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ABSTRACT

Factorization homology is a homology theory on manifolds with coefficients in suitable E_n-algebras. In this paper, we use the minimal categorical background and maximal concreteness to study equivariant factorization homology in the V-framed case.

We work with a finite group G and an n-dimensional orthogonal G-representation V. The main results are:

(1) We construct a GTop-enriched category $\mathrm{Mfld}_n^{\mathrm{fr}_V}$. Its objects are V-framed G-manifolds of dimension n. The endomorphism operad of the object V is equivalent to the little V-disk operad.

(2) With this category, we define the equivariant factorization homology $\int_M A$ by a monadic bar construction.

(3) We prove the nonabelian Poincaré duality theorem using a geometrically-seen scanning map, which establishes a weak G-equivalence between $\int_M A$ and $\mathrm{Map}_*(M^+, \mathbf{B}^V A)$.

Here, M is a V-framed manifold, and M^+ is its one-point compactification. In the language of Guillou-May [GM17], the coefficient A is an algebra over the little V-disks operad and $\mathbf{B}^V A$ is the V-fold deloop of A.

The approach in this paper follows the non-equivariant treatment in [Mil15]. It is a global generalization of the delooping machines of [May72, GM17]. The nonabelian Poincaré duality theorem gives a simplicial filtration on the mapping space $\mathrm{Map}_*(M^+, \mathbf{B}^V A)$, thus offering a calculational tool.

There are other approaches of a different flavor to equivariant factorization homology, developed by [Hor19, Wee18]. In joint work with Horev and Klang, we give an alternative proof of the nonabelian Poincaré duality theorem in [HKZ] in Horev's context, together with an application to Thom G-spectra.

CHAPTER 1: INTRODUCTION

1.1 Factorization homology: history and equivariant

The language of factorization homology has been used to formulate and solve questions in many areas of mathematics. Among others, there are homological stability results in [KM18, Knu18], a reconstruction of the cyclotomic trace in [AMGR17] and the study of quantum field theory in [BZBJ18, CG16].

Non-equivariantly, factorization homology has multiple origins. The most well-known approach started in Bellinson-Drinfeld's study of an algebraic geometry approach to conformal field theory [BD04] under the name of Chiral Homology. Lurie [Lur, 5.5] and Ayala–Francis [AF15] introduced and extensively studied the algebraic topology analogue, named as factorization homology. This route relies heavily on ∞-categorical foundations. An alternative geometric model is Salvatore's configuration spaces with summable labels [Sal01]. This construction is close to the geometric intuition, but not homotopical. Yet another model, using the bar construction and developed by Andrade [And10], Miller [Mil15] and Miller–Kupers [KM18], is homotopically well-behaved while staying close to the geometric intuition of configuration spaces.

We take the third approach in this paper. To give an idea of the concept, we start with the non-equivariant story.

Classically, the Dold–Thom theorem states that the symmetric product is a homology theory. For a based CW-complex M with base point $*$, the symmetric product on M is $\mathrm{Symm}(M) = \left(\coprod_{k\geq 0} M^k/\Sigma_k\right)/\sim$, where $\sim$ is the base-point identification $(m_1, \cdots, m_k, *) \sim (m_1, \cdots, m_k)$. The Dold-Thom theorem states that when M is connected, there are natural isomorphisms $\pi_*(\mathrm{Symm}(M)) \cong \widetilde{H}_*(M, \mathbb{Z})$.

Factorization homology, viewed as a functor on manifolds, generalizes the homology theory of topological spaces. It uses the manifold structure to work with coefficients in the noncommutative setting. Essentially, $\int_M A$ is the configuration space on M with summable labels in an E_n-algebra A; the local Euclidean chart offers the way to sum the labels. Rigorously, [KM18] defines the factorization homology on M to be:

$$\int_M A = \mathrm{B}(\mathrm{D}_M, \mathrm{D}_n, A), \tag{1.1.1}$$

where D_n is the reduced monad associated to the little n-disks operad and D_M is the functor associated to embeddings of disks in M. We give the details in Chapter 2 using an elementary categorical framework of Λ-objects, developed in more detail in [MZZ20].

This bar construction definition is a concrete point-set level model of the ∞-categorical definition [Lur, AF15]. We can construct a topological category $\mathrm{Mfld}_n^{\mathrm{fr}}$ of framed smooth n-dimensional manifolds and framed embeddings. It is a symmetric monoidal category under taking disjoint union. Let $\mathrm{Disk}_n^{\mathrm{fr}}$ be the full subcategory spanned by objects equivalent to $\sqcup_k \mathbb{R}^n$ for some $k \geq 0$. An E_n-algebra A is just a symmetric monoidal topological functor out of $\mathrm{Disk}^{\mathrm{fr}}$. The factorization homology is the derived symmetric monoidal topological left Kan extension of A along the inclusion:

$$\begin{array}{ccc} \mathrm{Disk}_n^{\mathrm{fr}} & \xrightarrow{A} & (\mathrm{Top}, \times) \\ \big\downarrow & \nearrow_{\int_{-} A} & \\ \mathrm{Mfld}_n^{\mathrm{fr}} & & \end{array} \tag{1.1.2}$$

Horel [Hor13, 7.7] shows the equivalence of (1.1.1) and (1.1.2).

We could also view factorization homology as a functor on the algebra. This gives a geometric interpretation of some classical invariants of structured rings and a way to produce more. For example, THH of an associative ring is equivalent to the factorization homology on S^1. We will not make use of this perspective in this paper.

We point out one technicality of this bar construction that takes some effort equivariantly, namely, how to give the morphism space of $\mathrm{Mfld}_n^{\mathrm{fr}}$. On the one hand, from the definition of the little V-disks operad, we want to include only the "rectilinear" embeddings as framed embeddings; on the other hand, we lose control of any rectilinearily once we throw the little disks into the wild category of framed manifolds.

The solution is to allow all embeddings but add in path data to correct the homotopy type. This idea goes back to Steiner [Ste79] where he used paths to construct an especially useful E_n-operad. Miller–Kupers [KM18] used paths in the framing space to define framed embedding spaces so that they do not see the unwanted rotations.

An equivariant version of E_n-algebra is given by Guillou–May's little V-disks operad $\mathscr{D}_V$ and E_V-algebras in [GM17]. The E_V-algebras give the correct coefficient input of equivariant factorization homology on V-framed manifolds.

In Section 5.1 to Section 5.3, we construct the category $\mathrm{Mfld}_n^{\mathrm{fr}_V}$ of V-framed smooth G-manifolds of dimension n. A V-framing of M is a trivialization $\phi_M : \mathrm{T}M \cong M \times V$ of its tangent bundle. We put the V-framing into a general framework of tangential structures $\theta : B \to B_G O(n)$ and define the θ-framed embedding space of θ-framed manifolds.

In Section 5.4, we use the $G\mathrm{Top}$-enriched category $\mathrm{Mfld}_n^{\mathrm{fr}_V}$ to build V-framed factorization homology by a monadic bar construction. The ingredients to set up the bar construction are the V-framed little disks operad $\mathscr{D}_V^{\mathrm{fr}_V}$, the monad $\mathrm{D}_V^{\mathrm{fr}_V}$ and the functor $\mathrm{D}_M^{\mathrm{fr}_V}$ that is a right module over $\mathrm{D}_V^{\mathrm{fr}_V}$ (Definition 5.4.1).

Definition 1.1.3. (Definition 5.4.3) The equivariant factorization homology is:

$$\int_M A = \mathbf{B}(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A).$$

In Section 5.5, we study the homotopy type of the defined embedding space in $\mathrm{Mfld}_n^{\mathrm{fr}_V}$ and show that $\mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M)$, the V-framed embedding space, has the same homotopy type as $\mathscr{F}_M(k)$, the ordered configuration space of k points in M:

Theorem 1.1.4. *(Corollary 5.5.9(1)) Evaluating at* 0 *of the embedding gives a* $(G \times \Sigma_k)$*-homotopy equivalence:*

$$ev_0 : \mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M) \xrightarrow{\simeq} \mathscr{F}_M(k).$$

In particular, the V-framed little disks operad is equivalent to the Guillou–May little V-disks operad (Proposition 5.5.12), so it is an E_V-operad.

1.2 Nonabelian Poincaré duality theorem

Our main theorem is:

Theorem 1.2.1. *(Theorem 6.2.3) Let* M *be a* V*-framed manifold and* A *be a* G*-connected* $\mathrm{D}_V^{\mathrm{fr}_V}$*-algebra in* $G\mathrm{Top}$*. There is a weak* G*-equivalence:*

$$\int_M A \simeq \mathrm{Map}_*(M^+, \mathbf{B}^V A).$$

The proof of Theorem 1.2.1 is inspired by [Mil15]. There are two main ingredients: the recognition principal in [May72, GM17] for the local result, and the scanning map that has been studied non-equivariantly in [McD75, BM88, MT14].

In Section 6.1, we construct the scanning map, a natural transformation of right $\mathrm{D}_V^{\mathrm{fr}_V}$-functors:

$$s : \mathrm{D}_M^{\mathrm{fr}_V}(-) \to \mathrm{Map}_c(M, \Sigma^V -),$$

and compare it to the scanning maps in the literature in Appendix A.

In Section 6.2 to Section 6.5, we prove Theorem 1.2.1.

1.3 Equivariant bundle theory

Both the framing of a G-manifold and the construction of $\mathrm{Mfld}_n^{\mathrm{fr}_V}$ in this paper use the notion of equivariant bundles. This approach has the advantage of being very concrete. However, to work with it, we have to get our hands dirty in the quagmire of equivariant bundle theory, where everything is supposed to work like its non-equivariant analogue, but proofs are hard to find. In this paper, we summarize the literature and work out some missing details, which may be of independent interest.

Let G and Π be compact Lie groups, where G is the ambient action group and Π is the structure group. Fix a group extension $1 \to \Pi \to \Gamma \to G \to 1$.

In Chapter 3, we clarify different definitions of equivariant bundles. There are two concepts of G-fiber bundles that are in general different: with structure group Π and fiber F with Π-action as in Definition 3.2.1 or with structure group Π, total group Γ and fiber F with Γ-action as in Definition 3.2.12. One forthright, one ad hoc. Both of them have structure theorems (Theorem 3.2.6 and Theorem 3.2.19). In fact, a refinement of the first concept by specifying an extension Γ and the Γ-action on the fiber F is a special case of the second concept (Proposition 3.3.4).

In Chapter 4, we study the classifying space $B_G\Pi$. Theorem 4.3.3 gives an example of a naturally arising classifying space for equivariant principal bundles, but in the sense of the seemingly ad hoc Definition 3.2.8. Theorem 4.4.10 shows a weak G-equivalence between the free loop space $LB_G\Pi$ and the adjoint bundle $Ad(E_G\Pi) := E_G\Pi \times_\Pi \Pi_{\mathrm{ad}}$.

1.4 Notations

- GTop is the Top-enriched category of G-spaces and G-equivariant maps.
- Top_G is the GTop-enriched category of G-spaces and non-equivariant maps where G acts by conjugation on the mapping space.

We note the following facts:

(1) GTop is the underlying Top-enriched category of Top_G :

$$G\mathrm{Top}(X,Y) \cong G\mathrm{Top}(\mathrm{pt}, \mathrm{Top}_G(X,Y)).$$

(2) GTop is a closed Cartesian monoidal category. The interal hom G-space is given by the morphism in Top_G.

For orthogonal G-representations V and W, we use the following notations for the mapping spaces, all of which are G-subspaces of $\mathrm{Top}_G(V, W)$:

- $\mathrm{Hom}(V, W)$ for linear maps;
- $\mathrm{Iso}(V, W)$ for linear isomorphisms of vector spaces;
- $\mathrm{O}(V, W)$ for linear isometries;
- $\mathrm{O}(V)$ for $\mathrm{O}(V, V)$.

For a fiber bundle $E \to B$,

- $\mathrm{Aut}_B(E)$ is the space of bundle automorphisms of E that project to the identity map on B.

For a space X and $b \in X$,

- $P_b X$ is the path space of X at the base point b;
- $\Omega_b X$ is the loop space of X at the base point b;
- $\mathbf{\Lambda}_b X$ is the Moore loop space of X at the base point b, defined to be

$$\mathbf{\Lambda}_b X = \{(l, \alpha) \in \mathbb{R}_{\geq 0} \times X^{\mathbb{R}_{\geq 0}} | \alpha(0) = b, \quad \alpha(t) = b \text{ for } t \geq l\}.$$

For a space X, a vector space V and a map $\phi : V \to X$,

- $\Omega_\phi X$ is $\Omega_{\phi(0)} X$; $\mathbf{\Lambda}_\phi X$ is $\mathbf{\Lambda}_{\phi(0)} X$.

For based spaces X, Y and an unbased space M,

- $\mathrm{Map}_*(Y, X)$ is the space of based maps;
- $\mathrm{Map}_c(M, X) = \{f \in \mathrm{Map}(M, X) | \overline{f^{-1}(X \setminus *)} \text{ is compact}\}$ is the space of compactly supported maps.

For a space M and a fiber bundle $E \to M$,

- $\mathscr{F}_M(k)$ is the ordered configuration space of k points in M.
- $\mathscr{F}_{E\downarrow M}(k)$ is the ordered configuration space of k points in E whose images are k distinct points in M.

For an operad $\mathscr{C}$ in a symmetric monoidal category $\mathscr{V}$ and a monoid M in $\mathscr{V}$,

- $\mathscr{C}[\mathscr{V}]$ is the category of $\mathscr{C}$-algebras in $\mathscr{V}$.
- $M[\mathscr{V}]$ is the category of M-modules in $\mathscr{V}$. (So $G\mathrm{Top} = G[\mathrm{Top}]$.)

CHAPTER 2: Λ-SEQUENCES AND OPERADS

In a separate paper with May and Zhang [MZZ20], we study unital operads, reduced monads and bar constructions. This section is a summary of the relevant content for this paper.

Let Λ be the category of based finite sets $\mathbf{n} = \{0, 1, 2, \cdots, n\}$ with base point 0 and based injections. The morphisms of Λ are generated by permutations and the ordered injections $s_i^k : \mathbf{k-1} \to \mathbf{k}$ that skip i for $1 \leq i \leq k$. It is a symmetric monoidal category with wedge sum as the symmetric monoidal product. Let $(\mathscr{V}, \otimes, \mathcal{I})$ be a bicomplete symmetric monoidal category with initial object $\varnothing$, terminal object $*$. Let $\mathscr{V}_{\mathcal{I}}$ be the category under the unit. Later we will mostly be concerned about $(G\mathrm{Top}, \times, \mathrm{pt})$ which is Cartesian monoidal, so $G\mathrm{Top}_{\mathrm{pt}} = G\mathrm{Top}_*$ is the category of pointed G-spaces.

Definition 2.0.1. A Λ-sequence in $\mathscr{V}$ is a functor $\mathscr{E} : \Lambda^{op} \to \mathscr{V}$. It is called unital if $\mathscr{E}(\mathbf{0}) = \mathcal{I}$. The category of all Λ-sequences in $\mathscr{V}$ is denoted $\Lambda^{op}(\mathscr{V})$, where morphisms are natural transformations of functors. The category of all unital Λ-sequences in $\mathscr{V}$ is denoted $\Lambda^{op}_{\mathcal{I}}(\mathscr{V})$, where morphisms are natural transformations of functors that are identity at level zero.

The category $\Lambda^{op}[\mathscr{V}]$ admits a symmetric monoidal structure $(\Lambda^{op}[\mathscr{V}], \boxtimes, \mathscr{I}_0)$. It is the Day convolution of functors on the closed symmetric monoidal category Λ^{op}. The unit is given by

$$\mathscr{I}_0(n) = \begin{cases} \mathcal{I}, & n = 0; \\ \varnothing, & n > 0; \end{cases}$$

We write $\Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$ for the category of objects under the unit $\mathscr{I}_0$. The symmetric monoidal product $\boxtimes$ on $\Lambda^{op}[\mathscr{V}]$ induces a symmetric monoidal product on $\Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$ and its subcategory $\Lambda^{op}_{\mathcal{I}}[\mathscr{V}]$, which we still denote by $\boxtimes$.

Remark 2.0.2. To clarify a possible confusion with notation, note that $\mathscr{E} \in \Lambda^{op}_{\mathcal{I}}[\mathscr{V}]$ is a unital Λ-sequence with $\mathscr{E}(\mathbf{0}) = I$, while $\mathscr{F} \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$ comes with a specified map

$\mathcal{I} \to \mathscr{F}(\mathbf{0})$. $\mathscr{F}$ is called a *unitary* Λ-sequence in [MZZ20].

Both categories highlighted in the remark above admit a (nonsymmetric) monoidal product $\odot$ in addition to $\boxtimes$. It is analogous to Kelly's circle product on symmetric sequences [Kel05]. The unit for $\odot$ is given by

$$\mathscr{I}_1(n) = \begin{cases} \mathcal{I}, & n = 0, 1; \\ \varnothing, & n > 1; \end{cases}$$

where the only non-trivial morphism $\mathscr{I}_1(1) \to \mathscr{I}_1(0)$ is the identity. For a brief definition of $\odot$, see Construction 2.0.9 (2); for a detailed definition, see [MZZ20].

For a Λ-sequence $\mathscr{E}$, the spaces $\mathscr{E}(\mathbf{k})$ admit Σ_k-actions, so $\mathscr{E}$ has an underlying symmetric sequence. Though not relevant to this paper, it is surprising that the Day convolution of Λ-sequences agrees with the Day convolution of symmetric sequences:

Theorem 2.0.3. *([MZZ20, Theorem 3.3]) For $\mathscr{D}, \mathscr{E} \in \Lambda^{op}[\mathscr{V}]$, there is an isomorphism of symmetric sequences $\mathscr{D} \boxtimes_\Sigma \mathscr{E} \to \mathscr{D} \boxtimes_\Lambda \mathscr{E}$.* □

Of course, Kelly's circle product on symmetric sequences does not agree with the circle product on Λ-sequences.

An operad in $\mathscr{V}$, as defined in [May97], gives an example of a symmetric sequence in $\mathscr{V}$. If the operad is unital, meaning the 0-space of the operad is the unit, it has the structure of a Λ-sequence in $\mathscr{V}$. In fact, We have the unital variant of Kelly's observation [Kel05]:

Theorem 2.0.4. *([MZZ20, Theorem 0.10]) A unital operad in $\mathscr{V}$ is a monoid in the monoidal category $(\Lambda^{op}_{\mathcal{I}}[\mathscr{V}], \odot, \mathscr{I}_1)$.* □

When $\mathscr{V} = \mathrm{Top}$ or $\mathscr{V} = G\mathrm{Top}$, a unital operad is also called a reduced operad in [May97].

We give a construction which will be used in the definition of equivariant factorization homology: the associated functor of a unital Λ-sequence. This construction specializes to the reduced monad associated to a reduced operad of [May97] when $\mathscr{V}$ is Cartesian monoidal; it also appears in the definition of the circle product $\odot$.

Construction 2.0.5. Let $(\mathscr{W}, \otimes, \mathcal{J})$ be a symmetric monoidal category and $X \in \mathscr{W}_{\mathcal{J}}$ be an object under the unit. Define $X^* : \Lambda \to \mathscr{W}$ to be the covariant functor that sends $\mathbf{n}$ to $X^{\otimes n}$. On morphisms, it sends the permutations to permutations of the X's and sends the injection $s_i^k : \mathbf{k-1} \to \mathbf{k}$ for $1 \leq i \leq k$ to the map

$$(s_i^k)_* : X^{\otimes k-1} \cong X^{\otimes i-1} \otimes \mathcal{J} \otimes X^{\otimes k-i} \xrightarrow{\mathrm{id}^{i-1} \otimes \eta \otimes \mathrm{id}^{k-i}} X^{\otimes k},$$

where $\eta : \mathcal{J} \to X$ is the unit map of X. By convention, $X^{\otimes 0} = \mathcal{J}$.

This defines a functor $(-)^* : \mathscr{W}_{\mathcal{J}} \to \mathrm{Fun}^{\otimes}(\Lambda, \mathscr{W})$. Here, $\mathrm{Fun}^{\otimes}(\Lambda, \mathscr{W})$ is the category of strong symmetric monoidal functors from Λ to $\mathscr{W}$.

Remark 2.0.6. The above defined functor $(-)^*$ is indeed an equivalence with an inverse given by the forgetful functor $\mathrm{Fun}^{\otimes}(\Lambda, \mathscr{W}) \to \mathscr{W}_{\mathcal{J}}$ that sends $\mathcal{X}$ to $\mathcal{X}(\mathbf{1})$.

Assume that $(\mathscr{W}, \otimes, \mathcal{J})$ is a cocomplete symmetric monoidal category tensored over $\mathscr{V}$. Then one can form the categorical tensor product over Λ of the contravariant functor $\mathscr{E}$ and the covariant functor X^*.

Construction 2.0.7. Let $\mathscr{E} \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$ be a unitary Λ-sequence. The functor

$$\mathrm{E} : \mathscr{W}_{\mathcal{J}} \to \mathscr{W}_{\mathcal{J}}$$

associated to $\mathscr{E}$ is defined to be

$$\mathrm{E}(X) = \mathscr{E} \otimes_{\Lambda} X^* = \coprod_{k \geq 0} \mathscr{E}(k) \otimes X^{\otimes k} / \approx,$$

where $(\alpha^* f, \mathbf{x}) \approx (f, \alpha_* \mathbf{x})$ for all $f \in \mathscr{E}(m)$, $\mathbf{x} \in X^{\otimes n}$ and $\alpha \in \Lambda(\mathbf{n}, \mathbf{m})$. The unit map of $\mathrm{E}(X)$ is given by $\mathcal{J} \cong \mathcal{I} \otimes \mathcal{J} \to \mathscr{E}(0) \otimes X^{\otimes 0} \to \mathrm{E}(X)$.

Remark 2.0.8. It is sometimes useful to take the quotient in two steps and use the following alternative formula for E:

$$\mathrm{E}(X) = \coprod_{k \geq 0} \mathscr{E}(k) \otimes_{\Sigma_k} X^{\otimes k} / \sim,$$

where $[(s_i^k)^* f, \mathbf{x}] \sim [f, (s_i^k)_* \mathbf{x}]$ for all $f \in \mathscr{E}(k)$, $\mathbf{x} \in X^{\otimes k-1}$. We will use $\approx$ or $\sim$ for the equivalence relation to be clear which formula we are using and refer to $\sim$ as the base point identification.

Construction 2.0.9. We focus on the following context of Construction 2.0.7.

(1) Let $\mathscr{W} = \mathscr{V}$. The associated functor is $\mathrm{E} : \mathscr{V}_{\mathcal{I}} \to \mathscr{V}_{\mathcal{I}}$. In particular, taking $\mathscr{V} = G\mathrm{Top}$, one gets for a reduced G-operad $\mathscr{C} \in \Lambda_*^{op}(G\mathrm{Top})$ the *reduced monad*

$$\mathrm{C} : G\mathrm{Top}_* \to G\mathrm{Top}_*.$$

(2) Let $\mathscr{W} = (\Lambda^{op}[\mathscr{V}], \boxtimes, \mathscr{I}_0)$ via the Day monoidal structure. Then $\mathscr{W}$ is tensored over $\mathscr{V}$ in the obvious way by levelwise tensoring. One gets the *circle product* for $\mathscr{E} \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$ and $\mathscr{F} \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$ by:

$$\mathscr{E} \odot \mathscr{F} := \mathscr{E} \otimes_{\Lambda} \mathscr{F}^* \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}.$$

These two cases are further related: the 0-th level functor

$$\imath_0 : \mathscr{V} \to \Lambda^{op}[\mathscr{V}],\ (\imath_0 X)(n) = \begin{cases} X, & n = 0; \\ \varnothing, & n > 0; \end{cases}$$

gives an inclusion of a full symmetric monoidal subcategory, so we have

$$\imath_0(\mathrm{E}X) \cong \imath_0(\mathscr{E} \otimes_\Lambda X^*) \cong \mathscr{E} \otimes_\Lambda (\imath_0(X)^*) \cong \mathscr{E} \odot \imath_0 X. \tag{2.0.10}$$

In words, the reduced monad construction is what happens at the 0-space of the circle product. Using this, one can show

Proposition 2.0.11. *([MZZ20, Proposition 6.2]) Let* $\mathrm{E}, \mathrm{F} : \mathscr{V}_\mathcal{I} \to \mathscr{V}_\mathcal{I}$ *be the functors associated to* $\mathscr{E}$ *and* $\mathscr{F}$*. Then the functor associated to* $\mathscr{E} \odot \mathscr{F}$ *is* $\mathrm{E} \circ \mathrm{F}$. □

A monad is a monoid in the functor category. Using the associativity of the circle product and (2.0.10), it is easy to prove that when $\mathscr{C}$ is an operad, the associated functor C in Construction 2.0.7 is a monad.

The following construction gives examples of monoids and modules in $(\Lambda^{op}_\mathcal{I}[\mathscr{V}], \odot)$:

Construction 2.0.12. ([MZZ20, Section 8]) Suppose that we have a $\mathscr{V}$-enriched symmetric monoidal category $(\mathscr{W}, \otimes, \mathcal{I}_\mathscr{W})$ such that $\underline{\mathscr{W}}(\mathcal{I}_\mathscr{W}, Y) \cong \mathcal{I}_\mathscr{V}$ for all objects Y of $\mathscr{W}$. Then we can construct a $\Lambda^{op}_{\mathcal{I}_\mathscr{V}}[\mathscr{V}]$-enriched category $\mathcal{H}_\mathscr{W}$. The objects are the same as those of $\mathscr{W}$, while the enrichment is given by

$$\underline{\mathcal{H}_\mathscr{W}}(X, Y) = \underline{\mathscr{W}}(X^{\otimes *}, Y).$$

The definition of the composition in $\mathcal{H}_\mathscr{W}$ is similar to the structure maps of an endomorphism operad. So, for any objects X, Y, Z of $\mathscr{W}$, $\underline{\mathcal{H}_\mathscr{W}}(Y, Y)$ is monoid in $(\Lambda^{op}_\mathcal{I}[\mathscr{V}], \odot)$, $\underline{\mathcal{H}_\mathscr{W}}(X, Y)$ is a left module over it, and $\underline{\mathcal{H}_\mathscr{W}}(Y, Z)$ is a right module. In the light of Theorem 2.0.4, $\underline{\mathcal{H}_\mathscr{W}}(Y, Y)$ is a unital operad, the endomorphism operad. The assumption $\underline{\mathscr{W}}(\mathcal{I}_\mathscr{W}, Y) \cong \mathcal{I}_\mathscr{V}$ is automatically satisfied if $\mathscr{W}$ is coCartesian monoidal.

We will use that the circle product is strong symmetric monoidal in the first variable:

Proposition 2.0.13. *([MZZ20, Proposition 4.7]) For any $\mathscr{E} \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$, the functor $-\odot\mathscr{E}$ on $(\Lambda^{op}(\mathscr{V})_{\mathscr{I}_0}, \boxtimes, \mathscr{I}_0)$ is strong symmetric monoidal. That is, the circle product distributes over the Day convolution: for any $\mathscr{D}, \mathscr{D}' \in \Lambda^{op}(\mathscr{V})_{\mathscr{I}_0}$, we have*

$$(\mathscr{D} \boxtimes \mathscr{D}') \odot \mathscr{E} \cong (\mathscr{D} \odot \mathscr{E}) \boxtimes (\mathscr{D}' \odot \mathscr{E}). \quad \square$$

Just for comparison, the circle product is lax symmetric monoidal in the second variable if $\mathscr{V}$ is Cartesian monoidal:

Proposition 2.0.14. *Assume that $\mathscr{V}$ is Cartesian monoidal. Then for any $\mathscr{E} \in \Lambda^{op}[\mathscr{V}]_{\mathscr{I}_0}$, the functor $\mathscr{E} \odot -$ on $(\Lambda^{op}(\mathscr{V})_{\mathscr{I}_0}, \boxtimes, \mathscr{I}_0)$ is lax symmetric monoidal, but it is not strong monoidal in general. That is, for any $\mathscr{D}, \mathscr{D}' \in \Lambda^{op}(\mathscr{V})_{\mathscr{I}_0}$, we have natural transformation*

$$\mathscr{E} \odot (\mathscr{D} \boxtimes \mathscr{D}') \to (\mathscr{E} \odot \mathscr{D}) \boxtimes (\mathscr{E} \odot \mathscr{D}'),$$

but it is not an isomorphism.

Proof. We have

$$\begin{aligned}
\mathscr{E} \odot (\mathscr{D} \boxtimes \mathscr{D}') &\cong \int^{\mathbf{p}\in\Lambda} \mathscr{E}(\mathbf{p}) \otimes \left(\mathscr{D}^{\boxtimes \mathbf{p}} \boxtimes \mathscr{D}'^{\boxtimes \mathbf{p}}\right); \\
(\mathscr{E} \odot \mathscr{D}) \boxtimes (\mathscr{E} \odot \mathscr{D}') &\cong \int^{(\mathbf{q},\mathbf{r})\in\Lambda\times\Lambda} \left(\mathscr{E}(\mathbf{q}) \otimes \mathscr{D}^{\boxtimes \mathbf{q}}\right) \boxtimes \left(\mathscr{E}(\mathbf{r}) \otimes \mathscr{D}'^{\boxtimes \mathbf{r}}\right); \\
&\cong \int^{(\mathbf{q},\mathbf{r})\in\Lambda\times\Lambda} \mathscr{E}(\mathbf{q}) \otimes \mathscr{E}(\mathbf{r}) \otimes \left(\mathscr{D}^{\boxtimes \mathbf{q}} \boxtimes \mathscr{D}'^{\boxtimes \mathbf{r}}\right).
\end{aligned}$$

The natural transformation is the composite

$$\int^{\mathbf{p}\in\Lambda} \mathscr{E}(\mathbf{p}) \otimes \left(\mathscr{D}^{\boxtimes\mathbf{p}} \boxtimes \mathscr{D}'^{\boxtimes\mathbf{p}}\right) \to \int^{\mathbf{p}\in\Lambda} \mathscr{E}(\mathbf{p}) \otimes \mathscr{E}(\mathbf{p}) \otimes \left(\mathscr{D}^{\boxtimes\mathbf{p}} \boxtimes \mathscr{D}'^{\boxtimes\mathbf{p}}\right)$$
$$\to \int^{(\mathbf{q},\mathbf{r})\in\Lambda\times\Lambda} \mathscr{E}(\mathbf{q}) \otimes \mathscr{E}(\mathbf{r}) \otimes \left(\mathscr{D}^{\boxtimes\mathbf{q}} \boxtimes \mathscr{D}'^{\boxtimes\mathbf{r}}\right),$$

where the first map is induced by the diagonal $\mathscr{E}(\mathbf{p}) \to \mathscr{E}(\mathbf{p}) \otimes \mathscr{E}(\mathbf{p})$ and the second map is induced by the diagonal $\Delta : \Lambda \to \Lambda \times \Lambda$.

For a counter example, take an object $Y \in \mathscr{V}$ and take

$$\mathscr{E}(n) = \begin{cases} \mathcal{I} = *, & n = 0; \\ Y, & n = 1; \\ \varnothing, & n > 1. \end{cases}$$

It can be computed directly that

$$\mathscr{E} \odot (\mathscr{I}_1 \boxtimes \mathscr{I}_1)(\mathbf{2}) \cong Y \otimes \mathcal{I}[\Sigma_2];$$
$$(\mathscr{E} \boxtimes \mathscr{E})(\mathbf{2}) \cong (Y \otimes Y) \otimes \mathcal{I}[\Sigma_2],$$

and the natural transformation

$$\mathscr{E} \odot (\mathscr{I}_1 \boxtimes \mathscr{I}_1) \to \mathscr{E} \boxtimes \mathscr{E} \cong (\mathscr{E} \odot \mathscr{I}_1) \boxtimes (\mathscr{E} \odot \mathscr{I}_1)$$

is induced by $Y \to Y \otimes Y$ on the object $\mathbf{2}$. So it is not an isomorphism in general. □

CHAPTER 3: EQUIVARIANT BUNDLES

3.1 Non-equivariant bundles

We start with a review of non-equivariant bundles.

A *fiber bundle* with fiber F is a map $p : E \to B$ with an open cover $\{U_i\}$ of B and homeomorphisms $\phi_i : p^{-1}(U_i) \cong U_i \times F$. The U_i are called *coordinate neighborhoods* and the ϕ_i are called *local trivializations*.

The structure group of a fiber bundle gives information about how local trivializations change under changes of coordinate neighborhoods. Let Π be a topological group with an effective action on F. Here, *effective* means $\Pi \to \mathrm{Aut}(F)$ is an injection. A bundle with fiber F is said to have *structure group* Π, if for any two local trivialization $U_i \cap U_j \neq \varnothing$, the composite $\phi_i\phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ is given by $(b, f) \mapsto (b, g_{ij}(b)(f))$ for some continuous function $g_{ij} : U_i \cap U_j \to \Pi$, called a *coordinate transformation*. We always topologize $\mathrm{Aut}(F)$ with the compact-open topology of mapping spaces. If F is a compact Hausdorff space, $\mathrm{Aut}(F)$ is a topological group; If F is only locally compact, there are more technical assumptions for the inverse map to be continuous due to Arens (See [Ste51, I.5.4]). Morally, a fiber bundle with fiber F is automatically a fiber bundle with the implicit structure group $\mathrm{Aut}(F)$. Having an explicit structure group Π is extra data to reduce the structure group to a smaller one.

