

THE UNIVERSITY OF CHICAGO

BESICOVITCH SETS, RECTIFIABILITY, AND PROJECTIONS

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This thesis is dedicated to my family, and also to you, the reader!

“We will use this process to generate our monster, which will have a tiny heart and many arms.”

– Elias Stein, *Harmonic Analysis* [51, §X.1: The Besicovitch set]

# TABLE OF CONTENTS

LIST OF FIGURES . . . . .	viii
ACKNOWLEDGMENTS . . . . .	ix
ABSTRACT . . . . .	x
1 INTRODUCTION . . . . .	1
1.1 Besicovitch sets and the Kakeya conjecture . . . . .	1
1.2 Rectifiability and projections . . . . .	2
1.3 Main results of each chapter . . . . .	2
1.3.1 Chapter 2: Kakeya needle problem and Besicovitch sets . . . . .	2
1.3.2 Chapter 3: Small unions of affine subspaces and skeletons . . . . .	3
1.3.3 Chapter 4: Analytic capacity and projections . . . . .	5
2 THE KAKEYA NEEDLE PROBLEM AND THE EXISTENCE OF BESICOVITCH AND NIKODYM SETS FOR RECTIFIABLE SETS . . . . .	7
2.1 Introduction . . . . .	7
2.1.1 History . . . . .	8
2.1.2 Translations . . . . .	9
2.1.3 Rotations . . . . .	10
2.1.4 Besicovitch and Nikodym sets . . . . .	13
2.1.5 The sharpness of our results, and dilations . . . . .	15
2.2 Main ideas of the proof of Theorems 2.1.2 and 2.5.1 . . . . .	16
2.2.1 The first key idea . . . . .	16
2.2.2 The second key idea . . . . .	17
2.2.3 Combining the key ideas . . . . .	17
2.3 Preliminaries . . . . .	19
2.3.1 Tangents of rectifiable sets . . . . .	19
2.3.2 Rotations . . . . .	20
2.3.3 The “second key idea” for translations . . . . .	21
2.3.4 The “second key idea” for rotations . . . . .	22
2.4 Kakeya needle problem for translations . . . . .	24
2.4.1 Notation . . . . .	24
2.4.2 Basic zigzag . . . . .	25
2.4.3 Venetian blind . . . . .	25
2.4.4 Main construction . . . . .	27
2.4.5 Choosing the parameters $\beta_{\mathbf{i}}, \gamma_{\mathbf{i}}$ . . . . .	27
2.4.6 Choosing the signs $\sigma_{\mathbf{i}}$ . . . . .	28
2.4.7 Stopping time . . . . .	29
2.4.8 Fineness of the zigzags, and the small neighborhood lemma . . . . .	30
2.4.9 Area estimate . . . . .	31
2.5 Kakeya needle problem for rotations . . . . .	33
2.5.1 Basic zigzags, deconstructed . . . . .	33

2.5.2	The structure of intrinsic compositions . . . . .	34
2.5.3	Basic zigzag construction for rotations . . . . .	36
2.5.4	Iterating the basic zigzag . . . . .	37
2.5.5	Turning the translations into rotations . . . . .	40
2.5.6	Ignoring small parts of $E$ and of $\tilde{P}$ . . . . .	41
2.5.7	Area estimates . . . . .	42
2.5.8	Further remarks . . . . .	44
2.6	Besicovitch and Nikodym sets . . . . .	45
2.6.1	Construction of a Besicovitch set for translations . . . . .	45
2.6.2	Construction of a Besicovitch set for rotations . . . . .	48
2.6.3	The main theorem . . . . .	49
2.6.4	Construction of a Nikodym set . . . . .	51
2.7	Dilations and similarity transformations . . . . .	53
2.7.1	Circles . . . . .	57
3	SMALL UNIONS OF AFFINE SUBSPACES AND SKELETONS VIA BAIRE CATEGORY . . . . .	60
3.1	Introduction . . . . .	60
3.2	Scaled copies . . . . .	64
3.3	Scaled and rotated copies . . . . .	70
3.4	Rotated copies: dimension . . . . .	73
3.5	Rotated copies: measure . . . . .	77
3.5.1	Translating cones . . . . .	78
3.5.2	Proof of Theorem 3.5.1 . . . . .	82
4	ANALYTIC CAPACITY AND PROJECTIONS . . . . .	87
4.1	Introduction . . . . .	87
4.2	Notation and preliminaries . . . . .	93
4.2.1	Generalities . . . . .	93
4.2.2	Measures and rectifiability . . . . .	94
4.2.3	Cauchy and Riesz transform, and capacities . . . . .	95
4.3	The Fourier calculation . . . . .	97
4.3.1	The planar case . . . . .	97
4.3.2	The higher dimensional case . . . . .	100
4.4	The dyadic lattice of David and Mattila . . . . .	105
4.5	The corona decomposition . . . . .	110
4.6	The construction of an approximate Lipschitz graph . . . . .	112
4.6.1	The stopping cubes . . . . .	112
4.6.2	Preliminary estimates . . . . .	113
4.6.3	A key estimate . . . . .	114
4.6.4	An algorithm to construct a Lipschitz graph close to the stopping cubes . . . . .	118
4.6.5	The small measure of the low density set . . . . .	121
4.6.6	The approximate Lipschitz graph . . . . .	125
4.7	The family of $\text{Top}$ cubes . . . . .	126
4.7.1	The family $\text{Top}$ . . . . .	126

4.7.2	The family of cubes $ID$	127
4.8	The packing condition	128
4.9	Proof of Lemma 4.5.1	132
4.10	Application to curvature, Riesz transforms, and capacities	133
4.10.1	Curvature of measures and Riesz transforms	133
4.10.2	Analytic capacity	135
4.10.3	The capacities $\Gamma_{d,n}$	137
4.A	The reverse inequality	137
REFERENCES		142

## LIST OF FIGURES

1	Words of encouragement. (Produced by Kevin Chang and used with permission.)	ix
2.1	Two steps in the Venetian blind construction, starting with (a) as the initial segment. . . . .	27



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Figure 1: Words of encouragement. (Produced by Kevin Chang and used with permission.)

## ABSTRACT

In this thesis, we use the connections between projections and rectifiability to study problems in geometric measure theory, harmonic analysis, and complex analysis.

In Chapter 2 (joint work with Marianna Csörnyei), we study “curved” versions of the Kakeya needle problem and Besicovitch sets; i.e., we consider what happens if we replace the line segment with a  $C^1$  curve, or more generally, a rectifiable set  $E \subset \mathbb{R}^2$ . Roughly speaking, our main result states that we can move a rectifiable set  $E$  between any two prescribed positions in a set of measure zero, provided that at each time moment  $t$  of the movement we are allowed to “hide” a set  $E_t \subset E$  of linear measure zero.

In Chapter 3 (joint work with Marianna Csörnyei, Kornélia Héra and Tamás Keleti), our aim is to find the minimal Hausdorff dimension of the union of scaled and/or rotated copies of the  $k$ -skeleton of a fixed polytope centered at the points of a given set. For many of these problems, we show that a typical arrangement in the sense of Baire category gives minimal Hausdorff dimension. In particular, this proves a conjecture of R. Thornton. Our results also show that Nikodym sets are typical among all sets which contain, for every  $x \in \mathbb{R}^n$ , a punctured hyperplane  $H \setminus \{x\}$  through  $x$ . With similar methods we also construct a Borel subset of  $\mathbb{R}^n$  of Lebesgue measure zero containing a hyperplane at every positive distance from every point.

In Chapter 4 (joint work with Xavier Tolsa), we study the connection between the analytic capacity of a set and the size of its orthogonal projections. More precisely, we prove that if  $E \subset \mathbb{C}$  is compact and  $\mu$  is a Borel measure supported on  $E$ , then the analytic capacity of  $E$  satisfies

$$\gamma(E) \geq c \frac{\mu(E)^2}{\int_I \|P_\theta \mu\|_2^2 d\theta},$$

where  $c$  is some positive constant,  $I \subset [0, \pi)$  is an arbitrary interval, and  $P_\theta \mu$  is the image measure of  $\mu$  by  $P_\theta$ , the orthogonal projection onto the line  $\{re^{i\theta} : r \in \mathbb{R}\}$ . This result is related to an old conjecture of Vitushkin about the relationship between the Favard length

and analytic capacity.

# CHAPTER 1

## INTRODUCTION

### 1.1 Besicovitch sets and the Kakeya conjecture

In the early 20th century, Besicovitch proved the following results.

**Theorem 1.1.1** ([3, 4]).

1. (*Solution to the Kakeya needle problem*) For all  $\varepsilon > 0$ , we can rotate a unit line segment in the plane 180 degrees within a set of area less than  $\varepsilon$ .
2. (*Existence of Besicovitch sets*) There exists a set in  $\mathbb{R}^2$  of Lebesgue measure zero which contains a unit line segment in every direction.

A set in  $\mathbb{R}^n$  of measure zero which contains a unit line segment in every direction is called a *Besicovitch set* (or *Kakeya set*). Theorem 1.1.1 implies the existence of Besicovitch sets in  $\mathbb{R}^n$  for any  $n \geq 2$ . The *Kakeya conjecture* asserts that Besicovitch sets in  $\mathbb{R}^n$  must have Hausdorff dimension  $n$ .

Charles Fefferman showed that the existence of Besicovitch sets implies the unboundedness of the ball multiplier operator on  $L^p(\mathbb{R}^n)$  for  $p \neq 2$  and  $n \geq 2$  [25]. Since then, a wide variety of problems have been related to the Kakeya conjecture, for example:

local smoothing conj.  $\Rightarrow$  Bochner–Riesz conj.  $\Rightarrow$  restriction conj.  $\Rightarrow$  Kakeya conj.

The conjectures mentioned above have applications ranging from regularity of solutions to the wave equation to estimates on Fourier operators. The Kakeya conjecture has also found connections to many other areas, including incidence geometry, additive combinatorics, and analytic number theory.

## 1.2 Rectifiability and projections

Recall that a set  $E \subset \mathbb{R}^n$  is *purely unrectifiable* if every rectifiable curve intersects  $E$  in a set of linear measure zero. Besicovitch proved the following theorem about purely unrectifiable sets.

**Theorem 1.2.1** (Besicovitch projection theorem in  $\mathbb{R}^2$ ). *Suppose  $E \subset \mathbb{R}^2$  has finite  $\mathcal{H}^1$ -measure and is purely unrectifiable. Then almost every projection of  $E$  has measure zero.*

A standard example of a set satisfying the hypothesis of Theorem 1.2.1 is the *four corner Cantor set*. Let  $K_0 \subset \mathbb{R}^2$  be the unit square. Subdivide  $K_0$  into 16 squares, and let  $K_1$  be the union of the four corner squares. To construct  $K_2$ , repeat this process with each of the four squares in  $K_1$  to get a total 16 squares, and proceed similarly with  $K_3, K_4, \dots$ . The four corner Cantor set is defined to be  $K = \bigcap_n K_n$ .

It is not difficult to show that  $K$  is purely unrectifiable. By applying Theorem 1.2.1 to  $K$  and using point-line duality in  $\mathbb{R}^2$ , Besicovitch gave another construction of Besicovitch sets [5], providing a connection between purely unrectifiable sets of finite length and Besicovitch sets. This type of connection is implicitly used in Chapter 2 and Chapter 3 of this thesis.

## 1.3 Main results of each chapter

### 1.3.1 Chapter 2: *Keakeya needle problem and Besicovitch sets*

In Chapter 2, we study “curved” versions of the Keakeya needle problem and Besicovitch sets; i.e., we consider what happens in Theorem 1.1.1 if we replace the line segment with a  $C^1$  curve, or more generally, a rectifiable set  $E \subset \mathbb{R}^2$ .

To explain our result, first, let  $\text{Isom}^+(\mathbb{R}^2)$  denote the space of orientation-preserving isometries of  $\mathbb{R}^2$ . If  $\rho \in \text{Isom}^+(\mathbb{R}^2)$ , then a movement from  $E$  to  $\rho(E)$  can be described by a path in  $\text{Isom}^+(\mathbb{R}^2)$  from the identity map to  $\rho$ . If  $t \mapsto \phi_t$  is such a path, then  $\bigcup_t \phi_t(E)$  is the set covered when we move  $E$  “along” this path. We prove the following. Here,  $\mathcal{H}^1$

denotes the 1-dimensional Hausdorff measure (or linear measure).

**Theorem 1.3.1** (Theorem 2.6.6 of this thesis). *Let  $E \subset \mathbb{R}^2$  be an arbitrary rectifiable set. Let  $\rho \in \text{Isom}^+(\mathbb{R}^2)$ . Then there is a path  $t \mapsto \phi_t$  in  $\text{Isom}^+(\mathbb{R}^2)$  from the identity map to  $\rho$ , and for each  $t$ , there exists a subset  $E_t \subset E$  of  $\mathcal{H}^1$ -measure zero, such that  $\bigcup_t \phi_t(E \setminus E_t)$  has Lebesgue measure zero.*

Roughly speaking, Theorem 1.3.1 states that we can move a rectifiable set  $E \subset \mathbb{R}^2$  between any two prescribed positions in a set of measure zero, provided that at each time moment  $t$  of the movement we are allowed to “hide” a set  $E_t$  of linear measure zero. On the other hand, by a result of Csörnyei, Héra, and Laczkovich [11], if  $E$  is any closed and connected set which is not a subset of a circle or a line, then  $\bigcup_t \phi_t(E)$  cannot have arbitrarily small Lebesgue measure. In other words, our theorem is not true if we do not “hide” subsets of  $E$ .

Our proof of Theorem 1.3.1 is constructive; it gives us the path  $t \mapsto \phi_t$  as well as the sets  $E_t$ . As an example, if  $E$  is a full line or a parabola, then each  $E_t$  consists of at most one point. In other words, we can move a line or a parabola between any two prescribed positions in a set of measure zero if we are allowed to hide just one point at each time moment of the movement. We also remark that a variant of Theorem 1.3.1 in Chapter 2 contains Theorem 1.1.1 as a special case.

To prove Theorem 1.3.1, we combine several ideas, including the structure of the projective plane, the geometric properties of rectifiable sets, the nonabelian structure of the Lie group  $\text{Isom}^+(\mathbb{R}^2)$ , and the technique of iterated Venetian blinds from geometric measure theory.

### 1.3.2 Chapter 3: Small unions of affine subspaces and skeletons

Some results of Stein, Bourgain, and Marstrand combined imply that any subset of  $\mathbb{R}^n$  ( $n \geq 2$ ) which contains an  $(n - 1)$ -sphere centered at each point of  $\mathbb{R}^n$  must have positive

Lebesgue measure [50, 6, 37]. In fact, we can draw the same conclusion if the sphere is replaced by other sets with curvature.

However, the situation is very different if we consider sets without curvature. In Chapter 3, we study sets which contain scaled and/or rotated copies of polytopes around subsets of  $\mathbb{R}^n$  of a given dimension. For most cases, we determine the minimal dimension of such sets. An example of such a result is the following:

**Theorem 1.3.2** (Corollary 3.1.1 of this thesis). *For any integers  $0 \leq k < n$ , the minimal dimension of a Borel set  $A \subset \mathbb{R}^n$  that contains the  $k$ -skeleton of*

1. *a scaled copy of a cube around every point of  $\mathbb{R}^n$  is  $n - 1$ ;*
2. *a scaled and rotated copy of a cube around every point of  $\mathbb{R}^n$  is  $k$ ;*
3. *a rotated copy of a cube around every point of  $\mathbb{R}^n$  is  $k + 1$ ;*
4. *a rotated cube of every size around every point of  $\mathbb{R}^n$  is  $k + 1$ .*

Theorem 1.3.2, part (1) with  $n = 2$  and  $k = 1$  had already been known prior to our work, thanks to constructions by Keleti, Nagy, and Shmerkin (for  $n = 2$ ) and Thornton (for  $n \geq 3$ ) [34, 52]. Our approach differs from theirs. For each of the four situations of Theorem 1.3.2, we show that the minimum is attained by a “typical” set which contains the scaled and/or rotated copies, in the sense of Baire category. For these problems, Baire category arguments turn out to be much easier than explicit constructions. Furthermore, these arguments are more flexible; we also obtain Theorem 1.3.2 for other polytopes, and in fact for subsets of countable unions of  $k$ -planes.

We also show that Nikodym sets are typical in  $\mathbb{R}^n$ , again in the sense of Baire category; i.e., a typical set which contains a punctured hyperplane through every point in  $\mathbb{R}^n$  has Lebesgue measure zero. In fact, we show that sets which contain even more hyperplanes than Nikodym sets typically have measure zero:

**Theorem 1.3.3** (Corollary 3.5.2 of this thesis). *There is a set of measure zero in  $\mathbb{R}^n$  which*

(\*) contains a hyperplane at every positive distance from every point as well as a punctured hyperplane through every point.

In fact, in an appropriate Baire category sense, a “typical” set which satisfies (\*) has measure zero.

### 1.3.3 Chapter 4: Analytic capacity and projections

A compact set  $E \subset \mathbb{C}$  is *removable for bounded analytic functions* (or *removable*, for short) if for any open set  $\Omega$  containing  $E$ , every bounded analytic function on  $\Omega \setminus E$  has an analytic extension to  $\Omega$ . *Painlevé’s problem* is to find a geometric characterization of removable sets.

In [1], Ahlfors introduced the notion of *analytic capacity* of a set  $E \subset \mathbb{C}$ , denoted  $\gamma(E)$ , and he showed that  $E$  is removable if and only if  $\gamma(E) = 0$ . The analytic capacity is defined purely in complex analytic terms, and hence does not provide a geometric characterization of removable sets. (See (4.1) for the definition of  $\gamma(E)$ .)

The analytic capacity is closely connected to the Cauchy transform, and techniques from non-homogeneous Calderón-Zygmund theory have been useful in studying this quantity. Multiscale characterizations of rectifiability, introduced by Jones in his proof of the analyst traveling salesman theorem [31], led to the David–Semmes theory of uniform rectifiability and singular integrals [13, 16]. This in turn led to Tolsa’s characterization of removable sets via curvature of measures [53], ultimately resolving the Painlevé problem.

There is another geometric question related to analytic capacity that is still open. The *Favard length* of a Borel set  $E \subset \mathbb{C}$  is defined as

$$\text{Fav}(E) = \int_0^\pi \mathcal{H}^1(P_\theta(E)) d\theta, \tag{1.1}$$

where  $P_\theta$  is orthogonal projection onto the line  $\{re^{i\theta} : r \in \mathbb{R}\}$ . In the 1960s, Vitushkin conjectured that  $\gamma(E) = 0$  if and only if  $\text{Fav}(E) = 0$ . However, Mattila [41] proved that Vitushkin’s conjecture is false. Jones and Murai [32] (and later Joyce and Mörters [33])



constructed a compact set  $E \subset \mathbb{C}$  with  $\text{Fav}(E) = 0$  and  $\gamma(E) > 0$ . It is not known yet if the other implication of Vitushkin's conjecture holds. Namely, does positive Favard length imply positive analytic capacity? We can formulate a quantitative version of this as follows.

**Conjecture 1.3.4** (Vitushkin's conjecture, quantitative form). There is an absolute constant  $c > 0$  such that  $\gamma(E) \geq c \text{Fav}(E)$  for every compact  $E \subset \mathbb{C}$ .

In Chapter 4, we show that if one strengthens the assumption of positive Favard length in a suitable way, then the answer is positive. Our result is the following:

**Theorem 1.3.5** (Theorem 4.1.1 of this thesis). *Let  $I \subset [0, \pi)$  be an interval. There is a constant  $c > 0$  depending only on  $\mathcal{H}^1(I)$  such that for any compact set  $E \subset \mathbb{C}$  and any Borel probability measure  $\mu$  supported on  $E$ , we have  $\gamma(E) \geq c (\int_I \|P_\theta \mu\|_2^2 d\theta)^{-1}$ . (Here,  $P_\theta \mu$  denotes the pushforward measure of  $\mu$  by the projection  $P_\theta$ .)*

Our result can be viewed as a step towards Conjecture 1.3.4, since we give a lower bound on  $\gamma(E)$  in terms of the projections of  $E$ . To prove Theorem 1.3.5, first we relate the projections of a Radon measure  $\mu$  to a quantity which we call the *conical 1-Riesz energy* of  $\mu$ :

$$\iint_{x-y \in K_s} \frac{d\mu x d\mu y}{|x-y|} \leq \frac{1}{2} \int_{-s}^s \|P_\theta \mu\|_2^2 d\theta. \quad (1.2)$$

Here,  $K_s \subset \mathbb{R}^2$  is a double cone with vertex at the origin and aperture  $2s$ . To obtain Theorem 1.3.5, we combine (1.2) with two other ingredients: first, a geometric technique of [38] to find a “big piece of a Lipschitz graph” inside  $\text{spt } \mu$ , and second, a “corona decomposition” (a type of multi-scale decomposition) of [53].

**CHAPTER 2**  
**THE KAKEYA NEEDLE PROBLEM AND THE EXISTENCE**  
**OF BESICOVITCH AND NIKODYM SETS FOR**  
**RECTIFIABLE SETS**

This chapter is joint work with Marianna Csörnyei and originally appeared in [8].

**2.1 Introduction**

Let  $E \subset \mathbb{R}^2$  be a rectifiable set. Our aim in this paper is to show that the classical results about rotating a line segment in arbitrarily small area, and the existence of a Besicovitch and a Nikodym set hold if we replace the line by the set  $E$ . We will explain our results in more details below, but first we present two illustrative examples.

1. If  $E$  is the graph of a convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , our results imply the following:  *$E$  can be rotated continuously by  $360^\circ$  covering only a set of zero Lebesgue measure, if at each time moment  $t$  we are allowed to delete just one point from the rotated copy of  $E$ .*
2. If  $f$  is not just convex but strictly convex, then:  *$E$  can be moved continuously, using only translations, to any other shifted position, covering a set of measure zero, if at each time moment  $t$  we are allowed to delete just one point from the translated copy of  $E$ .*

*Remark 2.1.1.* In the two examples given above, our movement  $t \mapsto E_t$  is continuous, but the point  $x_t \in E_t$  we delete cannot be chosen continuously. However, all our constructions in this paper are Borel.

In the first example, if we take  $E$  to be a general rectifiable set, the result still holds, but instead of a single point, we need to delete an  $\mathcal{H}^1$ -null subset of  $E$  (see Theorem 2.6.6). For the generalization of the second example to rectifiable sets, see Theorem 2.6.2.

In the first example,  $\bigcup_t(E_t \setminus \{x_t\})$  is Lebesgue null. Therefore,  $\bigcup_t(E_t \setminus \{x_t\})$  is a *Besicovitch set*: in each direction it contains not just a “unit line segment of the line  $E$ ” but a whole copy of the set  $E$  except for one of its points.

On the other hand, since  $\bigcup_t E_t$  has non-empty interior, we can cover  $\mathbb{R}^2$  by taking a countable union of copies of  $\bigcup_t E_t$ . Therefore, the countable union of copies of  $\bigcup_t(E_t \setminus \{x_t\})$  is a *Nikodym set*: it has measure zero, and through each point  $x \in \mathbb{R}^2$ , it contains a copy of the set  $E$  with one point removed.

For the case when  $E$  is a line, see, e.g., [40] for both classical and recent results.

### 2.1.1 History

The *Keakeya needle* problem for sets other than the line segment has been studied before. R.O. Davies proved in [18] that not only one but any finite union of parallel line segments can be rotated by  $360^\circ$  covering arbitrarily small area. He also showed that the line segments must be parallel: if a set contains two line segments that are not parallel to each other, then it can no longer be moved.

In [11] the authors introduced the following definitions: a planar set  $E$  has the *Keakeya property* if there exist two different positions of  $E$  such that  $E$  can be moved continuously from the first position to the second in such a way that the area covered by  $E$  along the movement is arbitrarily small. A planar set  $E$  has the *strong Keakeya property* if it can be moved in the plane continuously to any other shifted or rotated position in a set of arbitrarily small area.

In [11] it is shown that if  $E$  is a closed connected set that has the *Keakeya property*, then  $E$  must be a subset of a line or of a circle. Moreover, if  $E$  is an arbitrary closed set that has the *Keakeya property*, then the union of the non-trivial connected components of  $E$  must be a subset of parallel lines or of concentric circles.

In [27] the authors show that short enough circular arcs of the unit circle possess the *strong Keakeya property*. (For topological reasons, it is clear that a full circle does not have

the strong Kakeya property.)

### 2.1.2 Translations

Let us consider a related question for circular arcs: can we *translate* a full circle continuously to any other position covering arbitrarily small area, if at each point of the translation, we are allowed to delete an arc of the circle of a given length? How long must the deleted arc be? Because of rotational symmetry, the question of which circular arcs have the strong Kakeya property is equivalent to this one, as long as we choose the deleted arc piecewise continuously.

In this paper, we will answer this “piecewise continuous question” for an arbitrary rectifiable set  $E$  of finite  $\mathcal{H}^1$ -measure in the following way: we only need to delete points whose tangent directions lie in a small interval.

Let us state our results precisely. We will use the following notation and terminology.

We let  $\mathbb{P}^1 \simeq \mathbb{R}/\pi\mathbb{Z}$  denote the set of all directions in  $\mathbb{R}^2$ . We will use the standard embedding of  $\mathbb{R}^2$  into the projective plane  $\mathbb{P}^2$ , so that  $\mathbb{P}^2 = \mathbb{R}^2 \cup \mathbb{P}^1$ . The arc-length metric on the unit sphere  $S^2$  together with the quotient map  $S^2 \rightarrow \mathbb{P}^2$  gives us a metric on  $\mathbb{P}^2$ . Let  $(\mathbb{P}^2)^*$  denote all the lines in  $\mathbb{P}^2$ .

We denote by  $|\cdot|$  the Lebesgue measure on  $\mathbb{R}^2$  or  $\mathbb{P}^1$ , and by  $\mathcal{H}^1$  the 1-dimensional Hausdorff measure on  $\mathbb{R}^2$ . As usual,  $B(x, r)$  denotes the open ball centered at  $x$  of radius  $r$ , and  $B(S, r)$  denotes the open  $r$ -neighborhood of a set  $S$ . We denote by  $\text{cl}S$  the closure of  $S$ . We write  $A \lesssim B$  to mean  $A \leq CB$  for some absolute constant  $C > 0$ .

Recall that every rectifiable set  $E \subset \mathbb{R}^2$  has a tangent field, which is defined for  $\mathcal{H}^1$ -almost every  $x \in E$  (see Section 2.3.1). We let  $\theta_x \in \mathbb{P}^1$  denote the tangent of  $E$  at  $x$ , and we let  $\nu_x \in (\mathbb{P}^2)^*$  denote the normal line of  $E$  at  $x$ . (The direction of  $\nu_x$  is the one orthogonal to  $\theta_x$ .) Note that  $\nu_x$  is the normal *line* passing through the point  $x$ , and not just a normal vector.

We will start by proving the following theorem:

**Theorem 2.1.2** (Kakeya needle problem for translations). *Let  $E \subset \mathbb{R}^2$  be a rectifiable set of finite  $\mathcal{H}^1$ -measure. Let  $\varepsilon > 0$  be arbitrary. Then between the origin and any prescribed point in  $\mathbb{R}^2$ , there exists a polygonal path  $P = \bigcup_{i=1}^n L_i$  with each  $L_i$  a line segment, and for each  $i$  there exists a direction  $\theta_i \in \mathbb{P}^1$ , such that*

$$\left| \bigcup_i \bigcup_{p \in L_i} (p + \{x \in E : \theta_x \notin B(\theta_i, \varepsilon)\}) \right| < \varepsilon. \quad (2.1)$$

Although the tangent field of a rectifiable set is defined only  $\mathcal{H}^1$ -almost everywhere, for the statement of Theorem 2.1.2 (and for all other results in this paper), we need to define it pointwise. We will show that regardless of which pointwise representation we choose, the results remain true (see Section 2.3.1).

Theorem 2.1.2 has an immediate corollary:

**Corollary 2.1.3.** *If we remove an arbitrary neighborhood of two diametrically opposite points from a circle, the resulting set can be moved continuously to any other position in the plane in arbitrarily small area.*

This strengthens the previously known result [27] that sufficiently short circular arcs have the strong Kakeya property.

### 2.1.3 Rotations

We note that Theorem 2.1.2 does not handle the classical Kakeya needle problem: clearly it is not possible to translate a line segment to every other position in small area. We can still apply Theorem 2.1.2 with  $E$  a line segment, but since every point of  $E$  has the same tangent direction, it allows us to delete the entire line segment at every point  $p \in P$ . To obtain a more meaningful statement for line segments, we need to consider what happens if we allow rotations as well as translations.

In order to unify translations and rotations, it is helpful to consider the projective plane

$\mathbb{P}^2$ .

We can consider a translation in direction  $\theta \in \mathbb{P}^1$  to be a “rotation” around the infinite point  $\theta^\perp \in \mathbb{P}^1 \subset \mathbb{P}^2$  (see Section 2.3.2).

We need to generalize the notion of a polygonal path from a path in  $\mathbb{R}^2$  to one in  $\text{Isom}^+(\mathbb{R}^2)$ , the space of all orientation-preserving isometries of  $\mathbb{R}^2$ . (This space is also known as the *special Euclidean group*  $SE(2)$ .) The polygonal path in Theorem 2.1.2 can be viewed as a sequence of vectors, each indicating in which direction and how far to translate. Then, a polygonal path of rotations should be a sequence of rotations, indicating around which point and how much to rotate.

Specifying a sequence of rotations is slightly trickier than a sequence of translations: when we rotate a set around a point, the centers of all the other rotations move. To avoid this problem, we will find it much more convenient to specify our sequence in the *intrinsic coordinate system*. That is, with  $\rho_i$  denoting rotations around  $z_i \in \mathbb{R}^2$ , our continuous movement will be to rotate first with center  $z_1$ , then with center  $\rho_1(z_2)$ , and so on.

Our *polygonal path*  $P$  will be specified by the *intrinsic sequence*  $\rho_i$ , but it will still lie in the space  $\text{Isom}^+(\mathbb{R}^2)$ , and its points will be isometries not in the intrinsic but in the standard coordinate system.

For each sequence  $\{\rho_i\}$  we obtain a  $P = \bigcup_i L_i$ . For each “line segment”  $L_i$  in  $P$ , the rotations in  $\{p' \circ p^{-1} : p, p' \in L_i\}$  all have the same center. (It is important to remember that this center depends not only on  $z_i$  but also on the previous rotations.)

Also, we find it much more convenient to specify a rotation not by the point that we rotate around, but by the image of this point in the projective plane when we embed  $\mathbb{R}^2$  into  $\mathbb{P}^2$ . We will call this the *projective center* of  $\rho$  (both for translations and rotations).

First we will prove a preliminary result (see Theorem 2.5.1). The exact statement is quite technical, but essentially says that instead of using translations, we can move our set  $E$  using rotations whose projective centers are almost aligned: if we want to connect  $\rho \in \text{Isom}^+(\mathbb{R}^2)$  to the identity map by a polygonal path, we can choose a line  $\ell \in (\mathbb{P}^2)^*$  that passes through

the center of  $\rho$ , and choose the (intrinsic) rotations so that their projective centers lie in  $B(\ell, \varepsilon)$ . We also obtain, for each  $i$ , a  $u_i \in \ell$  such that:

$$\left| \bigcup_i \bigcup_{p \in L_i} p(\{x \in E : \nu_x \cap \ell \cap B(u_i, \varepsilon) = \emptyset\}) \right| < \varepsilon. \quad (2.2)$$

Theorem 2.1.2 can be viewed as a special case of Theorem 2.5.1 by taking  $\rho$  to be a translation and then taking  $\ell$  to be  $\mathbb{P}^1$ . In this case, the centers lie on  $\ell$ , not just in an  $\varepsilon$ -neighborhood of  $\ell$ . The reason we need an  $\varepsilon$ -neighborhood for rotations is that, unlike for translations, the composition of a rotation around  $z_1$  and a rotation around  $z_2$  does not equal a rotation around a point on the line through  $z_1, z_2$ . (Recall the centers are specified with intrinsic coordinates.) This makes the statement and the proof of Theorem 2.5.1 more complicated than those of Theorem 2.1.2. We will need careful error estimates on how far the centers move, and, consequently, how large area the set  $E$  covers during its movement.

The essential observation for the error estimates is the following: the composition structure of translations is linear, i.e., given by vector addition. The composition structure of rotations is not linear, but it is “linear up to a quadratic error,” using an appropriate parametrization of  $\text{Isom}^+(\mathbb{R}^2)$  (see Lemma 2.5.4).

*Remark 2.1.4.* Let  $E$  be a countable union of parallel line segments which is bounded and has finite total length. It is easy to see that Theorem 2.5.1 implies that we can rotate  $E$  inside a set of arbitrarily small area. This strengthens the result of Davies mentioned at the beginning of this introduction, who proved the same result when  $E$  is a finite union of parallel line segments [18].

*Remark 2.1.5.* Theorem 2.1.2 and Theorem 2.5.1 also provide a new insight into the other results mentioned in Section 2.1.1, that the non-trivial connected components of a closed set with the Kakeya property can be covered by parallel lines or by concentric circles [11]. It turns out that the key property of the line and the circle is that they are homogeneous: by rotating around the center of the circle, any sub-arc can be mapped onto any other sub-arc of

the same length, by a continuous movement that covers only zero area. The same is true for lines with shifts. Therefore our piecewise continuous deletion of the line segments/sub-arcs in Theorem 2.1.2/Theorem 2.5.1 can be replaced by a continuous one. No rectifiable set except for the union of parallel lines or concentric circles has this property.

*Remark 2.1.6.* The set  $E$  in Theorem 2.5.1 needs to be bounded. Take, for example,  $E$  to be a union of countably many circles with centers  $z_i$  and radius  $r_i$ , such that  $\sum r_i < \infty$  and  $\sum r_i |z_i| = \infty$ . Then it is a rectifiable set with finite  $\mathcal{H}^1$ -measure, but every continuous rotation with a fixed center covers infinite area, even with a normal line removed. However, for the limit version Theorem 2.6.6 (explained below), in which the centers of the rotations no longer need to be piecewise constant along the path, we can drop the boundedness condition.

#### 2.1.4 Besicovitch and Nikodym sets

In Section 2.6, we study what happens in the limit as  $\varepsilon \rightarrow 0$ . By taking a sequence of  $\varepsilon$  tending to zero, the balls  $B(u_i, \varepsilon)$  shrink to a single point in  $\mathbb{P}^2$ , and the area covered shrinks to zero. We obtain in the limit a continuous movement  $P \subset \text{Isom}^+(\mathbb{R}^2)$  such that the set  $E$  covers only *zero* area, where at each time moment we only need to delete a subset of  $\mathcal{H}^1$ -measure zero (see Theorem 2.6.6). The resulting set of zero area is an analogue of a Besicovitch set for  $E$ .

Consider, e.g., the special case where there is a line  $\ell \in (\mathbb{P}^2)^*$  such that there is a neighborhood of  $\ell$  in which no two normal lines of  $E$  intersect. Then Theorem 2.6.6 says that we can rotate  $E$  continuously by  $360^\circ$ , covering a set of zero Lebesgue measure, where at each time moment, we only need to delete *one point*. This happens, e.g., in the special case when  $E$  is the graph of a convex function; by choosing the line  $\ell$  to lie below the graph, there is a neighborhood of  $\ell$  where no two normal lines meet.

If  $E$  is strictly convex, then we can apply Theorem 2.6.6 with  $\ell = \mathbb{P}^1$  and hence translate  $E$  to an arbitrary position in the plane in a set of zero Lebesgue measure, deleting one point at each time moment.



For moving a circle, we can choose any line  $\ell$ . In this case we need to delete, at each time moment, not just one but two diametrically opposite points of the circle, since they have the same normal line.

By the continuity of  $P$ , we can construct, from these Besicovitch sets, analogues of Nikodym sets, using the technique outlined at the beginning of this introduction. We state these more precisely in Section 2.6.

*Remark 2.1.7.* There is not only one continuous  $P$ , but residually many, in the sense of Baire category (see Remark 2.6.8). For results of similar nature when  $E$  is a line, see, e.g., [35] and [9].

*Remark 2.1.8.* It is well-known that there are no sets in  $\mathbb{R}^n$  ( $n \geq 2$ ) which have measure zero and contain a circle centered at every point. Stein first proved this for  $n \geq 3$  by his estimates on spherical maximal functions [50]. Bourgain and Marstrand independently showed the same non-existence result holds for  $n = 2$  around the same time [6, 37]. Bourgain’s paper actually treats smooth curves with non-vanishing curvature. More work has been done on such curves, e.g., [45, 59, 60].

The non-existence results concern placing a copy of  $E$  around every point in  $\mathbb{R}^2$ . For our Nikodym result, we instead place a copy of  $E$  *through* every point of  $\mathbb{R}^2$ . With this change, such a construction is now possible.

*Remark 2.1.9.* Somewhat surprisingly, “Besicovitch sets” for rectifiable sets in  $\mathbb{R}^2$  do not necessarily have dimension 2.

Trivially, if  $E$  is a countable union of concentric circles, then we can rotate them around their common center without increasing the dimension. More interestingly, there are also other, less trivial examples. For example, if  $E$  is a countable union of circles (not necessarily concentric ones), then there is a 1-dimensional set which contains a rotated copy of  $E$  in each direction: since a residual set contains a shifted copy of any countable configuration of points, by putting countably many circles around the points of a 0-dimensional residual set (of the same sizes as the circles in  $E$ ), we obtain a 1-dimensional Besicovitch set for  $E$ .

### 2.1.5 *The sharpness of our results, and dilations*

We do not know whether the sizes of the sets we delete are sharp. While Theorem 2.6.6 tells us that we only need to delete an  $\mathcal{H}^1$ -nullset of  $E$  at each time moment, perhaps it is possible to delete much fewer points than specified by the theorem. (For more precise information on the size of the sets we delete, see also Proposition 2.6.1, Remark 2.6.3, Proposition 2.6.4 and Remark 2.6.7.)

For example, it would be interesting to know whether it is possible to translate a circle in a set of Lebesgue measure zero, deleting only one point at every time moment. This is still an open problem.

Cunningham proved that if we remove an arbitrary neighborhood of one point from a circle, the resulting circular arc can be *shrunk* to a point using translations, rotations, and dilations [12]. His construction is based on the classical straight line results, and the stereographic projection between the plane and the sphere; it works only for circles.

Motivated by Cunningham's result, in Section 2.7, we include a brief note about what happens for general rectifiable sets when we consider the space  $\text{Sim}^+(\mathbb{R}^2)$  of all orientation-preserving similarity transformations of  $\mathbb{R}^2$ . This space consists of  $\text{Isom}^+(\mathbb{R}^2)$  as well as dilations and transformations which rotate and dilate simultaneously, moving points along logarithmic spirals. The techniques in Section 2.5 carry over to this setting because the composition structure of  $\text{Sim}^+(\mathbb{R}^2)$ , like that of  $\text{Isom}^+(\mathbb{R}^2)$ , is “linear up to a quadratic error.”

In (2.2), we consider the intersection of the normal lines  $\nu_x$  with balls in the projective plane. In the similarity transformations setting, we need to consider instead the intersections with *rotated* normal lines. The angle by which we rotate normal lines depends on the “pitch angle” of the logarithmic spirals of the similarity transformations.

As an illustrative example, in Section 2.7.1 we present the following application. (Compare this with Corollary 2.1.3.)

**Corollary 2.1.10.** *Any circular arc which is not the full circle can be moved continuously to any other position in the plane (of the same size) in arbitrarily small area via similarity transformations, such that the size of the circular arc always remains arbitrarily close its initial size.*

We also obtain a Nikodym set for circles, i.e., a set in the plane of Lebesgue measure zero which contains a punctured circle through every point.

**Corollary 2.1.11.** *There exists a set  $A \subset \mathbb{R}^2$  of Lebesgue measure zero such that for each  $x \in \mathbb{R}^2$ , there is a circle  $C$  such that  $x \in C$  and  $C \setminus A$  has at most one point.*

## 2.2 Main ideas of the proof of Theorems 2.1.2 and 2.5.1

Our proof of Theorem 2.1.2 relies on two key ideas.

### 2.2.1 The first key idea

Our first key idea is the “small neighborhood lemma”: suppose we move a compact set  $E \subset \mathbb{R}^2$  along a path of isometries  $P \subset \text{Isom}^+(\mathbb{R}^2)$ . If we perturb  $P$  by a small amount, the area covered by the perturbed movement will not increase very much because the new region covered is contained in a small neighborhood of the original. This simple and obvious fact turns out to be extremely useful.

**Lemma 2.2.1** (Small neighborhood lemma). *Let  $E \subset \mathbb{R}^2$  be any compact set, and let  $P$  be an arbitrary path in  $\text{Isom}^+(\mathbb{R}^2)$ . Then for every  $\varepsilon > 0$  there exists a neighborhood  $U \subset \text{Isom}^+(\mathbb{R}^2)$  of  $P$  such that*

$$\left| \bigcup_{p \in U} p(E) \right| \leq \left| \bigcup_{p \in P} p(E) \right| + \varepsilon. \quad (2.3)$$

(In this paper, a path is the image of a continuous map on a compact interval.)

### 2.2.2 The second key idea

The second key idea is more technical, so we give only an informal presentation here and defer the precise details to Section 2.3.3 and Section 2.3.4. First, note that:

**Lemma 2.2.2.** *For any polygonal path  $P \subset \mathbb{R}^2$  and for an arbitrary  $E \subset \mathbb{R}^2$ , if we translate  $E$  along  $P$ , then the area covered is  $\lesssim \mathcal{H}^1(E)\mathcal{H}^1(P)$ .*

*Proof.* If  $B$  is a ball of radius  $r$ , where  $r$  is smaller than the line segments in the polygonal path, then for each line segment  $L \subset P$ , by translating  $B$  along  $L$  we cover a set of area  $\lesssim r\mathcal{H}^1(L)$ . Adding these up for all line segments  $L$  and approximating  $E$  by a union of small balls, we obtain Lemma 2.2.2.  $\square$

*Remark 2.2.3.* Lemma 2.2.2 shows that in our proof of Theorem 2.1.2 we can ignore small subsets of  $E$ , since in the movements these will cover only small area. Also, we can ignore small subsets of  $P$ .

Our second key idea is the simple observation that the estimate in Lemma 2.2.2 can be improved if we also take into account the directions of the tangents of  $E$ . For simplicity, suppose that  $E$  is a  $C^1$  curve. Then we can cover  $E$  with thin rectangles that approximate the curve. Each thin rectangle  $R$  has the property that translating  $E \cap R$  along a line segment  $L$  in the direction of the long side of  $R$  covers area  $\lesssim \delta\mathcal{H}^1(L)$ , where  $\delta$  is the length of the short side of the rectangle. If the rectangle is thin enough, then this is a much better estimate than the estimate  $\mathcal{H}^1(E \cap R)\mathcal{H}^1(L)$  that we obtain from Lemma 2.2.2.

*Remark 2.2.4.* For general rectifiable sets  $E$ , instead of thin rectangles, we will choose  $R \subset E$  such that  $\theta_x$  is almost constant on  $R$ . The key idea remains the same (see Section 2.3.3).

### 2.2.3 Combining the key ideas

We combine these two “key ideas” to construct polygonal paths in a Venetian blind-type construction. (For Venetian blinds, see, e.g., [22, Theorem 6.9] or [40, Lemma 11.8].) Again, we give an informal presentation. See Section 2.4 for the precise details.

The method is as follows. Suppose that  $E$  is a  $C^1$  curve, which we cover by thin rectangles. Suppose our initial path is a translation along a horizontal segment. Let  $R, R'$  be two rectangles from our cover and let  $\theta, \theta'$  be the directions of their long sides, with  $\theta \neq \theta'$ .

1. First, we replace our horizontal segment by a zigzag so that every other segment has direction  $\theta$ . Then  $R \cap E$  will cover small area when translated along these segments.
2. Now we repeat the previous step, replacing each segment in direction  $\theta$  with a new zigzag such that every other segment has direction  $\theta'$ . Then  $R' \cap E$  will cover small area when translated along the segments in direction  $\theta'$ . Furthermore, if we make these new zigzags sufficiently “fine” (many turns and small enough segments), then these zigzags will remain close to the segments of direction  $\theta$  that we just replaced. Then by the small neighborhood lemma,  $R \cap E$  also covers small area when moved along the segments in direction  $\theta'$ .

By the end of step (2), we now have a “Venetian blind.” The line segments in direction  $\theta'$  are the “good” segments, because translations along these segments cover small area for both rectangles  $R$  and  $R'$ . By iterating with more angles, we can increase the number of rectangles for which translations along the good segments cover small area.

