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ABSTRACT

We consider an intersection pairing of cycles, as well as the corresponding biextension. More specifically, we construct a pairing $L : Z^p(X) \times Z^q(X) \rightarrow Z^1(S)$ between all codimensions p and q cycles on a variety X of relative dimension d over a base S , both over a field F . The main question we consider is under what conditions on the codimension q cycle D on X , all rational equivalences between two codimension p cycles A and A' on X become the same rational equivalence between the divisors $L(A, D)$ and $L(A', D)$ on S . For cycles D that are algebraically trivial on the generic fiber X_η , the divisors on S do not depend on the rational equivalence of the codimension p cycles. Nevertheless, for numerically trivial divisors and zero cycles, the image does depend on the rational equivalence of the zero cycles. Therefore, Bloch's biextension of $\mathrm{CH}_{hom}^p(X) \times \mathrm{CH}_{hom}^q(X)$ by F^\times can not be extended to the numerically trivial cycles. As a part of the proof of the numerically trivial case, we give an explicit expression of the Suslin-Voevodsky isomorphism.

CHAPTER 1

INTRODUCTION

1.1 Overview

For a smooth projective variety X of dimension d over a base field F there is a well-known intersection index $\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \rightarrow \mathbb{Z}$ between the Chow groups of cycles of codimensions p and q such that $p + q = d$. A generalization of this situation is the case when X is no longer a variety over a field F but a scheme over a Dedekind domain such as the ring of integers \mathcal{O}_F of a number field F . In this case the intersection pairing of arithmetic cycles on arithmetic schemes relates to height pairings. This leads to studying the intersection pairing of cycles on X no longer of complementary codimensions i.e. $p + q \geq d + 1$. In this paper we study a similar intersection pairing when both X and S are irreducible smooth quasi-projective variety over a field F such that the proper structural morphism $\pi : X \rightarrow S$ is of relative dimension $d = \dim X - \dim S$ and S of an arbitrary dimension. Here we want an intersection pairing $\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \rightarrow \mathrm{CH}^1(S) = \mathrm{Pic}(S)$ for $p + q = d + 1$, instead of the classical pairing $\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \rightarrow \mathrm{CH}^0(S) = \mathbb{Z}$ for $p + q = d$.

Usually intersecting pairings are studied on the level of equivalence classes of cycles via the moving lemma. For cycles A and D on X of codimensions p and q respectively intersecting properly (in the expected codimension), the pairing is defined as $\pi_*(A \cdot D)$. However, in this paper we construct a pairing $L : Z^p(X) \times Z^q(X) \rightarrow Z^1(S)$ between all cycles on X of codimensions p and q . Our main tool is Bloch's cubical complex $C^p(X, \cdot)$, which allows us to define the pairing on the level of cycles themselves. A key point is that we can do this even when the cycles A and D have a bad intersection; when the intersection is proper the image $L(A, D)$ is the usual $\pi_*(A \cdot D)$.

Additionally, using Bloch's complexes allows us to study what happens to rational equivalences between cycles on X under the intersection pairing. In other words, if A and A' are rationally equivalent cycles on X of codimensions p and D is a cycle of codimension q

on X , the divisors $L(A, D)$ and $L(A', D)$ on S are also rationally equivalent with rational equivalence coming out from the rational equivalence of A and A' , and we can ask under what conditions on D the equivalence between $L(A, D)$ and $L(A', D)$ is independent of the equivalence of A and A' . This would imply that whenever $A - A' = \text{div } f = \text{div } g$ for two different functions f and g on subvarieties W_f and W_g of codimension $p - 1$ on X , the corresponding functions on S giving us the rational equivalence between the divisors $L(A, D)$ and $L(A', D)$ on S are the same.

To motivate the question, consider the special case when the codimension p cycle A on X is principal and hence the divisor $L(A, D)$ on S is also principal. In this case the main question of the paper asks: under what conditions on the codimension q cycle D on X the resulting cycle $L(A, D)$ is independent of the choice of rational equivalence between the codimension p cycles. However, even in the case when $S = \text{Spec } F$ and $Z^1(S)$ is the single object F the question of isomorphism between the results for different rational equivalences remains. This question relates to the Poincare biextensions of $\text{CH}_{hom}^p(X) \times \text{CH}_{hom}^q(X)$ by F^\times discussed in [21], [6], and [12]. The main result of this paper will imply that the biextension can not be extended from the homologically trivial cycles to the numerically trivial ones.

To answer this question we appropriate the language of categories. Thus, we construct Chow categories $\text{Cat}(\text{CH}^p(X))$ and a functor $L(\cdot, \cdot) : \text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X)) \rightarrow \text{Cat}(\text{Pic}(S))$. The construction is based on the Bloch's cubical complex $C^p(X, \cdot)$. The objects of $\text{Cat}(\text{CH}^p(X))$ are the elements in $C^p(X, 0)$ i.e. the cycles on X of codimension p and the morphisms come from $C^p(X, 1)/\partial C^p(X, 2)$. More specifically, for any objects $\alpha, \beta \in \text{Cat}(\text{CH}^d(X))$ we define

$$\text{Mor}(\alpha, \beta) = \frac{\{W \in C^p(X, 1) | \partial W = \beta - \alpha\}}{\partial C^p(X, 2)} \simeq \frac{\{(W_i, f_i) | \text{codim } W_i = p - 1, \sum \text{div } f_i = \beta - \alpha\}}{\text{im } Tame}$$

Informally we want the functor $L(\cdot, \cdot) : \text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X)) \rightarrow \text{Cat}(\text{Pic}(S))$

to satisfy:

- For objects A and D in the categories $\text{Cat}(\text{CH}^p(X))$ and $\text{Cat}(\text{CH}^q(X))$ i.e. subvarieties $A, D \subset X$ of $\text{codim } A = p$ and $\text{codim } D = q$, we want $L(A, D) = \pi_*(A \cdot D)$
- For a morphism $A' \xrightarrow{f} A$ in the category $\text{Cat}(\text{CH}^p(X))$ i.e. $A - A' = \text{div}(f)$ for $f \in k(W)^\times$ with $W \subset X$ of $\text{codim } W = p - 1$, we want $L(A', D) \xrightarrow{L(f, D)} L(A, D)$ such that $L(f, D) = f(W \cdot D)$

We construct the functor $L(\cdot, \cdot) : \text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ from a true morphism of complexes $C^p(X, \cdot) \otimes C^q(X, \cdot) \rightarrow C^1(S, \cdot)$.

Let us paraphrase the main question of this paper in the language of categories: for a fixed codimension q cycle D on X , when do **all** morphisms of $\text{Cat}(\text{CH}^p(X))$ between any two codimension p cycles α, β on X become the same morphism in $\text{Cat}(\text{Pic}(S))$. That is, for two different morphisms $\alpha \xrightarrow{f} \beta$ and $\alpha \xrightarrow{g} \beta$ in $\text{Cat}(\text{CH}^p(X))$, under what conditions on D are the corresponding morphisms $L(f, D)$ and $L(g, D)$ the same morphism $L(\alpha, D) \rightarrow L(\beta, D)$ in $\text{Cat}(\text{Pic}(S))$. The answer is that is true when D is algebraically trivial on the generic fiber X_η , while this is almost always false when D is only numerically trivial on the generic fiber X_η . In other words we have the following proposition:

Proposition 1.1.1. *The functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ sends all morphisms in $\text{Mor}_{\text{Cat}(\text{CH}^p(X))}(\alpha, \beta)$ to the same morphism in $\text{Mor}_{\text{Cat}(\text{Pic}(S))}(L(\alpha, D), L(\beta, D))$ when the codimension q cycle D is algebraically trivial on the generic fiber X_η .*

Furthermore, specializing to the case $p = d$ and $q = 1$ the above proposition has a converse, the theorem below.

Theorem 1.1.2. *Let the base $S = \text{Spec } F$ be a point and F be an algebraically closed field. Let D be a numerically trivial divisor. If $\text{char } F = 0$, then the functor $L(\cdot, D) : \text{Cat}(\text{CH}^d(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ sends all morphisms in $\text{Mor}_{\text{Cat}(\text{CH}^d(X))}(\alpha, \beta)$ to the same morphism in $\text{Mor}_{\text{Cat}(\text{Pic}(S))}(L(\alpha, D), L(\beta, D))$ if and only if D is algebraically*

trivial. If $\text{char} F = p > 0$, then a necessary and sufficient condition for the functor $L(\cdot, D) : \text{Cat}(\text{CH}^d(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ to send at least two distinct morphisms in $\text{Mor}_{\text{Cat}(\text{CH}^d(X))}(\alpha, \beta)$ to distinct morphisms in $\text{Mor}_{\text{Cat}(\text{Pic}(S))}(L(\alpha, D), L(\beta, D))$ is for the equivalence class $[D] \in \text{Pic}(X)$ not to be of the form $[D] = [E] + [T]$ where $[E]$ is equivalence class of algebraically trivial divisors and $[T]$ is a torsion element of $\text{Pic}(X)$ of order p^k on X .

Informally the theorem states that for a numerically trivial divisor D and any two codimension p cycles α, β on X , at least 2 of the morphisms in $\text{Mor}_{\text{Cat}(\text{CH}^d(X))}(\alpha, \beta)$ remain distinct as morphisms in $\text{Mor}_{\text{Cat}(\text{Pic}(S))}(L(\alpha, D), L(\beta, D))$, unless D is actually algebraically trivial.

Remark 1.1.3. It must be noted that there is a stronger version of the proposition: the map $L(\cdot, D) : C^p(X, 1) \rightarrow C^1(S, 1)$ is trivial for a homologically trivial codimension q cycle D , proven by Bloch for a field F of positive characteristic using etale cohomology in [6] and by Müller-Stach for a field F of characteristic zero using Deligne cohomology in [21]. Nevertheless, our result is still interesting because it gives a purely algebraic proof answering Gorchinskiy's question 2.4 in [12] - he suggested that since by Grothendieck's standard conjectures, homologically trivial cycles are the same as numerically trivial modulo torsion, there must be a purely algebraic proof using intersection in the numerically trivial case. We show that the map $L(\cdot, D)$ is not trivial for numerically trivial cycles D and provide a purely algebraic proof in the case of algebraically trivial cycles. The key is to consider exactly these torsion cycles D that are numerically trivial but not algebraically trivial.

The idea of the proof of the theorem is to use the proposition to reduce to the case of torsion divisors i.e. divisors D for which there is a natural number $n > 0$ such that nD is rationally trivial. Then using the map on the complexes we get a homomorphism $L : \text{Pic}(X)[n] \xrightarrow{\phi} \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F))$. We will show that the group homomorphism L factors through an isomorphism $\text{Pic}(X)[n] \xrightarrow{\phi} \text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F))$ such that the corresponding homomorphism $\text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F)) \rightarrow \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F))$ is the

one coming from the universal coefficients theorem. That is we will prove that we have a commutative diagram

$$\begin{array}{ccc}
 \text{Pic}(X)[n] & \xrightarrow{L} & \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F)) \\
 & \searrow \phi & \nearrow \psi \\
 & & \text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F))
 \end{array}$$

Thus $\ker L \simeq \ker \psi$ and we can find the size of $\ker \psi$ from the universal coefficients theorem to prove that $\ker \psi = \phi(\text{CH}_{alg}^1(X)[n])$. To construct the homomorphism ϕ we use that $\text{Pic}(X)[n]$ is isomorphic to the group of finite etale μ_n -covers of X . Then via Tsen's theorem and surjectivity of norms for C_1 fields we construct a homomorphism $H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n) \rightarrow \pi_1^{ab}(X) \otimes \mathbb{Z}/n$. We can give more explicit expression of the values of the dual homomorphism $\kappa : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n)$ in terms of Kummer theory. In particular, when X is a smooth projective curve, the homomorphism κ corresponds to the isomorphism coming from the Weil pairing. The proof that the homomorphism is an isomorphism in the general case for X of arbitrary dimension follows from functoriality and weak Lefschetz.

Moreover, using the quasi-isomorphism between Suslin and Bloch's complexes when X is projective, as a part of the proof of the theorem we reach the following corollary.

Corollary 1.1.4. *We construct an explicit map $\phi : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, G) \rightarrow \text{Hom}(H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), G)$ for any finite n -torsion abelian group G . Here $\text{Sus}_\bullet(X) = \text{Cor}(\Delta^\bullet, X)$ is the chain complex calculating the Suslin homology of X . For the specific case when $G = \mu_n$, the map ϕ is an isomorphism. By duality we get an isomorphism $H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n) \xrightarrow{\cong} \pi_1^{ab}(X) \otimes \mathbb{Z}/n$.*

Remark 1.1.5. The homomorphism $\kappa : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), \mu_n)$ coincides with the isomorphism $H_{et}^1(X, \mu_n) \simeq H^1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n)$ constructed Suslin and Voevodsky in [25]. Nevertheless, our way of describing of the homomorphism is the

best for our purposes. The reason for this is that while, it is known that the Suslin-Voevodsky homomorphism is an isomorphism, it is not expressed in terms that make it clear that the induced homomorphism $\psi : \text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F)) \rightarrow \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F))$ is the one coming from the universal coefficients theorem. This property of the homomorphism ψ was key in our calculation of the size of the kernel of L . Thus, it was a trade off between building the homomorphism ϕ from scratch and showing that it is an isomorphism and using this as a given but then working harder to show that ψ is coming from the universal coefficients theorem. Moreover, the proof that the homomorphism κ coincides with the Suslin-Voevodsky isomorphism relies on the fact that κ is a functorial isomorphism.

1.2 Knudsen-Mumford determinant line bundle

Before we consider the approach using Bloch's complex, let us briefly discuss another possible definition of the functor L coming from K-theory. More specifically, if we take the structure sheaves of the cycles on X and S , the construction of the intersection pairing in the previous section is related to the Knudsen-Mumford determinant line bundle constructed in [15]. Given coherent sheaves \mathcal{F} and \mathcal{G} on X we can define an invertible sheaf on S as

$$L(\mathcal{F}, \mathcal{G}) = \det R\pi_*(\mathcal{F} \otimes^L \mathcal{G})$$

Here as usual the derived tensor product $\mathcal{F} \otimes^L \mathcal{G} = \sum_{i=0}^{\infty} (-1)^i \text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in G_0(X)$ which is a finite sum since X is regular. Similarly, $R\pi_* A = \sum_{i=0}^{\infty} (-1)^i R^i \pi_* A \in G_0(S)$ for a coherent \mathcal{O}_X -sheaf A . Finally, for a coherent \mathcal{O}_S -sheaf \mathcal{H} we calculate the determinant $\det(\mathcal{H})$ by taking a finite locally free resolution $0 \rightarrow \mathcal{E}_m \rightarrow \mathcal{E}_{m-1} \rightarrow \dots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{H}$ and setting $\det(\mathcal{H}) = \bigotimes_{i=0}^{i=m} \det(\mathcal{E}_i)^{(-1)^m}$ where for a locally free sheaf \mathcal{E} , $\det(\mathcal{E}) = \wedge^{\text{rank } \mathcal{E}} \mathcal{E}$. Thus, to each pair $(\mathcal{F}, \mathcal{G})$ of coherent sheaves on X we give an invertible sheaf $L(\mathcal{F}, \mathcal{G})$ on S .

Note that because of the functoriality of Tor and right derived functors for a given exact

sequence of coherent \mathcal{O}_X -sheaves

$$0 \rightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \rightarrow 0$$

and another coherent \mathcal{O}_X -sheaf \mathcal{G} , we have a natural isomorphism

$$\det(\alpha, \beta) : L(\mathcal{F}', \mathcal{G}) \otimes L(\mathcal{F}'', \mathcal{G}) \xrightarrow{\cong} L(\mathcal{F}, \mathcal{G})$$

In particular, if $h : \mathcal{F}_1 \xrightarrow{\cong} \mathcal{F}_2$ then we have an isomorphism $\det(h) : L(\mathcal{F}_1, \mathcal{G}) \xrightarrow{\cong} L(\mathcal{F}_2, \mathcal{G})$.

Similarly, by the symmetry of Tor functor, we have a canonical isomorphism $L(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} L(\mathcal{G}, \mathcal{F})$

The issue with using the structure sheaves of cycles is that, in most cases there is no isomorphism in K-theory between the equivalence classes $2[\mathcal{O}_A]$ and $[\mathcal{O}_{2A}]$ for a cycle A on X . The only case when there is an isomorphism between $2[\mathcal{O}_A]$ and $[\mathcal{O}_{2A}]$ is when A is an effective divisor and X a curve. However, even then this isomorphism is not canonical. Hence, it is easier to work more directly with the cycles themselves, instead of with their structure sheaves. For this we use Bloch's complex.

CHAPTER 2

BLOCH'S COMPLEX AND CHOW CATEGORIES

2.1 Bloch's complex

In this section we briefly recall the construction and some of the basic properties of Bloch's complex, focusing on the moving lemmas. The advantage of using Bloch's complex is that it allows us to imitate working directly with cycles, while benefiting from the machinery of complexes and providing a better definition of proper intersection.

As a matter of notation, a scheme will always mean a separated Noetherian scheme. In this paper all schemes are additionally quasi-projective of finite type over a field F , equi-dimensional over the field F . Similarly, a subvariety W of a scheme Y is an integral closed subscheme.

For a scheme X Bloch's simplicial complex $Z^p(X, \cdot)$, defined in [5], is of the form

$$\cdots \rightarrow Z^p(X, n) \rightarrow Z^p(X, n-1) \rightarrow \cdots \rightarrow Z^p(X, 0)$$

The algebraic n -simplex Δ^n is the scheme $\Delta^n = \text{Spec } F[t_0, \dots, t_n] / (\sum_{i=0}^n t_i = 1)$. A face O of Δ^n is a closed subscheme defined by equations of the form $t_{i_1} = \cdots = t_{i_s} = 0$. Each face $O \subset \Delta^n$ is isomorphic to Δ^m for some $m \leq n$. The group $Z^p(X, n)$ is generated by all subvarieties $A \subset X \times \Delta^n$ of codimension p meeting $X \times O$ properly for all faces O of Δ^n . Here meeting properly means $\text{codim}(W \cap X \times O) \geq \text{codim } W + \text{codim } X \times O$. The face maps $\partial_i : Z^p(X, n) \rightarrow Z^p(X, n-1)$ are defined via the pullbacks of the homomorphisms $\Delta^{n-1} \rightarrow \Delta^n$ defined as $\partial_i : (t_0, \dots, t_{i-1}, t_i, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$. Then the differential is defined as $\partial = \sum (-1)^i \partial_i$.

Instead of working with simplicial definition, it is easier to work with cubical complexes. For this replace $(\mathbb{A}^1, 0, 1)$ by $(\mathbb{P}^1 - \{1\}, 0, \infty)$ and similarly for $\square^n = (\mathbb{P}^1 - \{1\})^n$. The face maps are defined by setting $x_i = 0$ and $x_i = \infty$. Thus the differential is the alternating sum

$\partial = \sum (-1)^i \partial_i$ where $\partial_i = \partial_i^0 - \partial_i^\infty$. Then $C^p(X, n)$ is defined as the free abelian group $z^p(X, n)$ generated by all closed integral subvarieties of $X \times \square^n$ of codimension p meeting all cubical faces properly modulo the subgroup $z^p(X, n)_{degn}$ of degenerate cycles, i.e. those which arise from pulling back along the projection maps $\square^n \rightarrow \square^{n-1}$. Note that since the generators of the free abelian group $z^p(X, n)_{degn}$ are some of the generators of the free abelian group $z^p(X, n)$, the quotient group $C^p(X, n)$ is isomorphic to a subgroup of $z^p(X, n)$. For more details see [16].

Theorem 2.1.1 (Theorem 4.7 of [17]). *Let X be a scheme over a field F . The resulting cubical complex $C^p(X, \cdot)$ is quasi-isomorphic to the simplicial version. Hence, the higher Chow groups are calculated as $CH^p(X, n) = H_n(C^p(X, \cdot))$.*

We now define the subcomplex $C^p(X, \cdot)$ for a proper intersection with a fixed cycle D on X and we study its properties.

Definition 2.1.2. A fixed cycle $D \in Z^q(X)$ defines a complex $C^p(X, \cdot)'$, a subcomplex of the Bloch's complex $C^p(X, \cdot)$. The complex $C^p(X, \cdot)'$ consists of the cycles in $C^p(X, \cdot)$ intersecting the cycle D properly in the sense of Bloch. Two cycles α, β intersect properly if any irreducible component of the support of α intersects any irreducible component of the support of β at the expected codimension. That is

$$z^p(X, n) = \{A \in z^p(X, n) \mid \text{codim}(A \cap D \times O) \geq \text{codim } A + \text{codim } D \times O \text{ for all faces } O \subset \square^n\}$$

and $C^p(X, n)' = z^p(X, n) / z^p(X, n)'_{degn}$ where $z^p(X, n)'_{degn}$ is the subgroup of degeneracies.

Moreover, working on the level of homology groups we can work only with proper intersections as the following theorem shows.

Theorem 2.1.3 ([2] [16], [4]). *Let F be a field and X a smooth quasi-projective scheme equi-dimensional over F . For a fixed cycle D of codimension a on X , the inclusion of complexes $C^p(X, \cdot)' \rightarrow C^p(X, \cdot)$ is a quasi-isomorphism.*

There is a stronger version of this moving lemma which takes into account the closures of the cycles. However, the stronger version requires X to be projective.

Definition 2.1.4. For an integral subscheme or closed subset $Z \subset X \times \square^n$, let \overline{Z} denote the closure of Z in $X \times \overline{\square^n} = X \times (\mathbb{P}^1)^n$. Let \mathcal{C} be a finite set of locally closed subsets of X . Let $C^p(X, n)^{s'}$ be the subgroup of $C^p(X, n)$ generated by integral $Z \in z^p(X, n)$ such that for each face O of \square^n and each element $C \in \mathcal{C}$ we have

$$\text{codim}_{C \times \overline{O}} \overline{Z \cap X \times \overline{O} \cap C \times \overline{O}} \geq p$$

modulo degeneracies.

Theorem 2.1.5. (*Lemma 1.15 of [16]*) *Let F be a field and X a smooth projective scheme, equi-dimensional over F . For a fixed cycle D of codimension a on X , the inclusion $C^p(X, \cdot)^{s'} \rightarrow C^p(X, \cdot)$ is a quasi-isomorphism.*

Additionally, when the scheme X is smooth we know all the homology groups of the complex $C^1(X, \cdot)$.

Lemma 2.1.6. (*Theorem 6.1 of [5]*) *For X a regular Noetherian quasi-projective scheme of a finite type over a field F , we have*

$$H_q(C^1(X, \cdot)) = \text{CH}^1(X, q) \simeq \begin{cases} \text{Pic}(X) & q = 0 \\ \Gamma(X, \mathcal{O}_X^\times) & q = 1 \\ 0 & q \geq 2 \end{cases}$$

2.2 Construction of Chow categories

From now on let X and S be irreducible smooth quasi-projective varieties over a field F with a proper dominant generically smooth morphism $\pi : X \rightarrow S$. Denote by $d = \dim X - \dim S$

the relative dimension of the morphism π . Let p and q be nonnegative integers such that $p + q = d + 1$.

Based on the above Bloch's complexes, we have the following *Picard categories*. We refer the reader to the appendix for a more detailed treatment on the correspondence between complexes and strict Picard categories and their properties.

First, a Chow category $\text{Cat}(\text{CH}^p(X))$ with objects the elements of $C^p(X, 0)$ and morphisms $\alpha \mapsto \beta$ given by $W \in C^p(X, 1)$ such that $\partial W = \beta - \alpha$. Two morphisms $W_1, W_2 \in C^p(X, 1)$ are the same if there is $E \in C^p(X, 2)$ such that $\partial E = W_1 - W_2$.

More specifically, given two objects $\alpha, \beta \in C^p(X, 0)$ there is a morphism between them if $\bar{\alpha} = \bar{\beta} \in \text{CH}^p(X)$ and in this case

$$\text{Mor}(\alpha, \beta) = \frac{\{W \in C^p(X, 1) \mid \partial W = \beta - \alpha\}}{\partial C^p(X, 2)}$$

The monoidal structure on $\text{Cat}(\text{CH}^p(X))$ is naturally defined by the group structure of the complex $C^p(X, \cdot)$ formed by adding algebraic cycles. Note that the underlying category is a groupoid because all morphisms are actually isomorphisms. Moreover, the set of isomorphic objects $\pi_0(\text{Cat}(\text{CH}^p(X))) = \text{CH}^p(X) = H_0(C^p(X, \cdot))$ is a group and so the category is Picard. Also, the set of endomorphisms of the unit object is $\pi_1(\text{Cat}(\text{CH}^p(X))) = \text{CH}^p(X, 1) = H_1(C^p(X, \cdot))$. Thus, for any two objects $\alpha, \beta \in C^p(X, 0)$ any two morphisms in $\text{Mor}(\alpha, \beta)$ differ by an automorphism of α i.e. by an element of

$$\text{Aut}(\alpha) = H_1(C^p(X, \cdot)) = \frac{\{W \in C^p(X, 1) \mid \partial W = 0\}}{\partial C^p(X, 2)}$$

Second, for the fixed cycles $D \in C^q(X)$ a category $\text{Cat}(\text{CH}^p(X)')$, a subcategory of $\text{Cat}(\text{CH}^p(X))$, with objects the elements of $C^p(X, 0)'$ and morphisms $\alpha \mapsto \beta$ given by $W \in C^p(X, 1)'$ such that $\partial W = \beta - \alpha$. Hence,

$$\text{Mor}_{\text{Cat}(\text{CH}^p(X)')}(\alpha, \beta) = \frac{\{W \in C^p(X, 1)' \mid \partial W = \beta - \alpha\}}{\partial C^p(X, 2)'}$$

Third, a category $\text{Cat}(\overline{\text{CH}}^p(X))$ with objects the elements of $C^p(X, 0)$ such that if $\alpha, \beta \in C^p(X, 0)$ give rise to $\bar{\alpha} = \bar{\beta} \in \text{CH}^p(X)$, then there exists a unique morphism $\alpha \rightarrow \beta$. This means that we identify all isomorphisms between α, β in the usual Chow category $\text{Cat}(\text{CH}^p(X))$.

Finally, a category $\text{Cat}(\text{Pic } S)$ with objects the elements of $C^1(S, 0)$. The morphisms are as expected

$$\text{Mor}_{\text{Cat}(\text{Pic}(S))}(\alpha, \beta) = \frac{\{W \in C^1(S, 1) \mid \partial W = \alpha - \beta\}}{\partial C^1(S, 2)}$$

In the next corollary we discuss how the properties of the Bloch's complex and complex for proper intersection translate into the language for categories.

Corollary 2.2.1. *For a fixed cycle $D \in Z^q(X)$, the subcategory $\text{Cat}(\text{CH}^p(X)')$ is adjoint equivalent to the category $\text{Cat}(\text{CH}^p(X))$. This means that the inclusion $M : \text{Cat}(\text{CH}^p(X)') \hookrightarrow \text{Cat}(\text{CH}^p(X))$ has a right adjoint inverse $G : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{CH}^p(X)')$ such that the functor G when restricted to the subcategory $\text{Cat}(\text{CH}^p(X)')$ is the identity. Moreover, any two inverse functors G_1, G_2 are naturally isomorphic such that the isomorphism becomes the identity natural transformation when restricted to the subcategory $\text{Cat}(\text{CH}^p(X)')$.*

Proof. By Theorem 2.1.3 for a fixed cycle D the inclusion $C^p(X, \cdot)' \hookrightarrow C^p(X, \cdot)$ is quasi-isomorphism of complexes of free abelian groups. Moreover, $C^p(X, n)' = z^p(X, n)' / z^p(X, n)'_{\text{degn}}$ is not only a subgroup of $C^p(X, n) = z^p(X, n) / z^p(X, n)_{\text{degn}}$ but its set of generators is a subset of the set of generators of $C^p(X, n)$ as the generators of $z^p(X, n)'_{\text{degn}}$ are the generators of $z^p(X, n)_{\text{degn}}$ inside $z^p(X, n)'$. Then the quotient complex $C^p(X, \cdot) / C^p(X, \cdot)'$ is a complex of free abelian groups with generators the generators of $z^p(X, n)$ not in $z^p(X, n)'$.

Thus the inclusion of complexes $C^p(X, \cdot)' \hookrightarrow C^p(X, \cdot)$ is quasi-isomorphism of complexes of free abelian groups such that the quotient complex $C^p(X, \cdot) / C^p(X, \cdot)'$ is a complex of free

abelian groups. Hence, the existence of an inverse chain homomorphism and corresponding functor $G : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{CH}^p(X))'$ and their uniqueness follows from Corollary 7.0.12 in the Appendix. \square

Thus, we can always work with proper intersections.

CHAPTER 3

CONSTRUCTION OF THE FUNCTOR

$$L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic } S)$$

Let F be a field and X and S be smooth quasi-projective irreducible varieties over F with a proper dominant generically smooth morphism $\pi : X \rightarrow S$. The main objective is to define an intersection functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ for a fixed cycle $D \in C^q(X)$. Denote by $\text{Cat}(\text{CH}^p(X)')$ the subcategory for proper intersection with the cycle D , constructed from the complex for proper intersection $C^p(X, \cdot)'$. We first define the desired functor $L(\cdot, D)$ on the subcategory $\text{Cat}(\text{CH}^p(X)')$ for proper intersection and then extend to whole $\text{Cat}(\text{CH}^p(X))$ by composing with the inverse functor $G : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{CH}^p(X)')$ from Corollary 2.2.1. The functor on categories is constructed from the map on complexes.

