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BARGAINING WITH HETEROGENEOUS BELIEFS

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To my parents

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ABSTRACT

I consider a reputation based bargaining model with uncertainty of players' initial beliefs, which comes from access to an exogenous signaling device. I characterize the set of non-trivial equilibria satisfying initial posture constraints. The analysis yields a unique equilibrium with a phased war of attrition structure — players holding different beliefs have disjoint supports of conceding strategies. The result implies that learning about each other's type leads to players departing from the strategies induced by the prior, which creates either efficiency improvement or further delay depending on the realized posteriors. When ex ante probabilities of behavioral types go to zero, the delay and inefficiency persist even if we allow initial demands as strategic choices due to the expectation of the learning opportunity. We also characterize the case with a rich set of posterior beliefs.

Bargaining with Heterogeneous Beliefs

1 Introduction

When players participate in bargaining games, they do not always have perfect information about their opponents. Rather, players may acquire information and learn about each other, consequently, they behave differently after getting different information. For example, a negotiator may be informed about an opponent's "stubbornness" (whether the opponent would ever concede) before the negotiation starts and acts accordingly, a positive signal about the opponent being less stubborn would make the negotiator more likely to delay conceding. The goal of this paper is to show that additional information or the expectation of additional information leads to players departing from the strategy induced by the prior, but not necessarily in an inefficient way when compared to the model without the learning opportunity. Depending on the realized posteriors, the signal could incur an ex post efficiency improvement by reducing delay, or further delay could be induced by the expectation of an informative signal.

To capture the role of exogenous information, I consider a two-person reputational bargaining model in which both players have access to an exogenous signal revealing each other's type after initial demands were made and before the bargaining began. The signal structure is assumed to be common knowledge, in the baseline model, the posteriors may be either high or low. We construct a sequential equilibrium of the game and prove that it is the unique equilibrium in which players with different posteriors never concede simultaneously at any time after time zero.

In the equilibrium, players act according to their posteriors, a positive signal would incur further delays of conceding, specifically, players holding optimistic beliefs (that the opponent is more likely to be rational) would strictly prefer to wait until a time after which the possibility of a rational opponent holding pessimistic belief is ruled out. Players holding optimistic beliefs then proceed with a positive concession rate until they know for sure that their opponents must be of the behavioral types. The main intuition of this model is that both players learning from the signal are privately informed, this knowledge combined with the secrecy of their realized posterior beliefs are encouraging optimistic players to mimic a behavioral type while waiting for the possibly pessimistic opponents to concede.

Extensions of the model further explain the role of exogenous signal and starting prior. We allow strategic choices of initial demands with perfect commitment, which can be justified by the fact that a deviation from the initial demand would immediately reveal a player's type and incur a utility loss. While similar to Kambe (1999)'s framework, our results depend not only on the starting prior, but also on the signal structure. We find that while prior is a dominating factor in determining players' conceding strategies in the limit, the expectation of an exogenous signal also plays a crucial role in determining the bargaining postures, players who expect a more accurate exogenous signal revealing their opponents' types are more likely to make higher initial demands instead of taking the immediate settlement. In addition, we find that our results work with a continuum of priors/posteriors, where players are ranked by their posteriors and each concedes at a specific point in time.

The rest of this paper is structured as follows: Section 2 reviews related literature; Section 3 presents the baseline model; Section 4 discusses the details of the signaling device and limiting cases; Section 5 investigates the case where players can strategically choose their initial demands; Section 6 extends the set of posteriors to a continuum; Section 7 provides the conclusions.

2 Related Literature

The basic framework of noncooperative bargaining theory was originated from Rubinstein (1982), where a complete information model was studied and the equilibrium was determined by players' impatience. As the seminal paper on reputational bargaining, Abreu and Gul (2000) replaces the impatience between offers with uncertainty about opponents' strategic postures, specifically, they follow a number of literatures in reputation games and allow for ex ante probability of irrationality. By emphasizing the modeling of strategic postures, the reputational bargaining model departs from earlier literature on bargaining with asymmetric information where the moving parts are more specific factors such as a player's discount factor or reservation value (see for example Sobel and Takahashi (1983), Chatterjee and Samuelson (1987) and Rubinstein (1985)). In the unique sequential equilibrium, players mix between mimicking different behavioral types, where the mixing probability was determined by the prior and the initial stage was ensued by a war of attrition. One of the significant findings of Abreu and Gul (2000) is that the equilibrium is independent of the bargaining protocol, as long as the offers can be made frequently by both sides. Building on the model of Abreu and Gul (2000), Abreu et al. (2015) introduced behavioral perturbations with additional one-sided asymmetric information about discount rates, and behavioral types can differ in announcement time of their demands. In their model, non-Coasean equilibria can occur since patient rational players want to delay their initial demands to separate themselves from impatient players.

This paper shares many features with the above models, including the equilibrium concept and the requirement of transparency of types, where players announce their initial demands (bargaining postures) at time zero, specifically, rational players pick a type to mimic and players of behavioral types truthfully report their types.

In earlier works, one-sided reputation formation in bargaining was first developed in Myerson (1991) under the name "r-insistent strategy", Abreu and Gul (2000) extends the reputation concerns to both parties of the bargaining and showed that rational agents' strategies reduce to either imitating a behavioral type or conceding, in their results, the limiting equilibrium entails delay and inefficiency unless the ex ante probability of a behavioral type goes to zero. Abreu and Gul (2000)'s results also rely heavily on the structure of war of attrition once the initial offer has been made, among those are Hendricks et al. (1988), Chatterjee and Samuelson (1987) and Chatterjee and Samuelson (1988). Abreu and Gul (2000)'s model generates a unique equilibrium and simultaneous concession behavior within a finite time, which are the key departures from the previous literature.

Reputational bargaining models' important antecedents include the literature on reputation effects in repeated games started with Kreps et al. (1982), Kreps and Wilson (1982) and Milgrom and Roberts (1982). These papers considered models where a long-run incumbent faces a series of short-run entrants and showed that by maintaining the reputation of being "tough", it can deter the entrants from entering the market given an initial probability of "toughness". Fudenberg and Levine (1989) improved on this literature by allowing for many different types and making the conclusions robust to changes in information structure. Many recent developments built upon Abreu and Gul (2000) benefit from this line of work.

There are various developments in the area of reputational bargaining: Kambe (1999) was the first to introduce endogenous commitment demands, where each player starts with being rational, but after making the initial demands, players commit to the demands with some probability. In Section 5 we discuss an extension of our model where strategic choices of initial demands are allowed before the exogenous signal reveals information about players' types. More recently, Abreu and Pearce (2007) extended the model by allowing for complex types in the context of infinitely repeated game where agents bargain over an enforceable long-term contract. Fanning (2016) introduced a stochastically arriving deadline and found high frequency of deals prior to the deadline. Our model fills in the literature by allowing heterogeneous initial beliefs, which leads to phased concession events, this setting can be expanded to any finite number of initial beliefs and the intuitions remain robust to those changes. Our model includes an exogenous signaling device allowing players to learn about their opponents' types, on that front, our model is related to Daley and Green (2020), which studies a bargaining model with one-sided incomplete information, however, in their model, it is the uninformed party (the buyer) who makes frequent offers, while simultaneously learning gradually about the seller's type from stochastically arriving news. In our model, both players learn from an exogenous signaling device independently after initial demands were made.

3 The Model

There are two players, $i = 1, 2$, who bargain over the division of a unit of good in continuous time. Each player can be one of the two types: rational or behavioral. A behavioral type of player i is defined by $\alpha^i \in (0,1)$, who always demands α^i and accepts any offer larger than or equal to α^i. Denote $C^i \subset (0,1)$ the finite set of behavioral types, and $\pi^i(\alpha^i)$ the conditional distribution of type α^i given that the player is behavioral. A rational player holds one of two initial beliefs: z_h and z_l, which are the probabilities of their opponents being behavioral and $z_h > z_l$ by assumption. We assume that the distributions of initial beliefs are symmetric between two players, the distribution is common knowledge while no one has private information about the other player at time zero.

Throughout this section, the initial beliefs are assigned by nature before time zero, denote the probability of a player being "optimistic" (holding belief z_l) as p_l, and the probability of a player being "pessimistic" (holding belief z_h) as p_h. Later we will see that starting from a common prior, players receive an exogenous signal and update their beliefs through Bayes' rule, we defer this generating process of initial beliefs z_h and z_l until Section 4, for now, we take the probabilities p_h and p_l as given.

At the moment, we assume that there is one behavioral type for each player. At time 0, player 1 chooses the demand α^1 to mimic if she was rational, after observing the offer α^1, the rational player 2 chooses either to accept or make a demand α^2, both players of behavioral types choose the demands according to their type(s). We assume $\alpha^1 + \alpha^2 > 1$ to eliminate trivial equilibria. After player 2 makes his demand, player 1 can choose either to concede or enter the stage of war of attrition. Finally, player i's discount rate is r^i. This bargaining game is denoted as $B = \{(C^i, z^i_h, z^i_l, \pi^i, r^i, p^i_l, p^i_h)^2_{i=1}\}$.

We focus on the analysis of equilibrium in the case of a single behavioral type α^i. Rational player i's strategy is represented by the cumulative distribution F^i on $\mathbb{R}_+$. When player i concedes at time t before player j, he/she gets $(1-\alpha^j)\exp(-r^i t)$, and player j gets $\alpha^j \exp(-r^j t)$.

Our model inherits many features from Abreu and Gul (2000), for instance, the set of equilibria is characterized by the properties that at most one player concedes with strictly positive probability at time zero, after time zero, players' mixed strategies make their opponents indifferent between conceding within a certain interval and there exists a finite time after which all types of players concede with probability one simultaneously. However, in our model, only the most optimistic players concede with probability one at the same time, players holding more pessimistic beliefs concede at earlier times without overlapping with the support of optimistic types' conceding strategies.

Given the strategy profile $(F^i_h, F^i_l)^2_{i=1}$ and the distribution of players' initial beliefs p_l and

p_h, the expected utility for optimistic player i who concedes at time t is:

$$\begin{aligned} u_l^i(t) =&(1-z_l)\left\{\int_{x=0}^{t}\alpha^i\exp(-r^ix)\,\mathrm{d}\,\mathrm{E}\,F^j(x)+\left(1-\alpha^j\right)\exp(-r^it)\left(1-\mathrm{E}\,F^j(t)\right)\right\} \\ &+z_l\exp(-r^it)(1-\alpha^j) \end{aligned} \tag{1}$$

where $\mathrm{E}\,F^j := p_lF_l^j+p_hF_h^j$ is defined as the expected strategy of player j. The term multiplied by the probability $(1-z_l)$ is the expected utility conditional on the opponent being rational, the term multiplied by the probability z_l is the expected utility conditional on the opponent being behavioral, since behavioral opponents never concede, player i always gets $(1-\alpha^j)$ discounted by factor r^i. In order to make rational player i indifferent between conceding at any time t, player j's strategy has to be at a concession rate such that player i's utility is invariant with respect to time t. The differentiability of u^i implies that F is differentiable, let the time derivatives be f_l^j and f_h^j, we get the differential equation:

$$\begin{aligned} 0 =&(1-z_l)\Big\{\alpha^i\exp\left(-r^it\right)\mathrm{E}\,f^j(t)-\left(1-\alpha^j\right)\mathrm{E}\,f^j(t)\exp\left(-r^it\right) \\ &-\left(1-\alpha^j\right)r^i\left(1-\mathrm{E}\,F^j(t)\right)\exp\left(-r^it\right)\Big\}-z_lr^i\exp\left(-r^it\right)(1-\alpha^j) \end{aligned}$$

In standard form:

$$\mathrm{E}\,f^j(t)+\mathrm{E}\,F^j(t)\frac{(1-\alpha^j)r^i}{(\alpha^i-(1-\alpha^j))}=\frac{(1-\alpha^j)r^i}{(1-z_l)(\alpha^i-(1-\alpha^j))}$$

Solving the equation we have:

$$\mathrm{E}\,F_l^j(t)=\frac{\exp(\lambda^it)-1}{\exp(\lambda^it)(1-z_l)}+\frac{c_l^j}{\exp(\lambda^it)} \tag{2}$$

where $\lambda^i=\frac{r^i(1-\alpha^j)}{[\alpha^i-(1-\alpha^j)]}$. [1] Condition (2) is the condition for a player i with optimistic belief z_l to be indifferent between conceding at different times. Symmetrically, for a player i with pessimistic belief z_h to be indifferent, we have

$$\mathrm{E}\,F_h^j(t)=\frac{\exp(\lambda^it)-1}{\exp(\lambda^it)(1-z_h)}+\frac{c_h^j}{\exp(\lambda^it)} \tag{3}$$

for $i\in\{1,2\}$ and $j\in\{1,2\}\setminus i$. Condition (2) and Condition (3) are the conditions for a

[1] See Appendix A for the calculations.

rational player i to be indifferent between conceding at each time t when he/she holds the belief of z_l or z_h, respectively. These expressions are not unique given each player's type, belief, time preference and initial demands, specifically, the constants in the solutions of the differential equations are not uniquely pinned down yet. However, additional conditions can help us determine the moving parts, the additional requirements come from the fact that each player has a *reputation* for rationality, since only rational players concede and rates of concession remain constant through the support, eventually, the posterior probability of each player being behavioral must reach 1 at a finite time.

