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COMPLETED COHOMOLOGY AND IWASAWA THEORY

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ABSTRACT

We compare two different constructions of cyclotomic p-adic L-functions for modular forms and their relationship to Galois cohomology: one using Kato's Euler system and the other using Emerton's p-adically completed cohomology of modular curves. At a more technical level, we prove the equality of two elements of a local Iwasawa cohomology group, one arising from Kato's Euler system, and the other from the theory of modular symbols and p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. We show that this equality holds even in the cases when the construction of p-adic L-functions is still unknown (i.e. when the modular form f is supercuspidal at p). Thus, we are able to give some representation-theoretic descriptions of Kato's Euler system.

We also compare two different constructions of anti-cyclotomic p-adic L-functions for modular forms on quaternion algebras: one defined by Bertolini and Darmon in [3] and the other using Emerton's p-adically completed cohomology of Shimura sets.

CHAPTER 1

INTRODUCTION

1.1 Summary of main results

Let f be a cuspidal newform of weight $k \geq 2$ and level $\Gamma_0(p^n) \cap \Gamma_1(N)$, V_f be the p-adic Galois representation attached to f. In [11], Kato constructed for every cuspidal Hecke eigenform f an Euler system $\mathbf{z}_{\mathrm{Kato}}(f)$ in the *global* Iwasawa cohomology of the dual Galois representation V_f^* attached to f. The element $\mathbf{z}_{\mathrm{Kato}}(f)$ is the key object for studying both the p-adic interpolation properties of classical L-functions and the p-adic arithmetic information of the modular form f. For example, an important property of $\mathbf{z}_{\mathrm{Kato}}(f)$ we will be using in this paper is that the images of $\mathbf{z}_{\mathrm{Kato}}(f)$ under various dual exponential maps compute the special values of the classical L-functions of f and its twists by characters of p-power conductors. We will describe the precise statement in Theorem 1.1.3 below.

Another approach to studying the p-adic interpolation properties of classical L-functions of modular forms is via modular symbols. In [14] and [15], Emerton rephrases this approach by regarding the modular symbol $\{0 - \infty\}$ as a functional on p-adically complected cohomology of modular curves. Combining the construction of this functional with the p-adic local-global compatibility [16] and Colmez's theory of p-adic local Langlands correspondence [18], we may regard $\{0 - \infty\}$ as an element $\mathbf{z}_M(f)$ (the letter "M" stands for modular symbols) in the local Iwasawa cohomology of V_f^*.

The precise definition of $\mathbf{z}_M(f)$ is as follows: according to [5] and [18], there are isomorphisms

$$\mathrm{Exp}^* : \quad \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*) \xrightarrow{\cong} \mathbf{D}(V_f^*)^{\psi=1}$$

and

$$\mathfrak{C} : \quad \big(\Pi(V_f)^*\big)^{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)=1} \xrightarrow{\cong} \mathbf{D}(V_f^*)^{\psi=1},$$

where $\mathbf{D}(V_f^*)$ is the (φ, Γ)-module attached to V_f^*, $\Pi(V_f)$ is the p-adic Banach space representation attached to V_f by the p-adic local Langlands correspondence[1].

Using Emerton's local-global compatibility [16], the modular form f gives a (pair of) $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding (which is actually a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism)

$$\Phi_f^{\pm} : \quad \Pi(V_f) \hookrightarrow \widetilde{H}^1_c(K^p)^{\pm, f} \tag{1.1.1}$$

where K^p is the tame level of f, $\widetilde{H}^1_c(K^p)$ is the completed cohomology of tame level K^p.

We denote the composite

$$\Pi(V_f) \xrightarrow{\Phi_f^{\pm}} \widetilde{H}^1_c(K^p)^{\pm, f} \xrightarrow{\{0-\infty\}} \mathbb{C}_p$$

as an element in $\Pi(V_f)^*$ by $\mathcal{M}_f^{\pm}$. In fact, we have (Lemma 2.2.3)

$$\mathcal{M}_f^{\pm} \in \big(\Pi(V_f)^*\big)^{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)=1}.$$

1. Our notation is slightly different from [18]: the $\Pi(V)$ in this paper is denoted by $\Pi(V(1))$ in [18].

We define $\mathbf{z}_M^{\pm}(f) := (\mathrm{Exp}^*)^{-1} \circ \mathfrak{C}(\mathcal{M}_f^{\pm}) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$, and $\mathbf{z}_M(f) := \mathbf{z}_M^+(f) - \mathbf{z}_M^-(f) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$.

The cohomology classes $\mathbf{z}_{\mathrm{Kato}}(f)$ and $\mathbf{z}_M(f)$ arise from very different perspectives of the p-adic arithmetic of the modular form f. On the other hand, both of them lie in the Iwasawa cohomology group attached to f, and in fact both encode special L-values of f and its twists. Thus it is natural to ask what the relationship between these two elements is. Our main result answers this question:

Theorem 1.1.1. *If $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible, then $\mathbf{z}_M(f) = \mathbf{z}_{\mathrm{Kato}}(f)$ as elements in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$.*

From the perspective of the special L-values they encode, $\mathbf{z}_{\mathrm{Kato}}(f)$ comes from the Rankin-Selberg method, and $\mathbf{z}_M(f)$ comes from the Mellin transforms, so the above theorem compares these two different integral formulas for L-functions of modular forms. It would be interesting to have a direct comparison, but our approach is to use the equality of special values under dual exponential maps (computed either way for $\mathbf{z}_{\mathrm{Kato}}(f)$ and $\mathbf{z}_M(f)$) as the basic input.

The strategy of proving Theorem 1.1.1 is to show that the images of both $\mathbf{z}_M(f)$ and $\mathbf{z}_{\mathrm{Kato}}(f)$ under various dual exponential maps are the same, and then use Proposition II.3.1 of [13] to conclude that $\mathbf{z}_M(f) = \mathbf{z}_{\mathrm{Kato}}(f)$ as elements in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$.

Theorem 1.1.2. *If $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible, then the element $\mathbf{z}_M(f) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ satisfies the following property: For any $0 \leq j \leq k-2$, any finite* *order character* ϕ *of* $\mathbb{Z}_p^* \cong \Gamma_{\mathbb{Q}_p}$ *with conductor* p^n*,*

$$\exp^*\left(\int_{\mathbb{Z}_p^*}\phi(x)x^{-j}\cdot \mathbf{z}_M(f)\right) = \frac{1}{\tau(\phi)}\cdot\frac{\widetilde{\Lambda}_{(p)}(f,\phi,j+1)}{j!}\cdot \overline{f}_\phi\cdot t^j,$$

where

$$\widetilde{\Lambda}_{(p)}(f,\phi,j+1) = \frac{\Gamma(j+1)}{(2\pi i)^{j+1}}\cdot\frac{L_{(p)}(f,\phi,j+1)}{\Omega_f^{\pm}} \in \mathbb{Q}(f,\mu_{p^n}),$$

$L_{(p)}(f,\phi,j+1)$ *is obained by removing the Euler factor at* p *from the classical* L*-function of* f *twisted by* ϕ*,* $\tau(\phi)$ *is the Gauss sum of the character* ϕ*, and*

$$\overline{f}_\phi\cdot t^j := \left(\sum_a \phi(a)\overline{f}\otimes\zeta_{p^n}^a\right)\cdot t^j = \overline{f}\cdot e_\phi\cdot t^j \in \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f^*(\phi)(-j)).$$

Two key ingredients used in proving Theorem 1.1.2 are the theory of p-adic Kirillov models of locally algebraic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ introduced in [18] (Section VI.2.5), and an "explicit reciprocity law" introduced in [18] (Proposition VI.3.4), [10] (Théorème 8.3.1) and [9] (Theorem 5.4.3).

On the other hand, Kato in [11] showed that $\mathbf{z}_{\mathrm{Kato}}(f)$ has the following interpolation property:

Theorem 1.1.3 (Kato, [11], Theorem 12.5)**.** *The element* $\mathbf{z}_{\mathrm{Kato}}(f) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ *satisfies the following property:*

For any $0 \le j \le k-2$*, any finite order character* ϕ *of* $\mathbb{Z}_p^* \cong \Gamma_{\mathbb{Q}_p}$ *with conductor* p^n*,*

$$\exp^*\left(\int_{\mathbb{Z}_p^*}\phi(x)x^{-j}\cdot \mathbf{z}_{\mathrm{Kato}}(f)\right) = \frac{1}{\tau(\phi)}\cdot\frac{\widetilde{\Lambda}_{(p)}(f,\phi,j+1)}{j!}\cdot \overline{f}_\phi\cdot t^j.$$

Comparing the formulas in Theorem 1.1.2 and Theorem 1.1.3, and using Proposition II.3.1 of [13], we obtain Theorem 1.1.1.

Similar approach (using Emerton's completed cohomology) can be used to study the anti-cyclotomic p-adic L-functions, which are originally defined in [3] for quaternionic forms of weight 2 and in [7] and [6] for quaternionic forms of higher weight. It is described in [4] that certain elements in the dual of completed H^0 of Shimura sets can be used to construct the anti-cyclotomic p-adic L-functions.

We describe an example of this situation in Chapter 3. For simplicity, consider a modular form f of weight 2 on a quaternion algebra over $\mathbb{Q}$ which is ramified exactly at l and ∞, where l is a prime number different from p. We assume also that f is Steinberg at prime p.

Using Emerton's local-global compatibility [16] again, the quaternionic form f gives a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding (which is actually a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism)

$$\Phi_f: \quad \Pi \cong \left(\mathbf{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right)/1 \hookrightarrow \widetilde{H}_c^0(R^p)^f \tag{1.1.2}$$

where R^p is the tame level of f, $\widetilde{H}_c^0(R^p)$ is the completed H^0 of the tower of Shimura sets with tame level R^p. The choice of the Borel subgroup $B(\mathbb{Q}_p) \subset \mathrm{GL}_2(\mathbb{Q}_p)$ is determined by choosing an imaginary quadratic field K that splits at p and an embedding of K into the quaternion algebra.

There is a distinguished point $\mathbb{1}$ on the tower of Shimura set, whose coordinate at every place is the identity element when we write the Shimura set as a double quotient (see Section 3.1). Notice that $\widetilde{H}_c^0(R^p)$ is actually the collection of p-adically continuous functions defined on the tower of Shimura set, we can then evaluate functions in $\widetilde{H}_c^0(R^p)$ at the point $\mathbb{1}$.

We denote the composite

$$\Pi \xrightarrow{\Phi_f} \widetilde{H}_c^1(R^p)^f \xrightarrow{\mathrm{ev}_{\mathbb{1}}} \mathbb{C}_p$$

as an element in Π^* by $\mathrm{ev}_{\mathbb{1},f}$. Since Π is actually the collection of all p-adically continuous functions on $P^1(\mathbb{Q}_p)$ which vanishes at infinity, restricting $\mathrm{ev}_{\mathbb{1},f}$ to $\mathbb{Z}_p^*$ thus gives us a measure on $\mathbb{Z}_p^*$. Our main theorem in Chapter 3 is then:

Theorem 1.1.4. $\mathrm{ev}_{\mathbb{1},f}$ *restricting to* $\mathbb{Z}_p^*$ *equals (half of) the anti-cyclotomic* p*-adic* L*-function* $\widetilde{\mathcal{L}}_f$ *as elements in* $\mathbb{Z}_p[[\mathbb{Z}_p^*]]$*, where* $\widetilde{\mathcal{L}}_f$ *is defined in [3] (notations as in [3] as well).*

We can also notice that $\mathrm{ev}_{\mathbb{1},f} \in \Pi^*$ is fixed by $\begin{pmatrix} pu & 0 \\ 0 & 1 \end{pmatrix}$, where $u \in \mathbb{Z}_p^*$ is a p-adic unit. We can thus define $\mathbf{z}_{\mathbb{1}} \in \mathbf{D}(V_f^*)^{\psi=\sigma_u^{-1}}$ the image of $\mathrm{ev}_{\mathbb{1}}$ under the isomorphism $\Pi^{*,\begin{pmatrix} pu & 0 \\ 0 & 1 \end{pmatrix}=1} \cong \mathbf{D}(V_f^*)^{\psi=\sigma_u^{-1}}$ provided by Colmez, where V_f be the p-adic Galois representation attached to f, and $\mathbf{D}(V_f^*)$ is the (φ, Γ)-module associated to V_f^*.

The author of this thesis is planning to study the relationship between the element $\mathbf{z}_{\mathbb{1}}$ and the Euler system constructed in [3] and [6] in future work.

1.2 Outline of this thesis

We work with modular forms on modular curves in Chapter 2:

We compute in Section 2 the evaluations of the modular symbol $\{0-\infty\}$ on locally algebraic vectors of $\Pi(V_f)$ under the embedding $\Phi_f^{\pm}:\Pi(V_f)\hookrightarrow \hat{H}_c^1(K^p)^{\pm,f}$ given by the modular form f.

Section 3 - 7 compute the images of $\mathbf{z}_M(f)$ under various dual exponential maps, with the assumption that $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible. These sections are divided into two parts: Section 3 - 5 deal with the supercuspidal case (meaning that the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f is supercuspidal), and Section 6 & 7 deal with principal series case (meaning that the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f is a principal series).

In Section 3, we introduce the theory of p-adic Kirillov models for locally algebraic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and use the explicit formulas of the p-adic Kirillov models to compute the images of locally algebraic vectors of $\Pi\left(V_f\right)$ under the maps ι_m^-. In section 4, we describe an "explicit reciprocity law", and use it to compute the image of $\mathbf{z}_M^{\pm}(f)$ under the maps ι_m^-. In section 5, we present and give a proof of the formulas for the images of $\mathbf{z}_M(f)$ under various dual exponential maps in the supercuspidal case. In section 6, we describe explicitly the p-adic Local Langlands correspondence, and then use results from [14] to describe explicitly the image of $\mathbf{z}_M^{\pm}(f)$ in $\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}\mathbf{D}_{\mathrm{crys}}(V_f^*)$. In section 7, we deduce the formulas for the images of $\mathbf{z}_M(f)$ under various dual exponential maps in the principal series case.

In Section 8, we conclude $\mathbf{z}_M(f)=\mathbf{z}_{\mathrm{Kato}}(f)$ in both the supercuspidal and the principal series cases, under the assumption that $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible.

We work with an example of quaternionic forms on Shimura sets in Chapter 3:

In Section 3.1 we show that $\mathrm{ev}_{\mathbb{1},f}$ restricting to $\mathbb{Z}_p^*$ equals (half of) the anti-cyclotomic p-adic L-function $\widetilde{\mathcal{L}}_f$ as elements in $\mathbb{Z}_p[[\mathbb{Z}_p^*]]$, and in Section 3.2 we phrase the same example using more representation theoretic viewpoint.

CHAPTER 2

COMPLETED COHOMOLOGY OF MODULAR CURVES AND KATO'S EULER SYSTEM

2.1 Notations and conventions

We denote Γ the Galois group $\mathrm{Gal}\left(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p\right)$, H the group $\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty})\right)$. We identify Γ with $\mathbb{Z}_p^*$ via the cyclotomic character $\mathrm{cycl} : \Gamma \xrightarrow{\cong} \mathbb{Z}_p^*$.

$P\left(\mathbb{Q}_p\right)$ is the subgroup of $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$ consisting of matrices of the form $\begin{pmatrix} \mathbb{Q}_p^* & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$.

Throughout this chapter, f denotes a (normalized) classical cuspidal newform of weight $k \geq 2$ and level $\Gamma_0(p^n) \cap \Gamma_1(N)$. V_f is the cohomological Galois representation of $G_{\mathbb{Q}}$ attached to f. Thus the restriction $V_f|_{G_{\mathbb{Q}_p}}$ has Hodge-Tate weight 0 and $1-k$.[1]

Let F be a finite extension of $\mathbb{Q}$ which is sufficiently large, that is, containing all the Fourier coefficients of f and values of χ when a finite order character χ is chosen. We choose λ a place of F lying over p, and denote $L := F_\lambda$ the localization of F at λ. For any integer $n \geq 0$, we write $L_n := L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$. Here, $\left\{\zeta_{p^n}\right\}_{n \geq 1}$ is chosen to be a compatible system of p-power roots of unity. We also write $\mathbb{Q}_{p,n} := \mathbb{Q}_p(\zeta_{p^n})$

We denote $\pi_p^{\mathrm{sm}}(f)$ the p-adic smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f, and $\Pi(V_f)$ the p-adic Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f. Our notation is slightly different from [18]: the $\Pi(V)$ in this paper is denoted by $\Pi(V(1))$ in [18].

1. Here, the convention is that the cyclotomic character has Hodge-Tate weight 1.

We will only work with the cases when $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible. This happens precisely when $\pi_p^{\mathrm{sm}}(f)$ is supercuspidal, or when $\pi_p^{\mathrm{sm}}(f)$ is some twist of an unramified principal series.

The modular form f is a holomorphic section of some line bundle on the modular curve, therefore can be viewed naturally as an element in $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f(k-1))$. Similarly, the complex conjugate of f, which we denote by $\overline{f}$, can be viewed naturally as an element in $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\overline{f}}(k-1)) = \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f^*)$.

There is a natural pairing

$$[\ ,\ ]_{\mathrm{dR}}\colon\quad \mathbf{D}_{\mathrm{dR}}\left(V_f^*\right) \times \mathbf{D}_{\mathrm{dR}}\left(V_f(1)\right) \to \mathbf{D}_{\mathrm{dR}}(L(1)) \cong \frac{1}{t}L,$$

under which $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}\left(V_f^*\right)$ and $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}\left(V_f(1)\right)$ are orthogonal complement of each other. We define $\overline{f}^*$ the unique element in $\mathbf{D}_{\mathrm{dR}}\left(V_f(1)\right)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}\left(V_f(1)\right)$ such that $[\overline{f}, \overline{f}^*]_{\mathrm{dR}} = \frac{1}{t}$.

2.2 Special values of L-function in terms of modular symbols

In this section, we compute the modular symbol $\{0-\infty\}$ evaluated at locally algebraic vectors of $\Pi(V_f)$ under the embedding (1.1.1).

Let f be a cuspidal newform of weight $k \geq 2$ and level $\Gamma_0(p^n) \cap \Gamma_1(N)$. We denote K^p the tame level of f, i.e.

$$K^p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}^p) \,\middle|\, c \equiv 0 \bmod N,\, d \equiv 1 \bmod N \right\},$$

and $Y(Np^n)$ the modular curve defined over $\mathbb{Q}$ whose $\mathbb{C}$ points are given by

$$Y(Np^n)(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q}) \Big\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \Big/ \Gamma(p^n) \cdot K^p \cdot \mathbb{C}^\times.$$

Let $Y(Np^n)_{\mathbb{Q}(\zeta_{Np^n})}$ denote the base change to $\mathbb{Q}(\zeta_{Np^n})$ of $Y(Np^n)$, and $Y^\circ(Np^n)$ denote the connected component of $Y(Np^n)_{\mathbb{Q}(\zeta_{Np^n})}$. We also write $X^\circ(Np^n)$ (resp. $X(Np^n)$) the compactification of $Y^\circ(Np^n)$ (resp. $Y(Np^n)$).

Fix an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. There is an action of complex conjugation τ on $Y(Np^n)_{\mathbb{Q}(\zeta_{Np^n})}$. The action of τ on $\pi_0\left(Y(Np^n)_{\mathbb{Q}(\zeta_{Np^n})}\right) \subset (\mathbb{Z}/(Np^n))^\times$ coincides with the one induced from multiplication by -1 on $(\mathbb{Z}/(Np^n))^\times$.

We let F be a finite extension of $\mathbb{Q}$ that contains all Fourier coefficients of f and values of χ when a finite order character χ is chosen in the future. Choose λ a place of F lying over p, and denote $L := F_\lambda$ the localization of F at λ. The completed cohomology of the modular curve with tame level K^p and coefficient L is defined as

$$\widetilde{H}^1_c(K^p) := L \otimes_{\mathcal{O}_L} \varprojlim_m \varinjlim_n H^1_{\text{ét},c}\left(Y(Np^n)_{\overline{\mathbb{Q}}},\ \mathcal{O}_L/p^m\right).$$

This is a p-adic Banach space equiped with a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathcal{H}^p \times \{\mathrm{Id}, \tau\}$.

