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# DUALITY AND AXION COUPLINGS IN TWO-DIMENSIONAL SUPERSYMMETRIC QUANTUM FIELD THEORIES

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To Krisztina and my parents

I want to make it mean something to you. That you are in the cosmos. That you are of the cosmos. That you are born from stardust and to stardust you will return. That you are a way for the universe to be in awe of itself.

— Katherine J. Mack

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## ABSTRACT

One of the main open questions in string theory is understanding how to characterize generic string compactifications. The answer to this question has far-reaching implications on our ability to extract physical predictions from string theory and connect it to the observable universe. While a great deal of work has been done on compactifying string theory on specific backgrounds, such as Calabi-Yau spaces, a general picture of the space of compactifications remains elusive. In this work, we explore the parameter space of gauged linear sigma models (GLSM) with $(2,2)$ and $(0,2)$ supersymmetry and study the geometries that they can describe. First we provide a more general picture of the essential couplings that should be considered in a $(2,2)$ GLSM. We then explore analogues of $(2,2)$ quantum cohomology rings for $(0,2)$ theories which are not obtainable as deformations of theories on the $(2,2)$ locus. Finally, we explore abelian duality of chiral $(0,2)$ GLSM. The duals of these models generically involve charged fields, unlike those of their $(2,2)$ counterparts.

# CHAPTER 1

# INTRODUCTION

At the present time, string theory remains our best hope to obtain the standard model of particle physics and general relativity from one unified framework. And even in the case that the correct description of the fundamental physics of our universe turns out not to include string theory, a better understanding of string theory still has a lot to teach us about what a consistent quantum theory of gravity may look like, since it is the only specimen of this class that we currently know of. Over the last few decades, efforts to better understand string theory have also proved useful to reach a better understanding of quantum field theory, especially beyond the perturbative regime in which most particle physics calculations are carried out.

One of the largest still outstanding questions in string theory is a generic characterization of how the theory leads to a spacetime with four macroscopic dimensions, as the one we can observe today. In most of its well-understood limits, string theory gives rise to a ten-dimensional spacetime, and therefore the reduction to four dimensions involves a wrapping of the remaining six dimensions into a *string compactification.* However, not any compact six-dimensional space is a potential string compactification. The theory of a string propagating in a ten-dimensional geometry is expressed in terms of a two-dimensional non-linear sigma model summed over all 2d surfaces (representing the *worldsheet* of the string, similar to the worldline of a relativistic particle), which for consistency of the string theory must be a conformal field theory. Imposing conformality leads to a set of partial differential equations on the defining data of the geometry that are impossible to solve in its full generality.

Turning off some of the possible ingredients, however, leads to large classes of solutions, the most famous of which are Calabi-Yau three-folds (so named because they are spaces parametrized by three complex dimensions). Adding quantized fluxes to these Calabi-Yau manifolds often leads to large estimates for the number of possible string compactifications, the most famous of which is $\mathcal{O}(10^{500})$.

While the study of Calabi-Yau compactifications has led to significant improvements in our understanding of string theory and the possibilities contained in string phenomenology, it is important to emphasize that they do not constitute a generic class of compactifications. In fact, it behooves us to search for compactifications farther away from the lamp post and towards examples containing more generic features. Some such examples have been found, for example by employing a duality chain from established examples obtained in different limits of string theory [1].

When the compactified geometry can no longer be described by a Calabi-Yau manifold, we lose access to the algebraic geometry tools that allow us to provide a full description of those cases. Therefore, we must search for different and more generalizable tools. In this work, we will make use of gauged linear sigma models (GLSM), in the form introduced by Witten [2]. These are two-dimensional gauge theories in the same universality class as the non-linear sigma models we are interested in studying. Since GLSM are linear gauge theories, calculations are much easier to carry out in the gauge theory. Certain protected quantities can then be argued to be invariant under renormalization group flow. These facts allow us to learn more about the non-linear sigma models from a simpler setting.

Central to the existence of such protected quantities is the fact that the GLSM we are interested in are supersymmetric. Supersymmetry is a symmetry in quantum field theory which transforms between bosonic and fermionic fields. In order for the spectrum of the string theory to include fermionic states, the two-dimensional model must have some amount of supersymmetry. For the string theory to lead to a theory with minimal supersymmetry in four dimensions, the worldsheet theory must have two distinct supersymmetry charges, usually denoted as $\mathcal{N} = (0, 2)$ since the two supersymmetric charges have the same chirality.[1]

For some time, it was believed that $(0, 2)$ theories of the type relevant to string theory would be generically plagued by instabilities caused by worldsheet instantons, and therefore we would need more symmetry to attain control [3, 4]. The Calabi-Yau theories referred to

1. This notation will be fully explained in Chapter 2.

above, for instance, are non-chirally $\mathcal{N} = (2,2)$ supersymmetric. It was later realized that the feared instabilities in $(0,2)$ theories do not arise for large classes of theories, including those that can be described by a $(0,2)$ GLSM [5–7].