We also need the total length of the remaining “bad” segments to be strictly smaller than the initial segment, so that the size of the bad segments tends to zero when we iterate the Venetian blind construction. (For this reason, we cannot deviate too far from the initial horizontal direction. This leads to condition (2.10).) After sufficiently many iterations, we can ignore the bad segments by Remark 2.2.3.

The main ideas of the proof of Theorem 2.5.1, where we use rotations, are similar. As in the proof for translations, we combine the small neighborhood lemma with the covering of  $E$  by sets  $R$  such that rotating  $R$  around an appropriate point  $z$  covers only a small area. We still use a Venetian blind construction, but now our zigzags will be in  $\text{Isom}^+(\mathbb{R}^2)$ . The

general ideas of the argument are the same, but, as we explained in the introduction, they will require more delicate estimates than for translations.

## 2.3 Preliminaries

### 2.3.1 Tangents of rectifiable sets

Recall that a set  $E \subset \mathbb{R}^2$  is called *rectifiable* if  $\mathcal{H}^1$ -a.e. point of  $E$  can be covered by countably many  $C^1$  curves. For any two  $C^1$  curves, their tangent directions agree at  $\mathcal{H}^1$ -a.e. point of their intersection. Therefore, there exists a tangent field to  $E$ , i.e., a map  $x \mapsto \theta_x$  from  $E$  to  $\mathbb{P}^1$  such that for any  $C^1$  curve  $\Gamma$ , the tangent direction to  $\Gamma$  at  $x$  agrees with  $\theta_x$  for  $\mathcal{H}^1$ -a.e.  $x \in \Gamma \cap E$ . This gives one of the (many equivalent) descriptions of a tangent field of a rectifiable set.

Of course, the tangent field is uniquely defined only up to an  $\mathcal{H}^1$ -null subset of  $E$ . That is, if we change the tangent field along a set that meets each  $C^1$  curve in a set of  $\mathcal{H}^1$ -measure zero, it is still a tangent field.

In order to prove Theorem 2.1.2 and also our other results, we fix a particular tangent field  $x \mapsto \tilde{\theta}_x$  on  $E$  as follows: first we fix a subset  $E' \subset E$  of full  $\mathcal{H}^1$ -measure and a cover of  $E'$  by countably many  $C^1$  curves  $\{\Gamma_i\}$ . Next, for each  $x \in E$ , if all the curves  $\Gamma_i$  that go through  $x$  have the same tangent direction at that point, then we let  $\tilde{\theta}_x \in \mathbb{P}^1$  be that direction. (This also defines the normal line  $\tilde{\nu}_x$  at  $x$ .)

Consider the set of those  $x \in E$  where our  $\tilde{\theta}_x, \tilde{\nu}_x$  either (1) are not defined, or (2) are defined but do not agree with the  $\theta_x, \nu_x$  from the statements of our theorems. This is a set of zero  $\mathcal{H}^1$ -measure; hence we can ignore it by Remark 2.2.3 when we work with translations, and we will be able to ignore it by Lemma 2.3.4 (below) when we work with rotations. Hence, for the remainder of this paper, we may assume that  $\theta_x$  is the particular tangent field  $\tilde{\theta}_x$  from the previous paragraph (and make the analogous assumption for  $\nu_x$ ).

### 2.3.2 Rotations

We denote by  $\text{Isom}^+(\mathbb{R}^2)$  the space of all orientation preserving isometries of  $\mathbb{R}^2$ . Each element of  $\text{Isom}^+(\mathbb{R}^2)$  is either a translation by a vector  $v$ , or a rotation around a point  $z \in \mathbb{R}^2$  by angle  $\phi$ . Using complex notation, such a rotation is the map  $u \mapsto e^{i\phi}(u - z) + z$ . The image of 0 under this mapping is  $z(1 - e^{i\phi})$ , so it is natural to denote

$$v := z(1 - e^{i\phi}). \quad (2.4)$$

We can see from (2.4) that  $v = z(-i\phi + O(\phi^2))$ . We denote

$$w = \begin{cases} z\phi & \phi \neq 0 \\ iv & \phi = 0. \end{cases} \quad (2.5)$$

The motivation behind our notation is that, for small  $\phi$  and near the origin, the rotation acts, to first order, like translation by  $-iw$ .

Both translations and rotations can now be specified by an ordered pair  $(w, \phi) \in \mathbb{R}^2 \times \mathbb{R}$ . (We have  $\phi = 0$  for translations.) From now on, we will refer to translations as rotations as well.

For  $\rho \neq \text{id}$ , we define the *projective center* of  $\rho$  to be the image of  $(w, \phi)$  under the quotient map  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ . We still use  $(w, \phi)$  to denote the image in  $\mathbb{P}^2$ . If  $\phi \neq 0$ , then, using homogeneous coordinates, this reduces to  $(z\phi, \phi) = (z, 1)$ , as expected. If  $\phi = 0$ , then  $(w, 0) = (iv, 0) = (v^\perp, 0)$ , which is indeed the point at infinity orthogonal to the direction of  $v$ .

*Remark 2.3.1.* Even though we now view translations as rotations around infinite points, translations and rotations are still different, even when viewed in  $\mathbb{P}^2$ . For example, a rotation with angle  $\phi \neq 0$  fixes just one point in  $\mathbb{P}^2$  (its projective center) whereas a translation fixes an entire line ( $\mathbb{P}^1 \subset \mathbb{P}^2$ ).

We will use the notation  $\rho(w, \phi)$  for a rotation whose projective center is  $(w, \phi) \in \mathbb{P}^2$  and whose angle is  $\phi$ . That is, we assign to each point  $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$  the rotation  $\rho(x)$  whose:

- projective center is the image of  $(x_1, x_2, x_3)$  under the projection  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ ;
- angle is the last coordinate  $x_3$ .

*Remark 2.3.2.* We will use the same notation  $\rho = \rho(w, \phi)$  for the mapping  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and for the continuous movement that rotates  $\mathbb{R}^2$  around a point. For example, if  $\phi = 2\pi$  then the former is the identity mapping and the latter is not. It will be always clear from the context which one we mean.

### 2.3.3 The “second key idea” for translations

We fix a small  $\delta > 0$ , and a direction  $\theta$ . Let  $R$  be a subset of  $E$  such that  $|\theta_x - \theta| \lesssim \delta$  for every  $x \in R$ . Our aim is to estimate how large area we cover if we translate  $R$  by a vector  $v$  of direction  $\theta$ .

For each  $x \in R$  there is a  $C^1$  curve  $\Gamma_i$  from Section 2.3.1 that goes through the point  $x$ . We choose a decomposition  $R = \bigcup R_i$  such that  $R_i \subset \Gamma_i$  for each  $i$ . Then locally, i.e., in a neighborhood of  $x \in R_i$ ,  $\Gamma_i$  is the graph of a Lipschitz function  $f$  in the  $(\theta, \theta^\perp)$  coordinate system, with Lipschitz constant  $\lesssim \delta$ . Without loss of generality we can assume that  $\theta = 0$ .

Now, when we translate  $R_i$  by the horizontal vector  $v$ , for each fixed  $t \in \mathbb{R}$  we obtain  $\#\{x \in \mathbb{R} : f(x) = t, (x, f(x)) \in R_i\}$  many (not necessary disjoint) horizontal line segments on the line  $y = t$ , each of length  $|v|$ . Therefore, by Fubini’s theorem, the area covered is at most

$$|v| \int \#\{x \in \mathbb{R} : f(x) = t, (x, f(x)) \in R_i\} dt. \quad (2.6)$$

Next, recall that the coarea formula (e.g., [24, Theorem 3.2.22]) implies that for any measurable function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , for any Lipschitz  $h : \mathbb{R} \rightarrow \mathbb{R}$ , and for any measurable



$S \subset \mathbb{R}$ ,

$$\int_{\mathbb{R}} g(t) \#\{x \in S : h(x) = t\} dt = \int_S g(h(x)) |h'(x)| dx. \quad (2.7)$$

Using (2.7), we can bound (2.6) by

$$\lesssim \delta \mathcal{H}^1(R_i) |v|$$

since  $f$  has Lipschitz constant  $\lesssim \delta$ . Summing over  $i$ , we obtain the following:

**Lemma 2.3.3.** *Let  $\delta > 0$  be sufficiently small, and let  $\theta$  be an arbitrary direction. Let  $R$  be a subset of  $E$  such that  $|\theta_x - \theta| \lesssim \delta$  for every  $x \in R$ . Then if we translate  $R$  by a vector  $v$  of direction  $\theta$ , the total area covered is  $\lesssim \delta \mathcal{H}^1(R) |v|$ .*

Note that  $|\theta_x - \theta| \lesssim \delta$  if and only if  $\nu_x$  meets a  $\lesssim \delta$ -neighborhood of  $\theta^\perp$  in  $\mathbb{P}^1$ . Using this observation, we generalize Lemma 2.3.3 to rotations in the next section.

### 2.3.4 The “second key idea” for rotations

Let  $z \in \mathbb{R}^2$ , and let  $\phi$  be an arbitrary angle. If we rotate the set  $E$  around the center  $z$  by angle  $\phi$ , then each point  $x \in E$  moves along a circular arc of length  $|x - z| |\phi|$ . Therefore, the trivial estimate we get is that by rotating  $E$ , the area covered is

$$\leq |\phi| \int_{\mathbb{R}} r \#\{x \in E : |x - z| = r\} dr \leq |\phi| \int_E |x - z| d\mathcal{H}^1(x). \quad (2.8)$$

The first inequality follows from Fubini’s theorem. The second follows from the coarea formula (2.7) and the fact that if we parametrize the curve  $\Gamma_i$  by arc-length, the mapping  $t \mapsto |x(t) - z|$  is Lipschitz, with Lipschitz constant at most 1.

For a general rectifiable set, the right-hand side of (2.8) can be infinite (cf. Remark 2.1.6). From now on, in this section we assume that  $E$  is bounded. More precisely, we assume that  $E \subset B(0, r) \subset \mathbb{R}^2$  (here, we used the Euclidean metric). We will show that there is a constant  $c$  (that depends only on  $r$ ) such that the following two lemmas hold.

**Lemma 2.3.4.** *Let  $y = (w, \phi) \in \mathbb{R}^2 \times \mathbb{R}$ , let  $\rho = \rho(y)$  be a rotation, and let  $R \subset E$  be arbitrary. Then, if we rotate  $R$  by  $\rho$ , the area covered is  $\lesssim c\mathcal{H}^1(R)|y|$ .*

**Lemma 2.3.5.** *Let  $\delta > 0$  be sufficiently small (depending on  $r$ ). Let  $y = (w, \phi) \in \mathbb{R}^2 \times \mathbb{R}$  and let  $\rho = \rho(y)$  be a rotation with projective center  $z$ . Let  $R \subset E$  be such that, for each  $x \in R$ ,  $\nu_x \cap B(z, \delta) \neq \emptyset$ . (Here, the ball  $B(z, \delta)$  is defined with respect to the metric on  $\mathbb{P}^2$ .) Then, when we rotate  $R$  by  $\rho$ , the area covered is*

$$\lesssim c\delta\mathcal{H}^1(R)|y|.$$

*Proof of Lemma 2.3.4.* By Lemma 2.2.2, we know that Lemma 2.3.4 holds (with  $c = 1$ ) when  $\rho$  is a translation. Now suppose that  $z \in \mathbb{R}^2$  and  $\phi \neq 0$ . Then there is a constant  $c_1$  (that depends only on  $r$ ) such that  $|x - z| \leq r + |z| \leq c_1\sqrt{1 + |z|^2}$  for every  $x \in E$ . Since  $|y| = \sqrt{|z|^2\phi^2 + \phi^2} = |\phi|\sqrt{1 + |z|^2}$ , therefore Lemma 2.3.4 follows from the trivial estimate (2.8), with  $E$  replaced by  $R$ , and  $|x - z|$  replaced by  $c_1\sqrt{1 + |z|^2}$ .  $\square$

*Proof of Lemma 2.3.5.* First, suppose that  $z \in \mathbb{R}^2$  and  $\phi \neq 0$ . We note that we can improve the estimate (2.8) by noticing that the derivative of  $t \mapsto |x(t) - z|$  is  $\langle \dot{x}(t), \frac{x(t) - z}{|x(t) - z|} \rangle = \frac{1}{|x(t) - z|} \text{dist}(\nu_{x(t)}, z)$ . (Here,  $\text{dist}$  denotes the Euclidean distance.) Therefore, by the coarea formula (2.7), rotating the set  $R$  covers area

$$\leq |\phi| \int_R \text{dist}(\nu_x, z) d\mathcal{H}^1(x). \quad (2.9)$$

Thus, it suffices to show that if  $\nu_x$  intersects the  $\delta$ -neighbourhood of  $z$  in  $\mathbb{P}^2$ , then  $\text{dist}(\nu_x, z) \leq c\delta\sqrt{1 + |z|^2}$  in  $\mathbb{R}^2$ . If  $|z| \leq 2r$  and  $\delta$  is sufficiently small, then the Euclidean and projective distances are comparable, and  $\sqrt{1 + |z|^2}$  is comparable to 1 (where the implied constants depend only on  $r$ ), so there is nothing to prove. Now, suppose  $|z| > 2r$ . Since  $E \subset B(0, r)$ , therefore, for  $\delta$  sufficiently small, the projective ball  $B(z, \delta)$  is bounded away from  $x \in E$ . Let  $\pi$  denote the quotient map  $\pi : S^2 \rightarrow \mathbb{P}^2$ . Then there is a constant  $c_2$  (that

depends only on  $r$ ) such that the angle between any two great circles through  $\pi^{-1}x$  that meet  $\pi^{-1}(B(z, \delta))$  is  $\leq c_2\delta$ . Then there is a constant  $c_3$  (that depends only on  $r$ ) such that the angle between  $x - z$  and  $\nu_x$  in  $\mathbb{R}^2$  is  $\leq c_2c_3\delta$ . With  $c_1$  as in the proof of Lemma 2.3.4, we have  $\text{dist}(\nu_x, z) \leq c_2c_3\delta|x - z| \leq c_1c_2c_3\delta\sqrt{1 + |z|^2}$ , as desired.

Now, we prove Lemma 2.3.5 when  $z \in \mathbb{P}^1$  (i.e., when  $\rho$  is a translation). Again, for  $\delta$  sufficiently small, the projective ball  $B(z, \delta)$  is bounded away from  $E$ . Thus, if  $\nu_x$  intersects  $B(z, \delta)$ , then the angle between  $\nu_x$  and  $\mathbb{P}^1$  is bounded away from zero. Therefore, there is a constant  $c_4$  (that depends only on  $r$ ) such that if  $\nu_x$  intersects  $B(z, \delta)$  for some  $x \in E$ , then the projective distance between  $z$  and  $\nu_x \cap \mathbb{P}^1$  is  $\leq c_4\delta$ . Hence, we can apply Lemma 2.3.3 to obtain our desired result.  $\square$

## 2.4 Keakeya needle problem for translations

In this section  $E$  is an arbitrary rectifiable set of finite  $\mathcal{H}^1$ -measure. Without loss of generality we assume that  $\mathcal{H}^1(E) = 1$ , and that  $\theta_x$  is defined for each  $x \in E$ , as in Section 2.3.1.

### 2.4.1 Notation

We say that a subset of  $\mathbb{P}^1$  is an *interval* if it is connected. For  $\theta_1, \theta_2 \in \mathbb{P}^1$  with  $|\theta_1 - \theta_2| < \pi/2$ , we denote by  $[\theta_1, \theta_2]$  the interval in  $\mathbb{P}^1$  whose endpoints are  $\theta_1, \theta_2$  and has length less than  $\pi/2$ . (When we use this notation, we do not specify which one is the left and which one is the right endpoint.)

The symbol  $\mathbf{i}$  will always denote a finite binary sequence, i.e., a sequence  $i_1i_2 \dots i_k$ , where  $k \geq 0$  and each term  $i_j$  is 0 or 1. The empty sequence  $\emptyset$  corresponds to  $k = 0$ . The length of  $\mathbf{i}$  is denoted by  $|\mathbf{i}|$ . We denote  $\mathbf{i}' = i_1i_2 \dots i_{k-1}$  (note that  $\emptyset'$  is not defined). We will say that  $\mathbf{i}'$  is the *parent* of  $\mathbf{i}$ , and  $\mathbf{i}$  is a *child* of  $\mathbf{i}'$ , respectively. The *ancestors* and the *descendants* of an  $\mathbf{i}$  are defined in the obvious way. We will also say that a sequence is *bad* if it ends with a 0 and *good* if it ends with a 1. (The empty sequence  $\emptyset$  is also good.)

### 2.4.2 Basic zigzag

A *basic zigzag* is a polygonal path which is made up of  $N$  congruent and equally spaced segments in direction  $\theta_0$  interlaced with  $N$  congruent segments in direction  $\theta_1$ . (See Figure 2.1(b) for an example.)

The fundamental procedure in our construction is taking a line segment  $L$  and replacing it with a basic zigzag with the same endpoints. The key properties of basic zigzags are the following two geometrically obvious facts:

- With  $L, \theta_0, \theta_1$  fixed, we can ensure that the basic zigzag lies in an arbitrarily small neighborhood of  $L$  by making the zigzag sufficiently “fine,” i.e., making  $N$  sufficiently large.
- The total length of each of the two parallel pieces of the basic zigzag depends only on  $L, \theta_0, \theta_1$  and not on the fineness of the zigzag.

### 2.4.3 Venetian blind

Like the basic zigzag, a Venetian blind is a polygonal path of line segments of two fixed directions. These segments are constructed by iterating the basic zigzag construction. We fix a line segment  $L$ , small parameters  $0 < \gamma \leq \beta < \frac{\pi}{4}$  and a sign  $\sigma \in \{-1, 1\}$ . The Venetian blind construction is as follows.

Let  $\theta_L \in \mathbb{P}^1$  denote the direction of  $L$ . In our first step, we replace  $L$  by a basic zigzag with directions  $\theta_L - \sigma\beta, \theta_L + \sigma\gamma$ . (See Figure 2.1(b).) Let  $G_1$  denote the union of the line segments of the basic zigzag in direction  $\theta_L + \sigma\gamma$ . Iteratively, in our  $i^{\text{th}}$  step for  $i \geq 2$ , we replace each line segment in  $G_{i-1}$  by a basic zigzag of directions  $\theta_L - \sigma\beta, \theta_L + i\sigma\gamma$ , and let  $G_i$  denote the union of the line segments in direction  $\theta_L + i\sigma\gamma$ . (See Figure 2.1(c) for the line segments obtained after the second step.) We stop this procedure after  $k$  steps, where  $k$  is defined by

$$k\gamma \in [\pi/2 - 2\beta - \gamma, \pi/2 - 2\beta). \quad (2.10)$$

The zigzag we end up with is what we call our *Venetian blind*. We denote by  $L_1$  the final set  $G_k$  obtained by this construction, and we denote by  $L_0$  the rest of the Venetian blind. That is, the Venetian blind is the polygonal path  $L_0 \cup L_1$  where  $L_0$  and  $L_1$  are unions of line segments of directions  $\theta_L - \sigma\beta$  and  $\theta_L + k\sigma\gamma$  respectively. We call  $L_0$  the *bad* part of the Venetian blind and  $L_1$  the *good* part.

We say that the directions  $\theta_L - \sigma\beta, \theta_L, \theta_L + \sigma\gamma, \dots, \theta_L + k\sigma\gamma$  have been *used* in the construction of this Venetian blind. This terminology will be used in Section 2.4.8.

*Remark 2.4.1.* Usually, in the literature, a Venetian blind consists only of the “good” line segments. In our definition of a Venetian blind, it contains both  $L_0$  and  $L_1$ .

*Remark 2.4.2.* The lengths of  $L_0$  and  $L_1$  depend only on  $\mathcal{H}^1(L)$  and  $\beta, \gamma$ . They *do not* depend on the fineness of the zigzags. Furthermore, condition (2.10) ensures that there is a constant  $c(\beta) < 1$  such that

$$\mathcal{H}^1(G_j) \leq \mathcal{H}^1(L), \quad \mathcal{H}^1(L_i) \leq c(\beta)\mathcal{H}^1(L) \quad (2.11)$$

for each  $j = 1, 2, \dots, k$  and  $i = 0, 1$ .

*Remark 2.4.3.* We consider two natural ways to partition  $L_0$  into line segments. The first way is into the maximal disjoint line segments of  $L_0$ . Note that a line segment in this partition could be made up of segments from multiple basic zigzags in our construction. (For example, the right-most segment in Figure 2.1(c) has this property.)

The second partition is a refinement of the first. We subdivide each segment from the first partition into the individual segments from the basic zigzags, i.e., each segment in the second partition is a segment from some basic zigzag used in the Venetian blind construction.

In Section 2.4.4 (below), we describe how to iterate the Venetian blind construction on each line segment of  $L_0$ . We can interpret the word “each” in two different ways, corresponding to the two decompositions above. In this section, it does not matter which way we choose, but in Section 2.5, we must choose the second one.

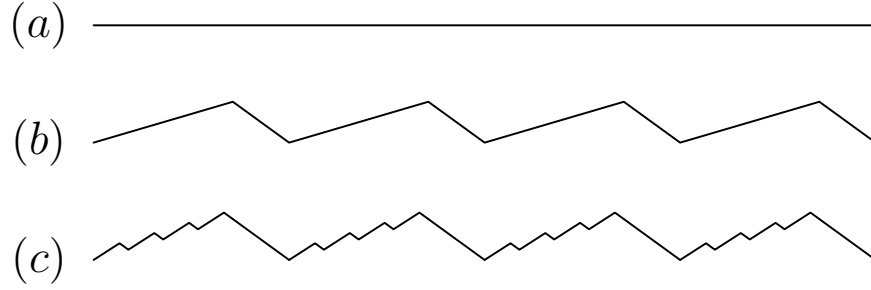


Figure 2.1: Two steps in the Venetian blind construction, starting with (a) as the initial segment.

#### 2.4.4 Main construction

Our strategy of proving Theorem 2.1.2 is to iterate the Venetian blind construction. Given a point in  $\mathbb{R}^2$ , we construct a polygonal path from the origin to this point. We start with the line segment  $L_\emptyset$  joining the origin to this point, and then, iteratively, for each finite sequence  $\mathbf{i}$ , we apply the Venetian blind construction to each segment in  $L_{\mathbf{i}}$  with some parameters  $\beta = \beta_{\mathbf{i}}, \gamma = \gamma_{\mathbf{i}}, \sigma = \sigma_{\mathbf{i}}$ . We let  $L_{\mathbf{i}0}$  be the union of all the bad parts of the Venetian blinds and  $L_{\mathbf{i}1}$  be the union of all the good parts (as defined in Section 2.4.3). Since we use the same  $\beta_{\mathbf{i}}, \gamma_{\mathbf{i}}, \sigma_{\mathbf{i}}$  on each line segment in  $L_{\mathbf{i}}$ , it follows by induction that every  $L_{\mathbf{i}}$  is a union of parallel line segments of some direction  $\theta_{\mathbf{i}}$ .

We also iteratively assign, to each  $\mathbf{i}$ , an interval  $I_{\mathbf{i}} \subset \mathbb{P}^1$  by the following simple method. We put  $I_\emptyset = \emptyset$ . Then, for each finite sequence  $\mathbf{i} \neq \emptyset$ , we define  $I_{\mathbf{i}} := I_{\mathbf{i}'} \cup [\theta_{\mathbf{i}}, \theta_{\mathbf{i}'}]$ . Then clearly, by induction, we can see that for every  $\mathbf{i} \neq \emptyset$ ,  $I_{\mathbf{i}}$  is an interval and  $\theta_{\mathbf{i}} \in I_{\mathbf{i}}$ .

#### 2.4.5 Choosing the parameters $\beta_{\mathbf{i}}, \gamma_{\mathbf{i}}$

For each  $\mathbf{i}$ , we fix a small  $\varepsilon_{\mathbf{i}}$  that we will specify later. They will depend only on  $\varepsilon$  (where  $\varepsilon$  is from the statement of Theorem 2.1.2). We denote the number of 1's in the sequence  $\mathbf{i}$  by  $n_{\mathbf{i}}$ . Then we can choose our parameters  $\beta_{\mathbf{i}} \geq \gamma_{\mathbf{i}} > 0$  in our Venetian blind constructions such that they satisfy:

1.  $\beta_{\mathbf{i}} \leq \beta_{\mathbf{i}'}$  for every  $\mathbf{i}$ ;

2.  $\beta_{\mathbf{i}} = \beta_{\mathbf{k}}$ , where  $\mathbf{k}$  is the last (i.e., youngest) good sequence among  $\mathbf{i}$  and its ancestors;
3.  $\beta_{\mathbf{i}} \leq 1/n_{\mathbf{i}}$  for every  $\mathbf{i}$ ;
4.  $\beta_{\mathbf{i}} \mathcal{H}^1(L_{\mathbf{i}}) \leq \varepsilon_{\mathbf{i}}$  if  $\mathbf{i}$  is good;
5.  $\gamma_{\mathbf{i}} \mathcal{H}^1(L_{\mathbf{i}}) \leq \varepsilon_{\mathbf{i}}$  for every  $\mathbf{i}$ .

We can indeed make these choices, since  $\mathcal{H}^1(L_{\mathbf{i}})$  is determined by the  $\beta$ 's and  $\gamma$ 's of its ancestors.

We will also use the notation:

- (6)  $\alpha_{\mathbf{i}} = \beta_{\mathbf{i}'}$  if  $\mathbf{i}$  is bad, and  $\alpha_{\mathbf{i}} = \gamma_{\mathbf{i}'}$  if  $\mathbf{i}$  is good.

#### 2.4.6 Choosing the signs $\sigma_{\mathbf{i}}$

We choose each sign  $\sigma_{\mathbf{i}}$  such that it makes  $I_{\mathbf{i}\mathbf{1}} = I_{\mathbf{i}} \cup [\theta_{\mathbf{i}}, \theta_{\mathbf{i}\mathbf{1}}]$  as large as possible. That is, if  $\theta_{\mathbf{i}}$  is in the right half of the interval  $I_{\mathbf{i}}$  (where we embed  $I_{\mathbf{i}}$  into  $\mathbb{R}$ ), then we choose  $\sigma_{\mathbf{i}} = 1$ ; otherwise, we choose the  $\sigma_{\mathbf{i}} = -1$ . (If  $\mathbf{i} = \emptyset$ , or if  $\theta_{\mathbf{i}}$  is in the middle of the interval  $I_{\mathbf{i}}$ , or if  $I_{\mathbf{i}} = \mathbb{P}^1$ , then we can choose the sign arbitrarily.) Our choice of  $\sigma_{\mathbf{i}}$  ensures that the length of the interval  $I_{\mathbf{i}\mathbf{1}}$  can be estimated by

$$|I_{\mathbf{i}\mathbf{1}}| \geq |I_{\mathbf{i}}|/2 + |\theta_{\mathbf{i}} - \theta_{\mathbf{i}\mathbf{1}}| \geq |I_{\mathbf{i}}|/2 + \pi/2 - 2\beta_{\mathbf{i}}.$$

The second inequality follows from (2.10). This can be re-written as:

$$\pi - |I_{\mathbf{i}\mathbf{1}}| \leq (\pi - |I_{\mathbf{i}}|)/2 + 2\beta_{\mathbf{i}}.$$

Suppose  $\mathbf{k}$  is the last good sequence among  $\mathbf{i}$  and its ancestors, and  $\mathbf{m}$  is the second-to-last one. Then since the intervals  $I_{\mathbf{i}}$  are increasing and the parameters  $\beta_{\mathbf{i}}$  are decreasing along

each family line, we have  $\pi - |I_{\mathbf{i}}| \leq \pi - |I_{\mathbf{k}}|$  and hence

$$\pi - |I_{\mathbf{i}}| \leq (\pi - |I_{\mathbf{m}}|)/2 + 2\beta_{\mathbf{m}}. \quad (2.12)$$

### 2.4.7 Stopping time

We need to define when we stop our Venetian blind constructions on various family lines. In order to construct our polygonal path  $P$  (for Theorem 2.1.2), we need to ensure that ultimate extinction occurs. We will, of course, define the polygonal path  $P$  as the union of those  $L_{\mathbf{i}}$  where the construction stops.

First of all, we stop our Venetian blind construction at  $L_{\mathbf{i}}$  if  $\mathcal{H}^1(L_{\mathbf{i}}) \leq \varepsilon_{\mathbf{k}}$ , where  $\mathbf{k}$  is the last good sequence among  $\mathbf{i}$  and its ancestors. By (2.11) and condition (2) in Section 2.4.5,

$$\mathcal{H}^1(L_{\mathbf{i}}) \geq c(\beta_{\mathbf{i}})^{-1}\mathcal{H}^1(L_{\mathbf{i}0}) \geq c(\beta_{\mathbf{i}})^{-2}\mathcal{H}^1(L_{\mathbf{i}00}) \geq \dots \quad (2.13)$$

This ensures that, for each  $\mathbf{i}$ , the family line  $\mathbf{i}, \mathbf{i}0, \mathbf{i}00, \dots$  dies out after finitely many generations, where the number of generations depends only on  $\beta_{\mathbf{i}}$ . Therefore  $\min\{n_{\mathbf{i}} : |\mathbf{i}| = k\} \rightarrow \infty$  as  $k \rightarrow \infty$ , and consequently, by condition (3),  $\max\{\beta_{\mathbf{i}} : |\mathbf{i}| = k\} \rightarrow 0$ . Using this and (2.12), if  $k$  is large enough, then

$$\max\{\pi - |I_{\mathbf{i}}| : |\mathbf{i}| = k\} < \varepsilon. \quad (2.14)$$

We stop our whole construction after  $k$  generations, where  $k$  is so large that (2.14) holds.

By choosing the parameters  $\varepsilon_{\mathbf{i}}$  such that  $\sum_{\mathbf{i}} \varepsilon_{\mathbf{i}}$  is small enough, by Lemma 2.2.2, Remark 2.2.3 and our assumption  $\mathcal{H}^1(E) = 1$ , we can ignore those  $L_{\mathbf{i}}$  for which  $\mathcal{H}^1(L_{\mathbf{i}}) \leq \varepsilon_{\mathbf{k}}$ , where  $\mathbf{k}$  is the last good sequence among  $\mathbf{i}$  and its ancestors. (This is because each  $\mathbf{i}$  at which we stop our construction has a different “last good sequence among  $\mathbf{i}$  and its ancestors.”) For the line segments that belong to the remaining part of the polygonal path, we have  $|I_{\mathbf{i}}| > \pi - \varepsilon$  by (2.14).

Using the previous paragraph, we choose our balls  $B(\theta_i, \varepsilon)$  for Theorem 2.1.2 as follows.



If we ignore  $L_{\mathbf{i}}$  (as described in the previous paragraph), then for each line segment  $L_i$  in  $L_{\mathbf{i}}$ , we let  $\theta_i$  to be any point we like. If we do not ignore  $L_{\mathbf{i}}$ , then for each  $L_i$  in  $L_{\mathbf{i}}$ , we choose  $\theta_i$  so that  $\mathbb{P}^1 \setminus B(\theta_i, \varepsilon) \subset I_{\mathbf{i}}$ . We can do this because  $|I_{\mathbf{i}}| > \pi - \varepsilon$ . In both cases, since  $\theta_{\mathbf{i}} \in I_{\mathbf{i}}$  for all  $\mathbf{i}$ , we can also choose each  $\theta_i$  so that

$$\theta_{\mathbf{i}} \notin B(\theta_i, \varepsilon). \quad (2.15)$$

In order to finish the proof of Theorem 2.1.2, it suffices to show that

$$A := \bigcup_{L_{\mathbf{i}} \subset P} (L_{\mathbf{i}} + \{x \in E : \theta_x \in I_{\mathbf{i}}\}) \quad (2.16)$$

has small measure.

*Remark 2.4.4.* So far, our definition of the polygonal path  $P$  did not depend on the set  $E$ . In what follows, we will show that if the zigzags we use are sufficiently fine (depending on the set  $E$ ) then indeed the set  $A$  in (2.16) has small measure. Note that the fineness of the zigzags is the only remaining parameter we need to specify. The parameters  $\beta_{\mathbf{i}}, \gamma_{\mathbf{i}}, \sigma_{\mathbf{i}}$ , the lengths  $\mathcal{H}^1(L_{\mathbf{i}})$ , the stopping time, and the intervals  $I_{\mathbf{i}}$  are all independent of the fineness of the zigzags and of  $E$ .

#### 2.4.8 *Fineness of the zigzags, and the small neighborhood lemma*

We have already chosen all the directions we use in all the basic zigzags to construct  $P$ . These directions divide  $\mathbb{P}^1$  into finitely many intervals, which we call *elementary intervals*. By an elementary interval we mean a *closed* interval  $I \subset \mathbb{P}^1$  such that its endpoints are directions used in our construction, and such that  $I$  does not contain any other such direction. (See Section 2.4.3 for what it means for a direction to be “used in our construction.”)

Since we already know the length  $\mathcal{H}^1(P)$ , we also know how large subset of  $E$  we may ignore by Lemma 2.2.2 and Remark 2.2.3. Therefore, by throwing away a sufficiently small

subset of  $E$  if necessary, we can assume that  $E$  is compact and also that  $x \mapsto \theta_x$  is a continuous function on  $E$ .

For an elementary interval  $I$ , we denote

$$E_I = \{x \in E : \theta_x \in I\}.$$

Because of our assumptions above,  $E_I$  is also compact.

Here is our strategy for choosing the fineness of the zigzags. Suppose that for some  $E_I$  and for some line segment  $L$  in our construction, we have obtained the estimate  $|L + E_I| < \eta$  for some  $\eta$ . Then, we require all zigzags descending from  $L$  to be fine enough so that they stay in a sufficiently small neighborhood of  $L$ . This ensures that by the small neighborhood lemma, translating  $E_I$  along the descendants of  $L$  still covers area  $< \eta$ .

In the next section, we obtain finitely many estimates of the form  $|L + E_I| < \eta$ . We make the zigzags sufficiently fine at each step so that these estimates are preserved by the descendants of  $L$ , as explained above.

#### 2.4.9 Area estimate

We fix an elementary interval  $I$  and the corresponding set  $E_I$ , and revisit the Venetian blind construction. Our aim is to estimate the measure of the set

$$A_I := \bigcup_{L_i \subset P \text{ s.t. } I \subset I_i} (L_i + E_I). \quad (2.17)$$

Our final goal is to show  $|\bigcup_I A_I| < \varepsilon$ . In the next two paragraphs, we will use the same notations as in Section 2.4.3.

First assume that the elementary interval  $I$  is contained in the interval  $[\theta_L + (j-1)\sigma\gamma, \theta_L + j\sigma\gamma]$  for some  $j = 1, 2, \dots, k$ . Since the line segments of  $G_j$  are of direction  $\theta_L + j\sigma\gamma$ , it follows from Lemma 2.3.3 (and the estimate (2.11)) that translating  $E_I$  along the line

segments of  $G_j$  covers area  $\lesssim \gamma \mathcal{H}^1(E_I) \mathcal{H}^1(G_j) \leq \gamma \mathcal{H}^1(E_I) \mathcal{H}^1(L)$ . By our remarks in the previous section about choosing the fineness of the zigzags, the same estimate  $\gamma \mathcal{H}^1(E_I) \mathcal{H}^1(L)$  remains true if we translate  $E_I$  along the line segments of  $G_k$ .

We can argue similarly when the elementary interval is contained in  $[\theta_L, \theta_L - \sigma\beta]$ . Therefore, we proved the following lemma:

**Lemma 2.4.5.** *Suppose that an elementary interval  $I$  is contained in  $[\theta_{\mathbf{i}}, \theta_{\mathbf{i}'}]$ . Then  $|L_{\mathbf{i}} + E_I| \lesssim \alpha_{\mathbf{i}} \mathcal{H}^1(E_I) \mathcal{H}^1(L_{\mathbf{i}'})$ .*

Now consider an  $L_{\mathbf{i}} \subset P$  with  $I \subset I_{\mathbf{i}}$ . Since the intervals  $I_{\mathbf{i}}, I_{\mathbf{i}'}, I_{\mathbf{i}''}, \dots$  are decreasing, there is a  $\mathbf{k}$  among  $\mathbf{i}$  and its ancestors such that  $I \subset I_{\mathbf{k}} \setminus I_{\mathbf{k}'} \subset [\theta_{\mathbf{k}}, \theta_{\mathbf{k}'}]$ . By Lemma 2.4.5,

$$|L_{\mathbf{k}} + E_I| \lesssim \alpha_{\mathbf{k}} \mathcal{H}^1(E_I) \mathcal{H}^1(L_{\mathbf{k}'}). \quad (2.18)$$

The estimates (2.18) are precisely those that we would like to maintain when we replace the set  $L_{\mathbf{k}}$  by all of its final descendants in  $P$ , as described in the previous section. Thus, we make the zigzags sufficiently fine so that these estimates are preserved.

Therefore, instead of taking the sum of the estimates (2.18) for *all* finite sequences  $\mathbf{k}$ , it is sufficient to take the sum for *some*  $\mathbf{k}$ , each of which belongs to a different family line. Let  $\mathbf{k}_1, \mathbf{k}_2, \dots$  be arbitrary sequences from different family lines.

We distinguish two cases: if  $\mathbf{k}_m$  is good, then by (5) and (6) in Section 2.4.5,

$$\alpha_{\mathbf{k}_m} \mathcal{H}^1(E_I) \mathcal{H}^1(L_{\mathbf{k}_m}') \leq \varepsilon_{\mathbf{k}_m'} \mathcal{H}^1(E_I). \quad (2.19)$$

With the bad  $\mathbf{k}_m$ , the same trivial bound does not work. Nonetheless, because of the “different family lines condition,” each bad  $\mathbf{k}_m$  has a different “last good among  $\mathbf{k}_m$  and its ancestors.” Therefore by (2), (4), and (6) in Section 2.4.5 and (2.11), we have

$$\sum_{\mathbf{k}_m \text{ bad}} \alpha_{\mathbf{k}_m} \mathcal{H}^1(E_I) \mathcal{H}^1(L_{\mathbf{k}_m}') \leq \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \mathcal{H}^1(E_I), \quad (2.20)$$

where the summation on the right is taken over all  $\mathbf{k}$ . Adding together the estimates (2.19) for all good  $\mathbf{k}_m$  and (2.20), we have

$$|A_I| \leq 2 \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \mathcal{H}^1(E_I). \quad (2.21)$$

Since each  $x \in E$  belongs to at most two of the sets  $E_I$ , by summing over  $I$  and choosing  $\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}$  small enough, the proof of Theorem 2.1.2 is finished.

## 2.5 Keakeya needle problem for rotations

Our aim in this section is to prove the following theorem, which can be thought of as a direct analogue of Theorem 2.1.2. Recall when we create a polygonal path  $P \subset \text{Isom}^+(\mathbb{R}^2)$  from a sequence of rotations  $\{\rho_i\}$ , we always interpret the rotations in the *intrinsic* coordinate system. We will occasionally use the phrase *intrinsic rotation* to remind ourselves of this convention.

**Theorem 2.5.1.** *Let  $E \subset \mathbb{R}^2$  be a bounded rectifiable set of finite  $\mathcal{H}^1$ -measure. Let  $\varepsilon > 0$ , and let  $\rho \in \text{Isom}^+(\mathbb{R}^2)$  be arbitrary. Let  $\ell \subset \mathbb{P}^2$  be a line through the projective center  $z$  of  $\rho$ .*

*Then there are intrinsic rotations  $\rho_i = \rho(x_i)$  with projective centers  $z_i \in B(\ell, \varepsilon) \subset \mathbb{P}^2$  such that the corresponding polygonal path  $P = \bigcup_i L_i \subset \text{Isom}^+(\mathbb{R}^2)$  connects the identity and  $\rho$ , and for each  $i$ , there exists a  $u_i \in \ell$  such that*

$$\left| \bigcup_i \bigcup_{p \in L_i} p(\{x \in E : \nu_x \cap \ell \cap B(u_i, \varepsilon) = \emptyset\}) \right| < \varepsilon. \quad (2.22)$$

### 2.5.1 Basic zigzags, deconstructed

The heart of the matter in our proof of Theorem 2.1.2 was that we repeatedly replaced line segments by basic zigzags. Each line segment  $L$  represented a translation. In our proof of Theorem 2.5.1 we will do an analogue construction with rotations instead of translations.

However, this is a bit more delicate, so first, we present the basic zigzag construction for translations in more detail than before. We decompose this construction into two steps.

The first step of the basic zigzag construction for translations divides a line segment  $L$  into  $N$  equal parts. In the second step, for translations, we replace each of the  $N$  line segments by two line segments of given directions. We can represent these two steps by the two equations

$$\begin{aligned} v &= (v/N) + \cdots + (v/N) \\ &= (v_0/N) + (v_1/N) + \cdots + (v_0/N) + (v_1/N), \end{aligned}$$

where  $v_0$  and  $v_1$  are vectors in the two given directions and such that  $v = v_0 + v_1$ .

The first step for rotations is easy to understand: we replace a rotation  $\rho = \rho(x)$  by  $N$  copies of  $\rho(x/N)$ , which are rotations around the same projective center as  $\rho$  but with angle reduced by a factor of  $N$ . In the intrinsic coordinate system, if we apply  $\rho(x/N)$  repeatedly  $N$  times, then indeed we obtain  $\rho(x)$ .

The second step for rotations would be to replace each  $\rho(x/N)$  by  $\rho(x_0/N)$  and  $\rho(x_1/N)$ , for some  $x_0$  and  $x_1$ . We need to determine the necessary condition on  $x, x_0, x_1$ , i.e., the analogue of  $v = v_0 + v_1$ . It is not as simple as  $x = x_0 + x_1$ ; the composition of  $\rho(x_0)$  followed by  $\rho(x_1)$  is not necessarily  $\rho(x_0 + x_1)$ . Therefore, first we need to understand which rotations a given  $\rho$  can be replaced by. We do this in the next section.

### 2.5.2 *The structure of intrinsic compositions*

Using the notation  $\rho = \rho(w, \phi)$  and  $v$  from Section 2.3.2, we can see that  $\rho_3$  can be replaced by  $\rho_1, \rho_2$  if

$$\phi_1 + \phi_2 = \phi_3 \tag{2.23}$$

and

$$v_1 + e^{i\phi_1}v_2 = v_3. \tag{2.24}$$

Indeed, (2.23) says that by applying  $\rho_1$  and  $\rho_2$ , we rotate  $\mathbb{R}^2$  by angle  $\phi_1 + \phi_2$ . And (2.24) says that the image of 0 after applying  $\rho_1$  and  $\rho_2$  will be  $v_1 + e^{i\phi_1}v_2$ . To see this, the first rotation,  $\rho_1$ , displaces 0 by  $v_1$ . Then,  $\rho_2$  displaces it further. This displacement is  $v_2$  in the intrinsic coordinate system and  $e^{i\phi_1}v_2$  in the extrinsic coordinate system, where the extra factor of  $e^{i\phi_1}$  is due to the offset in directions between the intrinsic and extrinsic coordinate systems introduced by  $\rho_1$ . If two rotations have the same angle and they map 0 to the same point, then they are the same rotation.

For  $x_j \in \mathbb{R}^3 \setminus \{0\}$ , we will use the notation  $x_3 = x_1 \star x_2$  if (2.23) and (2.24) hold for  $\rho_j = \rho(x_j)$ .

*Remark 2.5.2.* Note that (2.23) and (2.24) hold if and only if  $\rho_3 = \rho_1 \circ \rho_2$ . In general, the composition of two *intrinsic* rotations  $\rho_1$  and  $\rho_2$  (in that order) is  $\rho_1 \circ \rho_2$ , *not*  $\rho_2 \circ \rho_1$ .

*Remark 2.5.3.* We do not need the following fact in this paper, but the conditions (2.23) and (2.24) imply that  $\star$  is a group operation on  $\mathbb{R}^3$ . The group  $(\mathbb{R}^3, \star)$  has the structure of the semidirect product  $\mathbb{R}^2 \rtimes \mathbb{R}$ , where  $\phi \in \mathbb{R}$  acts on  $v \in \mathbb{R}^2$  by  $v \mapsto e^{i\phi}v$ .

The extra difficulty in our proof for rotations is essentially due to the failure of  $\star$  to agree with  $+$ . Nonetheless, we can modify the proof for translations to obtain a proof for rotations because for small  $x_1, x_2$ ,  $\star$  is “close enough” to  $+$ , as we show in the next section.