Since optimistic players concede at later times, the weaker player is the one whose reputation for being behavioral reaches 1 at an earlier time conditional on holding the optimistic belief, as a result, the player concedes with sufficient probability at time zero so that both optimistic players' reputation for being behavioral reaches 1 at the same time.

Define an agent's *exhaustion time* as the time when the reputation of being behavioral reaches probability 1 conditional on the agent holding initial belief z_h or z_l and not conceding at time zero. Let T_h^i denote the exhaustion time of a rational player i holding belief z_h (i.e., conditional on being rational and pessimistic, player i concedes on or before T_h^i with probability 1). Similarly, T_l^i is defined as the exhaustion time of a rational player i holding belief z_l.

At the exhaustion time of each type of player, the probability of concession conditional on rationality and a specific type of belief must equal to 1, starting with the pessimistic players, by Bayes' rule:

$$\begin{aligned} 1 &= \mathbb{P}\left(\text{ player } i \text{ concedes before } T_h^i \mid \text{ player } i \text{ is rational and pessimistic}\right) \\ &= \frac{\mathbb{P}\left(\text{ player } i \text{ concedes before } T_h^i\right)}{\mathbb{P}(\text{ player } i \text{ is rational }) * \mathbb{P}(\text{ player } i \text{ is pessimistic })} \\ &= \frac{F_h^i\left(T_h^i\right)}{p_h(1-z_h)} \end{aligned} \tag{4}$$

we have $F_h^i(T_h^i) = p_h(1-z_h)$. By the same argument, for optimistic player i holding belief z_l, $F_l^i(T_l^i) = p_l(1-z_l)$.

Starting with the pessimistic players, if a player does not concede at time zero, then $F^i(0) = 0$, which requires the constant $c^i = 0$, therefore, to calculate T_h^i, set $c_h^i = 0$ in Condition (3)[2] and solve:

$$\frac{\exp(\lambda^j T_h^i) - 1}{\exp(\lambda^j T_h^i)(1-z_h)} = p_h(1-z_h)$$

[2]The superscripts i and j in the expressions are interchangeable so they can represent either player.

The solution is:

$$T_h^i = \frac{-\log(1 - p_h(1 - z_h)^2)}{\lambda^j} \tag{5}$$

for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus i$. Similarly, from the equation

$$\frac{\exp(\lambda^j T_l^i) - 1}{\exp(\lambda^j T_l^i)(1 - z_l)} = p_l(1 - z_l)$$

the exhaustion times of optimistic players are:

$$T_l^i = \frac{-\log(1 - p_l(1 - z_l)^2)}{\lambda^j} \tag{6}$$

for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus i$.

For each rational player, the one with larger exhaustion time conditional on being optimistic must concede with sufficient probability at time zero, which is dictated by the constant c_h^i, since at the end of bargaining, the player having longer exhaustion time has no incentive to delay concession once he/she knows for sure that the other player is behavioral, we require that $F^1(T_l) = F^2(T_l) = 1 - z_l$, so that both players concede with probability 1 at $T_l := \min\{T_l^1, T_l^2\}$, the calculation is done in Appendix B.

To pin down constant c_l^i, note that the strategy F^i must be continuous at any time t, otherwise players would strictly prefer to wait an instance before conceding around the discontinuity (this argument is detailed in the proof of Proposition 1), thus as a part of the strategy at a later phase, we require that when the optimistic player i starts conceding at the cutoff time T_h^i, a mass concession that equals to $p_h(1 - z_h)$ must be made:

$$F_l^i(T_h^i) = \frac{\exp(\lambda^j T_h^i) - 1}{\exp(\lambda^j T_h^i)(1 - z_l)} + \frac{c_l^i}{\exp(\lambda^j T_h^i)} = p_h(1 - z_h) \tag{7}$$

substitute Equation (5) into Equation (7), we get:

$$c_l^i = p_h(1 - z_h)\exp(\lambda^j T_h^i) - \frac{\exp(\lambda^j T_h^i) - 1}{(1 - z_l)} \tag{8}$$

for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus i$.

Now we have uniquely pinned down constants c_h^i and c_l^i. Combining the previous results, we propose that in equilibrium, each player chooses a mixed strategy such that it makes their opponents indifferent between conceding at any time within their respective stages, which are determined by their exhaustion times. Therefore, at any given time, for each side, only one type of player holding one type of belief concedes with strictly positive probability.

All players know their own types and the beliefs they are holding, so they choose to concede at the previously calculated rates until their reputation for being behavioral reaches 1 conditional on the beliefs they are holding. The conditional distribution of concession is common knowledge, but players do not know whether their opponents are rational or behavioral, nor do they know the belief that each other's holding.

From player 2's perspective, conditional on player 1 being rational:

$$F^1 = \begin{cases} \frac{\exp(\lambda^2 t)-1}{\exp(\lambda^2 t)(1-z_h)} + \frac{c_h^1}{\exp(\lambda^2 t)} & \text{for } t \in [0, T_h^2] \\ \frac{\exp(\lambda^2 t)-1}{\exp(\lambda^2 t)(1-z_l)} + \frac{c_l^1}{\exp(\lambda^2 t)} & \text{for } t \in (T_h^2, T_l^2] \end{cases} \tag{9}$$

and from player 1's perspective conditional on player 2 being rational:

$$F^2 = \begin{cases} \frac{\exp(\lambda^1 t)-1}{\exp(\lambda^1 t)(1-z_h)} + \frac{c_h^2}{\exp(\lambda^1 t)} & \text{for } t \in [0, T_h^1] \\ \frac{\exp(\lambda^1 t)-1}{\exp(\lambda^1 t)(1-z_l)} + \frac{c_l^2}{\exp(\lambda^1 t)} & \text{for } t \in (T_h^1, T_l^1] \end{cases} \tag{10}$$

where $c_l^i = p_h(1-z_h)\exp(\lambda^j T_h^i) - \left(\exp(\lambda^j T_h^i) - 1\right)/(1-z_l)$, $\lambda^i = r^i\left(1-\alpha^j\right)/\left(\alpha^i - \left(1-\alpha^j\right)\right)$ and c_h^i is determined by Equation (31).

To illustrate the dynamics of this bargaining game, we plot the timeline showing the reputation and concession rates stipulated in this section, take the case where $\lambda^1 < \lambda^2$ as an example:

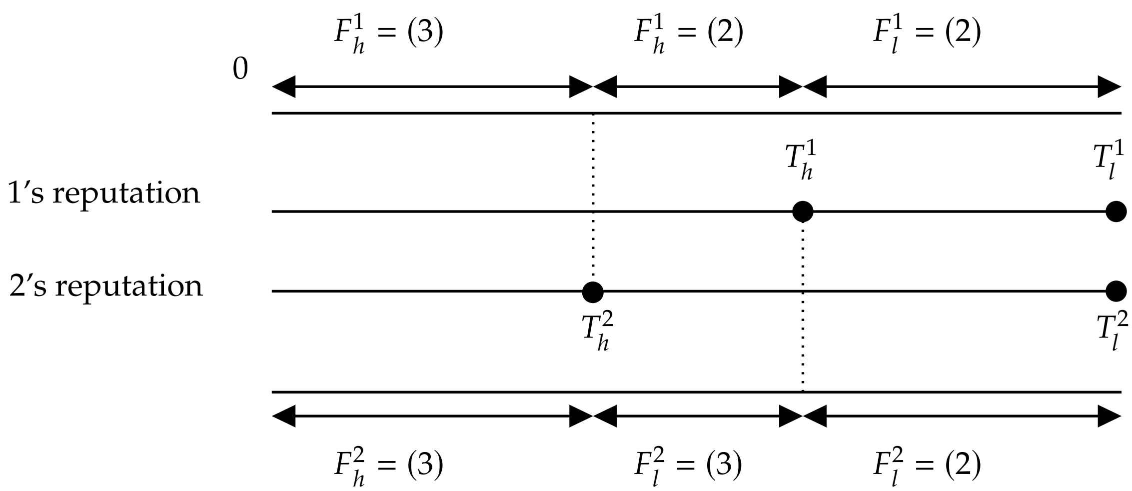

Figure 1: Equilibrium timeline when $\lambda^1 < \lambda^2$

In Figure 1, we can see that each player concedes with a rate such that the rational opponent that the player potentially faces at any given time is made indifferent. The concession rate switches at the exhaustion time of each player's opponent, while the identities of players implementing the strategies switch at each player's own exhaustion time.

As next steps we want to prove the following two results: 1. Players with z_l strictly prefers to wait until their pessimistic opponents' exhaustion time. 2. The proposed strategy profile forms the unique sequential equilibrium.

We want to show that players holding optimistic belief z_l will prefer to wait until their pessimistic opponents drop out before conceding:

Lemma 1. *Given a strategy F^j such that a pessimistic player i with the belief z_h is made indifferent between conceding at any $t \in [0, T_h^i]$, an optimistic player i with the belief z_l strictly prefers to wait until T_h^i before conceding.*

Proof. The utility difference between conceding at time T_h^i and t for any $t \in [0, T_h^i]$ for player i holding z_h is:

$$\begin{aligned} u_h^i(T_h^i) - u_h^i(t) =&(1-z_h)\Bigg\{\alpha^i \left(\int_{x=0}^{T_h^i} \exp(-r^i x)\,\mathrm{d}\,\mathrm{E}\,F^j(x) - \int_{x=0}^{t} \exp(-r^i x)\,\mathrm{d}\,\mathrm{E}\,F^j(x)\right) \\ &+ (1-\alpha^j)\left[\exp(-r^i T_h^i)\left(1-\mathrm{E}\,F^j(T_h^i)\right) - \exp(-r^i t)\left(1-\mathrm{E}\,F^j(t)\right)\right]\Bigg\} \\ &+ z_h(1-\alpha^j)\left[\exp(-r^i T_h^i) - \exp(-r^i t)\right] \end{aligned}$$

By Condition (3), we know that player i holding z_h is indifferent between conceding at any $t \in [0, T_h^i]$ under F^j, thus $u_h^i(T_h^i) - u_h^i(t) = 0$. Note that since the second term $z_h(1-\alpha^j) \cdot \left[\exp(-r^i T_h^i) - \exp(-r^i t)\right]$ is negative, the first term must be positive. When the belief z_h is switched to z_l, the first term is larger as the multiplier $(1-z_h)$ goes up to $(1-z_l)$, the second term is also larger as the multiplier z_h goes down to z_l, this means the overall utility difference is positive and an optimistic player i holding z_l would strictly prefer to wait when player j's strategy takes the form in Equation (10).