The modular symbol $\{0-\infty\}$ is contained in

$$H_1(X^\circ(Np^n)(\mathbb{C}), \text{cusps}; \mathbb{Z}) \subset H_1(X(Np^n)(\mathbb{C}), \text{cusps}; \mathbb{Z})$$

for all n, and are compatible with respect to the map $H_1(X(Np^{n+1})(\mathbb{C}), \text{cusps}; \mathbb{Z}) \to H_1(X(Np^n)(\mathbb{C}), \text{cusps}; \mathbb{Z})$ induced from the natural projection $X(Np^{n+1}) \to X(Np^n)$. Note that the homology theory we are using here is the singular homology.

We then have for every positive integer m,

$$\begin{aligned}\{0-\infty\} \in \varprojlim_n H_1(X(Np^n)(\mathbb{C}),&\text{cusps}; \mathbb{Z}/p^m) \\ &= \operatorname{Hom}\left(\varinjlim_n H_c^1\left(Y(Np^n)(\mathbb{C}), \mathbb{Z}/p^m\right), \mathbb{Z}/p^m\right).\end{aligned}$$

Under the identification

$$H^1\left(Y(Np^n)(\mathbb{C}), \mathbb{Z}/p^m\right) \cong H^1_{\text{ét},c}\left(Y(Np^n)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^m\right),$$

we can view

$$\{0-\infty\} \in \operatorname{Hom}\left(\varinjlim_n H^1_{\text{ét},c}\left(Y(Np^n)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^m\right), \mathbb{Z}/p^m\right) \tag{2.2.1}$$

for every m, hence

$$\{0-\infty\} \in \left(\widetilde{H}^1_c(K^p)\right)'_{\text{b}} \tag{2.2.2}$$

where the right hand side is the bounded dual of the p-adic Banach space $\widetilde{H}^1_c(K^p)$.

We denote W_{k-2} the contragredient to the $(k-2)$-nd symmetric power of the standard representation of GL_2. Since $f \in H^0\left(Y(Np^n), \omega^{k-2} \otimes \Omega^1\right)$ is a holomorphic cuspidal newform of weight $k \geq 2$, we have a period map:

$$\mathrm{per} : H^0\left(Y(Np^n), \omega^{k-2} \otimes \Omega^1\right) \to H^1_c\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\overline{\mathbb{Q}})}\right) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

If we denote by i^* the restriction

$$\begin{aligned} i^* : H^1_c\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\overline{\mathbb{Q}})}\right) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to & H^1_c\left(Y^\circ(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\overline{\mathbb{Q}})}\right) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \\ & = H^1_c\left(\Gamma, \check{W}_{k-2}(\mathbb{C})\right), \end{aligned}$$

then we have

$$i^*\mathrm{per}(f) = \left\{ \gamma \in \Gamma \mapsto \sum_{i+j=k-2} \int_\gamma f(z) z^i dz \cdot e_1^i e_2^j \in \check{W}_{k-2}(\mathbb{C}) \right\}. \tag{2.2.3}$$

Here, we consider $\mathbb{Q}^2$ having the standard basis e_1 and e_2, and the action of any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{Q}^2$ is given by formulas $\begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1 = ae_1 + ce_2$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} e_2 = be_1 + de_2$. The basis of $\check{W}_{k-2}(\mathbb{Q}) = \mathrm{Sym}^{k-2}\mathbb{Q}^2$ is chosen as $e_1^i e_2^j$ for any $i + j = k - 2$.

Remark 2.2.1. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ normalizes Γ, then there is an action of g on the group $H^1_c\left(Y^\circ(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\mathbb{C})}\right) \cong H^1_c\left(\Gamma, \check{W}_{k-2}(\mathbb{C})\right)$ given by formula:

$(g.c)(\gamma) = g.\left(c(g^{-1}\gamma g)\right)$.

In particular, we have:

$$(g.\left(i^*\mathrm{per}(f)\right))\left(\{0-\infty\}\right) = \sum_{i+j=k-2} \int_{-\frac{d}{c}}^{-\frac{b}{a}} f(z)(cz+d)^{k-2}\left(\frac{az+b}{cz+d}\right)^i dz \cdot e_1^i e_2^j. \tag{2.2.4}$$

The complex conjugation τ acting on $Y(Np^n)(\mathbb{C})$ induces an action of complex conjugation on $H_c^1\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\overline{\mathbb{Q}})}\right)$, which we still denote by τ. We have

$$i^*\tau\left(\mathrm{per}(f)\right) = \left\{\gamma \in \Gamma \mapsto \sum_{i+j=k-2} \int_{\tau(\gamma)} f(z)(-z)^i dz \cdot e_1^i e_2^j.\right\} \tag{2.2.5}$$

Here, we are identifying $Y^\circ(Np^n)(\mathbb{C})$ as a quotient of the complex upper half plane, and the action of τ on the upper half plane is given by reflecting along the imaginary axis.

We write $H_c^1\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\overline{\mathbb{Q}})}\right)^\pm$ the $\pm$ eigenspace of the complex conjugation. We then have the $\pm$ period maps:

$$\mathrm{per}^\pm : H^0\left(Y(Np^n), \omega^{k-2}\otimes\Omega^1\right)^f \to H_c^1\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(\overline{\mathbb{Q}})}\right)^{\pm,f} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}, \tag{2.2.6}$$

one has for any $0 \le j \le k-2$,

$$i^*\mathrm{per}^{\pm}(f) = \begin{cases} \left\{\gamma \in \Gamma \mapsto \sum_{i+j=k-2} \frac{\int_\gamma f(z)z^j dz + \int_{\tau(\gamma)} f(z)z^j dz}{2} \cdot e_1^i e_2^j\right\}, & \text{if } (-1)^j = \pm 1; \\ \left\{\gamma \in \Gamma \mapsto \sum_{i+j=k-2} \frac{\int_\gamma f(z)z^j dz - \int_{\tau(\gamma)} f(z)z^j dz}{2} \cdot e_1^i e_2^j\right\}, & \text{if } (-1)^j = \mp 1. \end{cases} \tag{2.2.7}$$

It is well known (see for example, [17], page 11) that there are complex numbers $\Omega_f^{\pm} \in \mathbb{C}$ such that for any $0 \le j \le k-2$,

$$\frac{\int_\gamma f(z)z^j dz \pm \int_{\tau(\gamma)} f(z)z^j dz}{2\Omega_f^{\pm}} \in F, \tag{2.2.8}$$

where the signs in the numerator and the denominator are the same if $(-1)^j = 1$, different if otherwise $(-1)^j = -1$. Therefore, we can define:

$$\mathrm{per}_F^{\pm} : H^0\left(Y(Np^n), \omega^{k-2} \otimes \Omega^1\right)^f \to H_c^1\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(F)}\right)^{\pm, f}, \tag{2.2.9}$$

using the formula:

$$i^*\mathrm{per}_F^{\pm}(f) = \begin{cases} \left\{\gamma \in \Gamma \mapsto \sum_{i+j=k-2} \frac{\int_\gamma f(z)z^j dz + \int_{\tau(\gamma)} f(z)z^j dz}{2\Omega_f^{\pm}} \cdot e_1^i e_2^j\right\}, & \text{if } (-1)^j = \pm 1 \\ \left\{\gamma \in \Gamma \mapsto \sum_{i+j=k-2} \frac{\int_\gamma f(z)z^j dz - \int_{\tau(\gamma)} f(z)z^j dz}{2\Omega_f^{\pm}} \cdot e_1^i e_2^j\right\}, & \text{if } (-1)^j = \mp 1 \end{cases} \tag{2.2.10}$$

where $0 \le j \le k-2$.

Remark 2.2.2. We then have:

$$\operatorname{per}(f) = \Omega_f^+ \cdot \operatorname{per}_F^+(f) + \Omega_f^- \cdot \operatorname{per}_F^-(f); \tag{2.2.11}$$

$$\tau(\operatorname{per}(f)) = \Omega_f^+ \cdot \operatorname{per}_F^+(f) - \Omega_f^- \cdot \operatorname{per}_F^-(f). \tag{2.2.12}$$

Recall that λ is a place of F lying above p, and $L = F_\lambda$. Let $\operatorname{per}_L^\pm$ denote the composite of isomorphism

$$H_c^1\left(Y(Np^n)(\mathbb{C}), \mathcal{V}_{\check{W}_{k-2}(F)}\right)^\pm \otimes_F L \cong H_{\text{ét},c}^1\left(Y(Np^n)_{\overline{\mathbb{Q}}}, \mathcal{V}_{\check{W}_{k-2}(L)}\right)^\pm$$

with $\operatorname{per}_F^\pm$.

Let V_f be the cohomological Galois representation of $G_{\mathbb{Q}}$ attached to f with coefficients in L. Thus the restriction $V_f|_{G_{\mathbb{Q}_p}}$ has Hodge-Tate weights 0 and $1-k$. Let $\Pi\left(V_f\right)$ (with convention same as in [16]) be the p-adic Banach space representation of $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$ attached to $V_f|_{G_{\mathbb{Q}_p}}$ via the p-adic local Langlands correspondence.

As is mentioned in the introduction, according to Emerton's local-global compatibility [16], the newform f gives a (pair of) $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding, which are in fact isomorphisms:

$$\Phi_f^\pm : \quad \Pi(V_f) \hookrightarrow \widetilde{H}_c^1(K^p)^{\pm,f} \subset \widetilde{H}_c^1(K^p)$$

by choosing a new vector at every place away from p.

We denote the composite

$$\Pi(V_f) \xrightarrow{\Phi_f^{\pm}} \widetilde{H}_c^1(K^p)^{\pm,f} \xrightarrow{\{0-\infty\}} \mathbb{C}_p$$

as an element in $\Pi(V_f)^*$ by $\mathcal{M}_f^{\pm}$.

Notice that if $g \in \mathrm{GL}_2(\mathbb{Q})$, then there is an action of g on $\widetilde{H}_c^1(K^p)$ as a diagonal element in $\mathrm{GL}_2\left(\mathbb{Q}_p\right) \times \mathcal{H}^p \times \{\mathrm{Id}, \tau\}$ (induced from the action of $\mathrm{GL}_2\left(\mathbb{A}_{\mathbb{Q}}\right)$ on $\widetilde{H}_c^1 := \varinjlim_{K^p} \widetilde{H}_c^1(K^p)$), where the action of g through the last factor is Id if $\det(g)$ is positive, and τ if $\det(g)$ is negative. We have the following lemma:

Lemma 2.2.3. *The actions of matrices of the form* $\begin{pmatrix} p^{\mathbb{Z}} & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ *on* $\widetilde{H}_c^1(K^p)$ *as a diagonal element in* $\mathrm{GL}_2\left(\mathbb{Q}_p\right) \times \mathcal{H}^p \times \{\mathrm{Id}, \tau\}$ *and as an element in* $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$ *coincide. In particular, we have* $\mathcal{M}_f^{\pm} \in \left(\Pi(V_f)^*\right)^{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}=1}$.

Proof. The first assertion follows from the fact that $\begin{pmatrix} p^{\mathbb{Z}} & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ belongs to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Z}_l)$ for every $l \neq p$ and has positive determinant.

The second assertion follows from the fact that the modular symbol $\{0 - \infty\}$ is fixed by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ as an element in $\mathrm{GL}_2\left(\mathbb{Q}_p\right) \times \mathcal{H}^p \times \{\mathrm{Id}, \tau\}$. □

We can then define

$$\mathbf{z}_M^{\pm}(f) := (\mathrm{Exp}^*)^{-1} \circ \mathfrak{C}(\mathcal{M}_f^{\pm}) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*),$$

where the isomorphisms

$$\mathrm{Exp}^*: \quad \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*) \xrightarrow{\cong} \mathbf{D}(V_f^*)^{\psi=1}$$

and

$$\mathfrak{C}: \quad \left(\Pi(V_f)^*\right)^{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)=1} \xrightarrow{\cong} \mathbf{D}(V_f^*)^{\psi=1}$$

are as described in [5] and [18]. Here, $\mathbf{D}(V_f^*)$ is the (φ, Γ)-module attached to V_f^* as defined in [18].

There is an isomorphism:

$$W_{k-2}(L) \otimes_L \varinjlim_n H^1_{\text{ét},c}\left(Y(Np^n), \mathcal{V}_{\check{W}_{k-2}(L)}\right) \xrightarrow{\cong} \widetilde{H}^1_c(K^p)_{W_{k-2}-\mathrm{lalg}}. \tag{2.2.13}$$

This isomorphicm is equivariant for the complex conjugation τ, where the action of τ on the left is given by $\mathrm{Id} \otimes \tau$.

We still use $\mathrm{per}_L^{\pm}$ to denote the following composition:

$$\begin{aligned} H^0\left(Y(Np^n), \omega^{k-2} \otimes \Omega^1\right) &\xrightarrow{\mathrm{per}_L^{\pm}} H^1_{\text{ét},c}\left(Y(Np^n)_{\overline{\mathbb{Q}}}, \mathcal{V}_{\check{W}_{k-2}(L)}\right)^{\pm} \\ &\to \left(\varinjlim_n H^1_{\text{ét},c}\left(Y(Np^n), \mathcal{V}_{\check{W}_{k-2}(L)}\right)\right)^{\pm}. \end{aligned} \tag{2.2.14}$$

Classical Eichler-Shimura theory tells us that for any choice of sign $\pm$, there is a unique $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding

$$\pi_p^{\mathrm{sm}}(f) \hookrightarrow \left(\varinjlim_n H^1_{\text{ét},c}\left(Y(Np^n), \mathcal{V}_{\check{W}_{k-2}(L)}\right)\right)^{\pm,f}$$

where $\pi_p^{\mathrm{sm}}(f)$ is the p-adic smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f (or equivalently, attached to Weil-Deligne representation associated to $V_f|_{G_{\mathbb{Q}_p}}$ via the local Langlands correspodence), under which the image of $v_{\mathrm{new}} \in \pi_p^{\mathrm{sm}}(f)$ equals $\mathrm{per}_L^{\pm}(f)$.

Thus we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathbb{C}_p \\ & \nearrow^{\mathcal{M}_f^{\pm}} & \uparrow{\scriptstyle \{0-\infty\}} \\ W_{k-2}(L) \otimes_L \left(\varinjlim_n H^1_{\text{ét},c}\left(Y(Np^n), \mathcal{V}_{\check{W}_{k-2}(L)}\right)\right)^{\pm,f} & \xrightarrow{\cong} & \left(\tilde{H}^1_c(K^p)_{W_{k-2}-\mathrm{lalg}}\right)^{\pm,f} \\ \uparrow{\scriptstyle \mathrm{Id}\otimes\text{Eichler-Shimura}} & & \uparrow{\scriptstyle \Phi_f^{\pm}} \\ W_{k-2}(L) \otimes_L \pi_p & \xrightarrow{\cong} & \Pi(V_f)_{W_{k-2}-\mathrm{lalg}} \end{array}$$

Write the weight vectors of W_{k-2} to be

$$v_0 = v_{\mathrm{hw}} = (e_2^{k-2})^*,\ v_1 = (e_1 e_2^{k-3})^*,\ \cdots,\ v_{k-3} = (e_1^{k-3}e_2)^*,\ v_{k-2} = v_{\mathrm{lw}} = (e_1^{k-2})^*.$$

We then have the following lemma:

Lemma 2.2.4. *For any* $0 \le j \le k-2$,

$$\begin{aligned} \mathcal{M}_f^{\pm}\left(v_j \otimes v_{\mathrm{new}}\right) &= \left\langle \{0-\infty\}, v_j \otimes \mathrm{per}_L^{\pm}(f) \right\rangle_{(e_1^j e_2^{k-2-j})} \\ &= \begin{cases} \dfrac{\int_{i\infty}^0 f(z) z^j \, dz}{\Omega_f^{\pm}}, & \text{if } (-1)^j = \pm 1; \\ 0, & \text{if } (-1)^j = \mp 1. \end{cases} \end{aligned} \qquad (2.2.15)$$

□

According to [17], for any finite order chraracter χ of conductor p^n and any $0 \le j \le k-2$, we have the formula:

$$
\begin{aligned}
& L(f, \overline{\chi}, j+1) \\
= & \frac{(-2\pi i)^j}{j!} \cdot p^{n(\frac{k}{2}-j-2)} \cdot \tau(\overline{\chi}) \cdot \sum_{a \bmod p^n} \chi(a) \cdot 2\pi i \cdot \int_{i\infty}^{0} \left(f\Big|^{k}_{\begin{pmatrix} 1 & -a \\ 0 & p^n \end{pmatrix}} \right)(z) \cdot z^j dz \\
= & \frac{(-2\pi i)^j}{j!} \cdot p^{n(-j-1)} \cdot \tau(\overline{\chi}) \cdot \sum_{a \bmod p^n} \chi(a) \cdot 2\pi i \cdot \int_{i\infty}^{-\frac{a}{p^n}} f(z) \cdot (p^n z + a)^j dz.
\end{aligned}
$$

Notice that if we denote $\{-\frac{a}{p^n} - \infty\}$ by γ, then by equation (2.2.10), we have:

$$
\begin{aligned}
& \int_{i\infty}^{-\frac{a}{p^n}} f(z) \cdot (p^n z + a)^j dz + \int_{i\infty}^{+\frac{a}{p^n}} f(z) \cdot (p^n z - a)^j dz \\
= & \int_{\gamma} f(z) \cdot (p^n z + a)^j dz + \int_{\tau(\gamma)} f(z) \cdot (p^n z - a)^j dz \\
= & 2\Omega_f^{\pm} \cdot \left\langle \left\{ -\frac{a}{p^n} - \infty \right\}, \left(\sum_{t=0}^{j} C_j^t p^{nt} a^{j-t} v_t \right) \otimes \mathrm{per}_F^{\pm}(f) \right\rangle, \quad \text{if } (-1)^j = \pm 1,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
& \int_{i\infty}^{-\frac{a}{p^n}} f(z) \cdot (p^n z + a)^j dz - \int_{i\infty}^{+\frac{a}{p^n}} f(z) \cdot (p^n z - a)^j dz \\
= & \int_{\gamma} f(z) \cdot (p^n z + a)^j dz - \int_{\tau(\gamma)} f(z) \cdot (p^n z - a)^j dz \\
= & 2\Omega_f^{\pm} \cdot \left\langle \left\{ -\frac{a}{p^n} - \infty \right\}, \left(\sum_{t=0}^{j} C_j^t p^{nt} a^{j-t} v_t \right) \otimes \mathrm{per}_F^{\pm}(f) \right\rangle, \quad \text{if } (-1)^j = \mp 1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&L(f,\overline{\chi},j+1)\\
=&\frac{(-2\pi i)^j}{j!}\cdot p^{n(-j-1)}\cdot\tau(\overline{\chi})\cdot\sum_{a \bmod p^n}\chi(a)\cdot 2\pi i\cdot\int_{i\infty}^{-\frac{a}{p^n}} f(z)\cdot(p^n z+a)^j dz\\
=&\frac{(-2\pi i)^j}{j!}\cdot p^{n(-j-1)}\cdot\tau(\overline{\chi})\cdot\frac{1}{2}\sum_{a \bmod p^n}\chi(a)\cdot 2\pi i\cdot\left(\int_{i\infty}^{-\frac{a}{p^n}} f(z)\cdot(p^n z+a)^j dz\right.\\
&\left.\qquad\qquad\qquad\qquad\qquad\qquad+\chi(-1)\int_{i\infty}^{+\frac{a}{p^n}} f(z)\cdot(p^n z-a)^j dz\right)\\
=&\frac{(-2\pi i)^j}{j!}\cdot p^{n(-j-1)}\cdot\tau(\overline{\chi})\cdot\frac{1}{2}\sum_{a \bmod p^n}\chi(a)\cdot 2\pi i\cdot 2\Omega_f^{\pm}\cdot\Bigg\langle\left\{-\frac{a}{p^n}-\infty\right\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left(\sum_{t=0}^{j}C_j^t p^{nt}a^{j-t}v_t\right)\otimes \mathrm{per}_F^{\pm}(f)\Bigg\rangle
\end{aligned}
$$

(Here the sign $\pm$ is chosen such that $\pm 1=\chi(-1)\cdot(-1)^j$.)

$$=(-1)^j \cdot \frac{(2\pi i)^{j+1}}{\Gamma(j+1)} \cdot \frac{\Omega_f^{\pm}}{p^{n(j+1)}} \cdot \tau(\overline{\chi}) \cdot \sum_{a \bmod p^n} \chi(a) \cdot \left\langle \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix}^{-1} \cdot \{0-\infty\}, \right.$$

$$\left. \left(\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix}^{-1} \cdot v_j \right) \otimes \mathrm{per}_F^{\pm}(f) \right\rangle$$

$$=(-1)^j \cdot \frac{(2\pi i)^{j+1}}{\Gamma(j+1)} \cdot \frac{\Omega_f^{\pm}}{p^{n(j+1)}} \cdot \tau(\overline{\chi}) \cdot \sum_{a \bmod p^n} \chi(a) \cdot \left\langle \{0-\infty\}, \right.$$

$$\left. v_j \otimes \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \mathrm{per}_F^{\pm}(f) \right\rangle$$

(Here we are using Remark 2.2.1 about the action of $\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix}$ on image(per).)

$$=(-1)^j \cdot \frac{(2\pi i)^{j+1}}{\Gamma(j+1)} \cdot \frac{\Omega_f^{\pm}}{p^{n(j+1)}} \cdot \tau(\overline{\chi}) \cdot \sum_{a \bmod p^n} \chi(a) \cdot \mathcal{M}_f^{\pm}\left(v_j \otimes \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} v_{\mathrm{new}} \right).$$

Here in the last equality, we are using Lemma 2.2.3 to identify the (global) action of the matrix $\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix}$ on the modular symbols and the (local) action of it on $\Pi(V_f)$.