One can associate a principal Π-bundle to a fiber bundle with structure group Π. An *admissible map* of the bundle is a homeomorphism $\psi : F \to p^{-1}(b)$ for some $b \in U_i$, satisfying $\phi_i\psi \in \Pi$. The *associated principal* Π*-bundle* of p is the space of admissible maps.

The following immediate observation about admissible maps hides the local trivializations in the background.

Lemma 3.1.1. *A map $\psi : F \to F_b$ is admissible if and only if for any admissible map $\zeta : F \to F_b$, the composite $\zeta^{-1}\psi$ is in Π.* □

Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be two fiber bundles with fiber F and structure group Π. A *morphism* between them is a bundle map $\chi : E_1 \to E_2$ such that for any local trivializations $\phi_U : p_1^{-1}(U) \cong U \times F$ and $\phi_V : p_2^{-1}(V) \cong V \times F$, the composite

$$\phi_V \chi \phi_U^{-1} : (U \cap \chi^{-1}(V)) \times F \to (\chi(U) \cap V) \times F \tag{3.1.2}$$

is given by $(b, f) \mapsto (\chi(b), g_{VU}(b)(f))$, where $g_{VU} : U \cap \chi^{-1}(V) \to \Pi$ is some continuous function. Such a morphism induces a morphism between the two associated principal Π-bundles.

We pause to clarify a possible confusion regarding how to check that a bundle map χ is a morphism, that is, it respects the structure group. It seems as if one only need to check that χ sends an admissible map to an admissible map. However, this is not true, since the set of admissible maps does not see the topology.

Steenrod [Ste51, I.5] studied this difference carefully and concluded that the following Assumption 3.1.3 will resolve the discrepancy. We include some explanation here for completeness: What the set of admissible maps sees is an Ehresmann-Feldbau bundle with structure group Π, which has now become an obsolete notion. An Ehresmann-Feldbau bundle is a bundle $p : E \to B$ with fiber F and a set of homeomorphism $\psi : F \cong p^{-1}(b)$ for all $b \in B$, called admissible maps. It is required that for any $b \in U_i$, the composite $F = \{b\} \times F \to U_i \times F \xrightarrow{\phi_i^{-1}} p^{-1}(U_i)$ is admissible, and that for any $b \in B$ and any admissible map $\psi : F \to p^{-1}(b)$, all the admissible maps $F \to p^{-1}(b)$ are exactly $\psi \circ \nu$ for some $\nu \in \Pi$. While this aligns with Lemma 3.1.1 when the bundle has a structure group Π, there is a difference of the two notions, which lies exactly in that an Ehresmann-Feldbau bundle does not require Π to have a topology. In other words, the coordinate transformations g_{ij} are not asked to be continuous, which is equivalent to putting the trivial topology on Π. If Π does start life with a different topology, the coordinate transformations g_{ij} obtained from an Ehresmann-Feldbau bundle may not be continuous. However, [Ste51, I.5.4] shows that if

Π has the subspace topology in $\mathrm{Aut}(F)$, the g_{ij}'s are automatically continuous.

Assumption 3.1.3. We always assume that Π has the subspace topology of $\mathrm{Aut}(F)$.

With this assumption, a fiber bundle has structure group Π if and only if the the admissible maps satisfy Lemma 3.1.1. We have the following criteria:

Proposition 3.1.4. *A bundle map $\chi : E_1 \to E_2$ is a morphism of fiber bundles with structure group Π if and only if either of the two equivalent conditions is true:*

(1) If F_1 is a fiber in E_1 and F_2 is a fiber in E_2 such that χ maps F_1 to F_2, then the composite $\zeta^{-1}\chi\psi$ is in Π for any admissible maps $\psi : F \to F_1$ and $\zeta : F \to F_2$.

(2) For any admissible map $\psi : F \to F_1$ to a fiber in E_1, the composite $\chi\psi$ is an admissible map to a fiber in E_2.

Proof. We need to check that for any ϕ_U, ϕ_V as in (3.1.2), the desired g_{VU} exists. With Assumption 3.1.3, it suffices to check that for any $b \in U \cap \chi^{-1}(V)$, there exists a desired $g_{VU}(b) \in \Pi$. This is part (1). Part (2) follows from Lemma 3.1.1. □

Example 3.1.5. The most familiar case is when F is a vector space ($\mathbb{R}^n$ or $\mathbb{C}^n$) and $\Pi = GL_n$ is the corresponding general linear group. By definition of the general linear group, χ being a bundle map is equivalent to it being fiberwise linear and non-degenerate.

The following well-known structure theorem turns the problem of classifying fiber bundles into classifying principal bundles.

Theorem 3.1.6. *Let Π be a compact Lie group. Let B, F be spaces. Assume that Π acts effectively on F. Then there is an equivalence of categories between $\{$fiber bundles over B with fiber F and structure group $\Pi\}$ and $\{$principal Π-bundles over $B\}$.*

Proof. We have already shown how to construct a principal Π bundle from a fiber bundle with fiber F and structural group Π at the beginning of this section. In the other direction, given a principal Π-bundle $P \to B$, the map $P \times_\Pi F \to B$ is a fiber bundle with fiber F and structure group Π. These two constructions are functorial and inverse of each other. Indeed, [Ste51, I] described both types of bundles using local transformations, called coordinate bundles, where the equivalence becomes transparent. □

3.2 Definitions of equivariant bundles

When it comes to the equivariant story, there are definitions of different generality, both on the fiber bundle side and on the principal bundle side. The reason is that the ambient group G could interact non-trivially with the structure group Π. We start with the simplest definition where "G and Π commute" [Las82]. Let G, Π be compact Lie groups in this section.

Definition 3.2.1. A G-fiber bundle with fiber F and structure group Π is a map $p : E \to B$ such that the following statements hold:

(1) The map p is a non-equivariant fiber bundle with fiber F and structure group Π;

(2) Both E and B are G-spaces and p is G-equivariant;

(3) The G-action is given by morphisms of bundles with structure group Π.

Proposition 3.2.2. *The requirement in (3) above is equivalent to the following: for any $g \in G$ and admissible map $\psi : F \to F_b$, the composite $F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb}$ is also admissible.*

Proof. By Proposition 3.1.4. □

Remark 3.2.3. Let G be a finite group. We take $F = \mathbb{R}^n$ and $\Pi = \mathrm{GL}_n(\mathbb{R})$ in Definition 3.2.1. Although $\mathrm{GL}_n(\mathbb{R})$ is not compact, the definition still works and we obtain a G-n-vector bundle.

Definition 3.2.4. A principal G-Π-bundle is a map $p : P \to B$ such that the following statements hold:

(1) The map p is a non-equivariant principal Π-bundle;

(2) Both P and B are G-spaces and p is G-equivariant;

(3) The actions of G and Π commute on P.

Remark 3.2.5. This is called a principal (G, Π)-bundle in [LMSM86, IV1].

As in the non-equivariant case, we write the Π-action on the right of a principal G-Π-bundle P; for convenience of diagonal action, we consider P to have a left Π-action, that is, $\nu \in \Pi$ acts on $z \in P$ by $\nu(z) = z\nu^{-1}$.

The structure theorem formally passes to this equivariant context.

Theorem 3.2.6. *Let G, Π be compact Lie groups and F, B be spaces. Assume that Π acts effectively on F. Then there is an equivalence of categories between $\{G$-fiber bundles over B with fiber F and structure group $\Pi\}$ and $\{$principal G-Π-bundles over $B\}$.*

Proof. The two types of G-bundles in Definitions 3.2.1 and 3.2.4 are indeed objects with a G-action in the corresponding non-equivariant category. So the equivalence in the non-equivariant structure theorem restricts to give an equivalence on the G-objects. □

However, Definitions 3.2.1 and 3.2.4 are not ideal for studying some interesting cases. In the most general scenario, we want to study a map $p : E \to B$ that happens to be both a fiber bundle with structure group Π and a G-map between G-spaces. It is true that p is a G-fiber bundle with structure group $\mathrm{Aut}(F)$, but p is usually not a G-fiber bundle with structure group Π. In other words, we can't reduce the structure group even though we know non-equivariantly it reduces to Π. Below, we give two concrete examples of this sort.

The first example is Atiyah's Real vector bundles [Ati66]. Let $G = C_2$. A Real vector bundle is a map $p : E \to B$ such that

- The map p is a complex vector bundle of dimension n;

- The non-trivial element of C_2 acts anti-complex-linearly.

In this case, p is a C_2-bundle with structure group $O(2n)$, but not $U(n)$.

The second simple but illuminating example is from [LMSM86].

Example 3.2.7. For G-spaces B and F, the projection $p : B \times F \to B$ is not a G-bundle with structure group e unless G acts trivially on F.

Proof. The admissible maps for p are only the inclusions of fibers

$$\psi_b : \{b\} \times F \to B \times F.$$

An element $g \in G$ acts by a bundle map if and only if for all b, the composite

$$\{b\} \times F \xrightarrow{\psi_b} p^{-1}(b) \xrightarrow{g} p^{-1}(gb) \xrightarrow{\psi_{gb}^{-1}} \{gb\} \times F$$

is in the structure group. But this map is merely the g action on F. □

Consequently, we would like a more general version than Definitions 3.2.1 and 3.2.4. To work with Real vector bundles, tom Dieck [TD69] introduced a complex conjugation action of C_2 on $U(n)$. Lashof–May [LM86] had the idea to further introduce a total group that is the extension of the structure group Π by G. Tom Dieck's work became a special case of a split extension, or equivalently a semidirect product. One good, but rather brief and sketchy, early reference for both is [LMSM86, IV1]; we shall flesh out that source and come back to the two examples afterwards.

We start with the well studied principal bundle story.

Definition 3.2.8. ([LM86]) Let $1 \to \Pi \to \Gamma \to G \to 1$ be an extension of compact Lie groups. A principal $(\Pi; \Gamma)$-bundle is a map $p : P \to B$ such that the following statements hold:

(1) The map p is a non-equivariant principal Π-bundle;

(2) The space P is a Γ-space; B is a G-space. Viewing B as a Γ-space by pulling back the action, the map p is Γ-equivariant.

Definition 3.2.9. A morphism between two principal $(\Pi;\Gamma)$-bundles $p_1 : P_1 \to B_1$ and $p_2 : P_2 \to B_2$ is a pair of maps $(\bar{f}, f)$ fitting in the commutative diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{f}} & P_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

such that f is G-equivariant and $\bar{f}$ is Γ-equivariant.

Example 3.2.10. Let $y \in \Gamma$ be with image $g \in G$. The action map (y, g) is an automorphism.

Taking $\Gamma = \Pi \times G$, we recover the principal G-Π-bundles of Definition 3.2.4. In this case we have two names for the same thing. This could be confusing, but since a "principal G-Π-bundle" looks more natural than a "principal $(\Pi; \Pi \times G)$-bundle" for this thing, we will keep both names.

Taking Γ to be a split extension, or equivalently $\Gamma = \Pi \rtimes_\alpha G$ for some group homomorphism $\alpha : G \to \mathrm{Aut}(\Pi)$, we recover tom Dieck's principal (G, α, Π)-bundles.

Remark 3.2.11. To be useful later, we write the elements of $\Gamma = \Pi \rtimes_\alpha G$ as (ν, g) for $\nu \in \Pi, g \in G$ and write $\alpha(g) \in \mathrm{Aut}(\Pi)$ as α_g. We have the following facts:

- The product in Γ is given by $(\nu, g)(\mu, h) = (\nu\alpha_g(\mu), gh)$ (That is, g acts on μ when they interchange);
- The identity element is (e, e);
- The inverse is $(\nu, g)^{-1} = (\alpha_{g^{-1}}(\nu^{-1}), g^{-1})$;

- The elements (e, g) form a subgroup of Γ that is canonically isomorphic to G;

- A space with Γ-action is a space with both Π and G actions such that

$$\nu(g(-)) = g(\alpha_g(\nu)(-)), \text{ which is indeed } (\nu, g)(-).$$

The fiber bundle story is not as satisfactory, as the appropriate fiber of an equivariant fiber bundle is not just the preimage of any point, but rather with a preassigned action of Γ. This is unnatural at first glance, for example in a G-vector bundle we won't expect there to be an $(O(n) \times G)$-action on the fiber $\mathbb{R}^n$. We will explain why this is necessary and how G-vector bundles fit in this context later. Let us start with the definition:

Definition 3.2.12. ([LMSM86, IV1]) Let $1 \to \Pi \to \Gamma \to G \to 1$ be an extension of compact Lie groups and F be a space with Γ-action. A G-fiber bundle with fiber F, structure group Π and total group Γ is a map $p : E \to B$ such that the following statements hold:

(1) The map p is a non-equivariant fiber bundle with fiber F and structure group Π;

(2) Both E, B are G-spaces and p is a G-map;

(3) For any $g \in G$ and admissible maps $\psi : F \to F_b$ and $\zeta : F \to F_{gb}$, the composite

$$F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb} \xrightarrow{\zeta^{-1}} F$$

is a lift $y \in \Gamma$ of $g \in G$. In other words, the y in the following diagram is asked to be a lift of $g \in G$ in Γ:

$$\begin{array}{ccc} F & \overset{y}{\dashrightarrow} & F \\ \psi \downarrow \cong & & \cong \downarrow \zeta \\ F_b & \xrightarrow{g} & F_{gb} \end{array}$$

Proposition 3.2.13. *The requirement (3) above is equivalent to the following: For each $y \in \Gamma$ with image $g \in G$ and admissible map $\psi : F \to F_b$, the composite*

$$F \xrightarrow{y^{-1}} F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb}$$

is also admissible.

Proof. For any two lifts y and y' of g, $y'y^{-1}$ is a lift of $e \in G$, so it is in Π. The claim then follows from Lemma 3.1.1. □

Taking $g = e$ in (3) or Proposition 3.2.13, the lifts y are exactly elements of the structure group Π, so we just see the non-equivariant structure group (compare with Lemma 3.1.1); Taking general g, the assignment $\psi \mapsto g\psi y^{-1}$ is mimicking the action by an element of Π on the admissible map ψ, but it changes the fiber from over b to over gb. In this sense, the definition uses the extension of the structure group Π to the total group Γ to regulate admissible maps to fibers over elements of the orbit of b.

Definition 3.2.14. Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be two G-fiber bundles with fiber F, structure group Π and total group Γ. A morphism between them is a pair of maps $(\bar{f}, f)$ fitting in the commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

such that the following statements hold:

(1) The pair $(\bar{f}, f)$ is a non-equivariant morphism between bundles with fiber F and structure group Π.

(2) Both $\bar{f}$ and f are G-equivariant.

Remark 3.2.15. By Proposition 3.1.4, the requirement (1) above is explicitly the following: For any admissible map $\psi : F \to F_1$ to a fiber in E_1, the composite $\bar{f}\psi$ is an admissible map to a fiber in E_2.

We do not have a requirement on a morphism regarding the condition Definition 3.2.12 (3) because it is automatic: if ψ is admissible, we have that $g\psi y^{-1}$ is admissible and so is $\bar{f}(g\psi y^{-1})$. But $\bar{f}g = g\bar{f}$, so $g(\bar{f}\psi)y^{-1}$ is also admissible.

As opposed to Definition 3.2.1, in Definition 3.2.12 the Γ-action on the total space E can restrict to a G-action only when there is a splitting of the extension given by $G \to \Gamma$. The following example illustrates that varying the splitting map can give different G-fiber bundle descriptions of the same bundle. It will be discussed in Corollary 3.3.11.

Example 3.2.16. A G-n-vector bundle is both a G-fiber bundle with fiber $\mathbb{R}^n$, structure group $O(n)$ and total group $O(n) \times G$ and a G-fiber bundle with fiber V, structure group $O(V)$ and total group $O(V) \rtimes G$. (Here, we take $\Gamma = O(n) \times G \cong O(V) \rtimes G$.)

Example 3.2.17. A Real vector bundle is a C_2-fiber bundle with fiber $\mathbb{C}^n$, structure group $U(n)$ and total group $\Gamma = U(n) \rtimes_\alpha C_2$, where $\alpha : C_2 \to \mathrm{Aut}(U(n))$ sends the non-trivial element of C_2 to the entry-wise complex-conjugation of $U(n)$.

Proof. Let the non-trivial element a of C_2 act by complex conjugation on $\mathbb{C}^n$. This extends the $U(n)$-action to a Γ-action by Remark 3.2.11. We only need to check that Definition 3.2.12 (3) holds for $g = a$. An automorphism X of $\mathbb{C}^n$ is anti-complex-linear if and only if $A = X \circ a$, the pre-composition of X with conjugation, is complex-linear. So A is an element of $U(n)$, and $X = (A, a)$ is the lift of a in $U(n) \rtimes_\alpha C_2$. □

Example 3.2.18. For G-spaces B and F, the projection $B \times F \to B$ is a G-fiber bundle with fiber F, structure group e and total group $\Gamma = G$.

Proof. The proof in Example 3.2.7 verifies Definition 3.2.12 (3). □

It is unexpected that even when $\Gamma = \Pi \times G$, Definitions 3.2.1 and 3.2.12 are different. On the one hand, a G-fiber bundle in the first sense needs extra data to be one in the second sense, as we will show shortly in Proposition 3.3.4. On the other hand, as we saw in Example 3.2.7, if G acts non-trivially on F, then the projection $B \times F \to F$ is not a G-bundle with structure group e in the first sense, but it is a G-fiber bundle with structure group e and total group G in the second sense.

We have the following structure theorem in the context of Definitions 3.2.8 and 3.2.12:

Theorem 3.2.19. *([LMSM86, IV1]) For any Π-effective Γ-space F and G-space B, there is an equivalence of categories between $\{G$-fiber bundles with structure group Π, total group Γ and fiber F over $B\}$ and $\{$principal $(\Pi;\Gamma)$-bundles over $B\}$.*

Proof. This is an expansion of the sketchy proof in the reference. For brevity, we refer to the two categories as equivariant fiber bundles and equivariant principal bundles when there is no confusion.

Given an equivariant fiber bundle $E \to B$, we take the non-equivariant associated principal bundle $\mathrm{Pr}(E) \to B$. It suffices to give a Γ-action on $\mathrm{Pr}(E)$ such that $\mathrm{Pr}(E) \to B$ is a G-map. For $y \in \Gamma$ with image $g \in G$ and an admissible map $\psi : F \to F_b$, let $y(\psi) = g\psi y^{-1}$. By Proposition 3.2.13, $g\psi y^{-1}$ is an admissible map to the fiber over gb. This shows that $\mathrm{Pr}(E) \to B$ is an equivariant principal bundle.

Given an equivariant principal bundle $P \to B$, let $E = (P \times F)/\Pi \to B$ be the fiber bundle with admissible maps $\psi_p : F \to E$ of the form $\psi_p(f) = [p, f]$ for some $p \in P$. We verify the three conditions for $E \to B$ to be an equivariant fiber bundle. Firstly, $E \to B$ is a non-equivariant fiber bundle with structure group Π. Secondly, we describe the G-action on E. Take the diagonal Γ-action on $P \times F$. For any space with Γ-action X, we can define a $\Gamma/\Pi \cong G$-action on X/Π by lifting $g \in G$ to $y \in \Gamma$ and let $g[x] = [yx]$ for $x \in X$. Since Π is a normal subgroup of Γ, this is a well defined action independent of choice of y or representative x. For $X = P \times F$, this gives $(P \times F)/\Pi$ a G-action. Since $P \to B$ is

Γ-equivariant, it can be checked that $E \to B$ is G-equivariant. Thirdly, we show that the condition in Proposition 3.2.13 is satisfied. In fact, for $y \in \Gamma$ lifting $g \in G$ and $p \in P$, we have $g\psi_p y^{-1} = \psi_{yp}$. To see this, evaluating on $f \in F$, we have

$$\begin{aligned} g\psi_p y^{-1}(f) &= g[p, y^{-1}f] && \text{definition of } \psi; \\ &= [yp, yy^{-1}f] && \text{definition of } G\text{-action}; \\ &= [yp, f] = \psi_{yp}(f) && \text{definition of } \psi. \end{aligned}$$

These two constructions give inverse functors. Given an equivariant fiber bundle $E \to B$, we have a map

$$\xi : (\mathrm{Pr}(E) \times F)/\Pi \to E,\ \xi([\psi, f]) = \psi(f).$$

Non-equivariantly we already know that (ξ, id_B) is a morphism of fiber bundles with structure group Π and that ξ is a homeomorphism. To check that ξ is G-equivariant, suppose $g \in G$ lifts to $y \in \Gamma$. Then

$$g([\psi, f]) = [y(\psi), yf] = [g\psi y^{-1}, yf]$$

and $\xi([g\psi y^{-1}, yf]) = (g\psi y^{-1})(yf) = g(\psi(f))$. So (ξ, id_B) is a morphism of equivariant fiber bundles by Definition 3.2.14. It is an isomorphism because the non-equivariant inverse is also an equivariant inverse as it is a homeomorphism. Given an equivariant principal bundle $P \to B$, we have a map which we abusively denote by

$$\psi : P \to \mathrm{Pr}((P \times F)/\Pi),\ p \mapsto \psi_p.$$

Here, ψ_p is the previously defined admissible map of $(P \times F)/\Pi$, thus an element of its associated principal bundle. Again, non-equivariantly we know that the map ψ is a homeomorphism (the Π-effectiveness is needed to assure that if $p \neq q$ in P, then $\psi_p \neq \psi_q$). To check that ψ is Γ-equivariant, the definition of the Γ-action on admissible maps gives

$y\psi_p = g\psi_p y^{-1}$ and we have verified $g\psi_p y^{-1} = \psi_{yp}$, so we have $y\psi_p = \psi_{yp}$. Thus, (ψ, id_B) is a morphism of equivariant principal bundles. It is also an isomorphism. □

We can see in the proof that it is essential for F to have a Γ-action. If P is a principal $(\Pi;\Gamma)$-bundle and the fiber F only had a Π-action, then the associated fiber bundle $(P \times F)/\Pi$ would not have a G-action. If we insist on our notion of a G-fiber bundle to be a G-map between G-spaces, this is the price to pay.

3.3 Comparisons of definitions

We have two concepts of G-fiber bundles. One is the G-fiber bundle with fiber F and structure group Π as in Definition 3.2.1; the other is the G-fiber bundle with fiber F, structure group Π and total group Γ for a specific extension of compact Lie groups $1 \to \Pi \to \Gamma \to G \to 1$, as in Definition 3.2.12. The differences between the concepts are two-fold: in the first one, G acts by bundle maps, but in the second one, the G-action is regulated by Γ; in the first one, F has only a Π-action, but in the second one, F has a Γ-action. We compare these two concepts and show that the first concept is a special case of the second where $\Gamma \cong \Pi \times G$ and Γ acts on F via the projection $\Pi \times G \to \Pi$ (Proposition 3.3.4).

We start with some simple group theory observations that will come into play.

Definition 3.3.1. A retraction $\Gamma \to \Pi$ is a group homomorphism that restricts to the identity on the subgroup Π.

It turns out that Γ admits a retraction to Π if and only if it is isomorphic to $\Pi \times G$. We prove this explicitly in the case of a semidirect product first, then for general Γ.

Proposition 3.3.2. *Let $\Gamma = \Pi \rtimes_\alpha G$ be a split extension. Then*

(1) The retractions $\tilde{\beta} : \Gamma \to \Pi$ are in bijection to homomorphisms $\beta : G \to \Pi$ satisfying $\alpha_g(\nu) = \beta(g)\nu\beta(g)^{-1}$ for all $g \in G$ and $\nu \in \Pi$. (Note that for a given $\alpha : G \to \mathrm{Aut}(\Pi)$, the homomorphism β may not exist.)

(2) Each β in (1) specifies an isomorphism $\Pi \rtimes_\alpha G \cong \Pi \times G$.

Proof. To see (1), we use the explicit expression for semidirect product as in Remark 3.2.11. Let $\beta(g)$ be the image $\tilde{\beta}(e, g)$. Then β is a group homomorphism. We have $\tilde{\beta}(\nu, e) = \nu$ and

$$\tilde{\beta}(\nu, g) = \tilde{\beta}((\nu, e)(e, g)) = \nu\beta(g).$$

In order for $\tilde{\beta}$ to be a homomorphism, it is required that the following two elements are equal for all $g, h \in G$ and $\nu, \mu \in \Pi$:

$$\tilde{\beta}(\nu\alpha_g(\mu), gh) = \nu\alpha_g(\mu)\beta(gh);$$
$$\tilde{\beta}(\nu, g)\tilde{\beta}(\mu, h) = \nu\beta(g)\mu\beta(h).$$

Comparing the two lines gives the conclusion.

Given such a β, the group isomorphism in (2) is given by

$$\Pi \rtimes_\alpha G \cong \Pi \times G, \ (\nu, g) \mapsto (\nu\beta(g), g).$$ ☐

Proposition 3.3.3. *There is a bijective correspondence between {retractions $\tilde{\beta} : \Gamma \to \Pi$} and {isomorphisms of extensions $\Gamma \cong \Pi \times G$}.*

Proof. Consider Π as a subgroup of Γ and denote by q the surjection $\Gamma \to G$. Given a retraction $\tilde{\beta} : \Gamma \to \Pi$, the map $(\tilde{\beta}, q) : \Gamma \to \Pi \times G$ is a group isomorphism, and vice versa.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \overset{\tilde{\beta}}{\leftarrow\!\!\cdots\!\!\rightarrow} & \Gamma & \overset{q}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle (\tilde{\beta}, q)} & & \| & & \\
1 & \longrightarrow & \Pi & \longrightarrow & \Pi \times G & \longrightarrow & G & \longrightarrow & 1
\end{array}$$ ☐

We now compare Definitions 3.2.1 and 3.2.12 in the following propositions. Note that we can think about a retraction $\Gamma \to \Pi$ as a chosen isomorphism $\Gamma \cong \Pi \times G$ of extensions by

Proposition 3.3.3.

Proposition 3.3.4. *Let F be a space with an effective Π-action and $1 \to \Pi \to \Gamma \to G \to 1$ be an extension of compact Lie groups. Then one can extend the Π-action on F to a Γ-action such that a G-fiber bundle of Definition 3.2.1 is always a G-fiber bundle of Definition 3.2.12 if and only if there is a retraction $\Gamma \to \Pi$ and the Γ-action on F is via the retraction.*

Proof. Suppose we have $p : E \to B$ as in Definition 3.2.1 and F has an extended Γ-action. Then the only thing to check for p to be a G-fiber bundle of Definition 3.2.1 is whether it satisfies the condition in Proposition 3.2.13. That is, it suffices to show for each $y \in \Gamma$ with image $g \in G$ and admissible homeomorphism $\psi : F \to F_b$, the composite $g\psi y^{-1}$ is also admissible. By Proposition 3.2.2, $g\psi$ is admissible. So by Lemma 3.1.1, for $y \in \Gamma$, $g\psi y^{-1}$ is admissible if and only if y acts on F as an element in Π. In other words, the group homomorphism $\Gamma \to \mathrm{Aut}(F)$ factors through $\Pi \to \mathrm{Aut}(F)$. □

The converse is also true.

Proposition 3.3.5. *Let $1 \to \Pi \to \Gamma \to G \to 1$ be an extension of compact Lie groups and F be a Π-effective Γ-space. Then a G-fiber bundle of Definition 3.2.12 is always a G-fiber bundle of Definition 3.2.1 if and only if Γ acts on F via a retraction $\Gamma \to \Pi$.*

Proof. We can reverse the argument in Proposition 3.3.4. Suppose we have $p : E \to B$ as in Definition 3.2.12; to check whether p is a G-fiber bundle of Definition 3.2.1, we only need to check whether the condition in Proposition 3.2.2 holds. Take any admissible homeomorphism $\psi : F \to F_b$. By Proposition 3.2.13, for any $y \in \Gamma$ with image $g \in G$, $g\psi y^{-1}$ is admissible. By Lemma 3.1.1, $g\psi$ is admissible if and only if y acts on F as an element in Π, □

Using Propositions 3.3.4 and 3.3.5, we can identity some special cases when the two notions of fiber bundles do agree.

Example 3.3.6. Let $\Gamma = \Pi \times G$ and F be a space with an effective Π-action. We give F the trivial G-action. Equivalently, this is viewing F as a space with Γ-action via the projection $\Gamma \to \Pi$. In this perspective, the structure theorem Theorem 3.2.6 is a special case of Theorem 3.2.19.

Example 3.3.7. In particular, let $\Gamma = O(n) \times G$ and give $\mathbb{R}^n$ the usual $O(n)$-action and the trivial G-action. We have an equivalence of the two concepts:

- G-vector bundles with fiber $\mathbb{R}^n$ (the classical G-equivariant vector bundles);
- G-fiber bundles with fiber $\mathbb{R}^n$, structure group $O(n)$ and total group $O(n) \times G$.

Example 3.3.8 (non-example)**.** For a Real vector bundle as in Example 3.2.17, Γ does not act on $\mathbb{C}^n$ via $U(n)$ for any n. So a Real vector bundle is not a C_2-fiber bundle with fiber $\mathbb{C}^n$ and structure group $U(n)$.

Proof. There is no retraction $\Gamma \to U(n)$, because otherwise by Proposition 3.3.2, we would need an element $\beta(a)$ of $U(n)$ such that $\beta(a)A = \bar{A}\beta(a)$ for all $A \in U(n)$, where $\bar{A}$ is the complex conjugation of A. But this does not exist for any n. □

In the extension $1 \to \Pi \to \Gamma \to G \to 1$, the group G is redundant because it is just Γ/Π. However, due to the special role of the group G in equivariant homotopy theory, we would like to understand the G-action wherever applicable. Since the total space of a principal $(\Pi;\Gamma)$-bundle has only a Γ-action, we now focus on the case of split extensions, when we have a specified group homomorphism $G \to \Gamma$. This becomes relevant at the end of this section when we define and study the V-framing bundle of a G-vector bundle for representations V. It turns out that $\mathrm{Pr}_V(E)$ and $\mathrm{Pr}_{\mathbb{R}^n}(E)$ are the same even as principal $(\Pi;\Gamma)$-bundles, but they have different G-actions.

Using Example 3.3.6 and Proposition 3.3.2, one can do some yoga with the fiber F. Fix a group homomorphism $\beta : G \to \Pi$. Let $\alpha : G \to \mathrm{Aut}(\Pi)$ be the group homomorphism given by

$$\alpha_g(\nu) = \beta(g)\nu\beta(g)^{-1}, \tag{3.3.9}$$

and the β determines an isomorphism (Proposition 3.3.2)

$$\Pi \rtimes_\alpha G \cong \Pi \times G. \tag{3.3.10}$$

Let F be a space with an effective Π-action. We can let the groups in (3.3.10) act on F via the retraction to Π. For clarity, we denote this space by F'. Explicitly, $(\Pi \times G)$ acts on F' by G acting trivially; $(\Pi \rtimes_\alpha G)$ acts on F' by

$$(\nu, g)(x) = \nu\big(\rho(g)(x)\big) \text{ for } x \in F'.$$

We point out that inclusion to the second coordinate gives a canonical inclusion of G into both $\Pi \times G$ and $\Pi \rtimes_\alpha G$, but this is not compatible with the isomorphism (3.3.10). The second image is in fact the graph subgroup $\Lambda_\beta = \{(\beta(g), g) | g \in G\}$. Consequently, the two G-actions on F' are different.

In summary, we have a commutative diagram of split extensions in the situation:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Pi \times G & \underset{\overset{(e,g)\leftarrow\!\shortmid g}{\longleftarrow}}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & \| & & \downarrow\cong & & \| & & \\
1 & \longrightarrow & \Pi & \longrightarrow & \Pi \rtimes_\alpha G & \overset{\longrightarrow}{\underset{(\beta(g),g)\leftarrow\!\shortmid g}{\longleftarrow}} & G & \longrightarrow & 1
\end{array}$$

As a consequence, we get the following trivial corollary of Propositions 3.3.4 and 3.3.5:

Corollary 3.3.11. *In the context above, for a group homomorphism* $\alpha : G \to \mathrm{Aut}(\Pi)$ *given by (3.3.9) with associated isomorphism (3.3.10), the following categories are equivalent:*

- *A G-fiber bundle with fiber F and structure group Π;*
- *A G-fiber bundle with fiber F', structure group Π and total group $\Pi \times G$;*

- *A G-fiber bundle with fiber F', structure group Π and total group $\Pi \rtimes_\alpha G$.* $\square$

Similarly, a principal $(\Pi; \Pi \times G)$-bundle is literally the same thing as a principal $(\Pi; \Pi \rtimes_\alpha G)$-bundle, but they have different specified G-actions.

Notation 3.3.12. For a principal G-Π-bundle, we call it a principal $(\Pi; \Pi \times G)$-bundle if we let G act on the total space by $G \subset \Pi \times G$; we call it a principal $(\Pi; \Pi \rtimes G)$-bundle if we let G act on the total space by $\Lambda_\beta \subset \Pi \times G$. And similarly for a G-fiber bundle with fiber F and structure group Π.

