, L_1, \dots, L_{27}) \mid L_i \subset S, L_i \cap L_j \neq \emptyset \text{ iff } (i, j) \in R\} \subset X \times \text{PConf}^{27} \text{Gr}(2, 4).$$

Note that Y is an algebraic variety over X and, by above, $Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is a Galois cover with Galois group $W(E_6) \cong \text{Stab}(R) < \mathfrak{S}_{27}$. This finite subgroup is of order 51840 and has a simple subgroup of index 2; we describe some of its representation theory in Section 4.3.

There are also actions of $\text{PGL}(4) = \text{Aut}(\mathbb{P}^3)$ on X and on Y such that the covering map is equivariant. Both the actions over \mathbb{C} have closed orbits and finite stabilizers, which are subgroups of $W(E_6)$.

In the following, the summands on the right denote irreducible representations of $W(E_6)$; for details and the notation see Section 4.3 and Table 4.7. In particular, V_d or U_d has dimension d and V_1 is the trivial representation. Bergvall and Gounelas prove the following:

Theorem 4.4 ([BG19, Theorem 1.2]). *Let $H^i = H^i(Y(\mathbb{C})/\text{PGL}(4, \mathbb{C}); \mathbb{Q})$. Then as representations*

of $W(E_6)$:

$$H^0 = V_1;$$

$$H^1 = V_{15,2};$$

$$H^2 = V_{81};$$

$$H^3 = V_{15,1} \oplus U_{80} \oplus U_{90};$$

$$H^4 = V_{30} \oplus V'_{30} \oplus U_{10} \oplus U_{80}.$$

For dimensional reasons, $H^i = 0$ for other i .

This is sufficient to determine $H^*(Y(\mathbb{C}); \mathbb{Q})$ as a $W(E_6)$ representation since it follows from the work of Peters and Steenbrink [PS03] that

$$H^*(Y(\mathbb{C}); \mathbb{Q}) \cong H^*(Y(\mathbb{C})/\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q}) \otimes H^*(\mathrm{PGL}(4, \mathbb{C}); \mathbb{Q}),$$

where the action on $\mathrm{PGL}(4, \mathbb{C})$ (and its cohomology) is trivial.

A subgroup $G < W(E_6)$ corresponds to an intermediate cover $Y/G \rightarrow X$. These often also correspond to marking a (labeled) subset of lines on $S \in X$, when G is the (pointwise) stabilizer of such a set. For example, marking one line L corresponds to $\mathrm{Stab}(L) \cong W(D_5)$ (see Chapter 2), and marking a ‘tritangent’ T corresponds to $\mathrm{Stab}(T) \cong W(F_4)$ (see [Nar82]). Note that $Y = Y/\{1\}$ and $X = Y/W(E_6)$ are the ‘trivial’ examples.

We denote the *incidence variety* of points in \mathbb{P}^3 and $S \in X$ by

$$U := \{(S, p) \mid p \in S\} \subset X \times \mathbb{P}^3.$$

Then $U \rightarrow X$ is the ‘universal family’ of smooth cubic surfaces and $U(\mathbb{C})$ is a fiber bundle over $X(\mathbb{C})$ with fiber $S \subset \mathbb{P}^3(\mathbb{C})$ over $S \in X(\mathbb{C})$ (see Chapter 3). We can also construct various associated spaces over X : $\Pi_X^n U$ with fiber S^n , $\mathrm{Sym}_X^n U$ with fiber $\mathrm{Sym}^n S$, $\mathrm{PConf}_X^n U$ with fiber $\mathrm{PConf}^n S$, etc.

Finally, we can combine the two constructions above by taking more fiber products over X , for instance:

$$(Y/W(D_5)) \times_X (\Pi_X^2 U) = \{(S, L, p_1, p_2) \mid L \subset S, p_i \in S\} \subset X \times \text{Gr}(2, 4) \times (\mathbb{P}^3)^2.$$

Of course,

$$(Y/G) \times_X U = (Y \times_X U)/G,$$

where the action of $G \subset W(E_6)$ on U is trivial, and similarly for $(Y/G) \times_X (\Pi_X^n U)$ etc. It is worth noting that in these examples there is no enforced relation between the points and the lines marked, in particular we do not insist that $p_i \in L$.