Lemma 3.0.1. *There is a unique functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{Pic}(S))$ such that for any object $\alpha \in \text{Obj } \text{Cat}(\text{CH}^p(X)')$ we have $L(\alpha, D) = \alpha \cdot D \in \text{Obj } \text{Cat}(\text{Pic}(S))$ and for any morphism $W \in \text{Mor}_{\text{Cat}(\text{CH}^p(X)')}(\alpha, \beta)$ we have $L(W, D) = W \cdot D \times \square^1 \in \text{Mor}_{\text{Cat}(\text{Pic}(S))}(L(\alpha, D), L(\beta, D))$.*

Proof. We define the desired functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{Pic}(S))$ via the map on complexes

$$C^p(X, \cdot)' \xrightarrow{\cdot D} C^{p+q}(X, \cdot) \xrightarrow{\pi_*} C^1(S, \cdot)$$

The first map is intersection with D and the second is push-forward for the proper morphism $\pi : X \rightarrow S$. The map on the cubical complexes $C^p(X, \cdot)' \xrightarrow{\cdot D} C^{p+q}(X, \cdot)$ exists because X is smooth and hence the diagonal $\Delta X \hookrightarrow X$ is a regular embedding and intersection multiplicities are defined. This gives us a functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{Pic}(S))$ satisfying the desired properties. □

Proposition 3.0.2. *For a fixed cycle $D \in Z^q(X)$, there is a functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ which when restricted to the subcategory $\text{Cat}(\text{CH}^p(X)')$ agrees with functor L defined in Lemma 3.0.1. Moreover, any two such functors L_1, L_2 are naturally isomorphic so that the isomorphism becomes the identity natural transformation when restricted to the subcategory $\text{Cat}(\text{CH}^p(X)')$.*

Proof. To define the functor $L(\cdot, D)$ we use the diagram

$$\begin{array}{ccc} \text{Cat}(\text{CH}^p(X)') & \rightarrow & \text{Cat}(\text{Pic}(S)) \\ \uparrow G & \nearrow & \\ \text{Cat}(\text{CH}^p(X)) & & \end{array}$$

We want to define the diagonal functor $\text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$. For this we define the functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{Pic}(S))$ as in Lemma 3.0.1 and we define the functor $\text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ as the composite of $L(\cdot, D)$ with the inverse functor $G : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{CH}^p(X)')$ of the inclusion $M : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{CH}^p(X))$. By Corollary 2.2.1, the inverse functor G exists and is unique up to a natural isomorphism.

Since that the composite functor $GM : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{CH}^p(X)')$ is the identity, we know that the functor $\text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ agrees with the functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)') \rightarrow \text{Cat}(\text{Pic}(S))$ when restricted to the subcategory $\text{Cat}(\text{CH}^p(X)') \hookrightarrow \text{Cat}(\text{CH}^p(X))$. The uniqueness of the functor L follows from Corollary 7.0.10. Moreover, the functor L is a symmetric monoidal functor of Picard categories because of its construction from a chain map between complexes of free abelian groups. \square

Remark 3.0.3. Observe that for a fixed cycle D , the categorical functor of Proposition 3.0.2 satisfies the desired properties from the introduction. Thus, on the level of objects for $A \in C^p(X, 0)'$ the image $L(A, D)$ in $C^1(S, 0)$ is $\pi_*(A \cdot D)$. Similarly for morphisms consider the element $W \in C^p(X, 1)'$ representing the graph $\Gamma(f)$ of function $f \in k(W')^\times$ for an irreducible subvariety $W' \subset X$ of codimension $p - 1$ such that $\text{div } f = \beta - \alpha$. Then

the corresponding morphism $L(\alpha, D) \xrightarrow{L(W, D)} L(\beta, D)$ in $\text{Cat}(CH^1(S))$ is the graph of the function $\text{Norm}_{k(W' \cdot D)/k(S)}(f)$ where the norm is taken with respect to the multiplicities of the irreducible components of the intersection $W' \cdot D$. Denote by $L(f, D)$ the morphism $L(\Gamma(f), D) = L(W, D)$.

CHAPTER 4

CONSTRUCTION OF THE PRODUCT CATEGORY AND FUNCTOR

We construct the *product* category $\text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X))$ from the product of complexes $C^p(X, \cdot) \otimes C^q(X, \cdot)$. This means that we construct the category from the total complex $\text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))$ in the usual way we construct a category from a complex. Objects are the elements of

$$\text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))_0 = C^p(X, 0) \otimes C^q(X, 0)$$

i.e. a pair (α, γ) of objects $\alpha \in \text{Cat}(\text{CH}^p(X))$ and $\gamma \in \text{Cat}(\text{CH}^q(X))$. Morphisms are the elements of

$$\text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))_1 = C^p(X, 0) \otimes C^q(X, 1) \oplus C^p(X, 1) \otimes C^q(X, 0)$$

i.e. a pair (α, E) of an object $\alpha \in \text{Cat}(\text{CH}^p(X))$ and a morphism $E \in \text{Cat}(\text{CH}^q(X))$ or a pair (W, γ) of a morphism $W \in \text{Cat}(\text{CH}^p(X))$ and an object $\gamma \in \text{Cat}(\text{CH}^q(X))$; modulo the image of $\text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))_2$.

Nevertheless, we do not give the product category $\text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X))$ the structure of Picard category, because on the level of objects we want

$$([\alpha_1], [\gamma]) + ([\alpha_2], [\gamma]) = ([\alpha_1] + [\alpha_2], [\gamma])$$

rather than $([\alpha_1] + [\alpha_2], 2[\gamma])$.

Hence, we call a functor $L : \text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X)) \rightarrow \mathcal{D}$ from the product category $\text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X))$ to a Picard category \mathcal{D} , *bi-additive* if $L([\alpha_1][\gamma]) + L([\alpha_2], [\gamma]) = L([\alpha_1] + [\alpha_2], [\gamma])$ and $L([\alpha], [\gamma_1]) + L([\alpha], [\gamma_2]) = L([\alpha], [\gamma_1] + [\gamma_2])$ and similarly

for morphisms. In the following proposition we extend the intersection functor $L(\cdot, D)$ defined in Chapter 3 when one of cycles D is fixed, to the case when both cycles vary.

Proposition 4.0.1. *Let F be a field and X and S be smooth quasi-projective irreducible varieties over F with a proper dominant generically smooth morphism $\pi : X \rightarrow S$. We have a bi-additive product functor $\text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X)) \rightarrow \text{Cat}(\text{Pic}(S))$. Moreover, when restricted to $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ for a fixed object $D \in \text{Cat}(\text{CH}^q(X))$, the functor L agrees with the functor $L(\cdot, D)$ from Chapter 3. Similarly for a fixed object $E \in \text{Cat}(\text{CH}^p(X))$.*

Proof. Step 1 The map on complexes:

The functor $L(\cdot, \cdot)$ is defined via the map on the complexes in the derived category

$$\begin{aligned} C^p(X, \cdot) \otimes C^q(X, \cdot) &\xrightarrow{\text{ext}} C^{p+q}(X \times X, \cdot) \xleftarrow{\simeq} C^{p+q}(X \times X, \cdot)' \\ &\xrightarrow{(\Delta X)^*} C^{p+q}(X, \cdot) \xrightarrow{\pi_*} C^1(S, \cdot) \end{aligned}$$

The first homomorphism is the product of complexes defined in Section 5 of [17] and the complex $C^{p+q}(X \times X, \cdot)'$ is defined for proper intersection with the diagonal $\Delta X \subset X \times X$. The second homomorphism is quasi-isomorphism by Theorem 2.1.3. The third homomorphism is pull back along local complete intersection $\Delta X \hookrightarrow X \times X$. The last homomorphism is the push-forward to S .

We apply the Corollary 2.2.1 to the quasi-isomorphism $f : C^{p+q}(X \times X, \cdot)' \xrightarrow{\simeq} C^{p+q}(X \times X, \cdot)$ of complexes of free abelian groups for which the quotient complex $C^{p+q}(X \times X, \cdot) / C^{p+q}(X \times X, \cdot)'$ a complex of free abelian groups. Hence, there is an inverse chain map $g : C^{p+q}(X \times X, \cdot) \rightarrow C^{p+q}(X \times X, \cdot)'$ such that g is the identity when restricted to the subcomplex $C^{p+q}(X \times X, \cdot)'$. Therefore, we have an actual map on complexes $L(\cdot, \cdot) = \pi_*(\Delta X)^*g \circ \text{ext} : C^p(X, \cdot) \otimes C^q(X, \cdot) \rightarrow C^1(S, \cdot)$.

Note that for all elements $(\alpha, \gamma) \in \text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))_0$ and $(W, \gamma) \in \text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))_1$ such that $\alpha, W \in C^p(X, \cdot)'$ - the subcomplex for proper

intersection with γ , the product functor L agrees with the already defined functor $L(\cdot, \gamma) : C^p(X, \cdot)' \rightarrow C^1(S, \cdot)$ from Chapter 3.

Step 2 The categorical functor:

To show that the functor L is a functor from the product category $\text{Cat}(CH^p(X)) \times \text{Cat}(CH^q(X))$ we need to show that L commutes with morphisms. This means that when we apply the functor L to the following commutative diagram in $\text{Cat}(CH^p(X)) \times \text{Cat}(CH^q(X))$

$$\begin{array}{ccc} (\alpha, \gamma) & \xrightarrow{W} & (\beta, \gamma) \\ E \downarrow & & \downarrow E \\ (\alpha, \delta) & \xrightarrow{W} & (\beta, \delta) \end{array}$$

we want the following diagram in $\text{Cat}(\text{Pic } S)$ to commute:

$$\begin{array}{ccc} L(\alpha, \gamma) & \xrightarrow{L(W, \gamma)} & L(\beta, \gamma) \\ L(\alpha, E) \downarrow & & \downarrow L(\beta, E) \\ L(\alpha, \delta) & \xrightarrow{L(W, \delta)} & L(\beta, \delta) \end{array}$$

Here $W \in C^p(X, 1)$ with $\partial W = \beta - \alpha \in C^p(X, 0)$ and $E \in C^q(X, 1)$ with $\partial E = \delta - \gamma$.

To show commutativity we need

$$L(W, \gamma) + L(\beta, E) - L(\alpha, E) - L(W, \delta) = 0 \in H_1(C^1(S, \cdot))$$

Note that after rearranging we get

$$(W, \gamma) + (\beta, E) - (\alpha, E) - (W, \delta) = -(W, \partial E) + (\partial W, E) = \partial(W, E)$$

for $(W, E) \in \text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot))_2$. That is, $\partial(W, E) = 0 \in H_1(\text{Tot}(C^p(X, \cdot) \otimes C^q(X, \cdot)))$

$C^q(X, \cdot)$). Hence, the commutativity follows from induced map on homology $H_1(C^p(X, \cdot) \otimes C^q(X, \cdot)) \rightarrow H_1(C^1(S, \cdot))$.

Step 3 Bi-additivity of the functor L :

Claim 4.0.2. The functor $L(\cdot, \cdot) : \text{Cat}(\text{CH}^p(X)) \times \text{Cat}(\text{CH}^q(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ is bi-additive. That is for any objects $\alpha, \beta \in \text{Cat}(\text{CH}^p(X))$ and $D, D_1, D_2 \in \text{Cat}(\text{CH}^q(X))$ we have

$$L(\alpha, D_1)L(\alpha, D_2) = L(\alpha, D_1 + D_2)$$

and

$$L(\alpha + \beta, D) = L(\alpha, D)L(\beta, D)$$

Proof. It follows from the construction of L as a map on complexes. □

□

Remark 4.0.3. In the special case when $S = \text{Spec } F$, the proof of the proposition implies Weil reciprocity for a smooth projective irreducible curve X , over an algebraically closed field F . In this case $p = q = d = 1$. Let f and g be rational functions on X such that $\text{div } f$ and $\text{div } g$ have disjoint support. Then their graphs $\Gamma(f), \Gamma(g) \in C^1(X, 1)$ intersect properly. Thus, we have $\alpha, \beta \in C^1(X)$ such that $\text{div } f = -\alpha + \beta$ for $\Gamma(f) \in C^1(X, 1)$ and $\gamma, \delta \in C^1(X)$ such that $\text{div } g = -\gamma + \delta$ for $\Gamma(g) \in C^1(X, 1)$. Then applying the functor L to the following commutative diagram in $\text{Cat}(\text{CH}^1(X)) \times \text{Cat}(\text{CH}^1(X))$

$$\begin{array}{ccc} (\alpha, \gamma) & \xrightarrow{f} & (\beta, \gamma) \\ g \downarrow & & g \downarrow \\ (\alpha, \delta) & \xrightarrow{f} & (\beta, \delta) \end{array}$$

in $\text{Cat}(\text{Pic}(F))$, we get the commutative diagram:

$$\begin{array}{ccc}
L(\alpha, \gamma) & \xrightarrow{L(f, \gamma)} & L(\beta, \gamma) \\
L(\alpha, g) \downarrow & & \downarrow L(\beta, g) \\
L(\alpha, \delta) & \xrightarrow{L(f, \delta)} & L(\beta, \delta)
\end{array}$$

By commutativity, we have

$$L(f, \gamma)L(\beta, g) = L(\alpha, g)L(f, \delta)$$

By the definitions of $L(f, \cdot)$ and $L(\cdot, g)$, this is equivalent to

$$g(\operatorname{div} f) = g(\beta - \alpha) = L(\beta, g)L(\alpha, g)^{-1} = L(f, \delta)L(f, \gamma)^{-1} = f(\delta - \gamma) = f(\operatorname{div} g)$$

Thus $g(\operatorname{div} f) = f(\operatorname{div} g)$ which is exactly the statement of the Weil reciprocity for the curve X and the functions f and g .

Remark 4.0.4. Even if $D_\eta = 0$ i.e. $\pi(D) \subsetneq S$, the image $\operatorname{Obj}(L(\cdot, D) : \operatorname{Cat}(\operatorname{CH}^p(X)) \rightarrow \operatorname{Cat}(\operatorname{Pic}(S)))$ can be nontrivial. For example, let X be a surface and S be a curve. Then $p = q = d = 1$ and for a closed point $s \in S$ we can have $\pi^{-1}(s) = C_1 \cup C_2$ where C_i are irreducible smooth curves intersecting transversely in a point. Considering $C_1 \in C^p(X)$ and $C_2 \in C^q(X)$ i.e. C_2 plays the role of D with $D_\eta = 0$, we get $L(C_1, C_2) = \pi_*(C_1 \cdot C_2) = s$. This means that vertical fibers do matter and we can not restrict to working only over the generic fiber i.e. to a paring $\operatorname{Cat}(\operatorname{CH}^p(X_\eta)) \times \operatorname{Cat}(\operatorname{CH}^q(X_\eta))$

Remark 4.0.5. More generally, when $p + q = d + r$ with $r \geq 1$, we still have a map on complexes $C^p(X, \cdot) \otimes C^q(X, \cdot) \rightarrow C^r(S, \cdot)$ and hence is a bi-additive functor $L(\cdot, \cdot) : \operatorname{Cat}(\operatorname{CH}^p(X)) \times \operatorname{Cat}(\operatorname{CH}^q(X)) \rightarrow \operatorname{Cat}(\operatorname{CH}^r(S))$. In particular, when $S = \operatorname{Spec} F$ for $r = 2$ we get $C^2(F, 0) = C^2(F, 1) = 0$ and hence the functor $L(\cdot, \cdot) : \operatorname{Cat}(\operatorname{CH}^p(X)) \times \operatorname{Cat}(\operatorname{CH}^q(X)) \rightarrow \operatorname{Cat}(\operatorname{CH}^2(\operatorname{Spec} F))$ is trivial.

CHAPTER 5

THE CASE OF AN ALGEBRAICALLY TRIVIAL CYCLE D

Before we proceed let us observe that when the base S is a field, the quasi-projective variety X is actually projective since $\pi : X \rightarrow S$ is proper.

For now let $S = \text{Spec}(F)$ and X be a smooth projective irreducible variety of dimension $\dim X = d$ with $p + q = d + 1$. We will show that the functor $L(\cdot, D) : \text{Cat}(CH^p(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ factors via $\text{Cat}(\overline{CH}^p(X))$ when the fixed cycle $D \in C^q(X)$ is algebraically trivial. For this we need to show that for any two objects $\alpha, \beta \in \text{Cat}(CH^p(X))$ all morphisms in $\text{Mor}_{\text{Cat}(CH^p(X))}(\alpha, \beta)$ map to the same morphism in $\text{Cat}(\text{Pic}(F))$. Since any two morphisms in $\text{Mor}_{\text{Cat}(CH^p(X))}(\alpha, \beta)$ differ by an automorphism of α , this is equivalent to showing $H_1(C^p(X, \cdot)) = \text{Aut}(\alpha)$ maps to the trivial morphism in $\text{Mor}(\text{Cat}(\text{Pic } F)) = F^\times$. That is, $H_1(C^p(X, \cdot)) \mapsto 1 \in F^\times$ when D is algebraically trivial.

Remark 5.0.1. For the purpose of this paper when the base field F is not algebraically closed, a cycle $D \in C^q(X)$ is algebraically trivial if the pullback $D_{\overline{F}} \in C^q(X_{\overline{F}})$ is algebraically trivial using the usual definition. Any other definition of algebraically trivial cycle D over non-algebraically closed field F should have the property that under base change to the algebraic closure \overline{F} , the resulting cycle $D_{\overline{F}} \in C^q(X_{\overline{F}})$ is algebraically trivial. Hence, our definition gives us the largest possible set of algebraically trivial cycles to work with.

The plan of the proof is the following. Consider an irreducible cycle $Z \in C^q(X \times C, 0)$, flat over an irreducible smooth projective curve C . For any homology class $[\alpha] \in H_1(C^p(X, \cdot))$ we will construct a set theoretic function $\phi([\alpha]) : C^{(0)} \rightarrow F^\times$ on the set of closed points of the curve C , defined as $s \mapsto L(W, Z_s)$ for $W \in [\alpha]$. After this by using semi-continuity and the stronger version of the moving lemma, we will show that for any closed point $s \in C$, the function $\phi([\alpha])$ agrees on a neighborhood V_s of s with a regular function g_s on V_s . Hence, these regular functions g_s glue to a regular function g on all of the projective curve C and so g is a constant.

Proposition 5.0.2. *The functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ factors through $\text{Cat}(\overline{\text{CH}}^p(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ when $D \in C^q(X, 0)$ is algebraically trivial.*

Proof. Let us first consider the case when F is algebraically closed field. Let C be a smooth projective irreducible curve over F . Since F is algebraically closed, for all closed points $s \in C$ the fibers $X \times s$ of $X \times C$ are canonically identified with the original variety X .

Fix an irreducible cycle $Z \in C^q(X \times C, 0)$, flat over C . Thus, for all points $s \in C$, we have $Z_s \in C^q(X, 0)$. Fix a homology class $[\alpha] \in H_1(C^p(X, \cdot))$. Consider an arbitrary closed point $s \in C$. Let $C^p(X, \cdot)'$ be the Bloch's subcomplex for proper intersections with $Z_s \in C^q(X)$. The projection $p : X \times C \rightarrow C$ induces a map on complexes $L(\cdot, Z_s) : C^p(X, \cdot)' \xrightarrow{\cdot Z_s} C^{p+q}(X, \cdot) \xrightarrow{p_*} C^1(s, \cdot)$ and hence a homomorphism on homology groups $H_1(C^p(X, \cdot)') \rightarrow H_1(C^1(s, \cdot)) \simeq k(s)^\times = F^\times$. By Theorem 2.1.3 we have $H_1(C^p(X, \cdot)') \simeq H_1(C^p(X, \cdot))$ and so we can choose a representative $W \in C^p(X, 1)'$ for homology class $[\alpha]$ and the homomorphism $H_1(C^p(X, \cdot)') \simeq H_1(C^p(X, \cdot)) \rightarrow F^\times = H_1(C^1(s, \cdot))$ is independent of the choice of representative $W \in C^p(X, 1)'$ for homology class $[\alpha]$. This gives us a set theoretic function $\phi([\alpha]) : C^{(0)} \rightarrow F^\times$ defined as $s \mapsto L(W, Z_s)$.

Claim 5.0.3. For any homology class $[\alpha] \in H_1(C^p(X, \cdot))$, the set theoretic function $\phi([\alpha]) : C^{(0)} \rightarrow F^\times$ is regular on all of C and hence a constant c .

Proof. We want to show that there is a neighborhood V_s of the fixed point s such that the chosen representative $W \in C^p(X, 1)'$ for homology class $[\alpha]$ intersects not only Z_s properly but also $Z_{s'}$ for every $s' \in V_s$. Since $X \times \square^1$ is not projective, to be able to use semi-continuity we need the stronger version (Theorem 2.1.5) of Bloch's moving lemma which takes into account the endpoints of $\overline{\square^1} = \mathbb{P}^1$.

Let $C^p(X, \cdot)^{s'}$ be the Bloch subcomplex for strong proper intersections with $Z_s \in C^q(X)$. For the precise definition see Definition 2.1.4. Given $[\alpha] \in H_1(C^p(X, \cdot))$, by Theorem 2.1.5 we have $H_1(C^p(X, \cdot)^{s'}) \simeq H_1(C^p(X, \cdot))$ and so we can choose a representative $W \in C^p(X, 1)^{s'}$ for the homology class $[\alpha]$. Note this means that any irreducible component of the intersec-

tion $\overline{W} \cap Z_s \times \mathbb{P}^1$ has dimension 0 for \overline{W} the closure of W in $X \times \overline{\square}^1 = X \times \mathbb{P}^1$. Moreover, the theorem also implies that $\overline{W \cap X \times 0} \cap Z_s \times 0$ intersect properly. Here the closure $\overline{W \cap X \times 0}$ is taken in $X \times \overline{\square}^1 = X \times \mathbb{P}^1$. Hence, $\overline{W \cap X \times 0} = \overline{W_0} \times 0$ where $W_0 \times 0 = W \cap X \times 0$. Thus, $\overline{W \cap X \times 0} \cap Z_s \times 0$ intersecting properly implies that $\overline{W_0} \times 0 \cap Z_s \times 0 = \emptyset$. Similarly, $\overline{W_\infty} \times \infty \cap Z_s \times \infty = \emptyset$.

The next step is to show that since W intersects Z_s properly in the strong sense of Definition 2.1.4, this is also true for all points s' in a neighborhood V_s of s . Consider the intersection $\overline{W} \times C \cap Z \times \mathbb{P}^1 \subset X \times C \times \mathbb{P}^1$. The projection $\overline{W} \times C \cap Z \times \mathbb{P}^1 \rightarrow C$ is proper because $\overline{W} \subset X \times \mathbb{P}^1$ is closed and hence projective. This is the reason why we needed the stronger moving lemma as stated in Theorem 2.1.5 rather than the usual Bloch's moving lemma of Theorem 2.1.3. Thus, we have semi-continuity on fibers of the projection. In particular, for the fixed point $s \in C$, there exists an open neighborhood $s \in V_1 \subset C$ such that for all $s' \in V_1$ we have

$$\dim \overline{W} \cap Z_{s'} \times \mathbb{P}^1 = \dim p^{-1}(s') \leq \dim p^{-1}(s) = \dim \overline{W} \cap Z_s \times \mathbb{P}^1 = 0$$

Similarly, the intersection $\overline{W_0} \times C \cap Z \subset X \times C$ induces a proper morphism $p : \overline{W_0} \times C \cap D \rightarrow C$. Again by semi-continuity, since $\overline{W_0} \cap Z_s = \emptyset$, there is an open neighborhood $s \in V_2$ such that for all $s' \in V_2$ we have $\overline{W_0} \cap Z_{s'} = \emptyset$. Finally, repeating this argument for W_∞ we get an open neighborhood $s \in V_3$ such that for all $s' \in V_3$ we have $\overline{W_\infty} \cap Z_{s'} = \emptyset$.

Set $V_s = \cap V_i$. This is an open neighborhood of s such that for each $s' \in V_s$ we have $\dim \overline{W} \cap Z_{s'} \times \mathbb{P}^1 = 0$. Thus $\dim W \times V_s \cap Z_{V_s} \times \square^1 \leq 1$. Similarly, for each $s' \in V_s$ we have $W \times V_s \cap Z_{V_s} \times 0 = W_0 \times V_s \times 0 \cap Z_{V_s} \times 0 \subset \overline{W_0} \times V_s \cap Z_{V_s} = \emptyset$. Analogously, $W \times V_s \cap Z_{V_s} \times \infty$ is also empty.

Combining this we get that $W \times V_s \in C^p(X \times V_s, 1)$ intersects both Z_{V_s} and $X \times s$ properly, $W \times V_s \cap X \times s$ intersects Z_s properly, and $W \times V_s \cap Z_{V_s}$ intersects $X \times s$ properly as in Definition 2.1.2. Then using that $X \times V_s$ and $X \times s$ are smooth we get $(W \times V_s \cdot Z_{V_s} \times$

$\square^1) \cdot X \times s \times \square^1 = W \times s \cdot Z_s \times \square^1$. Similarly for $Z_s \times 0$ and $Z_s \times \infty$. Moreover, since $W \in Z_1(C^p(X, 1)) = \ker(C^p(X, 1) \xrightarrow{\partial} C^p(X, 0))$ we have $W \times V_s \in Z_1(C^p(X \times V_s, 1))$.

Denote by A the image of $W \times V_s$ under the composition of the homomorphisms

$$Z_1(C^p(X \times V_s, 1))' \xrightarrow{\cdot Z_{V_s}} Z_1(C_{Z_{V_s}}^{p+q}(X \times V_s, 1)) \xrightarrow{p_*} Z_1(C^1(V_s, 1))$$

Since $A \in Z_1(C^1(V_s, 1))$, there is a homology class in $H_1(C^1(V_s, \cdot)) \simeq \Gamma(V_s, \mathcal{O}_{V_s}^\times)$ i.e. an invertible regular function g_s on V_s . Similarly, for any $s' \in V_s$ the pullback cycle $A_{s'} = A \cdot s' \times \square^1 \in Z^1(C^1(s', 1))$ corresponds to a regular function on s' i.e. to a scalar $c' \in k(s')^\times \simeq F^\times$.

Since X and $X \times V_s$ are smooth and $p : X \times V_s \rightarrow V_s$ and $X \times s \rightarrow s$ proper, we are in situation to apply Fulton's Theorems 1.7 and 6.2. Hence, $\phi([\alpha])(s) = p_*(W \times V_s \cdot Z_{V_s} \times \square^1) \in C^1(V_s)$ agrees with $A = p_*(W \cdot Z_s \times \square^1) \in C^1(s)$ when restricted the point s . This means that $A_s = \phi([\alpha])(s)$. Similarly, for any point $s' \in V_s$ we have $A_{s'} = \phi([\alpha])(s')$.

We claim $g_s(s') = A_{s'}$ for any point $s' \in V_s$. Let $C^1(V_s, \cdot)'$ be the subcomplex for proper intersection with s' . Since $A \in H_1(C^1(V_s, \cdot)) \simeq \Gamma(V_s, \mathcal{O}_{V_s}^\times)$ i.e. A corresponds to an invertible regular function g_s on V_s , the graph $\Gamma(g_s)$ is an element of the subcomplex $C^1(V_s, \cdot)'$ for proper intersection with s' . By the proofs of Proposition 6.3.1 and Lemma 2.1.6, A and $\Gamma(g_s)$ represent the same homology class in $H_1(C^1(V_s, \cdot)) \simeq H_1(C^1(V_s, \cdot)')$. The map on complexes $C^1(V_s, \cdot)' \rightarrow C^1(s', \cdot)$ induces a homomorphism on homology groups $H_1(C^1(V_s, \cdot)') \rightarrow H_1(C^1(s', \cdot))$ under which any $[B] \in H_1(C^1(V_s, \cdot)')$ maps to $[B \cdot s' \times \square^1] \in H_1(C^1(s', \cdot))$. Combining, we get

$$[A_{s'}] = [\Gamma(g_s) \cdot s' \times \square^1] = g_s(s') \in F^\times \simeq H_1(C^1(s', \cdot))$$

This means that $g_s(s') = c' = \phi([\alpha])(s')$ for all points $s' \in V_s$. If we repeat the same process for another point $\bar{s} \in C$ we will again get a regular function $g_{\bar{s}}$ on an open subset $V_{\bar{s}} \subset C$. Moreover, the two functions g_s and $g_{\bar{s}}$ will agree on the intersection $V_s \cap V_{\bar{s}}$ as for any $s' \in V_s \cap V_{\bar{s}}$ we have $g_s(s') = \phi([\alpha])(s') = g_{\bar{s}}(s')$. Therefore, we can cover the curve

C by open subsets V_s and on each open subset we get a regular function g_s such that the functions g_s agree on the intersection $V_s \cap V_{\bar{s}}$ of the corresponding open subsets. Hence, the various regular functions g_s glue to one regular function g on the whole curve C . By construction this is the function $\phi([\alpha])$. Since C is an irreducible projective curve over an algebraically closed field F , this means that $\phi([\alpha])$ is a nonzero constant $c \in F$. \square

Because of the claim, we see that for any representative W of $[\alpha] \in H_1(C^p(X, \cdot))$ and any closed point $s \in C$, the corresponding value is

$$L(W, Z_s) = \phi([\alpha])(s) = c$$

Then for any two closed points $s_1, s_2 \in C$ we have $L(W, Z_{s_1} - Z_{s_2}) = 1$

Since any algebraically trivial cycle D of codimension q on X is of the form $D = \sum_{i=1}^r Z_{s_{i1}}^i - Z_{s_{i2}}^i$, for $i = 1, \dots, r$, for cycles $Z^i \in C^q(X \times C)$ as above and closed points $s_{ij} \in C$, we get $L(W, D) = 1 \in F^\times$. Moreover, as W was a representative for an arbitrary homology class $[\alpha] \in H_1(C^p(X, \cdot))$, this means that for Z an algebraically trivial codimension q cycle the homomorphism $L(\cdot, D) : H_1(C^p(X, \cdot)) \rightarrow F^\times$ is the constant map 1.

