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RECONSTRUCTION OF THE MAGNETIC FIELD FOR A SCHRÖDINGER OPERATOR IN A CYLINDRICAL SETTING

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DANIEL CAMPOS SALAS

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A mis padres

"Lo mejor de nosotros es el paso.

La huella misma,

la huella que es como un triunfo sobre el barro.

El golpe del tacón que suena a golpe

de lucha en el asfalto."

—Jorge Debravo, *El paso*

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ABSTRACT

In this thesis we consider a magnetic Schrödinger inverse problem over a compact domain contained in an infinite cylindrical manifold. We show that, under certain conditions on the electromagnetic potentials, we can recover the magnetic field from boundary measurements in a constructive way. A fundamental tool for this procedure is a global Carleman estimate for the magnetic Schrödinger operator. We prove this by conjugating the magnetic operator essentially into the Laplacian, and using the Carleman estimates for it proven by Kenig–Salo–Uhlmann in the anisotropic setting, see [KSU11a]. The conjugation is achieved through pseudodifferential operators over the cylinder, for which we develop the necessary results.

The main motivations to attempt this question are the following results concerning the magnetic Schrödinger operator: first, the solution to the uniqueness problem in the cylindrical setting in [DSFKSU09], and, second, the reconstruction algorithm in the Euclidean setting from [Sal06]. We will also borrow ideas from the reconstruction of the electric potential in the cylindrical setting from [KSU11b]. These two new results answer partially the Carleman estimate problem (Question 4.3.) proposed in [Sal13] and the reconstruction for the magnetic Schrödinger operator mentioned in the introduction of [KSU11b]. To our knowledge, these are the first global Carleman estimates and reconstruction procedure for the magnetic Schrödinger operator available in the cylindrical setting.

CHAPTER 1

INTRODUCTION

Let us present the notion of an inverse problem through the following contrasting settings. A direct problem aims to determine, from the knowledge of the internal properties of a system, the reaction of it to certain stimuli. For example, knowing the conductivity of a medium and the voltage potential at the boundary we can determine the voltage induced in the interior of the domain and, therefore, the current flowing through the boundary. In contrast, an inverse problem looks to deduce properties of the system from the knowledge of the reactions to the stimuli. For instance, in his seminal paper [Cal80], Calderón proposes to study the uniqueness and the subsequent reconstruction of the conductivity of a medium from the voltage–to–current measurements at the boundary. This problem came to be known as the Calderón inverse conductivity problem. Since then, this and other related problems have attracted a great deal of attention; see the survey [Uhl14a]. Various examples of inverse problems are also presented in [Uhl14b] and [Isa17].

For a domain $M \subseteq \mathbb{R}^d$, the isotropic conductivity equation can be expressed as the boundary value problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases}$$

where the unknown conductivity γ is a function in M. The known data is the boundary measurement $\Lambda_\gamma : f \mapsto \gamma \partial_\nu u|_{\partial M}$, which maps the voltage potential at the boundary to the current flowing through the boundary due to the induced voltage in the interior. As mentioned before, the Calderón inverse problem consists in recovering the function γ from the map Λ_γ.

After a change of variables the conductivity equation can be expressed in the form $H_{0,W} v := (D^2 + W)v = 0$, where $D = -i\nabla$ is the gradient, $D^2 := D \cdot D = -\operatorname{div} \cdot \nabla$ is the (negative) Laplacian, and W is a function; we refer to $H_{0,W}$ as a (*electric*) Schrödinger operator. In greater generality, we can consider a *magnetic* Schrödinger operator, which has a structure similar to the previous operator but contains first order terms in the form $H_{V,W} := (D + V)^2 + W$. In any of these cases, the inverse problem consists in recovering information about either (or both) of the electromagnetic potentials V and W, in the interior of the domain, from boundary measurements. We elaborate this with more detail in the following section. One of the reasons why this problem is interesting and relevant is its relation to the inverse scattering problem at fixed energy from quantum mechanics; see the introduction of the Ph.D. thesis by Haberman [Hab15] for a detailed presentation on this.

As mentioned before, there is a significant body of work surrounding these problems. In the Euclidean setting, the uniqueness (or identifiability) problem for the *electric* Schrödinger operator was explicitly addressed by Nachman–Sylvester–Uhlmann in [NSU88], but it was implicitly used in the proof of the uniqueness for the conductivity problem by Sylvester–Uhlmann in [SU87]. Their proof uses the construction of many special solutions inspired by the complex exponential solutions introduced by Calderón in [Cal80]; this method of construction relies on a global Carleman estimate for the Laplacian. The Carleman estimates are a kind of parameter–dependent weighted inequalities, originally introduced in the setting of unique continuation problems. The reconstruction of the electric potential is due to Nachman, see [Nac88], and uses the uniqueness for the global Carleman estimate from [SU87] in two ways. First, it is shown that the uniqueness "at infinity" implies a uniqueness property at the boundary, and this allows to determine the boundary values of the special solutions. Second, the smallness that is established in the estimate makes it possible to disregard certain correction terms. Later we will elaborate more carefully on this. For the *magnetic* operator, the uniqueness has been established in a series of papers under different assumptions. This was started with the work of Sun, in [Sun93], under smallness conditions on the magnetic field; then the smallness condition was replaced by a smoothness condition by Nakamura–Sun–Uhlmann in [NSU95]. Further improvements of these include the results by Salo in [Sal06] and Krupchyk–Uhlmann in [KU14]. For a more detailed account of the available results, see [Hab15]. Moreover, in [Sal06], Salo carries out a constructive procedure to recover the electromagnetic parameters. As before, the reconstruction uses the existence of many special solutions which are constructed through a Carleman estimate for the magnetic Schrödinger operator. We will follow closely the arguments from this paper.

Moving away from the Euclidean setting, the Calderón problem, or its corresponding problem for the Schrödinger operator, can also be formulated in the context of Riemannian manifolds. This problem arises as a model for electrical imaging in anisotropic media, and it is one of the most basic inverse problems in a geometric setting; for the basic results in this context we refer to [Sal13]. Motivated by the results in the Euclidean setting, we are interested in proving analogous Carleman estimates on manifolds. Looking to deduce such an estimate, in [DSFKSU09] it is proven that the existence of a limiting Carleman weight implies some kind of product structure on the manifold. Since then, it has been usual to consider a cylindrical manifold, as we will do with $T = \mathbb{R} \times \mathbb{T}^d$, and the Carleman weight x_1; for instance, see [KSU11a] or [KSU11b]. Our setting will be slightly different from the so–called *admissible Riemannian manifolds* from [DSFKSU09]. The solution to the uniqueness problem for the magnetic operator was established by Dos Santos Ferreira–Kenig–Salo–Uhlmann in [DSFKSU09], and the reconstruction problem for the electric Schrödinger operator is elaborated in [KSU11b]. For a more complete exposition of the results either in the Euclidean or Riemannian setting we refer to the surveys [Uhl14a] and [Uhl14b].

In this thesis we prove a global Carleman estimate for the magnetic Schrödinger operator and propose a reconstruction procedure for the magnetic field. The main motivations to attempt this question are the following results concerning the magnetic Schrödinger operator: first, the solution to the uniqueness problem in the cylindrical setting in [DSFKSU09], and, second, the reconstruction algorithm in the Euclidean setting from [Sal06]. We will also borrow ideas from the reconstruction of the electric potential in the cylindrical setting from [KSU11b]. These two new results answer partially the Carleman estimate problem (Question 4.3.) proposed in [Sal13] and the reconstruction for the magnetic Schrödinger operator mentioned in the introduction of [KSU11b]. To our knowledge, these are the first global

Carleman estimates and reconstruction algorithms for the magnetic Schrödinger operator available in the cylindrical setting.

1.1 Setting and main results

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus with standard metric g_0 and let e be the Euclidean metric on $\mathbb{R}$. Consider the cylinder $T = \mathbb{R} \times \mathbb{T}^d$ with the standard product metric $g = e \oplus g_0$. We denote the points in the cylinder T by (x_1, x'), meaning that $x_1 \in \mathbb{R}$ and $x' \in \mathbb{T}^d$. Let $(M, g) \subseteq T$ be a smooth connected compact $(d+1)$-submanifold. Let ∂M denote its smooth d-dimensional boundary, let $M_- := M \setminus \partial M$ and $M_+ = T \setminus M$. We call M_- and M_+ the interior and exterior of M, respectively.

Let $D_{x_1} = \partial_{x_1}/(2\pi i)$ and $D_{x'} = \nabla_{x'}/(2\pi i)$, and define the gradient $D := (D_{x_1}, D_{x'})$ and Laplacian $-\Delta_g := D^2 = D_{x_1}^2 + D_{x'}^2$. We denote $-\Delta_{g_0} := D_{x'}^2$, so that its eigenvalues on $\mathbb{T}^d$ consist of the set $\mathrm{Spec}(-\Delta_{g_0}) := \{|k|^2 : k \in \mathbb{Z}^d\}$. Let F, $G_1, \ldots, G_d$, W be functions in M, and consider the vector field $V := (F, G) := (F, G_1, \ldots, G_d)$. We call V and W the magnetic and electric potentials, respectively. We consider the magnetic Schrödinger operator

$$H_{V,W} := (D+V)^2 + W = D^2 + 2V \cdot D + (V^2 + D \cdot V + W),$$

and its associated Dirichlet problem

$$\begin{cases} H_{V,W} u = 0 & \text{in } M_-, \\ u = f & \text{on } \partial M. \end{cases} \tag{$*$}$$

In *Chapter 2. Prelimaries* we will introduce the necessary notation and motivate the following definitions. For $f \in H^{1/2}(\partial M)$, we say that $u \in H^1(M)$ is a weak solution to the Dirichlet problem $(*)$ if $\mathrm{tr}^-(u) = f$ and

$$\int_M -Du \cdot D\varphi + V \cdot (\varphi Du - uD\varphi) + (V^2 + W)u\varphi = 0, \tag{1.1}$$

for all test functions $\varphi \in H_0^1(M)$. Under certain conditions on the potentials, which we later elaborate, there exists a unique weak solution to the Dirichlet problem $(*)$. We define the Dirichlet-to-Neumann (DN) map $\Lambda_{V,W}$ as follows: if $f, g \in H^{1/2}(\partial M)$ and $v \in H^1(M)$ is any function extending g, i.e. $\mathrm{tr}^-(v) = g$, then

$$\langle \Lambda_{V,W} f, g \rangle := \int_M -Du \cdot Dv + V \cdot (vDu - uDv) + (V^2 + W)uv, \tag{1.2}$$

where $u \in H^1(M)$ is the weak solution of $(*)$. Formally, the DN map corresponds to the boundary measurement

$$\Lambda_{V,W} f = \frac{i}{2\pi} \nu \cdot (D + V)u\big|_{\partial M}.$$

The reconstruction problem then consists in using measurements at the boundary of the domain, such as the DN map $\Lambda_{V,W}$, to recover information about the potentials in the interior of it.

Before we proceed to formulate the results, let us recall the gauge invariance of the DN map observed in [Sun93]. The conjugation identity $e^{-2\pi i\varphi} D e^{2\pi i\varphi} = D + \nabla\varphi$ gives that $H_{V+\nabla\varphi,W} = e^{-2\pi i\varphi} H_{V,W} e^{2\pi i\varphi}$, which implies that if 0 is not an eigenvalue of $H_{V,W}$ on M and $\varphi \in C^\infty(T)$, then 0 is also not an eigenvalue of the operator $H_{V+\nabla\varphi,W}$. Indeed, $\widetilde{u}$ is a solution of the Dirichlet problem

$$\begin{cases} H_{V+\nabla\varphi,W}\widetilde{u} = 0 & \text{in } M_-, \\ \widetilde{u} = g & \text{on } \partial M, \end{cases}$$

if and only if $u = e^{2\pi i\varphi}\widetilde{u}$ solves

$$\begin{cases} H_{V,W} u = 0 & \text{in } M_-, \\ u = e^{2\pi i\varphi} g & \text{on } \partial M. \end{cases}$$

A routine computation yields that $\Lambda_{V+\nabla\varphi,W} = e^{-2\pi i\varphi}|_{\partial M}\Lambda_{V,W}e^{2\pi i\varphi}$ which gives the gauge invariance $\Lambda_{V+\nabla\varphi,W} = \Lambda_{V,W}$ if $\varphi|_{\partial M} = 0$. Therefore, it is not possible to determine the magnetic potential V from the knowledge of $\Lambda_{V,W}$. Let us note, however, that the magnetic fields are the same, i.e. $\operatorname{curl} V = \operatorname{curl}\,(V + \nabla\varphi)$. The main result from the thesis is that it is possible to reconstruct the magnetic field $\operatorname{curl} V$ under the following smoothness, support, and vanishing moment conditions:

$$V \in C_c^\infty(M_-), \quad W \in L^\infty(M), \quad \operatorname{supp}(W) \subseteq M, \quad \int_{\mathbb{R}} V(x_1, x')dx_1 = 0 \text{ for all } x' \in \mathbb{T}^d. \quad (\dagger)$$

Theorem 1.1. *Let $M \subseteq T$ be as before, with $d \geq 3$. Assume that the potentials V, W satisfy* $(\dagger)$ *and* 0 *is not an eigenvalue of $H_{V,W}$ in M. Then the magnetic field $\operatorname{curl} V$ can be reconstructed from the knowledge of the Dirichlet-to-Neumann map $\Lambda_{V,W}$.*

A fundamental step in the reconstruction of $\operatorname{curl} V$ from the DN map $\Lambda_{V,W}$ is the construction of many special solutions to the equation $H_{V,W}u = 0$. Following Sylvester–Uhlmann's method of complex geometric optics (CGOs), see [SU87], the solutions consist in appropriate corrections, depending on a large parameter, of harmonic functions. The standard technique to perform these constructions has been the use of Carleman estimates. Following [Nac88] and [KSU11b], we use a uniqueness result for these kind of estimates, as mentioned in the previous section, for a twofold purpose: first, to characterize the boundary values of the CGOs from the DN map; second, to "disregard" the correction terms as the parameter grows. The other main result of the thesis is the following Carleman estimate, which holds under the following conditions on the potentials:

$$V \in C_c^\infty(T), \quad \operatorname{supp}(V) \subseteq [-R, R] \times \mathbb{T}^d, \quad \langle x_1 \rangle^{2\delta} W \in L^\infty(T),$$

$$\int_{\mathbb{R}} V(x_1, x')dx_1 = 0 \text{ for all } x' \in \mathbb{T}^d. \qquad (\star)$$

Theorem 1.2. *Let $1/2 < \delta < 1$ and let V, W satisfy $(\star)$. There exists $\tau_0 \geq 1$, such that if $|\tau| \geq \tau_0$ and $\tau^2 \notin Spec(-\Delta_{g_0})$, then for any $f \in L^2_\delta(T)$ there exists a unique $u \in H^2_{-\delta}(T)$ which solves*

$$e^{2\pi\tau x_1} H_{V,W} e^{-2\pi\tau x_1} u = f.$$

Moreover, this solution satisfies the estimates

$$\|u\|_{H^s_{-\delta}(T)} \lesssim |\tau|^{s-1} \|f\|_{L^2_\delta(T)},$$

for $s = 0, 1, 2$. The constant of the inequality is independent of τ.

In the next chapter we introduce the weighted Sobolev spaces $L^2_\delta(T)$ and $H^s_{-\delta}(T)$. The solution to this equation is based on a reduction to the case of the Laplacian, i.e. when there are no electromagnetic potentials. The gain of one derivative in the estimate, meaning the constant τ^{-1}, allows to deduce the estimate of Theorem 1.2 in the presence of an electric potential alone through perturbative methods. The reconstruction procedure of the electric potential has been given in [Nac88] for the Euclidean case and in [KSU11b] for the cylindrical case. However, the gain of one derivative is not enough to deal with the magnetic potential beyond the perturbative regime, i.e. when the norm of the magnetic potential may not be small. Following the ideas in [NU94], [NSU95], and especially [Sal06], we prove this by conjugating the equation through pseudodifferential operators in order to "essentially eliminate" the magnetic potential. To do this we consider the small parameter $\hbar = \tau^{-1}$ and use the results for semiclassical analysis on $\mathbb{R}$.

1.2 Structure of the thesis

In *Chapter 2. Preliminaries* we recall some definitions and results on Fourier analysis, introduce the function spaces that will appear through the problem, and present the basic facts necessary to formulate the magnetic Schrödinger inverse problem.

In *Chapter 3. Semiclassical pseudodifferential operators over* $\mathbb{R}\times\mathbb{T}^d$ we define these operators over T and prove the usual results specific to our cylindrical setting. These results do not seem to be explicitly stated in the standard references, see [Tay81] or [Zwo12], so, we elaborate the necessary theory for it. For zero order pseudodifferential operators we prove an analog of the Calderón–Vaillancourt L^2–boundedness theorem, as well as a norm estimate for the first order expansion of the composition of two such operators.

In *Chapter 4. Conjugation and Carleman estimate* we carry out the construction of the conjugation as well as the proof of Theorem 1.2. The conjugation requires the solution of a first order differential equation, together with the appropriate estimates. In our cylindrical setting, through the expansion in Fourier series, this equation can be reduced to the solution of multiple ODEs. The ideas follow closely the results from [Sal06].

In *Chapter 5. Equivalent formulations and boundary characterization* we use Theorem 1.2 to construct many solutions of the equation $H_{V,W}u = 0$. Starting from a harmonic solution, we construct a unique solution (CGO) to the equation that "behaves like" it at infinity. We show that the uniqueness at infinity implies a uniqueness property at the boundary, and so the boundary values of the CGOs can be characterized as solutions to boundary integral equations involving only the knowledge of the DN map $\Lambda_{V,W}$ and not the unknown electromagnetic potentials. We follow the presentation from [KSU11b].

In *Chapter 6. Reconstruction of the magnetic field* we restrict the attention to CGOs that result from correcting the harmonic functions $e^{\pm 2\pi|m|x_1}e_m(x')$. We prove that such CGOs can also be written in the form $e^{\pm 2\pi|m|x_1}e_m(x')a_m + e^{-2\pi\tau x_1}r_{m,\tau}$, for an appropriate amplitude a_m making the correction term have better estimates. Then we define an analog of the scattering transform from [Nac88] and [Sal06], and use it together with the correction estimates to obtain integrals that are basically a mixed (Laplace–Fourier) transform of terms involving the magnetic potential. Finally, we show that it is possible to recover the magnetic field curl V from these integrals. These steps require some linear algebra lemmas over $\mathbb{Q}$

and the reconstruction formula for an entire function, which we prove in the appendix of the chapter. This is perhaps the most interesting chapter: not only the methods require playful ideas, but the results obtained are somewhat different from analogous previous ones.

CHAPTER 2

PRELIMINARIES

Consider the cylinder $T = \mathbb{R} \times \mathbb{T}^d$ with standard product metric $g = e \oplus g_0$. The points in T are denoted by $x = (x_1, x')$, meaning that $x_1 \in \mathbb{R}$ and $x' \in \mathbb{T}^d$. Let $(M, g) \subseteq T$ is a smooth connected compact $(d+1)$-submanifold with boundary ∂M. We denote the volume element in T and M by $dx = dx_1 dx'$ and the surface measure in ∂M by $d\sigma$.

Let $D_{x_1} = \partial_{x_1}/(2\pi i)$ and $D_{x'} = \nabla_{x'}/(2\pi i)$, and define the gradient $D := (D_{x_1}, D_{x'})$ and Laplacian $-\Delta_g := D^2 = D_{x_1}^2 + D_{x'}^2$. For a multiindex $\alpha = (\alpha_1, \alpha') = (\alpha_1, \alpha_1', \ldots, \alpha_d')$, we denote $|\alpha| = \alpha_1 + \alpha_1' + \ldots + \alpha_d'$ and $D^\alpha = D_{x_1}^{\alpha_1} D_{x_1'}^{\alpha_1'} \ldots D_{x_d'}^{\alpha_d'}$.

In what follows we define several functions spaces over T, and we mention when the definitions allow for analogous spaces over $\mathbb{R}$, $\mathbb{T}^d$, or M. Most of the definitions and results from this chapter can be found in [Sal06], [Tay11], [Tay81], [Zwo12].

2.1 Fourier analysis and distributions

2.1.1 Distributions and Sobolev spaces

We consider the space of smooth compactly supported functions $\mathcal{D}(T) := C_c^\infty(T)$ with the family of seminorms

$$\|f\|_{k,l} = \sup\{|D^\alpha f(x_1, x')| : |x_1| \leq k,\ x' \in \mathbb{T}^d,\ |\alpha| \leq l\},$$

with $k, l \in \mathbb{N}$. We say a linear functional $\varphi : \mathcal{D}(T) \to \mathbb{C}$ is continuous, if for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ and $C > 0$, both possibly depending of k, such that $|\langle \varphi, f \rangle| \leq C\|f\|_{k,l}$ for all $f \in \mathcal{D}(T)$. We call *distributions* to these functionals and denote its space by $\mathcal{D}'(T)$.

In addition, we define the space of Schwartz functions $\mathcal{S}(T)$ as the space of rapidly decaying smooth functions with the family of seminorms

$$\|f\|_k = \sup\{\langle x_1\rangle^k |D^\alpha f(x_1,x')| : (x_1,x') \in T,\ 0 \le |\alpha| \le k\},$$

with $k \in \mathbb{N}$. We say that $f_j \to f$ in $\mathcal{S}(T)$ if $\|f_j - f\|_k \to 0$ for all k. We say a linear functional $\varphi : \mathcal{S}(T) \to \mathbb{C}$ is continuous if $\langle \varphi, f_j\rangle \to \langle \varphi, f\rangle$ whenever $f_j \to f$ in $\mathcal{S}(T)$. We call tempered distributions to these functionals and denote its space by $\mathcal{S}'(T)$. We define that $\varphi_j \rightharpoonup \varphi$ in $\mathcal{S}'(T)$ if $\langle \varphi_j, f\rangle \to \langle \varphi, f\rangle$ for all $f \in \mathcal{S}(T)$. A well–known result in functional analysis says that if $\varphi \in \mathcal{S}'(T)$, then there exist $k \in \mathbb{N}$ and $C > 0$ such that $|\langle \varphi, f\rangle| \le C\|f\|_k$ for all $f \in \mathcal{S}(T)$. The space of *tempered distributions* $\mathcal{S}'(T)$ is a subspace of the distributions $\mathcal{D}'(T)$. The definitions of the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are analogous.

For $1 \le p \le \infty$, let $L^p(T) = L^p(T, dx_1dx')$ denote the standard L^p space in T. For a nonnegative integer s, we consider the L^p Sobolev spaces $W^{s,p}(T)$ with norm given by $\|f\|_{W^{s,p}(T)} := \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^p(T)}$. Similarly, we also consider the spaces $W^{s,p}(\mathbb{R})$ and $W^{s,p}(\mathbb{T}^d)$.

2.1.2 Fourier analysis on smooth functions

For a function $f \in L^1(\mathbb{R})$ we define its Fourier transform by $\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i x_1 \xi} f(x_1)dx$.

Proposition 2.1 ([Tay11], [Zwo12]). *If $f \in \mathcal{S}(\mathbb{R})$, then its Fourier transform $\widehat{f}$ satisfies the following:*

a). the transform and its derivatives have polynomial decay bounds $|D^\alpha_\xi \widehat{f}(\xi)| \lesssim \langle \xi\rangle^{-2m} \|x_1^\alpha f\|_{W^{2m,1}(\mathbb{R})}$, where $\langle \xi\rangle := (1+\xi^2)^{1/2}$ and the constant of the inequality may depend on m,

b). $\widehat{f} \in \mathcal{S}(\mathbb{R})$ and we have the inversion formula $f(x_1) = \int_{\mathbb{R}} e^{2\pi i x_1\xi} \widehat{f}(\xi)d\xi$, with pointwise absolute uniform convergence, as well as for its derivatives,

c). Plancherel's identity holds, $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})}$.

For $k \in \mathbb{Z}^d$, let $e_k(x') := e^{2\pi i k\cdot x'}$. For a function $f \in L^1(\mathbb{T}^d)$ we define its k-th Fourier coefficient by $f_k := \int_{\mathbb{T}^d} e_{-k}(x')f(x')dx'$.

Proposition 2.2 ([Tay11])**.** *If $f \in C^\infty(\mathbb{T}^d)$, then its Fourier coefficients and series satisfy the following:*

a). the coefficients have polynomial decay bound $|f_k| \lesssim \langle k\rangle^{-2m}\|f\|_{W^{2m,1}(\mathbb{T}^d)}$, where $\langle k\rangle := (1+|k|^2)^{1/2}$ and the constant of the inequality may depend on m and d,

b). there is pointwise absolute uniform convergence of the Fourier series $f(x') = \sum_{k\in\mathbb{Z}^d} f_k e_k(x')$, as well as for of its derivatives,

c). Plancherel's identity holds, $\|f\|^2_{L^2(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d} |f_k|^2$.

Similarly, for a function $f \in \mathcal{S}(T)$ we define its k-th Fourier coefficient by $f_k(x_1) := \int_{\mathbb{T}^d} e_{-k}(x')f(x_1, x')dx'$. The previous results can be combined as follows.

Proposition 2.3. *If $f \in \mathcal{S}(T)$, then its Fourier coefficients f_k are in $\mathcal{S}(\mathbb{R})$. Moreover, these satisfy the following:*

a). the coefficients have polynomial decay bounds $\|f_k\|_{L^1(\mathbb{R})} \lesssim \langle k\rangle^{-2m}\|f\|_{W^{2m,1}(T)}$, where the constant of the inequality may depend on m and d,

b). the transform of the coefficients and its derivatives have polynomial decay bounds $|D^\alpha_\xi \widehat{f}_k(\xi)| \lesssim \langle \xi, k\rangle^{-2m}\|x_1^\alpha f\|_{W^{2m,1}(T)}$, where $\langle \xi, k\rangle := (1+\xi^2+|k|^2)^{1/2}$ and the constant of the inequality may depend on m and d,

c). the inversion formula holds,

$$f(x_1, x') = \sum_{k\in\mathbb{Z}^d} f_k(x_1)e_k(x') = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1\xi} e_k(x')\widehat{f}_k(\xi)d\xi,$$

with pointwise absolute uniform convergence, as well as for its derivatives,

d). Plancherel's identity holds, $\|f\|^2_{L^2(T)} = \sum_{k\in\mathbb{Z}^d} \|f_k\|^2_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{Z}^d} \|\widehat{f}_k\|^2_{L^2(\mathbb{R})}$.

e). for any $k \in \mathbb{Z}^d$, the function $f_k(x_1)e_k(x')$ is in $\mathcal{S}(T)$, and we have that $\|f_k e_k\|_l \lesssim \langle k\rangle^{-(2m-l)}\|f\|_{l+2m}$, where the constant may depend on l, m, and d. Moreover, the partial sums of the Fourier series, $S_N f(x_1, x') := \sum_{|k|\leq N} f_k(x_1)e_k(x')$, converge to f in $\mathcal{S}(T)$,

Proof. For $\alpha \leq m$ we have that $\langle x_1\rangle^m |D^\alpha_{x_1} f_k| \leq \|\langle x_1\rangle^m D^\alpha_{x_1} f(x_1, \cdot)\|_{L^1(\mathbb{T}^d)} \leq \|f\|_m$, where $\|f\|_m$ is the seminorm defined above for functions in $\mathcal{S}(T)$. This proves that the Fourier coefficients f_k are in $\mathcal{S}(\mathbb{R})$. Moreover, using the identity $\langle k\rangle^{2m} e_{-k}(x') = \langle D_{x'}\rangle^{2m} e_{-k}(x')$ and integrating by parts yields that

$$
\begin{aligned}
\langle k\rangle^{2m}|f_k(x_1)| &= \left|\int_{\mathbb{T}^d} \langle D_{x'}\rangle^{2m}(e_{-k}(x'))f(x_1, x')dx'\right| \\
&= \left|\int_{\mathbb{T}^d} e_{-k}(x')\langle D_{x'}\rangle^{2m} f(x_1, x')dx'\right| \lesssim \int_{\mathbb{T}^d} |\langle D_{x'}\rangle^{2m} f(x_1, x')|dx'.
\end{aligned}
$$

Integrating over $\mathbb{R}$ gives the first result. Similarly, using the identities

$$
D^\alpha_\xi e^{-2\pi i x_1 \xi} = (-x_1)^\alpha e^{-2\pi i x_1 \xi}, \quad \langle \xi, k\rangle^{2m}(e^{-2\pi i x_1 \xi} e_{-k}(x')) = \langle D\rangle^{2m}(e^{-2\pi i x_1 \xi} e_{-k}(x')),
$$

and integrating by parts we conclude that

$$
\begin{aligned}
\langle \xi, k\rangle^{2m}|D^\alpha_\xi \widehat{f}_k(\xi)| &= \left|\int_T \langle D\rangle^{2m}(e^{-2\pi i x_1 \xi} e_{-k}(x'))x_1^\alpha f dx_1 dx'\right| \\
&= \left|\int_T e^{-2\pi i x_1 \xi} e_{-k}(x')\langle D\rangle^{2m}(x_1^\alpha f)dx_1 dx'\right| \lesssim \|x_1^\alpha f\|_{W^{2m,1}(T)}.
\end{aligned}
$$

Moreover, this result proves that the Fourier decomposition converges absolutely, and so the inversion and Plancherel formulas follow from Proposition 2.1 and Proposition 2.2. Finally, we can bound

$$
\begin{aligned}
\|f_k e_k\|_l &= \sup\{\langle x_1\rangle^l |D^\alpha(f_k e_k)| : |\alpha| \leq l\} \\
&\lesssim \langle k\rangle^l \sup\{\langle x_1\rangle^l |D^{\alpha_1}_{x_1} f_k| : \alpha_1 \leq l\} \lesssim \langle k\rangle^{-(2m-l)}\|f\|_{l+2m},
\end{aligned}
$$

where we have used Proposition 2.2 for the last step. The convergence of the partial sums in $\mathcal{S}(T)$ follows from this. $\square$

2.1.3 Fourier analysis on tempered distributions

From Fubini's theorem we have that $\int_{\mathbb{R}} f\widehat{g} = \int_{\mathbb{R}} \widehat{f}g$, for $f, g \in \mathcal{S}(\mathbb{R})$. This suggests to define the Fourier transform of $\varphi \in \mathcal{S}'(\mathbb{R})$ by $\langle \widehat{\varphi}, f\rangle := \langle \varphi, \widehat{f}\rangle$. To see indeed that $\widehat{\varphi} \in \mathcal{S}'(\mathbb{R})$ we use Proposition 2.1 to get that $\widehat{f_n} \to \widehat{f}$ in $\mathcal{S}(\mathbb{R})$ if $f_n \to f$ in $\mathcal{S}(\mathbb{R})$. It is clear that this definition extends the above definition of Fourier transform in $\mathcal{S}(\mathbb{R})$.

Finally, we proceed to define the Fourier coefficients of a tempered distribution. Let us consider the operators $\pi_k : \mathcal{S}(T) \to \mathcal{S}(\mathbb{R})$ and $\psi_k : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(T)$ given by $\pi_k f := f_k$ and $\psi_k g := g(x_1)e_k(x')$. The Fourier inversion formula on $\mathcal{S}(T)$ can be written formally as $I = \sum_{k\in\mathbb{Z}^d} \psi_k \pi_k$. Moreover, proceeding as in the proof of Proposition 2.3 we have that $\|\pi_k f\|_l \lesssim \langle k\rangle^{-2m}\|f\|_{l+2m}$ and $\|\psi_k g\|_l \lesssim \langle k\rangle^l \|g\|_l$, for all $l, m \geq 0$. By duality, this gives rise to the adjoint operators $\pi_k^* : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(T)$ and $\psi_k^* : \mathcal{S}'(\mathbb{T}) \to \mathcal{S}'(\mathbb{R})$, defined by $\langle \pi_k^* \phi, f\rangle := \langle \phi, \pi_k f\rangle$ and $\langle \psi_k^* \varphi, g\rangle := \langle \varphi, \psi_k g\rangle$. For a distribution $\varphi \in \mathcal{S}'(T)$, we define its k-th Fourier coefficient by $\varphi_k := \psi_{-k}^* \varphi \in \mathcal{S}'(\mathbb{R})$. This definition extends that of Fourier coefficients for functions in $\mathcal{S}(T)$. Below, we prove the formal dual of the inversion formula above, which reads $I = \sum_{k\in\mathbb{Z}^d} \pi_k^* \psi_k^* = \sum_{k\in\mathbb{Z}^d} \pi_{-k}^* \psi_{-k}^*$.

Proposition 2.4. *Let $f \in \mathcal{S}(T)$, $\varphi \in \mathcal{S}'(T)$. The Fourier coefficients satisfy the following:*

a). the usual differentiation properties hold, i.e. $(D_{x'}^\alpha \varphi)_k = k^\alpha \varphi_k$,

b). Parseval's identity holds, $\langle \varphi, \overline{f}\rangle = \sum_{k\in\mathbb{Z}^d} \langle \varphi_k, \overline{f_k}\rangle$, with absolute convergence.

c). the partial sums of the Fourier series, $S_N \varphi = \sum_{|k|\leq N} \pi_{-k}^* \varphi_k$, converge to φ in $\mathcal{S}'(T)$.

Proof. If $g \in \mathcal{S}(\mathbb{R})$, then we have that

$$\langle (D_{x'}^\alpha \varphi)_k, g\rangle = \langle D_{x'}^\alpha \varphi, ge_{-k}\rangle = (-1)^\alpha \langle \varphi, D_{x'}^\alpha (ge_{-k})\rangle = (-1)^\alpha \langle \varphi, (-k)^\alpha ge_{-k}\rangle = k^\alpha \langle \varphi_k, g\rangle,$$

i.e. $(D^\alpha_{x'}\varphi)_k = k^\alpha \varphi$. To prove Parseval's identity we first prove that the series converges absolutely. We know that there exists $l \in \mathbb{N}$ such that $|\langle \varphi, g\rangle| \lesssim \|g\|_l$ for all $g \in \mathcal{S}(T)$. From a remark above we have that

$$|\langle \varphi_k, \overline{f_k}\rangle| = |\langle \varphi, \overline{f_k e_k}\rangle| \lesssim \|f_k e_k\|_l \lesssim \langle k\rangle^{-(2m-l)}\|f\|_{l+2m},$$

so it follows that the series converges absolutely by choosing m large. Recalling from Proposition 2.3 that $S_N f \to f$ in $\mathcal{S}(T)$, we conclude that,

$$\langle \varphi, \overline{f}\rangle = \lim_{N\to+\infty} \sum_{|k|\le N} \langle \varphi, \overline{f_k e_k}\rangle = \lim_{N\to+\infty} \sum_{|k|\le N} \langle \varphi_k, \overline{f_k}\rangle = \sum_{k\in\mathbb{Z}^d} \langle \varphi_k, \overline{f_k}\rangle.$$

The convergence of the partial sums $S_N\varphi \rightharpoonup \varphi$ in $\mathcal{S}'(T)$ follows from Plancherel's identity and the fact that $\overline{f_k} = (\overline{f})_{-k}$. □

2.2 Function spaces

Recall that for a nonnegative integer s we considered the Sobolev spaces $W^{s,p}(T)$ with norm $\|f\|_{W^{s,p}(T)} := \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^p(T)}$. For $p = 2$ we denote $H^s(T) := W^{s,2}(T)$. The definitions of these spaces over $\mathbb{R}$, $\mathbb{T}^d$, and M are analogous. In the case of T and $\mathbb{R}$, these spaces are also the completions of the corresponding space of Schwartz functions under the respective norm, while for M these spaces are the completions of restrictions to M of the Schwartz functions $\mathcal{S}(T)$ under the $W^{s,p}(M)$ norm.