We write for all $0 \leq j \leq k-2$,

$$\widetilde{\Lambda}(f,\chi,j+1) = \frac{\Gamma(j+1)}{(2\pi i)^{j+1}} \cdot \frac{L(f,\chi,j+1)}{\Omega_f^{\pm}} \tag{2.2.16}$$

where the sign $\pm$ is chosen so that $\chi(-1) \cdot (-1)^j = \pm 1$. Thus we have obtained the following lemma:

Lemma 2.2.5. *For any finite order character χ of conductor p^n and any integer $0 \le j \le k-2$, we have*

$$\widetilde{\Lambda}(f, \overline{\chi}, j+1) = \frac{(-1)^j \tau(\overline{\chi})}{p^{n(j+1)}} \cdot \sum_{a \bmod p^n} \chi(a) \cdot \mathcal{M}_f^{\pm} \left(v_j \otimes \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} v_{\text{new}} \right), \quad (2.2.17)$$

where the sign $\pm$ is chosen to satisfy $\chi(-1) \cdot (-1)^j = \pm 1$.

□

We define

$$F_\chi := \sum_{a \text{ mod } p^n} \chi(a) \cdot \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} v_{\text{new}}$$

when the conductor of χ is p^n. Then Lemma 2.2.5 can also be stated as:

Lemma 2.2.6. *For any $0 \le j \le k-2$, any finite order character χ of conductor p^n with $n \ge 0$, let*

$$F_\chi := \sum_{a \bmod p^n} \chi(a) \cdot \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} v_{\text{new}},$$

then we have for all $0 \le j \le k-2$,

$$\mathcal{M}_f^{\pm} \left(v_j \otimes F_\chi \right) = (-1)^j \cdot \frac{p^{n(j+1)}}{\tau(\overline{\chi})} \cdot \widetilde{\Lambda}(f, \overline{\chi}, j+1). \quad (2.2.18)$$

Here, the sign $\pm$ is chosen such that

$$\chi(-1) = \pm(-1)^j.$$

$\square$

Remark 2.2.7. If the sign $\pm$ is such that $\chi(-1) \neq \pm(-1)^j$, then $\mathcal{M}_f^{\pm}\left(v_j \otimes F_\chi\right) = 0$.

2.3 The p-adic Kirillov model, supercuspidal case

Throughout this section, we assume $\pi_p^{\mathrm{sm}}(f)$, the smooth p-adic representation of $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$ attached to the modular form f, is supercuspidal. We recall the theory of p-adic Kirillov models developed in [18], chapter VI.

If Π is a p-adic locally algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with coefficient field L that can be written as $\Pi = \det^a \otimes \mathrm{Sym}^b \otimes \pi$ where a, $b \in \mathbb{Z}$, $b \geq 0$ and π is a smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the p-adic Kirillov model of Π is the unique $P(\mathbb{Q}_p)$-equivariant embedding

$$\mathscr{K} : \Pi \to \prod_{\mathbb{Z}} t^a L_\infty[t]/t^{a+b+1},$$

where the action of $P(\mathbb{Q}_p)$ on the RHS is given by the formula

$$\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} S\right)_n = (1 \otimes \varepsilon\left(bp^n\right)) \cdot \exp\left(bp^n t\right) \cdot (1 \otimes \sigma_{a^*}) \left(S_{v(a)+n}\right) \tag{2.3.1}$$

for any $S \in \prod_{\mathbb{Z}} t^a L_\infty[t]/t^{a+b+1}$ and any $n \in \mathbb{Z}$. Here, $a^* := |a|_p \cdot a$, and ε is defined in the following way: we let $\varepsilon(x) = \zeta_{p^n}^r$ if $x \equiv \frac{r}{p^n} \mod \mathbb{Z}_p$, where $\left\{\zeta_{p^n}\right\}_{n \geq 0}$ is the compatible system of p-power roots of unity chosen as in Section 2.1.

Remark 2.3.1. The usual definition of the Kirillov model of a p-adic locally algebraic representation (as in [18]) is a $P(\mathbb{Q}_p)$ equivariant map $\mathscr{K}$ from $\Pi = \det^a \otimes \mathrm{Sym}^b \otimes \pi$ to $\mathrm{LP}^{[a,a+b]}\left(\mathbb{Q}_p^*, t^a L_\infty[t]/t^{a+b+1}\right)$, where $L_\infty = L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^\infty})$. There is an action of Γ on $\mathrm{LP}^{[a,a+b]}\left(\mathbb{Q}_p^*, t^a L_\infty[t]/t^{a+b+1}\right)$ given by formula

$$\sigma_u(S)(x) = (1 \otimes \sigma_u)^{-1}(S(ux)) \tag{2.3.2}$$

for any $u \in \mathbb{Z}_p^*$ and $x \in \mathbb{Q}_p^*$, which commutes with the action of $P(\mathbb{Q}_p)$ on

$$\mathrm{LP}^{[a,a+b]}\left(\mathbb{Q}_p^*, t^a L_\infty[t]/t^{a+b+1}\right).$$

Therefore the uniqueness of the p-adic Kirillov model forces the image of $\mathscr{K}$ to be contained in $\mathrm{LP}^{[a,a+b]}\left(\mathbb{Q}_p^*, t^a L_\infty[t]/t^{a+b+1}\right)^\Gamma$, which can then be identified as $\prod_{\mathbb{Z}} t^a L_\infty[t]/t^{a+b+1}$ by sending every

$$S \in \mathrm{LP}^{[a,a+b]}\left(\mathbb{Q}_p^*, t^a L_\infty[t]/t^{a+b+1}\right)^\Gamma$$

to $\{S(p^n)\}_{n\in\mathbb{Z}}$. In the rest of this paper, we will always regard a p-adic Kirillov function as an element in $\prod_{\mathbb{Z}} t^a L_\infty[t]/t^{a+b+1}$.

In the case when $\Pi = \Pi(V_f)^{\mathrm{lalg}}$ is the collection of locally algebraic vectors of the p-adic Banach space representation $\Pi(V_f)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, we have $\Pi(V_f)^{\mathrm{lalg}} = \left(\mathrm{Sym}^{k-2} L^2\right)^\vee \otimes_L \pi_p^{\mathrm{sm}}(V_f)$, thus its p-adic Kirillov model is a $P(\mathbb{Q}_p)$-equivariant embedding

$$\mathscr{K} : \Pi(V_f)^{\mathrm{lalg}} \to \prod_{\mathbb{Z}} \frac{1}{t^{k-2}} L_\infty[[t]]/tL_\infty[[t]],$$

where we continue using the notations $F = \mathbb{Q}(f)$, $L = F_\lambda$ where λ is a place lying over p.

Remark 2.3.2. Since in our case $\pi_p^{\mathrm{sm}}(f)$ is supercuspidal, the map $\mathscr{K}$ is in fact a $P(\mathbb{Q}_p)$-equivariant isomorphism onto $\bigoplus_{\mathbb{Z}} \frac{1}{t^{k-2}} L_\infty[[t]]/tL_\infty[[t]]$.

We assume $\pi_p^{\mathrm{sm}}(f)$ has central character δ. Then the explicit formula of the Borel action on $\prod_{\mathbb{Z}} \frac{1}{t^{k-2}} L_\infty[[t]]/tL_\infty[[t]]$ is:

$$\begin{aligned}
\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} S\right)_n &= \left(d^{2-k}\delta(d) \otimes \varepsilon\left(\frac{bp^n}{d}\right)\right) \cdot \exp\left(\frac{bp^n t}{d}\right) \cdot \left(1 \otimes \sigma_{\left(\frac{a}{d}\right)^*}\right) \left(S_{v(a)-v(d)+n}\right) \\
&= d^{2-k}\delta(d) \cdot \left[(1+T)^{\frac{bp^n}{d}}\right] \cdot \left(1 \otimes \sigma_{\left(\frac{a}{d}\right)^*}\right) \left(S_{v(a)-v(d)+n}\right).
\end{aligned}$$

If we write the weight vectors of $\left(\mathrm{Sym}^{k-2} L^2\right)^\vee = \mathrm{Sym}^{k-2} L^2 \otimes_L \det^{2-k}$ to be

$$v_0 = v_{\mathrm{hw}},\ v_1,\ \cdots,\ v_{k-3},\ v_{k-2} = v_{\mathrm{lw}}$$

as in the previous sections, then by [18] Page 146, we have

$$\left(\mathscr{K}\left(v_j \otimes v\right)\right)_n = j!\, p^{-nj} \cdot \left(\mathscr{K}^{\mathrm{sm}}(v)\right)_n \cdot t^{-j}, \tag{2.3.3}$$

where $\mathscr{K}^{\mathrm{sm}}$ is the p-adic Kirillov model for smooth representations.

For every integer n, we let $e_n \in \prod_{\mathbb{Z}} L_\infty$ be the vector whose coordinate at the

n-th place is 1, and 0 everywhere else.

Recall that we are in the case when $\pi_p^{\mathrm{sm}}(f)$ is supercuspidal, hence $\mathscr{K}^{\mathrm{sm}}(v_{\mathrm{new}}) = \mathbb{1}_{\mathbb{Z}_p^*} = e_0$. Thus

$$\mathscr{K}^{\mathrm{sm}}\left(F_\chi\right) = \left(\sum_{a \bmod p^n} \chi(a) \otimes \zeta_{p^n}^a\right) \cdot e_{-n}, \tag{2.3.4}$$

where χ is any finite order character of conductor p^n with $n \geq 0$, and F_χ is defined as in section 2.2.

Therefore, we have for all $0 \leq j \leq k-2$,

$$\mathscr{K}\left(v_j \otimes F_\chi\right) = j! \cdot p^{nj} \cdot \left(\sum_{a \bmod p^n} \chi(a) \otimes \zeta_{p^n}^a\right) \cdot t^{-j} \cdot e_{-n}. \tag{2.3.5}$$

In the rest of this section, we will describe the p-adic Kirillov model of $\Pi(V_f)$ from another point of view.

Recall the following theorem from [18]:

Theorem 2.3.3 (Colmez, [18], Corollaire II.2.9)**.** *If $\pi_p^{\mathrm{sm}}(f)$ is supercuspidal, then there is an isomorphism as $P\left(\mathbb{Q}_p\right)$-modules:*

$$\widetilde{\mathbf{D}}(V_f(1)) \Big/ \widetilde{\mathbf{D}}^+(V_f(1)) \xrightarrow{\cong} \Pi(V_f). \tag{2.3.6}$$

Remark 2.3.4. The action of $P\left(\mathbb{Q}_p\right)$ on $\widetilde{\mathbf{D}}(V_f(1)) \Big/ \widetilde{\mathbf{D}}^+(V_f(1))$ is defined as follows:

- The matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ acts on $\widetilde{\mathbf{D}}(V_f(1)) \Big/ \widetilde{\mathbf{D}}^+(V_f(1))$ via the operator φ;

- The matrix $\begin{pmatrix} \mathbb{Z}_p^* & 0 \\ 0 & 1 \end{pmatrix}$ acts on $\widetilde{\mathbf{D}}(V_f(1)) \big/ \widetilde{\mathbf{D}}^+(V_f(1))$ via the operator Γ;
- The matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} 1 & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$ acts on $\widetilde{\mathbf{D}}(V_f(1)) \big/ \widetilde{\mathbf{D}}^+(V_f(1))$ via multiplying $(1+T)^a$.

We denote the inverse of the above isomorphism by η. We then have the following lemma:

Lemma 2.3.5.

The image of $\Pi(V_f)^{\mathrm{lalg}}$ *under the map* η *is contained in*

$$\widetilde{\mathbf{D}}^+(V_f(1)) \left[\frac{1}{\varphi^r(T)},\ r \geq 0 \right] \bigg/ \widetilde{\mathbf{D}}^+(V_f(1)).$$

Proof. For any $v_{\mathrm{sm}} \in \pi_p^{\mathrm{sm}}(f)$, $v_0 \otimes v_{\mathrm{sm}}$ is fixed by $\begin{pmatrix} 1 & p^r \\ 0 & 1 \end{pmatrix}$ if r is sufficiently large. This means

$$(1+T)^{p^r} \cdot \eta(v_0 \otimes v_{\mathrm{sm}}) = \eta(v_0 \otimes v_{\mathrm{sm}}),$$

that is, $\eta(v_0 \otimes v_{\mathrm{sm}}) \in \frac{1}{\varphi^r(T)} \widetilde{\mathbf{D}}^+(V_f(1)) \big/ \widetilde{\mathbf{D}}^+(V_f(1))$.

We then proceed by induction: if $0 \leq j < k-2$ and

$$\eta(v_i \otimes v_{\mathrm{sm}}) \in \frac{1}{(\varphi^r(T))^j} \widetilde{\mathbf{D}}^+(V_f(1)) \big/ \widetilde{\mathbf{D}}^+(V_f(1))$$

for all $0 \le i < j$ and some r, then we will show that

$$\eta(v_j \otimes v_{\mathrm{sm}}) \in \frac{1}{(\varphi^r(T))^{j+1}} \widetilde{\mathbf{D}}^+(V_f(1)) \Big/ \widetilde{\mathbf{D}}^+(V_f(1)).$$

To see this, notice that for sufficiently large r,

$$\begin{pmatrix} 1 & p^r \\ 0 & 1 \end{pmatrix} (v_j \otimes v_{\mathrm{sm}}) = \begin{pmatrix} 1 & p^r \\ 0 & 1 \end{pmatrix} v_j \otimes v_{\mathrm{sm}} = v_j \otimes v_{\mathrm{sm}} + \sum_{0 \le i < j} \binom{j}{i} p^{r(j-i)} v_i \otimes v_{\mathrm{sm}},$$

so

$$\varphi^r(T) \cdot \eta(v_j \otimes v_{\mathrm{sm}}) \in \frac{1}{(\varphi^r(T))^j} \widetilde{\mathbf{D}}^+(V_f(1)) \Big/ \widetilde{\mathbf{D}}^+(V_f(1))$$

and hence $\eta(v_j \otimes v_{\mathrm{sm}}) \in \frac{1}{(\varphi^r(T))^{j+1}} \widetilde{\mathbf{D}}^+(V_f(1)) \Big/ \widetilde{\mathbf{D}}^+(V_f(1))$. □

Definition 2.3.6. We define a map

$$\mathscr{I} : \widetilde{\mathbf{D}}^+(V_f(1)) \left[\frac{1}{\varphi^r(T)}, r \ge 0 \right] \Big/ \widetilde{\mathbf{D}}^+(V_f(1)) \longrightarrow \prod_{\mathbb{Z}} \widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1)) / \widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V_f(1))$$

as follows:

- The 0-th coordinate $\mathscr{I}_0 := \iota_0^-$, which is the natural map

  $$\iota_0^- : \widetilde{\mathbf{D}}^+(V_f(1)) \left[\frac{1}{\varphi^r(T)}, r \ge 0 \right] \Big/ \widetilde{\mathbf{D}}^+(V_f(1)) \longrightarrow \widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1)) / \widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V_f(1))$$

  induced by the inclution $\widetilde{\mathbf{B}}^+ \left[\frac{1}{\varphi^r(T)}, r \ge 0 \right] \subset \mathbf{B}_{\mathrm{dR}}$.

- The n-th coordinate $\mathscr{I}_n$ is defined as $\iota_0^- \circ \varphi^n$ if $n \ge 0$.

- The $(-n)$-th coordinate $\mathscr{I}_n$ is defined as ι_n^- if $n \geq 0$: we choose a compatible system of p-power roots of unity $\left\{\zeta_{p^i}\right\}_{i\geq 0}$ as before, and for any positive integer n and any function $f(T) \in \widetilde{\mathbf{B}}^+$, we define

$$\iota_n\left(f(T)\right) := f\left(\zeta_{p^n}\exp\left(\frac{t}{p^n}\right) - 1\right),$$

  The map ι_n extends naturally to a map from $\widetilde{\mathbf{B}}^+\left[\frac{1}{\varphi^r(T)}, r \geq 0\right]$ to $\mathbf{B}_{\mathrm{dR}}$, which induces a map ι_n from $\widetilde{\mathbf{D}}^+(V_f(1))\left[\frac{1}{\varphi^r(T)}, r \geq 0\right]$ to $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))$.

  The map ι_n^- is the composite of the natural projection from $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))$ to $\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V_f(1))$ with ι_n.

We equip $\prod_{\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))/\widetilde{\mathbf{D}}^+_{\mathrm{dif}}(V_f(1))$ with an action of $P(\mathbb{Q}_p)$ in the following way:

- The matrix $\begin{pmatrix}\mathbb{Z}_p^* & 0\\ 0 & 1\end{pmatrix}$ acts coordinate wise through the action of Γ;
- A matrix $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \in \begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & 1\end{pmatrix}$ acts on the n-th coordinate by multiplying $\varepsilon\left(bp^n\right)\cdot\exp\left(bp^n t\right)$;
- $\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$ acts by shifting to the left.

We then have the following observation:

Proposition 2.3.7. *The map*

$$\mathscr{I}:\ \widetilde{\mathbf{D}}^{+}(V_f(1))\left[\frac{1}{\varphi^r(T)},\ r\geq 0\right]\Big/\widetilde{\mathbf{D}}^{+}(V_f(1))\longrightarrow\prod_{\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))/\widetilde{\mathbf{D}}^{+}_{\mathrm{dif}}(V_f(1))$$

is $P(\mathbb{Q}_p)$-equivariant. Moreover, the following diagram is commutative:

$$\begin{array}{ccc}\widetilde{\mathbf{D}}^{+}(V_f(1))\left[\frac{1}{\varphi^r(T)},\ r\geq 0\right]\Big/\widetilde{\mathbf{D}}^{+}(V_f(1)) & \xrightarrow{\mathscr{I}} & \prod_{\mathbb{Z}}\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))/\widetilde{\mathbf{D}}^{+}_{\mathrm{dif}}(V_f(1)) \\ \eta\big\uparrow & & \big\uparrow \\ \Pi(V_f)^{\mathrm{lalg}} & \xrightarrow{\mathscr{K}} & \prod_{\mathbb{Z}}\frac{1}{t^{k-2}}L_\infty[[t]]/tL_\infty[[t]]\end{array}$$

where the vertical map on the right on each coordinate is given by the composite of the following maps:

$$\begin{aligned}\frac{1}{t^{k-2}}L_\infty[[t]]/tL_\infty[[t]]&\xrightarrow{\cong}\mathbf{D}_{\mathrm{dR}}(V_f(1))\otimes_L\frac{1}{t^{k-2}}L_\infty[[t]]\Big/\mathbf{D}^{+}_{\mathrm{dif}}(V_f(1))\\&\hookrightarrow\mathbf{D}_{\mathrm{dif}}(V_f(1))/\mathbf{D}^{+}_{\mathrm{dif}}(V_f(1))\\&\subset\widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f(1))/\widetilde{\mathbf{D}}^{+}_{\mathrm{dif}}(V_f(1)).\end{aligned}$$

The first isomorphism above is defined by sending every $g(t)$ to $\overline{f}^*\otimes g(t)$. See Section 2.1 for the precise definition of $\overline{f}^*\in\mathbf{D}_{\mathrm{dR}}(V_f(1))/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f(1))$.

Proof. The $P(\mathbb{Q}_p)$-equivariance of the map $\mathscr{I}$ can be checked by direct computation.