Our main estimate is the following:

**Lemma 2.5.4.** *Let  $x_j = (w_j, \phi_j) \in \mathbb{R}^3 \setminus \{0\}$  with  $x_3 = x_1 \star x_2$  and  $|\phi_j| \lesssim 1$  for each  $j$ .*

*Then*

$$|w_1 + w_2 - w_3| \lesssim |w_2\phi_1| + |w_1\phi_1| + |w_2\phi_2| + |w_3\phi_3|. \quad (2.25)$$

*Proof.* Observe that  $|v_j| \leq |w_j|$  and  $|v_j + iw_j| \lesssim |z_j\phi_j^2| = |w_j\phi_j|$ . (For the second inequality, we used  $|\phi_j| \lesssim 1$ .) By (2.24),  $|v_1 + v_2 - v_3| = |v_2(1 - e^{i\phi_1})| \leq |w_2\phi_1|$ . Thus indeed,

$$\begin{aligned} |w_1 + w_2 - w_3| &\leq |v_1 + v_2 - v_3| + |v_1 + iw_1| + |v_2 + iw_2| + |v_3 + iw_3| \\ &\lesssim |w_2\phi_1| + |w_1\phi_1| + |w_2\phi_2| + |w_3\phi_3|. \end{aligned} \quad \square$$

### 2.5.3 Basic zigzag construction for rotations

Now we are ready to define our basic zigzag construction in general. This construction, for given  $x, x_0, x_1 \in \mathbb{R}^3 \setminus \{0\}$  with  $x = x_0 + x_1$  and a given  $N$ , replaces the rotation  $\rho(x)$  by the sequence of intrinsic rotations  $\rho(y_0), \rho(y_1), \dots, \rho(y_0), \rho(y_1)$ . We define  $y_0 = x_0/N$ , and then  $y_1 = \tilde{x}_1/N$  is defined by  $y_0 \star y_1 = x/N$ .

The key properties of the construction are the following.

**Lemma 2.5.5.** *For any given  $\varepsilon > 0$ , if  $N$  is sufficiently large, then:*

1.  $|y_j| < \varepsilon$  for  $j = 0, 1$ ;
2.  $|\tilde{x}_1 - x_1| < \varepsilon$ .

*Proof.* Since  $y_0 = x_0/N$ , property (1) for  $j = 0$  is obvious. For  $j = 1$ , this property follows from  $y_1 = \tilde{x}_1/N$  and from (2).

Let  $x = (w, \phi)$ ,  $x_j = (w_j, \phi_j)$ , and  $\tilde{x}_1 = (\tilde{w}_1, \tilde{\phi}_1)$ . To prove (2), it suffices to show that  $\tilde{w}_1 \rightarrow w_1$  as  $N \rightarrow \infty$ , since  $\tilde{\phi}_1 = \phi_1$ . If  $N$  is large enough, then  $\frac{\phi}{N}, \frac{\phi_j}{N} \lesssim 1$ , so we can apply Lemma 2.5.4 for  $x/N = (x_0/N) \star (\tilde{x}_1/N)$  to obtain:

$$|\tilde{w}_1/N + w_0/N - w/N| \lesssim \frac{1}{N^2} (|w_0\phi_1| + |\tilde{w}_1\phi_1| + |w_0\phi_0| + |w\phi|).$$

Therefore

$$\begin{aligned} |\tilde{w}_1 - w_1| &= |\tilde{w}_1 + w_0 - w| \lesssim \frac{1}{N} (|w_0\phi_1| + |\tilde{w}_1\phi_1| + |w_0\phi_0| + |w\phi|) \\ &\leq \frac{1}{N} (c_1 + c_2|\tilde{w}_1|) \end{aligned}$$

for some  $c_1, c_2$  independent of  $N$  (since  $w, w_0, w_1, \phi, \phi_0, \phi_1$  do not depend on  $N$ ). Therefore  $\tilde{w}_1 \rightarrow w_1$  as  $N \rightarrow \infty$ . □

Property (1) allows the polygonal path for  $y_0 \star y_1 \star \dots \star y_0 \star y_1$  to stay within an arbitrarily small neighborhood of the line segment defined by  $x$ . This is because by decomposing  $x$  into

$(x/N) \star \cdots \star (x/N)$ , we divide the line segment into  $N$  equal segments. When we replace each segment by  $y_0 \star y_1$ , we stay in a small neighborhood of it.

#### 2.5.4 Iterating the basic zigzag

In our proof of Theorem 2.1.2, we started from a line segment  $L$  and then, iteratively, we replaced each line segment by a Venetian blind; the indices  $\mathbf{i}$  indexed the Venetian blinds. However, in this section, we need to focus also on basic zigzags, hence we introduce a new set of indices  $\mathbf{j}$  (finite binary sequences) to index the basic zigzags. For  $\mathbf{j} = j_1 \cdots j_k$ , we denote  $\mathbf{j}' = j_1 \cdots j_{k-1}$ .

Our construction from Section 2.4 is an iteration of basic zigzag constructions. That is, we begin with a line segment and replace it a basic zigzag. Then we iterate this by replacing each line segment of our basic zigzag with a basic zigzag. (For this to be an accurate description of our construction from Section 2.4, we must use the “second partition” from Remark 2.4.3.)

Given such an iteration of basic zigzags, we can describe it as follows. We start with a line segment  $L_\emptyset$ , which corresponds to a translation by a vector  $v_\emptyset \in \mathbb{R}^2$ . In our first basic zigzag, we chose two directions  $\theta_0, \theta_1$ . Then we can uniquely decompose  $v_\emptyset = v_0 + v_1$ , where  $v_j$  is in direction  $\theta_j$ . If the fineness is  $N$ , we can represent the basic zigzag as

$$v_\emptyset = (v_0/N) + (v_1/N) + \cdots + (v_0/N) + (v_1/N).$$

This gives us  $N$  copies of the segments  $v_0/N$  and  $v_1/N$ . We set  $M_\emptyset = M_1 = N$ .

Now suppose we have  $M_{\mathbf{j}}$  copies of  $v_{\mathbf{j}}/M_{\mathbf{j}}$ . To apply a basic zigzag on every copy, we write  $v_{\mathbf{j}} = v_{\mathbf{j}0} + v_{\mathbf{j}1}$  and choose a fineness  $N_{\mathbf{j}}$ . Then our basic zigzag is

$$\frac{v_{\mathbf{j}}}{M_{\mathbf{j}}} = \frac{v_{\mathbf{j}0}}{M_{\mathbf{j}}N_{\mathbf{j}}} + \frac{v_{\mathbf{j}1}}{M_{\mathbf{j}}N_{\mathbf{j}}} + \cdots + \frac{v_{\mathbf{j}0}}{M_{\mathbf{j}}N_{\mathbf{j}}} + \frac{v_{\mathbf{j}1}}{M_{\mathbf{j}}N_{\mathbf{j}}}.$$



Here we have  $N_{\mathbf{j}}$  copies of  $\frac{v_{\mathbf{j}0}}{M_{\mathbf{j}}N_{\mathbf{j}}} + \frac{v_{\mathbf{j}1}}{M_{\mathbf{j}}N_{\mathbf{j}}}$  and  $M_{\mathbf{j}0} = M_{\mathbf{j}1} = M_{\mathbf{j}}N_{\mathbf{j}}$ .

We let  $L_{\mathbf{j}} \subset \mathbb{R}^2$  be the union of the  $M_{\mathbf{j}}$  congruent and parallel line segments corresponding to the  $M_{\mathbf{j}}$  copies of  $v_{\mathbf{j}}/M_{\mathbf{j}}$ . We let  $\theta_{\mathbf{j}}$  be the direction of these segments.

*Remark 2.5.6.* The vectors  $v_{\mathbf{j}}$  do not depend on the fineness of the zigzags. Note also that  $|v_{\mathbf{j}}| = \mathcal{H}^1(L_{\mathbf{j}})$ .

*Remark 2.5.7.* As noted earlier, the indices  $\mathbf{j}$  index the basic zigzag constructions from Section 2.4.2, whereas the indices  $\mathbf{i}$  index the Venetian blind constructions from Section 2.4.3. Note that each  $L_{\mathbf{i}}$  is a union of  $L_{\mathbf{j}}$  over some set of indices  $\mathbf{j}$ .

Fix some  $\mathbf{i}$  and  $\mathbf{j}$  with  $L_{\mathbf{j}} \subset L_{\mathbf{i}}$ . When we apply the Venetian blind construction to  $L_{\mathbf{i}}$ , we obtain  $L_{\mathbf{i}0}$  and  $L_{\mathbf{i}1}$ . As part of this procedure, we iterate the basic zigzag construction  $k$  times on  $L_{\mathbf{j}}$ , where in the  $i$ th step ( $1 \leq i \leq k$ ), we replace  $L_{\mathbf{j}1^{i-1}}$  with the basic zigzags  $L_{\mathbf{j}1^{i-1}0} \cup L_{\mathbf{j}1^{i-1}1}$ . (Here,  $1^i$  denotes a string of  $i$  1s.) In the end, we obtain

$$\bigcup_{i=0}^{k-1} L_{\mathbf{j}1^i0} \cup L_{\mathbf{j}1^k}, \quad \text{with} \quad \bigcup_{i=0}^{k-1} L_{\mathbf{j}1^i0} \subset L_{\mathbf{i}0} \quad \text{and} \quad L_{\mathbf{j}1^k} \subset L_{\mathbf{i}1}.$$

For each  $i = 0, \dots, k-1$  we say that the index  $\mathbf{j}1^i0$  is *between*  $\mathbf{i}$  and  $\mathbf{i}0$  and that the index  $\mathbf{j}1^i1$  is *between*  $\mathbf{i}$  and  $\mathbf{i}1$ . (Note that by this definition, if  $\mathbf{i} \neq \emptyset$ , then  $\mathbf{j}$  is between  $\mathbf{i}'$  and  $\mathbf{i}$ .)

In the paragraphs above, we showed how to construct  $\{v_{\mathbf{j}}\}$  given an iteration of basic zigzags. Conversely, we could start with a collection  $\{v_{\mathbf{j}}\}$  satisfying  $v_{\mathbf{j}} = v_{\mathbf{j}0} + v_{\mathbf{j}1}$  and turn this into instructions for iterating the basic zigzags. (We would also need to specify the fineness  $N_{\mathbf{j}}$  at each step.)

The analogue of the above scheme for rotations is the following. Suppose that we are given some points  $x_{\mathbf{j}} \in \mathbb{R}^3 \setminus \{0\}$ , where the  $\mathbf{j}$  are finite binary sequences, such that  $x_{\mathbf{j}} = x_{\mathbf{j}0} + x_{\mathbf{j}1}$  for each  $\mathbf{j}$ . We also fix a small  $r > 0$ .

In our first step of the construction, we choose a sufficiently large  $N$  and choose  $y_0, y_1$

as in the previous section. That is, we replace  $x$  by  $N$  copies of  $(x_0/N) \star (\tilde{x}_1/N)$ :

$$x = (x_0/N) \star (\tilde{x}_1/N) \star \cdots \star (x_0/N) \star (\tilde{x}_1/N).$$

We choose  $N$  so large that  $|\tilde{x}_1 - x_1| < r$ . (We can do this by Lemma 2.5.5(2).) We also put  $\tilde{x} = x$ ,  $\tilde{x}_0 = x_0$ , and  $M_0 = M_1 = N$ .

Now suppose that we have already chosen  $\tilde{x}_{\mathbf{j}}$  and an  $M_{\mathbf{j}}$  for some sequence  $\mathbf{j}$ , and  $|\tilde{x}_{\mathbf{j}} - x_{\mathbf{j}}| < r$ . Then we apply a basic zigzag construction with  $x$  replaced by  $\tilde{x}_{\mathbf{j}}/M_{\mathbf{j}}$ ,  $x_0$  replaced by  $x_{\mathbf{j}0}/M_{\mathbf{j}}$  and  $x_1$  replaced by  $(x_{\mathbf{j}1} + \tilde{x}_{\mathbf{j}} - x_{\mathbf{j}})/M_{\mathbf{j}}$ , and with fineness  $N_{\mathbf{j}}$ . That is, we replace  $\tilde{x}_{\mathbf{j}}/M_{\mathbf{j}}N_{\mathbf{j}} = (x_{\mathbf{j}0}/M_{\mathbf{j}}N_{\mathbf{j}}) \star (\tilde{x}_{\mathbf{j}1}/M_{\mathbf{j}}N_{\mathbf{j}})$ , giving us

$$\frac{\tilde{x}_{\mathbf{j}}}{M_{\mathbf{j}}} = \left( \frac{x_{\mathbf{j}0}}{M_{\mathbf{j}}N_{\mathbf{j}}} \right) \star \left( \frac{\tilde{x}_{\mathbf{j}1}}{M_{\mathbf{j}}N_{\mathbf{j}}} \right) \star \cdots \star \left( \frac{x_{\mathbf{j}0}}{M_{\mathbf{j}}N_{\mathbf{j}}} \right) \star \left( \frac{\tilde{x}_{\mathbf{j}1}}{M_{\mathbf{j}}N_{\mathbf{j}}} \right).$$

If  $N_{\mathbf{j}}$  is very large, then  $\tilde{x}_{\mathbf{j}1}/M_{\mathbf{j}}$  will be very close to  $(x_{\mathbf{j}1} + \tilde{x}_{\mathbf{j}} - x_{\mathbf{j}})/M_{\mathbf{j}}$ , which means that  $\tilde{x}_{\mathbf{j}1}$  will be very close to  $x_{\mathbf{j}1} + \tilde{x}_{\mathbf{j}} - x_{\mathbf{j}}$ . Therefore, by choosing  $N_{\mathbf{j}}$  large enough,  $|\tilde{x}_{\mathbf{j}1} - x_{\mathbf{j}1}| < r$  holds. We put  $\tilde{x}_{\mathbf{j}0} = x_{\mathbf{j}0}$  and  $M_{\mathbf{j}0} = M_{\mathbf{j}1} = M_{\mathbf{j}}N_{\mathbf{j}}$ .

Using this procedure, we obtain an  $\tilde{x}_{\mathbf{j}}$  and an  $M_{\mathbf{j}}$  for each  $\mathbf{j}$ , such that  $|\tilde{x}_{\mathbf{j}} - x_{\mathbf{j}}| < r$ , and for  $y_{\mathbf{j}} := \tilde{x}_{\mathbf{j}}/M_{\mathbf{j}}$ :

$$y_{\mathbf{j}} = y_{\mathbf{j}0} \star y_{\mathbf{j}1} \star \cdots \star y_{\mathbf{j}0} \star y_{\mathbf{j}1}$$

(where we have  $N_{\mathbf{j}}$  copies of  $y_{\mathbf{j}0} \star y_{\mathbf{j}1}$ ).

In this way, we have shown how to take a collection  $\{x_{\mathbf{j}}\}$  with  $x_{\mathbf{j}} = x_{\mathbf{j}0} + x_{\mathbf{j}1}$ , together with fineness  $N_{\mathbf{j}}$ , and turn this data into a sequence of rotations, the composition of which is the original rotation  $\rho(x)$ .

*Remark 2.5.8.* For translations, the sequence  $\{v_{\mathbf{j}}\}$  tells us every direction we will translate in, even before the fineness  $N_{\mathbf{j}}$  are chosen. However, for rotations, the sequence  $\{x_{\mathbf{j}}\}$  alone does not tell us the projective centers of the rotations we will use. The centers are given by  $\{\tilde{x}_{\mathbf{j}}\}$ , which depend on  $N_{\mathbf{j}}$ . The  $N_{\mathbf{j}}$  in turn depend on  $\{x_{\mathbf{j}}\}$  and  $r$  (in the way explained

above) as well as on the area estimates in the following sections.

### 2.5.5 Turning the translations into rotations

In the previous section, we showed how to turn a collection  $\{x_{\mathbf{j}}\}$  into a sequence of rotations, but we did not say which sequence  $\{x_{\mathbf{j}}\}$  to start with. We specify that now. The construction of  $\{x_{\mathbf{j}}\}$  is actually very simple: we use a rotation in  $\mathbb{R}^3$  to “transform” a sequence of vectors  $\{v_{\mathbf{j}}\}$  in  $\mathbb{R}^2$  into our desired sequence  $\{x_{\mathbf{j}}\}$ .

Let  $\rho(x)$  and  $\varepsilon$  be as in the statement of Theorem 2.5.1. Then we can apply the results of Section 2.3.4 to  $E$ ; let  $c$  be the constant in Lemma 2.3.4 and Lemma 2.3.5. Without loss of generality we can assume that  $\varepsilon$  is small enough, so that the conclusion of Lemma 2.3.5 holds for every  $\delta < \varepsilon$ .

Let  $v \in \mathbb{R}^2$  be an arbitrary vector with  $|v| = |x|$ . We can follow the steps in Section 2.4 to construct the vectors  $v_{\mathbf{j}}$  with  $v_{\emptyset} = v$  as well as the stopping time.

Our aim is to “turn” the sequence  $\{v_{\mathbf{j}}\}$  into a sequence of rotations. Let  $Q : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear rotation that maps  $(v, 0)$  to  $x$ , and that maps the plane  $\phi = 0$  (i.e., those  $x = (w, \phi) \in \mathbb{R}^3$  for which  $\rho(x)$  is a translation) onto the plane of  $\ell$  (i.e., those  $x = (w, \phi) \in \mathbb{R}^3$  for which the projective center of  $\rho(x)$  lies in  $\ell$ ). We define  $x_{\mathbf{j}} := Q(v_{\mathbf{j}}, 0)$  for each  $\mathbf{j}$ . Since  $Q$  is linear, we do indeed have  $x_{\mathbf{j}} = x_{\mathbf{j}0} + x_{\mathbf{j}1}$ .

We denote by  $z_{\mathbf{j}}$  the projective image of  $x_{\mathbf{j}}$  onto  $\mathbb{P}^2$ . Then  $z_{\mathbf{j}} \in \ell$ . A trivial but very important property we have is this: since  $Q$  is an isometry, the distance between any two  $z_{\mathbf{j}}$  is the same as the angle between the corresponding vectors  $v_{\mathbf{j}}$ . If  $\mathbf{j}$  is between  $\mathbf{i}'$  and  $\mathbf{i}$  (see Remark 2.5.7), we denote  $\alpha_{\mathbf{j}} := \alpha_{\mathbf{i}}$  and let

$$B_{\mathbf{j}} = B(z_{\mathbf{j}}, 2\alpha_{\mathbf{j}}) \subset \mathbb{P}^2. \quad (2.26)$$

*Remark 2.5.9.* Suppose  $\mathbf{j}$  is between  $\mathbf{i}$  and  $\mathbf{i}'$ . If  $\mathbf{i}$  is good, then the ball  $B_{\mathbf{j}}$  contains  $[z_{\mathbf{j}}, z_{\mathbf{j}'}] \subset \ell$ , which is the image of  $[\theta_{\mathbf{j}}, \theta_{\mathbf{j}'}] \subset \mathbb{P}^1$  under the rotation  $Q$ . If  $\mathbf{i}$  is bad, then  $B_{\mathbf{j}}$  contains

$$[z_{\mathbf{j}}, z_{\mathbf{i}'}] = [z_{\mathbf{i}}, z_{\mathbf{i}'}] \subset \ell.$$

So far, none of the objects we defined depend on the fineness of the zigzags; they depend only on  $E$ ,  $\ell$ ,  $\rho(x)$  and  $\varepsilon$ .

Now we use the basic zigzag iteration process in Section 2.5.4 to obtain  $\{\tilde{x}_{\mathbf{j}}\}$  with  $N_{\mathbf{j}}$  large enough (that we will specify in the next section). We denote the projective center of the rotations by  $\tilde{z}_{\mathbf{j}}$ . That is,  $\tilde{z}_{\mathbf{j}}$  is the image of  $y_{\mathbf{j}}$  (which is the same as the image of  $\tilde{x}_{\mathbf{j}}$ ) under the projection  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ . We will also denote  $x_{\mathbf{j}} = (w_{\mathbf{j}}, \phi_{\mathbf{j}})$  and  $\tilde{x}_{\mathbf{j}} = (\tilde{w}_{\mathbf{j}}, \phi_{\mathbf{j}})$ . (Caution: we do not use the notation  $v_{\mathbf{j}}$  as in (2.4). Instead, the  $v_{\mathbf{j}}$  satisfy  $x_{\mathbf{j}} = Q(v_{\mathbf{j}}, 0)$ .)

Recall from Section 2.5.4 that  $\tilde{x}_{\mathbf{j}} \in B(x_{\mathbf{j}}, r) \subset \mathbb{R}^3$ , where we can choose  $r$  as small as we wish. We choose  $r$  small enough so that  $r \leq \frac{1}{2} \min_{\mathbf{j}} |x_{\mathbf{j}}|$  and so that for each  $\mathbf{j}$ , the image of  $B(x_{\mathbf{j}}, r)$  under the projection  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$  is contained in  $B_{\mathbf{j}} \cap B(\ell, \varepsilon)$ . It follows that  $|\tilde{x}_{\mathbf{j}}| \lesssim |x_{\mathbf{j}}|$  and  $\tilde{z}_{\mathbf{j}} \in B(z_{\mathbf{j}}, 2\alpha_{\mathbf{j}}) \cap B(\ell, \varepsilon)$  for each  $\mathbf{j}$ .

In the end, we have two polygonal paths. One is  $P = \bigcup_{\mathbf{j}} L_{\mathbf{j}} \subset \mathbb{R}^2$ , corresponding to  $\{v_{\mathbf{j}}\}$ ; the other is  $\tilde{P} = \bigcup_{\mathbf{j}} \tilde{L}_{\mathbf{j}} \subset \text{Isom}^+(\mathbb{R}^2)$ , corresponding to  $\{x_{\mathbf{j}}\}$ . In both cases, we use the same fineness  $N_{\mathbf{j}}$  (still to be specified). (We also have the same stopping time since that is encoded in the sequences  $\{v_{\mathbf{j}}\}$ ,  $\{x_{\mathbf{j}}\}$ .)

Thus,  $Q$  “transforms” a polygonal path  $P = \bigcup_{\mathbf{j}} L_{\mathbf{j}} \subset \mathbb{R}^2$  into a polygonal path  $\tilde{P} = \bigcup_{\mathbf{j}} \tilde{L}_{\mathbf{j}} \subset \text{Isom}^+(\mathbb{R}^2)$  by “transforming”  $L_{\mathbf{j}}$  into  $\tilde{L}_{\mathbf{j}}$ . Our next aim is to turn the estimates for  $P$  we obtained in Section 2.4 into estimates for  $\tilde{P}$ .

### 2.5.6 Ignoring small parts of $E$ and of $\tilde{P}$

Recall the definition of the intervals  $I_{\mathbf{i}}$ , the elementary intervals  $I$ , and the sets  $E_I$  from Section 2.4. Because of the rotation  $Q$ , the relevant objects are now  $J_{\mathbf{i}} := QI_{\mathbf{i}}$ ,  $J := QI \subset \ell$ , and  $E_J := \{x \in E : \nu_x \cap J \neq \emptyset\}$ .

We made the sets  $E_I$  compact by “ignoring” a sufficiently small subset of  $E$ . Since we knew the length of the final polygon  $P$  (this depended on the stopping time, but *not* on the fineness of the zigzags) we also knew from Lemma 2.2.2 that during our movement, small

enough subsets of  $E$  will automatically cover small area. By the same reason, we could also “ignore” those  $L_{\mathbf{i}}$  for which  $\mathcal{H}^1(L_{\mathbf{i}}) \leq \varepsilon_{\mathbf{k}}$ , where  $\mathbf{k}$  is the last good sequence among  $\mathbf{i}$  and its ancestors.

We now obtain the analogue estimates for rotations, by applying Lemma 2.3.4 in place of Lemma 2.2.2. Indeed, since  $|\tilde{x}_{\mathbf{j}}| \lesssim |x_{\mathbf{j}}| = |v_{\mathbf{j}}| = \mathcal{H}^1(L_{\mathbf{j}})$  for each  $\mathbf{j}$ , therefore every subset  $R \subset E$  will cover, during the movement by  $\tilde{P}$ , an area  $\lesssim c\mathcal{H}^1(R) \sum_{\mathbf{j}} |\tilde{x}_{\mathbf{j}}| \lesssim c\mathcal{H}^1(R) \sum_{\mathbf{j}} \mathcal{H}^1(L_{\mathbf{j}}) = c\mathcal{H}^1(R)\mathcal{H}^1(P)$ , where the sums are over all  $\mathbf{j}$  with  $L_{\mathbf{j}} \subset P$  (or, equivalently,  $\tilde{L}_{\mathbf{j}} \subset \tilde{P}$ ). That is, we obtain a  $c$  times larger estimate than in Lemma 2.2.2. Similarly, when we move any  $R$  by  $\tilde{L}_{\mathbf{i}}$ , we cover an area at most  $c\mathcal{H}^1(R)|\tilde{x}_{\mathbf{i}}| \lesssim c\mathcal{H}^1(R)\mathcal{H}^1(L_{\mathbf{i}})$  instead of  $\mathcal{H}^1(R)\mathcal{H}^1(L_{\mathbf{i}})$ .

Since  $Q$  is a rotation,  $|J_{\mathbf{i}}| = |I_{\mathbf{i}}|$ . Similarly as in Section 2.4, for each line segment  $\tilde{L} \subset \tilde{L}_{\mathbf{i}}$  appearing in the final polygon  $\tilde{P}$ , we choose  $B(u_i, \varepsilon)$  of Theorem 2.5.1 so that  $\ell \setminus B(u_i, \varepsilon) \subset J_{\mathbf{i}}$  whenever  $|J_{\mathbf{i}}| = |I_{\mathbf{i}}| \geq \pi - \varepsilon$ . If  $|J_{\mathbf{i}}| < \pi - \varepsilon$ , we can choose  $B(u_i, \varepsilon)$  arbitrarily.

### 2.5.7 Area estimates

For each  $J$ , let  $A_J$  denote the set covered by moving  $E_J$  along those  $\tilde{L}_{\mathbf{i}} \subset \tilde{P}$  for which  $J \subset J_{\mathbf{i}}$  (cf. (2.17)). Our final goal is to show  $\sum_J |A_J| < \tilde{c}\varepsilon$ , for some  $\tilde{c}$  independent of  $\varepsilon$ . This would imply that (2.22) holds with  $\varepsilon$  replaced by  $\tilde{c}\varepsilon$  in its right hand side.

First we prove the following analogue of Lemma 2.4.5.

**Lemma 2.5.10.** *By making the basic zigzags sufficiently fine, we can achieve the following: if  $J$  is an elementary interval contained in  $[z_{\mathbf{i}}, z_{\mathbf{i}'}]$ , then the area covered by moving  $E_J$  along  $\tilde{L}_{\mathbf{i}}$  is  $\lesssim c\alpha_{\mathbf{i}}\mathcal{H}^1(E_J)\mathcal{H}^1(L_{\mathbf{i}'})$ .*

*Proof.* Suppose  $J$  is an elementary interval contained in  $[z_{\mathbf{i}}, z_{\mathbf{i}'}]$ . If  $\mathbf{i}$  is good, then there is a  $\mathbf{j}$  between  $\mathbf{i}$  and  $\mathbf{i}'$  such that  $J \subset [z_{\mathbf{j}}, z_{\mathbf{j}'}]$ . If  $\mathbf{i}$  is bad, then for all  $\mathbf{j}$  between  $\mathbf{i}$  and  $\mathbf{i}'$ ,  $J \subset [z_{\mathbf{j}}, z_{\mathbf{j}'}]$ .

Suppose that  $J \subset [z_{\mathbf{j}}, z_{\mathbf{j}'}]$ . Applying Lemma 2.3.5 with  $\rho = \rho(y_{\mathbf{j}})$  and  $R = E_J$  (noting Remark 2.5.9), we see that if we move  $E_J$  by the rotation  $\rho(y_{\mathbf{j}})$ , the area covered is  $\lesssim c\alpha_{\mathbf{j}}\mathcal{H}^1(E_J)|\tilde{x}_{\mathbf{j}}|/M_{\mathbf{j}}$ . Hence the total area covered by moving  $E_J$  by all  $M_{\mathbf{j}}$  copies of  $\rho(y_{\mathbf{j}})$  is

$\lesssim c\alpha_{\mathbf{j}}\mathcal{H}^1(E_J)|\tilde{x}_{\mathbf{j}}| \lesssim c\alpha_{\mathbf{j}}\mathcal{H}^1(E_J)\mathcal{H}^1(L_{\mathbf{j}})$ . We make the zigzags so fine in our constructions that the same estimate

$$\lesssim c\alpha_{\mathbf{j}}\mathcal{H}^1(E_J)\mathcal{H}^1(L_{\mathbf{j}}) \quad (2.27)$$

remains true when we rotate the set  $E_J$  by the descendants of the  $M_{\mathbf{j}}$  copies of  $y_{\mathbf{j}}$ .

Now, we break into two cases. If  $\mathbf{i}$  is good, then  $L_{\mathbf{i}}$  descends from  $L_{\mathbf{j}}$ , so the statement of the lemma follows from  $\mathcal{H}^1(L_{\mathbf{j}}) \leq \mathcal{H}^1(L_{\mathbf{i}'})$ .

If  $\mathbf{i}$  is bad, we use the fact that  $L_{\mathbf{i}} = \bigcup_{\mathbf{j}} L_{\mathbf{j}}$  and  $\tilde{L}_{\mathbf{i}} = \bigcup_{\mathbf{j}} \tilde{L}_{\mathbf{j}}$ , where the unions are over all  $\mathbf{j}$  between  $\mathbf{i}$  and  $\mathbf{i}'$ . Then summing over the estimate (2.27) for each such  $\mathbf{j}$ , we have that moving along  $\tilde{L}_{\mathbf{i}}$ , the area is

$$\lesssim c\alpha_{\mathbf{i}}\mathcal{H}^1(E_J) \sum_{\mathbf{j}} \mathcal{H}^1(L_{\mathbf{j}}) = c\alpha_{\mathbf{i}}\mathcal{H}^1(E_J)\mathcal{H}^1(L_{\mathbf{i}}) \leq c\alpha_{\mathbf{i}}\mathcal{H}^1(E_J)\mathcal{H}^1(L_{\mathbf{i}'})$$

which completes the proof. □

Having established this estimate, the proof continues in the same way as in Section 2.4, to obtain  $|A_J| \lesssim 2c \sum_{\mathbf{i}} \varepsilon_{\mathbf{i}} \mathcal{H}^1(E_J)$ , the analogue of (2.21). We explain some details below.

Consider an  $\tilde{L}_{\mathbf{i}} \subset \tilde{P}$  with  $J \subset J_{\mathbf{i}}$ . Since the intervals  $J_{\mathbf{i}}, J_{\mathbf{i}'}, J_{\mathbf{i}''}, \dots$  are decreasing, there is a  $\mathbf{k}$  among  $\mathbf{i}$  and its ancestors such that  $J \subset J_{\mathbf{k}} \setminus J_{\mathbf{k}'} \subset [z_{\mathbf{k}}, z_{\mathbf{k}'}]$ . By Lemma 2.5.10, the total area covered when we move  $E_J$  along  $\tilde{L}_{\mathbf{k}}$  is

$$\lesssim c\alpha_{\mathbf{k}}\mathcal{H}^1(E_J)\mathcal{H}^1(L_{\mathbf{k}'}) \quad (2.28)$$

By making the zigzags sufficiently fine, the same estimate remains true when we move  $E_J$  along all the descendants of  $\tilde{L}_{\mathbf{k}}$  in  $\tilde{P}$ .

Therefore, similarly as in section 4, the area of  $A_J$  can be estimated by summing the estimate (2.28) for those ancestors that are on different family lines. Let  $\mathbf{k}_1, \mathbf{k}_2, \dots$  be arbitrary sequences from different family lines.

We distinguish two cases: if  $\mathbf{k}_m$  is good, then

$$\alpha_{\mathbf{k}_m} \mathcal{H}^1(E_J) \mathcal{H}^1(L_{\mathbf{k}_m}') \leq \varepsilon_{\mathbf{k}_m'} \mathcal{H}^1(E_J). \quad (2.29)$$

With the bad  $\mathbf{k}_m$ , because of the different family lines condition, each bad  $\mathbf{k}_m$  has a different “last good among  $\mathbf{k}_m$  and its ancestors” so

$$\sum_{\mathbf{k}_m \text{ is bad}} \alpha_{\mathbf{k}_m} \mathcal{H}^1(E_J) \mathcal{H}^1(L_{\mathbf{k}_m}') \leq \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \mathcal{H}^1(E_J), \quad (2.30)$$

where the summation on the right is taken over all  $\mathbf{k}$ . Adding together the estimates (2.29) for all good  $\mathbf{k}_m$  and (2.30), we proved that

$$|A_J| \lesssim 2c \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \mathcal{H}^1(E_J). \quad (2.31)$$

Let  $c'$  be the implied constant in (2.31). Since each  $x \in E$  belongs to at most two of the sets  $E_J$ , we proved that  $\sum_J |A_J|$  is at most  $cc'$  times larger than the bound of  $\varepsilon$  for  $\sum_I |A_I|$  that we obtained in Section 2.4. In other words, we showed  $\sum_J |A_J| < cc'\varepsilon$ . The constant  $cc'$  depends only  $\ell$  and  $E$  (and not on  $\varepsilon$ ). This completes the proof.

### 2.5.8 Further remarks

*Remark 2.5.11.* In both Section 2.4 and Section 2.5, we constructed a polygonal path that replaced a continuous movement with a fixed intrinsic projective center by a sequence of intrinsic rotations. By choosing all the zigzags sufficiently fine in our constructions, we can stay in an arbitrarily small neighborhood of the initial movement in  $\text{Isom}^+(\mathbb{R}^2)$ .

*Remark 2.5.12.* It is possible to choose the  $u_i$  in Theorem 2.5.1 so that  $z$ , the initial center of rotation, is not in any of the closed balls  $\text{cl } B(u_i, \varepsilon)$ .

By applying Lemma 2.3.5 to the initial rotation  $\rho$  and a sufficiently small ball  $B(z, \eta)$ , we see that rotating the set  $R = \{x \in E : \nu_x \cap \ell \cap B(z, \eta)\}$  by  $\rho$  covers small area. By

making the zigzags sufficiently fine and using the small neighborhood lemma, the set  $R$  still covers small area when moved by the final polygonal path. Thus, (2.22) holds with  $B(u_i, \varepsilon)$  replaced by  $B(u_i, \varepsilon) \setminus B(z, \eta)$ , so we can reselect the  $u_i$  so that  $z \notin \text{cl } B(u_i, \varepsilon)$ .

This property will be used in the proof of Theorem 2.6.6.

## 2.6 Besicovitch and Nikodym sets

We conclude this paper by showing that when we iterate the polygonal constructions in Section 2.4 and Section 2.5 and “take the limit,” we obtain the analogues of Besicovitch and Nikodym sets for rectifiable sets.

### 2.6.1 Construction of a Besicovitch set for translations

We start with the following, somewhat technical conditions. Afterwards, we will discuss some interesting special cases.

Suppose that we are given some rectifiable sets  $E_1 \subset E_2 \subset \dots$ , and a tangent field  $x \mapsto \theta_x$  of  $\bigcup E_n$ , satisfying the following:

1. each  $E_n$  is compact, and has finite  $\mathcal{H}^1$ -measure;
2. each  $E_n$  has a subset  $E'_n$  of full  $\mathcal{H}^1$ -measure, such that the restriction of the tangent  $\theta$  to  $E'_n$  is continuous, and for each  $y \in E_n$ ,

$$\theta_y \in \bigcap_{r>0} \text{cl}(\theta(B(y, r) \cap E'_n)). \quad (2.32)$$

We will prove the following proposition:

**Proposition 2.6.1.** *Suppose that the sets  $E_n$  satisfy the assumptions above. Let  $P_0$  be an arbitrary path in  $\mathbb{R}^2$ . Then for any neighborhood of  $P_0$ , there is a path  $P$  in this neighborhood*



with the same endpoints as  $P_0$ , and there is a Borel mapping  $p \mapsto \theta_p \in \mathbb{P}^1$  such that

$$\left| \bigcup_{p \in P} (p + \{x \in \bigcup E_n : \theta_x \neq \theta_p\}) \right| = 0. \quad (2.33)$$

*Proof.* Given any neighborhood of  $P_0$ , let  $P^0$  be a polygonal path in this neighborhood with the same endpoints as  $P_0$ . For each  $n$ , we choose an  $\varepsilon_n > 0$  with  $\sum \varepsilon_n < \infty$ . Then iteratively, for each  $n \geq 1$  we apply Theorem 2.1.2 to each segment  $L \subset P^{n-1}$  with  $E$  replaced by  $E'_n$  and  $\varepsilon$  replaced by some  $\varepsilon_L > 0$  such that  $\sum_{L \subset P^{n-1}} \varepsilon_L < \varepsilon_n$ . This gives us a polygonal path  $P^n = \bigcup_i L_i^n$  and directions  $\theta_i^n$  such that

$$\left| \bigcup_i \bigcup_{p \in L_i^n} (p + E_i^n) \right| < \varepsilon_n, \quad (2.34)$$

where

$$E_i^n := \text{cl} \{x \in E'_n : \theta_x \notin B(\theta_i^n, \varepsilon_n)\}. \quad (2.35)$$

Although Theorem 2.1.2 gives us the sets  $E_i^n$  without their closure, we can take the closure in (2.35) since, by our assumptions, doing so does not increase their measure. (In particular, by assumption (2), we have  $E_i^n \setminus \{x \in E'_n : \theta_x \notin B(\theta_i^n, \varepsilon_n)\} \subset E_n \setminus E'_n$  and  $\mathcal{H}^1(E_n \setminus E'_n) = 0$ .) We know that moving an  $\mathcal{H}^1$ -null set along a polygonal path covers only zero area, so indeed, (2.34) holds.

We construct  $P^{n+1}$  by replacing each line segment  $L_i^n$  of  $P^n$  by a polygonal path that stays in such a small neighborhood of  $L_i^n$  that the area estimate in (2.34) remains true when, instead of  $L_i^n$ , we shift the sets  $E_i^n$  along the line segments that we replace  $L_i^n$  with. (Here we used Remark 2.5.11 and that the sets  $E_i^n$  are compact.)

Also, we choose the neighborhoods small enough so that the polygonal paths  $P^n$  converge to a continuous limit curve  $P$ . For each  $p \in P$ , and for each fixed  $n$ , we have an  $i = i(p, n)$

such that

$$\left| \bigcup_{p \in P} (p + E_{i(p,n)}^n) \right| < \varepsilon_n$$

holds. We denote

$$E^p := \limsup_{n \rightarrow \infty} E_{i(p,n)}^n. \quad (2.36)$$

Then

$$\left| \bigcup_{p \in P} (p + E^p) \right| \leq \left| \bigcup_{p \in P} \left( p + \bigcup_{m \geq n} E_{i(p,m)}^m \right) \right| = \left| \bigcup_{m \geq n} \bigcup_{p \in P} (p + E_{i(p,m)}^m) \right| \leq \sum_{m \geq n} \varepsilon_m.$$

Since this is true for every  $n$ , it follows that  $\bigcup_{p \in P} (p + E^p)$  is Lebesgue null.

By the definition (2.35), if a point  $y \in \bigcup E_n$  does not belong to  $E^p$ , then for every large enough  $n$ , it has a neighborhood disjoint from  $\{x \in E'_n : \theta_x \notin B(\theta_i^n, \varepsilon_n)\}$ . That is, there is an  $r > 0$  such that  $\theta_x \in B(\theta_i^n, \varepsilon_n)$  for every  $x \in B(y, r) \cap E'_n$ . Hence, by our assumption (2.32),  $\theta_y \in \text{cl}(\theta(B(y, r) \cap E'_n)) \subset \text{cl} B(\theta_i^n, \varepsilon_n)$ . That is,  $\theta_y$  is in  $\liminf_{n \rightarrow \infty} \text{cl} B(\theta_{i(p,n)}^n, \varepsilon_n)$ , which has at most one point. For  $p \in P$ , if this set has one point, then we let  $\theta_p$  denote that point. Otherwise, we let  $\theta_p$  be arbitrary.

Then for each  $p$ ,  $\{x \in \bigcup E_n : \theta_x \neq \theta_p\} \subset E^p$ , and the proof is finished.  $\square$

For every rectifiable set  $E$ , we can choose the sets  $E_n = E'_n$  such that they satisfy the requirements at the beginning of this section, and such that  $\bigcup E_n$  is a subset of  $E$  of full  $\mathcal{H}^1$ -measure. Therefore we obtain the following theorem:

**Theorem 2.6.2** (Besicovitch set for translations). *Let  $E$  be an arbitrary rectifiable set, and let  $x \mapsto \theta_x$  be an arbitrary tangent field of  $E$ . Then there is an  $E_0 \subset E$  of full  $\mathcal{H}^1$ -measure in  $E$  for which the following holds.*

*For every path  $P_0$  in  $\mathbb{R}^2$ , and for any neighborhood of  $P_0$ , there is a path  $P$  in this neighborhood with the same endpoints as  $P_0$ , and there is a Borel mapping  $p \mapsto \theta_p \in \mathbb{P}^1$  such*

that

$$\left| \bigcup_{p \in P} (p + \{x \in E_0 : \theta_x \neq \theta_p\}) \right| = 0. \quad (2.37)$$

*Remark 2.6.3.* Another interesting corollary of Proposition 2.6.1 is the following. Suppose that  $E$  can be covered by a finite union of (not necessarily disjoint)  $C^1$  curves, or  $E$  is the graph of a convex function. In these cases there is an  $E_0 \subset E$  of full measure so that the tangent is continuous on  $E_0$ . Moreover, we can define the tangent on  $E \setminus E_0$  (in a natural way) and find the sets  $E_n, E'_n$  so that they satisfy our requirements and so that  $\bigcup_n E_n$  covers  $E$ . Therefore the statement of Theorem 2.6.2 holds with  $E_0$  replaced by  $E$ .

For example, if  $E$  is the graph of a strictly convex function, then it is enough to delete at most one point for each  $p \in P$ , as we claimed in the introduction.

### 2.6.2 Construction of a Besicovitch set for rotations

The main ideas for rotations are the same as for translations.

**Proposition 2.6.4.** *Suppose that the sets  $E_n$  satisfy the assumptions as in the beginning of Section 2.6.1. Let  $P_0$  be an arbitrary path in  $\text{Isom}^+(\mathbb{R}^2)$ . Then for any neighborhood of  $P_0$ , there is a path  $P$  in the neighborhood of  $P_0$  with the same endpoints as  $P_0$ , and there is a Borel mapping  $p \mapsto x_p \in \mathbb{P}^2$  such that*

$$\left| \bigcup_{p \in P} p(\{x \in \bigcup E_n : x_p \notin \nu_x\}) \right| = 0. \quad (2.38)$$

*Proof.* We begin with choosing  $P^0$  to be an arbitrary polygonal path in the neighborhood of  $P_0$  with the same endpoints as  $P_0$ . We iterate Theorem 2.5.1 to construct the polygonal paths  $P^n$  in  $\text{Isom}^+(\mathbb{R}^2)$ , each lying in a small neighborhood of the previous one. Here, the details are now a bit more technical, and we need to be careful when we specify our parameters for Theorem 2.5.1.

As before, we choose an  $\varepsilon_n > 0$  for each  $n$  such that  $\sum_n \varepsilon_n < \infty$ . Each line segment  $L_i^n \subset$

$P^n$  corresponds to a rotation  $\rho_i^n$  with projective center  $z_i^n$ . We choose a line  $\ell_i^n$  containing  $z_i^n$  and a  $0 < \delta_i^n < \varepsilon_n$ . (We will impose additional conditions on  $\ell_i^n, \delta_i^n$  in Section 2.6.3.) Then we replace  $\rho_i^n$  by a sequence of intrinsic rotations by applying Theorem 2.5.1 and Remark 2.5.12 with  $E$  replaced by  $E'_{n+1}$ ,  $\ell$  replaced by  $\ell_i^n$ , and  $\varepsilon$  replaced by  $\delta_i^n$ .

Choosing each of the parameters  $\delta_i^n$  sufficiently small, we obtain the balls  $B(u_i^n, \delta_i^n)$  and:

$$|\bigcup_i \bigcup_{p \in L_i^n} p(E_i^n)| < \varepsilon_n, \quad (2.39)$$

where

$$E_i^n := \text{cl}\{x \in E'_n : \nu_x \cap B(u_i^n, \delta_i^n) = \emptyset\}. \quad (2.40)$$

We define  $i(p, n)$  as in the previous section, and again take  $E^p := \limsup_{n \rightarrow \infty} E_{i(p, n)}^n$ . Then as in the previous section, the movement  $\bigcup_{p \in P} p(E^p)$  covers only a null set.

Since  $\delta_{i(p, n)}^n \rightarrow 0$ , we know  $\liminf_{n \rightarrow \infty} \text{cl} B(u_{i(p, n)}^n, \delta_{i(p, n)}^n)$  can have at most one point. If it has one point, let  $x_p$  be that point. Otherwise, let  $x_p$  be arbitrary.