With this in mind, this thesis will focus on achieving a better understanding of the space of $(0,2)$ GLSM that preserve supersymmetry at lower energies. In particular, one ingredient of non-linear sigma models that we are interested in getting a better handle on through GLSM constructions is *torsion*, which is the flux of the antisymmetric two-form field in a non-linear sigma model. Along the way, we will also find $(2,2)$ analogues of previously constructed $(0,2)$ torsional GLSM, leading to a more complete picture of the most general $(2,2)$ GLSM construction.

The concept of *duality* between different theories will play a key role throughout this work. Duality refers to a quantum equivalence between theories that have different semi-classical descriptions. We have referred above to duality chains that were found between different limits of string theory, allowing us to argue that there is a unique string theory, despite it having several different descriptions valid in different regimes. Many developments in the last decades have also found dualities between different quantum field theories, allowing us to form a more complete picture of the mathematical structure underlying our fundamental theories.

In particular, abelian duality between quantum field theories will be used very often throughout the dissertation. In four-dimensional gauge theories, this is a duality between descriptions using different one-form fields, which is commonly known as electric-magnetic duality. In two-dimensional theories, in its simplest form, abelian duality exchanges a circle of size $R$ for a circle of size $1/R$ in units of the string scale, and it is known as T-duality. When applied to $(2,2)$ GLSM, abelian duality is related to mirror symmetry [8], a mathematical relationship between the topological numbers of different manifolds which was first observed by studying the non-linear sigma models describing those manifolds.

The outline of this dissertation is as follows. In Chapter 2 we will review the essential elements of $(0,2)$ and $(2,2)$ GLSM, as well as abelian duality in its well-understood cases. This will be followed by constructions of certain torsional spaces in a $(2,2)$ context in Chapter 3, providing a more general picture of the essential ingredients that should be considered in such a theory. In Chapter 4, we will explore the parameter space of theories that can be constructed as $(0,2)$ GLSM, finding new examples of such theories that move farther away from the locus of $(2,2)$ theories than previously constructed examples. We will characterize the vacua of such theories and calculate their elliptic genera in order to argue that supersymmetry remains unbroken in such constructions. Finally, in Chapter 5, we will present abelian duality on $(0,2)$ GLSM. As these are chiral gauge theories, path integral manipulations are more subtle than on $(2,2)$ GLSM, and we find that the duals still involve charged fields.

# CHAPTER 2

# $(0, 2)$ AND $(2, 2)$ SUPERSYMMETRY IN TWO DIMENSIONS

In this work, we will deal with aspects of two-dimensional supersymmetric quantum field theories. While in other numbers of dimensions, one number is enough to specify the amount of supersymmetry, in two dimensions (and others of the form $d = 4k + 2$) we need two numbers, one number for the amount of supersymmetry of each chirality. Fermions have chirality in any number of even dimensions, but in dimensions $d = 4k$ action by $CPT$ will bring us from one chirality to the other, so we cannot define supersymmetries of one chirality independently of the other, whereas in $d = 4k + 2$ this is possible. The amount of supersymmetry in a two-dimensional theory is then denoted as $\mathcal{N} = (p, q)$, with $p$ the number of left-moving supersymmetries, and $q$ the number of right-moving supersymmetries. All theories here will have either $\mathcal{N} = (2, 2)$ or $\mathcal{N} = (0, 2)$.

Our metric will be Lorentzian unless otherwise stated, and the spacetime coordinates will be written as $x^{\pm} = \frac{1}{2}(x^0 \pm x^1)$ and $\partial_{\pm} = \partial_0 \pm \partial_1$ so that $\partial_{\pm} x^{\pm} = 1$, $\partial_{\pm} x^{\mp} = 0$, and the non-zero components of the metric are $\eta_{+-} = \eta_{-+} = -2$. Here a $+$ index denotes the right-moving component of a vector, while a $-$ index denotes a left-moving vector. In a slight abuse of notation, we will use the same indices for fermions of that chirality, with the distinction being made by context. This will be helped by the fact that fermions will almost always be denoted by greek letters, while vectors will be denoted by latin letters. In order to ease the building of supersymmetric theories, it is common to add Grassmann-valued coordinates to the spacetime coordinates. In that spirit, we use so-called superspace coordinates with one Grassmann coordinate for each supersymmetry

$$(x^+, x^-, \theta^{\pm}, \bar{\theta}^{\pm})$$

for $(2, 2)$ theories, setting $\theta^- = \bar{\theta}^- = 0$ to reduce to $(0, 2)$. The superspace integration measure for $(2, 2)$ theories will be $d^4\theta = d\bar{\theta}^+ d\theta^+ d\bar{\theta}^- d\theta^-$, defined so that $\int d^4\theta \, \theta^- \bar{\theta}^- \theta^+ \bar{\theta}^+ =$