This finishes the case when the base field F is algebraically closed. Assume now that F is an arbitrary field. For any algebraically trivial cycle $D \in C^q(X)$ we have the commutative diagram

$$\begin{array}{ccc} H_1(C^p(X_{\bar{F}})) & \xrightarrow{L(\cdot, D_{\bar{F}})} & H_1(C^1(\bar{F}, \cdot)) = \bar{F}^\times \\ \uparrow & & \uparrow \downarrow \\ H_1(C^p(X_F, \cdot)) & \xrightarrow{L(\cdot, D)} & H_1(C^1(F, \cdot)) = F^\times \end{array}$$

where the vertical arrows are pullback after base change to the algebraic closure \bar{F} . By Lemma 2.1.6 we have canonical isomorphisms $H_1(C^1(\bar{F}, \cdot)) = \bar{F}^\times$ and $H_1(C^1(F, \cdot)) = F^\times$. Hence the right vertical arrow is the usual inclusion $F^\times \hookrightarrow \bar{F}^\times$ and in particular it is injective.

Note that since D is algebraically trivial, by the argument above for any homological class $[W] \in H_1(C^p(X_F, \cdot))$ we have $L([W]_{\overline{F}}, D_{\overline{F}}) = 1 \in \overline{F}^\times$. Then from the commutative diagram, we have $L([W], D)_{\overline{F}} = 1 \in \overline{F}^\times$. Using the injectivity of the right vertical arrow $F^\times \hookrightarrow \overline{F}^\times$, we get $L([W], D) = 1 \in F^\times$.

This shows that for any algebraically trivial cycle $D \in C^q(X)$ and any two morphism $W_1, W_2 : \alpha \rightarrow \beta$ in $\text{Cat}(\text{CH}^p(X))$ we have the same morphism $L(W_1, D) = L(W_2, D) : L(\alpha, D) \rightarrow L(\beta, D)$ in $\text{Cat}(\text{Pic}(F))$. Hence, the functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(X))$ factors through $\text{Cat}(\overline{\text{CH}}^p(X)) \rightarrow \text{Cat}(\text{Pic}(X))$ when $D \in C^q(X, 0)$ is algebraically trivial. \square

The following example illustrates the reason why we need the stronger version (Theorem 2.1.5) of the moving lemma instead of the usual version (Theorem 2.1.3). It gives an example of a variety X , a curve C with a point $s \in C$, and a cycle $W \in C^p(X, 1)$ properly intersecting Z_s and yet there is no open neighborhood $s \in V \subset C$ such that $W \times V \in C^p(X \times V, \cdot)$ properly intersecting Z_V . This means that we can not apply the upper semi-continuity theorem to $\overline{W} \times C \cap Z \times \mathbb{P}^1 \rightarrow C$, where \overline{W} is the closure of W in $X \times \mathbb{P}^1$. The reason for this that even though W intersects $Z_s \times \square^1$ properly, taking closures in $X \times \mathbb{P}^1$ the intersection $\overline{W} \cap Z_s \times \mathbb{P}^1$ is no longer proper because \overline{W}_∞ and $Z_s \times \{\infty\}$ intersect in an excessive dimension.

Example 5.0.4. Let $X = \mathbb{P}^3$ and the curve $C = \mathbb{P}^1$. Take $p = q = 2$. Fix two lines L_1 and L_2 in \mathbb{P}^3 , which do not intersect. Denote by F_i and G_i the equations cutting out the line L_i . Consider the one parameter family $\overline{W} \subset X \times \mathbb{P}^1$ such that the fiber above the point $[a : b] \in \mathbb{P}^1$ is the line cut out by the equations $aF_1 + bF_2$ and $aG_1 + bG_2$. Take $W = \overline{W}_{\square^1}$ so that \overline{W} is the closure of W in $X \times \mathbb{P}^1$. Let $Z \subset X \times C$ be defined in the same way as \overline{W} using that $C = \mathbb{P}^1$. Thus, for any point $s \in C$ the fiber Z_s is the line of the fiber \overline{W}_s and so $Z_s \subset X$ does not intersect any other fiber $\overline{W}_{s'} \subset X$.

Take the point $s = 1 \in C$. Then $\overline{W} \cap Z_1 \times \mathbb{P}^1 = L \times \{1\}$ where L is the line, which is the fiber of \overline{W} over $1 \in \mathbb{P}^1$. Hence, $W \cap Z_1 \times \square^1$ is empty and hence a proper intersection.

Similarly, for any other point $s' \in C$ we have $W \cap Z_{s'} \times \square^1 = \overline{W} \cap Z_{s'} \times \mathbb{P}^1 = \overline{W}_{s'} \times \{s'\}$ i.e. the intersection is of dimension 1 and hence not proper. Thus, there is no open neighborhood $s \in V \subset C$ such that $W \times V$ intersects $Z_V \times \square^1$ properly.

Combining Theorem 5.0.2 and Proposition 4.0.1 we have the more general result:

Corollary 5.0.5. *There is a bi-additive product functor $L(\cdot, \cdot) : \text{Cat}(\overline{\text{CH}}^p(X)) \times \text{Cat}(\text{CH}_{\text{alg}}^q(X)) \rightarrow \text{Cat}(\text{Pic}(F))$.*

Now let us consider the more general case when the base S is of an arbitrary dimension.

Definition 5.0.6. Let $\eta \in S$ be the generic point of the base S and let X_η be the generic fiber. Define

$$C_{\eta\text{-alg}}^q(X) = \{D \in C^q(X) : D|_{X_\eta} \text{ is algebraically trivial}\}$$

Similarly, define $\text{Cat}(\text{CH}_{\eta\text{-alg}}^q(X))$ to be the full subcategory of $\text{Cat}(\text{CH}^q(X))$ with objects in $C_{\eta\text{-alg}}^q(X)$.

Theorem 5.0.7. *For a fixed cycle $D \in C_{\eta\text{-alg}}^q(X)$ the functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ factors through the category $\text{Cat}(\overline{\text{CH}}^p(X))$.*

Proof. First, note that the inclusion $\eta \hookrightarrow S$ induces a map on complexes $C^1(S, \cdot) \rightarrow C^1(k(S), \cdot)$ and hence a functor on categories $\text{Cat}(\text{Pic}(S)) \rightarrow \text{Cat}(\text{Pic}(k(S)))$.

Claim 5.0.8. For any two objects $\alpha, \beta \in C^1(S)$ we have an inclusion on the morphisms $\text{Mor}_S(\alpha, \beta) \hookrightarrow \text{Mor}_\eta(\alpha_\eta, \beta_\eta)$

Proof. If the cycles α and β are not in the same equivalence class in $\text{Pic}(S)$, then $\text{Mor}_S(\alpha, \beta) = \emptyset$ and the inclusion $\text{Mor}_S(\alpha, \beta) \hookrightarrow \text{Mor}_\eta(\alpha_\eta, \beta_\eta)$ is trivial. If the cycles α and β are in the same equivalence class in $\text{Pic}(S)$, then the map $\text{Mor}_S(\alpha, \beta) \rightarrow \text{Mor}_\eta(\alpha_\eta, \beta_\eta)$ is given by $W \mapsto W_\eta$ for any $W \in C^1(S, 1)$ with $\partial W = \beta - \alpha$. To show that $\text{Mor}_S(\alpha, \beta) \rightarrow \text{Mor}_\eta(\alpha_\eta, \beta_\eta)$ is injective, as before it suffices to show that $H_1(C^1(S, \cdot)) \rightarrow H_1(C^1(\eta, \cdot))$ is

injective. By Lemma 2.1.6 this is equivalent to showing $\Gamma(S, \mathcal{O}_S^\times) \rightarrow \Gamma(\eta, k(\eta)^\times)$ is injective, which is true because localization is injective. \square

Second, for a fixed $D \in C_{\eta\text{-alg}}^q(X)$ by Proposition 5.0.2 applied to the morphism $X_\eta \rightarrow \eta$, the functor $L(\cdot, D_\eta) : \text{Cat}(\text{CH}^p(X_\eta)) \rightarrow \text{Cat}(\text{Pic}(k(S)))$ factors through $\text{Cat}(\overline{\text{CH}}^p(X_\eta))$ and hence we have the commutative diagram

$$\begin{array}{ccc}
\text{Cat}(\text{CH}^p(X)) & \xrightarrow{L(\cdot, D)} & \text{Cat}(\text{Pic}(S)) \\
\downarrow & & \downarrow \\
\text{Cat}(\text{CH}^p(X_\eta)) & \xrightarrow{L(\cdot, D_\eta)} & \text{Cat}(\text{Pic}(k(S))) \\
\downarrow & \nearrow & \\
\text{Cat}(\overline{\text{CH}}^p(X_\eta)) & &
\end{array}$$

Third, combine the previous two results with the commutative diagram

$$\begin{array}{ccc}
\text{Cat}(\text{CH}^p(X)) & \longrightarrow & \text{Cat}(\overline{\text{CH}}^p(X)) \\
\downarrow & & \downarrow \\
\text{Cat}(\text{CH}^p(X_\eta)) & \longrightarrow & \text{Cat}(\overline{\text{CH}}^p(X_\eta))
\end{array}$$

Then using that the functor $\text{Cat}(\text{Pic}(S)) \rightarrow \text{Cat}(\text{Pic}(k(S)))$ is injective on morphisms by Claim 5.0.8, we can lift the functor $\text{Cat}(\overline{\text{CH}}^p(X)) \rightarrow \text{Cat}(\text{Pic}(k(S)))$ to $\text{Cat}(\text{Pic}(S))$. Thus, the functor $\text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(S))$ factors through $\text{Cat}(\overline{\text{CH}}^p(X))$. \square

Combining the previous theorem and Proposition 4.0.1 we get the following result.

Corollary 5.0.9. *There is a bi-additive product functor $L(\cdot, \cdot) : \text{Cat}(\overline{\text{CH}}^p(X)) \times \text{Cat}(\text{CH}_{\eta\text{-alg}}^q(X)) \rightarrow \text{Cat}(\text{Pic}(S))$.*

CHAPTER 6

THE CASE OF A NUMERICALLY TRIVIAL CYCLE D

The main question of this chapter is if the fixed cycle D is numerically trivial, when does the functor $L(\cdot, D) : \text{Cat}(\text{CH}^p(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ factor through $\text{Cat}(\overline{\text{CH}}^p(X))$? For this chapter only assume the base field F to be algebraically closed. Throughout the chapter $S = \text{Spec}(F)$ and $d = \dim X$. Note the quasi-projective variety X is actually projective because $\pi : X \rightarrow S$ is proper. To state the main theorem of this paper we need first the following lemma

Lemma 6.0.1. *Let X be a smooth projective variety over algebraically closed field F . Let D be a numerically trivial divisor. Then the equivalence class $[D] \in \text{Pic}(X)$ can be expressed as $[D] = [E] + [T]$ where $[E]$ is an equivalence class of algebraically trivial divisors and $[T]$ is a torsion element of $\text{Pic}(X)$.*

Proof. Let D be a numerically trivial divisor, which is not algebraically trivial. By Theorem 4.6 of Expose XIII of [1], there is a natural number m such that $m[D] = [P]$ where $[P] \in \text{Pic}_{alg}(X)$, the group of equivalence classes of algebraically trivial divisors. Choose the smallest natural number m , satisfying this property. Since $\text{Pic}_{alg}(X)$ is divisible group, there is $[E] \in \text{Pic}_{alg}(X)$ such that $m[E] = [P]$. Consider the class $[T] = [D] - [E]$. The class $[T]$ is the desired torsion element as $m[T] = m[D] - m[E] = [P] - [P] = 0$. Moreover, since we choose m to be the smallest m such that $m[D] \in \text{Pic}_{alg}(X)$, for any $0 < t < m$ we have $t[D] \notin \text{Pic}_{alg}(X)$ which is the same as $t[T] = t[D] - t[E] \notin \text{Pic}_{alg}(X)$ since $[E] \in \text{Pic}_{alg}(X)$. \square

Now we are ready to state the main theorem of the paper:

Theorem 6.0.2. *Let F be algebraically closed field and D a numerically trivial divisor on a smooth projective variety X over F . If $\text{char} F = 0$, then the functor $L(\cdot, D)$ factors through $\text{Cat}(\overline{\text{CH}}^p(X))$ only if D is algebraically trivial. If $\text{char} F = p > 0$, by Lemma 6.0.1 we*

can express the equivalence class $[D]$ as $[D] = [E] + [T]$ where $[E] \in \text{Pic}_{alg}(X)$ and $[T]$ is a torsion element of $\text{Pic}(X)$ of order m such that $t[T] \notin \text{Pic}_{alg}(X)$ for any $0 < t < m$. Express $m = p^k n$ where $\gcd(n, p) = 1$, then the functor $L(\cdot, D)$ does not factor through $\text{Cat}(\overline{\text{CH}}^p(X))$ if and only if $n \geq 2$ i.e. $m \neq p^k$.

From Chapter 5 we know that for algebraically trivial divisors D the functor $L(\cdot, D) : \text{Cat}(\text{CH}^d(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ factors through $\text{Cat}(\overline{\text{CH}}^d(X))$. Hence, we only need to check what happens to morphisms when we have a divisor T such that mT becomes principal for some $m > 0$. By Proposition 5.0.2 the action of the functor $L(\cdot, D) : \text{Cat}(\text{CH}^d(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ on the morphisms depends only on the equivalence class $[D]$ of the divisor D . Moreover, torsion classes $[T] \in \text{Pic}(X)$ correspond to finite covers $\pi : Y \rightarrow X$ and morphisms in $\text{Cat}(\text{CH}^d(X))$ correspond to homology classes in $H_1(C^d(X, \cdot))$. Hence, we will focus on the action of finite covers $Y \rightarrow X$ and the corresponding torsion class $[D] \in \text{Pic}(X)$ on the group $H_1(C^d(X, \cdot))$. We will show that if the order of $[D]$ is coprime to the characteristic of the field F , the induced homomorphism $H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot)) = F^\times$ is trivial if and only if $[D] \in \text{Pic}_{alg}(X)$. If the class $[D]$ is of order p^k where $p = \text{char} F$, then by additivity the induced homomorphism $H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot)) = F^\times$ factors through $\mu_{p^k}(F) = 1$ and hence is trivial.

Before we present the proof of the main theorem let us first consider the following example.

6.1 Example

In this example we will construct a specific variety X , a torsion divisor class $[D]$ that is not algebraically trivial, and a homology class $[W] \in H_1(C^d(X, \cdot))$ such that $L(W, D) \neq 1 \in F^\times = H_1(C^1(F, \cdot))$.

Example 6.1.1. Let p be a prime, coprime to the characteristic of the base field F . Consider the morphism $\phi : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ given by $\phi([x_0, x_1, x_2, x_3]) = [x_0^p, x_1^p, x_2^p, x_3^p]$. Then using the

hyperplane $H \subset \mathbb{P}^3$ given by the equation $\sum x_i = 0$, we define the smooth projective Fermat surface $\phi^{-1}H = Y$ given by the equation $\sum x_i^p = 0$.

Note that μ_p^4 acts on \mathbb{P}^3 by $(\eta_1, \eta_2, \eta_3, \eta_4) \cdot [x_0, x_1, x_2, x_3] = [\eta_1 x_0, \eta_2 x_1, \eta_3 x_2, \eta_4 x_3]$. Fix an element $0 \neq \sigma = (\eta_1, \eta_2, \eta_3, \eta_4) \in \mu_p^4$ such that $\eta_i \neq \eta_j$ for all $i \neq j$ and set $G = \langle \sigma \rangle \simeq \mu_p$ the group generated by σ . The action of G is free on Y as the only points on \mathbb{P}^3 fixed by the action are $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$ and these points do not belong to Y . Consider the quotient $X = Y/G$ with quotient map $\pi : Y \rightarrow X$. Since the action of G on the projective variety Y is free, the quotient $X = Y/G$ is also a smooth projective variety. Moreover, the morphism $\pi : Y \rightarrow X$ is a finite etale covering with Galois group G .

On Y we have $3p^2$ Fermat curves given by the equations $x_i^p + x_j^p = 0$ for $i \neq j \in \{0, 1, 2, 3\}$. Consider the p^2 Fermat curves of the type $x_0^p + x_1^p = x_2^p + x_3^p = 0$. They are rational curves parametrized as $[x_0, \xi_1 x_0, x_2, \xi_2 x_2]$ for ξ_1, ξ_2 p -th roots of -1 . Two such curves $C_{\xi_1, \xi_2} = [x_0, \xi_1 x_0, x_2, \xi_2 x_2]$ and $C_{\xi'_1, \xi'_2} = [x_0, \xi'_1 x_0, x_2, \xi'_2 x_2]$ intersect in at most 1 point of the type

$$C_{\xi_1, \xi_2} \cdot C_{\xi'_1, \xi'_2} = \begin{cases} [1, \xi_1, 0, 0] & \xi_1 = \xi'_1 \\ [0, 0, 1, \xi_2] & \xi_2 = \xi'_2 \\ \emptyset & \xi_1 \neq \xi'_1, \xi_2 \neq \xi'_2 \end{cases}$$

Hence each curve C_{ξ_1, ξ_2} intersects the other $p - 1$ curves of the type C_{ξ_1, ξ'_2} and the other $p - 1$ curves of the type $C_{\xi'_1, \xi_2}$. Further, each intersection point has 0 for either the first 2 or the last 2 coordinates. Moreover, there are exactly p curves of of type C_{ξ_1, ξ_2} passing through the point $[1, \xi_1, 0, 0]$ and similarly for the point $[0, 0, 1, \xi_2]$. Combining, we get that there are a total of $2p$ intersection points - p of the type $[1, \xi_1, 0, 0]$ and p of the type $[0, 0, 1, \xi_2]$.

Let us consider the action of the group $G = \langle \sigma \rangle$ on the curves C_{ξ_1, ξ_2} . The generator $\sigma = (\eta_1, \eta_2, \eta_3, \eta_4)$ acts as $\sigma \cdot C_{\xi_1, \xi_2} = C_{\xi'_1, \xi'_2}$ where $\xi'_1 = (\mu_2 \mu_1^{-1}) \xi_1$ and $\xi'_2 = (\mu_4 \mu_3^{-1}) \xi_2$. Thus, $G = \langle \sigma \rangle = \mu_p$ identifies the p^2 curves of the type $x_0^p + x_1^p = x_2^p + x_3^p = 0$ into p groups of p curves $(C_{\xi_1, \xi_2}, \sigma \cdot C_{\xi_1, \xi_2}, \dots, \sigma^{p-1} \cdot C_{\xi_1, \xi_2})$ and so there are only p distinct curves of this type in $X = Y/G$. Moreover, G identifies all p intersection points of the type $[1, \xi_1, 0, 0]$

into one point and all p intersection points of the type $[0, 0, 1, \xi_2]$ into another point on X . Hence, on X we have p rational curves, every two intersecting in the 2 points $[1, \xi_1, 0, 0]$ and $[0, 0, 1, \xi_2]$.

Fix a primitive p -th root μ of 1. We can then express σ as $\sigma = (\mu^a, \mu^b, \mu^c, \mu^d)$ with a, b, c, d pairwise distinct. Now consider the function

$$g(x_0, x_1, x_2, x_3) = \frac{x_0^c x_1^d + x_2^a x_3^b}{x_0^d x_1^c + x_2^b x_3^a}$$

If we assume that $c + d = a + b = m$, we get that the denominator and numerator of g are homogeneous polynomials of degree m and hence $g \in k(Y)^\times$. Note that

$$\sigma \cdot g = \frac{\mu^{-(ac+bd)}}{\mu^{-(ad+bc)}} g$$

and hence g^p is σ -invariant. Hence, $g^p \in k(X)^\times$. Thus, $\text{div } g^p = pD$ for some divisor D on X . Set $E = \pi^* D$. Note that $E = \text{div } g$ on Y and $E = E_1 - E_2$ where E_1 is cut by $x_0^c x_1^d + x_2^a x_3^b = 0$ and E_2 is cut by $x_0^d x_1^c + x_2^b x_3^a = 0$.

Choose two distinct curves $C = C_{\xi_1, \xi_2}$ and $C' = C_{\xi'_1, \xi'_2}$ on X . Since the curves C and C' are rational, there are functions $f = \frac{x_0}{x_2} \in k(C)^\times$ and $f' = \frac{x_2}{x_0} \in k(C')^\times$. Note $\text{div } f + \text{div } f' = 0$ and hence, using the graphs of the functions, we have a corresponding element $[W] = \Gamma(f) + \Gamma(f') \in H_1(C^d(X, \cdot))$.

Fix irreducible components $\tilde{C} = \tilde{C}_{\xi_1, \xi_2}$ and $\tilde{C}' = \tilde{C}_{\xi'_1, \xi'_2}$ respectively of the covers of C and C' on Y . Note that $\pi : \tilde{C} \rightarrow C$ and $\pi : \tilde{C}' \rightarrow C'$ are isomorphisms and $\pi_* \tilde{C} = C$ and $\pi_* \tilde{C}' = C'$. Hence, there are functions $\tilde{f} \in k(\tilde{C})$ such that $f = \text{Norm}_{k(\tilde{C})/k(C)} \tilde{f}$ and $\tilde{f}' \in k(\tilde{C}')$ such that $f' = \text{Norm}_{k(\tilde{C}')/k(C')} \tilde{f}'$. This implies that $\pi_* \text{div}(\tilde{f}) = \text{div}(f)$ and $\pi_* \text{div}(\tilde{f}') = \text{div}(f')$.

Note that D intersects the curves C and C' properly on X and moreover, D does not pass through the intersection points $[1, \xi_1, 0, 0]$ and $[0, 0, 1, \xi_2]$. Thus, D intersects $\text{div } f = [0, 0, 1, \xi_2] - [1, \xi_1, 0, 0]$ properly in the sense of Definition 2.1.2 and hence f is a rational

function on the intersection $C \cdot D$, meaning f is a unit in the generic points of the irreducible components of the intersection. Similarly for \tilde{f} and $\tilde{C} \cdot E$. Then

$$L(f, D) = f(C \cdot D) = (\text{Norm}_{k(\tilde{C})/k(C)} \tilde{f})(D|_C) = \tilde{f}(\pi^* D|_{\tilde{C}}) = \tilde{f}(E \cdot \tilde{C}) = L(\tilde{f}, E)$$

Then

$$L(f, D) = L(\tilde{f}, E) = L(\tilde{f}, E_1)L(\tilde{f}, E_2)^{-1}$$

and

$$L(f', D) = L(\tilde{f}', E) = L(\tilde{f}', E_1)L(\tilde{f}', E_2)^{-1}$$

Let us calculate $L(\tilde{f}, E_1)$. By definition $L(\tilde{f}, E_1) = \tilde{f}(\tilde{C} \cdot E_1)$. The m points p_j of intersection of $\tilde{C} = \tilde{C}_{\xi_1, \xi_2}$ and $E_1 = [x_0^c x_1^d + x_2^a x_3^b = 0]$ satisfy

$$0 = x_0^c x_1^d + x_2^a x_3^b = x_0^c (\xi_1 x_1)^d + x_2^a (\xi_2 x_2)^b = x_0^{c+d} \xi_1^d + x_2^{a+b} \xi_2^b = x_0^m \xi_1^d + x_2^m \xi_2^b$$

Hence, $\left(\frac{x_0}{x_2}\right)^m = -\frac{\xi_2^b}{\xi_1^d}$ and so for the m points of intersection p_j we have

$$\tilde{f}(p_j) = \frac{x_0}{x_2} = \left(m \sqrt[m]{ -\frac{\xi_2^b}{\xi_1^d} } \right) t^j$$

for $j = 0, \dots, m-1$ and t a primitive m -th root of 1. Thus,

$$\tilde{f}(\tilde{C} \cdot E_1) = \prod_{j=0}^{m-1} \tilde{f}(p_j) = \prod_{j=0}^{m-1} \left(m \sqrt[m]{ -\frac{\xi_2^b}{\xi_1^d} } \right) t^j = -\frac{\xi_2^b}{\xi_1^d} t^{m(m-1)/2}$$

Similarly, $\tilde{C} = \tilde{C}_{\xi_1, \xi_2}$ and $E_2 = [x_0^d x_1^c + x_2^b x_3^a = 0]$ intersect in m points q_j satisfying

$$\tilde{f}(q_j) = \frac{x_0}{x_2} = \left(m \sqrt[m]{ -\frac{\xi_2^a}{\xi_1^c} } \right) t^j$$

for $j = 0, \dots, m-1$. Hence,

$$L(\tilde{f}, E_2) = \tilde{f}(\tilde{C} \cdot E_2) = \prod_{j=0}^{m-1} \tilde{f}(q_j) = \prod_{j=0}^{m-1} \left(m \sqrt{-\frac{\xi_2^a}{\xi_1^c}} \right) t^j = -\frac{\xi_2^a}{\xi_1^c} t^{m(m-1)/2}$$

Combining we have

$$L(\tilde{f}, E_1)L(\tilde{f}, E_2)^{-1} = \frac{-\frac{\xi_2^b}{\xi_1^d} t^{m(m-1)/2}}{-\frac{\xi_2^a}{\xi_1^c} t^{m(m-1)/2}} = \frac{\xi_1^{c-d}}{\xi_2^{a-b}}$$

Redoing the calculations for $\tilde{f}' = \frac{x_2}{x_0}$ we get

$$L(\tilde{f}', E_1)L(\tilde{f}', E_2)^{-1} = \frac{(\xi_2')^{a-b}}{(\xi_1')^{c-d}}$$

Then in particular if we choose $\xi_1 = \xi_1'$,

$$L(f, D)L(f', D) = L(\tilde{f}, E)L(\tilde{f}', E) = \frac{\xi_1^{c-d} (\xi_2')^{a-b}}{\xi_2^{a-b} (\xi_1')^{c-d}} = \frac{(\xi_2')^{a-b}}{\xi_2^{a-b}} = \left(\frac{\xi_2'}{\xi_2} \right)^{a-b}$$

Note that $\frac{\xi_2'}{\xi_2}$ is a primitive p root of 1 and so $\left(\frac{\xi_2'}{\xi_2} \right)^{a-b}$ is also a primitive p root of 1 in $F^\times = H_1(C^1(F, \cdot))$ as $a \neq b$. Thus, we get

$$L(W, D) = L(f, D)L(f', D) = \left(\frac{\xi_2'}{\xi_2} \right)^{a-b} \neq 1 \in F^\times = H_1(C^1(F, \cdot))$$

Remark 6.1.2. Note that the condition $c+d = a+b$ is not necessarily. Consider the linear system

$$\begin{aligned} \alpha + \beta &= \gamma + \delta \\ a\gamma + b\delta &= c\alpha + d\beta \end{aligned}$$

This is a system of 2 equations with 4 unknowns and hence it has a 2 'dimensional' set

of solutions. We can choose 2 'linearly independent' solutions $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ for the powers of x_0, x_1, x_2, x_3 in the numerator and denominator of $g = \frac{x_0^\gamma x_1^\delta + x_2^\alpha x_3^\beta}{x_0^{\gamma'} x_1^{\delta'} + x_2^{\alpha'} x_3^{\beta'}}$ such that the numerator and the denominator are homogeneous polynomials of the same degree (the first equation) and $\sigma \cdot g = \frac{\mu^{a\gamma+b\delta}}{\mu^{a\gamma'+b\delta'}} g = \frac{\mu^{c\alpha+d\beta}}{\mu^{c\alpha'+d\beta'}} g$ (the second equation).

6.2 The set up for the proof of the main theorem

For a given divisor $D \in C^1(X, 0)$, showing that all morphisms in $\text{Mor}_{\text{Cat}(CH^d(X))}(\alpha, \beta)$ become the same morphism in $\text{Mor}_{\text{Cat}(\text{Pic}(S))}(L(\alpha, D), L(\beta, D))$ is equivalent as in Chapter 5 to showing that the homomorphism $L(\cdot, D) : H_1(C^d(X, \cdot)) \rightarrow F^\times$ is trivial. Hence, we want to find the kernel of the homomorphism $L : C^1(X, 0) \rightarrow \text{Hom}(H_1(C^d(X, \cdot)), F^\times)$ given by $D \mapsto L(\cdot, D)$.

The strategy of the proof of Theorem 6.0.2 the following. The chain homomorphism $L : C^1(X, \cdot) \otimes C^d(X, \cdot) \rightarrow C^1(F, \cdot)$ induces a group homomorphism on homologies $H_0(C^1(X, \cdot)) \otimes H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$. Equivalently, we have a homomorphism $CH^1(X) \otimes H_1(C^d(X, \cdot)) \rightarrow F^\times$. Rearranging this gives us a homomorphism $CH^1(X) \xrightarrow{L} \text{Hom}(H_1(C^d(X, \cdot)), F^\times)$.

For the rest of this chapter let us fix an integer n coprime to $\text{char}(F)$. Restricting to the n -torsion equivalence classes of divisors on X , we get a group homomorphism $\text{Pic}(X)[n] \xrightarrow{L} \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F))$. Here as usual $\text{Pic}(X)[n]$ is the group of equivalence classes of divisors $[D]$ such that $n[D] = 0 \in \text{Pic}(X)$.