The commutativity of the diagram follows from the uniqueness of the p-adic Kirillov model. □

Corollary 2.3.8. *Let χ be any finite order character of conductor p^n, where $n\geq 0$. Let F_χ be as defined in section 2.2. Then $\mathscr{I}_m\circ\eta\left(F_\chi\otimes v_j\right)=0$ for any $m\neq -n$,* *and* $\mathscr{I}_{-n} \circ \eta\left(F_\chi \otimes v_j\right)$ *is contained in* $\mathbf{D}_{\mathrm{dif},n}\left(V_f(1)\right)/\mathbf{D}^+_{\mathrm{dif},n}\left(V_f(1)\right)$, *for any* $0 \leq j \leq k-2$.

In other words, we have:

- $\iota_m^- \circ \eta\left(v_j \otimes F_\chi\right) = 0$ *for all integer* $m \neq n$,
- $\iota_n^- \circ \eta\left(v_j \otimes F_\chi\right) \in \mathbf{D}_{\mathrm{dif},n}\left(V_f(1)\right)/\mathbf{D}^+_{\mathrm{dif},n}\left(V_f(1)\right)$.

Proof. This follows immediately from Proposition 2.3.7, together with the explicit formula of the map $\mathscr{K}$ in equation (2.3.5). □

2.4 An explicit reciprocity law, supercuspidal case

In this section, we continue assuming that the (p-adic) smooth representation of $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$ attached to the modular form f is supercuspidal. We review an "explicit reciprocity law" proved in [10] and [9] as a generalization of Proposition VI.3.4 in [18].

We first introduce two pairings that will be used later. The reference for the following definitions is Page 151 of [18].

The pairing "{ , }"

The pairing

$$V_f(1) \times V_f^* \to L(1)$$

gives rise to the following pairing:

$$[\ ,\ ]: \quad \widetilde{\mathbf{D}}\left(V_f(1)\right) \times \widetilde{\mathbf{D}}(V_f^*) \to \widetilde{\mathbf{D}}(L(1)) \cong \widetilde{B}_L.$$

We define $\{\ ,\ \}$ to be $\mathrm{Res}_{T=0}\left([\ ,\ ] \cdot \frac{dT}{1+T}\right)$.

The key properties of the pairing $\{\ ,\ \}$ are summarized in the following proposition, whose proof can be found in [18].

Proposition 2.4.1.

1. *$\widetilde{\mathbf{D}}\left(V_f(1)\right)/\widetilde{\mathbf{D}}^+\left(V_f(1)\right)$ and $\left(\mathbf{D}^\natural(V_f^*)\boxtimes\mathbb{Q}_p\right)_{\mathrm{b}} := \left(\varprojlim_{\psi} \mathbf{D}^\natural(V_f^*)\right)_{\mathrm{b}}$ are topological dual to each other. We have*

$$\widetilde{\mathbf{D}}^+\left(V_f^*\right) = \left(\mathbf{D}^\natural(V_f^*) \boxtimes \mathbb{Q}_p\right)_{\mathrm{pc}} \subset \left(\mathbf{D}^\natural(V_f^*) \boxtimes \mathbb{Q}_p\right)_{\mathrm{b}},$$

   and the pairing between $\widetilde{\mathbf{D}}\left(V_f(1)\right)/\widetilde{\mathbf{D}}^+\left(V_f(1)\right)$ and $\left(\mathbf{D}^\natural(V_f^*) \boxtimes \mathbb{Q}_p\right)_{\mathrm{b}}$ restriced to $\widetilde{\mathbf{D}}^+\left(V_f^*\right) = \left(\mathbf{D}^\natural(V_f^*) \boxtimes \mathbb{Q}_p\right)_{\mathrm{pc}}$ is the pairing $\{\ ,\ \}$ defined above.

2. *We have the following compatibility:*

$$\begin{array}{ccccc}
\widetilde{\mathbf{D}}^+\left(V_f(1)\right)\left[\frac{1}{\varphi^i(T)},\ i\geq 0\right]/\widetilde{\mathbf{D}}^+(V_f(1)) & \times & \mathbf{D}(V_f^*)^{\psi=1} & \xrightarrow{\{\ ,\ \}} & L \\
\uparrow \eta & & \mathfrak{C}\uparrow\cong & & \Vert \\
\Pi(V_f)^{\mathrm{lalg}} & \times & \Pi(V_f)^{*,\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)=1} & \longrightarrow & L
\end{array}$$

   where the pairing on the bottom row is induced by the usual pairing between $\Pi(V_f)$ and $\Pi(V_f)^*$.

Proof. (1) is [18] Proposition I.3.20. and Proposition I.3.21. □

The pairings "$\langle\ ,\ \rangle_{\mathrm{dif}}$" and "$\langle\ ,\ \rangle_{\mathrm{dif},m}$"

The pairing $V_f(1) \times V_f^* \to L(1)$ again gives rise to the following pairings (we still use the notation "[,]" here):

$$[\ ,\ ]: \quad \widetilde{\mathbf{D}}_{\mathrm{dif}}\left(V_f(1)\right) \times \widetilde{\mathbf{D}}_{\mathrm{dif}}\left(V_f^*\right) \to \widetilde{\mathbf{D}}_{\mathrm{dif}}(L(1)) \cong B_{\mathrm{dR}}^H \otimes_{\mathbb{Q}_p} L;$$

and for every integer $m \geq 0$,

$$[\ ,\ ]_m: \quad \mathbf{D}_{\mathrm{dif},m}\left(V_f(1)\right) \times \mathbf{D}_{\mathrm{dif},m}\left(V_f^*\right) \to \mathbf{D}_{\mathrm{dif},m}(L(1)) \cong L_m((t)).$$

We define $\langle\ ,\ \rangle_{\mathrm{dif}}$ (resp. $\langle\ ,\ \rangle_{\mathrm{dif},m}$) as

$$\mathrm{Tr} \circ \mathrm{Res}_{t=0}\left([\ ,\ ] \cdot \frac{dt}{t}\right)$$

(resp. $\mathrm{Tr} \circ \mathrm{Res}_{t=0}\left([\ ,\ ]_m \cdot \frac{dt}{t}\right)$), where Tr is the normalized trace map.

The key properties of the pairings $\langle\ ,\ \rangle_{\mathrm{dif}}$ and $\langle\ ,\ \rangle_{\mathrm{dif},m}$ are summarized in the following proposition, whose proof can again be found in [18], Chapter VI, Section 3.4.

Proposition 2.4.2.

1. *The pairings* $\langle\ ,\ \rangle_{\mathrm{dif}}$ *and* $\langle\ ,\ \rangle_{\mathrm{dif},m}$ *are compatible for every integer* $m \geq 0$. *In other words, we have the following commutative diagram:*

$$\begin{array}{ccccc} \mathbf{D}_{\mathrm{dif},m}\big(V_f(1)\big) & \times & \mathbf{D}_{\mathrm{dif},m}(V_f^*) & \xrightarrow{\langle\ ,\ \rangle_{\mathrm{dif},m}} & L \\ \big\downarrow & & \big\downarrow & & \big\| \\ \widetilde{\mathbf{D}}_{\mathrm{dif}}\big(V_f(1)\big) & \times & \widetilde{\mathbf{D}}^{+}_{\mathrm{dif}}(V_f^*) & \xrightarrow{\langle\ ,\ \rangle_{\mathrm{dif}}} & L \end{array}$$

2. $\widetilde{\mathbf{D}}^{+}_{\mathrm{dif}}\big(V_f(1)\big)$ *and* $\widetilde{\mathbf{D}}^{+}_{\mathrm{dif}}(V_f^*)$ *are orthogonal complement of each other under the pairing* $\langle\ ,\ \rangle_{\mathrm{dif}}$.

3. *For every integer* $m \geq 0$, $\mathbf{D}^{+}_{\mathrm{dif},m}\big(V_f(1)\big)$ *and* $\mathbf{D}^{+}_{\mathrm{dif},m}(V_f^*)$ *are orthogonal compliment of each other under the pairing* $\langle\ ,\ \rangle_{\mathrm{dif},m}$.

4. *We have the following compatibility:*

$$\begin{array}{ccccc} \mathbf{D}_{\mathrm{dR}}(V_f(1)) \otimes_{\mathbb{Q}_p} \mathbf{B}^{H}_{\mathrm{dR}} & \times & \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_{\mathbb{Q}_p} \mathbf{B}^{H}_{\mathrm{dR}} & \xrightarrow{\widetilde{\mathscr{B}}} & L \\ \cong\big\uparrow & & \cong\big\uparrow & & \big\| \\ \widetilde{\mathbf{D}}_{\mathrm{dif}}\big(V_f(1)\big) & \times & \widetilde{\mathbf{D}}_{\mathrm{dif}}(V_f^*) & \xrightarrow{\langle\ ,\ \rangle_{\mathrm{dif}}} & L \end{array}$$

where the pairing $\widetilde{\mathscr{B}}$ *on the top row is defined by the formula*

$$\widetilde{\mathscr{B}}(x \otimes f(t), y \otimes g(t)) := \mathrm{Tr} \circ \mathrm{Res}_{t=0}\left([x, y]_{\mathrm{dR}} \cdot f(t)g(t)dt\right),$$

and $[\ ,\ ]_{\mathrm{dR}}$ *is the pairing*

$$\mathbf{D}_{\mathrm{dR}}(V_f(1)) \times \mathbf{D}_{\mathrm{dR}}(V_f^*) \xrightarrow{[\ ,\ ]_{\mathrm{dR}}} \mathbf{D}_{\mathrm{dR}}(L(1)) = \frac{1}{t}L.$$

5. *Similarly, we have the following compatibility for every integer* $m \geq 0$*:*

$$\begin{array}{ccccc} \mathbf{D}_{\mathrm{dR}}(V_f(1)) \otimes_L L_m((t)) & \times & \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L L_m((t)) & \xrightarrow{\mathscr{B}_m} & L \\ \cong\uparrow & & \cong\uparrow & & \| \\ \mathbf{D}_{\mathrm{dif},m}\left(V_f(1)\right) & \times & \mathbf{D}_{\mathrm{dif},m}(V_f^*) & \xrightarrow{\langle\ ,\ \rangle_{\mathrm{dif},m}} & L \end{array}$$

where the pairing $\mathscr{B}_m$ *on the top row is defined by the formula*

$$\mathscr{B}_m(x \otimes f(t), y \otimes g(t)) := \mathrm{Tr} \circ \mathrm{Res}_{t=0} \left([x,y]_{\mathrm{dR}} \cdot f(t)g(t)dt\right).$$

The following theorem, which we shall call the “explicit reciprocity law” relating the pairing $\langle\ ,\ \rangle_{\mathrm{dif}}$ and the pairing $\{\ ,\ \}$, is proved by Dospinescu:

Theorem 2.4.3 (Dospinescu [10], Théorème 8.3.1)**.** *Let* Π *be a continuous Banach space representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$*,* Π^{lalg} *be the collection of locally algebraic vectors of* Π*. Assume* $\Pi^{\mathrm{lalg}} = W \otimes \pi$ *with* W *algebraic and* π *a supercuspidal smooth representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$*. Let* $v \in \Pi^{\mathrm{lalg}}$ *with* $\mathscr{K}_j(v) = 0$ *for all but finitely many* j*. Assume* m *is sufficiently large and* $\mathscr{I}_j \circ \eta(v) \in \mathbf{D}_{\mathrm{dif},m}/\mathbf{D}^+_{\mathrm{dif},m}$ *for all* j *(using the notations as in Proposition 2.3.7). Then for any* $z \in \Pi^{*,\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)=1}$*, we have*

$$\{v, z\} = \sum_{j \in \mathbb{Z}} \left\langle \mathscr{I}_j \circ \eta(v),\ \iota_m(\mathfrak{C}(z)) \right\rangle_{\mathrm{dif},m}.$$

In particular, the RHS of the above equation is independent of m *(as long as* m *is sufficiently large).*

□

Recall that we have defined the element $\mathcal{M}_f^{\pm} \in \Pi(V_f)^{*,\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)=1}$ and $\mathbf{z}_M^{\pm}(f) = (\mathrm{Exp}^*)^{-1}\,\mathfrak{C}(\mathcal{M}_f^{\pm}) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$. Since $\mathrm{Exp}^*(\mathbf{z}_M^{\pm}) \in \mathbf{D}(V_f^*)^{\psi=1}$, ι_n is defined on $\mathrm{Exp}^*(\mathbf{z}_M^{\pm})$ for all $n \geq 1$ and

$$\iota_n \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm}) \in \mathbf{D}^+_{\mathrm{dif},n}(V_f^*).$$

Moreover, one has

$$\frac{1}{p}\mathrm{Tr}_{L_n}^{L_{n+1}} \iota_{n+1} \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm}) = \iota_n \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm}) \tag{2.4.1}$$

for all $n \geq 1$.

Corollary 2.4.4. *Let χ be a finite order character of $\mathbb{Z}_p^*$ with conductor p^n, $n \geq 0$. $F_\chi \in \pi_p^{\mathrm{sm}}(f)$ be as in section 2.2. For every $0 \leq j \leq k-2$,*

$$\mathcal{M}_f^{\pm}\left(v_j \otimes F_\chi\right) = \begin{cases} \left\langle \iota_n^- \circ \eta(v_j \otimes F_\chi)\ ,\ \iota_n \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm})\right\rangle_{\mathrm{dif},n}, & \text{if } n \geq 1; \\ \left\langle \iota_0^- \circ \eta(v_j \otimes F_\chi)\ ,\ \frac{1}{p-1}\mathrm{Tr}_{L_0}^{L_1} \iota_1 \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm})\right\rangle_{\mathrm{dif},0}, & \text{if } n = 0. \end{cases} \tag{2.4.2}$$

Proof. Using Theorem 2.4.4 and Corollary 2.3.8, we conclude that

$$\mathcal{M}_f^{\pm}\left(v_j \otimes F_\chi\right) = \left\langle \iota_n^- \circ \eta(v_j \otimes F_\chi)\ ,\ \iota_m \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm})\right\rangle_{\mathrm{dif},m}$$

for all m sufficiently large, where we view $\iota_n^- \circ \eta(v_j \otimes F_\chi)$ as an element in

$$\mathbf{D}_{\mathrm{dif},m}(V_f(1))/\mathbf{D}^+_{\mathrm{dif},m}(V_f(1))$$

via the natural inclusion

$$\mathbf{D}_{\mathrm{dif},n}(V_f(1))/\mathbf{D}^+_{\mathrm{dif},n}(V_f(1)) \hookrightarrow \mathbf{D}_{\mathrm{dif},m}(V_f(1))/\mathbf{D}^+_{\mathrm{dif},m}(V_f(1)).$$

Since

$$\left\langle \iota_n^- \circ \eta(v_j \otimes F_\chi)\ ,\ \iota_m \circ \mathrm{Exp}^*(\mathbf{z}_M^\pm) \right\rangle_{\mathrm{dif},m} = \left\langle \iota_n^- \circ \eta(v_j \otimes F_\chi)\ ,\ \mathrm{Tr}_{L_n} \iota_m \circ \mathrm{Exp}^*(\mathbf{z}_M^\pm) \right\rangle_{\mathrm{dif},n}$$

where Tr_{L_n} denotes the normalized trace map to L_n, and

$$\mathrm{Tr}_{L_n} \iota_m \circ \mathrm{Exp}^*(\mathbf{z}_M^\pm) = \iota_n \circ \mathrm{Exp}^*(\mathbf{z}_M^\pm)$$

for all $n \geq 1$, the conclusion follows. □

Notice that V_f^* has Hodge-Tate weights 0 and $k-1$, we have

$$\begin{aligned} D^+_{\mathrm{dif},n}\left(V_f^*\right) &= \mathrm{Fil}^0\left(\mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L L_n((t))\right) \\ &= \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L t^{k-1} L_n[[t]] + \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L L_n[[t]]. \end{aligned}$$

We make the following definition:

Definition 2.4.5. Let $\overline{f} \in \mathrm{Fil}^0\left(\mathbf{D}_{\mathrm{dR}}(V_f^*)\right)$ be defined as in Section 2.1.

For any $0 \leq j \leq k-2$ and any integer $r \geq 1$, we write $C^{\pm}_{r,j} \in L_r = L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\left(\zeta_{p^r}\right)$ the coefficient of $\iota_r \circ \mathrm{Exp}^*(\mathbf{z}^{\pm}_M)$ in front of t^j:

$$\iota_r \circ \mathrm{Exp}^*(\mathbf{z}^{\pm}_M) = \overline{f} \otimes \sum_{j=0}^{k-2} C^{\pm}_{r,j} t^j + \text{ “Something in } \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L t^{k-1} L_r[[t]]\text{”}. \quad (2.4.3)$$

Further more, under the decomposition

$$L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\left(\zeta_{p^r}\right) = \bigoplus_{\phi} L(\phi^{-1})$$

where the direct sum is taken among all characters ϕ of $(\mathbb{Z}/p^r)^{\times}$, we write

$$C^{\pm}_{r,j} = \sum_{\phi} C^{\pm}_{r,j,\phi} \cdot e_{\phi} \quad (2.4.4)$$

with $C^{\pm}_{r,j,\phi} \in L$.

Here, we are viewing each $L(\phi^{-1})$ as a 1-dimensional L-subspace contained in $L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\left(\zeta_{p^r}\right)$, with basis

$$e_{\phi} = \sum_{a \bmod p^r} \phi(a) \otimes \zeta^a_{p^l} \in L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\left(\zeta_{p^r}\right),$$

where p^l is the conductor of ϕ.

Remark 2.4.6. Since $\mathrm{Exp}^*(\mathbf{z}^{\pm}_M)$ is fixed by ψ, we have

$$C^{\pm}_{r,j} = \frac{1}{p} \mathrm{Tr}^{\mathbb{Q}_{p,r+1}}_{\mathbb{Q}_{p,r}} C^{\pm}_{r+1,j}$$

for all integers $r \geq 1$ and $j \geq 0$.

Remark 2.4.7. It can be checked easily that

$$C^{\pm}_{r,j,\phi} = \frac{1}{p^{r-1}(p-1)} \sum_{a \bmod p^r} \phi(a)\sigma_a(C^{\pm}_{r,j})$$

for all integers $r \geq 1$, $j \geq 0$ and ϕ a character of $(\mathbb{Z}/p^r)^{\times}$.

We define

$$C^{\pm}_{0,j} = \frac{1}{p-1}\mathrm{Tr}^{\mathbb{Q}_p(\zeta_p)}_{\mathbb{Q}_p} C^{\pm}_{1,j} \tag{2.4.5}$$

for all $j \geq 0$.

Proposition 2.4.8. *Let χ be any finite order character of conductor p^n, $n \geq 0$.*

We have for all $0 \leq j \leq k-2$,

$$\mathrm{Tr} \circ \mathrm{Res}_{t=0}\left(\mathscr{K}_{-n}\left(v_j \otimes F_\chi\right) \cdot \sum_{i=0}^{k-2} C^{\pm}_{n,j} t^i \cdot \frac{dt}{t}\right) = \mathcal{M}^{\pm}_f\left(v_j \otimes F_\chi\right), \tag{2.4.6}$$

where Tr *is the normalized trace map to L.*

Proof. By Theorem 2.4.3, for any integer m sufficiently large,

$$\langle \iota_n^- \circ \eta(v_j \otimes F_\chi)\ ,\ \iota_m \circ \mathfrak{C}\left(\mathcal{M}^{\pm}_f\right)\rangle_{\mathrm{dif},m} = \mathcal{M}^{\pm}_f\left(v_j \otimes F_\chi\right).$$

Now by Proposition 2.3.7,

$$\iota_n^- \circ \eta(v_j \otimes F_\chi) = \overline{f}^* \otimes \mathscr{K}_{-n}\left(v_j \otimes F_\chi\right) \in \frac{\mathbf{D}_{\mathrm{dR}}(V_f(1))}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f(1))} \otimes_L \frac{t^{2-k}L_n[[t]]}{tL_n[[t]]};$$

and by Definition 2.4.5,

$$
\begin{aligned}
\iota_m \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm}) &= \iota_m \circ \mathfrak{C}\left(\mathcal{M}_f^{\pm}\right) \\
&= \overline{f} \otimes \sum_{i=0}^{k-2} C_{m,i}^{\pm} t^j + \text{ ``Something in } \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L t^{k-1} L_m[[t]]\text{''} \\
&\in \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L L_m[[t]] + \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L t^{k-1} L_m[[t]]
\end{aligned}
$$

whenever $m \geq 1$.