Alternatively, to see this,

$$\begin{aligned} &\left(u_l^i(T_h^i) - u_l^i(t)\right) - \left(u_h^i(T_h^i) - u_h^i(t)\right) \\ =&(z_h - z_l)\Bigg\{\alpha^i \left(\int_{x=t}^{T_h^i} \exp(-r^i x)\,\mathrm{d}\,\mathrm{E}\,F^j(x) - \int_{x=0}^{t} \exp(-r^i x)\,\mathrm{d}\,\mathrm{E}\,F^j(x)\right) \\ &+ (1-\alpha^j)\left[\exp(-r^i T_h^j)\left(1-\mathrm{E}\,F^j(T_h^i)\right) - \exp(-r^i t)\left(1-\mathrm{E}\,F^j(t)\right)\right]\Bigg\} \\ &+ (z_l - z_h)(1-\alpha^j)\left[\exp(-r^i T_h^j) - \exp(-r^i t)\right] \\ =&u_l^i(T_h^i) - u_l^i(t) > 0 \end{aligned}$$

where the last equality was due to the fact that $u^i_h(T^j_h) - u^i_h(t) = 0$ for $t \in [0, T^j_h]$.

Q.E.D.

Let $\left(\hat{F}^1, \hat{F}^2\right)$ be the unique strategy profile characterized by Equation (9), Equation (10), Equation (8) and Equation (31).

Proposition 1. *If $C^i = \{\alpha^i\}$ for $i = 1, 2$, then the unique sequential equilibrium of B is $\left(\hat{F}^i_h, \hat{F}^i_l\right)_{i=1}^2$.*

Proof. We want to show that the equilibrium must have the specified form and it is indeed an equilibrium.

Claim (a) $T^1_l = T^2_l$, meaning the most optimistic players must concede at the same time, if not, the one with higher T_l will be strictly better off to concede immediately at T_l.

Claim (b) For all $t > 0$ such that $F^1(t) - F^1(t^-) > 0$, we have that $F^2(t) - F^2(t^-) = 0$. This claim follows because player j would get higher expected utility by not conceding at all on the interval $(t - \varepsilon, t]$ for small $\varepsilon > 0$ and instead conceding an instant later because player i has a positive probability of conceding at time t.

Claim (c) For $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus i$, if $F^j(t^-) = F^j(t)$, then u^i is continuous at t if $t \in (0, T^i_h) \cup (T^i_h, T^i_l)$, since:

$$\begin{aligned}
\lim_{s \to t^-} \left|u^i(t) - u^i(s)\right| &= \lim_{s \to t^-} \left|(1-z)\alpha^i \int_s^t \exp(-r^i x)\, \mathrm{d}F^j(x)\right| \\
&+ \lim_{s \to t^-} \left|(1-z)\alpha^i(1-\alpha^j)\left[\exp(-r^i t)\left(1 - F^j(t)\right) - \exp(-r^i s)\left(1 - F^j(s)\right)\right]\right| \\
&+ \lim_{s \to t^-} \left|z(1-\alpha^j)\left[\exp(-r^i t) - \exp(-r^i s)\right]\right| \\
&= 0
\end{aligned}$$

where the second term equals to zero due to continuity of strategies F. This works similarly for $s \to t^+$.

Claim (d) Let $T^0 := T^1_l = T^2_l$, there is no interval (s, s') such that $0 \le s < s' \le T^0$ $F^1(s) = F^1(s')$ and $F^2(s) = F^2(s')$.

Proof of Claim (d). Assume by way of contradiction that such an interval exists and without loss of generality, let:

$$s^* = \sup\left\{s' \le T^0 : F^i(s) = F^i(s') \text{ for } i = 1, 2 \text{ and } s < s'\right\}$$

Fix $s' \in (s, s^*)$, note that for a small $\varepsilon > 0$, for any $t \in (s^* - \varepsilon, s^*)$, $\exists \delta > 0$ such that:

$$u^i(s') - \delta \ge u^i(t)$$

for $i = 1, 2$ (since the other player is not conceding). Note that by (b) and (c) there exists an i such that $u^i(\cdot)$ is continuous at s^*, since if $F^j(s^*) - F^j((s^*)^-) > 0$, we have that $F^i(s^*) = F^i((s^*)^-)$. Thus for this player i and for some $\eta > 0$, $u^i(t) < u^i(s')$ for any $t \in (s^*, s^* + \eta)$.

Therefore, F^i must be constant on $(s^*, s^* + \eta)$ since F^i is optimal. Not conceding at s' means that you shouldn't concede at any t for which $u^i(s') > u^i(t)$. But then F^j is also constant on $(s^*, s^* + \eta)$, and thus s^* could not have been the supremum as defined. ■

Claim (e) If $0 \leq s < s' \leq T^0$ then $F^i(s) < F^i(s')$ for $i = 1, 2$. This follows since if F^i is constant at some interval, then the optimality of F^j implies that F^j is constant at the same interval, contradicting claim (d).

Claim (f) F^i is continuous at time $t > 0$ for $i = 1, 2$. If F^i has a jump at time t (i.e. $F^i(t) - F^i(t^-) > 0$), then F^j is constant on the interval $(t - \varepsilon, t)$, contradicting claim (e).

The optimality and continuity requirements have uniquely pinned down constants c_l^1, c_l^2, c_h^1 and c_h^2. Given the equilibrium strategies $\left(\hat{F}^1, \hat{F}^2\right)$, u^1 and u^2 are constants within their respective intervals $[0, T_h^1]$, $(T_h^1, T_l^1]$ and $[0, T_h^2]$, $(T_h^2, T_l^2]$. We also know that $u_l^i(s) < u_l^i(T_h^2)\ \forall s < T_h^j$, $i = 1, 2$, hence any mixed strategy on the support is optimal for all types of players 1 and 2, which shows that $\left(\hat{F}^1, \hat{F}^2\right)$ is indeed an equilibrium.

Q.E.D.

In the unique equilibrium above, rational players with different beliefs concede during different time intervals. Optimistic players would strictly prefer to wait until they know for sure that their opponents are either behavioral or holding optimistic beliefs. Pessimistic players, on the other hand, would start conceding immediately on or after time zero and finish concession before they know for sure that their opponents are either behavioral or holding optimistic beliefs. Rational players' payoffs are also dependent on the signal they receive, a player i receiving a pessimistic signal about his/her opponent's type holds the belief z_h, in equilibrium, the player is indifferent between conceding at any time $t \in (0, T_h^1]$, his/her expected payoff is:

$$(1 - z_h)\left(\alpha^i p_h F_h^j(0) + (1 - \alpha^j)(1 - p_h F_h^j(0))\right) + z_h(1 - \alpha^j)$$

which is exactly $(1 - \alpha^j)$ if player i is the one who concedes at time zero. Note that players receiving positive signals have payoffs higher than $(1 - \alpha^j)$ in equilibrium by Lemma 1, this means that in expectation, players holding optimistic beliefs are strictly better off than not receiving any signal, which is the degenerate case where each player holds the prior belief with certainty.

4 Signal Structure and Limiting Cases

To utilize the results we get in the previous section, we need an alternative way of thinking about how the beliefs z_h and z_l were formed. In Section 3, we assumed that the initial beliefs were assigned by nature before time zero, now we discuss in detail how the initial beliefs were formed starting from the prior. Instead of nature assigning the beliefs as well as types, before the bargaining begins, nature assigns players' types only according to a prior distribution such that a player is behavioral with probability z_0, which is common knowledge.

Assuming that the players have access to an exogenous signal before time zero, denoted as y, distributed according to:

$$y = \begin{cases} y_h, & \text{with probability } p(y_h|b) \text{ if } i \text{ is behavioral;} \\ & \text{with probability } p(y_h|r) \text{ if } i \text{ is rational} \\ y_l, & \text{with probability } p(y_l|b) \text{ if } i \text{ is behavioral;} \\ & \text{with probability } p(y_l|r) \text{ if } i \text{ is rational} \end{cases}$$

where $p(y_h|b) \geq p(y_l|b)$, $p(y_l|r) \geq p(y_h|r)$, $p(y_h|b) + p(y_l|b) = 1$ and $p(y_h|r) + p(y_l|r) = 1$.

Player i updates his/her belief after observing signal y, let prior be z_0, by Bayesian updating, there are two possible posteriors generated under each fixed signal structure:

$$z_h := p(b|y_h) = \frac{z_0 p(y_h|b)}{z_0 p(y_h|b) + (1-z_0)p(y_h|r)}$$

$$z_l := p(b|y_l) = \frac{z_0 p(y_l|b)}{z_0 p(y_l|b) + (1-z_0)p(y_l|r)}$$

Therefore, the probabilities of holding beliefs z_h and z_l are $p_h := z_0 p(y_h|b)+(1-z_0)p(y_h|r)$ and $p_l := z_0 p(y_l|b)+(1-z_0)p(y_l|r)$, respectively. The two probabilities took part in the analysis in Section 3, and the simplified notation does not change the results derived above.

The signaling device gives us a foundation for the analysis of limiting cases in terms of the prior z_0. Fixing the signal structure, we want to show the dynamics of equilibrium payoffs and strategies.

To approach this problem, we start with the model in Section 3, again, the indifference condition told us that to make the other party indifferent, one must have the concession rate:

$$F_z^j(t) = \frac{\exp(\lambda^i t) - 1}{\exp(\lambda^i t)(1-z)} + \frac{c_z^j}{\exp(\lambda^i t)} \tag{11}$$

where z denotes any belief a player's holding. When the prior belief z_0 goes to 0 (i.e. players are more likely to be rational), z_h and z_l go to 0 as well, the exhaustion time is now determined by:

$$F_z^j(t) = \frac{\exp(\lambda^i t) - 1}{\exp(\lambda^i t)(1-z)} = 1 - z$$

Setting $z_0 = 0$, we get that the exhaustion times for both types of players go to infinity: knowing the opponent's full rationality, each player is more patient and never concedes with probability 1 at any finite time t. We demonstrate the equilibrium timeline in Figure 2:

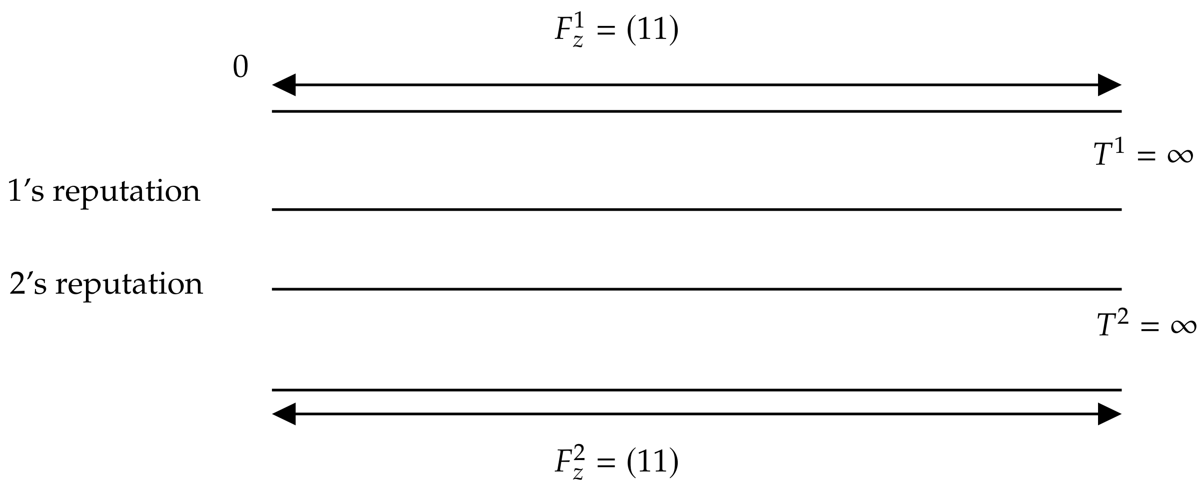

Figure 2: Equilibrium timeline when $z_0 = 0$

In this case, every player is holding the same posterior belief $z = 0$, meaning that both sides are certain of their opponents' rationality. Consequently, the two intervals we have seen in the previous section collapse to the positive real line, and the bargaining game becomes the war of attrition in the limit, as the signal realizations no longer matter and each player concedes with strictly positive probability over $\mathbb{R}_{++}$.