Remark 4.5. We should not expect the results and techniques of this chapter to apply if we do insist e.g. that $p_i \in L$. For one, the space

$$\{(S, L, p) \mid p \in L \subset S\} \subset (X \times \text{Gr}(2, 4) \times \mathbb{P}^3)(\mathbb{C})$$

is *not* a fiber bundle over $X(\mathbb{C})$ due to the existence of *Eckardt points*, i.e. triple intersections of lines, on some special $S \in X$.

The bundle $U \rightarrow X$ and the associated constructions each have monodromy: $H^*(S)$ is a $\pi_1(X, S)$ representation. However, the monodromy representation factors through $W(E_6)$ (see Eq. (4.1) for an explicit description of the representation) and consequently in the pullback bundle $Y \times_X U \rightarrow Y$ the monodromy action of $\pi_1(Y)$ on $H^*(S)$ is trivial. The same holds for $Y \times_X (\Pi_X^n U)$ etc.

Theorem 4.6. *Let Z be $\Pi_X^n U$ or $\text{Sym}_X^n U$. Let F be the fiber of $Z(\mathbb{C}) \rightarrow X(\mathbb{C})$ over S (i.e. S^n or $\text{Sym}^n S$ respectively). Then*

$$H^*((Y \times_X Z)(\mathbb{C}); \mathbb{Q}) \cong H^*(Y(\mathbb{C}); \mathbb{Q}) \otimes H^*(F; \mathbb{Q})$$

as both $W(E_6)$ representations and mixed Hodge structures.

Proof. We will suppress the field \mathbb{C} for brevity. In the quotient $Y \times_X \Pi_X^n U \rightarrow Y \times_X \text{Sym}_X^n U$, the \mathfrak{S}_n action is trivial on the Y factor (and also trivial on the $H^*(Y)$ factor in the action on $H^*(Y; \mathbb{Q}) \otimes H^*(\Pi_X^n U; \mathbb{Q})$). Thus by transfer, it is enough to restrict to the cases Π^n .

Since, as we noted, the monodromy $\pi_1(Y) : H^*(F)$ is trivial, the associated Serre spectral sequence converging to $H^*(Y \times_X Z; \mathbb{Q})$ has E_2 page

$$E_2^{p,q} \cong H^p(Y; \mathbb{Q}) \otimes H^q(F; \mathbb{Q}).$$

It is then enough to show that this spectral sequence degenerates immediately. Since $H^*(F) = H^*(S^n) \cong H^*(S)^{\otimes n}$, and the differentials satisfy the Leibniz rule, we reduce to the $n = 1$ case. Note that the differentials in these spectral sequences must be $W(E_6)$ -equivariant, since the respective bundles are.

To describe $H^*(S)$ as a $W(E_6)$ representation, identify S as the blowup of \mathbb{P}^2 at 6 points, with the exceptional divisors constituting 6 of the 27 lines. It follows that $H^2(S)$ is generated by the canonical class of S and the classes of these 6 lines, and implies that as $W(E_6)$ representations,

$$H^2(S; \mathbb{Q}) \cong V_1 \oplus V_6. \quad (4.1)$$

But there is no copy of the irreducible fundamental representation V_6 in $E^{2,1} = 0$ or $E^{3,0} \cong H^3(Y; \mathbb{Q})$. Thus it remains to show that the differentials vanish on the $W(E_6)$ -invariant classes in $H^*(S)$, or equivalently (by transfer) that the differentials in the Serre spectral sequence for the bundle $U \rightarrow X$ vanish. The latter is shown in the proof of Corollary 3.5. \square

Corollary 4.7. *For Z and F as above, and $G < W(E_6)$,*

$$H^*((Y/G) \times_X Z) \cong (H^*(Y(\mathbb{C}); \mathbb{Q}) \otimes H^*(F; \mathbb{Q}))^G.$$

In particular, taking $G = W(E_6)$,

$$H^*(Z) \cong (H^*(Y(\mathbb{C}); \mathbb{Q}) \otimes H^*(F; \mathbb{Q}))^{W(E_6)} = H^*(Y(\mathbb{C}); \mathbb{Q}) \otimes_{W(E_6)} H^*(F; \mathbb{Q}).$$

Thus computations of these cohomology groups reduce to elementary representation theory, namely character theory. For convenience, a character table of $W(E_6)$ is reproduced in Table 4.7.