This trivial observation allows us to define and study the V-framing bundle of an equivariant vector bundle. Let V be an orthogonal G-representation given by $\rho : G \to O(n)$. In the remainder of this section, we write $O(V)$ for the group $O(n)$ with the data $G \to \mathrm{Aut}(O(n))$ given by $g(\nu) = \rho(g)\nu\rho(g)^{-1}$ for $g \in G$ and $\nu \in O(n)$, so it is clear what $O(V) \rtimes G$ means. This convention coincides with the conjugation G-action on $O(V)$ thought of as a mapping space in Top_G. In this case, taking $F = \mathbb{R}^n$ and pointing aloud the G-action on F', Corollary 3.3.11 reads: A G-n-vector bundle is a G-fiber bundle with fiber $\mathbb{R}^n$, structure group $O(n)$ and total group $O(n) \times G$, as well as a G-fiber bundle with fiber V, structure group $O(V)$ and total group $O(V) \rtimes G$.

Definition 3.3.13. Let $p : E \to B$ be a G-n-vector bundle. Let $\mathrm{Pr}_V(E)$ be the space of the admissible maps with the G-action $g(\psi) = g\psi\rho(g)^{-1}$.

In other words, $\mathrm{Pr}_V(E)$ has the same underlying space as $\mathrm{Pr}_{\mathbb{R}^n}(E)$, but we think of admissible maps as mapping out of V instead of $\mathbb{R}^n$.

Proposition 3.3.14. *$\mathrm{Pr}_V(E)$ is a principal $(O(V); O(V) \rtimes G)$-bundle and we have isomorphisms of G-vector bundles:*

$$E \cong (\mathrm{Pr}_V(E) \times V)/O(n).$$

Proof. This is a corollary of the structure theorem Theorem 3.2.19. Namely, Corollary 3.3.11 and the explanation afterwards have turned the vector bundle $p : E \to B$ into a G-fiber bundle with fiber V, structure group $O(V)$ and total group $O(V) \rtimes G$. By examination, $\mathrm{Pr}_V(E)$ with the natural $O(n)$-action on admissible maps and the specified G-action agrees with the construction $\mathrm{Pr}(E)$ in the structure theorem. □

3.4 Fixed point theorems

Non-equivariantly, the long exact sequence of the homotopy groups of a fiber sequence is a useful tool to study the homotopy group of one term, knowing the other two. To do this equivariantly, we need to know what taking-fixed-points does to equivariant bundles. We focus on $\Gamma = \Pi \times G$ in this section; [LM86] gives the analogue of Theorem 3.4.2 for general Γ.

Let $\mathrm{Rep}(G, \Pi)$ be the set:

$$\mathrm{Rep}(G, \Pi) = \{\text{group homomorphism } \rho : G \to \Pi\}/\Pi\text{-conjugation}.$$

Any subgroup $H \subset G$ with a group homomorphism $\rho : H \to \Pi$ gives a subgroup Λ_ρ of $(\Pi \times G)$ via its graph. That is,

$$\Lambda_\rho = \{(\rho(h), h) | h \in H\}.$$

For each $\rho : H \to \Pi$, denote the centralizer of the image of ρ in Π by

$$\mathrm{Z}_\Pi(\rho) = \{\nu \in \Pi | \nu\rho(h) = \rho(h)\nu \text{ for all } h \in H\}.$$

Proposition 3.4.1. *Let Π be a compact Lie group and H be a subgroup. Then $\mathrm{Z}_\Pi(H)$ is a closed subgroup of Π, thus also a compact Lie group.*

Proof. Fix an element $h \in H$. Then the map $c_h : \Pi \to \Pi$, $\nu \mapsto \nu h \nu^{-1}$ is continuous. Since the singleton $\{h\} \in \Pi$ is closed, the set $c_h^{-1}(\{h\}) = \{\nu \in \Pi | \nu h = h\nu\}$ is also closed. So $\mathrm{Z}_\Pi(H) = \bigcap_{h \in H} c_h^{-1}(\{h\})$ is closed. □

Theorem 3.4.2. *([LM86, Theorem 12]) Let G and Π be compact Lie groups. Let $p : E \to B$ be a principal G-Π-bundle and $H \subset G$ be a subgroup. Assume that E is completely regular.*

(1) On the base,

$$B^H = \coprod_{[\rho] \in \mathrm{Rep}(H,\Pi)} p(E^{\Lambda_\rho}).$$

(2) As sets, the preimages over each component of B^H are

$$p^{-1}(p(E^{\Lambda_\rho})) = \coprod_{\{\rho' : \Pi\text{-conjugate to } \rho\}} E^{\Lambda_{\rho'}}.$$

As spaces,

$$p^{-1}(p(E^{\Lambda_\rho})) \cong \Pi \times_{Z_\Pi(\rho)} E^{\Lambda_\rho}.$$

(3) For a fixed representative ρ of $[\rho]$, we have a principal $Z_\Pi(\rho)$-bundle:

$$Z_\Pi(\rho) \to E^{\Lambda_\rho} \xrightarrow{p} p(E^{\Lambda_\rho}).$$

(4) In particular, the following is a principal Π-bundle:

$$\Pi \to E^H \xrightarrow{p} p(E^H).$$

Explanation. In words, part (1) says that the H-fixed points of B are the images of the Λ-fixed points of E for all subgroups $\Lambda \subset \Pi \times G$ that are graphs of a homomorphism $H \to \Pi$. Furthermore, E^Λ and $E^{\Lambda'}$ share the same projection image when Λ and Λ' are Π-conjugate, or equivalently the corresponding representations $H \to \Pi$ are Π-conjugate. The assumption that E is completely regular implies that if Λ and Λ' are not Π-conjugate, the images of E^Λ and $E^{\Lambda'}$ are disjoint.

Parts (2) and (3) imply that E restricted on each component of B^H has a reduction of the structure group from Π to $Z_{\Pi}(\rho)$. In the proof of Theorem 4.2.8(1), we will describe in an example how to find the representations ρ when $H = G$. The idea is that the fiber over an H-fixed base has an H-action, and ρ tells what this action is in terms of the native Π-action as a principal bundle. Note that the representation ρ is dependent on the choice of a base point z in the fiber; a different choice gives a conjugate representation. From the description of the action, a point in the same fiber, written uniquely as $z\nu$ for some $\nu \in \Pi$, is Λ_ρ-fixed if and only if $\rho(h)\nu\rho(h)^{-1} = \nu$ for all $h \in H$. This justifies the first statement of part (2) as well as part (3).

For the second statement of part (2), which is not in the reference, we use the map:

$$\Pi \times_{Z_{\Pi}(\rho)} E^{\Lambda_\rho} \to E, \ (\nu, x) \mapsto x\nu^{-1}.$$

Here, $Z_{\Pi}(\rho)$ is a subgroup of Π and acts on the right of Π by multiplication; the left Π-action on E restricts to a left $Z_{\Pi}(\rho)$-action on E^{Λ_ρ}. It is a homeomorphism to its image, which is exactly $p^{-1}(p(E^{\Lambda_\rho}))$:

We have $\Lambda_e = H$ for the trivial representation $e : H \to \Pi$. Part (4) follows from taking $\rho = e$ in part (3). □

Remark 3.4.3. From Theorem 3.4.2, for a principal G-Π-bundle $p : E \to B$ and a subgroup $H \subset G$, each component B_0 of B^H has an associated representation class $[\rho] \in \mathrm{Rep}(H, \Pi)$. It is characterized by the fact that for any representation $\rho' : H \to \Pi$,

$$\left(p^{-1}(B_0)\right)^{\Lambda_{\rho'}} \neq \varnothing \text{ if and only if } [\rho'] = [\rho].$$

The restricted principal Π-bundle $p^{-1}(B_0) \to B_0$ has a reduction of the structure group from Π to $Z_{\Pi}(\rho)$.

Non-equivariantly, a map between two principal G-bundles that is an underlying equivalence on the total spaces will give an equivalence on the base spaces, as can be shown by the long exact sequence of homotopy groups. Equivariantly, we also want this tool of knowing when a map of two principal G-Π-bundles gives a G-equivalence on the base spaces.

Theorem 3.4.4. *Let $i : \Pi \to \Pi'$ be an inclusion of compact Lie groups. Let E, E' be principal G-Π- and G-Π'- bundles respectively of spaces of G-CW homotopy types. Then E' has a $(\Pi \times G)$-action by i.*

Suppose that there is a $(\Pi \times G)$-map $\bar{f} : E \to E'$ over a G-map $f : B \to B'$, as in the following commutative diagram:

$$\begin{array}{ccc} \Pi & \xrightarrow{i} & \Pi' \\ \downarrow & & \downarrow \\ E & \xrightarrow{\bar{f}} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

such that

(1) The map i includes Π as a deformation retract of Π' in groups, that is, there exists a group homomorphism $j : \Pi' \to \Pi$ such that $j \circ i = \mathrm{id}$ and $i \circ j \simeq \mathrm{id}$ rel $i(\Pi)$ in topological groups;

(2) On the total spaces, the map $\bar{f}$ is a Λ-equivalence for any subgroup $\Lambda \subset G \times \Pi$ such that $\Lambda \cap \Pi = e$.

Then, on the base spaces, $f : B \to B'$ is a G-equivalence.

Proof. To simply notation in this proof, we use the same letters to denote the restrictions of the corresponding maps to a subspace. By the equivariant Whitehead theorem, it suffices to show that:

$$\text{For any subgroup } H \subset G, \text{ the map } f : \ B^H \to (B')^H \text{ is an equivalence.}$$

We make the following two claims comparing Π and Π':

(a) For any group H, the induced map $i_* : \mathrm{Rep}(H, \Pi) \to \mathrm{Rep}(H, \Pi')$ is a bijection.

(b) For any subgroup K of Π, the inclusion $i : Z_\Pi K \to Z_{\Pi'} i(K)$ is a homotopy equivalence;

These two claims follow from the assumption (1). For (a), we take the functor $F = \mathrm{Rep}(H, -)$ from the category of groups to sets. It has equivalent images on Π and Π', and we skip the details. For (b), we take the functor $F = Z_{(-)}K$ from the category of groups containing K as a subgroup. It also has equivalent images on Π and Π', and the details come later in Lemma 3.4.8.

By Theorem 3.4.2 (1) and (a), it suffices to show that:

$$\text{For any } H \text{ and } \rho \in \mathrm{Rep}(H, \Pi), \text{ the map } f : p(E^{\Lambda_\rho}) \to p'((E')^{\Lambda_\rho}) \text{ is an equivalence.}$$

By Theorem 3.4.2 (3), taking the Λ_ρ-fixed points of E and E' yields a map between principal bundles:

$$\begin{array}{ccc} Z_\Pi(\rho) & \xrightarrow{\;i\;} & Z_{\Pi'}(\rho) \\ \downarrow & & \downarrow \\ E^{\Lambda_\rho} & \xrightarrow{\;\bar{f}\;} & (E')^{\Lambda_\rho} \\ \downarrow p & & \downarrow p' \\ p(E^{\Lambda_\rho}) & \xrightarrow{\;f\;} & p'((E')^{\Lambda_\rho}) \end{array}$$

By the claim (b) and the assumption (2), both i and $\bar{f}$ are equivalences. The long exact sequence of homotopy groups shows that f is an equivalence. □

Remark 3.4.5. In Theorem 3.4.4, the assumption (1) is true in our applications with $\Pi' = \Pi$ or $\Pi' = \Pi^I$. The assumption (2) is satisfied when $\bar{f}$ is a $(G \times \Pi)$-equivalence, but is weaker. The weaker version is needed in our applications.

From the proof, we also have a version of Theorem 3.4.4 relaxing the assumption (2).

Corollary 3.4.6. *Suppose we have $(i, \bar{f}, f)$ in the context of Theorem 3.4.4, except that instead of the assumption (2), $\bar{f} : E \to E'$ is only a Λ_ρ-equivalence for a fixed representation*

$\rho : H \to \Pi$. *Then on the base spaces,* $f : p(E^{\Lambda_\rho}) \to p((E')^{\Lambda_\rho})$ *is an equivalence.*

Note that $p(E^{\Lambda_\rho})$ is the space of components of B^H that are associated to ρ as described in Remark 3.4.3. In particular, if $(B')^H$ is connected for all subgroups $H \subset G$, then $(B')^H$ has only one associated representation ρ_H. Moreover, ρ_H has to be the restriction of ρ_G. We have:

Corollary 3.4.7. *Let* B' *be a* G*-connected space as explained above and* ρ_G *be the associated representation. Suppose we have* $(i, \bar{f}, f)$ *in the context of Corollary 3.4.6, such that* $\bar{f}$ *is a* Λ_{ρ_G}*-equivalence. Then on the base spaces,* $f : B \to B'$ *is a* G*-equivalence.*

Proof. Since the map $f : B^H \to (B')^H$ preserves the associated representation, we know that B^H only has one associated representation ρ_H as well. The claim then follows by applying Corollary 3.4.6 to $\rho = \rho_H$ for all H. $\square$

The following is a lemma for Theorem 3.4.4:

Lemma 3.4.8. *Assume* $i : \Pi \to \Pi'$ *is an inclusion of topological groups with a deformation retract* $j : \Pi' \to \Pi$*, that is, they satisfy condition (1) in Theorem 3.4.4. Then for any subgroup* K *of* Π*, the inclusion* $i : Z_\Pi K \to Z_{\Pi'} i(K)$ *is a homotopy equivalence.*

Proof. We first check that in general, given any group homomorphism $f : G \to G'$ and subgroup $K \subset G$, the map f restricts to a map $f_0 : Z_G K \to Z_{G'}(f(K))$ on subspaces. This is because $xk = kx$ for all $k \in K$ implies $f(x)f(k) = f(k)f(x)$ for all $f(k) \in f(K)$. So, we have

$$i_0 : Z_\Pi K \to Z_{\Pi'}(i(K)) \text{ and } j_0 : Z_{\Pi'}(i(K)) \to Z_\Pi(ji(K)) = Z_\Pi K.$$

The map j_0 gives deformation retract data of the inclusion i_0. It is obvious that $j_0 i_0 = \mathrm{id}$. It remains to show $i_0 j_0 \simeq \mathrm{id}$. The image of i_0 is the subspace $Z_{i(\Pi)}(i(K)) \subset Z_{\Pi'}(i(K))$. The homotopy $ij \simeq \mathrm{id}$ rel $i(\Pi)$ restricts to a homotopy $i_0 j_0 \simeq \mathrm{id}$ rel $Z_{i(\Pi)}(i(K))$. $\square$

CHAPTER 4: CLASSIFYING SPACES

4.1 V-trivial bundles

An equivariant bundle $E \to B$ is V-trivial for some n-dimensional G-representation V if there is a G-vector bundle isomorphism $E \cong B \times V$. Such an isomorphism is a V-framing of the bundle. This is analogous to the case of non-equivariant vector bundles, except that equivariance adds in the complexity of a representation V that's part of the data.

However, the representation V in the equivariant trivialization of a fixed vector bundle may not be unique. We give a lemma to recognize when two trivial bundles are isomorphic, then a counterexample.

Let $\mathrm{Iso}(V, W)$ be the space of linear isomorphisms $V \to W$ with the conjugation G-action for G-representations V and W.

Lemma 4.1.1. *For a G-space B, there exists a G-vector bundle isomorphism $B \times V \cong B \times W$ if and only if there exists a G-map $f : B \to \mathrm{Iso}(V, W)$.*

Proof. Let $F : B \times V \to B \times W$ be a vector bundle map. For $b \in B$, let $F_b : V \to W$ be such that $F_b(v) = F(b, v)$. Then F is a G-vector bundle isomorphism if and only if

(1) F is fiberwise isomorphism: $F_b \in \mathrm{Iso}(V, W)$;

(2) F is a G-map: $gF(b, v) = F(gb, gv)$, or equivalently, $F_{gb} = gF_b g^{-1}$, for all $g \in G$.

Taking $f(b) = F_b$, it follows that F is an isomorphism if and only if f is a G-map. □

Corollary 4.1.2. *If B has a G-fixed point, then $B \times V \cong B \times W$ only when $V \cong W$.*

Proof. The equivariant map $f : B \to \mathrm{Iso}(V, W)$ induces $f^G : B^G \to \mathrm{Iso}_G(V, W)$. The source being nonempty implies that the target is nonempty. □

Remark 4.1.3. More generally, for any two n-dimensional G-vector bundles E, E' over B, one can form the non-equivariant bundle $\mathcal{H}om_B(E, E')$ which consists of all bundle maps

$E \to E'$ over B (not necessarily fiberwise isomorphisms). It has a G-action by conjugation and is indeed an n^2-dimensional G-vector bundle over B. Let $\mathcal{I}so_B(E,E')$ be the subspace consisting of only fiberwise isomorphisms. It is a GL_n-bundle over B. Then tautologically $E \cong E'$ if there is a G-invariant section of $\mathcal{I}so_B(E,E')$.

Example 4.1.4 (Counterexample)**.** Let $G = C_2$, σ be the sign representation. The unit sphere, $S(2\sigma)$, is S^1 with the 180 degree rotation action. As C_2-vector bundles,

$$S(2\sigma) \times \mathbb{R}^2 \cong S(2\sigma) \times 2\sigma.$$

Proof. By Lemma 4.1.1, it suffices to construct a C_2-map $S(2\sigma) \to \mathrm{Iso}(\mathbb{R}^2, 2\sigma) \cong GL_2$, where the nontrivial element of C_2 acts on GL_2 by multiplying by $-\mathrm{Id}$. We give $S(2\sigma)$ a G-CW decomposition of a 0-cell C_2/e and a 1-cell $C_2/e \times \mathrm{D}^1$ and construct the map by skeleton. It is obvious that any equivariant map on the 0-skeleton extends to the 1-skeleton if and only if the two images lie in the same path component of GL_2, which is true in this case as $-\mathrm{Id}$ and Id lie in the same path component. □

Example 4.1.5. (Counterexample, Gus Longerman) Take G to be any compact Lie group and V and W to be any two representation of G that are of the same dimension. Then $G \times V \cong G \times W$, because $\mathrm{Map}_G(G, \mathrm{Iso}(V,W)) \cong \mathrm{Map}(\mathrm{pt}, \mathrm{Iso}(V,W)) \neq \varnothing$. Indeed, the isomorphism can be constructed explicitly by $F(g,x) = (g, \rho_W(g)\rho_V(g)^{-1}x)$, where $\rho_V, \rho_W : G \to O(n)$ are matrix representations of V, W.

4.2 Universal equivariant bundles

The universal principal $(\Pi;\Gamma)$-bundle was constructed and studied by tom Dieck [TD69] and Lashof–May [Las82, LM86]. It can be recognized by the following property:

Theorem 4.2.1. *([LM86, Theorem 9]) A principal* $(\Pi;\Gamma)$*-bundle* $p : E \to B$ *is universal if* *and only if*

$$E^{\Lambda} \simeq *, \text{ for all subgroups } \Lambda \subset \Gamma \text{ such that } \Lambda \cap \Pi = e.$$

Notation 4.2.2. The universal $(\Pi;\Gamma)$-bundle is denoted $E(\Pi;\Gamma) \to B(\Pi;\Gamma)$.

Remark 4.2.3. When $\Gamma = \Pi \times G$, such a subgroup Λ comes in the form of

$$\{(\rho(h), h) | h \in H\}, \text{ for } H \subset G \text{ and } \rho : H \to \Pi \text{ is a group homomorphism.}$$

This group is denoted Λ_ρ in Theorem 3.4.2.

When $\Gamma = \Pi \rtimes_\alpha G$, such a subgroup Λ comes in the form of

$$\{(\rho(h), h) | h \in H\}, \text{ for } H \subset G \text{ and } \rho : H \to \Pi \text{ such that } \rho(h_1 h_2) = \rho(h_1) \cdot \alpha_{h_1}(\rho(h_2)).$$

We mostly specialize to the case $\Gamma = G \times O(n)$, when a principal $(\Pi;\Gamma)$ is also a principal G-$O(n)$-bundle.

Notation 4.2.4. We denote the universal principal G-$O(n)$-bundle by $E_G O(n) \to B_G O(n)$.

It is universal in the sense that the equivalence classes of principal G-$O(n)$-bundles over a G-space B are classified by G-homotopy classes of G-maps $B \to B_G O(n)$. We denote the universal G-n-vector bundle by $\zeta_n \to B_G O(n)$ where

$$\zeta_n = E_G O(n) \times_{O(n)} \mathbb{R}^n.$$

As an immediate corollary of Theorems 4.2.1 and 3.4.2, one gets the G-homotopy type of the universal base. Recall that

$$\begin{aligned} \operatorname{Rep}(G, O(n)) &= \{\rho : G \to O(n) \text{ group homomorphism }\}/O(n)\text{-conjugation;} \\ &\cong \{V : n\text{-dimensional orthogonal representation of } G\}/\text{isomorphism} \end{aligned}$$

and $\mathrm{Z}_{O(n)}(\rho) = \{a \in O(n) | a\rho(g) = \rho(g)a, \text{ for all } g \in G\}$ is the centralizer of the image of ρ in $O(n)$.

Theorem 4.2.5. *([Las82, Theorem 2.17])*

$$\begin{aligned}(B_G O(n))^G &\simeq \coprod_{[\rho]\in \mathrm{Rep}(G,O(n))} B\mathrm{Z}_{O(n)}(\rho);\\ &\simeq \coprod_{[V]\in \mathrm{Rep}(G,O(n))} B(O(V)^G).\end{aligned}$$

Example 4.2.6. Take $H = G = C_2$ and $\Pi = O(2)$. Then

$$\mathrm{Rep}(C_2, O(2)) = \{\text{id}, \text{ rotation}, \text{ reflection}\}.$$

For $\rho = \text{id}$ or $\rho = \text{rotation}$, $Z_\Pi(\rho) = O(2)$. For $\rho = \text{reflection}$, $Z_\Pi(\rho) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. So

$$(B_{C_2} O(n))^{C_2} \simeq BO(2) \sqcup BO(2) \sqcup B(\mathbb{Z}/2 \times \mathbb{Z}/2).$$

From Theorem 4.2.5, one can make explicit the classifying maps of V-trivial bundles as follows.

Proposition 4.2.7. *A G-map $\theta : \mathrm{pt} \to B_G O(n)$ lands in one of the G-fixed components of $B_G O(n)$, indexed by $[V]$. Then the pullback of the universal bundle is $\theta^* \zeta_n \cong V$.*

Proof. It follows from part (1) of the following Theorem 4.2.8 that

$$\theta^* \zeta_n \cong O(\mathbb{R}^n, V) \times_{O(n)} \mathbb{R}^n \cong V.$$

In fact, the n-planes in a complete G-universe with the tautology n-plane bundle is a model for $B_G O(n)$ and ζ_n; $\theta(\mathrm{pt})$ is just a G-representation isomorphic to V. □

Theorem 4.2.8. *Take a G-fixed base point $b \in B_GO(n)$ in the component indexed by $[V]$. Let $p : E_GO(n) \to B_GO(n)$ be the universal principal G-$O(n)$-bundle. Then*

(1) The fiber over b, $p^{-1}(b)$, is homeomorphic to $O(\mathbb{R}^n, V)$ as an $(O(n) \times G)$-space. Here, $(G \times O(n))$ acts on $O(\mathbb{R}^n, V)$ by G acting on V and $O(n)$ acting on $\mathbb{R}^n$.

(2) The loop space of $B_GO(n)$ at the base point b, $\Omega_b B_GO(n)$, is G-homotopy equivalent to $O(V)$, the isometric self maps of V with G acting by conjugation.

Proof. (1) This is due to Lashof and we explain how to find the representation V here. Choose and fix a base point $z \in p^{-1}(b)$. We construct a group homomorphism $\rho_z : G \to O(n)$ as follows. For any $g \in G$, there exists a unique element, $\rho_z(g) \in O(n)$ such that $gz = z\rho_z(g)$. It is easy to check that $g \mapsto \rho_z(g)$ gives a group homomorphism. Suppose z' is another base point in $p^{-1}(b)$, and $z' = z\nu$ for some unique $\nu \in O(n)$. Then

$$gz' = gz\nu = z\rho_z(g)\nu = z'(\nu^{-1}\rho_z(g)\nu).$$

So $\rho_{z'} = \nu^{-1}\rho_z\nu$ is $O(n)$-conjugate to ρ_z. The different ρ_z's are the matrix representations of some vector space representation V. From the proof of Theorem 2.17 of [Las82], this is exactly the index V. Without loss of generality, we take V to be given by ρ_z as matrix representation.

The following map gives a non-equivariant homeomorphism:

$$\begin{array}{rcl} O(\mathbb{R}^n, V) \ \cong O(n) & \overset{\cong}{\to} & p^{-1}(b), \\ \nu & \mapsto & z\nu. \end{array}$$

It suffices to check it is an equivariant homeomorphism with the described action. Let $(\mu, g) \in O(n) \times G$. Then

$$z((\mu, g) \circ \nu) = z(\rho_z(g)\nu\mu^{-1}) = (z\rho_z(g))(\nu\mu^{-1}) = (gz)(\nu\mu^{-1}) = (\mu, g) \circ z\nu.$$

(2) The idea is to compare the path space fibration with the universal bundle. Equivariantly, the base point should be G-fixed. Since the space involved is not G-connected, base points from different components might give inequivalent loop spaces. We use subscripts in path spaces and loop spaces to indicate the base point. For example,

$$P_b B_G O(n) = \{\alpha \in \mathrm{Map}([0,1], B_G O(n)) | \alpha(0) = b\}.$$

Fix $z \in p^{-1}(b)$ and $\rho = \rho_z : G \to O(n)$ as above. Take z to be the base point of $E_G O(n)$. It is a Λ-fixed point, where

$$\Lambda = \{(\rho(g), g) | g \in G\} \subset O(n) \times G.$$

We prove that $E_G O(n)$ is Λ-contractible. In fact, let Λ' be any subgroup of Λ. Then $\Lambda' \cap O(n) = e$, so by Theorem 4.2.1, $(E_G O(n))^{\Lambda'}$ is contractible.

So, the contraction map gives a based Λ-equivariant homotopy:

$$E_G O(n) \wedge I \to E_G O(n).$$

Here, $I = [0, 1]$ is based at 0 and has the trivial Λ-action. (The map sends $x \wedge 0$ and $z \wedge t$ to z for all $x \in E_G O(n)$ and $t \in I$.) We take the adjoint of this homotopy to get $E_G O(n) \to P_z E_G O(n)$, and then compose with $P_z E_G O(n) \to P_b B_G O(n)$ induced by $p : E_G O(n) \to B_G O(n)$. The composite is

$$f : E_G O(n) \to P_z E_G O(n) \to P_b B_G O(n).$$

It sends a point $x \in E_G O(n)$ to a path in $B_G O(n)$ that starts at b and ends at $p(x)$. This yields a commutative diagram:

$$\begin{array}{ccc} E_GO(n) & \xrightarrow{f} & P_bB_GO(n) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ B_GO(n) & = & B_GO(n) \end{array}$$

Moreover, this diagram is G-equivariant, where the G-action on $P_bB_GO(n)$ is by pointwise action on the path. It is worth noting that the G-action we take on $E_GO(n)$ is not the original one, but via the identification $q : \Lambda \cong G$. In other words, $g \in G$ acts by what $(\rho(g), g)$ acts. The two vertical maps are non-equivariant fibrations and f maps the fiber of p over $b \in B_GO(n)$, denoted F_1, to the fiber of p_1 over b, denoted F_2.

We first identify the fibers F_1 and F_2. From part (1), $F_1 \cong O(\mathbb{R}^n, V)$ as $(O(n) \times G)$-spaces. So $F_1 \cong O(V)$ as G-spaces. It is clear that $F_2 \cong \Omega_b B_GO(n)$ as G-spaces.

We claim that f restricts to a G-equivalence $F_1 \to F_2$. The strategy is to show that it induces an isomorphism on homotopy groups of H-fixed points for all $H \subset G$, using the long exact sequences of homotopy groups of fiber sequences. Without dealing with general G-fibrations, it suffices to work out the following:

- Denote by $\Lambda' = q^{-1}(H)$, the subgroup of Λ that is isomorphic to H. The commutative diagram above restricts to the following commutative diagram:

$$\begin{array}{ccc} (F_1)^H & \longrightarrow & (F_2)^H \\ \downarrow & & \downarrow \\ (E_GO(n))^{\Lambda'} & \xrightarrow{f^H} & (P_bB_GO(n))^H \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ p((E_GO(n))^{\Lambda'}) & \longrightarrow & p_1((P_bB_GO(n))^H) \end{array}$$

- On the total space level, f^H induces isomorphism on homotopy groups. This is true because $E_GO(n)$ is Λ-contractible and $P_bB_GO(n)$ is G-contractible.

- The base spaces are equal. In fact, it is easy to see that they are both the component of $(B_G O(n))^H$ indexed by $[V]$ from Theorems 3.4.2 and 4.2.5.

- The two vertical lines are fiber sequences. For the first, we use Theorem 3.4.2 (3) with $(F_1)^H = (O(V))^H = Z_\Pi(\rho|_H)$; for the second, it is merely the path space fibration $\Omega_b X \to P_b X \to X$, where X denotes the component of $(B_G O(n))^H$ containing b. $\square$

4.3 The gauge group of an equivariant principal bundle

This section is a detour to prove Theorem 4.3.3 and Lemma 4.3.4. They are used later in Section 5.2 to understand the space of bundle maps and θ-framed bundle maps. They are also interesting in their own right.

Let $EO(n) \to BO(n)$ be the universal principal $O(n)$-bundle and $p : P \to B$ be any principal $O(n)$-bundle. The gauge group of P, $\mathrm{Aut}_B(P)$, is the space of bundle automorphisms of P that are identity on the base space B ([Hus94, Chap 7, Definition 1.1]). The space of principal bundle maps, $\mathrm{Hom}(P, EO(n))$, turns out to be also universal: The map

$$\mathrm{Hom}(P, EO(n)) \to \mathrm{Map}_p(B, BO(n)) \tag{4.3.1}$$

that restricts a bundle map to its base spaces is known to be the universal principal $\mathrm{Aut}_B(P)$-bundle. Here, $\mathrm{Map}_p(B, BO(n))$ denotes the component of the classifying map of p in $\mathrm{Map}(B, BO(n))$. A proof of this result can be found in [Hus94, Chap 7, Corollary 3.5].

We show in Theorem 4.3.3 the equivariant generalization: Let $E_G O(n) \to B_G O(n)$ be the universal principal G-$O(n)$-bundle and $p : P \to B$ be any principal G-$O(n)$-bundle. The restricting-to-the-base map

$$\pi : \mathrm{Hom}(P, E_G O(n)) \to \mathrm{Map}_p(B, B_G O(n)) \tag{4.3.2}$$

is a G-map lifting (4.3.1). Here, $\mathrm{Map}_p(B, B_G O(n))$ is the (non-equivariant) component of the classifying map of p in $\mathrm{Map}(B, B_G O(n))$; G acts by conjugation on both sides of (4.3.2). Let $\Gamma = \mathrm{Aut}_B P \rtimes G$, where G acts on $\mathrm{Aut}_B P$ by conjugation. Then the map π in (4.3.2) is a universal principal $(\mathrm{Aut}_B(P); \Gamma)$-bundle. Note that this is an equivariant principal bundle not in the sense of Definition 3.2.4, but of Definition 3.2.8 - the total group is a non-trivial extension of $\mathrm{Aut}_B(P)$ by G.

Theorem 4.3.3. *In the context above, the map*

$$\pi : \mathrm{Hom}(P, E_G O(n)) \to \mathrm{Map}_p(B, B_G O(n))$$

is a universal principal $(\mathrm{Aut}_B P; \Gamma)$*-bundle.*

Proof. As stated above, it is known non-equivariantly that π is a universal principal $\mathrm{Aut}_B P$-bundle. One can use the conjugation G-action to get a principal $(\mathrm{Aut}_B P; \Gamma)$-bundle structure on π. However, later in this proof we want a Γ-action on the bundle P, so at the risk of elaborating the obvious, we describe the Γ-action on $\mathrm{Hom}(P, E_G O(n))$ by putting a Γ-action on both P and $E_G O(n)$. The group $\mathrm{Aut}_B P$ naturally has a left action on P; take its trivial action on $E_G O(n)$. The group G acts on P and $E_G O(n)$ because they are G-vector bundles. One can check by Remark 3.2.11 that this gives a Γ-action on P and $E_G O(n)$, thus by conjugation on $\mathrm{Hom}(P, E_G O(n))$. Explicitly,

$$\begin{array}{ccc} (\mathrm{Aut}_B P \rtimes G) \times \mathrm{Hom}(P, E_G O(n)) & \to & \mathrm{Hom}(P, E_G O(n)) \\ ((s, g), f) & \mapsto & g f g^{-1} s^{-1}. \end{array}$$

Since $s \in \mathrm{Aut}_B P$ restricts to identity on B, we have

$$\pi(g f g^{-1} s^{-1}) = g \pi(f) g^{-1}.$$

By Definition 3.2.8, the map π is a principal $(\mathrm{Aut}_B P; \Gamma)$-bundle.

It remains to show that π is universal. Although $\mathrm{Aut}_B(P)$ can be fairly large, its size does not matter that much: By Theorem 4.2.1, it suffices to show that

$$\mathrm{Hom}(P, E_G O(n))^{\Lambda} \simeq * \text{ for any } \Lambda \subset \Gamma \text{ such that } \Lambda \cap \mathrm{Aut}_B P = e.$$

This follows from various application of the postponed Lemma 4.3.4, and it is essentially a consequence of the universality of $E_G O(n)$.

To see it, we first consider the case $\Lambda = H$, that is, the case where $\rho(h) = e$ for all $h \in H$ in Remark 4.2.3. By restricting the G-action to an H-action, $E_G O(n)$ is also the universal principal H-$O(n)$-bundle. Then $\mathrm{Hom}(P, E_G O(n))^H \simeq *$ by taking $\Pi = O(n)$, $G = H$ and $\Gamma = O(n) \times H$ in Lemma 4.3.4.