1. For $(0,2)$ theories we use $d^2\theta^+ = d\bar{\theta}^+ d\theta^+$. The Levi-Civita tensor is defined by $\epsilon^{01} = 1$ so $\epsilon^{-+} = \frac{1}{2}$ and $\epsilon_{+-} = 2$. Throughout this dissertation, unless otherwise indicated, we use actions normalized with $\alpha' = 1$. This avoids relative factors of $4\pi$ in T-duality relations and one-loop corrections. The supersymmetry charges and super-derivatives can be written in terms of superspace coordinates as follows:

$$Q_\pm = \partial_{\theta^\pm} + i\bar{\theta}^\pm \partial_\pm, \qquad \bar{Q}_\pm = -\bar{\partial}_{\theta^\pm} - i\theta^\pm \partial_\pm, \tag{2.1}$$

$$D_\pm = \partial_{\theta^\pm} - i\bar{\theta}^\pm \partial_\pm, \qquad \bar{D}_\pm = -\bar{\partial}_{\theta^\pm} + i\theta^\pm \partial_\pm. \tag{2.2}$$

These operators satisfy the supersymmetry algebras

$$\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm, \qquad \{D_\pm, \bar{D}_\pm\} = 2i\partial_\pm. \tag{2.3}$$

Armed with superspace coordinates, we can define superfields which are functions of both spacetime and the $\theta$ coordinates. Since the $\theta$ are nilpotent, the power series expansion on them terminates, and a superfield can be deconstructed into component fields which are a function of spacetime only. A general superfield $X$ in $(2,2)$ superspace will have $2^4 = 16$ such components. The supersymmetry variation of these components can be obtained by acting with supersymmetry charges on the superfield,

$$\sqrt{2}\delta X = [\epsilon^+ Q_+ - \epsilon^- Q_- - \bar{\epsilon}^+ \bar{Q}_+ + \bar{\epsilon}^- \bar{Q}_-, X]. \tag{2.4}$$

Since the set of component fields in a superfield forms a representation of supersymmetry, superfields are also called multiplets. Using superfields allows us to more easily write supersymmetric theories because the supersymmetric variation of a superspace integral of any function of superfields $X$ vanishes,

$$Q_\pm \int d^2x d^4\theta f(X) = 0 \tag{2.5}$$

and the theory is therefore automatically supersymmetric. As we will see shortly, we can also use these tools to find irreducible representations of supersymmetry actions, by working with superfields obeying certain constraints that anti-commute with the action of supersymmetry.

We will be working mostly with gauged linear sigma models, or GLSM. These were introduced by Witten [2] as ultraviolet-complete quantum field theories that descend at lower energies to theories of interest. These are very useful because, in supersymmetric theories, there are quantities we can compute that are protected from changing under renormalization group flow. GLSM then provide us with a simpler, linear theory where such quantities can be calculated. Throughout this work, we normalize scalar field actions with conventions that correspond to taking $\alpha' = 1$ in the Polyakov action.

In the remainder of this chapter, we will start by presenting the ingredients needed in the $(2,2)$ theories presented, and then decompose those ingredients into their $(0,2)$ components, with a focus on the additional freedom allowed by a smaller amount of supersymmetry. We then summarize T-duality as an abelian duality [9], and its form in gauge theories that led to the "physics proof" of $(2,2)$ mirror symmetry found in [8]. We will take special care to ensure the correct normalization for the Lagrange multiplier terms so that circles in both the original and dual descriptions have $2\pi$ periodicity.

## 2.1 $(2,2)$ supersymmetry

A $(2,2)$ GLSM is then constructed from a collection of constrained superfields. The ingredients we use most often in this work are chiral superfields, $\Phi$, which satisfy $\bar{D}_\pm \Phi = 0$. They contain a complex scalar $\phi$ and two Weyl fermions $\psi_+$ and $\psi_-$, as well as a complex auxiliary field $F$:

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- + 2\theta^-\theta^+ F + \dots, \tag{2.6}$$

with all other terms involving derivatives of these fields. We also use twisted chiral superfields, $\widehat{\Phi}$, satisfying the conditions $\bar{D}_+\widehat{\Phi} = D_-\widehat{\Phi} = 0$. The $\theta$-expansion of these superfields takes the form:

$$\widehat{\Phi} = \widehat{\phi} + \sqrt{2}\theta^+ \widehat{\psi}_+ + \sqrt{2}\bar{\theta}^- \widehat{\psi}_- + 2\bar{\theta}^- \theta^+ \widehat{F} + \dots . \tag{2.7}$$

To gauge an abelian global symmetry acting on chiral superfields, we introduce a $U(1)$ vector superfield $V$, which is a real superfield. Its components include the vector field $A_\mu$, a complex scalar $\sigma$, Weyl fermions $\lambda_+$ and $\lambda_-$, and a real auxiliary field $D$. In Wess-Zumino gauge, the vector superfield has the expansion:

$$\begin{aligned} V = {} & \theta^+\bar{\theta}^+ A_+ + \theta^-\bar{\theta}^- A_- - \theta^-\bar{\theta}^+\sigma + \bar{\theta}^-\theta^+\bar{\sigma} + \sqrt{2}\theta^-\theta^+\bar{\theta}^+\lambda_+ - \sqrt{2}\bar{\theta}^-\theta^+\bar{\theta}^+\bar{\lambda}_+ \\ & + \sqrt{2}\theta^-\bar{\theta}^-\bar{\theta}^+\lambda_- - \sqrt{2}\theta^-\bar{\theta}^-\theta^+\bar{\lambda}_- + 2\theta^-\bar{\theta}^-\theta^+\bar{\theta}^+ D. \end{aligned} \tag{2.8}$$