Since T has no boundary we can define the dual space $H^{-1}(T) := (H^1(T))^*$; we leave the definition of $H^{-1}(M)$ to the next section. By Plancherel's theorem, from Proposition 2.3, we see that if s is a nonnegative integer and $f \in \mathcal{S}(T)$, then

$$\|f\|^2_{H^s(T)} \simeq \sum_{|\alpha|\le s} \|D^\alpha f\|^2_{L^2(T)} \simeq \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} \langle \xi, k\rangle^{2s} |\widehat{f}_k(\xi)|^2 d\xi.$$

This allows to extend the definition of the spaces $H^s(T)$ to any $s \in \mathbb{R}$. Moreover, observe that this extension coincides also with the previous definition of $H^{-1}(T)$. Analogous extensions can also be defined for $\mathbb{R}$ and $\mathbb{T}^d$.

On the boundary ∂M, we consider the usual L^2 space $L^2(\partial M, d\sigma)$ and its corresponding Sobolev spaces $H^s(\partial M)$; we elaborate more on the Sobolev spaces in the next section. We also define the Sobolev subspaces

$$H^s_{loc}(T) := \{f : f \in H^s([-R, R] \times \mathbb{T}^d) \text{ for any } R > 0\},$$

$$H^s_c(T) := \{f \in H^s(T) : f(x_1, x') = 0 \text{ when } |x_1| \geq R \text{ for some } R > 0\},$$

and its corresponding analogs over $\mathbb{R}$. For $\delta \in \mathbb{R}$ we define the L^2 weighted spaces $L^2_\delta(T) := \{f : \langle x_1\rangle^\delta f \in L^2(T)\}$, with the norm $\|f\|_{L^2_\delta(T)} := \|\langle x_1\rangle^\delta f\|_{L^2(T)}$. Similarly, we also define $L^2_\delta(\mathbb{R})$. It follows from Proposition 2.2 that we also have Plancherel's identity for weighted spaces,

$$\begin{aligned} \|f\|_{L^2_\delta(T)} &= \int_T \langle x_1\rangle^{2\delta} |f(x_1, x')|^2 dx_1 dx' \\ &= \int_{\mathbb{R}} \langle x_1\rangle^{2\delta} \sum_{k \in \mathbb{Z}^d} |f_k(x_1)|^2 dx_1 = \sum_{k \in \mathbb{Z}^d} \|f_k\|^2_{L^2_\delta(\mathbb{R})}. \end{aligned} \tag{2.1}$$

For a nonnegative integer s the weighted Sobolev spaces have two equivalent definitions,

$$H^s_\delta(T) := \{f \in L^2_\delta(T) : D^\alpha f \in L^2_\delta(T) \text{ for } |\alpha| \leq s\} = \{f \in L^2_\delta(T) : \langle x_1\rangle^\delta f \in H^s(T)\}.$$

We consider the norm $\|f\|_{H^s_\delta(T)}$ as any of the two equivalent norms: $\sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2_\delta(T)}$ or $\|\langle x_1\rangle^\delta f\|_{H^s(T)}$. We also consider the analogs of these spaces over $\mathbb{R}$. By (2.1) we get that

$$\|f\|^2_{H^s_\delta(T)} \simeq \sum_{|\alpha| \leq s} \|D^\alpha f\|^2_{L^2_\delta(T)} \simeq \sum_{m=0}^{s} \sum_{k \in \mathbb{Z}^d} \langle k\rangle^{2s-2m} \|D^m_{x_1} f_k\|^2_{L^2_\delta(T)}.$$

We can endow the space $H^s_\delta(T)$ with another norm by considering a (small) real parameter $\hbar$ and defining

$$\|f\|_{H^s_{\delta,\hbar}(T)} := \sum_{|\alpha|\leq s} \|(\hbar D)^\alpha f\|_{L^2_\delta(T)} \simeq \left(\sum_{m=0}^{s} \sum_{k\in\mathbb{Z}^d} \langle \hbar k\rangle^{2s-2m} \|(\hbar D_{x_1})^m f_k\|^2_{L^2_\delta(T)}\right)^{1/2}.$$

We call this the *semiclassical* weighted Sobolev space. Analogously, we define the semiclassical Sobolev spaces $W^{s,p}_\hbar(T)$ and their norm.

2.3 Dirichlet problem: definitions and basic facts

In this section we introduce the necessary definitions give a precise formulation of the Dirichlet problem

$$\begin{cases} H_{V,W} u = 0 & \text{in } M_-, \\ u = f & \text{on } \partial M, \end{cases} \tag{$*$}$$

where $H_{V,W} = (D+V)^2 + W = D^2 + 2V\cdot D + (V^2 + D\cdot V + W)$.

2.3.1 Trace operators and Sobolev spaces

We call the trace operator that which restricts functions in the cylinder T to its boundary values in ∂M, and we denote it by tr. For $s > 1/2$, the operator $\mathrm{tr} : H^s(T) \to H^{s-1/2}(\partial M)$ is continuous. For functions defined only either in the interior or exterior of M, denoted by M_- and M_+ respectively, there are also the operators $\mathrm{tr}^\pm : H^s(M_\pm) \to H^{s-1/2}(\partial M)$ for $s > 1/2$.

On the boundary we define the dual space $H^{-1/2}(\partial M) = (H^{1/2}(\partial M))^*$. The continuity of the trace operator $\mathrm{tr} : H^1(T) \to H^{1/2}(\partial M)$ gives the existence of the adjoint operator $\mathrm{tr}^* : H^{-1/2}(\partial M) \to H^{-1}(T)$. If $\varphi \in H^1(T)$ is supported away from ∂M, then $\mathrm{tr}(\varphi) = 0$; this implies that the adjoint tr^* actually maps $H^{-1/2}(\partial M)$ into $H^{-1}_c(T)$. The adjoint is supported on ∂M and, formally, we have $\mathrm{tr}^*\varphi = \varphi d\sigma$.

We also consider the space $H_0^1(M) := \{u \in H^1(M) : \mathrm{tr}^-(u) = 0\}$ and its dual $H^{-1}(M) := (H_0^1(M))^*$. The space $H_0^1(M)$ is also the closure of $C_c^\infty(M_-)$ under the $H^1(M)$ norm.

2.3.2 Weak solutions

The definitions below of weak solution and Dirichlet-to-Neumann map are natural after we formally integrate by parts,

$$\begin{aligned}\int_M (H_{V,W}u)v &= \int_M D\cdot[(D+V)u]v + V\cdot[(D+V)u]v + Wuv \\ &= \int_M -[(D+V)u]\cdot Dv + V\cdot[(D+V)u]v + Wuv + \frac{1}{2\pi i}\int_{\partial M}\nu\cdot[(D+V)u]v \\ &= \int_M -Du\cdot Dv + V\cdot(vDu - uDv) + (V^2+W)uv - \frac{i}{2\pi}\int_{\partial M}\nu\cdot[(D+V)u]v.\end{aligned}$$

For $f \in H^{1/2}(\partial M)$, we say that $u \in H^1(M)$ is a weak solution to the Dirichlet problem $(*)$ if $\mathrm{tr}^-(u) = f$ and

$$\int_M -Du\cdot D\varphi + V\cdot(\varphi Du - uD\varphi) + (V^2+W)u\varphi = 0, \tag{1.1}$$

for all test functions $\varphi \in H_0^1(M)$.

2.3.3 Inhomogeneous problem and extension operator

The first step towards solving the Dirichlet problem $(*)$ is the solution to the inhomogeneous boundary value problem

$$D^2u = f \in H^{-1}(M), \quad u \in H_0^1(M). \tag{$**$}$$

We say that u is a solution to $(**)$ if for any $\varphi \in H_0^1(M)$ we have

$$\langle f, \varphi\rangle = \int_M -Du\cdot D\varphi.$$

Proposition 2.5 ([Tay11]). *For any $f \in H^{-1}(M)$ there exists a unique solution $u \in H_0^1(M)$ to the boundary value problem $D^2 u = f$. If $Tf := u$ denotes the solution operator, then $T : H^s(M) \to H^{s+2}(M) \cap H_0^1(M)$ is bounded for any $s \geq -1$.*

In [Tay11] it is shown an explicit construction of a bounded extension operator $E : H^{s-1/2}(\partial M) \to H^s(M)$ for all $s \geq 1$, such that $\mathrm{tr}^- \circ E = I$. Moreover, for any $N \in \mathbb{N}$ and $s \leq N$ there is an extension $H^s(M) \to H^s(T)$ (that may depend on N), so that we have an extension $E : H^{s-1/2}(\partial M) \to H^s(T)$ with $\mathrm{tr} \circ E = I$; in particular, the trace operator $\mathrm{tr} : H^s(T) \to H^{s-1/2}(\partial M)$ is surjective for $s \geq 1$. Moreover, by cutting off the extension with an appropriate fixed smooth function we can assume that Ef is supported on some fixed compact set of T, containing M, for any $f \in H^s(\partial M)$.

Remark. *We will be concerned with values of s in a fixed range, so we will avoid to refer constantly to the integer associated to the extension.*

2.3.4 Solution to the Dirichlet problem

The existence of the extension allows to turn the Dirichlet problem $(*)$ into the boundary value problem $(**)$ for which we know the existence, uniqueness, and regularity properties.

Proposition 2.6 ([Tay11]). *Assume that the potentials satisfy $V, W \in L^\infty(M)$ and $D \cdot V \in L^\infty(M)$. If 0 is not a Dirichlet eigenvalue of $H_{V,W}$ in M, then for any $f \in H^{1/2}(\partial M)$ there exists a unique weak $u \in H^1(M)$ solution to the Dirichlet problem $(*)$. If we denote $D_{V,W} f := u$, then $D_{V,W} : H^s(\partial M) \to H^{s+1/2}(M)$ is bounded for $1/2 \leq s \leq 3/2$.*

Proof. Under the conditions on the potentials we have that the first order differential operator $X := H_{V,W} - D^2 = 2V \cdot D + (V^2 + D \cdot V + W)$ maps $H^1(M)$ into $L^2(M)$. We first consider the case $f \in H^{1/2}(\partial M)$, so that $Ef \in H^1(M)$. Then, $u \in H^1(M)$ solves the Dirichlet problem $(*)$ if and only if $v := u - Ef \in H_0^1(M)$ solves the boundary value problem $H_{V,W} v = -H_{V,W} Ef$. From Proposition 2.5 we can look for a solution of the form $v = Tw$, with $w \in H^{-1}(M)$, leaving us to solve the equation $(I + XT)w = -H_{V,W} Ef \in H^{-1}(M)$.

From Proposition 2.5 and the conditions on the potentials we know that the operator $XT : H^{-1}(M) \to L^2(M)$ is continuous, and by Rellich's theorem we have that XT is a compact operator on $H^{-1}(M)$. If 0 is not a Dirichlet eigenvalue of $H_{V,W}$ in M, then the Dirichlet problem $(*)$ has at most one solution and therefore $I + XT$ is injective. It follows then from Fredholm's alternative that $I+XT$ is bijective, and by the Open Mapping theorem that its inverse is continuous. Then,

$$\|v\|_{H_0^1(M)} \lesssim \|w\|_{H^{-1}(M)} \lesssim \|H_{V,W}Ef\|_{H^{-1}(M)} \lesssim \|Ef\|_{H^1(M)} \lesssim \|f\|_{H^{1/2}(\partial M)}.$$

Therefore, $u = v + Ef \in H^1(M)$ and $\|u\|_{H^1(M)} \lesssim \|v\|_{H^1(M)} + \|Ef\|_{H^1(M)} \lesssim \|f\|_{H^{1/2}(\partial M)}$, as desired. To prove the higher–order regularity of the solutions, all we need to modify in the proof is the fact that for $f \in H^{3/2}(\partial M)$ we have $Ef \in H^2(M)$, and therefore $H_{V,W}Ef \in L^2(M)$. The higher–order regularity properties of T from Proposition 2.5 imply that $T : L^2(M) \to H^1(M)$ is compact, and so XT is compact on $L^2(M)$. After this the proof carries out exactly as before. □

2.3.5 Dirichlet-to-Neumann map and normal derivatives

We define the Dirichlet-to-Neumann (DN) map $\Lambda_{V,W}$ as follows: if $f, g \in H^{1/2}(\partial M)$ and $v \in H^1(M)$ is any function extending g, i.e. $\mathrm{tr}^-(v) = g$, then

$$\langle \Lambda_{V,W} f, g\rangle := \int_M -Du \cdot Dv + V \cdot (vDu - uDv) + (V^2 + W)uv, \tag{1.2}$$

where $u = D_{V,W}f \in H^1(M)$ is the weak solution of $(*)$. The definition of the weak solution implies that the DN map is well-defined, i.e. it depends only on g and not on the choice of the extension. Formally we have that

$$\Lambda_{V,W} f = \frac{i}{2\pi} \nu \cdot (D + V)u\big|_{\partial M}.$$

We record a Green's identity that will be useful now and in Chapter 5. This is just slightly more general than saying that the divergence theorem holds for vector fields in $W^{1,1}(M)$.

Proposition 2.7. *If supp*$(V) \subseteq M_-$, $V \in L^\infty(M)$, $D \cdot V \in L^\infty(M)$, $w \in W^{1,1}(M)$, *then*

$$\int_M V \cdot Dw + (D \cdot V)w = 0.$$

Proof. This proof is taken from Lemma 5.2 in [Sal06]. Let $N = d+1$. Since $L^\infty(M)$ does not have good approximation properties, we start proving it for $V \in L^N(M)$, $D\cdot V \in L^{N/2}(M)$, $w = W^{1,N/(N-1)}(M)$. From the Sobolev embedding we have that $w \in W^{N/(N-2)}(M)$, so that the integral is in fact convergent.

Given that $\text{supp}(V) \subseteq M_-$ we can find a compact set $K \subseteq M_-$ and smooth functions $\{V_k\}$ such that $\text{supp}(V_k) \subseteq K$, $V_k \to V$ in $L^N(M)$ and $D \cdot V_k \to D \cdot V$ in $L^{N/2}(M)$. Moreover, $V_k w \in W^{1,N/(N-1)}(M)$ and $\text{supp}(V_k w) \subseteq K$. The divergence theorem holds for vector fields in $W^{1,1}(M)$, and so we get

$$\begin{aligned}\int_M V \cdot Dw + (D \cdot V)w &= \lim_{k\to+\infty} \int_M V_k \cdot Dw + (D \cdot V_k)w \\ &= \lim_{k\to+\infty} \int_M D \cdot (V_k w) = \lim_{k\to+\infty} \frac{1}{2\pi i} \int_{\partial M} \nu \cdot (V_k w) = 0.\end{aligned}$$

The conditions $V \in L^N(M)$ and $D\cdot V \in L^{N/2}(M)$ are satisfied if we assume $V \in L^\infty(M)$ and $D \cdot V \in L^\infty(M)$. Finally, the integral only takes place in $\text{supp}(V) \subseteq M_-$. We know that there exist smooth functions $\{w_k\}$ such that $w_k \to w$ in $L^1(M)$ and $Dw_k \to Dw$ in $L^1(\text{supp}(V))$, and thus the conclusion follows. □

Remark. *The condition supp*$(V) \subseteq M_-$ *is not necessary; in [Sal06] this is proven under weaker conditions whose analogs would be supp*$(V) \subseteq M$ *and* $D \cdot V \in L^\infty(T)$.

Before we continue, we need to define the interior and exterior normal derivative of a function. This represents no problem if the function u is in $H^2(M)$ or $H^2_{loc}(M_+)$, as the gradient Du is in $H^1(M)$ or $H^1_{loc}(M_+)$ and so its trace is in $H^{1/2}(\partial M)$. Moreover, for

$\varphi \in C_c^\infty(T)$ it satisfies either

$$\int_{\partial M} (\partial_\nu^\pm u)\varphi = \mp 4\pi^2 \int_{M_\pm} (D^2 u)\varphi + Du \cdot D\varphi.$$

These identities suggest that we can define the normal derivatives for harmonic functions in $H^1(M)$ or $H^1_{loc}(M_+)$. We say u, in $H^1(M)$ or $H^1_{loc}(M_+)$, is harmonic if for any $\varphi \in C_c^\infty(M_\pm)$ we have

$$\int_{M_\pm} Du \cdot D\varphi = 0, \tag{2.2}$$

as it corresponds. By continuity these definitions extend to all test functions $\varphi \in H^1(M_\pm)$ with $\mathrm{tr}^\pm(\varphi) = 0$. If $f \in H^{1/2}(\partial M)$ and $v \in H^1(M_\pm)$ is any function extending f, i.e. $\mathrm{tr}^\pm(v) = f$, then we define the normal derivatives as the functionals

$$\langle \partial_\nu^\pm u, f\rangle := \mp 4\pi^2 \int_{M_\pm} Du \cdot Dv. \tag{2.3}$$

The condition (2.2) ensures that this is well-defined, i.e. it depends only on f and not on the choice of the extension. In particular, taking $v = Ef \in H_c^1(T)$ and using the boundedness and support properties of Ef we can conclude that $\partial_\nu^\pm u \in H^{-1/2}(\partial M)$.

Proposition 2.8. *Assume that the potentials satisfy $V, W \in L^\infty(M)$ and $D \cdot V \in L^\infty(M)$. Suppose in addition that supp$(V) \subseteq M_-$. If 0 is not a Dirichlet eigenvalue of $H_{V,W}$ in M, then the Dirichlet-to-Neumann map $\Lambda_{V,W} : H^s(\partial M) \to H^{s-1}(\partial M)$ is bounded for $1/2 \le s \le 3/2$. Moreover, if $f \in H^{3/2}(\partial M)$ and $u = D_{V,W} f \in H^2(M)$, then we have*

$$\Lambda_{V,W} f = \frac{1}{4\pi^2} \partial_\nu^- u\big|_{\partial M}.$$

Proof. We first prove that $\Lambda_{V,W} : H^{1/2}(\partial M) \to H^{-1/2}(\partial M)$ is bounded. If $f, g \in H^{1/2}(\partial M)$, then we have to show that $|\langle \Lambda_{V,W} f, g\rangle| \lesssim \|f\|_{H^{1/2}(\partial M)} \|g\|_{H^{1/2}(\partial M)}$. For $u, v \in H^1(M)$ we have that

$$\left|\int_M -Du\cdot Dv + V\cdot(vDu - uDv) + (V^2+W)uv\right| \lesssim \|u\|_{H^1(M)}\|v\|_{H^1(M)}.$$

In particular, taking $u = D_{V,W}f \in H^1(M)$ and $v = Eg \in H^1(M)$, we conclude from (1.2) that

$$|\langle \Lambda_{V,W}f, g\rangle| \lesssim \|u\|_{H^1(M)}\|v\|_{H^1(M)} \lesssim \|f\|_{H^{1/2}(\partial M)}\|g\|_{H^{1/2}(\partial M)},$$

where we used in the last inequality the boundedness of $D_{V,W}$ and E.

Now we prove the result when $f \in H^{3/2}(\partial M)$. Let g, v be as before. From Proposition 2.6 we have that $u = D_{V,W}f \in H^2(M)$, and so $\partial_\nu^- u \in H^{1/2}(\partial M)$. Moreover, we can integrate by parts to obtain

$$\int_M (D^2u)v = \int_M -Du\cdot Dv - \frac{1}{4\pi^2}\int_{\partial M}(\partial_\nu^- u)g$$

In addition, for $u \in H^2(M)$, $v \in H^1(M)$ we have that $uv \in W^{1,1}(M)$, so that we obtain $\int_M D\cdot(Vu)v = -\int_M V\cdot(uDv)$. from Proposition 2.7. From the previous identities and $H_{V,W}u = 0$ we get that

$$\begin{aligned} 0 &= \int_M D\cdot[(D+V)u]v + V\cdot[(D+V)u]v + Wuv \\ &= \int_M -Du\cdot Dv + V\cdot(vDu - uDv) + (V^2+W)uv - \frac{1}{4\pi^2}\int_{\partial M}(\partial_\nu^- u)g, \end{aligned}$$

i.e. $\Lambda_{V,W}f = \partial_\nu^- u/4\pi^2$, and $\|\Lambda_{V,W}f\|_{H^{1/2}(\partial M)} \lesssim \|Du\|_{H^1(M)} \lesssim \|u\|_{H^2(M)} \lesssim \|f\|_{H^{3/2}(\partial M)}$, as we wanted to prove. □

An important application of the previous theorem is the case of the Laplacian $H_{0,0} = D^2$. We know that 0 is not a Dirichlet eigenvalue of the Laplacian in M, and so we have the DN map $\Lambda_{0,0}$ defined by

$$\langle \Lambda_{0,0}f, g\rangle := \int_M -Du\cdot Dv, \tag{2.4}$$

where $u = D_{0,0}f \in H^1(M)$ and $v \in H^1(M)$ is any function extending $g \in H^{1/2}(\partial M)$. We will not use the result for $s > 3/2$, but it can be shown that for $s \geq 1/2$, the map $\Lambda_{0,0} : H^s(\partial M) \to H^{s-1}(\partial M)$ is bounded. Moreover, the symmetry in (2.4) implies the symmetry of the DN map, i.e. we have $\langle \Lambda_{0,0} f, g \rangle = \langle \Lambda_{0,0} g, f \rangle$ for $f, g \in H^{1/2}(\partial M)$.

CHAPTER 3

SEMICLASSICAL PSEUDODIFFERENTIAL OPERATORS OVER $\mathbb{R} \times \mathbb{T}^d$

We denote the points in the cylinder $T = \mathbb{R} \times \mathbb{T}^d$ by (x_1, x'), meaning that $x_1 \in \mathbb{R}$ and $x' \in \mathbb{T}^d$. As it has been usual in the inverse problem literature, instead of the large parameter τ (appearing in the Carleman estimate) we consider a small parameter $\hbar = 1/\tau > 0$, and use the standard notation and results from semiclassical analysis. We use the notation $\hbar$ instead of h to prevent confusion with the later use of h for a harmonic function.

In this chapter, we define and prove the necessary results for pseudodifferential operators on the cylinder $T = \mathbb{R} \times \mathbb{T}^d$. We will use the definition and basic properties of these operators on $\mathbb{R}$ and $\mathbb{T}^d$, for which we refer to [Ste93], [Zwo12], [Sal06], [Sob99], [RT10].

Some of the results below may be valid in greater generality than that we consider here. We will restrict to prove the results that we will need.

3.1 Definitions and basic facts

3.1.1 Semiclassical Fourier transform

We will use the ideas from semiclassical analysis only for the real variable x_1, as the term $\tau x_1 = x_1/\hbar$ appears in the limiting Carleman weight, and expressions of the form

$$e^{2\pi\tau x_1} D_{x_1} e^{-2\pi\tau x_1} = D_{x_1} + i\tau = \tau(\hbar D_{x_1} + i)$$

will continue to appear through the problem. For this reason, throughout the present chapter, we define the semiclassical Fourier transform, for functions in $L^1(\mathbb{R})$, by

$$\widehat{f}^{\hbar}(\xi) := \int_{\mathbb{R}} e^{-2\pi i x_1 \xi/\hbar} f(x_1) dx_1,$$

i.e. $\widehat{f^\hbar}(\hbar\xi) = \widehat{f}(\xi)$. We can rewrite the results from Proposition 2.3 as follows.

Proposition 3.1. *If $f \in \mathcal{S}(T)$, then its Fourier coefficients f_k are in $\mathcal{S}(\mathbb{R})$. Moreover, these satisfy the following:*

a). the transform of the coefficients and its derivatives have polynomial decay bounds $\langle \xi, \hbar k\rangle^{-2m} |(\hbar D_\xi)^\alpha \widehat{f_k^\hbar}(\xi)| \lesssim \|x_1^\alpha f\|_{W_\hbar^{2m,1}(T)}$, where $\langle \xi, \hbar k\rangle := (1 + \xi^2 + |\hbar k|^2)^{1/2}$ and the constant of the inequality may depend on m and d,

b). the inversion formula holds,

$$f(x_1, x') = \sum_{k\in\mathbb{Z}^d} f_k(x_1) e_k(x') = \frac{1}{\hbar} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} e_k(x') \widehat{f_k^\hbar}(\xi) d\xi,$$

with pointwise absolute uniform convergence, as well as for its derivatives,

c). Plancherel's identity holds $\|f\|^2_{L^2(T)} = \sum_{k\in\mathbb{Z}^d} \|f_k\|^2_{L^2(\mathbb{R})} = \hbar^{-1} \sum_{k\in\mathbb{Z}^d} \|\widehat{f_k^\hbar}\|^2_{L^2(\mathbb{R})}$.

3.1.2 Semiclassical pseudodifferential operators

For the differential operator $a_{\alpha,\beta}(x_1, x')(\hbar D_{x_1})^\alpha (\hbar D_{x'})^\beta$ on T and $f \in \mathcal{S}(T)$ we have the Fourier inversion relation

$$\begin{aligned}
&[a_{\alpha,\beta}(x_1, x')(\hbar D_{x_1})^\alpha (\hbar D_{x'})^\beta] f(x_1, x') \\
&\qquad = \frac{1}{\hbar} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} e_k(x') [a_{\alpha,\beta}(x_1, x') \xi^\alpha (\hbar k)^\beta] \widehat{f_k^\hbar}(\xi) d\xi.
\end{aligned}$$

We refer to the function $a(x_1, x', \xi, k) = a_{\alpha,\beta}(x_1, x')\xi^\alpha (\hbar k)^\beta$ as the symbol of the differential operator. In what follows we show that we can admit symbols more general than polynomials (in the dual variables ξ and k). Finally, although we only need to define the symbol over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{Z}^d$, it may be convenient also to allow for symbols over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$. We denote the points in $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$ by (x_1, x', ξ, t), and we call ξ and t the dual real and toroidal variables, respectively.

Definition 3.2. *We say that $a = a(x_1, \xi; \hbar)$ is a (semiclassical) m-th order symbol over $\mathbb{R} \times \mathbb{R}$ if there exists $\hbar_0$ such that if $0 < \hbar \leq \hbar_0$, then for any $M \geq 0$ there exists a constant A_M such that*

$$|D^{\alpha}_{x_1} D^{\beta}_{\xi} a(x_1, \xi; \hbar)| \leq A_M \langle \xi \rangle^m,$$

whenever $\alpha + |\beta| \leq M$. The associated pseudodifferential operator is defined by

$$Af(x_1) := Op_{\hbar}(a) f(x_1) = \frac{1}{\hbar} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} a(x_1, \xi; \hbar) \widehat{g^{\hbar}}(\xi) d\xi.$$

Definition 3.3. *We say that $a = a(x_1, x', \xi, t; \hbar)$ is a (semiclassical) m-th order symbol over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$ if there exists $\hbar_0$ such that if $0 < \hbar \leq \hbar_0$, then for any $M \geq 0$ there exists a constant A_M such that*

$$|D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} a(x_1, x', \xi, t; \hbar)| \leq A_M \langle \xi, \hbar t \rangle^m,$$

whenever $\alpha + |\beta| + \gamma \leq M$. The associated pseudodifferential operator is defined by

$$Af(x_1, x') := Op_{\hbar}(a) f(x_1, x') := \frac{1}{\hbar} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} e_k(x') a(x_1, x', \xi, k; \hbar) \widehat{f^{\hbar}_k}(\xi) d\xi.$$

Remark. *Observe that we do not require the order of the factor $\langle \xi \rangle$ or $\langle \xi, \hbar t \rangle$ to decrease whenever we differentiate with respect to ξ. This would be the case if the symbol were a polynomial or a rational function, but we will be considering more general symbols. In the notation of [Ste93], these would correspond to symbols in $S^m_{0,0}$.*

Remark. *Note that we do not require any condition on the differences (or derivatives) with respect to the dual toroidal variables. In a later section, Composition, we will need these symbols and refer to them as special.*

Remark. *To avoid unnecessary notation, we may occasionally drop the dependance of the symbol on the semiclassical parameter and just write $a(x_1, x', \xi, t)$ instead of $a(x_1, x', \xi, t; \hbar)$.*

Example. *With this definition, the functions* ξ *and* $\hbar t_j$ *are symbols of order* 1*. Moreover, we have that* $\hbar D_{x_1} = Op_\hbar(\xi)$ *and* $\hbar D_{x'_j} = Op_\hbar(\hbar t_j)$ *as*

$$(\hbar D_{x_1})f(x_1, x') = \frac{1}{\hbar}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} e_k(x')(\xi)\widehat{f_k^\hbar}(\xi)d\xi,$$

$$(\hbar D_{x'_j})f(x_1, x') = \frac{1}{\hbar}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} e_k(x')(\hbar k_j)\widehat{f_k^\hbar}(\xi)d\xi.$$

Example. *The function* $\langle\xi, \hbar t\rangle^{-2} := 1/(\xi^2 + |\hbar t|^2 + 1)$ *is a symbol of order* -2*.*

Proposition 3.4. *If* a*,* b *are symbols of order* m *and* n*, then* $D^\alpha_{x_1} D^\beta_{x'} D^\gamma_\xi a$*,* $a + b$*, and* ab *are symbols of order* m*,* $\max\{m, n\}$*, and* $m + n$*, respectively. The seminorms of each of these symbols are bounded by those of* a*, the maximum of those of* a *and* b*, and products of those of* a *and* b*, respectively.*

Proof. This is a routine argument. □

Proposition 3.5. *If* $A = Op_\hbar(a)$ *is a pseudodifferential operator over* T*, then* A *maps the space of Schwartz functions* $\mathcal{S}(T)$ *into itself.*

Proof. For this proof we will use the notation $D_x = (D_{x_1}, D_{x'})$. Let $f \in \mathcal{S}(T)$. The polynomial control of the symbol a and its derivatives, together with the rapid decay of $\widehat{f_k^\hbar}(\xi)$ from Proposition 3.1 give that $Af \in C^\infty(T)$, and it is bounded together with its derivatives. Moreover, differentiating the expression we see that (the vector) $(\hbar D_x)Af$ equals

$$\begin{aligned}\hbar D_x Af(x_1, x') &= \frac{1}{\hbar}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} \hbar D_x(e^{2\pi i x_1\xi/\hbar} e_k(x')a)\widehat{f_k^\hbar}(\xi)d\xi \\ &= \frac{1}{\hbar}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} e_k(x')[(\xi, \hbar k)a + \hbar D_x a]\widehat{f_k^\hbar}(\xi)d\xi,\end{aligned}$$

and so it is a pseudodifferential operator corresponding to the symbol $(\xi, \hbar t)a + \hbar D_x a$. By induction the same is true for higher order derivatives. Therefore, in order to show that

$Af \in \mathcal{S}(T)$, it suffices to show that $\langle x_1 \rangle^{2m} |Af| \le C_m$ for all $m \ge 0$. Integrating by parts we obtain that

$$\begin{aligned}\langle x_1 \rangle^{2m} Af(x_1, x') &= \frac{1}{\hbar} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}} \langle \hbar D_\xi \rangle^{2m} (e^{2\pi i x_1 \xi/\hbar}) e_k(x') a \widehat{f_k^\hbar}(\xi) d\xi \\ &= \frac{1}{\hbar} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} e_k(x') \langle \hbar D_\xi \rangle^{2m} [a \widehat{f_k^\hbar}(\xi)] d\xi.\end{aligned}$$

Again, the polynomial control of the symbol a and its derivatives, together with the rapid decay of the derivatives of $\widehat{f_k^\hbar}(\xi)$ from Proposition 3.1 give that this is bounded, from where the conclusion follows. □

Proposition 3.6. *Let $A = Op_\hbar(a)$ be a pseudodifferential operator over T. Then, it satisfies the following identities,*

$$\hbar D_{x_1} A = Op_\hbar(\xi a + \hbar D_{x_1} a), \quad \hbar^2 D_{x_1}^2 A = Op_\hbar(\xi^2 a + 2\hbar \xi D_{x_1} a + \hbar^2 D_{x_1}^2 a),$$

$$\hbar D_{x'_j} A = Op_\hbar(\hbar t_j a + \hbar D_{x'_j} a), \quad \hbar^2 D_{x'}^2 A = Op_\hbar(|\hbar t|^2 a + 2\hbar(\hbar t \cdot D_{x'} a) + \hbar^2 D_{x'}^2 a),$$

$$A \circ \hbar D_{x_1} = Op_\hbar(\xi a), \quad A \circ \hbar^2 D_{x_1}^2 = Op_\hbar(\xi^2 a), \quad A \circ \hbar^2 D_{x'}^2 = Op_\hbar(|\hbar t|^2 a).$$

Proof. These results follow directly from the definition. □

The simplest case when dealing with pseudodifferential operators in $\mathbb{R}^d$, is when the symbol has spatial compact support, see Chapter 6, Section 2.1 in [Ste93]. This is always the case for symbols on the torus, so in analogy to [Ste93], we decompose the symbol in its Fourier series. With uniform convergence (in x_1, x', ξ, and l), we have that

$$a(x_1, x', \xi, l; \hbar) = \sum_{k \in \mathbb{Z}^d} a_k(x_1, \xi, l; \hbar) e_k(x'),$$

so we can rewrite the operator $A = \mathrm{Op}_\hbar(a)$ as

$$
\begin{aligned}
Af(x_1,x') &= \frac{1}{\hbar}\sum_{l\in\mathbb{Z}^d}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} e_l(x') a(x_1,x',\xi,l)\widehat{f_l^{\hbar}}(\xi)d\xi \\
&= \frac{1}{\hbar}\sum_{k,l\in\mathbb{Z}^d}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} e_{k+l}(x') a_k(x_1,\xi,l)\widehat{f_l^{\hbar}}(\xi)d\xi \\
&= \sum_{k,l\in\mathbb{Z}^d}\left(\frac{1}{\hbar}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} a_{k-l}(x_1,\xi,l)\widehat{f_l^{\hbar}}(\xi)d\xi\right)e_k(x'). \qquad (3.1)
\end{aligned}
$$

If a is a symbol over $\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}\times\mathbb{R}^d$, then for fixed $k,l\in\mathbb{Z}^d$ we can define a symbol over $\mathbb{R}\times\mathbb{R}$ by $a_{k,l}(x_1,\xi;\hbar) := a_{k-l}(x_1,\xi,l;\hbar)$. We will elaborate below on the properties of this symbol. Let us define $A^{kl} := \mathrm{Op}_\hbar(a_{k,l})$ on $\mathcal{S}(\mathbb{R})$. For $f\in\mathcal{S}(T)$, we define $A_{kl}f(x_1,x') := A^{kl}f_l(x_1)e_k(x')$, so that (3.1) can be expressed as the decomposition

$$
Af(x_1,x') = \sum_{k,l\in\mathbb{Z}^d} A^{kl}f_l(x_1)e_k(x') = \sum_{k,l\in\mathbb{Z}^d} A_{kl}f(x_1,x'). \qquad (3.2)
$$

3.2 Boundedness

In this section we prove a weighted version of the Calderón–Vaillancourt theorem for pseudodifferential operators over T. It is interesting to observe that we do not need to control the differences over the dual toroidal variables; this had already been noted in [Sob99], [RT10].

Recall from before that for the symbol $a(x_1,x',\xi,l;\hbar)$ over $\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}\times\mathbb{Z}^d$, we defined the symbol $a_{k,l}(x_1,\xi;\hbar) := a_{k-l}(x_1,\xi,l;\hbar)$ over $\mathbb{R}\times\mathbb{R}$.