In the general case, Λ is isomorphic to a subgroup $H \subset G$ by the projection map $\Gamma \to G$, with a possibly non-trivial map ρ in Remark 4.2.3. Here is the crux: the elements in $\mathrm{Aut}_B P$ are $O(n)$-equivariant maps, so the $(\Gamma = \mathrm{Aut}_B P \rtimes G)$-action on P defined at the beginning of this proof commutes with the $O(n)$-action; and we have $\Lambda \subset \Gamma$. In other words, P is also a principal Λ-$O(n)$-bundle. Since Λ acts by H on $E_G O(n)$, the space $E_G O(n)$ is also the universal principal Λ-$O(n)$-bundle. Now we are basically in the first case again: $\mathrm{Hom}(P, E_G O(n))^{\Lambda} \simeq *$ by taking $\Pi = O(n)$, $G = \Lambda$ and $\Gamma = O(n) \times \Lambda$ in Lemma 4.3.4.

□

The following lemma is a consequence of the universality:

Lemma 4.3.4. *Let $1 \to \Pi \to \Gamma \to G \to 1$ be an extension of groups. Let*

$$p_{\Pi;\Gamma} : E(\Pi; \Gamma) \to B(\Pi; \Gamma)$$

be the universal principal $(\Pi; \Gamma)$-bundle and let $p : P \to B$ be any principal $(\Pi; \Gamma)$-bundle. Then $\big(\mathrm{Hom}(P, E(\Pi; \Gamma))\big)^G$ is contractible.

Proof. To clarify the notations, $\mathrm{Hom}(P, E(\Pi;\Gamma))$ is the space of maps of (nonequivariant) principal Π-bundles. By definition,

$$\mathrm{Hom}(P, E(\Pi;\Gamma)) \cong \mathrm{Map}_{\Pi}(P, E(\Pi;\Gamma)).$$

The space $\mathrm{Hom}(P, E(\Pi;\Gamma))$ has a Γ-action by conjugation. Since $\Pi \subset \Gamma$ acts trivially, it descends to a G-action, and

$$\big(\mathrm{Hom}(P, E(\Pi;\Gamma))\big)^G \cong \mathrm{Map}_{\Gamma}(P, E(\Pi;\Gamma)).$$

By definition, the space $\mathrm{Map}_{\Gamma}(P, E(\Pi;\Gamma))$ is in fact the space of morphisms between principal $(\Pi;\Gamma)$-bundles. It is non-empty because it consists of the classifying map of p. It is further path-connected because any two Γ-maps $P \to E(\Pi;\Gamma)$ will both restrict to a classifying map $B \to B(\Pi;\Gamma)$ of p, so they are G-homotopic. The pull back of $p_{\Pi;\Gamma}$ along this homotopy gives a homotopy, or path, between the two maps.

Using the arbitrariness of P in the above argument, one can further show that the space $\mathrm{Map}_{\Gamma}(P, E(\Pi;\Gamma))$ is contractible as follows. Let Y be a random G-space. We denote by $Y \times P$ the principal $(\Pi;\Gamma)$-bundle $Y \times P \to Y \times B$. Here, Γ acts on Y by pulling back the G-action and acts $Y \times P$ diagonally. Then we have an adjunction:

$$\mathrm{Map}_G(Y, \mathrm{Hom}(P, E(\Pi;\Gamma))) \cong \mathrm{Map}_{\Gamma}(Y \times P, E(\Pi;\Gamma)). \tag{4.3.5}$$

By what has been shown, the right hand side, thus the left hand side of (4.3.5) is always non-empty and path-connected for any Y. Taking $Y = \mathrm{Hom}(P, E(\Pi;\Gamma))$, we obtain that $\mathrm{Map}_G(Y, Y)$ is path-connected. In particular, the identity map and the constant map to a point in Y^G are connected by a path. This implies the contractibility of $Y^G = \big(\mathrm{Hom}(P, E(\Pi;\Gamma))\big)^G$. □

4.4 Free loop spaces and adjoint bundles

We end this chapter by another detour to show an equivariant equivalence of the free loop space $LB_G\Pi$ and the adjoint bundle $\mathrm{Ad}(E_G\Pi)$ in Theorem 4.4.10. This gives Corollary 4.4.11, which appears later in Theorem 5.3.2 for understanding the automorphism space of the disk V. Our proof follows the non-equivariant treatment in the appendix of Gruher's thesis [Gru07] and uses Theorem 3.4.4.

We start with G-fibrations.

Definition 4.4.1. A G-map $p: E \to B$ between G-spaces is a G-fibration if for all subgroups $H \subset G$, the map $p^H : E^H \to B^H$ is a Hurewicz fibration.

Example 4.4.2. Let $p: E \to B$ be a principal G-Π-bundle as in Definition 3.2.4. Then p is also a G-fibration by Theorem 3.4.2 (4). However, $p: E^H \to B^H$ is not necessarily surjective. In contrast to the other parts of Theorem 3.4.2, we do not have control over the components of B^H that are not hit by $p(E^H)$, at least not obviously. In this sense, the notion of a G-fibration is not as rich as a principal G-Π-bundle.

Example 4.4.3. Let F be an effective Π-space and $q: E' \to B'$ be a G-fiber bundle with fiber F, structure group Π as in Definition 3.2.1. Then q is also a G-fibration.

Lemma 4.4.4. *We have the following results on the fiber of a G-fibration:*

(1) Let $p: E \to B$ be a G-fibration and $b \in B^H$ be an H-fixed point, then the maps $(p^{-1}(b))^H \to E^H \to B^H$ form a fiber sequence.

(2) Let $p: D \to B$ and $q: E \to B$ be two G-fibrations and $f: D \to E$ be a G-map over B. Take an H-fixed point $b \in B^H$. If f is a G-equivalence, then $p^{-1}(b) \to q^{-1}(b)$ is an H-equivalence.

Proof. Non-equivariantly ($G = \{e\}$), this is the fact that a map over B and homotopy equivalence is a homotopy equivalence of fibrations over B (See [May99, 7.5-7.6]). Equivariantly, the first claim is immediate from the definition; the second claim reduces to the non-equivariant case for each subgroup $H' \subset H$. □

We adopt the language of fiberwise monoids in [Gru07, Definition 4.2.1].

Definition 4.4.5. A G-fibration $p : E \to B$ is a G-fiberwise monoid if there is a unit section map $\eta : B \to E$ and a multiplication map $m : E \times_B E \to E$ over B, both G-equivariant, that satisfy the unital and associativity conditions. In other words, E is a monoid in the category of G-fibrations over B.

We can relax the strict associativity condition and define G-fiberwise A_∞-monoids as well. Let $\mathscr{A}$ be a reduced A_∞-operad in Top ($\mathscr{A}(0) = *$).

Definition 4.4.6. A G-fibration $p : E \to B$ is a G-fiberwise A_∞-monoid if it is an algebra over $\mathscr{A}$ in the category of G-fibrations over B. In concrete words, there are G-equivariant structure maps over B for each $k \geq 0$

$$\gamma_k : \mathscr{A}(k) \times_{\Sigma_k} \big(\underbrace{E \times_B E \times_B \cdots \times_B E}_{k \text{ times}}\big) \to E$$

that satisfy the unital, associativity and Σ-equivariance conditions of an algebra over an operad. Here, $\mathscr{A}(k)$ is thought to have the trivial G-action; for $k = 0$, $\gamma_0 : B \to E$ is just a section of p.

Definition 4.4.7. A morphism of G-fiberwise A_∞-monoids over B is a morphism of A_∞-monoids in the category of G-fibrations over B. An equivalence is a morphism and G-equivalence on the total space.

By a G-monoid, we mean a monoid in G-spaces, and similarly for a G-A_∞-monoid. Notice that the fiber of a G-fiberwise (A_∞)-monoid over a point $b \in B$ is not a G-(A_∞)-monoid. Instead, it is a G_b-(A_∞)-monoid, where G_b is the isotropy subgroup of b. A morphism of fiberwise G-(A_∞)-monoids induces a morphism of G_b-(A_∞)-monoids on the fibers over b; An equivalence induces a G_b-equivalence on the fibers by Lemma 4.4.4.

To clarify this notion, we make the following remarks:

(1) A G-fiberwise monoid is a G-fiberwise A_∞-monoid. In this case, the unit section map η is γ_0 and the multiplication map m is γ_2.

(2) The relevant examples of G-fiberwise A_∞-monoids here are mostly G-fibrations over B whose fibers are some sort of loops. The structure maps come from fiberwise-A_∞ structure of loop spaces. We will abuse terms to refer to the structure maps as the unit map and the multiplication map.

(3) A G-fiberwise monoid or a G-monoid here is not a "genuinely equivariant algebra" as it does not have G-set indexed multiplications.

Construction 4.4.8. For a G-space X, the free loop space $LX = X^{S^1}$ is a G-fibration over X by evaluating at a base point of S^1. It is also a G-fiberwise A_∞-monoid with the unit map given by the constant loop and the multiplication map given by the concatenation of loops.

Construction 4.4.9. For a principal G-Π-bundle $E \to B$, the adjoint bundle of E is $Ad(E) = E \times_\Pi \Pi_{\mathrm{ad}}$. Here, Π_{ad} is the left Π-space Π with adjoint action: for elements $\mu \in \Pi$ and $\nu \in \Pi_{\mathrm{ad}}$, μ acts on ν by $\mu(\nu) = \mu\nu\mu^{-1}$. As defined, $Ad(E)$ is a G-fiber bundle over B with fiber Π, but no longer a principal G-Π-bundle. We claim that $Ad(E)$ has the structure of a G-fiberwise monoid over B. First, $Ad(E)$ is the fiberwise automorphism bundle $\mathcal{I}so_B(E,E)$, so naturally a fiberwise monoid over B. This is the bundle version of the observation that for a right Π-space S homeomorphic to Π, there is a homeomorphism

$$\begin{aligned} \mathrm{Aut}_\Pi(S) &\cong S \times_\Pi \Pi_{\mathrm{ad}} \\ f(s) = s\nu &\leftrightarrow [(s,\nu)]. \end{aligned}$$

Moreover, $Ad(E) \cong \mathcal{I}so_B(E,E)$ as G-spaces over B, where G acts on $Ad(E)$ by acting on E and on $\mathcal{I}so_B(E,E)$ by conjugation. This breaks down to commuting the action of G and Π on E. Just to clarify,

$$\mathrm{Aut}_B(E) = \mathrm{Iso}_B(E,E) \cong \mathrm{Section}(\mathcal{I}so_B(E,E)).$$

Theorem 4.4.10. *Let G, Π be compact Lie groups. Then there is a G-fiberwise A_∞-monoid $(\widetilde{P}E_G\Pi)/\Pi$ over $B_G\Pi$ and equivalences as G-fiberwise A_∞-monoids over $B_G\Pi$:*

$$LB_G\Pi \xleftarrow[\simeq]{\xi} (\widetilde{P}E_G\Pi)/\Pi \xrightarrow[\simeq]{\psi} Ad(E_G\Pi)$$

Proof. We first construct the space and the map

$$\widetilde{p} : (\widetilde{P}E_G\Pi)/\Pi \to B_G\Pi.$$

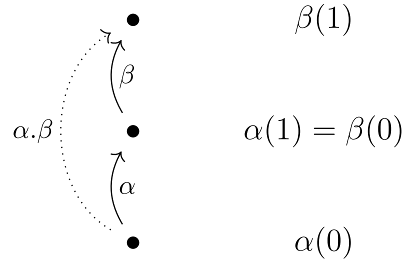
Recall that $p : E_G\Pi \to B_G\Pi$ is the universal principal G-Π bundle. Denote the space of paths in $E_G\Pi$ that start and end in the same fiber over $B_G\Pi$ to be

$$\widetilde{P}E_G\Pi = \{\alpha : I \to E_G\Pi \,|\, p(\alpha(0)) = p(\alpha(1))\}.$$

Then $\widetilde{P}E_G\Pi$ inherits an $(\Pi \times G)$-action from $E_G\Pi$. The quotient $(\widetilde{P}E_G\Pi)/\Pi$ is a G-space over $B_G\Pi$ by $\widetilde{p}(\alpha) = p(\alpha(0))$.

The map $\widetilde{p}$ has the structure of a G-fiberwise A_∞-monoid. The unit map η is given by the constant path in the fiber of p. There is only one constant path in each fiber since we have taken quotient of the Π-action. The multiplication map m is given as follows: for two classes of paths $[\alpha], [\beta] \in (\widetilde{P}E_G\Pi)/\Pi$, we may choose representatives such that $\alpha(1) = \beta(0)$.

Let $m([\alpha],[\beta]) = [\alpha.\beta]$ be the concatenation of the paths:



The class $[\alpha.\beta]$ does not depend on the choice of α, β. Both η and m are G-equivariant.

Next, we compare both $LB_G\Pi$ and $Ad(E_G\Pi)$ with $(\widetilde{P}E_G\Pi)/\Pi$.

On one hand, we have $LB_G\Pi = (\widetilde{P}E_G\Pi)/\Pi^I$. Here, Π^I is the group $\mathrm{Map}([0,1],\Pi)$ and acts on $\widetilde{P}E_G\Pi \subset (E_G\Pi)^I$ pointwise in I. The projection $\widetilde{P}E_G\Pi \to LB_G\Pi$ is a principal G-Π^I-bundle, as the Π^I action commutes with the G-action on $\widetilde{P}E_G\Pi$.

The projection $\xi : (\widetilde{P}E_G\Pi)/\Pi \to (\widetilde{P}E_G\Pi)/\Pi^I$ commutes with the unit map and multiplication map, so it is a map of G-fiberwise A_∞-monoids. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc}
\Pi & \longrightarrow & \Pi^I \\
\downarrow & & \downarrow \\
\widetilde{P}E_G\Pi & = & \widetilde{P}E_G\Pi \\
\downarrow & & \downarrow \\
(\widetilde{P}E_G\Pi)/\Pi & \xrightarrow{\xi} & (\widetilde{P}E_G\Pi)/\Pi^I = LB_G\Pi
\end{array}$$

By Theorem 3.4.4, ξ is a G-equivalence. (The idea is that Π and Π^I are not so different.)

On the other hand, we may define a $(\Pi \times G)$-equivariant map

$$\begin{array}{rccc}
\bar{\psi}: & \widetilde{P}E_G\Pi & \to & E_G\Pi \times \Pi_{\mathrm{ad}} \\
& \alpha & \mapsto & (\alpha(1), \nu)
\end{array}$$

where $\nu \in \Pi$ is the unique element such that $\alpha(1) = \alpha(0)\nu^{-1}$. We give $E_G\Pi \times \Pi_{\mathrm{ad}}$ the

G-action on $E_G\Pi$ and the diagonal Π-action. To check the equivariance of $\bar{\psi}$, take any $(\mu, g) \in \Pi \times G$, then $(\mu, g) \circ \alpha(t) = g\alpha(t)\mu^{-1}$ for $t \in [0,1]$. So,

$$\bar{\psi}((\mu, g) \circ \alpha) = (g\alpha(1)\mu^{-1}, \mu\nu\mu^{-1}) = (\mu, g) \circ \bar{\psi}(\alpha).$$

Since $Ad(E_G\Pi) = (E_G\Pi \times \Pi_{\text{ad}})/\Pi$, we get a map $\psi : (\widetilde{P}E_G\Pi)/\Pi \to Ad(E_G\Pi)$. It is easy to check that ψ commutes with the unit and multiplication maps, and is thus a map of G-fiberwise A_∞-monoids.

To show that ψ is a G-equivalence, we consider the following morphism of principal G-Π-bundles:

$$\begin{array}{ccc}
\Pi & = & \Pi \\
\downarrow & & \downarrow \\
\widetilde{P}E_G\Pi & \xrightarrow{\bar{\psi}} & E_G\Pi \times \Pi_{\text{ad}} \\
\downarrow & & \downarrow \\
(\widetilde{P}E_G\Pi)/\Pi & \xrightarrow{\psi} & Ad(E_G\Pi)
\end{array}$$

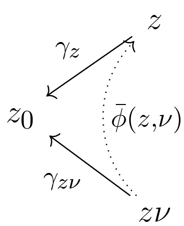
By Theorem 3.4.4, it suffices to show that $\bar{\psi}$ is a Λ-equivalence for any $\Lambda \subset \Pi \times G$ with $\Lambda \cap \Pi = e$.

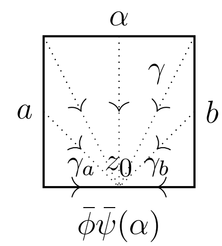
We can construct a Λ-homotopy inverse for $\bar{\psi} : \widetilde{P}E_G\Pi \to E_G\Pi \times \Pi_{\text{ad}}$, called $\bar{\phi}$. The idea is already in Gruher's proof [Gru07]. But in the equivariant case, $\bar{\phi}$ is dependent on the subgroup Λ. (In particular, it is not meant to be a $(\Pi \times G)$-homotopy inverse.) Recall that $\bar{\psi}$ records the two endpoints of a path. So an inverse $\bar{\phi}$ is going to choose a canonical path between any two points in a continuous way. This choice of canonical path exists because of the Λ-contractibility of $E_G\Pi$; it is not meant to be a canonical choice.

The construction of $\bar{\phi}$ is as follows: Since $E_G\Pi$ is Λ-contractible, $(E_G\Pi)^\Lambda$ is non-empty. We fix a Λ-fixed base point $z_0 \in E_G\Pi$. Let $E_G\Pi \times I \to E_G\Pi$ be a Λ-equivariant contraction of $E_G\Pi$ to z_0; the adjoint of it gives a Λ-map $\gamma : E_G\Pi \to P_{z_0}E_G\Pi$. For $z \in E_G\Pi$, we write $\gamma(z)$ as γ_z. It is a path connecting z to z_0. Now, recall that for an element $(z, \nu) \in E_G\Pi \times \Pi_{\text{ad}}$, the image $\bar{\phi}(z,\nu) \in \widetilde{P}E_G\Pi$ wants to be a path from $z\nu$ to z in $E_G\Pi$. We define it to be

$$\bar{\phi}(z,\nu) = \text{concatenation of } \gamma_{z\nu} \text{ and the reverse of } \gamma_z,$$

as illustrated in the picture on the left:





It remains to verify that $\bar{\phi}$ is Λ-homotopy inverse of $\bar{\psi}$. It is clear that $\bar{\psi}\bar{\phi} = \text{id}$. The illustration above on the right shows how a Λ-equivariant homotopy $\bar{\phi}\bar{\psi} \simeq \text{id}$ is defined: For a path $\alpha \in \widetilde{P}E_G\Pi$ going from a point a to a point b, the path $\bar{\phi}\bar{\psi}(\alpha)$ is the concatenation of γ_a and the reverse of γ_b. A homotopy of paths $\bar{\phi}\bar{\psi}(\alpha) \simeq \alpha$ is a map H out of the square, such that the value of H has been given on the border as indicated. To fill the interior, we connect every point x on the border to the point labeled by z_0 with line segments and use the map $\gamma_{H(x)}$ on each segment. This homotopy H is "functorial" for elements $\alpha \in \widetilde{P}E_G\Pi$, so it extends to a homotopy $\bar{\phi}\bar{\psi} \simeq \text{id}$; it is Λ-equivariant because the map γ is. □

As a corollary, we can upgrade Theorem 4.2.8 (2) into an equivalence of G-A_∞-monoids $\Omega_b B_G O(n) \simeq O(V)$. Strictifying $\Omega_b B_G O(n)$ to the Moore loop space $\mathbf{\Lambda}_b B_G O(n)$, there is an equivalence of G-monoids $\mathbf{\Lambda}_b B_G O(n) \simeq O(V)$:

Corollary 4.4.11. *Take a G-fixed base point $b \in B_G O(n)$ in the component indexed by V. Then $\mathbf{\Lambda}_b B_G O(n)$ is equivalent to $O(V)$ as a G-monoid. Here, G acts on $\mathbf{\Lambda}_b B_G O(n)$ by acting on $B_G O(n)$ and acts on $O(V)$ by conjugation.*

Proof. We explain how the G-A_∞-monoid statement is a corollary. Take the fiber over b in Theorem 4.4.10 for $\Pi = O(n)$. Then there are equivalences of the fibers as G-A_∞-monoids by Lemma 4.4.4. The fiber of $LB_G O(n)$ is $\Omega_b B_G O(n)$. By Theorem 4.2.8 (1), the fiber of $Ad(E_G O(n))$ is $O(\mathbb{R}^n, V) \times_{O(n)} O(n)_{\text{ad}} \cong O(V)$ as G-monoid. So there is a zig-zag of equivalences of G-A_∞-monoids between $\Omega_b B_G O(n)$ and $O(V)$. For the G-monoid statement, just replace the free loop space and path space in Theorem 4.4.10 by the Moore version, and the proof stays intact.

Explicitly, the zigzag of G-monoids is given by

$$\mathbf{\Lambda}_b B_G O(n) \xleftarrow[\simeq]{\xi} (\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi \xrightarrow[\simeq]{\psi} O(V). \tag{4.4.12}$$

We use p to denote the universal principal G-$O(n)$-bundle $E_G O(n) \to B_G O(n)$. We define

$$\widetilde{\mathbf{\Lambda}}_b E_G O(n) = \{(l, \alpha) | l \in \mathbb{R}_{\geq 0}, \alpha : \mathbb{R}_{\geq 0} \to E_G O(n), p(\alpha(0)) = p(\alpha(t)) = b \text{ for } t \geq l\},$$

so that $(\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi = [l, \alpha]$ where the equivalence relation is

$$(l, \alpha) \sim (l, \beta) \text{ if there is } \nu \in O(n) \text{ such that } \alpha(t) = \beta(t)\nu \text{ for all } t \geq 0.$$

While $\widetilde{\mathbf{\Lambda}}_b E_G O(n)$ does not have the structure of a G-monoid, $(\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi$ does.

Fix a base point $z \in p^{-1}(b) \subset E_G O(n)$. The maps are given by

$$\xi([l, \alpha]) = (l, p(\alpha)) \in \mathbf{\Lambda}_b B_G O(n);$$

$$\psi([l, \alpha]) \in O(V) \text{ is determined by } \alpha(0)\psi([l, \alpha]) = \alpha(l). \qquad \square$$

CHAPTER 5: TANGENTIAL STRUCTURES AND FACTORIZATION HOMOLOGY

5.1 Equivariant tangential structures

Fix an integer n and a finite group G. A tangential structure is a G-map $\theta : B \to B_GO(n)$. Here, $B_GO(n)$ is the classifying space as in Notation 4.2.4. A morphism of two tangential structures is a G-map over $B_GO(n)$. All tangential structures form a category $\mathcal{TS}$, which is simply the over category $G\mathrm{Top}/_{B_GO(n)}$.

In this section we construct two covariant functors from $\mathcal{TS}$ to categories. The first one sends θ to Vec^θ, the category of n-dimensional θ-framed bundles with θ-framed bundle maps as morphisms. The second one sends θ to Mfld^θ, the category of smooth n-dimensional θ-framed manifolds and θ-framed embeddings. The category Mfld^θ is a subcategory of Vec^θ; both Mfld^θ and Vec^θ are enriched over $G\mathrm{Top}$.

Denote by ζ_n the universal G-n-vector bundle over $B_GO(n)$. Pulling back along the tangential structure $\theta : B \to B_GO(n)$ gives a bundle $\theta^*\zeta_n$ over B. This is meant to be the universal θ-framed vector bundle. For an n-dimensional smooth G-manifold M, the tangent bundle of M is a G-n-vector bundle. It is classified by a G-map up to G-homotopy:

$$M \xrightarrow{\tau} B_GO(n).$$

Definition 5.1.1. A θ-framing on a G-n-vector bundle $E \to M$ is a G-n-vector bundle map $\phi : E \to \theta^*\zeta_n$. A θ-framing on a smooth G-manifold M is a θ-framing ϕ_M on its tangent bundle. We abuse notations and refer to the map on the base spaces as ϕ_M as well.

A bundle has a θ-framing if and only if its classifying map τ has a factorization up to G-homotopy through B; see diagram (5.1.2). However, a factorization $\tau_B : M \to B$ does not uniquely determine a θ-framing $E \to \theta^*(\zeta_n)$. Indeed, a bundle map from E to $\theta^*(\zeta_n)$ is the same data as a map $\tau_B : M \to B$ on the base plus a homotopy between the two classifying maps from M to $B_G O(n)$. For a detailed proof, see Corollary 5.2.9 with Definition 5.2.4.

$$\begin{array}{ccc} & & B \\ & \overset{\tau_B}{\nearrow} \; h \curvearrowright & \downarrow \theta \\ M & \xrightarrow[\tau]{} & B_G O(n) \end{array} \tag{5.1.2}$$

Example 5.1.3. When B is a point, a tangential structure $\theta : \mathrm{pt} \to B_G O(n)$ picks out in its image a G-fixed component of $B_G O(n)$. This component is indexed by some n-dimensional G-representation V up to isomorphism. Then $\theta^* \zeta_n \cong V$ as a G-vector space over pt (Proposition 4.2.7). We denote this tangential structure by $\mathrm{fr}_V : \mathrm{pt} \to B_G O(n)$ and call it a V-framing. A V-framing on a vector bundle $E \to M$ is just an equivariant trivialization $E \cong M \times V$. We emphasize that the V-framing tangential structure is not only a space $B = \mathrm{pt}$ but also a map fr_V.

Definition 5.1.4. Given two θ-framed bundles E_1, E_2 with framings ϕ_1, ϕ_2, the space of θ-framed bundle maps between them is defined as:

$$\mathrm{Hom}^\theta(E_1, E_2) := \mathrm{hofib}\big(\mathrm{Hom}(E_1, E_2) \xrightarrow{\phi_2 \circ -} \mathrm{Hom}(E_1, \theta^* \zeta_n)\big), \tag{5.1.5}$$

where $\mathrm{Hom}(E_1, \theta^* \zeta_n)$ is based at ϕ_1. Explicitly, a θ-framed bundle map is a bundle map f and a homotopy connecting the two resulting θ-framings ϕ_1 and $\phi_2 f$ of E_1:

$$\mathrm{Hom}^\theta(E_1, E_2) = \{(f, \alpha) \in \mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_1, \theta^* \zeta_n)^I | \alpha(0) = \phi_1, \alpha(1) = \phi_2 f\}.$$

The unit in $\mathrm{Hom}^\theta(E, E)$ is given by $(\mathrm{id}_E, \phi_{\mathrm{const}})$; the composition of two θ-bundle maps is defined as:

$$\begin{array}{rcl} \mathrm{Hom}^\theta(E_2, E_3) \times \mathrm{Hom}^\theta(E_1, E_2) & \to & \mathrm{Hom}^\theta(E_1, E_3); \\ \big((g, \beta), (f, \alpha)\big) & \mapsto & (g \circ f, \lambda), \end{array}$$

$$\text{where } \lambda(t) = \begin{cases} \alpha(2t), & \text{when} \quad 0 \le t \le 1/2; \\ \beta(2t-1) \circ f, & \text{when} \quad 1/2 < t \le 1. \end{cases}$$

As defined above, the composition is unital and associative only up to homotopy. One can modify $\mathrm{Hom}^\theta(E_1, E_2)$ using Moore paths in the homotopy to make the composition strictly unital and associative; see [KM18, Definition 17] or Definition 5.2.4 for a construction in the same spirit. We omit the details here and assume we have built a category Vec^θ of θ-framed bundles and θ-framed embeddings. As such, an element of $\mathrm{Hom}^\theta(E_1, E_2)$ has a third data of the length of the path, which is a locally constant function on $\mathrm{Hom}(E_1, E_2)$, but for brevity we sometimes do not write it.

In the definition of $\mathrm{Hom}^\theta(E_1, E_2)$, everything is taken non-equivariantly. The spaces $\mathrm{Hom}(E_1, E_2)$ and $\mathrm{Hom}(E_1, \theta^*\zeta_n)$ have G-actions by conjugation. Since ϕ_1 and ϕ_2 are G-maps, the homotopy fiber $\mathrm{Hom}^\theta(E_1, E_2)$ inherits the conjugation G-action.

Definition 5.1.6. The space of θ-framed embeddings between two θ-framed manifolds is defined as the pullback displayed in the following diagram of G-spaces:

$$\begin{array}{ccc} \mathrm{Emb}^\theta(M, N) & \longrightarrow & \mathrm{Hom}^\theta(\mathrm{T}M, \mathrm{T}N) \\ \downarrow & & \downarrow \\ \mathrm{Emb}(M, N) & \xrightarrow{d} & \mathrm{Hom}(\mathrm{T}M, \mathrm{T}N) \end{array} \tag{5.1.7}$$

Here, $\mathrm{Emb}(M, N)$ is the space of smooth embeddings and the map d takes an embedding to its derivative.

Remark 5.1.8. Most of the time, we drop the Moore-path-length data and write an element of $\mathrm{Emb}^\theta(M, N)$ as a package of a map f and a homotopy $\bar{f} = (f, \alpha)$, with $f \in \mathrm{Emb}(M, N)$ and $\alpha : [0, 1] \to \mathrm{Hom}(\mathrm{T}M, \mathrm{T}N)$ satisfying $\alpha(0) = \phi_M$ and $\alpha(1) = \phi_N \circ df$. There is a functor $\mathrm{Mfld}^\theta \to \mathrm{Mfld}$ by forgetting the tangential structure. It sends $\bar{f} \in \mathrm{Emb}^\theta(M, N)$ to $f \in \mathrm{Emb}(M, N)$.

Let $\sqcup$ be the disjoint union of θ-framed vector bundles or manifolds and $\varnothing$ be the empty bundle or manifold. Both $(\mathrm{Vect}^\theta, \sqcup, \varnothing)$ and $(\mathrm{Mfld}^\theta, \sqcup, \varnothing)$ are $G\mathrm{Top}$-enriched symmetric monoidal categories. In both categories, $\varnothing$ is the initial object. In Vect^θ, $\sqcup$ is the coproduct, but not in Mfld^θ.

Remark 5.1.9. We need the length of the Moore path to be locally constant as introduced in [KM18, Definition 17] as opposed to constant for the enrichment to work. Namely, the map

$$\mathrm{Hom}^\theta(E_1, E_1') \times \mathrm{Hom}^\theta(E_2, E_2') \to \mathrm{Hom}^\theta(E_1 \sqcup E_2, E_1' \sqcup E_2')$$

is given by first post-composing with the obvious θ-framed map $E_i' \to E_1' \sqcup E_2'$ for $i = 1, 2$, then using a homeomorphism, as follows:

$$\begin{aligned} \mathrm{Hom}^\theta(E_1, E_1') \times \mathrm{Hom}^\theta(E_2, E_2') &\to \mathrm{Hom}^\theta(E_1, E_1' \sqcup E_2') \times \mathrm{Hom}^\theta(E_2, E_1' \sqcup E_2') \\ &\cong \mathrm{Hom}^\theta(E_1 \sqcup E_2, E_1' \sqcup E_2') \end{aligned}$$

If the length of the Moore path were constant, the displayed homeomorphism would only be a homotopy equivalence, as the length of a Moore path can be different on the two parts.

To set up factorization homology in Section 5.4, we fix an n-dimensional orthogonal G-representation V; in addition, we suppose that V is θ-framed and fix a θ-framing

$$\phi : \mathrm{T}V \to \theta^* \zeta_n$$

on V. Since $\mathrm{T}V \cong V$ as G-vector bundles, the space of θ-framings on V is

$$\mathrm{Hom}(\mathrm{T}V, \theta^*\zeta_n)^G \simeq \mathrm{Hom}(V, \theta^*\zeta_n)^G = \mathrm{Hom}(\mathbb{R}^n, \theta^*\zeta_n)^{\Lambda_\rho} \simeq (\theta^* E_G O(n))^{\Lambda_\rho}, \qquad (5.1.10)$$

where $\Lambda_\rho = \{(\rho(g), g) \in O(n) \times G | g \in G\}$ and $\rho : G \to O(n)$ is a matrix representation of V.

(Here, the change of the group G to Λ_ρ that accompanies the change of V to $\mathbb{R}^n$ is the same phenomena as changing from $\mathrm{Pr}_{\mathbb{R}^n}(E)$ to $\mathrm{Pr}_V(E)$ of Definition 3.3.13.) By Theorem 3.4.2,

$$(\theta^* E_G O(n))^{\Lambda_\rho} \cong \theta^*(E_G O(n))^{\Lambda_\rho}.$$

So the spaces in (5.1.10) are non-empty, or a θ-framing on V exists, if and only if the intersection of $\theta(B)$ and the V-indexed component of $(B_G O(n))^G$ as introduced in Theorem 4.2.5 is non-empty.

We also describe the change of tangential structures, which is not studied in this paper. Let q be a morphism from $\theta_1 : B_1 \to B_G O(n)$ to $\theta_2 : B_2 \to B_G O(n)$, equivalently, a G-map $q : B_1 \to B_2$ satisfying $\theta_2 q = \theta_1$. Then a θ_1-framed vector bundle $E \to B$ with $\phi_E : E \to \theta_1^* \zeta_n$ is θ_2-framed by

$$E \to \theta_1^* \zeta_n = q^* \theta_2^* \zeta_n \to \theta_2^* \zeta_n.$$

The morphism q also induces a map on framed-morphisms. So we have a functor

$$q_* : \mathrm{Vec}^{\theta_1} \to \mathrm{Vec}^{\theta_2}, \text{ and similarly } q_* : \mathrm{Mfld}^{\theta_1} \to \mathrm{Mfld}^{\theta_2}.$$

5.2 The θ-framed maps

In Section 5.1, we defined the θ-framed embedding space $\mathrm{Emb}^\theta(M, N)$ for two θ-framed manifolds M and N. In this section, we give an alternative definition in Proposition 5.2.10 following Ayala–Francis [AF15, Definition 2.7].