Now suppose that  $y \in E_n$  and  $y \notin E_{i(p, n)}^n$ . Then  $\nu_y \cap \text{cl} B(u_{i(p, n)}^n, \delta_{i(p, n)}^n) \neq \emptyset$ . Therefore indeed  $\{x \in \bigcup E_n : x_p \notin \nu_x\} \subset E^p$ , and the proof is finished.  $\square$

### 2.6.3 The main theorem

In Proposition 2.6.4, the points on  $E$  that we hide at each  $p \in P$  are those whose normal line passes through a particular point  $x_p$ . Since we would like to hide as little of  $E$  as possible, it would be undesirable if an  $x_p$  from our construction has the property that the normal line of positively many points of  $E$  pass through  $x_p$ .

Fortunately such points are very rare:

**Lemma 2.6.5.** *There are at most countably many points with the property that the normal line of positively many points of  $E$  pass through this point.*

*Proof.* Note that for any two such points there is only one common line, and there can be

only an  $\mathcal{H}^1$ -nullset of points of  $E$  which have a given normal line. Since  $E$  has  $\sigma$ -finite  $\mathcal{H}^1$ -measure, it cannot have more than countably many subsets of positive measure such that their pairwise intersections are null.  $\square$

We denote the exceptional points above by  $x_1, x_2, \dots$ . In what follows, we show how to choose the parameters in our construction more carefully to avoid these points, i.e., so that  $x_p \notin \{x_1, x_2, \dots\}$  for any  $p \in P$ .

We use the notation from the previous section. For each  $n \geq 1$  and for each  $L_i^n \subset P^n$ , let  $S_i^n$  denote the strip  $B(\ell, \delta)$  assigned to *the parent* of  $L_i^n$ , i.e., to the line segment in  $P^{n-1}$  that we replaced by a polygon in the construction of  $L_i^n$ . Then we choose  $\ell_i^n, \delta_i^n$  such that  $B(\ell_i^n, \delta_i^n) \subset S_i^n$  and such that  $\text{cl } B(\ell_i^n, \delta_i^n) \setminus \{z_i^n\}$  does not contain any of the points  $x_m$  with  $m \leq n$ .

Then  $\liminf_{n \rightarrow \infty} \text{cl } B(u_{i(p,n)}^n, \delta_{i(p,n)}^n)$  is either empty, or contains one point. Suppose it contains a point  $x_p$ . Since  $u_{i(p,n)}^n \in \ell_{i(p,n)}^n$  and the strips  $\{B(\ell_{i(p,n)}^n, \delta_{i(p,n)}^n)\}_n$  are nested, it follows that  $x_p \in \bigcap_n \text{cl } B(\ell_{i(p,n)}^n, \delta_{i(p,n)}^n)$ . By Remark 2.5.12,

$$z_{i(p,n)}^n \notin \text{cl } B(u_{i(p,n+1)}^{n+1}, \delta_{i(p,n+1)}^{n+1}),$$

so  $x_p \neq z_{i(p,n)}^n$  for all  $n$ .

Thus we have shown the following. This is the main theorem in our paper.

**Theorem 2.6.6** (Besicovitch set for rotations). *Let  $E$  be an arbitrary rectifiable set, and let  $x \mapsto \theta_x$  be an arbitrary tangent field of  $E$ . Then there is an  $E_0 \subset E$  of full  $\mathcal{H}^1$ -measure in  $E$  for which the following holds.*

*For every path  $P_0$  in  $\text{Isom}^+(\mathbb{R}^2)$ , and for any neighborhood of  $P_0$ , there is a path  $P$  in the neighborhood of  $P_0$  with the same endpoints as  $P_0$ , and there is a Borel mapping  $p \mapsto x_p \in \mathbb{P}^2$  such that*

$$\left| \bigcup_{p \in P} p(\{x \in E_0 : x_p \notin \nu_x\}) \right| = 0. \quad (2.41)$$

Furthermore, for each  $p$ , the set  $\{x \in E_0 : x_p \notin \nu_x\}$  has full  $\mathcal{H}^1$ -measure in  $E$ .

*Remark 2.6.7.* By the same argument as at the end of the previous section, we can get a stronger statement if the set  $E$  has nice geometric properties. For instance, if it is covered by finitely many  $C^1$  curves, or if it is the graph of a convex function, then the statement holds with  $E_0$  replaced by  $E$ .

As mentioned in the introduction, consider the special case where there is a line  $\ell \in (\mathbb{P}^2)^*$  such that there is a neighborhood of  $\ell$  in which no two normal lines of  $E$  intersect. Then by choosing all the lines  $\ell_i^n$  to lie inside this neighborhood, we can ensure that all the  $x_p$  do as well. Hence, Theorem 2.6.6 says that we can rotate  $E$  continuously by  $360^\circ$ , covering a set of zero Lebesgue measure, where at each time moment, we only need to delete *one point*.

*Remark 2.6.8.* By the small neighborhood lemma, we can see that (2.39) holds (with the same sets  $E_i^n$ ) not only for the path  $P^n$  but for every continuous path  $P$  sufficiently close to  $P^n$ . Using this observation, we obtain a dense open set of curves, and then, by taking the limit, a residual set of continuous paths  $P$  connecting the endpoints of  $P_0$ , for which the statement of Theorem 2.6.6 holds.

#### 2.6.4 Construction of a Nikodym set

We conclude this paper by explaining how the continuous Besicovitch sets can be used to construct Nikodym sets for rectifiable curves.

Let  $E \subset \mathbb{R}^2$  be an arbitrary rectifiable set. We fix an arbitrary (continuous) rectifiable curve  $\Gamma \subset \mathbb{R}^2$  (if  $E$  contains such a curve, we can choose  $\Gamma$  to be that curve). By “putting a copy of  $E$  onto a point  $y$ ,” we mean that the corresponding copy of  $\Gamma$  (i.e., the same isometry applied to  $\Gamma$ ) goes through  $y$ .

For every continuous rectifiable curve  $\Gamma$ , there is a path  $P_0 \subset \text{Isom}^+(\mathbb{R}^2)$  and a neighborhood of  $P_0$  such that  $\Gamma$  covers a set of non-empty interior along any path  $P$  which lies in this neighborhood and has the same endpoints as  $P_0$ . (For example, if  $\Gamma$  is a circle, we

make sure that it is not possible for  $P$  to be a rotation around the circle's center.)

We apply Theorem 2.6.6 with  $E$  and with this neighborhood of  $P_0$  to obtain a path  $P$ , and for each  $p \in P$  to obtain a subset  $E^p \subset E$  of full  $\mathcal{H}^1$ -measure so that  $|\bigcup_{p \in P} p(E^p)| = 0$ .

By our choice of  $P_0$ , we know that  $\bigcup_{p \in P} p(\Gamma)$  has nonempty interior. Thus,

$$\bigcup_{q \in \mathbb{Q}^2} \bigcup_{p \in P} (q + p(\Gamma)) = \mathbb{R}^2,$$

whereas

$$A := \bigcup_{q \in \mathbb{Q}^2} \bigcup_{p \in P} (q + p(E^p)) \tag{2.42}$$

has measure zero. Thus, we have shown the following.

**Theorem 2.6.9.** *Let  $E$  be a rectifiable set and  $\Gamma$  a rectifiable curve. Then the set  $A$  defined by (2.42) is a Nikodym set for  $E$ :*

1.  $A$  has Lebesgue measure zero;
2. Through each point  $y \in \mathbb{R}^2$ ,  $A$  contains a copy of  $\mathcal{H}^1$ -a.e. point of  $E$ . That is, for all  $y \in \mathbb{R}^2$ , there is an  $E_y \subset E$  and a  $p_y \in \text{Isom}^+(\mathbb{R}^2)$  such that  $\mathcal{H}^1(E \setminus E_y) = 0$ ,  $y \in p_y(\Gamma)$ , and  $p_y(E_y) \subset A$ .

With Theorem 2.6.2 in place of Theorem 2.6.6 we can prove a result about placing translated copies of  $E$  at each point  $y \in \mathbb{R}^2$ .

By essentially the same arguments as above, we now obtain a path  $P \subset \mathbb{R}^2$  and  $\theta_p \in \mathbb{P}^1$  such that  $\bigcup_{p \in P} (p + E^p)$  has Lebesgue measure zero, where  $E^p = \{x \in E_0 : \theta_x \neq \theta_p\}$ , and such that  $\bigcup_{p \in P} (p + \Gamma)$  has nonempty interior. Thus,  $\bigcup_{q \in \mathbb{Q}^2} \bigcup_{p \in P} (q + p + \Gamma) = \mathbb{R}^2$ , whereas

$$A := \bigcup_{q \in \mathbb{Q}^2} \bigcup_{p \in P} (q + p + E^p) \tag{2.43}$$

has Lebesgue measure zero. To ensure that  $E^p$  has full  $\mathcal{H}^1$ -measure in  $E$ , it is sufficient to assume that  $\{x \in E : \theta_x = \theta\}$  is  $\mathcal{H}^1$ -null for every  $\theta \in \mathbb{P}^1$ .

**Theorem 2.6.10.** *Let  $E$  be a rectifiable set and  $\Gamma$  a rectifiable curve. Suppose that for every direction  $\theta \in \mathbb{P}^1$ , the set  $\{x \in E : \theta_x = \theta\}$  is  $\mathcal{H}^1$ -null. Then the set  $A$  defined by (2.43) satisfies the following:*

1.  $A$  has Lebesgue measure zero;
2. Through each point  $y \in \mathbb{R}^2$ ,  $A$  contains a translated copy of  $\mathcal{H}^1$ -a.e. point of  $E$ . That is, for all  $y \in \mathbb{R}^2$ , there is an  $E_y \subset E$  and a  $p_y \in \mathbb{R}^2$  such that  $\mathcal{H}^1(E \setminus E_y) = 0$ ,  $y \in p_y + \Gamma$ , and  $p_y + E_y \subset A$ .

## 2.7 Dilations and similarity transformations

In this section, we show how the techniques of Section 2.5 can be applied to analyze similarity transformations. Let  $\text{Sim}^+(\mathbb{R}^2)$  denote the space of all orientation-preserving similarity transformations in  $\mathbb{R}^2$ .

Elements in  $\text{Isom}^+(\mathbb{R}^2)$  were specified by the parameters  $(w, \phi) \in \mathbb{R}^2 \times \mathbb{R}$ . To index elements in  $\text{Sim}^+(\mathbb{R}^2)$ , we introduce a new parameter  $\alpha \in \mathbb{R}$ . (In the special case of isometries we can take  $\alpha = 0$ .)

For  $\alpha, \phi \in \mathbb{R}$ , define  $\phi_\alpha = e^{i\alpha}\phi$ . For  $\phi \neq 0$ , we let  $\rho_\alpha(w, \phi)$  denote the similarity transformation  $u \mapsto e^{i\phi_\alpha}(u - z) + z$ , where  $z = w/\phi_\alpha$ . Then it is natural to let  $\rho_\alpha(w, 0)$  denote translation by  $-iw$ . For any  $(w, \phi) \neq (0, 0)$ , we define the *projective center* of  $\rho_\alpha(w, \phi)$  to be the image of  $(e^{-i\alpha}w, \phi) \in \mathbb{R}^3$  under the quotient map  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ .

*Remark 2.7.1.* The center of a translation now depends on  $\alpha$ . This is natural because a single translation can be viewed, e.g., as a rotation around some point at infinity and also as a dilation around some other point at infinity.

*Remark 2.7.2.* When  $\alpha \equiv 0 \pmod{\pi}$ , the transformation  $\rho_\alpha(w, \phi)$  is an isometry. When  $\alpha \equiv \pi/2 \pmod{\pi}$ , the transformation is a dilation. For all other  $\alpha$ , the trajectory of a point



$x$  under  $\rho_\alpha(w, \phi)$  is a logarithmic spiral centered at  $z$ . Since

$$e^{i\psi\alpha} = e^{i\psi(\cos\alpha + i\sin\alpha)} = e^{-\psi\sin\alpha} e^{i\psi\cos\alpha},$$

the trajectory consists of those points  $u$  for which  $|u - z| = e^{-\psi\sin\alpha}|x - z|$  and  $\arg(u - z) = \psi\cos\alpha + \arg(x - z)$  for some  $\psi \in [0, \phi]$ . For future reference, note that

$$\arg(u - z) = -\cot\alpha(\log|u - z| - \log|x - z|) + \arg(x - z). \quad (2.44)$$

When studying similarity transformations, it turns out that instead of the normal line  $\nu_x$ , it is much more relevant to look at the normal line rotated by angle  $\alpha$  around  $x$ . We denote this line by  $(\nu_x)_\alpha$ . We will prove the following generalization of Theorem 2.5.1.

**Theorem 2.7.3.** *Let  $E \subset \mathbb{R}^2$  be a bounded rectifiable set of finite  $\mathcal{H}^1$ -measure. Let  $\varepsilon > 0$ , and let  $\rho$  be a similarity transformation with parameter  $\alpha$ . Let  $\ell \subset \mathbb{P}^2$  be a line through the projective center  $z$  of  $\rho$ .*

*Then there are intrinsic similarity transformations  $\rho_i = \rho_\alpha(x_i)$  with projective centers  $z_i \in B(\ell, \varepsilon) \subset \mathbb{P}^2$  such that the corresponding polygonal path  $P = \bigcup_i L_i \subset \text{Sim}^+(\mathbb{R}^2)$  connects the identity and  $\rho$ , and for each  $i$ , there exists a  $u_i \in \ell$  such that*

$$\left| \bigcup_i \bigcup_{p \in L_i} p(\{x \in E : (\nu_x)_\alpha \cap \ell \cap B(u_i, \varepsilon) = \emptyset\}) \right| < \varepsilon. \quad (2.45)$$

*Furthermore, if  $\alpha \equiv \pi/2 \pmod{\pi}$ , then we can take  $z_i \in \ell$ .*

Throughout this section, we fix an  $\alpha \not\equiv 0 \pmod{\pi}$ . To prove Theorem 2.7.3, we first establish the analogues of Lemma 2.2.2 and Lemma 2.3.5. As in Section 2.3.4, assume  $E \subset B(0, r) \subset \mathbb{R}^2$ . We will show that there is a constant  $c$  that depends only on  $r$  such that the following two lemmas hold.

**Lemma 2.7.4.** *Let  $y = (w, \phi) \in \mathbb{R}^2 \times \mathbb{R}$  with  $|\phi| \lesssim 1$ . Let  $\rho = \rho_\alpha(y)$  be a similarity transformation and let  $R \subset E$  be arbitrary. Then, if we transform  $R$  by  $\rho$ , the area covered is  $\lesssim c\mathcal{H}^1(R)|y|$ .*

**Lemma 2.7.5.** *Let  $\delta > 0$  be sufficiently small (depending on  $r$ ). Let  $y = (w, \phi) \in \mathbb{R}^2 \times \mathbb{R}$  with  $|\phi| \lesssim 1$ . Let  $\rho = \rho_\alpha(y)$  be a transformation with projective center  $z$ . Let  $R \subset E$  be such that, for each  $x \in R$ ,  $(\nu_x)_\alpha \cap B(z, \delta) \neq \emptyset$ . (Here, the ball  $B(z, \delta)$  is defined with respect to the metric on  $\mathbb{P}^2$ .) Then, when we transform  $R$  by  $\rho$ , the area covered is*

$$\lesssim c\delta\mathcal{H}^1(R)|y|.$$

*Proof of Lemma 2.7.4 and Lemma 2.7.5.* Let  $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the measure preserving map that rotates each circle  $|u - z| = \text{const}$  around  $z$  by angle  $\cot \alpha \log |u - z|$ . By (2.44),  $\Psi$  takes the spiral trajectories of  $\rho$  to straight lines through  $z$ . In particular,  $\Psi$  takes the trajectory of point  $x$  under  $\rho$  to the line segment

$$[z + e^{i \cot \alpha \log |x-z|}(x - z), z + e^{-\phi \sin \alpha} e^{i \cot \alpha \log |x-z|}(x - z)]. \quad (2.46)$$

In other words,  $z + \lambda e^{i\theta}$  belongs to this line segment if and only if  $\theta = \cot \alpha \log |x - z| + \arg(x - z)$  and  $\lambda$  belongs to the interval  $I_x \subset \mathbb{R}$ , whose endpoints are  $|x - z|$  and  $e^{-\phi \sin \alpha} |x - z|$ . (We do not specify which endpoint is the left and which is the right.)

Let  $S \subset \mathbb{R}^2$  be the region covered by applying  $\rho$  to  $R$ . We have

$$|S| = |\Psi(S)| = \int_0^{2\pi} \int_{\{\lambda: z + \lambda e^{i\theta} \in \Psi(S)\}} \lambda d\lambda d\theta.$$

To simplify the inner integral, observe that  $\Psi(S)$  is the union of the line segments (2.46) over all  $x \in R$ . Thus

$$\int_{\{\lambda: z + \lambda e^{i\theta} \in \Psi(S)\}} \lambda d\lambda \leq \sum_{x \in R: \arg(x-z) + \cot \alpha \log |x-z| = \theta} \int_{I_x} \lambda d\lambda,$$

where

$$\int_{I_x} \lambda d\lambda = |e^{-2\phi \sin \alpha} - 1| \cdot |x - z|^2 \lesssim |\phi \sin \alpha| \cdot |x - z|^2.$$

Let  $t \mapsto x(t)$  be a parametrization of  $R$  by arclength. Note that the derivative of  $t \mapsto \arg(x(t) - z) + \cot \alpha \log |x(t) - z|$  is

$$\left\langle \frac{i(x(t) - z)}{|x(t) - z|^2}, \dot{x}(t) \right\rangle + \cot \alpha \left\langle \frac{x(t) - z}{|x(t) - z|^2}, \dot{x}(t) \right\rangle.$$

Using the estimates above and the coarea formula, we have

$$\begin{aligned} |S| &\lesssim |\phi \sin \alpha| \int_0^{2\pi} \sum_{x \in R: \arg(x-z) + \cot \alpha \log |x-z| = \theta} |x - z|^2 d\theta \\ &= |\phi| \int \left\langle e^{i\alpha}(x(t) - z), \dot{x}(t) \right\rangle dt \\ &= |\phi| \int \text{dist}((\nu_x)_\alpha, z) d\mathcal{H}^1(x). \end{aligned}$$

To prove Lemma 2.7.4, we use the trivial estimate  $\text{dist}((\nu_x)_\alpha, z) \leq |x - z|$  and proceed as in the proof of Lemma 2.3.4. To prove Lemma 2.7.5, we proceed as in the proof of Lemma 2.3.5.  $\square$

For  $x_1, x_2$ , we define  $x_3 = x_1 \star_\alpha x_2$  if  $\rho_\alpha(x_3)$  can be replaced by  $\rho_\alpha(x_1), \rho_\alpha(x_2)$ . Explicitly, this means  $\phi_1 + \phi_2 = \phi_3$  and  $v_1 + e^{i(\phi_1)\alpha} v_2 = v_3$ , where  $v_j = z_j(1 - e^{i(\phi_j)\alpha})$ .

It is easy to check that with  $\star_\alpha$  in place of  $\star$ , the arguments in Section 2.5 still hold with very little modification, giving us a proof of Theorem 2.7.3. (One small issue is that since the size of  $E$  can change, we need to apply a correction factor to Lemma 2.7.4 and Lemma 2.7.5. However, we can ensure that at any point in the transformations, our set is never more than twice its initial size, so that the correction factor is bounded by an absolute constant.)

The statement at the end of Theorem 2.7.3 about  $\alpha \equiv \pi/2 \pmod{\pi}$  follows from the fact that for such  $\alpha$ , if  $x_3 = x_1 \star_\alpha x_2$ , then the centers of the three dilations are collinear.

### 2.7.1 Circles

We briefly sketch the proof of Corollary 2.1.10.

*Proof of Corollary 2.1.10.* Let  $E$  be a circle. Let  $\varepsilon > 0$  (to be specified later). By Theorem 2.1.2, there is a polygonal path  $P = \bigcup_{i=1}^n L_i \subset \mathbb{R}^2$  with each  $L_i$  a line segment, and for each  $i$  there exists a direction  $\theta_i \in \mathbb{P}^1$ , such that

$$\left| \bigcup_i \bigcup_{p \in L_i} (p + \{x \in E : \theta_x \notin B(\theta_i, \varepsilon)\}) \right| < \varepsilon. \quad (2.47)$$

By (2.15), we can assume that  $\theta_{L_i} \notin B(\theta_i, \varepsilon)$  (recall that  $\theta_{L_i}$  is the direction of the line segment  $L_i$ ). By the fact that the tangent direction changes continuously as we move around the circle, there is an  $\varepsilon' < \varepsilon$  such that

$$\left| \bigcup_i \bigcup_{p \in L_i} (p + \{x \in E : \theta_x \notin B(\theta_i, \varepsilon')\}) \right| < 2\varepsilon. \quad (2.48)$$

Since  $\varepsilon' < \varepsilon$ , we have  $\theta_{L_i} \notin \text{cl } B(\theta_i, \varepsilon')$ .

Fix  $\alpha = \pi/2$ . Then  $\theta_{L_i}$  is the projective center of the translation along  $L_i$ , and  $(\nu_x)_\alpha$  is the tangent line at  $x \in E$ .

Let  $\ell_i \in (\mathbb{P}^2)^*$  be the line through the center of  $E$  of direction  $\theta_{L_i}$ . We fix an  $i$ , and for each  $x \in E$  we denote by  $\bar{x}$  the reflection of the point  $x \in E$  across the line  $\ell_i \cap \mathbb{R}^2$ .

Let  $\varepsilon_i > 0$  (to be specified later). We can apply Theorem 2.7.3 to the translation along  $L_i$ , with  $\alpha = \pi/2$  and line  $\ell_i$ , to replace this translation by a polygonal path of dilations  $\bigcup_j L_{i,j} \subset \text{Sim}^+(\mathbb{R}^2)$  such that

$$\left| \bigcup_j \bigcup_{p \in L_{i,j}} p(\{x \in E : (\nu_x)_\alpha \cap \ell_i \cap B(u_{i,j}, \varepsilon_i) = \emptyset\}) \right| < \varepsilon_i \quad (2.49)$$

for some  $u_{i,j} \in \ell_i$ .

Those points  $x$  of the circle  $E$  for which the tangent line  $(\nu_x)_\alpha$  intersects  $\ell_i \cap B(u_{i,j}, \varepsilon_i)$

lie on circular arcs that are symmetric with respect to  $\ell_i \cap \mathbb{R}^2$ . Therefore we can find some  $y_{i,j} \in E$  and  $\varepsilon'_i$  such that

$$\left| \bigcup_j \bigcup_{p \in L_{i,j}} p(E \setminus (B(y_{i,j}, \varepsilon'_i) \cup B(\bar{y}_{i,j}, \varepsilon'_i))) \right| < \varepsilon_i. \quad (2.50)$$

By the analogue of Remark 2.5.11 and the small neighborhood lemma, we can also ensure that

$$\left| \bigcup_{i,j} \bigcup_{p \in L_{i,j}} p(\{x \in E : \theta_x \notin B(\theta_i, \varepsilon')\}) \right| < 3\varepsilon. \quad (2.51)$$

Since  $\theta_{L_i} \notin \text{cl } B(\theta_i, \varepsilon')$ , it follows that if  $r_i$  is small enough, then for each  $y \in E$ ,  $\{x \in E : \theta_x \in B(\theta_i, \varepsilon')\} \cap (B(y, r_i) \cup B(\bar{y}, r_i))$  is either empty or is one arc of angle at most  $\varepsilon'$ . Hence, if we choose  $\varepsilon_i$  so small that  $\varepsilon'_i < r_i$ , then

$$E_{i,j} := \{x \in E : \theta_x \notin B(\theta_i, \varepsilon')\} \cup (E \setminus (B(y_{i,j}, \varepsilon'_i) \cup B(\bar{y}_{i,j}, \varepsilon'_i)))$$

is a circular arc of angle at least  $2\pi - \varepsilon'$ .

Thus, by combining (2.50) and (2.51), we have

$$\left| \bigcup_{i,j} \bigcup_{p \in L_{i,j}} p(E_{i,j}) \right| < 3\varepsilon + \sum_i \varepsilon_i. \quad (2.52)$$

This gives us a movement of a subarc of  $E$  of angle  $2\pi - \varepsilon'$  covering area less than  $3\varepsilon + \sum_i \varepsilon_i$ . Therefore by choosing the parameters small enough, we can move an arbitrarily large sub-arc covering arbitrarily small area.

Furthermore, as for isometries, we can construct not just one but a dense open set of movements. Therefore we can ensure that the radius of the circular arc remains very close to the radius of the original arc during the movement.  $\square$

By repeated applications of Theorem 2.7.3, we obtain in the limit a Besicovitch set and a Nikodym set. The proofs proceed in the same way as in Section 2.6. The result for

Besicovitch sets in the special case when  $E$  is a circle is stated below; for Nikodym sets, see Corollary 2.1.11.

**Corollary 2.7.6.** *Let  $E$  be a circle. For every path  $P_0$  in  $\text{Isom}^+(\mathbb{R}^2)$ , and for any neighborhood of  $P_0$  in  $\text{Sim}^+(\mathbb{R}^2)$ , there is a path  $P$  in the neighborhood of  $P_0$  with the same endpoints as  $P_0$ , and there is a Borel mapping  $p \mapsto x_p \in E$  such that*

$$\left| \bigcup_{p \in P} p(E \setminus \{x_p\}) \right| = 0.$$

# CHAPTER 3

## SMALL UNIONS OF AFFINE SUBSPACES AND SKELETONS VIA BAIRE CATEGORY

This chapter is joint work with Marianna Csörnyei, Kornélia Héra and Tamás Keleti and originally appeared in [9].

### 3.1 Introduction

E. Stein [50] proved in 1976 that for any  $n \geq 3$ , if a set  $A \subset \mathbb{R}^n$  contains a sphere centered at each point of a set  $C \subset \mathbb{R}^n$  of positive Lebesgue measure, then  $A$  also has positive Lebesgue measure. It was shown by Mitsis [45] that the same holds if we only assume that  $C$  is a Borel subset of  $\mathbb{R}^n$  of Hausdorff dimension greater than 1. The analogous results are also true in the case  $n = 2$ ; this was proved independently by Bourgain [6] and Marstrand [37] for circles centered at the points of an arbitrary set  $C \subset \mathbb{R}^2$  of positive Lebesgue measure, and by Wolff [60] for  $C \subset \mathbb{R}^2$  of Hausdorff dimension greater than 1. In fact, Bourgain proved a stronger result, which extends to other curves with non-zero curvature.

Inspired by these results, the authors in [34] studied what happens if the circles are replaced by axis-parallel squares. They constructed a closed set  $A$  of Hausdorff dimension 1 that contains the boundary of an axis-parallel square centered at each point in  $\mathbb{R}^2$  (see [34, Theorem 1.1]). Thornton studied in [52] the higher dimensional versions: the problem when  $0 \leq k < n$  and  $A \subset \mathbb{R}^n$  contains the  $k$ -skeleton of an  $n$ -dimensional axis-parallel cube centered at every point of a compact set of given dimension  $d$  for some fixed  $d \in [0, n]$ . (Recall that the  $k$ -skeleton of a polytope is the union of its  $k$ -dimensional faces.) He found the smallest possible dimension of such a compact  $A$  in the cases when we consider box dimension and packing dimension. He conjectured that the smallest possible Hausdorff dimension of  $A$  is  $\max(d - 1, k)$ , which would be the generalization of [34, Theorem 1.4], which addresses the case  $n = 2, k = 0$ .

In this paper we prove Thornton’s conjecture not only for cubes but for general polytopes of  $\mathbb{R}^n$ . It turns out that it plays an important role whether 0 is contained in one of the  $k$ -dimensional affine subspaces defined by the  $k$ -skeleton of the polytope (see Theorem 3.2.1). This is even more true if instead of just scaling, we also allow rotations. In this case, we ask what the minimal Hausdorff dimension of a set is that contains a scaled and rotated copy of the  $k$ -skeleton of a given polytope centered at each point of  $C$ . Obviously, it must have dimension at least  $k$  if  $C$  is nonempty. It turns out that this is sharp: we show that there is a Borel set of dimension  $k$  that contains a scaled and rotated copy of the  $k$ -skeleton of a polytope centered at each point of  $\mathbb{R}^n$ , *provided that 0 is not in any of the  $k$ -dimensional affine subspaces defined by the  $k$ -skeleton*. On the other hand, if 0 belongs to one of these affine subspaces, then the problem becomes much harder (see Remark 3.3.3).

As mentioned above at the end of the second paragraph, a (very) special case of Theorem 3.2.1, namely, when  $n = 2$  and  $S$  consists of the 4 vertices of a square centered at the origin, was already proved in [34]. Our proof of Theorem 3.2.1 is much simpler than the proof in [34]. In fact, in all our results mentioned above, we will show that, in the sense of Baire category, the minimal dimension is attained by residually many sets. As it often happens, it is much easier to show that some properties hold for residually many sets than to try to construct a set for which they hold. In our case, after proving residuality for  $k$ -dimensional affine subspaces, we automatically obtain residuality for countable unions of  $k$ -dimensional subsets of  $k$ -dimensional affine subspaces, hence  $k$ -skeletons.

If we allow rotations but do not allow scaling, the question becomes: what is the minimal Hausdorff dimension of a set that contains a rotated copy of the  $k$ -skeleton of a given polytope centered at each point of  $C$ ? We do not know the answer to this question for a general compact set  $C$ . However, as the following simple example shows, it is no longer true that a typical construction has minimal dimension.

Let  $C \subset \mathbb{R}^2$  denote the unit circle centered at 0, and let the “polytope” be a single point of  $C$ . Then  $\{0\}$  is a set of dimension 0 that contains, centered at each point of  $C$ , a rotated



copy of our “polytope”. (That is, it contains a point at distance 1 from each point of  $C$ .) On the other hand, it is easy to show that, if  $A$  contains a *nonzero* point at distance 1 from each point of  $C$ , then  $A$  has dimension at least 1. In particular, a “typical”  $A$  has dimension 1 and not 0. The same example also shows that the minimal dimension can be different depending on whether the “polytope” consists of one point or two points.

However, we will show that a typical construction does have minimal dimension, provided that  $C$  has full dimension, i.e.,  $\dim C = n$  for  $C \subset \mathbb{R}^n$ . In this case, the minimal (as well as typical) dimension of a set  $A$  that contains a rotated copy of the  $k$ -skeleton of a polytope centered at each point of  $C$  is  $k + 1$ . Somewhat surprisingly, we obtain that the smallest possible dimension (and also the typical dimension) is still  $k + 1$  if we want the  $k$ -skeleton of a rotated copy of the polytope of *every size* centered at every point.

Let us state our results more precisely. Throughout this paper, by a *scaled copy* of a fixed set  $S \subset \mathbb{R}^n$  we mean a set of the form  $x + rS = \{x + rs : s \in S\}$ , where  $x \in \mathbb{R}^n$  and  $r > 0$ . We say that  $x + rS$  is a scaled copy of  $S$  *centered at*  $x$ . (That is, the center of  $S$  is assumed to be the origin.) Similarly, a *rotated copy* of  $S$  centered at  $x \in \mathbb{R}^n$  is  $x + T(S) = \{x + T(s) : s \in S\}$ , where  $T \in SO(n)$ . Combining these two, we define a *scaled and rotated copy* of  $S$  centered at  $x \in \mathbb{R}^n$  by  $x + rT(S) = \{x + rT(s) : s \in S\}$ , where  $r > 0$  and  $T \in SO(n)$ .

In this paper we will consider only Hausdorff dimension, and we will denote by  $\dim E$  the Hausdorff dimension of a set  $E$ . We list here the special cases of our results when the polytope is a cube and the set of centers is  $\mathbb{R}^n$ . (The first statement was already proved in [52].)

**Corollary 3.1.1.** *For any integers  $0 \leq k < n$ , the minimal dimension of a Borel set  $A \subset \mathbb{R}^n$  that contains the  $k$ -skeleton of*

1. *a scaled copy of a cube centered at every point of  $\mathbb{R}^n$  is  $n - 1$ ;*
2. *a scaled and rotated copy of a cube centered at every point of  $\mathbb{R}^n$  is  $k$ ;*

3. a rotated copy of a cube centered at every point of  $\mathbb{R}^n$  is  $k + 1$ ;

4. a rotated cube of every size centered at every point of  $\mathbb{R}^n$  is  $k + 1$ .

In fact, the same results hold if the  $k$ -skeleton of a cube is replaced by any  $S \subset \mathbb{R}^n$  with  $\dim S = k$  that can be covered by a countable union of  $k$ -dimensional affine subspaces that do not contain 0.

For  $k = n - 1$  it is natural to ask if, in addition to dimension  $k + 1 = n$ , we can also guarantee positive Lebesgue measure in the settings (3) and (4). As we will see, we cannot guarantee positive measure. We show that there are residually many *Nikodym sets*, i.e., sets of measure zero which contain a punctured hyperplane through every point. The existence of Nikodym sets in  $\mathbb{R}^n$  for every  $n \geq 2$  was proved by Falconer [20]. We also obtain residually many sets of measure zero which contain a hyperplane at every positive distance from every point. By combining our these two results, we get the following.

**Corollary 3.1.2.** *Let  $S \subset \mathbb{R}^n$  ( $n \geq 2$ ) be a set that can be covered by countably many hyperplanes and suppose that  $0 \notin S$ . Then there exists a set of Lebesgue measure zero that contains a scaled and rotated copy of  $S$  of every scale centered at every point of  $\mathbb{R}^n$ .*

Note that here we need only the assumption  $0 \notin S$  (which clearly cannot be dropped), while in Corollary 3.1.1 we needed the stronger assumption that the covering affine subspaces do not contain 0. Also, Corollary 3.1.2 is clearly false for  $n = 1$ .

One can ask what happens for those sets  $S$  to which neither the classical results nor our results can be applied. One of the simplest such case is when, say,  $n = 1$  and  $S = C - 1/2$ , where  $C$  is the classical triadic Cantor set in the interval  $[0, 1]$ . We do not know how large a set  $A$  can be that contains a scaled copy of  $S$  centered at each  $x \in \mathbb{R}$ . Does it always have positive Lebesgue measure, or Hausdorff dimension at least 1? In [36] Laba and Pramanik construct random Cantor sets for which such a set must have positive Lebesgue measure, and by the result of Máthé [39], there exist Cantor sets for which such a set  $A$  can have

zero measure. Hochman [29] and Bourgain [7] prove that for any porous Cantor set  $C$  with  $\dim C > 0$ , such a set  $A$  must have Hausdorff dimension strictly larger than  $\dim C$  and at least  $1/2$ .

Finally we remark that T. W. Körner [35] observed in 2003 that small Kakeya-type sets can be constructed using Baire category argument. He proved that if we consider the Hausdorff metric on the space of all compact sets that contain line segments in every possible direction between two fixed parallel line segments, then in this space, residually many sets have zero Lebesgue measure. As we will see, in our results we obtain residually many sets in a different type of metric space: we consider Hausdorff metric in a “code space”.

### 3.2 Scaled copies

In this section we consider only scaled (not rotated) copies of  $S$ . We will prove the following theorem:

**Theorem 3.2.1.** *Let  $S$  be the  $k$ -skeleton of an arbitrary polytope in  $\mathbb{R}^n$  for some  $0 \leq k < n$ , and let  $d \in [0, n]$  be arbitrary.*

- (i) *Suppose that  $0$  is not contained in any of the  $k$ -dimensional affine subspaces defined by  $S$ . Then the smallest possible dimension of a compact set  $A$  that contains a scaled copy of  $S$  centered at each point of some  $d$ -dimensional compact set  $C$  is  $\max(d - 1, k)$ .*
- (ii) *Suppose that  $0$  is contained in at least one of the  $k$ -dimensional affine subspaces defined by  $S$ . Then the smallest possible dimension of a compact set  $A$  that contains a scaled copy of  $S$  centered at each point of some  $d$ -dimensional compact set  $C$  is  $\max(d, k)$ .*

Thornton’s conjecture mentioned in the introduction is clearly a special case of part (i) of this theorem.

In fact, our main goal is to study a slightly different problem, from which we can deduce the results above. Our aim is to find for a given “skeleton”  $S$  and for a given nonempty

compact set of centers  $C$  (instead of a given  $S$  and a given  $\dim C$ ) the smallest possible value of  $\dim A$ , where  $A$  contains a scaled copy of  $S$  centered at each point of  $C$ .

We will study the case when  $S$  is the  $k$ -skeleton of a polytope, or more generally, the case when  $S$  is a countable union  $S = \bigcup S_i$ , where each  $S_i$  is contained in an affine subspace  $V_i$ . We will assume that  $C$  is compact and nonempty. Our aim is to show that, in the sense of Baire category, a typical set  $A$  that contains a scaled copy of  $S$  centered at each point of  $C$  has minimal dimension.

Let us make this more precise. Fix a nonempty compact set  $C \subset \mathbb{R}^n$  and a non-degenerate closed interval  $I \subset (0, \infty)$ . In what follows, we view  $C \times I$  as a parametrization of the space of certain scaled copies of a given set  $S \subset \mathbb{R}^n$ ; in particular,  $(x, r) \in C \times I$  corresponds to the copy centered at  $x$  and scaled by  $r$ . Let  $\mathcal{K}$  denote the space of all compact sets  $K \subset C \times I$  that have full projection onto  $C$ . (That is, for each  $x \in C$  there is an  $r \in I$  with  $(x, r) \in K$ .) We equip  $\mathcal{K}$  with the Hausdorff metric. Clearly,  $\mathcal{K}$  is a closed subset of the space of all compact subsets of  $C \times I$ , and hence it is a complete metric space. In particular, the Baire category theorem holds for  $\mathcal{K}$ , so we can speak about a typical  $K \in \mathcal{K}$  in the Baire category sense: a property  $P$  holds for a typical  $K \in \mathcal{K}$  if  $\{K \in \mathcal{K} : P \text{ holds for } K\}$  is residual in  $\mathcal{K}$ , or equivalently, if there exists a dense  $G_\delta$  set  $\mathcal{G} \subset \mathcal{K}$  such that the property holds for every  $K \in \mathcal{G}$ .

Let  $A$  be an arbitrary set that contains a scaled copy of  $S \subset \mathbb{R}^n$  centered at each point of  $C$ . First we show an easy lower estimate on  $\dim A$ , which in some important cases will turn out to be sharp. Let  $C'$  denote the orthogonal projection of  $C$  onto  $W := \text{span}\{S\}^\perp$ . (As usual, we denote by  $\text{span}\{S\}$  the linear span of  $S$ , so it always contains the origin.) For every point  $x' \in C'$  there exists an  $x \in C$  such that the projection of  $x$  onto  $W$  is  $x'$ , and there exists an  $r > 0$  such that  $x + rS \subset A$  and hence  $x + rS \subset (x' + \text{span}\{S\}) \cap A$ . Since for any  $x' \in C' \subset W = \text{span}\{S\}^\perp$  the set  $(x' + \text{span}\{S\}) \cap A$  contains a scaled copy of  $S$ ,

we obtain by the general Fubini type inequality (see e.g. in [19] or [22])

$$\dim A \geq \dim C' + \dim S. \quad (3.1)$$

Now let  $K \in \mathcal{K}$  and  $S \subset \mathbb{R}^n$  and consider

$$A = A_{K,S} := \bigcup_{(x,r) \in K} x + rS. \quad (3.2)$$

Note that  $A_{K,S}$  contains a scaled copy of  $S$  centered at each point of  $C$ , so by the previous paragraph,

$$\dim A_{K,S} \geq \dim C' + \dim S. \quad (3.3)$$

The following lemma shows that for a typical  $K \in \mathcal{K}$  we have equality in (3.3) if  $S$  is an affine subspace.

**Lemma 3.2.2.** *Let  $V$  be an affine subspace of  $\mathbb{R}^n$ , let  $\emptyset \neq C \subset \mathbb{R}^n$  be compact, and let  $C'$  denote the projection of  $C$  onto  $\text{span}\{V\}^\perp$ . Then for a typical  $K \in \mathcal{K}$ , and for  $A_{K,V}$  defined by (3.2),*

$$\dim A_{K,V} = \dim C' + \dim V.$$

We postpone the proof of this lemma and first study some of its corollaries. Suppose that  $S$  is a countable union  $S = \bigcup S_i$ , where each  $S_i$  is a subset of an affine subspace  $V_i$ . Let  $C'_i$  denote the orthogonal projection of  $C$  onto  $W_i := \text{span}\{V_i\}^\perp$ . Since a countable intersection of residual sets is residual, and since the Hausdorff dimension of a countable union of sets is the supremum of the Hausdorff dimension of the individual sets, it follows that for a typical  $K \in \mathcal{K}$ ,

$$\dim A_{K,S} = \dim \left( \bigcup_i A_{K,S_i} \right) \leq \sup_i (\dim C'_i + \dim V_i).$$

On the other hand, if  $A$  contains a scaled copy of  $S = \bigcup_i S_i$  centered at each  $x \in C$ , then applying (3.1) to each  $S_i$ , we get  $\dim A \geq \dim C'_i + \dim S_i$  for each  $i$  and thus  $\dim A \geq$

$\sup_i(\dim C'_i + \dim S_i)$ . Therefore, we obtain the following theorem:

**Theorem 3.2.3.** *Let  $C$  be an arbitrary nonempty compact subset in  $\mathbb{R}^n$ , and let  $S = \bigcup_{i=1}^{\infty} S_i$ , where each  $S_i$  is a subset of an affine subspace  $V_i$ . Let  $C'_i$  denote the orthogonal projection of  $C$  onto  $\text{span}\{V_i\}^{\perp}$ . Then:*

(i) *For every set  $A$  that contains a scaled copy of  $S$  centered at each point of  $C$ ,*

$$\dim A \geq \sup_i(\dim C'_i + \dim S_i).$$

(ii) *For a typical  $K \in \mathcal{K}$ , the set  $A = A_{K,S}$  defined by (3.2) contains a scaled copy of  $S$  centered at each point of  $C$  and*

$$\dim A \leq \sup_i(\dim C'_i + \dim V_i).$$

*Furthermore, if  $S$  is compact then so is  $A$ .*

Let  $W_i = \text{span}\{V_i\}^{\perp}$ . Note that if  $0 \notin V_i$  then  $\dim W_i = n - \dim V_i - 1$ . Therefore if  $\dim C = n$ ,  $k < n$ ,  $\dim S = k$ , and for every  $i$  we have  $0 \notin V_i$  and  $\dim V_i = k$ , then  $\sup_i \dim S_i = k$  and  $\dim C'_i = n - k - 1$  for every  $i$ , so Theorem 3.2.3 gives  $\dim A = n - 1$ , which proves the general version of (1) of Corollary 3.1.1.

So far we studied the problem of finding the minimal Hausdorff dimension of a set  $A$  that contains a copy of a given set  $S$  centered at each point of a given set  $C$ . Now we turn to the problem when, instead of  $S$  and  $C$ , we are only given  $S$  and  $d = \dim C$ . We suppose that  $\dim S_i = \dim V_i$  for each  $i$ , so the lower and upper estimates in (i) and (ii) agree.

Since clearly  $\dim C'_i \geq \max(0, \dim C - \text{codim } W_i)$ , where  $\text{codim } W_i$  denotes the co-dimension of the linear space  $W_i$ , therefore Theorem 3.2.3(i) gives

$$\dim A \geq \sup_i(\max(0, d - \text{codim } W_i) + \dim S_i).$$

In order to show that this estimate is sharp when  $\dim S_i = \dim V_i$ , by Theorem 3.2.3(ii), it is enough to find a compact set  $C \subset \mathbb{R}^n$  for which  $\dim C'_i = \max(0, \dim C - \text{codim } W_i)$  holds for each  $i$ . This can be done by the following claim, which we will prove later.

**Claim 3.2.4.** *For each  $i \in \mathbb{N}$ , let  $W_i$  be a linear subspace of  $\mathbb{R}^n$  of co-dimension  $l_i \in \{0, 1, \dots, n\}$ . Then for every  $d \in [0, n]$  there exists a  $d$ -dimensional compact set  $C \subset \mathbb{R}^n$  whose projection onto  $W_i$  has dimension  $\max(0, d - l_i)$  for each  $i$ .*

Therefore Theorem 3.2.3 and Claim 3.2.4 give the following.

**Corollary 3.2.5.** *Suppose that  $S = \bigcup_{i=1}^{\infty} S_i$ , where each  $S_i$  is a subset of an affine subspace  $V_i$  with  $\dim S_i = \dim V_i$ . For each  $i$ , let  $W_i = \text{span}\{V_i\}^\perp$ . Let  $d \in [0, n]$  be arbitrary. Then the smallest possible dimension of a set  $A$  that contains a scaled copy of  $S$  centered at each point of some  $d$ -dimensional set  $C$  is  $\sup_i (\max(0, d - \text{codim } W_i) + \dim S_i)$ .*

Now we claim that Theorem 3.2.1 is a special case of Corollary 3.2.5. Indeed, if  $S$  is a  $k$ -skeleton of a polytope, then for each  $i$  we have  $\dim S_i = \dim V_i = k$ , and  $W_i$  has co-dimension either  $k + 1$  if  $0 \notin V_i$ , or  $k$  if  $0 \in V_i$ . Thus  $\max(0, d - \text{codim } W_i) + \dim S_i$  is either  $\max(k, d - 1)$  if  $0 \notin V_i$ , or  $\max(k, d)$  if  $0 \in V_i$ .