A gauge transformation acts by sending,

$$V \to V + \frac{i}{2}(\bar{\Lambda} - \Lambda), \qquad \Phi \to e^{iQ\Lambda}\Phi, \tag{2.9}$$

where $\Lambda$ is a chiral superfield, and $\Phi$ has charge $Q$. In Wess-Zumino gauge, most of this gauge freedom becomes fixed, reducing to a more familiar $U(1)$ gauge symmetry. From $V$ we can build the field strength superfield

$$\Sigma = \bar{D}_+ D_- V = \sigma + \sqrt{2}\theta^+\lambda_+ + \sqrt{2}\bar{\theta}^-\lambda_- + \bar{\theta}^-\theta^+(-2D + iF_{-+}) + \dots , \tag{2.10}$$

which is gauge-invariant and twisted chiral by construction. Similarly, if we want to gauge a global symmetry acting on twisted chiral superfields, we need to introduce a chiral vector superfield $\widehat{V}$.

Armed with these ingredients, we can describe the basic couplings of a $(2,2)$ GLSM. The canonical kinetic term of a chiral field $\Phi$ is given by

$$\begin{aligned} S &= \frac{1}{16\pi}\int d^2x d^4\theta\, \bar{\Phi} e^{2QV}\Phi, \\ &= \frac{1}{4\pi}\int d^2x \big[ -|\mathcal{D}_\mu\phi|^2 + i\bar{\psi}_+\mathcal{D}_-\psi_+ + i\bar{\psi}_-\mathcal{D}_+\psi_- + |F|^2 + QD|\phi|^2 - Q^2|\sigma|^2|\phi|^2 \\ &\qquad + Q\lambda_-\bar{\phi}\psi_+ + Q\bar{\psi}_+\phi\bar{\lambda}_- + Q\lambda_+\phi\bar{\psi}_- + Q\psi_-\bar{\phi}\bar{\lambda}_+ + Q\psi_+\sigma\bar{\psi}_- + Q\psi_-\bar{\sigma}\bar{\psi}_-\big], \end{aligned} \tag{2.11}$$

where $\mathcal{D}$ denotes a covariant derivative, $\mathcal{D}_\mu = \partial_\mu + iQA_\mu$. Note the auxiliary field $F$ has no kinetic terms. Kinetic terms for the gauge-field are built from $\Sigma$,

$$\begin{aligned} S &= -\frac{1}{8e^2}\int d^2x d^4\theta\, \bar{\Sigma}\Sigma, \\ &= \frac{1}{2e^2}\int d^2x \left[ -|\partial_\mu\sigma|^2 + i\bar{\lambda}_-\partial_+\lambda_- + i\bar{\lambda}_+\partial_-\lambda_+ + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right]. \end{aligned} \tag{2.12}$$

While scalar fields in two dimensions with canonical kinetic terms (such as $\phi$) have classical mass dimension 0, $\Sigma$ and its bottom component $\sigma$ have mass dimension 1, since the gauge coupling $e$ has mass dimension 1. We could make the kinetic term of the scalar field canonical if we rescaled it by $e$, so we can take $\sigma$ to contain a factor of the scale $e$.

Chiral fields allow one different type of supersymmetry-invariant coupling, namely holomorphic combinations of chiral fields integrated over half of superspace,

$$\begin{aligned} S_W &= \frac{1}{8\pi}\int d^2x d\theta^+ d\theta^-\, W(\Phi) + \text{c.c.} \\ &= \frac{1}{4\pi}\int d^2x \left[\partial_i W(\phi) F^i + \partial_{ij} W \psi_+^i \psi_-^j\right] + \text{c.c.} \end{aligned} \tag{2.13}$$

This type of coupling is usually called *superpotential*. When the auxiliary fields $F^i$ are integrated out, this results in a potential of the form

$$V_W = \sum_i |\partial_i W|^2 \tag{2.14}$$

For holomorphic combinations of twisted chiral superfields, we can build an analogous twisted chiral superpotential $\widetilde{W}$. A standard example is a superpotential involving $\Sigma$, named Fayet-Iliopoulos (FI) coupling for an abelian gauge-field, given by

$$S_{FI} = -\frac{it}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \,\Sigma + \text{c.c.} = \frac{1}{4\pi}\int d^2x \left[-rD + \theta\epsilon^{\mu\nu}F_{\mu\nu}\right], \tag{2.15}$$

where

$$t = \frac{ir}{2} + \theta. \tag{2.16}$$

The auxiliary field $D$ has no kinetic terms, and when it is integrated out we find from the above terms a potential of the form

$$V = \frac{e^2}{2}\left(\sum_i Q_i|\phi^i|^2 - r\right)^2 + \sum_i Q_i^2|\sigma|^2|\phi^i|^2. \tag{2.17}$$