Notice that anything in $\mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L t^{k-1} L_m[[t]]$ paired with $\overline{f}^* \otimes \mathscr{K}_{-n}\left(v_j \otimes F_\chi\right)$ equals 0 because the latter element has coefficient 0 in front of t^i for all $i \leq 1-k$. Thus

$$
\begin{aligned}
&\left\langle \iota_n^- \circ \eta(v_j \otimes F_\chi) \ , \ \iota_m \circ \mathfrak{C}\left(\mathcal{M}_f^{\pm}\right) \right\rangle_{\mathrm{dif},m} \\
&\qquad = \left\langle \overline{f}^* \otimes \mathscr{K}_{-n}\left(v_j \otimes F_\chi\right) \ , \ \mathrm{Tr}_{L_n}\left(\iota_m \circ \mathfrak{C}\left(\mathcal{M}_f^{\pm}\right)\right) \right\rangle_{\mathrm{dif},n} \\
&\qquad = \left\langle \overline{f}^* \otimes \mathscr{K}_{-n}\left(v_j \otimes F_\chi\right) \ , \ \overline{f} \otimes \sum_{i=0}^{k-2} C_{n,i}^{\pm} t^j \right\rangle_{\mathrm{dif},n} \\
&\qquad = \mathrm{Tr} \circ \mathrm{Res}_{t=0}\left([\overline{f}^*, \overline{f}]_{\mathrm{dR}} \cdot \mathscr{K}_{-n}\left(v_j \otimes F_\chi\right) \cdot \sum_{i=0}^{k-2} C_{n,i}^{\pm} t^j \cdot dt \right) \\
&\qquad = \mathrm{Tr} \circ \mathrm{Res}_{t=0}\left(\mathscr{K}_{-n}\left(v_j \otimes F_\chi\right) \cdot \sum_{i=0}^{k-2} C_{n,j}^{\pm} t^i \cdot \frac{dt}{t} \right).
\end{aligned}
$$

Here, Tr_{L_n} denotes the normalized trace map to L_n, and Tr is the normalized trace map to L. $\square$

Corollary 2.4.9. *Let χ, j be as in Propostion 2.4.8. We have:*

$$j!\cdot p^{nj}\cdot\Big(\mathrm{Id}\otimes\mathrm{Tr}\Big)\left(C^{\pm}_{n,j}\cdot\sum_{a\ mod\ p^n}\chi(a)\otimes\zeta^a_{p^n}\right)=\mathcal{M}^{\pm}_f\left(v_j\otimes F_\chi\right)$$
$$=(-1)^j\cdot\frac{p^{n(j+1)}}{\tau(\overline{\chi})}\cdot\widetilde{\Lambda}(f,\overline{\chi},j+1)$$

when the sign $\pm$ on the LHS are chosen so that $\chi(-1)=\pm(-1)^j$, and

$$j!\cdot p^{nj}\cdot\Big(\mathrm{Id}\otimes\mathrm{Tr}\Big)\left(C^{\pm}_{n,j}\cdot\sum_{a\ mod\ p^n}\chi(a)\otimes\zeta^a_{p^n}\right)=0$$

if the sign $\pm$ doesn't satisfy $\chi(-1)=\pm(-1)^j$. Here, Tr *denotes the normalized trace map to $\mathbb{Q}_p$.*

Proof. This follows directly from Proposition 2.4.8, equation (2.3.5), Lemma 2.2.6 and Remark 2.2.7. □

Corollary 2.4.10. *Let χ be any finite order character of conductor p^n, $n\geq 1$ as before, and $C^{\pm}_{n,j,\chi}$ be as defined in Definition 2.4.5. Then for every $0\leq j\leq k-2$,*

$$C^{\pm}_{n,j,\chi}=\pm\frac{p}{p-1}\cdot\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}(f,\chi,j+1)}{j!}\qquad(2.4.7)$$

when the sign $\pm$ are chosen so that $\chi(-1)=\pm(-1)^j$.

If the sign $\pm$ doesn't satisfy $\chi(-1)=\pm(-1)^j$, then $C^{\pm}_{n,j,\chi}=0$.

Proof. For any character ϕ of $(\mathbb{Z}/p^n)^*$ with conductor p^l ($l \leq n$), using the notation in Definition 2.4.5, we have a basis vector e_ϕ of the 1-dimensional vector space $L(\phi^{-1})$:

$$e_\phi := \sum_{a \bmod p^n} \phi(a) \otimes \zeta_{p^l}^a \in L(\phi^{-1}) \subset L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\left(\zeta_{p^n}\right).$$

We can check easily that

$$\Big(\mathrm{Id} \otimes \mathrm{Tr}\Big)(e_\phi) = \begin{cases} p^{n-1}(p-1), & \text{If } \phi = 1 \text{ is the trivial character;} \\ \quad 0 \quad, & \text{If } \phi \text{ has conductor } p^l \text{ with } l \geq 1 \end{cases} \tag{2.4.8}$$

where Tr is the normalized trace map to $\mathbb{Q}_p$. Since

$$C_{n,j}^{\pm} \cdot \sum_{a \bmod p^n} \chi(a) \otimes \zeta_{p^n}^a = \left(\sum_{\chi} C_{n,j,\phi}^{\pm} \cdot e_\phi\right) \cdot e_\chi$$

(here χ is the character of conductor p^n in the statement of Corollary 2.4.10), we have

$$
\begin{aligned}
&\left(\mathrm{Id}\otimes\mathrm{Tr}\right)\left(C^{\pm}_{n,j}\cdot\sum_{a \bmod p^n}\chi(a)\otimes\zeta^{a}_{p^n}\right)\\
=&\left(\mathrm{Id}\otimes\mathrm{Tr}\right)\left(C^{\pm}_{n,j,\chi^{-1}}\cdot e_{\chi^{-1}}\cdot e_{\chi}\right)\\
=&C^{\pm}_{n,j,\chi^{-1}}\cdot\left(\mathrm{Id}\otimes\mathrm{Tr}\right)\left(\sum_{a,\ b\in(\mathbb{Z}/p^n)^*}\chi(ab^{-1})\otimes\zeta^{a+b}_{p^n}\right)\\
=&C^{\pm}_{n,j,\chi^{-1}}\cdot\left(\mathrm{Id}\otimes\mathrm{Tr}\right)\left(\sum_{c\in(\mathbb{Z}/p^n)^*}\chi(c)\otimes\sum_{b\in(\mathbb{Z}/p^n)^*}\zeta^{bc+b}_{p^n}\right)\\
=&p^{n-1}(p-1)\cdot\chi(-1)\cdot C^{\pm}_{n,j,\chi^{-1}}.
\end{aligned}
$$

Combining this with Corollary 2.4.9 gives equation (2.4.7). □

2.5 Images of $\mathbf{z}^{\pm}_{M}(f)$ under dual exponential maps, supercuspidal case

In this section, we continue assuming that the local (p-adic) smooth representation of $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$ attached to the modular form f is supercuspidal.

Before we state our main theorem, let's recall the definition of integrating classes in the local Iwasawa cohomology.

For any class $c\in H^1(\mathbb{Q}_p, V_f^*\otimes_{\mathbb{Z}_p}\Lambda)$ where Λ is the Iwasawa algebra viewed as the space of measures on $\mathbb{Z}_p^*$:

- If ϕ is a locally constant function on $\mathbb{Z}_p^*$ which factors through $(\mathbb{Z}/p^n)^*$, we can define

$$\int_{\mathbb{Z}_p^*} \phi \cdot c := \left\{ \gamma \in G_{\mathbb{Q}_p(\zeta_{p^n})} \mapsto \int_{\mathbb{Z}_p^*} \phi(x) \cdot c(\gamma) \in V_f^* \right\} \in H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V_f^*\right). \tag{2.5.1}$$

It is easy to check that the above map from $H^1(\mathbb{Q}_p, V_f^* \otimes_{\mathbb{Z}_p} \Lambda)$ to $H^1(\mathbb{Q}_p(\zeta_{p^n}), V_f^*)$ is well defined for any locally constant function ϕ which factors through $(\mathbb{Z}/p^n)^*$.

- If χ is a continuous character of $\mathbb{Z}_p^*$, then we can define the integration which takes value in $H^1(\mathbb{Q}_p, V_f^*(\chi))$:

$$\int_{\mathbb{Z}_p^*} \chi \cdot c := \left\{ \gamma \in G_{\mathbb{Q}_p} \mapsto \int_{\mathbb{Z}_p^*} \chi(x) \cdot c(\gamma) \in V_f^* \right\} \in H^1\left(\mathbb{Q}_p, V_f^*(\chi)\right). \tag{2.5.2}$$

It is easy to check that the above map from $H^1(\mathbb{Q}_p, V_f^* \otimes_{\mathbb{Z}_p} \Lambda)$ to $H^1(\mathbb{Q}_p, V_f^*(\chi))$ is well defined for any continuous character χ of $\mathbb{Z}_p^*$.

- The above two definitions are compatible for finite order characters ϕ of $\mathbb{Z}_p^*$ with conductor p^n in the following sense:

There is the following commutative diagram:

$$\begin{array}{ccc} H^1\left(\mathbb{Q}_p, V_f^* \otimes_{\mathbb{Z}_p} \Lambda\right) & \xrightarrow{\text{First definition of } \int_{\mathbb{Z}_p^*} \phi} & H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V_f^*\right) \\ & \searrow^{\text{Second definition of } \int_{\mathbb{Z}_p^*} \phi} & \uparrow \text{res} \\ & & H^1\left(\mathbb{Q}_p, V_f^*(\chi)\right) \end{array}$$

In the rest of this paper, for notational convenience, we will not mention which of the above definitions we are using.

Recall that Shapiro's lemma gives an isomorphism between $H^1(\mathbb{Q}_p, V_f^* \otimes_{\mathbb{Z}_p} \Lambda)$ and $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$. We can regard the above integrations as defined on $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$.

Lemma 2.5.1. *We denote $c_{j,n}$ the projection from $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ to $H^1(\mathbb{Q}_p(\zeta_{p^n}), V_f^*(-j))$. Then for any $\mathbf{z} \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ and any $a \in (\mathbb{Z}/p^n)^*$:*

$$\int_{a+p^n\mathbb{Z}_p} x^{-j} \cdot \mathbf{z} = \sigma_a\left(c_{j,n}(\mathbf{z})\right). \tag{2.5.3}$$

Therefore, for any locally constant function ϕ of conductor p^n,

$$\int_{\mathbb{Z}_p^*} \phi(x) x^{-j} \cdot \mathbf{z} = \sum_{a \ mod \ p^n} \phi(a) \cdot \sigma_a\left(c_{j,n}(\mathbf{z})\right). \tag{2.5.4}$$

Proof. We first recall the explicit description of Shapiro's lemma.

Let G be a profinite group, and $H \leq G$ be a subgroup of finite index such that G/H is commutative, V a representation of the group G with coefficients in L. The isomorphism given by Shapiro's lemma

$$S: \quad H^1\Big(G, \mathrm{Hom}_{L[H]}(L[G], V)\Big) \xrightarrow{\cong} H^1(H, V)$$

has formula $S(c) = \{h \in H \mapsto c(h)([1])\} \in H^1(H, V)$, where "1" is the identity element in G. Here, the action of G on $\mathrm{Hom}_{L[H]}(L[G], V)$ is given by formula $(g.F)([g']) = F[g'g]$.

Since V a representation of the group G, there is an isomorphism of G-representations

$$\alpha : \quad \mathrm{Hom}_{L[H]}(L[G],\, V) \xrightarrow{\cong} V \otimes_L L\,[G/H]$$

given by formula

$$\alpha(f) = \sum_{\overline{z} \in G/H} \Big(z^{-1} \cdot f([z]) \Big) \otimes [\overline{z}^{-1}],$$

where z is any lift of $\overline{z}$ in G. Here, the action of G on $\mathrm{Hom}_{L[H]}(L[G],\, V)$ is still as described as before, and the action of G on $V \otimes_L L\,[G/H]$ is given by formula $g.(v \otimes [g']) = (g.v) \otimes [gg']$.

Thus the isomorphism α induces an isomorphism

$$\alpha_* : \quad H^1\left(G, \mathrm{Hom}_{L[H]}(L[G],\, V)\right) \xrightarrow{\cong} H^1\left(G,\, V \otimes_L L\,[G/H]\right),$$

so we have the composite:

$$\alpha_* \circ S^{-1} : \quad H^1\left(H,\, V\right) \xrightarrow{\cong} H^1\left(G,\, V \otimes_L L\,[G/H]\right).$$

If H' is a subgroup of H such that G/H' is commutative, then we have the following commutative diagram:

$$\begin{array}{ccc} H^1\left(H',\, V\right) & \xrightarrow[\cong]{\alpha_* \circ S^{-1}} & H^1\left(G,\, V \otimes_L L\left[G/H'\right]\right) \\ \big\downarrow \text{cores} & & \big\downarrow \text{Proj} \\ H^1\left(H,\, V\right) & \xrightarrow[\cong]{\alpha_* \circ S^{-1}} & H^1\left(G,\, V \otimes_L L\,[G/H]\right) \end{array}$$

where the vertical map on the right is induced by the natural projection from G/H'

to G/H.

The inverse limit of the map $\alpha_* \circ S^{-1}$ with respect to corestrictions and projections gives the isomorphism between $\mathrm{H}^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, V\right)$ and $H^1\left(\mathbb{Q}_p, V \otimes_{\mathbb{Z}_p} \Lambda\right)$, which we will denote by $\mathscr{S} : \mathrm{H}^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, V\right) \xrightarrow{\cong} H^1\left(\mathbb{Q}_p, V \otimes_{\mathbb{Z}_p} \Lambda\right)$.

To prove equation (2.5.3) for $j = 0$, we let

$$G = \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$$

and

$$H = \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\left(\zeta_{p^n}\right)\right).$$

Notice that the following composite

$$H^1\left(G, \mathrm{Hom}_{L[H]}(L[G], V)\right) \xrightarrow[\cong]{\alpha_*} H^1\left(G, V \otimes_L L\left[G/H\right]\right) \xrightarrow{\int_{a+p^n\mathbb{Z}_p} 1} H^1\left(H, V\right)$$

sends any class $c \in H^1\left(G, \mathrm{Hom}_{L[H]}(L[G], V)\right)$ to

$$\left\{h \in H \mapsto \sigma_a \cdot \left(c(h)([\sigma_a^{-1}])\right)\right\} = \sigma_a \cdot \{h \in H \mapsto c(h)([1])\} = \sigma_a \cdot S(c) \ \in H^1\left(H, V\right)$$

using the explicit formula of the map α_* above. Here, the action of σ_a on $H^1\left(H, V\right)$ is through the usual action of G/H on $H^1\left(H, V\right)$. Thus we have

$$\sigma_a\left(c_{0,n}(\mathbf{z})\right) = \int_{a+p^n\mathbb{Z}_p} 1 \cdot \left(\alpha_* \circ S^{-1}\left(c_{0,n}(\mathbf{z})\right)\right) = \int_{a+p^n\mathbb{Z}_p} 1 \cdot \mathbf{z}.$$

We can then deduce equation (2.5.3) for general j by using the following commutative diagram:

$$
\begin{array}{ccccc}
H^1\left(\mathbb{Q}_p, V\otimes_{\mathbb{Z}_p}\Lambda\right) & \xleftarrow[\cong]{\mathscr{S}} & \mathrm{H}^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, V\right) & \searrow^{\int_{\mathbb{Z}_p^*} x^j\phi(x)} & \\
\downarrow\cong & & \downarrow\cong & & \\
H^1\left(\mathbb{Q}_p, V(j)\otimes_{\mathbb{Z}_p}\Lambda\right) & \xleftarrow[\cong]{\mathscr{S}} & \mathrm{H}^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, V(j)\right) & \xrightarrow[\int_{\mathbb{Z}_p^*}\phi(x)]{} & H^1\left(\mathbb{Q}_p\left(\zeta_{p^n}, V(j)\right)\right)
\end{array}
$$

where the vertical isomorphism on the left is induced by sending $v\otimes[\sigma_a]\in V\otimes_{\mathbb{Z}_p}\Lambda$ to $v\otimes a^j\otimes[\sigma_a]\in V(j)\otimes_{\mathbb{Z}_p}\Lambda$, and the locally constant function ϕ factors through $(\mathbb{Z}/p^n)^*$.

□

The follwing lemma will be useful later on:

Lemma 2.5.2. *Assume F is a finite extension of $\mathbb{Q}_p$. If V is a p-adic continuous representation of G_F, and K/F is a finite extension, then following diagrams are commutative:*

$$
\begin{array}{ccc}
H^1(K,V) & \xrightarrow{\exp^*} & \mathbf{D}_{\mathrm{dR},K}(V) \\
\downarrow{\scriptstyle\text{cores}} & & \downarrow{\scriptstyle\mathrm{Tr}_F^K} \\
H^1(F,V) & \xrightarrow{\exp^*} & \mathbf{D}_{\mathrm{dR},F}(V)
\end{array}
\qquad\qquad
\begin{array}{ccc}
H^1(K,V) & \xrightarrow{\exp^*} & \mathbf{D}_{\mathrm{dR},K}(V) \\
{\scriptstyle\text{res}}\uparrow & & \uparrow{\scriptstyle\textit{natural inclusion}} \\
H^1(F,V) & \xrightarrow{\exp^*} & \mathbf{D}_{\mathrm{dR},F}(V)
\end{array}
$$

Where the trace map Tr_F^K appeared above is the unnormalized one.

Proof. Recall the definition of $\exp^*$ is as the following commutative diagram:

$$
\begin{array}{ccc}
H^0\left(F, V\otimes\mathbf{B}_{\mathrm{dR}}\right) & \xrightarrow[\cong]{\smile\log(\mathrm{cycl})} & H^1\left(F, V\otimes\mathbf{B}_{\mathrm{dR}}\right) \\
& \nwarrow^{\exp^*} & \uparrow \\
& & H^1(F,V)
\end{array}
$$

where $\log(\mathrm{cycl}) \in H^1\left(F, \mathbb{Q}_p\right)$.

Therefore, Lemma 2.5.2 follows from the following two commutative diagrams for cup products, when $F \subset K$:

$$\begin{array}{ccccc} H^0\left(F, V \otimes \mathbf{B}_{\mathrm{dR}}\right) & \times & H^1\left(F, \mathbb{Q}_p\right) & \xrightarrow{\smile} & H^1\left(F, V \otimes \mathbf{B}_{\mathrm{dR}}\right) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ H^0\left(K, V \otimes \mathbf{B}_{\mathrm{dR}}\right) & \times & H^1\left(K, \mathbb{Q}_p\right) & \xrightarrow{\smile} & H^1\left(K, V \otimes \mathbf{B}_{\mathrm{dR}}\right) \end{array}$$

$$\begin{array}{ccccc} H^0\left(F, V \otimes \mathbf{B}_{\mathrm{dR}}\right) & \times & H^1\left(F, \mathbb{Q}_p\right) & \xrightarrow{\smile} & H^1\left(F, V \otimes \mathbf{B}_{\mathrm{dR}}\right) \\ {\scriptstyle \mathrm{cores}}\uparrow & & \downarrow{\scriptstyle \mathrm{res}} & & {\scriptstyle \mathrm{cores}}\uparrow \\ H^0\left(K, V \otimes \mathbf{B}_{\mathrm{dR}}\right) & \times & H^1\left(K, \mathbb{Q}_p\right) & \xrightarrow{\smile} & H^1\left(K, V \otimes \mathbf{B}_{\mathrm{dR}}\right) \end{array}$$

□

We can now state our main theorem in the case when $\pi_p^{\mathrm{sm}}(f)$ is supercuspidal:

Theorem 2.5.3. *The element $\mathbf{z}_M^{\pm}(f) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ has the following property:*

For any $0 \leq j \leq k-2$, any locally constant character χ of $\mathbb{Z}_p^*$ with conductor p^n $(n \geq 0)$,

$$\exp^*\left(\int_{\mathbb{Z}_p^*} \chi(x) x^{-j} \cdot \mathbf{z}_M^{\pm}(f)\right) = \pm \frac{1}{\tau(\chi)} \cdot \frac{\widetilde{\Lambda}(f, \chi, j+1)}{j!} \cdot \overline{f}_\chi \cdot t^j, \tag{2.5.5}$$

where

$$\overline{f}_\chi \cdot t^j := \left(\sum_a \chi(a) \overline{f} \otimes \zeta_{p^n}^a\right) \cdot t^j = \overline{f} \cdot e_\chi \cdot t^j \in \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V_f^*(\chi)(-j)).$$

Here, the sign $\pm$ for $\mathbf{z}_M^{\pm}(f)$ is chosen such that $\chi(-1) = \pm(-1)^j$. If the sign doesn't match, then the LHS of equation (2.5.5) equals 0.