It remains to prove Claim 3.2.4 and Lemma 3.2.2. The following simple proof is based on an argument that was communicated to us by K. J. Falconer.

*Proof of Claim 3.2.4.* We can clearly suppose that  $d > 0$  and  $l_i \in \{1, \dots, n - 1\}$ . For  $0 < s \leq n$ , Falconer [21] introduced  $\mathcal{G}_n^s$  as the class of those  $G_\delta$  subsets  $F \subset \mathbb{R}^n$  for which  $\bigcap_{i=1}^{\infty} f_i(F)$  has Hausdorff dimension at least  $s$  for all sequences of similarity transformations  $\{f_i\}_{i=1}^{\infty}$ . Among other results, Falconer proved that  $\mathcal{G}_n^s$  is closed under countable intersection, and if  $F_1 \in \mathcal{G}_n^s$  and  $F_2 \in \mathcal{G}_m^t$  then  $F_1 \times F_2 \in \mathcal{G}_{n+m}^{s+t}$ . Examples of sets of  $\mathcal{G}_n^s$  with Hausdorff dimension exactly  $s$  are also shown in [21] for every  $0 < s \leq n$ .

For  $l < d$ , let  $E_l \in \mathcal{G}_{n-l}^{d-l}$  with  $\dim E_l = d - l$ , and for  $l \geq d$  let  $E_l$  be a dense  $G_\delta$  subset of  $\mathbb{R}^{n-l}$  with  $\dim E_l = 0$ . Let  $F_l = E_l \times \mathbb{R}^l \subset \mathbb{R}^{n-l} \times \mathbb{R}^l$ . Clearly, the projection of  $F_l$  onto  $\mathbb{R}^{n-l}$  has Hausdorff dimension  $\max(0, d - l)$ .

Now we show that  $F_l \in \mathcal{G}_n^d$ . This follows from the product rule mentioned above if  $l < d$ . In the case  $l \geq d$ , we need to prove that  $\dim(\bigcap_{i=1}^{\infty} f_i(E_l \times \mathbb{R}^l)) \geq d$  for any sequence of similarity transformations  $\{f_i\}_{i=1}^{\infty}$ . Let  $V$  be an  $(n-l)$ -dimensional subspace of  $\mathbb{R}^n$  which is generic in the sense that it intersects all the countably many  $l$ -dimensional affine subspaces  $f_i(\{0\} \times \mathbb{R}^l)$  in a single point. Then for each translate  $V+x$  of  $V$ , the set  $f_i(E_l \times \mathbb{R}^l) \cap (V+x)$  is similar to the dense  $G_\delta$  set  $E_l$ , hence  $(\bigcap_{i=1}^{\infty} f_i(E_l \times \mathbb{R}^l)) \cap (V+x)$  is nonempty for each  $x$ , which implies that indeed  $\dim(\bigcap_{i=1}^{\infty} f_i(E_l \times \mathbb{R}^l)) \geq l \geq d$ .

For each  $i$ , let  $H_i$  be a rotated copy of  $F_{l_i}$  with projection of Hausdorff dimension  $\max(0, d - l_i)$  onto  $W_i$ . Since each  $H_i$  is of class  $\mathcal{G}^d$ , the intersection  $D := \bigcap_{i=1}^{\infty} H_i$  is of class  $\mathcal{G}^d$ . In particular, its Hausdorff dimension is at least  $d$ . It is also clear that the projection of  $D$  onto each  $W_i$  has Hausdorff dimension at most  $\max(0, d - l_i)$ .

Now  $D$  has all the required properties except that it might have Hausdorff dimension larger than  $d$ , and it is not compact but  $G_\delta$ . If  $\dim D > d$ , then let  $C$  be a compact subset of  $D$  with Hausdorff dimension  $d$ . Then for each  $i$ , the projection of  $C$  onto  $W_i$  is at most  $\max(0, d - l_i)$ , but it cannot be smaller since  $W_i$  has co-dimension  $l_i$ . If  $\dim D = d$  then let  $D_j$  be compact subsets of  $D$  with  $\dim D_j \rightarrow d$  and let  $C$  be a disjoint union of shrunken converging copies of  $D_j$  and their limit point.  $\square$

*Proof of Lemma 3.2.2.* By (3.3), it is enough to show that  $\dim A_{K,V} \leq \dim C' + \dim V$  holds for a typical  $K \in \mathcal{K}$ . Write  $V = v + V_0$  where  $V_0$  is a  $k$ -dimensional linear subspace,  $v \in \mathbb{R}^n$  and  $v \perp V_0$ . Without loss of generality we can assume that  $v = 0$  or  $|v| = 1$ . Let  $x'$  denote the projection of a point  $x$  onto  $\text{span}\{V\}^\perp$ , and let  $\text{proj } x \in \mathbb{R}$  denote the projection of  $x$  onto  $\mathbb{R}v$ . (Clearly, if  $v = 0$ , then  $\text{proj } x = 0$ .)

Let  $\mathcal{K}^n$  denote the space of all nonempty compact subsets of  $\mathbb{R}^n$ , equipped with the Hausdorff metric. Then

$$A = A_{K,V} = \bigcup_{(x,r) \in K} x' + (\text{proj } x + r)v + V_0,$$



so

$$\dim A = \dim V_0 + \dim \left( \bigcup_{(x,r) \in K} x' + (\text{proj } x + r)v \right) = k + \dim F(K),$$

where  $F : \mathcal{K} \rightarrow \mathcal{K}^n$  is defined by

$$F(K) = \bigcup_{(x,r) \in K} x' + (\text{proj } x + r)v.$$

It is easy to see that  $F$  is continuous.

Since for every open set  $G \subset \mathbb{R}^n$  and for every compact set  $K \subset G$  we have  $\text{dist}(\mathbb{R}^n \setminus G, K) > 0$ , it follows that for any open set  $G \subset \mathbb{R}^n$ ,  $\{K \in \mathcal{K}^n : K \subset G\}$  is an open subset of  $\mathcal{K}^n$ . Consequently, for any  $s, \delta, \varepsilon > 0$ , the set of those compact sets  $K \in \mathcal{K}^n$  that have an open cover  $\bigcup G_i$  where  $\sum_i (\text{diam } G_i)^s < \varepsilon$  and  $\text{diam } G_i < \delta$  for each  $i$  is an open subset of  $\mathcal{K}^n$ . Therefore for any  $s > 0$ ,  $\{K \in \mathcal{K}^n : \dim K \leq s\}$  is a  $G_\delta$  subset of  $\mathcal{K}^n$ . Since  $F$  is continuous,  $\{K \in \mathcal{K} : \dim F(K) \leq s\}$  is a  $G_\delta$  subset of  $\mathcal{K}$ .

We finish the proof by showing that  $\{K \in \mathcal{K} : \dim F(K) \leq \dim C'\}$  is dense. To obtain this, for every compact set  $L \in \mathcal{K}$  we construct another compact set  $K \in \mathcal{K}$  arbitrary close to  $L$ , such that  $\{\text{proj } x + r : (x, r) \in K\}$  is finite and so  $F(K)$  is covered by a finite union of copies of  $C'$ . For a given  $L \in \mathcal{K}$ , such a  $K \in \mathcal{K}$  can be constructed by choosing a sufficiently small  $\varepsilon > 0$  and letting

$$K := \{(x, r) : \exists r' \text{ s.t. } (x, r') \in L, \text{proj } x + r \in \varepsilon\mathbb{Z}, |r - r'| \leq \varepsilon\}. \quad \square$$

### 3.3 Scaled and rotated copies

In this section, we study the problem when we are allowed to *scale and rotate* copies of  $S$ . That is, now our aim is to find for a given set  $S \subset \mathbb{R}^n$  and a nonempty compact set of centers  $C \subset \mathbb{R}^n$  the minimal possible value of  $\dim A$ , where  $A$  contains a scaled and rotated copy of  $S$  centered at each point of  $C$ . (That is, for every  $x \in C$ , there exist  $r > 0$  and  $T \in SO(n)$ )

such that  $x + rT(S) \subset A$ .)

For a fixed nonempty compact set  $C \subset \mathbb{R}^n$  and a closed interval  $I \subset (0, \infty)$ , let  $\mathcal{K}'$  denote the space of all compact sets  $K \subset C \times I \times SO(n)$  that have full projection onto  $C$ . We fix a metric on  $SO(n)$  that induces the natural topology and equip  $\mathcal{K}'$  with the Hausdorff metric. Then  $\mathcal{K}'$  is also a complete metric space, so again we can talk about typical  $K \in \mathcal{K}'$  in the Baire category sense. Now for  $K \in \mathcal{K}'$  and  $S \subset \mathbb{R}^n$ , we let

$$A'_{K,S} := \bigcup_{(x,r,T) \in K} x + rT(S). \quad (3.4)$$

Note that  $A'_{K,S}$  contains a scaled and rotated copy of  $S$  centered at each point of  $C$ .

Again, first we consider the case when  $S$  is an affine subspace, but we now exclude the case when  $S$  contains 0.

**Lemma 3.3.1.** *Let  $V$  be an affine subspace of  $\mathbb{R}^n$  such that  $0 \notin V$  and let  $C \subset \mathbb{R}^n$  be an arbitrary nonempty compact set. Then for a typical  $K \in \mathcal{K}'$ , and for  $A'_{K,V}$  defined by (3.4),*

$$\dim A'_{K,V} = \dim V.$$

*Proof.* Clearly it is enough to show that  $\dim A'_{K,V} \leq \dim V$  holds for a typical  $K \in \mathcal{K}'$ . For any  $N \in \mathbb{N}$ , we define  $F'_N : \mathcal{K}' \rightarrow \mathcal{K}^n$  by  $F'_N(K) = A'_{K,V} \cap [-N, N]^n$ . It is easy to see that  $F'_N$  is continuous. Then exactly the same argument as in the proof of Lemma 3.2.2 gives that  $\{K \in \mathcal{K}' : \dim F'_N(K) \leq s\}$  is a  $G_\delta$  subset of  $\mathcal{K}'$ , which implies that  $\{K \in \mathcal{K}' : \dim A'_{K,V} \leq s\}$  is also  $G_\delta$ .

So it remains to prove that  $\{K \in \mathcal{K}' : \dim A'_{K,V} \leq \dim V\}$  is dense. Fix  $\varepsilon > 0$ . Then, by compactness and since  $0 \notin V$ , there exists an  $N = N(\varepsilon) \in \mathbb{N}$  and  $(\dim V)$ -dimensional affine subspaces  $V_1, \dots, V_N$  such that for any  $(x, r, T) \in C \times I \times SO(n)$  there exists  $(r', T') \in I \times SO(n)$  within  $\varepsilon$  distance of  $(r, T)$  such that  $x + r'T'(V) = V_i$  for some

$i \leq N$ . Thus, given any compact set  $L \in \mathcal{K}'$  and  $\varepsilon > 0$ , we can take

$$K = \{(x, r', T') : x + r'T'(V) \in \{V_1, \dots, V_N\}\} \cap L_\varepsilon,$$

where

$$L_\varepsilon = \{(x, r', T') : \exists (r, T) \text{ s.t. } (x, r, T) \in L, \text{dist}((r', T'), (r, T)) \leq \varepsilon\}.$$

It follows that  $K \in \mathcal{K}'$  and the (Hausdorff) distance between  $K$  and  $L$  is at most  $\varepsilon$ . Furthermore,  $\dim A'_{K,V} = \dim V$ , since  $A'_{K,V}$  can be covered by finitely many  $(\dim V)$ -dimensional affine spaces.  $\square$

By taking a countable intersection of residual sets we obtain the following corollary of Lemma 3.3.1, which clearly implies the general form of (2) of Corollary 3.1.1.

**Theorem 3.3.2.** *Let  $C$  be an arbitrary nonempty compact subset in  $\mathbb{R}^n$ ,  $k < n$  and let  $S \subset \mathbb{R}^n$  be a  $k$ -Hausdorff-dimensional set that can be covered by a countable union of  $k$ -dimensional affine subspaces that do not contain 0. Then for a typical  $K \in \mathcal{K}'$ , the set  $A'_{K,S}$  contains a scaled and rotated copy of  $S$  centered at every point of  $C$ , and  $\dim A'_{K,S} = \dim S$ .*

*Remark 3.3.3.* If  $0 \in V$  and  $V$  is  $k$ -dimensional then a scaled and rotated copy of  $V$  centered at  $x$  is a  $k$ -dimensional affine subspace that contains  $x$ . Therefore a set  $A$  that contains a scaled and rotated copy of  $V$  centered at every point of  $C$  is a set that contains a  $k$ -dimensional affine subspace through every point of  $C$ . The Lebesgue measure of such an  $A$  is clearly bounded below by the Lebesgue measure of  $C$ . By generalizing the planar result of Davies [17] to higher dimensions, Falconer [20] proved there is such an  $A$  which attains this lower bound. In Section 3.5 we show that the Lebesgue measure of a typical such  $A$  is in fact this minimum. On the other hand, to find the minimal dimension of such an  $A$  is closely related to the Kakeya problem, especially in the special case  $k = 1$ , and for some nontrivial  $C$  this problem is as hard as the Kakeya problem.

### 3.4 Rotated copies: dimension

Now we study what happens if we allow rotation but do not allow scaling. As we mentioned in the introduction, it is *not* true that for a general nonempty compact set of centers  $C$ , a typical construction has minimal dimension. However, we will show that this is true provided that  $C$  has full dimension.

The following lower estimate can be found in [28]:

**Fact 3.4.1.** *Let  $0 \leq k < n$  be integers, and let  $S \subset \mathbb{R}^n$  be a  $k$ -Hausdorff-dimensional set that can be covered by a countable union of  $k$ -dimensional affine subspaces that do not contain 0. Let  $\emptyset \neq C \subset \mathbb{R}^n$  and  $A \subset \mathbb{R}^n$  be such that for every  $x \in C$ , there exists a rotated copy of  $S$  centered at  $x$  contained in  $A$ . Then  $\dim A \geq \max\{k, k + \dim C - (n - 1)\}$ .*

*In particular, if  $\dim C = n$  then  $\dim A \geq k + 1$ .*

*Remark 3.4.2.* If instead of fixing  $C$ , we fix only the dimension  $d$  of  $C$ , and  $S$  can be covered by one  $k$ -dimensional affine subspace  $V$ , then the following simple examples show that the estimate in Fact 3.4.1 is sharp. Without loss of generality we can assume that  $V$  is at unit distance from 0. For  $d \leq n - 1$ , we can take  $A = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$  and take  $C$  to be a  $d$ -dimensional subset of  $\mathbb{R}^k \times S^{n-k-1}$ , where  $S^m$  denotes the unit sphere in  $\mathbb{R}^{m+1}$  centered at 0. For  $d = n - 1 + s$ , where  $s \in [0, 1]$ , let  $E \subset \mathbb{R}^{n-k}$  be an  $s$ -dimensional subset of a line and let  $F \subset \mathbb{R}^{n-k}$  be the set with a copy of  $S^{n-k-1}$  centered at every point of  $E$ . It is easy to show that  $\dim F = n - k - 1 + s$ . Let  $C = \mathbb{R}^k \times F$  and  $A = \mathbb{R}^k \times E$ . In both cases  $A$  contains a rotated copy of  $S$  centered at every point of  $C$ ,  $\dim C = d$  and  $\dim A = \max\{k, k + \dim C - (n - 1)\}$ .

If  $S$  can be covered by two distinct  $k$ -dimensional affine subspaces but cannot be covered by one, then this question becomes much more difficult. Consider, for example, the case when  $S$  consists of two points, both at distance 1 from 0, so now  $A$  contains two distinct points at distance 1 from every point of a 1-dimensional set  $C \subset \mathbb{R}^2$ . The discussion in the introduction implies that if we take  $C = S^1$ , then  $\dim A \geq 1$ . We do not know if there exists

a set  $C$  with  $\dim C = 1$  for which there is such a set  $A$  with  $\dim A < 1$ .

Our goal is to show that for every fixed  $C$  with  $\dim C = n$ , the estimate  $\dim A \geq k + 1$  in Fact 3.4.1 is always sharp. Moreover, we construct sets of Hausdorff dimension  $k + 1$  that contain the  $k$ -skeleton of an  $n$ -dimensional rotated polytope of *every* size centered at *every* point. More precisely, we want to construct a set  $A$  that contains a rotated copy of every positive size of a given set  $S \subset \mathbb{R}^n$  centered at every point of a given nonempty compact set  $C$ . (That is, for every  $x \in C$  and  $r > 0$  there exists  $T \in SO(n)$  such that  $x + rT(S) \subset A$ .) Instead of every  $x \in C$  and  $r > 0$  we will guarantee only every  $(x, r)$  from each fixed nonempty compact set  $J \subset \mathbb{R}^n \times (0, \infty)$ . By taking countable unions, we get the desired construction for every  $(x, r) \in \mathbb{R}^n \times (0, \infty)$ .

For a fixed nonempty compact set  $J \subset \mathbb{R}^n \times (0, \infty)$ , let  $\mathcal{K}''$  denote the space of all compact sets  $K \subset J \times SO(n)$  that have full projection onto  $J$ . Again, by taking a metric on  $SO(n)$  that induces the natural topology and equipping  $\mathcal{K}''$  with the Hausdorff metric,  $\mathcal{K}''$  is also a complete metric space, so again we can talk about typical  $K \in \mathcal{K}''$  in the Baire category sense.

Now for any  $K \in \mathcal{K}''$  and  $S \subset \mathbb{R}^n$ , the set

$$A''_{K,S} := \bigcup_{(x,r,T) \in K} x + rT(S) \tag{3.5}$$

contains a rotated copy of  $S$  of scale  $r$  centered at  $x$  for every  $(x, r) \in J$ . Note that taking  $J = C \times \{1\}$  gives us the special case when only rotation is used.

Again, we start with the case when  $S$  is a  $k$ -dimensional ( $0 \leq k < n$ ) affine subspace of  $\mathbb{R}^n$  that does not contain the origin. Note that if  $d = \text{dist}(S, 0)$  then  $x + rT(S)$  is at distance  $rd$  from  $x$ . This motivates the following easy deterministic  $(k + 1)$ -dimensional construction.

**Proposition 3.4.3.** *For any integers  $0 \leq k < n$  there exists a Borel set  $B \subset \mathbb{R}^n$  of Hausdorff dimension  $k + 1$  that contains a  $k$ -dimensional affine subspace at every positive distance from every point of  $\mathbb{R}^n$ .*

*Proof.* Let  $W_1, W_2, \dots$  be a countable collection of  $(k+1)$ -dimensional affine subspaces of  $\mathbb{R}^n$  such that  $B := \bigcup_i W_i$  is dense. Then  $B$  is clearly a Borel set  $B \subset \mathbb{R}^n$  of Hausdorff dimension  $k+1$ , so all we need to show is that for any fixed  $x \in \mathbb{R}^n$  and  $r > 0$  the set  $B$  contains a  $k$ -dimensional affine subspace at distance  $r$  from  $x$ . Choose  $i$  such that  $W_i$  intersects the interior of the ball  $B(x, r)$ . Then the intersection of  $W_i$  and the sphere  $S(x, r)$  is a sphere in the  $(k+1)$ -dimensional affine space  $W_i$ , and any  $k$ -dimensional affine subspace of  $W_i \subset B$  that is tangent to this sphere is at distance  $r$  from  $x$ .  $\square$

The proof of the following lemma is based on the same idea as in the construction above.

**Lemma 3.4.4.** *Let  $0 \leq k < n$  be integers and  $V$  be a  $k$ -dimensional affine subspace of  $\mathbb{R}^n$  such that  $0 \notin V$ . Let  $J \subset \mathbb{R}^n \times (0, \infty)$  be an arbitrary nonempty compact set. Then for a typical  $K \in \mathcal{K}''$ , and for  $A''_{K,V}$  defined by (3.5),*

$$\dim A''_{K,V} \leq k + 1.$$

*Proof.* Without loss of generality we can assume that  $V$  is at distance 1 from the origin.

Let  $A(n, k+1)$  be the space of all  $(k+1)$ -dimensional affine subspaces of  $\mathbb{R}^n$ , equipped with a natural metric (for example the metric defined in [42, 3.16]), and let  $W_1, W_2, \dots$  be a countable dense set in  $A(n, k+1)$ . Let  $B = \bigcup_i W_i$ .

Exactly the same argument as in the proof of Lemma 3.3.1 gives that  $\{K \in \mathcal{K}'' : \dim A''_{K,V} \leq s\}$  is  $G_\delta$  for any  $s$ , so again it remains to prove that  $\{K \in \mathcal{K}'' : \dim A''_{K,V} \leq k+1\}$  is dense in  $\mathcal{K}''$ . Since  $\dim B = k+1$ , it is enough to show that  $\{K \in \mathcal{K}'' : A''_{K,V} \subset B\}$  is dense in  $\mathcal{K}''$ .

First we show that for any  $(x, r, T) \in J \times SO(n)$  and  $\varepsilon > 0$ , there exist  $i \in \mathbb{N}$  and  $T' \in SO(n)$  such that  $\text{dist}(T, T') < \varepsilon$  and  $x + rT'(V) \subset W_i$ . We will also see from the proof that for the given  $\varepsilon > 0$  and the above chosen  $i$ , there exists a neighborhood of  $(x, r, T)$  such that for any  $(x^*, r^*, T^*)$  from that neighborhood, there exists  $T^{*'} \in SO(n)$  such that  $\text{dist}(T^*, T^{*'}) < \varepsilon$  and  $x^* + r^*T^{*'}(V) \subset W_i$ . Hence, by the compactness of  $J \times SO(n)$ , for

a given  $\varepsilon > 0$ , there exists an  $N$  such that we can choose an  $i \leq N$  for every  $(x, r, T) \in J \times SO(n)$ .

So fix  $(x, r, T) \in J \times SO(n)$  and  $\varepsilon > 0$ . Let  $W$  be a  $(k+1)$ -dimensional affine subspace of  $\mathbb{R}^n$  that contains  $V$  such that  $0 < \text{dist}(W, 0) < \text{dist}(V, 0) = 1$ . We denote by  $v$  be the point of  $V$  closest to the origin, and let  $V_0 = x + rT(V)$ ,  $v_0 = x + rT(v)$  and  $W_0 = x + rT(W)$ . Then  $S_0 := W_0 \cap S(x, r)$  is a sphere in  $W_0$ , and  $V_0$  is the tangent of  $S_0$  at the point  $v_0$ . If  $W_i$  is sufficiently close to  $W_0$ , then we can pick a point  $v'_0 \in S'_0 := W_i \cap S(x, r)$  close to  $v_0$ , and a  $k$ -dimensional affine subspace  $V'_0 \subset W_i$  close to  $V_0$  that is the tangent of  $S'_0$  at  $v'_0$ . Then  $V'_0$  is at distance  $r$  from  $x$  and it is as close to  $V_0 = x + rT(V)$  as we wish, so  $V'_0 = x + rT'(V)$  for some  $T' \in SO(n)$  and  $T'$  can be chosen arbitrarily close to  $T$ , which completes the proof of the claim of the previous paragraph.

Thus, for a given  $L \in \mathcal{K}''$  and  $\varepsilon > 0$ , if we let

$$K = \{(x, r, T') : \exists i \leq N, \exists T \text{ s.t. } (x, r, T) \in L, \text{dist}(T, T') \leq \varepsilon, x + rT'(V) \subset W_i\},$$

then  $K \in \mathcal{K}''$  and the Hausdorff distance between  $K$  and  $L$  is at most  $\varepsilon$ . Furthermore,  $A''_{K,V} \subset \bigcup_{i=1}^N W_i \subset B$ , which completes the proof.  $\square$

The same statements hold if, instead of  $S = V$ , we consider any subset  $S \subset V$ . By taking a countable intersection of residual sets we obtain the following.

**Theorem 3.4.5.** *Let  $0 \leq k < n$  be integers and let  $S = \bigcup_{i=1}^{\infty} S_i$ , where each  $S_i$  is a subset of a  $k$ -dimensional affine subspace  $V_i$  with  $0 \notin V_i$ . Let  $J \subset \mathbb{R}^n \times (0, \infty)$  be an arbitrary nonempty compact set. Recall that  $\mathcal{K}''$  denotes the space of all compact sets  $K \subset J \times SO(n)$  that have full projection onto  $J$ .*

*Then for a typical  $K \in \mathcal{K}''$ , the set  $A''_{K,S}$  defined by (3.5) is a closed set with  $\dim A''_{K,S} \leq k+1$ , and for every  $(x, r) \in J$ , there exists a  $T \in SO(n)$  such that  $x + rT(S) \subset A''_{K,S}$ .*

We can see from Fact 3.4.1 that the estimate  $k+1$  above is sharp, provided that  $\dim S = k$  and  $J \supset C \times \{r\}$  for some  $r > 0$  and  $C \subset \mathbb{R}^n$  with  $\dim C = n$ . This gives the general version

of (3) and (4) of Corollary 3.1.1.

*Remark 3.4.6.* In Theorem 3.4.5 we obtain a rotated and scaled copy of  $S$  for every  $(x, r) \in J$  inside a set of Hausdorff dimension  $k + 1$ . We claim that using a similar argument as in [30, Remark 1.6] we can also move  $S$  *continuously* inside a set of Hausdorff dimension  $k + 1$  so that during this motion we get  $S$  in every required position. Indeed, let  $K$  be a fixed (typical) element of  $\mathcal{K}''$  guaranteed by Theorem 3.4.5 such that  $\dim A''_{K,S} \leq k + 1$ . Since  $K$  is a nonempty compact subset of the metric space  $\text{conv}(J) \times SO(n)$ , where  $\text{conv}$  denotes the convex hull, there exists a continuous function  $g : C_{1/3} \rightarrow \text{conv}(J) \times SO(n)$  on the classical Cantor set  $C_{1/3}$  such that  $g(C_{1/3}) = K$ . All we need to do is to extend this map continuously to  $[0, 1]$  such that  $\dim A''_{g([0,1]),S} \leq k + 1$ . For each complementary interval  $(a, b)$  of the Cantor set, we define  $g$  on  $(a, b)$  in such a way that  $g$  is smooth on  $[a, b]$  and that the diameter of  $g([a, b])$  is at most a constant multiple of the distance between  $g(a)$  and  $g(b)$ . This gives the desired extension since the union of the sets of the form  $x + rT(S)$  ( $(x, r, T) \in g((a, b))$ ) will be a countable union of smooth  $k + 1$ -dimensional manifolds, so  $\dim A''_{g((a,b)),S} = k + 1$ .

Note that if  $J = C \times \{1\}$  then we get only congruent copies. So in particular, for any  $k < n$ , the  $k$ -skeleton of a unit cube can be continuously moved by rigid motions in  $\mathbb{R}^n$  within a set of Hausdorff dimension  $k + 1$  in such a way that the center of the cube goes through every point of  $C$ , or by joining such motions, through every point of  $\mathbb{R}^n$ .

### 3.5 Rotated copies: measure

In this section, we study what happens when we place a rotated punctured hyperplane through every point. We show that typical arrangements of this kind have Lebesgue measure zero and are hence Nikodym sets. Using similar methods, we also show that typical arrangements of placing a rotated hyperplane at every positive distance from every point have measure zero. We use  $|\cdot|$  to denote the Lebesgue measure.

Let  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$  and  $H = \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_1 = 0\}$ . By a *rotated*



hyperplane at distance  $r \in [0, \infty)$  from  $x \in \mathbb{R}^n$ , we mean a set of the form  $x + rT(e_1) + T(H)$  for some  $T \in SO(n)$ . Note that we now allow  $r$  to be 0, and that  $x + rT(e_1) + T(H)$  differs from  $x + rT(e_1 + H)$  when  $r = 0$ .

Fix a nonempty compact set  $J \subset \mathbb{R}^n \times [0, \infty)$ . As in Section 3.4, we let  $\mathcal{K}''$  denote the space of compact sets  $K \subset J \times SO(n)$  that have full projection onto  $J$ .

In this section we prove the following result.

**Theorem 3.5.1.** *For a typical  $K \in \mathcal{K}''$ , the set*

$$\bigcup_{(x,r,T) \in K} (x + rT(e_1) + T(H)) \setminus \{x\}$$

has measure zero.

Note that if  $r = 0$ , then  $(x + rT(e_1) + T(H)) \setminus \{x\}$  is  $x + T(H \setminus \{0\})$ , so we are placing a rotated copy of the *punctured hyperplane*  $H \setminus \{0\}$  through  $x$ . Thus, if we consider the case  $J = C \times \{0\}$  for some compact set  $C \subset \mathbb{R}^n$ , we see that typical arrangements give rise to Nikodym sets. We also obtain our claim in Remark 3.3.3 that if we place an un-punctured hyperplane through every point in  $C$ , the typical arrangement of this kind has Lebesgue measure equal to  $|C|$ .

By taking countable unions of sets of the form in Theorem 3.5.1, we obtain the following:

**Corollary 3.5.2.** *There is a set of measure zero in  $\mathbb{R}^n$  which contains a hyperplane at every positive distance from every point as well as a punctured hyperplane through every point.*

### 3.5.1 Translating cones

In this section we introduce the main geometric construction for proving Theorem 3.5.1. This construction is done in  $\mathbb{R}^2$  and we will later see how to apply it to the  $n$ -dimensional problem. Our geometric arguments are similar to those used to construct Kakeya needle sets of arbitrarily small measure, see e.g. [4].

For  $-\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}$ , we define

$$D(\phi_1, \phi_2) = \{(r \sin \theta, r \cos \theta) : r \in \mathbb{R}, \theta \in [\phi_1, \phi_2]\}.$$

In other words,  $D(\phi_1, \phi_2) \subset \mathbb{R}^2$  denotes the double cone bounded by the lines through the origin of signed angles  $\phi_1, \phi_2$  with respect to the  $y$ -axis. (Note in particular that our sign convention measures the angles in the clockwise direction.)

Our geometric construction begins by partitioning  $D = D(\phi_1, \phi_2)$  into finitely many double cones  $\{D_i\}$ . Next, we translate each  $D_i$  downwards to a new vertex  $v_i \in D_i \cap \{y_2 < 0\}$  to obtain  $\tilde{D}_i := v_i + D_i$ . Our goal is to choose the  $\{D_i\}$  and the  $\{v_i\}$  so that the resulting double cones  $\{\tilde{D}_i\}$  satisfy the following three properties.

First, the  $\{\tilde{D}_i\}$  should have considerable overlap (and hence small measure) in a strip below the  $x$ -axis.

Second, we would like our construction to preserve certain distances to lines. To be more precise, first let

$$D^\perp(\phi_1, \phi_2) = \{(r \sin \theta, r \cos \theta) : r \geq 0, \theta \in [\phi_2 - \frac{\pi}{2}, \phi_1 + \frac{\pi}{2}]\}.$$

Our second desired property is that for any point  $p \in D^\perp(\phi_1, \phi_2)$  and any line  $\ell \subset D$ , there is a line in some  $\tilde{D}_i$  which has the same distance to  $p$  as  $\ell$  does.

For a non-horizontal line  $\ell \subset \mathbb{R}^2$  and  $p \in \mathbb{R}^2$ , we define  $d(p, \ell)$  to be the signed distance from  $p$  to  $\ell$ . The sign is positive if  $p$  is on the left of  $\ell$ , and negative if  $p$  is on the right. In our construction, we will always consider only lines whose direction belongs to the original cone  $D(\phi_1, \phi_2)$ . In particular, they are never horizontal so the signed distance is defined. The essential property of  $D^\perp(\phi_1, \phi_2)$  is that for any  $p \in D^\perp(\phi_1, \phi_2)$ , the map  $\ell \mapsto d(p, \ell)$  is an increasing function as  $\ell$  rotates from one boundary line  $\ell_1$  of  $D$  to the other boundary line  $\ell_2$ . Hence,

$$\{d(p, \ell) : \ell \subset D\} = [d(p, \ell_1), d(p, \ell_2)].$$

Before stating the third and final property, we observe that since  $v_i \in D_i \cap \{y_2 < 0\}$  for all  $i$ , we have  $D \cap \{y_2 \geq 0\} \subset (\bigcup_i \tilde{D}_i) \cap \{y_2 \geq 0\}$ . The third desired property is that the reverse containment holds if we thicken  $D$  slightly. That is,  $(\bigcup_i \tilde{D}_i) \cap \{y_2 \geq 0\}$  should be contained in a small neighborhood of  $D \cap \{y_2 \geq 0\}$ .

The following lemma asserts that it is indeed possible to partition  $D$  and translate the pieces to achieve the three desired properties above.

**Lemma 3.5.3.** *Let  $-\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}$ ,  $D = D(\phi_1, \phi_2)$ ,  $R > 0$ , and  $\varepsilon > 0$ . Then we can choose the partition  $D = \bigcup D_i$  and the translates  $\tilde{D}_i = v_i + D_i$  so that*

1.  $|(\bigcup_i \tilde{D}_i) \cap \{-R \leq y_2 \leq 0\}| < \varepsilon$ .
2. If  $p \in D^\perp$  and  $\ell_0 \subset D$  is a line, then there is a line  $\tilde{\ell}$  in some  $\tilde{D}_i$  such that  $d(p, \tilde{\ell}) = d(p, \ell_0)$ .
3.  $(\bigcup_i \tilde{D}_i) \cap \{y_2 \geq 0\}$  is contained in the  $\varepsilon$ -neighborhood of  $D \cap \{y_2 \geq 0\}$ .

To prove this lemma, we first need a more elementary construction, in which we translate each  $D_i$  downwards by only a small amount  $\delta$ .

**Lemma 3.5.4.** *Let  $-\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}$ ,  $D = D(\phi_1, \phi_2)$ ,  $\delta > 0$ , and  $\varepsilon > 0$ . Then we can choose the partition  $D = \bigcup D_i$  and the translates  $\tilde{D}_i = v_i + D_i$  so that*

1.  $|(\bigcup_i \tilde{D}_i) \cap \{-\delta \leq y_2 \leq 0\}| \leq c\delta^2$ , where  $c = |D \cap \{0 \leq y_2 \leq 1\}|$ .
2. If  $p \in D^\perp$  and  $\ell_0 \subset D$  is a line, then there is a line  $\tilde{\ell}$  in some  $\tilde{D}_i$  such that  $d(p, \tilde{\ell}) = d(p, \ell_0)$ .
3. For each  $i$ ,  $v_i \in \{y_2 = -\delta\}$ .
4. For each  $i$ ,  $D^\perp \subset \tilde{D}_i^\perp$ .
5.  $(\bigcup_i \tilde{D}_i) \cap \{y_2 \geq 0\}$  is contained in the  $\varepsilon$ -neighborhood of  $D \cap \{y_2 \geq 0\}$ .

(If  $D_i = D(\psi_1, \psi_2)$ , then  $D_i^\perp := D^\perp(\psi_1, \psi_2)$  and  $\tilde{D}_i^\perp := v_i + D_i^\perp$ .)

*Proof.* We claim that for any partition  $D = \bigcup_i D_i$ , if we choose any  $v_i \in \{y_2 = -\delta\} \cap D_i \cap (-D_i^\perp)$ , then we have (1), (2), (3), and (4). Indeed, (3) is immediate. Since  $-v_i \in D_i^\perp$ , we have  $D^\perp \subset D_i^\perp \subset \tilde{D}_i^\perp$ , so (4) holds. And (3) implies (1) since  $|(\bigcup_i \tilde{D}_i) \cap \{-\delta \leq y_2 \leq 0\}| \leq \sum_i |\tilde{D}_i \cap \{-\delta \leq y_2 \leq 0\}| = \sum_i |D_i \cap \{0 \leq y_2 \leq \delta\}| = c\delta^2$ .

To show (2) holds, let  $p \in D^\perp$  and  $\ell_0 \subset D$ . Then  $\ell_0$  is in some  $D_i$ . Let  $\ell_1, \ell_2$  be the two boundary lines of  $D_i$  with  $d(p, \ell_1) < d(p, \ell_2)$ . Recall that  $\ell \subset D_i$  if and only if  $v_i + \ell \subset \tilde{D}_i$ . Since  $p \in D_i^\perp$  and  $p \in \tilde{D}_i^\perp$ , we have  $\{d(p, \ell) : \ell \subset D_i\} = [d(p, \ell_1), d(p, \ell_2)]$  and  $\{d(p, \ell) : \ell \subset \tilde{D}_i\} = [d(p, v_i + \ell_1), d(p, v_i + \ell_2)]$ . Since  $-v_i \in D_i \cap \{y_2 \geq 0\}$ , we have  $[d(p, \ell_1), d(p, \ell_2)] \subset [d(p, v_i + \ell_1), d(p, v_i + \ell_2)]$ . Thus,

$$d(p, \ell_0) \in \{d(p, \ell) : \ell \subset D_i\} \subset \{d(p, \ell) : \ell \subset \tilde{D}_i\},$$

so there is some  $\tilde{\ell} \subset \tilde{D}_i$  such that  $d(p, \tilde{\ell}) = d(p, \ell_0)$ , which completes the proof of (2) and hence our claim. Finally, by making the partition  $\bigcup_i D_i$  sufficiently fine and choosing  $v_i$  as above, we can ensure that (5) holds.  $\square$

*Proof of Lemma 3.5.3.* We fix a large  $N$  and repeatedly apply Lemma 3.5.4 with  $\delta = R/N$  until the vertex of each double cone lies in  $\{y_2 = -R\}$ . That is, we apply Lemma 3.5.4 once on  $D$  to get  $E_1$ , a union of double cones with vertices in  $\{y_2 = -\delta\}$  and such that  $|E_1 \cap \{-\delta \leq y_2 \leq 0\}| < c'\delta^2$ , where  $c' = 2|D \cap \{0 \leq y_2 \leq 1\}|$ . Next, we apply Lemma 3.5.4 to every double cone in  $E_1$  to get  $E_2$ , a union of double cones with vertices in  $\{y_2 = -2\delta\}$  and such that  $|E_2 \cap \{-2\delta \leq y_2 \leq -\delta\}| < c'\delta^2$ . By Lemma 3.5.4(5), we can also ensure that  $|E_2 \cap \{-\delta \leq y_2 \leq 0\}| < c'\delta^2$ .

We continue in this way to obtain  $E_1, \dots, E_N$ , such that  $|E_k \cap \{-j\delta \leq y_2 \leq -(j-1)\delta\}| < c'\delta^2$  for  $1 \leq j \leq k \leq N$ . Because of Lemma 3.5.4(5), we can also ensure that  $E_k \cap \{y_2 \geq 0\}$  is in the  $\varepsilon$ -neighborhood of  $D \cap \{y_2 \geq 0\}$  for each  $k$ . Ultimately, we have  $|E_N \cap \{-R \leq y_2 \leq 0\}| \leq Nc'\delta^2 = c'R^2/N$ . By choosing  $N$  sufficiently large, we can make this quantity as small as we wish. Writing  $E_N$  as  $\bigcup_i \tilde{D}_i$ , we obtain (1) and (3). Furthermore,

every time we translate downwards by  $\delta$ , Lemma 3.5.4(4) allows us to apply Lemma 3.5.4(2). Thus, (2) holds.  $\square$

### 3.5.2 Proof of Theorem 3.5.1

First we apply our main geometric construction from the previous section to prove the following lemma.

**Lemma 3.5.5.** *Let  $n \geq 2$ ,  $(x_0, r_0, T_0) \in \mathbb{R}^n \times [0, \infty) \times SO(n)$ , let  $B \subset \mathbb{R}^n$  be a closed ball, and suppose that  $x_0 + r_0 T_0(e_1) \notin B$ . Let  $\eta > 0$  be arbitrary. Then there is a (relatively) open neighborhood  $\tilde{U}$  of  $(x_0, r_0)$  in  $\mathbb{R}^n \times [0, \infty)$  such that for each  $\varepsilon > 0$ , there is a set  $\tilde{D} \subset \mathbb{R}^n$  such that:*

1. *For all  $(x, r) \in \tilde{U}$ , there is an affine hyperplane  $V \subset \tilde{D}$  of distance  $r$  from  $x$  and such that the angle between  $V$  and  $T_0(H)$  is at most  $\eta$ .*
2.  $|B \cap \tilde{D}| < \varepsilon$ .

*Proof.* First we show the lemma for  $n = 2$ . Since  $x_0 + r_0 T_0(e_1) \notin B$ , without loss of generality, we may assume that  $T_0 \in SO(2)$  is the identity, that  $x_0 + r_0 e_1 \in \{y_1 = 0, y_2 > 0\}$ , and that  $B$  does not intersect  $\{y_1 = 0, y_2 \geq 0\}$ . We can also assume that  $B$  lies in  $\{y_2 \geq -2 \text{ diam } B\}$ . It follows that  $x_0$  lies in the upper half-plane  $\{y_2 > 0\}$ .

Using the notation from Section 3.5.1, let  $D = D(-\phi, \phi)$  be a double cone, where  $\phi \in (0, \eta)$  is small enough so that  $x_0 \in D^\perp$  and  $B \cap D \cap \{y_2 \geq 0\} = \emptyset$ . The boundary of  $D$  is made up of two lines,  $\ell_1, \ell_2$ , with  $d(x_0, \ell_1) < r_0 < d(x_0, \ell_2)$ . Let  $\rho > 0$  be sufficiently small so that  $d(x_0, \ell_1) < r_0 - \rho$  and  $r_0 + \rho < d(x_0, \ell_2)$ . Let  $\tilde{U}$  be a (relatively) open neighborhood of  $(x_0, r_0)$  contained in

$$\{y \in D^\perp : d(y, \ell_1) < r_0 - \rho \text{ and } r_0 + \rho < d(y, \ell_2)\} \times (r_0 - \rho, r_0 + \rho).$$

Then for any  $(x, r) \in \tilde{U}$ , there is a line  $\ell \subset D$  of signed distance  $r$  from  $x$ . Given  $\varepsilon > 0$ ,

we apply Lemma 3.5.3 to get  $\tilde{D} := \bigcup_i \tilde{D}_i$  with  $|\tilde{D} \cap \{-2 \operatorname{diam} B \leq y_2 \leq 0\}| < \varepsilon$  and  $B \cap \tilde{D} \cap \{y_2 \geq 0\} = \emptyset$ . It follows that  $|B \cap \tilde{D}| < \varepsilon$ . By Lemma 3.5.3(2), for every  $(x, r) \in \tilde{U}$ , there is some line  $\ell \subset \tilde{D}$  of distance  $r$  from  $x$ . Every line  $\ell \subset \tilde{D}$  is a translate of some line in  $D$ , so the angle between  $\ell$  and  $H$  is at most  $\phi < \eta$ . This completes the proof in dimension  $n = 2$ .

For an arbitrary  $n \geq 2$ , we can assume without loss of generality that  $T_0$  is the identity, that  $x_0$  (and hence also  $x_0 + r_0 e_1$ ) is contained in the two-dimensional plane  $\mathbb{R}^2 \subset \mathbb{R}^n$  defined by the first two coordinate axes, and that the same assumptions hold as in the first paragraph of our proof. Then, if we project the ball  $B$  into  $\mathbb{R}^2$ , take the sets  $\tilde{U}, \tilde{D} \subset \mathbb{R}^2$  constructed above, and multiply them by  $\mathbb{R}^{n-2}$ , the resulting sets satisfy the requirements of the statement of Lemma 3.5.5 with  $\varepsilon$  replaced by  $\varepsilon \operatorname{diam}(B)^{n-2}$ .  $\square$

**Lemma 3.5.6.** *Let  $(x_0, r_0, T_0) \in \mathbb{R}^n \times [0, \infty) \times SO(n)$ , and let  $B \subset \mathbb{R}^n$  be a closed ball. Let  $G$  be a (relatively) open neighborhood of  $(x_0, r_0, T_0)$  in  $\mathbb{R}^n \times [0, \infty) \times SO(n)$ . Then there is an open neighborhood  $U \subset \mathbb{R}^n \times [0, \infty)$  of  $(x_0, r_0)$  such that for each  $\varepsilon > 0$ , there is a compact set  $K \subset G$  with full projection onto  $U$  and such that*

$$B \cap \bigcup_{\substack{(x,r,T) \in K \\ x+rT(e_1) \notin 2B}} (x + rT(e_1) + T(H)) \quad (3.6)$$

*has measure less than  $\varepsilon$ . (Here  $2B$  denotes the closed ball with the same center as  $B$  and with twice the radius.)*

*Proof.* Without loss of generality, we may assume  $G = G_1 \times G_2$ , where  $G_1$  and  $G_2$  are open sets in  $\mathbb{R}^n \times [0, \infty)$  and  $SO(n)$ , respectively.