When $z \to 0$, no player concedes at time zero, the corresponding payoffs in the limit becomes:

$$\begin{aligned}
\lim_{z \to 0} u_z^i(t) =& (1-z)\left\{\int_{x=0}^{t} \alpha^i \exp(-r^i x)\,\mathrm{d}F^j(x) + \left(1-\alpha^j\right)\exp(-r^i t)\left(1 - F^j(t)\right)\right\} \\
&+ z\exp(-r^i t)(1-\alpha^j) \\
=& \int_{x=0}^{\infty} \alpha^i \exp(-r^i x)\,\mathrm{d}F^j(x)
\end{aligned} \tag{12}$$

The structure degenerates to a perfect information war of attrition model with known values, however, the degenerate case arises because we have fixed initial demands while assuming full commitment, next, we allow for strategic demands where players make the choices with only the knowledge of the common prior z_0 and the signal structure y.

5 Strategic Demands

Similar to Kambe (1999), we now allow for strategic choices of initial demands, which means instead of taking demands (α^1, α^2) as given, we allow players to strategically choose their initial demands based on their prior z_0 but before getting the exogenous signal y to maximize their ex ante expected utilities.

At time zero, both players simultaneously announce their initial demands (α^1, α^2), where $\alpha^i \in [0, 1)$. If $\alpha^1 + \alpha^2 \leq 1$, players split the surplus equally (i.e., player i gets $\alpha^i + (1 - \alpha^i - \alpha^j)/2$), though the tie-breaking assumption does not change the results. We call any pair of demands (α^1, α^2) *compatible* when $\alpha^1 + \alpha^2 \leq 1$ and *just compatible* if $\alpha^1 + \alpha^2 = 1$. After time zero, if $\alpha^1 + \alpha^2 > 1$, players enter the next stage where they decide whether and when to concede. In the setting of this section, after making the initial demands, nature assigns the type of each player, which can be either rational or behavioral, if a player becomes behavioral, he/she never concedes. The assignment of the types follows a prior distribution z_0, which means each player turns behavioral with probability z_0. Both players know the prior distribution and have access to the exogenous signal that reveals information about each other's type, the signal distribution is the same as in Section 4. The equilibrium concept, conceding strategies and form of utility functions after the making of initial demands are identical to those in Section 3.

The natural candidate for an equilibrium outcome is the pair of just compatible demands such that optimistic players' exhaustion times are aligned without requiring mass concession at time zero from one of the players. As before, the exhaustion times are essential elements reflecting each player's reputation and determining the identity of the player who concedes at time zero, for player i:

$$T_h^i = \frac{-\log(1 - p_h(1 - z_h)^2)}{\lambda^j}$$
$$T_l^i = \frac{-\log(1 - p_l(1 - z_l)^2)}{\lambda^j}$$

where $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus i$ and $\lambda^j = r^j \left(1 - \alpha^i\right) / \left[\alpha^j - \left(1 - \alpha^i\right)\right]$.

Let (α_*^1, α_*^2) be a pair of just compatible demands that aligns the exhaustion times of optimistic players, for reasons discussed in Section 3, the one who concedes at time zero is the one with longer exhaustion time after receiving a positive signal and hence holding an optimistic belief z_l.

Setting the ratio of exhaustion times $T_l^i/T_l^j = 1$, we have:

$$\begin{aligned}
&\frac{-\log(1-p_l(1-z_l)^2)}{\lambda^j}\Bigg/\frac{-\log(1-p_l(1-z_l)^2)}{\lambda^i} = 1\\
\Longrightarrow &\lambda^i/\lambda^j = 1\\
\Longrightarrow &\frac{r^i\left(1-\alpha^j\right)}{\left[\alpha^i-\left(1-\alpha^j\right)\right]}\Bigg/\frac{r^j\left(1-\alpha^i\right)}{\left[\alpha^j-\left(1-\alpha^i\right)\right]} = 1\\
\Longrightarrow &r^i\left(1-\alpha^j\right) = r^j\left(1-\alpha^i\right)
\end{aligned}$$

combined with the just compatibility: $\alpha^1 + \alpha^2 = 1$, we get:

$$\alpha_*^i = \frac{r^j}{r^i + r^j}$$

for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus i$. We call this pair of initial demands the Nash demands as it coincides with the case where players have no uncertainty about each other's rationality. Under the initial demands (α_*^i, α_*^j), no player concedes at time zero since the exhaustion times are automatically aligned. Due to the symmetry of (optimistic) beliefs, we can see that the belief z_l does not enter the expressions here, we defer the discussion of asymmetric beliefs to a later section. The following proposition shows that despite the reputation alignment feature of the just compatible demands (α_*^i, α_*^j), an equilibrium is not guaranteed.

For every fixed pair of incompatible initial demands $\overline{\alpha} = (\alpha^1, \alpha^2)$, let $F_{\overline{\alpha}}^i$ be the equilibrium strategies conditional on $\overline{\alpha}$ specified by Equation (9) and Equation (10). The expected utility for optimistic player i who concedes at time t is denoted as $u_l^i(t)$ with the same expression as Equation (1).

Proposition 2. *The immediate settlement at Nash demands (α_*^1, α_*^2) is an equilibrium outcome if and only if $\forall \alpha^i \in (\alpha_*^i, 1]$, $i = 1, 2$,*

$$\frac{(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}{1-(1-z_0)p(y_h|r)-z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)+p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right)} \leq 1-\alpha_*^j \tag{13}$$

where $\overline{\alpha} := (\alpha^i, \alpha_*^j)$.

Proof. In the proof, we let player i represent any one of player 1 and 2, and $j \in 1, 2 \setminus i$. First, player i has no incentive to decrease the demand when facing α_*^j, as it will strictly decrease the resulting immediate settlement and hence the player's utility. Assume player

i unilaterally deviates to a higher demand $\alpha^i > \alpha^i_*$, we first calculate the mass of initial concession at time zero by the pessimistic player, which is exactly c^i_h.

Since player i demands more than α^i_*, this means $T^i_l > T^j_l$ and player i needs to make a mass concession at time zero. To see this, first notice that $\partial T^i_l / \partial \lambda^j < 0$, by the expression of $\lambda^j = r^j \left(1-\alpha^i\right) / \left[\alpha^j - \left(1-\alpha^i\right)\right]$, it is clear that $\partial \lambda^j / \partial \alpha^j < 0$, this leads to $\partial T^i_l / \partial \alpha^j > 0$.

Then, player i's ex ante expected utility becomes:

$$
\begin{aligned}
\mathrm{E}\, u^i_{z_0} =& c^i_h(1-z_0)(1-\alpha^j_*) + (1-c^i_h)(1-z_0)\left(p(y_h|r)\int_0^{T^i_h} u^i_h(x)\,\mathrm{d}F^i_{\bar{\alpha}}(x) + p(y_l|r)\int_{T^i_h}^{T^j_l} u^i_l(x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right) \\
&+ z_0 c^i_h(1-\alpha^j_*) + z_0(1-c^i_h)\left(p(y_h|b)\int_0^{T^i_h}(1-\alpha^j_*)\exp(-r^i x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right. \\
&\qquad\left. + p(y_l|b)\int_{T^i_h}^{T^j_l}(1-\alpha^j_*)\exp(-r^i x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right) \\
=& c^i_h(1-\alpha^j_*) + (1-c^i_h)\left[(1-z_0)\left(p(y_h|r)\int_0^{T^i_h} u^i_h(x)\,\mathrm{d}F^i_{\bar{\alpha}}(x) + p(y_l|r)\int_{T^i_h}^{T^j_l} u^i_l(x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right)\right. \\
&\left. + z_0\left(p(y_h|b)\int_0^{T^i_h}(1-\alpha^j_*)\exp(-r^i x)\,\mathrm{d}F^i_{\bar{\alpha}}(x) + p(y_l|b)\int_{T^i_h}^{T^j_l}(1-\alpha^j_*)\exp(-r^i x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right)\right] \\
=& c^i_h(1-\alpha^j_*) + (1-c^i_h)\left[(1-z_0)\left(p(y_h|r)(1-\alpha^j_*) + p(y_l|r)\int_{T^i_h}^{T^j_l} u^i_l(x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right)\right. \\
&\left. + z_0\left(p(y_h|b)\int_0^{T^i_h}(1-\alpha^j_*)\exp(-r^i x)\,\mathrm{d}F^i_{\bar{\alpha}}(x) + p(y_l|b)\int_{T^i_h}^{T^j_l}(1-\alpha^j_*)\exp(-r^i x)\,\mathrm{d}F^i_{\bar{\alpha}}(x)\right)\right]
\end{aligned}
\tag{14}
$$

which takes into account of the potential signals that a player receives and the corresponding posterior beliefs. To compare $\mathrm{E}\, u^i_{z_0}$ with the utility from the immediate settlement $(1-\alpha^j_*)$, we only need to compare $(1-\alpha_*^j)$ with the term

$$
\begin{aligned}
&(1-z_0)\left(p(y_h|r)(1-\alpha_*^j)+p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\right)\\
&+z_0\left(p(y_h|b)\int_0^{T_h^i}(1-\alpha_*^j)\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)+p(y_l|b)\int_{T_h^i}^{T_l^j}(1-\alpha_*^j)\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\right)\\
=&(1-\alpha_*^j)\left[(1-z_0)p(y_h|r)+z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)+p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\right)\right]\\
&+(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x) \qquad (15)
\end{aligned}
$$

When player j demands α_*^j, the best response for player i is demanding $\alpha_*^i=(1-\alpha_*^j)$ if and only if Equation (15) is less or equal than $(1-\alpha_*^j)$, which gives:

$$
\begin{aligned}
&(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\\
\leq&(1-\alpha_*^j)\left(1-(1-z_0)p(y_h|r)-z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)+p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\right)\right)
\end{aligned}
$$

where $u_l^i(x)$ is given by Equation (1). This is the desired condition for (α_*^1,α_*^2) to be an equilibrium outcome, both terms in the right-hand side are strictly positive, so Condition (13) is well defined. If the condition is satisfied for all $\alpha^i\in(\alpha_*^i,1]$, there exists no profitable unilateral deviation, and immediate settlement at (α_*^1,α_*^2) is an equilibrium outcome.

As the dual of the main condition, we have the condition for player i to strictly prefer to enter the war of attrition stage: $\exists\alpha^i\in(\alpha_*^i,1]$, $i=1,2$ such that,

$$
\frac{(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)}{1-(1-z_0)p(y_h|r)-z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)+p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\right)} > 1-\alpha_*^j \qquad (16)
$$

where $u_l^i(x)$ is given by Equation (1). If Condition (16) is satisfied, (α_*^1,α_*^2) cannot be an equilibrium outcome since there exist profitable unilateral deviations $\alpha^1>\alpha_*^1$ for player i, and the game goes to the war of attrition stage.

Q.E.D.

The above proposition showed that despite the demands being deterministic and just compatible, players still cannot reach immediate agreement at Nash demands unless additional conditions were satisfied, the below proposition summarizes this finding. The intuition behind this phenomenon is that ex ante, players are expecting access to an exogenous signal, the accuracy of this signal combined with the prior collectively determine players' behaviors in equilibrium, e.g., if a signal accurate enough was expected and the prior z_0 was not too high, players would strictly prefer to wait instead of settling at time zero. The conditions derived above are the starting points for explorations of comparative statics on equilibrium outcomes with respect to different sets of parameters such as the prior and signal accuracy, which will be discussed next.

The above-discussed observations formulated the key departure from Kambe (1999): the just compatible demands are no longer guaranteeing an equilibrium, the exogenous signaling device is generating additional payoff by improving the accuracy of each player's information, we first discuss the role of the prior z_0 and then proceed to defining and discussing the accuracy of the exogenous signal.