Proof. We denote $c_{j,n}$ the projection from $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ to $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_f^*(-j))$.

According to [18] Lemma VIII.2.1, $\exp^*\left(c_{j,n}(\mathbf{z}_M^{\pm}(f))\right)$ equals $\frac{1}{p^n}$ times the image of $\iota_m\left(\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f))\right)$ modulo $\gamma_n - \varepsilon(\gamma_n)^j$ for all m sufficiently large, where ε is the cyclotomic character.

In other words, $\exp^*\left(c_{j,n}(\mathbf{z}_M^{\pm}(f))\right)$ equals $\frac{1}{p^n}$ times Tr_{L_n} applied to the coefficient of t^j of $\iota_m\left(\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f))\right)$, where Tr_{L_n} is the normalized trace map to L_n. Since

$$\iota_m \circ \mathrm{Exp}^*(\mathbf{z}_M^{\pm}) = \overline{f} \otimes \sum_{j=0}^{k-2} C_{m,j}^{\pm} t^j + \text{ “Something in } \mathbf{D}_{\mathrm{dR}}(V_f^*) \otimes_L t^{k-1} L_m[[t]]\text{”},$$

we have

$$\exp^*\left(c_{j,n}(\mathbf{z}_M^{\pm}(f))\right) = \frac{1}{p^n} \cdot C_{n,j}^{\pm} \cdot \overline{f} \cdot t^j \quad \in \mathbf{D}_{\mathrm{dR}}(V_f^*(-j)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n}). \tag{2.5.6}$$

Using Lemma 2.5.1, we conclude that

$$\begin{aligned}
\exp^*\left(\int_{\mathbb{Z}_p^*} \chi(x)x^{-j}\cdot \mathbf{z}_M^{\pm}(f)\right) &= \frac{1}{p^n}\sum_{a \bmod p^n} \chi(a)\cdot\sigma_a.\exp^*\left(c_{j,n}(\mathbf{z}_M^{\pm}(f))\right)\\
&= \frac{1}{p^n}\sum_{a \bmod p^n} \chi(a)\cdot\sigma_a\left(C_{n,j}^{\pm}\right)\cdot\overline{f}\cdot t^j\\
&= \frac{1}{p^n}\sum_{a \bmod p^n}\sum_{\chi'} C_{n,j,\chi'}^{\pm}\cdot\chi(a)\chi'(a)^{-1}\cdot\overline{f}\cdot e_{\chi'}\cdot t^j\\
&= \frac{p-1}{p}\cdot C_{n,j,\chi}^{\pm}\cdot\overline{f}\cdot e_{\chi}\cdot t^j\\
&\quad\text{(Using equation (2.4.7))}\\
&= \pm\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}(f,\chi,j+1)}{j!}\cdot\overline{f}_{\chi}\cdot t^j.
\end{aligned}$$

This finishes the proof. □

Remark 2.5.4. Since we are working with the case when the modular form f is supercuspidal at p, we know f necessarily has bad reduction at p. Hence $\widetilde{\Lambda}(f,\chi,j+1) = \widetilde{\Lambda}_{(p)}(f,\chi,j+1)$ for all χ and j. Therefore, the formula in Theorem 2.5.3 can also be written as

$$\exp^*\left(\int_{\mathbb{Z}_p^*} \chi(x)x^{-j}\cdot \mathbf{z}_M^{\pm}(f)\right) = \pm\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}_{(p)}(f,\chi,j+1)}{j!}\cdot\overline{f}_{\chi}\cdot t^j. \tag{2.5.7}$$

2.6 Explicit p-adic Local Langlands correspondence, principal series case

The main purpose of this section is to compare the notations used in [1], [2], [18] with the one used in this paper, and then describe explicitly the isomorphism (as representations of $B(\mathbb{Q}_p)$)

$$\mathscr{F}:\ \Pi\left(V_f\right)^* \xrightarrow{\cong} \left(\varprojlim_{\psi} \mathbf{D}(V_f^*)\right)_{\mathrm{b}} \tag{2.6.1}$$

in the cases when $\pi_p^{\mathrm{sm}}(f) \cong \mathrm{ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\mathrm{unr}(\alpha)\otimes\mathrm{unr}(\beta p^{-1})\right)$ with $|\alpha\beta| = |p^{k-1}|$, $\alpha/\beta \neq p^{\pm 1}$ and $0 < v_p(\alpha), v_p(\beta) < k-1$. This is equivalent to the condition that $V_f|_{G_{\mathbb{Q}_p}}$ is crystaline and absolutely irreducible. Here, ind means the smooth induction, $\overline{B}(\mathbb{Q}_p)$ is the group of lower triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$, and $\mathrm{unr}(\lambda)$ is the unramified character of $\mathbb{Q}_p^*$ that sends $p \in \mathbb{Q}_p^*$ to λ.

The main reference for this section is [18] (Section II.3.3), [1] and [2].

Recall that the theory of intertwining operators gives an isomorphism

$$I:\ \mathrm{ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\mathrm{unr}(\alpha)\otimes\mathrm{unr}(\beta p^{-1})\right) \xrightarrow{\cong} \mathrm{ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\mathrm{unr}(\beta)\otimes\mathrm{unr}(\alpha p^{-1})\right). \tag{2.6.2}$$

We define

$$\pi^{\mathrm{la}}(\alpha) := \mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\mathrm{unr}(\alpha)\otimes\mathrm{unr}(\beta p^{-1})x^{2-k}\right)$$

and

$$\pi^{\mathrm{la}}(\beta) := \mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(\mathrm{unr}(\beta) \otimes \mathrm{unr}(\alpha p^{-1}) x^{2-k} \right),$$

where Ind means the locally analytic induction.

It can be seen easily that $\pi^{\mathrm{la}}(\alpha)$ contains $\Pi(V_f)^{\mathrm{lalg}} := \left(\mathrm{Sym}^{k-2}\right)^{\vee} \otimes \pi_p^{\mathrm{sm}}(f)$ as a subrepresentation. Using the intertwining operator, we know $\pi^{\mathrm{la}}(\beta)$ also contains $\Pi(V_f)^{\mathrm{lalg}}$. We thus have an inclusion (see Corollary 7.2.5 of [1]):

$$\mathscr{I}_{\mathrm{E}} : \ \pi^{\mathrm{la}}(\alpha) \oplus_{\Pi(V_f)^{\mathrm{lalg}}} \pi^{\mathrm{la}}(\beta) \hookrightarrow \Pi(V_f), \tag{2.6.3}$$

and in fact, Liu has proved in [19] that the above map identifies the source with the space of locally analytic vectors of $\Pi(V_f)$.

Let $\mathrm{LA}_c\left(\mathbb{Q}_p, L\right)$ be the space of locally analytic functions with compact support on $\mathbb{Q}_p$ taking values in L, where L is the coefficient of the representation V_f.

We define a map

$$i_\alpha : \ \mathrm{LA}_c\left(\mathbb{Q}_p, L\right) \hookrightarrow \pi^{\mathrm{la}}(\alpha)$$

by sending every function $h \in \mathrm{LA}_c\left(\mathbb{Q}_p, L\right)$ to a function $i_\alpha(h)$ defined on $\mathrm{GL}_2\left(\mathbb{Q}_p\right)$, such that

$$i_\beta(h)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = h(x).$$

We define $i_\beta : \ \mathrm{LA}_c\left(\mathbb{Q}_p, L\right) \hookrightarrow \pi^{\mathrm{la}}(\beta)$ in the same way.

Recall that the induction from the lower triangular Borel $\overline{B}(\mathbb{Q}_p)$ and the induction from the upper triangular Borel $B(\mathbb{Q}_p)$ are related by the following isomorphism:

$$\begin{aligned} W_\alpha :\ &\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{unr}(\alpha) \\ &\otimes \mathrm{unr}(\beta p^{-1})x^{2-k}\Big) \xrightarrow{\cong} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{unr}(\beta p^{-1})x^{2-k} \otimes \mathrm{unr}(\alpha)\Big), \end{aligned} \tag{2.6.4}$$

where for every $h \in \mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{unr}(\alpha) \otimes \mathrm{unr}(\beta p^{-1})x^{2-k}\Big)$, $W_\alpha(h)$ is a function on $\mathrm{GL}_2(\mathbb{Q}_p)$ such that for every $g \in \mathrm{GL}_2\big(\mathbb{Q}_p\big)$,

$$W_\alpha(h)(g) = h\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g\right).$$

We then define a map

$$j_\alpha :\ \mathrm{LA}_c\big(\mathbb{Q}_p, L\big) \hookrightarrow \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{unr}(\beta p^{-1})x^{2-k} \otimes \mathrm{unr}(\alpha)\Big)$$

by sending every function $h \in \mathrm{LA}\big(\mathbb{Q}_p, L\big)$ to a function $j_\alpha(h)$ defined on $\mathrm{GL}_2\big(\mathbb{Q}_p\big)$, such that

$$j_\beta(h)\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right) = h(x).$$

We then have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{LA}_c\big(\mathbb{Q}_p, L\big) & \xhookrightarrow{\ i_\alpha\ } & \mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{unr}(\alpha) \otimes \mathrm{unr}(\beta p^{-1})x^{2-k}\Big) \\ & \searrow^{j_\alpha} & \cong\Big\downarrow W_\alpha \\ & & \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{unr}(\beta p^{-1})x^{2-k} \otimes \mathrm{unr}(\alpha)\Big) \end{array}$$

All the above definitions and discussions hold when we replace α by β.

Notice that the central character of $\Pi\left(V_f\right)$ is $\delta(z) = z^{2-k}\left(\mathrm{unr}(\frac{\alpha\beta}{p})(z)\right)$, which is a unitary character; while the determinant of $V_f|_{G_{\mathbb{Q}_p}}$ is $(z|z|)^{-1}\delta(z)$ for $z \in \mathbb{Q}_p^*$, where we are identifying $\mathbb{Q}_P^*$ with the Weil group $W_{\mathbb{Q}_p}$ by sending $p \in \mathbb{Q}_p^*$ to the inverse of Frobenius. Hence we have $V_f^*(1) \cong V_f \otimes z^2|z|^2\delta^{-1}(z)$ and $\Pi\left(V_f^*(-1)\right) \cong \Pi\left(V_f\right) \otimes \left(\delta^{-1} \circ \det\right)$.

In [1], Berger and Breuil defined

$$LA(\alpha) := \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\mathrm{unr}(\alpha^{-1}) \otimes x^{k-2}\mathrm{unr}(p\beta^{-1})\right)$$

and

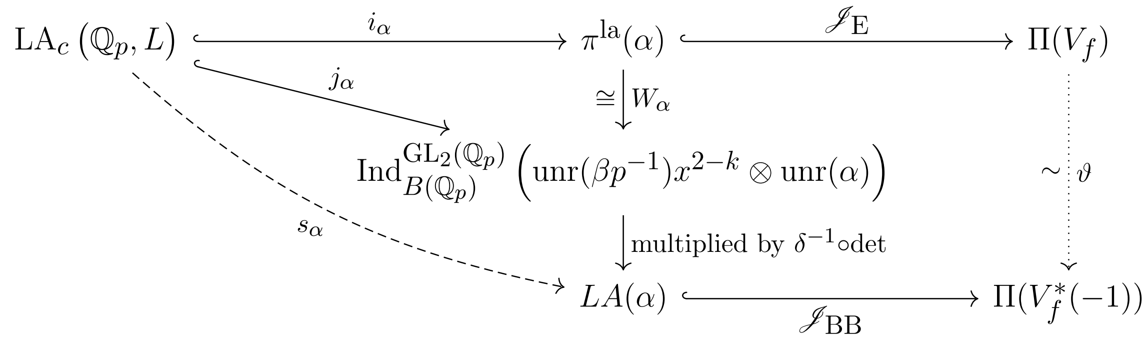
$$LA(\beta) := \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\mathrm{unr}(\beta^{-1}) \otimes x^{k-2}\mathrm{unr}(p\alpha^{-1})\right)$$

Similar to the inclusion (2.6.3), we have

$$\mathscr{I}_{\mathrm{BB}}:\ LA(\alpha) \oplus_{\Pi(V_f^*(-1))^{\mathrm{lalg}}} LA(\beta) \hookrightarrow \Pi(V_f^*(-1)), \tag{2.6.5}$$

see Corollary 7.2.5 of [1]. (There is a difference of notations for Banach space representations between [1] and this paper, which will be explained in Remark 2.6.7 later in this section.)

We now have the following commutative diagram, and we denote the map of the dashed arrow by s_α:



Here, we write "multiplied by $\delta^{-1} \circ \det$" (which is not a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map) because we are viewing both the source and target of this map as some space of functions on the group $\mathrm{GL}_2(\mathbb{Q}_p)$.

We have similar diagram when we replace α by β, and the dotted arrow ϑ from $\Pi(V_f)$ to $\Pi(V_f^*(-1))$ is induced by the two diagrams for α and β.

Remark 2.6.1. The map $\vartheta : \Pi(V_f) \to \Pi(V_f^*(-1))$ is just an isomorphism of topological vector spaces. It is not $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant.

We have the following lemma, whose proof is just a careful tracking of the diagram above:

Lemma 2.6.2. *For any $z \in \big(\Pi(V_f)\big)^*$, if we define $\nu_\alpha(z) := i_\alpha^* \circ \mathscr{J}_{\mathrm{E}}^*(z) \in \mathscr{D}(\mathbb{Q}_p)$ and $\mu_\alpha(z) := s_\alpha^* \circ \mathscr{J}_{\mathrm{BB}}^* \circ \vartheta_*(z) \in \mathscr{D}(\mathbb{Q}_p)$, then $\nu_\alpha(f)$ and $\mu_\alpha(f)$ are related by the following formula:*

$$\int_{\mathbb{Q}_p} f(x)\nu_\alpha(z) = \int_{\mathbb{Q}_p} f(-x)\mu_\alpha(z) \tag{2.6.6}$$

for any function $f(x) \in \mathrm{LA}_c\big(\mathbb{Q}_p, L\big)$.

We have similar results when we replace α by β.

□

Remark 2.6.3. The distribution $\mu_\alpha(z)$ is the distribution on $\mathbb{Q}_p$ corresponding to z defined in [1].

Definition 2.6.4. We equip $\varprojlim_{\psi} \mathbf{D}(V_f^*)$ with an action of $B(\mathbb{Q}_p)$ as follows:

- The matrix $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ in the center acts by the scalar $\delta^{-1}(z)$ on each copy of $\mathbf{D}(V_f^*)$.
- The matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ acts via ψ^{-1}, that is, shifting to the right.
- The matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{Z}_p^*$ acts by σ_a on each copy of $\mathbf{D}(V_f^*)$.
- The matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ where $b \in \mathbb{Q}_p$ acts via multiplying $(1+T)^b$ on the last copy of $\mathbf{D}(V_f^*)$.

Remark 2.6.5. If we take $V = V_f^*$ and $\chi = \delta^{-1}$, then $\varprojlim_{\psi} \mathbf{D}(V_f^*)$ with the action of $B\left(\mathbb{Q}_p\right)$ as defined in [2] (Definition 3.4.3) is isomorphic to $\varprojlim_{\psi} \mathbf{D}(V_f^*) \otimes (\delta \circ \det)$ with the action of $B\left(\mathbb{Q}_p\right)$ as defined above in Definition 2.6.4.

In [1] and [2], Berger and Breuil proved the following theorem (See Theorem 8.1.1 in [1], or Théorème 5.2.7 in [2]), which we present here using our notation:

Theorem 2.6.6 (Berger & Breuil). *Assume that*

$$\pi_p^{\mathrm{sm}}(f) \cong \mathrm{ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \Big(\mathrm{unr}(\alpha) \otimes \mathrm{unr}(\beta p^{-1})\Big)$$

with $|\alpha\beta|= |p^{k-1}|$, $\alpha/\beta \neq p^{\pm 1}$ *and* $0 < v_p(\alpha), v_p(\beta) < k-1$. *Then under the* $B(\mathbb{Q}_p)$*-equivariant isomorphism of topological vector spaces (as defined by Colmez in [18])*

$$\mathscr{F} :\ \Pi\left(V_f\right)^* \xrightarrow{\cong} \Big(\varprojlim_{\psi} \mathbf{D}(V_f^*)\Big)_{\mathrm{b}}, \tag{2.6.7}$$

we have the explicit formula

$$\mathscr{F}(z) = \Big(\alpha^{-N}\mu_{\alpha,N}(z) \otimes e_\alpha + \beta^{-N}\mu_{\beta,N}(z) \otimes e_\beta\Big)_N. \tag{2.6.8}$$

Here, we are identifying $\Big(\varprojlim_{\psi} \mathbf{D}(V_f^*)\Big)_{\mathrm{b}}$ *with* $\varprojlim_{\psi} \mathbf{N}(V_f^*)$ *(see Proposition 8.1.2 in [1]), and we view* $\mathbf{N}(V_f^*)$ *as a subspace contained in* $\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes \mathbf{D}_{\mathrm{crys}}(V_f^*)$ *(see Proposition 5.5.3 in [1]). Also,* e_α *(resp.* e_β*) is a Frobenius eigenvector in* $\mathbf{D}_{\mathrm{crys}}(V_f^*)$ *with eigenvalue* α^{-1} *(resp.* β*) such that* $\overline{f} = e_\alpha + e_\beta \in \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f^*)$*, and* $\mu_{\alpha,N}(z)$*,* $\mu_{\beta,N}(z)$ *are distributions on* $\mathbb{Z}_p$ *(viewed as elements in* $\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ *via the Amice transform[2]) defined as follows:*

For any $z \in \Pi\left(V_f\right)^*$*, let* $\mu_\alpha(z)$ *and* $\mu_\beta(z)$ *be as defined in Lemma 2.6.2. We define for every* $N \in \mathbb{Z}_{\geq 0}$ *two distributions* $\mu_{\alpha,N}(z)$ *and* $\mu_{\beta,N}(z)$ *on* $\mathbb{Z}_p$ *using the*

2. Later in this and the next section, we will always identify $\mathscr{D}(\mathbb{Z}_p)$ with $\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ using the Amice transform without mentioning explicitly. This should not cause any confusion.

formulas

$$\int_{\mathbb{Z}_p} f(x)\mu_{\alpha,N}(z) := \int_{\frac{1}{p^N}\mathbb{Z}_p} f(p^N x)\mu_\alpha(z)$$

and

$$\int_{\mathbb{Z}_p} f(x)\mu_{\beta,N}(z) := \int_{\frac{1}{p^N}\mathbb{Z}_p} f(p^N x)\mu_\beta(z)$$

for all $f(x) \in \mathrm{LA}_c\left(\mathbb{Z}_p, L\right)$.

□

Remark 2.6.7. The notations we are using here is related to the notations used in [1] and [2] in the following way: for any continuous representation V of $G_{\mathbb{Q}_p}$, the $\Pi(V)$ in this paper is the $\Pi(V(1))$ in [1] and [2].

In [1] and [2], Theorem 2.6.6 is formulated as

$$\Pi\left(V_f^*\right)^* \xrightarrow{\cong} \left(\varprojlim_{\psi} \mathbf{D}(V_f^*)\right)_{\mathrm{b}},$$

with their notation of $\Pi\left(V_f^*\right)$ and their definition of $B\left(\mathbb{Q}_p\right)$ acting on $\left(\varprojlim_{\psi} \mathbf{D}(V_f^*)\right)_{\mathrm{b}}$. If we change both sides of the above isomorphism into our notations (see Lemma 2.6.5), we would get

$$\Pi\left(V_f^*(-1)\right)^* \xrightarrow{\cong} \left(\varprojlim_{\psi} \mathbf{D}(V_f^*)\right)_{\mathrm{b}} \otimes (\delta \circ \det).$$

Notice that we have $\Pi\left(V_f^*(-1)\right)^* \otimes (\delta \circ \det)^{-1} \cong \left(\Pi\left(V_f^*(-1)\right) \otimes (\delta \circ \det)\right)^*$ and $\Pi\left(V_f^*(-1)\right) \cong \Pi\left(V_f\right) \otimes (\delta(z) \circ \det)^{-1}$, we see that our isomophism (2.6.7) coincides with the isomorphism introduced in [1] and [2].

The notation used in [18] is also compatible with the notation in [1] and [2], which has already been verified in [19].