Proposition 3.7. *If $a(x_1,x',\xi,l;\hbar)$ is a semiclassical zero order symbol over $\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}\times\mathbb{Z}^d$, then $a_{k,l}(x_1,\xi;\hbar)$ is a semiclassical zero order symbol over $\mathbb{R}\times\mathbb{R}$ with seminorm bounds*

$$
|D^\alpha_{x_1}D^\beta_\xi a_{k,l}(x_1,\xi;\hbar)| \lesssim A_{M+2N}\langle k-l\rangle^{-2N},
$$

whenever $\alpha+\beta\le M$ and any $N\ge 0$.

Proof. From Proposition 2.2 we have that

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta} a_{k,l}(x_1,\xi)| = |D_{x_1}^{\alpha} D_{\xi}^{\beta} a_{k-l}(x_1,\xi,l)| \lesssim \langle k-l\rangle^{-2N} \|D_{x_1}^{\alpha} D_{\xi}^{\beta} a(x_1,\cdot,\xi,l)\|_{W^{2N,1}(\mathbb{T}^d)}.$$

Given that a is a zero order symbol, then for any $N \geq 0$ we can bound $\|D_{x_1}^{\alpha} D_{\xi}^{\beta} a(x_1,\cdot,\xi,l)\|_{W^{2N,1}(\mathbb{T}^d)} \lesssim A_{M+2N}$, whenever $\alpha+\beta \leq M$, as we wanted to prove. □

We use this to show that the decomposition from (3.2) actually converges. The first step is to recall the standard boundedness properties of pseudodifferential operators on weighted spaces over $\mathbb{R}$. We will elaborate a little more on the quantitative aspect of the bound in the appendix at the end of the chapter.

Proposition 3.8 ([Sal06])**.** *Let $0 < \hbar \leq 1$. Let $a(x_1,\xi;\hbar)$ be a semiclassical zero order symbol over $\mathbb{R}\times\mathbb{R}$. For any $\delta \in \mathbb{R}$ the operator $Op_{\hbar}(a)$ is bounded in $L^2_{\delta}(\mathbb{R})$. Moreover, if $|\delta| \leq \delta_0$, then the operator norms $\|Op_{\hbar}(a)\|_{L^2_{\delta}(\mathbb{R})\to L^2_{\delta}(\mathbb{R})}$ are uniformly bounded (in δ and $\hbar$) by a multiple (depending on δ_0) of some seminorm of a.*

Remark. *From Proposition 3.7 and Proposition 3.8 we obtain that if $|\delta| \leq \delta_0$, then we can uniformly bound $\|A^{kl}\|_{L^2_{\delta}(\mathbb{R})\to L^2_{\delta}(\mathbb{R})} \lesssim A_{M+2N}\langle k-l\rangle^{-2N}$, for some value of M and any $N \geq 0$.*

Let us consider the elliptic differential operator $\langle \hbar D\rangle^2 = (\hbar D)^2 + 1 = \mathrm{Op}_{\hbar}(\langle \xi, \hbar t\rangle^2)$, and the multiplier operator $\langle \hbar D\rangle^{-2} := \mathrm{Op}_{\hbar}(\langle \xi, \hbar t\rangle^{-2})$. These are pseudodifferential operators, so by Proposition 3.5 they map $\mathcal{S}(T)$ to itself. Moreover, these are inverses to each other on $\mathcal{S}(T)$. The proof the following result is presented in the appendix.

Proposition 3.9. *Let $0 < \hbar \leq 1$ and let $|\delta| \leq \delta_0$. The differential operator $\langle \hbar D\rangle^2 : H^2_{\delta,\hbar}(T) \to L^2_{\delta}(T)$ and the multiplier operator $\langle \hbar D\rangle^{-2} : L^2_{\delta}(T) \to H^2_{\delta,\hbar}(T)$ are uniformly bounded (in δ and $\hbar$) operators, and inverses to each other. The bounds of the operators may depend in δ_0.*

Proposition 3.10. *Let $0 < \hbar \leq 1$. If $A = Op_\hbar(a)$ is an m-th order pseudodifferential operator, then $\langle \hbar D\rangle^2 A \langle \hbar D\rangle^{-2}$ is also an m-th order pseudodifferential operator with symbol*

$$\widetilde{a} := a + \frac{2\hbar(\xi, \hbar t)\cdot(D_{x_1}a, D_{x'}a) + \hbar^2(D_{x_1}^2 a + D_{x'}^2 a)}{\langle \xi, \hbar t\rangle^2}.$$

Moreover, the seminorms of $\widetilde{a}$ are bounded by seminorms of a.

Proof. The first part is a direct computation,

$$\begin{aligned}
&\langle \hbar D\rangle^2 A \langle \hbar D\rangle^{-2} f(x_1, x') \\
&= \langle \hbar D\rangle^2 \left(\frac{1}{\hbar} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} e_k(x') a \frac{\widehat{f_k^\hbar}(\xi)}{\langle \xi, \hbar k\rangle^2} d\xi \right) \\
&= \frac{1}{\hbar} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} \langle \hbar D\rangle^2 (e^{2\pi i x_1 \xi/\hbar} e_k(x') a) \frac{\widehat{f_k^\hbar}(\xi)}{\langle \xi, \hbar k\rangle^2} d\xi \\
&= \frac{1}{\hbar} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\pi i x_1 \xi/\hbar} e_k(x') \left(a + \frac{2\hbar(\xi, \hbar k)\cdot(D_{x_1}a, D_{x'}a) + \hbar^2(D_{x_1}^2 a + D_{x'}^2 a)}{\langle \xi, \hbar k\rangle^2} \right) \widehat{f_k^\hbar}(\xi) d\xi.
\end{aligned}$$

The symbols $(\xi, \hbar t)/(\xi^2 + |\hbar t|^2 + 1)$ and $1/(\xi^2 + |\hbar t|^2 + 1)$ have order -1 and -2, respectively, with uniformly bounded (in $\hbar$) seminorms. The conclusion follows from Proposition 3.4. □

Theorem 3.11. *Let $0 < \hbar \leq 1$. Let a be a zero order symbol over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$. For $s = 0, 2$ and $|\delta| \leq \delta_0$, the operator $Op_\hbar(a)$ is uniformly bounded (in δ and $\hbar$) on $H^s_{\delta,\hbar}(T)$. The bounds depend on d, δ_0, and (linearly) in some seminorm of the symbol, but are independent of $\hbar$.*

Proof. We prove first the result on the weighted spaces $L^2_\delta(T)$, and then conjugate by $\langle \hbar D\rangle^2$ to show the result in $H^2_{\delta,\hbar}(T)$. Recall the decomposition $A = \sum_{k,l} A_{kl}$ from (3.2), where $A_{kl} f(x_1, x') := A^{kl} f_l(x_1) e_k(x')$. Choosing $N = N(d)$ large enough and using the bounds for the operators A^{kl}, from the remark after Proposition 3.8, we obtain that

$$\sup_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\|A^{kl}\|_{L^2_\delta(\mathbb{R})\to L^2_\delta(\mathbb{R})},\quad \sup_{l\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\|A^{kl}\|_{L^2_\delta(\mathbb{R})\to L^2_\delta(\mathbb{R})}\lesssim A_{M+2N}.$$

By Plancherel's theorem and Schur's criterion it follows that

$$\begin{aligned}\|Af\|^2_{L^2_\delta(T)} &= \sum_{k\in\mathbb{Z}^d}\left\|\sum_{l\in\mathbb{Z}^d}A^{kl}f_l\right\|^2_{L^2_\delta(\mathbb{R})}\\ &\le \sum_{k\in\mathbb{Z}^d}\left(\sum_{l\in\mathbb{Z}^d}\|A^{kl}\|_{L^2_\delta(\mathbb{R})\to L^2_\delta(\mathbb{R})}\|f_l\|_{L^2_\delta(\mathbb{R})}\right)^2\\ &\lesssim A^2_{M+2N}\sum_{l\in\mathbb{Z}^d}\|f_l\|^2_{L^2_\delta(\mathbb{R})} = A^2_{M+2N}\|f\|^2_{L^2_\delta(T)}.\end{aligned}$$

This gives that A is a bounded operator on $L^2_\delta(T)$ for $|\delta|\le\delta_0$. From Proposition 3.9 we get that the boundedness of A on $H^2_{\delta,\hbar}(T)$ is equivalent to the boundedness of $\langle\hbar D\rangle^2 A\langle\hbar D\rangle^{-2}$ on $L^2_\delta(T)$. We know from Proposition 3.10 that this is a zero order pseudodifferential operator with seminorms bounded by those of a, and thus the conclusion follows. □

Remark. *The result proven above will be sufficient for our purposes, but the result can be extended to $H^s_{\delta,\hbar}(T)$ for any $0\le s\le 2$ by complex interpolation.*

3.3 Composition

Let a,b be zero order symbols, and let $A=\mathrm{Op}_\hbar(a)$ and $B=\mathrm{Op}_\hbar(b)$. We know from Proposition 3.4 that ab is also a zero order symbol. The following result provides a relation between the operator $\mathrm{Op}_\hbar(ab)$ and the composition AB. This will be used to obtain the invertibility of pseudodifferential operators corresponding to certain zero order symbols. In contrast to Theorem 3.11, in this case we need our symbols to satisfy bounds for the differences in the dual toroidal variables. For $(u,v,w)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ we denote $\langle u,v,w\rangle:=(1+u^2+|v|^2+|w|^2)^{1/2}$.

Definition 3.12. *We say a zero order symbol $a=a(x_1,x',\xi,t;\hbar)$ is special if, in addition* *to the conditions from Definition 3.3, for any $M \geq 0$ there exists a constant A'_M such that*

$$|D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi}(a(\cdot, t_1) - a(\cdot, t_2))| \leq \hbar A'_M |t_1 - t_2|,$$

whenever $\alpha + |\beta| + \gamma \leq M$.

Remark. *If the symbol is differentiable with respect to t and satisfies*

$$|D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} D_t a| \leq \hbar A'_M,$$

whenever $\alpha + |\beta| + \gamma \leq M$, then the symbol is be special.

Theorem 3.13. *Let $0 < \hbar \leq 1$. Let a, b be zero order symbols over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$. There exists a zero order symbol c such that $Op_{\hbar}(a) Op_{\hbar}(b) = Op_{\hbar}(c)$. If the symbols are special, then, for $s = 0, 2$ and $|\delta| \leq \delta_0$, we have*

$$\| Op_{\hbar}(a) Op_{\hbar}(b) - Op_{\hbar}(ab) \|_{H^s_{\delta,\hbar}(T) \to H^s_{\delta,\hbar}(T)} = \| Op_{\hbar}(c - ab) \|_{H^s_{\delta,\hbar}(T) \to H^s_{\delta,\hbar}(T)} \lesssim \hbar,$$

where the constant of the inequality is a multiple (depending on d, δ_0) of the product of some seminorm of the symbols, but is independent of $\hbar$.

Proof. Let $A = \mathrm{Op}_{\hbar}(a)$ and $B = \mathrm{Op}_{\hbar}(b)$. Let us decompose $A = \sum A_{jk}$ and $B = \sum B_{lm}$ as in (3.2). We have that $A_{jk} B_{lm} = 0$ if $k \neq l$, so that $ABf(x_1, x') = \sum A_{jk} B_{kl} f(x_1, x')$, and $A_{jk} B_{kl} f(x_1, x') = A^{jk} B^{kl} f_l(x_1) e_j(x')$. We know that there exists a zero order symbol $c_{j,k,l}$ over $\mathbb{R} \times \mathbb{R}$ such that $A^{jk} B^{kl} = \mathrm{Op}_{\hbar}(c_{j,k,l})$, see [Ste93], [Zwo12]. Thus,

$$\begin{aligned} ABf(x_1, x') &= \sum_{j,k,l \in \mathbb{Z}^d} A_{jk} B_{kl} f(x_1, x') \\ &= \sum_{j,k,l \in \mathbb{Z}^d} A^{jk} B^{kl} f_l(x_1) e_j(x') = \sum_{j,k,l \in \mathbb{Z}^d} \mathrm{Op}_{\hbar}(c_{j,k,l}) f_l(x_1) e_j(x'). \end{aligned}$$

From Proposition 3.7 and Proposition 3.19, which we prove in the appendix, there exists some K such that for any $N \geq 0$ we have that

$$|D^\alpha_{x_1} D^\beta_\xi c_{j,k,l}| \lesssim A_{K+M+2N} B_{K+M+2N} \langle j-k\rangle^{-2N} \langle k-l\rangle^{-2N}, \tag{3.3}$$

$$|D^\alpha_{x_1} D^\beta_\xi (c_{j,k,l} - a_{j,k} b_{k,l})| \lesssim \hbar A_{K+M+2N} B_{K+M+2N} \langle j-k\rangle^{-2N} \langle k-l\rangle^{-2N}, \tag{3.4}$$

whenever $\alpha+\beta \leq M$. Recall that for a symbol s we denote $s_{k,l}(x_1,\xi) = s_{k-l}(x_1,\xi,l)$. Using this notation and the decomposition from (3.2), we see that if c were a symbol such that $AB = \mathrm{Op}_\hbar(c)$, then we must have $c_{j,l}(x_1,\xi) = \sum_{k\in\mathbb{Z}^d} c_{j,k,l}(x_1,\xi)$. Thus we define

$$\begin{aligned} c(x_1,x',\xi,l) &= \sum_{j\in\mathbb{Z}^d} c_j(x_1,\xi,l) e_j(x') \\ &= \sum_{j\in\mathbb{Z}^d} c_{j+l,l}(x_1,\xi) e_j(x') = \sum_{j\in\mathbb{Z}^d} c_{j,l}(x_1,\xi) e_{j-l}(x') := \sum_{j,k\in\mathbb{Z}^d} c_{j,k,l}(x_1,\xi) e_{j-l}(x'). \end{aligned}$$

It follows from (3.3) that c is a zero order symbol. Moreover,

$$\begin{aligned} c(x_1,x',\xi,l) - &a(x_1,x',\xi,l) b(x_1,x',\xi,l) \\ &= \sum_{j,k\in\mathbb{Z}^d} (c_{j,k,l}(x_1,\xi) - a_{j-k}(x_1,\xi,l) b_{k-l}(x_1,\xi,l)) e_{j-l}(x') \\ &= \sum_{j,k\in\mathbb{Z}^d} (c_{j,k,l}(x_1,\xi) - a_{j,k}(x_1,\xi) b_{k,l}(x_1,\xi)) e_{j-l}(x') \\ &\quad + (a_{j-k}(x_1,\xi,k) - a_{j-k}(x_1,\xi,l)) b_{k,l}(x_1,\xi) e_{j-l}(x'). \end{aligned}$$

From (3.4) and the inequality $\langle x+y\rangle \lesssim \langle x\rangle\langle y\rangle$, we obtain that the first difference is a zero order symbol with seminorms bounded by (appropriate) multiples of $\hbar$. For the second difference we use that the symbol is special and Proposition 2.2 (as in the proof of Proposition 3.7) to obtain

$$|D^\alpha_{x_1} D^\beta_\xi (a_{j-k}(x_1,\xi,k) - a_{j-k}(x_1,\xi,l))| \lesssim \hbar |k-l| A'_{M+2N} \langle j-k\rangle^{-2N},$$

any $N \geq 0$, whenever $\alpha + \beta \leq M$. Therefore,

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta}[(a_{j-k}(x_1, \xi, k) - a_{j-k}(x_1, \xi, l))b_{k,l}(x_1, \xi)]| \lesssim \hbar A'_{M+2N} B_{M+2N} \langle j-k \rangle^{-2N} \langle k-l \rangle^{-2N+1},$$

for any $N \geq 0$, whenever $\alpha + \beta \leq M$. As before, we conclude that the second difference is a zero order symbol with seminorms bounded by (appropriate) multiples of $\hbar$, and the result follows from Theorem 3.11. □

Proposition 3.14. *Let a and b be special zero order symbols over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$. Then,*

a). $D_{x_1}^{\alpha} D_{x'}^{\beta} D_{\xi}^{\gamma} a$, $a + b$, and ab are also special zero order symbols. Moreover, their seminorms are controlled by the products of the seminorms of a and b.

b). the function e^a is also a special zero order symbol,

c). for small enough $\hbar$, depending on a, the function $\log(1+\hbar a)$ is also a special zero order symbol.

Proof. This is a routine argument. □

Corollary 3.15. *Let a be a special zero order symbol over $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$. Then there exists $\hbar_0 > 0$, such that if $0 < \hbar \leq \hbar_0$, then $Op_\hbar(e^a)$ is an invertible operator in $H^s_{\delta,\hbar}(T)$ for $|\delta| \leq \delta_0$ and $s = 0, 2$. Moreover, the norms in $H^s_{\delta,\hbar}(T)$ of the operator and its inverse are uniformly bounded (in δ and $\hbar$).*

Proof. We know from Proposition 3.14 that $e^{\pm a}$ are zero order symbols. From Theorem 3.13 we have that

$$\|\mathrm{Op}_\hbar(e^a)\mathrm{Op}_\hbar(e^{-a}) - I\|_{H^s_{\delta,\hbar}(T) \to H^s_{\delta,\hbar}(T)}, \|\mathrm{Op}_\hbar(e^{-a})\mathrm{Op}_\hbar(e^a) - I\|_{H^s_{\delta,\hbar}(T) \to H^s_{\delta,\hbar}(T)} \lesssim \hbar.$$

This implies that $\mathrm{Op}_\hbar(e^a)$ has left and right inverses and the conclusion follows. □

3.4 Appendices

3.4.1 Some facts about weighted spaces

Let us recall the multiplier operator $\langle \hbar D\rangle^{-2} := \mathrm{Op}_\hbar(\langle \xi, \hbar t\rangle^{-2})$, i.e.

$$\begin{aligned}\langle \hbar D\rangle^{-2} f(x_1, x') &= \frac{1}{\hbar}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} e^{2\pi i x_1\xi/\hbar} e_k(x')\frac{1}{\xi^2+|\hbar k|^2+1}\widehat{f_k^\hbar}(\xi)d\xi \\ &= \sum_{k\in\mathbb{Z}^d}\left(\frac{1}{\hbar}\int_{\mathbb{R}}\frac{e^{2\pi i x_1\xi/\hbar}}{\xi^2+|\hbar k|^2+1}\widehat{f_k^\hbar}(\xi)d\xi\right)e_k(x')\end{aligned}$$

If $\lambda > 0$, then we have the classical Fourier transform

$$\int_{\mathbb{R}}\frac{e^{2\pi i x_1\xi}}{\xi^2+\lambda^2}d\xi = \frac{\pi}{\lambda}e^{-2\pi\lambda|x_1|},$$

so that the multiplier operator is also given by a convolution with the convergent Fourier series

$$\frac{\pi}{\hbar}\sum_{k\in\mathbb{Z}^d}\frac{1}{\langle \hbar k\rangle}e^{-2\pi\langle \hbar k\rangle|x_1|/\hbar}e_k(x'). \tag{3.5}$$

In the following results we study the properties of convolutions with functions of the form $e^{-\lambda|x_1|}$ to give a proof to Proposition 3.9.

Proposition 3.16. *Let $|\delta| \le \delta_0$ and $g \in L^1_{\delta_0}(\mathbb{R})$. For any $f \in L^2_\delta(\mathbb{R})$ we have that $\|f * g\|_{L^2_\delta} \lesssim \|f\|_{L^2_\delta}\|g\|_{L^1_{\delta_0}}$, where the constant of the inequality may depend on δ_0.*

Proof. Using that $\langle a\rangle\langle b\rangle^{-1} \lesssim \langle a-b\rangle$ and $\langle a\rangle \ge 1$ we obtain that

$$\begin{aligned}\langle x_1\rangle^\delta |f*g|(x_1) &\le \langle x_1\rangle^\delta(|f|*|g|)(x_1) \\ &= \int_{\mathbb{R}}\langle y_1\rangle^\delta|f(y_1)|\langle x_1-y_1\rangle^{\delta_0}|g(x_1-y_1)|[\langle x_1\rangle^\delta\langle y_1\rangle^{-\delta}\langle x_1-y_1\rangle^{-\delta_0}]dy_1 \\ &\lesssim (\langle\cdot\rangle^\delta|f|)*(\langle\cdot\rangle^{\delta_0}|g|)(x_1).\end{aligned}$$

The conclusion then follows from Young's inequality. □

Proposition 3.17. *Let $\lambda \geq 1$. Then $e^{-\lambda|x_1|} \in L^1_{\delta_0}(\mathbb{R})$ for all $\delta_0 \geq 0$, and satisfies $\|e^{-\lambda|x_1|}\|_{L^1_{\delta_0}} \lesssim \lambda^{-1}$, with the constant depending on δ_0.*

Proof. Integrating by parts we obtain that if $n \geq 0$ is an integer, then

$$\int_0^\infty e^{-\lambda x_1} x_1^n dx_1 = \frac{n!}{\lambda^{n+1}} \lesssim \frac{1}{\lambda}.$$

We have that $\langle x_1 \rangle^n \lesssim 1 + |x_1|^n$, so that if $\delta_0 \leq n$, then

$$\int_{\mathbb{R}} e^{-\lambda|x_1|} \langle x_1 \rangle^{\delta_0} dx_1 \leq \int_{\mathbb{R}} e^{-\lambda|x_1|} \langle x_1 \rangle^n dx_1 \lesssim \int_0^\infty e^{-\lambda x_1}(1 + x_1^n) dx_1 \lesssim \frac{1}{\lambda},$$

as we wanted to prove. □

Proposition 3.18. *Let $0 < \hbar \leq 1$, $|\delta| \leq \delta_0$, and $\lambda \geq 1$. If $f \in L^2_\delta(\mathbb{R})$ and we define $T_\lambda f := e^{-2\pi\lambda|x_1|/\hbar} * f$, then $T_\lambda f \in H^2_{\delta,\hbar}(\mathbb{R})$ and $\|(\hbar D_{x_1})^m T_\lambda f\|_{L^2_\delta} \lesssim \hbar\lambda^{m-1}\|f\|_{L^2_\delta}$ for $0 \leq m \leq 2$.*

Proof. Differentiating we observe that

$$\hbar D_{x_1}(e^{-2\pi\lambda|x_1|/\hbar}) = \lambda i \mathrm{sgn}(x_1) e^{-2\pi\lambda|x_1|/\hbar}, \quad (\hbar D_{x_1})^2(e^{-2\pi\lambda|x_1|/\hbar}) = \frac{\hbar\lambda}{\pi}\delta_0 - \lambda^2 e^{-2\pi\lambda|x_1|/\hbar},$$

so that we have

$$\hbar D_{x_1} T_\lambda f = \lambda i(\mathrm{sgn}(x_1) e^{-2\pi\lambda|x_1|/\hbar}) * f, \quad (\hbar D_{x_1})^2 T_\lambda f = \frac{\hbar\lambda}{\pi} f - \lambda^2 T_\lambda f.$$

From Proposition 3.16 and Proposition 3.17 we obtain that

$$\|T_\lambda f\|_{L^2_\delta} \lesssim \frac{\hbar}{\lambda}\|f\|_{L^2_\delta}, \quad \|\hbar D_{x_1} T_\lambda f\|_{L^2_\delta} \lesssim \hbar\|f\|_{L^2_\delta}, \quad \|(\hbar D_{x_1})^2 T_\lambda f\|_{L^2_\delta} \lesssim \hbar\lambda\|f\|_{L^2_\delta}.$$

□

Proposition 3.9. *Let $0 < \hbar \le 1$ and let $|\delta| \le \delta_0$. The differential operator $\langle \hbar D\rangle^2 : H^2_{\delta,\hbar}(T) \to L^2_\delta(T)$ and the multiplier operator $\langle \hbar D\rangle^{-2} : L^2_\delta(T) \to H^2_{\delta,\hbar}(T)$ are uniformly bounded (in δ and $\hbar$) operators, and inverses to each other. The bounds of the operators may depend in δ_0.*

Proof. It is clear that the differential operator $\langle \hbar D\rangle^2 : H^2_{\delta,\hbar}(T) \to L^2_\delta(T)$ is a bounded operator. Using the notation of Proposition 3.18, we get from (3.5) that

$$F(x_1, x') := \langle \hbar D\rangle^{-2} f(x_1, x') = \frac{\pi}{\hbar} \sum_{k\in\mathbb{Z}^d} \frac{1}{\langle \hbar k\rangle} T_{\langle \hbar k\rangle} f_k(x_1) e_k(x').$$

Moreover, we know that

$$\|F\|^2_{H^2_{\delta,\hbar}(T)} \simeq \sum_{k\in\mathbb{Z}^d} \langle \hbar k\rangle^4 \|F_k\|^2_{L^2_\delta(\mathbb{R})} + \langle \hbar k\rangle^2 \|\hbar D_{x_1} F_k\|^2_{L^2_\delta(\mathbb{R})} + \|(\hbar D_{x_1})^2 F_k\|^2_{L^2_\delta(\mathbb{R})}.$$

Therefore, the bound $\|F\|^2_{H^2_{\delta,\hbar}(T)} \lesssim \|f\|^2_{L^2_\delta(T)}$ follows from Proposition 3.18. We have proven that both of these maps are uniformly bounded. The fact that these maps are inverses to each other on $\mathcal{S}(T)$, together with the density of $\mathcal{S}(T)$ in $L^2_\delta(T)$ and $H^s_{\delta,\hbar}(T)$, implies the desired result. $\square$

3.4.2 Some facts about semiclassical pseudodifferential operators over $\mathbb{R}$

In the results above we needed some quantitative results for the bounds of the symbols and operators over $\mathbb{R}$. They are implicitly hinted in the literature, but, for the sake of completeness, we state them explicitly. We start with the operator bounds for zero order pseudodifferential operators. To avoid unnecessary notation, we denote $x_1 \in \mathbb{R}$ simply by x.

Proposition 3.8 ([Sal06])**.** *Let $0 < \hbar \le 1$. Let $a(x_1, \xi; \hbar)$ be a semiclassical zero order symbol over $\mathbb{R} \times \mathbb{R}$. For any $\delta \in \mathbb{R}$ the operator $Op_\hbar(a)$ is bounded in $L^2_\delta(\mathbb{R})$. Moreover, if $|\delta| \le \delta_0$, then the operator norms $\| Op_\hbar(a)\|_{L^2_\delta(\mathbb{R})\to L^2_\delta(\mathbb{R})}$ are uniformly bounded (in δ and $\hbar$) by a multiple (depending on δ_0) of some seminorm of a.*

Proof. Let us write

$$\mathrm{Op}_\hbar(a)f(x) := \frac{1}{\hbar}\int_{\mathbb{R}} e^{2\pi i x\xi/\hbar} a(x,\xi)\widehat{f^\hbar}(\xi)d\xi = \int_{\mathbb{R}} e^{2\pi i x\xi} a(x,\hbar\xi)\widehat{f}(\xi)d\xi.$$

The symbols $a_\hbar(x,\xi) := a(x,\hbar\xi)$ satisfy the same seminorm estimates $|D_x^\alpha D_\xi^\beta a_\hbar| \leq A_M$, whenever $\alpha+\beta \leq M$. Therefore, it suffices to prove this estimate for the case $\hbar = 1$. The case $\delta = 0$, i.e. in $L^2(\mathbb{R})$, is the Calderón–Vaillancourt theorem, and the bound for it in terms of the seminorms is stated in [Ste93] at the end of Section 2.4, Chapter 6, or Section 4.5 in [Zwo12] (in the semiclassical setting for the Weyl quantization, but the method of proof is the same). We prove first the case $\delta > 0$ for $\delta = 2n$, with n a positive integer. Using the identities

$$\langle x\rangle^{2n} e^{2\pi i x\xi} = \langle D_\xi\rangle^{2n} e^{2\pi i x\xi}, \quad D_\xi^k \widehat{f}(\xi) = (-1)^k \widehat{(x^k f)}(\xi),$$

and integrating by parts we obtain that if $f \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}\langle x\rangle^{2n}\mathrm{Op}(a)f &= \int_{\mathbb{R}} \langle D_\xi\rangle^{2n}(e^{2\pi i x\xi})a(x,\xi)\widehat{f}(\xi)d\xi \\ &= \int_{\mathbb{R}} e^{2\pi i x\xi}\langle D_\xi\rangle^{2n}(a(x,\xi)\widehat{f}(\xi))d\xi \\ &= \sum_{k=0}^{2n}\int_{\mathbb{R}} e^{2\pi i x\xi}a_k(x,\xi)\widehat{(x^k f)}(\xi)d\xi = \sum_{k=0}^{2n}\mathrm{Op}(a_k)(x^k f),\end{aligned}$$

for some zero order symbols $a_k(x,\xi)$, with seminorms controlled by those of a. From this and the Calderón–Vaillancourt theorem we get that

$$\|\langle x\rangle^{2n}\mathrm{Op}(a)f\|_{L^2(\mathbb{R})} \leq \sum_{k=0}^{2n}\|\mathrm{Op}(a_k)(x^k f)\|_{L^2(\mathbb{R})} \lesssim \sum_{k=0}^{2n}\|x^k f\|_{L^2(\mathbb{R})} \lesssim \|\langle x\rangle^{2n} f\|_{L^2(\mathbb{R})},$$

with the constant of the inequality depending on n and some seminorm of a. We have shown that $\mathrm{Op}(a)$ is bounded on $L^2_{2n}(\mathbb{R})$. The intermediate values $0 < \delta < 2n$ are obtained by complex interpolation.

Now, let us consider the case $\delta < 0$ for $\delta = -2n$, with n a positive integer. Integrating by parts we obtain that if $f \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}\langle x\rangle^{-2n}\mathrm{Op}(a)\langle x\rangle^{2n}f &= \langle x\rangle^{-2n}\int_{\mathbb{R}} e^{2\pi i x\xi}a(x,\xi)\langle D_\xi\rangle^{2n}\widehat{f}(\xi)d\xi \\ &= \langle x\rangle^{-2n}\int_{\mathbb{R}}\langle D_\xi\rangle^{2n}(e^{2\pi i x\xi}a(x,\xi))\widehat{f}(\xi)d\xi = \mathrm{Op}(\widetilde{a})f,\end{aligned}$$

for some zero order symbol $\widetilde{a}(x,\xi)$, with seminorms controlled by those of a. This identity can be rewritten as $\langle x\rangle^{-2n}\mathrm{Op}(a) = \mathrm{Op}(\widetilde{a})\langle x\rangle^{-2n}$, and so the Calderón–Vaillancourt theorem gives the boundedness on $L^2_{-2n}(\mathbb{R})$. Again, the intermediate values $-2n < \delta < 0$ are obtained by complex interpolation. □

We also prove the following result for the symbol of the composition, which we used in the proof of Theorem 3.13.

Proposition 3.19. *Let $a(x,\xi;\hbar)$ and $b(x,\xi;\hbar)$ be symbols over $\mathbb{R}\times\mathbb{R}$ satisfying*

$$|D_x^\alpha D_\xi^\beta a| \le A_M, \quad |D_x^\alpha D_\xi^\beta b| \le B_M,$$

whenever $\alpha+\beta \le M$. If $c = c(x_1,\xi;\hbar)$ is the symbol such that $Op_\hbar(c) = Op_\hbar(a)Op_\hbar(b)$, then there exists some K such that for any $M \ge 0$ the symbol satisfies

$$|D_x^\alpha D_\xi^\beta c| \lesssim A_{K+M}B_{K+M}, \quad |D_x^\alpha D_\xi^\beta (c-ab)| \lesssim \hbar A_{K+M}B_{K+M},$$

whenever $\alpha+\beta \le M$, where the constants of the inequalities may depend on M but are independent of $\hbar$.

Proof. Proceeding as in [Ste93], see Chapter 6, Section 3, it suffices to show the estimates for compactly supported symbols and prove that these are independent of the size of the support. Let us recall the integral kernel representation of a pseudodifferential operator,

$$\mathrm{Op}_\hbar(s)f(x) = \frac{1}{\hbar}\int_{\mathbb{R}} e^{2\pi i x\xi/\hbar}s(x,\xi)\widehat{f^\hbar}(\xi)d\xi = \frac{1}{\hbar}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{2\pi i(x-y)\xi/\hbar}s(x,\xi)d\xi\right)f(y)dy.$$

Then, the composition has an integral kernel representation given by

$$\begin{aligned}\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(b)f(x) &= \frac{1}{\hbar}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{2\pi i(x-y)\xi/\hbar}a(x,\xi)d\xi\right)\mathrm{Op}_\hbar(b)f(y)dy \\ &= \frac{1}{\hbar}\int_{\mathbb{R}}\left(\frac{1}{\hbar}\int_{\mathbb{R}^2} e^{2\pi i(x-y)\xi/\hbar}e^{2\pi i(y-z)\eta/\hbar}a(x,\xi)b(y,\eta)d\xi d\eta dy\right)f(z)dz \\ &= \frac{1}{\hbar}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{2\pi i(x-z)\eta/\hbar}c(x,\eta)d\eta\right)f(z)dz,\end{aligned}$$

and therefore the symbol of $\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(b)$ is equal to

$$\begin{aligned}c(x,\eta) :=& \frac{1}{\hbar}\int_{\mathbb{R}^2} e^{2\pi i(x-y)(\xi-\eta)/\hbar}a(x,\xi)b(y,\eta)d\xi dy \\ =& \frac{1}{\hbar}\int_{\mathbb{R}^2} e^{-2\pi iy\xi/\hbar}a(x,\eta+\xi)b(x+y,\eta)d\xi dy.\end{aligned}$$

From the inversion formula we have that

$$a(x,\eta) = \frac{1}{\hbar}\int_{\mathbb{R}^2} e^{-2\pi iy\xi/\hbar}a(x,\eta+\xi)d\xi dy,$$

which implies that

$$c(x,\eta) - a(x,\eta)b(x,\eta) = \frac{1}{\hbar}\int_{\mathbb{R}^2} ye^{-2\pi iy\xi/\hbar}a(x,\eta+\xi)\cdot\frac{b(x+y,\eta)-b(x,\eta)}{y}d\xi dy.$$

Now we use the identity

$$\frac{D_\xi\langle\hbar D_\xi\rangle^2}{\langle y\rangle^2}e^{-2\pi iy\xi/\hbar} = -\frac{1}{\hbar}ye^{-2\pi iy\xi/\hbar}$$

and integrate by parts to obtain that

$$\begin{aligned}c(x,\eta) - a(x,\eta)b(x,\eta) &= \int_{\mathbb{R}^2} e^{-2\pi iy\xi/\hbar}(D_\xi\langle\hbar D_\xi\rangle^2 a(x,\eta+\xi))\left(\frac{b(x+y,\eta)-b(x,\eta)}{y\langle y\rangle^2}\right)d\xi dy \\ &=: \int_{\mathbb{R}^2} e^{-2\pi iy\xi/\hbar}A(x,\eta,\xi)B(x,\eta,y)d\xi dy.\end{aligned}$$

Let us observe that the integral above is absolutely convergent because A has compact support in ξ and B is integrable in y. Therefore, we can exchange the order of integration and obtain that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2} e^{-2\pi i y\xi/\hbar} A(x,\eta,\xi)B(x,\eta,y)d\xi dy\Big| \\
&\qquad = \hbar\Big|\int_{\mathbb{R}^2} e^{-2\pi i y\mu} A(x,\eta,\hbar\mu)B(x,\eta,y)dyd\mu\Big| \\
&\qquad = \hbar\Big|\int_{\mathbb{R}} A(x,\eta,\hbar\mu)B(x,\eta,\widehat{\mu})d\mu\Big| \lesssim \hbar\|A(x,\eta,\cdot)\|_{L^\infty(\mathbb{R})}\|\langle D_y\rangle^2 B(x,\eta,\cdot)\|_{L^1(\mathbb{R})},
\end{aligned}$$

where we used in the last inequality that

$$\int_{\mathbb{R}} |\widehat{f}(\mu)|d\mu = \int_{\mathbb{R}} \frac{|\langle\mu\rangle^2 \widehat{f}(\mu)|}{\langle\mu\rangle^2}d\mu = \int_{\mathbb{R}} \frac{|\widehat{\langle D\rangle^2 f}(\mu)|}{\langle\mu\rangle^2}d\mu \lesssim \|\widehat{\langle D\rangle^2 f}\|_{L^\infty(\mathbb{R})} \lesssim \|\langle D\rangle^2 f\|_{L^1(\mathbb{R})}.$$

Similarly, for $\alpha+\beta \le M$ we can bound

$$\begin{aligned}
&|D_x^\alpha D_\eta^\beta(c-ab)| \\
&\qquad \lesssim \hbar\Big|\int_{\mathbb{R}} D_x^\alpha D_\eta^\beta (A(x,\eta,\hbar\mu)B(x,\eta,\widehat{\mu}))d\mu\Big| \\
&\qquad \lesssim \hbar \sup_{\alpha_0+\beta_0\le M} \|D_x^{\alpha_0} D_\eta^{\beta_0} A(x,\eta,\cdot)\|_{L^\infty(\mathbb{R})} \sup_{\substack{\alpha_1+\beta_1\le M\\ \gamma_1\le 2}} \|D_x^{\alpha_1} D_\eta^{\beta_1} D_y^{\gamma_1} B(x,\eta,\cdot)\|_{L^1(\mathbb{R})}.
\end{aligned}$$

To finish the estimate we use that

$$|D_x^\alpha D_\eta^\beta A(x,\eta,\xi)| \lesssim A_{M+3}, \quad |D_x^\alpha D_\eta^\beta D_y^\gamma B(x,\eta,y)| \lesssim \frac{B_{M+3}}{\langle y\rangle^2},$$

whenever $\alpha+\beta \le M$ and $\gamma \le 2$. The differential inequalities for the difference $c-ab$ imply the results for the symbol c, and this completes the proof. □

CHAPTER 4

CONJUGATION AND CARLEMAN ESTIMATE

The results of this chapter are an adaptation of those from [Sal06] in the case of $\mathbb{R}^d$ to the case of the cylinder $T = \mathbb{R} \times \mathbb{T}^d$. As briefly mentioned in Chapter 1, the proof of the magnetic Carleman estimate Theorem 1.2 is reduced to the case with no potentials. The proof of this Carleman estimate in [KSU11a] for $\mathbb{R} \times M_0$, where M_0 is a Riemannian manifold with boundary, is realized by an eigenfunction expansion and the solution of first order linear constant coefficient ODEs. This can be carried out in the exact same way for the torus $\mathbb{T}^d$, so we have the following result.