Proposition 3. *As long as $p(y_l|r) < 1$ and $p(y_h|b) < 1$, there exists z_* such that when $z_0 < z_*$, Nash demands (α_*^1, α_*^2) are not an equilibrium outcome.*

Proof. We only need to show that Condition (16) is satisfied when $z_0 \to 0$. Rearranging the terms gives us:

$$\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)$$
$$> \frac{(1-\alpha_*^j)\left(1-(1-z_0)p(y_h|r) - z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x) + p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)\right)\right)}{(1-z_0)p(y_l|r)}$$

By Bayes' rule:

$$\begin{aligned}
&\lim_{z_0\to 0}\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x) \qquad (17)\\
&= \lim_{z_l\to 0}\int_{T_h^i}^{T_l^j}\left[(1-z_l)\left\{\int_0^x \alpha^i\exp(-r^i h)\,\mathrm{d}\,\mathrm{E}\,F^j(h) + \left(1-\alpha^j\right)\exp(-r^i x)\left(1-\mathrm{E}\,F^j(x)\right)\right\}\right.\\
&\qquad\qquad \left. + z_l\exp(-r^i x)(1-\alpha^j)\right]\mathrm{d}F_{\bar{\alpha}}^i(x)\\
&= \int_{T_h^i}^{T_l^j}\left\{\int_{h=0}^x \alpha^i\exp(-r^i h)\,\mathrm{d}\,\mathrm{E}\,F^j(h) + \left(1-\alpha^j\right)\exp(-r^i x)\left(1-\mathrm{E}\,F^j(x)\right)\right\}\mathrm{d}F_{\bar{\alpha}}^i(x)
\end{aligned}$$

Taking the limit of the right-hand side gives:

$$\lim_{z_0\to 0}\frac{(1-\alpha_*^j)\left(1-(1-z_0)p(y_h|r)-z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)+p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right)\right)}{(1-z_0)p(y_l|r)}$$

$$=\frac{(1-\alpha_*^j)\,(1-p(y_h|r))}{p(y_l|r)}$$

$$=(1-\alpha_*^j)$$

Notice that $u_l^i(x)$ is a constant on the interval $[T_h^i, T_l^j]$, Equation (17) must be larger than $(1-\alpha_*^j)$, if not, player i with optimistic belief z_l would never wait to concede on $[T_h^i, T_l^j]$, a contradiction to Proposition 1.

Q.E.D.

When the prior z_0 goes to zero, rational players' identities are almost fully revealed, in this case, both sides have less incentive to settle immediately as they expect to face rational opponents, eventually, each player strictly prefers to wait.

Example 1. Figure 3 demonstrates the ex ante expected utility and how it varies with initial demands and the prior z_0. In this example, both players have a discount factor of $r^i = 0.8$, resulting in a pair of symmetric Nash demands $(0.5, 0.5)$, assuming player 1 deviates to a higher demand, we plot player 1's expected utility under different signal structures, the left panel shows her utility under an informative signal structure with $p(y_l|r) = p(y_h|b) = 0.75$ and $p(y_h|r) = p(y_l|b) = 0.25$ while the right panel shows the uninformative signal structure with $p(y_l|r) = p(y_h|b) = p(y_h|r) = p(y_l|b) = 0.5$ that degenerates to a case similar to that in Kambe (1999), in this example, a player's expected utility improves under optimal demands with and without an informative signal. Under an informative signal, the expected utility for the player who deviates is single-peaked, specifying an optimal departure from his/her Nash demand within the domain.

In addition to the prior belief of the other player being behavioral, the informativeness of the exogenous signal y is also playing a crucial part in characterizing the equilibrium. A measure of "distance" between signals and the definition of accuracy and informativeness is required for subsequent analyses.

Let y^* denote the perfectly informative signal such that conditional on the type of a player, there is only one signal realization within the support, without loss of generality, we

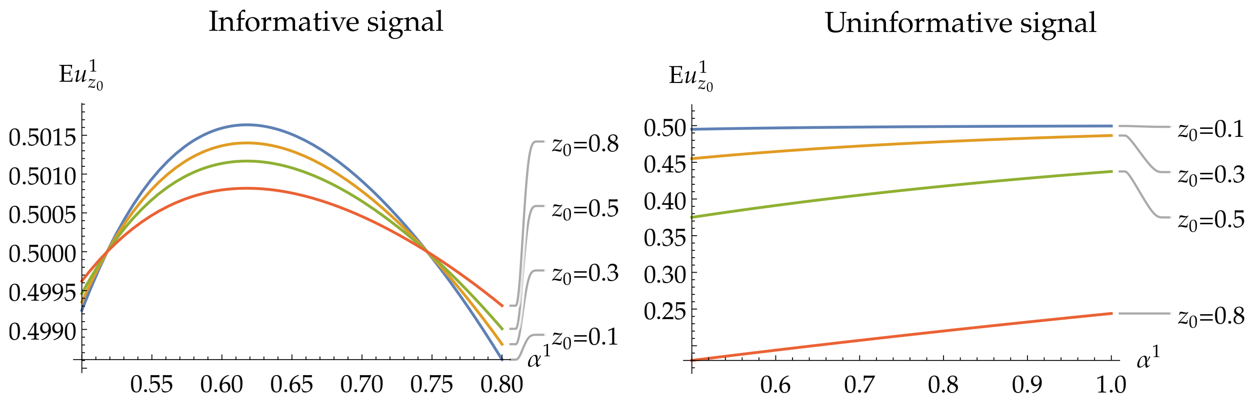

Figure 3: Expected utility and initial demands

assume the following signal structure:

$$y^* = \begin{cases} y_h, & \text{with probability 1 if } i \text{ is behavioral;} \\ y_l, & \text{with probability 1 if } i \text{ is rational} \end{cases}$$

And the informativeness of a signal y is measured by the relative entropy between the signal structures of y and y^*.

Definition 1 (Relative entropy, Cover and Thomas (2006))**.** The relative entropy or Kullback-Leibler distance between two probability mass functions $p(x)$ and $q(x)$ is defined as

$$\begin{aligned} D(p\|q) &:= \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)} \\ &= E_p \log\frac{p(X)}{q(X)} \end{aligned}$$

We follow the convention that $0\log\frac{0}{0}=0$ and the convention that $0\log\frac{0}{q}=0$ and $p\log\frac{p}{0}=\infty$. Therefore, if there exists $x\in\mathcal{X}$ such that $p(x)>0$ and $q(x)=0$ then $D(p\|q)=\infty$.

Before we proceed to the relationship between signal structures and equilibrium outcomes, we study how the bound of z_0 for Condition (16) to be satisfied moves with the conditional probability of each signal realization $p(y_h|r)$, $p(y_l|r)$, $p(y_h|b)$ and $p(y_l|b)$. Given the fact that $p(y_h|r)=1-p(y_l|r)$ and $p(y_l|b)=1-p(y_h|b)$, with a slight abuse of notation, let $f(y|\overline{\alpha}) \overset{\text{def}}{=} f\left(p(y_l|r), p(y_h|b)|\overline{\alpha}\right)$ be defined by:

$$
\begin{aligned}
f(y|\overline{\alpha}) =&(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) - (1-\alpha_*^j)\Bigg[1-(1-z_0)p(y_h|r) \\
&- z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right)\Bigg]
\end{aligned}
$$

Lemma 2. *$f(y|\overline{\alpha})$ is strictly increasing with $p(y_l|r)$ and $p(y_h|b)$.*

Proof. Holding everything other than $p(y_l|r)$ and $p(y_h|b)$ fixed, since $p(y_h|r) = 1 - p(y_l|r)$ and $p(y_l|b) = 1 - p(y_h|b)$, we can rewrite $f(y|\overline{\alpha})$ as:

$$
\begin{aligned}
f(y|\overline{\alpha}) =&(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) - (1-\alpha_*^j)\Bigg[1-(1-z_0)(1-p(y_l|r)) \\
&- z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + (1-p(y_h|b))\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right)\Bigg]
\end{aligned}
$$

Taking the partial derivative with respect to $p(y_l|r)$ gives:

$$
\begin{aligned}
\frac{\partial}{\partial p(y_l|r)}f(y|\overline{\alpha}) =&(1-z_0)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + (1-z_0)p(y_l|r)\frac{\partial}{\partial p(y_l|r)}\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) \\
&- (1-\alpha_*^j)(1-z_0) + z_0(1-\alpha_*^j)\frac{\partial}{\partial p(y_l|r)}\int_0^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)
\end{aligned}
$$

Here, we evaluate each term, first notice that $(1-z_0)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) - (1-\alpha_*^j)(1-z_0) > 0$ since $\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) > (1-\alpha_*^j)$, otherwise, players with the posterior belief z_l would not wait to concede during the second phase. To determine the sign of $\partial\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)/\partial p(y_l|r)$, it is sufficient to evaluate the sign of $\partial u_l^i/\partial p(y_l|r)$ since in equilibrium player i is indifferent between conceding at each time $t \in [T_h^i, T_l^j]$.

$$
\begin{aligned}
&\frac{\partial}{\partial p(y_l|r)} u_l^i(x) \\
=&\frac{\partial}{\partial p(y_l|r)} \left[(1 - z_l) \left\{ \int_0^x \alpha^i \exp(-r^i h) \, \mathrm{d} \, \mathrm{E} \, F^j(h) + \left(1 - \alpha_*^j\right) \exp(-r^i x) \left(1 - \mathrm{E} \, F^j(x)\right) \right\} \right. \\
&\qquad \left. + z_l \exp(-r^i t)(1 - \alpha_*^j) \right] \\
=& - \left\{ \int_0^x \alpha^i \exp(-r^i h) \, \mathrm{d} \, \mathrm{E} \, F^j(h) + \left(1 - \alpha_*^j\right) \exp(-r^i x) \left(1 - \mathrm{E} \, F^j(x)\right) \right\} \frac{\partial z_l}{\partial p(y_l|r)} \\
&+ \exp(-r^i x)(1 - \alpha_*^j) \frac{\partial z_l}{\partial p(y_l|r)} \\
=& \left\{ \exp(-r^i x)(1 - \alpha_*^j) - \int_0^x \alpha^i \exp(-r^i h) \, \mathrm{d} \, \mathrm{E} \, F^j(h) - \left(1 - \alpha_*^j\right) \exp(-r^i x) \left(1 - \mathrm{E} \, F^j(x)\right) \right\} \frac{\partial z_l}{\partial p(y_l|r)} \\
>& 0
\end{aligned}
$$

where the last inequality comes from the definition of z_l:

$$
\frac{\partial z_l}{\partial p(y_l|r)} = \frac{\partial}{\partial p(y_l|r)} \frac{z_0 p(y_l|b)}{z_0 p(y_l|b) + (1 - z_0) p(y_l|r)} < 0
$$

and

$$
\exp(-r^i x)(1 - \alpha_*^j) < \int_0^x \alpha^i \exp(-r^i h) \, \mathrm{d} \, \mathrm{E} \, F^j(h) + \left(1 - \alpha_*^j\right) \exp(-r^i x) \left(1 - \mathrm{E} \, F^j(x)\right)
$$

Finally, from the definitions of T_h^i and T_l^j, we get

$$
\frac{\partial}{\partial p(y_l|r)} \int_0^{T_l^j} \exp(-r^i x) \, \mathrm{d} F_{\overline{\alpha}}^i(x) > 0
$$

Therefore, $\partial f(y|\overline{\alpha})/\partial p(y_l|r) > 0$, next we take the partial derivative with respect to $p(y_h|b)$:

$$\begin{aligned}\frac{\partial f(y|\overline{\alpha})}{\partial p(y_h|b)} =& (1-z_0)p(y_l|r)\frac{\partial}{\partial p(y_h|b)}\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) \\ &+ z_0(1-\alpha_*^j)\Bigg[\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + p(y_h|b)\frac{\partial}{\partial p(y_h|b)}\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) \\ &- \int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + (1-p(y_h|b))\frac{\partial}{\partial p(y_h|b)}\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\Bigg]\end{aligned}$$

Note that $\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) > \int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)$, $\partial\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)/\partial p(y_h|b) > 0$ and $\partial\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)/\partial p(y_h|b) > 0$, this concludes the proof.