Other superfields that will play a role in our work are neutral twisted chiral superfields with a periodicity

$$Y \sim Y + 2\pi i. \tag{2.18}$$

We will always denote these periodic fields by $Y$. They have a component expansion of the form

$$Y = y + \sqrt{2}\theta^+\chi_+ + \sqrt{2}\bar{\theta}^-\chi_- + 2\bar{\theta}^-\theta^+ G + \ldots. \tag{2.19}$$

and kinetic terms given by

$$\begin{aligned} S &= -\frac{1}{16\pi b}\int d^2x d^4\theta\, \bar{Y}Y, \\ &= \frac{1}{4\pi b}\int d^2x \left[-|\partial_\mu y|^2 + i\bar{\chi}_+\partial_-\chi_+ + i\bar{\chi}_-\partial_+\chi_- + |G|^2\right]. \end{aligned} \tag{2.20}$$

Finally we also consider shift-charged (Stueckelberg-like) chiral superfields, with action

$$\begin{aligned} S &= \frac{b}{32\pi}\int d^2x d^4\theta (P+\bar{P}+2QV)^2 \\ &= \frac{b}{4\pi}\int d^2x \big[ -|\mathcal{D}_\mu p|^2 + i\bar{\psi}_+\partial_-\psi_+ + i\bar{\psi}_-\partial_+\psi_- + |F|^2 + QD(p+\bar{p}) - Q^2|\sigma|^2 \\ &\qquad + Q\lambda_-\psi_+ + Q\bar{\psi}_+\bar{\lambda}_- + Q\lambda_+\bar{\psi}_- + Q\psi_-\bar{\lambda}_+\big]. \end{aligned} \tag{2.21}$$

Specific examples of twisted superpotentials central to our discussion involve the gauge-field strength $\Sigma$ and a $Y$ field, taking the form

$$\begin{aligned} S_{\widetilde{W}} &= -\frac{k}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \, Y\Sigma + \text{c.c.}, \\ &= \frac{k}{4\pi}\int d^2x \left[2\,\text{Re}(y)D + \text{Im}(y)F_{-+} - (\sigma G + \chi_+\lambda_- + \lambda_+\chi_- + \text{c.c.})\right]. \end{aligned} \tag{2.22}$$

or

$$\begin{aligned} S_{\widetilde{W}} &= -\frac{\kappa}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \, e^Y\Sigma + \text{c.c.}, \\ &= \frac{\kappa}{4\pi}\int d^2x \left[2\,\text{Re}(e^y)D + \text{Im}(e^y)F_{-+} - (e^y(\sigma G + \chi_+\lambda_- + \lambda_+\chi_- + \sigma\chi_+\chi_-) + \text{c.c.})\right]. \end{aligned} \tag{2.23}$$

Note that compatibility of these couplings with the periodicity $Y \sim Y + 2\pi i$ imposes $k \in \mathbb{Z}$, but there is no such restriction on $\kappa$.

### 2.2 $(0,2)$ supersymmetry

We will now decompose the $(2,2)$ supersymmetry multiplets into $(0,2)$ multiplets, and outline where the additional freedom from having fewer supersymmetries manifests itself. In $(0,2)$ superspace we are left only with $\theta^+$ and $\bar{\theta}^+$. The integration measure will be given by $d^2\theta = d\bar{\theta}^+ d\theta^+$, such that $\int d^2\theta\theta^+\bar{\theta}^+ = 1$.

A $(2,2)$ chiral superfield can be decomposed in terms of $(0,2)$ chiral and Fermi superfields as

$$\Phi^{(2,2)} = \Phi^{(0,2)} + \sqrt{2}\theta^- \Gamma_- - i\theta^- \bar{\theta}^- \partial_- \Phi^{(0,2)}, \tag{2.24}$$

$$\Phi^{(0,2)} = \phi + \sqrt{2}\theta^+ \psi_+ - i\theta^+ \bar{\theta}^+ \partial_+ \phi, \tag{2.25}$$

$$\Gamma_- = \gamma_- + \sqrt{2}\theta^+ F - i\theta^+ \bar{\theta}^+ \partial_+ \gamma_- - \sqrt{2}\bar{\theta}^+ \left[ E(\phi) + \sqrt{2}\theta^+ \partial_i E(\phi) \psi_+^i \right]. \tag{2.26}$$

$(0,2)$ chirals $\Phi$ obey $\bar{D}_+ \Phi = 0$, while Fermi superfields have a more general condition

$$\bar{D}_+ \Gamma_- = \sqrt{2} E(\Phi), \tag{2.27}$$

where $E$ is a holomorphic function of chiral superfields. On the $(2,2)$ locus, $E$ is fixed to be

$$E(\Phi) = Q \Sigma \Phi. \tag{2.28}$$

For a gauge field, we can decompose the $(2,2)$ vector multiplet $V$ in $(0,2)$ components as

$$V = A - \theta^- \bar{\theta}^+ \Sigma + \bar{\theta}^- \theta^+ \bar{\Sigma} + \theta^- \bar{\theta}^- V_-, \tag{2.29}$$

where the $(0,2)$ superfields have the component expansions

$$A = \theta^+ \bar{\theta}^+ A_+, \tag{2.30}$$

$$\Sigma = \sigma + \sqrt{2}\theta^+ \lambda_+ - i\theta^+ \bar{\theta}^+ \partial_+ \sigma, \tag{2.31}$$

$$V_- = A_- + \sqrt{2}\bar{\theta}^+ \lambda_- - \sqrt{2}\theta^+ \bar{\lambda}_- + 2\theta^+ \bar{\theta}^+ D, \tag{2.32}$$

and we can define the gauge-invariant field strength multiplet as

$$\Upsilon_- = \frac{i}{\sqrt{2}} \bar{D}_+ (\partial_- A + i V_-) = \lambda_- - i\theta^+ \bar{\theta}^+ \partial_+ \lambda_- - \sqrt{2}\theta^+ \left( D + \frac{i}{2} F_{+-} \right). \tag{2.33}$$