We will show that the group homomorphism L factors through an isomorphism $\text{Pic}(X)[n] \xrightarrow{\phi} \text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F))$ such that the corresponding homomorphism $\text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F)) \rightarrow \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F))$ is the one coming from the universal coefficients theorem. That is we will prove that we have a commutative diagram

$$\begin{array}{ccc}
\text{Pic}(X)[n] & \xrightarrow{L} & \text{Hom}(H_1(C^d(X, \cdot)), \mu_n(F)) \\
& \searrow \phi & \nearrow \psi \\
& & \text{Hom}(H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n), \mu_n(F))
\end{array}$$

Thus $\ker L \simeq \ker \psi$ and we can find the size of $\ker \psi$ from the universal coefficients theorem. Thus we will prove that $\ker \psi = \phi(\text{CH}_{alg}^1(X)[n])$. The proof occupies the rest of the chapter and is split into several parts. The first step is to show that there is a complex $K_\bullet^d(X)$ based on Milnor K-theory that is quasi-isomorphic to Bloch's complex in low degrees, which while it lacks some functorial properties, is easier for calculations. Using this complex we construct a group homomorphism $H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ for a finite etale μ_n -cover of X , corresponding to an equivalence class $[D] \in \text{Pic}(X)[n]$. This will allow us to define the homomorphism ϕ . Weil pairing for curves and Lefschetz hyperplane theorem will imply that ϕ is an isomorphism. To determine the kernel of the homomorphism ψ we use the result of Proposition 5.0.2 together with Roitman's theorem and Universal coefficient theorem.

6.3 New interpretation of the morphisms of the Chow categories

When the base field F is infinite and X is smooth, we can replace Bloch's complex $C^d(X, \cdot)$ by a quasi-isomorphic in low degrees complex $K_\bullet^d(X)$ consisting of Milnor K-theory groups. The new complex will allow us to calculate more explicitly the functor L and related homomorphisms. Nevertheless, the new complex $K_\bullet^d(X)$ is not a complex of free abelian groups and hence we need to prove a few additional results in order to have functoriality and in particular to have the universal coefficients theorem for $H_1(K_\bullet^d(X))$.

Recall that for any field E the Milnor K-theory groups satisfy the properties $K_0^M(E) = \mathbb{Z}$, $K_1^M(E) = E^\times$ and $K_2^M(E) = E^\times \otimes E^\times / a \times (1-a)$ for $1 \neq a \in E^\times$. Then the Milnor complex $K_\bullet^p(X)$ for codimension p on X comes from the Gersten resolution for Milnor K-theory and

truncated to the last 3 terms is

$$\bigoplus_{\text{codim } Z=p-2} K_2^M(k(Z)) \xrightarrow{\text{Tame}} \bigoplus_{\text{codim } Z=p-1} k(Z)^\times \xrightarrow{\text{div}} \bigoplus_{\text{codim } Z=p} \mathbb{Z}$$

The first map is the tame symbol map and the second is the divisor of a function.

Proposition 6.3.1. *Let X be a smooth quasi-projective variety over an infinite field F . Then there is a map of complexes $C^p(X, \cdot) \rightarrow K_\bullet^p(X)$ between Bloch's complex and Milnor's complex. The induced homomorphism on homologies $H_i(C^p(X, \cdot)) \rightarrow H_i(K_\bullet^p)$ is an isomorphism for $0 \leq i \leq 2$.*

Proof. Note that for $i = 0$ the isomorphism is the Bloch's formula $\text{CH}^p(X) \simeq H^0(K_\bullet^p)$. Here, we sketch the proof for the case $i = 1$ following Corollary 5.3 of [22] by Stefan Müller-Stach.

Truncating both complexes to the last 3 terms we have the following commutative diagram

$$\begin{array}{ccccc} C^p(X, 2) & \xrightarrow{\partial} & C^p(X, 1) & \xrightarrow{\partial} & C^p(X, 0) \\ \text{Norm} \downarrow & & \text{Norm} \downarrow \uparrow \Gamma & & \downarrow = \\ \bigoplus_{\text{codim } Z=p-2} K_2^M(k(Z)) & \xrightarrow{\text{Tame}} & \bigoplus_{\text{codim } Z=p-1} k(Z)^\times & \xrightarrow{\text{div}} & \bigoplus_{\text{codim } Z=p} \mathbb{Z} \end{array}$$

The middle vertical arrow is the norm map, which is defined in the following way. We will define the Norm homomorphism $C^p(X, 1) \rightarrow \bigoplus_{\text{codim } Z=p-1} k(Z)^\times$ on the generators of the free abelian group $C^p(X, 1)$. Consider an irreducible $W \subset X \times \square^1$. Denote by W' the closure of the projection $pr_X(W)$ in X . Restricting the projection $X \times \square^1 \rightarrow \square^1$ to W defines a function f on W . Then we set $\text{Norm}(W) = \text{Norm}_{k(W)/k(W')}(f) \in k(W')^\times$. The norm map has a set theoretical inverse via the graph map Γ i.e. $\text{Norm}(\Gamma(Z, f)) = (Z, f)$ for any subvariety $Z \subset X$ of codimension $p - 1$ and function $f \in k(Z)^\times$. On the left, the homomorphism Norm is defined in a similar way.

Since the boundary map $C^p(X, 2) \xrightarrow{\partial} C^p(X, 1)$ coincides with the tame symbol, we have a homomorphism $\phi : \text{CH}^p(X, 1) \rightarrow H_1(K_\bullet^p)$. We need to show that ϕ is bijective. This is proved in the reference. \square

Since in the situation we are considering, the base field F is algebraically closed and hence infinite, the isomorphism $\text{CH}^d(X, 1) \simeq H_1(K_\bullet^d(X))$ in Proposition 6.3.1 holds. Hence for any objects $\alpha, \beta \in \text{Cat}(\text{CH}^d(X))$ we have

$$\text{Mor}(\alpha, \beta) = \frac{\{W \in C^d(X, 1) \mid \partial W = \beta - \alpha\}}{\partial C^d(X, 2)} \simeq \frac{\{(W_i, f_i) \mid \text{codim } W_i = d - 1, \sum \text{div } f_i = \beta - \alpha\}}{\text{im } Tame}$$

There are several issues with working directly with intersection of cycles and using functions as the definition for morphisms. The most serious disadvantage of working with the Milnor complex K_\bullet^p is that we can't work directly with cycles in $C^q(X, 0)$ and instead we need to work with equivalence classes in order to use moving lemmas. Thus, we can't define a pairing $K_\bullet^p(X) \otimes K_\bullet^q(X) \rightarrow K_\bullet^1(S)$ on the level of the complexes themselves. The main advantage of the Bloch's complex is that it is a complex of free abelian groups and hence we had an inverse of the quasi-isomorphism with the subcomplex for a proper intersection. We loose this functoriality with the Milnor complex and yet we gain a more intuitive understanding of the elements of the complexes. Therefore, we will use Bloch's cubical complex when defining the functors $L(\cdot, \cdot)$ and $L(\cdot, D)$ for a fixed divisor D . After we establish that the functors define homomorphisms such as $H_1(C^1(X, 0)) \rightarrow \text{Hom}(H_1(C^d(X, \cdot)), F^\times)$, we can apply moving lemmas and use functions as morphisms to do explicit computations.

Another serious problem with working with Milnor's complex $K_\bullet^d(X)$ is the somewhat problematic notion of a proper intersection. For example, a function f on a given subvariety W may not be a rational function on the intersection $W \cdot D$ meaning f may not be a unit in the generic points of the irreducible components of the intersection, even if $\text{div } f$ and D intersect properly as illustrated in the following example.

Example 6.3.2. (Example 1.10 of [13]) Consider $X = \mathbb{P}^2$ with affine coordinates x, y ,

$S = \text{Spec } F$, and let D be the hyperplane $x = 0$. On the singular cubic W given by the affine equation $y^2 = x^2(x + 1)$, consider the function $f(x, y) = \frac{y-x}{y+x}$. Note that f has both a simple pole and zero at the node $P = (0, 0)$ of W and hence $\text{div}(f) = 0$. This means that D intersects $\text{div}(f)$ properly at the expected dimension 0. But since $W \cap D$ is the node P and the point at infinity of W , the scalar $f(W \cdot D)$ is not well-defined as f does not have a value at P .

The issue is that W is not normal. Hence, to fix the issue we need to talk about 'bigger' cycle $D(f)$ instead of $\text{div}(f)$ involving the normalization \widetilde{W} of W .

Definition 6.3.3. Let W be a variety with a rational function $f \in k(W)^\times$. Consider the normalization $v : \widetilde{W} \rightarrow W$ and the pullback function v^*f of f to \widetilde{W} . Then the normalized support of f is $D(f) = v(\text{supp}(\text{div } v^*f))$. Note $\text{supp}(\text{div } f) \subset D(f)$ and if W is normal then $\text{supp}(\text{div } v^*f) = D(f)$.

Note that for W, f, D as in Example 6.3.2, we have $P \in D(f)$ and hence D doesn't intersect $D(f)$ properly at the expected dimension. In general, when D intersects properly both the subvariety W and the normalized support $D(f)$ for a function $f \in k(W)^\times$, then f restricts to a rational function on the intersection $W \cdot D$. Unfortunately, there are functorial issues with using this definition of proper intersection when treating the Chow groups as categories and constructing functors on product categories.

The more technical construction of Bloch's complex has the advantage over directly taking the graphs of functions as in Proposition 6.3.1 in that it avoids the issues with proper intersection from the example. In particular, let X, W, f, D be as in the Example 6.3.2, the cycle corresponding to the graph of f is in $\Gamma(f) \in C^p(X, 1)$ but not in the proper intersection subcomplex $C^p(X, 1)'$. The reason for this is that $\Gamma(f)$ does not intersect $D \times \square^1$ properly in the sense of Definition 2.1.2 as $P \times \{0\} \in \Gamma(f) \cap D \times \{0\}$ and so $\dim \Gamma(f) \cap D \times \{0\} = 0$ while the expected dimension is $\dim W + \dim D - \dim(X \times \square^1) = -1$. With the definition of proper intersection as in $C^p(X, 1)'$, for any X, D, W and any function $f \in k(W)^\times$ with $\Gamma(f) \in C^p(X, 1)'$, the function f restricts to a rational function on the intersection $D \cdot W$.

The following two results prove that two complexes Bloch's complex $C^d(X, \cdot)$ and Milnor's complex $K_\bullet^d(X)$ remain quasi-isomorphic in low degree after tensoring with \mathbb{Z}/n . That is $H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n) \simeq H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)$.

Lemma 6.3.4. *Let C_\bullet be a homological complex such that the zero term C_0 is a free abelian group and all negative degree terms C_i for $i < 0$ vanish. Then $H_1(C_\bullet \otimes^L G) \xrightarrow{\simeq} H_1(C_\bullet \otimes G)$ for any abelian group G .*

Proof. For any complex C_\bullet there is a first quadrant spectral sequence with first page terms $E_{p,q}^1 = \text{Tor}_q^{\mathbb{Z}}(C_p, G)$ converging to the homology $H_{p+q}(C_\bullet \otimes^L G)$ for an abelian group G .

The first homology H_1 is calculated from the two terms $E_{1,0}^\infty$ and $E_{0,1}^\infty$. Note that $E_{1,0}^2 = E_{1,0}^\infty$ because even on the second page all differential coming and going to $E_{1,0}^2$ vanish as $0 = E_{3,-1}^2 \rightarrow E_{1,0}^2 \rightarrow E_{-1,0}^2 = 0$. Moreover,

$$E_{1,0}^2 = \frac{\ker(E_{1,0}^1 \rightarrow E_{0,0}^1)}{\text{im}(E_{2,0}^1 \rightarrow E_{1,0}^1)} = \frac{\ker(C_1 \otimes G \rightarrow C_0 \otimes G)}{\text{im}(C_2 \otimes G \rightarrow C_1 \otimes G)} = H_1(C_\bullet \otimes G)$$

Similarly, $E_{0,1}^r = \text{Tor}_r^{\mathbb{Z}}(C_0, G) = 0$ for all $r > 0$ because C_0 is a free abelian group. Then $E_{0,1}^\infty = E_{0,1}^1 = 0$.

Finally using that $E_{1,0}^\infty = F_1 H_1 / F_0 H_1$ and $E_{0,1}^\infty = F_0 H_1 / F_{-1} H_1 = F_0 H_1$ and $F_1 H_1 = H_1$ we get $0 \rightarrow F_0 H_1 \rightarrow F_1 H_1 \rightarrow F_1 H_1 / F_0 H_1 \rightarrow 0$ and so $0 \rightarrow E_{0,1}^\infty \rightarrow H_1 \rightarrow E_{1,0}^\infty \rightarrow 0$ and $0 \rightarrow 0 \rightarrow H_1(C_\bullet \otimes^L G) \rightarrow H_1(C_\bullet \otimes G) \rightarrow 0$ i.e. $H_1(C_\bullet \otimes^L G) \xrightarrow{\simeq} H_1(C_\bullet \otimes G)$. \square

Corollary 6.3.5. *The Milnor complex $K_\bullet^d(X)$*

$$\bigoplus_{\text{codim } Z=d-2} K_2^M(k(Z)) \xrightarrow{\text{Tame}} \bigoplus_{\text{codim } Z=d-1} k(Z)^\times \xrightarrow{\text{div}} \bigoplus_{\text{codim } Z=d} \mathbb{Z}$$

satisfies the universal coefficients theorem for homology in degree 1. That is, we have a natural short exact sequence

$$0 \rightarrow H_1(K_\bullet^d(X)) \otimes \mathbb{Z}/n \rightarrow H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n) \rightarrow 0$$

Moreover, we have natural isomorphisms

$$H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \xleftarrow{\cong} H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n) \xleftarrow{\cong} H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n)$$

Proof. There is a second spectral sequence which converges to $H_{p+q}(C_\bullet \otimes^L G)$ with terms on the second page $E_{p,q}^2 = \text{Tor}_p^{\mathbb{Z}}(H_q(C_\bullet), G)$ for G abelian group. The spectral sequence gives us the short exact sequence $0 \rightarrow H_1(C_\bullet) \otimes G \rightarrow H_1(C_\bullet \otimes^L G) \rightarrow \text{Tor}_1(H_0(C_\bullet), G) \rightarrow 0$. Applying this to the Bloch's complex and Milnor's complex $K_\bullet^d(X)$, we get the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n) & \longrightarrow & H_1(C^d(X, \cdot) \otimes^L \mathbb{Z}/n) & \longrightarrow & \text{Tor}_1(H_0(C^d(X, \cdot)), \mathbb{Z}/n) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) & \longrightarrow & H_1(K_\bullet^d(X) \otimes^L \mathbb{Z}/n) & \longrightarrow & \text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n) \longrightarrow 0 \end{array}$$

The first and the last vertical arrows are isomorphisms by Proposition 6.3.1. By the five lemma, the middle arrow is an isomorphism. Thus, $H_1(C^d(X, \cdot) \otimes^L \mathbb{Z}/n) \xrightarrow{\cong} H_1(K_\bullet^d(X) \otimes^L \mathbb{Z}/n)$. By Lemma 6.3.4 this implies as well $H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n) \xrightarrow{\cong} H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)$. The isomorphism $H_1(C^d(X, \cdot) \otimes \mathbb{Z}/n) \xleftarrow{\cong} H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n)$ follows from $H_1(C^d(X, \cdot)) \xleftarrow{\cong} H_1(\text{Sus}_\bullet(X))$ because $\text{Sus}_\bullet(X) \rightarrow C^d(X, \cdot)$ is a chain map of complexes of free abelian groups. □

6.4 A homomorphism $\kappa : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ for G -covers.

In this and the following subsection we construct the desired isomorphism $\phi : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ via the homomorphism $\kappa : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$. To get ϕ from κ we will need to compose with the automorphism $\tau \mapsto \tau^{-1}$ of μ_n . The homomorphism κ will turn out to coincide

with the homomorphism $H^1(X, \mu_n) \rightarrow H^1(\text{Sus}_\bullet(X), \mu_n)$ Suslin and Voevodsky constructed in [25]. To construct the homomorphism κ , we define more generally a homomorphism $\kappa(Y) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ on the cocycles for any finite etale cover $\pi : Y \rightarrow X$ with n -torsion Galois group G , with the additional property that equivariant cover morphisms $Y \rightarrow Z$ induce compatible homomorphisms $\kappa(Y) \rightarrow \kappa(Z)$. Then we will show that for any finite etale μ_n -cover the boundary $B_1(K_\bullet^d(X)) \mapsto 1$ under the homomorphism κ for the cover and hence we have a well-defined homomorphism $H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \pi_1^{ab}(X) \otimes \mathbb{Z}/n$ on the whole homology group.

The main ingredients in the construction of the homomorphisms κ are Tate cohomology, Tschen's theorem for function fields of curves, and the fact that principal divisors have degree 0.

Throughout this subsection let $\pi : Y \rightarrow X$ be a finite abelian etale cover with n -torsion Galois group G . Here X is a smooth projective variety over an algebraically closed field F and n is coprime to the characteristic of F .

Note that by direct calculations we have

$$Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) = \frac{\{(C_i, f_i) \mid f_i \in k(C_i)^\times, \sum \text{div } f_i = nE\}}{\{f^n \mid f \in k(C)^\times\}}$$

where C_i are curves on X and E is some 0-cycle. Denote by B the group

$$B = \{(C_i, f_i) : \sum \text{div } f_i = nE\}$$

where C_i are curves on X and E is some 0-cycle. Moreover,

$$B_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) = B_1(K_\bullet^d(X)) = (\text{Tame}_{T_i}(f, g))$$

where T_i are surfaces on X .

Before we define the homomorphism $\kappa(Y) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ we need a couple

supporting facts.

Lemma 6.4.1. *Let $\pi : Y \rightarrow X$ be as above. For any irreducible curve $C \subset X$ and a function $f \in k(C)^\times$, on any irreducible component \tilde{C} of $\pi^{-1}C = C \times_X Y$ there is a function $u \in k(\tilde{C})^\times$ such that $\text{Norm}_{k(\tilde{C})/k(C)}(u) = f$. Given \tilde{C} , the choice of u is unique up to a product of functions of the form $\frac{v}{gv}$ for g in the stabilizer of \tilde{C} and a function $v \in k(\tilde{C})^\times$. For a different choice of irreducible component \tilde{C}' of the preimage $\pi^{-1}C = C \times_X Y$, we replace (\tilde{C}, u) by (\tilde{C}', u') with $\tilde{C}' = g\tilde{C}$ and $u' = gu$ for $g \in G$.*

Proof. For a given irreducible curve $C \subset X$, choose the curve \tilde{C} to be any of the irreducible components of $\pi^{-1}C = C \times_X Y$. Since the base field F is algebraically closed, by Tsen's theorem the function field of any irreducible curve over F is C_1 field. In particular, the function field $k(C)$ is C_1 field. Hence, by Proposition 8 of Section 3.1 by Serre in [24] the norm map $\text{Norm} : k(\tilde{C})^\times \rightarrow k(C)^\times$ is surjective. So for any function $f \in k(C)^\times$ there is a function $u \in k(\tilde{C})^\times$ such that $\text{Norm}(u) = f$. Then $\pi_* \text{div}(u) = \text{div}(f)$.

Furthermore, by Proposition 5 of Section 3.2 by Serre in [24], the norm map being surjective implies that $k(\tilde{C})^\times$ is cohomologically trivial H module for $H = \text{Gal}(k(\tilde{C})/k(C)) \subset G$ and hence all Tate cohomology groups are trivial. Hence, by Chapter IV of [8], we have the exact sequence

$$0 \rightarrow \hat{H}_0(H, k(\tilde{C})^\times) \rightarrow H_0(H, k(\tilde{C})^\times) \rightarrow H^0(H, k(\tilde{C})^\times) \rightarrow \hat{H}^0(H, k(\tilde{C})^\times) \rightarrow 0$$

which has a trivial kernel and cokernel. Therefore, $H_0(H, k(\tilde{C})^\times) \rightarrow H^0(H, k(\tilde{C})^\times)$ is isomorphism.

Considering the commutative diagram

$$\begin{array}{ccc} H_0(H, k(\tilde{C})^\times) & \xrightarrow{\cong} & H^0(H, k(\tilde{C})^\times) \\ \uparrow & \nearrow & \\ k(\tilde{C})^\times & & \end{array}$$

and $H^0(H, k(\tilde{C})^\times) = \left(k(\tilde{C})^\times\right)^H = k(C)^\times$, we get that $\ker(\text{Norm} : k(\tilde{C})^\times \rightarrow k(C)^\times) = \ker(k(\tilde{C})^\times \rightarrow H_0(H, k(\tilde{C})^\times))$. But by definition, $H_0(H, k(\tilde{C})^\times) = k(\tilde{C})^\times / Ik(\tilde{C})^\times$, where I be the augmented ideal of $\mathbb{Z}[H]$. Thus, the kernel of the Norm homomorphism is generated by elements v which look like $v = \frac{w}{g(w)}$ for $w \in k(\tilde{C})^\times$ for $g \in H$ i.e. $v \in Ik(\tilde{C})^\times$. Combining, this means for any function $f \in k(C)^\times$ we can choose a function $u \in k(\tilde{C})^\times$ such that $\text{Norm}(u) = v$ which is well-defined up to a product of functions $v = \frac{w}{g(w)}$ for $w \in k(\tilde{C})^\times$.

Another thing to consider is what happens when we choose a different irreducible component \tilde{C}' of $\pi^{-1}(C)$. Since $X = Y/G$, this means that all irreducible components of $\pi^{-1}C = C \times_X Y$ are isomorphic and the Galois group G acts transitively on the components. Thus, $\tilde{C}' = g\tilde{C}$ for some $g \in G$. Then the new component \tilde{C}' has a copy gu of the function $u \in k(\tilde{C})^\times$ such that $\text{div } gu = g \text{div } u$. Repeating the same argument for functions on the new component \tilde{C}' as for functions on \tilde{C} we see that any other function u' on \tilde{C}' with $\text{Norm}(u') = \text{Norm}(gu) = f$ differs from gu by $v = \frac{w}{h(w)}$ for $w \in k(\tilde{C}')^\times$ and $h \in H = \text{Gal}(k(\tilde{C}')/k(C)) = \text{Gal}(k(\tilde{C})/k(C))$.

Combing the results for functions on \tilde{C} and \tilde{C}' we get the desired conclusion that the difference between any two functions u, u' on irreducible components \tilde{C}, \tilde{C}' of $\pi^{-1}(C)$ such that $\text{Norm}_{k(\tilde{C}')/k(C)}(u') = \text{Norm}_{k(\tilde{C})/k(C)}(u) = f$ is generated by elements of the type $\frac{v}{gv}$ for $g \in G$ and v a function on an irreducible component of $\pi^{-1}(C)$. \square

For a given point $p \in X$, choose an arbitrary lift $q \in Y$. Using that as a set $\pi^{-1}(p) = Gq$, for any ring R there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R[G] & \xrightarrow{\epsilon} & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & R[\pi^{-1}(p)] & \xrightarrow{\epsilon} & R & \longrightarrow & 0 \end{array}$$

Here I is the augmented ideal of $R[G]$ with elements of the form $\sum_{g \in G} m_g g$ with $\sum_{g \in G} m_g =$

$0 \in R$ and J is the $R[G]$ -module with elements of the form $\sum_{g \in G} m_g(gq)$ with $\sum_{g \in G} m_g = 0 \in R$. Moreover, there is an isomorphism $R[G] \rightarrow R[\pi^{-1}(p)]$ of $R[G]$ -modules given by $1 \mapsto q$. Hence, we have an isomorphism $I \xrightarrow{\cong} J$ of $R[G]$ -modules, which induces an isomorphism $I/I^2 \rightarrow J/IJ$.

From the canonical isomorphism $I/I^2 \rightarrow R \otimes G_{ab}$ we define an isomorphism $J/IJ \rightarrow R \otimes G_{ab}$. The isomorphism is given by $\sum_{g \in G} m_g(gq) \mapsto 1 \otimes \prod_{g \in G} g^{m_g}$. In our situation, $R = \mathbb{Z}/n$ and G a finite n -torsion abelian group we have a natural isomorphism $R \otimes G_{ab} = G$. Hence, there is an isomorphism $\theta_q : J/IJ \rightarrow G$ given by

$$\sum_{g \in G} m_g(gq) \mapsto \prod_{g \in G} g^{m_g}$$

Claim 6.4.2. For any other choice q' of lift of p , the two isomorphisms $\theta_q, \theta_{q'} : J/IJ \rightarrow G$ coincide.

Proof. For any other lift q' of $p \in X$, since set-theoretically $\pi^{-1}(p) = Gq$ we have that $q' = hq$ for some $h \in G$. Then for any element $\sum_{g \in G} m_g(gq) \in J$ we have

$$\begin{aligned} \theta_{q'} \left(\sum_{g \in G} m_g(gq) \right) &= \theta_{q'} \left(\sum_{g \in G} m_g(gh^{-1}q') \right) = \prod_g (gh^{-1})^{m_g} = \\ &= h^{-\sum_g m_g} \prod_g g^{m_g} = \prod_g g^{m_g} = \theta_q \left(\sum_{g \in G} m_g(gq) \right) \end{aligned}$$

using that $\sum_g m_g \equiv 0 \pmod{n}$. This shows that $\theta_{q'} = \theta_q$ and hence the isomorphism $J/IJ \rightarrow G$ is independent of the choice of lift q . \square

We are finally ready to define the homomorphism $\kappa(Y) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow G$.

Proposition 6.4.3 (Construction). *Let $\pi : Y \rightarrow X$ be a finite etale cover with n -torsion Galois group G . There is a group homomorphism $\kappa(Y) : B \rightarrow G$ defined via the homomorphisms θ .*

Proof. Given an element (C_i, f_i) of the group B , denote by $n(C_i)$ the normalization of the curve C_i . Then we have the morphism $\coprod n(C_i) \xrightarrow{v} X$ with $\sum v_* \operatorname{div} f_i = nE$ where E is a 0-cycle.

Let $p \in X$ be an arbitrary point and set $k_p = \operatorname{ord}_p E$. On the disjoint union of the normalizations $\coprod n(C_i)$ there are finitely many points x_j in the preimage $v^{-1}(p)$. Working only on the fiber $\pi^{-1}(p)$, we have $\sum \operatorname{div} f_i|_{v^{-1}(p)} = \sum_j n_j x_j$ such that $\sum n_j = nk_p$ because $v_*(\sum_j n_j x_j) = (nE)|_p$.

Consider the fiber product digram

$$\begin{array}{ccc} \coprod \widetilde{n(C_i)} & \xrightarrow{\tilde{v}} & Y \\ \bar{\pi} \downarrow & & \downarrow \pi \\ \coprod n(C_i) & \xrightarrow{v} & X \end{array}$$

Here the smooth curves $\widetilde{n(C_i)}$ on Y are the corresponding covers of the smooth curves $n(C_i)$ on X . Note that the curves $\coprod \widetilde{n(C_i)}$ are the disjoint union of the normalizations of the curves $\widetilde{C_i} \subset Y$ which lie above the curves $C_i \subset X$.

As a set, the fiber $\pi^{-1}(p)$ consists of the points Gq where q is an arbitrary lift of p . Similarly, for each j , the fiber $\bar{\pi}^{-1}(x_j)$ consists of points Gy_j where y_j is a chosen lift of x_j . Moreover, each gy_j corresponds under \tilde{v} to gq in the fiber of $\pi^{-1}(p)$ i.e. $\tilde{v}(gy_j) = gq$ because the action of G on Y is compatible with the induced action of G on $\coprod \widetilde{n(C_i)}$.

By Lemma 6.4.1 we can choose functions u_i on one of the irreducible components of the corresponding curves $\widetilde{n(C_i)}$ such that $\operatorname{Norm}(u_i) = f_i$ and $\pi_* \operatorname{div}(u_i) = \operatorname{div}(f_i)$. Each choice of a function u_i for a function $f_i \in k(n(C_i))^\times$ is well-defined up to a product of functions $\frac{v}{gv}$ for $g \in G$ and a function v on a possibly different irreducible component of $\widetilde{n(C_i)}$. Then

$$\sum \operatorname{div} u_i|_{\bar{\pi}^{-1}v^{-1}(p)} = \sum_g \sum_j m_{gj}(gy_j)$$

for some coefficients $n_{gj} \in \mathbb{Z}$. Note that as $\bar{\pi}_*(\sum \operatorname{div} u_i) = \sum \operatorname{div} f_i$ combining the coefficients on the fiber $\bar{\pi}^{-1}(x_j)$ we get $\sum_g m_{gj} = n_j$. Moreover, $(\tilde{v}_* \sum \operatorname{div} u_i)|_{\pi^{-1}(p)} = \sum_g \sum_j m_{gj}(gq)$.

Consider the composition of maps

$$\Theta : \bigoplus_{p \in X} J \rightarrow \bigoplus_{p \in X} J/IJ \xrightarrow{\bigoplus \theta_q} \bigoplus_{p \in X} G \rightarrow G$$

where $q \in \pi^{-1}(x) \subset Y$ is any choice of a lift of p . The first map is the obvious quotient and the last map $(g_p)_{p \in X} \mapsto \prod g_p$. The map Θ is independent of the choice of a lift q for each point $p \in X$ because by Claim 6.4.2 each of the homomorphisms $\theta_q : J \rightarrow J/IJ \xrightarrow{\theta_q} G$ is independent of the choice of lift q .