Q.E.D.

Proposition 4. *For a bargaining game $B = \{(z_0, r^i, y)_{i=1}^2\}$, the immediate settlement at Nash demands (α_*^1, α_*^2) is an equilibrium outcome if and only if one of the following conditions holds:*

(a)

$$D(y\|y^*) = 0$$

(b) $D(y\|y^*) > 0$ *and* $\forall \alpha^i \in (\alpha_*^i, 1], i = 1, 2,$ *and* $\overline{\alpha} := (\alpha^i, \alpha_*^j)$,

$$z_0 \geq \frac{1-\alpha_*^j - \int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}{(1-\alpha_*^j)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) - \int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}.$$

Proof. By definition, let $\mathcal{X} = \{y_h, y_l\}$, denote the generic element of $\mathcal{X}$ as x and consider the two distributions over $\mathcal{X}$, $y(x)$ and $y^*(x)$, since y^* represents the perfectly informative signal which always takes the value y_h (y_l) when the respective player is behavioral (rational), the two random variables y and y^* have the following joint distribution:

$y \backslash y^*$	y_h	y_l
y_h	$z_0 p(y_h\|b)$	$(1-z_0)p(y_h\|r)$
y_l	$z_0 p(y_l\|b)$	$(1-z_0)p(y_l\|r)$

Their relative entropy is given by:

$$\begin{aligned}
D(y\|y^*) &= \sum_{x\in\mathcal{X}} y(x)\log\frac{y(x)}{y^*(x)} \\
&=(z_0 p(y_h|b) + (1-z_0)p(y_h|r))\log\frac{z_0 p(y_h|b) + (1-z_0)p(y_h|r)}{z_0} \\
&\quad + (z_0 p(y_l|b) + (1-z_0)p(y_l|r))\log\frac{z_0 p(y_l|b) + (1-z_0)p(y_l|r)}{1-z_0}
\end{aligned} \tag{18}$$

Since both y and y^* have full support, then

$$\begin{aligned}
-D(y\|y^*) &= -\sum_{x\in\mathcal{X}} y(x)\log\frac{y(x)}{y^*(x)} \\
&= \sum_{x\in\mathcal{X}} y(x)\log\frac{y^*(x)}{y(x)} \\
&\leq \log\sum_{x\in\mathcal{X}} y(x)\frac{y^*(x)}{y(x)} \qquad (19)\\
&= \log\sum_{x\in\mathcal{X}} y^*(x) \\
&= \log 1 \\
&=0
\end{aligned}$$

where Inequality (19) comes from Jensen's inequality. Now we know that $D(y\|y^*) \geq 0$ whenever y and y^* have full supports. Next, we want to show that the equality holds if and only if $y(x) = y^*(x)\ \forall x \in \mathcal{X}$. Notice that when $y \xrightarrow{d} y^*$, setting $p(y_h|b) = p(y_l|r) = 1$ in Equation (18), we have

$$\begin{aligned}
D(y\|y^*) &= z_0\log\frac{z_0}{z_0} + (1-z_0)\log\frac{1-z_0}{1-z_0} \\
&=0
\end{aligned}$$

Since we also know the function $\log x$ is strictly concave in x, the equality holds in Inequality (19) if and only if $y^*(x)/y(x)$ is constant everywhere (i.e., $y^*(x) = cy(x)\ \forall x \in \mathcal{X}$ for some $c \in \mathbb{R}_{++}$). Meanwhile, we know that $\log\sum_{x\in\mathcal{X}} y(x)\frac{y^*(x)}{y(x)} = \log c\sum_{x\in\mathcal{X}} y(x) = \log\sum_{x\in\mathcal{X}} y^*(x)$, therefore $c = 1$. We now have established that the global minimum of $D(y\|y^*)$ is attained if and only if $y(x) = y^*(x)\ \forall x \in \mathcal{X}$.

Thus, we have established the relationship that $D(y\|y^*) = 0 \Leftrightarrow p(y_l|r) = 1, p(y_h|b) = 1$.

Combined with $D(y\|y^*) \geq 0$, the relationship $D(y\|y^*) > 0 \Leftrightarrow \min\{p(y_l|r), p(y_h|b)\} < 1$ is also true, this allows us to work directly with $p(y_l|r)$ and $p(y_h|b)$ (resp. $p(y_h|r)$ and $p(y_l|b)$).

Next, we want to show that the bound for z_0 indeed has the specified form. To achieve this, it first needs to be shown that for any

$$z_0 < \frac{1 - \alpha_*^j - \int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}{(1-\alpha_*^j)\int_0^{T_h^i} \exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) - \int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)} =: B_0,$$

we can always find a signal y with $D(y\|y^*) > 0$ such that Condition (16) is satisfied and players prefer to go on to the war of attrition stage and B_0 is the supremum of all such bounds.

By Lemma 2, $f(y|\overline{\alpha})$ is strictly increasing with $p(y_l|r)$ and $p(y_h|b)$, the optimizers are $\arg\max_{(p(y_l|r),p(y_h|b))} f(y|\overline{\alpha}) = (1,1)$. Assume that we have almost perfect signals, let $p(y_l|r) = 1-\epsilon$, $p(y_h|r) = \epsilon$, $p(y_h|b) = 1-\delta$ and $p(y_l|b) = \delta$, by Condition (16), players strictly prefer to enter the war of attrition stage in any equilibrium if and only if:

$$\begin{aligned}
&(1-z_0)p(y_l|r)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) \\
>&(1-\alpha_*^j)\Bigg[1-(1-z_0)p(y_h|r) \\
&\qquad - z_0\left(p(y_h|b)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + p(y_l|b)\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right)\Bigg] \\
\Longrightarrow&(1-z_0)(1-\epsilon)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) \\
>&(1-\alpha_*^j)\left[1-\epsilon(1-z_0) - z_0\left((1-\delta)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + \delta\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right)\right] \\
\Longrightarrow& z_0 < \frac{(1-\epsilon)(1-\alpha_*^j) - (1-\epsilon)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}{\begin{array}{r}(1-\alpha_*^j)\left((1-\delta)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) + \delta\int_{T_h^i}^{T_l^j}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\right) \\ -\epsilon(1-\alpha_*^j) - (1-\epsilon)\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)\end{array}} =: B_{\epsilon,\delta}
\end{aligned}$$

Since $B_{\epsilon,\delta}$ is monotonically decreasing in ϵ and δ, we have that for any fixed $\overline{\alpha}$:

$$\sup_{\substack{\epsilon>0\\ \delta>0}} B_{\epsilon,\delta} = \limsup_{\substack{\epsilon\to 0\\ \delta\to 0}} B_{\epsilon,\delta} = \frac{1-\alpha_*^j - \int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}{(1-\alpha_*^j)\int_0^{T_h^i}\exp(-r^i x)\,\mathrm{d}F_{\overline{\alpha}}^i(x) - \int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\overline{\alpha}}^i(x)}$$

Therefore, for any $z_0 < B_0$, by continuity and monotonicity, there exist $\epsilon > 0$ and $\delta > 0$ such that $z_0 < B_{\epsilon,\delta}$, and we can find the corresponding signal y with $D(y\|y^*) > 0$ (since $p(y_l|r) = 1-\epsilon > 0$ and $p(y_h|b) = 1-\delta > 0$) that satisfies Condition (16) and players prefer to enter the war of attrition over immediate settlement.

An immediate corollary of Lemma 2 is that $-f(y|\overline{\alpha})$ is strictly decreasing with $p(y_l|r)$ and $p(y_h|b)$, by the same argument, when $z_0 \geq B_0$, players strictly prefer the immediate settlement at Nash demands, the equality holds since $D(y\|y^*)$ is bounded away from zero and Condition (13) is satisfied when $z_0 = B_0$ by monotonicity of $-f(y|\overline{\alpha})$.

Finally, when $D(y\|y^*) = 0$, since the signals are perfect, players have common knowledge of rationality and the multiplicity of equilibria occurs in the perfect information model where the Nash demands are among the equilibrium outcomes.

To establish the sufficiency, first notice that condition (a) guarantees an equilibrium at Nash demands since the game degenerates to a perfect information bargaining model; when condition (b) is satisfied, we know that a player has no incentive to unilaterally deviate to a higher demand, thus Nash demands form an equilibrium outcome. *Q.E.D.*

Proposition 4 highlights two facts: First, even though additional incentives were generated, the exogenous signal alone cannot guarantee that players would prefer to delay into a war of attrition stage in equilibrium, not even when the signal is almost perfectly informative. Second, we have the least restrictive bound on z_0 for a war of attrition stage to happen when the exogenous signal goes to perfect informativeness.

In the following example, I illustrate the relationship between the signal accuracy and players' ex ante utilities.

Example 2. Consider a bargaining problem where players have incompatible demands $\alpha^1 = 0.7$, $\alpha^2 = 0.8$ and a common prior $z_0 = 0.2$. Signal structure is assumed to be symmetric across different types of players for ease of demonstration, for example, $p(y_h|r) = 0.3$, $p(y_l|r) = 0.7$, $p(y_h|b) = 0.7$ and $p(y_l|b) = 0.3$ would be a signal structure that has symmetric conditional probabilities across the types. Discount factors are set to be $r^1 = 0.2$, $r^2 = 0.1$.

In this case, $T_l^1 > T_l^2$, player 1 should be conceding at time zero with rate c_h^i. At the limit, when we have a perfect signal revealing each player's type (i.e., $D(y\|y^*) = 0$), the game admits a Nash bargaining solution. As we can see from Figure 4, the expected utility is strictly increasing with signal accuracy for each player, when $D(y\|y^*) \to 0$, the ex ante expected utilities for both players exceed those from immediate settlement at the Nash solution. When the accuracy decreases, players have lower expected utilities and are set to settle at time zero once they hit their respective thresholds. At any time, players can opt out and take their reservation utilities by conceding, which are $\underline{\alpha}^1 = 0.2$ and $\underline{\alpha}^2 = 0.3$, however, as long as signals are sufficiently informative, players would strictly prefer to wait instead of settling immediately at time zero.

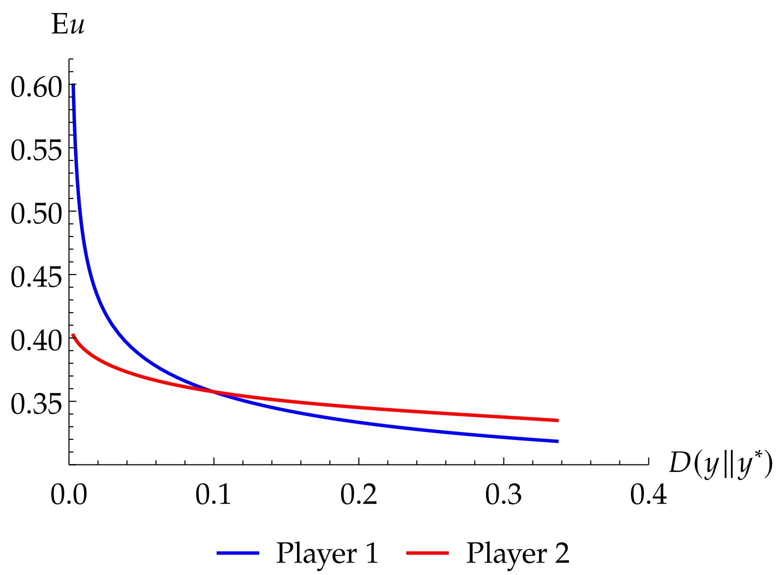

Figure 4: Ex ante expected utilities and signal accuracy

Proposition 5. *Given $z_0 < 1$, there exists y with $D(y\|y^*) > 0$ such that immediate settlement at (α_*^1, α_*^2) cannot be an equilibrium outcome.*