Theorem 4.1 ([KSU11a]). *Let $\delta > 1/2$. There exists $\tau_0 \geq 1$ such that if $|\tau| \geq \tau_0$ and $\tau^2 \notin Spec(-\Delta_{g_0})$, then for any $f \in L^2_\delta(T)$ there exists a unique $u \in H^1_{-\delta}(T)$ which solves*

$$e^{2\pi\tau x_1} D^2 e^{-2\pi\tau x_1} u = f.$$

Moreover, this solution is in $H^2_{-\delta}(T)$ and satisfies the estimates

$$\|u\|_{H^s_{-\delta}(T)} \lesssim |\tau|^{s-1} \|f\|_{L^2_\delta(T)},$$

for $0 \leq s \leq 2$, with the constant of the inequality independent of τ.

Remark. *It is important to note that the constant in the inequality only requires the condition that τ^2 does not belong to $Spec(-\Delta_{g_0})$; it is not necessary to ensure any distance condition to the spectrum. We can easily see that the uniqueness would fail if $\tau^2 \in Spec(-\Delta_{g_0})$ as $u = e_m(x') \in H^2_{-\delta}(T)$, with $\tau^2 = |m|^2$, is a solution of the homogeneous problem.*

The theorem above allows to define the operator $G_\tau : L^2_\delta(T) \to H^2_{-\delta}(T)$ by $G_\tau f := u$, so that $\Delta_\tau G_\tau = I$ on $L^2_\delta(T)$, where $\Delta_\tau = e^{2\pi\tau x_1} D^2 e^{-2\pi\tau x_1}$.

The reduction from our problem to this one is accomplished through a conjugation by two invertible pseudodifferential operators, i.e. essentially through the construction of an integrating factor. The construction of these operators is the main content of this chapter.

Let us consider the relevant terms from the expression $e^{2\pi\tau x_1}H_{V,W}e^{-2\pi\tau x_1}$:

$$\Delta_\tau := e^{2\pi\tau x_1}D^2e^{-2\pi\tau x_1} = D_{x_1}^2 + 2i\tau D_{x_1} - \tau^2 + D_{x'}^2,$$

$$V_\tau := e^{2\pi\tau x_1}(V\cdot D)e^{-2\pi\tau x_1} = e^{2\pi\tau x_1}(FD_{x_1} + G\cdot D_{x'})e^{-2\pi\tau x_1} = F(D_{x_1}+i\tau) + G\cdot D_{x'}.$$

Remark. *Observe that we have absorbed the negative sign of the Laplacian into the definition of Δ_τ.*

If we use semiclassical notation, with $\hbar = 1/\tau$ a small parameter, we can denote

$$\Delta_\hbar := \tau^{-2}\Delta_\tau = \hbar^2D_{x_1}^2 + 2i\hbar D_{x_1} - 1 + \hbar^2D_{x'}^2, \qquad V_\hbar := \tau^{-1}V_\tau = F(\hbar D_{x_1}+i) + G\cdot\hbar D_{x'}.$$

Equivalently, we could have defined

$$\Delta_\hbar := e^{2\pi x_1/\hbar}(\hbar D)^2e^{-2\pi x_1/\hbar}, \qquad V_\hbar := e^{2\pi x_1/\hbar}[V\cdot(\hbar D)]e^{-2\pi x_1/\hbar}.$$

Then we have that $\hbar^2(\Delta_\tau + 2V_\tau) = \Delta_\hbar + 2\hbar V_\hbar$. A significant part of this chapter is devoted to prove that we can conjugate this operator into the Laplacian plus a suitable error, as we state next. This construction follows closely the ideas from [Sal06].

Theorem 4.2. *Let $1/2 < \delta < 1$. Let V satisfy $(\star)$. There are $\varepsilon > 0$ and $0 < \hbar_0 \le 1$ such that for $0 < |\hbar| \le \hbar_0$ there exist zero order semiclassical pseudodifferential operators A, B, R over the cylinder T, so that the following conjugation identity holds,*

$$(\Delta_\hbar + 2\hbar V_\hbar)A = B\Delta_\hbar + \hbar^{1+\varepsilon}R.$$

Moreover, the operators A and B are invertible, uniformly bounded (in $\hbar$) together with its inverses in $H^s_{\pm\delta,\hbar}(T)$, for $s = 0, 2$, and $R : L^2_{-\delta}(T) \to L^2_\delta(T)$ is uniformly bounded (in $\hbar$).

With the semiclassical notation we have that

$$\Delta_\hbar = \mathrm{Op}_\hbar(\xi^2 + 2i\xi - 1 + |\hbar t|^2) = \mathrm{Op}_\hbar((\xi + i)^2 + |\hbar t|^2).$$

Moreover, from Proposition 3.6 we have that

$$\begin{aligned}
\Delta_\hbar A &= (\hbar^2 D_{x_1}^2 + 2i\hbar D_{x_1} - 1 + \hbar^2 D_{x'}^2)A \\
&= \mathrm{Op}_\hbar(\xi^2 a + 2\hbar\xi D_{x_1} a + \hbar^2 D_{x_1}^2 a) + 2i\mathrm{Op}_\hbar(\xi a + \hbar D_{x_1} a) - \mathrm{Op}_\hbar(a) \\
&\quad + \mathrm{Op}_\hbar(|\hbar t|^2 a + 2\hbar(\hbar t \cdot D_{x'} a) + \hbar^2 D_{x'}^2 a) \\
&= \mathrm{Op}_\hbar([(\xi + i)^2 + |\hbar t|^2] a) + 2\hbar\mathrm{Op}_\hbar((\xi + i) D_{x_1} a + \hbar t \cdot D_{x'} a) + \hbar^2 \mathrm{Op}_\hbar(D^2 a) \\
&= A\Delta_\hbar + 2\hbar\mathrm{Op}_\hbar((\xi + i) D_{x_1} a + \hbar t \cdot D_{x'} a) + \hbar^2 \mathrm{Op}_\hbar(D^2 a),
\end{aligned}$$

$$\begin{aligned}
V_\hbar A &= (F(\hbar D_{x_1} + i) + G \cdot \hbar D_{x'})A \\
&= \mathrm{Op}_\hbar((\xi + i)Fa + \hbar F \cdot D_{x_1} a) + \mathrm{Op}_\hbar(\hbar t \cdot Ga + \hbar G \cdot D_{x'} a) \\
&= \mathrm{Op}_\hbar([(\xi + i)F + \hbar t \cdot G] a) + \hbar\mathrm{Op}_\hbar(V \cdot Da),
\end{aligned}$$

so, we obtain

$$\begin{aligned}
(\Delta_\hbar + 2\hbar V_\hbar)A = {} & A\Delta_\hbar + 2\hbar\mathrm{Op}_\hbar((\xi + i) D_{x_1} a + \hbar t \cdot D_{x'} a + (\xi + i)Fa + \hbar t \cdot Ga) \\
& + \hbar^2 \mathrm{Op}_\hbar(D^2 a + 2V \cdot Da).
\end{aligned}$$

If a has nice properties, then the last operator already has the form we look for the remainder term in Theorem 4.2. Then, roughly speaking, we are left to make the operator

$$2\hbar\mathrm{Op}_\hbar((\xi + i) D_{x_1} a + \hbar t \cdot D_{x'} a + (\xi + i)Fa + \hbar t \cdot Ga) \tag{4.1}$$

suitably small. In order to do that, we split it in two parts: we make one part of it vanish and the remainder will be supported on a set where the operator $\Delta_\hbar$ is elliptic. The remainder will be subsumed by the expression $A\Delta_\hbar$ becoming into $B\Delta_\hbar$.

In order for the operator A to be invertible it is usual to look for the symbol to be of the form $a = e^{-u}$, so that the symbol (4.1) becomes

$$(\xi+i)D_{x_1}a+\hbar t\cdot D_{x'}a+(\xi+i)Fa+\hbar t\cdot Ga = a[-(\xi+i)D_{x_1}u-\hbar t\cdot D_{x'}u+(\xi+i)F+\hbar t\cdot G],$$

leaving us to solve the equation

$$(\xi+i)D_{x_1}u+\hbar t\cdot D_{x'}u = (\xi+i)F+\hbar t\cdot G, \tag{4.2}$$

for $t\in\mathbb{Z}^d$. In the following sections we deal with appropriate existence and uniqueness of solutions to these equations, as well as with its estimates. Recall that the symbol of $\Delta_\hbar$ is $(\xi+i)^2+|\hbar t|^2$, and note that this vanishes if and only if $\xi=0$ and $|\hbar t|=1$. The symbol is elliptic away from this set. Therefore, for the construction of the solution to the equation we will be mostly interested in working in a neighborhood of this vanishing set.

Finally, let us mention some difficulties of our problem which do not seem to be present in the Euclidean setting, like in [Sun93] or [Sal06]. Observe that (4.2) can be rewritten as

$$(\xi+i,\hbar t)\cdot Du = (\xi+i,\hbar t)\cdot V.$$

Near the vanishing set of the symbol of $\Delta_\hbar$, i.e. $\xi=0$ and $|\hbar t|=1$, this equation resembles a higher dimensional version of the $\overline{\partial}$–equation. It has been usual to reduce all such equations to a $\overline{\partial}$–equation through a rotation, see for instance [Sun93] or [Sal06]. In our setting this is not immediately possible, in part because $\mathbb{Z}^d$ does not admit non–trivial rotations. One way to try to remedy this could be as follows. In [Wei11], it is shown that for any $k\in\mathbb{Z}^d\setminus\{0\}$ there is a matrix $A\in SL_n(\mathbb{Z}^d)$, i.e. a linear automorphism of $\mathbb{T}^d$, such that $Ak=\gcd(k)e_1$. This allows to make a change of variables so that the directional derivative $\hbar k\cdot D_{x'}$ becomes an “exact” partial derivative $\hbar\,\mathrm{gcd}(k)D_{y_1'}$, and so the equation reduces from $\mathbb{R}\times\mathbb{T}^d$ to $(\mathbb{R}\times\mathbb{T}^1)\times\mathbb{T}^{d-1}$, where the last $(d-1)$ toroidal variables do not intervene. In this case, the coefficient $\hbar\,\mathrm{gcd}(k)$ plays a role in the estimates, and this seems difficult to handle. In addition to the previous inconvenient, we will to need to estimate of the differences of the solutions when k varies; since it is not clear how the change of variables (i.e. the matrix) depends on k, we will refrain from using this idea. Instead, we will proceed using the decomposition in Fourier series.

4.1 Equation

We start recalling the assumptions on the magnetic potential $V=(F,G)$:

$$V\in C_c^\infty(T),\quad \mathrm{supp}(V)\subseteq[-R,R]\times\mathbb{T}^d,\quad \int_{\mathbb{R}} V(x_1,x')dx_1=0 \text{ for all } x'\in\mathbb{T}^d. \tag{$\star$}$$

Under these conditions, we will see that there is no difference in working over $\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}\times\mathbb{Z}^d$ or $\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}\times\mathbb{R}^d$, and so, to avoid suggesting that there is something special about the former, we will work over the latter. In this section we state the properties of the solution the equation (4.2) and motivate the reasons for assuming ($\star$).

Let us recall that for $(u,v)\in\mathbb{R}\times\mathbb{R}^d$ and $(u,v,w)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ we use the notation $\langle u,v\rangle := (1+u^2+|v|^2)^{1/2}$ and $\langle u,v,w\rangle := (1+u^2+|v|^2+|w|^2)^{1/2}$.

Theorem 4.3. *Let $\hbar>0$ and let $V=(F,G)$ satisfy ($\star$). For fixed $(\xi,t)\in\mathbb{R}\times\mathbb{R}^d$, the equation*

$$(\xi+i)D_{x_1}u+\hbar t\cdot D_{x'}u=(\xi+i)F+\hbar t\cdot G, \tag{4.2}$$

has a unique solution $u(\cdot,\xi,t;\hbar)\in C^\infty(T)$ with the decay condition $\|u(x_1,\cdot,\xi,t)\|_{L^\infty(\mathbb{T}^d)}\to 0$ as $x_1\to\pm\infty$. Moreover, we have that $u(\cdot,t)\in C^\infty(\mathbb{R}\times\mathbb{T}^d\times\mathbb{R})$ and it satisfies the bounds

$$|D_{x_1}^\alpha D_{x'}^\beta D_\xi^\gamma u|\lesssim\langle\xi,\hbar t\rangle,\quad |D_{x_1}^\alpha D_{x'}^\beta D_\xi^\gamma(u(\cdot,\xi,t_1)-u(\cdot,\xi,t_2))|\lesssim\hbar|t_1-t_2|\langle\xi,\hbar t_1,\hbar t_2\rangle. \tag{4.3}$$

For $|x_1| \geq 2R$ we have the linear decay bound

$$|D_{x_1}^{\alpha} D_{x'}^{\beta} D_{\xi}^{\gamma} u| \lesssim \frac{\langle \xi, \hbar t \rangle}{|x_1|}. \tag{4.4}$$

The constants of the inequalities may depend on α, β, γ, d, R, $\|V\|_{W^{N,1}(T)}$ for some $N = N(\alpha, \beta, d)$, but are independent of $\hbar$, ξ, t.

Remark. *Under some conditions on t, like $t \in \mathbb{Z}^d$ or other arithmetic properties, it may be possible to show that $u(\cdot, \xi, t) \in \mathcal{S}(T)$. We do not need such a strong result, so we do not intend to prove it.*

The equation (4.2) has constant coefficients in (x_1, x'), so we can decompose u, F, G in its Fourier series

$$u = \sum_{m \in \mathbb{Z}^d} u_m(x_1, \xi, t; \hbar) e_m(x'),$$

$$F(x_1, x') = \sum_{m \in \mathbb{Z}^d} F_m(x_1) e_m(x'), \quad G(x_1, x') = \sum_{m \in \mathbb{Z}^d} G_m(x_1) e_m(x'),$$

and look for u_m to solve the equation

$$(\xi + i) D_{x_1} u_m + \hbar t \cdot m u_m = (\xi + i) F_m + \hbar t \cdot G_m. \tag{4.5}$$

For instance, in order to prove (4.3), it would suffice to show inequalities of the form

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m| \lesssim \langle m \rangle^{-M} \langle \xi, \hbar t \rangle,$$

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta} (u_m(\cdot, \cdot, t_1) - u_m(\cdot, \cdot, t_2))| \lesssim \hbar \langle m \rangle^{-M} |t_1 - t_2| \langle \xi, \hbar t_1, \hbar t_2 \rangle,$$

for some sufficiently large M. We will prove these in a later section.

Before we proceed, let us motivate the conditions that we are requiring for V. Considering the Fourier transform (no longer semiclassical) in (4.5) gives that

$$\widehat{u_m}(\eta) = \frac{(\xi + i)\widehat{F_m}(\eta) + \hbar t \cdot \widehat{G_m}(\eta)}{(\xi + i)\eta + \hbar t \cdot m}.$$

The denominator vanishes only if $\eta = 0$ and $\hbar t \cdot m = 0$. This suggests that the case $\hbar t \cdot m \neq 0$ may be less problematic than the case $\hbar t \cdot m = 0$. Indeed, we already see from (4.5) that there is not a unique solution, and even when defining such a solution it may not decay. The simplest way to avoid the problem of the denominator vanishing is to require $\widehat{F_m}(0) = \widehat{G_m}(0) = 0$, which is the same as the vanishing moments from ($\star$). Uniqueness and decay are not necessarily required, but we will see that the decay estimates play a role in the construction of the conjugation (namely on the properties of R) and the reconstruction procedure.

4.2 Lemmas: ODEs and calculus

In this section we prove some boundedness estimates for the solution of an ODE of the form (4.5), as well as some other necessary calculus facts. To avoid unnecessary notation, in this section we will denote the variable x_1 simply by x.

We start with the most elementary estimate for solutions of an ODE, and then improve it under the hypothesis of a vanishing moment. The reason why we will be dealing only with L^1 and L^∞ estimates is that these are useful to iterate and they relate through the Fundamental Theorem of Calculus (i.e. as a 1-dimensional version of the Gagliardo–Nirenberg–Sobolev inequality).

Lemma 4.4. *Let $a, \xi \in \mathbb{R}$, $a \neq 0$, and let $H \in \mathcal{S}(\mathbb{R})$. Consider the equation*

$$(\xi + i)D_x v + av = H.$$

Then there exists a unique solution in the sense of tempered distributions. Moreover, the *solution belongs to* $\mathcal{S}(\mathbb{R})$ *and satisfies the estimates*

$$\|v\|_{L^1} \leq \frac{\langle \xi \rangle}{|a|} \|H\|_{L^1}, \quad \|v\|_{L^\infty} \lesssim \|D_x v\|_{L^1} \lesssim \|H\|_{L^1}.$$

The constants in the inequality are independent of a*,* ξ*,* H*.*

Proof. Taking the Fourier transform yields that

$$\widehat{v}(\eta) = \frac{\widehat{H}(\eta)}{(\xi + i)\eta + a}.$$

We can bound the norm of the denominator by

$$|(\xi + i)\eta + a|^2 = \langle \xi \rangle^2 \eta^2 + 2\xi\eta a + a^2 = \left(\langle \xi \rangle \eta + \frac{\xi a}{\langle \xi \rangle} \right)^2 + \frac{a^2}{\langle \xi \rangle^2} \geq \frac{a^2}{\langle \xi \rangle^2} > 0.$$

Therefore, the denominator is a non-vanishing smooth function, and we obtain the existence and uniqueness in the sense of distributions. Moreover, the rapid decay of $\widehat{H}(\eta)$ and the bound for the denominator give that $v \in \mathcal{S}(\mathbb{R})$. Let $\mu := 2\pi i a/(\xi + i) = 2\pi a(1 + i\xi)/\langle \xi \rangle^2$, so that $\mathrm{Re}(\mu) = 2\pi a/\langle \xi \rangle^2$. The form of the solution depends on the sign of $\mathrm{Re}(\mu)$. The cases $a > 0$ and $a < 0$ are analogous, so we only consider one of these. If we assume that $a > 0$, then the solution is given by

$$v(x) = \frac{2\pi i}{\xi + i} \int_{-\infty}^{x} e^{-\mu(x-s)} H(s) ds,$$

as $\mathrm{Re}(\mu) = 2\pi a/\langle \xi \rangle^2 > 0$. This gives that

$$\begin{aligned} \|v\|_{L^1} &\leq \frac{2\pi}{\langle \xi \rangle} \int_{\mathbb{R}} \int_{-\infty}^{x} e^{-2\pi a(x-s)/\langle \xi \rangle^2} |H(s)| ds dx \\ &= \frac{2\pi}{\langle \xi \rangle} \int_{\mathbb{R}} \int_{s}^{\infty} e^{-2\pi a(x-s)/\langle \xi \rangle^2} |H(s)| dx ds = \frac{\langle \xi \rangle}{a} \|H\|_{L^1}. \end{aligned} \tag{4.6}$$

Using this together with the equation gives

$$\|D_x v\|_{L^1} \le \frac{1}{\langle \xi \rangle}(a\|v\|_{L^1} + \|H\|_{L^1}) \le \frac{\langle \xi \rangle + 1}{\langle \xi \rangle}\|H\|_{L^1} \lesssim \|H\|_{L^1},$$

as we wanted. The bound $\|v\|_{L^\infty} \lesssim \|D_x v\|_{L^1}$ follows from the Fundamental Theorem of Calculus and the fact that $v \in \mathcal{S}(\mathbb{R})$. □

Remark. *It is also possible to show that*

$$\|v\|_{L^\infty} \lesssim \frac{\langle \xi \rangle}{|a|}\|H\|_{L^\infty}.$$

The following result shows that a vanishing moment assumption allows to consider the missing case $a = 0$ and also gives an improvement of the L^1-estimates for the solution.

Lemma 4.5. *Let $a, \xi \in \mathbb{R}$ and let $H \in \mathcal{S}(\mathbb{R})$. The equation*

$$(\xi + i)D_x v + av = D_x H.$$

has a unique solution $v \in \mathcal{S}(\mathbb{R})$ and it satisfies

$$\|v\|_{L^1} \lesssim \|H\|_{L^1}, \quad \|v\|_{L^\infty} \lesssim \|D_x v\|_{L^1} \lesssim \|D_x H\|_{L^1}.$$

The constants in the inequality are independent of a, ξ, H. Moreover, if $a \neq 0$, then there exists $w \in \mathcal{S}(\mathbb{R})$ such that $v = D_x w$.

Proof. If $a = 0$ then $v = H/(\xi + i) \in \mathcal{S}(\mathbb{R})$, and the result follows immediately. For $a \neq 0$ the existence and uniqueness follow from Lemma 4.4. The cases $a > 0$ and $a < 0$ are analogous, so we only consider one of these. Assume that $a > 0$ and let $\mu := 2\pi a/(\xi + i)$ be as in the previous proof. Integrating by parts yields that

$$v(x) = \frac{2\pi i}{\xi + i}\int_{-\infty}^{x} e^{-\mu(x-s)} D_s H(s) ds = \frac{1}{\xi + i}\left(H(x) - \mu \int_{-\infty}^{x} e^{-\mu(x-s)} H(s) ds\right). \quad (4.7)$$

We use the estimate (4.6) for the integral term, to conclude that

$$\|v\|_{L^1} \leq \frac{1}{\langle\xi\rangle}\left(1+|\mu|\frac{\langle\xi\rangle^2}{a}\right)\|H\|_{L^1} \lesssim \|H\|_{L^1}.$$

The L^∞ bound follows trivially if $a = 0$, and from Lemma 4.4 if $a \neq 0$. Finally, if $a \neq 0$, then

$$v = \frac{1}{a}D_x(H - (\xi + i)v) = D_x w,$$

with $w = (H - (\xi + i)v)/a \in \mathcal{S}(\mathbb{R})$. □

Remark. *The solutions in* $C^1(\mathbb{R})$ *to the equation* $(\xi+i)D_x v + av = 0$ *are multiples of* $e^{-\mu x}$*. Therefore, there is also uniqueness under the weaker conditions* $v \in C^1(\mathbb{R})$ *and* $v(x) \to 0$ *as* $x \to \pm\infty$*.*

Remark. *From* (4.7) *it is also possible to show that*

$$\|v\|_{L^\infty} \leq \frac{1}{\langle\xi\rangle}\left(1+\frac{|\mu|\langle\xi\rangle^2}{a}\right)\|H\|_{L^\infty} \lesssim \|H\|_{L^\infty}.$$

In the proof of the main result of the next section we will remark why focusing only in the $L^\infty \to L^\infty$ *estimates may not be so convenient.*

In the following proposition we show that the exponential function appearing from the integrating factor of the differential equations is bounded, together with all its derivatives. In proving the boundedness results from the following section we will use this result, as well as the idea of the proof.

Lemma 4.6. *Let* $\eta, \xi \in \mathbb{R}$ *with* $\eta \geq 0$*. Then, for any polynomial* p *we have*

$$\left| e^{-i\eta/(\xi+i)} p\left(\frac{\eta}{\langle\xi\rangle^2}\right)\right| \leq C(p),$$

for some constant $C(p)$ *depending on the polynomial. Moreover,* $|D_\xi^\beta(e^{-i\eta/(\xi+i)})| \leq C_\beta$ *for any* $\beta \geq 0$*. The constants* $C(p)$ *and* C_β *are independent of* η *and* ξ*.*

Proof. Let $\mu = i\eta/(\xi+i) = \eta(1+i\xi)/\langle\xi\rangle^2$, so that $\mathrm{Re}(\mu) = \eta/\langle\xi\rangle^2 > 0$. The first inequality follows from the triangle inequality and the bound $e^{-x}x^n \leq n!$ for $x \geq 0$. In order to differentiate with respect to ξ, we first observe that $D_\xi \mu = -\mu/(2\pi i(\xi+i))$. We show by induction that

$$D_\xi^\beta(e^{-\mu}) = \frac{e^{-\mu}P_\beta(\mu)}{(\xi+i)^\beta},$$

where P_β is a polynomial of degree β, whose coefficients depend only on β. For $\beta = 0$ it is clear. Moreover,

$$D_\xi\left(\frac{e^{-\mu}P_\beta(\mu)}{(\xi+i)^\beta}\right) = e^{-\mu}\left(\frac{\mu}{2\pi i(\xi+i)}\frac{P_\beta(\mu)}{(\xi+i)^\beta} - \frac{\mu}{2\pi i(\xi+i)}\frac{P_\beta'(\mu)}{(\xi+i)^\beta} - \frac{\beta P_\beta(\mu)}{2\pi i(\xi+i)^{\beta+1}}\right).$$

Thus, by defining the polynomial $P_{\beta+1}(z) := (zP_\beta(z) - zP_\beta'(z) - \beta P_\beta)/(2\pi i)$ we complete the induction. Since $|1/(\xi+i)| \leq 1$ and P_β has degree β, we obtain that there exists some polynomial $\widetilde{P}_\beta$ of degree β such that

$$\left|\frac{P_\beta(\mu)}{(\xi+i)^\beta}\right| \leq \widetilde{P}_\beta\left(\frac{\eta}{\langle\xi\rangle^2}\right).$$

With this we conclude that

$$|D_\xi^\beta e^{-\mu}| = \left|\frac{e^{-\mu}P_\beta(\mu)}{(\xi+i)^\beta}\right| \leq \left|e^{-\mu}\widetilde{P}_\beta\left(\frac{\eta}{\langle\xi\rangle^2}\right)\right| \leq C_\beta.$$

□

4.3 Estimates for the solutions of the equations

The purpose of this section is to finally prove Theorem 4.3. We start by proving the estimates for the ODEs (4.5) which result from expanding in Fourier series the equation (4.2). We start with the following elementary result.

Proposition 4.7. *If $f \in \mathcal{S}(\mathbb{R})$ is such that $\int_{\mathbb{R}} f(x_1)dx_1 = 0$, then there exists a unique*

$g \in \mathcal{S}(\mathbb{R})$ such that $D_{x_1} g = f$. Moreover, if f is compactly supported, then so is g. Similarly, if $f \in \mathcal{S}(T)$ is such that $\int_{\mathbb{R}} f(x_1, x')dx_1 = 0$ for all $x' \in \mathbb{T}^d$, then there exists $g \in \mathcal{S}(T)$ such that $D_{x_1} g = f$. Moreover, its Fourier coefficients $f_k \in \mathcal{S}(\mathbb{R})$ satisfy that $\int_{\mathbb{R}} f_k(x_1) = 0$ and $D_{x_1} g_k = f_k$.

Proof. Let us first consider the case $f \in \mathcal{S}(\mathbb{R})$. The uniqueness follows from the Schwartz condition. For the existence we define

$$g(x_1) := \int_{-\infty}^{x_1} f(y)dy = -\int_{x_1}^{\infty} f(y)dy.$$

To show that $g \in \mathcal{S}(\mathbb{R})$ we use that if $L \geq 1$, then for any positive $m > 0$ we have

$$\int_{-\infty}^{-L} \frac{1}{\langle y \rangle^{m+1}} dy, \ \int_{L}^{\infty} \frac{1}{\langle y \rangle^{m+1}} dy \lesssim \frac{1}{L^m}.$$

If f were compactly supported, then the definition above shows that g shares the same support. For the case of T, the first part of the proof follows exactly as before. Moreover, by Fubini's theorem we have that

$$\int_{\mathbb{R}} f_k(x_1)dx_1 = \int_{\mathbb{T}^d} e_{-k}(x') \left(\int_{\mathbb{R}} f(x_1, x')dx_1 \right) dx' = 0,$$

and

$$D_{x_1} g_k(x_1) = \int_{\mathbb{T}^d} D_{x_1} g(x_1, x') e_{-k}(x')dx' = \int_{\mathbb{T}^d} f(x_1, x') e_{-k}(x')dx' = f_k(x_1).$$

□

If f and g are as in Proposition 4.7, then we define $D_{x_1}^{-1} f := g$.

Theorem 4.8. *Let $\hbar > 0$ and let $V = (F, G)$ satisfy $(\star)$. For fixed $(m, \xi, t) \in \mathbb{Z}^d \times \mathbb{R} \times \mathbb{R}^d$ the equation*

$$(\xi + i) D_{x_1} u_m + \hbar t \cdot m u_m = (\xi + i) F_m + \hbar t \cdot G_m \tag{4.5}$$

has a unique solution $u_m(\cdot,\xi,t;\hbar) \in C^1(\mathbb{R})$ with the decay condition $u_m(x_1,\xi,t) \to 0$ as $x_1 \to \pm\infty$. Moreover, we have that $u_m(\cdot,\xi,t) \in \mathcal{S}(\mathbb{R})$, $u_m(\cdot,\cdot,t) \in C^\infty(\mathbb{R}\times\mathbb{R})$, and for any $\alpha,\beta \geq 0$ it satisfies that $D_{x_1}^\alpha D_\xi^\beta u_m(\cdot,\xi,t) \in \mathcal{S}(\mathbb{R})$. If $\hbar t\cdot m = 0$, then u_m is supported on $|x_1| \leq R$. If $\hbar t \cdot m \neq 0$, then u_m vanishes in one of the components of $|x_1| > R$, and decays exponentially (depending on the product $\hbar t \cdot m$) on the other component. In addition, it satisfies the bounds

$$|D_{x_1}^\alpha D_\xi^\beta u_m| \lesssim \langle \xi, \hbar t\rangle \|D_{x_1}^\alpha V_m\|_{L^1}, \tag{4.8}$$

$$\begin{aligned} &|D_{x_1}^\alpha D_\xi^\beta (u_m(\cdot,t_1) - u_m(\cdot,t_2))| \\ &\qquad \lesssim \hbar|t_1-t_2|(\|D_{x_1}^\alpha V_m\|_{L^1} + \langle \xi, \hbar t_1, \hbar t_2\rangle |m| \|D_{x_1}^{\alpha-1} V_m\|_{L^1}). \end{aligned} \tag{4.9}$$

Moreover, for $|x_1| \geq 2R$ we have the linear decay bound

$$|D_{x_1}^\alpha D_\xi^\beta u_m(x_1)| \lesssim \frac{\langle \xi, \hbar t\rangle}{|x_1|} \|D_{x_1}^{\alpha-1} V_m\|_{L^1}. \tag{4.10}$$

The constants in the inequalities may depend on α, β, d, but are independent of $\hbar$, m, ξ, t, and R.

Proof. The uniqueness of such a solution follows from the remark after Lemma 4.5. From ($\star$) and Lemma 4.5 we obtain the existence and that it belongs to $\mathcal{S}(\mathbb{R})$. As in the previous proofs, let $\mu = 2\pi i(\hbar t\cdot m)/(\xi + i)$, so that $\mathrm{Re}(\mu) = 2\pi(\hbar t \cdot m)/\langle \xi \rangle^2$. We know that the form of the solution depends on the sign of $\mathrm{Re}(\mu)$; more explicitly, proceeding as in the proof of Lemma 4.5, we have that:

1. if $\hbar t \cdot m = 0$, then

$$u_m(x_1,\xi,t) = \frac{1}{\xi+i}[(\xi+i)D_{x_1}^{-1}F_m(x_1) + \hbar t \cdot D_{x_1}^{-1} G_m(x_1)],$$

2. if $\hbar t \cdot m > 0$, then

$$u_m(x_1, \xi, t) = \frac{1}{\xi + i}\Big[(\xi + i)D_{x_1}^{-1}F_m(x_1) + \hbar t \cdot D_{x_1}^{-1}G_m(x_1) - \mu \int_{-\infty}^{x_1} e^{-\mu(x_1 - y_1)}[(\xi + i)D_{y_1}^{-1}F_m(y_1) + \hbar t \cdot D_{y_1}^{-1}G_m(y_1)]dy_1\Big],$$

3. if $\hbar t \cdot m < 0$, then

$$u_m(x_1, \xi, t) = \frac{1}{\xi + i}\Big[(\xi + i)D_{x_1}^{-1}F_m(x_1) + \hbar t \cdot D_{x_1}^{-1}G_m(x_1) + \mu \int_{x_1}^{\infty} e^{-\mu(x_1 - y_1)}[(\xi + i)D_{y_1}^{-1}F_m(y_1) + \hbar t \cdot D_{y_1}^{-1}G_m(y_1)]dy_1\Big].$$

From Proposition 4.7 we know that $(\xi + i)D_{x_1}^{-1}F_m + \hbar t \cdot D_{x_1}^{-1}G_m$ is a smooth compactly supported function. In particular, this implies that in the first case the solution is compactly supported. In the second and third case, this implies that the solutions vanish if $x_1 < -R$ and $x_1 > R$, respectively, and are decaying exponentials if $x_1 > R$ and $x_1 < -R$, respectively. Moreover, all the terms involved ($\xi + i$, μ, $e^{-\mu(x_1 - y_1)}$, and $(\xi + i)D_{x_1}^{-1}F_m + \hbar t \cdot D_{x_1}^{-1}G_m$) are differentiable with respect to x_1 and ξ, and the possible different cases depend only on m and t (namely on the sign of $\hbar t \cdot m$), and not on x_1 or ξ. This implies that $u_m(\cdot, \cdot, t) \in C^\infty(\mathbb{R} \times \mathbb{R})$ and $D_{x_1}^\alpha D_\xi^\beta u_m(\cdot, \xi, t) \in \mathcal{S}(\mathbb{R})$.