2.7 Images of $\mathbf{z}_M^{\pm}(f)$ under dual exponential maps, principal series case

We continue using the same notations and assumptions as in the previous section. For example, $\pi_p^{\mathrm{sm}}(f)$ is still assumed to be a principal series. In [14], Emerton has proved the following result (Proposition 4.9 in [14]):

Theorem 2.7.1 (Emerton, [14])**.** *With the notations as in Lemma 2.6.2, we have* $\nu_\alpha(\mathcal{M}_f^{\pm})\Big|_{\mathbb{Z}_p^*}$ *(resp.* $\nu_\beta(\mathcal{M}_f^{\pm})\Big|_{\mathbb{Z}_p^*}$*) coincides with the p-adic L-function* $L_{p,\alpha}(f)$ *(resp.* $L_{p,\beta}(f)$*) as defined in [17].*

□

We can now prove our main theorem in the case when $\pi_p^{\mathrm{sm}}(f)$ is an unramified principal series:

Theorem 2.7.2. *The element* $\mathbf{z}_M^{\pm}(f) \in \mathrm{H}^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, V_f^*\right)$ *has the following property:*

For any $0 \leq j \leq k-2$*, any locally constant character* χ *of* $\mathbb{Z}_p^*$ *with conductor* p^n

$(n \geq 0)$,

$$\exp^*\left(\int_{\mathbb{Z}_p^*} \chi(x)x^{-j} \cdot \mathbf{z}_M^{\pm}(f)\right) = \pm\frac{1}{\tau(\chi)} \cdot \frac{\widetilde{\Lambda}_{(p)}(f,\chi,j+1)}{j!} \cdot \overline{f}_\chi \cdot t^j \qquad \textit{(2.7.1)}$$
$$\in \ \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f^*(\chi)(-j)).$$

Here, the sign $\pm$ *for* $\mathbf{z}_M^{\pm}(f)$ *is chosen such that* $\chi(-1) = \pm(-1)^j$*. If the sign doesn't match, then the LHS of equation (2.7.1) equals 0.*

Proof. As in the proof of Theorem 2.5.3, we again denote $c_{j,n}$ the projection from $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ to $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_f^*(-j))$,

and we use Lemma VIII.2.1 in [18] that $\exp^*\left(c_{j,n}(\mathbf{z}_M^{\pm}(f))\right)$ equals $\frac{1}{p^n}$ times the image of $\iota_n\left(\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f))\right)$ modulo $\gamma_n - \varepsilon(\gamma_n)^j$, where ε is the cyclotomic character.

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{D}(V_f^*)^{\psi=1} & \xrightarrow{\quad \iota_n \quad} & L_n[[t]] \otimes_L \mathbf{D}_{\mathrm{dR}}(V_f^*) \\
\downarrow F & \nearrow_{\iota_n \otimes \varphi^{-n}} & \\
\left(\mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes \mathbf{D}_{\mathrm{crys}}(V_f^*)\right)^{\psi=1} & &
\end{array}$$

where the vertical map F is induced from the isomorphism

$$\mathbf{D}^+_{\mathrm{rig}}(V_f^*)[\frac{1}{t}] \cong \mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p}[\frac{1}{t}] \otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{crys}}(V_f^*)$$

as described in [12], together with the fact that $\mathbf{D}(V_f^*)^{\psi=1} \subset \mathbf{N}(V_f^*) \subset \mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes \mathbf{D}_{\mathrm{crys}}(V_f^*)$ (see Theorem A.3 in [12] and Proposition 5.5.3 in [1], remember that we have always assumed $V_f|_{G_{\mathbb{Q}_p}}$ to be absolutely irreducible).

We write

$$F\left(\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f))\right) = \Upsilon_\alpha \otimes e_\alpha + \Upsilon_\beta \otimes e_\beta \ \in \mathbf{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes \mathbf{D}_{\mathrm{crys}}(V_f^*).$$

Since $\Upsilon_\alpha \otimes e_\alpha + \Upsilon_\beta \otimes e_\beta$ is fixed by ψ, we have $\psi\left(\Upsilon_\alpha\right) = \alpha^{-1}\Upsilon_\alpha$ and $\psi\left(\Upsilon_\beta\right) = \beta^{-1}\Upsilon_\beta$.

Notice that $\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f)) = \mathfrak{C}\left(\mathcal{M}_f^{\pm}\right) = \mathscr{F}\left(\mathcal{M}_f^{\pm}\right)$ is fixed by ψ, so if we write

$$\mathscr{F}\left(\mathcal{M}_f^{\pm}\right) = \left(\alpha^{-N}\mu_{\alpha,N}(\mathcal{M}_f^{\pm}) \otimes e_\alpha + \beta^{-N}\mu_{\beta,N}(\mathcal{M}_f^{\pm}) \otimes e_\beta\right)_N,$$

then

$$\begin{aligned} F\left(\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f))\right) &= \mu_{\alpha,0}(\mathcal{M}_f^{\pm}) \otimes e_\alpha + \mu_{\beta,0}(\mathcal{M}_f^{\pm}) \otimes e_\beta \\ &= \mu_\alpha(\mathcal{M}_f^{\pm})\Big|_{\mathbb{Z}_p} \otimes e_\alpha + \mu_\beta(\mathcal{M}_f^{\pm})\Big|_{\mathbb{Z}_p} \otimes e_\beta, \end{aligned}$$

hence $\Upsilon_\alpha = \mu_\alpha(\mathcal{M}_f^{\pm})\Big|_{\mathbb{Z}_p}$ and $\Upsilon_\beta = \mu_\beta(\mathcal{M}_f^{\pm})\Big|_{\mathbb{Z}_p}$.

Now since

$$\begin{aligned} \iota_n\left(\Upsilon_\alpha\right) &= \Upsilon_\alpha\left(\zeta_{p^n}\exp\left(\frac{t}{p^n}\right) - 1\right) \\ &= \int_{\mathbb{Z}_p}\exp\left(\frac{tx}{p^n}\right) \otimes \zeta_{p^n}^x \cdot \mu_\alpha(\mathcal{M}_f^{\pm}), \end{aligned}$$

we have

$$
\begin{aligned}
A_{j,n,\alpha} := \text{Coefficient of } t^j \text{ of } \iota_n(\Upsilon_\alpha) &= \frac{1}{p^{jn}j!}\int_{\mathbb{Z}_p} x^j \otimes \zeta_{p^n}^x \cdot \mu_\alpha(\mathcal{M}_f^\pm) \\
&= \frac{1}{p^{jn}j!}\sum_{i=0}^{+\infty}\int_{p^i\mathbb{Z}_p^*} x^j \otimes \zeta_{p^n}^x \cdot \mu_\alpha(\mathcal{M}_f^\pm) \\
&= \frac{1}{p^{jn}j!}\sum_{i=0}^{+\infty} p^i \int_{\mathbb{Z}_p^*} x^j \otimes \zeta_{p^{n-i}}^x \cdot \psi^i \cdot \mu_\alpha(\mathcal{M}_f^\pm) \\
&= \frac{1}{p^{jn}j!}\sum_{i=0}^{+\infty} \frac{p^i}{\alpha^i} \int_{\mathbb{Z}_p^*} x^j \otimes \zeta_{p^{n-i}}^x \cdot \mu_\alpha(\mathcal{M}_f^\pm).
\end{aligned}
$$

Thus, for any locally constant character χ of $\mathbb{Z}_p^*$ with conductor p^n $(n \geq 0)$,

$$
\begin{aligned}
&\alpha^n \sum_{a\in(\mathbb{Z}/p^n)^*} \chi(a)\sigma_a\left(A_{j,n,\alpha}\right) \\
&\qquad = \frac{\alpha^n}{p^{jn}j!}\sum_{i=0}^{+\infty}\frac{p^i}{\alpha^i}\int_{\mathbb{Z}_p^*} x^j \left(\sum_{a\in(\mathbb{Z}/p^n)^*} \chi(a)\otimes\zeta_{p^{n-i}}^{ax}\right)\cdot\mu_\alpha(\mathcal{M}_f^\pm).
\end{aligned}
$$

It is easily seen that the Gauss sum $\sum_{a\in(\mathbb{Z}/p^n)^*}\chi(a)\otimes\zeta_{p^{n-i}}^{ax} = 0$ whenever $i > 0$ because χ has conductor p^n. Therefore, if we pick the sign $\pm$ such that $\pm 1 = (-1)^j\chi(-1)$, then we have

$$
\begin{aligned}
&\alpha^n \sum_{a\in(\mathbb{Z}/p^n)^*} \chi(a)\sigma_a\left(A_{j,n,\alpha}\right)\\
&= \frac{\alpha^n \sum_a \chi(a)\otimes\zeta_{p^n}^a}{p^{jn} j!} \int_{\mathbb{Z}_p^*} x^j \chi^{-1}(x)\mu_\alpha(\mathcal{M}_f^\pm)\\
&= (-1)^j\chi(-1)\frac{\alpha^n \sum_a \chi(a)\otimes\zeta_{p^n}^a}{p^{jn} j!} \int_{\mathbb{Z}_p^*} x^j \chi^{-1}(x)\nu_\alpha(\mathcal{M}_f^\pm) \quad \text{(Lemma 2.6.2)}\\
&= \pm\frac{\alpha^n \sum_a \chi(a)\otimes\zeta_{p^n}^a}{p^{jn} j!} \int_{\mathbb{Z}_p^*} x^j \chi^{-1}(x)\nu_\alpha(\mathcal{M}_f^\pm)\\
&= \pm\frac{\alpha^n \sum_a \chi(a)\otimes\zeta_{p^n}^a}{p^{jn} j!} \cdot L_{p,\alpha}(f)\left(x^j\chi^{-1}(x)\right) \quad \text{(Theorem 2.7.1)}\\
&= \pm\frac{\alpha^n \sum_a \chi(a)\otimes\zeta_{p^n}^a}{p^{jn} j!} \cdot j!\cdot p^{n(j+1)}\cdot\alpha^{-n}\cdot\tau(\chi)^{-1}\cdot(2\pi i)^{-(j+1)}\cdot\frac{L_{(p)}(f,\chi,j+1)}{\Omega_f^\pm}\\
&\qquad\qquad \text{(See for example, [11], Page 269)}\\
&= \pm p^n\cdot\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}_{(p)}\left(f,\chi,j+1\right)}{j!}\cdot\sum_a \chi(a)\otimes\zeta_{p^n}^a.
\end{aligned}
$$

Similarly, $\beta^n \sum_{a\in(\mathbb{Z}/p^n)^*} \chi(a)\sigma_a\left(A_{j,n,\beta}\right) = \pm p^n\cdot\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}_{(p)}(f,\chi,j+1)}{j!}\cdot\sum_a\chi(a)\otimes\zeta_{p^n}^a$, from which we can see that the RHS doesn't depend on α or β.

Hence,

$$
\begin{aligned}
&\exp^*\left(\int_{\mathbb{Z}_p^*}\chi(x)x^{-j}\cdot\mathbf{z}_M^{\pm}(f)\right)\\
&=\sum_{a\in(\mathbb{Z}/p^n)^*}\chi(a)\exp^*\left(c_{j,n}(\mathbf{z}_M^{\pm}(f))\right)\\
&=\frac{1}{p^n}\cdot\sum_{a\in(\mathbb{Z}/p^n)^*}\chi(a)\sigma_a\Big(\text{Coefficient of } t^j \text{ of } \iota_n\left(\mathrm{Exp}^*(\mathbf{z}_M^{\pm}(f))\right)\Big)\cdot t^j\\
&=\frac{1}{p^n}\cdot\sum_{a\in(\mathbb{Z}/p^n)^*}\chi(a)\sigma_a\left(A_{j,n,\alpha}\otimes\alpha^n e_\alpha+A_{j,n,\beta}\otimes\beta^n e_\beta\right)\cdot t^j\\
&=\pm\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}_{(p)}\left(f,\chi,j+1\right)}{j!}\cdot\left(\sum_a\chi(a)\otimes\zeta_{p^n}^a\right)\otimes\left(e_\alpha+e_\beta\right)\cdot t^j\\
&=\pm\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}_{(p)}\left(f,\chi,j+1\right)}{j!}\cdot\left(\sum_a\chi(a)\otimes\zeta_{p^n}^a\right)\otimes\overline{f}\cdot t^j\\
&=\pm\frac{1}{\tau(\chi)}\cdot\frac{\widetilde{\Lambda}_{(p)}\left(f,\chi,j+1\right)}{j!}\cdot\overline{f}_\chi\cdot t^j\qquad\in\ \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f^*(\chi)(-j)).
\end{aligned}
$$

□

2.8 Comparison with Kato's Euler system

Let f be any cuspidal newform of weight $k\geq 2$ and level $\Gamma_0(p^n)\cap\Gamma_1(N)$, V_f the (cohomological) p-adic Galois representation of $G_{\mathbb{Q}}$ attached to f. We assume throughout this section that $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible.

We have exactly two cases when the above assumption can happen, namely, when $\pi_p^{\mathrm{sm}}(f)$ is supercuspidal (as in section 3 to 5) or a twist of unramified principal series (as in Section 6 and 7).

Corollary 2.8.1. *Let f be as above. We define* $\mathbf{z}_M(f) := \mathbf{z}_M^+(f) - \mathbf{z}_M^-(f) \in \mathrm{H}^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, V_f^*\right)$. *Then we have for any $0 \leq j \leq k-2$, any locally constant character χ of $\mathbb{Z}_p^*$ with conductor p^n $(n \geq 0)$,*

$$\exp^*\left(\int_{\mathbb{Z}_p^*} \chi(x)x^{-j} \cdot \mathbf{z}_M(f)\right) = \frac{1}{\tau(\chi)} \cdot \frac{\widetilde{\Lambda}_{(p)}(f,\chi,j+1)}{j!} \cdot \overline{f}_\chi \cdot t^j, \tag{2.8.1}$$

where

$$\widetilde{\Lambda}_{(p)}(f,\chi,j+1) = \frac{\Gamma(j+1)}{(2\pi i)^{j+1}} \cdot \frac{L_{(p)}(f,\chi,j+1)}{\Omega_f^\pm} \in \mathbb{Q}(f,\mu_{p^n})\ ,$$

$$\overline{f}_\chi \cdot t^j \in \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_f^*(\chi)(-j))$$

are defined as in equation (2.2.16) and Theorem 2.5.3.

Proof. This follows immediately from Theorem 2.5.3 (for the supercuspidal case) and Theorem 2.7.2 (for the principal series case). □

Recall the following theorem by Kato:

Theorem 2.8.2 (Kato, [11]). *There exists a* unique *element* $\mathbf{z}_{\mathrm{Kato}}(f) \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}, V_f^*)$ (*in the* global *Iwasawa cohomology group*), *which is obtained by global methods using Siegel unites on modular curves, such that for any $0 \leq j \leq k-2$ and any locally* *constant character χ of $\mathbb{Z}_p^* \cong \Gamma_{\mathbb{Q}_p}$ with conductor p^n $(n \geq 0)$*, we have

$$\exp^*\left(\int_{\mathbb{Z}_p^*} \chi(x)x^{-j} \cdot \mathbf{z}_{\mathrm{Kato}}(f)\right) = \frac{1}{\tau(\chi)} \cdot \frac{\widetilde{\Lambda}_{(p)}(f,\chi,j+1)}{j!} \cdot \overline{f}_\chi \cdot t^j. \tag{2.8.2}$$

□

We still use $\mathbf{z}_{\mathrm{Kato}}(f)$ to denote its image in the *local* Iwasawa cohomology group $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$. Combining Corollary 2.8.1 and Theorem 2.8.2, we can prove the following:

Theorem 2.8.3. *If $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible, then $\mathbf{z}_M(f) = \mathbf{z}_{\mathrm{Kato}}(f)$ as elements in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$.*

Proof. We define $\mathrm{H}^1_{\mathrm{Iw},e}(\mathbb{Q}_p, V_f^*)$ the collection of elements in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ whose projection to $H^1(\mathbb{Q}_p(\zeta_{p^n}), V_f^*)$ belongs to $H^1_e(\mathbb{Q}_p(\zeta_{p^n}), V_f^*)$ for all $n \geq 0$, and we define $\mathrm{H}^1_{\mathrm{Iw},g}(\mathbb{Q}_p, V_f^*)$ the collection of elements in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$ whose projection to $H^1(\mathbb{Q}_p(\zeta_{p^n}), V_f^*)$ belongs to $H^1_g(\mathbb{Q}_p(\zeta_{p^n}), V_f^*)$ for all $n \geq 0$, where

$$\begin{aligned}
H^1_e\left(\mathbb{Q}_p(\zeta_{p^n}), V\right) &= \ker\left(H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V\right) \to H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}}^{\varphi=1}\right)\right) \\
&= \ker\left(H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V\right) \xrightarrow{\exp^*} \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V)\right) \\
&\subset H^1_g\left(\mathbb{Q}_p(\zeta_{p^n}), V\right) \\
&\quad := \ker\left(H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V\right) \to H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}\right)\right) \\
&\quad \subset H^1\left(\mathbb{Q}_p(\zeta_{p^n}), V\right)
\end{aligned}$$

for any continuous p-adic Galois representation V of $G_{\mathbb{Q}_p}$.

By Corollary 2.8.1 and Theorem 2.8.2, we know that in both cases, we have for any $0 \leq j \leq k-2$ and any finite order character ϕ of $\mathbb{Z}_p^*$,

$$\exp^*\left(\int_{\mathbb{Z}_p^*} \phi(x)x^{-j}\left(\mathbf{z}_M(f) - \mathbf{z}_{\mathrm{Kato}}(f)\right)\right) = 0, \tag{2.8.3}$$

This means

$$\mathbf{z}_M(f) - \mathbf{z}_{\mathrm{Kato}}(f) \in \mathrm{H}^1_{\mathrm{Iw},e}(\mathbb{Q}_p, V_f^*).$$

In the case when $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible, Proposition II.3.1 of [13] says

$$\mathrm{H}^1_{\mathrm{Iw},e}(\mathbb{Q}_p, V_f^*) \subset \mathrm{H}^1_{\mathrm{Iw},g}(\mathbb{Q}_p, V_f^*) = 0, \tag{2.8.4}$$

hence $\mathbf{z}_M(f) - \mathbf{z}_{\mathrm{Kato}}(f) = 0$, which proves (1).

□

Remark 2.8.4. In fact, we only used equation (2.8.3) for $j = 0$ in order to conclude $\mathbf{z}_M(f) = \mathbf{z}_{\mathrm{Kato}}(f)$ as elements in $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f^*)$.

CHAPTER 3

COMPLETED COHOMOLOGY OF SHIMURA SETS AND ANTI-CYCLOTOMIC P-ADIC L-FUNCTIONS: AN EXAMPLE

3.1 An example: Steinberg case

Fix $p \geq 3$. Let B be a quaternion algebra defined over $\mathbb{Q}$ that is ramified exactly at $l \neq p$ and ∞. K be an imaginary quadratic field embedded in B such that l is innert in K and p splits in K.

We fix an isomorphism $B \cong K \oplus K \cdot J$, where $J^2 = \beta$ with $\beta < 0$, and $Jt = \bar{t}J$ for every $t \in K$.

We define

$$\begin{aligned} i_K : \quad & B \hookrightarrow M_2(K) \\ & a + bJ \mapsto \begin{pmatrix} a & b\beta \\ \bar{b} & \bar{a} \end{pmatrix} \end{aligned}$$

For every q that splits in K, we fix an isomorphism $i_q : \ B(\mathbb{Q}_q) \xrightarrow{\cong} M_2(\mathbb{Q}_q)$ by defining $i_q = \mathrm{pr}_1 \circ (i_K \otimes \mathbb{Q}_p)$, where pr_1 is induced from the natural projection $K \otimes \mathbb{Q}_q \cong \mathbb{Q}_q \oplus \mathbb{Q}_q \xrightarrow{\mathrm{pr}_1} \mathbb{Q}_q$.

For every $q \neq l$ that innert in K, we fix an isomorphism $i_q : \ B(\mathbb{Q}_q) \xrightarrow{\cong} M_2(\mathbb{Q}_q)$.