4.2.1 The Grothendieck–Lefschetz trace formula and point counts

The varieties above are all smooth and quasiprojective. Therefore their point counts over the finite field \mathbb{F}_q can be obtained from the action of Frob_q on their étale cohomology via the Grothendieck–Lefschetz trace formula:

$$\#Z(\mathbb{F}_q) = q^{\dim Z} \sum_{i \geq 0} (-1)^i \text{Tr}(\text{Frob}_q : H_{\text{ét}}^i(Z_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)^\vee).$$

This formula holds for a smooth Z and a prime ℓ not dividing q .

The Cayley–Salmon theorem holds over any algebraically closed field, in particular $\overline{\mathbb{F}}_q$. The identification of the Galois group of $Y \rightarrow X$ with $W(E_6)$ implies that if S is defined over \mathbb{F}_q , then Frob_q permutes the 27 lines by some element of $W(E_6) < \mathfrak{S}_{27}$. We denote this by $\text{Frob}_{q,S}$. This also determines how Frob_q acts on $H_{\text{ét}}^*(S)$ as follows.

Recall that a smooth cubic surface S is the blowup of \mathbb{P}^2 at 6 points and

$$H^2(S; \mathbb{Q}) \cong V_1 \oplus V_6. \quad (4.2)$$

Since Frob_q acts on $H_{\text{ét}}^{2i}(\mathbb{P}^n_{\overline{\mathbb{F}}_q})^\vee$ by the scalar q^{-i} , we deduce that Frob_q acts on $H^0(S)^\vee$ as identity, on $H^2(S)^\vee$ by the action of $q^{-1} \cdot \text{Frob}_{q,S} \in \mathbb{Q}[W(E_6)]$ on $V_1 \oplus V_6$ and on $H^4(S)^\vee$ by the scalar q^{-2} . Here (and henceforth) we use that each irreducible representation of $W(E_6)$ are self-dual. This also proves Corollary 4.2. It remains to prove Theorem 4.1.

Remark 4.8. The action of Frob_q on $H^*(S)$ determines the action of Frob_q on $H^*(S^n)$ and $H^*(\text{Sym}^n S)$. Totaro [Tot96] describes $H^*(\text{PConf}^n S)$ as the cohomology of a DGA generated by $H^*(S^n)$ along with classes in degree 3 which can be identified with the generator of $H^3(\mathbb{A}^2 - 0)$. This is enough to determine the action of Frob_q on $H^*(\text{PConf}^n S)$ and, since this description is \mathfrak{S}_n -equivariant, on $H^*(\text{UConf}^n S)$. In particular, the conjugacy class of $\text{Frob}_{q,S}$ determines the number of \mathbb{F}_q points on S^n , $\text{Sym}^n S$, etc, again via the Grothendieck–Lefschetz trace formula.

Proof of Theorem 4.1. We want to count points of X depending on the conjugacy class of Frob_q in $W(E_6)$. Thus we will apply a version of the Grothendieck–Lefschetz trace formula with local

coefficients (see e.g. [DL76, §3]), specifically for representations of $\pi_1(X)$ that factor through the finite group $W(E_6)$. By transfer, this exactly corresponds to the cohomology of Y ; more explicitly, for a $W(E_6)$ -representation V ,

$$H^*(X; V) \cong H^*(Y; \mathbb{Q}) \otimes_{\mathbb{Q}[W(E_6)]} V.$$

To combine this with the Grothendieck–Lefschetz trace formula, we need knowledge of how $W(E_6)$ and Frob_q acts on $H_{\text{ét}}^*(Y)$.

Even though Theorem 4.4 is stated for singular cohomology of the complex points, the same identifications with $W(E_6)$ representations (now over \mathbb{Q}_ℓ) can be made for $H_{\text{ét}}^*(Y_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ for every q . Further, it is an important part of the results of Bergvall–Gounelas in [BG19] that $H^*(Y/\text{PGL}(4))$ is *minimally pure*, i.e. that Frob_q acts on $(H_{\text{ét}}^i)^\vee$ by the scalar q^{-i} . In fact, the argument in [BG19] in some sense reverses the steps here, to go from point counts to étale cohomology to singular cohomology.