Since G is n -torsion, the module J consists of zero cycles of the type $\sum m_g(gq)$ with $\sum m_g \equiv 0 \pmod{n}$. Note $\sum \tilde{v}_* \operatorname{div} u_i \in \bigoplus_{p \in X} J$ since $\pi_* \tilde{v}_*(\sum \operatorname{div} u_i) = \sum v_* \bar{\pi}_* \operatorname{div} u_i = \sum v_* \operatorname{div}(f_i) = nE$. Hence, we can evaluate

$$\Theta \left(\sum \tilde{v}_* \operatorname{div} u_i \right) = \Theta \left(\sum_p \sum_g \sum_j m_{gj}(gq) \right) = \prod_p \prod_g g^{\sum_j m_{gj}} \in G$$

Define the map $\kappa(Y) : B \rightarrow G$ as $(C_i, f_i) \mapsto \Theta(\sum \tilde{v}_* \operatorname{div} u_i)$.

Claim 6.4.4. The map $\kappa(Y)$ is independent of the choice of functions u_i on irreducible components of $\pi^{-1}(C_i)$.

Proof. We need to check that $\Theta(\sum \tilde{v}_* \operatorname{div} u_i) \in G$ is independent of the choice functions u_i . By Lemma 6.4.1 it is sufficient to show that $\Theta(\tilde{v}_*(\operatorname{div} gw - \operatorname{div} w)) = 1 \in G$ for any function w on an irreducible component of $\widetilde{n(C_i)}$ and $g \in G$. To see this, note that for $\operatorname{div} w = \sum_y m_y y$, we have $\operatorname{div} gw - \operatorname{div} w = \sum_y m_y (gy - y)$. Moreover, $\sum_y m_y = 0$ because any irreducible

component of $\widetilde{n(C_i)}$ is a projective curve and hence $0 = \deg(\operatorname{div} w) = \sum_y m_y$. Then

$$\begin{aligned} \Theta(\tilde{v}_*(\operatorname{div} gw - \operatorname{div} w)) &= \Theta\left(\sum_y m_y \tilde{v}_*(gy - y)\right) = \Theta\left(\sum_y m_y (g\tilde{v}_*y - \tilde{v}_*y)\right) \\ &= \prod_y g^{m_y} = g^{\sum_y m_y} = g^0 = 1 \in G \end{aligned}$$

□

Claim 6.4.5. The map $\kappa(Y) : B \rightarrow G$ is a homomorphism.

Proof. We want to show $\kappa(Y)((C_i, g_i) \cdot (C'_j, g'_j)) = \kappa(Y)(C_i, g_i)\kappa(Y)(C'_j, g'_j)$. By construction of the homomorphism $\kappa(Y)$, the result is tautological if $C_i \neq C'_j$ for all i, j . If we have the same curve repeated in both summands, the result follows from $\operatorname{Norm}(u_i u'_j) = \operatorname{Norm}(u_i)\operatorname{Norm}(u'_j)$ and $\operatorname{div} u_i u'_j = \operatorname{div} u_i + \operatorname{div} u'_j$. □

□

Claim 6.4.6. For any curve $C \subset X$ and function $f \in k(C)^\times$ we have $\kappa(Y)(f^n) = 1 \in G$. Hence, $\kappa(Y)$ is defined on $Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) = \frac{\{(C_i, f_i) : \sum \operatorname{div} f_i = nE\}}{\{f^n | f \in k(C)^\times\}}$.

Proof. Let $\tilde{C}_1 \subset Y$ be any irreducible component of the cover $\pi^{-1}(C)$ of C . Then by Proposition 6.4.1, there is a function u on the curve \tilde{C}_1 such that $\operatorname{Norm}(u) = f$ and hence $\operatorname{Norm}(u^n) = f^n$. Using the notations of Proposition 6.4.3, for any point $p \in X$ when we restrict to the fiber $\pi^{-1}(p)$ we have $(\tilde{v}_* \operatorname{div} u)|_{\pi^{-1}(p)} = \sum_g m_g(gq)$. Then on the same fiber

$$(\tilde{v}_* \operatorname{div} u^n)|_{\pi^{-1}(p)} = \sum_g (nm_g)(gq) \in J \subset \mathbb{Z}/n[\pi^{-1}(p)]$$

Moreover, the contribution of $\tilde{v}_* \operatorname{div} u^n$ above the point p to $\Theta : \bigoplus_{p \in X} J \rightarrow G$ is trivial as $\prod_g g^{nm_g} = \prod_g 1 = 1 \in G$. The point p was arbitrary, hence $\kappa(Y)(C, f^n) = \Theta(\tilde{v}_* \operatorname{div} u^n) = 1 \in G$. □

Proposition 6.4.3 and Claim 6.4.6 imply that constructed homomorphism $\kappa(Y) : Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ is independent of the made choices. Hence, we get the following corollary:

Corollary 6.4.7. *For any finite etale cover $\pi : Y \rightarrow X$ with n -torsion Galois group G , the associated homomorphism $\kappa(Y) : Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ depends only on the isomorphic class of cover Y .*

Remark 6.4.8. Let us apply the homomorphism $\kappa(Y)$ with $G = \mu_p$ to the situation in Example 6.1.1 with the same notation as in the example. Then $\kappa(Y)((C, f) + (C', f')) = \Theta(\operatorname{div} \tilde{f} + \operatorname{div} \tilde{f}')$. For $\tilde{f} = \frac{x_0}{x_2} \in k(\tilde{C}_{\xi_1, \xi_2})^\times$ we have $\operatorname{div} \tilde{f} = [0, 0, 1, \xi_2] - [1, \xi_1, 0, 0]$ and similarly, for $\tilde{f}' = \frac{x_2}{x_0} \in k(\tilde{C}'_{\xi'_1, \xi'_2})^\times$ we have $\operatorname{div} \tilde{f}' = [1, \xi'_1, 0, 0] - [0, 0, 1, \xi'_2]$. Using that $\xi_1 = \xi'_1$, we have $\operatorname{div} \tilde{f} + \operatorname{div} \tilde{f}' = [0, 0, 1, \xi_2] - [0, 0, 1, \xi'_2]$. Note $\sigma \cdot [0, 0, 1, \xi_2] = [0, 0, 1, \mu^{d-c}\xi_2]$. Then as $\frac{\xi'_2}{\xi_2} = \mu^s$ and $c \neq d$ we can always find a solution $r \in \{1, \dots, p-1\}$ such that $(d-c)r = s \pmod{p}$ and $\sigma^r \cdot [0, 0, 1, \xi_2] = [0, 0, 1, \xi'_2]$. Using that, we get $\Theta(\operatorname{div} \tilde{f} + \operatorname{div} \tilde{f}') = \sigma^r \in G$. This shows again that for the chosen cycle $(C, f) + (C', f') \in Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n)$ the image under $\kappa(Y)$ is not trivial.

6.5 Relations between covers of X and the homomorphisms κ

By Claim 6.4.6 for any finite etale cover $Y \rightarrow X$ with n -torsion Galois group G , there is a homomorphism $\kappa(Y) : Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow G$. We will show that the homomorphisms κ are compatible with equivariant morphisms of covers of X . Hence, it will be sufficient to consider only the maximal unramified cover $\pi : Y \rightarrow X$ with Galois group $G = \pi_1^{ab}(X) \otimes \mathbb{Z}/n$. Let us briefly recall the construction of the cover Y and some basic facts about the finite etale covers of a variety X .

Here $\pi_1(X)$ is the etale fundamental group of X . It is the automorphism group on the fibre functor of a geometric point \bar{x} of X and is independent of the choice of the geometric point up to conjugation. When X is connected, the fundamental group $\pi_1(X)$ is profinite

group with the profinite topology. Then the abelianization $\pi_1^{ab}(X)$ is the maximal abelian quotient. Finally, $G = \pi_1^{ab}(X) \otimes \mathbb{Z}/n$ is a finite abelian group by Corollary 5.8.8 of [26].

By p.113 in [19] for any finite abelian group M the group $\text{Hom}_{\text{cont}}(\pi_1(X), M)$ is isomorphic to the group of finite etale abelian covers of X with Galois group M .

In particular for the quotient homomorphism $q : \pi_1(X) \rightarrow \pi_1^{ab}(X) \otimes \mathbb{Z}/n$ the finite etale cover Y we get is connected. This is the maximal unramified cover of X with n -torsion Galois group, which equals G .

Lemma 6.5.1. *Let $G = \pi_1^{ab}(X) \otimes \mathbb{Z}/n$ and M be a finite n -torsion abelian groups. There is a bijective correspondence between the group $\text{Hom}(G, M)$ and the set of isomorphic classes of finite etale covers of X with Galois group M . Moreover, to each homomorphism $\beta \in \text{Hom}(G, M)$, there is a corresponding equivariant morphism $f : Y \rightarrow Z$ of covers of X . The cover $\pi : Y \rightarrow X$ is the maximal unramified cover Y with Galois group G and the cover $\pi' : Z \rightarrow X$ is the finite etale cover corresponding to β .*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & \searrow \pi & \downarrow \pi' \\ & & X \end{array}$$

Equivariant homomorphisms $f : Y \rightarrow Z$ of finite etale covers of X with Galois groups G and M respectively naturally induce a relation between the homomorphisms $\kappa(Y) : Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ and $\kappa(Z) : Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow M$ defined in Claim 6.4.6.

Lemma 6.5.2. *Consider a finite etale cover $\pi' : Z \rightarrow X$ with a Galois group M corresponding to an equivariant morphism of covers*

$$\begin{array}{ccc}
Y & \xrightarrow{f} & Z \\
& \searrow \pi & \downarrow \pi' \\
& & X
\end{array}$$

where Y is the maximal unramified cover of X with Galois group $G = \pi_1^{ab}(X) \otimes \mathbb{Z}/n$. Denote by the β the homomorphism $\beta : G \rightarrow M$ inducing the morphism $f : Y \rightarrow Z$ of covers of X . Then the homomorphism $\kappa(Z) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow M$ factors as $\kappa(Z) = \beta \circ \kappa(Y)$.

Proof. Let $(C_i, g_i) \in Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)$ be an arbitrary element. Then to calculate $\kappa(Z)(C_i, g_i)$ for each i we choose an irreducible component \tilde{C}'_i of $\pi^{-1}(C_i)$ and functions u'_i on \tilde{C}'_i such that $\text{Norm}_{k(\tilde{C}'_i)/k(C_i)}(u'_i) = g_i$. Using that $f : Y \rightarrow Z$ is finite, there are irreducible curves \tilde{C}_i on Y such that $f(\tilde{C}_i) = \tilde{C}'_i$ and functions u_i on \tilde{C}_i such that $\text{Norm}_{k(\tilde{C}_i)/k(\tilde{C}'_i)}(u_i) = u'_i$. By the properties of norm for a tower of field extensions, we get $\text{Norm}_{k(\tilde{C}_i)/k(C_i)}(u_i) = g_i$ and $\pi(\tilde{C}_i) = C_i$.

Then $\kappa(Y)(C_i, g_i) = \Theta^Y(\sum \text{div } u_i) \in G$ and $\kappa(Z)(C_i, g_i) = \Theta^Z(\sum \text{div } u'_i) \in M$ where Θ^Y, Θ^Z are the homomorphisms Θ from Proposition 6.4.3 for the cover $Y \rightarrow X$ and $Z \rightarrow X$ with Galois groups G and M respectively. As in the Proposition, the divisors $\text{div } u_i$ and $\text{div } u'_i$ are calculated on the normalization of the corresponding curve. Moreover, again as in Proposition 6.4.3 the normalizations of the curves $\tilde{C}_i \subset Y$ lie above the normalizations of the curves $\tilde{C}'_i \subset Z$ and hence $f_*(\sum \text{div } u_i) = \sum \text{div } u'_i$.

Note

$$\sum \text{div } u_i = \sum_{p \in X} \sum_{\sigma \in G} m_\sigma(\sigma y)$$

for a fixed generator y of the fiber $\pi^{-1}(p) = Gy$. Similarly

$$\sum \text{div } u'_i = \sum_{p \in X} \sum_{\tau \in M} m_\tau(\tau z)$$

for $z = f(y)$ a generator of the fiber $\pi'^{-1}(p) = Mz$. From $f(y) = z$, we get that $f_*(y) = z$ and $f_*(\sigma y) = \beta(\sigma)z$ for any $\sigma \in G$. Hence, $m_\tau = \sum_{\beta(\sigma)=\tau} m_\sigma$ for $\tau \in M$. Then

$$\sum \operatorname{div} u'_i = f_* \left(\sum \operatorname{div} u_i \right) = f_* \left(\sum_{p \in X} \sum_{\sigma \in G} m_\sigma(\sigma y) \right) = \sum_{p \in X} \sum_{\tau \in M} \sum_{\beta(\sigma)=\tau} m_\sigma(\tau z)$$

Hence

$$\begin{aligned} \Theta^Z \left(\sum \operatorname{div} u'_i \right) &= \Theta^Z \left(\sum_{p \in X} \sum_{\tau \in M} \sum_{\beta(\sigma)=\tau} m_\sigma(\tau z) \right) = \prod_p \prod_{\tau \in M} \prod_{\beta(\sigma)=\tau} \tau^{m_\sigma} = \\ \prod_p \prod_{\sigma \in G} \beta(\sigma)^{m_\sigma} &= \beta \left(\Theta^Y \left(\sum_{p \in X} \sum_{\sigma \in G} m_\sigma(\sigma y) \right) \right) = \beta \left(\Theta^Y \left(\sum \operatorname{div} u_i \right) \right) \end{aligned}$$

Thus we get

$$\kappa(Z)(C_i, g_i) = \Theta^Z \left(\sum \operatorname{div} u'_i \right) = \beta \left(\Theta^Y \left(\sum \operatorname{div} u_i \right) \right) = \beta(\kappa(Y))(C_i, g_i)$$

for an arbitrary element $(C_i, g_i) \in Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)$. Therefore, we get the desired relation $\kappa(Z) = \beta \circ \kappa(Y)$. \square

Using the previous lemma we see that constructed homomorphisms $\kappa(Z) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ are compatible with equivariant morphisms of covers $Z \rightarrow X$. We use that to define the desired homomorphism $\kappa : \operatorname{Hom}(G, M) \rightarrow \operatorname{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), M)$ on the group of isomorphic classes of M -covers of X .

Corollary 6.5.3. *Consider the constructed in Proposition 6.4.3 homomorphism $\kappa(Y) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow G$ for Y the maximal unramified cover of X with Galois group $G = \pi_1^{ab}(X) \otimes \mathbb{Z}/n$. The homomorphism $\kappa(Y)$ induces a group homomorphism $\kappa : \operatorname{Hom}(G, M) \rightarrow \operatorname{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), M)$ given by $\beta \mapsto \beta \circ \kappa(Y)$. Moreover, via the correspondence between the group of isomorphic classes of finite etale covers of X with n -torsion Galois group M and $\operatorname{Hom}(G, M)$, the map $Z \mapsto \kappa(Z)$ is compatible with the group*

structure of the group of covers.

Proof. In Lemma 6.5.1 we have shown that any finite etale cover Z of X with Galois group M is induced by a homomorphism $\beta : G \rightarrow M$ and in this case by Lemma 6.5.2 the homomorphism $\kappa(Z) : Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow M$ is actually the composition $\kappa(Z) = \beta \circ \kappa(Y)$.

Furthermore, the homomorphism $\kappa : \text{Hom}(G, M) \rightarrow \text{Hom}(Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), M)$ is compatible with the group structure of the group of finite etale covers of X with Galois group M . The product of two covers Z_1 and Z_2 with Galois group M corresponding to homomorphisms $\beta_1, \beta_2 \in \text{Hom}(G, M)$ is a cover Z with Galois group M and homomorphism $\beta = \beta_1 \cdot \beta_2$. Then $\kappa(Z) = \beta(\kappa(Y)) = \beta_1(\kappa(Y))\beta_2(\kappa(Y)) = \kappa(Z_1)\kappa(Z_2)$. \square

6.6 The homomorphism κ in terms of Kummer theory for μ_n -covers.

Let us consider the case of primary interest when $M = \mu_n$. Before we express the homomorphism $\text{Hom}(G, \mu_n) \rightarrow \text{Hom}(Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ in terms of Kummer theory, let us first recall some identifications of the group of μ_n -covers.

Lemma 6.6.1. *The group of isomorphic classes of finite etale covers of X with Galois group μ_n is isomorphic to $H_{et}^1(X, \mu_n) \simeq \text{Hom}_{conts}(\pi_1^{et, ab}(X) \otimes \mathbb{Z}/n, \mu_n) \simeq \text{Pic}(X)[n]$. All of the isomorphisms are natural and compatible with one another.*

Proof. The group of isomorphic classes of finite etale μ_n -covers of X is by definition isomorphic to $H_{et}^1(X, \mu_n) \simeq \text{Hom}_{conts}(\pi_1^{et, ab}(X) \otimes \mathbb{Z}/n, \mu_n)$. We get the isomorphism with $\text{Pic}(X)[n]$ from the exact sequence $1 \rightarrow \mathcal{O}_X^{\times}(X) \otimes \mathbb{Z}/n \rightarrow H_{et}^1(X, \mu_n) \rightarrow \text{Pic}(X)[n] \rightarrow 1$, using that X is an irreducible projective variety over an algebraically closed field F and so $\mathcal{O}_X^{\times}(X) \otimes \mathbb{Z}/n = 0$. \square

Remark 6.6.2. The isomorphism $\text{Pic}(X)[n] \rightarrow$ the group of isomorphic classes of finite etale covers of X with Galois group μ_n is given in the following way. For any $[D] \in \text{Pic}(X)[n]$ we

have $nD = \text{div } h$ for $h \in k(X)^\times$ with h well-defined up to a function f^n for $f \in k(X)^\times$. In this case, the corresponding cover $Y \rightarrow X$ is the normalization of X into the function field $K(X)[\sqrt[e]{h}]$. If $h = f^e$ for $f \in k(X)^\times$ and $e|n$, then the cover $Y \rightarrow X$ is $\mu_{n/e}$ cover and to get a μ_n cover the cover we want is the disjoint union of n/e copies of the normalization of X into $K(X)[\sqrt[e]{h}]$.

We will also use often the following well-known fact (Lemma 8.3.13 of [10]):

Claim 6.6.3. Let $\pi : V \rightarrow W$ be a finite dominant morphism between smooth irreducible varieties over algebraically closed field F . Then for any function $f \in k(V)$ and zero-cycle α on W we have $f(\pi^*\alpha) = \text{Norm}_{k(V)/k(W)}(f)(\alpha)$ as long as f restricts to a unit at each point of the support of $\pi^*\alpha$.

Now we are finally ready to prove the main result of this subsection:

Proposition 6.6.4. *Given a finite etale μ_n -cover $\pi : Z \rightarrow X$ we can express the homomorphism $\kappa(Z) : B \rightarrow \mu_n$ in terms of Kummer theory in the following way. Let (C_a, g_a) be an arbitrary element in B i.e. irreducible curves C_a on X with functions g_a such that $\sum \text{div } g_a = nE$. Then $\kappa(Z)(C_a, g_a) = \frac{h(E)}{\prod g_a(D \cdot C_a)}$ for a suitable choice of representative D of the class $[D] \in \text{Pic}(X)[n]$ corresponding to the cover $\pi : Z \rightarrow X$ and a function h on X such that $\text{div } h = nD$.*

Proof. Let us first consider the special case when the cover Z is irreducible.

Let (C_a, g_a) be an arbitrary element of B . This means that $C_a \subset X$ are irreducible curves, not necessarily smooth with functions $g_a \in k(C_a)^\times$ such that $\sum \text{div}(g_a) = nE$. Since the curves C_a are not necessarily smooth, we calculate each divisor $\text{div}(g_a)$ on the normalization $n(C_a)$ of the curve C_a . Thus, $v_*(\sum_a \text{div}(g_a)) = nE$ where $v : \coprod_a n(C_a) \rightarrow \coprod_a C_a \hookrightarrow X$.

Let us evaluate $\kappa(Z)(C_a, g_a)$ using the method described in Proposition 6.4.3. We form the fiber product

$$\begin{array}{ccc}
\coprod \widetilde{n(C_a)} & \xrightarrow{\tilde{v}} & Z \\
\bar{\pi} \downarrow & & \downarrow \pi \\
\coprod n(C_a) & \xrightarrow{v} & X
\end{array}$$

By base change the curves $\widetilde{n(C_a)}$ are normal. For an arbitrary point $p \in X$ we have finitely many points x_j in the preimage $v^{-1}(p)$. Note that for each of them $v_*(x_j) = p$. We also have $\pi^*p = \sum_{\tau \in \mu_n} (\tau q)$ where $\pi(q) = p$. Similarly, we have $\bar{\pi}^*x_j = \sum_{\tau} (\tau y_j)$. Moreover, for any τ the fiber $\tilde{v}^{-1}(\tau q)$ consists of (τx_j) for all j and for each j we have $\tilde{v}_*(\tau y_j) = (\tau q)$.

Denote $\text{ord}_p(E) = k_p$. Then

$$\left(\sum \text{div } g_a \right)_{v^{-1}(p)} = \sum_j n_j x_j$$

such that

$$\sum_j n_j = \text{ord}_p(nE) = nk_p$$

By Lemma 6.4.1 for each a there is a function \tilde{g}_a on an irreducible component of the smooth curve $\widetilde{n(C_a)}$ such that $\text{Norm}(\tilde{g}_a) = g_a$ and so $\bar{\pi}_* (\sum_a \text{div } \tilde{g}_a) = \sum_a \text{div } g_a$. Thus,

$$\left(\sum_a \text{div } \tilde{g}_a \right)_{(\bar{\pi}v)^{-1}(p)} = \sum_j \sum_{\tau} n_{\tau j} (\tau y_j)$$

with

$$\sum_{\tau} n_{\tau j} = n_j \quad \text{and} \quad \sum_j n_{\tau j} = m_j$$

so that

$$\left(\tilde{v}_* \sum_a \text{div } \tilde{g}_a \right)_{\pi^{-1}(p)} = \sum_{\tau} m_{\tau} (\tau q)$$

Since the cover $\pi : Z \rightarrow X$ is a finite etale μ_n -cover, it corresponds to an equivalence

class $[D] \in \text{Pic}(X)[n]$. By the usual moving lemma, there is a representative $D \in [D]$ such that $D \cap C_a \cap C_b = \emptyset$ for $a \neq b$, $D \cap \text{sing}(C_a) = \emptyset$ and $D \cap C_a$ transversely for any curve C_a and the same conditions for π^*D and the curves $\pi^{-1}(C_a)$. Moreover, we can choose the divisor D such that $\pi^*(D)$ is disjoint from each $\text{div } \tilde{g}_a$.

On the other hand, the divisor D is a n -torsion, there is a function h on X such that $nD = \text{div } h$. Set $h' = \sqrt[n]{h} \in k(Z)^\times$. Note that $\text{div } h' = \pi^*D$. Additionally, any $\tau \in \mu_n$ acts on h' as a scalar via the homomorphism $\beta : G \rightarrow \mu_n$ associated to the cover $Z \rightarrow X$, where as before $G = \pi_1^{ab}(X) \otimes \mathbb{Z}/n$. Thus

$$\frac{\tau h'}{h'} = \frac{\beta(\sigma)h'}{h'} = \beta(\sigma)$$

and

$$h'(\tau q) = \tau^{-1}h'(q) = \beta(\sigma)^{-1}h'(q)$$

for any $\sigma \in G$ for which $\beta(\sigma) = \tau$.

Similarly, by construction, for any j we have

$$(h')(\tau y_j) = h'(\tau y_1) = \beta(\sigma)^{-1}h'(q) = \tau^{-1}h'(q)$$

We will show that

$$\frac{h(E)}{\prod g_a(D \cdot C_a)} = \kappa(Z)(C_a, g_a) \in \mu_n$$

Note $\text{Norm}(\tilde{g}_a) = g_a$ and by the choice of representative $D \in [D]$, we get $g_a(D|_{C_a}) = g_a(v^*D|_{n(C_a)}) = \tilde{g}_a((\bar{\pi}v)^*D|_{\widetilde{n(C_a)}}) = \tilde{g}_a(\text{div } h'|_{\widetilde{n(C_a)}})$. The middle equality follows by Claim 6.6.3. Then by Weil reciprocity on the smooth curve $\widetilde{n(C_a)}$, we get $\tilde{g}_a(\text{div } h'|_{\widetilde{n(C_a)}}) = h'(\text{div } \tilde{g}_a)$. Thus,

$$g_a(D \cdot C_a) = \tilde{g}_a(\text{div } h'|_{\widetilde{n(C_a)}}) = h'(\text{div } \tilde{g}_a)$$

On the fiber $(\bar{\pi}v)^{-1}(p)$ we get

$$\begin{aligned} h' \left(\sum_a \operatorname{div} \tilde{g}_a \right) |_{(\bar{\pi}v)^{-1}(p)} &= h' \left(\sum_{\tau} \sum_j n_{\tau j} (\tau y_j) \right) = \prod_j \prod_{\tau} [h'(\tau y_j)]^{n_{\tau j}} = \\ &= \prod_j \prod_{\tau} \tau^{-n_{\tau j}} [h'(y_j)]^{n_{\tau j}} = \prod_j \left(\prod_{\tau} \tau^{-n_{\tau j}} \right) [h'(y_j)]^{\sum_{\tau} n_{\tau j}} = \prod_j \left(\prod_{\tau} \tau^{-n_{\tau j}} \right) [h'(y_j)]^{n_j} = \\ &= \left(\prod_{\tau} \tau^{-\sum_j n_{\tau j}} \right) [h'(y_1)]^{\sum_j n_j} = \left(\prod_{\tau} \tau^{-m_{\tau}} \right) [h'(q)]^{nk_p} \end{aligned}$$

Since the cover Z is irreducible, we have $\operatorname{Norm}(h') = (-1)^{n-1}h$. Moreover, $n \deg E = \deg nE = \deg (v_* \sum \operatorname{div} g_a) = 0$ implies that $\deg E = 0$ and so $[(-1)^{n-1}h](E) = h(E)$. Then as h' is regular at each point of the preimage of E , we get $h'(\pi^*E) = h(E)$ by Claim 6.6.3.

Similarly again on the fiber $\pi^{-1}(p)$,

$$\begin{aligned} h'(v^*E)|_{\pi^{-1}(p)} &= h' \left(k_p \sum_{\tau} (\tau q) \right) = \left(\prod_{\tau} h'(\tau q) \right)^{k_p} = \\ &= \left(\prod_{\tau} \tau^{-1} h'(q) \right)^{k_p} = \left[\left(\prod_{\tau} \tau^{-1} \right) (h'(q)^n) \right]^{k_p} = \left(\prod_{\tau} \tau^{-k_p} \right) (h'(q)^{k_p n}) \end{aligned}$$

Thus, any point $p \in X$ contributes

$$\frac{h(E)}{\prod_a g_a(D \cdot C_a)} |_{\pi^{-1}(p)} = \frac{h'(v^*E)}{h'(\sum_a \operatorname{div} \tilde{g}_a)} |_{\pi^{-1}(p)} = \frac{\left(\prod_{\tau} \tau^{-k_p} \right) (h'(q)^{k_p n})}{\left(\prod_{\tau} \tau^{-m_{\tau}} \right) [h'(q)]^{nk_p}} = \frac{\prod_{\tau} \tau^{-k_p}}{\prod_{\tau} \tau^{-m_{\tau}}}$$

Hence

$$\frac{h(E)}{\prod_a g_a(D \cdot C_a)} = \prod_{p \in X} \frac{\prod_{\tau} \tau^{-k_p}}{\prod_{\tau} \tau^{-m_{\tau}}} = \frac{\prod_{\tau} \tau^{-\sum_p k_p}}{\prod_p \prod_{\tau} \tau^{-m_{\tau}}}$$

Note that $\sum_{p \in X} k_p = \deg E = 0$. Hence,

$$\frac{h(E)}{\prod_a g_a(D \cdot C_a)} = \frac{1}{\prod_p \prod_{\tau} \tau^{-m_{\tau}}} = \prod_p \prod_{\tau} \tau^{m_{\tau}}$$

Moreover,

$$\prod_p \prod_{\tau} \tau^{m_{\tau}} = \prod_p \prod_{\tau} \tau^{\sum_{\beta(\sigma)=\tau} m_{\sigma}} = \prod_p \prod_{\sigma \in G} \beta(\sigma)^{m_{\sigma}}$$

Observe that by Lemma 6.5.2 we have

$$\kappa(Z)(C_a, g_a) = \beta(\Theta^Y(\sum \operatorname{div} u_i)) = \prod_p \prod_{\sigma \in G} \beta(\sigma)^{m_{\sigma}}$$

Combining the last two equalities we get

$$\frac{h(E)}{\prod_a g_a(D \cdot C_a)} = \kappa(Z)(C_a, g_a) \in \mu_n$$

The proof works similarly when the cover Z has $e > 1$ irreducible components which happens exactly when $h = s^e$ for $s \in k(X)$ and $de = n$. In particular, the function h' is defined on one of the irreducible components Z_{λ} and $\operatorname{Norm}(h') = (-1)^{d-1}s$. Then the Galois group of $Z_{\lambda} \rightarrow X$ is $\mu_d = \langle \eta^e \rangle = \operatorname{im} \beta$ for η a primitive n -th root of 1. We can choose the functions g_a on an irreducible components of the curve $\widetilde{n(C_a)}$ contained in the same irreducible component Z_{λ} of Z . In this case the key calculation about the contribution of any point $p \in X$ is:

$$\begin{aligned} h(E)|_p &= s^e(E)|_p = s(eE)|_p = h'(e\pi^*E)|_{\pi^{-1}(p)} = h' \left(ek_p \sum_{\delta \in \mu_d} h'(\delta q) \right) = \left(\prod_{\delta} h'(\delta q) \right)^{ek_p} = \\ &= \left(\prod_{\delta} \delta^{-1} h'(q) \right)^{ek_p} = \left[\left(\prod_{\delta} \delta^{-1} \right) \left(h'(q)^d \right) \right]^{ek_p} = \left(\prod_{\delta} \delta^{-ek_p} \right) \left(h'(q)^{k_p n} \right) \end{aligned}$$

□

Definition 6.6.5. Define the homomorphism $\phi : \operatorname{Pic}(X)[n] \rightarrow \operatorname{Hom}(Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ such that the diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(\pi_1^{ab}(X)/\pi_1^{ab}(X)^n, \mu_n) & \xrightarrow{\kappa} & \mathrm{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n) \\
\cong \downarrow & & \downarrow \chi \\
\mathrm{Pic}(X)[n] & \xrightarrow{\phi} & \mathrm{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)
\end{array}$$

Here $\chi : \mu_n \rightarrow \mu_n$ is the automorphism given by $\tau \mapsto \tau^{-1}$.