Using the above notation, we see that the isomorphism i_p when restricted to K_p

is given by the following formula:

$$\mathbb{Q}_p \oplus \mathbb{Q}_p \cong K_p \hookrightarrow B(\mathbb{Q}_p) \xrightarrow{i_p} \mathrm{GL}_2(\mathbb{Q}_p)$$
$$(t_1, t_2) \longmapsto \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

We define

$$S = B^\times(\mathbb{Q}) \Big\backslash B^\times(\mathbb{A}_{\mathbb{Q}}^f) \Big/ R^p \cdot \mathbb{A}_{\mathbb{Q}}^{f,\times} \xrightarrow{\cong} \Gamma \backslash \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times,$$

where the isomorphism above is induced by the strong approximation and the isomorphism i_p.

We let $\mathrm{Cont}(S)$ denote the collection of p-adically continous functions on S taking values in $\mathbb{C}_p$.

Let $I(p^n) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ be the standard Iwahori subgroup. That is,

$$I(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \Big| c \equiv 0 \mod p^n \right\}.$$

We denote $I(p)$ by I. S/I is a finite set, which is a quotient of the collection of oriented edges $\overrightarrow{E(\mathcal{T})}$ of the Bruhat-Tits tree $\mathcal{T}$ by the group Γ. That is, we have $S/I \cong \Gamma \backslash \overrightarrow{E(\mathcal{T})}$.

Recall that the Bruhat-Tits tree $\mathcal{T}$ has vertices

$$V(\mathcal{T}) = \left\{ (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} p^i & a \\ 0 & p^j \end{pmatrix} \,\middle|\, \begin{pmatrix} p^i & a \\ 0 & p^j \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \backslash \mathrm{GL}_2(\mathbb{Q}_p) / \mathbb{Q}_p^* \right\},$$

where $i, j \geq 0$, $a \in \mathbb{Z}/p^i\mathbb{Z}$ if $j = 0$, and $a \in \left(\mathbb{Z}/p^i\mathbb{Z}\right)^*$ if $j > 0$.

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $V(\mathcal{T})$ is given by the formula

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \left((\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} p^i & a \\ 0 & p^j \end{pmatrix} \right) = (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p^i & a \\ 0 & p^j \end{pmatrix}.$$

Define

$$\widetilde{G}_\infty = K^\times \Big\backslash \mathbb{A}_K^{f,\times} \Big/ \Pi_{\mathfrak{q} \nmid p} \mathcal{O}_{K_\mathfrak{q}}^\times \cdot \mathbb{A}_\mathbb{Q}^{f,\times},$$

and if we assume K has trivial class group and $\mathrm{disc}(K) > 6$, then we have

$$\begin{pmatrix} \mathbb{Z}_p^* & 0 \\ 0 & 1 \end{pmatrix} \cong \left(\mathcal{O}_{K_{\mathfrak{p}_1}}^\times \times \mathcal{O}_{K_{\mathfrak{p}_2}}^\times \right) / \mathbb{Z}_p^\times \xrightarrow{\cong} \widetilde{G}_\infty,$$

where the isomorphism on the left is induced from i_p. Using the above identification, the group $\widetilde{G}_\infty$ then acts on $\mathcal{T}$.

We denote by $\mathrm{Acyc} : \widetilde{G}_\infty \xrightarrow{\cong} \mathbb{Z}_p^*$ the composite of the inverse of the above isomorphism with projection to the top-left entry of the 2-by-2 matrix. Define

$$\xi_a = \mathrm{Acyc}^{-1}(a) \in \widetilde{G}_\infty$$

for every $a \in \mathbb{Z}_p^*$.

For every $n \geq 1$, we also define

$$U_n = \mathrm{Acyc}^{-1}(1 + p^n\mathbb{Z}_p) \subset \widetilde{G}_\infty,$$

and $\widetilde{G}_n = \widetilde{G}_\infty / U_n$.

It is easy to see the following:

Lemma 3.1.1. *$\widetilde{G}_\infty$ fixes every vertex of the form $(\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} p^i & 0 \\ 0 & p^j \end{pmatrix}$, where $i, j \in \mathbb{Z}$ and $ij = 0$. We call the collection of the above vertices "main axis".*

U_n fixes all vertices with distance at most n to the "main axis", for every $n \geq 1$.

Furthermore, $\widetilde{G}_\infty$ acts transitively on the collection of rays

$$\left\{ (\mathbf{e}_1, \mathbf{e}_2) — (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} p & a \\ 0 & 1 \end{pmatrix} — \cdots \middle| a \in (\mathbb{Z}/p\mathbb{Z})^* \right\}$$

and U_n acts transitively on the collection of rays

$$\left\{ (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} — (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} p^{n+1} & a' \\ 0 & 1 \end{pmatrix} — \cdots \middle| a' \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^*,\ a \equiv a' \mod p^n \right\}$$

for every $a \in (\mathbb{Z}/p^n\mathbb{Z})^*$.

□

We define for every $n \geq 1$ an oriented edge

$$E_n = \left[(\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} p^{n-1} & 1 \\ 0 & 1 \end{pmatrix} \rightarrow (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} p^n & 1 \\ 0 & 1 \end{pmatrix} \right].$$

It is then easily seen that

$$\begin{pmatrix} p^{n-1} & a \\ 0 & 1 \end{pmatrix} .E_1 = \xi_a.E_n \tag{3.1.1}$$

for all $a \in \mathbb{Z}_p^*$.

Now let f be a weight 2 modular form on $S/I \cong \Gamma \backslash \overrightarrow{E(\mathcal{T})}$. The pull back of f as a function on $\overrightarrow{E(\mathcal{T})}$ will still be denoted by f.

Using the definition from [3], we have an element $\widetilde{\mathcal{L}}_f \in \mathbb{Z}_p[[\widetilde{G}_\infty]]$ defined as

$$\widetilde{\mathcal{L}}_f = \varprojlim_n \widetilde{\mathcal{L}}_{f,n} = \varprojlim_n \sum_{\xi_a \in \widetilde{G}_n} f(\xi_a.E_n) \cdot \xi_a. \tag{3.1.2}$$

On the other hand, $f \in \mathrm{Cont}(S)$ topologically generates a continuous Steinberg representation

$$\Phi_f : \quad \left(\mathbf{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right)/1 \hookrightarrow \mathrm{Cont}(S).$$

Here, the continuous induction $\left(\mathbf{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right)/1$ can be identified with the collection of continuous functions on $\mathbb{Q}_p$ with limit 0 at infinity. We denote the latter space by $\mathrm{Cont}_0(\mathbb{Q}_p)$, and we denote $\left(\mathbf{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right)/1$ by $\mathbf{St}$. More precisely, for any $\phi \in \mathrm{Cont}_0(\mathbb{Q}_p)$, the corresponding function $\widetilde{\phi} \in \left(\mathbf{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right)/1$ is a function on $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\widetilde{\phi}\left(\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}\right) = \phi(x)$ for all $x \in \mathbb{Q}_p$.

The modular form $f \in \mathrm{Cont}(S)$ is then the image of $1_{\mathbb{Z}_p} \in \mathrm{Cont}_0(\mathbb{Q}_p)$ under then map Φ_f.

Let $\mathbb{1} = (\cdots 1, 1, \cdots) \in S$ be a points with every coordinate equal to 1. Then

$$\mathrm{ev}_{\mathbb{1}} \in \mathrm{Cont}(S)'_{\mathrm{b}}.$$

After composing with Φ_f and the inclusion $\mathrm{Cont}(\mathbb{Z}_p^*) \hookrightarrow \left(\mathbf{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right)/1 \cong \mathrm{Cont}_0(\mathbb{Q}_p)$, we can view $\mathrm{ev}_{\mathbb{1}}$ as an element in $\mathrm{Cont}(\mathbb{Z}_p^*)'_{\mathrm{b}} \cong \mathbb{Z}_p[[\mathbb{Z}_p^*]]$, which we denote by $\mathrm{ev}_{\mathbb{1},f}$.

Proposition 3.1.2. $\mathrm{ev}_{\mathbb{1}} = \widetilde{\mathcal{L}}_f$ *as elements in* $\mathbb{Z}_p[[\mathbb{Z}_p^*]]$, *under the identification* $\mathrm{Acyc} : \widetilde{G}_\infty \xrightarrow{\cong} \mathbb{Z}_p^*$.

Proof. The element $\mathrm{ev}_{\mathbb{1}} \in \mathbb{Z}_p[[\mathbb{Z}_p^*]]$ is a measure on $\mathbb{Z}_p^*$. We have for all $a \in \mathbb{Z}_p^*$ and all integer $n \geq 1$,

$$\begin{aligned}
\int_{a+p^n\mathbb{Z}_p} 1 \cdot \mathrm{ev}_{\mathbb{1}} &= \left\langle \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} .\mathbb{1}_{\mathbb{Z}_p}, \mathrm{ev}_{\mathbb{1}} \right\rangle = \left\langle \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} .f, \mathrm{ev}_{\mathbb{1}} \right\rangle = f\left(\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \right) \\
&= f\left(\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \left[(\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \to (\mathbf{e}_1, \mathbf{e}_2) \right] \right) \\
&= f\left(\left[(\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} p^{n-1} & a \\ 0 & 1 \end{pmatrix} \to (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \right] \right) \\
&= f(\xi_a.E_n) \qquad \text{(using equation 3.1.1)} \\
&= \int_{a+p^n\mathbb{Z}_p} 1 \cdot \widetilde{\mathcal{L}}_f \qquad \text{(using the definition of } \widetilde{\mathcal{L}}_f.)
\end{aligned}$$

This proves $\mathrm{ev}_{\mathbb{1}} = \widetilde{\mathcal{L}}_f$ as measures on $\mathbb{Z}_p^*$. □

3.2 Another way of phrasing the above example

Let $I \subset \mathrm{GL}_2(\mathbb{Z}_p)$ be the standard Iwahori subgroup.

By "extension of scalars", there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map

$$\Phi : \quad \left(\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1 \right) \otimes \mathrm{Cont}(S/I) \longrightarrow \mathrm{Cont}(S),$$

where c-ind means the smooth compact induction.

We denote $\mathrm{St}^{\mathrm{sm}} = \left(\mathrm{ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1 \right) /1$ the smooth Steinberg representation. There is a surjection from $\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1$ to $\mathrm{St}^{\mathrm{sm}}$, which is taking quotient by $U_p - 1$.

The weight 2 Hecke eigenform f is an element in $\mathrm{Cont}(S/I)$. It is known that f necessarily has U_p eigenvalue ± 1, and we assume $U_p f = f$ in this example. The restriction of Φ to $\left(\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right) \otimes f$ then factors through the quotient $\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1 \twoheadrightarrow \mathrm{St}^{\mathrm{sm}}$, and we denote the induced map from St (and from its completion $\mathbf{St}$) to $\mathrm{Cont}(S)$ by Φ_f:

$$
\begin{array}{ccccc}
 & \left(\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right) \otimes \mathrm{Cont}(S/I) & \xrightarrow{\Phi} & \mathrm{Cont}(S) & \\
 & \uparrow & \nearrow^{\Phi_f} & \uparrow^{\Phi_f} & \\
 & \left(\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right) \otimes f & & & \\
 & \downarrow & & & \\
\mathrm{LC}_0(\mathbb{Q}_p) \xrightarrow{\cong} & \mathrm{St}^{\mathrm{sm}} & \hookrightarrow & \mathbf{St} & \xleftarrow{\cong} \mathrm{Cont}_0(\mathbb{Q}_p)
\end{array}
$$

Here, we identifies $\mathbf{St}$ with $\mathrm{Cont}_0(\mathbb{Q}_p)$ as in the previous section, and we identifies $\mathrm{St}^{\mathrm{sm}}$ with $\mathrm{LC}_0(\mathbb{Q}_p)$ (the space of locally constant functions on $\mathbb{Q}_p$ with limit 0 at infinity).

We now describe the map Φ_f explicitly. First notice that $\mathrm{St}^{\mathrm{sm},I}$ is 1 dimensional and is spanned by the image of $1_{\mathbb{Z}_p} \in \mathrm{LC}_0(\mathbb{Q}_p)$, we see the image of $1_{I.\mathbb{Q}_p^*} \otimes f \in \left(\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right) \otimes f$ in $\mathrm{St}^{\mathrm{sm}}$ equals the image of $1_{\mathbb{Z}_p} \in \mathrm{LC}_0(\mathbb{Q}_p)$ in $\mathrm{St}^{\mathrm{sm}}$.

More generally, for any $a \in \mathbb{Z}_p^*$ and any integer $n \geq 1$, the image of the element $1_{a+p^n\mathbb{Z}_p} = \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} .1_{\mathbb{Z}_p} \in \mathrm{LC}_0(\mathbb{Q}_p)$ in $\mathrm{St}^{\mathrm{sm}}$ equals the image of $1_{I\cdot\mathbb{Q}_p^*\cdot\begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix}} \otimes f \in \left(\text{c-ind}_{I\cdot\mathbb{Q}_p^*}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1\right) \otimes f$ in $\mathrm{St}^{\mathrm{sm}}$.

Therefore, we have

$$\Phi_f(1_{a+p^n\mathbb{Z}_p}) = \Phi\left(1_{I\cdot\mathbb{Q}_p^*\cdot\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}} \otimes f\right) = \begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix} f \quad \in \mathrm{Cont}(S). \tag{3.2.1}$$

Let $\mathbb{1} \in S$ be as defined in section 3.1. We can then restrict $\mathrm{ev}_{\mathbb{1}} \in \mathrm{Cont}(S)'_{\mathrm{b}}$ to $\mathrm{Cont}(\mathbb{Z}_p^*)$ using the map Φ_f and the identification $\mathbf{St} \cong \mathrm{Cont}_0(\mathbb{Q}_p) \supset \mathrm{Cont}(\mathbb{Z}_p^*)$, hence view $\mathrm{ev}_{\mathbb{1}}$ as an element in $\mathrm{Cont}(\mathbb{Z}_p^*)'_{\mathrm{b}} \cong \mathbb{Z}_p[[\mathbb{Z}_p^*]]$, which we denote by $\mathrm{ev}_{\mathbb{1},f}$.

Using equation 3.2.1, we see for any $a \in \mathbb{Z}_p^*$ and any integer $n \geq 1$,

$$\int_{a+p^n\mathbb{Z}_p^*} 1 \cdot \mathrm{ev}_{\mathbb{1},f} = \left(\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix} f\right)(1) = f\left(\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}\right). \tag{3.2.2}$$

Recall that we have fixed an isomorphism $i_p : B^\times(\mathbb{Q}_p) \xrightarrow{\cong} \mathrm{GL}_2(\mathbb{Q}_p)$ such that it is compatible with the embedding $K \hookrightarrow B$ in the way that the following diagram commutes:

$$\begin{array}{ccc} (K \otimes \mathbb{Q}_p)^\times & \xrightarrow{\cong} & \mathbb{Q}_p^* \times \mathbb{Q}_p^* \\ \big\downarrow & & \big\downarrow \\ B^\times(\mathbb{Q}_p) & \xrightarrow[i_p]{\cong} & \mathrm{GL}_2(\mathbb{Q}_p) \end{array}$$

where the injection on the right is given by sending (t_1, t_2) to the diagonal matrix $\mathrm{Diag}(t_1, t_2) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Under the assumption that K has trivial class group and $\mathrm{disc}(K) > 6$, we can view $\widetilde{G}_\infty$ as a subgroup of $(K \otimes \mathbb{Q}_p)^* / \mathbb{Q}_p^*$ via global class field theory. We define

$$\Xi_K : \quad \widetilde{G}_\infty \longrightarrow \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^* \twoheadrightarrow S$$

be the following composite of the above two maps. (In fact Ξ_K only depends on the local embeddng $K_p \hookrightarrow B(\mathbb{Q}_p)$.) Using the notation in section 3.1, we have $\Xi_K(\xi_a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma\backslash \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^* = S$ for all $a \in \mathbb{Z}_p^*$.

Define $\Upsilon_{f,K}$ be the composition of the following maps:

$$\mathrm{Cont}(\widetilde{G}_\infty) \xleftarrow[\mathrm{Acyc}^*]{\cong} \mathrm{Cont}(\mathbb{Z}_p^*) \hookrightarrow \mathbf{St} \xrightarrow{\Phi_f} \mathrm{Cont}(S) \xrightarrow{\Xi_K^*} \mathrm{Cont}(\widetilde{G}_\infty).$$

It is then not hard to see (using equation 3.2.1 and notations in section 3.1) that

$$\Upsilon_{f,K}\left(1_{\xi_a+U_n}\right) = \left\{ \xi_x \mapsto f\left(\left(\begin{pmatrix} p^n & ax \\ 0 & 1 \end{pmatrix}\right)\right) \right\} \tag{3.2.3}$$

for all $a \in \mathbb{Z}_p^*$ and integer $n \geq 1$.

For any $x \in \mathbb{Z}_p^* \subset P^1(\mathbb{Q}_p)$, let $\mathscr{E}_x := \{E_1 - E_2 - \cdots\}$ be a sequence of consecutive edges connecting the center of the Bruhat-Tits tree $\mathcal{T}$ to the point x on the boundary. Associated to $\mathscr{E}_x$, there is an element $\widetilde{\mathcal{L}}_{f,x} \in \mathbb{Z}_p[[\widetilde{G}_\infty]]$ defined as in [3]. We then have the following:

Lemma 3.2.1. *For any $x \in \mathbb{Z}_p^*$, we have*

$$\widetilde{\mathcal{L}}_{f,x} = \Upsilon_{f,K}^*\left(\delta_{\xi_x}\right)$$

where δ_{ξ_x} *is the "delta measure" on* $\widetilde{G}_\infty$ *at* ξ_x*. Moreover, we have*

$$\widetilde{\mathcal{L}}_{f,1} = \mathrm{ev}_{\mathbb{1},f}$$

under the identification $\mathrm{Acyc} : \widetilde{G}_\infty \xrightarrow{\cong} \mathbb{Z}_p^*$.

□

3.3 Relations to (φ, Γ)-modules

Assume $p = \varpi\overline{\varpi}$ in K, and $\varpi \otimes 1$ is identified with (pu_1, u_2) under the identification $K \otimes \mathbb{Q}_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$. The isomorphism $i_p : B(\mathbb{Q}_p) \xrightarrow{\cong} M_2(\mathbb{Q}_p)$ then sends ϖ to the matrix $\begin{pmatrix} pu_1 & 0 \\ 0 & u_2 \end{pmatrix}$. Since R^p is the product of all maximal open compact subgroups away from p, we have $\varpi \in R^p$.

Therefore, if we define $u = u_1 u_2^{-1} \in \mathbb{Z}_p^*$, we see that $\mathbb{1} \in S$ is invariant under the action of $\begin{pmatrix} pu & 0 \\ 0 & 1 \end{pmatrix}$ via $\mathrm{GL}_2(\mathbb{Q}_p) \cong B(\mathbb{Q}_p)$. We thus have the following lemma:

Lemma 3.3.1. $\mathrm{ev}_{\mathbb{1}} \in \mathrm{Cont}(S)_{\mathrm{b}}'$ *is fixed under the action of* $\begin{pmatrix} pu & 0 \\ 0 & 1 \end{pmatrix}$ *via* $\mathrm{GL}_2(\mathbb{Q}_p) \cong B(\mathbb{Q}_p)$*. Thus the image of* $\mathrm{ev}_{\mathbb{1}}$ *under the map* Φ_f^* *is in* $\mathbf{St}^{*,\begin{pmatrix} pu & 0 \\ 0 & 1 \end{pmatrix}=1}$.

□

Remark 3.3.2. The number pu defined above is called "a fundamental p-unit of K"

in [8], i.e. a generator of the group of elements in $\mathcal{O}_K[\frac{1}{p}]^*$ of norm one. It is denoted by u_p in [8].

We define $\mathbf{z}_{\mathbb{1}} \in \mathbf{D}(V_f^*)^{\psi=\sigma_u^{-1}}$ the image of $\mathrm{ev}_{\mathbb{1}}$ under the isomorphism

$$\mathbf{St}^{*,\begin{pmatrix} pu & 0 \\ 0 & 1 \end{pmatrix}=1} \cong \mathbf{D}(V_f^*)^{\psi=\sigma_u^{-1}}$$

provided by Colmez.

The following questions will be studied in the future work following this thesis:

Question 3.3.3.

1. Is it true that $(1-\varphi\sigma_u^{-1})\mathbf{D}(V_f^*)^{\psi=\sigma_u^{-1}}$ *is free of rank* 2 *over* Λ*?*

2. What is the relationship between $\mathbf{D}(V_f^*)^{\psi=\sigma_u^{-1}}$ *and the Iwasawa cohomology of* V_f^**?*

3. What is the relationship between $\mathbf{z}_{\mathbb{1}}$ *and the Euler system constructed in [3]?*

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