Equipped with this knowledge and using some linear algebra, one obtains

$$\#\{S \in X \mid \text{Frob}_{q,S} \in c\} = \#\text{PGL}(4, \mathbb{F}_q) \sum_{i \geq 0} (-1)^i q^{4-i} \langle H^i, \chi_c \rangle_{W(E_6)}$$

where χ_c is the characteristic function of the conjugacy class c and H^i are as in Theorem 4.4. This formula is enough to compute Table 4.1 and also explains the normalizing factors in its second column. \square

We can also state a point-count version of Theorem 4.6. We already saw in Remark 4.8 that $\#S^n(\mathbb{F}_q)$, $\#\text{PConf}^n S(\mathbb{F}_q)$ etc only depend on $\text{Frob}_{q,S}$. More elementary is the fact that the cardinality of the fiber of $Y/G \rightarrow X$ over $S \in X(\mathbb{F}_q)$ is determined by $\text{Frob}_{q,S}$ for any $G < W(E_6)$, since this fiber is isomorphic to $W(E_6)/G$ as a G -set. Thus we get the following statement whose proof is obvious.

Theorem 4.9. *Let Z be $\Pi_X^n U$, $\text{Sym}_X^n U$, $\text{PConf}_X^n U$ or $\text{UConf}_X^n U$ and $G < W(E_6)$. Let c be any conjugacy class of $W(E_6)$ and set $X_c = \{S \in X(\mathbb{F}_q) \mid \text{Frob}_{q,S} \in c\}$. For any $S \in X_c$, let $d =$*

$\#(Y/G)_S(\mathbb{F}_q)$ and let F be the fiber of $Z \rightarrow X$ over S (i.e. S^n , $\text{Sym}^n S$, $\text{PConf}^n S$ or $\text{UConf}^n S$ respectively). Then

$$\#[((Y \times_X Z)/G)(\mathbb{F}_q) \times_{X(\mathbb{F}_q)} X_c] = d \times (\#F(\mathbb{F}_q)) \times (\#X_c).$$

Since we know the distribution of $\text{Frob}_{q,S}$, i.e. $\#X_c$ for each c , we can find the distribution of $d \times \#F$. Two examples of such distributions are tabulated in Tables 4.3 and 4.4. Specifically, for Table 4.4, $H^*(\text{UConf}^2(S); \mathbb{Q})$ as a $W(E_6)$ representation can be computed to be the following using the spectral sequence in [Tot96]:

$$H^0 \cong V_1; \quad H^2 \cong V_1 \oplus V_6; \quad H^4 \cong V_1^{\oplus 2} \oplus V_6 \oplus V_{20}; \quad H^i = 0 \text{ for other } i.$$

Remark 4.10. It is possible to deduce Theorem 4.9 by applying the trace formula to Theorem 4.6 for $Z = \Pi_X^n U$ or $\text{Sym}_X^n U$, and to the Serre spectral sequence of the bundle $Y \times_X Z \rightarrow Y$ for $Z = \text{PConf}_X^n U$ or $Z = \text{UConf}_X^n U$. The differentials in this spectral sequence are irrelevant for the purpose of point-counting, since the trace formula uses an alternating sum of traces, similar to the Euler characteristic. We leave the details to experts.

4.3 Character table of $W(E_6)$

In this section we explain our notation for the conjugacy classes and irreducible representations, borrowing heavily from [Fra51]. The conjugacy classes will be denoted by “virtual cycle types” as determined by their action on the 6-dimensional *fundamental representation* V_6 . In more detail, elements from different conjugacy classes in $W(E_6)$ remain non-conjugate in the representation V_6 . Thus a conjugacy class c can be identified by the action of any $g \in c$ on V_6 , and since this action of $\{1, g, g^2, \dots\} \cong \mathbb{Z}/n$ (i.e. n is the order of g) is defined over \mathbb{Q} , it can be decomposed as a (virtual) direct sum

$$\bigoplus_{d|n} \mathbb{Q}[\mathbb{Z}/d]^{\oplus i_d}.$$

Table 4.3: Distribution of tritangents over $S \in X(\mathbb{F}_q)$. The average is 1.