Remark 6.6.6. We need to compose with the inverse automorphism $\chi : \mu_n \rightarrow \mu_n$ given by $\tau \mapsto \tau^{-1}$ because the action of the Galois group on the space of points of Z and the space of the functions on Z are cancel out.

6.7 The homomorphism L factors through ϕ

Recall that we wanted the homomorphism $\phi : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ to satisfy two conditions:

- (i) the homomorphism ϕ descends to an isomorphism $\phi : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$
- (ii) the homomorphism $L : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ factors as ϕ and the natural homomorphism $\mathrm{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n) \rightarrow \mathrm{Hom}(H_1(K_\bullet^d(X))), \mu_n)$

Now that we have defined the homomorphism ϕ , we will show the second property first and use it to prove the first.

Lemma 6.7.1. *The homomorphism $L : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(H_1(C^d(X, \cdot)), \mu_n)$ constructed in Section 6.2 factors via the homomorphism $\phi : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ constructed in Proposition 6.4.3.*

Proof. Using the isomorphism $H_1(K_\bullet^d(X)) \simeq H_1(C^d(X, \cdot))$ and the homomorphism $L : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(H_1(C^d(X, \cdot)), F^\times)$, we can define a homomorphism $\phi^L : \mathrm{Pic}(X)[n] \rightarrow \mathrm{Hom}(H_1(K_\bullet^d(X)), F^\times)$.

For each equivalence class $[D] \in \text{Pic}(X)[n]$ we get a homomorphism $L([D]) : H_1(C^d(X, \cdot)) \rightarrow F^\times$. For each $W \in Z_1(C^d(X, \cdot))$ the image of the homomorphism $L([D])(W) \in F^\times$ can be calculated in the following way. There is a representative D of the equivalence class $[D]$ for which W and $D \times \square^1$ intersect properly. Then $L([D])(W)$ is the homology class of $\pi_*(W \cdot D \times \square^1) \in C^1(F, 1)$ in $H_1(C^1(F, \cdot)) \simeq F^\times$. This scalar is calculated by intersecting W and $D \times \square^1$ in finitely many points, push-forwarding the intersection onto \square^1 , and finally multiplying the points we get on \square^1 considered as scalars. Because W and $D \times \square^1$ intersect properly, the image $L([D])(W)$ is an invertible scalar in F^\times .

In particular, if $W = \Gamma(g)$ for $g \in k(C)^\times$ and $C \subset X$ a curve, we get

$$L([D])(\Gamma(g)) = g(C \cdot D)$$

Note that as $L([D]) : H_1(C^d(X, \cdot)) \rightarrow F^\times$ for any $W \in B_1(C^d(X, \cdot))$ we have $L([D])(W) = 1$. Hence, for any element $(g_i, C_i) \in B_1(K_\bullet^d(X))$, by the isomorphism $H_1(K_\bullet^d(X)) \simeq H_1(C^d(X, \cdot))$ we get $\sum \Gamma(g_i) \in B_1(C^d(X, \cdot))$ and so $L([D])(\sum \Gamma(g_i)) = 1$. In particular, for each divisor D intersecting each of the graphs $\Gamma(g_i)$ properly, we get

$$1 = L([D]) \left(\sum \Gamma(g_i) \right) = \prod g_i(D \cdot C_i)$$

Consider the diagram

$$\begin{array}{ccc} \text{Pic}(X)[n] & \xrightarrow{\phi^L} & \text{Hom}(Z_1(K_\bullet^d(X)), F^\times) \\ & \searrow \phi & \nearrow \psi \\ & & \text{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), F^\times) \end{array}$$

All homomorphisms exist: ϕ^L by the above discussion, ϕ by Claim 6.4.3, and ψ by the natural homomorphism $Z_1(K_\bullet^d(X)) \rightarrow Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)$. We will show that the homomorphisms

$\phi^L([D]), (\psi \circ \phi)[D] \in \text{Hom}(Z_1(K_{\bullet}^d(X)), F^\times)$ are the same for any isomorphic class $[D] \in \text{Pic}(X)[n]$.

For each element $(C_i, g_i) \in Z_1(K_{\bullet}^d(X))$, by Proposition 6.6.4 the image $(\psi \circ \phi)([D])(C_i, g_i) \in F^\times$ can be calculated as

$$(\psi \circ \phi)([D])(C_i, g_i) = \prod g_i(C_i \cdot D) \in F^\times$$

after choosing a suitable representative D of the class $[D]$. Similarly, for ϕ^L the image $\phi^L([D])(C_i, g_i) \in F^\times$ can be calculated as

$$\phi^L([D])(C_i, g_i) = \prod g_i(C_i \cdot D) \in F^\times$$

after choosing a suitable representative D of the class $[D]$.

Therefore for a divisor D satisfying both sets of conditions the homomorphisms $\phi^L([D]), (\psi \circ \phi)[D]$ agree for each element $(C_i, g_i) \in Z_1(K_{\bullet}^d)$. Hence, they are the same and the diagram commutes. Since the equivalence class $[D] \in \text{Pic}(X)[n]$ was arbitrary we have $\phi^L = \psi \circ \phi$.

Thus using that the group $\text{Pic}(X)[n]$ is n -torsion, we have the commutative diagram

$$\begin{array}{ccc} \text{Pic}(X)[n] & \xrightarrow{L} & \text{Hom}(Z_1(C^d(X, \cdot)), \mu_n) \\ & \searrow \phi & \nearrow \psi \\ & & \text{Hom}(Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n) \end{array}$$

□

Paraphrasing the previous lemma we get the following result.

Corollary 6.7.2. *The homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ descends to $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$. Thus, we have the commutative diagram*

$$\begin{array}{ccc}
\text{Pic}(X)[n] & \xrightarrow{L} & \text{Hom}(H_1(C^d(X, \cdot)), \mu_n) \\
& \searrow \phi & \nearrow \psi \\
& & \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)
\end{array}$$

which shows that the homomorphism L constructed in Section 6.2 factors via the homomorphism ϕ constructed in Proposition 6.4.3 as desired.

Proof. The proof of Lemma 6.7.1 showed that for any $(C_i, g_i) \in B_1(K_\bullet^d(X))$ and any $[D] \in \text{Pic}(X)[n]$, we have $L([D])(\Gamma(g_i)) = \prod g_i(C_i \cdot D) = 1$ for a suitable choice of $D \in [D]$. Note also that as $(C_i, g_i) \in B_1(K_\bullet^d(X))$ we have $\sum \text{div } g_i = 0$ and hence

$$\phi([D])(C_i, g_i) = \frac{\prod g_i(C_i \cdot D)}{h(0)} = \prod g_i(C_i \cdot D) = 1$$

for a suitable choice of $D \in [D]$ and $\text{div } h = nD$. Therefore as $H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \simeq \frac{Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)}{B_1(K_\bullet^d(X))}$, for any equivalence class $[D] \in \text{Pic}(X)[n]$ the homomorphism $\phi([D]) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ descends to $\phi([D]) : H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$.

More generally, the homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ descends to $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$. Combining with Lemma 6.7.1, we get the desired commutative diagram. \square

We can also show that ϕ sends $B_1(K_\bullet^d(X))$ to $1 \in \mu_n$ by hand using Tame symbols. The idea is the Tame symbol on a normal surface, restricted to any curve is very similar to the way we defined the Weil pairing in Proposition 6.8.1.

Claim 6.7.3. Let $Y \rightarrow X$ be a finite etale μ_n -cover corresponding to $[D] \in \text{Pic}(X)[n]$. Then boundary $B_1(K_\bullet^d(X)) \mapsto 1 \in \mu_n$ under the homomorphism $\phi([D]) : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ and hence we have a well-defined map $H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$.

Proof. Any element in $B_1(K_\bullet^d(X))$ is constructed in the following way from Tame symbols of a surface $T \subset X$ with 2 functions $f, g \in k(T)^\times$. Let $n(T)$ be the normalization of the

surface T . Note that since $n(T)$ is a normal surface, it is regular in codimension one and hence the valuations $v_C : k(T) \rightarrow \mathbb{Z}$ are defined for each curve $C \subset n(T)$. Then the Tame symbol map

$$(f, g) \mapsto T_C\{f, g\} = (-1)^{v_C(f)v_C(g)} \frac{f^{v_C(g)}}{g^{v_C(f)}} \in k(C)^\times$$

is also defined.

For each pair of functions $(f, g) \in k(T)^\times \times k(T)^\times$ there are only finitely many curves C_i such that $T_{C_i}\{f, g\} \neq 1$. Let $v : n(T) \rightarrow T$ be the normalization $n(T)$ of T .

Then the element of $B_1(K_\bullet^d(X))$ corresponding to the functions f and g is the collection $(v(C_i), T_i)$, where T_i are the functions on the curves $v(C_i)$ on the surface T corresponding to the functions $T_{C_i}\{f, g\}$ on the curves C_i on the normalization $n(T)$ of the surface. The idea of the proof is to choose such a representative $D \in [D]$ such that both of the following statements are well-defined and hence true

1. $T_{C_i}\{f, g\}(C_i \cdot v^*D) = T_i(v(C_i) \cdot D) = \phi([D])(v(C_i), T_i)$ i.e. we can work on the normal surface $n(T)$ instead of the potentially very singular surface T
2. $\prod_i T_{C_i}\{f, g\}(C_i \cdot v^*D) = \prod_i T_{v^*(D)}\{f|_{v^*(D)}, g|_{v^*(D)}\}(C_i \cdot v^*D) = 1$ where $T_{v^*(D)}\{f|_{v^*(D)}, g|_{v^*(D)}\} \in k(v^*D)^\times$ is the Weil symbol on the curve v^*D for the functions $f|_{v^*(D)}, g|_{v^*(D)}$.

Combining the two statements we will get the desired statement that $\phi([D])(v(C_i), T_i) = \prod_i T_i(v(C_i) \cdot D) = 1$ for any element $(v(C_i), T_i) \in B_1(K_\bullet^d(X))$.

Since X is a smooth variety, the divisor D is Cartier and hence we can choose a representative $D \in [D]$ such that D restricts to a Cartier divisor (W_k, f_k) on the surface T . Then we can pullback D as a Cartier divisor $(v^{-1}W_k, f_k \circ v)$ to the normalization $n(T)$. Since $n(T)$ is regular in codimension 1, the pullback v^*D is a Weil divisor as well. By the usual moving lemma, there is a representative $D \in [D]$ such that both D and the pullback v^*D miss finitely many undesirable points. Thus there is a representative $D \in [D]$ which intersects each of the curves $v(C_i)$ transversely i.e. $D \cap v(C_i) = \{P\}$ and $\text{mult}_P D \cdot v(C_i) = 1$. By

Proposition 8.2(c) of [9] this means that P is a smooth point of both D and $v(C_i)$. We can also arrange for each $i \neq j$ to have $D \cap v(C_i) \cap v(C_j) = \emptyset$. Similarly, we can also arrange for each $i \neq j$ to have $v^*D \cap C_i \cap C_j = \emptyset$. Finally, we can also arrange that v^*D misses the finitely many singular points of $n(T)$.

The normalization $v : n(T) \rightarrow T$ is an isomorphism on a dense open subset $U \subset n(T)$. Denote the complement by $Z = n(T) - U$. For any of the curves C_i not contained in Z , the corresponding curve $v(C_i)$ on T is birational to the curve C_i and we can arrange that D intersects the curves $v(C_i)$ only at points $P \in v(C_i)$ for which $v^{-1}(P) \subset U$. Then the function T_i on $v(C_i)$ is the same as the function $T_{C_i}\{f, g\}$ on C_i . Combining we get that the curves v^*D and C_i intersect transversely at smooth points on v^*D and moreover, $T_i(v(C_i) \cdot D) = T_{C_i}\{f, g\}(C_i \cdot v^*D)$.

Let us now consider the curves C_i contained in Z . The map $v : n(T) \rightarrow T$ is finite and hence $v_*C_i = d_i v(C_i)$ where $d_i = \deg k(C_i)/k(v(C_i))$. Moreover, the functions T_i on $v(C_i)$ are the norms $\text{Norm}_{k(C_i)/k(v(C_i))} T_{C_i}\{f, g\}$. Each of the curves $v(C_i)$ has a dense open subset U_i such that each point $x \in U_i$ has exactly d_i preimages under $v : C_i \rightarrow v(C_i)$. We can arrange that D intersects each of the curves $v(C_i)$ transversally and the points of intersection lie inside the open subsets U_i . Using that $T_i = \text{Norm}_{k(C_i)/k(v(C_i))} T_{C_i}\{f, g\}$, by Claim 6.6.3 for any point P in the intersection $C_i \cap D$ we have $T_i(P) = T_{C_i}\{f, g\}(v^*P) = \prod_{j=1}^{d_i} T_{C_i}\{f, g\}(Q_j)$ where Q_j are the preimages of P . Here we apply Claim 6.6.3 to the smooth parts of the curves C_i and $v(C_i)$.

Moreover, by the projection formula for $v : n(T) \rightarrow T$ we have $v_*(C_i \cdot v^*D) = (v_*C_i) \cdot D = d_i(v(C_i) \cdot D)$. Let us consider the contribution of the intersection point $P \in v(C_i) \cdot D$ to the equality. Since D intersects each curve $v(C_i)$ transversally on the right hand we have P with multiplicity d_i . Note that by construction of the pullback, the Weil divisor v^*D has a support the whole preimage $v^{-1}(D)$. Then in $C_i \cdot v^*D$ we have all d_i preimages Q_j of P with coefficients at least 1. Therefore as $v_*Q_j = P_i$, to get equality we need the coefficients of all d_i points Q_j to be exactly 1. This means that $v^*(v(C_i) \cdot D) = Q_1 + Q_2 + \dots + Q_{d_i} = C_i \cdot v^*D$,

where on the left hand side we pullback along v restricted to the smooth parts of the curves C_i and $v(C_i)$. By Proposition 8.2(c) of [9] intersection with multiplicity 1 implies that Q_j is a regular point on v^*D and C_i . Combining with $T_i(P) = T_{C_i}\{f, g\}(v^*P)$, once again we get $T_i(v(C_i) \cdot D) = T_{C_i}\{f, g\}(C_i \cdot v^*D)$.

On the other hand, the formula

$$v_Q(g|_{v^*(D)}) = v_{C_i}(g) \text{mult}_Q(v^*(D) \cdot C_i)$$

follows from 2.12 of [13] since $v^*(D) \cap C_i \cap C_j = \emptyset$ for $i \neq j$ and v^*D and C_i intersect only at smooth points of $n(T)$. Then combining the two cases for curves C_i contained or not in Z , we get that for each curve C_i we have $v_{Q_j}(g|_{v^*(D)}) = v_{C_i}(g)$ for any point Q_j in the intersection $v^*D \cap C_i$. This allows us to relate the Tame symbol for f, g on the normalized surface $n(T)$ and the Weil symbol for $f|_{v^*(D)}, g|_{v^*(D)}$ on the curve v^*D .

Hence,

$$\begin{aligned} T_{C_i}\{f, g\}(C_i \cdot v^*(D))|_{Q_j} &= (-1)^{v_{C_i}(f)v_{C_i}(g)} \frac{f^{v_{C_i}(g)}}{g^{v_{C_i}(f)}}(Q_j) = \\ (-1)^{v_{Q_j}(f|_{v^*(D)})v_{Q_j}(g|_{v^*(D)})} &\frac{f^{v_{Q_j}(g|_{v^*(D)})}}{g^{v_{Q_j}(f|_{v^*(D)})}}(Q_j) = T_{v^*(D)}\{f|_{v^*(D)}, g|_{v^*(D)}\}(Q_j) \end{aligned}$$

Therefore

$$\begin{aligned} \prod_{C \subset n(T)} T_C\{f, g\}(C \cdot v^*(D)) &= \prod_{C_i} T_{C_i}\{f, g\}(C_i \cdot v^*(D)) = \\ \prod_{Q_j} T_{v^*(D)}\{f|_{v^*(D)}, g|_{v^*(D)}\}(Q_j) &= \prod_{Q \in v^*(D)} T_{v^*(D)}\{f|_{v^*(D)}, g|_{v^*(D)}\}(Q) = 1 \end{aligned}$$

For the last equality we use Weil reciprocity for the curve $v^*(D)$. Weil reciprocity holds for the potentially singular curve v^*D because $D(f|_{v^*(D)}) \cup D(g|_{v^*(D)}) \subset \cup_i (C_i \cap v^*D)$ is contained in the smooth part of the curve v^*D by the choice of divisor D .

Overall, we get

$$\prod_i T_i(v(C_i) \cdot D) = \prod_i T_{C_i}\{f, g\}(C_i \cdot v^*(D)) = 1$$

□

6.8 The case of smooth projective curves

In the previous subsection we showed that we have a well-defined homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$. In this subsection we will show that when X is a smooth projective curve then $H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \simeq \text{Pic}(X)[n]$ and the induced homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(\text{Pic}(X)[n], \mu_n)$ is the one coming from the Weil pairing. For this let us recall the following formulation of the Weil pairing.

Proposition 6.8.1. *For a smooth projective curve C over an algebraically closed field F the Weil pairing $\text{Pic}(C)[n] \times \text{Pic}(C)[n] \rightarrow \mu_n(F)$ can be expressed in the following way. Let $D \in [D]$ and $E \in [E]$ be representatives of the equivalence classes $[D], [E] \in \text{Pic}(C)[n]$ and hence there are rational functions f, g on C such that $\text{div } h = nD$ and $\text{div } g = nE$. Then under the Weil pairing the pair of equivalence classes $([D], [E])$ goes to*

$$\prod_P (-1)^{n(\text{ord}_P D)(\text{ord}_P E)} \frac{g^{\text{ord}_P E}}{h^{\text{ord}_P D}}(P)$$

where P ranges over all geometric points of C . Using Weil reciprocity, when D and E have disjoint support, the above expression can be rewritten as $(D, E) \mapsto \frac{g(D)}{h(E)}$.

Proof. This is Theorem 1.1 in [14]. □

Proposition 6.8.1 implies that for any smooth projective curve C there is an isomorphism $\Phi^W : \text{Pic}(C)[n] \xrightarrow{\cong} \text{Hom}(\text{Pic}(C)[n], \mu_n)$ such that for any equivalence class $[D] \in \text{Pic}(C)[n]$ the homomorphism $\Phi^W([D]) \in \text{Hom}(\text{Pic}(C)[n], \mu_n)$ can be calculated in the following way.

Fix a representative $D \in [D]$. Then for any equivalence class $[E] \in \text{Pic}(C)[n]$ choose any representative $E \in [E]$ with support disjoint from D . Hence $\Phi^W([D])([E]) = \frac{g(D)}{h(E)}$ where $\text{div } g = nE$ and $\text{div } h = nD$ for some functions g and h .

The resulting scalar $\Phi^W([D])([E]) = \frac{g(D)}{h(E)}$ in μ_n is independent of the choices of representatives D, E and the choices of functions g and h . To check that $\frac{g(D)}{h(E)} \in \mu_n(F)$ note

$$g(D)^n = g(nD) = g(\text{div } h) = h(\text{div } g) = h(nE) = h(E)^n$$

For the middle equality we used the Weil reciprocity for the smooth curve C .

Before we show that the isomorphism Φ^W coincides with the homomorphism ϕ when X is a curve, we need the following easy claim.

Claim 6.8.2. There is a natural isomorphism $Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n) \rightarrow \text{Pic}(X)[n]$ for any smooth projective curve X over an algebraically closed field.

Proof. The isomorphism follows from the exact sequence

$$0 \rightarrow k(X)^\times \otimes \mathbb{Z}/n \rightarrow B \rightarrow \text{Pic}(X)[n] \rightarrow 0$$

Here B is the group of rational functions g on X such that $\text{div } g = nE$ for some 0-cycle E .

The group homomorphism $B \rightarrow \text{Pic}(X)[n]$ is defined as $g \mapsto [E]$. By the definition of $\text{Pic}(X)[n]$, the homomorphism is surjective. It remains to determine the kernel. The kernel consists of functions g of the type $\text{div } g = nE$ for E a rational divisor. Then $\text{div } g = nE = n \text{div } f = \text{div } f^n$ for a rational function f . Thus, g and f^n are two rational functions with the same divisor. Since X is a smooth projective curve, $\frac{g}{f^n}$ is an invertible regular function on X and hence a constant $k \in F^\times$. Nevertheless, as F is an algebraically closed field, there is $t \in F$ for which $t^n = k$ and hence $g = (ft)^n \in k(X)^\times \otimes \mathbb{Z}/n$. Moreover, clearly the subgroup $k(X)^\times \otimes \mathbb{Z}/n$ of B is in the kernel of the homomorphism $B \rightarrow \text{Pic}(X)[n]$. Thus, $\text{Pic}(X)[n] \simeq B / (k(X)^\times \otimes \mathbb{Z}/n) = Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n)$. \square

We are finally ready to show that $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ is isomorphism in the special case when X is a curve.

Proposition 6.8.3. *Let X be a smooth projective irreducible curve over algebraically closed field F . Then the homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ is an isomorphism coming from the Weil pairing isomorphism $\Phi^W : \text{Pic}(X)[n] \xrightarrow{\cong} \text{Hom}(\text{Pic}(X)[n], \mu_n)$.*

Proof. Note that when X is a curve, $K_2^1(X) = 0$ and hence $H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) = Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n)$.

Using the Weil pairing isomorphism $\Phi^W : \text{Pic}(X)[n] \xrightarrow{\cong} \text{Hom}(\text{Pic}(X)[n], \mu_n)$ and the isomorphism $\text{Pic}(X)[n] \simeq Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n)$, we get an isomorphism $\phi^W : \text{Pic}(X)[n] \xrightarrow{\cong} \text{Hom}(Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n), \mu_n)$. By Proposition 6.8.1, the homomorphism $\phi^W([D]) : Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ can be calculated in the following way for any equivalence class $[D] \in \text{Pic}(X)[n]$. Given a function $g \in Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n)$ with $\text{div } g = nE$, choose any representative $D \in [D]$ with support disjoint from E . Then $\text{div } h = nD$ for some function h and hence $\phi^W([D])(g) = \frac{g(D)}{h(E)}$.

On the other hand, in Proposition 6.4.3 we also constructed a homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n), \mu_n)$. By Proposition 6.6.4 the homomorphism ϕ has the property that for any equivalence class $[D] \in \text{Pic}(X)[n]$ and a function $g \in Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n)$ with $\text{div } g = nE$, there is a nice representative $D \in [D]$ with $\text{div } h = nD$ such that $\phi([D])(g) = \frac{g(D)}{h(E)}$.

This shows that for any equivalence class $[D] \in \text{Pic}(X)[n]$ the homomorphisms $\phi^W([D])$ and $\phi([D])$ coincide. Hence, ϕ and ϕ^W coincide as homomorphisms $\text{Pic}(X)[n] \rightarrow \text{Hom}(Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n), \mu_n)$. Since ϕ^W is an isomorphism, so is ϕ . Thus, when X is of dimension 1, we get an isomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(Z_1(K_\bullet^1(X) \otimes \mathbb{Z}/n), \mu_n)$. Naturally, this becomes an isomorphism $\phi : H^1(X, \mu_n) \xrightarrow{\cong} \text{Hom}(H_1(K_\bullet^1(X) \otimes \mathbb{Z}/n), \mu_n)$. \square

6.9 Functoriality and the proof of ϕ is isomorphism

In this subsection we will generalize the result of the previous subsection and show that the homomorphism $\phi : H^1(X, \mu_n) = \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_{\bullet}^d \otimes \mathbb{Z}/n), \mu_n)$ is isomorphism for a variety X of an arbitrary dimension. We already know that ϕ is a homomorphism. Hence, we only need to show that ϕ is bijective. In the following proposition, we will show that ϕ is injective by using Lefschetz hyperplane theorem and the result from the previous subsection.

Proposition 6.9.1. *For a smooth projective variety X of positive dimension over algebraically closed field F , the homomorphism $\phi : H^1(X, \mu_n) = \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ is injective.*

Proof. In the general case, when X is not necessarily a curve, the Lefschetz hyperplane section theorem implies that $H^1(X, \mu_n) \hookrightarrow H^1(Y, \mu_n)$ for any smooth hyperplane $Y \subset X$ of positive dimension. When $\dim X \geq 2$, a general hyperplane Y is both irreducible and smooth. Hence, after consecutive intersecting with general hyperplanes, we can produce a smooth irreducible complete intersection curve $C \subset X$ with the property that $H^1(X, \mu_n) \hookrightarrow H^1(C, \mu_n)$.

By Proposition 6.8.3 for the smooth projective curve C we have an isomorphism $H^1(C, \mu_n) \xrightarrow{\cong} \text{Hom}(H_1(K_{\bullet}^1(C) \otimes \mathbb{Z}/n), \mu_n)$. Furthermore, we have a natural diagram:

$$\begin{array}{ccc} H^1(C, \mu_n) & \xrightarrow{\phi^C} & \text{Hom}(H_1(K_{\bullet}^1(C) \otimes \mathbb{Z}/n), \mu_n) \\ \uparrow & & \uparrow \\ H^1(X, \mu_n) & \xrightarrow{\phi^X} & \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n) \end{array}$$

The diagram is commutative for the following reasons. Any μ_n -cover of X induces a μ_n -cover of the curve $C \subset X$. For this curve C any element $g \in H_1(K_{\bullet}^1(C) \otimes \mathbb{Z}/n)$ has the same image in μ_n under both the homomorphism in $\text{Hom}(H_1(K_{\bullet}^1(C) \otimes \mathbb{Z}/n), \mu_n)$ corresponding to the induced cover of C when g is considered an element of $(H_1(K_{\bullet}^1(C) \otimes \mathbb{Z}/n))$, and by the homomorphism in $\text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ corresponding to the cover of X when (C, g) is considered an element of $H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n)$. This follows by construction of the homomorphism κ in Proposition 6.4.3.

By commutativity of the diagram, the homomorphism

$$H_{et}^1(X, \mu_n) \hookrightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$$

is injective. □

It remains to show that the homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ is surjective. We will get surjectivity because we work with finite groups of the same order, as shown in the following proposition.

Proposition 6.9.2. *The homomorphism*

$$H_{et}^1(X, \mu_n) \hookrightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$$

is an isomorphism. By duality, we also get an isomorphism

$$H_1(K_{\bullet}^d \otimes \mathbb{Z}/n) \xrightarrow{\cong} \pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n)$$

Proof. We have already proved injectivity of $H_{et}^1(X, \mu_n) \hookrightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$. It remains to prove surjectivity.

By [25] we know that the groups $H_{et}^1(X, \mathbb{Z}/n)$ and $H^1(\text{Hom}(\text{Sus}_{\bullet}(X), \mathbb{Z}/n)) = \text{Hom}(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n, \mathbb{Z}/n)$ are isomorphic. The universal coefficients theorem gives us the exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}/n}^1(H_0(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n), \mathbb{Z}/n) \rightarrow H^1(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n) \rightarrow \text{Hom}(H_1(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n), \mathbb{Z}/n) \rightarrow 0$$

where by definition $H^1(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n) = H^1(\text{Hom}(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n, \mathbb{Z}/n))$.

Since \mathbb{Z}/n is an injective module over itself, the first term $\text{Ext}_{\mathbb{Z}/n}^1(H_0(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n), \mathbb{Z}/n)$ in the short exact sequence vanishes. Thus, $H^1(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n) \xrightarrow{\cong}$

$\text{Hom}(H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), \mathbb{Z}/n)$. Therefore, the groups

$$H_{et}^1(X, \mu_n) \simeq H^1(X, \mathbb{Z}/n) \simeq H^1(\text{Hom}(\text{Sus}_\bullet(X), \mathbb{Z}/n)) \simeq \text{Hom}(H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), \mu_n)$$

are isomorphic.

By Corollary VI.2.8 of [20], the etale cohomology groups $H_{et}^1(X, \mu_n)$ are finite for X over separably closed field F . Then $H_{et}^1(X, \mu_n)$ and $\text{Hom}(H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), \mu_n)$ are finite groups of the same order since they are isomorphic. Since $H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n) \simeq H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n)$ by Lemma 6.3.5, the group $\text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ is finite of the same order as $\text{Hom}(H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), \mu_n)$.