If  $x_0 + r_0 T_0(e_1) \in B$ , then we can choose  $K \subset G$  to contain a neighborhood of  $(x_0, r_0, T_0)$  and such that  $x + rT(e_1) \in 2B$  for all  $(x, r, T) \in K$ . Then the set (3.6) is empty, so the lemma holds trivially.

Now suppose  $x_0 + r_0 T_0(e_1) \notin B$ . We can apply the previous lemma with  $\eta$  sufficiently

small (depending on  $G_2$ ) to get a set  $\tilde{U}$ . We take  $U$  to be an open neighborhood of  $(x_0, r_0)$  inside  $\tilde{U}$  and compactly contained in  $G_1$ . Then for each  $\varepsilon > 0$ , the previous lemma gives a set  $\tilde{D}$ . We take  $K$  to be the closure of

$$\{(x, r, T) \in U \times SO(n) : x + rT(e_1) + T(H) \subset \tilde{D}\},$$

and by the properties of  $\tilde{D}$  given by the previous lemma, this  $K$  has the desired properties.  $\square$

For  $B \subset \mathbb{R}^n$  a closed ball, let  $\mathcal{A}(B)$  be the set of all  $K \in \mathcal{K}''$  such that

$$B \cap \bigcup_{\substack{(x,r,T) \in K \\ x+rT(e_1) \notin 2B}} (x + rT(e_1) + T(H))$$

has measure zero.

**Lemma 3.5.7.**  $\mathcal{A}(B)$  is residual in  $\mathcal{K}''$ .

*Proof.* For  $\varepsilon > 0$ , let  $\mathcal{A}(B, \varepsilon)$  be the set of those  $K \in \mathcal{K}''$  for which there is an  $\eta > 0$  such that

$$B \cap \bigcup_{\substack{(x,r,T) \in K^\eta \\ x+rT(e_1) \notin 2B}} (x + rT(e_1) + T(H))$$

has measure less than  $\varepsilon$ , where  $K^\eta$  denotes the open  $\eta$ -neighborhood of  $K$ . Since  $\mathcal{A}(B) = \bigcap_{m=1}^{\infty} \mathcal{A}(B, \frac{1}{m})$ , it is enough to show that  $\mathcal{A}(B, \varepsilon)$  is open and dense in  $\mathcal{K}''$  for each  $\varepsilon > 0$ .

Fix  $\varepsilon > 0$ .  $\mathcal{A}(B, \varepsilon)$  is clearly open in  $\mathcal{K}''$ . To show that it is dense, let  $L \in \mathcal{K}''$  be arbitrary. Our aim is to find a  $K \in \mathcal{A}(B, \varepsilon)$  arbitrarily close to  $L$ . For each  $(x, r, T) \in L$ , we take a neighborhood  $G_{(x,r,T)}$  of  $(x, r, T)$ , which we choose sufficiently small (to be specified later). Then we apply Lemma 3.5.6 to  $(x, r, T)$  to get a neighborhood  $U_{(x,r,T)} \subset \mathbb{R}^n \times [0, \infty)$  of  $(x, r)$ . By compactness, there is a finite collection  $\{(x_i, r_i, T_i)\} \subset L$  such that  $\{U_{(x_i, r_i, T_i)}\}$  covers  $L$ .

Choose  $\varepsilon_i$  so that  $\sum_i \varepsilon_i < \varepsilon$ . We apply Lemma 3.5.6 to each  $U_{(x_i, r_i, T_i)}$  with  $\varepsilon_i$  in place of  $\varepsilon$  to get a compact  $K_i \subset G_{(x_i, r_i, T_i)}$  with full projection onto  $U_{(x_i, r_i, T_i)}$ . Let  $\tilde{K}_i = K_i \cap (J \times SO(n))$ . Let  $K$  be the union of  $\bigcup_i \tilde{K}_i$  together with a finite  $\delta$ -net of  $L$ . Then  $K \in \mathcal{A}(B, \varepsilon)$ . By choosing  $\delta$  and all the  $G_{(x, r, T)}$  sufficiently small, we can make  $K$  and  $L$  arbitrarily close to each other in the Hausdorff metric.  $\square$

Now we are ready to prove Theorem 3.5.1. It follows easily from Lemma 3.5.7 that, for a typical  $K \in \mathcal{K}''$ ,

$$\bigcup_{(x, r, T) \in K} (x + rT(e_1) + T(H \setminus \{0\})) \quad (3.7)$$

has measure zero. Indeed, let  $\{B_i\}$  be a countable collection of balls such that every point in  $\mathbb{R}^n$  is covered by a ball of arbitrarily small diameter, and suppose that  $K \in \bigcap_i \mathcal{A}(B_i)$ . For every  $(x, r, T) \in K$  and for every  $y \in H \setminus \{0\}$  there is a  $B_i$  which contains  $x + rT(e_1) + T(y)$  and has diameter less than  $|y|/2$ . Then  $x + rT(e_1) \notin 2B_i$ , so  $x + rT(e_1) + T(y)$  belongs to the null set

$$B_i \cap \bigcup_{\substack{(x, r, T) \in K \\ x + rT(e_1) \notin 2B_i}} (x + rT(e_1) + T(H)).$$

To complete the proof of Theorem 3.5.1, we need to show that we can remove the puncture from  $H \setminus \{0\}$  when the distance  $r$  is nonzero. By adapting the argument in the proof of Lemma 3.4.4, we can show that for any  $r_0 > 0$ , for a typical  $K \in \mathcal{K}''$ , the set

$$\bigcup_{\substack{(x, r, T) \in K \\ r \geq r_0}} x + rT(e_1) \quad (3.8)$$

has dimension at most 1, hence measure zero. By taking a countable intersection of  $r_0$  tending to 0, we see that for a typical  $K \in \mathcal{K}''$ , the set

$$\bigcup_{\substack{(x, r, T) \in K \\ r \neq 0}} x + rT(e_1)$$



has measure zero. This completes the proof of Theorem 3.5.1.

*Remark 3.5.8.* The argument in the proof of Lemma 3.4.4 cannot be applied directly to show that the set (3.8) has dimension at most 1 for a typical  $K \in \mathcal{K}''$ . There is a slight complication due to the fact that for  $r_0 > 0$ , the function  $\mathcal{K}'' \rightarrow \mathcal{K}^n$  defined by  $K \mapsto \bigcup_{(x,r,T) \in K, r \geq r_0} x + rT(e_1)$  is not necessarily continuous. However, the technical modifications required are straightforward, so we leave this to the reader.

# CHAPTER 4

## ANALYTIC CAPACITY AND PROJECTIONS

This chapter is joint work with Xavier Tolsa and originally appeared in [10].

### 4.1 Introduction

The objective of this paper is to study the connection between the analytic capacity of a set and the size of its projections onto lines. First we introduce some notation and definitions. A compact set  $E \subset \mathbb{C}$  is said to be *removable for bounded analytic functions* if for any open set  $\Omega$  containing  $E$ , every bounded function analytic on  $\Omega \setminus E$  has an analytic extension to  $\Omega$ . In order to study removability, in the 1940s, Ahlfors [1] introduced the notion of analytic capacity. The *analytic capacity* of a compact set  $E \subset \mathbb{C}$  is

$$\gamma(E) = \sup |f'(\infty)|, \tag{4.1}$$

where the supremum is taken over all analytic functions  $f : \mathbb{C} \setminus E \rightarrow \mathbb{C}$  with  $|f| \leq 1$  on  $\mathbb{C} \setminus E$ , and  $f'(\infty) = \lim_{z \rightarrow \infty} z(f(z) - f(\infty))$ . In [1], Ahlfors showed that  $E$  is removable for bounded analytic functions if and only if  $\gamma(E) = 0$ .

In the 1960s, Vitushkin conjectured that a compact set in the plane is non-removable for bounded analytic functions (or equivalently, has positive analytic capacity) if and only if its orthogonal projections have positive length in a set of directions of positive measure, or in other words, if and only if it has positive Favard length. The *Favard length* of a Borel set  $E \subset \mathbb{C}$  is

$$\text{Fav}(E) = \int_0^\pi \mathcal{H}^1(P_\theta(E)) d\theta, \tag{4.2}$$

where, for  $\theta \in [0, \pi)$ ,  $P_\theta$  denotes the orthogonal projection onto the line  $L_\theta := \{re^{i\theta} : r \in \mathbb{R}\}$ , and  $\mathcal{H}^1$  stands for the 1-dimensional Hausdorff measure. In 1986, Mattila [41] showed that Vitushkin's conjecture is false. Indeed, he proved that the property of having zero

Favard length is not invariant under conformal mappings while removability for bounded analytic functions remains invariant. Mattila's result didn't tell which implication in the above conjecture is false. This question was partially solved in 1988, when Jones and Murai [32] constructed a set with zero Favard length and positive analytic capacity. Later on, Joyce and Mörters [33] later obtained an easier example using curvature of measures.

Although Vitushkin's conjecture is not true in full generality, it turns out that it holds in the particular case of sets with finite length. This was proved by G. David [14] in 1998. Indeed, he showed that such sets are removable if and only if they are purely unrectifiable, which is equivalent to having zero Favard length, by the Besicovitch projection theorem. For sets with arbitrary length, removability can be characterized in terms of curvature of measures, by [53] (see Theorem 4.2.1 below for more details).

As mentioned above, one of the directions of Vitushkin's conjecture is false. However, it is not known yet if the other implication holds. Namely, does positive Favard length imply positive analytic capacity? In a sense, the main result of this paper asserts that if one strengthens the assumption of positive Favard length in a suitable way, then the answer is positive. See the survey [57] for more information on this and other related questions.

Given a Borel measure  $\mu$  in  $\mathbb{R}^2$ , we denote by  $P_\theta\mu$  the image measure of  $\mu$  by the orthogonal projection  $P_\theta$  from  $\mathbb{R}^2$  onto the line  $L_\theta := \{re^{i\theta} : r \in \mathbb{R}\}$ . Our main result is the following:

**Theorem 4.1.1.** *Let  $I \subset [0, \pi)$  be an interval. For any compact set  $E \subset \mathbb{C}$  and any Borel measure  $\mu$  on  $\mathbb{C}$  with  $\int_I \|P_\theta\mu\|_2^2 d\theta > 0$ , we have*

$$\gamma(E) \geq c \frac{\mu(E)^2}{\int_I \|P_\theta\mu\|_2^2 d\theta}, \quad (4.3)$$

where  $c$  is some positive constant depending only on  $\mathcal{H}^1(I)$ .

In the theorem above,  $\|P_\theta\mu\|_2$  stands for the  $L^2$  norm of the density of  $P_\theta\mu$  with respect to length in  $L_\theta$ , and the  $L^2$  norm is computed with respect to length in  $L_\theta$  too.

Observe that if the Hausdorff dimension of  $E$  is larger than 1, by Frostman's lemma there exists a nonzero measure  $\mu$  supported on  $E$  such that  $\mu(B(x, r)) \leq r^s$ , with  $s > 1$ . Then it holds that

$$\int_0^\pi \|P_\theta \mu\|_2^2 d\theta \leq c \iint \frac{1}{|x-y|} d\mu(x) d\mu(y) < \infty. \quad (4.4)$$

See Theorem 9.7 in [42], for example. So (4.3) implies the well-known fact that  $\gamma(E) > 0$  in this case. In fact, from (4.4) and (4.3) we get the sharper (also well-known) inequality  $\gamma(E) \geq c C_1(E)$ , where  $C_1$  is the capacity associated with the Riesz kernel  $1/|x|$ .

On the other hand, there are sets  $E$  in the plane with  $C_1(E) = 0$  which support a nonzero Borel measure  $\mu$  such that  $\int_I \|P_\theta \mu\|_2^2 d\theta < \infty$  for some interval  $I \subset [0, \pi)$ . This is the case, for example, of any rectifiable set  $E$  with  $\mathcal{H}^1(E) < \infty$ . To see this, just consider a Lipschitz graph  $\Gamma$  such that  $\mathcal{H}^1(E \cap \Gamma) > 0$  and let  $\mu = \mathcal{H}^1|_{E \cap \Gamma}$ . If  $L_{\theta_0}$  denotes the horizontal axis with respect to which  $\Gamma$  is constructed and  $I$  is a sufficiently small neighborhood of  $\theta_0$ , then  $\int_I \|P_\theta \mu\|_2^2 d\theta < \infty$ , as desired.

It is worth mentioning that the Favard length of  $E$  satisfies an estimate very similar to (4.3):

$$\text{Fav}(E) \geq c_I \frac{\mu(E)^2}{\int_I \|P_\theta \mu\|_2^2 d\theta}, \quad (4.5)$$

for some constant  $c_I > 0$ . This follows easily from the Cauchy-Schwarz inequality. Indeed, for any Borel measure  $\mu$  supported on  $E$  and  $\theta \in [0, \pi)$ ,

$$\mu(E) = \|P_\theta \mu\|_1 \leq \|P_\theta \mu\|_2 \mathcal{H}^1(P_\theta E)^{1/2}. \quad (4.6)$$

Thus,

$$\mu(E) \mathcal{H}^1(I) \leq \int_I \|P_\theta \mu\|_2 \mathcal{H}^1(P_\theta E)^{1/2} d\theta \leq \left( \int_I \|P_\theta \mu\|_2^2 d\theta \right)^{1/2} \text{Fav}(E)^{1/2},$$

which yields (4.5) with  $c_I = \mathcal{H}^1(I)^2$ . So a natural question is the following: does there exist

some absolute constant  $c > 0$  such that

$$\gamma(E) \geq c \text{Fav}(E)? \tag{4.7}$$

If the answer were positive, then Theorem 4.1.1 would be an immediate consequence of this and (4.5).

The question (4.7) is widely open, and we think that Theorem 4.1.1 can be interpreted as a contribution that supports a positive answer. However, we remark that the assumption that  $\text{Fav}(E) > 0$  is strictly stronger than the existence of a measure  $\mu$  supported on  $E$  satisfying  $P_\theta\mu \in L^2$  for a.e.  $\theta$  in some interval  $I \subset [0, \pi)$ . Indeed, there exists a set  $E \subset \mathbb{R}^2$  such that the following hold:

1.  $\mathcal{H}^1(P_\theta E) > 0$  for a.e.  $\theta \in [0, \pi)$ .
2. For all Borel measures  $\mu$  such that  $\mu(E) > 0$  and all intervals  $I \subset [0, \pi)$ , we have  $\mathcal{H}^1(\{\theta \in I : P_\theta\mu \notin L^2\}) > 0$ .

To construct a set  $E$  satisfying (1) and (2), let  $\{I_k\}_{k=1}^\infty$  be a sequence of subsets of  $[0, \pi)$  such that  $\mathcal{H}^1(\bigcap_k I_k) = 0$  and such that  $\mathcal{H}^1(I \cap I_k) > 0$  for all  $k$  and all intervals  $I \subset [0, \pi)$ . (For example, we can take each  $I_k$  to be open and dense in  $[0, \pi)$ , with  $\mathcal{H}^1(I_k) \rightarrow 0$ .) By the digital sundial theorem [20], for each  $k$ , there is a set  $E_k \subset \mathbb{R}^2$  such that  $\mathcal{H}^1(P_\theta E_k) = 0$  for a.e.  $\theta \in I_k$  and  $\mathcal{H}^1(P_\theta E_k) > 0$  for a.e.  $\theta \notin I_k$ . Let  $E = \bigcup_k E_k$ .

Property (1) immediately follows from the facts that  $\mathcal{H}^1(P_\theta E) > 0$  for a.e.  $\theta \notin \bigcap_k I_k$  and that  $\mathcal{H}^1(\bigcap_k I_k) = 0$ . To check (2), let  $\mu$  be a Borel measure such that  $\mu(E) > 0$ . Then  $\mu(E_k) > 0$  for some  $k$ , so using (4.6), we see that if  $\theta \in [0, \pi)$  is such that  $P_\theta(\mu|_{E_k}) \in L^2$ , then  $\mathcal{H}^1(P_\theta E_k) > 0$ . Hence, for any interval  $I \subset [0, \pi)$ , we have

$$\mathcal{H}^1(\{\theta \in I : P_\theta\mu \notin L^2\}) \geq \mathcal{H}^1(\{\theta \in I : \mathcal{H}^1(P_\theta E_k) = 0\}) = \mathcal{H}^1(I_k \cap I) > 0,$$

and we have verified (2).

The result stated in Theorem 4.1.1 extends to higher dimensions. In  $\mathbb{R}^d$  the role of the analytic capacity  $\gamma$  is played by the capacities  $\Gamma_{d,n}$  associated to the vector-valued Riesz kernels  $x/|x|^{n+1}$ . Given an integer  $0 < n < d$  and a compact  $E \subset \mathbb{R}^d$ , one sets

$$\Gamma_{d,n}(E) = \sup |\langle T, 1 \rangle|,$$

where the supremum is taken over all real distributions  $T$  supported in  $E$  such that

$$\frac{x}{|x|^{n+1}} * T \in L^\infty(\mathbb{R}^d) \quad \text{and} \quad \left\| \frac{x}{|x|^{n+1}} * T \right\|_{L^\infty(\mathbb{R}^d)} \leq 1.$$

In the case  $n = 1$ ,  $d = 2$ ,  $\Gamma_{2,1}$  is the real version of the analytic capacity  $\gamma$ , and from [53] it holds that  $\gamma \approx \Gamma_{2,1}$ .

In the codimension 1 case (i.e.,  $n = d - 1$ ),  $\Gamma_{d,d-1}$  is the so-called Lipschitz harmonic capacity introduced by Paramonov [48]. The analogue of Vitushkin's conjecture also holds for sets with finite  $\mathcal{H}^n$ -measure. That is,  $E$  removable for Lipschitz harmonic functions in  $\mathbb{R}^{n+1}$  if and only if  $E$  is purely  $n$ -unrectifiable, or equivalently, the orthogonal projections of  $E$  on almost all hyperplanes have  $\mathcal{H}^n$ -measure zero. See [46] and [47]. The analogous result for  $1 < n < d - 1$  is still an open problem.

The higher dimensional extension of Theorem 4.1.1 is the following:

**Theorem 4.1.2.** *For an integer  $n$  with  $0 < n < d$ , let  $V_0 \subset \mathbb{R}^d$  be an  $n$ -plane through the origin and let  $s > 0$ . For any compact set  $E \subset \mathbb{R}^d$  and any Borel measure  $\mu$  on  $\mathbb{R}^d$  with  $\int_{B(V_0,s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) > 0$ , we have*

$$\Gamma_{d,n}(E) \geq c \frac{\mu(E)^2}{\int_{B(V_0,s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V)},$$

where  $c$  is some positive constant depending only on  $s$ ,  $n$ , and  $d$ .

In this theorem,  $P_V$  is the orthogonal projection onto the  $n$ -dimensional subspace  $V$ , and

$\|P_V\mu\|_2$  is the  $L^2$  norm (with respect to  $n$ -dimensional Lebesgue measure) in  $V$  of  $P_V\mu$  (we identify  $P_V\mu$  with its density with respect to  $n$ -dimensional Lebesgue measure in  $V$ ). Also,  $\gamma_{d,n}$  is the natural probability measure on  $G(d,n)$  (see [42, Chapter 3]), and  $B(V_0, s)$  is a ball of radius  $s$  in  $G(d,n)$ , where  $G(d,n)$  denotes the Grassmanian space of  $n$ -dimensional linear subspaces of  $\mathbb{R}^d$ . See Section 4.3.2 below for the definition of the metric in  $G(d,n)$ .

The first fundamental step towards the proof of Theorems 4.1.1 and 4.1.2 is a Fourier calculation which shows that there exist constants  $c, \lambda > 1$  (depending only on  $s, n, d$ ) such that

$$\iint_{x-y \in K(V_0^\perp, \lambda^{-1}s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} \leq c \int_{B(V_0, s)} \|P_V\mu\|_2^2 d\gamma_{d,n}(V), \quad (4.8)$$

where, given  $U \in G(d, m)$  and  $t > 0$ ,  $K(U, t)$  is the cone

$$K(U, t) = \{x \in \mathbb{R}^d : \text{dist}(x, U) < t|x|\},$$

and  $V_0^\perp$  is the subspace orthogonal to  $V_0$ . In the planar case  $n = 1, d = 2$ , the calculation is particularly clean and we obtain a more precise result. See Corollaries 4.3.3 and 4.3.11 for more details. Our inspiration for proving the estimate (4.8) comes from the work of Martikainen and Orponen [38]. In that paper, the authors characterize the “big pieces of Lipschitz graph” condition on  $n$ -AD-regular sets in terms of an integral of the form  $\int_{B(V_0, s)} \|P_V\mu\|_2^2 d\gamma_{d,n}(V)$ , with  $\mu$  equal to  $n$ -dimensional Hausdorff measure  $\mathcal{H}^n$  restricted to a suitable subset  $E$ . Roughly speaking, in the proof of the main lemma (Lemma 1.10) of [38], the authors obtain a variant of the estimate (4.8), but with the left-hand side replaced by a discretized version of the integral. They do not use the Fourier transform, but instead use more geometric arguments.

The second step in the proof of Theorems 4.1.1 and 4.1.2 is the construction of a corona type decomposition which will allow us to bound the  $L^2(\mu)$  norm of the Riesz transform

$$\mathcal{R}^n \mu(x) = \int \frac{x-y}{|x-y|^{n+1}} d\mu(y),$$

assuming that  $\mu$  satisfies the growth condition

$$\mu(B(x, r)) \leq c_0 r^n \quad \text{for all } x \in \mathbb{R}^d, r > 0.$$

Using this corona decomposition we will deduce that

$$\|\mathcal{R}^n \mu\|_{L^2(\mu)}^2 \lesssim \mu(\mathbb{R}^d) + \iint_{x-y \in K(U_0, s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} \quad (4.9)$$

for any  $U_0 \in G(d, d-n)$  and  $s > 0$ . In the case  $n = 1$ , we will get the following estimate involving the curvature of  $\mu$ :

$$\iiint \frac{1}{R(x, y, z)^2} d\mu(x) d\mu(y) d\mu(z) \lesssim \mu(\mathbb{C}) + \iint_{x-y \in K(U_0, s)} \frac{d\mu(x) d\mu(y)}{|x-y|}, \quad (4.10)$$

where  $R(x, y, z)$  stands for the radius of the circumference passing through  $x, y, z$ . In both (4.9) and (4.10), the implicit constant depends only on  $s, c_0, n$  and  $d$ . See Theorem 4.10.2 for more details. We remark that other related corona decompositions have already appeared in [54] and [2], for example. However, the use of the conical Riesz-type energy in (4.8) in connection with corona decompositions is totally new, as far as we know.

Using (4.8), (4.9), (4.10), and the characterization of analytic capacity in terms of curvature of measures from [53] and the characterization of the capacities  $\Gamma_{d,n}$  in terms of  $L^2$  estimates of Riesz transforms from [58] and [49], we will obtain Theorem 4.1.1 and Theorem 4.1.2 respectively.

## 4.2 Notation and preliminaries

### 4.2.1 Generalities

We write  $a \lesssim b$  if there is a  $C > 0$  such that  $a \leq Cb$ , and we write  $a \lesssim_t b$  if the constant  $C$  depends on the parameter  $t$ . We write  $a \approx b$  to mean  $a \lesssim b \lesssim a$  and define  $a \approx_t b$  similarly.



We denote the open ball of radius  $r$  centered at  $x$  by  $B(x, r)$ . For a ball  $B = B(x, r)$  and  $\delta > 0$  we write  $r(B)$  for its radius and denote  $\delta B = B(x, \delta r)$ .

Given an  $m$ -plane  $V \subset \mathbb{R}^d$ ,  $z \in \mathbb{R}^d$ , and  $s > 0$ , we consider the (open) cone

$$K(z, V, s) = \{x \in \mathbb{R}^d : \text{dist}(x - z, V) < s|x - z|\}.$$

In the case  $z = 0$ , we also write  $K(V, s) = K(0, V, s)$ .

### 4.2.2 Measures and rectifiability

The Lebesgue measure of a set  $A \subset \mathbb{R}^d$  is denoted by  $\mathcal{L}^d(A)$ . Given  $0 < \delta \leq \infty$ , we set

$$\mathcal{H}_\delta^n(A) = \inf \left\{ \sum_i \text{diam}(A_i)^n : A_i \subset \mathbb{R}^d, \text{diam}(A_i) \leq \delta, A \subset \bigcup_i A_i \right\}.$$

We define the  $n$ -dimensional Hausdorff measure as

$$\mathcal{H}^n(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^n(A).$$

A set  $E \subset \mathbb{R}^d$  is called  $n$ -rectifiable if there are Lipschitz maps  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^d$ ,  $i = 1, 2, \dots$ , such that

$$\mathcal{H}^n \left( E \setminus \bigcup_i f_i(\mathbb{R}^n) \right) = 0. \quad (4.11)$$

On the other hand,  $E$  is called *purely  $n$ -unrectifiable* if any  $n$ -rectifiable subset  $F \subset E$  has zero  $\mathcal{H}^n$ -measure.

Also, one says that a Radon measure  $\mu$  on  $\mathbb{R}^d$  is  $n$ -rectifiable if  $\mu$  vanishes out of an  $n$ -rectifiable set  $E \subset \mathbb{R}^d$  and moreover  $\mu$  is absolutely continuous with respect to  $\mathcal{H}^n|_E$ .

A measure  $\mu$  is called  $n$ -AD-regular (or just AD-regular or Ahlfors-David regular) if there exists some constant  $c > 0$  such that

$$c^{-1}r^n \leq \mu(B(x, r)) \leq cr^n \quad \text{for all } x \in \text{supp}(\mu) \text{ and } 0 < r \leq \text{diam}(\text{supp}(\mu)).$$

### 4.2.3 Cauchy and Riesz transform, and capacities

Given a signed Radon measure  $\nu$  in  $\mathbb{C}$ , the Cauchy transform is defined by

$$\mathcal{C}\nu(x) = \int \frac{1}{z-w} d\nu(w),$$

whenever the integral makes sense. The  $\varepsilon$ -truncated version is

$$\mathcal{C}_\varepsilon\nu(x) = \int_{|z-w|>\varepsilon} \frac{1}{z-w} d\nu(w).$$

For a signed Radon measure  $\nu$  in  $\mathbb{R}^d$ , we consider the  $n$ -dimensional Riesz transform

$$\mathcal{R}^n\nu(x) = \int \frac{x-y}{|x-y|^{n+1}} d\nu(y),$$

whenever the integral makes sense. For  $\varepsilon > 0$ , its  $\varepsilon$ -truncated version is given by

$$\mathcal{R}_\varepsilon^n\nu(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} d\nu(y).$$

The curvature of a non-negative Borel measure  $\mu$  in  $\mathbb{C}$  is defined by

$$c^2(\mu) = \iiint \frac{1}{R(x,y,z)^2} d\mu(x) d\mu(y) d\mu(z),$$

where  $R(x,y,z)$  stands for the radius of the circumference passing through  $x,y,z$ . For  $\varepsilon > 0$ , its  $\varepsilon$ -truncated version is

$$c_\varepsilon^2(\mu) = \iiint_{\substack{|x-y|>\varepsilon \\ |x-z|>\varepsilon \\ |y-z|>\varepsilon}} \frac{1}{R(x,y,z)^2} d\mu(x) d\mu(y) d\mu(z).$$

As shown in [44], if  $\mu$  is a finite Borel measure in  $\mathbb{C}$  satisfying the linear growth condition

$$\mu(B(z, r)) \leq c_0 r \quad \text{for all } z \in \mathbb{C}, r > 0,$$

then

$$\|\mathcal{C}_\varepsilon \mu\|_{L^2(\mu)}^2 = \frac{1}{6} c_\varepsilon^2(\mu) + O(\mu(\mathbb{C})),$$

where  $|O(\mu(\mathbb{C}))| \lesssim \mu(\mathbb{C})$ , with the implicit constant depending only on  $c_0$ . The connection between the Cauchy kernel and curvature of measures was first observed by Melnikov while studying analytic capacity [43].

We denote by  $L_n(E)$  the set of positive Borel measures  $\mu$  supported on  $E$  satisfying

$$\mu(B(x, r)) \leq r^n \quad \text{for all } x \in E, r > 0.$$

The following theorem characterizes analytic capacity in terms of measures from  $L_1(E)$  with finite curvature.

**Theorem 4.2.1.** *Let  $E \subset \mathbb{C}$  be compact. Then we have:*

$$\gamma(E) \approx \sup\{\mu(E) : \mu \in L_1(E), c^2(\mu) \leq \mu(E)\}. \quad (4.12)$$

The fact that  $\gamma(E)$  is bigger than a constant multiple of the supremum is due to Melnikov [43], and the more difficult converse estimate to Tolsa [53].

The extension of the preceding result to the capacities  $\Gamma_{d,n}$  is the following:

**Theorem 4.2.2.** *Let  $E \subset \mathbb{R}^d$  be compact. Then we have:*

$$\Gamma_{d,n}(E) \approx \sup\left\{\mu(E) : \mu \in L_n(E), \sup_{\varepsilon>0} \|\mathcal{R}_\varepsilon^n \mu\|_{L^2(\mu)}^2 \leq \mu(E)\right\}. \quad (4.13)$$

Theorem 4.2.2 was proved by Volberg in the case  $n = d - 1$ , and it was later extended to all the values  $0 < n < d$  by Prat [49].

## 4.3 The Fourier calculation

### 4.3.1 The planar case

We think it is worth first studying the planar case because it is simpler than the higher dimensional case, and the result is more precise.

Recall that for any Schwartz function  $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$ , we have

$$\widehat{P_\theta \phi}(x) = \widehat{\phi}(x) \quad \text{for all } x \in L_\theta, \quad (4.14)$$

where  $(P_\theta \phi)(x) = \int_{x+L_\theta^\perp} \phi d\mathcal{L}^1 = \int_{L_\theta^\perp} \phi(x+y) d\mathcal{L}^1 y$  and  $\widehat{P_\theta \phi}$  denotes the 1-dimensional Fourier transform on  $L_\theta$ . (Note that  $P_\theta \phi$  is the density of  $P_\theta \nu$  where  $\nu = \phi(x) dx$ .) To see (4.14), observe that for  $\xi \in L_\theta$ ,

$$\widehat{P_\theta \phi}(\xi) = \int_{L_\theta} P_\theta \phi(x) e^{-2\pi i x \cdot \xi} dx = \int_{L_\theta} \int_{L_\theta^\perp} \phi(x+y) e^{-2\pi i x \cdot \xi} dx dy = \widehat{\phi}(\xi),$$

where we use the fact that  $y \cdot \xi = 0$  for  $y \in L_\theta^\perp$ .

**Lemma 4.3.1.** *Let  $K_I$  be the cone  $K_I = \{re^{i\theta} : r \in \mathbb{R}, \theta \in I\}$ . Let  $I^\perp = I + \frac{\pi}{2} \pmod{\pi}$  and define the cone  $K_{I^\perp}$  similarly. Then the (distributional) Fourier transform of  $\chi_{K_I}(x) |x|^{-1}$  is  $\chi_{K_{I^\perp}}(x) |x|^{-1}$ .*

*Proof.* Let  $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$  be a Schwartz function. By applying the identity (4.14) to  $\phi$  and using the Fourier inversion formula, we have  $\int_{L_\theta} \widehat{\phi} d\mathcal{L}^1 = (P_\theta \phi)(0) = \int_{L_\theta^\perp} \phi d\mathcal{L}^1$ . Thus, by polar coordinates,

$$\int_{K_I} |x|^{-1} \widehat{\phi}(x) dx = \int_I \int_{L_\theta} \widehat{\phi} d\mathcal{L}^1 d\theta = \int_I \int_{L_\theta^\perp} \phi d\mathcal{L}^1 d\theta = \int_{K_{I^\perp}} |x|^{-1} \phi(x) dx,$$

which completes the proof. □

**Proposition 4.3.2.** *Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a Schwartz function. Then, for any set  $I \subset [0, \pi]$ ,*

we have

$$\int_{I^\perp} \|P_\theta \psi\|_2^2 d\theta = \iint_{x-y \in K_I} \frac{\psi(x) \psi(y)}{|x-y|} dx dy.$$

*Proof.* Let  $k(x) = \chi_{K_I}(x) |x|^{-1}$ , so that  $\widehat{k}(x) = \chi_{K_{I^\perp}}(x) |x|^{-1}$  and

$$\iint_{x-y \in K_I} \frac{\psi(x) \psi(y)}{|x-y|} dx dy = \int (k * \psi) \psi dx.$$

Since  $\psi$  is a real-valued Schwartz function, we have  $\int (k * \psi) \psi dx = \int \widehat{k} |\widehat{\psi}|^2 dx$ , so it follows that

$$\iint_{x-y \in K_I} \frac{\psi(x) \psi(y)}{|x-y|} dx dy = \int (k * \psi) \psi dx = \int \widehat{k} |\widehat{\psi}|^2 dx = \int_{K_{I^\perp}} |x|^{-1} |\widehat{\psi}(x)|^2 dx.$$

Finally, by Plancherel and the identity (4.14) applied to  $\psi$ , we have

$$\int_{I^\perp} \|P_\theta \psi\|_2^2 d\theta = \int_{I^\perp} \int_{L_\theta} |\widehat{P_\theta \psi}|^2 d\mathcal{L}^1 d\theta = \int_{I^\perp} \int_{-\infty}^{\infty} |\widehat{\psi}(r e^{i\theta})|^2 dr d\theta \quad (4.15)$$

$$= \int_{K_{I^\perp}} |x|^{-1} |\widehat{\psi}(x)|^2 dx, \quad (4.16)$$

which completes the proof.  $\square$

**Corollary 4.3.3.** Let  $\mu$  be a finite Borel measure in  $\mathbb{C}$  with compact support and  $I \subset [0, \pi]$  an arbitrary open set. Then we have

$$\iint_{x-y \in K_I \setminus \{0\}} \frac{1}{|x-y|} d\mu(x) d\mu(y) \leq \int_{I^\perp} \|P_\theta \mu\|_2^2 d\theta,$$

where  $K_I$  is the cone  $K_I = \{r e^{i\theta} : r \in \mathbb{R}, \theta \in I\}$  and  $I^\perp = I + \frac{\pi}{2} \pmod{\pi}$ .

*Proof.* We assume that the integral on the right hand side above is finite. Fix  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  a  $C^\infty$  radial bump function and for  $\varepsilon > 0$ , let  $\phi_\varepsilon(x) = \frac{1}{\varepsilon^2} \phi(\frac{x}{\varepsilon})$ . Denote  $\mu_\varepsilon = \mu * \phi_\varepsilon$ . It is straightforward to check that the identity (4.15) holds both for  $\mu$  and  $\mu_\varepsilon$ . Then, by the

dominated convergence theorem we deduce that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{I^\perp} \|P_\theta \mu_\varepsilon\|_2^2 d\theta &= \lim_{\varepsilon \rightarrow 0} \int_{K_{I^\perp}} |x|^{-1} |\widehat{\mu}(x) \widehat{\phi}(\varepsilon x)|^2 dx \\ &= \int_{K_{I^\perp}} |x|^{-1} |\widehat{\mu}(x)|^2 dx = \int_{I^\perp} \|P_\theta \mu\|_2^2 d\theta. \end{aligned}$$

Hence, by Proposition 4.3.2 we have

$$\limsup_{\varepsilon \rightarrow 0} \iint_{x-y \in K_I} \frac{1}{|x-y|} d\mu_\varepsilon(x) d\mu_\varepsilon(y) \leq \int_{I^\perp} \|P_\theta \mu\|_2^2 d\theta.$$

So it suffices to show that

$$\iint_{x-y \in K_I \setminus \{0\}} \frac{1}{|x-y|} d\mu(x) d\mu(y) \leq \limsup_{\varepsilon \rightarrow 0} \iint_{x-y \in K_I} \frac{1}{|x-y|} d\mu_\varepsilon(x) d\mu_\varepsilon(y). \quad (4.17)$$

To this end, consider an arbitrary non-negative continuous, compactly supported, function  $f(x) \leq \chi_{K_I}(x) |x|^{-1}$ . We have that

$$\int f * \mu_\varepsilon d\mu_\varepsilon = \int (f * \mu * \phi_\varepsilon * \phi_\varepsilon) d\mu.$$

Since  $f * \mu$  is compactly supported and continuous,  $f * \mu * \phi_\varepsilon * \phi_\varepsilon$  converges uniformly to  $f * \mu$  as  $\varepsilon \rightarrow 0$ , and thus

$$\int f * \mu d\mu = \lim_{\varepsilon \rightarrow 0} \int f * \mu_\varepsilon d\mu_\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} \iint_{x-y \in K_I} \frac{1}{|x-y|} d\mu_\varepsilon(x) d\mu_\varepsilon(y).$$

As this holds uniformly for any continuous compactly supported function  $f$  such that  $0 \leq f \leq \chi_{K_I}(x) |x|^{-1}$  and  $K_I \setminus \{0\}$  is open, (4.17) follows by the monotone convergence theorem.  $\square$

### 4.3.2 The higher dimensional case

Let  $G(d, n)$  denote the Grassmanian space of  $n$ -dimensional linear subspaces of  $\mathbb{R}^d$ . Let  $\gamma_{d,n}$  denote the natural probability measure on  $G(d, n)$ . For  $V \in G(d, n)$ , let  $P_V : \mathbb{R}^d \rightarrow V$  be the orthogonal projection onto  $V$ . We define a metric on  $G(d, n)$  by  $d(V, W) = \|P_V - P_W\|$ , where  $\|\cdot\|$  denotes the operator norm.

Recall that for any Schwartz function  $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$  and any  $V \in G(d, n)$ , we have

$$\widehat{P_V \phi}(x) = \widehat{\phi}(x) \quad \text{for all } x \in V, \quad (4.18)$$

where  $(P_V \phi)(x) = \int_{x+V^\perp} \phi d\mathcal{L}^{d-n}$  and  $\widehat{P_V \phi}$  denotes the  $n$ -dimensional Fourier transform on  $V$ . The proof is identical to the proof of (4.14).

The following is the higher dimensional analogue of Lemma 4.3.1.

**Lemma 4.3.4.** *Let  $B \subset G(d, n)$ . Let  $\sigma, \nu$  be measures on  $\mathbb{R}^d$  given by*

$$\int f d\sigma = \int_B \int_V f d\mathcal{L}^n d\gamma_{d,n}(V), \quad (4.19)$$

$$\int f d\nu = \int_B \int_{V^\perp} f d\mathcal{L}^{d-n} d\gamma_{d,n}(V). \quad (4.20)$$

Then the (distributional) Fourier transform of  $\sigma$  is  $\nu$ .

*Proof.* Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$  be a Schwartz function. Then, as in the proof of Lemma 4.3.1,

$$\begin{aligned} \int \widehat{\phi} d\sigma &= \int_B \int_V \widehat{\phi} d\mathcal{L}^n d\gamma_{d,n}(V) = \int_B \int_V \widehat{P_V \phi} d\mathcal{L}^n d\gamma_{d,n}(V) \\ &= \int_B P_V \phi(0) d\gamma_{d,n}(V) = \int_B \int_{V^\perp} \phi d\mathcal{L}^{d-n} d\gamma_{d,n}(V) = \int \phi d\nu. \quad \square \end{aligned}$$

**Lemma 4.3.5.** *Let  $B \subset G(d, n)$ . Let  $\sigma$  be given by (4.19). Then  $\text{supp } \sigma \subset \bigcup_{V \in \bar{B}} V$ ,  $\sigma \ll \mathcal{L}^d$ , and  $\frac{d\sigma}{dx}(x) \leq \frac{c(d,n)}{|x|^{d-n}}$ .*

*Proof.* From the definition of  $\sigma$ , it is immediate that  $\text{supp } \sigma \subset \bigcup_{V \in \bar{B}} V$ . The next two

properties follow from the following identity:

$$\int_{G(d,n)} \int_V f d\mathcal{L}^n d\gamma_{d,n}(V) = c(d,n) \int_{\mathbb{R}^d} \frac{f(x)}{|x|^{d-n}} d\mathcal{L}^d.$$

For a proof of this identity, see (24.2) in [40].  $\square$

**Lemma 4.3.6.** *Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a Schwartz function. Then for any set  $B \subset G(d,n)$ , we have*

$$\int_B \|P_V \psi\|_2^2 d\gamma_{d,n}(V) = \iint \frac{d\nu}{dx}(x-y) \psi(x) \psi(y) dx dy.$$

*Proof.* The proof is identical to the proof of Proposition 4.3.2.  $\square$

To make Lemma 4.3.6 more useful, we obtain a lower bound on  $\frac{d\nu}{dx}$  via the following three lemmas.

**Lemma 4.3.7.** *For all  $V \in G(d,n)$  and for  $\delta \lesssim_{d,n} 1$ ,*

$$\gamma_{d,n}(B(V, \delta)) \approx_{d,n} \delta^{n(d-n)}$$

*Proof.* This is Proposition 4.1 of [23].  $\square$

**Lemma 4.3.8.** *Let  $x \in \mathbb{R}^d \setminus \{0\}$ . Then  $G_x := \{V \in G(d,n) : x \in V\}$  and  $G(d-1, n-1)$  are isomorphic as metric spaces.*

*Proof.* Let  $A : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$  be a linear map satisfying  $A^T A = \text{id}$  and whose image is the orthogonal complement of  $x$ . Consider the map  $\Psi : G(d-1, n-1) \rightarrow G_x$  given by  $V \mapsto \text{span}(AV, x)$ . We will show  $\Psi$  is an isometry. First we make two observations.

1. For any  $V \in G_x$ , we have  $P_V x = x$ .
2. For any  $z \in \mathbb{R}^{d-1}$  and  $V \in G(d-1, n-1)$ , we have  $P_{\Psi V} A z = A P_V z$ .

Let  $V, W \in G(d-1, n-1)$ . We need to show

$$\|P_{\Psi V} - P_{\Psi W}\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} = \|P_V - P_W\|_{\mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}}.$$



Let  $y \in \mathbb{R}^d$  and write  $y = \lambda x + Az$ , where  $\lambda \in \mathbb{R}$  and  $z \in \mathbb{R}^{d-1}$ . Note that  $z = A^T y$ . Using the two observations above, we have

$$(P_{\Psi V} - P_{\Psi W})y = (P_{\Psi V} - P_{\Psi W})Az = A(P_V - P_W)z = A(P_V - P_W)A^T y.$$

Hence  $P_{\Psi V} - P_{\Psi W} = A(P_V - P_W)A^T$ . Since  $\|A\|_{\mathbb{R}^{d-1} \rightarrow \mathbb{R}^d} = \|A^T\|_{\mathbb{R}^d \rightarrow \mathbb{R}^{d-1}} = 1$ , it follows that  $\|P_{\Psi V} - P_{\Psi W}\| \leq \|P_V - P_W\|$ . To show the reverse inequality, note that for any  $z \in \mathbb{R}^{d-1}$ , we have

$$\begin{aligned} |(P_V - P_W)z| &= |A(P_V - P_W)z| \\ &= |(P_{\Psi V} - P_{\Psi W})Az| \\ &\leq \|P_{\Psi V} - P_{\Psi W}\| |Az| \\ &= \|P_{\Psi V} - P_{\Psi W}\| |z|, \end{aligned}$$

which implies  $\|P_V - P_W\| \leq \|P_{\Psi V} - P_{\Psi W}\|$ . □

**Lemma 4.3.9.** *Let  $B = B(V_0, r) \subset G(d, n)$ . Let  $\sigma$  be given by (4.19). Then there is a  $c = c(d, n, r) > 0$  such that*

$$\frac{d\sigma}{dx}(x) \geq \frac{c}{|x|^{d-n}} \quad \text{on the cone } \bigcup_{V \in B(V_0, \frac{1}{2}r)} V.$$

*Proof.* Note that  $\sigma(\lambda A) = \lambda^n \sigma(A)$ , which implies  $\frac{d\sigma}{dx}(\lambda x) = \lambda^{n-d} \frac{d\sigma}{dx}(x)$ . Hence, it suffices to show  $\frac{d\sigma}{dx}(x) \gtrsim_{d,n,r} 1$  for all  $x \in \bigcup_{V \in B(V_0, \frac{1}{2}r)} V$  with  $|x| = 1$ .

Fix  $x \in \bigcup_{V \in B(V_0, \frac{1}{2}r)} V$  with  $|x| = 1$ . Let  $G_x = \{V \in G(d, n) : x \in V\}$ , and let  $(G_x)^\delta \subset G(d, n)$  denote the  $\delta$ -neighborhood of  $G_x$ .