N	$\frac{\#W(E_6)}{\#\text{PGL}(4, \mathbb{F}_q)} \times \#\{S \mid S \text{ has } N \text{ tritangents}\}$
0	$576(38q^3 - 5q^2 + 5)q$
1	$540(39q^4 + 3q^3 + 3q^2 - 3q - 10)$
2	$2160(q^2 - q + 1)(q + 1)q$
3	$240(17q^4 - 25q + 24)$
4	$1440(q^2 + q - 1)(q + 1)q$
5	$270(q + 1)^2(q - 2)(q - 3)$
6	$240(q^2 - q + 1)(q + 1)q$
7	$540(q^2 - 3)(q - 1)q$
9	$80(q^2 + q - 3)(q + 1)^2$
13	$45(q^2 - 2q - 7)(q - 2)(q - 3)$
15	$36(q^2 - 4q + 5)(q - 1)q$
45	$(q - 2)(q - 3)(q - 5)^2$

Table 4.4: Distribution of $\#\text{UConf}^2(S)$ over $S \in X(\mathbb{F}_q)$. The average is $q^2(q^2 + q + 2)$.

N	$\frac{\#W(E_6)}{\#\text{PGL}(4, \mathbb{F}_q)} \times \#\{S \mid \#\text{UConf}^2(S) = N\}$
$q^4 - 2q^3 + q^2$	$80(q + 1)^2(q^2 + q - 3)$
$q^4 - q^3 + q^2$	$2880q(q^3 - 3)$
$q^4 - q^3 + 2q^2$	$540q(q - 1)^3$
$q^4 - q^3 + 4q^2$	$45(q - 2)(q - 3)(q^2 - 2q - 7)$
$q^4 + q^2$	$864(q + 1)(11q^3 - 6q^2 + 5)$
$q^4 + 2q^2$	$2160q(q + 1)(q^2 - q + 1)$
$q^4 + q^3 + q^2$	$960(11q^4 - 6q^3 + 5q + 6)$
$q^4 + q^3 + 2q^2$	$3240(q + 1)(3q - 2)(q^2 + 1)$
$q^4 + q^3 + 4q^2$	$540q(q - 1)(q^2 - 3)$
$q^4 + 2q^3 + q^2$	$720(q + 1)(q^3 - 2q^2 + 2q - 3)$
$q^4 + 2q^3 + 2q^2$	$4320q(q - 1)(q^2 + q + 1)$
$q^4 + 2q^3 + 3q^2$	$5184q^2(q^2 + 1)$
$q^4 + 2q^3 + 4q^2$	$2880q^4$
$q^4 + 3q^3 + 4q^2$	$540(q + 1)^3(q - 2)$
$q^4 + 3q^3 + 6q^2$	$1620q(q + 1)^2(q - 1)$
$q^4 + 3q^3 + 8q^2$	$270(q + 1)^2(q - 2)(q - 3)$
$q^4 + 4q^3 + 10q^2$	$240q(q + 1)(q^2 - q + 1)$
$q^4 + 5q^3 + 16q^2$	$36q(q - 1)(q^2 - 4q + 5)$
$q^4 + 7q^3 + 28q^2$	$(q - 2)(q - 3)(q - 5)^2$

Then we will denote the conjugacy class c by the tuple $(d^{i_d})_{i_d \neq 0}$. For example, the tuple

$$(1, 2^{-2}, 3^{-1}, 6^2)$$

denotes a conjugacy class of order 6 whose elements act on V_6 with eigenvalues

$$(\zeta, \zeta, \zeta^2, \zeta^4, \zeta^5, \zeta^5),$$

where ζ is a primitive 6th root of unity. We assure the reader that the tuples below do in fact represent actual (i.e. not just virtual) representations of dimension 6, in particular they satisfy $\sum d i_d = 6$.

The group $W(E_6)$ contains a simple subgroup $W(E_6)^+$ of index 2. Call the conjugacy classes in $W(E_6)$ that are contained in $W(E_6)^+$ *even*, the others *odd*. In particular, $(1^4, 2)$ is odd. Every irreducible character of $W(E_6)$ that does not remain irreducible upon restriction to $W(E_6)^+$ vanishes on odd conjugacy classes. There are 5 such characters, and they have different dimensions, so we will denote the one of dimension n by U_n .