Combining, we get that the homomorphism $H_{et}^1(X, \mu_n) \hookrightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ we constructed is an injective homomorphism between finite groups of the same order and hence an isomorphism. Thus

$$\phi : \text{Hom}(\pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n), \mu_n) \simeq H_{et}^1(X, \mu_n) \xrightarrow{\cong} \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$$

By duality, we also get an isomorphism

$$\phi^* : H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \xrightarrow{\cong} \pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n)$$

□

Remark 6.9.3. Note that because we had to compose $\kappa : \text{Hom}(\pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n), \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ with the automorphism $\chi : \mu_n \rightarrow \mu_n$ given by $\tau \mapsto \tau^{-1}$, the isomorphism $\phi^* : H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \xrightarrow{\cong} \pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n)$ is the homomorphism $\kappa(Y) : H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n)$ composed with the automorphism $\pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n) \rightarrow \pi_1^{ab}(X)^{ab}/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n)$ given by $\sigma \mapsto \sigma^{-1}$. Here as before Y is the maximal unramified cover of X with finite abelian group $\pi_1^{ab}(X)/(\pi_1^{ab}(X) \otimes \mathbb{Z}/n)$.

6.10 The homomorphism κ coincides with the Suslin-Voevodsky isomorphism

Note the homomorphism κ coincides with the homomorphism $H_{et}^i(X, \mathbb{Z}/n) \rightarrow \text{Hom}(H_i(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n), \mu_n)$ constructed by Suslin and Voevodsky in [25] for $i = 1$ when X is nice as proved in the following proposition. Thus, the Kummer version of the homomorphism κ gives an explicit expression of the Suslin-Voevodsky isomorphism under nice conditions.

Proposition 6.10.1. *Let X be a smooth projective variety over an algebraically closed field F of characteristic coprime to n . The homomorphism $\kappa : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ coincides with homomorphism $\text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ defined by Suslin and Voevodsky in [25].*

Proof. In [11] Geisser and Schmidt defined a homomorphism $rec : H_1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n) \rightarrow \pi_1^{ab}(X) \otimes \mathbb{Z}/n$ and proved that it is dual to homomorphism $H^1(X, \mathbb{Z}/n) \rightarrow H^1(\text{Sus}_\bullet(X) \otimes \mathbb{Z}/n)$ defined by Suslin and Voevodsky in [25]. We will show that the homomorphism $\kappa : \text{Hom}(\pi_1^{ab}(X) \otimes \mathbb{Z}/n, \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ coincides with the dual homomorphism.

Let us first consider the case of a smooth projective curve X . By [14] the Weil pairing is induced by Kummer theory from the following diagram

$$\begin{array}{ccc} Y & \longrightarrow & \text{Jac}(X) \\ \downarrow & & \downarrow^{\times n} \\ X & \longrightarrow & \text{Jac}(X) \end{array}$$

Here $\text{Jac}(X)$ is the Jacobian of the smooth projective curve X . The right vertical arrow is multiplication by n and hence a finite etale cover with Galois group $\text{Jac}(X)[n]$. The homomorphism $X \rightarrow \text{Jac}(X)$ is the Abel-Jacobi homomorphism and Y is the variety that

forms the Cartesian square. Note that Y is the maximal unramified cover of X with n -torsion Galois group G . Thus we have an isomorphism $t : \text{Jac}(X)[n] \rightarrow G$ as translation by an element of $\text{Jac}(X)[n]$ induces an automorphism of Y over X . Moreover, the function field $k(Y)$ is constructed from the function field $k(X)$ by adjoining all functions $f \in k(X)$ such that $\text{div } f = nE$ for some zero cycle E , which are not n -th power of functions on X . This means that $k(Y)$ is constructed from adjoining to $k(X)$ the elements of $Z_1(K_\bullet^d(X))$. Recall we have an isomorphism $\text{div} : Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \text{Pic}(X)[n]$ given by $g \mapsto E$ where $g \in k(X)^\times$ and $\text{div } g = nE$.

In op.cit Howe showed that the Weil pairing $\text{Jac}(X)[n] \times \text{Jac}(X)[n] \rightarrow \mu_n$ using the isomorphisms div and t has an expression of Kummer theory $G \times Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ under which $(\sigma, g) \mapsto \frac{\sigma(g^{1/n})}{g^{1/n}}$. Moreover, if $\text{div } g = nE$ and $t^{-1}(\sigma) = [D]$ then the Weil pairing gives us $\frac{\sigma(g^{1/n})}{g^{1/n}} = \frac{g(D)}{h(E)}$. Here $h \in k(X)^\times$ such that $\text{div } h = nD$.

$$\begin{array}{ccc} \text{Jac}(X)[n] \times & \text{Jac}(X)[n] & \longrightarrow \mu_n \\ t \downarrow & \uparrow \text{div} & = \downarrow [F][a'] \\ G & \times Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) & \longrightarrow \mu_n \end{array}$$

Then the Weil pairing gives us the following isomorphism

$$\text{Jac}(X)[n] \xleftarrow{\text{div}} Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \text{Hom}(G, \mu_n) \xrightarrow{t} \text{Hom}(\text{Jac}(X)[n], \mu_n)$$

under which $[D] \mapsto \left([E] \mapsto \frac{h(E)}{g(D)} \right)$. The homomorphism $Z_1(K_\bullet^d(X) \otimes \mathbb{Z}/n) \rightarrow \text{Hom}(G, \mu_n)$ given by $h \mapsto \left(\sigma \mapsto \frac{\sigma(h^{1/n})}{h^{1/n}} \right)$ is the one induced by the pairing. In Proposition 6.6.4 we showed that for μ_n -cover Z corresponding to the equivalence class $[D] \in \text{Pic}(X)[n]$ we have $\kappa(Z)(g) = \frac{h(E)}{g(D)}$ as well. Thus we similarly get an isomorphism $\mathcal{K} :$

$$\text{Jac}[n] \xleftarrow{\text{div}} Z_1(K_\bullet^d \otimes \mathbb{Z}/n) \rightarrow \text{Hom}(G, \mu_n) \xrightarrow{\kappa} \text{Hom}(Z_1(K_\bullet^d \otimes \mathbb{Z}/n), \mu_n) \xleftarrow{\text{div}} \text{Hom}(\text{Jac}[n], \mu_n)$$

which agrees with the one from the Weil pairing. Then the isomorphism $\text{Hom}(G, \mu_n) \xrightarrow{t}$ $\text{Hom}(\text{Jac}(X)[n], \mu_n)$ is the same as the isomorphism $\text{Hom}(G, \mu_n) \xrightarrow{\kappa} \text{Hom}(Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n) \xleftarrow{\text{div}} \text{Hom}(\text{Jac}(X)[n], \mu_n)$.

In [11] Geisser and Schmidt using the homomorphism *rec* constructed a perfect pairing $H_1(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n) \times H^1(X, \mathbb{Z}/n) \rightarrow \mu_n$ which under the identifications $H_1(\text{Sus}_{\bullet}(X) \otimes \mathbb{Z}/n) = Z_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \xrightarrow{\text{div}} \text{Jac}(X)[n]$ and $H^1(X, \mathbb{Z}/n) = \text{Hom}(G, \mu_n) \xrightarrow{t} \text{Hom}(\text{Jac}(X)[n], \mu_n)$ becomes the evaluation pairing $\text{Jac}(X)[n] \times \text{Hom}(\text{Jac}(X)[n], \mu_n) \rightarrow \mu_n$. The pairing induces the identity isomorphism $\text{Hom}(\text{Jac}(X)[n], \mu_n) \xrightarrow{id} \text{Hom}(\text{Jac}(X)[n], \mu_n)$. Thus, we get an isomorphism *SV* :

$$\text{Jac}(X)[n] \xleftarrow{\text{div}} Z_1(K_{\bullet}^d \otimes \mathbb{Z}/n) \rightarrow \text{Hom}(G, \mu_n) \xrightarrow{t} \text{Hom}(\text{Jac}(X)[n], \mu_n) \xrightarrow{=} \text{Hom}(\text{Jac}(X)[n], \mu_n)$$

Thus using that $t = \text{div}^{-1} \circ \kappa$ we get that the isomorphism coincides with the one we constructed from Proposition 6.6.4.

The case of a general smooth projective variety *X* follows from the case of a smooth curve for the following reason. Let *C* be a smooth complete intersection curve on *X*. Then we have an injection $\text{Pic}(X)[n] \hookrightarrow \text{Pic}(C)[n]$ which by functoriality fits into the diagram

$$\begin{array}{ccc} \text{Pic}(C)[n] & \xrightarrow{\cong} & \text{Hom}(\text{Pic}(C)[n], \mu_n) \\ \uparrow & & \uparrow \\ \text{Pic}(X)[n] & \xrightarrow{\cong} & \text{Hom}(\text{Pic}(X)[n], \mu_n) \end{array}$$

We know that Suslin-Voevodsky isomorphism *SV* also gives an analogous commutative diagram and for curves the homomorphisms \mathcal{K}_C and SV_C coincide. Hence, by commutativity, they coincide for a general smooth projective variety *X*. □

6.11 The proof of the main theorem

In the previous subsection we finally showed the homomorphism $\phi : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)$ is an isomorphism and composed with the natural homomorphism $H_1(K_{\bullet}^d(X)) \rightarrow H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n)$ coincides with the homomorphism $\phi^L : \text{Pic}(X)[n] \rightarrow \text{Hom}(H_1(K_{\bullet}^d(X)), \mu_n)$. The homomorphism $H_1(K_{\bullet}^d(X)) \rightarrow H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n)$ fits into the short exact sequence of the Universal coefficients theorem applied to $H_1(K_{\bullet}^d(X))$. This fact combined with some homological algebra and Roitman's theorem will give us that the finite $\ker \phi^L$ has the same size as $\text{Pic}_{\text{alg}}(X)[n]$ which we already know is in the kernel. The proof is given in the next few easy lemmas.

Lemma 6.11.1. *For a given nontrivial element $[D] \in \text{Pic}(X)[n]$, the homomorphism $\phi([D]) : H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ is trivial if and only if $\phi([D]) : H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ factors via $\text{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n) \rightarrow \mu_n$*

Proof. From the sequence of isomorphisms $\text{Pic}(X)[n] \simeq H^1(X, \mu_n) \simeq \text{Hom}(H_1(K_{\bullet}^d \otimes \mathbb{Z}/n), \mu_n)$ we get a nontrivial element $\xi_{[D]} \in \text{Hom}(H_1(K_{\bullet}^d \otimes \mathbb{Z}/n), \mu_n)$ corresponding to the nontrivial $[D] \in \text{Pic}(X)[n]$.

Moreover, by Lemma 6.7.1 we have a commutative diagram

$$\begin{array}{ccc}
 \text{Pic}(X)[n] & \xrightarrow{\phi^L} & \text{Hom}(H_1(K_{\bullet}^d(X)), \mu_n) \\
 & \searrow \phi & \nearrow \psi \\
 & & \text{Hom}(H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n), \mu_n)
 \end{array}$$

The natural homomorphism $H_1(K_{\bullet}^d(X)) \rightarrow H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n)$ inducing ψ is the one coming from the sequence

$$H_1(K_{\bullet}^d(X)) \xrightarrow{n} H_1(K_{\bullet}^d(X)) \rightarrow H_1(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow \text{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n) \rightarrow 0$$

The sequence is exact by Corollary 6.3.5.

Thus, we have the exact sequence

$$0 \rightarrow \text{Hom}(\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n), \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X)), \mu_n)$$

By exactness, $\ker(\text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X)), \mu_n)) = \text{im}(\text{Hom}(\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n), \mu_n) \rightarrow \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n))$. Therefore, $\xi_{[D]} \in \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ becomes trivial in $\text{Hom}(H_1(K_\bullet^d(X)), \mu_n)$ if and only if $\xi_{[D]}$ factors via $\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n)$. \square

Proposition 6.11.2. *Using the notation of Lemma 6.11.1, for a given $[D] \in \text{Pic}(X)[n]$ the homomorphism $\xi_{[D]} \in \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n)$ factors through $\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n)$ if and only if $[D]$ is algebraically equivalent to zero.*

Proof. Denote by FT the set of equivalence classes $[D] \in \text{Pic}(X)[n]$ for which the corresponding homomorphism $\xi_{[D]} : H_1(K_\bullet^d(X)) \rightarrow \mu_n$ is trivial. By Lemma 6.11.1 we know the homomorphism $\xi_{[D]}$ is trivial exactly when it factors as a homomorphism $\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n) \rightarrow \mu_n$. Thus we have the following commutative diagram

$$\begin{array}{ccc} \text{Pic}(X)[n] & \xrightarrow{\cong} & \text{Hom}(H_1(K_\bullet^d(X) \otimes \mathbb{Z}/n), \mu_n) \\ \uparrow & & \uparrow \\ FT & \xrightarrow{\cong} & \text{Hom}(\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n), \mu_n) \end{array}$$

From Proposition 5.0.2 we know that $\text{Pic}_{alg}(X)[n]$ is a subgroup of the group FT of equivalence classes of divisors for which the corresponding homomorphisms $\xi_{[D]}$ are trivial. In the next claim we will show that the inclusion $\text{Pic}_{alg}(X)[n] \hookrightarrow FT$ is actually an isomorphism.

Claim 6.11.3. The injective homomorphism $\text{Pic}_{alg}(X)[n] \rightarrow \text{Hom}(\text{Tor}_1(H_0(K_\bullet^d(X)), \mathbb{Z}/n), \mu_n)$ is an isomorphism.

Proof. There is a canonical isomorphism $\mathrm{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n) \simeq H_0(K_{\bullet}^d(X))[n]$ i.e. the n -torsion in $H_0(K_{\bullet}^d(X))$. By Corollary 6.3.5 and [5] for smooth projective X over algebraically closed field F , we have $H_0(K_{\bullet}^d(X)) \simeq H_0(C^d(X, \cdot)) \simeq \mathrm{CH}^d(X)$, and so

$$\mathrm{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n) \simeq \mathrm{CH}^d(X)[n]$$

Since F is algebraically closed and $\gcd(\mathrm{char}(F), n) = 1$ by Roitman's theorem proved in [3] we have an isomorphism on torsion $\mathrm{CH}^d(X)[n] \simeq \mathrm{Alb}(X)[n]$. Also by Proposition on p.64 of [23] since $\mathrm{Alb}(X)$ is an abelian variety and $\gcd(\mathrm{char}(F), n) = 1$, the n -torsion $\mathrm{Alb}(X)[n]$ is a finite group and in particular isomorphic to $(\mathbb{Z}/n)^{2 \dim \mathrm{Alb}(X)}$. Combining we have

$$\mathrm{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n) \simeq \mathrm{Alb}(X)[n] \simeq (\mathbb{Z}/n)^{2 \dim \mathrm{Alb}(X)}$$

Hence

$$\mathrm{Hom}(\mathrm{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n), \mu_n) \simeq \mathrm{Hom}((\mathbb{Z}/n)^{2 \dim \mathrm{Alb}(X)}, \mu_n) \simeq \mathbb{Z}/n^{2 \dim \mathrm{Alb}(X)}$$

Similarly, $\mathrm{Pic}_{alg}(X)$ is an abelian variety. Then again by Proposition on p.64 of [23] since $\gcd(\mathrm{char}(F), n) = 1$, the n -torsion $\mathrm{Pic}_{alg}(X)[n]$ is isomorphic to

$$(\mathbb{Z}/n)^{2 \dim \mathrm{Pic}_{alg}(X)}$$

Furthermore, we know that $\mathrm{Pic}_{alg}(X)$ and $\mathrm{Alb}(X)$ are dual abelian varieties and hence of the same dimension.

Thus, the injective homomorphism

$$\mathrm{Pic}_{alg}(X)[n] \rightarrow \mathrm{Hom}(\mathrm{Tor}_1(H_0(K_{\bullet}^d(X)), \mathbb{Z}/n), \mu_n)$$

becomes an injective homomorphism between finite groups of the same order

$(\mathbb{Z}/n)^{2\text{Pic}_{alg}(X)} \hookrightarrow (\mathbb{Z}/n)^{2\dim\text{Alb}(X)}$ and hence an isomorphism. This means that $\text{Pic}_{alg}(X)[n]$ is exactly the whole group of equivalence classes of divisors $[D]$ for which the composition homomorphism $H_0(K_{\bullet}^d(X)) \rightarrow H_0(K_{\bullet}^d(X) \otimes \mathbb{Z}/n) \rightarrow \mu_n$ is trivial. \square

\square

Finally we are ready to finish the proof of the main Theorem 6.0.2.

Theorem 6.11.4 (=Theorem 6.0.2). *Let F be algebraically closed field and M a numerically trivial divisor on a smooth projective variety X over F . If $\text{char}F = 0$, then the functor $L(\cdot, M)$ factors through $\text{Cat}(\overline{CH^p}(X))$ only if M is algebraically trivial. If $\text{char}F = p > 0$, by Lemma 6.0.1 we can express the equivalence class $[M]$ as $[M] = [E] + [T]$ where $[E] \in \text{Pic}_{alg}(X)$ and $[T]$ is a torsion element of $\text{Pic}(X)$ of order m such that $t[T] \notin \text{Pic}_{alg}(X)$ for any $0 < t < m$. Express $m = p^k n$ where $\text{gcd}(n, p) = 1$, then the functor $L(\cdot, M)$ does not factor through $\text{Cat}(\overline{CH^p}(X))$ if and only if $n \geq 2$ i.e. $m \neq p^k$.*

Proof. Let M be a numerically trivial divisor. By Lemma 6.0.1 we know we can express the equivalence class $[M] \in \text{Pic}(X)$ as $[M] = [E] + [T]$ where $[E]$ is an equivalence class of algebraically trivial divisors and $[T]$ is a torsion element of $\text{Pic}(X)$ of order m such that $t[T] \notin \text{Pic}_{alg}(X)$ for any $0 < t < m$. Note that by Proposition 6.11.2 the homomorphism $L(\cdot, M) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ depends only on the equivalence class $[M]$ and hence there is a well-defined homomorphism $L(\cdot, [M]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$. Again by Proposition 6.11.2, the homomorphism $L(\cdot, [M]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ agrees with the homomorphism $L(\cdot, [T]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$.

If $\text{char}(F) = 0$, then as long as $[T] \neq 0 \in \text{Pic}(X)$, the equivalence class $[T]$ is of positive order coprime to characteristic of the field F . Hence, by Lemma 6.11.1 and Proposition 6.11.2, the homomorphism $L(\cdot, [T]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ is not trivial. Thus, the homomorphism $L(\cdot, [M]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ is trivial only if $[T] = 0 \in \text{Pic}(X)$. Thus, $L(\cdot, M)$ factors through $\text{Cat}(\overline{CH^d}(X))$ exactly when M is algebraically trivial.

If $\text{char}(F) = p > 0$ we can factor m uniquely as $m = np^k$ with n coprime to p . Assume $n \geq 2$ i.e. $m \neq p^k$. Hence, $p^k[T]$ is a torsion class of $\text{Pic}(X)$ of order n with n coprime to $p = \text{char}F$. Moreover, $p^k[T] \notin \text{Pic}_{\text{alg}}(X)$ because by construction $t[T] \notin \text{Pic}_{\text{alg}}(X)$ for any $0 < t < m$. Then by Proposition 6.11.2 the homomorphism $L(\cdot, p^k[T]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ is not trivial. Hence, the homomorphism $L(\cdot, [T]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ is also not trivial. Since $[M] = [E] + [T]$ with $[E]$ is an equivalence class of algebraically trivial divisors, by Proposition 5.0.2 the homomorphism $L(\cdot, [M]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot))$ is also not trivial. Therefore, the functor $L(\cdot, M) : \text{Cat}(\text{CH}^d(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ is not trivial on morphisms and hence the functor $L(\cdot, M)$ does not factor through $\text{Cat}(\overline{\text{CH}^d}(X))$.

We are left considering the case $n = 1$. Then $m = p^k$ and $[T]$ is of order p^k . Thus, the homomorphism $L(\cdot, p^k[T]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot)) = F^\times$ is trivial. By additivity the homomorphism $L(\cdot, [T]) : H_1(C^d(X, \cdot)) \rightarrow H_1(C^1(F, \cdot)) = F^\times$ factors through $\mu_{p^k}(F) = 1$ because Frobenius is injective. Hence, the homomorphism $L(\cdot, [T]) : H_1(C^d(X, \cdot)) \rightarrow \mu_{p^k}(F)$ is tautologically trivial. Therefore, the functor $L(\cdot, M) : \text{Cat}(\text{CH}^d(X)) \rightarrow \text{Cat}(\text{Pic}(F))$ is trivial on morphisms and hence the functor $L(\cdot, M)$ factors through $\text{Cat}(\overline{\text{CH}^d}(X))$. \square

This finishes the proof of the main theorem 6.0.2.

CHAPTER 7

APPLICATION TO BLOCH'S BIEXTENSION

In [6] Bloch defines a biextension of $\mathrm{CH}_{hom}^p(X) \times \mathrm{CH}_{hom}^q(X)$ by F^\times for a smooth projective variety X of dimension $d = p + q - 1$ over an algebraically closed field F . In this chapter we will consider when Bloch's biextension extends to a larger biextension of Chow groups. In particular, we will show that for $p = d$ and $q = 1$ Bloch's biextension can not be extended to any subgroups of $\mathrm{CH}^*(X)$ larger than $\mathrm{CH}_{hom}^*(X)$.

Let us briefly recall the construction of quotient biextensions given by Gorchinskiy in Chapter 3 of [12].

Definition 7.0.1. Let A, B , and N be additive abelian groups. A subset $T \subset A \times B$ is a *bisubgroup* if for all elements $(a, b), (a, b')$ and (a', b) in T , the elements $(a + a', b), (a, b + b'), (a - a', b), (a, b - b')$ belong to T . For a bisubgroup $T \subset A \times B$, a bilinear map $\gamma : T \rightarrow N$ is a map of sets such that for all elements $(a, b), (a, b')$ and (a', b) in T , we have $\gamma(a, b) + \gamma(a', b) = \gamma(a + a', b)$ and $\gamma(a, b) + \gamma(a, b') = \gamma(a, b + b')$.

• Construction of quotient biextensions

Let $T \subset A \times B$ be a subgroup and $\alpha : A \rightarrow \bar{A}, \beta : B \rightarrow \bar{B}$ be surjective group homomorphisms such that $(\alpha \times \beta)(T) = \bar{A} \times \bar{B}$. Consider the subgroup

$$S = T \cap (\ker(\alpha) \times B \cup A \times \ker(\beta)) \subset A \times B$$

Then the quotient P_γ of $N \times T$ by the following equivalence relation has the structure of a biextension of (\bar{A}, \bar{B}) by N . The equivalence relation on $N \times T$ is the transitive closure of the isomorphisms $N \times \{(a, b)\} \xrightarrow{\gamma(a', b)} N \times \{(a + a', b)\}$ and $N \times \{(a, b)\} \xrightarrow{\gamma(a, b')}$ $N \times \{(a, b + b')\}$ for all $(a, b) \in T$ and $(a', b), (a, b') \in S$.

• Construction via a true morphism of complexes

As indicated in Section 3.2 of op.cit., the above method can be applied to a *true* morphism

of homological complexes $\delta : A_\bullet \otimes B_\bullet \rightarrow N[-1]$. Here we present the construction in more details in the special case when $A_i = B_i = 0$ for $i < 0$ and explain why we can not construct a biextension of $\text{CH}^p(X) \times \text{CH}^q(X)$ by F^\times . But instead we need to consider the subgroups of $\text{CH}^*(X) = H_0(C^*(X, \cdot))$ corresponding to $\ker(\text{CH}^*(X) \rightarrow \text{Hom}(\text{CH}^{d+1-*}(X, 1), F^\times))$.

If we naively try use the above method to construct a biextension of all of $H_0(A_\bullet) \times H_0(B_\bullet)$ by N , i.e. $\bar{A} = H_0(A_\bullet)$ and $\bar{B} = H_0(B_\bullet)$, we need to define the maps $\alpha : A_0 \rightarrow H_0(A_\bullet)$ and $\beta : B_0 \rightarrow H_0(B_\bullet)$. Then $\ker \alpha = \partial A_1$ and $\ker \beta = \partial B_1$. For $T = A_0 \times B_0$ the group S is

$$S = \partial A_1 \times B_0 \cup A_0 \times \partial B_1$$

Note that $\delta : A_\bullet \otimes B_\bullet \rightarrow N[-1]$ induces homomorphisms on homologies $[\delta] : H_0(A_\bullet) \otimes H_1(B_\bullet) \rightarrow H_1(N[-1]) = N$ and $[\delta] : H_1(A_\bullet) \otimes H_0(B_\bullet) \rightarrow H_1(N[-1]) = N$. We can use these homomorphisms to define $\gamma : S \rightarrow N$ by setting

$$\gamma(a_0, \partial b_1) = \delta(a_0 \otimes b_1)$$

$$\gamma(\partial a_1, b_0) = \delta(a_1 \otimes b_0)$$

for $(a_0, \partial b_1) \in A_0 \times \partial B_1$ and $(\partial a_1, b_0) \in \partial A_1 \times B_0$.

In order for γ to be a well-defined homomorphism we need $\delta(a_0 \otimes b_1) = \delta(a_0 \otimes b'_1)$ for any $b_1, b'_1 \in B_1$ such that $\partial b_1 = \partial b'_1$. Thus, we need $\partial(a_0 \otimes b_1 - b'_1) = 0 \in N$. Note that $\partial b_1 = \partial b'_1$ implies that $b_1 - b'_1 \in Z_1(B_\bullet)$. Thus, the largest subgroup \widetilde{A}_0 of A_0 we can work with is

$$\widetilde{A}_0 = \ker(A_0 \rightarrow \text{Hom}(Z_1(B_\bullet), N))$$

Note that from the map $H_0(A_\bullet) \otimes H_1(B_\bullet) \rightarrow H_1(N[-1]) = N$ we get that $\partial A_1 \subset \widetilde{A}_0$. Similarly for $\gamma(\partial a_1, b_0)$ to be well-defined we need $b_0 \in \widetilde{B}_0 = \ker(B_0 \rightarrow \text{Hom}(Z_1(A_\bullet), N))$ and $\partial B_1 \subset \widetilde{B}_0$.

For γ to be a well-defined homomorphism we additionally need

$$\delta(\partial a_1 \otimes b_1) = \gamma(\partial a_1, \partial b_1) = \delta(a_1 \otimes \partial b_1)$$

Note $\partial a_1 \otimes b_1 - a_1 \otimes \partial b_1 = \partial(a_1 \otimes b_1) \in \partial \text{Tot}(A_\bullet \otimes B_\bullet)_2$. Since $\phi : A_\bullet \otimes B_\bullet \rightarrow N[-1]$ is map on complexes we get $\phi(\partial a_1 \otimes b_1 - a_1 \otimes \partial b_1) \in \partial N[-1]_2 = 0$. Thus, we have the desired equality.

Therefore, to use the above method to get a biextension the biggest subgroups of $H_0(A_\bullet) \times H_0(B_\bullet)$ we can use are $(\widetilde{A}_0/\partial A_1) \times (\widetilde{B}_0/\partial B_1)$ for $T = \widetilde{A}_0 \times \widetilde{B}_0$.

• **Bloch's biextension via true morphism on complexes**

We will apply the above method to the case $A_\bullet = C^p(X, \cdot)$, $B_\bullet = C^q(X, \cdot)$, and $N = F^\times$. We define the homomorphism $\delta : C^p(X, \cdot) \otimes C^q(X, \cdot) \rightarrow F^\times[-1]$ as the composition of maps on complexes $C^p(X, \cdot) \otimes C^q(X, \cdot) \xrightarrow{L} C^1(F, \cdot) \xrightarrow{h} F^\times[-1]$. The map $h : C^1(F, \cdot) \rightarrow F^\times$ is well-defined because $C^1(F, 0) = 0$ and $H_1(C^1(F, \cdot)) = F^\times$. Then the largest possible biextension of Chow groups by F^\times is on $\widetilde{A} \times \widetilde{B}$ where

$$\widetilde{A} = \ker(L : H_0(C^p(X, \cdot)) \rightarrow \text{Hom}(H_1(C^q(X, \cdot)), F^\times)) \subset \text{CH}^p(X)$$

$$\widetilde{B} = \ker(L : H_0(C^q(X, \cdot)) \rightarrow \text{Hom}(H_1(C^p(X, \cdot)), F^\times)) \subset \text{CH}^q(X)$$

By Bloch in [6] and Müller-Stach in [21], we have $\text{CH}_{hom}^p(X) \subset \widetilde{A}$ and $\text{CH}_{hom}^q(X) \subset \widetilde{B}$. Bloch's biextension in [6] is the same biextension of $\text{CH}_{hom}^p(X) \times \text{CH}_{hom}^q(X)$ by F^\times as the one constructed from the true morphism on complexes.

The following lemma will imply that there is no larger biextension of Chow groups by F^\times using the above method than the one of $\text{CH}_{num}^p(X) \times \text{CH}_{num}^q(X)$.