To prove the estimates (4.8) we succesively differentiate the equation (4.5) to show that $D_{x_1}D_\xi u_m(\cdot, \xi, t)$ (which we know is in $\mathcal{S}(\mathbb{R})$) solves certain ODE, and then use the estimates for the unique solution in $\mathcal{S}(\mathbb{R})$ from Lemma 4.5. Differentiating the equation, we see that if $\alpha \geq 0$ then $D_{x_1}^\alpha u_m$ solves the equation

$$(\xi + i)D_{x_1}[D_{x_1}^\alpha u_m] + \hbar t \cdot m[D_{x_1}^\alpha u_m] = (\xi + i)D_{x_1}^\alpha F_m + \hbar t \cdot D_{x_1}^\alpha G_m. \tag{4.11}$$

By $(\star)$ and Lemma 4.5 we can bound

$$\|D_{x_1}^{\alpha} u_m\|_{L^1} \lesssim \langle \xi \rangle \|D_{x_1}^{\alpha-1} F_m\|_{L^1} + |\hbar t| \|D_{x_1}^{\alpha-1} G_m\|_{L^1} \lesssim \langle \xi, \hbar t \rangle \|D_{x_1}^{\alpha-1} V_m\|_{L^1}, \tag{4.12}$$

$$|D_{x_1}^{\alpha} u_m| \lesssim \|D_{x_1}^{\alpha+1} u_m\|_{L^1} \lesssim \langle \xi, \hbar t \rangle \|D_{x_1}^{\alpha} V_m\|_{L^1}. \tag{4.13}$$

Differentiating (4.11) with respect to ξ gives that $D_{x_1}^{\alpha} D_{\xi} u_m$ solves the equation

$$(\xi + i) D_{x_1}[D_{x_1}^{\alpha} D_{\xi} u_m] + \hbar t \cdot m[D_{x_1}^{\alpha} D_{\xi} u_m] = \frac{1}{2\pi i}(D_{x_1}^{\alpha} F_m - D_{x_1}^{\alpha+1} u_m). \tag{4.14}$$

By $(\star)$, Lemma 4.5, and (4.12) we can bound

$$\|D_{x_1}^{\alpha} D_{\xi} u_m\|_{L^1} \lesssim \|D_{x_1}^{\alpha-1} F_m\|_{L^1} + \|D_{x_1}^{\alpha} u_m\|_{L^1} \lesssim \langle \xi, \hbar t \rangle \|D_{x_1}^{\alpha-1} V_m\|_{L^1}, \tag{4.15}$$

$$|D_{x_1}^{\alpha} D_{\xi} u_m| \lesssim \|D_{x_1}^{\alpha+1} D_{\xi} u_m\|_{L^1} \lesssim \langle \xi, \hbar t \rangle \|D_{x_1}^{\alpha} V_m\|_{L^1}. \tag{4.16}$$

By induction on $\beta \geq 2$, differentiating (4.14) with respect to ξ gives that $D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m$ solves the equation

$$(\xi + i) D_{x_1}[D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m] + \hbar t \cdot m[D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m] = \frac{-\beta}{2\pi i} D_{x_1}^{\alpha+1} D_{\xi}^{\beta-1} u_m. \tag{4.17}$$

From (4.17), Lemma 4.5, and (4.15) we obtain

$$\|D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m\|_{L^1} \lesssim \|D_{x_1}^{\alpha} D_{\xi}^{\beta-1} u_m\|_{L^1} \lesssim \ldots \lesssim \|D_{x_1}^{\alpha} D_{\xi} u_m\|_{L^1} \lesssim \langle \xi, \hbar t \rangle \|D_{x_1}^{\alpha-1} V_m\|_{L^1}, \tag{4.18}$$

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m| \lesssim \|D_{x_1}^{\alpha+1} D_{\xi}^{\beta} u_m\|_{L^1} \lesssim \langle \xi, \hbar t \rangle \|D_{x_1}^{\alpha} V_m\|_{L^1}. \tag{4.19}$$

We have shown (4.8) through (4.13), (4.16), and (4.19). Now we prove (4.9). Let us denote $u_m^j(x_1, \xi) := u_m(x_1, \xi, t_j)$ and recall that $D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m^j(\cdot, \xi) \in \mathcal{S}(\mathbb{R})$ for any $\alpha, \beta \geq 0$. There are two cases to consider: when both products $\hbar t_j \cdot m$ vanish, and when at least one of them does not vanish. In the first case we have that

$$u_m^j(x_1,\xi) = \frac{1}{\xi+i}[(\xi+i)D_{x_1}^{-1}F_m(x_1) + \hbar t_j \cdot D_{x_1}^{-1}G_m(x_1)],$$

and so

$$u_m^1(x_1,\xi) - u_m^2(x_1,\xi) = \frac{\hbar(t_1-t_2)}{\xi+i} \cdot D_{x_1}^{-1}G_m(x_1).$$

It follows directly from this and the Fundamental Theorem of Calculus that

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta}(u_m^1 - u_m^2)| \lesssim \hbar|t_1-t_2| \|D_{x_1}^{\alpha} V_m\|_{L^1}. \tag{4.20}$$

Suppose now that $\hbar t_1 \cdot m \neq 0$. Substracting the equations (4.11) for $D_{x_1}^{\alpha} u_m^j$ we obtain that

$$\begin{aligned}(\xi+i)D_{x_1}[D_{x_1}^{\alpha}(u_m^1-u_m^2)] &+ \hbar t_2 \cdot m[D_{x_1}^{\alpha}(u_m^1-u_m^2)] \\ &= \hbar(t_1-t_2)\cdot D_{x_1}^{\alpha}G_m - \hbar(t_1-t_2)\cdot m D_{x_1}^{\alpha}u_m^1. \end{aligned} \tag{4.21}$$

By $(\star)$ and Lemma 4.5 we have that the condition $\hbar t_1 \cdot m \neq 0$ implies that $u_m^1 = D_{x_1} w$ for some $w \in \mathcal{S}(\mathbb{R})$. Therefore, the difference $D_{x_1}^{\alpha}(u_m^1 - u_m^2)$ (which we know is in $\mathcal{S}(\mathbb{R})$) is the unique solution in $\mathcal{S}(\mathbb{R})$ to a differential equation, (4.21), as in the setting of Lemma 4.5. By $(\star)$, (4.21), Lemma 4.5, and (4.12) we obtain

$$\begin{aligned}|D_{x_1}^{\alpha}(u_m^1-u_m^2)| &\lesssim \|D_{x_1}^{\alpha+1}(u_m^1-u_m^2)\|_{L^1} \\ &\lesssim \hbar|t_1-t_2|(\|D_{x_1}^{\alpha}G_m\|_{L^1} + |m|\|D_{x_1}^{\alpha}u_m^1\|_{L^1}) \\ &\lesssim \hbar|t_1-t_2|(\|D_{x_1}^{\alpha}V_m\|_{L^1} + \langle \xi, \hbar t_1\rangle |m| \|D_{x_1}^{\alpha-1}V_m\|_{L^1}). \end{aligned} \tag{4.22}$$

Remark. *This step shows why it is useful to have at disposal the $L^1 \to L^\infty$ estimates and not the $L^\infty \to L^\infty$ estimates alone.*

Similarly, substracting the equations (4.14) for $D_{x_1}^{\alpha} D_{\xi} u_m^j$ we obtain that

$$(\xi + i) D_{x_1}[D_{x_1}^{\alpha} D_{\xi}(u_m^1 - u_m^2)] + \hbar t_2 \cdot m[D_{x_1}^{\alpha} D_{\xi}(u_m^1 - u_m^2)] \\ = -\frac{1}{2\pi i} D_{x_1}^{\alpha+1}(u_m^1 - u_m^2) - \hbar(t_1 - t_2) \cdot m D_{x_1}^{\alpha} D_{\xi} u_m^1.$$

For $\beta \geq 2$ the equation takes the same form. Indeed, substracting the equations (4.17) for $D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m^j$ we obtain that

$$(\xi + i) D_{x_1}[D_{x_1}^{\alpha} D_{\xi}^{\beta}(u_m^1 - u_m^2)] + \hbar t_2 \cdot m[D_{x_1}^{\alpha} D_{\xi}^{\beta}(u_m^1 - u_m^2)] \\ = -\frac{\beta}{2\pi i} D_{x_1}^{\alpha+1} D_{\xi}^{\beta-1}(u_m^1 - u_m^2) - \hbar(t_1 - t_2) \cdot m D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m^1. \quad (4.23)$$

Since $u_m^1 = D_{x_1} w$, then we are in the setting of Lemma 4.5 as before. By (4.23), Lemma 4.5, (4.15), and (4.18) we can bound

$$\|D_{x_1}^{\alpha+1} D_{\xi}^{\beta}(u_m^1 - u_m^2)\|_{L^1} \lesssim \|D_{x_1}^{\alpha+1} D_{\xi}^{\beta-1}(u_m^1 - u_m^2)\|_{L^1} + \hbar|t_1 - t_2||m|\|D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m^1\|_{L^1} \\ \lesssim \|D_{x_1}^{\alpha+1} D_{\xi}^{\beta-1}(u_m^1 - u_m^2)\|_{L^1} + \hbar|t_1 - t_2|\langle \xi, \hbar t_1 \rangle |m| \|D_{x_1}^{\alpha-1} V_m\|_{L^1}.$$

Iterating this and using (4.22) we obtain that

$$\|D_{x_1}^{\alpha+1} D_{\xi}^{\beta}(u_m^1 - u_m^2)\|_{L^1} \lesssim \ldots \lesssim \|D_{x_1}^{\alpha+1}(u_m^1 - u_m^2)\|_{L^1} + \hbar|t_1 - t_2|\langle \xi, \hbar t_1 \rangle |m| \|D_{x_1}^{\alpha-1} V_m\|_{L^1} \\ \lesssim \hbar|t_1 - t_2|(\|D_{x_1}^{\alpha} V_m\|_{L^1} + \langle \xi, \hbar t_1 \rangle |m| \|D_{x_1}^{\alpha-1} V_m\|_{L^1}).$$

From this and the Fundamental Theorem of Calculus we conclude that

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta}(u_m^1 - u_m^2)| \lesssim \|D_{x_1}^{\alpha+1} D_{\xi}^{\beta}(u_m^1 - u_m^2)\|_{L^1} \\ \lesssim \hbar|t_1 - t_2|(\|D_{x_1}^{\alpha} V_m\|_{L^1} + \langle \xi, \hbar t_1 \rangle |m| \|D_{x_1}^{\alpha-1} V_m\|_{L^1}). \quad (4.24)$$

We have shown (4.9) through (4.20), (4.22), and (4.24). Finally, we prove the decay estimates

(4.10) for the solution. In the case $\hbar t \cdot m = 0$ there is nothing to prove, as the solution is compactly supported on $|x_1| \leq R$. The other two cases are analogous, so we only consider the case $\hbar t \cdot m > 0$, so that $\mathrm{Re}(\mu) > 0$. For this one we know the solution vanishes if $x_1 < -R$, so we only have to deal with $x_1 > R$. We can rewrite the solution for $x_1 > R$ as

$$u_m(x_1, \xi, t) = \frac{-\mu e^{-\mu(x_1 - R)}}{\xi + i} \int_{-R}^{R} e^{-\mu(R-y_1)}[(\xi + i)D_{y_1}^{-1} F_m(y_1) + \hbar t \cdot D_{y_1}^{-1} G_m(y_1)]dy_1.$$

Integrating by parts we obtain that

$$\begin{aligned} D_{x_1}^{\alpha} u_m &= \left(\frac{-\mu}{2\pi i}\right)^{\alpha} u_m \\ &= \frac{-\mu e^{-\mu(x_1 - R)}}{\xi + i} \int_{-R}^{R} [(-D_{y_1})^{\alpha} e^{-\mu(R-y_1)}][(\xi + i)D_{y_1}^{-1} F_m(y_1) + \hbar t \cdot D_{y_1}^{-1} G_m(y_1)]dy_1 \\ &= \frac{-\mu e^{-\mu(x_1 - R)}}{\xi + i} \int_{-R}^{R} e^{-\mu(R-y_1)}[(\xi + i)D_{y_1}^{\alpha-1} F_m(y_1) + \hbar t \cdot D_{y_1}^{\alpha-1} G_m(y_1)]dy_1 =: \varphi\psi. \end{aligned}$$

We will prove that φ and its derivatives (with respect to ξ) have the required decay, while ψ and its derivatives remain bounded. Let us observe that $\mu(R - y_1) = i\eta/(\xi + i)$ with $\eta \geq 0$ for $y_1 \in [-R, R]$, so that we are in the setting of Lemma 4.6. It follows from this that $|D_{\xi}^{\beta}\psi| \lesssim \langle \xi, \hbar t\rangle \|D_{x_1}^{\alpha-1} V_m\|_{L^1}$, where we allow the constant of the inequality to depend on β. By a similar induction as in the proof of Lemma 4.6, we obtain that

$$D_{\xi}^{\beta}\left(\frac{\mu e^{-\mu s}}{\xi + i}\right) = \frac{\mu e^{-\mu s} Q_{\beta}(\mu s)}{(\xi + i)^{\beta+1}},$$

where Q_{β} is a polynomial of degree β with coefficients depending only on β. Multiplying by s we obtain

$$sD_{\xi}^{\beta}\left(\frac{\mu e^{-\mu s}}{\xi + i}\right) = \frac{e^{-\mu s} Q_{\beta}^{*}(\mu s)}{(\xi + i)^{\beta+1}},$$

where $Q_{\beta}^{*}(z) = zQ_{\beta}(z)$ is a polynomial of degree $\beta + 1$. Again, as $|\xi + i| \geq 1$, there exists a polynomial $\widetilde{Q}^*_\beta$ of degree $\beta+1$ such that

$$\left|\frac{Q^*_\beta(\mu s)}{(\xi+i)^{\beta+1}}\right| \le \widetilde{Q}^*_\beta\left(\frac{|\mu s|}{\langle\xi\rangle}\right).$$

We have that $\mu(x_1-R)=i\eta/(\xi+i)$ with $\eta\ge 0$ for $x_1\ge R$, so that we are in the setting of Lemma 4.6. It follows from the previous inequalities that

$$|(x_1-R)D^\beta_\xi\varphi| \le \left|e^{-i\eta/(\xi+i)}\widetilde{Q}^*_\beta\left(\frac{\eta}{\langle\xi\rangle^2}\right)\right| \lesssim 1,$$

where we allow the constant in the last inequality to depend on β. In particular, for $x_1\ge 2R$ we have that $x_1-R\ge x_1/2$, so that the previous inequality gives $|D^\beta_\xi\varphi|\lesssim 1/|x_1|$. Combining the bounds for ψ and φ gives the decay estimate that we wanted. □

Remark. *It may not be relevant for this particular problem, but the condition $m\in\mathbb{Z}^d$ does not seem to intervene in the proof of the result.*

Remark. *It does not seem relevant, but the constants in the inequalities are independent of R. It only plays a role when we need that $|x_1|\ge 2R$ to have the decay.*

Let us discuss briefly why the vanishing moment conditions were important in the previous proof. First, they appear in proving the differences estimates. If $t_1,t_2\in\mathbb{R}^d$ are such that $\hbar t_1\cdot m>0>\hbar t_2\cdot m$, then for $x_1>R$ we would have $u^2_m(x_1)=0$ and

$$u^1_m(x_1)=\frac{2\pi i e^{-\mu_1(x_1-R)}}{\xi+i}\int_{-R}^{R} e^{-\mu_1(R-y_1)}[(\xi+i)F_m(y_1)+\hbar t_1\cdot G_m(y_1)]dy_1.$$

It seems that there is no way to estimate $u^1_m-u^2_m$ in terms of the difference $\hbar|t_1-t_2|$. For instance, letting $\mu_1\to 0$, we obtain that the difference would be approximately

$$\frac{2\pi i}{\xi+i}\int_{-R}^{R}[(\xi+i)F_m(y_1)+\hbar t_1\cdot G_m(y_1)]dy_1,$$

which suggests the need of the vanishing moment condition. They also show up with a crucial improvement for the decay estimates. In the case $\hbar t \cdot m > 0$ and $x_1 > R$, without the vanishing moment condition, we would only have the exponential decay

$$u_m(x_1) = \frac{2\pi i e^{-\mu(x_1-R)}}{\xi + i} \int_{-R}^{R} e^{-\mu(R-y_1)}[(\xi+i)F_m(y_1) + \hbar t \cdot G_m(y_1)]dy_1.$$

It may happen that μ is small, for instance if $t \cdot m = 1$, making the exponential decay very slow. In this case, the best estimates we seem to obtain are

$$|u_m| \lesssim \frac{\langle \xi \rangle}{\mu |x_1 - R|},$$

with $1/\mu$ potentially being as big as $1/\hbar$. In our proof, what allows us to get better estimates is the presence of the factor μ in front of the integral. This factor comes from integrating by parts using the vanishing moment condition.

We rewrite the previous estimates to depend on V and no longer on its Fourier coefficients.

Corollary 4.9. *Let $V = (F, G)$ satisfy $(\star)$, and let u_m be the solution from Theorem 4.8. Then, for any $M \geq 0$ we have*

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m| \lesssim \langle \xi, \hbar t \rangle \langle m \rangle^{-2M} \|V\|_{W^{\alpha+2M,1}(T)}, \tag{4.25}$$

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta}(u_m(\cdot, t_1) - u_m(\cdot, t_2))| \lesssim \hbar |t_1 - t_2| \langle \xi, \hbar t_1, \hbar t_2 \rangle \langle m \rangle^{-2M+1} \|V\|_{W^{\alpha+2M,1}(T)}. \tag{4.26}$$

Moreover, for $|x_1| \geq 2R$ we have the linear decay bound

$$|D_{x_1}^{\alpha} D_{\xi}^{\beta} u_m(x_1)| \lesssim \frac{\langle \xi, \hbar t \rangle}{|x_1|} \langle m \rangle^{-2M} \|V\|_{W^{\alpha+2M,1}(T)}. \tag{4.27}$$

The constants in the inequalities may depend on α, β, M, d, R, but are independent of $\hbar$, m, ξ, t.

Proof. We use that V is compactly supported and the Fundamental Theorem of Calculus to get that

$$\|D_{x_1}^{\alpha-1}V_m\|_{L^1} \lesssim \|D_{x_1}^{\alpha-1}V_m\|_{L^\infty} \lesssim \|D_{x_1}^{\alpha}V_m\|_{L^1},$$

where we allow the constant of the inequality to depend on R. For $\alpha \geq 0$ and any $M \geq 0$, we have from Proposition 2.3 that

$$\|D_{x_1}^{\alpha}V_m\|_{L^1(\mathbb{R})} \lesssim \langle m\rangle^{-2M}\|D_{x_1}^{\alpha}V\|_{W^{2M,1}(T)} \lesssim \langle m\rangle^{-2M}\|V\|_{W^{\alpha+2M,1}(T)},$$

where we allow the constant of the inequality to depend on M. Then the conclusion follows from Theorem 4.8. □

Proof of Theorem 4.3. Let $u(\cdot,\xi,t;\hbar) \in C^\infty(T)$ solve the equation (4.2) and satisfy the decay condition $\|u(x_1,\cdot,\xi,t)\|_{L^\infty(\mathbb{T}^d)} \to 0$ as $x_1 \to \pm\infty$. Then its Fourier coefficients must solve the ODEs (4.5) and satisfy the decay conditions as $x_1 \to \pm\infty$. Under $(\star)$, we know from Theorem 4.8 the existence and uniqueness of such solutions. Let $\{u_m\}$ be such and define

$$u(x_1,x',\xi,t;\hbar) := \sum_{m\in\mathbb{Z}^d} u_m(x_1,\xi,t;\hbar)e_m(x').$$

The control on derivatives of u_m from Corollary 4.9 implies that $u(\cdot,t) \in C^\infty(\mathbb{R}\times\mathbb{T}^d\times\mathbb{R})$ and it solves (4.2). Moreover, we can bound

$$|D_{x_1}^{\alpha}D_{x'}^{\beta}D_{\xi}^{\gamma}u| \leq \sum_{m\in\mathbb{Z}^d}|m|^{\beta}|D_{x_1}^{\alpha}D_{\xi}^{\gamma}u_m| \lesssim \sum_{m\in\mathbb{Z}^d}|m|^{\beta}\langle m\rangle^{-2M}\langle\xi,\hbar t\rangle \lesssim \langle\xi,\hbar t\rangle,$$

by taking $M = M(\beta,d)$ sufficiently large. The constants in the inequalities may depend on the admissible quantities, but are independent of $\hbar$, ξ, t. In the same way we prove the difference estimate from (4.3) and the decay estimate (4.4); finally, the existence of a solution satisfying the decay condition follows from (4.4). □

4.4 Explicit definition of the symbol and properties

The purpose of this section is to prove the conjugation from Theorem 4.2. Recall that we have

$$(\Delta_\hbar + 2\hbar V_\hbar)A = A\Delta_\hbar + 2\hbar \mathrm{Op}_\hbar((\xi + i)D_{x_1}a + \hbar t \cdot D_{x'}a + (\xi + i)Fa + \hbar t \cdot Ga)$$
$$+ \hbar^2 \mathrm{Op}_\hbar(D^2 a + 2V \cdot Da).$$

Let us define $a = e^{-u\phi}$, where $u(x_1, x', \xi, t; \hbar)$ is the solution from Theorem 4.3 to

$$(\xi + i)D_{x_1}u + \hbar t \cdot D_{x'}u = (\xi + i)F + \hbar t \cdot G,$$

and $\phi(x_1, \xi, t; \hbar)$ is defined as follows. Let $\psi(s)$ be a (nonnegative, even, decreasing) smooth function such that $\psi(s) \equiv 1$ for $|s| \leq 1$ and $\psi(s) \equiv 0$ for $|s| \geq 2$. Define

$$\phi_0(\xi, t; \hbar) := \psi(\xi)\psi(4(|\hbar t| - 1)), \quad \phi(x_1, \xi, t; \hbar) := \psi(\hbar^\theta x_1)\phi_0(\xi, t; \hbar),$$

with $\theta > 0$ to be defined later.

Remark. *Note that ϕ_0 and ϕ vanish if $|\hbar t| \leq 1/2$, in particular they vanish for t near the origin and so these are smooth in all their variables. Moreover, it is a special zero order symbol.*

Remark. *The cutoff in x_1 is unnecessary for the properties of A and B, but it is actually needed for the properties of R.*

The estimates from Theorem 4.3 give that $u\phi$ is a special zero order symbol. From Corollary 3.15 we obtain that, for small $\hbar$, the operator $A := \mathrm{Op}_\hbar(a)$ is invertible and its inverse is uniformly bounded (in $\hbar$) in $H^s_{\pm\delta,\hbar}(T)$. We also have

$$
\begin{aligned}
(\xi+i)D_{x_1}a + \hbar t\cdot D_{x'}a + (\xi+i)Fa + \hbar t\cdot Ga& \\
&= a[-(\xi+i)D_{x_1}(u\phi) - \hbar t\cdot D_{x'}(u\phi) + (\xi+i)F + \hbar t\cdot G] \\
&= a(1-\phi)[(\xi+i)F + \hbar t\cdot G] - (\xi+i)auD_{x_1}\phi \\
&= a(1-\phi)[(\xi+i)F + \hbar t\cdot G] - \frac{1}{2\pi i}\hbar^\theta(\xi+i)au\psi'(\hbar^\theta x_1)\phi_0.
\end{aligned}
$$

Note that $(1-\phi)$ vanishes if $|x_1|\le \hbar^{-\theta}$, $|\xi|\le 1$ and $||\hbar t|-1|\le 1/4$. Let us rewrite

$$
1-\phi = (1-\phi_0) + \phi_0\cdot(1-\psi(\hbar^\theta x_1)),
$$

and observe that $1-\psi(\hbar^\theta x_1)$ vanishes for $|x_1|\le \hbar^{-\theta}$. Because F and G are supported on $|x_1|\le R$, then, for small enough $\hbar$, say $\hbar^{-\theta}\ge R$, we have

$$
a(1-\phi)[(\xi+i)F + \hbar t\cdot G] = a(1-\phi_0)[(\xi+i)F + \hbar t\cdot G].
$$

The function $(1-\phi_0)$ vanishes if $|\xi|\le 1$ and $||\hbar t|-1|\le 1/4$, and outside of this set the operator $\Delta_\hbar$ is elliptic, i.e. its symbol $(\xi+i)^2+|\hbar t|^2$ does not vanish.

Proposition 4.10. *Outside of the set* $\{|\xi|\le 1\}\cap\{||\hbar t|-1|\le 1/4\}$*, we can bound*

$$
|(\xi+i)^2+|\hbar t|^2| \gtrsim \langle \xi, \hbar t\rangle^2
$$

Proof. Let $s := (\xi+i)^2+|\hbar t|^2$. Let us observe that

$$
|s| = ([\xi^2+|\hbar t|^2-1]^2+4\xi^2)^{1/2} = (\xi^4+2\xi^2(|\hbar t|^2+1)+(|\hbar t|^2-1)^2)^{1/2} \ge \xi^2+||\hbar t|^2-1|.
$$

This proves that $|s|\ge \xi^2$. If $|\xi|\ge 1$, this also proves $|s|\ge 1$. If $|\xi|\le 1$, then we must have $||\hbar t|-1|\ge 1/4$, and this gives $|s|\ge ||\hbar t|+1|/4\ge 1/4$. We have shown that $|s|\gtrsim 1$. Thus, all that remains to prove is that $|s|\gtrsim |\hbar t|^2$. If $|\hbar t|\le 2$, then this follows from before. If $|\hbar t|\ge 2$, then $||\hbar t|^2-1|\ge |\hbar t|^2/2$, and the conclusion follows. □

Let us consider the function

$$r(x_1, x', \xi, t; \hbar) := \frac{1-\phi_0}{(\xi+i)^2 + |\hbar t|^2}[(\xi+i)F + \hbar t \cdot G]$$
$$= \frac{(1-\phi_0)\langle \xi, \hbar t\rangle^2}{(\xi+i)^2 + |\hbar t|^2} \cdot \frac{(\xi+i)F + \hbar t \cdot G}{\langle \xi, \hbar t\rangle^2} =: q_1 \cdot q_2.$$

The functions $\xi/\langle \xi, \hbar t\rangle^2$ and $\hbar t/\langle \xi, \hbar t\rangle^2$ are special zero order symbols, and so q_2 is a special zero order symbol. We know that q_1 is supported outside of $\{|\xi| \le 1\} \cap \{||\hbar t| - 1| \le 1/4\}$ and $1-\phi_0$ is a special zero order symbol. By induction we can show that

$$D_\xi^\alpha D_t^\beta \frac{\langle \xi, \hbar t\rangle^2}{(\xi+i)^2 + |\hbar t|^2} = \frac{\hbar^\beta P_{\alpha,\beta}(\xi, \hbar t)}{((\xi+i)^2 + |\hbar t|^2)^{\alpha+\beta+1}},$$

for some polynomial $P_{\alpha,\beta}$ of degree at most $\alpha+\beta+2$. Outside of $\{|\xi| \le 1\} \cap \{||\hbar t| - 1| \le 1/4\}$, from Proposition 4.10, we can bound

$$\left| D_\xi^\alpha D_t^\beta \frac{\langle \xi, \hbar t\rangle^2}{(\xi+i)^2 + |\hbar t|^2} \right| = \left| \frac{\hbar^\beta P_{\alpha,\beta}(\xi, \hbar t)}{((\xi+i)^2 + |\hbar t|^2)^{\alpha+\beta+1}} \right| \lesssim \frac{\hbar^\beta \langle \xi, \hbar t\rangle^{\alpha+\beta+2}}{\langle \xi, \hbar t\rangle^{2(\alpha+\beta+1)}} \le \hbar^\beta.$$

This proves that q_1 is a special zero order symbol, and by Proposition 3.14 so is r. Let us define the symbol b by

$$b := a + 2\hbar a \frac{1-\phi_0}{(\xi+i)^2 + |\hbar t|^2}[(\xi+i)F + \hbar t \cdot G] = a(1 + 2\hbar r).$$

Since r is a special zero order symbol, by Proposition 3.14 we have that, for small enough $\hbar$, $v := \log(1+2\hbar r)$ is also a special zero order symbol. Therefore $b = a(1+2\hbar r) = e^{-u\phi+v}$ and $-u\phi + v$ a special zero order symbol. From Corollary 3.15, we conclude that for small $\hbar$, the operator $B := \mathrm{Op}_\hbar(b)$ is invertible and its inverse is uniformly bounded (in $\hbar$) in $H^s_{\pm\delta,\hbar}(T)$.

Finally, we are left to consider the expression

$$\hbar^2(D^2 a + 2V \cdot Da) - \frac{2}{2\pi i}\hbar^{1+\theta}(\xi+i)au\psi'(\hbar^\theta x_1)\phi_0 =: r_1 + r_2.$$

In order to prove that the pseudodifferential operator R, from Theorem 4.2, is bounded from $L^2_{-\delta}(T)$ to $L^2_{\delta}(T)$, we will show that $\langle x_1\rangle^{2\delta} r_i$ are zero order symbols. Recall that $u\phi$ is supported on $|x_1| \leq 2\hbar^{-\theta}$ and so $a = e^{-u\phi} \equiv 1$ if $|x_1| \geq 2\hbar^{-\theta}$. Therefore, r_1 is supported on $|x_1| \leq 2\hbar^{-\theta}$, and we have the bounds

$$|D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} r_1| \lesssim \hbar^2, \quad |D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} (\langle x_1\rangle^{2\delta} r_1)| \lesssim \hbar^{2-2\delta\theta}.$$

The term $\psi'(\hbar^{\theta} x_1)$ gives that r_2 is supported on $\hbar^{-\theta} \leq |x_1| \leq 2\hbar^{-\theta}$. The decay estimate from Theorem 4.3 gives that $|D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} u| \lesssim \hbar^{\theta}\langle \xi, \hbar t\rangle$. Moreover, a is a zero order symbol and $(\xi + i)\psi'(\hbar^{\theta} x_1)\phi_0$ is a zero order symbol supported on $\{|\xi| \leq 2\} \cap \{||\hbar t| - 1| \leq 1/2\}$. We obtain that

$$|D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} r_2| \lesssim \hbar^{1+2\theta}, \quad |D^{\alpha}_{x_1} D^{\beta}_{x'} D^{\gamma}_{\xi} (\langle x_1\rangle^{2\delta} r_2)| \lesssim \hbar^{1+2\theta-2\delta\theta}.$$

We can ensure that $2 - 2\delta\theta, 1 + 2\theta - 2\delta\theta > 1$ by taking $\theta = 1/2$ and $\delta < 1$. If we let $\varepsilon = 1 - \delta > 0$, then we obtain the conjugation identity from Theorem 4.2.

Remark. *The restriction $\delta < 1$ does not appear in [Sal06]. This may be due that in our case we look for the operator R to be bounded from $L^2_{-\delta}(T)$ to $L^2_{\delta}(T)$, so we need a gain of a factor $\langle x_1\rangle^{2\delta}$. In [Sal06], it is only needed from $L^2_{\delta}(\mathbb{R}^d)$ to $L^2_{\delta+1}(\mathbb{R}^d)$, so just a factor $\langle x\rangle$ is required.*

4.5 Proof of the Carleman estimate

Let us rewrite the statements of Theorem 4.1 and Theorem 1.2 in semiclassical notation.

Theorem 4.11. *Let $\delta > 1/2$. There exists $\hbar_0 \leq 1$ such that if $0 < |\hbar| \leq \hbar_0$ and $\hbar^{-2} \notin Spec(-\Delta_{g_0})$, then for any $f \in L^2_{\delta}(T)$ there exists a unique $u \in H^1_{-\delta,\hbar}(T)$ which solves*

$$e^{2\pi x_1/\hbar}(\hbar D)^2 e^{-2\pi x_1/\hbar} u = \hbar^2 f.$$

Moreover, this solution is in $H^2_{-\delta,\hbar}(T)$ and satisfies the estimates

$$\|u\|_{H^2_{-\delta,\hbar}(T)} \lesssim \hbar\|f\|_{L^2_\delta(T)},$$

with the constant of the inequality independent of $\hbar$.

This theorem allows to define $G_\hbar : L^2_\delta(T) \to H^2_{-\delta,\hbar}(T)$ by $G_\hbar f := u$, so that $\Delta_\hbar G_\hbar = \hbar^2 I$ on $L^2_\delta(T)$ and $\|G_\hbar\|_{L^2_\delta(T)\to H^2_{-\delta,\hbar}(T)} \lesssim \hbar$.

Theorem 4.12. *Let $1/2 < \delta < 1$ and let V, W satisfy $(\star)$. There exists $\hbar_0 \geq 1$ such that if $0 < |\hbar| \leq \hbar_0$ and $\hbar^{-2} \notin Spec(-\Delta_{g_0})$, then for any $f \in L^2_\delta(T)$ there exists a unique $u \in H^2_{-\delta,\hbar}(T)$ which solves*

$$e^{2\pi x_1/\hbar}(\hbar^2 H_{V,W})e^{-2\pi x_1/\hbar}u = \hbar^2 f.$$

Moreover, this solution satisfies the estimates

$$\|u\|_{H^2_{-\delta,\hbar}(T)} \lesssim \hbar\|f\|_{L^2_\delta(T)},$$

with the constant of the inequality independent of $\hbar$.

Proof. We prove this only for $\hbar > 0$, as the other case is analogous. Let us write

$$(\Delta_\hbar + 2\hbar V_\hbar + \hbar^2\widetilde{W})u = e^{2\pi x_1/\hbar}(\hbar^2 H_{V,W})e^{-2\pi x_1/\hbar}u = \hbar^2 f, \tag{4.28}$$

where $\Delta_\hbar = e^{2\pi x_1/\hbar}(\hbar D)^2 e^{-2\pi x_1/\hbar}$, $V_\hbar = e^{2\pi x_1/\hbar}[V\cdot(\hbar D)]e^{-2\pi x_1/\hbar}$, $\widetilde{W} := V^2 + D\cdot V + W$. To avoid repetition throughout the proof we recall from Theorem 4.2 that A and B are uniformly bounded invertible operators on $H^s_{\pm\delta,\hbar}(T)$ with uniformly bounded inverses, $R : L^2_{-\delta}(T) \to L^2_\delta(T)$ is bounded. Also, $\widetilde{W} : L^2_{-\delta}(T) \to L^2_\delta(T)$ is bounded, because it is bounded and compactly supported.