The classification theorem says that isomorphism classes of bundles are in bijection to homotopy classes of maps to a classifying space. Passing to the classification maps seem to lose the information about morphisms between bundles, but it turns out not to. We show that the automorphism space of a bundle is equivalent to the space of homotopies of a chosen classifying map in Corollary 5.2.9. To this end, we first define a suitable "over category up to homotopy".

Let B be a G-space. A typical example is to take $B = B_G O(n)$. Then we have a Top-enriched over category $G\mathrm{Top}/_B$: the objects are G-spaces over B, and the morphisms are G-maps over B. Explicitly, for G-spaces over B given by G-maps $\phi_M : M \to B$ and $\phi_N : N \to B$, the space $\mathrm{Hom}_{G\mathrm{Top}/_B}(M, N)$ is the pullback displayed in the following diagram: (note that we have $\mathrm{Hom}_{G\mathrm{Top}} = \mathrm{Map}_G$)

$$\begin{array}{ccc} \mathrm{Hom}_{G\mathrm{Top}/_B}(M,N) & \longrightarrow & \mathrm{Map}_G(M,N) \\ \big\downarrow & & \big\downarrow {\scriptstyle \phi_N \circ -} \\ * & \xrightarrow[\{\phi_M\}]{} & \mathrm{Map}_G(M,B) \end{array} \tag{5.2.1}$$

Now we want to work with G-spaces over B up to homotopy. We modify the morphism space by taking the homotopy pullback in (5.2.1). Just like the difference between $G\mathrm{Top}$ and Top_G, we have two versions: the Top-enriched $G\mathrm{Top}^{\mathrm{h}}/_B$ and the GTop-enriched $\mathrm{Top}^{\mathrm{h}}_G/_B$. That is, we have homotopy pullback diagrams of spaces in (5.2.2) and of G-spaces in (5.2.3):

$$\begin{array}{ccc} \mathrm{Hom}_{G\mathrm{Top}^{\mathrm{h}}/_B}(M,N) & \longrightarrow & \mathrm{Map}_G(M,N) \\ \big\downarrow & & \big\downarrow {\scriptstyle \phi_N \circ -} \\ * & \xrightarrow[\{\phi_M\}]{} & \mathrm{Map}_G(M,B) \end{array} \tag{5.2.2}$$

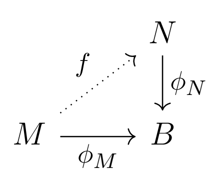
$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/_B}(M,N) & \longrightarrow & \mathrm{Map}(M,N) \\ \big\downarrow & & \big\downarrow {\scriptstyle \phi_N \circ -} \\ * & \xrightarrow[\{\phi_M\}]{} & \mathrm{Map}(M,B) \end{array} \tag{5.2.3}$$

Using the Moore path space model for the homotopy fiber as given in the following definition, one can define unital and associative compositions to make $G\mathrm{Top}^{\mathrm{h}}/_B$ and $\mathrm{Top}^{\mathrm{h}}_G/_B$ categories.

Definition 5.2.4. For $\phi_M : M \to B$ and $\phi_N : N \to B$, the space $\mathrm{Hom}_{G\mathrm{Top}^{\mathrm{h}}/_B}(M, N)$ and the G-space $\mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/_B}(M, N)$ are given by:

$$
\begin{aligned}
\mathrm{Hom}_{G\mathrm{Top}^{\mathrm{h}}/_B}(M, N) = \{(f, \alpha, l) | & f \in \mathrm{Map}_G(M, N), \alpha \in \mathrm{Map}(\mathbb{R}_{\geq 0}, \mathrm{Map}_G(M, B)), \\
& l \in \mathrm{Map}(\mathrm{Map}_G(M, N), \mathbb{R}_{\geq 0}) \text{ such that} \\
& l \text{ is locally constant,} \\
& \alpha(0) = \phi_M, \alpha(t) = \phi_N \circ f \text{ for } t \geq l(f)\}. \\
\mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/_B}(M, N) = \{(f, \alpha, l) | & f \in \mathrm{Map}(M, N), \alpha \in \mathrm{Map}(\mathbb{R}_{\geq 0}, \mathrm{Map}(M, B)), \\
& l \in \mathrm{Map}(\mathrm{Map}(M, N), \mathbb{R}_{\geq 0}) \text{ such that} \\
& l \text{ is locally constant,} \\
& \alpha(0) = \phi_M, \alpha(t) = \phi_N \circ f \text{ for } t \geq l(f)\}.
\end{aligned}
$$

Remark 5.2.5. Roughly speaking, a point in the morphism space $G\mathrm{Top}^{\mathrm{h}}/_B$ is a G-map $f \in \mathrm{Map}_G(M, N)$ and a G-homotopy from ϕ_M to $\phi_N \circ f$ in the following diagram:



A point in the morphism space $\mathrm{Top}^{\mathrm{h}}_G/_B$ is a map $f \in \mathrm{Map}(M, N)$ and a homotopy from ϕ_M to $\phi_N \circ f$; the map f is not necessarily a G-map, but we do require ϕ_M and ϕ_N to be G-maps. And we have

$$\mathrm{Hom}_{G\mathrm{Top}^{\mathrm{h}}/_B}(M, N) \cong (\mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/_B}(M, N))^G.$$

The category $\mathrm{Top}^{\mathrm{h}}_{G}/_{B}$ models θ-framed bundles:

Proposition 5.2.6. *For $i = 1, 2$, let $E_i \to B_i$ be G-n-vector bundles with θ-framings $\phi_i : E_i \to \theta^*\zeta_n$. We have the following equivalences of G-spaces that are natural with respect to the two variables as well as the tangential structure:*

$$\beta : \mathrm{Hom}^{\theta}(E_1, E_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_{G}/_{B}}(B_1, B_2).$$

Proof. One can restrict bundle maps to get maps on the base spaces. We denote this restriction map by π. From our definition of Hom^{θ} in Definition 5.1.4 and $\mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_{G}/_{B}}$ in Definition 5.2.4, π induces the map β and they fit in the following commutative diagram of G-spaces:

$$\begin{array}{ccc}
\mathrm{Hom}^{\theta}(E_1, E_2) & \xrightarrow[\sim]{\beta} & \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_{G}/_{B}}(B_1, B_2) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(E_1, E_2) & \xrightarrow{\pi} & \mathrm{Map}(B_1, B_2) \\
{\scriptstyle \phi_2\circ -}\downarrow & \lrcorner & \downarrow{\scriptstyle \phi_2\circ -} \\
\mathrm{Hom}(E_1, \theta^*\zeta_n) & \xrightarrow{\pi} & \mathrm{Map}(B_1, B)
\end{array} \tag{5.2.7}$$

We claim that the bottom square is a pullback. Since each column is a homotopy fiber sequence, this implies immediately that β is a G-equivalence.

To show the claim, first we note that the isomorphism $\phi_2 : E_2 \cong \phi_2^*\theta^*\zeta_n$ establishes E_2 as a pullback of $\theta^*\zeta_n$ over ϕ_2. So a bundle map $E_1 \to E_2$ is determined by a map on the base $f : B_1 \to B_2$ and a bundle map $(\bar{\varphi}, \varphi) : (E_1, B_1) \to (\zeta_n, B)$ satisfying $\varphi = \phi_2 f$.

$$\begin{array}{ccccc}
 & & \overset{\bar{\varphi}}{\curvearrowright} & & \\
E_1 & \dashrightarrow & E_2 & \longrightarrow & \theta^*\zeta_n \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
B_1 & \xrightarrow[f]{} & B_2 & \xrightarrow[\phi_2]{} & B
\end{array}$$

□

We remark that in Proposition 5.2.6, π is not a homotopy equivalence to its image (see

Theorem 4.3.3 for an in-depth analysis of this map). In other words, a vector bundle map is not just a map on the bases. In contrast, a θ-framed vector bundle map can be seen as a map on the bases as β is an equivalence.

The "classical" bundle maps are the θ-framed bundle maps for the tangential structure $\theta = \mathrm{id} : B_G O(n) \to B_G O(n)$:

Lemma 5.2.8. *For G-vector bundles $E_i \to B_i$, $i = 1, 2$, we have an equivalence of G-spaces:*

$$\alpha : \mathrm{Hom}^{\mathrm{id}}(E_1, E_2) \xrightarrow{\sim} \mathrm{Hom}(E_1, E_2).$$

Proof. By definition, $\mathrm{Hom}^{\mathrm{id}}(E_1, E_2)$ is the homotopy fiber of $\phi_2 \circ -$, so we have a homotopy fiber sequence of G-spaces:

$$\mathrm{Hom}^{\mathrm{id}}(E_1, E_2) \xrightarrow{\alpha} \mathrm{Hom}(E_1, E_2) \xrightarrow{\phi_2 \circ -} \mathrm{Hom}(E_1, \zeta_n) \; .$$

By Lemma 4.3.4, we know $\mathrm{Hom}(E_1, \zeta_n)$ is G-contractible. So α is a G-equivalence. □

Corollary 5.2.9. *For G-vector bundles $E_i \to B_i$, $i = 1, 2$, we have an equivalence of G-spaces:*

$$\mathrm{Hom}(E_1, E_2) \simeq \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G / B_G O(n)}(B_1, B_2).$$

Proof. This follows from Proposition 5.2.6 and Lemma 5.2.8. □

Proposition 5.2.10. *The G-space $\mathrm{Emb}^{\theta}(M, N)$ as defined in Definition 5.1.6 is the homotopy pullback displayed in the following diagram of G-spaces:*

$$\begin{array}{ccc} \mathrm{Emb}^{\theta}(M, N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G / B}(M, N) \\ \downarrow & & \downarrow \\ \mathrm{Emb}(M, N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G / B_G O(n)}(M, N) \end{array} \qquad (5.2.11)$$

Proof. The lower horizontal map in (5.2.11) is neither obvious nor canonical. We take it as the composite in the following commutative diagram with a chosen G-homotopy inverse to α. The maps α and β are G-equivalences by Proposition 5.2.6 and Lemma 5.2.8.

$$\begin{array}{ccccc}
\mathrm{Emb}^\theta(M,N) & \longrightarrow & \mathrm{Hom}^\theta(\mathrm{T}M,\mathrm{T}N) & \xrightarrow[\sim]{\beta} & \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/B}(M,N) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & \mathrm{Hom}^{\mathrm{id}}(\mathrm{T}M,\mathrm{T}N) & \xrightarrow[\sim]{\beta} & \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/B_GO(n)}(M,N) \\
 & & \alpha\Big\downarrow\sim & & \\
\mathrm{Emb}(M,N) & \xrightarrow{d} & \mathrm{Hom}(\mathrm{T}M,\mathrm{T}N) & &
\end{array} \tag{5.2.12}$$

As defined in Definition 5.1.6, $\mathrm{Emb}^\theta(M,N)$ is the pullback in the left square. It is clear that it is also equivalent to the homotopy pullback of the whole square. □

We can take (5.2.11) as an alternative definition to (5.1.7). In practice, (5.1.7) is easier to deal with. First, the right vertical map in the square is a fibration so the diagram is an actual pullback. Second, the map d is easy to describe. On the other hand, (5.2.11) has a conceptual advantage. It can be viewed as a comparison of the θ-framing to the trivial framing $\mathrm{id} : B_GO(n) \to B_GO(n)$.

5.3 Automorphism space of (V, ϕ)

With this alternative description of θ-framed mapping spaces in Section 5.2, we can identify the automorphism G-space $\mathrm{Emb}^\theta(V,V)$ of V in Mfld^θ by first identifying of the automorphism G-space $\mathrm{Hom}^\theta(\mathrm{T}V,\mathrm{T}V)$ of $\mathrm{T}V$ in Vec^θ.

Notation 5.3.1. As ϕ is an equivariant map, $\phi(0)$ for the origin $0 \in V$ is a G-fixed point in B. We denote by $\mathbf{\Lambda}_\phi B$ the Moore loop space of B at the base point $\phi(0)$.

Theorem 5.3.2. *We have the following:*

(1) There is an equivalence of monoids in G-spaces

$$\mathrm{Hom}^{\theta}(\mathrm{T}V, \mathrm{T}V) \xrightarrow{\sim} \mathbf{\Lambda}_{\phi} B,$$

which is natural with respect to tangential structures $\theta : B \to B_G O(n)$. Here, the group G acts on both sides by conjugation.

(2) The automorphism G-space $\mathrm{Emb}^{\theta}(V, V)$ of (V, ϕ) in Mfld^{θ} fits in the following homotopy pullback diagram of G-spaces:

$$\begin{array}{ccc} \mathrm{Emb}^{\theta}(V, V) & \longrightarrow & \mathbf{\Lambda}_{\phi} B \\ \downarrow & & \downarrow \\ \mathrm{Emb}(V, V) & \xrightarrow[d_0]{} & O(V) \end{array}$$

Proof. (1) We have $\mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/B}(V, V)$ from Definition 5.2.4 and showed in Proposition 5.2.6 that restriction-to-the-base gives a natural G-equivalence:

$$\beta : \mathrm{Hom}^{\theta}(\mathrm{T}V, \mathrm{T}V) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/B}(V, V).$$

Let pt be the G-space over B given by $\phi(0) : \mathrm{pt} \to B$. We claim that the two maps $\mathrm{inc} : 0 \to V$ and $\mathrm{proj} : V \to \mathrm{pt}$ can be lifted to give equivalences of $V \simeq \mathrm{pt}$ in $\mathrm{Top}^{\mathrm{h}}_G/B$. If so, pre-composing with inc and post-composing with proj give

$$\mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/B}(V, V) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_G/B}(\mathrm{pt}, \mathrm{pt}) \cong \mathbf{\Lambda}_{\phi} B.$$

It remains to verify the claim, which is a routine job. We choose the lifts of inc and proj given by

$$I = (\mathrm{inc}, \alpha_1, 0) \in \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_{G}/B}(\mathrm{pt}, V), \text{ where } \alpha_1(t) = \phi(0) \text{ for all } t \geq 0.$$

$$P = (\mathrm{proj}, \alpha_2, 1) \in \mathrm{Hom}_{\mathrm{Top}^{\mathrm{h}}_{G}/B}(V, \mathrm{pt}), \text{ where } \alpha_2(t) = \begin{cases} \phi \circ h_t, & 0 \leq t < 1; \\ \phi(0), & t \geq 1; \end{cases}$$

where $h_t : V \to V$ is any chosen homotopy from $h_0 = \mathrm{id}$ to $h_1 = \mathrm{proj}$. Then we have an obvious homotopy:

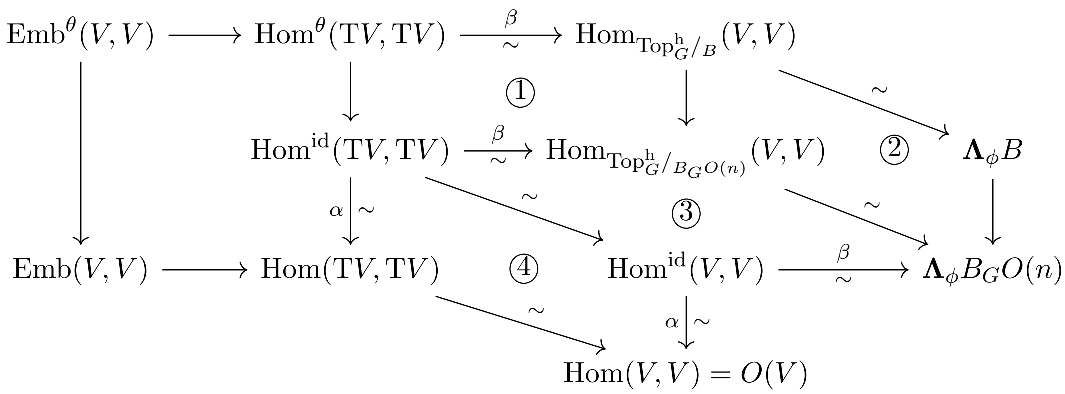
$$P \circ I = (\mathrm{id}, \mathrm{const}_{\phi(0)}, 1) \simeq (\mathrm{id}, \mathrm{const}_{\phi(0)}, 0) = \mathrm{id}_{\mathrm{pt}}$$

and using the contraction h_t, we can also construct a homotopy:

$$I \circ P = (\mathrm{proj}, \alpha_2, 1) \simeq (\mathrm{id}, \mathrm{const}_{\phi}, 0) = \mathrm{id}_V. \qquad \square$$

(2) This is an assembly of Proposition 5.2.10, (1) and part Corollary 4.4.11. However, we note that the map $\mathbf{\Lambda}_{\phi} B \to O(V)$ is only a non-canonical G-equivalence. The author does not know how to upgrade it to a map of G-monoids. So although all spaces displayed in the pullback diagram are G-monoids, it is not obvious whether one can write $\mathrm{Emb}^{\theta}(V, V)$ as a pullback of G-monoids.

To be more precise, we show how the quoted results assemble. We have the following large commutative diagram (5.3.3) expanding (5.2.12). Note that this is a commutative diagram of G-monoids.



(5.3.3)

The map α is studied in Lemma 5.2.8. The map β and the square ① are in Proposition 5.2.6. The diagonal unlabeled maps are all induced by the inclusion $V \to \mathrm{T}V$ and the projection $\mathrm{T}V \to V$. In particularly, the parallelogram ② is in part (1). Naturality of α and β gives the commutativity of ③ and ④. Now, d_0 in the theorem is the composite

$$\mathrm{Emb}(V,V) \xrightarrow{d} \mathrm{Hom}(\mathrm{T}V,\mathrm{T}V) \xrightarrow{\sim} \mathrm{Hom}(V,V).$$

It can be seen that the vertical map in the theorem involves choosing an inverse of the β displayed in the third line.

Remark 5.3.4. From (4.4.12), we have a zigzag of equivalences of G-monoids for any b in the V-indexed component of $(B_GO(n))^G$:

$$\mathbf{\Lambda}_b B_G O(n) \xleftarrow[\sim]{\xi} (\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi \xrightarrow[\sim]{\psi} O(V).$$

This zigzag is hidden in (5.3.3). Recall that we abusively use ϕ to denote both $\mathrm{T}V \to \zeta_n$ and $V \to B_GO(n)$. Firstly, $b = \phi(0) \in B_GO(n)$ is a point in the desired component, and we have

$$\mathrm{Hom}^{\mathrm{id}}(V,V) \cong (\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi.$$

This is because $\mathrm{Hom}(V,\zeta_n) = \mathrm{Pr}_V(B_G O(n)) \cong E_G O(n)$ (see Definition 3.3.13 for Pr_V), and the framing $\phi(0) \in \mathrm{Hom}(V,\zeta_n)$ corresponds to a chosen point $z \in E_G O(n)$ over b. The point z is G-fixed using the G-action on $\mathrm{Pr}_V(B_G O(n))$. We can identify the path data of an element of $\mathrm{Hom}^{\mathrm{id}}(V,V)$, as defined in Definition 5.1.4, to the path data of a representative element of $(\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi$ that starts at z, as described in Corollary 4.4.11. Secondly, the maps ψ and ξ are just the maps α and β. In other words, (4.4.12) can identified with the following part of (5.3.3):

$$\mathbf{\Lambda}_\phi B_G O(n) \xleftarrow[\sim]{\beta} \mathrm{Hom}^{\mathrm{id}}(V,V) \xrightarrow[\sim]{\alpha} \mathrm{Hom}(V,V) = O(V).$$

5.4 Equivariant factorization homology

In this section, we use the Λ-sequence machinery in Chapter 2 and the $G\mathrm{Top}$-enriched category Mfld^θ developed in Section 5.1 to define the equivariant factorization homology as a bar construction.

Recall from Section 5.1 that we have fix an n-dimensional orthogonal G-representation V and a θ-framing $\phi : \mathrm{T}V \to \theta^*\zeta_n$ on the G-manifold V.

Definition 5.4.1. For a θ-framed manifold M, we define the Λ-sequence $\mathscr{D}_M^\theta$ to be

$$\mathscr{D}_M^\theta = \mathrm{Emb}^\theta({}^*V, M) \in \Lambda_*^{op}(G\mathrm{Top}).$$

Here, ${}^*V$ is the symmetric monoidal functor $(\Lambda, \vee, \mathbf{0}) \to (\mathrm{Mfld}^\theta, \sqcup, \varnothing)$ that sends $\mathbf{1}$ to (V,ϕ) and sends $\mathbf{0} \to \mathbf{1}$ to the unique map $\varnothing \hookrightarrow V$.

Explicitly, on objects, we have

$$\mathscr{D}^\theta_M : \Lambda^{op} \to G\mathrm{Top},$$
$$\mathbf{k} \mapsto \mathrm{Emb}^\theta(\coprod_k V, M);$$

On morphisms, $\Sigma_k = \Lambda(\mathbf{k}, \mathbf{k})$ acts by permuting the copies of V, and $s^k_i : \mathbf{k-1} \to \mathbf{k}$ induces $(s^k_i)^* : \mathscr{D}^\theta_M(k) \to \mathscr{D}^\theta_M(k-1)$ by forgeting the i-th V in the embeddings for $1 \le i \le k$.

Plugging in V in the second variable, we have $\mathscr{D}^\theta_V$. Using Construction 2.0.9, we get associated functors of $\mathscr{D}^\theta_M$ and $\mathscr{D}^\theta_V$, which we denote by

$$\mathrm{D}^\theta_M, \mathrm{D}^\theta_V : G\mathrm{Top}_* \to G\mathrm{Top}_*;$$
$$\mathrm{D}^\theta_M(X) = \coprod_{k \ge 0} \mathscr{D}^\theta_M(k) \times_{\Sigma_k} X^{\times k}/\sim .$$

These Λ-sequences satisfy certain structures coming from the composition of morphisms in Mfld^θ. It is best described using the Kelly monoidal structure $(\Lambda^{op}_*(G\mathrm{Top}), \odot)$ as defined in Construction 2.0.9. Taking $\mathscr{V} = G\mathrm{Top}$ and $(\mathscr{W}, \otimes) = (\mathrm{Mfld}^\theta, \sqcup)$ in Construction 2.0.12, we can identify

$$\mathscr{D}^\theta_M = \underline{\mathcal{H}_\mathscr{W}}(V, M).$$

Consequently, $\mathscr{D}^\theta_V = \underline{\mathcal{H}_\mathscr{W}}(V, V)$ is a monoid in $(\Lambda^{op}_*(G\mathrm{Top}), \odot)$ and $\mathscr{D}^\theta_M$ is a right module over it.

Translating by Theorem 2.0.4, $\mathscr{D}^\theta_V$ is a reduced operad in $(G\mathrm{Top}, \times)$. This operad is close to the little V-disk operad $\mathscr{D}_V$ except it also allows θ-framed automorphisms of the embedded V-disks. In light of Theorem 5.3.2, we expect there to be something like an equivalence of G-operads: $\mathscr{D}^\theta_V \simeq \mathscr{D}_V \rtimes (\mathbf{\Lambda}_\phi B)$. This will be formulated in Appendix B and proved in Proposition B.2.2.

By Proposition 2.0.11, the right module map $\mathscr{D}^\theta_M \odot \mathscr{D}^\theta_V \to \mathscr{D}^\theta_M$ of Λ-sequences yields a natural transformation $\mathrm{D}^\theta_M \circ \mathrm{D}^\theta_V \to \mathrm{D}^\theta_M$; The monoid structure maps $\mathscr{I}_1 \to \mathscr{D}^\theta_V$ and $\mathscr{D}^\theta_V \odot \mathscr{D}^\theta_V \to \mathscr{D}^\theta_V$ yield natural transformations $\mathrm{id} \to \mathrm{D}^\theta_V$ and $\mathrm{D}^\theta_V \circ \mathrm{D}^\theta_V \to \mathrm{D}^\theta_V$.

The following is a standard definition from [May97]:

Definition 5.4.2. Let $\mathscr{C}$ be a reduced operad in $(G\mathrm{Top}, \times)$ and C be the associated reduced monad. An object $A \in G\mathrm{Top}_*$ is a $\mathscr{C}$-algebra if there is a map $\gamma : \mathrm{C}A \to A$ such that the following diagrams commute, where the unlabeled maps are the unit and multiplication map of the monad C:

$$\begin{array}{ccc} CCA & \xrightarrow{C\gamma} & CA \\ \downarrow & & \downarrow\gamma \\ CA & \xrightarrow{\gamma} & A \end{array} \quad ; \qquad \begin{array}{ccc} A & \longrightarrow & CA \\ & \searrow^{=} & \downarrow\gamma \\ & & A \end{array} \quad .$$

In what follows, let A be a $\mathscr{D}^\theta_V$-algebra in $G\mathrm{Top}_*$. We have a simplicial G-space, whose q-th level is

$$\mathbf{B}_q(\mathrm{D}^\theta_M, \mathrm{D}^\theta_V, A) = \mathrm{D}^\theta_M(\mathrm{D}^\theta_V)^q A.$$

The face maps are induced by the above-given structure maps

$$\mathrm{D}^\theta_M \mathrm{D}^\theta_V \to \mathrm{D}^\theta_M, \quad \mathrm{D}^\theta_V \mathrm{D}^\theta_V \to \mathrm{D}^\theta_V \text{ and } \gamma : \mathrm{D}^\theta_V A \to A.$$

The degeneracy maps are induced by $\mathrm{id} \to \mathrm{D}^\theta_V$.

We have the following definition after the idea of [And10, IX.1.5]:

Definition 5.4.3. The factorization homology of M with coefficient A is

$$\int_M^\theta A := \mathbf{B}(\mathrm{D}^\theta_M, \mathrm{D}^\theta_V, A).$$

Notation 5.4.4. Since we are not comparing tangential structures in this paper, we drop the θ in the notation and write $\int_M^\theta A$ as $\int_M A$.

The category of algebras $\mathscr{D}^\theta_V[G\mathrm{Top}_*]$ has a transfer model structure via the forgetful functor $\mathscr{D}_V^\theta[G\mathrm{Top}_*] \to G\mathrm{Top}_*$ ([BM03, 3.2, 4.1]), so that weak equivalences of maps between algebras are just underlying weak equivalences.

Proposition 5.4.5. *The functor* $\int_M - : \mathscr{D}_V^\theta[G\mathrm{Top}_*] \to G\mathrm{Top}_*$ *is homotopical.*

Proof. The proof is a formal argument assembling the literature and deferred. We show that the bar construction is Reedy cofibrant in Corollary 6.4.7 and geometric realization preserves levelwise weak equivalences between Reedy cofibrant simplicial G-sapces as quoted in Theorem 6.3.5. □

We have the following properties of the factorization homology.

Proposition 5.4.6.

$$\int_V A \simeq A.$$

Proof. This follows from the extra degeneracy argument of [May72, Proposition 9.8]. The extra degeneracy coming from the unit map of the first D_V^θ establishes A as a retract of $\mathbf{B}(\mathrm{D}_V^\theta, \mathrm{D}_V^\theta, A)$, which is just $\int_V A$. □

Proposition 5.4.7. *For θ-framed manifolds M and N,*

$$\int_{M \sqcup N} A \simeq \int_M A \times \int_N A.$$

Proof. Without loss of generality, we may assume that both M and N are connected. Then

$$\begin{aligned}
\mathscr{D}_{M\sqcup N}^\theta(k) &\cong \mathrm{Emb}^\theta(\sqcup_k V, M \sqcup N) \\
&\cong \coprod_{i=0}^{k} \big(\mathrm{Emb}^\theta(\sqcup_i V, M) \times \mathrm{Emb}^\theta(\sqcup_{k-i} V, N)\big) \times_{\Sigma_i \times \Sigma_{k-i}} \Sigma_k \\
&\cong \coprod_{i=0}^{k} \big(\mathscr{D}_M^\theta(i) \times \mathscr{D}_N^\theta(k-i)\big) \times_{\Sigma_i \times \Sigma_{k-i}} \Sigma_k
\end{aligned}$$

This is the formula of the Day convolution of $\mathscr{D}^\theta_M$ and $\mathscr{D}^\theta_N$. So we have

$$\mathscr{D}^\theta_{M\sqcup N} \cong \mathscr{D}^\theta_M \boxtimes \mathscr{D}^\theta_N. \tag{5.4.8}$$

We drop the θ in the rest of the proof. By (5.4.8) and iterated use of Proposition 2.0.13, there is an isomorphism in $\Lambda^{op}_*(G\mathrm{Top})$ for each q:

$$\mathbf{B}_q(\mathscr{D}_{M\sqcup N}, \mathscr{D}_V, \imath_0(A)) \cong \mathbf{B}_q(\mathscr{D}_M, \mathscr{D}_V, \imath_0(A)) \boxtimes \mathbf{B}_q(\mathscr{D}_N, \mathscr{D}_V, \imath_0(A)). \tag{5.4.9}$$

Iterated use of (2.0.10) identifies

$$\imath_0(\mathbf{B}_q(\mathrm{D}_M, \mathrm{D}_V, A)) \cong \mathbf{B}_q(\mathscr{D}_M, \mathscr{D}_V, \imath_0(A)),$$

so evaluating on the 0-th level of (5.4.9) gives equivalence of simplical G-spaces:

$$\mathbf{B}_*(\mathrm{D}_{M\sqcup N}, \mathrm{D}_V, A) \cong \mathbf{B}_*(\mathrm{D}_M, \mathrm{D}_V, A) \times \mathbf{B}_*(\mathrm{D}_N, \mathrm{D}_V, A).$$

The claim follows from passing to geometric realization and commuting the geometric realization with finite products. □

5.5 Relation to configuration spaces

Now we restrict our attention to the V-framed case for an orthogonal n-dimensional G-representation V. We give V the canonical V-framing $\mathrm{T}V \cong V \times V$ and let M be a G-manifold of dimension n. When M is V-framed, we denote the V-framing by $\phi_M : \mathrm{T}M \to V$.

In this section, we first prove that a smooth embedding of $\sqcup_k V$ into M is determined by its images and derivatives at the origin up to a contractible choice of homotopy (Proposition 5.5.5). The proof of the non-equivariant version can be found in Andrade's thesis

[And10, V4.5]. Then we proceed to prove that a V-framed embedding space of $\sqcup_k V$ into M as defined in (5.1.7) is homotopically the same as choosing the center points (Corollary 5.5.9).

To formulate the result, we first define the suitable equivariant configuration space related to a manifold, which will be "the space of points and derivatives".

We use $\mathscr{F}_E(k)$ to denote the ordered configuration space of k distinct points in E, topologized as a subspace of E^k. When E is a G-space, $\mathscr{F}_E(k)$ has a G-action by pointwise acting that commutes with the Σ_k-action by permuting the points.

Definition 5.5.1. For a fiber bunde $p : E \to M$, define $\mathscr{F}_{E\downarrow M}(k)$ to be configurations of k-ordered distinct points in E with distinct images in M. $\mathscr{F}_{E\downarrow M}(k)$ is a subspace of $\mathscr{F}_E(k)$ and inherits a free Σ_k-action. When p is a G-fiber bundle, $\mathscr{F}_{E\downarrow M}(k)$ is a G-space.

Example 5.5.2. When $k = 1$, $\mathscr{F}_{E\downarrow M}(1) \cong \mathscr{F}_E(1)$.

Example 5.5.3. When $E = M \times F$ is a trivial bundle over M with fiber F,

$$\mathscr{F}_{E\downarrow M}(k) \cong \mathscr{F}_M(k) \times F^k.$$

In general, we have the following pullback diagram:

$$\begin{array}{ccc} \mathscr{F}_{E\downarrow M}(k) & \hookrightarrow & E^k \\ \downarrow & & \downarrow p^k \\ \mathscr{F}_M(k) & \hookrightarrow & M^k. \end{array}$$

Now, we take $E = \mathrm{Fr}_V(\mathrm{T}M)$. Recall that $\mathrm{Fr}_V(\mathrm{T}M) = \mathrm{Hom}(V, \mathrm{T}M)$ is a G-bundle over M. For an embedding $\sqcup_k V \to M$, we take its derivative and evaluate at $0 \in V$. We will get k-points in $\mathrm{Fr}_V(\mathrm{T}M)$ with different images projecting to M. In other words, the composition

$$\mathrm{Emb}(\coprod_k V, M) \xrightarrow{d} \mathrm{Hom}(\coprod_k \mathrm{T}V, \mathrm{T}M) \xrightarrow{ev_0} \mathrm{Hom}(\coprod_k V, \mathrm{T}M) = \mathrm{Fr}_V(\mathrm{T}M)^k$$

factors as

$$\mathrm{Emb}(\coprod_k V, M) \overset{d_0}{\to} \mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}M)\downarrow M}(k) \hookrightarrow \mathrm{Fr}_V(\mathrm{T}M)^k. \tag{5.5.4}$$

Proposition 5.5.5. *The map d_0 in (5.5.4) is a G-Hurewicz fibration and $(G\times\Sigma_k)$-homotopy equivalence.*

Proof. It suffices to prove for $k = 1$, that is, for

$$d_0 : \mathrm{Emb}(V, M) \to \mathrm{Fr}_V(\mathrm{T}M),$$

since the general case will follow from the pullback diagram:

$$\begin{array}{ccc}
\mathrm{Emb}(\coprod_k V, M) & \hookrightarrow & \mathrm{Emb}(V, M)^k \\
\downarrow d_0 & & \downarrow (d_0)^k \\
\mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}M)\downarrow M}(k) & \hookrightarrow & \mathrm{Fr}_V(\mathrm{T}M)^k.
\end{array}$$

We show that d_0 is a G-Hurewicz fibration by finding an equivariant local trivialization. Fix an H-fixed point $x \in \mathrm{Fr}_V(\mathrm{T}M)$ and let $d_0^{-1}(x)$ be the fiber at x. Our goal is to find an H-invariant neighborhood $\bar{U}$ of x in $\mathrm{Fr}_V(\mathrm{T}M)$ and an H-equivariant homeomorphism

$$\bar{U} \times d_0^{-1}(x) \cong d_0^{-1}(\bar{U}) \subset \mathrm{Emb}(V, M).$$

First, we find the small neighborhood $\bar{U}$. Let x_0 be the image of x under the projection $\mathrm{Fr}_V(\mathrm{T}M) \to M$, then x_0 is also H-fixed. Consequently, $W = \mathrm{T}_{x_0}M$ is an H-representation. Using the exponential map, there is a local chart for M that is H-homeomorphic to W with $0 \in W$ mapping to x_0. We will refer to this local chart as W. On the chart, $\mathrm{Fr}_V(\mathrm{T}M)$ is homeomorphic to $W \times \mathrm{Hom}(V, W)$, and we may identify x with $(0, A) \in W \times \mathrm{Hom}(V, W)$ for some H-invariant A. For simplicity, we put a metric on W to make it an orthogonal representation. Choose an ϵ-ball $U_1 \subset W$ and a small enough H-invariant neighborhood

$A \in U_2 \subset \mathrm{Hom}(V, W)$ and set $\bar{U} = U_1 \times U_2$.