Note the $(2,2)$ field strength multiplet can then be expanded in terms of $(0,2)$ superfields as

$$\Sigma^{(2,2)} = \Sigma^{(0,2)} + \sqrt{2}\bar{\theta}^- \Upsilon_- + i\theta^- \bar{\theta}^- \partial_- \Sigma^{(0,2)}. \tag{2.34}$$

Gauge transformations act by

$$A \to A + \frac{\Lambda - \bar{\Lambda}}{2i}, \qquad\qquad V_- \to V_- - \frac{\partial_-(\Lambda + \bar{\Lambda})}{2}, \tag{2.35}$$

where $\Lambda$ is a chiral superfield. Note in $(0,2)$ $\Sigma$ is simply another chiral superfield in the adjoint representation of the gauge group (or for $U(1)$ gauging, a neutral superfield).

Charged chiral fields $\Phi$ have the action

$$\begin{aligned} S &= -\frac{i}{16\pi}\int d^2x d^2\theta^+ \bar{\Phi} e^{2QA} \nabla_- \Phi + \text{c.c.}, \\ &= \frac{1}{4\pi}\int d^2x \big[ -|\mathcal{D}_\mu \phi|^2 + i\bar{\psi}_+ \mathcal{D}_- \psi_+ + QD|\phi|^2 + Q\lambda_- \bar{\phi}\psi_+ + Q\bar{\psi}_+ \phi \bar{\lambda}_- \big] \end{aligned} \tag{2.36}$$

where $\nabla_- = \partial_- + iQV_-$, while the canonical kinetic terms of Fermi multiplets are given by

$$\begin{aligned} S &= -\frac{1}{8\pi}\int d^2x d^2\theta^+ \bar{\Gamma}_- e^{2QA} \Gamma_- \\ &= \frac{1}{4\pi}\int d^2x \left[ i\bar{\gamma}_- \mathcal{D}_+ \gamma_- + |F|^2 - |E|^2 + \psi^i_+ \partial_i E \bar{\gamma}_- + \gamma_- \partial_{\bar{\imath}} \bar{E} \bar{\psi}^{\bar{\imath}}_+ \right]. \end{aligned} \tag{2.37}$$

and in Wess-Zumino gauge, the vector superfield action is

$$\begin{aligned} S &= -\frac{1}{4e^2}\int d^2x d^2\theta^+\, \bar{\Upsilon}_- \Upsilon_-, \\ &= \frac{1}{2e^2}\int d^2x \left[ i\bar{\lambda}_- \partial_+ \lambda_- + D^2 - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right]. \end{aligned} \tag{2.38}$$

In order for the gauge symmetries to not be anomalous, we will always require that the charges of the chiral superfields $\Phi^i$ and of the Fermi superfields $\Gamma_-^a$ obey

$$\sum_i Q_i^2 = \sum_a Q_a^2. \tag{2.39}$$

A superpotential in $(0,2)$ is integrated over only one of the supercharges,

$$\begin{aligned} S_J &= \frac{\sqrt{2}}{8\pi}\int d^2x d\theta^+ \Gamma_- J(\Phi) + \text{c.c.} \\ &= \frac{1}{4\pi}\int d^2x \left[J(\phi)F + \psi_+^i \partial_i J \gamma_-\right] + \text{c.c.} \end{aligned} \tag{2.40}$$

As in $(2,2)$, integrating out $F$ leads to a potential equal to $|J|^2$. To preserve supersymmetry in a theory of Fermi multiplets with both $E$ and $J$ couplings, we must have

$$E \cdot J = 0, \tag{2.41}$$

where the inner product is taken over the space of Fermi multiplets. One can also see that after integrating out auxiliary fields, the physics of Fermi multiplets is invariant under

$$\gamma_- \longleftrightarrow \bar{\gamma}_-, \qquad E \longleftrightarrow J. \tag{2.42}$$

It is often useful to translate between these two descriptions, and we will derive this duality as a form of abelian duality.

The FI parameter coupling in $(0,2)$ notation has the form

$$S_{FI} = -\frac{\sqrt{2}it}{8\pi}\int d^2x d\theta^+ \Upsilon_- + \text{c.c.}, \tag{2.43}$$

with $t$ defined as in (2.16).

## 2.3 From GLSM to NLSM

In this section, we will review how one can obtain geometry from gauge theory, by taking a standard GLSM and descending to an energy scale below the mass scale of the gauge fields into a non-linear sigma model (NLSM). Schematically, at an energy scale $\mu \ll e$, the physics is well-described by the limit $e \to \infty$. When we take this limit, the potential condition

$$-\frac{D}{e^2} = \sum_i Q_i |\phi_i|^2 - r = 0 \tag{2.44}$$

is enforced exactly. In addition, the gauge field kinetic term vanishes in this limit, and therefore the vector field becomes a Lagrange multiplier that simply removes the gauge direction from the action. We see that one condition is imposed on the radial components of the fields $\phi$ and one angle is removed, maintaining the original complex structure.