We claim that

$$\sigma(B(x, s)) \gtrsim s^n \gamma_{d,n}((G_x)^{s/2} \cap B(V_0, r)) \quad \text{for all } s > 0. \quad (4.21)$$

To see this, note that if  $V \in (G_x)^{s/2}$ , then there is some  $W \in G_x$  such that  $d(V, W) < \frac{s}{2}$ . Then  $|x - P_V x| = |P_W x - P_V x| \leq \|P_W - P_V\| \leq d(W, V) < \frac{s}{2}$ , so  $\mathcal{L}^n(V \cap B(x, s)) \gtrsim_{d,n} s^n$ . Hence,

$$\begin{aligned} \sigma(B(x, s)) &= \int_{B(V_0, r)} \mathcal{L}^n(V \cap B(x, s)) d\gamma_{d,n}(V) \\ &\gtrsim_{d,n} s^n \gamma_{d,n}((G_x)^{s/2} \cap B(V_0, r)), \end{aligned}$$

which proves (4.21).

Next, we bound  $\gamma_{d,n}((G_x)^{s/2} \cap B(V_0, r))$  from below. Fix a  $V_1 \in G_x \cap B(V_0, \frac{1}{2}r)$ . (Since  $x \in \bigcup_{V \in B(V_0, \frac{1}{2}r)} V$ ,  $V_1$  exists.)

Suppose  $s < r$ . Let  $F_s$  be a maximal  $s$ -separated subset of  $G_x \cap B(V_1, \frac{1}{2}r)$ . It follows from the maximality of  $F_s$  that

$$G_x \cap B(V_1, \frac{1}{2}r) \subset \bigcup_{W \in F_s} (G_x \cap B(W, s)). \quad (4.22)$$

By (4.22), Lemma 4.3.8 and Lemma 4.3.7 applied to  $G(d-1, n-1)$ , it follows that for  $s \lesssim_{d,n,r} 1$ ,

$$\#F_s \gtrsim_{d,n,r} s^{-(n-1)(d-n)}. \quad (4.23)$$

Next, observe that the balls  $\{B(W, \frac{s}{2})\}_{W \in F_s}$  are pairwise disjoint and contained in  $(G_x)^{s/2} \cap B(V_0, r)$ , so

$$\gamma_{d,n}((G_x)^{s/2} \cap B(V_0, r)) \geq \sum_{W \in F_s} \gamma_{d,n}(B(W, \frac{s}{2})) \gtrsim_{d,n,r} s^{d-n} \quad \text{for } s \lesssim_{d,n,r} 1, \quad (4.24)$$

where we used (4.23) and Lemma 4.3.7 in the last inequality. Finally, (4.21) and (4.24) imply  $\frac{d\sigma}{dx}(x) \gtrsim_{d,n,r} 1$ , as desired.  $\square$

**Corollary 4.3.10.** Let  $V_0 \in G(d, n)$  and  $s > 0$ . Then there exist constants  $\lambda, c > 1$  such that for any Schwartz function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\begin{aligned} c^{-1} \iint_{x-y \in K(V_0^\perp, \lambda^{-1}s)} \frac{\psi(x) \psi(y)}{|x-y|^n} dx dy &\leq \int_{B(V_0, s)} \|P_V \psi\|_2^2 d\gamma_{d,n}(V) \\ &\leq c \iint_{x-y \in K(V_0^\perp, \lambda s)} \frac{\psi(x) \psi(y)}{|x-y|^n} dx dy. \end{aligned}$$

*Proof.* By Lemma 4.3.5 and Lemma 4.3.9 (applied to  $G(d, d-n)$ ),

$$c_1 \frac{\chi_{K_1}(x)}{|x|^n} \leq \frac{d\nu}{dx}(x) \leq c_2 \frac{\chi_{K_2}(x)}{|x|^n},$$

where  $K_1 = \bigcup_{V \in B(V_0, \frac{1}{2}s)} V^\perp$ ,  $K_2 = \bigcup_{V \in B(V_0, s)} V^\perp$ .

For  $\lambda$  sufficiently large, we have  $K(V_0^\perp, \lambda^{-1}s) \subset K_1 \subset K_2 \subset K(V_0^\perp, \lambda s)$ . Thus,

$$c_1 \frac{\chi_{K(V_0^\perp, \lambda^{-1}s)}(x)}{|x|^n} \leq \frac{d\nu}{dx}(x) \leq c_2 \frac{\chi_{K(V_0^\perp, \lambda s)}(x)}{|x|^n},$$

so this corollary follows easily from Lemma 4.3.6. □

**Corollary 4.3.11.** Let  $V_0 \in G(d, n)$  and  $s > 0$ . Then there exist constants  $\lambda, c > 1$  such that for any finite Borel measure  $\mu$  in  $\mathbb{R}^d$ ,

$$\iint_{x-y \in K(V_0^\perp, s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} \leq c \int_{B(V_0, \lambda s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V).$$

The proof of this corollary follows from Corollary 4.3.10, along the same lines as the one of Corollary 4.3.3, and so we skip it.

*Remark 4.3.12.* The converse inequality

$$\int_{B(V_0, s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) \leq c \iint_{x-y \in K(V_0^\perp, \lambda s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} \quad (4.25)$$

does not hold for arbitrary measures. Indeed, in the case  $d = 2$ ,  $n = 1$ , consider a segment  $L$  through the origin, and let  $K(V_0^\perp, \lambda s)$  be a cone such that  $L$  is not contained in the closure of the cone. Then with  $\mu = \mathcal{H}^1|_L$ , the integral on the left hand side is positive (and finite) while the one on the right hand side is zero.

However, if we modify (4.25) by adding an additional term to the right-hand side, we can make the inequality true. This result not needed in our paper, but we include the details in Appendix 4.A.

#### 4.4 The dyadic lattice of David and Mattila

Now we will introduce the dyadic lattice of cubes with small boundaries of David-Mattila associated with a Radon measure  $\mu$ . This lattice has been constructed in [15, Theorem 3.2]. Its properties are summarized in the next lemma.

**Lemma 4.4.1** (David, Mattila). *Let  $\mu$  be a compactly supported Radon measure in  $\mathbb{R}^d$ . Consider two constants  $C_0 > 1$  and  $A_0 > 5000 C_0$  and denote  $W = \text{supp } \mu$ . Then there exists a sequence of partitions of  $W$  into Borel subsets  $Q$ ,  $Q \in \mathcal{D}_{\mu,k}$ , with the following properties:*

- *For each integer  $k \geq 0$ ,  $W$  is the disjoint union of the “cubes”  $Q$ ,  $Q \in \mathcal{D}_{\mu,k}$ , and if  $k < l$ ,  $Q \in \mathcal{D}_{\mu,l}$ , and  $R \in \mathcal{D}_{\mu,k}$ , then either  $Q \cap R = \emptyset$  or else  $Q \subset R$ .*
- *The general position of the cubes  $Q$  can be described as follows. For each  $k \geq 0$  and each cube  $Q \in \mathcal{D}_{\mu,k}$ , there is a ball  $B(Q) = B(x_Q, r(Q))$  such that*

$$x_Q \in W, \quad A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k},$$

$$W \cap B(Q) \subset Q \subset W \cap 28 B(Q) = W \cap B(x_Q, 28r(Q)),$$

and

the balls  $5B(Q)$ ,  $Q \in \mathcal{D}_{\mu,k}$ , are disjoint.

- The cubes  $Q \in \mathcal{D}_{\mu,k}$  have small boundaries. That is, for each  $Q \in \mathcal{D}_{\mu,k}$  and each integer  $l \geq 0$ , set

$$N_l^{ext}(Q) = \{x \in W \setminus Q : \text{dist}(x, Q) < A_0^{-k-l}\},$$

$$N_l^{int}(Q) = \{x \in Q : \text{dist}(x, W \setminus Q) < A_0^{-k-l}\},$$

and

$$N_l(Q) = N_l^{ext}(Q) \cup N_l^{int}(Q).$$

Then

$$\mu(N_l(Q)) \leq (C^{-1}C_0^{-3d-1}A_0)^{-l} \mu(90B(Q)). \quad (4.26)$$

- Denote by  $\mathcal{D}_{\mu,k}^{db}$  the family of cubes  $Q \in \mathcal{D}_{\mu,k}$  for which

$$\mu(100B(Q)) \leq C_0 \mu(B(Q)). \quad (4.27)$$

We have that  $r(Q) = A_0^{-k}$  when  $Q \in \mathcal{D}_{\mu,k} \setminus \mathcal{D}_{\mu,k}^{db}$  and

$$\mu(100B(Q)) \leq C_0^{-l} \mu(100^{l+1}B(Q)) \text{ for all } l \geq 1 \text{ with } 100^l \leq C_0 \text{ and } Q \in \mathcal{D}_{\mu,k} \setminus \mathcal{D}_{\mu,k}^{db}. \quad (4.28)$$

We use the notation  $\mathcal{D}_\mu = \bigcup_{k \geq 0} \mathcal{D}_{\mu,k}$ . Observe that the families  $\mathcal{D}_{\mu,k}$  are only defined for  $k \geq 0$ . So the diameter of the cubes from  $\mathcal{D}_\mu$  are uniformly bounded from above. For  $Q \in \mathcal{D}_\mu$ , we set  $\mathcal{D}_\mu(Q) = \{P \in \mathcal{D}_\mu : P \subset Q\}$ . Given  $Q \in \mathcal{D}_{\mu,k}$ , we denote  $J(Q) = k$ , and

we set  $\ell(Q) = 56 C_0 A_0^{-k}$  and we call it the side length of  $Q$ . Notice that

$$C_0^{-1} \ell(Q) \leq \text{diam}(28B(Q)) \leq \ell(Q).$$

Observe that  $r(Q) \approx \text{diam}(Q) \approx \ell(Q)$ . Also we call  $x_Q$  the center of  $Q$ , and the cube  $Q' \in \mathcal{D}_{\mu, k-1}$  such that  $Q' \supset Q$  the parent of  $Q$ . We set  $B_Q = 28B(Q) = B(x_Q, 28r(Q))$ , so that

$$W \cap \frac{1}{28} B_Q \subset Q \subset B_Q.$$

We assume  $A_0$  big enough so that the constant  $C^{-1} C_0^{-3d-1} A_0$  in (4.26) satisfies

$$C^{-1} C_0^{-3d-1} A_0 > A_0^{1/2} > 10.$$

Then we deduce that, for all  $0 < \lambda \leq 1$ ,

$$\begin{aligned} \mu(\{x \in Q : \text{dist}(x, W \setminus Q) \leq \lambda \ell(Q)\}) + \mu(\{x \in 3.5B_Q \setminus Q : \text{dist}(x, Q) \leq \lambda \ell(Q)\}) \\ \leq c \lambda^{1/2} \mu(3.5B_Q). \end{aligned} \tag{4.29}$$

We denote  $\mathcal{D}_\mu^{db} = \bigcup_{k \geq 0} \mathcal{D}_{\mu, k}^{db}$ . Note that, in particular, from (4.27) it follows that

$$\mu(3B_Q) \leq \mu(100B(Q)) \leq C_0 \mu(Q) \quad \text{if } Q \in \mathcal{D}_\mu^{db}. \tag{4.30}$$

For this reason we will call the cubes from  $\mathcal{D}_\mu^{db}$  doubling. Given  $Q \in \mathcal{D}_\mu$ , we set  $\mathcal{D}_\mu^{db}(Q) = \mathcal{D}_\mu^{db} \cap \mathcal{D}_\mu(Q)$ .

As shown in [15, Lemma 5.28], every cube  $R \in \mathcal{D}_\mu$  can be covered  $\mu$ -a.e. by a family of doubling cubes:

**Lemma 4.4.2.** *Let  $R \in \mathcal{D}_\mu$ . Suppose that the constants  $A_0$  and  $C_0$  in Lemma 4.4.1 are chosen suitably. Then there exists a family of doubling cubes  $\{Q_i\}_{i \in I} \subset \mathcal{D}_\mu^{db}$ , with  $Q_i \subset R$*

for all  $i$ , such that their union covers  $\mu$ -almost all  $R$ .

The following result is proved in [15, Lemma 5.31].

**Lemma 4.4.3.** *Suppose that the constants  $A_0$  and  $C_0$  in Lemma 4.4.1 are chosen suitably. Let  $R \in \mathcal{D}_\mu$  and let  $Q \subset R$  be a cube such that all the intermediate cubes  $S$ ,  $Q \subsetneq S \subsetneq R$  are non-doubling (i.e. belong to  $\mathcal{D}_\mu \setminus \mathcal{D}_\mu^{db}$ ). Then*

$$\mu(100B(Q)) \leq A_0^{-10d(J(Q)-J(R)-1)} \mu(100B(R)). \quad (4.31)$$

We remark that for the preceding two lemmas to hold, we need to choose  $A_0$  much larger than  $C_0$ . From now on we assume this condition to hold.

Given a ball (or an arbitrary set)  $B \subset \mathbb{R}^d$  and a fixed  $n \geq 1$ , we consider its  $n$ -dimensional density:

$$\Theta_\mu(B) = \frac{\mu(B)}{\text{diam}(B)^n}.$$

From the preceding lemma we deduce:

**Lemma 4.4.4.** *Let  $Q, R \in \mathcal{D}_\mu$  be as in Lemma 4.4.3. Then*

$$\Theta_\mu(100B(Q)) \leq (C_0 A_0)^d A_0^{-9d(J(Q)-J(R)-1)} \Theta_\mu(100B(R))$$

and

$$\sum_{S \in \mathcal{D}_\mu: Q \subset S \subset R} \Theta_\mu(100B(S)) \leq c \Theta_\mu(100B(R)),$$

with  $c$  depending on  $C_0$  and  $A_0$ .

For the easy proof, see [56, Lemma 4.4], for example.

We will also need the following technical result.

**Lemma 4.4.5.** *Let  $R \in \mathcal{D}_\mu$  such that  $\mu(2B_R) \leq C_1 \mu(R)$ . Then there exists another cube  $Q \subsetneq R$  from  $\mathcal{D}_\mu^{db}$  such that*

$$\mu(Q) \approx \mu(R) \quad \text{and} \quad \ell(Q) \approx \ell(R),$$

*with the implicit constants depending on  $C_1$ .*

*Proof.* Suppose that  $R \in \mathcal{D}_{\mu,k}$ . For some  $N > 1$  to be fixed later, denote by  $I_N$  the family cubes from  $\mathcal{D}_{\mu,k+N}$  which are contained in  $R$ . Recall that the balls  $B(Q)$ ,  $Q \in I_N$ , are disjoint and that their radii satisfy

$$A_0^{-k-N} \leq r(Q) \leq C_0 A_0^{-k-N}.$$

All the balls from  $I_N$  intersect  $R$  and are contained in  $2B_R$  for  $N$  big enough. Thus we have

$$\#I_N \cdot C_d A_0^{(-k-N)d} \leq \mathcal{L}^d \left( \bigcup_{Q \in I_N} B(Q) \right) \leq C_d (2C_0 A_0^{-k})^d,$$

and so

$$\#I_N \leq 2^d C_0^d A_0^{Nd}.$$

Therefore, the cube  $Q' \in I_N$  with maximal measure satisfies

$$\mu(Q') \geq 2^{-d} C_0^{-d} A_0^{-Nd} \mu(R). \tag{4.32}$$

We claim now that if  $N$  is big enough then there exists some cube  $Q \in \mathcal{D}_\mu^{db}$  such that  $Q' \subset Q \subsetneq R$ . Indeed, if such cube  $Q$  does not exist, then denoting by  $R'$  the son of  $R$  that



contains  $Q'$ , we deduce

$$\begin{aligned} \mu(Q') &\leq \mu(100B(Q')) \leq A_0^{-10d(N-2)} \mu(100B(R')) \leq A_0^{-10d(N-2)} \mu(2B_R) \\ &\leq C_1 A_0^{-10d(N-2)} \mu(R), \end{aligned}$$

taking into account that  $100B(R') \subset 2B_R$ . For  $N$  big enough, this estimate contradicts (4.32), and thus the cube  $Q$  mentioned above exists. It is clear that this satisfies the estimates  $\mu(Q) \approx \mu(R)$  and  $\ell(Q) \approx \ell(R)$ , as wished.  $\square$

## 4.5 The corona decomposition

Let  $\mu$  a Borel measure in  $\mathbb{R}^d$  satisfying the growth condition

$$\mu(B(x, r)) \leq c_0 r^n \quad \text{for all } x \in \mathbb{R}^d, r > 0. \quad (4.33)$$

We consider the dyadic lattice  $\mathcal{D}_\mu$  of David-Mattila associated to  $\mu$ , and we assume that  $\text{supp } \mu \in \mathcal{D}_\mu$  is the biggest cube in this lattice. (To this end, we assume  $\mathcal{D}_{\mu, k}$  to be defined for  $k \geq k_0$ , with an appropriate  $k_0$ .) Sometimes we will also denote by  $R_0$  the initial cube  $\text{supp } \mu$ . We allow all constants  $c, C$ , and other implicit constants to depend on  $n, d$ , and the parameters in the definition of the David-Mattila cubes.

Let  $\text{Top}$  be a family of cubes from  $\mathcal{D}_\mu^{db}$  to be fixed below, with  $R_0 \in \text{Top}$ . For every  $R \in \text{Top}$ , denote by  $\text{Next}(R)$  the family of maximal cubes  $Q \in \text{Top}$  that are contained in  $R$ , and by  $\text{Tr}(R)$  the family of cubes  $Q \in \mathcal{D}_\mu$  that are contained in  $R$  and not contained in any  $Q' \in \text{Next}(R)$ . Then, define

$$\text{Good}(R) = R \setminus \bigcup_{Q \in \text{Next}(R)} Q,$$

and for  $Q, S \in \mathcal{D}_\mu$  with  $Q \subset S$ ,

$$\delta_\mu(Q, S) = \int_{2B_S \setminus 2B_Q} \frac{1}{|y - x_Q|^n} d\mu(y).$$

The next lemma is the main tool which will allow us to connect the energy

$$\iint_{x-y \in K} \frac{1}{|x - y|^n} d\mu(x) d\mu(y)$$

to the curvature of  $\mu$ .

**Lemma 4.5.1** (Corona decomposition). *Given  $V_0 \in G(d, d - n)$  and  $s > 0$ , consider the cone  $K := K(V_0, s) \subset \mathbb{R}^d$ . Let  $\mu$  be a Borel measure in  $\mathbb{R}^d$  satisfying the growth condition (4.33). There exists a family  $\text{Top} \subset \mathcal{D}_\mu^{db}$  as above such that, for all  $R \in \text{Top}$ , there exists an  $n$ -dimensional Lipschitz graph  $\Gamma_R$  with the slope depending only on  $s$  such that:*

- (a)  $\mu$ -almost all  $\text{Good}(R)$  is contained in  $\Gamma_R$ .
- (b) For all  $Q \in \text{Next}(R)$  there exists another cube  $\tilde{Q} \in \mathcal{D}_\mu$  such that  $\delta_\mu(Q, \tilde{Q}) \leq c \Theta_\mu(2B_R)$  and  $2B_{\tilde{Q}} \cap \Gamma_R \neq \emptyset$ .
- (c) For all  $Q \in \text{Tr}(R)$ ,  $\Theta_\mu(2B_Q) \leq c \Theta_\mu(2B_R)$ .

Furthermore, the following packing condition holds:

$$\sum_{R \in \text{Top}} \Theta_\mu(2B_R) \mu(R) \lesssim \mu(R_0) + \iint_{x-y \in K} \frac{1}{|x - y|^n} d\mu(x) d\mu(y), \quad (4.34)$$

with the implicit constant depending only on  $c_0$  and the aperture  $s$  of the cone  $K$ .

The next Sections 4.6-4.9 are devoted to the proof of this lemma. In these sections we assume that  $\mu$  is a measure in  $\mathbb{R}^d$  that satisfies (4.33) and that  $K = K(V_0, s)$  is a cone, with  $V_0 \in G(d, d - n)$ ,  $s > 0$ , such that

$$\iint_{x-y \in K} \frac{1}{|x - y|^n} d\mu(x) d\mu(y) < \infty.$$

## 4.6 The construction of an approximate Lipschitz graph

From now on we will allow all the constants denoted by  $c$  or  $C$ , and all the implicit constants in the relations  $\lesssim$  and  $\approx$  to depend on the parameters  $C_0$  and  $A_0$  of the David-Mattila lattice.

### 4.6.1 The stopping cubes

We consider constants  $A \gg 1$ ,  $0 < \varepsilon \ll \tau \ll 1$ , and  $0 < \eta \ll 1$  to be fixed below. For  $Q \in \mathcal{D}_\mu$ , we denote

$$\mathcal{E}_\mu(Q) = \frac{1}{\mu(Q)} \iint_{\substack{x \in 2B_Q \\ x-y \in K \\ \eta \ell(Q) \leq |x-y| \leq \eta^{-1} \ell(Q)}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y). \quad (4.35)$$

Observe that  $\mathcal{E}_\mu(Q)$  “scales” like  $\Theta_\mu(2B_Q)$ . (That is, both quantities have the same “physical dimensions” – two factors of  $\mu$  in the numerator and one factor of length in the denominator.)

Given a cube  $R \in \mathcal{D}_\mu^{db}$ , we consider the following families of cubes:

- The high density family  $\text{HD}_0(R)$ , which is made up of the cubes  $Q \in \mathcal{D}_\mu^{db}(R)$  which satisfy  $\Theta_\mu(2B_Q) \geq A \Theta_\mu(2B_R)$ .
- The low density family  $\text{LD}_0(R)$ , which is made up of the cubes  $Q \in \mathcal{D}_\mu(R)$  which satisfy  $\Theta_\mu(2B_Q) \leq \tau \Theta_\mu(2B_R)$ .
- The family  $\text{BCE}_0(R)$  of cubes with big conical Riesz energy, which is made up of the cubes  $Q \in \mathcal{D}_\mu(R) \setminus (\text{HD}_0(R) \cup \text{LD}_0(R))$  such that

$$\sum_{S \in \mathcal{D}_\mu: Q \subset S \subset R} \mathcal{E}_\mu(S) \geq \varepsilon \Theta_\mu(2B_R).$$

We denote by  $\text{Stop}(R)$  the maximal (and thus disjoint) subfamily from  $\text{HD}_0(R) \cup \text{LD}_0(R) \cup$

$\text{BCE}_0(R)$ , and we set

$$\text{HD}(R) = \text{HD}_0(R) \cap \text{Stop}(R)$$

$$\text{LD}(R) = \text{LD}_0(R) \cap \text{Stop}(R)$$

$$\text{BCE}(R) = \text{BCE}_0(R) \cap \text{Stop}(R).$$

Notice that the cubes from  $\text{HD}(R)$  are doubling, while the cubes from  $\text{LD}(R)$  and  $\text{BCE}(R)$  may be non-doubling.

We let  $\text{Tree}(R)$  denote the subfamily of the cubes from  $\mathcal{D}_\mu(R)$  which are not strictly contained in any cube from  $\text{Stop}(R)$ . (Note that it is possible for  $\text{Tree}(R)$  to contain only the cube  $R$  itself.)

#### 4.6.2 Preliminary estimates

In this subsection we assume that  $R \in \mathcal{D}_\mu^{db}$ . The following statement is an immediate consequence of the construction of  $\text{Stop}(R)$  and  $\text{Tree}(R)$ .

**Lemma 4.6.1.** *If  $Q \in \mathcal{D}_\mu$  and  $Q \in \text{Tree}(R) \setminus \text{Stop}(R)$ , then*

$$\tau \Theta_\mu(2B_R) \leq \Theta_\mu(2B_Q) \leq c A \Theta_\mu(2B_R).$$

*Further, the second inequality also holds if  $Q \in \text{Stop}(R)$ .*

*Proof.* The fact that  $\Theta_\mu(2B_Q) \geq \tau \Theta_\mu(2B_R)$  for all  $Q \in \text{Tree}(R) \setminus \text{Stop}(R)$  follows from the definition of the family  $\text{LD}(R)$ . To check that for such cubes it also holds  $\Theta_\mu(2B_Q) \leq c A \Theta_\mu(2B_R)$ , note first that this holds if  $Q \in \mathcal{D}_\mu^{db}(R)$  (with  $c = 1$ ). If  $Q \notin \mathcal{D}_\mu^{db}(R)$ , let  $P(Q) \in \mathcal{D}_\mu^{db}(R)$  be the smallest doubling cube that contains  $Q$  (such a cube exists because  $R \in \mathcal{D}_\mu^{db}(R)$ ), so that  $\Theta_\mu(2B_{P(Q)}) \leq A \Theta_\mu(2B_R)$ . For  $j \geq 0$ , denote by  $Q_j$  the  $j$ -th ancestor of  $Q$  (i.e.  $Q_j \in \mathcal{D}_\mu$  is such that  $Q \subset Q_j$  and  $\ell(Q_j) = A_0^j \ell(Q)$ ). Let  $i \geq 0$  be such that

$P(Q) = Q_i$ . Since the cubes  $Q_1, \dots, Q_{i-1}$  do not belong to  $\mathcal{D}_\mu^{db}$ , by Lemma 4.4.4 we have

$$\begin{aligned} \Theta_\mu(2B_Q) &\lesssim \Theta_\mu(100B(Q)) \leq c(C_0, A_0) A_0^{-9di} \Theta_\mu(100B(Q_i)) \approx A_0^{-9di} \Theta_\mu(2B_{Q_i}) \\ &\lesssim A \Theta_\mu(2B_R). \end{aligned}$$

Finally, if  $Q \in \text{Stop}(R)$ , we just need to take into account that  $\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_{\widehat{Q}})$ , where  $\widehat{Q}$  is the parent of  $Q$ .  $\square$

The next result is essentially proven in Lemma 6.3 from [2].

**Lemma 4.6.2.** *There is some constant  $C(A, \tau) > 0$  big enough so that for any cube  $Q \in \text{Tree}(R)$  there exists some cube  $Q' \supset Q$  such that  $Q' \in \mathcal{D}_\mu^{db} \cap \text{Tree}(R)$  and  $\ell(Q') \leq C(A, \tau) \ell(Q)$ . If  $R \notin \text{Stop}(R)$ , then we can take  $Q' \in \text{Tree}(R) \setminus \text{Stop}(R)$ .*

### 4.6.3 A key estimate

For  $x \in \mathbb{R}^d$  and  $\lambda > 0$  we denote

$$K_x^\lambda = K(x, V_0, \lambda s),$$

Given  $Q \in \mathcal{D}_\mu$ , we also set

$$K_Q^\lambda = \bigcup_{x \in 2B_Q} K_x^\lambda.$$

In the case  $\lambda = 1$ , we write  $K_x$  and  $K_Q$  instead of  $K_x^1$  and  $K_Q^1$ .

**Lemma 4.6.3.** *There exists some constant  $M > 1$  depending only on  $s$  such that the following are true.*

- (a) *Suppose  $Q \in \text{Tree}(R)$  and  $Q' \in \mathcal{D}_\mu(R)$  satisfy  $Q' \cap (K_Q^{1/2} \setminus MB_Q) \neq \emptyset$  and  $\ell(Q') \leq \frac{1}{M} \text{dist}(Q', Q)$ . Then  $Q' \notin \text{Tree}(R)$ .*

(b) Let  $J \subset \text{Tree}(R)$  be a family of pairwise disjoint cubes. Then

$$\mu\left(\bigcup_{Q \in J} K_Q^{1/2} \cap (R \setminus MB_Q)\right) \leq C(A, \tau, M) \varepsilon \mu(R). \quad (4.36)$$

*Proof of (a).* Let  $P \in \mathcal{D}_\mu(R)$  be such that  $Q' \subsetneq P \subset R$ ,  $P \subset K_Q^{3/4}$  and  $\ell(P) \approx \text{dist}(P, Q)$ . Let  $S \in \mathcal{D}_\mu(R)$  be such that  $Q \subsetneq S \subset R$ ,  $\ell(S) \approx \frac{1}{M} \ell(P)$ , and  $\text{dist}(P, S) \approx \ell(P)$ . For  $M$  big enough and an appropriate choice of the implicit constants, we have

$$2B_P \subset K_x \quad \text{for all } x \in 2B_S.$$

Therefore,

$$\mu(2B_P) \mu(2B_S) \frac{1}{\ell(P)^n} \lesssim \iint_{\substack{x \in 2B_S \\ x-y \in K \\ \eta \ell(S) \leq |x-y| \leq \eta^{-1} \ell(S)}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y) = \mu(S) \mathcal{E}_\mu(S), \quad (4.37)$$

assuming  $\eta$  small enough (depending on  $M$ ).

Since  $Q \in \text{Tree}(R)$  and  $Q \subsetneq S$ , we have  $S \notin \text{Stop}(R)$  and thus

$$\Theta_\mu(2B_P) \approx \frac{\mu(2B_P)}{\ell(P)^n} \lesssim \frac{\mu(S)}{\mu(2B_S)} \mathcal{E}_\mu(S) \leq \mathcal{E}_\mu(S) \leq \varepsilon \Theta_\mu(2B_R).$$

Thus if  $\varepsilon$  is small enough, then  $P \in LD_0(R)$ . Since  $Q' \subsetneq P$ , it follows that  $Q' \notin \text{Tree}(R)$ .  $\square$

*Proof of (b).* We can assume that the cubes of the family  $J$  cover  $R$ . Otherwise we replace  $J$  by a suitable enlarged family  $J'$ .

For a fixed  $Q \in J$ , if  $M$  is big enough, we can cover  $K_Q^{1/2} \cap (R \setminus MB_Q)$  by a family of cubes  $P \in \mathcal{D}_\mu(R)$  such that  $P \subset K_Q^{3/4}$ ,  $P \cap (R \setminus MB_Q) \neq \emptyset$ , and  $\ell(P) \approx \text{dist}(P, Q)$ . We denote by  $I_Q$  this family. We assign a cube  $S_{P,Q} \in \mathcal{D}_\mu(R)$  to each  $P \in I_Q$  such that  $Q \subsetneq S_{P,Q} \subset R$ ,  $\ell(S_{P,Q}) \approx \frac{1}{M} \ell(P)$ , and  $\text{dist}(P, S_{P,Q}) \approx \ell(P)$ . As in the proof of (4.37), for

$M$  big enough,  $\eta$  small enough, and an appropriate choice of the implicit constants, we have

$$\mu(2B_P) \mu(2B_{S_{P,Q}}) \frac{1}{\ell(P)^n} \lesssim \mu(S_{P,Q}) \mathcal{E}_\mu(S_{P,Q}).$$

Since  $Q \subsetneq S_{P,Q}$ , we have  $S_{P,Q} \notin \text{Stop}(R)$  and thus

$$\mu(2B_{S_{P,Q}}) \frac{1}{\ell(P)^n} \approx_M \mu(2B_{S_{P,Q}}) \frac{1}{\ell(S_{P,Q})^n} \approx_{A,\tau,M} \Theta_\mu(2B_R),$$

so

$$\Theta_\mu(2B_R) \mu(P) \lesssim_{A,\tau,M} \mu(S_{P,Q}) \mathcal{E}_\mu(S_{P,Q}).$$

Consider now a maximal subfamily  $\mathcal{A} \subset \bigcup_{Q \in J} I_Q$ , so that the cubes from  $\mathcal{A}$  are pairwise disjoint and

$$\bigcup_{Q \in J} K_Q^{1/2} \cap (R \setminus M B_Q) \subset \bigcup_{P \in \mathcal{A}} P.$$

For each  $P \in \mathcal{A}$  we choose a cube  $S(P) := S_{P,Q}$ , where  $Q$  is such that  $P \in I_Q$ . The precise choice of  $Q$  does not matter as long as  $P \in I_Q$ . Observe that for each cube  $S \in \mathcal{D}_\mu$  there is at most a bounded number (depending on  $M$ ) of cubes  $P \in \mathcal{A}$  such that  $S = S(P)$ , taking into account that  $\ell(S(P)) \approx \frac{1}{M} \ell(P)$  and  $\text{dist}(P, S(P)) \approx \ell(P)$ . Further, all the cubes  $\{S(P)\}_{P \in \mathcal{A}}$  belong to  $\text{Tree}(R) \setminus \text{Stop}(R)$ . As a consequence,

$$\begin{aligned} \mu\left(\bigcup_{Q \in J} K_Q^{1/2} \cap (R \setminus M B_Q)\right) &\leq \sum_{P \in \mathcal{A}} \mu(P) & (4.38) \\ &\lesssim_{A,\tau,M} \frac{1}{\Theta_\mu(2B_R)} \sum_{P \in \mathcal{A}} \mu(S(P)) \mathcal{E}_\mu(S(P)) \\ &\lesssim_{A,\tau,M} \frac{1}{\Theta_\mu(2B_R)} \sum_{\substack{S \in \text{Tree}(R) \setminus \text{Stop}(R) \\ S \supset Q \text{ for some } Q \in J}} \mu(S) \mathcal{E}_\mu(S). \end{aligned}$$

By Fubini, we get

$$\begin{aligned}
\sum_{\substack{S \in \text{Tree}(R) \setminus \text{Stop}(R) \\ S \supset Q \text{ for some } Q \in J}} \mu(S) \mathcal{E}_\mu(S) &= \sum_{\substack{S \in \text{Tree}(R) \setminus \text{Stop}(R) \\ S \supset Q \text{ for some } Q \in J}} \sum_{Q \in J: Q \subset S} \mu(Q) \mathcal{E}_\mu(S) \\
&= \sum_{Q \in J} \mu(Q) \sum_{\substack{S \in \text{Tree}(R) \setminus \text{Stop}(R) \\ Q \subset S \subset R}} \mathcal{E}_\mu(S).
\end{aligned}$$

Note now that for any  $Q \in J$ ,

$$\sum_{\substack{S \in \text{Tree}(R) \setminus \text{Stop}(R) \\ Q \subset S \subset R}} \mathcal{E}_\mu(S) \leq \varepsilon \Theta_\mu(2B_R),$$

because of the stopping condition involving the cubes from  $\text{BCE}(R)$ . Therefore,

$$\sum_{\substack{S \in \text{Tree}(R) \setminus \text{Stop}(R) \\ S \supset Q \text{ for some } Q \in J}} \mu(S) \mathcal{E}_\mu(S) \leq \varepsilon \Theta_\mu(2B_R) \sum_{Q \in J} \mu(Q) \leq \varepsilon \Theta_\mu(2B_R) \mu(R).$$

From (4.38) and the preceding estimate, the lemma follows.  $\square$

Denote

$$G_R = R \setminus \bigcup_{Q \in \text{Stop}(R)} Q \quad \text{and} \quad \tilde{G}_R = \bigcap_{k=1}^{\infty} \bigcup_{\substack{Q \in \text{Tree}(R) \\ \ell(Q) \leq A_0^{-k}}} 2MB_Q.$$

and observe that  $G_R \subset \tilde{G}_R$ . As an immediate consequence of the preceding lemma, we can show that  $\tilde{G}_R$  is contained in a Lipschitz graph.

**Lemma 4.6.4.** *For all  $x, y \in \tilde{G}_R$ , we have  $x - y \notin K^{1/2}$ . Hence,  $\tilde{G}_R$  is contained in an  $n$ -dimensional Lipschitz graph with Lipschitz constant depending only on  $s$ .*

*Proof.* Suppose for contradiction that  $x, y \in \tilde{G}_R$  and  $x - y \in K^{1/2}$ . Let  $Q, Q' \in \text{Tree}(R)$  be such that  $x \in 2MB_Q$  and  $y \in 2MB_{Q'}$ , with  $\ell(Q), \ell(Q')$  so small that  $Q' \cap (K_Q^{1/2} \setminus MB_Q) \neq \emptyset$



and  $\ell(Q') \leq \frac{1}{M} \text{dist}(Q', Q)$ . By Lemma 4.6.3(a), it follows that  $Q' \notin \text{Tree}(R)$ , which is a contradiction.  $\square$

#### 4.6.4 *An algorithm to construct a Lipschitz graph close to the stopping cubes*

Given  $t > 1$ , we say that two cubes  $Q, Q' \subset \mathcal{D}_\mu$  are  $t$ -neighbors if

$$t^{-1} \ell(Q') \leq \ell(Q) \leq t \ell(Q') \quad (4.39)$$

and

$$\text{dist}(Q, Q') \leq t(\ell(Q) + \ell(Q')). \quad (4.40)$$

We say that a family of cubes is  $t$ -separated if there is not any pair of cubes in this family which are  $t$ -neighbors.

Given a big constant  $t > M$  to be fixed below, we denote by  $\text{Sep}(R)$  a maximal  $t$ -separated subfamily of  $\text{Stop}(R)$ . It is easy to check that such subfamily exists. Next, we define

$$\widetilde{\text{Sep}}(R) = \{Q \in \text{Sep}(R) : 2MB_Q \cap \widetilde{G}_R = \emptyset \text{ and } \nexists Q' \in \text{Sep}(R) \text{ such that } 2MB_{Q'} \subset 2MB_Q\}.$$

**Lemma 4.6.5.** *Suppose  $t$  is sufficiently large (depending on  $M$ ). Then for all  $Q, Q' \in \widetilde{\text{Sep}}(R)$ , we have  $Q' \not\subset MB_Q$ .*

*Proof.* Suppose  $Q' \in \widetilde{\text{Sep}}(R)$  and  $Q' \subset MB_Q$ . We will show  $Q \notin \widetilde{\text{Sep}}(R)$ . If  $\ell(Q') > t^{-1}\ell(Q)$ , then  $Q' \subset MB_Q$  implies that  $Q, Q'$  are  $t$ -neighbors if  $t$  is sufficiently large. On the other hand, if  $\ell(Q') \leq t^{-1}\ell(Q)$ , then  $Q' \subset MB_Q$  implies that  $2MB_{Q'} \subset 2MB_Q$  if  $t$  is sufficiently large. Hence, in both cases, we have  $Q \notin \widetilde{\text{Sep}}(R)$ .  $\square$

**Lemma 4.6.6.** *The following holds:*

- (a) *For each  $Q \in \text{Stop}(R)$  there exists some cube  $Q' \in \text{Sep}(R)$  which is  $t$ -neighbor of  $Q$ .*

(b) For each  $Q \in \text{Sep}(R)$ , at least one of the following is true:

- $2MB_Q \cap \tilde{G}_R \neq \emptyset$ .
- There exists some  $P \in \widetilde{\text{Sep}}(R)$  such that  $P \subset 2MB_Q$ .

*Proof.* The first statement is obvious from the maximality of the separated family  $\text{Sep}(R)$ . For the second one, note first that the statement is clearly true if  $Q \in \widetilde{\text{Sep}}(R)$ . If  $Q \in \text{Sep}(R) \setminus \widetilde{\text{Sep}}(R)$  and  $2MB_Q \cap \tilde{G}_R = \emptyset$ , then there exists another cube  $Q_1 \in \text{Sep}(R)$  such that  $\ell(Q_1) \leq t^{-1}\ell(Q)$  and  $2MB_{Q_1} \subset 2MB_Q$ .

If  $Q_1 \in \widetilde{\text{Sep}}(R)$ , then we take  $P = Q_1$ . Otherwise, since  $2MB_{Q_1} \cap \tilde{G}_R \subset 2MB_Q \cap \tilde{G}_R = \emptyset$ , there exists some cube  $Q_2 \in \text{Sep}(R)$  such that  $\ell(Q_2) \leq t^{-1}\ell(Q_1)$  and  $2MB_{Q_2} \subset 2MB_{Q_1}$ . Iterating this process, we will get a sequence of cubes  $Q \equiv Q_0, Q_1, \dots, Q_m$  such that  $\ell(Q_j) \leq t^{-1}\ell(Q_{j-1})$  and  $2MB_{Q_j} \subset 2MB_{Q_{j-1}}$ , for  $j = 1, \dots, m$ .

If the process does not terminate, then  $\bigcap_{j=0}^{\infty} 2MB_{Q_j}$  is nonempty. By definition of  $\tilde{G}_R$ , we have  $\bigcap_{j=0}^{\infty} 2MB_{Q_j} \subset \tilde{G}_R$ , which contradicts our assumption that  $2MB_Q \cap \tilde{G}_R = \emptyset$ . Hence, this process terminates at some  $Q_m$  (i.e.,  $Q_m \in \widetilde{\text{Sep}}(R)$ ). We take  $P = Q_m$ , and obtain  $P \subset 2MB_P \subset 2MB_Q$ .  $\square$

**Lemma 4.6.7.** *Assume that  $t$  is chosen big enough (depending on  $M$ , but not on  $A$ ,  $\tau$ , or  $\varepsilon$ ). Then:*

(a) For all  $Q, Q' \in \widetilde{\text{Sep}}(R)$ , we have

$$Q \cap K_{Q'}^{1/2} = Q' \cap K_Q^{1/2} = \emptyset. \quad (4.41)$$

(b) For all  $x \in \tilde{G}_R$  and for all  $Q \in \widetilde{\text{Sep}}(R)$ , we have

$$x \notin K_Q^{1/2} \text{ and } Q \cap K_x^{1/2} = \emptyset. \quad (4.42)$$

*Proof of (a).* Suppose that (4.41) fails. Observe that this implies that both  $Q \cap K_{Q'}^{1/2}$  and

$Q' \cap K_Q^{1/2}$  are nonempty. Suppose that  $\ell(Q) \leq \ell(Q')$ . Since  $Q$  and  $Q'$  are not  $t$ -neighbors, then we must have  $\ell(Q') \leq t^{-1} \ell(Q)$ .

We claim that  $Q' \subset MB_Q$ . Suppose not. Then  $Q' \cap (K_Q^{1/2} \setminus MB_Q) \neq \emptyset$ , and  $\ell(Q') \leq t^{-1} \ell(Q) \leq \frac{1}{M} \text{dist}(Q, Q')$ . Hence it follows that  $Q' \notin \text{Tree}(R)$ , a contradiction. This shows that  $Q' \subset MB_Q$ . But that contradicts Lemma 4.6.5.  $\square$

*Proof of (b).* Suppose for contradiction that  $x \in K_Q^{1/2}$ . We claim that  $x \in MB_Q$ . Suppose not, so that  $x \in K_Q^{1/2} \setminus MB_Q$ . Let  $Q' \in \text{Tree}(R)$  be such that  $x \in 2MB_{Q'}$  with  $\ell(Q')$  so small that  $\ell(Q') \leq \frac{1}{M} \text{dist}(Q', Q)$ . By this inequality and the fact that  $Q' \cap (K_Q^{1/2} \setminus MB_Q) \neq \emptyset$ , it follows from Lemma 4.6.3(a) that  $Q' \notin \text{Tree}(R)$ , which is a contradiction. Hence  $x \in MB_Q$ , so  $2MB_Q \cap \widetilde{G}_R \neq \emptyset$ . But this contradicts  $Q \in \widetilde{\text{Sep}}(R)$ .  $\square$

**Lemma 4.6.8.** *Let  $\Lambda_0 > 0$  be big enough, depending on  $M$  and  $t$ . There is a Lipschitz graph  $\Gamma_R$  with slope depending only on  $s$  such that  $\widetilde{G}_R \subset \Gamma_R$  and  $\Lambda_0 B_Q \cap \Gamma_R \neq \emptyset$  for every  $Q \in \text{Tree}(R)$ .*

*Proof.* For each  $Q \in \widetilde{\text{Sep}}(R)$ , pick a point  $z_Q \in Q$ . Let  $F = \{z_Q : Q \in \widetilde{\text{Sep}}(R)\} \cup \widetilde{G}_R$ . By Lemma 4.6.4 and Lemma 4.6.7, it follows that

$$x - y \notin K^{1/2} \text{ for all } x, y \in F.$$

Hence there is a Lipschitz graph  $\Gamma_R$  containing  $F$ .

By Lemma 4.6.6, any cube  $P \in \text{Stop}(R)$  is  $t$ -neighbor of some cube  $P' \in \text{Sep}(R)$  and  $2MB_{P'} \cap F \neq \emptyset$ . So if  $Q \in \text{Tree}(R)$ , then either  $Q$  intersects  $\widetilde{G}_R$  or  $Q$  contains some cube  $P \in \text{Stop}(R)$  and thus

$$\text{dist}(Q, \Gamma_R) \leq \text{dist}(P, \Gamma_R) \leq C(t)\ell(P) \leq C(t)\ell(Q).$$

which implies  $\Lambda_0 B_Q \cap \Gamma_R \neq \emptyset$ .  $\square$

### 4.6.5 The small measure of the low density set

The goal in this section is to estimate the total measure of the cubes in the low density set.