Second, we construct an H-equivariant local trivialization of d_0 on $\bar{U}$,

$$\begin{aligned} \bar{\phi}: \quad \bar{U} \times d_0^{-1}(x) &\to \mathrm{Emb}(V, M), \\ (y, f) &\mapsto \phi(y) \circ f \end{aligned}$$

by utilizing a yet-to-be-constructed map $\phi : \bar{U} \to \mathrm{Diff}(M)$. The map ϕ needs to satisfy the following properties:

(1) ϕ is H-equivariant;

(2) $\phi(x) = \mathrm{id}$;

(3) For any $y \in \bar{U}$, $d(\phi(y)) \circ x = y$. (Recall that $x, y \in \mathrm{Fr}_V(\mathrm{T}M) = \mathrm{Hom}(V, \mathrm{T}M)$ and $d(\phi(y)) : \mathrm{T}M \to \mathrm{T}M$ is the derivative of $\phi(y)$.)

For any $\chi \in \mathrm{Diff}(M)$ and $g \in \mathrm{Emb}(V, M)$, $d_0(\chi \circ g) = d_{g(0)}(\chi) \circ d_0(g)$. We can check that $d_0(\phi(y) \circ f) = y$ and that for any $g \in \mathrm{Emb}(V, M)$ with $d_0(g) = y$, $d_0(\phi(y)^{-1} \circ g) = x$. So, the map $\phi(y) \circ -$ translates $d_0^{-1}(x)$, the fiber over x, to $d_0^{-1}(y)$, the fiber over y. This shows $\bar{\phi}$ is an H-equivariant homeomorphism to $d_0^{-1}(\bar{U})$.

Third, we describe only the idea of the construction of ϕ, as it is a bit technical. Noticing that the requirement (3) is local, we can construct $\phi_0 : \bar{U} \to \mathrm{Diff}(W)$ on the local chart W satisfying all the requirements using linear maps. Then we need to modify these diffeomorphisms of W equivariantly without changing them on the ϵ-ball U_1, so that they become compactly supported and still satisfy all the requirements. Finally, we extend the modified ϕ_0 by identity to get ϕ, diffeomorphisms of M. The technical part is the modification for ϕ_0. It can be done by (1) taking an H-invariant polytope P containing U_1, (2) taking a large enough multiple m such that mP contains the image of all $\phi_0(\bar{U})(U_1)$, (3) setting $\phi_0(y)$ to be id outside of mP, (4) extending by piecewise linear function between P and mP, and (5) smoothing it. It is because of this step that we have to choose a small enough neighborhood U_2, but it is good enough for our purpose.

To show d_0 is a G-homotopy equivalence, one can construct a section of d_0 by the exponential map:

$$\sigma : \mathrm{Fr}_V(\mathrm{T}M) \to \mathrm{Emb}(V, M).$$

Since there is a (contractible) choice of the radius at each point for the exponential map to be homeomorphism, σ is defined only up to homotopy. Using blowing-up-at-origin techniques, the section can be shown to indeed give a deformation retract of d_0.

To be useful later, the section exists up to homotopy for general k as well:

$$\sigma : \mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}M)\downarrow M}(k) \to \mathrm{Emb}(\coprod_k V, M). \tag{5.5.6}$$

□

Now we are ready to justify our desired equivalence of the V-framed embedding spaces from V to M and configuration spaces of M. Moreover, we show that this equivalence is compatible over $\mathrm{Emb}(\coprod_k V, M)$ in part (2). This will be used in later sections to compare different scanning maps.

Lemma 5.5.7. *For a V-framed manifold M, the projection*

$$\mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}M)\downarrow M}(k) \to \mathscr{F}_M(k)$$

is a trivial bundle with fiber $(\mathrm{Hom}(V, V))^k$. We call the section that selects $(\mathrm{id}_V)^k$ in each fiber the zero section z.

Proof. Regarding V as a bundle over a point, we may identify $\mathrm{Fr}_V(V) = \mathrm{Hom}(V, V)$. Since M is V-framed, $\mathrm{Fr}_V(\mathrm{T}M) \cong \mathrm{Fr}_V(M \times V) \cong M \times \mathrm{Fr}_V(V)$ as equivariant bundles. The claim follows from Example 5.5.3. □

We can restrict the exponential map (5.5.6) to the zero section in Lemma 5.5.7 to get

$$\sigma_0 : \mathscr{F}_M(k) \to \mathrm{Emb}(\coprod_k V, M). \tag{5.5.8}$$

Corollary 5.5.9. *For a V-framed manifold M, we have:*

(1) Evaluating at 0 of the embedding gives a $(G \times \Sigma_k)$-homotopy equivalence:

$$ev_0 : \mathscr{D}_M^{\mathrm{fr}_V}(k) \equiv \mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M) \to \mathscr{F}_M(k).$$

(2) The map ev_0 and σ_0 in (5.5.8) fit in the following $(G \times \Sigma_k)$-homotopy commutative diagram:

$$\begin{array}{ccc} \mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M) & \longrightarrow & \mathrm{Emb}(\coprod_k V, M) \\ {\scriptstyle ev_0}\downarrow & \nearrow_{\sigma_0} & \\ \mathscr{F}_M(k) & & \end{array}$$

Proof. (1) By Definitions 5.1.6 and 5.4.1, $\mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M)$ is the homotopy fiber of the composite:

$$D : \mathrm{Emb}(\coprod_k V, M) \xrightarrow{d} \mathrm{Hom}(\coprod_k \mathrm{T}V, \mathrm{T}M) \xrightarrow{(\phi_M)_*} \mathrm{Hom}(\coprod_k \mathrm{T}V, V).$$

We would like to restrict the composite at $\{0\} \sqcup \cdots \sqcup \{0\} \subset V \sqcup \cdots \sqcup V$. Since

$$\mathrm{Hom}(\coprod_k \mathrm{T}V, \mathrm{T}M) \cong \prod_k \mathrm{Hom}(\mathrm{T}V, \mathrm{T}M)$$

and $i_0 : V \to \mathrm{T}V$ is a G-homotopy equivalence of G-vector bundles,

$$ev_0 : \mathrm{Hom}(\coprod_k \mathrm{T}V, \mathrm{T}M) \xrightarrow{(i_0)^*} \prod_k \mathrm{Hom}(V, \mathrm{T}M) \cong (\mathrm{Fr}_V(\mathrm{T}M))^k$$

is a $(G \times \Sigma_k)$-homotopy equivalence. So in the following commutative diagram, the vertical maps are all $(G \times \Sigma_k)$-homotopy equivalences:

$$\begin{array}{ccccc}
\mathrm{Emb}(\coprod_k V, M) & \xrightarrow{d} & \mathrm{Hom}(\coprod_k \mathrm{T}V, \mathrm{T}M) & \xrightarrow{(\phi_M)_*} & \mathrm{Hom}(\coprod_k \mathrm{T}V, V) \\
{\scriptstyle d_0}\downarrow \simeq \text{ by Proposition 5.5.5} & & {\scriptstyle ev_0}\downarrow \simeq & & {\scriptstyle ev_0}\downarrow \simeq \\
\mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}M)\downarrow M}(k) & \hookrightarrow & \mathrm{Fr}_V(\mathrm{T}M)^k & \xrightarrow{(\phi_M)_*} & \mathrm{Fr}_V(V)^k \\
\downarrow \cong \text{ by Lemma 5.5.7} & & & & \| \\
\mathscr{F}_M(k) \times \mathrm{Fr}_V(V)^k & & \xrightarrow{proj_2} & & \mathrm{Fr}_V(V)^k.
\end{array}$$

We focus on the top composition D and the bottom map $proj_2$. The map ev_0 between their codomains is a based map. Indeed, the base point of $\mathrm{Hom}(\coprod_k \mathrm{T}V, V)$ is from the V-framing of $\coprod_k V$ and is $(G \times \Sigma_k)$-fixed. It is mapped to id^k, the base point of $\mathrm{Fr}_V(V)^k$. Consequently, there is a $(G \times \Sigma_k)$-homotopy equivalence between the homotopy fibers of these two maps.

$$\mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M) = \mathrm{hofib}(D) \xrightarrow{\simeq} \mathrm{hofib}(proj_2). \tag{5.5.10}$$

Our desired ev_0 in question is the composite of (5.5.10) and the following map:

$$X : \mathrm{hofib}(proj_2) \to \mathscr{F}_M(k) \times \mathrm{Fr}_V(V)^k \xrightarrow{proj_1} \mathscr{F}_M(k).$$

It suffices to show that X is a $(G \times \Sigma_k)$-equivalence. Indeed, X is the comparison of the homotopy fiber and the actual fiber of $proj_2$. Write temporarily $F = \mathscr{F}_M(k)$ and $B = \mathrm{Fr}_V(V)^k$ with the $(G \times \Sigma_k)$-fixed base point b. Then the map X is projection to F:

$$\mathrm{hofib}(proj_2) \cong P_b B \times F \to F.$$

The claim follows from the fact that $P_b B$ is $(G \times \Sigma_k)$-contractible.

(2) We have the following $(G \times \Sigma_k)$-homotopy commutative solid diagram, where z is the

zero section in Lemma 5.5.7:

$$\begin{array}{ccc} \mathrm{Emb}^{\mathrm{fr}_V}(\coprod_k V, M) & \longrightarrow & \mathrm{Emb}(\coprod_k V, M) \\ \Big\downarrow ev_0 & \overset{\sigma_0}{\nearrow} & \Big\downarrow d_0 \\ \mathscr{F}_M(k) & \xrightarrow{z} & \mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}M)\downarrow M}(k). \end{array}$$

The commutativity can be seen easily and is actually an extension of the big commutativity diagram in part (1) to (homotopy) fibers. As $\sigma_0 = \sigma \circ z$ and σ is a $(G\times\Sigma_k)$-homotopy inverse of d_0 by Proposition 5.5.5, the diagram with the dotted arrow is homotopy commutative. □

Remark 5.5.11. Part (1) of Corollary 5.5.9 gives a levelwise equivalence of objects in $\Lambda_*^{op}(G\mathrm{Top})$:

$$ev_0 : \mathscr{D}_M^{\mathrm{fr}_V} \to \mathscr{F}_M.$$

We conclude this section by comparing $\mathscr{D}_V^{\mathrm{fr}_V}$ to $\mathscr{D}_V$. For background, the little V-disks operad $\mathscr{D}_V$ is a well-studied notion introduced for recognizing V-fold loop spaces; see [GM17, 1.1]. It is an equivariant generalization of the little n-disks operad. Roughly speaking, $\mathscr{D}_V(k)$ is the space of non-equivariant embeddings of k copies of the open unit disks $\mathrm{D}(V)$ to $\mathrm{D}(V)$, each of which takes only the form $\mathbf{v} \mapsto a\mathbf{v} + \mathbf{b}$ for some $0 < a \leq 1$ and $\mathbf{b} \in \mathrm{D}(V)$, called rectilinear. In particular, the spaces are the same as those of little n-disks operad, and so are the structure maps. The G-action on $\mathscr{D}_V(k)$ is by conjugation. It is well-defined, commutes with the Σ_k-action and the structure maps are G-equivariant.

Proposition 5.5.12. *There is an equivalence of G-operads $\beta : \mathscr{D}_V \to \mathscr{D}_V^{\mathrm{fr}_V}$.*

Proof. To construct the map of operads β, we first define $\beta(1) : \mathscr{D}_V(1) \to \mathscr{D}_V^{\mathrm{fr}_V}(1)$. Take $e \in \mathscr{D}_V(1)$, we must give $\beta(1)(e) = (f, l, \alpha) \in \mathscr{D}_V^{\mathrm{fr}_V}(1)$. Explicitly,

$$e : \mathrm{D}(V) \to \mathrm{D}(V) \text{ is } e(\mathbf{v}) = a\mathbf{v} + \mathbf{b} \text{ for some } 0 < a \leq 1 \text{ and } \mathbf{b} \in \mathrm{D}(V).$$

Define

$$f : V \to V \quad \text{to be} \quad f(\mathbf{v}) = a\mathbf{v} + \mathbf{b};$$

$$l \in \mathbb{R}_{\geq 0} \quad \text{to be} \quad l = -\ln(a);$$

$$\alpha : \mathbb{R}_{\geq 0} \to \mathrm{Hom}(\mathrm{T}V, V) \quad \text{to be} \quad \alpha(t) = \begin{cases} \mathfrak{c}_{\exp(-t)\mathrm{I}} & \text{for } t \leq l; \\ \mathfrak{c}_{a\mathrm{I}} & \text{for } t > l. \end{cases}$$

For α, $\mathrm{Hom}(\mathrm{T}V, V) \cong \mathrm{Map}(V, O(V))$, I is the unit element of $O(V)$ and $\mathfrak{c}$ is the constant map to the indicated element. It can be checked that $\beta(1)$ as defined is a map of G-monoids.

Restricting $\beta(1)^k : \mathscr{D}_V(1)^k \to \mathscr{D}_V^{\mathrm{fr}_V}(1)^k$ to the subspace $\mathscr{D}_V(k) \subset \mathscr{D}_V(1)^k$, we get $\beta(k) : \mathscr{D}_V(k) \to \mathscr{D}_V^{\mathrm{fr}_V}(k)$. Then β is automatically a map of G-operads because $\mathscr{D}_V$ and $\mathscr{D}_V^{\mathrm{fr}_V}$ are suboperads of $\mathscr{D}_V(1)^-$ and $(\mathscr{D}_V^{\mathrm{fr}_V})^-$.

The composite $\mathrm{ev}_0 \circ \beta : \mathscr{D}_V \to \mathscr{D}_V^{\mathrm{fr}_V} \to \mathscr{F}_V$ is a levelwise homotopy equivalence by [GM17, Lemma 1.2]. We have shown ev_0 is a levelwise equivalence (Remark 5.5.11). So β is also a levelwise homotopy equivalence. □

CHAPTER 6: NONABELIAN POINCARÉ DUALITY FOR V-FRAMED MANIFOLDS

Configuration spaces have scanning maps out of them. It turns out that equivariantly the scanning map is an equivalence on G-connected labels X. Since the factorization homology is built up simplicially by the configuration spaces, we can upgrade the scanning equivalence to what is known as the nonabelian Poincaré duality theorem.

6.1 Scanning map for V-framed manifolds

In this section we construct the scanning map, a natural transformation of right $\mathrm{D}_V^{\mathrm{fr}_V}$-functors:

$$s : \mathrm{D}_M^{\mathrm{fr}_V}(-) \to \mathrm{Map}_c(M, \Sigma^V -). \tag{6.1.1}$$

In Appendix A, we compare our scanning map to the existing different constructions in the literature and utilize known results about equivariant scanning maps to give Theorem 6.1.5, a key input to the nonabelian Poincaré duality theorem in Section 6.2.

Assuming that the scanning map (6.1.1) has been constructed for a moment, the right $\mathrm{D}_V^{\mathrm{fr}_V}$-functor structure for $\mathrm{Map}_c(M, \Sigma^V -)$ is as follows: the scanning map for $M = V$ gives a map of monads $s : \mathrm{D}_V^{\mathrm{fr}_V} \to \Omega^V\Sigma^V$. It induces a natural map

$$\Sigma^V \mathrm{D}_V^{\mathrm{fr}_V} \xrightarrow{\Sigma^V s} \Sigma^V\Omega^V\Sigma^V \xrightarrow{\text{counit}} \Sigma^V.$$

Now we construct the scanning map. For any G-space X, recall that

$$\mathrm{D}_M^{\mathrm{fr}_V}(X) = \coprod_{k\geq 0} \mathscr{D}_M^{\mathrm{fr}_V}(k) \times_{\Sigma_k} X^k/\sim,$$

where $\sim$ is the base point identification. Take an element

$$P = [\bar{f}_1, \cdots, \bar{f}_k, x_1, \cdots, x_k] \in \mathscr{D}_M^{\mathrm{fr}_V}(k) \times_{\Sigma_k} X^k.$$

Here, each $\bar{f}_i = (f_i, \alpha_i)$ consists of an embedding $f_i : V \to M$ and a homotopy α_i of two bundle maps $\mathrm{T}V \to V$, see Definition 5.1.6. We use only the embeddings f_i to define an element $s_X(P) \in \mathrm{Map}_c(M, \Sigma^V X)$:

$$s_X(P)(m) = \begin{cases} f_i^{-1}(m) \wedge x_i & \text{when } m \in M \text{ is in the image of some } f_i; \\ * & \text{otherwise.} \end{cases} \tag{6.1.2}$$

Notice that if x_i is the base point, $f_i^{-1}(m) \wedge x_i$ is the base point regardless of what f_i is. So passing to the quotient, (6.1.2) yields a well-defined map

$$s_X : \mathrm{D}_M^{\mathrm{fr}_V}(X) \to \mathrm{Map}_c(M, \Sigma^V X). \tag{6.1.3}$$

In particular, taking $X = S^0$, we get

$$s_{S^0} : \coprod_{k \geq 0} \mathscr{D}_M^{\mathrm{fr}_V}(k)/\Sigma_k \to \mathrm{Map}_c(M, S^V), \tag{6.1.4}$$

and s_X is simply a labeled version of it. A more categorical construction of the scanning map s_X, as the composition of the Pontryagin-Thom collapse map and a "folding" map $\vee_k S^V \times X^k \to \Sigma^V X$ is given in [MZZ20, Section 9].

We use the following results of Rourke–Sanderson [RS00], which are proved using equivariant transversality. To translate from their context to ours, see Theorem A.0.2 and Theorem A.3.2.

Theorem 6.1.5. *The scanning map* $s_X : \mathrm{D}_M^{\mathrm{fr}_V} X \to \mathrm{Map}_c(M, \Sigma^V X)$ *is:*

(1) a weak G-equivalence if X is G-connected,

(2) or a weak group completion if $V \cong W \oplus 1$ and $M \cong N \times \mathbb{R}$. Here, W is a $(n-1)$-dimension G-representation and N is a W-framed compact manifold, so that $N \times \mathbb{R}$ is V-framed. □

6.2 Nonabelian Poincaré duality theorem

We have seen that the scanning map is an equivalence for G-connected labels X. Since the factorization homology is built up simplicially by the configuration spaces, we can upgrade the scanning equivalence to what is known as the nonabelian Poincaré duality theorem (NPD). The proof in this section follows the non-equivariant treatment by Miller [Mil15].

Let A be a $\mathrm{D}_V^{\mathrm{fr}_V}$-algebra in $G\mathrm{Top}$ throughout this section. Assume that A is non-degenerately based, meaning that the structure map $\mathscr{D}_V^{\mathrm{fr}_V}(0) = \mathrm{pt} \to A$ gives a non-degenrate base point of A. This is a mild assumption for homotopical purposes. We use the following V-fold delooping model of A.

Definition 6.2.1. The V-fold delooping of A, denoted as $\mathrm{B}^V A$, is the monadic two sided bar construction $\mathrm{B}(\Sigma^V, \mathrm{D}_V^{\mathrm{fr}_V}, A)$.

Here, $\mathrm{B}_q(\Sigma^V, \mathrm{D}_V^{\mathrm{fr}_V}, A) = \Sigma^V (\mathrm{D}_V^{\mathrm{fr}_V})^q A$. The first face map $\Sigma^V \mathrm{D}_V^{\mathrm{fr}_V} \to \Sigma^V$is induced by the scanning map of monads $\mathrm{D}_V^{\mathrm{fr}_V} \to \Omega^V \Sigma^V$. The last face map $\mathrm{D}_V^{\mathrm{fr}_V} A \to A$ is the structure maps of the algebra. The middle face maps and degeneracy maps are induced by the structure map of the monad $\mathrm{D}_V^{\mathrm{fr}_V} \mathrm{D}_V^{\mathrm{fr}_V} \to \mathrm{D}_V^{\mathrm{fr}_V}$ and $\mathrm{Id} \to \mathrm{D}_V^{\mathrm{fr}_V}$.

Remark 6.2.2. There is an equivalence of G-operads $\mathscr{D}_V \to \mathscr{D}_V^{\mathrm{fr}_V}$ from the little V-disk operad to the little V-framed disk operad. So a $\mathrm{D}_V^{\mathrm{fr}_V}$-algebra restricts to a D_V-algebra and there is an equivalence from the Guillou–May delooping [GM17] to our delooping: $\mathrm{B}(\Sigma^V, \mathrm{D}_V, A) \to \mathrm{B}(\Sigma^V, \mathrm{D}_V^{\mathrm{fr}_V}, A)$

Theorem 6.2.3. *(NPD) Let M be a V-framed manifold and A be a $\mathrm{D}_V^{\mathrm{fr}_V}$-algebra in $G\mathrm{Top}$. Then there is a G-map, which is a weak G-equivalence if A is G-connected:*

$$\int_M A \equiv |B_\bullet(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A)| \to \mathrm{Map}_*(M^+, \mathrm{B}^V A),$$

where M^+ is the one-point-compactification of M.

Proof. We will sketch the proof, assuming some lemmas that are proven in the remainder of this section. First, from (6.1.1), we have a scanning map for each $q \geq 0$:

$$\mathrm{D}_M^{\mathrm{fr}_V}(\mathrm{D}_V^{\mathrm{fr}_V})^q A \to \mathrm{Map}_c(M, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^q A).$$

They assemble to a simplicial scanning map, which is a levelwise weak G-equivalence as shown in Corollary 6.3.4:

$$\mathrm{B}(s, \mathrm{id}, \mathrm{id}) : \mathrm{B}_\bullet(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A) \to \mathrm{Map}_c(M, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A). \tag{6.2.4}$$

One can identify the space of compactly supported maps with the space of based maps out of the one point compactification:

$$\mathrm{Map}_c(M, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A) \xrightarrow{\simeq} \mathrm{Map}_*(M^+, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A).$$

With some cofibrancy argument in Theorem 6.3.5 and Corollary 6.4.7, this map induces is a weak G-equivalence on the geometric realization:

$$\mathrm{B}(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A) \to |\mathrm{Map}_*(M^+, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A)|.$$

Next, we change the order of the mapping space and the geometric realization. There is a natural map:

$$|\mathrm{Map}_*(M^+, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A)| \to \mathrm{Map}_*(M^+, |\Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A|).$$

Theorem 6.5.7, taking $X = M^+$ and $K_\bullet = \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A$, gives a sufficient connectivity condition for it to be a weak G-equivalence. This connectivity condition is then checked in Lemma 6.5.13.

Finally, $|\Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A| = \mathrm{B}^V A$ by Definition 6.2.1. This finishes the proof of the theorem. □

Remark 6.2.5. If we take $M = V$ in the theorem and use Proposition 5.4.6, we get that $A \simeq \Omega^V \mathrm{B}^V A$ for a G-connected E_V-algebra A. This recovers [GM17, Theorem 1.14] and justifies the definition of $\mathrm{B}^V A$.

6.3 Connectedness

Definition 6.3.1. A G-space X is G-connected if X^H is connected for all subgroups $H \subset G$.

To show that the scanning map is an equivalence in each simplicial level, we need:

Lemma 6.3.2. *If X is G-connected, then $\mathrm{D}_V^{\mathrm{fr}_V} X$ is also G-connected.*

Proof. By Corollary 5.5.9, $\mathrm{D}_V^{\mathrm{fr}_V} X$ is G-homotopy equivalent to $F_V X$. It suffices to show that $F_V X$ is G-connected. Fix any subgroup $H \subset G$; we must show that $(F_V X)^H$ is connected. This is the space of H-equivariant unordered configuration on V with based labels in X. Intuitively, this is true because the space of labels X is G-connected, so that one can always move the labels of a configuration to the base point. Nevertheless, we give a proof here by carefully writing down the fixed points of $F_V X$ in terms of the fixed points of $\mathscr{F}_V(k)$ and

X. We have:

$$(F_V X)^H = (\coprod_{k\geq 0} F_V(k) \times_{\Sigma_k} X^k / \sim)^H = \coprod_{k\geq 0} (F_V(k) \times_{\Sigma_k} X^k)^H / \sim_H$$

Here, $\sim$ is the equivalence relation in Remark 2.0.8 and $\sim_H$ is $\sim$ restricted on H-fixed points. They are explicitly forgetting a point in the configuration if the corresponding label is the base point in X. Notice that taking H-fixed points will not commute with $\approx$ in Construction 2.0.7, but commutes with $\sim$. This is because the H-action preserves the filtration and $\sim$ only identifies elements of different filtrations.

Since the Σ_k-action is free on $F_V(k) \times X^k$ and commutes with the G-action, we have a principal G-Σ_k-bundle

$$F_V(k) \times X^k \to F_V(k) \times_{\Sigma_k} X^k.$$

To get H-fixed points on the base space, we need to consider the Λ_α-fixed points on the total space for all the subgroups $\Lambda_\alpha \subset G \times \Sigma_k$ that are the graphs of some group homomorphisms $\alpha : H \to \Sigma_k$. More precisely, by Theorem 3.4.2, we have

$$(F_V(k) \times_{\Sigma_k} X^k)^H = \coprod_{[\alpha: H \to \Sigma_k]} \Big((F_V(k) \times X^k)^{\Lambda_\alpha} / Z_{\Sigma_k}(\alpha) \Big).$$

Here, the coproduct is taken over Σ_k-conjugacy classes of group homomorphisms and $Z_{\Sigma_k}(\alpha)$ is the centralizer of the image of α in Σ_k.

We would like to make the expression coordinate-free for k. A homomorphism α can be identified with an H-action on the set $\{1, \cdots, k\}$. For an H-set S, write $X^S = \mathrm{Map}(S, X)$ and $F_V(S) = \mathrm{Emb}(S, V)$. Then

$$(F_V(k) \times X^k)^{\Lambda_\alpha} = (F_V(S) \times X^S)^H \text{ and } Z_{\Sigma_k}(\alpha) = \mathrm{Aut}_H(S).$$

So we have:

$$(F_V(k)\times_{\Sigma_k} X^k)^H = \coprod_{[S]:\text{iso classes of } H\text{-set}, |S|=k} \Big((F_V(S)\times X^S)^H/\mathrm{Aut}_H(S)\Big).$$

If we take care of the base point identification, we end up with:

$$(F_V X)^H = \Big(\coprod_{[S]:\text{iso classes of finite } H\text{-set}} (F_V(S)\times X^S)^H/\mathrm{Aut}_H(S)\Big)/\sim_H. \tag{6.3.3}$$

Suppose that the H-set S breaks into orbits as $S = \amalg_i r_i(H/K_i)$ for $i = 1, \cdots, s$, where K_i's are in distinct conjugacy classes of subgroups of H and $r_i > 0$, then we know explicitly each coproduct component is:

$$\begin{aligned}(F_V(S)\times X^S)^H/\mathrm{Aut}_H S &= (\mathrm{Emb}_H(S,V)\times \mathrm{Map}_H(S,X))/\mathrm{Aut}_H S\\ &= (\mathrm{Emb}_H(\amalg_i r_i(H/K_i), V)\times \prod_i (X^{K_i})^{r_i})/\prod_i (W_H(K_i)\wr \Sigma_{r_i}).\end{aligned}$$

Since X^{K_i} are all connected, so are the spaces $\prod_i (X^{K_i})^{r_i}$. Each contains the base point of the labels $* = \prod_i \prod_{r_i} * \to \prod_i (X^{K_i})^{r_i}$. So after the gluing $\sim_H$, each component in (6.3.3) is in the same component as the base point of $F_V X$. Thus $(F_V X)^H$ is connected. □

Corollary 6.3.4. *The map* $B_\bullet(\mathrm{D}_M^{\mathrm{fr}_V}, \mathrm{D}_V^{\mathrm{fr}_V}, A) \to \mathrm{Map}_c(M, \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A)$ *in (6.2.4) is a levelwise weak G-equivalence of simplicial G-spaces if A is G-connected.*

Proof. This is a consequence of Theorem 6.1.5 and Lemma 6.3.2. □

For geometric realization, we have:

Theorem 6.3.5 (Theorem 1.10 of [MMOar])**.** *A levelwise weak G-equivalence between Reedy cofibrant simplicial objects realizes to a weak G-equivalence.*

6.4 Cofibrancy

We take care of the cofibrancy issues in this part, following details in [May72]. We first show that some functors preserve G-cofibrations. One who is willing take it as a blackbox may skip directly to Definition 6.4.5. The NDR data give a hands-on way to handle cofibrations.

Definition 6.4.1 (Definition A.1 of [May72])**.** A pair (X, A) of G-spaces with $A \subset X$ is an NDR pair if there exists a G-invariant map $u : X \to I = [0, 1]$ such that $A = u^{-1}(0)$ and a homotopy given by a map $h : I \to \mathrm{Map}_G(X, X)$ satisfying

- $h_0(x) = x$ for all $x \in X$;
- $h_t(a) = a$ for all $t \in I$ and $a \in A$;
- $h_1(x) \in A$ for all $x \in u^{-1}[0, 1)$.

The pair (h, u) is said to a representation of (X, A) as an NDR pair. A pair (X, A) of based G-spaces is an NDR pair if it is an NDR pair of G-spaces with the h_t being based maps for all $t \in I$.

Such a pair gives a G-cofibration $A \to X$. The function u gives an open neighborhood U of A by taking $U = u^{-1}[0, 1)$. The function h restricts on $I \times U$ to a neighborhood deformation retract of A in X. We refer to u as the neighborhood data and h as the retract data.

We have the following "*ad hoc* definition" for a functor F to preserve NDR-pairs in a functorial way:

Definition 6.4.2 (Definition A.7 of [May72])**.** A functor $F : G\mathrm{Top} \to G\mathrm{Top}$ is admissible if for any representation (h, u) of (X, A) as an NDR pair, there exists a representation (Fh, Fu) of (FX, FA) as an NDR pair such that:

(i) The map $Fh : I \to \mathrm{Map}_G(FX, FX)$ is determined by $(Fh)_t = F(h_t)$.

(ii) The map $Fu : FX \to [0,1]$ satisfies the following property: for any map $g : X \to X$ such that $ug(x) < 1$ whenever $x \in X$ and $u(x) < 1$, $(Fu)(Fg)(y) < 1$ whenever $y \in FX$ and $(Fu)(y) < 1$.

And similarly for functors $F : G\mathrm{Top}_* \to G\mathrm{Top}_*$.

In plain words, the retract data Fh for (FX, FA) are dictated by applying the functor F to h, but there is some room in choosing the neighborhood data Fu. Denote the open neighborhood of FA in FX by $U' = (Fu)^{-1}[0,1)$. Condition (ii) says that U' is a "robust open neighborhood" in the sense that a map of pairs $g : (X, U) \to (X, U)$ induces a map $Fg : (FX, U') \to (FX, U')$.

Remark 6.4.3. Suppose that F sends inclusions to inclusions and that we have (Fh, Fu) satisfying (i) and (ii).

- In order for (Fh, Fu) to be a representation of (FX, FA) as an NDR pair, we only need to check

$$(Fu)^{-1}(0) = FA,\ (Fu)^{-1}[0,1) \subset (Fh_1)^{-1}(FA).$$

- Since we have $U \subset h_1^{-1}(A)$, we get $FU \subset F(h_1^{-1}A) \subset (Fh_1)^{-1}(FA)$. That is, the neighborhood FU of FA retracts to FA, but it may not be open.

Admissible functors obviously preserve cofibrations. The elaboration of the NDR data gives a way to easily verify that a functor is admissible, at least in the following cases:

Lemma 6.4.4. *Any functor F associated to $\mathscr{F} \in \Lambda_*^{op}(G\mathrm{Top})$ is admissible. In particular, both $\mathrm{D}_V^{\mathrm{fr}_V}$ and $\mathrm{D}_M^{\mathrm{fr}_V}$ are admissible. The functors $\mathrm{Map}_c(M, -)$ and $\mathrm{Map}_*(M^+, -)$ are admissible. The functor Σ^V sends NDR pairs to NDR pairs.*

Proof. To show F is admissible, it suffices to find the neighborhood data Fu in each case.