This is particularly easy to see in $(0,2)$ notation, where in the limit $e \to \infty$, $V_-$ becomes a Lagrange multiplier enforcing the condition

$$\sum_i Q_i |\Phi^i|^2 e^{2Q_i A} = r. \tag{2.45}$$

On a patch of the manifold for which a specific $\Phi$ is non-zero, we can use the full chiral superfield gauge invariance to fix that $\Phi$ to 1, and then use this equation to solve for $A(|\Phi|^2)$.

The action can now be written in the form of a $(0,2)$ non-linear sigma model

$$S = \frac{1}{8\pi} \int d^2x d^2\theta^+ \left[ -\frac{i}{2} K_i(\Phi, \bar{\Phi}) \partial_- \Phi^i + \text{c.c.} - \bar{\Gamma}_-^{\bar{a}} h_{\bar{a}b}(\Phi, \bar{\Phi}) \Gamma_-^b \right]. \tag{2.46}$$

$K_i$ can be obtained in a simple form from the original couplings,

$$K_i = \bar{\Phi}_i e^{2Q_i A} + 2i\theta \partial_i A, \tag{2.47}$$

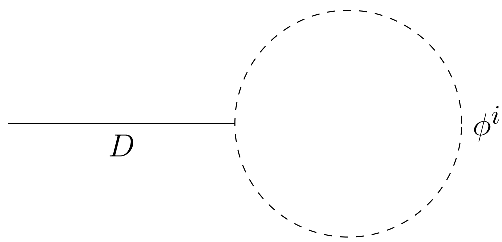


Figure 2.1: The divergent one-loop diagram that remains in a $(0,2)$ and $(2,2)$ GLSM. $\phi^i$ represents all charged chiral fields.

from which we can find the metric and $B$-field appearing in the component action,

$$S = \frac{1}{4\pi}\int d^2x \left[-G_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}} + \epsilon^{\mu\nu}B_{i\bar{\jmath}}\partial_\mu\phi^i\partial_\nu\bar{\phi}^{\bar{\jmath}}\right], \tag{2.48}$$

$$G_{i\bar{\jmath}} = \partial_{(i}K_{\bar{\jmath})}, \qquad\qquad B_{i\bar{\jmath}} = \partial_{[i}K_{\bar{\jmath}]}. \tag{2.49}$$

We can see from the above equations that with no extra couplings $B_{i\bar{\jmath}} = 0$, but we can reach non-zero $B$ if we let the $\theta$ couplings be field-dependent. This possibility will be explored in a $(2,2)$ context in Chapter 3. In $(2,2)$ theories built only from chiral superfields, $K_i = \partial_i K$ for some scalar function $K$, which also implies $B_{i\bar{\jmath}} = 0$, so we will need to include twisted chiral superfields.

## 2.4 Renormalization and Coulomb branch

In both $(0,2)$ and $(2,2)$ gauged linear sigma models, most divergences are cancelled out between boson and fermion contributions. Only one divergent diagram remains, from a loop of bosons coupled to $D$, pictured in Figure 2.1.

This diagram is proportional to the charge of the chiral field, since it multiplies the interaction $|\phi|^2 D$. Taking the boson to have a mass $M$, the result is a correction to the one-point function of $D$ of the form

$$\frac{1}{e^2}\langle D\rangle = Q_i \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M^2} = \frac{Q_i}{4\pi}\log\frac{\Lambda^2+M^2}{\mu^2+M^2}, \tag{2.50}$$

where we introduced a UV cutoff $\Lambda$ and an IR cutoff $\mu$, as this diagram is UV-divergent but also IR-divergent if $M = 0$.

If the scalars are all massless, this diagram can be thought of as giving a renormalization for the FI parameter,

$$t_a(\mu) = t_a(\Lambda) + i \left( \sum_i Q_i^a \right) \log \frac{\mu}{\Lambda}. \tag{2.51}$$

We see this renormalization does not have any effect in GLSM with vanishing sum of charges. If the sum of charges is positive, $r$ is large at large energies and becomes negative as the energy becomes smaller, and vice-versa if the sum of charges is negative.