Proof. Without loss of generality, for $i = 1, 2$, let player i be the one who deviates from (α_*^i, α_*^j), denote (α^i, α_*^j) as $\bar{\alpha}$, for each fixed pair of incompatible initial demands $\bar{\alpha}$, when $D(y\|y^*) \to 0$, by Proposition 4, we have the bound for the prior z_0:

$$B_{\bar{\alpha}} := \frac{1 - \alpha_*^j - \int_{T_h^i}^{T_l^j} u_l^i(x)\, \mathrm{d}F_{\bar{\alpha}}^i(x)}{(1 - \alpha_*^j) \int_0^{T_h^i} \exp(-r^i x)\, \mathrm{d}F_{\bar{\alpha}}^i(x) - \int_{T_h^i}^{T_l^j} u_l^i(x)\, \mathrm{d}F_{\bar{\alpha}}^i(x)}$$

When $z_0 \geq B_{\bar{\alpha}}$, the deviation α^i is not profitable conditional on an imperfect signal y with $D(y\|y^*) > 0$. Note that $(1 - \alpha_*^j) \int_0^{T_h^i} \exp(-r^i x)\, \mathrm{d}F_{\bar{\alpha}}^i(x) - \int_{T_h^i}^{T_l^j} u_l^i(x)\, \mathrm{d}F_{\bar{\alpha}}^i(x) < 1 - \alpha_*^j - \int_{T_h^i}^{T_l^j} u_l^i(x)\, \mathrm{d}F_{\bar{\alpha}}^i(x) < 0$, by the indifference conditions, $u_l^i(x)$ is constant over $[T_h^i, T_l^j]$ for a fixed

$\bar{\alpha}$ and $\int_{T_h^i}^{T_l^j} u_l^i(x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)$ is bounded for all $\bar{\alpha}$ by definition and the range of α^i. Differentiating $\int_0^{T_h^i} \exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)$ and applying Leibniz's rule, we get $\partial \int_0^{T_h^i} \exp(-r^i x)\,\mathrm{d}F_{\bar{\alpha}}^i(x)/\partial\alpha^i = \exp(-r^i T_h^i) f_{\bar{\alpha}}^i(T_h^i)\partial T_h^i/\partial\alpha^i + \int_0^{T_h^i} \partial \exp(-r^i x) f_{\bar{\alpha}}^i(x)/\partial\alpha^i < 0$. When $\alpha^i \to \alpha_*^i$, $B_{\bar{\alpha}} \to 1$, therefore, for each $z_0 < 1$, we can always find $\alpha^i = \alpha_*^i + \epsilon$ such that $B_{\bar{\alpha}} > z_0$, and makes player i strictly better off than choosing immediate settlement. Thus, when the signal can be arbitrarily close to perfect informativeness, for any prior $z_0 < 1$, immediate settlement cannot be an equilibrium outcome since there exists a profitable deviation $\alpha^i > \alpha_*^i$ arbitrarily close to α_*^i such that it is strictly preferable to the immediate settlement at Nash demands.

Q.E.D.

6 Continuous Beliefs

What would happen if there was a rich set of exogenous signals informing the players? This is the question we seek to answer in this section. Starting from the prior z_0, assume there exists a continuum of signal realizations, and we have a continuum of posteriors through Bayesian updating.

As the goal here is to establish the phased bargaining timeline rather than discuss the effects of different signals, we can start directly from the set of posteriors for simplicity. Now we assume there's a continuum of posterior beliefs $Z = [\underline{z}, \bar{z}]$, with the generic element denoted as z and $0 < \underline{z} < \bar{z} < 1$. Let $G : Z \to [0, 1]$ be the cumulative distribution function on posterior beliefs with full support, and its corresponding probability density function is $g : Z \to \mathbb{R}_{++}$. Let $F : \mathbb{R}_+ \to [0, 1]$ be the cumulative distribution function for the mass of players that concede until time t.

The first thing to notice is that the following condition must be satisfied:

$$1 - G^i(z^i) = F^i(t)$$

for $i = 1, 2$, which means for player i, the mass of players holding posterior beliefs larger than or equal to z must equal to the mass of players conceding before time t. This is essentially a corollary of Lemma 1, for any given pair of posterior beliefs z, we can always find an order so that applying the lemma in the discrete setting leads to the fact that a player concedes at the time ordered by the realization of posterior beliefs (ex post, only one belief gets realized).

The baseline setting in Section 3 works with little modification with continuous beliefs. To get the indifference condition and the strategies for concession, we first write the utility function for player $i = 1, 2$:

$$\begin{aligned} u_z^i(t) =& (1-z)\left\{\int_{x=0}^{t} \alpha^i \exp(-r^i x)\, \mathrm{d}F^j(x) + \left(1-\alpha^j\right)\exp(-r^i t)\left(1-F^j(t)\right)\right\} \\ &+ z\exp(-r^i t)(1-\alpha^j) \end{aligned} \tag{20}$$

where $j = \{1, 2\} \setminus i$ and F^i here denotes the mass of player i that would have conceded by time t. $u_z^i(t)$ is a combination of continuously differentiable functions, hence differentiable with respect to t. Differentiating $u_z^i(t)$,

$$\begin{aligned} \frac{\partial u_z^i(t)}{\partial t} =& (1-z)\left\{\alpha^i \exp(-r^i t) f^j(t) - \left(1-\alpha^j\right)\exp(-r^i t) f^j(t) - r^i\left(1-\alpha^j\right)\exp(-r^i t)\left(1-F^j(t)\right)\right\} \\ &- z r^i \exp(-r^i t)(1-\alpha^j) \end{aligned} \tag{21}$$

Set $\frac{\partial u_z^i(t)}{\partial t} = 0$, we have a similar solution as in Condition (2):

$$\begin{aligned} F_z^j(t) &= \frac{\exp(\kappa^i t) - 1}{\exp(\kappa^i t)(1-z)} + \frac{c_z^j}{\exp(\kappa^i t)} \\ \kappa^i &:= \frac{r^i\left(1-\alpha^j\right)}{\left[\alpha^i - \left(1-\alpha^j\right)\right]} \end{aligned} \tag{22}$$

which determines the optimal time t for player i to concede. To get the hazard rate of concession, we first find $f_z^j(t)$ by differentiating $F_z^j(t)$:

$$f_z^j(t) = \frac{\kappa^i \exp(-\kappa^i t)\left[1 - c_z^j(1-z)\right]}{1-z} \tag{23}$$

We now have the hazard rate:

$$\lambda_z^j := \frac{\kappa^i(1 - c_z^j + z)}{1 - z\exp(\kappa^i t) - (1-z)c_z^j} \tag{24}$$

Since at each time t, the mass of players that have conceded equals to the mass of players holding beliefs larger than or equal to z, in equilibrium, it is required that:

$$
\begin{aligned}
&F_z^j(t) = 1 - G(z(t)) \\
\Longrightarrow &f_z^j(t) = -g(z(t))z'(t) \\
\Longrightarrow &\lambda_z^j = \frac{f_z^j(t)}{1 - F_z^j(t)} = \frac{-g(z(t))z'(t)}{G(z(t))}
\end{aligned} \tag{25}
$$

Combined with Equation (24), we get the differential equation:

$$
\frac{f_z^j(t)}{1 - F_z^j(t)} = \frac{-g(z(t))z'(t)}{G(z(t))} = \frac{\kappa^i(1 - c_z^j + z)}{1 - z\exp(\kappa^i t) - (1 - z)c_z^j} \tag{26}
$$

where $c_z^j = z(1 - G(z))/(1 - (1 - G(z))(1 - z)^2))$

While there is no closed-form solution in sight, we provide a numerical example to illustrate the solution of this differential equation, $z(t)$, which specifies the type of players conceding at each time.

Example 3. Assume that the posterior beliefs follow a uniform distribution on $[0.1, 0.8]$:

$$
G(z) = \begin{cases} 0 & \text{for } z < 0.1 \\ \frac{z-0.1}{0.7} & \text{for } z \in [0.1, 0.8] \\ 1 & \text{for } z > 0.8 \end{cases}
$$

Let discount factors be $r^1 = 0.8$ and $r^2 = 0.7$. The initial demands of players are set to be $\alpha^1 = 0.7$ and $\alpha^2 = 0.6$.

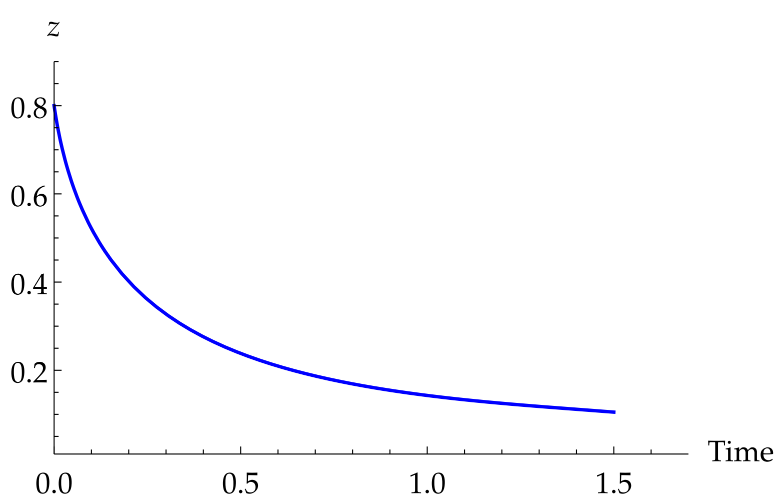

Figure 5: An example of function $z(t)$ with initial condition $z(0) = 0.8$

In Figure 5, the numerical solution of the function $z(t)$ was shown with the initial condition $z(0) = 0.8$ to ensure the most pessimistic type of player i concedes at time zero. We can see that as expected, players with more pessimistic beliefs concede at earlier times and their conceding times are strictly decreasing with their posterior beliefs after updating through the exogenous signal.

This example illustrated that with a rich set of signals, our results remain valid. In equilibrium, each player follows a strategy that maps each posterior belief to a time point at which concession happens with probability 1.

7 Conclusion

This paper considers a reputation based bargaining model with heterogeneous beliefs generated through a learning process. It is shown that starting from a common prior, when players receive different signals, they concede in different phases in equilibrium, specifically, players never concede at the same time when holding different posterior beliefs.

Departing from the results in Kambe (1999), we also find the prior and signal accuracy to be the deciding factors in determining equilibrium outcomes at limiting cases, when players strategically choose their initial demands, immediate settlement at Nash demands is no longer guaranteed as an equilibrium outcome, instead, it depends on the prior whose bound is determined by the informativeness of the exogenous signal.

In the case with a continuum of signals (hence a continuum of posterior beliefs), we can extend our findings in the binary signal realizations case, where more optimistic players concede at later times than those more pessimistic. In any case, the existence of the exogenous signal can substantially influence the conceding behaviors of players and create additional delay and inefficiency.

References

ABREU, D. AND F. GUL (2000): "Bargaining and Reputation," *Econometrica*, 68, 85–117.

ABREU, D. AND D. PEARCE (2007): "Bargaining, Reputation, and Equilibrium Selection in Repeated Games with Contracts," *Econometrica*, 75, 653–710.

ABREU, D., D. PEARCE, AND E. STACCHETTI (2015): "One-sided Uncertainty and Delay in Reputational Bargaining," *Theoretical Economics*, 10, 719–773.

CHATTERJEE, K. AND L. SAMUELSON (1987): "Bargaining with Two-sided Incomplete Information: An Infinite Horizon Model with Alternating Offers," *The Review of Economic Studies*, 54, 175–192.

——— (1988): "Bargaining under Two-sided Incomplete Information: The Unrestricted Offers Case," *Operations Research*, 36, 605–618.

COVER, T. M. AND J. A. THOMAS (2006): *Elements of Information Theory*, John Wiley & Sons.

DALEY, B. AND B. GREEN (2020): "Bargaining and News," *American Economic Review*, 110, 428–474.

FANNING, J. (2016): "Reputational Bargaining and Deadlines," *Econometrica*, 84, 1131–1179.