Let $\mathscr{F} \in \Lambda_*^{op}(G\mathrm{Top})$ be a unital Λ-sequence. The functor F associated to $\mathscr{F}$ as defined in Construction 2.0.7 sends $X \in G\mathrm{Top}_*$ to $FX = \left(\sqcup_k \mathscr{F}(k) \times_{\Sigma_k} X^k\right)/\sim$. Define $Fu(c, x_1, \cdots, x_j) = \max_{i=1,\cdots,j} u(x_i)$ for $c \in \mathscr{F}(k)$ and $x_i \in X$. This is well-defined and G-equivariant. We check that Fu satisfies Definition 6.4.2. For (ii), suppose we have $g : X \to X$ and $y = (c, x_1, \cdots, x_j) \in FX$ with $Fu(y) = \max_{i=1,\cdots,j} u(x_i) < 1$. Then

$$(Fu)(Fg)(y) = \max_{i=1,\cdots,j} u(gx_i) < 1.$$

To check the conditions in Remark 6.4.3, we have $Fu(c, x_1, \cdots, x_j) = 0$ if and only if $u(x_i) = 0$ for all i. This gives $(Fu)^{-1}(0) = FA$; $Fu(c, x_1, \cdots, x_j) < 1$ if and only if $u(x_i) < 1$ for all i. This gives $(Fu)^{-1}[0,1) \subset FU \subset (Fh_1)^{-1}(FA)$.

For $F = \mathrm{Map}_c(M, -)$, let $Fu(f) = \max_{m \in M} u(f(m))$ for $f \in \mathrm{Map}_c(M, X)$. This is well-defined since f is compactly supported. Fu is G-equivariant since u is. We check that Fu satisfies Definition 6.4.2. For (ii), suppose we have $g : X \to X$ and $f \in \mathrm{Map}_c(M, X)$ with $Fu(f) = \max_{m \in M} u(f(m)) < 1$. Then $(Fu)(Fg)(f) = \max_{m \in M} u(gf(m)) < 1$. For the conditions in Remark 6.4.3, $Fu(f) = 0$ if and only if $u(f(m)) = 0$ for all $m \in M$. This gives $(Fu)^{-1}(0) = \mathrm{Map}_c(M, A) = FA$; $Fu(f) < 1$ if and only if $u(f(m)) < 1$ for all $m \in M$. This gives $(Fu)^{-1}[0,1) \subset FU \subset (Fh_1)^{-1}(FA)$. The same argument works for $F = \mathrm{Map}_*(M^+, -)$.

The functor $F = \Sigma^V$ can not be admissible in the sense of Definition 6.4.2, because for the pair $(X, A) = (S^1, \mathrm{pt})$ and any NDR representation (h, u) of it,

$$(Fh_1)^{-1}(FA) = \Sigma^V(h_1^{-1}A)$$

does not contain an open neighborhood of the base point of $\Sigma^V X$, which leaves no room for U' to exist. Nevertheless, using the fact that (S^V, ∞) is an NDR pair, $(\Sigma^V X, \Sigma^V A)$ is still an NDR pair by a based version of [May72, Lemma A.3]. □

Definition 6.4.5 (Lemma 1.9 of [MMOar]). A simplicial G-space $X_\bullet$ is Reedy cofibrant if all degeneracy operators s_i are G-cofibrations.

The following lemma shows that monadic bar constructions are Reedy cofibrant.

Lemma 6.4.6 (adaptation of Proposition A.10 of [May72]). *Let $\mathscr{C}$ be a reduced operad in G-spaces such that the unit map $\eta : \mathrm{pt} \to \mathscr{C}(1)$ gives a non-degenerate base point. Let C be the reduced monad associated to $\mathscr{C}$. Let A be a C-algebra in $G\mathrm{Top}_*$ and $F : G\mathrm{Top}_* \to G\mathrm{Top}_*$ be a right-C-module functor. Suppose that F sends NDR pairs to NDR pairs. Then $B_\bullet(F, C, A)$ is Reedy cofibrant.*

Proof. We need to show that for any $n \geq 0$ and $0 \leq i \leq n$, the degeneracy map

$$s_n^i = FC^i \eta_{C^{n-i}A} : FC^n A \to FC^{n+1}A$$

is a G-cofibration. Write $X = C^{n-i}A$. By Lemma 6.4.4, C sends NDR pairs to NDR pairs. Start from the NDR pair (A, pt) and apply this functor $(n-i)$ times, we get an NDR pair $(C^{n-i}A, \mathrm{pt}) = (X, \mathrm{pt})$. Together with the assumption that $\mathscr{C}(1)$ is non-degenerately based, we can show (CX, X) is an NDR pair where X is identified with the image $\eta_X : X \to CX$ (see the proof of [May72, A.10]). Applying C another i times and then F, we get the NDR pair $\left(FC^{i+1}X, FC^i X\right) = \left(FC^{n+1}A, FC^n A\right)$. Thus $s_n^i = FC^i \eta_X$ is a G-cofibration. □

Corollary 6.4.7. *Let M, V, A be as in Theorem 6.2.3. Then the following are Reedy cofibrant simplicial G-spaces:*

$$\mathrm{B}_\bullet(\mathrm{D}_M^{\mathrm{fr}_V}, D_V^{\mathrm{fr}_V}, A),\ \mathrm{Map}_c(M, \Sigma^V (\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A) \text{ and } \mathrm{Map}_*(M^+, \Sigma^V (\mathrm{D}_V^{\mathrm{fr}_V})^\bullet A).$$

Proof. In Lemma 6.4.6, we take $C = \mathrm{D}_V^{\mathrm{fr}_V}$ and respectively $F = \mathrm{D}_M^{\mathrm{fr}_V}$, $F = \mathrm{Map}_c(M, \Sigma^V -)$ or $F = \mathrm{Map}_*(M^+, \Sigma^V -)$. By Lemma 6.4.4, each F does send NDR pairs to NDR pairs. □

6.5 Dimension

We start with an introduction to G-CW complexes and equivariant dimensions following [May96, I.3]. A G-CW complex X is a union of G-spaces X^n obtained by inductively gluing cells $G/K \times D^n$ for subgroups $K \subset G$ via G-maps along their boundaries $G/K \times S^{n-1}$ to the previous skeleton X^{n-1}. Conventionally, $X^{-1} = \varnothing$.

We shall look at functions from the conjugacy classes of subgroups of G to $\mathbb{Z}_{\geq -1}$ and typically denote such a function by ν. We say that a G-CW complex X has dimension $\leq \nu$ if its cells of orbit type G/H all have dimensions $\leq \nu(H)$, and that a G-space X is ν-connected if X^H is $\nu(H)$-connected for all subgroups $H \subset G$, that is, $\pi_k(X^H) = 0$ for $k \leq \nu(H)$. We allow $\nu(H) = -1$ for the case $X^H = \varnothing$.

For the purpose of induction in this paper, we use the following *ad hoc* definition:

Definition 6.5.1. A based G-CW complex is a union of G-spaces X^n obtained by inductively gluing cells to $X^{-1} = \mathrm{pt}$. We refer to the base point as $*$. And we do NOT count the point in X^{-1} as a cell for a based G-CW complex, excluding it from counting the dimension as well. This is not the same as a based G-CW complex in [May96, Page 18], where the base point is put in the 0-skeleton X^0.

Fix a subgroup $H \subset G$. We have the double coset formula

$$G/K \cong \coprod_{1 \leq i \leq |H\backslash G/K|} H/K_i \text{ as } H\text{-sets,} \tag{6.5.2}$$

where each $K_i = H \cap g_i K g_i^{-1}$ for some element $g_i \in G$. So a (based) G-CW structure on X restricts to a (based) H-CW structure on the H-space $\mathrm{Res}^G_H X$. A function ν from the conjugacy classes of subgroups of G to $\mathbb{Z}_{\geq -1}$ induces a function from the conjugacy classes of subgroups of H to $\mathbb{Z}_{\geq -1}$, which we still call ν. However, for X of dimension $\leq \nu$, $\mathrm{Res}^G_H X$ may not be of dimension $\leq \nu$, as we see in (6.5.2) that an H/K_i-cell can come from a G/K-cell for a larger group K. For a function ν, we define the function d_ν to be

$$d_\nu(K) = \max_{K \subset L} \nu(L).$$

Then $\mathrm{Res}^G_H X$ is of dimension $\leq d_\nu$.

Remark 6.5.3. More specifically, one can define the dimension of a (based) G-CW complex X to be the minimum ν such that X is of dimension $\leq \nu$. Suppose X has dimension ν. Then from (6.5.2), we get:

(i) The (based) H-CW complex $\mathrm{Res}^G_H X$ has dimension μ, where

$$\mu(K) = \max_{\substack{K \subset L \\ K = L \cap H}} \nu(L).$$

We have $\mu \leq d_\nu$, and it can be strictly less. (For a trivial example, take $H = G$.)

(ii) The (based) CW-complex X^H has dimension $\mu(H) = d_\nu(H)$. (In the based case, we also exclude the base point from counting the dimension of X^H.)

We define the dimension of a representation V to be $\dim(V)(H) = \dim(V^H)$ for H representing a conjugacy class of subgroups of G. Note that $d_{\dim(V)} = \dim(V)$.

The goal of this section is to give a sufficent condition for the following map (6.5.4) to be a weak G-equivalence. Let X be a finite based G-CW complex and $K_\bullet$ be a simplicial G-space. Then the levelwise evaluation is a G-map

$$|\mathrm{Map}_*(X, K_\bullet)| \wedge X \cong |\mathrm{Map}_*(X, K_\bullet) \wedge X| \to |K_\bullet|,$$

whose adjoint gives a G-map

$$|\mathrm{Map}_*(X, K_\bullet)| \to \mathrm{Map}_*(X, |K_\bullet|). \tag{6.5.4}$$

Non-equivariantly, it is one of the key steps in May's recognition principal [May72] to realize that (6.5.4) is a weak equivalence when the dimension of X is small compared to the connectivity of $K_\bullet$. May proved this using quasi-fibrations, a concept that goes back to Dold–Thom. Equivariantly, one has a similar result (see Theorem 6.5.7). It is due to Hauschild and written down by Costenoble–Waner [CW91].

Definition 6.5.5. A map $p : Y \to W$ of spaces is a quasi-fibration if p is onto and it induces an isomorphism on homotopy groups $\pi_*(Y, p^{-1}(w), y) \to \pi_*(W, w)$ for all $w \in W$ and $y \in p^{-1}(w)$. In other words, there is a long exact sequence on homotopy groups of the sequence $p^{-1}(w) \to Y \to W$ for any $w \in W$.

Usually, the geometric realization of a levelwise fibration is not a fibration. The following theorem gives conditions when it is a quasi-fibration, which is good enough for handling the homotopy groups.

Theorem 6.5.6. *([May72, Theorem 12.7]) Let $p : E_\bullet \to B_\bullet$ be a levelwise Hurewicz fibration of pointed simplicial spaces such that $B_\bullet$ is Reedy cofibrant and B_n is connected for all n. Set $F_\bullet = p^{-1}(*)$. Then the realization $|E_\bullet| \to |B_\bullet|$ is a quasi-fibration with fiber $|F_\bullet|$.* □

We need the following:

Theorem 6.5.7. *Let G be a finite group. If X is a finite based G-CW complex of dimension $\leq \nu$ and $K_\bullet$ is a simplicial G-space such that for any n, K_n is d_ν-connected, then the natural map (6.5.4)*

$$|\mathrm{Map}_*(X, K_\bullet)| \to \mathrm{Map}_*(X, |K_\bullet|)$$

is a weak G-equivalence.

Proof. Let $* = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{d_\nu(e)} = X$ be the G-CW skeleton of X. We use induction on k to show that

(i) $\mathrm{Map}_*(X^k, K_n)^H$ is connected for all n and $H \subset G$.

(ii) $|\mathrm{Map}_*(X^k, K_\bullet)|^H \to \mathrm{Map}_*(X^k, |K_\bullet|)^H$ is a weak equivalence for all $H \subset G$;

The base case $k = -1$ is obvious. For the inductive case, take the cofiber sequence

$$X^k \to X^{k+1} \to X^{k+1}/X^k$$

and map it into $K_\bullet$. We then apply (6.5.4) and get the following commutative diagram:

$$\begin{array}{ccccc} |\mathrm{Map}_*(X^{k+1}/X^k, K_\bullet)|^H & \longrightarrow & |\mathrm{Map}_*(X^{k+1}, K_\bullet)|^H & \longrightarrow & |\mathrm{Map}_*(X^k, K_\bullet)|^H \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_*(X^{k+1}/X^k, |K_\bullet|)^H & \longrightarrow & \mathrm{Map}_*(X^{k+1}, |K_\bullet|)^H & \longrightarrow & \mathrm{Map}_*(X^k, |K_\bullet|)^H \end{array} \tag{6.5.8}$$

Since maps out of a cofiber sequence form a fiber sequence, we have a fiber sequence in the second row and a realization of the following levelwise fiber sequence in the first row:

$$\mathrm{Map}_*(X^{k+1}/X^k, K_\bullet)^H \longrightarrow \mathrm{Map}_*(X^{k+1}, K_\bullet)^H \longrightarrow \mathrm{Map}_*(X^k, K_\bullet)^H \tag{6.5.9}$$

By the inductive hypothesis (i) and Theorem 6.5.6, it realizes to a quasi-fibration.

We first show the inductive case of (i). Suppose that we have

$$X^{k+1}/X^k = \vee_i (G/K_i)_+ \wedge S^{k+1},$$

where $\{K_i\}_i$ is a finite sequence of subgroups of G. This implies $\nu(K_i) \geq k+1$. From (6.5.2), we have $X^{k+1}/X^k \cong \vee_i \vee_j (H/K_{i,j})_+ \wedge S^{k+1}$ as a space with H-action, where each $K_{i,j}$ is G-conjugate to a subgroup of K_i. Since $d_\nu(K_{i,j}) \geq \nu(K_i)$, we have $d_\nu(K_{i,j}) \geq k+1$ and the following space is connected by assumption:

$$\mathrm{Map}_*(X^{k+1}/X^k, K_n)^H = \prod_i \mathrm{Map}_*(S^{k+1}, K_n^{K_{i,j}}).$$

This space is the fiber in (6.5.9). The connectedness of the base space by the inductive hypothesis (i) implies that of the total space.

We next show the inductive case of (ii). Commuting geometric realization with finite product and fixed point, the left vertical map of (6.5.8) is a product of maps

$$|\mathrm{Map}_*(S^{k+1}, K_\bullet^{K_{i,j}})| \to \mathrm{Map}_*(S^{k+1}, |K_\bullet^{K_{i,j}}|).$$

These maps are weak equivalences by [May72, Theorem 12.3]. By the inductive hypothesis (ii), the right vertical map is a weak equivalence. Comparing the long exact sequences of homotopy groups, this implies that the middle vertical map is also a weak equivalence. □

Remark 6.5.10. Non-equivariantly, supposing that $\dim(X) = m$, Miller [Mil15, Cor 2.22] observed that the theorem is also true if K_n is only $(m-1)$-connected for all n, since the only thing that fails in the proof is (i) for $k = m$. Equivariantly, one needs (i) to hold for $k < d_\nu(e)$. So an equivariant stingy man can only relax the assumption to K_n^H being $\min\{d_\nu(H), d_\nu(e) - 1\}$-connected for all n and H.

Just as a remark, the unbased version of Theorem 6.5.7 is the following:

Theorem 6.5.11. *([CW91, Lemma 5.4]) Let G be a finite group. If Y is a finite (unbased) G-CW complex and $K_\bullet$ is a simplicial G-space such that for any n, K_n is* $\dim(Y)$*-connected, then the natural map*

$$|\mathrm{Map}(Y, K_\bullet)| \to \mathrm{Map}(Y, |K_\bullet|)$$

is a weak G-equivalence.

Theorem 6.5.11 is a consequence of Theorem 6.5.7 by taking $X = Y \sqcup \{*\}$ and using Remark 6.5.3. Note that by adopting the strange convention of the dimension of a based G-CW complex, the dimension of Y is the same as X. On the other hand, we have the cofiber sequence $S^0 \to X_+ \to X$ for a based G-CW complex X as well as the identification of $\mathrm{Map}_*(X_+, K_\bullet)$ with $\mathrm{Map}(X, K_\bullet)$. If $K_\bullet$ is G-connected, we can use the quasi-fibration technique and take $Y = X$ in Theorem 6.5.11 to deduce Theorem 6.5.7. But there are also cases to apply Theorem 6.5.7 where $K_\bullet$ is not required to be G-connected, for example, when $X = (G/H)_+ \wedge S^n$ for $H \neq G$. So Theorem 6.5.7 is slightly finer than Theorem 6.5.11.

Finally, we prepare the following results for the application of Theorem 6.5.7 in the setting of nonabelian Poincaré duality Theorem 6.2.3. We need G-CW structures on G-manifolds M, which exist by work of Illman:

Theorem 6.5.12 (Theorem 3.6 of [Ill78])**.** *For a smooth G-manifold M and a closed smooth G-submanifold N, there exists a smooth G-equivariant triangulation of (M, N).* □

Lemma 6.5.13. *Let M be a V-framed manifold and A be a G-connected space, then*

(1) M^+ has the homotopy type of a G-CW complex of dimension $\leq \dim(V)$.

(2) $K_n = \Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^n A$ is $\dim(V)$-connected.

Proof. (1) Since M is a V-framed, the exponential maps give local coordinate charts of M^H as a (possibly empty) manifold of dimension $\dim(V^H)$. If M is compact we take $W = M$, otherwise we take a compact manifold W with boundary such that M is diffeomorphic to the interior of W. By Theorem 6.5.12, $(W, \partial W)$ has a G-equivariant triangulation. It gives a relative G-CW structure on $(W, \partial W)$ with relative cells of type G/H of dimension $\leq \dim(V^H)$. The quotient $W/\partial W$ gives the desired G-CW model for M^+.

(2) For any subgroup $H \subset G$, we have $K_n^H = (\Sigma^V(\mathrm{D}_V^{\mathrm{fr}_V})^n A)^H = \Sigma^{V^H}((\mathrm{D}_V^{\mathrm{fr}_V})^n A)^H$. By Lemma 6.3.2, $((\mathrm{D}_V^{\mathrm{fr}_V})^n A)^H$ is connected. So K_n^H is $\dim(V^H)$-connected. Thus, K_n is $\dim(V)$-connected. □

APPENDIX A: A COMPARISON OF SCANNING MAPS

The scanning map studied in Section 6.1 is a key input to the Nonabelian Poincaré duality theorem. In this chapter we compare our scanning map (6.1.3) to other constructions.

Notation A.0.1. For a G-manifold M, $\mathrm{Sph}(\mathrm{T}M)$ is the fiberwise one-point compatification of the tangent bundle of M. It is a G-fiber bundle over M with based fiber S^n, where the base point in each fiber is the point at infinity.

Non-equivariantly, people have used the name scanning map to refer to different but related constructions. In slogan, it is a map from the (fattened) configuration spaces of a manifold M to compactly defined sections of $\mathrm{T}M$, or compactly supported sections of $\mathrm{Sph}(\mathrm{T}M)$. McDuff [McD75] was probably the first to study the scanning map for general manifolds. She thought of it as the field of the point charges and proved homological stability properties of this map. In our case of $\mathrm{T}M \cong M \times V$, the situation is simpler and we have defined a scanning map in (6.1.4):

$$s_{S^0} : \coprod_{k \geq 0} \mathscr{D}_M^{\mathrm{fr}_V}(k)/\Sigma_k \to \mathrm{Map}_c(M, S^V).$$

The left hand side is a model of the configuration space as justified in Corollary 5.5.9 (1); the right hand side is equivalent to the compactly supported sections of $\mathrm{Sph}(\mathrm{T}M) \cong M \times S^V$.

We are interested in the scanning maps of Manthorpe–Tillman and McDuff, both of which can be made equivariant without pain. The following table is a summary of the natural domains and codomains of each construction:

scanning map	domain	codomain
this paper, s	framed embeddings V to M	maps M^+ to S^V
Manthorpe–Tillman, $\tilde{s}^{\mathrm{MT}}$	embeddings V to M	sections of $\mathrm{Sph}(\mathrm{T}M)$
McDuff, $\tilde{s}^{\mathrm{MD}}$	configuration of points of M	sections of $\mathrm{Sph}(\mathrm{T}M)$

In this chapter, we focus on the case of V-framed manifolds M. Then these maps have equivalent domains and codomains. We will show in Proposition A.1.4 and Proposition A.2.3 that:

Theorem A.0.2. *The scanning maps s_X, s_X^{MD} and s_X^{MT} are G-homotopic after the change of domain.* □

Notation A.0.3. In the above and subsequent paragraphs,

- We use the letter s for scanning maps without labels and s_X for labels in X.
- A tilde is put on s to denote when the codomain is the sections of $\mathrm{Sph}(\mathrm{T}M)$, that is, before composition with the framing.
- A superscript is put on s to distinguish between the different authors in the literature.

A.1 Scanning map from tubular neighborhood

Non-equivariantly, Manthorpe–Tillman [MT14, Section 3.1] gave a map

$$\gamma^+ : \Big(\coprod_{k\geq 0} \mathrm{Emb}(\sqcup_k \mathbb{R}^n, M) \times_{\Sigma_k} X^k\Big)/\sim\ \to \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X).$$

Here, $\mathrm{Section}_c$ is the space of compactly supported sections; τ_X is the constant parametrized base space $X \times M$ over M and $\mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X$ is the fiberwise smashing of $\mathrm{Sph}(\mathrm{T}M)$ with X. (To translate, take their $M_0 = \varnothing$, $Y = W \times X$. Their $E_k(M, \pi)$ is $\mathrm{Emb}(\coprod_k \mathbb{R}^n, M) \times_{\Sigma_k} X^k$, and their $\Gamma(W \setminus M_0, W \setminus M, \pi)$ is $\mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X)$.)

The key feature of their construction is to exploit the data of the tubular neighborhood, so a framing on M is not needed. For example, when $k = 1$, we start with an embedding $f \in \mathrm{Emb}(\mathbb{R}^n, M)$ and want to define $\gamma^+(f)$, a compactly supported section of $\mathrm{Sph}(\mathrm{T}M)$. The image of f is a tubular neighborhood of the image of $0 \in V$ in M, and f induces an inclusion of bundles $df : \mathrm{T}\mathbb{R}^n \to \mathrm{T}M$. There is a canonical diagonal section $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \cong \mathrm{T}\mathbb{R}^n$. Pushing this section by df gives $\gamma^+(f)$.

We can modify their γ^+ by replacing $\mathbb{R}^n$ by the representation V to get

$$\gamma_V^+ : \mathrm{Emb}_M(X) \equiv \Big(\coprod_{k\geq 0} \mathrm{Emb}(\sqcup_k V, M) \times_{\Sigma_k} X^k\Big)/\sim\ \to \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X).$$

We then precompose with the forgetting map $\mathrm{D}_M^{\mathrm{fr}_V}(X) \to \mathrm{Emb}_M(X)$ in Remark 5.1.8 to get

$$\tilde{s}_X^{\mathrm{MT}} : \mathrm{D}_M^{\mathrm{fr}_V}(X) \to \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X). \tag{A.1.1}$$

We describe how $\tilde{s}_X^{\mathrm{MT}}$ works on the subspace $k = 1$ and it is similar on the whole space. For the element $\bar{f} = (f, \alpha) \in \mathrm{Emb}^{\mathrm{fr}_V}(V, M)$, we take the embedding $f : V \to M$. The derivative map of f is $df : \mathrm{T}V \cong V \times V \to \mathrm{T}M$. For each $m \in \mathrm{image}(f)$, we need a vector $\tilde{s}^{\mathrm{MT}}(f) \in \mathrm{T}_m M$ that is determined by f. Denote $v = f^{-1}(m) \in V$. We have $df_v : V \cong \mathrm{T}_v V \to \mathrm{T}_m M$. Then the explicit formulas without or with labels are given by

$$\tilde{s}^{\mathrm{MT}}(\bar{f})(m) = df_v(v) \quad \text{and} \quad \tilde{s}_X^{\mathrm{MT}}(\bar{f}, x)(m) = df_v(v) \wedge x. \tag{A.1.2}$$

Both of them are G-maps.

The V-framing $\phi_M : \mathrm{T}M \to V$ induces $\mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X \cong M \times \Sigma^V X$. So we obtain a map which we still call the scanning map:

$$s_X^{\mathrm{MT}} : \mathrm{D}_M^{\mathrm{fr}_V}(X) \to \mathrm{Map}_c(M, \Sigma^V X). \tag{A.1.3}$$

A prior, this scanning map is different from the scanning map (6.1.2) in Section 6.1. For an element $\bar{f} = (f, \alpha)$ where $f : V \to M$ with $f(v) = m$, we have $s(\bar{f})(m) = v \in V$ in (6.1.2), while $s^{\mathrm{MT}}(\bar{f})(m) = df_v(v) \in \mathrm{T}_m M$ in (A.1.2). However, the data of a homotopy in defining the V-framed embedding ensure that the two approaches give homotopic scanning maps:

Proposition A.1.4. *The map s_X defined by (6.1.2) is G-homotopic to the map s_X^{MT} defined by (A.1.2).*

Proof. We show that $s \simeq s^{\mathrm{MT}} : \mathscr{D}_M^{\mathrm{fr}_V}(k) \to \mathrm{Map}_c(M, S^V)$. We write the homotopy explicitly for $k = 1$ and the case for general k is similar. To unravel the data, an element $\bar{f} = (f, \alpha) \in \mathscr{D}_M^{\mathrm{fr}_V}(1)$ consists of an embedding $f : V \to M$ and a homotopy α of two maps $\mathrm{T}V \to V$, where $\alpha(0)$ is the standard framing on V and $\alpha(1)$ is $\phi_M \circ df$.

The two scanning maps use the two endpoints of this homotopy. Namely, for m in $\mathrm{Image}(f)$, write $v = f^{-1}(m) \in V \cong \mathrm{T}_v V$. Then the first approach can be written as

$$s(\bar{f})(m) = v = \alpha(0)_v(v)$$

and the df-shifted-approach can be written as

$$s^{\mathrm{MT}}(\bar{f})(m) = \phi_M df_v(v) = \alpha(1)_v(v).$$

Now it is clear that we can define a homotopy

$$H : \mathscr{D}_M^{\mathrm{fr}_V}(1) \times I \to \mathrm{Map}_c(M, S^V);$$

$$H(\bar{f}, t)(m) = \alpha(t)_{f^{-1}(m)}(f^{-1}(m)).$$

It is G-equivariant and gives a homotopy between $H(-,0) = s$ and $H(-,1) = s^{\mathrm{MT}}$. The claim follows from observing that this homotopy is compatible with forgetting from k to $k-1$. □

A.2 Scanning map using geodesic

McDuff gave a geometric construction for

$$F_M(S^0) = \coprod_{k\geq 0} \mathscr{F}_M(k) \to \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M)),$$

Recall that $\mathscr{F}_M(k)$ is the configuration space of k points in M. Note that the base point in each fiber of $\mathrm{Sph}(\mathrm{T}M)$ is the point at infinity; so such a compactly supported section of $\mathrm{Sph}(\mathrm{T}M)$ is just a vector field defined in the interior of a compact set on M that blows up to infinity towards the boundary.

We first copy McDuff's construction and fit it into a neat comparison with the previously defined scanning maps.

We focus on the case of M without boundary. Then we can translate her M_ϵ to our M; her E_M can be identified with our $\mathrm{Sph}(\mathrm{T}M)$; her $\tilde{C}_M$ to our $F_M(S^0)$; her $\tilde{C}_\epsilon(M)$ to a subspace of our $\mathrm{Emb}_M(S^0)$.

In summary, the scanning map goes in two steps: fatten up the configurations ([McD75, Lemma 2.3]) and use geodesics to give compactly supported vector fields ([McD75, p95]).

$$\begin{array}{ccccc} \tilde{s}^{\mathrm{MD}} : F_M(S^0) & \xrightarrow{\text{fatten}} & \tilde{C}_\epsilon(M) & \xrightarrow{\phi_\epsilon} & \mathrm{Section}_c(M, E_M) \\ & & {\scriptstyle\text{include}}\big\downarrow & & \cong\big\downarrow{\scriptstyle\eta_1} \\ & & \mathrm{Emb}_M(S^0) & \xrightarrow{\gamma^+} & \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M)) \end{array} \tag{A.2.1}$$

The commutative (A.2.1) is central in this section. In the first row, fatten and ϕ_ϵ are the two steps in McDuff's scanning map. The map γ^+ is from Section A.1. We will define the undefined spaces and maps as we go along.

Define

$$\tilde{C}_\epsilon(M)_1 \equiv \{\exp_{m_0} : \mathrm{T}_{m_0}M \to M \text{ such that it is a diffeomorphism on the } \epsilon\text{-ball}\};$$
$$\tilde{C}_\epsilon(M) \equiv \{(\delta, e_1, \cdots, e_k) | 0 < \delta \leq \epsilon,\ k \in \mathbb{N}, e_i \in \tilde{C}_\epsilon(M)_1 \text{ for } 1 \leq i \leq k,$$
$$\text{images of } e_i \text{ on the } \delta\text{-balls are disjoint in } M\}.$$

For preparation, we write down an explicit homeomorphism

$$\eta_\epsilon : D_\epsilon(\mathbb{R}^n) \to \mathbb{R}^n;\ v \mapsto \tan\big(\frac{\pi|v|}{2\epsilon}\big)\frac{v}{|v|}.$$

Here, $D_\epsilon(\mathbb{R}^n)$ is the disk of radius ϵ in $\mathbb{R}^n$. Then, abusively we also have

$$\eta_1 : D_1(\mathrm{T}_m M)/\partial D_1(\mathrm{T}_m M) \cong \mathrm{T}_m M \cup \{\infty\} \equiv \mathrm{Sph}(\mathrm{T}_m M).$$

Define E_M to be the bundle over M whose fiber over m is $D_1(\mathrm{T}_m M)/\partial D_1(\mathrm{T}_m M)$, which is identified with $\mathrm{Sph}(\mathrm{T}_m M)$ through η_1. This is the right vertical map in (A.2.1).

We give the vertical map in the middle of (A.2.1). For an element $\exp_{m_0} \in \exp_{m_0}$, the composite $\exp_{m_0} \circ \eta_\epsilon^{-1}$ is an embedding $\mathbb{R}^n \to M$, so we can identify $\tilde{C}_\epsilon(M)_1$ with a subspace of $\mathrm{Emb}(\mathbb{R}^n, M)$. Similarly, we can include as subspace:

$$\begin{array}{ccc} \tilde{C}_\epsilon(M) & \to & \mathrm{Emb}_M(S^0) \\ (\delta, e_1, \cdots, e_k) & \mapsto & (e_1 \circ \eta_\delta^{-1}, \cdots, e_k \circ \eta_\delta^{-1}) \end{array}$$

In McDuff's first step, let us define ϕ_ϵ and compare it to the map γ^+ locally. Put a Riemannian metric on M. The input for ϕ_ϵ are the exponential maps in $\tilde{C}_\epsilon(M)_1$. Define

$$\phi_\epsilon(\exp_{m_0})(m) = \begin{cases} * & \text{if } \mathrm{dist}(m, m_0) > \epsilon; \\ \frac{\mathrm{dist}(m,m_0)}{\epsilon} \cdot t(m, m_0) & \text{if } \mathrm{dist}(m, m_0) \leq \epsilon. \end{cases}$$

Here, the values are vectors in $D_1(\mathrm{T}_m M)$; $t(m, m_0)$ is the unit tangent at m of the minimal geodesic from m_0 to m; $\mathrm{dist}(m, m_0)$ is the distance between m and m_0. Now, it can be easily verified that

$$\gamma^+(\exp_{m_0} \circ \eta_\epsilon^{-1}) = \eta_1 \circ \phi_\epsilon(\exp_{m_0}).$$

We can work the same way to extend ϕ_ϵ to $\tilde{C}_\epsilon(M)$ and we have the commutativity part of (A.2.1):

$$\gamma^+(e_1 \circ \eta_\delta^{-1}, \cdots, e_k \circ \eta_\delta^{-1}) = \eta_1 \circ \phi_\epsilon(\delta, e_1, \cdots, e_k).$$

In McDuff's second step, we describe the fattening map in (A.2.1). We can take a continuous positive function ϵ on M such that for any $m_0 \in M$, the exponential map $\exp_{m_0} : \mathrm{T}_{m_0} M \to M$ is always a diffeomorphism on the $\epsilon(m_0)$-ball. (We note the change here: $\epsilon(m_0)$ is going to serve as the ϵ in the first step. It does not harm to think as if $\epsilon(m_0) = \epsilon$ for all m_0.) Then, as is easily checked, we can choose a continuous positive function $\bar{\epsilon}$ on $F_M(S^0)$ such that at any $p = (m_1, \cdots, m_k) \in \mathscr{F}_M(k)$,

(i) for all $i = 1, \cdots, k$, $\bar{\epsilon}(p) \leq \epsilon(m_i)$;

(ii) the m_i's are at least $2\bar{\epsilon}(p)$ apart from each other.