Lemma 7.0.2. *Let $Z_{num}^q(X)$ be the subgroup of numerically trivial codimension q cycles on X . Then $\ker(L : C^q(X, 0) \rightarrow \text{Hom}(H_1(C^p(X, \cdot)), F^\times)) \subset Z_{num}^q(X)$.*

Proof. Let D be a cycle of codimension q on X . If D is not numerically trivial cycle, there is a subvariety W of complementary codimension $\dim X - q = p - 1$ on X such that W and D intersect properly and $\deg(W \cdot D) \neq 0$. This means that W and D intersect in finitely many points p_1, \dots, p_e with respective multiplicities m_1, \dots, m_e and $m = \sum m_i = \deg(W \cdot D) \neq 0$. Let $k \in K(W)^\times$ be a constant function on W such that $k \notin \mu_m(F)$ (such a constant k exists because F is an infinite field.). Then as $\operatorname{div} k = 0$ we have $(W, k) \in Z_1(C^p(X, \cdot))$. Then the scalar $L((W, k), D) \in F^\times$ is calculated via evaluating the function k at the intersection $W \cdot D$. Thus, $L((W, k), D) = k^{\sum m_i} \neq 1$ because $m = \sum m_i \neq 0$ and $k \notin \mu_m(F)$. Hence the homomorphism $L(\cdot, D) : H_1(C^p(X, \cdot)) \rightarrow F^\times$ is not trivial and so D is not in the kernel of the homomorphism $L : C^q(X, 0) \rightarrow \operatorname{Hom}(H_1(C^p(X, \cdot)), F^\times)$. Therefore, the kernel of the homomorphism $L : C^q(X, 0) \rightarrow \operatorname{Hom}(H_1(C^p(X, \cdot)), F^\times)$ is a subgroup of the group of numerically trivial codimension q cycles on X . \square

Question 7.0.3. *For which (p, q) satisfying $p + q = d + 1$, can we extend Bloch's biextension of $\operatorname{CH}_{hom}^p(X) \times \operatorname{CH}_{hom}^q(X)$ by F^\times to a biextension of larger subgroups of $\operatorname{CH}^*(X)$?*

The following lemma will partially answer the question. It shows that for $p = d$ and $q = 1$ the Bloch's biextension is the best we can get.

Lemma 7.0.4. *The largest quotient biextension of Chow groups for $p = d$ and $q = 1$ is the one on $\operatorname{CH}_{alg}^d(X) \times \operatorname{CH}_{alg}^1(X)$ by F^\times if $\operatorname{char} F = 0$ and the one on $\operatorname{CH}_{alg}^d(X) \times (\operatorname{CH}_{alg}^1(X) \cup p$ -power torsion cycles) by F^\times if $\operatorname{char} F = p$.*

Proof. By Theorem 6.0.2 we know that if $\operatorname{char} F = 0$ then

$$\operatorname{CH}_{alg}^1(X) = \ker(H_0(C^1(X, \cdot)) \rightarrow \operatorname{Hom}(H_1(C^d(X, \cdot)), F^\times))$$

and if $\operatorname{char} F = p$ then

$$\operatorname{CH}_{alg}^1(X) \cup p\text{-power torsion cycles} = \ker(H_0(C^1(X, \cdot)) \rightarrow \operatorname{Hom}(H_1(C^d(X, \cdot)), F^\times))$$

Claim 7.0.5. $\text{CH}_{num}^d(X) = \ker(L : H_0(C^d(X, \cdot)) \rightarrow \text{Hom}(H_1(C^1(X, \cdot)), F^\times))$.

Proof. Let $\rho = \sum m_i p_i$ be an arbitrary zero cycle with $p_i \in X$. Denote by $m = \deg \rho$. If $m \neq 0$, then as before for any constant $k \notin \mu_m$ we have $L(\rho)(k) = k^{\sum m_i} \neq 1$ and so $\rho \notin \ker(L : H_0(C^d(X, \cdot)) \rightarrow \text{Hom}(H_1(C^1(X, \cdot)), F^\times))$. Thus, $\ker(L : H_0(C^d(X, \cdot)) \rightarrow \text{Hom}(H_1(C^1(X, \cdot)), F^\times)) \subset \text{CH}_{num}^d(X)$. The inclusion is an equality because by Lemma 2.1.6 since X is projective, we have $H_1(C^1(X, \cdot)) = \Gamma(X, \mathcal{O}_X^\times) = F^\times$. Then

$$\ker(L : H_0(C^d(X, \cdot)) \rightarrow \text{Hom}(H_1(C^1(X, \cdot)), F^\times)) = \text{CH}_{alg}^d(X) = \text{CH}_{hom}^d(X) = \text{CH}_{num}^d(X)$$

□

Hence, the 'best' quotient biextension we can construct for $p = d$ and $q = 1$ is Bloch's biextension of $\text{CH}_{alg}^d(X) \times \text{CH}_{alg}^1(X)$ or $\text{CH}_{alg}^d(X) \times (\text{CH}_{alg}^1(X) \cup p\text{-power torsion cycles})$ by F^\times (depending on the characteristic). □

APPENDIX

In the appendix we elaborate on the correspondence between complexes and Picard categories by pointing out how the most common terms translate into the new language.

Let $\mathcal{C} = (\mathcal{C}, +, 0, as, \lambda, \rho, s)$ be a symmetric monoidal category where as is the associativity constraint, λ and ρ are the left and right unit constraints, and s is the symmetry constraint. Denote by $\pi_0(\mathcal{C})$ the set of isomorphic classes of objects of \mathcal{C} . It is an abelian monoid with operation induced by $+$. The set $\pi_1(\mathcal{C})$ of automorphisms of the identity object 0 is an abelian group. The category \mathcal{C} is strict if the natural isomorphisms as, λ, ρ are identities.

Definition 7.0.6. A symmetric monoidal category \mathcal{C} is a Picard groupoid if the underlying category \mathcal{C} is a groupoid and the set $\pi_0(\mathcal{C})$ of isomorphic classes of objects is a group.

By [7], any strictly commutative Picard groupoid is equivalent to one that arises from complexes in the following way. Consider a complex $A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0$ of abelian groups in the following way.

- (1) The objects $[a]$ of \mathcal{A} are the elements a of A_0
- (2) The morphisms between two objects $[a_1], [a_2] \in \mathcal{A}$ are given by $\text{Mor}_{\mathcal{A}}([a_1], [a_2]) = \frac{\{W \in A_1 \mid \partial W = a_2 - a_1\}}{\partial A_2}$. The identity morphism $id_{[a]}$ for any object $[a]$ is given by any ∂E with $E \in A_2$
- (3) The composition of morphisms is induced by the group addition in A_1
- (4) The monoidal structure of the category \mathcal{A} comes from the addition in A_0
- (5) The associativity and commutativity constraints are given by $\partial A_2 \subset A_1$.

Let \mathcal{A} and \mathcal{B} be Picard categories constructed from complexes A_{\bullet} and B_{\bullet} . Then we have the following correspondences between the worlds of chain complexes and categories.

• **Chain map = functor**

A chain map $f : A_{\bullet} \rightarrow B_{\bullet}$ induces a functor $[f] : \mathcal{A} \rightarrow \mathcal{B}$ of symmetric monoidal categories as $[f][a_1] + [f][a_2] = [f][a_1 + a_2]$ for any objects $[a_1], [a_2] \in \mathcal{A}$ and similarly for morphisms. That is, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{+} & \mathcal{A} \\ [f] \times [f] \downarrow & & \downarrow [f] \\ \mathcal{B} \times \mathcal{B} & \xrightarrow{+} & \mathcal{B} \end{array}$$

• **Chain homotopy of chain maps = natural isomorphism of functors**

Let $F : A_{\bullet} \rightarrow B_{\bullet}$ be a chain homotopy between the chain maps $f, g : A_{\bullet} \rightarrow B_{\bullet}$. This means as usual that $F : A_i \rightarrow B_{i+1}$ for any i and

$$\partial F + F\partial = f - g$$

Now for any element $a \in A_0$ we have $f(a) - g(a) = (\partial F + F\partial)(a) = \partial F(a)$ with $F(a) \in B_1$. Thus in the category \mathcal{B} we get an isomorphism $[F][a] : [g][a] \rightarrow [f][a]$. For any morphism $[h] : [a] \rightarrow [a']$ in \mathcal{A} i.e. $\partial h = a' - a$ with $h \in A_1$ we have

$$f(h) - g(h) = (\partial F + F\partial)(h) = \partial F(h) + F(a' - a) = \partial E + F(a') - F(a)$$

for $E = F(h) \in B_2$. This means that $F(a) + f(h) \equiv g(h) + F(a')$ modulo ∂B_2 and so we have a natural transformation of functors $[F] : [g] \rightarrow [f]$ as illustrated in the commutative diagram

$$\begin{array}{ccc} [g][a] & \xrightarrow{[g][h]} & [g][a'] \\ [F][a] \downarrow & & \downarrow [F][a'] \\ [f][a] & \xrightarrow{[f][h]} & [f][a'] \end{array}$$

The natural transformation $[F]$ commutes with the symmetric monoidal structure of the categories because on the level of complexes we have $F(a + a') = F(a) + F(a')$ and so on the level of categories we have the commutative diagram

$$\begin{array}{ccc} [g][a + a'] & \xrightarrow{=} & [g][a] + [g][a'] \\ [F][a+a'] \downarrow & & \downarrow [F][a]+[F][a'] \\ [f][a + a'] & \xrightarrow{=} & [f][a] + [f][a'] \end{array}$$

• **Chain quasi-isomorphism = adjoint equivalence of categories**

Proposition 7.0.7. *Let $f : A_{\bullet} \rightarrow B_{\bullet}$ be a quasi-isomorphism between complexes of free abelian groups. Then the functor $[f] : \mathcal{A} \rightarrow \mathcal{B}$ is an adjoint equivalence of categories.*

Proof. We will break the proof into a couple of smaller lemmas.

Lemma 7.0.8. *The functor on the categories $[f] : \mathcal{A} \rightarrow \mathcal{B}$ is an equivalence of categories if and only if $H_0(A_{\bullet}) \xrightarrow{\cong} H_0(B_{\bullet})$ and $H_1(A_{\bullet}) \xrightarrow{\cong} H_1(B_{\bullet})$.*

Proof. We will show that

- (a) The functor $[f]$ is faithful if and only if $H_1(A_{\bullet}) \hookrightarrow H_1(B_{\bullet})$
- (b) The functor $[f]$ is full if and only if $H_1(A_{\bullet}) \twoheadrightarrow H_1(B_{\bullet})$ and $H_0(A_{\bullet}) \hookrightarrow H_0(B_{\bullet})$
- (c) The functor $[f]$ is essentially surjective if and only if $H_0(A_{\bullet}) \twoheadrightarrow H_0(B_{\bullet})$

Proof of (a). Assume $[f]$ is faithful. Then for any object $[\alpha] \in \mathcal{A}$ we have

$$\text{End}_{\mathcal{A}}([\alpha]) \hookrightarrow \text{End}_{\mathcal{B}}([f][\alpha]) \text{ i.e. } H_1(A_{\bullet}) \hookrightarrow H_1(B_{\bullet})$$

Conversely, assume $H_1(A_{\bullet}) \hookrightarrow H_1(B_{\bullet})$. We want to show that for any objects $[\alpha], [\beta] \in \mathcal{A}$ we have

$$\text{Mor}_{\mathcal{A}}([\alpha], [\beta]) \hookrightarrow \text{Mor}_{\mathcal{B}}([f][\alpha], [f][\beta])$$

Assume not. Then there are two elements $W_1, W_2 \in A_1$ with the property $\partial W_i = \beta - \alpha$ which represent different morphisms in \mathcal{A} but become equivalent morphisms in \mathcal{B} . In this case $0 \neq W_1 - W_2 \in H_1(A_\bullet)$ maps to $0 \in H_1(B_\bullet)$, contradicting injectivity on H_1 . Hence, $H_1(A_\bullet) \hookrightarrow H_1(B_\bullet)$ implies that the inclusion $[f]$ is faithful.

Proof of (b). Assume $[f]$ is full. Then for any object $[\alpha] \in \mathcal{A}$ we have

$$\text{End}_{\mathcal{A}}([\alpha]) \twoheadrightarrow \text{End}_{\mathcal{B}}([f][\alpha]) \text{ and hence } H_1(A_\bullet) \twoheadrightarrow H_1(B_\bullet)$$

Consider two different objects $[\alpha], [\beta] \in \mathcal{A}$ such that $f(\alpha) - f(\beta) = 0 \in H_0(B_\bullet)$. This means that there is $W \in B_1$ such that $\partial W = f(\alpha) - f(\beta)$. Since $[f]$ is full,

$$\text{Mor}_{\mathcal{A}}([\alpha], [\beta]) \twoheadrightarrow \text{Mor}_{\mathcal{B}}([f][\alpha], [f][\beta])$$

and so there is $W' \in A_1$ such that $\alpha - \beta = \partial W'$ and hence $\alpha - \beta = 0 \in H_0(A_\bullet)$ as well. This shows the injectivity of $H_0(A_\bullet) \hookrightarrow H_0(B_\bullet)$.

Conversely, assume $H_1(A_\bullet) \twoheadrightarrow H_1(B_\bullet)$ and $H_0(A_\bullet) \hookrightarrow H_0(B_\bullet)$. From the surjectivity of $H_1(A_\bullet) \twoheadrightarrow H_1(B_\bullet)$ we get that for any object $[\alpha] \in \mathcal{A}$ we have $\text{End}_{\mathcal{A}}([\alpha]) \twoheadrightarrow \text{End}_{\mathcal{B}}([\alpha])$.

Now consider two different objects $[\alpha], [\beta] \in \mathcal{A}$ for which there is $W \in B_1$ such that $\partial W = f(\alpha) - f(\beta)$. Then $f(\alpha) - f(\beta) = 0 \in H_0(B_\bullet)$ and by injectivity of $H_0(A_\bullet) \hookrightarrow H_0(B_\bullet)$, there is $W' \in A_1$ such that $\alpha - \beta = \partial W'$. This does not necessarily mean that $f(W') = W$. Nevertheless $\partial(f(W') - W) = 0$ and so $f(W') - W$ represents a homology class in $H_1(B_1)$. By surjectivity of $H_1(A_1) \twoheadrightarrow H_1(B_1)$, there are $E \in A_1$ and $T \in B_2$ such that $f(E) = f(W') - W - T$. This gives us an element $W' - E \in A_1$ such that $f(W' - E) = W + T$. Thus, $[W' - E]$ is the desired morphism in $\text{Mor}_{\mathcal{A}}([\alpha], [\beta])$ which is sent to the morphism $[W] = [W + T] \in \text{Mor}_{\mathcal{B}}([f][\alpha], [f][\beta])$, proving the surjectivity

$$\text{Mor}_{\mathcal{A}}([\alpha], [\beta]) \twoheadrightarrow \text{Mor}_{\mathcal{B}}([f][\alpha], [f][\beta])$$

Proof of (c). By definitions, the surjectivity of $H_0(A_\bullet) \rightarrow H_0(B_\bullet)$ is equivalent to the fact that for any $\alpha \in B_0$, there is $\alpha' \in A_0$ and $W \in B_1$ such that $\alpha - f(\alpha') = \partial W$. This on its own is equivalent to any object $[\alpha] \in \mathcal{B}$ being isomorphic to an object of the form $[f][\alpha']$ with $[\alpha'] \in \mathcal{A}$ i.e. the definition of $[f]$ being essentially surjective. \square

Since $f : A_\bullet \rightarrow B_\bullet$ is a quasi-isomorphism between complexes of free abelian groups, by Theorem 2.2.6 of [27] there exists a chain map $g : B_\bullet \rightarrow A_\bullet$ such that $g = f^{-1}$ in the derived category and g is unique up to chain homotopy. By Lemma 7.0.8 the functor $[g] : \mathcal{B} \rightarrow \mathcal{A}$ is fully faithful and essentially surjective.

Claim 7.0.9. Let $f : A_\bullet \rightarrow B_\bullet$ is a quasi-isomorphism between complexes of free abelian groups and $g : B_\bullet \rightarrow A_\bullet$ be the chain homotopic inverse. Then the functors $[f] : \mathcal{A} \rightarrow \mathcal{B}$ and $[g] : \mathcal{B} \rightarrow \mathcal{A}$ are adjoint inverses.

Proof. Since the chain maps f and g are inverses of each other up to chain homotopy, there is a chain homotopy $H : A_\bullet \rightarrow A_\bullet$ such that $gf - id_{A_\bullet} = \partial H + H\partial$. Thus, there is a natural isomorphism $[H] : [id_{\mathcal{A}}] \rightarrow [gf]$ of functors. By Theorem 2 in Chapter of [18], in order to declare the functor $[g]$ a right adjoint of $[f]$, it is sufficient to check the universality of $[H]$. This is equivalent to checking that for any objects $[\alpha] \in \mathcal{A}$ and $[\beta] \in \mathcal{B}$ and any morphism $[\alpha] \xrightarrow{[h]} [g][\beta]$ in \mathcal{A} there exists a unique morphism $[W] : [f][\alpha] \rightarrow [\beta]$ in \mathcal{B} such that the diagram

$$\begin{array}{ccc} [\alpha] & \xrightarrow{[H][\alpha]} & [gf][\alpha] \\ & \searrow [h] & \downarrow [f][W] \\ & & [g][\beta] \end{array}$$

commutes. We get existence of the morphism $[W]$ by defining $[W] \in \text{Mor}_{\mathcal{B}}([f][\alpha], [\beta])$ to be the unique morphism in $\text{Mor}_{\mathcal{B}}([f][\alpha], [\beta])$ corresponding to the morphism $[h]([H][\alpha])^{-1} \in \text{Mor}_{\mathcal{A}}([gf][\alpha], [g][\beta])$. The uniqueness of $[W]$ follows from fully-faithfulness of $[g]$. \square

□

Proposition 7.0.10. *If in addition $f : A'_\bullet \rightarrow A_\bullet$ is a quasi-isomorphism between complexes of free abelian groups such that A'_\bullet is a subcomplex and the quotient complex $A_\bullet/A'_\bullet = A''_\bullet$ is also a complex of free abelian groups, then*

- (i) *The 'inverse' map $g : A'_\bullet \rightarrow A_\bullet$ can be chosen so that $g|_{A'_\bullet} = id_{A'_\bullet}$.*
- (ii) *Such a map g is unique modulo $\alpha \in \text{Hom}(A''_\bullet, A'_\bullet)$ and each α is chain homotopic to zero.*

Proof. of (i)

Lemma 7.0.11. *Let $0 \rightarrow A'_\bullet \rightarrow A_\bullet \rightarrow A''_\bullet \rightarrow 0$ be a short exact sequence of complexes of free abelian groups such that $A'_\bullet \rightarrow A_\bullet$ is a quasi-isomorphism. Then there's a map of complexes $h : A''_\bullet \rightarrow A_\bullet$ inducing an isomorphism of complexes $A''_\bullet \oplus A'_\bullet \rightarrow A_\bullet$ given by $(a'', a') \mapsto h(a'') + a'$.*

Proof. We will provide a quick sketch. Since A''_\bullet is a complex of free abelian groups, for any $q \in \mathbb{Z}$ we have a split short exact sequence $0 \rightarrow Z_q(A''_\bullet) \rightarrow A''_q \rightarrow B_{q-1}(A''_\bullet) \rightarrow 0$. Let $s : B_{q-1}(A''_\bullet) \rightarrow A''_q$ be the splitting homomorphism. Set $V_q = s(B_{q-1}(A''_\bullet))$. By the quasi-isomorphism $A'_\bullet \rightarrow A_\bullet$, we get $B_q(A''_\bullet) = Z_q(A''_\bullet)$. Hence, $A''_q \simeq V_q \oplus V_{q+1}$. Note that in this isomorphism the differential $\partial : A''_q \rightarrow A''_{q-1}$ for A''_\bullet acts as $\partial(v_q, v_{q+1}) = (0, v_q)$ on the direct sum $V_q \oplus V_{q+1}$.

We will define the homomorphisms $h : A''_\bullet \rightarrow A_\bullet$ in two steps. First, consider the splitting exact sequence $0 \rightarrow A'_q \rightarrow A_q \rightarrow V_q \oplus V_{q+1} \rightarrow 0$ with splitting morphism $t : V_q \oplus V_{q+1} \rightarrow A_q$. For any $q \in \mathbb{Z}$, define h on the direct summand $V_q \subset V_q \oplus V_{q+1}$ as t i.e. $h(v_q) = t(v_q)$ for all $v_q \in V_q$. Second, for any $v_{q+1} \in V_{q+1} \subset V_q \oplus V_{q+1}$ define $h(v_{q+1}) = \partial h(v_{q+1})$. Here ∂ is the differential $\partial : A_{q+1} \rightarrow A_q$ and h is calculated on the direct summand $V_{q+1} \subset V_{q+1} \oplus V_{q+2}$. By the recursive definition of h on each of the direct summands $A''_q \simeq V_q \oplus V_{q+1}$, we get a map on complexes $A''_\bullet \rightarrow A_\bullet$.

It can be checked that h is injective as $ph(v_q) + p\partial h(v_{q+1}) = v_q + \partial ph(v_{q+1}) = v_q + \partial v_{q+1} = v_q + v_{q+1}$. Moreover, the composition $V_q \oplus V_{q+1} \xrightarrow{h} A_q \xrightarrow{p} A''_q$ is injective because by chasing diagrams we get $ph(v_q) + p\partial h(v_{q+1}) = v_q + \partial ph(v_{q+1}) = v_q + \partial v_{q+1} = v_q + v_{q+1}$. Hence, $h(V_q \oplus V_{q+1}) \oplus A'_q$ is a subgroup of A_q . To see that $h(V_q \oplus V_{q+1}) \oplus A'_q \hookrightarrow A_q$ is surjective, consider $a \in A_q$ such that $a \notin A'_q$. Then $p(a) \neq 0$ and $php(a) = p(a)$. Hence, $hp(a) - a \in A'_q$ and so $a \in h(V_q \oplus V_{q+1}) \oplus A'_q$. Finally, combining the isomorphisms of groups $h(V_q \oplus V_{q+1}) \oplus A'_q \xrightarrow{\cong} A_q$ compatible with the maps on complexes $A'_\bullet \rightarrow A_\bullet$ and $h : A''_\bullet \rightarrow A_\bullet$, we get the desired isomorphism of complexes $A''_\bullet \oplus A'_\bullet \rightarrow A_\bullet$ given by $(a'', a') \mapsto h(a'') + a'$. \square

\square

Proof. of (ii)

Let $g_1, g_2 : A'_\bullet \rightarrow A_\bullet$ be two quasi-isomorphisms 'inverses' of f such that $g_i|_{A'_\bullet} = id_{A'_\bullet}$. Set $\tilde{g} = g_1 - g_2$. Then $\tilde{g} : A'_\bullet \rightarrow A_\bullet$ is a chain map such that $\tilde{g}|_{A'_\bullet} = 0_{A'_\bullet}$. Then \tilde{g} factors as

$$\begin{array}{ccc} A_\bullet & \xrightarrow{v} & A''_\bullet \\ \tilde{g} \downarrow & \swarrow \alpha & \\ A'_\bullet & & \end{array}$$

Thus we get $\alpha \in \text{Hom}(A''_\bullet, A'_\bullet)$. Conversely, any homomorphism $\alpha \in \text{Hom}(A''_\bullet, A'_\bullet)$ gives rise to $\tilde{g} : A'_\bullet \rightarrow A_\bullet$ such that $\tilde{g}|_{A'_\bullet} = 0_{A'_\bullet}$.

Furthermore, because $f : A'_\bullet \hookrightarrow A_\bullet$ is a quasi-isomorphism, we get $H_q(A''_\bullet) = 0$ for all q . Thus, since A''_\bullet is a complex of free abelian groups, we get that $id_{A''_\bullet}$ is chain homotopic to 0. Hence, $\alpha : A''_\bullet \rightarrow A'_\bullet$ is also chain homotopic to $0 = \alpha \circ 0$ as $\alpha = \alpha \circ id_{A''_\bullet}$. Therefore, there is a chain homotopy $H : A''_\bullet \rightarrow A'_\bullet$ such that $\partial H + H\partial = \alpha$. This fits into a commutative diagram:

$$\begin{array}{ccc}
A_r & \xrightarrow{v} & A_r'' \\
\tilde{H} \downarrow & \swarrow H & \\
A_{r+1}' & &
\end{array}$$

Then $\tilde{g} = v \circ \alpha = v(\partial H + H\partial) = (v\partial)H + (vH)\partial = (\partial v)H + (vH)\partial = \partial\tilde{H} + \tilde{H}\partial$ i.e. \tilde{g} is chain homotopic to 0. \square

Combining the previous results we get the following corollary which we used in Lemma 3.0.1 and Proposition 4.0.1.

Corollary 7.0.12. *Let $f : A'_\bullet \rightarrow A_\bullet$ be a quasi-isomorphism of complexes of free abelian groups such that A'_\bullet is a subcomplex of A_\bullet and the quotient complex $A_\bullet/A'_\bullet = A''_\bullet$ is also a complex of free abelian groups. Let $h' : A'_\bullet \rightarrow B_\bullet$ a chain map with B_\bullet another complex of free abelian groups. Then there is quasi-isomorphism $g : A_\bullet \rightarrow A'_\bullet$, an inverse of f in the derived category, which is unique up to chain homotopy. The chain map g is the identity on the subcomplex A'_\bullet . Moreover, using the composition $h = h'g$ we can define a functor of categories*

$$\begin{array}{ccc}
A' & \xrightarrow{[h']} & B \\
\uparrow [g] & \nearrow [h] & \\
A & &
\end{array}$$

The functor $[h]$ satisfies the following properties:

- (i) *It is a functor of Picard categories*
- (ii) $[h]|_{A'_\bullet} = [h']$
- (iii) *For any two different functors $[h_1], [h_2]$ corresponding to different chain maps g , there is a natural isomorphism $N : [h_1] \rightarrow [h_2]$ such that $N|_{A'_\bullet}$ is the identity transformation*

$$[h'] \rightarrow [h'].$$

Proof. Parts (i) and (ii) follow from Proposition 7.0.10. We will provide a proof of (iii). Let $g_1, g_2 : A'_\bullet \rightarrow A_\bullet$ be two quasi-isomorphisms 'inverses' of f such that $g_i|_{A'_i} = id_{A'_i}$, then they induce functors $[g_i] : \mathcal{A} \rightarrow \mathcal{A}'$ such that when restricting to the subcategory $\mathcal{A}' \hookrightarrow \mathcal{A}$ we have $[g_i]|_{\mathcal{A}'} = id_{\mathcal{A}'}$. Moreover, the chain homotopy $\tilde{H} : A'_\bullet \rightarrow A_\bullet$ with $g_1 - g_2 = \partial\tilde{H} + \tilde{H}\partial$ gives a natural transformation $[\tilde{H}] : [g_2] \rightarrow [g_1]$ between the functors. Moreover, when restricted to the subcategory \mathcal{A}' , the natural transformation $[\tilde{H}]|_{\mathcal{A}'} : id_{\mathcal{A}'} \rightarrow id_{\mathcal{A}'}$ is the identity natural transformation. The reason for this is that $[\tilde{H}] : A_0 \rightarrow A'_1$ factors through A''_2 and hence for any $a \in A'_0$ we have $\tilde{H}[a] = 0 \in A'_1$. Thus, the morphism $[\tilde{H}][a] : [g_1][a] \rightarrow [g_2][a]$ is the identity. This fits into the commutative diagram

$$\begin{array}{ccc} [g_1][a] & \xrightarrow{[g_1][h]} & [g_1][a] \\ [\tilde{H}][a] \downarrow & & \downarrow [\tilde{H}][a'] \\ [g_2][a] & \xrightarrow{[g_2][h]} & [g_2][a'] \end{array}$$

which shows that $[\tilde{H}]$ is the identity transformation between the identity functors $[g_1]|_{\mathcal{A}'}$ and $[g_2]|_{\mathcal{A}'}$. Composing with the functor $[h']$ we get the desired result. \square

• Construction of product category $\mathcal{A} \times \mathcal{B}$ of two Picard categories

Let \mathcal{A} and \mathcal{B} be two Picard categories constructed from the complexes A_\bullet and B_\bullet . We will construct the product category $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ from the product of complexes $A_\bullet \otimes B_\bullet$ as the category which corresponds to the complex $Tor(A_\bullet \otimes B_\bullet)$. Objects of \mathcal{C} are the elements of $Tot(A_\bullet \otimes B_\bullet)_0 = A_0 \otimes B_0$ i.e. a pair $([\alpha], [\gamma])$ of objects $[\alpha] \in \mathcal{A}$ and $[\gamma] \in \mathcal{B}$. Morphisms of \mathcal{C} are the elements of $Tot(A_\bullet \otimes B_\bullet)_1 = A_0 \otimes B_1 \oplus A_1 \otimes B_0$ i.e. a pair $([\alpha], [E])$ of an object $[\alpha] \in \mathcal{A}$ and a morphism $[E] \in \mathcal{B}$ or a pair $([W], [\gamma])$ of a morphism $[W] \in \mathcal{A}$ and an object $[\gamma] \in \mathcal{B}$; modulo the image of $Tot(A_\bullet \otimes B_\bullet)_2$.

Note that the product category $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ does not have Picard category structure compatible with the Picard category structures of \mathcal{A} and \mathcal{B} i.e. on the level of objects we

want $([\alpha_1], [\gamma]) + ([\alpha_2], [\gamma]) = ([\alpha_1] + [\alpha_2], [\gamma])$ rather than $([\alpha_1] + [\alpha_2], 2[\gamma])$. Using this given a functor $N : \mathcal{C} \rightarrow \mathcal{D}$ of Picard categories we call N bi-additive if $N([\alpha_1], [\gamma]) + N([\alpha_2], [\gamma]) = N([\alpha_1] + [\alpha_2], [\gamma])$ and $N([\alpha], [\gamma_1]) + N([\alpha], [\gamma_2]) = N([\alpha], [\gamma_1] + [\gamma_2])$ and similarly for morphisms.

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