On the other hand, each of the other irreducible characters χ , whose restriction to $W(E_6)^+$ *does* remain irreducible, satisfies $\chi(1^4, 2) \neq 0$. These occur in 10 pairs, differing in the sign of the character on odd conjugacy classes (and hence by a tensor product with the sign representation V'_1 pulled back from the non-trivial representation of the quotient $W(E_6)/W(E_6)^+$). Each pair is denoted V and V' with subscripts, where V has $\chi(1^4, 2) > 0$, and V' has $\chi(1^4, 2) < 0$. Distinct pairs have different dimensions, except two of the pairs have dimension 15. Hence we will denote them by V_n, V'_n for $n = \dim \neq 15$, and $V_{15,1}, V'_{15,1}, V_{15,2}, V'_{15,2}$. A full list of the 25 irreducible representations and the notation for them in other sources is listed in Table 4.6.

Finally, we include a character table of $W(E_6)$ as Table 4.7. To avoid redundancy, we omit the characters of the representations denoted V'_n ; these can be obtained by taking the character of the corresponding character V_n and negating the values on the odd conjugacy classes.

Table 4.5: Conjugacy classes of $W(E_6)$

		Notation for the class c		Properties of c and $g \in c$		
		Here	[CCN+85] [Swi67]	ord g	$\#c$	$\#Z(g)$
even classes: $g \in W(E_6)^+$	(1^6)	1A	C_1	1	1	51840
	$(1^2, 2^2)$	2B	C_2	2	270	192
	$(1^{-2}, 2^4)$	2A	C_3	2	45	1152
	$(1^3, 3)$	3D	C_6	3	240	216
	$(1^{-3}, 3^3)$	3A, 3B	C_{11}	3	80	648
	(3^2)	3C	C_9	3	480	108
	$(1^2, 2^{-2}, 4^2)$	4A	C_4	4	540	96
	$(2, 4)$	4B	C_5	4	3240	16
	$(1, 5)$	5A	C_{15}	5	5184	10
	$(1, 2, 3^{-1}, 6)$	6C, 6D	C_7	6	1440	36
	$(1^{-1}, 2^2, 3)$	6F	C_8	6	2160	24
	$(1^{-2}, 2, 6)$	6E	C_{10}	6	1440	36
	$(1, 2^{-2}, 3^{-1}, 6^2)$	6A, 6B	C_{12}	6	720	72
	$(3^{-1}, 9)$	9A, 9B	C_{14}	9	5760	9
$(1^{-1}, 2, 3, 4^{-1}, 6^{-1}, 12)$	12A, 12B	C_{13}	12	4320	12	
odd classes: $g \notin W(E_6)^+$	$(1^4, 2)$	2C	C_{16}	2	36	1440
	(2^3)	2D	C_{17}	2	540	96
	$(1^2, 4)$	4D	C_{18}	4	1620	32
	$(1^{-2}, 2^2, 4)$	4C	C_{19}	4	540	96
	$(1, 2, 3)$	6G	C_{21}	6	1440	36
	$(1^{-2}, 2, 3^2)$	6H	C_{22}	6	1440	36
	(6)	6I	C_{23}	6	4320	12
	$(2, 4^{-1}, 8)$	8A	C_{20}	8	6480	8
	$(1^{-1}, 2, 5)$	10A	C_{25}	10	5184	10
	$(1, 2^{-1}, 3^{-1}, 4, 6)$	12C	C_{24}	12	4320	12