Tthe fattening map in (A.2.1) sends $p = (m_1, \cdots, m_k) \in \mathscr{F}_M(k)$ to $(\bar{\epsilon}(p), \exp_{m_1}, \cdots, \exp_{m_k}) \in \tilde{C}_\epsilon(M)$. The continuity of $\tilde{s}^{\mathrm{MD}}$ follows from the continuity of $\bar{\epsilon}$.

Remark A.2.2. The composite

$$F_M(S^0) \xrightarrow{\text{fatten}} \tilde{C}_\epsilon(M) \xrightarrow{\text{include}} \mathrm{Emb}_M(S^0)$$

in (A.2.1) is up to homotopy the σ_0 in (5.5.8).

Equivariantly, we can take all of the Riemanian metric, ϵ and $\bar{\epsilon}$ to be G-invariant because G is finite: for example, replacing ϵ by $\Sigma_{g \in G} \epsilon(g-)/|G|$ will do. Then $\tilde{s}^{\mathrm{MD}}$ defined by (A.2.1) is G-equivariant. We can fiberwise smash with labels to get

$$\tilde{s}_X^{\mathrm{MD}} : F_M(X) \to \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X).$$

We note that there is no V involved in $\tilde{s}_X^{\mathrm{MD}}$. When M is V-framed, we can compose it with the V-framing on M to get

$$s_X^{\mathrm{MD}} : F_M(X) \to \mathrm{Map}_c(M, \Sigma^V X).$$

This scanning map s_X^{MD} is good only for studying the configuration spaces, possibly with labels. It depends on the fattening-up radius $\bar{\epsilon}$, which is not recorded explicitly in the data. The choice does not matter because a different choice of the fattening-up will give a homotopic scanning map. But for the purpose of a scanning map out of "configuration spaces with summable labels" or the factorization homology, remembering the radius is important to sum the labels.

We have seen three scanning maps so far: s_X in (6.1.2), s_X^{MT} in (A.1.2) and s_X^{MD} in (A.2.1). We have shown that s_X and s_X^{MT} are G-homotopic in Proposition A.1.4. We compare s_X^{MD} and s_X^{MT} in the following proposition.

Proposition A.2.3. *The following diagram is G-homotopy commutative:*

$$\begin{array}{ccc} \mathrm{D}_M^{\mathrm{fr}_V} X & \xrightarrow{s_X^{\mathrm{MT}}} & \mathrm{Map}_c(M, \Sigma^V X) \\ \downarrow{\scriptstyle ev_0} & \nearrow{\scriptstyle s_X^{\mathrm{MD}}} & \\ F_M X & & \end{array}$$

Proof. Recall that s_X^{MT} is the composite of the forgetting map and γ_V^+:

$$s_X^{\mathrm{MT}} : \mathrm{D}_M^{\mathrm{fr}_V} X \to \mathrm{Emb}_M(X) \xrightarrow{\gamma_V^+} \mathrm{Map}_c(M, \Sigma^V X).$$

By (A.2.1) and Remark A.2.2, we have a homotopy commutative diagram:

$$\begin{array}{ccc} \mathrm{Emb}_M(X) & \xrightarrow{\gamma_V^+} & \mathrm{Map}_c(M, \Sigma^V X) \\ \sigma_0 \uparrow & \nearrow_{s_X^{\mathrm{MD}}} & \\ F_M(X) & & \end{array}$$

By Corollary 5.5.9(2), $\sigma_0 \circ ev_0$ is G-homotopic to the forgetting map $\mathrm{D}_M^{\mathrm{fr}_V} X \to \mathrm{Emb}_M(X)$. So the claim follows. □

A.3 Scanning equivalence

We are interested in when the scanning map is an equivalence. In this section, we list Rourke–Sanderson's result in [RS00]. Their work is based on McDuff's scanning map. The $C_M X$ in their paper is our $(F_M X)^G$.

Definition A.3.1. Let C and C' be A_∞-G-spaces. An A_∞-G-map $f : C \to C'$ is called a weak group completion if for any subgroup $H \subset G$, there is a homotopy equivalence $\Omega\mathrm{B}(C^H) \simeq (C')^H$ and f^H is homotopic to $C^H \to \Omega\mathrm{B}(C^H) \simeq (C')^H$.

Note that when C is an A_∞-G-space and $H \subset G$, the fixed point space C^H is an A_∞-space; so f^H is up to homotopy a group completion of C^H.

Theorem A.3.2. *The scanning map $s_X^{\mathrm{MD}} : F_M X \to \mathrm{Map}_c(M, \Sigma^V X)$ is:*

(1) a weak G-equivalence if X is G-connected,

(2) or a weak group completion if $V \cong W \oplus 1$ and $M \cong N \times \mathbb{R}$. Here, W is a $(n-1)$-dimension G-representation and N is a W-framed compact manifold, so that $N \times \mathbb{R}$ is V-framed.

Proof. (1) is [RS00, Theorem 5]. For (2), we first note that when $M \cong N \times \mathbb{R}$, the map s_X^{MD}

factors in steps as:

$$F_M X = F_{\mathbb{R}}(F_N X) \to \mathrm{Map}_c(\mathbb{R}, \Sigma F_N(X)) \tag{A.3.3}$$

$$\to \mathrm{Map}_c(\mathbb{R}, F_N(\Sigma X)) \tag{A.3.4}$$

$$\to \mathrm{Map}_c(\mathbb{R}, \mathrm{Map}_c(N, \Sigma^{1+W} X)). \tag{A.3.5}$$

Here, (A.3.3) and (A.3.5) are scanning maps for manifolds $\mathbb{R}$ and N; (A.3.4) sends an element $p \wedge t$ for a configuration p on N with labels in X and $t \in S^1$ to the same configuration on N with labels suspended all by t in ΣX. All spaces presented have A_∞-structures from the factor $\mathbb{R}$ in M: for any space Y, both the labeled configuration space $F_{\mathbb{R}} Y$ and the mapping space $\mathrm{Map}_c(\mathbb{R}, Y) \simeq \Omega Y$ have obvious A_∞-structures.

The map (A.3.5) is a weak G-equivalence by applying part (1) with M replaced by N and X replaced by ΣX, which is G-connected. It suffices to show the composite of (A.3.3) and (A.3.4), denoted as j, is a weak group completion.

[RS00, Theorem 3] constructed a homotopy equivalence

$$q : \mathrm{B}\big((F_M X)^G\big) \simeq \big(F_N(\Sigma X)\big)^G.$$

Moreover, in Page 548, they established a homotopy commutative diagram:

$$\begin{array}{ccc} (F_M X)^G & \xrightarrow{\ j^G\ } & \mathrm{Map}_c(\mathbb{R}, \big(F_N(\Sigma X))\big)^G \\ \downarrow & & \| \\ \mathrm{Map}_c(\mathbb{R}, \mathrm{B}\big((F_M X)^G\big)) & \xrightarrow{\ \Omega q\ } & \mathrm{Map}_c(\mathbb{R}, \big(F_N(\Sigma X)\big)^G) \end{array}$$

The left column is the group completion map for the A_∞-space $(F_M X)^G$. Since q is a homotopy equivalence, j^G is a weak group completion. This remains true for any subgroup $H \subset G$ replacing G. Therefore, j is a weak group completion. □

Remark A.3.6. [RS00] does not assume the manifold M to be framed. Without the framing on M, Theorem A.3.2 is true in the following form:

The scanning map $\tilde{s}_X^{\mathrm{MD}} : F_M X \to \mathrm{Section}_c(M, \mathrm{Sph}(\mathrm{T}M) \wedge_M \tau_X)$ is

(1) a weak G-equivalence if X is G-connected,

(2) or a weak group completion if $M \cong N \times \mathbb{R}$.

APPENDIX B: THE θ-FRAMED LITTLE V-DISKS OPERAD

With an eye to future work on the non-framed case of equivariant factorization homology, we study the operad $\mathscr{D}_V^\theta$ and its algebras in this chapter. First, we generalize the semidirect product of operads by Salvatore-Wahl [SW] to a more equivariant setting and explain how it is a special case of an operad pair of May [May09a]. Then we identify equivariantly $\mathscr{D}_V^\theta$ with a semidirect product $\mathscr{D}_V \rtimes \Pi$ in the sense of Definition B.1.9 for suitable Π.

B.1 Seimidirect products

Semidirect products appear naturally as modules over G-monoids. Let G be a topological group throughout this section. By a G-monoid, we mean a monoid object Π in $G\mathrm{Top}$. That is, we have G-maps $e : \mathrm{pt} \to \Pi$ and $m : \Pi \times \Pi \to \Pi$ satisfying the unital and associativity diagrams. The G-action on Π gives a map $\alpha : G \to \mathrm{Homeo}(\Pi, \Pi)$. It can be verified that Π is a G-monoid if and only if it is an underlying monoid in Top and the map α is a group homomorphism landing in

$$\mathrm{Aut}(\Pi) = \{\text{continuous, invertible } g : \Pi \to \Pi |\ g(x)g(y) = g(xy) \text{ for all } x, y \in \Pi\}.$$

A Π-module in $G\mathrm{Top}$ is a G-space X with a G-map $\Pi \times X \to X$ satisfying the unital and associativity diagrams of an action. In particular, the space X is a module over the monoid $\Pi \in \mathrm{Top}$.

In this section, we are only interested in the following example of a G-monoid. For a topological group Π, we define

$$\mathrm{Inn}(\Pi) = \{g : \Pi \to \Pi |\text{ there is } \nu \in \Pi \text{ such that } g(x) = \nu x \nu^{-1} \text{ for all } x \in \Pi\}.$$

Example B.1.1. A group homomorphism $\alpha : G \to \mathrm{Inn}(\Pi)$ gives a G-monoid structure on Π since $\mathrm{Inn}(\Pi) \subset \mathrm{Aut}(\Pi)$ is a subgroup. In fact, Π is a group object in $G\mathrm{Top}$, but we will not use that.

Let X be a Π-module in $G\mathrm{Top}$. Then the underlying space $X \in \mathrm{Top}$ has both a G-action and a Π-action, and it turns out to be a $(\Pi \rtimes_\alpha G)$-space: (See Remark 3.2.11 for the definition of $(\Pi \rtimes_\alpha G)$.)

Proposition B.1.2. *Fix a homomorphism* $\alpha : G \to \mathrm{Inn}(\Pi)$. *The category of* Π*-modules in* $G\mathrm{Top}$ *is isomorphic to the category of* $(\Pi \rtimes_\alpha G)$*-modules in* Top*:*

$$\Pi[G\mathrm{Top}] \cong (\Pi \rtimes_\alpha G)[\mathrm{Top}]. \tag{B.1.3}$$

The morphisms of the mentioned categories are equivariant maps in the corresponding context.

Proof. Let X be a Π-module in $G\mathrm{Top}$ with G-equivariant Π-action map

$$\Pi \times X \to X,\ (\nu, x) \mapsto \nu(x).$$

From the G-equivariance, for $g \in G, \nu \in \Pi$ and $x \in X$, we have $g(\nu x) = (\alpha(g)(\nu))(gx)$. By Remark 3.2.11, X has a $(\Pi \rtimes_\alpha G)$-action. The converse is also true.

Let X, Y be two Π-modules in $G\mathrm{Top}$ and $f : X \to Y$ be a morphism. Then f is a map between two $(\Pi \rtimes_\alpha G)$-spaces which is both Π-equivariant and G-equivariant. So f is $(\Pi \rtimes_\alpha G)$-equivariant. The converse is also true. □

From a different viewpoint, a monoid is a special case of an (unreduced) operad. Indeed, from a monoid M in a bicomplete symmetric monoidal category $\mathcal{C}$, we can define an operad $\iota_1(M)$: let

$$\iota_1(M)(1) = M,\ \iota_1(M)(k) = \varnothing \text{ for } k \neq 1;$$

The structure maps

$$\iota_1(M)(k) \otimes \iota_1(M)(j_1) \otimes \cdots \otimes \iota_1(M)(j_k) \to \iota_1(M)(j_1 + \cdots + j_k)$$

are only non-trivial when $k = 1$ and $j_1 = 1$, and

$$\iota_1(M)(1) \otimes \iota_1(M)(1) \to \iota_1(M)(1)$$

is given by the monoid structure $M \otimes M \to M$. It is straightforward to check that a M-module is the same thing as an $\iota_1(M)$-algebra in $\mathcal{C}$:

$$M[\mathcal{C}] \cong \iota_1(M)[\mathcal{C}].$$

We can transform the setting of Proposition B.1.2 into this viewpoint with $\mathcal{C} = G\mathrm{Top}$ and $M = \Pi$. Then (B.1.3) becomes:

$$\iota_1(\Pi)[G\mathrm{Top}] \cong (\Pi \rtimes_\alpha G)[\mathrm{Top}]. \tag{B.1.4}$$

Salvatore-Wahl's semidirect product on an operad is a generalization of the story, where the role of the G-monoid Π is replaced by a G-operad $\mathscr{C}$. From the operad $\mathscr{C}$ in $G\mathrm{Top}$, they construct an operad $\mathscr{C} \rtimes G$ in Top with spaces

$$(\mathscr{C} \rtimes G)(k) = \mathscr{C}(k) \times G^k. \tag{B.1.5}$$

The forgetful map $G\mathrm{Top} \to \mathrm{Top}$ induces an isomorphism on the category of algebras:

$$\mathscr{C}[G\mathrm{Top}] \cong (\mathscr{C} \rtimes G)[\mathrm{Top}]. \tag{B.1.6}$$

The name "semidirect" accounts for the twist on $\mathscr{C}$ by the G factors in the structure maps. For details, see [SW, Definition 2.1, Proposition 2.3]; the structure maps of the operad can also be seen by taking $G = \{e\}$ and $\Pi = G$ in Definition B.1.9 below. Note that for $\mathscr{C} = \iota_1(\Pi)$, we have an isomorphism of G-operads

$$\iota_1(\Pi) \rtimes G \cong \iota_1(\Pi \rtimes_\alpha G),$$

so (B.1.4) is a special case of (B.1.6).

To sum up, the semidirect product $\Pi \rtimes G$, respectively $\mathscr{C} \rtimes G$, gives a construction such that the modules over Π, respectively $\mathscr{C}$, in the category of modules over G can be identified with modules over this one thing. These are isomorphisms (B.1.3) and (B.1.6).

Remark B.1.7. For an amusing aside, we discuss a further generalization to replace the group G by an operad $\mathscr{G}$. It sets us to the context of operad pairs developed by May in studying multiplicative infinite loop machine. Let $\mathscr{G} := \iota_1(G)$ as defined above be an operad. Then an operad $\mathscr{C}$ in GTop is the same thing as an operad pair $(\mathscr{C}, \mathscr{G})$ in Top, where $\mathscr{G}$ acts on $\mathscr{C}$ in the sense of [May09a, Definition 4.2], except that $\mathscr{G}$ is not reduced. (We leave out the proof of this claim, as it is not very illuminating. In brief, the λ there is for $k = 1$ the G-action on $\mathscr{C}(k)$ and vacuous for other k. (i) and (iii) are satisfied because G acts on $\mathscr{C}(k)$. (ii) and (iv) are satisfied because the structure map of $\mathscr{C}$ is G-equivariant. (v) and (vi) are trivial. We don't have (vii) and (viii) here.) For a general operad pair $(\mathscr{C}, \mathscr{G})$, [May09b, P225] gives the definition of $(\mathscr{C}, \mathscr{G})$-spaces and [May09a, Proposition 10.1] proves that a $(\mathscr{C}, \mathscr{G})$-space is just a $\mathscr{C}$-algebra in the category of $\mathscr{G}$-spaces. Relating this to the isomorphism (B.1.6), we see that in the case when $\mathscr{G} = \iota_1(G)$, $(\mathscr{C}, \mathscr{G})$-spaces are just $(\mathscr{C} \rtimes G)$-spaces. However, there is not a semidirect product construction for a general operad pair $(\mathscr{C}, \mathscr{G})$.

It turns out that algebras over $\mathscr{D}_n \rtimes SO(n)$ are the correct input for the coefficient for factorization homology of oriented manifolds. Here, $\mathscr{D}_n \rtimes SO(n)$ is constructed from the little n-disks operad $\mathscr{D}_n$ considered as an operad in $SO(n)$-spaces, with $SO(n)$ acting on the spaces $\mathscr{D}_n(k)$ $(k = 0, 1, \cdots)$ by conjugation.

Remark B.1.8. Usually, $\mathscr{D}_n \rtimes SO(n)$ is called the "framed n-disks operad". However, as pointed out in [AF15], this name is confusing in the context of factorization homology with tangential structures. The "framed n-disks operad" $\mathscr{D}_n \rtimes SO(n)$ is equivalent to $\mathscr{D}_n^{\mathrm{or}}$ where $\mathrm{or} : BSO(n) \to BO(n)$ is the oriented tangential structure; the "plain" little n-disks operad $\mathscr{D}_n$ is equivalent to $\mathscr{D}_n^{\mathrm{fr}}$ where $\mathrm{fr} : \mathrm{pt} \to BO(n)$ is the framed tangential structure. To avoid this, we will always refer to $\mathscr{D}_V^{\theta}$ or its equivalent as the θ-framed little V-disks operad.

We need an even more equivariant construction than this for equivariant factorization homology. Examples of G in (B.1.5) usually include $O(n)$ and $SO(n)$. It is the automorphism group of local disks $\mathbb{R}^n$ under the (lack of) tangential structure and gives the extra action a coefficient needs to have. For equivariant factorization homology, G is an ambient group; local disks are G-representations V; the automorphism of V is denoted as Π instead.

Definition B.1.9. Let G, Π be topological groups and $\alpha : G \to \mathrm{Inn}(\Pi)$ be a group homomorphism. Let $\mathscr{C}$ be a $(\Pi \rtimes_\alpha G)$-operad with structure map $\gamma_{\mathscr{C}}$. Define the G-operad $\mathscr{C} \rtimes \Pi$ as follows: its spaces are

$$(\mathscr{C} \rtimes \Pi)(k) = \mathscr{C}(k) \times \Pi^k,$$

where G acts diagonally on the left and Σ_k acts on the right on $\mathscr{C}(k)$ and permutes the coordinates on Π^k. The unit is $(\mathrm{id}, e) \in \mathscr{C}(1) \times \Pi$ where id is the unit of $\mathscr{C}(1)$ and e is the unit of Π.

The structure map is given by

$$\gamma : (\mathscr{C}(k) \times \Pi^k) \times (\mathscr{C}(j_1) \times \Pi^{j_1}) \times \cdots \times (\mathscr{C}(j_k) \times \Pi^{j_k}) \to \mathscr{C}(j) \times \Pi^j$$

$$\gamma((a, \underline{A}), (b_1, \underline{B^1}), \ldots, (b_k, \underline{B^k})) = (\gamma_{\mathscr{C}}(a, A_1 b_1, \ldots, A_k b_k), A_1 \underline{B^1}, \ldots, A_k \underline{B^k}),$$

for any $k \geq 1, j_s \geq 0$ and $j = j_1 + \cdots + j_k$, $\underline{A} = (A_1, \ldots, A_k)$, $\underline{B^s} = (B_1^s, \ldots, B_{j_s}^s)$.

Checking that γ is G-equivariant in the definition is routine, based on the following ingredients:

- For $g \in G$, $(\alpha(g)A_s)(gb_s) = g(A_s b_s)$ and $\gamma_{\mathscr{C}}$ is G-equivariant.
- For $g \in G$, $\alpha(g) : \Pi \to \Pi$ is a group homomorphism.

The associativity diagrams of γ can be checked by hand. Alternatively, we can identify a $(\Pi \rtimes_\alpha G)$-operad in Top to a Π-operad in GTop unraveling all the definitions and using Proposition B.1.2. Then, Definition B.1.9 can be viewed as a verbatim generalization of Salvatore–Wahl's construction of semidirect product (B.1.5): Instead of taking as input a group G in Top and a G-operad $\mathscr{C}$, we can take a monoid Π in GTop and a Π-operad $\mathscr{C}$. As a consequence, we also have the following isomorphism of categories of algebras:

Proposition B.1.10. *Let $G, \Pi, \mathscr{C}$ be as in Definition B.1.9. Then there is equivalence of algebras over $\mathscr{C}$ in $(\Pi \rtimes_\alpha G)[\mathrm{Top}]$ and algebras over $\mathscr{C} \rtimes \Pi$ in GTop:*

$$\mathscr{C}\big[(\Pi \rtimes_\alpha G)[\mathrm{Top}]\big] \cong (\mathscr{C} \rtimes \Pi)[G\mathrm{Top}]. \tag{B.1.11}$$

Proof. A quick proof is to first check

$$\mathscr{C} \rtimes (\Pi \rtimes_\alpha G) \cong (\mathscr{C} \rtimes \Pi) \rtimes G$$

as operads in Top, then use isomorphisms (B.1.3) and (B.1.6) to identify both categories in the claim to algebras over this operad. □

B.2 Operads built from $\mathscr{D}_V$

Now, we take the compact Lie group Π to be $O(V)$ for an orthogonal G-representation V. We claim that $O(V)$ is a G-monoid. Namely, the G-action on V gives a group homomorphism

$\rho : G \to O(V)$. An element $g \in G$ acts on $O(V)$ via conjugation of $\rho(g)$. This gives a homomorphism $\alpha : G \to \mathrm{Inn}(O(V))$, and we are in the situation of Example B.1.1. We can apply Definition B.1.9 to the little V-disk operad $\mathscr{D}_V$ to get a G-operad $\mathscr{D}_V \rtimes O(V)$. For this purpose, we need the following:

Proposition B.2.1. *$\mathscr{D}_V$ is an $O(V)$-operad in $G\mathrm{Top}$, equivalently an $(O(V) \rtimes_\alpha G)$-operad in* Top*.*

Proof. First, we let $O(V)$ act on spaces $\mathscr{D}_V(k) \subset \mathrm{Emb}(\sqcup_k \mathrm{D}, \mathrm{D})$ by conjugation. Here, D is the unit disk in V. Note that the group G is not involved and we are just saying that $O(n)$ acts on $\mathscr{D}_n$. We do a sanity check: Since $A \in O(V)$ is rotation on D, so we must verify that A does send a rectilinear embedding of the form $\mathbf{v} \mapsto a\mathbf{v} + b$ to a rectilinear one. In fact, the A-action sends it to the embedding $\mathbf{v} \mapsto A(aA^{-1}\mathbf{v} + b) = a\mathbf{v} + Ab$, keeping the same radius but possibly moving the center.

Thus, it suffices to check that the structure maps for $\mathscr{D}_V$ are $(O(V) \rtimes_\alpha G)$-equivariant. Since they are in nature compositions of mappings, they are clearly equivariant with respect to the conjugation action by $O(V)$. We claim that the $(O(V) \rtimes G)$-equivariance is formal from the $O(V)$-equivariance. Writing $H = O(V)$, it is standard that the map

$$\mu : H_{\mathrm{adj}} \rtimes H \to H, (h_1, h_2) \mapsto h_1 h_2$$

is a group homomorphism. The claim follows from observing that the $(O(V) \rtimes_\alpha G)$-action on $\mathscr{D}_V$ is given by pulling back the $O(V)$-action along

$$O(V) \rtimes_\alpha G \xrightarrow{\mathrm{id} \rtimes \rho} O(V)_{\mathrm{adj}} \rtimes O(V) \xrightarrow{\mu} O(V). \qquad \square$$

Proposition B.2.2. *There is an equivalence of G-operads $\iota : \mathscr{D}_V \rtimes O(V) \to \mathrm{Emb}_V$.*

Proof. We define ι by introducing rotation into the embeddings. On the k-th space, it is:

$$\begin{array}{rccc} \iota(k): & \mathscr{D}_V(k) \times (O(V))^k & \to & \mathrm{Emb}(\coprod_k V, V) \\ & (f_1, \cdots, f_k : V \to V, a_1, \cdots, a_k \in O(V)) & \mapsto & (f_i \circ a_i : V \to V). \end{array}$$

It is routine to check the $(G \times \Sigma_k)$-equivariance and compatibility with structure maps on both sides. It remains to check that $\iota(k)$ is a $(G \times \Sigma_k)$-homotopy equivalence. We do this by factoring $\iota(k)$ as a sequence of maps.

First, the inclusion $O(V) \to \mathrm{Iso}(V, V)$ is a G-equivalence. For this, we need to show for all subgroups $H \subset G$, the inclusion $O(V)^H \to \mathrm{Iso}(V, V)^H$ is an equivalence. It suffices to show for $H = G$, and take $\mathrm{Res}^G_H V$ as V for general H. Using representation theory for finite groups, we may assume that $V \cong \oplus_i V_i^{\oplus n_i}$, where V_i are distinct irreducible orthogonal representations of G. Then $\mathrm{Iso}(V, V)^G \cong \prod_i GL(n_i, \mathbb{R})$ and $O(V)^G \cong \prod_i O(n_i)$ as a subspace. The conclusion follows since $O(n) \to GL(n, \mathbb{R})$ is a homotopy equivalence for any n with a homotopy inverse given by Gram-Schmidt. So, we have a $(G \times \Sigma_k)$-homotopy equivalence

$$\mathscr{D}_V(k) \times O(V)^k \to \mathscr{D}_V(k) \times \mathrm{Iso}(V, V)^k. \tag{B.2.3}$$

Second, evaluation at 0 gives a $(G \times \Sigma_k)$-homotopy equivalence $\mathscr{D}_V(k) \to \mathscr{F}_V(k)$, as shown in [GM17, Lemma 1.2]. So we have a $(G \times \Sigma_k)$-homotopy equivalence

$$\mathscr{D}_V(k) \times \mathrm{Iso}(V, V)^k \to \mathscr{F}_V(k) \times \mathrm{Iso}(V, V)^k \cong \mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}V)\downarrow V}(k). \tag{B.2.4}$$

By Proposition 5.5.5, there is further a $(G \times \Sigma_k)$-homotopy equivalence:

$$\sigma : \mathscr{F}_{\mathrm{Fr}_V(\mathrm{T}V)\downarrow V}(k) \to \mathrm{Emb}(\coprod_k V, V) \tag{B.2.5}$$

So, the composite of (B.2.3), (B.2.4) and (B.2.5) is a $(G \times \Sigma_k)$-homotopy equivalence.

By examination, this composite differs from $\iota(k)$ only by a rescaling that happens in the exponential map σ, so they are $(G\times\Sigma_k)$-homotopic. This shows that $\iota(k)$ is also a $(G\times\Sigma_k)$-homotopy equivalence. $\square$

In the context of Chapter 5, we fix a tangential structure $\theta : B \to B_G O(n)$ such that V is θ-framed by $\phi : \mathrm{T}V \to \theta^*\zeta_n$. In the rest of this section, we define a G-operad that is intuitively "$\mathscr{D}_V \rtimes \mathbf{\Lambda}_\phi B$" and show it is equivalent to $\mathscr{D}_V^\theta$.

The tangential structure induces a map of G-monoids

$$\mathbf{\Lambda}\theta : \mathbf{\Lambda}_\phi B \longrightarrow \mathbf{\Lambda}_b B_G O(n).$$

Recall from Corollary 4.4.11 that there is a zigzag of equivalences of G-monoids

$$\mathbf{\Lambda}_b B_G O(n) \xleftarrow[\simeq]{\xi} (\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi \xrightarrow[\simeq]{\psi} O(V).$$

Definition B.2.6. Define $\widetilde{\mathbf{\Lambda}}B$ to be the pullback of G-spaces, which is also the pullback of G-monoids in the diagram:

$$\begin{array}{ccc} \widetilde{\mathbf{\Lambda}}B & \longrightarrow & \mathbf{\Lambda}_\phi B \\ \big\downarrow & & \big\downarrow \mathbf{\Lambda}\theta \\ (\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi & \xrightarrow[\xi]{\simeq} & \mathbf{\Lambda}_b B_G O(n) \end{array}$$

From our definition, $\widetilde{\mathbf{\Lambda}}B$ has the following properties:

- $\widetilde{\mathbf{\Lambda}}B \simeq \mathbf{\Lambda}_\phi B$ as G-monoids;
- $\widetilde{\mathbf{\Lambda}}B$ has an action on V by

$$\alpha : \widetilde{\mathbf{\Lambda}}B \to (\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi \xrightarrow{\psi} O(V). \tag{B.2.7}$$

The $(O(V)\rtimes G)$-operad structure of $\mathscr{D}_V$ in Proposition B.2.1 restricts to a $(\widetilde{\mathbf{\Lambda}}B\rtimes G)$-operad structure, and we have the G-operad $\mathscr{D}_V\rtimes\widetilde{\mathbf{\Lambda}}B$ from Definition B.1.9.

Proposition B.2.8. *There is an equivalence of G-operads $\mathscr{D}_V^\theta\simeq\mathscr{D}_V\rtimes\widetilde{\mathbf{\Lambda}}B$.*

Proof. We construct a map of G-operads $\mathscr{D}_V\rtimes\widetilde{\mathbf{\Lambda}}B\to\mathscr{D}_V^\theta$ and show it is a levelwise $(G\times\Sigma_k)$-equivalence. Before going into this, we make some remarks to be needed.

- We can consider any G-monoid Π as a reduced G-operad $\mathscr{O}_\Pi$ with $\mathscr{O}_\Pi(k)=\Pi^k$ and the obvious structure maps given by monoid multiplications.

- We also note that pullbacks of G-operads can be computed by levelwise pullback of G-spaces. In our case of reduced operads, this can be seen by identifying reduced operads to monoids in Λ-sequences $\Lambda_*^{op}[G\mathrm{Top}]$ (Theorem 2.0.4) and noticing that the forgetful functor from monoids in $\Lambda_*^{op}[G\mathrm{Top}]$ to $\Lambda_*^{op}[G\mathrm{Top}]$ preserves limits.

- The θ-framed version of Remark 5.3.4 is also true: we have a homeomorphism of G-spaces: $\widetilde{\mathbf{\Lambda}}B\cong\mathrm{Hom}^\theta(V,V)$. As a consequence, the map α in (B.2.7) is a G-fibration since $\mathrm{Hom}^\theta(V,V)$ is a homotopy fiber of $\mathrm{Hom}(V,V)\cong O(V)\to\mathrm{Hom}(V,\theta^*\zeta_n)$ (Definition 5.1.4). Moreover, we have the following commutative diagram (B.2.9). The horizontal maps are the diagonal maps in (5.3.3) and G-homotopy equivalences; the vertical maps are G-fibrations. So there is an equivalence from $\mathrm{Hom}^\theta(\mathrm{T}V,\mathrm{T}V)$ to the pullback of the diagram. In other words, by replacing $\mathbf{\Lambda}_\phi B$ in Theorem 5.3.2 (2) with $\widetilde{\mathbf{\Lambda}}B$, we get a pullback of G-monoids:

$$\begin{array}{ccccc} \mathrm{Hom}^\theta(\mathrm{T}V,\mathrm{T}V) & \longrightarrow & \mathrm{Hom}^\theta(V,V) & \cong & \widetilde{\mathbf{\Lambda}}B \\ \downarrow & & \downarrow\alpha & & \\ \mathrm{Hom}(\mathrm{T}V,\mathrm{T}V) & \longrightarrow & \mathrm{Hom}(V,V) & \cong & O(V) \end{array} \tag{B.2.9}$$

We have the following operadic version of the outer terms of (5.3.3) with $\widetilde{\mathbf{\Lambda}}B$ fitted in:

$$\begin{tikzcd}
\mathscr{D}_V^{\theta} \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{O}_{\widetilde{\mathbf{\Lambda}}B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{O}_{\mathbf{\Lambda}_{\phi}B} \arrow[d, "\mathscr{O}_{\mathbf{\Lambda}\theta}"] \\
 & \mathscr{O}_{(\widetilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi} \arrow[r] \arrow[d] & \mathscr{O}_{\mathbf{\Lambda}_b B_G O(n)} \\
\mathrm{Emb}_V \arrow[r, "d_0"'] & \mathscr{O}_{O(V)} &
\end{tikzcd}$$

The two squares are pullbacks: the right one from definition; the left one from pasting the definition of the θ-framed embedding space in Definition 5.1.6 along diagram (B.2.9).

From Proposition B.2.2, we have an equivalence of operads $\iota : \mathscr{D}_V \rtimes O(V) \to \mathrm{Emb}_V$. By inspection, the composite map $d_0\iota : \mathscr{D}_V \rtimes O(V) \to \mathscr{O}_{O(V)}$ is induced by projecting all spaces in $\mathscr{D}_V$ to a point. Then, $\mathscr{D}_V \rtimes \widetilde{\mathbf{\Lambda}}B$ is the pullback of the big square in the following diagram:

$$\begin{tikzcd}
\mathscr{D}_V \rtimes \widetilde{\mathbf{\Lambda}}B \arrow[r, dotted] \arrow[d] & \mathscr{D}_V^{\theta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{O}_{\widetilde{\mathbf{\Lambda}}B} \arrow[d] \\
\mathscr{D}_V \rtimes O(V) \arrow[r, "\simeq", "\iota"'] & \mathrm{Emb}_V \arrow[r, "d_0"'] & \mathscr{O}_{O(V)}
\end{tikzcd}$$

The dotted arrow exists from the universality of a pullback. Since ι is a levelwise equivalence and the vertical maps are levelwise fibrations, the dotted arrow is also a levelwise equivalence.

$\square$

As a corollary, we proved Proposition 5.5.12 again:

Corollary B.2.10. *The V-framed little V-disks operad $\mathscr{D}_V^{\mathrm{fr}_V}$ is equivalent to the little V-disks operad $\mathscr{D}_V$ as G-operads.*

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