We start showing the existence and the estimates for the solution. We look for a solution of the form $u = AG_\hbar g \in L^2_{-\delta}(T)$, with $g \in L^2_\delta(T)$, and use Theorem 4.2 to rewrite the expression as

$$(\Delta_\hbar + 2\hbar V_\hbar + \hbar^2 \widetilde{W})u = (\Delta_\hbar + 2\hbar V_\hbar + \hbar^2 \widetilde{W})AG_\hbar g = \hbar^2(B + \hbar^{-1+\varepsilon} RG_\hbar + \widetilde{W} AG_\hbar)g.$$

Let $C := \hbar^{-1+\varepsilon} RG_\hbar + \widetilde{W} AG_\hbar$. We claim that $C : L^2_\delta(T) \to L^2_\delta(T)$ is a small perturbation of the invertible operator B, so that $B + C$ is also invertible. Using the boundedness properties of $G_\hbar$, from Theorem 4.11 we obtain that

$$\|\hbar^{-1+\varepsilon} RG_\hbar\|_{L^2_\delta(T)\to L^2_\delta(T)} \lesssim \hbar^\varepsilon, \quad \|\widetilde{W} AG_\hbar\|_{L^2_\delta(T)\to L^2_\delta(T)} \lesssim \hbar.$$

We observe that $B + C = (I + CB^{-1})B$, from where we conclude that $B + C$ is invertible in $L^2_\delta(T)$ as claimed, and its inverse has uniformly bounded norms. Thus if we define $g := (B + C)^{-1} f \in L^2_\delta(T)$, then we obtain that

$$u := AG_\hbar g = AG_\hbar (B + C)^{-1} f$$

solves the equation (4.28). The estimates for $G_\hbar$ from Theorem 4.11 give

$$\|u\|_{H^2_{-\delta,\hbar}(T)} = \|AG_\hbar (B + C)^{-1} f\|_{H^2_{-\delta,\hbar}(T)} \lesssim \hbar \|f\|_{L^2_\delta(T)},$$

as we wanted. Now we address the uniqueness. Assume that $u \in H^2_{-\delta,\hbar}(T)$ solves

$$(\Delta_\hbar + 2\hbar V_\hbar + \hbar^2 \widetilde{W})u = 0.$$

Let $v := A^{-1}u \in H^2_{-\delta,\hbar}(T)$, so that v satisfies $(B\Delta_\hbar + \hbar^{1+\varepsilon} R + \hbar^2 \widetilde{W} A)v = 0$, or equivalently

$$\Delta_\hbar v = -\hbar^2 B^{-1}(\hbar^{-1+\varepsilon} R + \widetilde{W} A)v.$$

The right-hand side is in $L^2_\delta(T)$ and

$$\|\hbar^{-1+\varepsilon}B^{-1}Rv\|_{L^2_\delta(T)} \lesssim \hbar^{-1+\varepsilon}\|v\|_{L^2_{-\delta}(T)}, \quad \|B^{-1}\widetilde{W}Av\|_{L^2_\delta(T)} \lesssim \|v\|_{L^2_{-\delta}(T)}.$$

The uniqueness from Theorem 4.11 implies that $v = -G_\hbar B^{-1}(\hbar^{-1+\varepsilon}R + \widetilde{W}A)v$. Using the estimates for $G_\hbar$, from Theorem 4.11, and the bound from above we obtain that

$$\|v\|_{L^2_{-\delta}(T)} \lesssim \hbar \cdot \hbar^{-1+\varepsilon}\|v\|_{L^2_{-\delta}(T)} = \hbar^\varepsilon\|v\|_{L^2_{-\delta}(T)}.$$

Taking $\hbar$ small enough yields that $v \equiv 0$, from where we conclude that $u \equiv 0$. □

Remark. *The uniqueness does not follow directly from the equation and perturbative arguments: if we rewrite the equation as*

$$\Delta_\hbar u = -\hbar^2(2\hbar^{-1}V_\hbar + \widetilde{W})u,$$

then the right-hand side is in $L^2_\delta(T)$, so that $u = -G_\hbar(2\hbar^{-1}V_\hbar + \widetilde{W})u$, but we obtain no contradiction as we can only say $\|G_\hbar(2\hbar^{-1}V_\hbar u)\|_{L^2_{-\delta}(T)} \lesssim \|u\|_{L^2_{-\delta}(T)}$.

Remark. *Let $f \in L^2_\delta(T)$ and let $u \in H^2_{-\delta,\hbar}(T)$ be the unique solution to the equation $(\Delta_\hbar + 2\hbar V_\hbar + \hbar^2\widetilde{W})u = \hbar^2 f$. We can rewrite this as*

$$\Delta_\hbar u = \hbar^2 f - (2\hbar V_\hbar + \widetilde{W})u,$$

and observe that $(2\hbar V_\hbar + \hbar^2\widetilde{W})u \in L^2_\delta(T)$. Therefore, $u = G_\hbar \widetilde{g}$, for some $\widetilde{g} \in L^2_\delta(T)$. In the proof of existence of the solution, we showed that u takes the form $AG_\hbar g$ for some $g \in L^2_\delta(T)$. This and the last observation yield that the operator A maps the subspace $G_\hbar L^2_\delta(T) \subseteq L^2_{-\delta}(T)$ to itself.

CHAPTER 5

EQUIVALENT FORMULATIONS AND BOUNDARY CHARACTERIZATION

As mentioned in the introduction, in order to reconstruct the electromagnetic parameters we are interested in constructing many solutions to the equation $H_{V,W}u = 0$. The result from Theorem 1.2 can be used to construct a unique solution that "behaves like" a harmonic function at infinity. Indeed, let $h \in H^2_{loc}(T)$ be harmonic and let us look for a solution of the form $u = h + e^{-2\pi\tau x_1}r$; such u solves $H_{V,W}u = 0$ if and only if the correction term r solves the equation

$$e^{2\pi\tau x_1}H_{V,W}e^{-2\pi\tau x_1}r = -e^{2\pi\tau x_1}Xh,$$

where $X := H_{V,W} - D^2 = 2V \cdot D + (V^2 + D \cdot V + W)$ is a first order differential operator supported in M. The conditions (†) imply that $Xh \in L^2_c(T)$, and so $e^{2\pi\tau x_1}Xh \in L^2_\delta(T)$. From Theorem 1.2 we obtain a unique solution $r \in H^2_{-\delta}(T)$, and so there is a unique solution to $H_{V,W}u = 0$ which "behaves like" the harmonic function. As has been usual, we call these functions the complex geometrical optics (CGO) solutions.

The purpose of this section is to show that the boundary values of the CGO can be characterized as the unique solution to a certain boundary integral equation. The passage from the uniqueness problem at the boundary to a uniqueness problem at infinity was first explicitly noticed by Nachman in [Nac88], and has become standard since then; for instance, see [Sal06] or [KSU11b]. The uniqueness of this corrected solution is crucial for our problem; the lack of such is what prevents the local Carleman estimate for the magnetic Schrödinger operator in [DSFKSU09] from being useful in the reconstruction procedure.

In this section we follow closely the presentation from [Sal06], as the operators are translation invariant, and [KSU11b]. However, we have to proceed slightly different, as $0 \in \operatorname{Spec}(-\Delta_{g_0})$ and the Laplacian in T does not have a bounded inverse, i.e. for $f \in L^2(T)$ (or $f \in H^{-1}(T)$) there may not be $u \in H^2(T)$ (or $u \in H^1(T)$) such that $D^2u = f$.

5.1 Green functions, operators, and layer potentials

5.1.1 τ–dependent Green function and operator

The differential operator $\Delta_\tau = D_{x_1}^2 + 2i\tau D_{x_1} - \tau^2 + D_{x'}^2$ has constant coefficients, in particular it is translation invariant, and so it is its right inverse G_τ from Theorem 4.1. Since $G_\tau : L^2_\delta(T) \to L^2_{-\delta}(T)$ is bounded, then there exists a tempered distribution $g_\tau \in \mathcal{S}'(T)$ such that $G_\tau f = g_\tau * f$ for Schwartz functions $f \in \mathcal{S}(T)$, where the convolution is considered over the whole cylinder. The purpose of this section is to understand the properties of g_τ and other related distributions. We have that $\Delta_\tau g_\tau = \delta_T(0)$, so the Fourier expansion of this distribution is given by

$$\widehat{g_{\tau,k}}(\xi) = \frac{1}{\xi^2 + 2i\tau\xi - \tau^2 + |k|^2} = \frac{1}{(\xi + i\tau)^2 + |k|^2}, \tag{5.1}$$

and therefore

$$g_{\tau,k}(x_1) = \int_{\mathbb{R}} \frac{e^{2\pi i x_1 \xi}}{(\xi + i\tau)^2 + |k|^2} d\xi. \tag{5.2}$$

Let us note that this integral converges absolutely, as the denominator is quadratic in ξ and never vanishes because $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$. These integrals can be computed explicitly as follows.

Proposition 5.1. *The Fourier coefficients $g_{\tau,k}(x_1)$ of the distribution g_τ are given by*

$$g_{\tau,k}(x_1) = \pi e^{2\pi\tau x_1} \begin{cases} -2\pi(|x_1| - sgn(\tau)x_1) & \text{if } k = 0, \\ (e^{-2\pi|k||x_1|} - e^{-2\pi|k| sgn(\tau) x_1})/|k| & \text{if } 0 < |k| < |\tau|, \\ e^{-2\pi|k||x_1|}/|k| & \text{if } |k| > |\tau|. \end{cases}$$

The distribution g_τ is actually smooth away from $(0,0) \in T$. For any $\varepsilon > 0$ and $|x_1| > \varepsilon$, the function $g_\tau(x_1, x')$ and all its derivatives are uniformly bounded, i.e. we have $|D_{x_1}^\alpha D_{x'}^\beta g_\tau(x_1, x')| \le C$, for some constant $C = C(\alpha, \beta, \tau, \varepsilon)$.

Proof. Instead of (5.2), let us consider the expression

$$g_\tau(x_1,\lambda) := \int_{\mathbb{R}} \frac{e^{2\pi i x_1 \xi}}{(\xi + i\tau)^2 + \lambda^2} d\xi, \tag{5.3}$$

with $\lambda \geq 0$ and $\lambda \neq |\tau|$, so that the denominator does not vanish. We start with the following two observations: first, $g_{-\tau}(-x_1,\lambda) = g_\tau(x_1,\lambda)$, so it suffices to consider the case $\tau > 0$; second, for fixed τ and x_1, the function $g_\tau(x_1,\cdot)$ is continuous, so the case $\lambda = 0$ follows from the case $\lambda > 0$. We would like to relate (5.3) to the classical integral

$$\int_{\mathbb{R}} \frac{e^{2\pi i x_1 \xi}}{\xi^2 + \lambda^2} d\xi = \frac{\pi}{\lambda} e^{-2\pi\lambda|x_1|},$$

which can be obtained by direct computation and the inversion formula. Consider the meromorphic function $f(z) = e^{2\pi i x_1 z}/(z^2 + \lambda^2)$, with simple poles at $z = \pm\lambda i$, and the rectangular contour bounded by the lines $\mathrm{Re}(z) = \pm L$, for some large $L > 0$, and $\mathrm{Im}(z) = 0$, $\mathrm{Im}(z) = \tau$. The pole $-\lambda i$ is outside of this domain since we are assuming $\tau > 0$. The pole λi is inside this domain if $0 < \lambda < \tau$ and outside if $\lambda > \tau$. Moreover, the residue of f at $z = \lambda i$ is equal to $e^{-2\pi\lambda x_1}/(2\lambda i)$. The vertical segments of the contour have length τ, and over them the numerator of f is bounded (uniformly in L), while the denominator is of order L^2. From the residue theorem, after letting $L \to +\infty$, we deduce that

$$\begin{aligned} &\frac{\pi}{\lambda} e^{-2\pi\lambda|x_1|} - e^{-2\pi\tau x_1} g_\tau(x_1,\lambda) \\ &\qquad = \int_{\mathbb{R}} f(z)dz - \int_{\mathbb{R}} f(z + i\tau)dz = 2\pi i \begin{cases} e^{-2\pi\lambda x_1}/(2\lambda i) & \text{if } \ 0 < \lambda < \tau, \\ 0 & \text{if } \ \lambda > \tau, \end{cases} \end{aligned}$$

which gives that

$$g_\tau(x_1,\lambda) = \frac{\pi e^{2\pi\tau x_1}}{\lambda} \begin{cases} e^{-2\pi\lambda|x_1|} - e^{-2\pi\lambda x_1} & \text{if } \ 0 < \lambda < \tau, \\ e^{-2\pi\lambda|x_1|} & \text{if } \ \lambda > \tau, \end{cases}$$

For the case $\lambda = 0$, we let $\lambda \to 0^+$, to conclude

$$g_\tau(x_1, 0) = \pi e^{2\pi\tau x_1} \lim_{\lambda \to 0^+} \frac{e^{-2\pi\lambda|x_1|} - e^{-2\pi\lambda x_1}}{\lambda} = -2\pi^2 e^{2\pi\tau x_1}(|x_1| - x_1).$$

We have proven the formulas for the Fourier coefficients. To show the regularity, let us observe that

$$D^2(e^{-2\pi\tau x_1} g_\tau) = e^{-2\pi\tau x_1} \Delta_\tau g_\tau = \delta_T(0).$$

From Weyl's regularity lemma for distributions, see Chapter 10 in [Går97], it follows that $e^{-2\pi\tau x_1} g_\tau$ is a smooth function away from $(0, 0) \in T$. Therefore, g_τ is also smooth away from $(0, 0)$. Assuming that $\tau > 0$, for $x_1 > 0$ we have that

$$g_\tau(x_1, x') = \pi \sum_{|k| > \tau} \frac{e^{-2\pi(|k| - \tau)|x_1|}}{|k|} e_k(x'),$$

while for $x_1 < 0$ we have that

$$g_\tau(x_1, x') = 4\pi^2 e^{-2\pi\tau|x_1|} x_1 + \pi \sum_{|k| \neq 0} \frac{e^{-2\pi(\tau + |k|)|x_1|}}{|k|} e_k(x') - \pi \sum_{0 < |k| < \tau} \frac{e^{-2\pi(\tau - |k|)|x_1|}}{|k|} e_k(x').$$

The uniform boundedness of g_τ and its derivatives, for $|x_1| > \varepsilon$, follows from the fact that $g_\tau(x_1, x')$ is a sum of negative exponentials on each half of $|x_1| > \varepsilon$. □

Remark. *The coefficient $g_{\tau,k}(x_1)$ can also be computed by solving the equation*

$$(D_{x_1}^2 + 2i\tau D_{x_1} - |\tau|^2 + |k|^2) g_{\tau,k}(x_1) = \delta_{\mathbb{R}}(0).$$

This can be solved in each half $x_1 > 0$ and $x_1 < 0$ as sum of exponentials, and then using decay conditions $\lim_{|x_1| \to \pm\infty} g_{\tau,k}(x_1) = 0$ and the jump condition at $x_1 = 0$.

As in the previous proof, we consider the distribution $\Gamma_\tau := e^{-2\pi\tau x_1} g_\tau \in \mathcal{D}'(T)$, which is no longer a tempered distribution, and satisfies $D^2\Gamma_\tau = \delta_T(0)$. In principle, it may not make sense to talk about the Fourier transform of Γ_τ as it is not a tempered distribution. However, from Proposition 5.1, we could formally say that the Fourier coefficients of Γ_τ are given by

$$e^{-2\pi\tau x_1} g_{\tau,k}(x_1) = \begin{cases} -2\pi^2(|x_1| - \mathrm{sgn}(\tau)x_1) & \text{if } \ k = 0, \\ \pi(e^{-2\pi|k||x_1|} - e^{-2\pi|k|\mathrm{sgn}(\tau)x_1})/|k| & \text{if } \ 0 < |k| < |\tau|, \\ \pi e^{-2\pi|k||x_1|}/|k| & \text{if } \ |k| > |\tau|. \end{cases}$$

Based on the formal Fourier coefficients above, we consider the harmonic function

$$H_\tau(x_1, x') := 2\pi^2 \mathrm{sgn}(\tau)x_1 - \pi \sum_{0<|k|<|\tau|} \frac{e^{-2\pi|k|\mathrm{sgn}(\tau)x_1}}{|k|} e_k(x').$$

We have that $H_\tau \in \mathcal{D}'(T)$ because it is a smooth function, and so $\Gamma_0 := \Gamma_\tau - H_\tau \in \mathcal{D}'(T)$ as well. Let us define the distributions $\Gamma_0^0 := -2\pi^2|x_1|$ and $\Gamma_0^* := \Gamma_0 - \Gamma_0^0$. Formally, the Fourier coefficients of Γ_0 and Γ_0^* are given by

$$\Gamma_{0,k}(x_1) = \begin{cases} -2\pi^2|x_1| & \text{if } \ k = 0, \\ \pi e^{-2\pi|k||x_1|}/|k| & \text{if } \ k \neq 0, \end{cases}$$

$$\Gamma_{0,k}^*(x_1) = \begin{cases} 0 & \text{if } \ k = 0, \\ \pi e^{-2\pi|k||x_1|}/|k| & \text{if } \ k \neq 0. \end{cases} \tag{5.4}$$

As in the proof of Proposition 5.1, we have that if $k \neq 0$, then

$$\widehat{\Gamma_{0,k}^*}(\xi) = \frac{1}{\xi^2 + |k|^2}. \tag{5.5}$$

Proposition 5.2. *The distributions* Γ_0^0 *and* Γ_0^* *are tempered distributions, thus so is* Γ_0.

Proof. It is clear that Γ_0^0 is a tempered distribution. From (5.5) we actually obtain that $\Gamma_0^* \in H^s(T)$ for all $s < -(d-3)/2$, and the conclusion follows. □

Proposition 5.3. *Let $\Gamma_\tau(x,y) := \Gamma_\tau(x-y)$. Then, $D^2\Gamma_\tau(x,\cdot) = \delta_T(x)$, $\Gamma_\tau(\cdot,\cdot)$ is smooth in $T \times T$ away from the diagonal.*

Proof. As mentioned in the proof of Proposition 5.1, the fact that $D^2\Gamma_\tau = \delta_T(0)$ and Weyl's regularity lemma imply that Γ_τ is smooth away from $(0,0) \in T$. This gives that $D^2\Gamma_\tau(x,\cdot) = \delta_T(x)$ and the smoothness of $\Gamma_\tau(\cdot,\cdot)$ away from the diagonal. □

Despite the reasons not being apparent at this moment, we consider the following definition. We will see later how this operator appears naturally when we try to reformulate the differential equation the solution $H_{V,W}u = 0$ as an integral equation. For the moment, in Proposition 5.5 below, we show how this operator relates to the distribution Γ_τ.

Definition 5.4. *Let $|\tau| \geq \tau_0$, $\tau^2 \notin Spec(-\Delta_{g_0})$, so that Δ_τ has a right inverse G_τ from Theorem 4.1. For functions in $L^2_c(T)$, we define the operator $K_\tau := e^{-2\pi\tau x_1} G_\tau e^{2\pi\tau x_1}$.*

Proposition 5.5. *The operator K_τ maps $L^2_c(T)$ into $H^2_{loc}(T)$, is translation invariant, commutes with differentiation, and satisfies $D^2 K_\tau = I$ on $L^2_c(T)$ and $K_\tau D^2 = I$ on $H^2_c(T)$. Moreover, its distributional kernel is $\Gamma_\tau(\cdot,\cdot)$, i.e. $K_\tau f(x) = \Gamma_\tau * f(x) = \langle \Gamma_\tau(x,\cdot), f\rangle$ for $f \in C^\infty_c(T)$.*

Proof. The first claim follows because G_τ maps $L^2_\delta(T)$ into $H^2_{-\delta}(T)$. The translation invariance of K_τ follows from the conjugation structure and the translation invariance of G_τ. Indeed, if we denote the translation operators by $t_y f(x) := f(x+y)$ and note that $t_y(e^{\lambda x} f) = e^{\lambda(x+y)} t_y f$, then

$$
\begin{aligned}
t_y K_\tau &= e^{-2\pi\tau(x_1+y_1)} t_y G_\tau e^{2\pi\tau x_1} \\
&= e^{-2\pi\tau(x_1+y_1)} G_\tau t_y e^{2\pi\tau x_1} \\
&= e^{-2\pi\tau(x_1+y_1)} G_\tau e^{2\pi\tau(x_1+y_1)} t_y = e^{-2\pi\tau x_1} G_\tau e^{2\pi\tau x_1} t_y = K_\tau t_y,
\end{aligned}
$$

as we wanted to prove. The commutativity with differentiation follows from the translation invariance. In addition, if $f \in L^2_c(T)$, then $e^{2\pi\tau x_1} f \in L^2_\delta(T)$, and so

$$D^2 K_\tau f = e^{-2\pi\tau x_1}(e^{2\pi\tau x_1} D^2 e^{-2\pi\tau x_1}) G_\tau(e^{2\pi\tau x_1} f) = e^{-2\pi\tau x_1}(\Delta_\tau G_\tau)(e^{2\pi\tau x_1} f) = f.$$

If $f \in H^2_c(T)$, then $D^2 f \in L^2_c(T)$, and the commutativity with differentiation yields that $K_\tau D^2 f = D^2 K_\tau f = f$. Finally, for $f \in C^\infty_c(T)$ we have that $e^{2\pi\tau x_1} f \in C^\infty_c(T) \subseteq \mathcal{S}(T)$, and so

$$\begin{aligned} K_\tau f(x) &= e^{-2\pi\tau x_1} G_\tau(e^{2\pi\tau x_1} f)(x) \\ &= e^{-2\pi\tau x_1} \int_T g_\tau(x_1 - y_1, x' - y') e^{2\pi\tau y_1} f(y_1, y') dy_1 dy' \\ &= \int_T e^{-2\pi\tau(x_1 - y_1)} g_\tau(x_1 - y_1, x' - y') f(y_1, y') dy_1 dy' = \int_T \Gamma_\tau(x, y) f(y) dy, \quad (5.6) \end{aligned}$$

as we wanted to prove. □

The purpose of what follows is to show that the mapping properties of K_τ from $L^2_c(T)$ into $H^2_{loc}(T)$ can be extended to $H^{-1}_c(T)$ into $H^1_{loc}(T)$. To show this, let us consider the operators

$$K^0_0 f := \Gamma^0_0 * f, \quad K^*_0 f := \Gamma^*_0 * f, \quad R_\tau f := H_\tau * f,$$

with the above definitions for Γ^0_0, Γ^*_0, and H_τ. Given that $\Gamma_\tau = \Gamma^0_0 + \Gamma^*_0 + H_\tau$, we have that $K_\tau = K^0_0 + K^*_0 + R_\tau$, and so it suffices to show that each of these maps $H^{-1}_c(T)$ into $H^1_{loc}(T)$.

Proposition 5.6. *The operator K^0_0 maps $H^{-1}_c(T)$ into $H^1_{loc}(T)$.*

Proof. Let $\varphi \in H^{-1}_c(T)$, so that $\varphi_0 \in H^{-1}_c(\mathbb{R}) \subseteq H^{-1}_c(T)$. Since $|x_1|$ does not depend on x', we have that $|x_1| * \varphi = |x_1| * \varphi_0$, and it remains to show that $|x_1| * \varphi_0 \in H^1_{loc}(\mathbb{R})$. Let $\mathrm{supp}(\varphi_0) \subseteq [-L, L]$ and let $\phi \in C^\infty_c(\mathbb{R})$ be such that $\phi \equiv 1$ on $[-L, L]$ and $\phi \equiv 0$ outside of $[-2L, 2L]$. Let us show that for fixed $x_1 \in \mathbb{R}$, $\phi|x_1 - \cdot| \in H^1(\mathbb{R})$. Indeed, by Leibniz' rule we obtain

$$\|\phi|x_1 - \cdot|\|_{H^1(\mathbb{R})} \lesssim \|\phi\|_{C^1(\mathbb{R})} \||x_1 - \cdot|\|_{H^1(-2L,2L)} \lesssim L + |x_1|,$$

where we allow the constants of the inequality to depend on L and ϕ. Let $\Phi(x_1) := |x_1| * \varphi_0$. Then,

$$\begin{aligned}
|\Phi(x_1)| &= |\langle \varphi_0, |x_1 - \cdot| \rangle| \\
&= |\langle \phi\varphi_0, |x_1 - \cdot| \rangle| \\
&= |\langle \varphi_0, \phi |x_1 - \cdot| \rangle| \leq \|\varphi_0\|_{H^{-1}(\mathbb{R})} \|\phi |x_1 - \cdot|\|_{H^1(\mathbb{R})} \lesssim \|\varphi_0\|_{H^{-1}(\mathbb{R})} (L + |x_1|).
\end{aligned}$$

This shows that $\Phi \in L^\infty_{loc}(\mathbb{R}) \subseteq L^2_{loc}(\mathbb{R})$, which implies that $\Phi' \in H^{-1}_{loc}(\mathbb{R})$. Moreover, because $|x_1|'' = 2\delta_{\mathbb{R}}(0)$, then we have that $\Phi'' = 2\varphi_0 \in H^{-1}(\mathbb{R})$. Let $\eta \in C^\infty_c(\mathbb{R})$ and $\rho = \eta\Phi$, so that $\rho \in L^2_c(\mathbb{R})$. We have to show that $\rho \in H^1(\mathbb{R})$, which is equivalent to showing that $\langle \xi \rangle \widehat{\rho}(\xi) \in L^2(\mathbb{R})$. Since $\rho \in L^2(\mathbb{R})$, we have $\langle \xi \rangle \widehat{\rho}(\xi) \in L^2(|\xi| \leq 1)$. Moreover, because $\Phi \in L^2_{loc}(\mathbb{R})$ and $\Phi', \Phi'' \in H^{-1}_{loc}(\mathbb{R})$, then $\rho'' = \eta''\Phi + 2\eta'\Phi' + \eta\Phi'' \in H^{-1}(\mathbb{R})$, Therefore, $\langle \xi \rangle^{-1} \xi^2 \widehat{\rho}(\xi) \in L^2(\mathbb{R})$, from where we conclude that $\langle \xi \rangle \widehat{\rho}(\xi) \in L^2(|\xi| \geq 1)$. This proves the result. □

Proposition 5.7. *The operator $K_0^* : H^s(T) \to H^{s+2}(T)$ is bounded for any $s \in \mathbb{R}$.*

Proof. For $\varphi \in H^s(T)$, let us consider the Fourier series $\varphi(x_1, x') = \sum_{k \in \mathbb{Z}^d} \varphi_k(x_1) e_k(x')$. From (5.4) we have that $\widehat{(K_0^*\varphi)}_0(\xi) = \widehat{(\Gamma_0^* * \varphi)}_0(\xi) = 0$, and for $k \neq 0$ we have

$$|\widehat{(K_0^*\varphi)}_k(\xi)| = |\widehat{(\Gamma_0^* * \varphi)}_k(\xi)| = |\widehat{\Gamma^*_{0,k}}(\xi)||\widehat{\varphi_k}(\xi)| = \frac{1}{\xi^2 + |k|^2} |\widehat{\varphi_k}(\xi)| \lesssim \langle \xi, k \rangle^{-2} |\widehat{\varphi_k}(\xi)|,$$

where we used in the last step that $|k| \geq 1$ for all $k \neq 0$. Therefore we conclude that

$$\|K_0^*\varphi\|^2_{H^{s+2}(T)} = \sum_{k \neq 0} \int_{\mathbb{R}} \langle \xi, k \rangle^{2s+4} |\widehat{(K_0^*\varphi)_k}(\xi)|^2 d\xi \lesssim \sum_{k \neq 0} \int_{\mathbb{R}} \langle \xi, k \rangle^{2s} |\widehat{\varphi_k}(\xi)|^2 d\xi = \|\varphi\|^2_{H^s(T)}.$$

□

Proposition 5.8. *The operator $K_0 := K_0^0 + K_0^*$ maps $L^2_c(T)$ into $H^2_{loc}(T)$ and $H^{-1}_c(T)$ into $H^1_{loc}(T)$, satisfies $D^2 K_0 = I$ on $L^2_c(T)$ and $K_0 D^2 = I$ on $H^2_c(T)$, and is symmetric, i.e.*

$\langle K_0 f, g\rangle = \langle K_0 g, f\rangle$ *for any* $f, g \in H_c^{-1}(T)$.

Proof. The operator $R_\mathcal{T}$ maps $L_c^2(T)$ into $C^\infty(T)$, because the kernel $H_\mathcal{T}$ is a smooth function. Therefore, $K_0 = K_\mathcal{T} - R_\mathcal{T}$ also maps $L_c^2(T)$ into $H_{loc}^2(T)$. Moreover, we have that K_0^0 and K_0^* map $H_c^{-1}(T)$ into $H_{loc}^1(T)$ from Proposition 5.6 and Proposition 5.7, and therefore so does K_0. The identities with the Laplacian follow from those of Proposition 5.5, since the kernel of $R_\mathcal{T}$ is a harmonic function. Finally, the symmetry of the operator follows from the symmetry of the kernel $\Gamma_0 := \Gamma_0^0 + \Gamma_0^*$. □

Proposition 5.9. *The operator* $K_\mathcal{T}$ *maps* $H_c^{-1}(T)$ *into* $H_{loc}^1(T)$.

Proof. Recall that $K_\mathcal{T} = K_0 + R_\mathcal{T}$. The result follows from Proposition 5.8 and the fact that $R_\mathcal{T}$ maps $H_c^{-1}(T)$ into $C^\infty(T)$. □

5.1.2 τ–dependent single layer potential

Recall that we have the boundedness of $\mathrm{tr} : H^s(T) \to H^{s-1/2}(\partial M)$, for $s > 1/2$. In particular we have $\mathrm{tr} : H^1(T) \to H^{1/2}(\partial M)$ and its adjoint $\mathrm{tr}^* : H^{-1/2}(\partial M) \to H_c^{-1}(T)$. The results from Proposition 5.8 and Proposition 5.9 allow for the following definition.

Definition 5.10. *Define the single layer operator* $S_0 := K_0 tr^* : H^{-1/2}(\partial M) \to H_{loc}^1(T)$. *Similarly, for* $|\tau| \geq \tau_0$, $\tau^2 \notin Spec(-\Delta_{g_0})$, *we define the* τ*–dependent single layer operator*

$$S_\mathcal{T} := K_\mathcal{T} tr^* : H^{-1/2}(\partial M) \to H_{loc}^1(T).$$

Proposition 5.11. *Let* S *denote either of the single layer operators* S_0 *or* $S_\mathcal{T}$ *from Definition 5.10, and let* Γ *and* K *denote either of* Γ_0 *and* K_0 *or* $\Gamma_\mathcal{T}$ *and* $K_\mathcal{T}$*, as it corresponds. For* $\varphi \in H^{-1/2}(\partial M)$*, the single layer potential* $S\varphi \in H_{loc}^1(T)$ *satisfies the following properties:*

a). for $x \notin \partial M$ *we have the integral representation* $S\varphi(x) = \langle \varphi, tr(\Gamma(x, \cdot))\rangle$,

b). $S\varphi$ *is harmonic in* $M_\pm$, *c). $S\varphi$ has no jump at the boundary, $tr^+(S\varphi) = tr^-(S\varphi)$, and therefore has a well-defined trace,*

d). the normal derivatives of $S\varphi$ satisfy that $\partial_\nu^- S\varphi - \partial_\nu^+ S\varphi = 4\pi^2\varphi$ on ∂M,

e). if $\varphi \in H^{1/2}(\partial M)$, then $S\varphi|_M \in H^2(M)$, $S\varphi|_{M_+}$ has an extension in $H^2_{loc}(T)$, and $tr \circ S$ maps $H^{1/2}(\partial M)$ into $H^{3/2}(\partial M)$.

Proof. Let $\varphi \in H^{-1/2}(\partial M)$. For a fixed $x \notin \partial M$ there exists an open neighborhood $N \subseteq T$, such that $\partial M \subseteq N$ and $x \notin N$. From Proposition 5.3 and the fact that H_τ is smooth we have that $\Gamma(x, \cdot)$ is smooth in N and so $\mathrm{tr}(\Gamma(x, \cdot)) \in H^{1/2}(\partial M)$. Therefore,

$$\langle \varphi, \mathrm{tr}(\Gamma(x, \cdot))\rangle = \langle \mathrm{tr}^*\varphi, \Gamma(x, \cdot)\rangle = \Gamma * \mathrm{tr}^*\varphi(x) = K\mathrm{tr}^*\varphi(x) = S\varphi(x).$$

The harmonicity of $S\varphi$ in $M_\pm$ follows from the previous result as $\Gamma(\cdot, y)$ is harmonic in $M_\pm$ for any $y \in \partial M$. The existence of a well-defined trace follows from the fact that $S\varphi \in H^1_{loc}(T)$. Given that $S\varphi \in H^1_{loc}(T)$ is harmonic in $M_\pm$, there are well-defined normal derivatives as elements of $H^{-1/2}(\partial M)$. Moreover, $K_\tau - K_0 = R_\tau$ maps $H^{-1}_c(T)$ into $C^\infty(T)$, so it suffices to show the jump condition for $S = S_0$. Let $g \in H^{3/2}(\partial M)$ and let $v \in H^2_c(T)$ be some function extending g. The definition of normal derivatives (2.3), integration by parts, and Proposition 5.8 give that

$$\begin{aligned}
\langle (\partial_\nu^- - \partial_\nu^+) S_0\varphi, g\rangle &= 4\pi^2 \int_T -DS_0\varphi \cdot Dv \\
&= 4\pi^2 \int_T S_0\varphi D^2 v \\
&= 4\pi^2 \langle K_0 \mathrm{tr}^*\varphi, D^2 v\rangle = 4\pi^2 \langle \mathrm{tr}^*\varphi, K_0 D^2 v\rangle = 4\pi^2 \langle \mathrm{tr}^*\varphi, v\rangle = 4\pi^2 \langle \varphi, g\rangle.
\end{aligned}$$

The density of $H^{3/2}(\partial M)$ in $H^{1/2}(\partial M)$ implies the jump condition of the normal derivatives. Finally, as in [KSU11b], we invoke the transmission property from [McL00] to prove the higher regularity properties of the single layer potential. Namely, the harmonicity of $S\varphi|_{M_\pm}$

and the jump conditions at the boundary give that if $\varphi \in H^{1/2}(\partial M)$, then $S\varphi|_{M_\pm}$ is in $H^2(M_\pm \cap N)$, for some neighborhood $N \subseteq T$ of ∂M. The interior regularity of harmonic functions gives that $S\varphi|_M \in H^2(M)$ and $S\varphi|_{M_+} \in H^2_{loc}(M_+)$, and the boundary regularity allows to construct the extension of $S\varphi|_{M_+}$ to $H^2_{loc}(T)$. □

Remark. *We will not need this, but the map $tr \circ S_\tau : H^s(\partial M) \to H^{s+1}(\partial M)$ is bounded for $s \geq -1/2$.*

5.2 Equivalent formulations and boundary characterization

For the rest of the section we assume that 0 is not an eigenvalue of the magnetic Schrödinger operator $H_{V,W}$ on M. Let $|\tau| \geq \tau_0$, $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$ as in Theorem 1.2, and let $h \in H^2_{loc}(T)$ be a harmonic function.

Theorem 5.12. *All the following problems have a unique solution:*

(DE): $u = h + e^{-2\pi\tau x_1} r$, *with* $r \in H^2_{-\delta}(T)$, *solves the differential equation* $H_{V,W} u = 0$ *in* T,

(IE): $u \in H^2_{loc}(T)$ *solves the integral equation* $u + K_\tau X u = h$ *in* T,

(EP): $\widetilde{u} \in H^2_{loc}(M_+)$ *is harmonic, has an extension in* $H^2_{loc}(T)$ *of the form* $h + e^{-2\pi\tau x_1} r$ *with* $r \in H^2_{-\delta}(T)$, *and* $\partial^+_\nu \widetilde{u} = 4\pi^2 \Lambda_{V,W}(tr^+(\widetilde{u}))$,

(BE): $f \in H^{3/2}(\partial M)$ *solves the boundary equation* $(I + tr \circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0}))f = tr(h)$.

These problems are equivalent in the following sense:

(DE) ⇔ (IE): u solves (DE) if and only if u solves (IE),

(DE) ⇔ (EP): if u solves (DE), then $u|_{M_+}$ solves (EP), and if $\widetilde{u}$ solves (EP), then there exists an extension u to T that solves (DE),

(DE) ⇒ (BE): if u solves (DE), then $tr(u)$ solves (BE),

(BE) ⇒ (EP): if f solves (BE), then there is an extension $\widetilde{u}$ to M_+ that solves (EP).