**Lemma 4.6.9.** *We have*

$$\sum_{Q \in \text{LD}(R)} \mu(Q) \leq c(\tau + C(A, \tau, M)\varepsilon) \mu(R).$$

To prove Lemma 4.6.9, we will construct an auxiliary  $n$ -dimensional Lipschitz graph, by arguments quite similar to the ones for  $\Gamma_R$ . We denote by  $\widetilde{\text{Stop}}(R)$  the subfamily of cubes  $Q \in \text{Stop}(R)$  such that

$$Q \not\subset \bigcup_{P \in \text{Stop}(R)} K_P^{1/2} \cap (R \setminus MB_P),$$

so that, by Lemma 4.6.3(b),

$$\mu\left(\bigcup_{Q \in \text{Stop}(R) \setminus \widetilde{\text{Stop}}(R)} Q\right) \leq C(A, \tau, M) \varepsilon \mu(R). \quad (4.43)$$

We claim that we can choose a subfamily  $\text{LD}_{\text{Sep}}(R) \subset \text{LD}(R) \cap \widetilde{\text{Stop}}(R)$  which is  $t$ -separated and such that

$$\sum_{Q \in \text{LD}(R) \cap \widetilde{\text{Stop}}(R)} \mu(Q) \leq C(t) \sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q). \quad (4.44)$$

To this end we argue as follows: let  $\text{LD}_1(R)$  be a maximal  $t$ -separated subfamily of  $\text{LD}(R) \cap \widetilde{\text{Stop}}(R)$ . Let  $\text{LD}_2(R)$  be a maximal  $t$ -separated subfamily of  $\text{LD}(R) \cap \widetilde{\text{Stop}}(R) \setminus \text{LD}_1(R)$ . By induction, let  $\text{LD}_j(R)$  be a maximal  $t$ -separated subfamily of  $\text{LD}(R) \cap \widetilde{\text{Stop}}(R) \setminus (\text{LD}_1(R) \cup \dots \cup \text{LD}_{j-1}(R))$ . It turns out that there is bounded number  $N_0$  of non-empty families  $\text{LD}_j(R)$ , with  $N_0$  depending on  $t$ . Indeed, if  $Q \in \text{LD}_j(R)$ , then  $Q$  is a  $t$ -neighbor of some cubes  $Q_1 \in \text{LD}_1(R)$ ,  $Q_2 \in \text{LD}_2(R)$ ,  $\dots$ ,  $Q_{j-1} \in \text{LD}_{j-1}(R)$ , by the maximality of  $\text{LD}_k(R)$  for  $k = 1, \dots, j-1$ . Since the number of  $t$ -neighbors of any cube has some bound depending on

$t$ , we get  $j \leq N_0$ . Now we just let  $\text{LD}_{\text{Sep}}(R)$  be the family  $\text{LD}_j(R)$  for which  $\sum_{Q \in \text{LD}_j(R)} \mu(Q)$  is maximal, and then we have

$$\sum_{Q \in \text{LD}(R) \cap \widetilde{\text{Stop}}(R)} \mu(Q) \leq N_0 \sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q),$$

which proves our claim.

Next, we modify the family  $\text{LD}_{\text{Sep}}(R)$  as follows: if there are two cubes  $Q, Q' \in \text{LD}_{\text{Sep}}(R)$  such that

$$1.1B_Q \cap 1.1B_{Q'} \neq \emptyset \quad \text{and} \quad \ell(Q) < \ell(Q'), \quad (4.45)$$

then we eliminate  $Q'$ . We denote by  $\widetilde{\text{LD}}(R)$  the resulting family after eliminating all cubes  $Q'$  of this type in  $\text{LD}_{\text{Sep}}(R)$ . We have the following variant of Lemma 4.6.6(b).

**Lemma 4.6.10.** *For each  $Q \in \text{LD}_{\text{Sep}}(R)$ , at least one of the following is true:*

- $1.2B_Q \cap \widetilde{G}_R \neq \emptyset$
- *There exists some  $P \in \widetilde{\text{LD}}(R)$  such that  $P \subset 1.2B_Q$ .*

*Proof.* If  $Q, Q' \in \text{LD}_{\text{Sep}}(R)$  satisfy (4.45), then (4.40) holds, and thus (4.39) must fail. Therefore,  $Q$  must be much smaller than  $Q'$ , and so

$$\ell(Q) \leq t^{-1}\ell(Q') \quad \text{and} \quad 1.2B_Q \subset 1.2B_{Q'} \quad \text{if } \ell(Q) < \ell(Q'),$$

assuming  $t$  big enough to guarantee the last inclusion. Now we can copy the proof of Lemma 4.6.6(b) with  $2M$  replaced everywhere by 1.2. □

For each  $Q \in \widetilde{\text{LD}}(R)$  we choose a point

$$w_Q \in Q \setminus \bigcup_{P \in \text{Stop}(R)} K_P^{1/2} \cap (R \setminus M B_P).$$

**Lemma 4.6.11.** *There exists an  $n$ -dimensional Lipschitz graph  $\Gamma_0$  which passes through every point  $w_P$ ,  $P \in \widetilde{\text{LD}}(R)$ .*

*Proof.* It is enough to show that for any pair  $w_Q, w_{Q'}$ , with  $Q, Q' \in \widetilde{\text{LD}}(R)$ ,  $Q \neq Q'$ , we have

$$w_Q - w_{Q'} \notin K^{1/2}. \quad (4.46)$$

To show this, suppose that  $\ell(Q) \leq \ell(Q')$ . By the construction of the points  $w_P$ ,  $P \in \widetilde{\text{LD}}(R)$ , it follows that

$$w_{Q'} \notin K_Q^{1/2} \cap (R \setminus MB_Q) \quad \text{and} \quad w_Q \notin K_{Q'}^{1/2} \cap (R \setminus MB_{Q'}),$$

which implies that

$$w_{Q'} \notin K_{w_Q}^{1/2} \setminus B(w_Q, c_1 M \ell(Q)) \quad \text{and} \quad w_Q \notin K_{w_{Q'}}^{1/2} \setminus B(w_{Q'}, c_1 M \ell(Q')),$$

for some  $c_1 \approx 1$ .

So to conclude the proof of (4.46) it suffices to show that

$$w_{Q'} \notin B(w_Q, c_1 M \ell(Q)). \quad (4.47)$$

To this end, notice that if  $w_{Q'} \in B(w_Q, c_1 M \ell(Q))$ , then

$$\text{dist}(Q, Q') \leq |w_Q - w_{Q'}| \leq c_1 M \ell(Q) \leq t(\ell(Q) + \ell(Q')),$$

assuming  $t = t(M)$  big enough. Since  $Q$  and  $Q'$  are not  $t$ -neighbors, we must have

$$\ell(Q) \leq t^{-1} \ell(Q').$$

Together with the fact that  $1.1B_Q \cap 1.1B_{Q'} = \emptyset$ , and recalling that  $w_Q \in Q \subset B_Q$  and

$w_{Q'} \in Q' \subset B_{Q'}$ , this implies that

$$|w_Q - w_{Q'}| \geq \frac{1}{10} r(B_{Q'}) \geq \frac{ct}{10} r(B_Q) > c_1 M \ell(Q)$$

if  $t(M)$  is big enough again. So (4.47) holds, and the lemma follows.  $\square$

*Proof of Lemma 4.6.9.* Consider the family of balls  $\{1.5B_Q\}_{Q \in \text{LD}_{\text{Sep}}(R)}$ . By the covering Theorem 9.31 from [55], there exists a subfamily  $\mathcal{F} \subset \text{LD}_{\text{Sep}}(R)$  such that:

$$(i) \quad \bigcup_{Q \in \text{LD}_{\text{Sep}}(R)} 1.5B_Q \subset \bigcup_{Q \in \mathcal{F}} 2B_Q,$$

$$(ii) \quad \sum_{Q \in \mathcal{F}} \chi_{1.5B_Q} \leq C.$$

Then

$$\sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q) \leq \sum_{Q \in \mathcal{F}} \mu(2B_Q) \leq c\tau \Theta_\mu(2B_R) \sum_{Q \in \mathcal{F}} r(B_Q)^n.$$

Recall now that for each  $B_Q$ , with  $Q \in \mathcal{F} \subset \text{LD}_{\text{Sep}}(R)$ , there exists some point  $w_P \in \Gamma_0 \cap 1.2B_Q$  or some point  $x \in \tilde{G}_R \cap 1.2B_Q \subset \Gamma_R \cap 1.2B_Q$ . Then we have

$$\mathcal{H}^n(1.5B_Q \cap (\Gamma_0 \cup \Gamma_R)) \approx r(B_Q)^n.$$

So using the property (ii) of the covering, we obtain

$$\sum_{Q \in \mathcal{F}} r(B_Q)^n \lesssim \sum_{Q \in \mathcal{F}} \mathcal{H}^n(1.5B_Q \cap (\Gamma_0 \cup \Gamma_R)) \leq C \mathcal{H}^n(2B_R \cap (\Gamma_0 \cup \Gamma_R)) \leq C' \ell(R)^n.$$

Thus,

$$\sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q) \leq c\tau \Theta_\mu(2B_R) \ell(R)^n \leq c\tau \mu(R).$$

Together with (4.44), this yields

$$\sum_{Q \in \text{LD}(R) \cap \widetilde{\text{Stop}}(R)} \mu(Q) \leq C \tau \mu(R),$$

with  $C$  depending on  $t$ . Finally to conclude the lemma, we just take into account that, by (4.43), we have

$$\sum_{Q \in \text{LD}(R) \setminus \widetilde{\text{Stop}}(R)} \mu(Q) \leq C(A, \tau, M) \varepsilon \mu(R).$$

□

#### 4.6.6 The approximate Lipschitz graph

In the next lemma we gather some of the previous results and estimates.

**Lemma 4.6.12.** *Let  $R \in \mathcal{D}_\mu^{db}$ , and suppose that  $\tau, \eta, \varepsilon$  are small enough and  $\varepsilon \ll \tau$ . Then there exists an  $n$ -dimensional Lipschitz graph  $\Gamma_R$  whose slope is bounded above by some constant depending only on  $s$  such that the following holds:*

(a)  $R \setminus \bigcup_{Q \in \text{Stop}(R)} Q \subset \Gamma_R.$

(b) *There exists some constant  $\Lambda_0 > 1$  such that for all  $Q \in \text{Tree}(R)$ ,*

$$\Lambda_0 B_Q \cap \Gamma_R \neq \emptyset.$$

(c) *We have:*

$$\sum_{Q \in \text{LD}(R)} \mu(Q) \leq \tau^{1/2} \mu(R), \tag{4.48}$$

and also

$$\sum_{Q \in \text{BCE}(R)} \mu(Q) \leq \frac{1}{\varepsilon \Theta_\mu(2B_R)} \sum_{S \in \text{Tree}(R)} \mathcal{E}_\mu(S) \mu(S). \tag{4.49}$$



*Proof.* The statement (a) follows from Lemma 4.6.4, and (b) from Lemma 4.6.8. On the other hand, the estimate (4.48) follows from the analogous one proved in Lemma 4.6.9 choosing  $\varepsilon$  and  $\tau$  suitably small. Finally, concerning (4.49), recall that if  $Q \in \text{BCE}(R)$ , then

$$\sum_{S \in \mathcal{D}_\mu: Q \subset S \subset R} \mathcal{E}_\mu(S) \geq \varepsilon \Theta_\mu(2B_R).$$

Therefore,

$$\begin{aligned} \Theta_\mu(2B_R) \sum_{Q \in \text{BCE}(R)} \mu(Q) &\leq \frac{1}{\varepsilon} \sum_{Q \in \text{BCE}(R)} \mu(Q) \sum_{S \in \mathcal{D}_\mu: Q \subset S \subset R} \mathcal{E}_\mu(S) \\ &= \frac{1}{\varepsilon} \sum_{S \in \text{Tree}(R)} \mathcal{E}_\mu(S) \sum_{Q \in \text{BCE}(R): Q \subset S} \mu(Q) \\ &\leq \frac{1}{\varepsilon} \sum_{S \in \text{Tree}(R)} \mathcal{E}_\mu(S) \mu(S). \end{aligned}$$

□

## 4.7 The family of Top cubes

### 4.7.1 The family Top

We are going to construct a family of cubes  $\text{Top} \subset \mathcal{D}_\mu^{db}$  inductively. To this end, we need to introduce some additional notation. Given a cube  $Q \in \mathcal{D}_\mu$ , we denote by  $\mathcal{MD}(Q)$  the family of maximal cubes (with respect to inclusion) from  $\mathcal{D}_\mu^{db}(Q) \setminus \{Q\}$ . Recall that, by Lemma 4.4.2, this family covers  $\mu$ -almost all of  $Q$ . Moreover, by Lemma 4.4.4 it follows that if  $P \in \mathcal{MD}(Q)$ , then  $\Theta_\mu(2B_P) \leq c \Theta_\mu(2B_Q)$ . Given  $R \in \mathcal{D}_\mu^{db}$ , we denote

$$\text{Next}(R) = \bigcup_{Q \in \text{Stop}(R)} \mathcal{MD}(Q).$$

By the construction above, it is clear that the cubes in  $\text{Next}(R)$  are different from  $R$  (because  $\mathcal{MD}(Q) \neq \{Q\}$ ).

For the record, notice that if  $P \in \text{Next}(R)$ , then

$$\Theta_\mu(2B_S) \leq c(A) \Theta_\mu(2B_R) \quad \text{for all } S \in \mathcal{D}_\mu \text{ such that } P \subset S \subset R. \quad (4.50)$$

We are now ready to construct the aforementioned family  $\text{Top}$ . We will have  $\text{Top} = \bigcup_{k \geq 0} \text{Top}_k$ . First we set

$$\text{Top}_0 = \{R_0\}.$$

(Recall that  $R_0 \equiv \text{supp } \mu$ .) Assuming  $\text{Top}_k$  has been defined, we set

$$\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R).$$

Note that the families  $\text{Next}(R)$ , with  $R \in \text{Top}_k$ , are pairwise disjoint.

### 4.7.2 The family of cubes $ID$

We distinguish a special type of cubes from  $\text{Top}$ . For  $R \in \text{Top}$ , we write  $R \in ID$  (increasing density) if

$$\mu\left(\bigcup_{Q \in \text{HD}(R)} Q\right) \geq \frac{1}{2} \mu(R).$$

**Lemma 4.7.1.** *Suppose that  $A$  is big enough. If  $R \in ID$ , then*

$$\Theta_\mu(2B_R) \mu(R) \leq \frac{1}{2} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q) \mu(Q). \quad (4.51)$$

*Proof.* Recalling that  $\Theta_\mu(2B_Q) \geq A \Theta_\mu(2B_R)$  for every  $Q \in \text{HD}(R)$ , we deduce that

$$\Theta_\mu(2B_R) \mu(R) \leq 2 \sum_{Q \in \text{HD}(R)} \Theta_\mu(2B_R) \mu(Q) \leq 2A^{-1} \sum_{Q \in \text{HD}(R)} \Theta_\mu(2B_Q) \mu(Q).$$

Since the cubes from  $\text{HD}(R)$  belong to  $\mathcal{D}_\mu^{db}$ , it follows immediately from Lemma 4.4.5 that for any  $Q \in \text{HD}(R)$ ,

$$\Theta_\mu(2B_Q) \mu(Q) \lesssim \sum_{P \in \text{Next}(R): P \subset Q} \Theta_\mu(2B_P) \mu(P),$$

and then it is clear that (4.51) holds if  $A$  is taken big enough.  $\square$

## 4.8 The packing condition

**Lemma 4.8.1.** *Suppose that*

$$\mu(B(x, r)) \leq c_0 r^n \quad \text{for all } x \in \text{supp } \mu, r > 0. \quad (4.52)$$

*For all  $S \in \text{Top}$  we have*

$$\sum_{R \in \text{Top}: R \subset S} \Theta_\mu(2B_R) \mu(R) \lesssim_{\varepsilon, \eta, c_0} \mu(S) + \iint_{\substack{x \in 2B_S \\ x-y \in K}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y), \quad (4.53)$$

*assuming that the constants  $A, \tau, \varepsilon$ , and  $\eta$  have been chosen suitably.*

*Proof.* We denote  $\text{Top}(S) = \text{Top} \cap \mathcal{D}_\mu(S)$  and  $\text{Top}_j(S) = \text{Top}_j \cap \mathcal{D}_\mu(S)$ . For a given  $k \geq 0$ , we also write

$$\text{Top}_0^k(S) = \bigcup_{0 \leq j \leq k} \text{Top}_j(S),$$

and also

$$ID_0^k = ID \cap \text{Top}_0^k(S).$$

To prove (4.53), first we deal with the cubes from the  $ID$  family. By Lemma 4.7.1, for every  $R \in ID$  we have

$$\Theta_\mu(2B_R) \mu(R) \leq \frac{1}{2} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q) \mu(Q)$$

and hence we obtain

$$\sum_{R \in ID_0^k} \Theta_\mu(2B_R) \mu(R) \leq \frac{1}{2} \sum_{R \in ID_0^k} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q) \mu(Q) \leq \frac{1}{2} \sum_{Q \in \text{Top}_0^{k+1}(S)} \Theta_\mu(2B_Q) \mu(Q),$$

because the cubes from  $\text{Next}(R)$  with  $R \in \text{Top}_0^k(S)$  belong to  $\text{Top}_0^{k+1}(S)$ . Thus,

$$\begin{aligned} & \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) \\ &= \sum_{R \in \text{Top}_0^k(S) \setminus ID_0^k} \Theta_\mu(2B_R) \mu(R) + \sum_{R \in ID_0^k} \Theta_\mu(2B_R) \mu(R) \\ &\leq \sum_{R \in \text{Top}_0^k(S) \setminus ID_0^k} \Theta_\mu(2B_R) \mu(R) + \frac{1}{2} \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) + C c_0 \mu(S), \end{aligned}$$

where, for the last inequality, we took into account that  $\Theta_\mu(2B_R) \leq C c_0$  for every  $R \in \text{Top}_{k+1}(S)$  because of the assumption (4.52). Using that

$$\sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) \leq (k+1) C c_0 \mu(S) < \infty,$$

we deduce

$$\sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) \leq 2 \sum_{R \in \text{Top}_0^k(S) \setminus ID_0^k} \Theta_\mu(2B_R) \mu(R) + C c_0 \mu(S).$$

Letting  $k \rightarrow \infty$ , we derive

$$\sum_{R \in \text{Top}(S)} \Theta_{\mu}(2B_R) \mu(R) \leq 2 \sum_{R \in \text{Top}(S) \setminus ID} \Theta_{\mu}(2B_R) \mu(R) + C c_0 \mu(S). \quad (4.54)$$

To estimate the first term on the right hand side of (4.54) we use the fact that, for  $R \in \text{Top}(S) \setminus ID$ , we have

$$\mu\left(R \setminus \bigcup_{Q \in \text{HD}(R)} Q\right) \geq \frac{1}{2} \mu(R),$$

and then using Lemma 4.4.2, we get

$$\begin{aligned} \mu(R) &\leq 2 \mu\left(R \setminus \bigcup_{Q \in \text{Stop}(R)} Q\right) + 2 \mu\left(\bigcup_{Q \in \text{Stop}(R) \setminus \text{HD}(R)} Q\right) \\ &= 2 \mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) + 2 \sum_{Q \in \text{LD}(R)} \mu(Q) + 2 \sum_{Q \in \text{BCE}(R)} \mu(Q). \end{aligned} \quad (4.55)$$

Recall now that, by (4.48),

$$\sum_{Q \in \text{LD}(R)} \mu(Q) \leq \tau^{1/2} \mu(R).$$

Choosing  $\tau \leq 1/16$ , say, from (4.55) we infer that

$$\mu(R) \leq 4 \mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) + 4 \sum_{Q \in \text{BCE}(R)} \mu(Q).$$

So we deduce that

$$\begin{aligned} \sum_{R \in \text{Top}(S) \setminus ID} \Theta_{\mu}(2B_R) \mu(R) &\leq 4 \sum_{R \in \text{Top}(S)} \Theta_{\mu}(2B_R) \mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) \\ &\quad + 4 \sum_{R \in \text{Top}(S)} \Theta_{\mu}(2B_R) \sum_{Q \in \text{BCE}(R)} \mu(Q). \end{aligned} \quad (4.56)$$

To deal with the first sum on the right hand side above, we take into account that the sets

$R \setminus \bigcup_{Q \in \text{Next}(R)} Q$ , with  $R \in \text{Top}(S)$ , are pairwise disjoint, and also that  $\Theta_\mu(2B_R) \leq C c_0$ , by the condition (4.52). Then we get

$$\sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R) \mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) \leq C c_0 \mu(S). \quad (4.57)$$

To deal with the second sum in (4.56), we use (4.49) to obtain

$$\begin{aligned} \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R) \sum_{Q \in \text{BCE}(R)} \mu(Q) &\leq \frac{1}{\varepsilon} \sum_{R \in \text{Top}(S)} \sum_{P \in \text{Tree}(R)} \mathcal{E}_\mu(P) \mu(P) \\ &\leq \frac{1}{\varepsilon} \sum_{P \in \mathcal{D}_\mu(S)} \mathcal{E}_\mu(P) \mu(P). \end{aligned}$$

Denote by  $\ell_k$  the side length of the cubes from  $\mathcal{D}_{\mu,k}$ . By the definition of  $\mathcal{E}_\mu(P)$  (see (4.35)) and the finite overlapping of the balls  $2B_P$  among the cubes  $P$  of the same generation, we get

$$\begin{aligned} \sum_{P \in \mathcal{D}_\mu(S)} \mathcal{E}_\mu(P) \mu(P) &= \sum_k \sum_{P \in \mathcal{D}_{\mu,k}(S)} \iint_{\substack{x \in 2B_P \\ x-y \in K \\ \eta \ell(P) \leq |x-y| \leq \eta^{-1} \ell(P)}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y) \\ &\lesssim \sum_k \iint_{\substack{x \in 2B_S \\ x-y \in K \\ \eta \ell_k \leq |x-y| \leq \eta^{-1} \ell_k}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y) \\ &\lesssim_\eta \iint_{\substack{x \in 2B_S \\ x-y \in K}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y). \end{aligned}$$

Therefore,

$$\sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R) \sum_{Q \in \text{BCE}(R)} \mu(Q) \lesssim_{\varepsilon, \eta} \iint_{\substack{x \in 2B_S \\ x-y \in K}} \frac{1}{|x-y|^n} d\mu(x) d\mu(y).$$

Together with (4.54), (4.56), and (4.57), this yields (4.53).  $\square$

## 4.9 Proof of Lemma 4.5.1

We have to show that the family  $\mathbf{Top}$  satisfies the properties stated in Lemma 4.5.1. By the definition of the family  $\mathbf{Next}(R)$  and Lemma 4.4.2, we have

$$\bigcup_{Q \in \mathbf{Stop}(R)} Q = \bigcup_{Q \in \mathbf{Next}(R)} Q \quad \text{up to a set of } \mu\text{-measure } 0.$$

Thus, by Lemma 4.6.12(a),  $\mu$ -almost all of  $R \setminus \bigcup_{Q \in \mathbf{Next}(R)} Q$  is contained in  $\Gamma_R$ , and we have verified property (a) of Lemma 4.5.1.

Next we deal with the property (b). Given  $Q \in \mathbf{Next}(R)$  we have to check that there exists some  $\tilde{Q} \in \mathcal{D}_\mu$  such that  $\delta_\mu(Q, \tilde{Q}) \leq c \Theta_\mu(2B_R)$  and  $2B_{\tilde{Q}} \cap \Gamma_R \neq \emptyset$ . Let  $Q' \in \mathbf{Stop}(R)$  such that  $Q \subset Q'$ . By Lemma 4.6.12(b), there exists some constant  $\Lambda_0 > 1$  such that  $\Lambda_0 B_{Q'} \cap \Gamma_R \neq \emptyset$ . This implies that there exists one cube  $\tilde{Q} \in \mathbf{Tree}(R)$  such that  $Q' \subset \tilde{Q}$ ,  $\ell(\tilde{Q}) \approx \ell(Q')$ , and  $2B_{\tilde{Q}} \cap \Gamma_R \neq \emptyset$ . We split

$$\delta_\mu(Q, \tilde{Q}) = \int_{2B_{\tilde{Q}} \setminus 2B_{Q'}} \frac{1}{|y - x_Q|^n} d\mu(y) + \int_{2B_{Q'} \setminus 2B_Q} \frac{1}{|y - x_Q|^n} d\mu(y).$$

To estimate the first integral we use the fact that  $|y - x_Q| \approx \ell(Q') \approx \ell(\tilde{Q})$  in the domain of integration, and we derive

$$\int_{2B_{\tilde{Q}} \setminus 2B_{Q'}} \frac{1}{|y - x_Q|^n} d\mu(y) \lesssim \Theta_\mu(2B_{\tilde{Q}}) \lesssim_A \Theta_\mu(2B_R).$$

To estimate the second one we take into account that there are no doubling cubes strictly between  $Q$  and  $Q'$ . Then from Lemma 4.4.4 and standard estimates, it easily follows that

$$\int_{2B_{Q'} \setminus 2B_Q} \frac{1}{|y - x_Q|^n} d\mu(y) \lesssim \Theta_\mu(100B(Q')).$$

If  $Q'$  is doubling, then  $\Theta_\mu(100B(Q')) \lesssim \Theta_\mu(2B_{Q'}) \lesssim A \Theta_\mu(2B_R)$ . Otherwise  $Q' \neq R$  and

the parent of  $Q'$ , which we denote by  $\widehat{Q}'$ , belongs to  $\text{Tree}(R) \setminus \text{Stop}(R)$ . Thus, by Lemma 4.6.1,

$$\Theta_\mu(100B(Q')) \lesssim \Theta_\mu(2B_{\widehat{Q}'}) \lesssim A \Theta_\mu(2B_R),$$

taking into account that  $100B(Q') \subset 2B_{\widehat{Q}'}$  for the first inequality (since  $A_0 \gg 1$  in the David-Mattila lattice). Hence in any case,  $\delta_\mu(Q, \tilde{Q}) \lesssim_A \Theta_\mu(2B_R)$  and (b) in Lemma 4.5.1 holds.

Next, we observe that (c) in Lemma 4.5.1 follows from Lemma 4.6.1 in case that  $Q \in \text{Tree}(R)$ , and from (4.50) otherwise.

Finally, the packing condition (4.34) has been proved in Lemma 4.8.1.

## 4.10 Application to curvature, Riesz transforms, and capacities

### 4.10.1 Curvature of measures and Riesz transforms

To estimate the curvature of  $\mu$  we will use the following result:

**Lemma 4.10.1.** *Let  $\mu$  be a measure satisfying the growth condition (4.33). Suppose that there exists a family  $\text{Top} \subset \mathcal{D}_\mu^{db}$  as in Section 4.5 such that, for all  $R \in \text{Top}$ , there exists an  $n$ -dimensional Lipschitz graph  $\Gamma_R$  whose slope is uniformly bounded above by some constant independent of  $R$  such that:*

(a)  $\mu$ -almost all  $\text{Good}(R)$  is contained in  $\Gamma_R$ .

(b) For all  $Q \in \text{Next}(R)$  there exists another cube  $\tilde{Q} \in \mathcal{D}_\mu$  such that  $\delta_\mu(Q, \tilde{Q}) \leq c \Theta_\mu(2B_R)$  and  $2B_{\tilde{Q}} \cap \Gamma_R \neq \emptyset$ .

(c) For all  $Q \in \text{Tr}(R)$ ,  $\Theta_\mu(2B_Q) \leq c \Theta_\mu(2B_R)$ .

In the case  $n = 1$ , we have:

$$c^2(\mu) \lesssim \sum_{R \in \text{Top}} \Theta_\mu(2B_R)^2 \mu(R), \tag{4.58}$$



and for any integer  $n \in (0, d)$ ,

$$\sup_{\varepsilon > 0} \|\mathcal{R}_\varepsilon^n \mu\|_{L^2(\mu)}^2 \lesssim \sum_{R \in \text{Top}} \Theta_\mu(2B_R)^2 \mu(R), \quad (4.59)$$

with the implicit constant depending only on  $c_0$  in both estimates.

For  $n = 1, d = 2$ , a version of this result which uses the usual dyadic lattice of  $\mathbb{R}^2$  instead of the David-Mattila lattice is proven in [54]. For arbitrary  $n, d$ , another version in terms of the David-Mattila lattice is shown in [26], which in fact is valid for other singular integral operators with odd kernel, besides the Cauchy and Riesz transforms.

By combining Lemmas 4.5.1 and 4.10.1 we obtain the following result.

**Theorem 4.10.2.** *Given  $V_0 \in G(d, d-n)$  and  $s > 0$ , consider the cone  $K := K(V_0, s) \subset \mathbb{R}^d$ .*

*Let  $\mu$  be a Borel measure in  $\mathbb{R}^d$  satisfying the growth condition*

$$\mu(B(x, r)) \leq c_0 r^n \quad \text{for all } x \in \mathbb{R}^d, r > 0.$$

*In the case  $n = 1$  we have*

$$c^2(\mu) \lesssim \|\mu\| + \iint_{x-y \in K} \frac{1}{|x-y|} d\mu(x) d\mu(y),$$

*and for any integer  $n \in (0, d)$ ,*

$$\sup_{\varepsilon > 0} \|\mathcal{R}_\varepsilon^n \mu\|_{L^2(\mu)}^2 \lesssim \|\mu\| + \iint_{x-y \in K} \frac{1}{|x-y|^n} d\mu(x) d\mu(y).$$

*The implicit constants in both inequalities depend only on  $d, n, c_0$ , and  $s$ .*

*Proof.* From Lemmas 4.5.1 and 4.10.1, in the case  $n = 1$  we deduce the following:

$$\begin{aligned} c^2(\mu) &\lesssim \|\mu\| + \sum_{R \in \text{Top}} \Theta_\mu(2B_R)^2 \mu(R) \lesssim \|\mu\| + \sum_{R \in \text{Top}} \Theta_\mu(2B_R) \mu(R) \\ &\lesssim \|\mu\| + \iint_{x-y \in K} \frac{1}{|x-y|} d\mu(x) d\mu(y), \end{aligned}$$

and analogously, for any integer  $n \in (0, d)$ , with  $\sup_{\varepsilon > 0} \|\mathcal{R}_\varepsilon^n \mu\|_{L^2(\mu)}^2$  instead of  $c^2(\mu)$ .  $\square$

### 4.10.2 Analytic capacity

To prove Theorem 4.1.1, recall that by Theorem 4.2.1, for any compact set  $E \subset \mathbb{C}$  we have

$$\gamma(E) \approx \sup\{\sigma(E) : \sigma \in L_1(E), c^2(\sigma) \leq \sigma(E)\}, \quad (4.60)$$

where  $L_1(E)$  stands for the set of positive Borel measures supported on  $E$  satisfying  $\sigma(B(x, r)) \leq r$  for all  $x \in E$ ,  $r > 0$ .

Let  $\mu$  be a measure supported on  $E$  such that  $0 < \int_I \|P_\theta \mu\|_2^2 d\theta < \infty$ , and denote

$$\lambda = \frac{1}{\mu(E)} \int_I \|P_\theta \mu\|_2^2 d\theta.$$

We intend to construct a suitable measure  $\sigma$  with linear growth from  $\mu$ , and then we will apply (4.60) to  $\sigma$ .

Let  $\theta_0 \in I$  be such that

$$\|P_{\theta_0} \mu\|_2^2 \leq \frac{1}{\mathcal{H}^1(I)} \int_I \|P_\theta \mu\|_2^2 d\theta = \frac{\lambda \mu(E)}{\mathcal{H}^1(I)}.$$

Denote  $\eta = P_{\theta_0} \mu$  and let  $L_{\theta_0} = \{re^{i\theta_0} : r \in \mathbb{R}\}$ . Observe that the preceding estimate is equivalent to

$$\int_{L_{\theta_0}} \left| \frac{d\eta}{dr} \right|^2 dr = \int_{P_{\theta_0}(E)} \frac{d\eta}{dr} d\eta(r) \leq \frac{\lambda \eta(P_{\theta_0}(E))}{\mathcal{H}^1(I)}.$$

So by Chebyshev's inequality we have

$$\eta\left\{r \in L_{\theta_0} : \frac{d\eta}{dr}(r) > \frac{2\lambda}{\mathcal{H}^1(I)}\right\} \leq \frac{1}{2}\eta(P_{\theta_0}(E)).$$

Hence there exists a compact set  $F_0$  contained in

$$\left\{r \in L_{\theta_0} : \frac{d\eta}{dr}(r) \leq \frac{2\lambda}{\mathcal{H}^1(I)}\right\}$$

such that  $\eta(F_0) \geq \frac{1}{4}\eta(P_{\theta_0}(E)) = \frac{1}{4}\mu(E)$ . Clearly,

$$\eta(F_0 \cap B(x, s)) \leq \frac{4\lambda}{\mathcal{H}^1(I)}s \quad \text{for all } x \in \mathbb{C}, s > 0.$$

Next we consider the closed set  $F = P_{\theta_0}^{-1}(F_0) \cap \text{supp } \mu$ , and the measure

$$\sigma = \frac{\mathcal{H}^1(I)}{4\lambda} \mu|_F.$$

Note that

$$\sigma(F) = \frac{\mathcal{H}^1(I)}{4\lambda} \mu(F) = \frac{\mathcal{H}^1(I)}{4\lambda} \eta(F_0) \geq \frac{\mathcal{H}^1(I)}{16\lambda} \mu(E). \quad (4.61)$$

Further, for any  $x \in \text{supp } \sigma$  and  $s > 0$ ,

$$\sigma(B(x, s)) \leq \sigma(P_{\theta_0}^{-1}(P_{\theta_0}(B(x, s)))) = \frac{\mathcal{H}^1(I)}{4\lambda} \eta(P_{\theta_0}(F \cap B(x, s))) \leq s,$$

and so  $\sigma$  has linear growth with constant 1. Also, by the definition of  $\lambda$  and (4.61),

$$\begin{aligned} \int_I \|P_{\theta}\sigma\|_2^2 d\theta &= \left(\frac{\mathcal{H}^1(I)}{4\lambda}\right)^2 \int_I \|P_{\theta}(\mu|_F)\|_2^2 d\theta \\ &\leq \left(\frac{\mathcal{H}^1(I)}{4\lambda}\right)^2 \int_I \|P_{\theta}\mu\|_2^2 d\theta = \frac{\mathcal{H}^1(I)^2}{16\lambda} \mu(E) \leq \mathcal{H}^1(I) \sigma(F). \end{aligned}$$

Hence, by Theorem 4.10.2 and Corollary 4.3.3, we deduce that

$$c^2(\sigma) \lesssim \sigma(F) + \iint_{x-y \in K_{I^\perp}} \frac{1}{|x-y|} d\sigma(x) d\sigma(y) \lesssim \sigma(F) + \int_I \|P_\theta \sigma\|_2^2 d\theta \leq C_I \sigma(F),$$

where the constant  $C_I$  depends only on  $\mathcal{H}^1(I)$ . Then, from (4.60) and (4.61), we deduce that

$$\gamma(E) \geq \gamma(F) \gtrsim \sigma(F) \gtrsim \frac{\mu(E)}{\lambda} = \frac{\mu(E)^2}{\int_I \|P_\theta \mu\|_2^2 d\theta},$$

with the implicit constants depending on  $\mathcal{H}^1(I)$ . This concludes the proof of Theorem 4.1.1.

### 4.10.3 The capacities $\Gamma_{d,n}$

The proof of Theorem 4.1.2 is analogous to the one of Theorem 4.1.1. The only difference is that we have to replace the curvature  $c^2(\mu)$  by  $\sup_{\varepsilon>0} \|\mathcal{R}_\varepsilon^n \mu\|_{L^2(\mu)}^2$ , and use Theorem 4.2.2 and Corollary 4.3.11 instead of Theorem 4.2.1 and Corollary 4.3.3, respectively. We skip the details.

## 4.A The reverse inequality

If  $\mu$  is a Radon measure on  $\mathbb{R}^d$ , we define

$$\Theta^{n,*}(x, \mu) = \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} \quad \text{and} \quad M_n \mu(x) = \sup_{r > 0} \frac{\mu(B(x, r))}{r^n}.$$

In this appendix, we prove the following inequality, which is not used in the paper but may be of independent interest.

**Lemma 4.A.1.** *Let  $V_0 \in G(d, n)$  and  $s > 0$ . Then there exist  $\lambda > 1$  and  $c$  (depending on  $d, n, s$ ) such that the following holds: If  $\mu$  is a Radon measure on  $\mathbb{R}^d$  such that*

$$\int M_n \mu(x) d\mu(x) < \infty, \tag{4.62}$$

then

$$\int_{B(V_0, s)} \|P_V \mu\|_2^2 d\gamma_{d, n}(V) \leq c \iint_{x-y \in K(V_0^\perp, \lambda s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} + c \int \Theta^{n, *}(x, \mu) d\mu(x). \quad (4.63)$$

*Proof.* Fix  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  a  $C^\infty$  radial bump function with support in  $B(0, 1)$ . For  $\varepsilon > 0$ , let  $\phi_\varepsilon(x) = \frac{1}{\varepsilon^d} \phi(\frac{x}{\varepsilon})$ . Denote  $\mu_\varepsilon = \mu * \phi_\varepsilon$ . By Corollary 4.3.10, there exists  $\lambda, c > 1$  such that

$$\int_{B(V_0, s)} \|P_V \mu_\varepsilon\|_2^2 d\gamma_{d, n}(V) \leq c \iint_{x-y \in K(V_0^\perp, \lambda s)} \frac{d\mu_\varepsilon(x) d\mu_\varepsilon(y)}{|x-y|^n}. \quad (4.64)$$

First, we will prove that, for any  $\varepsilon > 0$ ,

$$\iint_{x-y \in K(V_0^\perp, \lambda s)} \frac{d\mu_\varepsilon(x) d\mu_\varepsilon(y)}{|x-y|^n} \quad (4.65)$$

$$\leq 2 \iint_{x-y \in K(V_0^\perp, 2\lambda s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} + C \int \frac{\mu(B(x, C\varepsilon))}{\varepsilon^n} d\mu(x). \quad (4.66)$$

(Here and in what follows,  $C$  is independent of  $\varepsilon$ , and it may change from line to line.)

For fixed constants  $\varepsilon > 0$  and  $A > 10$ , denote

$$f(y) = \chi_{K(V_0^\perp, \lambda s)}(y) \frac{1}{|y|^n}, \quad f_C(y) = f(y) \chi_{|y| \leq A\varepsilon}, \quad f_F(y) = f(y) \chi_{|y| > A\varepsilon},$$

so that

$$\begin{aligned} \iint_{x-y \in K(V_0^\perp, \lambda s)} \frac{d\mu_\varepsilon(x) d\mu_\varepsilon(y)}{|x-y|^n} &= \int f * \mu_\varepsilon d\mu_\varepsilon = \int f * \mu * \phi_\varepsilon * \phi_\varepsilon d\mu \\ &= \int (f_C + f_F) * \mu * \phi_\varepsilon * \phi_\varepsilon d\mu. \end{aligned}$$

We will show that, for any  $x \in \mathbb{R}^d$ ,

$$f_C * \mu * \phi_\varepsilon * \phi_\varepsilon(x) \leq C(A) \frac{\mu(B(x, C(A)\varepsilon))}{\varepsilon^n} \quad (4.67)$$

and

$$f_F * \mu * \phi_\varepsilon * \phi_\varepsilon(x) \leq 2 \int_{x-y \in K(V_0^\perp, 2\lambda s)} \frac{d\mu(y)}{|x-y|^n}, \quad (4.68)$$

if  $A$  is sufficiently large. Clearly, (4.65) follows from the two preceding estimates.

First we deal with the estimate (4.67). Let  $\psi = \phi * \phi$  and  $\psi_\varepsilon(x) = \frac{1}{\varepsilon^d} \psi(\frac{x}{\varepsilon})$ , so that  $\psi_\varepsilon = \phi_\varepsilon * \phi_\varepsilon$ . For any  $z \in \mathbb{R}^d$ ,

$$f_C * \psi_\varepsilon(z) \leq \int_{|z-y| \leq A\varepsilon} \frac{1}{|z-y|^n} \psi_\varepsilon(y) dy \leq \frac{C(A)}{\varepsilon^n} \chi_{B(0, (A+2)\varepsilon)}(z), \quad (4.69)$$

taking into account that  $\text{supp } \psi_\varepsilon \subset B(0, 2\varepsilon)$  and  $\|\psi_\varepsilon\|_\infty \leq c_\phi \varepsilon^{-d}$ . It is clear that (4.69) implies (4.67).

To prove (4.68), first observe that since  $\text{supp } f_F \subset K(V_0^\perp, \lambda s) \cap B(0, A\varepsilon)^c$  and  $\text{supp } \psi_\varepsilon \subset B(0, 2\varepsilon)$ , it follows that  $\text{supp}(f_F * \psi_\varepsilon)$  is contained in the  $2\varepsilon$ -neighborhood of  $K(V_0^\perp, \lambda s) \cap B(0, A\varepsilon)^c$ . Therefore, by geometric arguments, we have

$$\text{supp } f_F * \psi_\varepsilon \subset K(V_0^\perp, \lambda s + CA^{-1}) \subset K(V_0^\perp, 2\lambda s)$$

assuming  $A$  big enough.

Next, suppose  $x \in \text{supp}(f_F * \psi_\varepsilon)$  and  $x' \in B(x, 2\varepsilon)$ . Since  $|x| \geq (A-2)\varepsilon$ , we have  $|x'| \geq \frac{1}{2}|x|$  so  $f_F(x') \leq \frac{2}{|x|^n}$ . Hence, for all  $x \in \mathbb{R}^d$ ,

$$f_F * \psi_\varepsilon(x) \leq \sup_{x' \in B(x, 2\varepsilon)} f_F(x') \leq 2 \frac{\chi_{K(V_0^\perp, 2\lambda s)}(x)}{|x|^n},$$

which yields (4.68), and completes the proof of (4.65).

By Fatou's lemma applied to  $M_n(x) - \frac{\mu(B(x, C\varepsilon))}{\varepsilon^n}$  and hypothesis (4.62), we have

$$\limsup_{\varepsilon \rightarrow 0} \int \frac{\mu(B(x, C\varepsilon))}{\varepsilon^n} d\mu(x) \leq \int \limsup_{\varepsilon \rightarrow 0} \frac{\mu(B(x, C\varepsilon))}{\varepsilon^n} d\mu(x) = C \int \Theta^{n,*}(x, \mu) d\mu(x).$$

Taking the limsup in (4.65) as  $\varepsilon \rightarrow 0$  and using (4.64), we obtain

$$\limsup_{\varepsilon \rightarrow 0} \int_{B(V_0, s)} \|P_V \mu_\varepsilon\|_2^2 d\gamma_{d,n}(V) \quad (4.70)$$

$$\leq C \iint_{x-y \in K(V_0^\perp, 2\lambda s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n} + C \int \Theta^{n,*}(x, \mu) d\mu(x). \quad (4.71)$$

Now we claim that

$$\int_{B(V_0, s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) = \lim_{\varepsilon \rightarrow 0} \int_{B(V_0, s)} \|P_V \mu_\varepsilon\|_2^2 d\gamma_{d,n}(V). \quad (4.72)$$

Note that (4.70) and (4.72) together imply (4.63) with  $2\lambda$  in place of  $\lambda$ .

Let  $\sigma$  be the measure on  $\mathbb{R}^d$  given by

$$\int f d\sigma = \int_{B(V_0, s)} \int_V f d\mathcal{L}^n d\gamma_{d,n}(V),$$

Then arguing analogously as in the proof of (4.15), we have

$$\int_{B(V_0, s)} \|P_V \mu_\varepsilon\|_2^2 d\gamma_{d,n}(V) = \int |\widehat{\mu}(x) \widehat{\phi}(\varepsilon x)|^2 d\sigma(x), \quad (4.73)$$

and

$$\int_{B(V_0, s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) = \int |\widehat{\mu}(x)|^2 d\sigma(x). \quad (4.74)$$

We split the proof of (4.72) into two cases. Suppose first that  $\int_{B(V_0, s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) < \infty$ . In this case, the dominated convergence theorem gives us

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{B(V_0, s)} \|P_V \mu_\varepsilon\|_2^2 d\gamma_{d,n}(V) &= \lim_{\varepsilon \rightarrow 0} \int |\widehat{\mu}(x) \widehat{\phi}(\varepsilon x)|^2 d\sigma(x) \\ &= \int |\widehat{\mu}(x)|^2 d\sigma(x) = \int_{B(V_0, s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V), \end{aligned}$$

which proves (4.72) in this case.

Now we consider the case  $\int_{B(V_0,s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) = \infty$ . By Fatou's lemma,

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0} \int_{B(V_0,s)} \|P_V \mu_\varepsilon\|_2^2 d\gamma_{d,n}(V) &= \liminf_{\varepsilon \rightarrow 0} \int |\widehat{\mu}(x) \widehat{\phi}(\varepsilon x)|^2 d\sigma(x) \\
&\geq \int \liminf_{\varepsilon \rightarrow 0} |\widehat{\mu}(x) \widehat{\phi}(\varepsilon x)|^2 d\sigma(x) \\
&= \int |\widehat{\mu}(x)|^2 d\sigma(x) \\
&= \int_{B(V_0,s)} \|P_V \mu\|_2^2 d\gamma_{d,n}(V) \\
&= \infty,
\end{aligned}$$

which proves (4.72) in this case. This completes the proof of (4.72). □



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