Table 4.6: Various notations for irreducible representations of $W(E_6)$

Here	[Fra51]	[CCN+85]	[Car93]
V_1	1_p	χ_1	$\phi_{1,0}$
V_6	6_p	χ_4	$\phi_{6,1}$
$V_{15,1}$	15_p	χ_7	$\phi_{15,5}$
$V_{15,2}$	15_q	χ_8	$\phi_{15,4}$
V_{20}	20_p	χ_9	$\phi_{20,2}$
V_{24}	24_p	χ_{10}	$\phi_{24,6}$
V_{30}	30_p	χ_{11}	$\phi_{30,3}$
V_{60}	60_p	χ_{18}	$\phi_{60,5}$
V_{64}	64_p	χ_{19}	$\phi_{64,4}$
V_{81}	81_p	χ_{20}	$\phi_{81,6}$
V'_1	1_n		$\phi_{1,36}$
V'_6	6_n		$\phi_{6,25}$
$V'_{15,1}$	15_n		$\phi_{15,17}$
$V'_{15,2}$	15_m		$\phi_{15,16}$
V'_{20}	20_n		$\phi_{20,20}$
V'_{24}	24_n		$\phi_{24,12}$
V'_{30}	30_n		$\phi_{30,15}$
V'_{60}	60_n		$\phi_{60,11}$
V'_{64}	64_n		$\phi_{64,13}$
V'_{81}	81_n		$\phi_{81,10}$
U_{10}	10_s	$\chi_2 + \chi_3$	$\phi_{10,9}$
U_{20}	20_s	$\chi_5 + \chi_6$	$\phi_{20,10}$
U_{60}	60_s	$\chi_{12} + \chi_{13}$	$\phi_{60,8}$
U_{80}	80_s	$\chi_{14} + \chi_{15}$	$\phi_{80,7}$
U_{90}	90_s	$\chi_{16} + \chi_{17}$	$\phi_{90,8}$

Table 4.7: Character table of $W(E_6)$. See text for notation.

Class c	V_1	V_6	$V_{15,1}$	$V_{15,2}$	V_{20}	V_{24}	V_{30}	V_{60}	V_{64}	V_{81}	U_{10}	U_{20}	U_{60}	U_{80}	U_{90}
(1^6)	1	6	15	15	20	24	30	60	64	81	10	20	60	80	90
$(1^2, 2^2)$	1	2	-1	3	4	0	2	4	0	-3	2	-4	4	0	-6
$(1^{-2}, 2^4)$	1	-2	-1	7	4	8	-10	-4	0	9	-6	4	12	-16	-6
$(1^3, 3)$	1	3	3	0	5	0	3	-3	4	0	-2	2	-6	-4	0
$(1^{-3}, 3^3)$	1	-3	6	-3	2	6	3	6	-8	0	1	-7	-3	-10	9
(3^2)	1	0	0	3	-1	3	3	-3	-2	0	4	2	0	2	0
$(1^2, 2^{-2}, 4^2)$	1	2	3	-1	0	0	-2	0	0	-3	2	4	4	0	2
$(2, 4)$	1	0	-1	1	0	0	0	0	0	-1	-2	0	0	0	2
$(1, 5)$	1	1	0	0	0	-1	0	0	-1	1	0	0	0	0	0
$(1, 2, 3^{-1}, 6)$	1	1	-1	-2	1	2	-1	-1	0	0	0	-2	0	2	0
$(1^{-1}, 2^2, 3)$	1	-1	-1	0	1	0	-1	1	0	0	2	2	-2	0	0
$(1^{-2}, 2, 6)$	1	-2	2	1	1	-1	-1	-1	0	0	0	-2	0	2	0
$(1, 2^{-2}, 3^{-1}, 6^2)$	1	1	2	1	-2	2	-1	2	0	0	-3	1	-3	2	-3
$(3^{-1}, 9)$	1	0	0	0	-1	0	0	0	1	0	1	-1	0	-1	0
$(1^{-1}, 2, 3, 4^{-1}, 6^{-1}, 12)$	1	-1	0	-1	0	0	1	0	0	0	-1	1	1	0	-1
$(1^4, 2)$	1	4	5	5	10	4	10	10	16	9					
(2^3)	1	0	-3	1	2	4	-2	2	0	-3					
$(1^2, 4)$	1	2	1	-1	2	0	0	-2	0	-1					
$(1^{-2}, 2^2, 4)$	1	-2	1	3	2	0	-4	-2	0	3					
$(1, 2, 3)$	1	1	-1	2	1	-2	1	1	-2	0					
$(1^{-2}, 2, 3^2)$	1	-2	2	-1	1	1	1	1	-2	0					
(6)	1	0	0	1	-1	1	1	-1	0	0					
$(2, 4^{-1}, 8)$	1	0	-1	-1	0	0	0	0	0	1					
$(1^{-1}, 2, 5)$	1	-1	0	0	0	-1	0	0	1	-1					
$(1, 2^{-1}, 3^{-1}, 4, 6)$	1	1	1	0	-1	0	-1	1	0	0					

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