Proof. From Theorem 1.2 we know that (DE) has a unique solution. It remains to show the equivalence between the existence of solutions, as the equivalence of the uniqueness follows from this.

We start proving that the problems (DE) and (IE) are equivalent. Assume that a solution to the equation $H_{V,W}u = 0$, has the form $u = h + e^{-2\pi\tau x_1}r$, with $r \in H^2_{-\delta}(T)$. Then $u \in H^2_{loc}(T)$ and we see that r solves

$$\Delta_\tau r = e^{2\pi\tau x_1}D^2(u-h) = e^{2\pi\tau x_1}D^2u = -e^{2\pi\tau x_1}Xu,$$

and $e^{2\pi\tau x_1}Xu \in L^2_c(M)$. Since $r \in H^2_{-\delta}(T)$, the uniqueness from Theorem 1.2 implies that $r = -G_\tau(e^{2\pi\tau x_1}Xu)$, and so $h = u - e^{-2\pi\tau x_1}r = u + K_\tau Xu$. Conversely, if $u \in H^2_{loc}(T)$ satisfies $u + K_\tau Xu = h$, then $u = h + e^{-2\pi\tau x_1}r$ with $r := -G_\tau(e^{2\pi\tau x_1}Xu) \in H^2_{-\delta}(T)$. This gives that

$$e^{2\pi\tau x_1}D^2u = e^{2\pi\tau x_1}D^2(u-h) = \Delta_\tau r = -e^{2\pi\tau x_1}Xu,$$

and thus $H_{V,W}u = 0$.

Now we show that the problems (DE) and (EP) are equivalent. Assume that a solution to the equation $H_{V,W}u = 0$, has the form $u = h + e^{-2\pi\tau x_1}r$, with $r \in H^2_{-\delta}(T)$, so that $u \in H^2_{loc}(T)$. If we let $\widetilde{u} := u|_{M_+}$, then $\widetilde{u} \in H^2_{loc}(M_+)$, and, given that V, W are supported in M, we have that $\widetilde{u}$ is harmonic in M_+. If $g \in H^{1/2}(\partial M)$ and $v \in H^1_c(T)$ is some function extending g, then from the definitions (2.3) and (1.2) we have

$$\langle \partial^+_\nu \widetilde{u}, g\rangle = -4\pi^2 \int_{M_+} -D\widetilde{u}\cdot Dv,$$

$$\langle \Lambda_{V,W}(\mathrm{tr}^-(u)), g\rangle = \int_M -Du\cdot Dv + V\cdot(vDu - uDv) + (V^2+W)uv.$$

Since u is a solution to $H_{V,W}u = 0$ in T, and V, W are supported in M, we obtain that

$$-\int_{M_+} -D\widetilde{u} \cdot Dv = -\int_{M_+} -Du \cdot Dv = \int_{M} -Du \cdot Dv + V \cdot (vDu - uDv) + (V^2 + W)uv$$

which gives that $\partial_\nu^+ \widetilde{u} = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^-(u))$. Thus we conclude that

$$\partial_\nu^+ \widetilde{u} = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^-(u)) = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^+(u)) = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^+(\widetilde{u})).$$

Conversely, suppose that $\widetilde{u} \in H^2_{loc}(M_+)$ is harmonic in M_+, satisfies $\partial_\nu^+ \widetilde{u} = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^+(\widetilde{u}))$, and is such that $\widetilde{u}$ has an extension in $H^2_{loc}(T)$ of the form $h + e^{-2\pi\tau x_1} r$ on M_+ with $r \in H^2_{-\delta}(T)$. We want to extend $\widetilde{u}$ to the interior of M in order to solve $H_{V,W}u = 0$ in T. Let $\overline{u} = D_{V,W}(\mathrm{tr}^+(\widetilde{u})) \in H^2(M)$, i.e. the solution of the problem

$$\begin{cases} H_{V,W}\overline{u} = 0 & \text{in } M_-, \\ \overline{u} = \mathrm{tr}^+(\widetilde{u}) & \text{on } \partial M. \end{cases}$$

and define $u|_{M_+} = \widetilde{u} \in H^2_{loc}(M_+)$ and $u|_M = \overline{u} \in H^2(M)$. Then we have

$$\mathrm{tr}^+(u) = \mathrm{tr}^+(\widetilde{u}) = \mathrm{tr}^-(\overline{u}) = \mathrm{tr}^-(u),$$

$$\partial_\nu^+ u = \partial_\nu^+ \widetilde{u} = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^+(\widetilde{u})) = 4\pi^2 \Lambda_{V,W}(\mathrm{tr}^-(\overline{u})) = \partial_\nu^- \overline{u} = \partial_\nu^- u,$$

where we used the result from Proposition 2.8. This implies that u is in $H^2_{loc}(T)$ and solves $H_{V,W}u = 0$. Moreover, $u = h + e^{-2\pi\tau x_1}\overline{r}$ in T, with $\overline{r} \in H^2_{-\delta}(T)$, where $\overline{r}|_{M_+} = r|_{M_+}$ and $\overline{r}|_M = e^{2\pi\tau x_1}(u - h)|_M$.

Now we prove that (DE) implies (BE). Assume that a solution to the equation $H_{V,W}u = 0$ has the form $u = h + e^{-2\pi\tau x_1} r$, with $r \in H^2_{-\delta}(T)$. Then we have $u \in H^2_{loc}(T)$, and so $\mathrm{tr}(u) \in H^{3/2}(\partial M)$. The equivalence between (DE) and (IE) yields that $u + K_\tau X u = h$, which gives $\mathrm{tr}(u) + \mathrm{tr} K_\tau X u = \mathrm{tr}(h)$. Taking (exterior) traces in Proposition 5.13 below, gives that $\mathrm{tr}K_\tau Xu = \mathrm{tr}\circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0})\mathrm{tr}(u)$, which implies that $\mathrm{tr}(u)$ solves (BE).

Finally, we show that (BE) implies (EP). Suppose $f \in H^{3/2}(\partial M)$ solves the boundary equation $(I + \mathrm{tr}\circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0}))f = \mathrm{tr}(h)$. Motivated by Proposition 5.13 below, we define $\widetilde{u} := h - S_\tau(\Lambda_{V,W} - \Lambda_{0,0})f$. The boundary equation gives that $\mathrm{tr}(\widetilde{u}) = f$. From Proposition 5.11 we know that the restrictions $\widetilde{u}|_{M_\pm}$ are in $H^2_{loc}(M_\pm)$ and are harmonic in $M_\pm$, respectively. Since $\widetilde{u}|_M$ is harmonic in M and $\mathrm{tr}(\widetilde{u}) = f$, then we have $\partial_\nu^- \widetilde{u} = 4\pi^2\Lambda_{0,0}f$. Given that $h \in H^2_{loc}(T)$, the definition of $\widetilde{u}$, and the jump condition of the normal derivatives from Proposition 5.11 we obtain that

$$\partial_\nu^+ \widetilde{u} = \partial_\nu^- \widetilde{u} + 4\pi^2(\Lambda_{V,W} - \Lambda_{0,0})f = 4\pi^2\Lambda_{V,W}f.$$

From Proposition 5.11 we know that $\widetilde{u}|_{M_+}$ has an extension in $H^2_{loc}(T)$. All that remains is to show that $\widetilde{u}|_{M_+}$ has an extension in $H^2_{loc}(T)$ of the form $h + e^{-2\pi\tau x_1}r$, with $r \in H^2_{-\delta}(T)$. Given that $\widetilde{u} := h - S_\tau\phi$ with $\phi \in H^{1/2}(\partial M)$, all we have to show is that $e^{2\pi\tau x_1}S_\tau\phi|_{M_+}$ has an extension in $H^2_{-\delta}(T)$. From Proposition 5.11 we know that it has an extension in $H^2_{loc}(T)$, so it suffices to show that $e^{2\pi\tau x_1}S_\tau\phi$ is in $H^2_{-\delta}(|x_1| \geq L)$ for some large L. From the integral representation in Proposition 5.11 we see that

$$e^{2\pi\tau x_1}S_\tau\phi(x) = e^{2\pi\tau x_1}\langle\phi, \mathrm{tr}(\Gamma_\tau(x,\cdot))\rangle = \langle e^{2\pi\tau y_1}\phi, \mathrm{tr}(g_\tau(x,\cdot))\rangle,$$

where we have used that $\Gamma_\tau(x,y) = e^{-2\pi\tau(x_1-y_1)}g_\tau(x,y)$. From Proposition 5.1 we have that the restrictions $\{\mathrm{tr}(D_x^\alpha g_\tau(x,\cdot))\}$ are uniformly bounded for $|x_1| \geq L$ with L large. This implies that $D^\alpha(e^{2\pi\tau x_1}S_\tau\phi)$ is uniformly bounded for $|x_1| \geq L$, and we conclude that $e^{2\pi\tau x_1}S_\tau\phi \in H^2_{-\delta}(|x_1| \geq L)$, as desired. □

The following identity, which follows by integration by parts, is at the core of the results of this section, and we consider it interesting in its own.

Proposition 5.13. *Let $u \in H^2(M)$ satisfy $H_{V,W}u = 0$ in M. Let $J : H^2(M) \twoheadrightarrow H^1(M)$ be* *the compact embedding, and let* $E : L^2(M) \to L^2(T)$ *denote the extension by zero, so that* $EXJu \in L^2_c(T)$*. For* $x \in M_+$ *we have the identity*

$$K_\tau(EXJu)(x) = S_\tau[(\Lambda_{V,W} - \Lambda_{0,0})tr^-(u)](x). \tag{5.7}$$

Proof. Let $x \in M_+$ be fixed, so that $\Gamma_\tau(x, \cdot)$ is smooth and harmonic in a neighborhood M. From the integral representation (5.6) and the fact that $EXJu$ is supported in M we get that

$$\begin{aligned} K_\tau(EXJu)(x) &= \int_T \Gamma_\tau(x, \cdot)EXJu \\ &= \int_M \Gamma_\tau(x, \cdot)Xu = \int_M \Gamma_\tau(x, \cdot)(2V \cdot Du + (V^2 + D \cdot V + W)u). \end{aligned} \tag{5.8}$$

From the integral representation in Proposition 5.11 and the definition of the DN map (1.2) we have that

$$\begin{aligned} &S_\tau(\Lambda_{V,W}\mathrm{tr}^-(u))(x) \\ &\qquad = \langle \Lambda_{V,W}\mathrm{tr}^-(u), \mathrm{tr}(\Gamma_\tau(x, \cdot))\rangle \\ &\qquad = \int_M -Du \cdot D\Gamma_\tau(x, \cdot) + V \cdot (\Gamma_\tau(x, \cdot)Du - uD\Gamma_\tau(x, \cdot)) + (V^2 + W)u\Gamma_\tau(x, \cdot). \end{aligned}$$

From the integral representation in Proposition 5.11, the harmonicity of $\Gamma_\tau(x, \cdot)$ in M, the definition (2.4) of the DN map $\Lambda_{0,0}$ and its symmetry we have that

$$\begin{aligned} S_\tau(\Lambda_{0,0}\mathrm{tr}^-(u))(x) &= \langle \Lambda_{0,0}\mathrm{tr}^-(u), \mathrm{tr}(\Gamma_\tau(x, \cdot))\rangle \\ &= \langle \Lambda_{0,0}\mathrm{tr}(\Gamma_\tau(x, \cdot)), \mathrm{tr}^-(u)\rangle = \int_M -D\Gamma_\tau(x, \cdot) \cdot Du. \end{aligned}$$

Therefore, we obtain

$$S_\tau[(\Lambda_{V,W} - \Lambda_{0,0})\mathrm{tr}^-(u)](x) = \int_M V \cdot (\Gamma_\tau(x, \cdot)Du - uD\Gamma_\tau(x, \cdot)) + (V^2 + W)u\Gamma_\tau(x, \cdot). \tag{5.9}$$

From Proposition 2.7 we have that

$$\int_M \Gamma_\tau(x,\cdot)(V\cdot Du + (D\cdot V)u) + V\cdot(uD\Gamma_\tau(x,\cdot)) = 0,$$

which implies the equality of (5.8) and (5.9) as we wanted. □

Proposition 5.14. *The operator* $tr \circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0})$ *in* $H^{3/2}(\partial M)$ *is compact.*

Proof. Recall that for $f \in H^{3/2}(\partial M)$ we have $D_{V,W} f := u \in H^2(M)$ as the solution to the Dirichlet problem

$$\begin{cases} H_{V,W}u = 0 & \text{in } M_-, \\ u = f & \text{on } \partial M. \end{cases}$$

Let $J : H^2(M) \twoheadrightarrow H^1(M)$ be the compact embedding, and let $E : L^2(M) \to L^2(T)$ denote the extension by zero. Then we have $EXJu \in L^2_c(T)$, and so $K_\tau EXJu \in H^2_{loc}(T)$. For $x \in M_+$ we can rewrite the result from Proposition 5.13 as

$$S_\tau(\Lambda_{V,W} - \Lambda_{0,0})f(x) = K_\tau EXJu(x). \tag{5.10}$$

The trace of a single layer potential is well-defined, so we can take traces on both sides of (5.10) to obtain

$$\text{tr} \circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0}) = \text{tr}K_\tau EXJD_{V,W}. \tag{5.11}$$

To prove the result it suffices to express the right-hand side of (5.11) as a composition of bounded operators, together with the compact operator J. Recall that $K_\tau = e^{-2\pi\tau x_1} G_\tau e^{2\pi\tau x_1}$. All of the following are continuous operators,

$$D_{V,W} : H^{3/2}(\partial M) \to H^2(M), \quad J : H^2(M) \twoheadrightarrow H^1(M), \quad X : H^1(M) \to L^2(M),$$

$$e^{2\pi\tau x_1}E : L^2(M) \to L^2_\delta(T), \quad G_\tau : L^2_\delta(T) \to H^2_{-\delta}(T), \quad \text{tr} \circ e^{-2\pi\tau x_1} : H^2_{-\delta}(T) \to H^{3/2}(\partial M).$$

and this completes the proof. □

Corollary 5.15. *The operator* $I + tr \circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0})$ *in* $H^{3/2}(\partial M)$ *is continuous and invertible. In particular, the boundary values of the CGO, constructed as* $u = h + e^{-2\pi\tau x_1} r$, *can be determined by boundary measurements as*

$$tr(u) = (I + tr \circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0}))^{-1} tr(h).$$

Proof. The uniqueness of the solution to (BE) in Theorem 5.12 implies that the operator $I + \mathrm{tr} \circ S_\tau(\Lambda_{V,W} - \Lambda_{0,0})$ is injective. From Proposition 5.14 and Fredholm's alternative it follows that it is bijective, and therefore invertible by the Open Mapping Theorem. The fact that the boundary values of the CGO are given by the expression above follows from Theorem 5.12. □

CHAPTER 6

RECONSTRUCTION OF THE MAGNETIC FIELD

As mentioned in the introduction and the previous chapter, the purpose of proving the Carleman estimate Theorem 1.2 is using it to construct many special solutions to the equation $H_{V,W}u = 0$, in order to recover the magnetic field curl V. In contrast to the previous chapter, we restrict our attention to a particular kind of harmonic functions and show that we can find an amplitude, i.e. a correction factor, that gives appropriate estimates for the remainder term. The choice of the special harmonic functions $h = e^{\pm 2\pi|m|x_1} e_m(x')$, and not any arbitrary harmonic function, comes from the fact that $(Dh)^2 = 0$. We elaborate more on this in a remark after the construction in Proposition 6.2. These ideas follow the so-called WKB method, and are presented systematically for more general settings in Sections 2 and 5 in [DSFKSU09]; see also Section 4 in [KSU07] or Sections 2 and 3 in [DSFKSU07]. We proceed analogously to the proof Lemma 6.1. in [Sal06] in the Euclidean setting.

After the amplitude has been constructed, we define an analog of the scattering transform from [Nac88] and [Sal06], and show that the estimates for the remainder term allow to disregard them, so that from the boundary measurements we are able to recover integrals involving the magnetic potential. After some work, we will show that this allows for the reconstruction of the magnetic field.

The exposition here follows closely the method from [Sal06], until the part involving the analog of the scattering transform. The difference of the methods at this point is due to the fact that the integrals contain terms that are real exponentials (like in the Laplace transform), rather than complex exponentials (like in the Fourier transform). This difference seems difficult, if not impossible, to reconcile.

As in the previous chapter, we denote by $X := 2V \cdot D + (V^2 + D \cdot V + W)$ the compactly supported first order differential operator, so that $H_{V,W} = D^2 + X$. For the rest of the chapter, we fix the potentials V, W and the constants R, δ. Any quantities involving them, like the constants in the inequalities from the previous chapters, will be regarded as fixed.

6.1 Construction of CGOs

A special family of harmonic solutions in T is given by the products $e^{\pm 2\pi|m|x_1}e_m(x')$ for any $m \in \mathbb{Z}^d$. These solutions are analogous to the Calderón complex exponential solutions $e^{2\pi i\zeta\cdot x}$, where $\zeta \in \mathbb{C}^d$ and $\zeta\cdot\zeta = 0$. In our case, $\zeta \in \mathbb{C}^d$ is replaced by $(\pm i|m|, m) \in i\mathbb{R}\times\mathbb{Z}^d$. We construct the correction terms for these harmonic functions in order to solve the equation $H_{V,W}u = 0$, and make more explicit the corresponding estimates for the correction terms.

Proposition 6.1. *Let $1/2 < \delta < 1$ and assume that 0 is not an eigenvalue of $H_{V,W}$ in M. Let $m \in \mathbb{Z}^d$ and let $\tau > 0$ be such that $\tau^2 \notin Spec(-\Delta_{g_0})$. Then there exists a unique $r_{m,\tau} \in H^2_{-\delta}(T)$ such that*

$$u_{m,\tau} := e^{-2\pi|m|x_1}e_m(x') + e^{-2\pi\tau x_1}r_{m,\tau}$$

satisfies $H_{V,W}u_{m,\tau} = 0$. Moreover, the correction term satisfies the estimates

$$\|r_{m,\tau}\|_{L^2_{-\delta}(T)} \lesssim \frac{e^{2\pi|\tau-|m||R}\langle m\rangle}{\tau}, \quad \|r_{m,\tau}\|_{H^1_{-\delta}(T)} \lesssim e^{2\pi|\tau-|m||R}\langle m\rangle.$$

In particular, we obtain $\|r_{m,\tau}\|_{L^2_{-\delta}(T)} \lesssim 1$ if $|\tau - |m|| \lesssim 1$.

Proof. Given that $D^2(e^{-2\pi|m|x_1}e_m(x')) = 0$, we have that $u_{m,\tau}$ solves $H_{V,W}u_{m,\tau} = 0$ if and only if $r_{m,\tau}$ solves the equation

$$e^{2\pi\tau x_1}H_{V,W}e^{-2\pi\tau x_1}r_{m,\tau} = -e^{2\pi\tau x_1}H_{V,W}(e^{-2\pi|m|x_1}e_m(x')) = -e^{2\pi\tau x_1}X(e^{-2\pi|m|x_1}e_m(x')).$$

The right-hand side is compactly supported and thus in $L^2_\delta(T)$. Therefore, Theorem 1.2 gives the existence and uniqueness of a solution in $H^2_{-\delta}(T)$. Finally, we observe that the right-hand side equals

$$f := -e^{2\pi\tau x_1}X(e^{-2\pi|m|x_1}e_m(x')) = -[2V\cdot(i|m|, m) + (V^2 + D\cdot V + W)]e^{2\pi(\tau-|m|)x_1}e_m(x'),$$

so we can bound it by $\|f\|_{L^2_\delta(T)} \lesssim e^{2\pi|\tau-|m||R|}\langle m\rangle$. The estimate for the correction term $r_{m,\tau}$ follows from Theorem 1.2. □

The estimates of Proposition 6.1 for the correction term are not sharp enough to allow us to neglect them in a later "asymptotic expansion". In order to improve the estimates for the correction term, we need to modify the harmonic function $e^{-2\pi|m|x_1}e_m(x')$ appropriately as we show next.

Proposition 6.2. *Let $1/2 < \delta < 1$. There exist $\varepsilon, \sigma > 0$ such that for any $m \in \mathbb{Z}^d$, with $|m|$ sufficiently large, there is a smooth function $a_m(x_1, x')$, such that $a_m - 1$ is supported on $|x_1| \le 2|m|^\sigma$, and*

$$b_m(x_1, x') := e^{2\pi|m|x_1} H_{V,W} e^{-2\pi|m|x_1} e_m(x') a_m,$$

is supported on $|x_1| \le 2|m|^\sigma$ with $\|b_m\|_{L^2_\delta(T)} \lesssim |m|^{1-\varepsilon}$.

Remark. *For the rest of the chapter, the notation a_m, b_m does not represent the Fourier coefficients of some functions as in previous chapters.*

Proof. We compute the conjugated operators

$$e^{2\pi|m|x_1} D e^{-2\pi|m|x_1} e_m(x') = e_m(x')[(i|m|, m) + D],$$

$$e^{2\pi|m|x_1} D^2 e^{-2\pi|m|x_1} e_m(x') = e_m(x')[(i|m|, m) + D]^2 = e_m(x')[2(i|m|, m) \cdot D + D^2].$$

Therefore, we have the conjugation identity for operators

$$e^{2\pi|m|x_1} H_{V,W} e^{-2\pi|m|x_1} e_m(x') = e_m(x')[2(i|m|, m) \cdot (D + V) + H_{V,W}]. \qquad (6.1)$$

We could define $a_m := \exp(v_m)$, where v_m is the solution of the equation

$(i|m|, m) \cdot (Dv_m + V) = 0$. This equation can be rewritten as

$$iD_{x_1} v_m + \frac{m}{|m|} D_{x'} v_m = -\left(iF + \frac{m}{|m|} G\right).$$

From Theorem 4.3 we know that this equation has a unique solution which decays, and is bounded with bounded derivatives of all orders. The only inconvenient with this is that the term $D^2 v_m$ may not be in $L^2_\delta(T)$. Therefore, we are left to redefine $a_m := \exp(w_m)$, where $w_m := v_m \psi(x_1/|m|^\sigma)$, with $\sigma > 0$ to be determined and ψ a cutoff function such that $\psi(t) \equiv 1$ if $|t| \leq 1$ and $\psi(t) \equiv 0$ if $|t| \geq 2$. With this we have $a_m - 1$ is supported on $|x_1| \leq 2|m|^\sigma$ and

$$(D+V)a_m = a_m \left[(Dv_m + V)\psi\left(\frac{x_1}{|m|^\sigma}\right) + \left(1 - \psi\left(\frac{x_1}{|m|^\sigma}\right)\right) V + \frac{v_m}{2\pi i |m|^\sigma} \psi'\left(\frac{x_1}{|m|^\sigma}\right)(1, 0, \ldots, 0)\right].$$

Because V is compactly supported, we see that the second term vanishes if $|m|$ is sufficiently large. Moreover, the dot product of $(i|m|, m)$ with first term vanishes (by construction). From this and (6.1) we are left with

$$b_m = e_m(x')[2(i|m|, m) \cdot (D+V)a_m + H_{V,W} a_m] = e_m(x') \left[a_m \frac{2i|m| v_m}{2\pi i |m|^\sigma} \psi'\left(\frac{x_1}{|m|^\sigma}\right) + H_{V,W} a_m \right].$$

The first term is supported on $|m|^\sigma \leq |x_1| \leq 2|m|^\sigma$. Using the boundedness of a_m and the decay estimates for v_m from Theorem 4.3, we can bound the $L^2_\delta(T)$ norm of the first term by

$$\frac{|m|}{|m|^\sigma} \left(\int_{|m|^\sigma}^{2|m|^\sigma} \frac{1}{|x_1|^2} \langle x_1 \rangle^{2\delta} dx_1 \right)^{1/2} \lesssim \frac{|m|}{|m|^\sigma} \cdot |m|^{\sigma(2\delta - 1)/2} = |m|^{1 + \sigma(2\delta - 3)/2}.$$

For the second term, we use that $H_{V,W} a_m = (D^2 + X) a_m$. The term $X a_m$ represents no problem, as X is compactly supported (in $|x_1| \leq R$) and a_m has bounded derivatives of all orders by Theorem 4.3. We are left with $D^2 a_m = a_m (D^2 w_m + (D w_m)^2)$, which is supported on $|x_1| \leq 2|m|^\sigma$. In addition to the boundedness of a_m, from Theorem 4.3 we also know that $|Dw_m|, |D^2 w_m| \lesssim \langle x_1\rangle^{-1}$. Therefore we can bound the $L^2_\delta(T)$ norms of these terms by

$$\begin{aligned}
\|D^2 a_m\|_{L^2_\delta(T)} &\le \|a_m D^2 w_m\|_{L^2_\delta(T)} + \|a_m (Dw_m)^2\|_{L^2_\delta(T)} \\
&\lesssim \left(\int_0^{2|m|^\sigma} \frac{1}{\langle x_1\rangle^2}\langle x_1\rangle^{2\delta} dx_1\right)^{1/2} + \left(\int_0^{2|m|^\sigma} \frac{1}{\langle x_1\rangle^4}\langle x_1\rangle^{2\delta} dx_1\right)^{1/2} \\
&\lesssim \left(\int_0^{2|m|^\sigma} \frac{1}{\langle x_1\rangle^2}\langle x_1\rangle^{2\delta} dx_1\right)^{1/2} \lesssim |m|^{\sigma(2\delta-1)/2}.
\end{aligned}$$

Taking any $\sigma > 1$ we obtain that $\sigma(2\delta-1)/2 > 1 + \sigma(2\delta-3)/2$. For instance, if $\sigma = 2$, then we ensure that all these exponents are less than 1, as we wanted to prove. □

Remark. *In the setting of Proposition 6.1, the choice $a_m \equiv 1$ gives compact support for b_m, but we only obtain $\|b_m\|_{L^2_\delta(T)} \lesssim |m|$.*

Remark. *Observe that if $h = e^{-2\pi|m|x_1} e_m(x')$, then the condition $(Dh)^2 = 0$ makes the higher order terms in* (6.1) *disappear, leaving only to appropriately disregard the next order terms (in this case of order $|m|$).*

We use Proposition 6.2 to construct another solution to the equation $H_{V,W} u = 0$, whose correction term has small norm. We observe that the "main terms" ($e^{-2\pi|m|x_1} e_m(x')$ and $e^{-2\pi|m|x_1} e_m(x') a_m$) of the two solutions coincide for $|x_1| \ge 2|m|^\sigma$, and we later prove that the corrected solutions must coincide.

Proposition 6.3. *Let $1/2 < \delta < 1$, and let $\varepsilon, \sigma > 0$ be as in Proposition 6.2. Let $m \in \mathbb{Z}^d$, with $|m|$ sufficiently large, and let $\tau > 0$ such that $\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$. There exists a unique function $\widetilde{r}_{m,\tau} \in H^2_{-\delta}(T)$, such that*

$$\widetilde{u}_{m,\tau} := e^{-2\pi|m|x_1} e_m(x') a_m + e^{-2\pi\tau x_1} \widetilde{r}_{m,\tau},$$

satisfies $H_{V,W} \widetilde{u}_{m,\tau} = 0$. Moreover, the correction term satisfies the estimates

$$\|\widetilde{r}_{m,\tau}\|_{L^2_{-\delta}(T)} \lesssim \frac{e^{4\pi|\tau-|m|||m|^\sigma} |m|^{1-\varepsilon}}{\tau}, \quad \|\widetilde{r}_{m,\tau}\|_{H^1_{-\delta}(T)} \lesssim e^{4\pi|\tau-|m|||m|^\sigma} |m|^{1-\varepsilon}.$$

In particular, if $|\tau - |m|||m|^\sigma \lesssim 1$*, then*

$$\|\widetilde{r}_{m,\tau}\|_{L^2_{-\delta}(T)} \lesssim |m|^{-\varepsilon}, \quad \|\widetilde{r}_{m,\tau}\|_{H^1_{-\delta}(T)} \lesssim |m|^{1-\varepsilon}.$$

In addition, if $K \subseteq T$ *is a compact set, then*

$$\|e^{2\pi(|m|-\tau)x_1}\widetilde{r}_{m,\tau}\|_{L^2(K)} \lesssim |m|^{-\varepsilon}, \quad \|e^{2\pi|m|x_1}D(e^{-2\pi\tau x_1}\widetilde{r}_{m,\tau})\|_{L^2(K)} \lesssim |m|^{1-\varepsilon},$$

where the constant of the inequality may depend on K*.*

Proof. We have that $H_{V,W}\widetilde{u}_{m,\tau} = 0$ if and only if there exists $\widetilde{r}_{m,\tau}$ which solves

$$e^{2\pi\tau x_1}H_{V,W}e^{-2\pi\tau x_1}\widetilde{r}_{m,\tau} = -e^{2\pi(\tau-|m|)x_1}b_m.$$

The conclusion follows from Theorem 1.2 and Proposition 6.2. □

Proposition 6.4. *The solutions to the equation* $H_{V,W}u = 0$ *constructed in Proposition 6.1 and Proposition 6.3 are equal.*

Proof. We write

$$\begin{aligned}\widetilde{u}_{m,\tau} &= e^{-2\pi|m|x_1}e_m(x')a_m + e^{-2\pi\tau x_1}\widetilde{r}_{m,\tau} \\ &= e^{-2\pi|m|x_1}e_m(x') + e^{-2\pi\tau x_1}(\widetilde{r}_{m,\tau} + e^{2\pi(\tau-|m|)x_1}e_m(x')(a_m - 1)).\end{aligned}$$

We have that $a_m - 1$ is a smooth bounded function supported on $|x_1| \leq 2|m|^\sigma$; in particular, $e^{2\pi(\tau-|m|)x_1}e_m(x')(a_m - 1) \in H^2_{-\delta}(T)$. The fact that $H_{V,W}\widetilde{u}_{m,\tau} = 0$ and the uniqueness from Proposition 6.1 give that we must have $\widetilde{u}_{m,\tau} = u_{m,\tau}$. □

6.2 Transforms and integrals

Recall that for the Laplacian $H_{0,0} := D^2$ in M there is a well-defined Dirichlet-to-Neumann map $\Lambda_{0,0}$. Moreover, this map is symmetric. If u and ϕ are solutions to $H_{V,W}u = 0$ and $H_{0,0}\phi = 0$, respectively, then we have the integral identities

$$\langle \Lambda_{V,W}\mathrm{tr}(u), \mathrm{tr}(\phi)\rangle = \int_M -Du \cdot D\phi + V \cdot (\phi Du - uD\phi) + (V^2 + W)u\phi,$$

$$\langle \Lambda_{0,0}\mathrm{tr}(u), \mathrm{tr}(\phi)\rangle = \langle \Lambda_{0,0}\mathrm{tr}(\phi), \mathrm{tr}(u)\rangle = \int_M -Du \cdot D\phi,$$

and so we obtain

$$\langle (\Lambda_{V,W} - \Lambda_{0,0})\mathrm{tr}(u), \mathrm{tr}(\phi)\rangle = \int_M V \cdot (\phi Du - uD\phi) + (V^2 + W)u\phi. \tag{6.2}$$

Let $m, n \in \mathbb{Z}^d$, $m, n \neq 0$, be fixed. Let $m_N := Nm \in \mathbb{Z}^d$, where $N > 0$ is a large integer parameter. Observe first, that $m_N/|m_N| = m/|m|$. With the notation from the last section, we see from Theorem 4.3 that $v_{m_N} = v_m$, as $m_N/|m_N| = m/|m|$ and both functions are the decaying solutions to the equation

$$iD_{x_1}v + \frac{m}{|m|} \cdot D_{x'}v = -\left(iF + \frac{m}{|m|} \cdot G\right).$$

According to the construction in Proposition 6.2, if N is large enough (depending only on R and σ) and $|x_1| \leq R$, then

$$\begin{aligned} a_{m_N}(x_1, x') :=&\ \exp\left(v_{m_N}\psi\left(\frac{x_1}{|m_N|^\sigma}\right)\right) \\ =& \exp\left(v_m\psi\left(\frac{x_1}{|m_N|^\sigma}\right)\right) = \exp(v_m) =: \widetilde{a}_m(x_1, x'). \end{aligned} \tag{6.3}$$

Let $u_{m_N,\tau}$ be the solution to $H_{V,W}u = 0$ constructed in the previous section as the correction of the harmonic function $e^{-2\pi|m_N|x_1}e_{m_N}(x')$. We choose $\tau = \tau(m, N, \sigma)$ to satisfy

$|\tau - |m_N|||m_N|^\sigma \lesssim 1$, so we have the last estimates in Proposition 6.3 for the correction $\widetilde{r}_{m_N,\tau}$ on the compact set M. For the choice of test function we consider the harmonic function $\phi_{m_N,n} = e^{2\pi|m_N+n|x_1}e_{-(m_N+n)x'}$. Using (6.2) we define the transform

$$\begin{aligned}T(m,n,N) &:= \langle(\Lambda_{V,W} - \Lambda_{0,0})\mathrm{tr}(u_{m_N,\tau}), \mathrm{tr}(\phi_{m_N,n})\rangle \\ &= \int_M V\cdot(\phi_{m_N,n}Du_{m_N,\tau} - u_{m_N,\tau}D\phi_{m_N,n}) + (V^2+W)u_{m_N,\tau}\phi_{m_N,n},\end{aligned}$$

From Corollary 5.15 we obtain that the transform $T(m,n,N)$ is determined by the knowledge of M and $\Lambda_{V,W}$. In [Nac88] and [Sal06] this is referred as the *scattering transform*; that name does not seem appropriate in our setting. Let us look at each term of the previous expression on $M \subseteq [-R,R]\times\mathbb{T}^d$. From Proposition 6.3 and (6.3) we have that

$$u_{m_N,\tau} = e^{-2\pi|m_N|x_1}e_{m_N}(x')\widetilde{a}_m + e^{-2\pi\tau x_1}\widetilde{r}_{m_N,\tau},$$

$$Du_{m_N,\tau} = e^{-2\pi|m_N|x_1}e_{m_N}(x')\widetilde{a}_m[(i|m_N|,m_N) + Dv_m] + D(e^{-2\pi\tau x_1}\widetilde{r}_{m_N,\tau}),$$

where we have used that $\widetilde{a}_m := \exp(v_m)$ for the second expression. From Theorem 4.3 and Proposition 6.3 we obtain that

$$u_{m_N,\tau} = e^{-2\pi|m_N|x_1}e_{m_N}(x')\widetilde{a}_m + e^{-2\pi|m_N|x_1}R_1,$$

$$Du_{m_N,\tau} = e^{-2\pi|m_N|x_1}e_{m_N}(x')\widetilde{a}_m(i|m_N|,m_N) + e^{-2\pi|m_N|x_1}R_2,$$

with $\|R_i\|_{L^2(M)} = o(N)$. We also have $D\phi_{m_N,n} = (-i|m_N+n|, -(m_N+n))\phi_{m_N,n}$. Thus,

$$\phi_{m_N,n}Du_{m_N,\tau} = e^{2\pi(|m_N+n|-|m_N|)x_1}e_{-n}(x')\widetilde{a}_m(i|m_N|,m_N) + e^{2\pi(|m_N+n|-|m_N|)x_1}\widetilde{R}_1,$$

$$u_{m_N,\tau} D\phi_{m_N,n} = e^{2\pi(|m_N+n|-|m_N|)x_1} e_{-n}(x')\widetilde{a}_m(-i|m_N+n|, -(m_N+n)) + e^{2\pi(|m_N+n|-|m_N|)x_1}\widetilde{R}_2,$$

$$u_{m_N,\tau}\phi_{m_N,n} = e^{2\pi(|m_N+n|-|m_N|)x_1} e_{-n}(x')\widetilde{a}_m + e^{2\pi(|m_N+n|-|m_N|)x_1}\widetilde{R}_3,$$

with $\|\widetilde{R}_i\|_{L^2(M)} = o(N)$. Finally, observe that

$$|m_N+n| - |m_N| = \frac{(|m_N|^2 + 2m_N\cdot n + |n|^2) - |m_N|^2}{|m_N+n|+|m_N|} = \frac{N}{N}\cdot\frac{2m\cdot n + \frac{|n|^2}{N}}{|m+\frac{n}{N}|+|m|} \to \frac{m\cdot n}{|m|} =: \mu_{m,n},$$

as $N\to+\infty$. These computations and the estimates from Theorem 4.3 and Proposition 6.3 give that

$$\|\phi_{m_N,n} Du_{m_N,\tau} - e^{2\pi\mu_{m,n}x_1} e_{-n}(x')\widetilde{a}_m(i|m_N|, m_N)\|_{L^2(M)} = o(N),$$

$$\|u_{m_N,\tau} D\phi_{m_N,n} + e^{2\pi\mu_{m,n}x_1} e_{-n}(x')\widetilde{a}_m(i|m_N|, m_N)\|_{L^2(M)} = o(N),$$

$$\|(V^2+W)u_{m_N,\tau}\phi_{m,n}\|_{L^2(M)} = o(N).$$

Thus, from the knowledge of the transform we are able to obtain the integrals

$$I(m,n) := \lim_{N\to+\infty}\frac{T(m,n,N)}{2N} = \int_M e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|, m)\cdot V\widetilde{a}_m.$$

We regard these integrals as a "mixed non-linear transform", in the sense that we have Laplace and Fourier transforms in the real and toroidal variables, respectively, and an additional term $\widetilde{a}_m(x_1, x')$.