FUDENBERG, D. AND D. LEVINE (1989): "Reputation and Equilibrium Selection in Games with a Patient Player," *Econometrica*, 57, 759–78.

HENDRICKS, K., A. WEISS, AND C. WILSON (1988): "The War of Attrition in Continuous Time with Complete Information," *International Economic Review*, 663–680.

KAMBE, S. (1999): "Bargaining with Imperfect Commitment," *Games and Economic Behavior*, 28, 217–237.

KREPS, D. M., P. MILGROM, J. ROBERTS, AND R. WILSON (1982): "Rational Cooperation in the Finitely Repeated Prisoners' Dilemma," *Journal of Economic Theory*, 27, 245–252.

KREPS, D. M. AND R. WILSON (1982): "Reputation and Imperfect Information," *Journal of Economic Theory*, 27, 253–279.

MILGROM, P. AND J. ROBERTS (1982): "Predation, Reputation, and Entry Deterrence," *Journal of Economic Theory*, 27, 280–312.

MYERSON, R. B. (1991): *Game Theory*, Harvard University Press.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50, 97–109.

——— (1985): "A Bargaining Model with Incomplete Information about Time Preferences," *Econometrica*, 1151–1172.

Sobel, J. and I. Takahashi (1983): "A Multistage Model of Bargaining," *The Review of Economic Studies*, 50, 411–426.

Appendix

A *Solving Condition (2) and Condition (3)*

For simplicity, denote $\mathrm{E}\, f^j(t)$ as $f(t)$ and $\mathrm{E}\, F^j(t)$ as $F(t)$, take the standard form:

$$f(t) + F(t)\frac{(1-\alpha^2)r^1}{(\alpha^1 - (1-\alpha^2))} = \frac{(1-\alpha^2)r^1}{(1-z_l)(\alpha^1 - (1-\alpha^2))}$$

Denote $\frac{(1-\alpha^2)r^1}{(\alpha^1-(1-\alpha^2))}$ as λ and $\frac{(1-\alpha^2)r^1}{(1-z_l)(\alpha^1-(1-\alpha^2))}$ as b, we are solving $f + \lambda F = b$, multiply both sides by $\exp(\lambda t)$:

$$\exp(\lambda t) f + \lambda \exp(\lambda t) F = \exp(\lambda t) b$$

Notice that the left-hand side equals to $\frac{\mathrm{d} \exp(\lambda t) F}{\mathrm{d}t}$, we then have:

$$\frac{\mathrm{d} \exp(\lambda t) F}{\mathrm{d}t} = \exp(\lambda t) b$$

We now integrate both sides to get:

$$e^{\lambda t} F = \int_x e^{\lambda x} b \,\mathrm{d}x + c$$

or:

$$F = \frac{1}{e^{\lambda t}} \int_{x=0}^{t} e^{\lambda x} b \,\mathrm{d}x + \frac{c}{e^{\lambda t}}$$

The solution is:

$$\frac{b\,(\exp(\lambda t) - 1)}{\lambda \exp(\lambda t)} + \frac{c}{\exp(\lambda t)}$$

note that $b/\lambda = 1/(1-z_l)$, finally we have:

$$F = \frac{\exp(\lambda t) - 1}{\exp(\lambda t)(1-z_l)} + \frac{c}{\exp(\lambda t)}$$

where $\lambda := \frac{(1-\alpha^2)r^1}{(\alpha^1-(1-\alpha^2))}$.

B *Calculation of Constants c_h^1 and c_h^2*

Assume $\lambda^1 > \lambda^2$, the other case is similar. We require that $T_l^1 = T_l^2$. And $F^1(T_l^1) = F^2(T_l^2) = 1 - z_l$, starting with the second phase, we have the expressions:

$$\begin{aligned} \frac{\exp(\lambda^2 t) - 1}{\exp(\lambda^2 t)(1 - z_l)} + \frac{c_l^1}{\exp(\lambda^2 t)} = 1 - z_l \\ \frac{\exp(\lambda^1 t) - 1}{\exp(\lambda^1 t)(1 - z_l)} + \frac{c_l^2}{\exp(\lambda^1 t)} = 1 - z_l \end{aligned} \tag{27}$$

Focusing on the side of player 2, after plugging in Equation (8), we want to get the time that satisfies Condition (27), namely the exhaustion time T_l^2 for optimistic player 2:

$$\begin{aligned} &\frac{\exp(\lambda^1 t) - 1}{\exp(\lambda^1 t)(1 - z_l)} + \frac{c_l^2}{\exp(\lambda^1 t)} = 1 - z_l \\ \Longrightarrow &\exp(\lambda^1 t)\left[1 - (1 - z_l)^2\right] = \exp(\lambda^1 T_h^2)\left[1 - (1 - z_h)(1 - z_l)p_h\right] \\ \Longrightarrow &T_l^2 = T_h^2 + \frac{\log\left[1 - (1 - z_h)(1 - z_l)p_h\right] - \log\left[1 - (1 - z_l)^2\right]}{\lambda^1} \end{aligned}$$

Similarly, we have

$$T_l^1 = T_h^1 + \frac{\log\left[1 - (1 - z_h)(1 - z_l)p_h\right] - \log\left[1 - (1 - z_l)^2\right]}{\lambda^2}$$

Solving $T_l^1 = T_l^2$, we get the requirements on T_h^1 and T_h^2:

$$\begin{aligned} &T_h^1 + \frac{\log\left[1 - (1 - z_h)(1 - z_l)p_h\right] - \log\left[1 - (1 - z_l)^2\right]}{\lambda^2} \\ =&T_h^2 + \frac{\log\left[1 - (1 - z_h)(1 - z_l)p_h\right] - \log\left[1 - (1 - z_l)^2\right]}{\lambda^1} \end{aligned} \tag{28}$$

If no one concedes at time zero, by Equation (5), we know $T_h^1 = \frac{-\log(1 - p_h(1 - z_h)^2)}{\lambda^2}$ and $T_h^2 = \frac{-\log(1 - p_h(1 - z_h)^2)}{\lambda^1}$. By assumption, $\lambda^1 > \lambda^2$, which means that Condition (28) is violated, in this case player 1 must concede with appropriate probability at time zero in order for Condition (28) to be satisfied, this is achieved through constant c_h^1.

First, notice that only one player has the incentive to concede at time zero, in this case

$c_h^2 = 0$, plugging in $T_h^2 = \frac{-\log(1-p_h(1-z_h)^2)}{\lambda^1}$, we get the expression of T_h^1:

$$\begin{aligned} T_h^1 = & -\frac{\log(1-p_h(1-z_h)^2)}{\lambda^1} \\ & + \frac{\log\left[1-(1-z_h)(1-z_l)p_h\right] - \log\left[1-(1-z_l)^2\right]}{\lambda^1} \\ & - \frac{\log\left[1-(1-z_h)(1-z_l)p_h\right] - \log\left[1-(1-z_l)^2\right]}{\lambda^2} \end{aligned} \tag{29}$$

We calculate c_h^1 by first plugging in Equation (29) into $\exp(\lambda^2 T_h^1)$:

$$\begin{aligned} \exp(\lambda^2 T_h^1) = \exp\Bigg\{ & -\frac{\lambda^2}{\lambda^1}\log\left[1-p_h(1-z_h)^2\right] \\ & + \frac{\lambda^2}{\lambda^1}\Big(\log\left[1-(1-z_h)(1-z_l)p_h\right] - \log\left[1-(1-z_l)^2\right]\Big) \\ & - \log\left[1-(1-z_h)(1-z_l)p_h\right] + \log\left[1-(1-z_l)^2\right] \Bigg\} \end{aligned} \tag{30}$$

By Bayes' rule, $F^1(T_h^1) = p_h(1-z_h)$, rewriting Equation (9), we get:

$$\begin{aligned} & \frac{\exp(\lambda^2 T_h^1) - 1}{\exp(\lambda^2 T_h^1)(1-z_h)} + \frac{c_h^1}{\exp(\lambda^2 T_h^1)} = p_h(1-z_h) \\ \Longrightarrow & c_h^1 = \frac{p_h(1-z_h)^2 - 1}{1-z_h}\exp(\lambda^2 T_h^1) + \frac{1}{1-z_h} \end{aligned}$$

where

$$\begin{aligned} \exp(\lambda^2 T_h^1) = \exp\Bigg\{ & -\frac{\lambda^2}{\lambda^1}\log\left[1-p_h(1-z_h)^2\right] \\ & + \frac{\lambda^2}{\lambda^1}\Big(\log\left[1-(1-z_h)(1-z_l)p_h\right] - \log\left[1-(1-z_l)^2\right]\Big) \\ & - \log\left[1-(1-z_h)(1-z_l)p_h\right] + \log\left[1-(1-z_l)^2\right] \Bigg\} \end{aligned}$$

In a more general form, when $\lambda^i > \lambda^j$, we have:

$$c_h^i = \frac{p_h(1-z_h)^2 - 1}{1-z_h}\exp(\lambda^j T_h^i) + \frac{1}{1-z_h} \tag{31}$$

where

$$\exp(\lambda^j T_h^i) = \exp\Bigg\{-\frac{\lambda^j}{\lambda^i}\log\left[1-p_h(1-z_h)^2\right] + \frac{\lambda^j}{\lambda^i}\Big(\log\left[1-(1-z_h)(1-z_l)p_h\right]-\log\left[1-(1-z_l)^2\right]\Big) - \log\left[1-(1-z_h)(1-z_l)p_h\right]+\log\left[1-(1-z_l)^2\right]\Bigg\}$$

C *Further Discussions*

In the main body of the paper, we base our analyses on the assumption that players receive signals with the same distribution, starting from a common prior, the realized posteriors are symmetric among players (in other words, z_h and z_l are not dependent on players' identities). As a matter of fact, our results still hold under asymmetric beliefs, even if one of the players faces only one type of opponent, as long as both sides concede with probability 1 at a finite time. However, when we have asymmetric posteriors, it is no longer sufficient to infer one's reputation dynamics and associated bargaining power from initial demands and time preferences. For example, for the exhaustion time $T_l^i = \frac{-\log(1-p_l(1-z_l)^2)}{\lambda^j}$, under a symmetric signal setting, the relative length between T_l^1 and T_l^2 is only dependent on λ^1 and λ^2, however, with asymmetric signals, now we must take the signal structures into consideration in determining the identity of the player that concedes at time zero if existed.

In a setting with strategic demands, the direct impact is that the Nash demands are now dependent on the beliefs z_l^i, setting the ratio of exhaustion times $T_l^i/T_l^j = 1$, we have:

$$\frac{-\log(1-p_l^i(1-z_l^i)^2)}{\lambda^j}\Bigg/\frac{-\log(1-p_l^j(1-z_l^j)^2)}{\lambda^i} = 1$$

$$\Longrightarrow \lambda^j/\lambda^i = -\log(1-p_l^i(1-z_l^i)^2)/-\log(1-p_l^j(1-z_l^j)^2)$$

$$\Longrightarrow \frac{r^i\left(1-\alpha^j\right)}{\left[\alpha^i-\left(1-\alpha^j\right)\right]}\Bigg/\frac{r^j\left(1-\alpha^i\right)}{\left[\alpha^j-\left(1-\alpha^i\right)\right]} = \frac{-\log(1-p_l^i(1-z_l^i)^2)}{-\log(1-p_l^j(1-z_l^j)^2)}$$

combined with the just compatibility: $\alpha^1+\alpha^2=1$, we get:

$$\alpha_*^i = \frac{r^j\log(1-p_l^i(1-z_l^i)^2)}{r^j\log(1-p_l^i(1-z_l^i)^2)+r^i\log(1-p_l^j(1-z_l^j)^2)}$$

for $i=1,2$.

In this case, our Proposition 3 still works as the prior goes to zero, since z_l^1 and z_l^2 both converge to zero and the asymmetry of signal structures does not affect the asymptotic features with respect to z_0 as long as the signal is informative. In Proposition 4, special attention is needed in determining the identity of the player who concedes at time zero, as the exhaustion times are now heavily dependent on the signal structures and realized posterior beliefs.