6.3 Determination of the Fourier coefficients of the magnetic field

In order to reconstruct the curl of V, we could try to remove the "non-linear" term $\widetilde{a}_m$ from the mixed transform, i.e. to determine the integrals

$$J(m,n) := \int_M e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|,m)\cdot V,$$

and relate them to the integrals $I(m,n)$. These integrals contain real exponentials, instead of only complex exponentials as in [Sal06]. This will turn out in a significantly different result. In the appendix, we introduce the necessary notation and prove the following result.

Theorem 6.5. *We have the following cases depending on the sign of the dot product $m\cdot n$:*

1. *if $m\cdot n = 0$, then $J(m,n) = 0$,*
2. *if $m\cdot n > 0$, then*
$$J(m,n) = \sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{-2\pi}{|m|}\right)^{j-1} I_j^-(m,n)$$
3. *if $m\cdot n < 0$, then*
$$J(m,n) = \sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{2\pi}{|m|}\right)^{j-1} I_j^+(m,n)$$

Moreover, if $m\cdot n = \pm 1$, then $J(m,n) = I(m,n)$.

6.3.1 Relation between the families $\{I(m,n)\}$ and $\{J(m,n)\}$

In this subsection will not be concerned with the explicit relations between these two families of integrals, but rather on the existence of such relation. Let $[p,q] \subseteq \mathbb{R}$ be any interval containing $[-R,R]$ so that $M \subseteq [p,q]\times\mathbb{T}^d$. The condition $\operatorname{supp}(V)\subseteq M$ implies that

$$I(m,n) := \int_M e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|,m)\cdot V\widetilde{a}_m = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|,m)\cdot V\widetilde{a}_m.$$

Recall from (6.3) that $\widetilde{a}_m := \exp(v_m)$ and $(i|m|, m) \cdot (Dv_m + V) = 0$, so that $(i|m|, m) \cdot (D\widetilde{a}_m + V\widetilde{a}_m) = 0$. This and the fact that $(i|m|, m) \cdot D(e^{2\pi\mu_{m,n}x_1} e_{-n}(x')) = 0$ allow us to rewrite

$$\begin{aligned} I(m,n) &= -\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|, m) \cdot D\widetilde{a}_m \\ &= -\int_{[p,q]\times\mathbb{T}^d} (i|m|, m) \cdot D(e^{2\pi\mu_{m,n}x_1} e_{-n}(x')\widetilde{a}_m) \\ &= \frac{-|m|}{2\pi}\left(e^{2\pi\mu_{m,n}x_1} \int_{\mathbb{T}^d} e_{-n}(x')\widetilde{a}_m(x_1, x')dx' \right)\Bigg|_{x_1=p}^{x_1=q}, \end{aligned} \tag{6.4}$$

where the last equality follows from the Fundamental Theorem of Calculus and the fact that the torus $\mathbb{T}^d$ has no boundary. Recall that we are interested in determining the integrals

$$J(m,n) := \int_M e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|, m) \cdot V = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|, m) \cdot V.$$

Using that $(i|m|, m) \cdot (Dv_m + V) = 0$, we can proceed as before to obtain

$$\begin{aligned} J(m,n) &= -\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(i|m|, m) \cdot Dv_m \\ &= -\int_{[p,q]\times\mathbb{T}^d} (i|m|, m) \cdot D(e^{2\pi\mu_{m,n}x_1} e_{-n}(x')v_m) \\ &= \frac{-|m|}{2\pi}\left(e^{2\pi\mu_{m,n}x_1} \int_{\mathbb{T}^d} e_{-n}(x')v_m(x_1, x')dx' \right)\Bigg|_{x_1=p}^{x_1=q}. \end{aligned} \tag{6.5}$$

Now we show that we can determine the integrals in (6.5) from the knowledge of the integrals in (6.4). First, let us observe that these equalities hold for any p, q such that $M \subseteq [p, q] \times \mathbb{T}^d$. Therefore, if necessary we may only consider the case when p, q are large. In addition, observe that for determining the integrals in (6.5) it suffices to determine $v_m(x_1, x')$ for $|x_1|$ large. Moreover, by Theorem 4.3 we have $|v_m(x_1, x')| \to 0$ as $|x_1| \to +\infty$ (uniformly in x'), thus the knowledge of $\widetilde{a}_m(x_1, x') = \exp(v_m(x_1, x')) \to 1$ for $|x_1|$ large and the invertibility of $\exp(z)$ near $z = 0$ are sufficent to determine $v_m(x_1, x')$. More concretely, we can recover

$v_m(x_1, x')$ by the power series

$$v_m(x_1, x') = \log(\widetilde{a}_m(x_1, x')) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(\widetilde{a}_m(x_1, x') - 1)^j.$$

Then, the problem reduces to recover $\widetilde{a}_m(x_1, x')$ for $|x_1|$ large from the knowledge of the integrals in (6.4). Let us consider the Fourier series $v_m(x_1, x') = \sum_{k\in\mathbb{Z}^d} v_{m,k}(x_1)e_k(x')$, so that the Fourier coefficient $v_{m,k}(x_1)$ solves the equation

$$iD_{x_1}v_{m,k} + \frac{m\cdot k}{|m|}v_{m,k} = -\Big(iF_k + \frac{m}{|m|}\cdot G_k\Big).$$

By Theorem 4.8, the solution $v_{m,k}(x_1)$ vanishes in $(-\infty, -R]$ or $[R, +\infty)$ depending whether $m\cdot k \geq 0$ or $m\cdot k \leq 0$, respectively. Thus, for $|x_1| \geq R$ we have

$$v_m(x_1, x') = \begin{cases} v_m^+(x_1, x') := \sum_{m\cdot k>0} v_{m,k}(x_1)e_k(x') & \text{if } \ x_1 \geq R, \\ v_m^-(x_1, x') := \sum_{m\cdot k<0} v_{m,k}(x_1)e_k(x') & \text{if } \ x_1 \leq -R. \end{cases} \tag{6.6}$$

From this and (6.5) we obtain

$$J(m,n) = \frac{-|m|}{2\pi}\cdot\begin{cases} e^{2\pi\mu_{m,n}q}v_{m,n}(q) & \text{if } \ m\cdot n > 0, \\ -e^{2\pi\mu_{m,n}p}v_{m,n}(p) & \text{if } \ m\cdot n < 0, \\ 0 & \text{if } \ m\cdot n = 0. \end{cases} \tag{6.7}$$

Moreover, we also have

$$\widetilde{a}_m(x_1, x') = \exp(v_m(x_1, x')) = \begin{cases} \widetilde{a}_m^+(x_1, x') := \exp(v_m^+(x_1, x')) & \text{if } \ x_1 \geq R, \\ \widetilde{a}_m^-(x_1, x') := \exp(v_m^-(x_1, x')) & \text{if } \ x_1 \leq -R. \end{cases}$$

Let us consider the Fourier series $\widetilde{a}_m(x_1, x') = \sum_{k\in\mathbb{Z}^d} \widetilde{a}_{m,k}(x_1)e_k(x')$. Given the form of $v_m^{\pm}$ from (6.6) and the fact that the exponential is a power series, we conclude that

$$\widetilde{a}_m(x_1, x') = \begin{cases} \widetilde{a}_m^{+}(x_1, x') = 1 + \sum_{m\cdot k>0} \widetilde{a}_{m,k}(x_1)e_k(x') & \text{if } \ x_1 \geq R, \\ \widetilde{a}_m^{-}(x_1, x') = 1 + \sum_{m\cdot k<0} \widetilde{a}_{m,k}(x_1)e_k(x') & \text{if } \ x_1 \leq -R, \end{cases} \tag{6.8}$$

This and (6.4) give that

$$\begin{aligned} I(m,n) &= \frac{-|m|}{2\pi}\left(e^{2\pi\mu_{m,n}x_1}\int_{\mathbb{T}^d} e_{-n}(x')\widetilde{a}_m(x_1,x')dx'\right)\Bigg|_{x_1=p}^{x_1=q} \\ &= \frac{-|m|}{2\pi}\cdot\begin{cases} e^{2\pi\mu_{m,n}q}\widetilde{a}_{m,n}(q) & \text{if } \ m\cdot n > 0, \\ -e^{2\pi\mu_{m,n}p}\widetilde{a}_{m,n}(p) & \text{if } \ m\cdot n < 0, \\ 0 & \text{if } \ m\cdot n = 0. \end{cases} \end{aligned}$$

Recall that this holds for any p, q such that $[-R, R] \subseteq [p, q]$. From this and (6.8) we conclude that

$$\widetilde{a}_m(x_1, x') = 1 + \frac{2\pi}{|m|}\cdot\begin{cases} -\sum_{m\cdot n>0} I(m,n)e^{-2\pi\mu_{m,n}x_1}e_n(x') & \text{if } \ x_1 \geq R, \\ \sum_{m\cdot n<0} I(m,n)e^{-2\pi\mu_{m,n}x_1}e_n(x') & \text{if } \ x_1 \leq -R, \end{cases} \tag{6.9}$$

Therefore, we have shown that from the integrals $I(m,n)$ we are able to determine $\widetilde{a}_m(x_1, x')$ for $|x_1| \geq R$, which in turn determines $v_m(x_1, x')$ for $|x_1| \geq R$, and so the integrals $J(m,n)$. The explicit dependence of $J(m,n)$ on the family of integrals $\{I(m,k)\}$ is shown in the appendix.

6.3.2 Curl vectors and Laplace transform

Let us show how we can use the integrals $J(m,n)$ to recover the Fourier coefficients of curl V. Using that $\mathrm{supp}(V) \subseteq M$, we integrate by parts to compute the mixed transform of the terms involved in the magnetic field curl V,

$$
\begin{aligned}
&\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') D_{x_1} G_j \\
&= \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') i\mu_{m,n} G_j = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') \left(0, \frac{im\cdot n}{|m|}\delta_j\right)\cdot V, \\
&\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') D_{x'_j} F \\
&= \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') n_j F = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') (n_j, 0)\cdot V, \\
&\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') D_{x'_j} G_k \\
&= \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') n_j G_k = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') (0, n_j\delta_k)\cdot V,
\end{aligned}
$$

where $\delta_1, \ldots, \delta_d$ are the standard basis vectors in $\mathbb{R}^d$. This means that we are interested in determining the integrals

$$
\begin{aligned}
&\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') (D_{x_1} G_j - D_{x'_j} F) \\
&\qquad\qquad = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') \left(-n_j, \frac{im\cdot n}{|m|}\delta_j\right)\cdot V, \\
&\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') (D_{x'_j} G_k - D_{x'_k} G_j) \\
&\qquad\qquad = \int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x') (0, n_j\delta_k - n_k\delta_j)\cdot V.
\end{aligned}
$$

Therefore, the problem reduces to obtain the "curl vectors"

$$
\left\{ \left(-n_j, \frac{im\cdot n}{|m|}\delta_j\right), \quad (0, n_j\delta_k - n_k\delta_j) \right\}
$$

as linear combinations of vectors $\{(i|m|, m)\}$ while keeping n and $\mu_{m,n}$ fixed. Moreover, we would like to have this result for many values of μ. We show in Lemma 6.8, from the following section, that this is indeed the case, so that for fixed $n \neq 0$, the knowledge of the integrals $J(m,n)$ allows to determine the integrals

$$\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(D_{x_1}G_j - D_{x'_j}F)$$
$$= \int_p^q e^{2\pi\mu_{m,n}x_1}\left(\int_{\mathbb{T}^d} e_{-n}(x')(D_{x_1}G_j - D_{x'_j}F)dx'\right)dx_1,$$
$$\int_{[p,q]\times\mathbb{T}^d} e^{2\pi\mu_{m,n}x_1} e_{-n}(x')(D_{x'_j}G_k - D_{x'_k}G_j)$$
$$= \int_p^q e^{2\pi\mu_{m,n}x_1}\left(\int_{\mathbb{T}^d} e_{-n}(x')(D_{x'_j}G_k - D_{x'_k}G_j)dx'\right)dx_1.$$

for a sequence of values of $\mu_{m,n} = \gcd(n)/K$ converging to 0. For $f \in C_c^\infty([p,q])$, its Laplace transform

$$F(\mu) := \int_p^q e^{2\pi\mu x_1} f(x_1)dx_1$$

is an entire function, and therefore its knowledge along a convergent sequence is enough to recover the entire function F over all $\mathbb{C}$. We describe this reconstruction in Theorem 6.16 in the following section. The values of F over the imaginary axis correspond to the Fourier transform of f, and therefore it is possible to reconstruct f from the knowledge of F along a convergent sequence. This completes the reconstruction of the Fourier coefficients of curl V.

6.4 Appendices

6.4.1 Explicit relation between the families $\{I(m,n)\}$ and $\{J(m,n)\}$

Let us prove Theorem 6.5. We mentioned before that we were interested in computing $v_m = \log \widetilde{a}_m$ by a power series; in particular, we are concerned with expressions of the form $(\widetilde{a}_m - 1)^k$. In (6.9) we were able to express the Fourier series of $\widetilde{a}_m$ in terms of the integrals $I(m,n)$. In particular, for $x_1 \geq R$ we have

$$\widetilde{a}_m(x_1,x') - 1 = \frac{-2\pi}{|m|} \sum_{m\cdot k>0} I(m,k)e^{-2\pi\mu_{m,k}x_1} e_k(x').$$

Consider the set $T_j^+(m,k) = \{\kappa = (\kappa_1, \ldots, \kappa_j) \in (\mathbb{Z}^d)^j : m \cdot \kappa_i > 0, \quad \kappa_1 + \ldots + \kappa_j = k\}$. Observe that if $\kappa \in T_j^+(m,k)$, then

$$m \cdot k = m \cdot (\kappa_1 + \ldots + \kappa_j) \geq \gcd(m) + \ldots + \gcd(m) = j \gcd(m).$$

This implies that $T_j^+(m,k)$ is empty when $j > m \cdot k/\gcd(m)$; in particular it is empty when $m \cdot k < 0$. Let us define

$$I_j^+(m,k) = \sum_{\kappa \in T_j^+(m,k)} I(m,\kappa_1) \cdot I(m,\kappa_2) \cdot \ldots \cdot I(m,\kappa_j)$$

By our previous observation, we also see that $I_j^+(m,k) = 0$ if $j > m \cdot k/\gcd(m)$. Finally, using that $\mu_{m,k}$ is a linear function of k we obtain

$$(\widetilde{a}_m(x_1,x') - 1)^j = \left(\frac{-2\pi}{|m|}\right)^j \sum_{m \cdot k > 0} I_j^+(m,k) e^{-2\pi\mu_{m,k}x_1} e_k(x')$$

This implies that if $x_1 \geq R$, then

$$\begin{aligned} v_m(x_1,x') &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} (\widetilde{a}_m(x_1,x') - 1)^j \\ &= - \sum_{m \cdot k > 0} \left(\sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{2\pi}{|m|}\right)^j I_j^+(m,k) \right) e^{-2\pi\mu_{m,k}x_1} e_k(x'). \end{aligned}$$

From this and (6.7) we conclude that if $m \cdot n > 0$, then

$$J(m,n) = \frac{-|m|}{2\pi} e^{2\pi\mu_{m,n}q} v_{m,n}(q) = \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{2\pi}{|m|}\right)^{j-1} I_j^+(m,n) \tag{6.10}$$

Similarly, if $x_1 \leq -R$ we have

$$\widetilde{a}_m(x_1,x') - 1 = \frac{2\pi}{|m|} \sum_{m \cdot k < 0} I(m,k) e^{-2\pi\mu_{m,k}x_1} e_k(x').$$

We consider $T_j^-(m,k) = \{\kappa = (\kappa_1, \ldots, \kappa_j) \in (\mathbb{Z}^d)^j : m \cdot \kappa_i < 0, \quad \kappa_1 + \ldots + \kappa_j = k\}$. As before, we have that that $T_j^-(m,k)$ is empty when $j > -m \cdot k / \gcd(m)$; in particular it is empty when $m \cdot k > 0$. Let us define

$$I_j^-(m,k) = \sum_{\kappa \in T_j^-(m,k)} I(m,\kappa_1) \cdot I(m,\kappa_2) \cdot \ldots \cdot I(m,\kappa_j)$$

By our previous observation, we also see that $I_j^-(m,k) = 0$ if $j > -m \cdot k / \gcd(m)$. As before, we obtain

$$(\widetilde{a}_m(x_1,x') - 1)^j = \left(\frac{2\pi}{|m|}\right)^j \sum_{m \cdot k < 0} I_j^-(m,k) e^{-2\pi\mu_{m,k} x_1} e_k(x')$$

This implies that if $x_1 \leq -R$, then

$$\begin{aligned} v_m(x_1,x') &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} (\widetilde{a}_m(x_1,x') - 1)^j \\ &= \sum_{m \cdot k < 0} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{2\pi}{|m|}\right)^j I_j^-(m,k) \right) e^{-2\pi\mu_{m,k} x_1} e_k(x'). \end{aligned}$$

From this and (6.7) we conclude that if $m \cdot n < 0$, then

$$J(m,n) = \frac{|m|}{2\pi} e^{2\pi\mu_{m,n} p} v_{m,n}(p) = \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-2\pi}{|m|}\right)^{j-1} I_j^-(m,n) \tag{6.11}$$

It was shown above that $I_j^{\pm}(m,n)$ vanish when $j > |m \cdot n| / \gcd(m)$, so the sum above is actually a finite sum. In particular, if m,n are such that $m \cdot n = \pm 1$, which implies $\gcd(m) = 1$, then we obtain that $I_j^{\pm} = 0$ for $j \geq 2$. Therefore, $J(m,n) = I(m,n)$ if $m \cdot n = \pm 1$.

Remark. *The relation between these two families of integrals in other problems had already been noted by Eskin–Ralston in [ER97], and was also used in [Sal06]. In their setting the two families ended up being entirely equal, not as in our problem where this only seems to be true in certain cases.*

6.4.2 A linear algebra lemma

In the previous subsection we were concerned with determining the curl vectors

$$\left\{\left(-n_j, \frac{im\cdot n}{|m|}\delta_j\right),\ \ (0, n_j\delta_k - n_k\delta_j)\right\}$$

as linear combinations of vectors $\{(i|m|, m)\}$ while keeping n and $\mu_{m,n}$ fixed. We observe that

$$\left(-n_j, \frac{im\cdot n}{|m|}\delta_j\right) = \frac{i}{|m|}(i|m|n_j, (m\cdot n)\delta_j),$$

so we can regard the family of "curl vectors" as $\{(i|m|n_j, (m\cdot n)\delta_j),\ \ (0, n_j\delta_k - n_k\delta_j)\}$. Let $n \in \mathbb{Z}^d \setminus \{0\}$. Consider the set of points

$$U(K) := \{(i|m|, m) : m \in \mathbb{Z}^d,\ \ m\cdot n = \gcd(n),\ \ |m| = K\},$$

where $\gcd(n)$ denotes the greatest common divisor of all the entries of n. We will show that if $d \geq 3$, then we can construct infinitely many K such that linear combinations of elements in $U(K)$ generate all the curl vectors

$$\{(iKn_j, \gcd(n)\delta_j),\ \ (0, n_i\delta_j - n_j\delta_i)\}.$$

Remark. *It may suffice to generate each curl vector for infinitely many K, but we will show that we can do all of them simultaneously.*

The curl vectors and the conditions defining $U(K)$ are homogeneous functions of the entries of n, so we can assume without loss of generality that $\gcd(n) = 1$. Moreover, it suffices to generate the first family of curl vectors, as we can express

$$(0, n_i\delta_j - n_j\delta_i) = n_i(iKn_j, \delta_j) - n_j(iKn_i, \delta_i).$$

In addition, note that if $\delta_j = \alpha_1 k_1 + \ldots + \alpha_N k_N$, with $(i|k_l|, k_l) \in U(K)$, then

$$n_j = \delta_j \cdot n = (\alpha_1 k_1 + \ldots + \alpha_N k_N) \cdot n = \alpha_1 + \ldots + \alpha_N,$$

and so $(iKn_j, \delta_j) = \alpha_1(iK, k_1) + \ldots + \alpha_N(iK, k_N)$. Therefore it suffices to construct infinitely many K such that the set

$$V(K) := \{k \in \mathbb{Z}^d : k \cdot n = 1, \quad |k| = K\}$$

has d linearly independent vectors. In what follows, we refer to the rank of a finite set of vectors as the dimension of the subspace generated by them. We prove this in several steps.

Proposition 6.6. *Let $d \geq 3$ and let $n \in \mathbb{Z}^d$ be such that $\gcd(n) = 1$. Then there exist $m_1, m_2 \in \mathbb{Z}^d$ such that $\{m_1, m_2, n\}$ are linearly independent and*

$$m_1 \cdot n = m_2 \cdot n = 1, \quad |m_1| = |m_2|.$$

Proof. Let $p \in \mathbb{Z}^d$ be such that $p \cdot n = 1$ and is linearly independent with n. Since $d \geq 3$, there exists $q \in \mathbb{Z}^d \setminus \{0\}$ orthogonal to both n and p. We can define $m_1 := p - q$ and $m_2 := p + q$. With this we have that

$$m_i \cdot n = (p \pm q) \cdot n = 1 \pm 0 = 1, \quad |m_i|^2 = |p \pm q|^2 = |p|^2 \pm 2p \cdot q + |q|^2 = |p|^2 + |q|^2.$$

Moreover, the span of $\{m_1, m_2, n\}$ is the same as the span of $\{p, q, n\}$, from where we conclude that these vectors are linearly independent. □

Remark. *A curious observation is that the only vectors $n \in \mathbb{Z}^2$ for which there exist $m_1, m_2 \in \mathbb{Z}^2$ such that*

$$m_1 \cdot n = m_2 \cdot n = 1, \quad |m_1| = |m_2|,$$

are the eight vectors $\pm\{(1,0), (0,1), (1,1), (1,-1)\}$. This problem appeared at the Olimpiada

Iberoamericana de Matemática Universitaria 2018.

Proposition 6.7. *Let $d \geq 3$ and let $\{m_1, m_2, n\}$ be as in Proposition 6.6. Consider the integers $M := |m_1|^2 = |m_2|^2$, $N := |n|^2$, $P = m_1 \cdot m_2$. Then the vectors*

$$p_1 := (NP-1)m_1+(1-MN)m_2+(M-P)n, \quad p_2 := (1-MN)m_1+(NP-1)m_2+(M-P)n,$$

satisfy $p_i \cdot m_i = p_i \cdot n = 0$, $|p_1| = |p_2|$, and the rank of the set $\{m_1, m_2, p_1, p_2\}$ is 3.

Proof. The computations are direct:

$$\begin{aligned}
p_i \cdot m_i &= [(NP-1)m_i + (1-MN)m_j + (M-P)n] \cdot m_i \\
&= (NP-1)M + (1-MN)P + (M-P) = 0, \\
p_i \cdot n &= [(NP-1)m_i + (1-MN)m_j + (M-P)n] \cdot n \\
&= (NP-1) + (1-MN) + (M-P)N = 0, \\
|p_i|^2 &= |(NP-1)m_i + (1-MN)m_j + (M-P)n|^2 \\
&= (NP-1)^2M + (1-MN)^2M + (M-P)^2N \\
&\quad + 2[(NP-1)(1-MN)P + (NP-1)(M-P) + (1-MN)(M-P)].
\end{aligned}$$

To see that the rank is 3, we first observe that $(M - P) > 0$, as m_1 and m_2 are linearly independent. This implies that n is contained in the span of $\{m_1, m_2, p_1, p_2\}$. As $\{m_1, m_2, n\}$ are linearly independent the conclusion follows. □

Lemma 6.8. *Let $d \geq 3$ and let $n \in \mathbb{Z}^d \setminus \{0\}$. We can construct infinitely many K for which there are d linearly independent vectors in the set $V(K)$.*

Proof. We can assume without loss of generality that $\gcd(n) = 1$. Let $\{m_1, m_2, p_1, p_2\}$ be as in Proposition 6.7, and let $\{q_4, \ldots, q_d\}$ be an orthogonal set of vectors in $\mathbb{Z}^d$ perpendicular in addition to $\{m_1, m_2, n\}$, so that it is also perpendicular to the set $\{m_1, m_2, p_1, p_2\}$. For fixed nonzero integers α, $\alpha_4, \ldots, \alpha_d$ consider the vectors

$$m_1 \pm \alpha p_1 \pm \alpha_4 q_4 \pm \ldots \pm \alpha_d q_d, \quad m_2 \pm \alpha p_2 \pm \alpha_4 q_4 \pm \ldots \pm \alpha_d q_d.$$

Note that the dot product of any of these vectors with n equals 1, as $m_i \cdot n = 1$ and $p_i \cdot n = q_j \cdot n = 0$. Moreover, they have the same norm for any choice of signs, since all the terms are orthogonal to each other and $|m_1| = |m_2|$ and $|p_1| = |p_2|$. Therefore, all these vectors belong to the same $V(K)$ for any choice of signs. Linear combinations of these vectors allow to obtain the set $\{m_1, m_2, p_1, p_2, q_4, \ldots, q_d\}$. The sets $\{m_1, m_2, p_1, p_2\}$ and $\{q_4, \ldots, q_d\}$ are perpendicular and its combined rank is $3 + (d-3) = d$. Different choices of the integers α, $\alpha_4, \ldots, \alpha_d$ give the infinitely many values of K. □

6.4.3 Reconstruction of an entire function from values along a convergent sequence

Suppose $F : \mathbb{C} \to \mathbb{C}$ is an entire function and $\{z_n\} \subseteq \mathbb{C}$ is a known sequence such that $z_n \to 0$ and the sequence $\{F(z_n)\}$ is also known. The Taylor coefficients of F can be recovered recursively as follows,

$$F^{(0)}(0) = \lim_{n\to+\infty} F(z_n), \quad \frac{F^{(m)}(0)}{m!} = \lim_{n\to+\infty} \frac{1}{z_n^m}\left(F(z_n) - \sum_{k=0}^{m-1} \frac{F^{(k)}(0)}{k!} z_n^k\right),$$

and so F can be reconstructed from this. However, we would like to propose a different approach based on Newton's method of divided differences for interpolation polynomials.

Definition 6.9. *Let $F^{(0)}(z) := F(z)$ and define the divided differences*

$$F^{(n)}(z_1, \ldots, z_{n+1}) := \frac{1}{z_n - z_{n+1}}(F^{(n-1)}(z_1, \ldots, z_{n-1}, z_n) - F^{(n-1)}(z_1, \ldots, z_{n-1}, z_{n+1})).$$

Remark. *It is clear that divided differences are symmetric with respect to the last two elements, i.e.*

$$F^{(n)}(z_1,\dots,z_n,z_{n+1}) = F^{(n)}(z_1,\dots,z_{n+1},z_n).$$

We will not use this, but it is possible to show by the induction that the divided differences are indeed symmetric with respect to all its entries. We can see this in the particular following result.

Proposition 6.10. *Consider the power function $p_k(x) := x^k$. Then,*

$$p_k^{(n)}(z_1,\dots,z_{n+1}) = \sum_{|\alpha|=k-n} z_1^{\alpha_1}\dots z_{n+1}^{\alpha_{n+1}}.$$

In particular, $p_k^{(k)} = 1$ and $p_k^{(n)} = 0$ if $n > k$. The number of monomials in the expression equals the binomial coefficient $\binom{k}{n}$.

Proof. We prove this by induction. For $n = 0$ this is true. Then

$$\begin{aligned} p_k^{(n+1)}(z_1,\dots,z_{n+1},z_{n+2}) &= \sum_{|\alpha|=k-n} z_1^{\alpha_1}\dots z_n^{\alpha_n}\left(\frac{z_{n+1}^{\alpha_{n+1}} - z_{n+2}^{\alpha_{n+1}}}{z_{n+1}-z_{n+2}}\right) \\ &= \sum_{\substack{|\alpha|=k-n \\ |\beta|=\alpha_{n+1}-1}} z_1^{\alpha_1}\dots z_n^{\alpha_n} z_{n+1}^{\beta_1} z_{n+2}^{\beta_2} = \sum_{|\gamma|=k-n-1} z_1^{\gamma_1}\dots z_{n+2}^{\gamma_{n+2}}. \end{aligned}$$

□

Definition 6.11. *For an entire function F, we define the n-th derivative majorant by the convergent series*

$$|F|^{(n)}(R) := \frac{1}{n!}\sum_{k=0}^{\infty}\frac{|F^{(n+k)}(0)|}{k!}R^k.$$

Proposition 6.12. *Let F be an entire function F and $z_i \in \mathbb{C}$, $|z_i| \le R$. Then the n-th derivative majorant dominates the divided differences, i.e.*

$$|F^{(n)}(z_1,\dots,z_{n+1})| \le |F|^{(n)}(R).$$

Proof. Let $F(z) = \sum_{k=0}^{\infty} a_k z^k$. The divided differences are linear operators, so we obtain

$$|F^{(n)}(z_1, \ldots, z_{n+1})| \leq \sum_{k=0}^{\infty} |a_k||p_k^{(n)}(z_1, \ldots, z_{n+1})| = \sum_{k=0}^{\infty} |a_{n+k}||p_{n+k}^{(n)}(z_1, \ldots, z_{n+1})|,$$

where we used in the last equality that $p_k^{(n)} = 0$ if $k < n$ from Proposition 6.10. Also from Proposition 6.10 we obtain the bound

$$|p_{n+k}^{(n)}(z_1, \ldots, z_{n+1})| \leq \binom{n+k}{n} R^k.$$

Therefore we conclude that

$$|F^{(n)}(z_1, \ldots, z_{n+1})| \leq \sum_{k=0}^{\infty} \left| \frac{F^{(n+k)}(0)}{(n+k)!} \right| \binom{n+k}{n} R^k = |F|^{(n)}(R).$$

□

Theorem 6.13. *Let F be an entire function and let $z_i \to 0$. Then, the Taylor coefficients of F can be recovered by the divided differences:*

$$\lim_{m \to +\infty} F^{(n)}(z_{m+1}, \ldots, z_{m+n+1}) = \frac{F^{(n)}(0)}{n!}.$$

Proof. Proceeding as in the previous proof, we can bound

$$\begin{aligned} |F^{(n)}(z_{m+1}, \ldots, z_{m+n+1}) - a_n| &\leq \sum_{k=1}^{\infty} |a_{n+k}||p_{n+k}^{(n)}(z_{m+1}, \ldots, z_{m+n+1})| \\ &\leq |F|^{(n)}(\max\{|z_{m+1}|, \ldots, |z_{m+n+1}|\}) - |F|^{(n)}(0). \end{aligned}$$

Taking limits as $m \to +\infty$ gives the result. □

The previous result already allows for the reconstruction of the entire function F from the values along the convergent sequence. However, we provide a slightly more explicit reconstruction for F using Newton's divided differences interpolation polynomials.

Definition 6.14. *Given a function f and $z_1, \ldots, z_N$, we define the N-th interpolation polynomial by*

$$f_N(z; z_1, \ldots, z_N) = \sum_{n=0}^{N-1} f^{(n)}(z_1, \ldots, z_{n+1}) \prod_{m=1}^{n} (z - z_m).$$

Proposition 6.15. *The interpolation polynomials satisfy $f_N(z_k; z_1, \ldots, z_N) = f(z_k)$ for $k = 1, \ldots, N$.*

Proof. We prove the result by induction. For the base case we have $f_1(z) = f(z_1)$. Assume the result is true for N. We have that

$$f_{N+1}(z; w_1, \ldots, w_N, w_{N+1}) = f_N(z; w_1, \ldots, w_N) + f^{(N)}(w_1, \ldots, w_{N+1}) \prod_{m=1}^{N} (z - w_m).$$

This and the inductive hypothesis give that for $k = 1, \ldots, N$ we have

$$f_{N+1}(w_k; w_1, \ldots, w_N, w_{N+1}) = f_N(w_k; w_1, \ldots, w_N) = f(w_k).$$

In addition, directly from the definitions it follows that

$$f_{N+1}(z; z_1, \ldots, z_N, z_{N+1}) = f_{N+1}(z; z_1, \ldots, z_{N+1}, z_N).$$

These two observations imply the result. □

Theorem 6.16. *Let F be an entire function and suppose that $z_i \in \mathbb{C}$, $z_i \to 0$. Then, F can be recovered as a limit of the interpolation polynomials:*

$$F(z) = \lim_{N \to +\infty} F_N(z; z_1, \ldots, z_N) = \sum_{n=0}^{\infty} f^{(n)}(z_1, \ldots, z_{n+1}) \prod_{m=1}^{n} (z - z_m).$$

The series converges absolutely and uniformly over compact sets.

Proof. Let $|z_i| \leq R$. From Proposition 6.12 we can bound absolutely the series by

$$\begin{aligned}\sum_{n=0}^{\infty} \left| F^{(n)}(z_1, \ldots, z_{n+1}) \prod_{m=1}^{n} (z - z_m) \right| \\ &\leq \sum_{n=0}^{\infty} |F|^{(n)}(R)(|z| + R)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{m=0}^{\infty} \frac{|F^{(m+n)}(0)|}{m!} R^m \right) (|z| + R)^n = \sum_{k=0}^{\infty} \frac{|F^{(k)}(0)|}{k!} (|z| + 2R)^k,\end{aligned}$$

where we used the binomial theorem in the last equality. The right-hand side is a uniformly convergent series over compact sets. It follows from Weierstrass' test that the convergence of the series

$$\sum_{n=0}^{\infty} f^{(n)}(z_1, \ldots, z_{n+1}) \prod_{m=1}^{n} (z - z_m)$$

is absolute and uniform over compact sets. Since the partial sums are polynomials, in particular entire, then the limit $\widetilde{F}(z)$ must be entire as well. However, from Proposition 6.15 we know that $\widetilde{F}(z_k) = F(z_k)$. Thus, F and $\widetilde{F}$ are entire and coincide over a convergent sequence, and so $F \equiv \widetilde{F}$. $\square$

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