In $(2,2)$ GLSM in particular, when the scalars in field strength multiplets $\Sigma_a$ gain expectation values, we can see from equation (2.11) that all fields $\phi^i$ charged under the gauge symmetries become massive with mass $|\sum_a Q_i^a \Sigma_a|$ (note that this expression has dimensions of mass, since $\sigma$ does not have a canonical kinetic term, and can be thought of as containing a scale $e$). The scalar fields can then be integrated out at one loop, leading to a quantum correction to the twisted superpotential. The effective superpotential then has the form [2, 10, 11]

$$\begin{aligned} S_{\widetilde{W}} &= \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \sum_a \Sigma_a \left( \sum_i Q_i^a \left[ \log \left( \frac{\sum_b Q_i^b \Sigma_b}{\Lambda} \right) - 1 \right] - it_a \right) \\ &= \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \sum_a \Sigma_a \left( \sum_i Q_i^a \left[ \log \left( \frac{\sum_b Q_i^b \Sigma_b}{\mu} \right) - 1 \right] - it_a(\mu) \right) \end{aligned} \tag{2.52}$$

where $\Lambda$ is the UV renormalization scale, $t_a$ are the bare FI parameters, and $t_a(\mu)$ include the renormalization (2.51). Note that fields $P$ that are shift-charged under the gauge symmetries or fields $Y$ with a $Y\Sigma$ superpotential are not massed up in the same way. This expression is only valid at large values of $\Sigma_a$, where the masses of the fields that have been integrated out are large, so once a solution is obtained it should be checked that we are in the right regime. Varying this superpotential leads to the vacuum equations

$$\sum_i Q_i^a \log\left(\frac{\sum_b Q_i^b \Sigma_b}{\mu}\right) = it_a(\mu) \Rightarrow \prod_i \left(\frac{\sum_b Q_i^b \Sigma_b}{\mu}\right)^{Q_i^a} = e^{it_a(\mu)}. \tag{2.53}$$

Taking the exponential of both sides here does not alter the solutions to the equation, since the right-hand side includes $i\theta$, with its $2\pi i$ periodicity. Since we will be discussing generalizations of this structure, we note that supersymmetric vacua on the Coulomb branch are determined by solutions to

$$\exp\left(\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \Sigma_a}\right) = 1 \tag{2.54}$$

for all field strength multiplets $\Sigma_a$, where $\widetilde{W}_{\text{eff}}$ is the effective twisted chiral superpotential obtained by integrating out all the charged fields. If there are additional twisted chiral fields which are not field strength multiplets, like $Y$ fields, then we also impose the condition

$$\frac{\partial \widetilde{W}_{\text{eff}}}{\partial Y} = 0 \tag{2.55}$$

for each such field $Y$.

As an example, we can use this superpotential to find the vacuum structure of the $\mathbb{C}P^{n-1}$ gauged linear sigma model. The field content of this model consists of one $\Sigma$ and $n$ chiral fields $\Phi^i$ of charge 1. A straightforward application of the formulas above leads to the superpotential

$$S_{\widetilde{W}} = \frac{1}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \Sigma \left(n\left[\log\left(\frac{\Sigma}{\mu}\right) - 1\right] - it(\mu)\right), \tag{2.56}$$

giving $n$ critical points obeying

$$\Sigma^n = \mu^n e^{it(\mu)}. \tag{2.57}$$

As we move to lower energies, we can see from (2.51) that $\text{Re}(it(\mu)) = -r$ grows large, so the mass of the fields we integrated out is $|\sigma| \gg \mu$, and the calculation is justified.

## 2.5 T-duality

One interesting phenomenon in quantum field theory which has spurred research in many different directions over the last few decades is that of duality, wherein two distinct semi-classical descriptions give rise to the same quantum theory. One particularly well-known example of this occurs in theories of compact scalar fields in two dimensions, where theories of free periodic fields of lengths $R$ and $1/R$ (in appropriate units) turn out to be equivalent. In this section, we will derive this duality using a formalism that will prove useful later.

We will derive T-duality by proving equality of the path integrals over the two dual theories, through the construction of a theory with auxiliary fields that can be transformed into either of the dual theories. Schematically, we want to prove equivalence between a theory with action

$$S = -\frac{b}{4\pi}\int d^2x \partial_\mu\varphi\partial^\mu\varphi \tag{2.58}$$

and one given by

$$S = -\frac{1}{4\pi b}\int d^2x \partial_\mu\theta\partial^\mu\theta, \tag{2.59}$$

where both $\varphi$ and $\theta$ are $2\pi$-periodic fields. The strategy will be to rewrite the first theory replacing $d\varphi$ with an unconstrained one-form, adding the right Lagrange multiplier so that the theories are equivalent.

On a topologically non-trivial worldsheet whose first cohomology group has rank $n$, take $\{\omega_i\}$, $i = 1, \ldots, n$ to be a basis for the closed non-trivial one-forms dual to a basis of non-trivial cycles such that the matrix $\int \omega^i \wedge \omega^j = J^{ij}$ is an element of $SL(n, \mathbb{Z})$. A generic closed one-form can then be written

$$c = c_\mu dx^\mu = d\varphi_0 + \sum_i a_i \omega^i, \tag{2.60}$$

with $\varphi_0$ single-valued and $a_i$ real numbers.

To dualize $\varphi$ in (2.58), we will substitute $\partial_\mu\varphi \to c_\mu$ and add a Lagrange multiplier $\theta$

restricting $dc = 0$. The action then becomes

$$S_0 = -\frac{b}{4\pi}\int d^2x c_\mu c^\mu + \kappa \int \theta dc \tag{2.61}$$

where $\kappa$ is a multiplicative constant. When we integrate out $\theta$, the equation $dc = 0$ will allow us to write $c = d\varphi$ and return to the original theory. However, as in (2.60), $\varphi$ in this expression will not be a single-valued field if the worldsheet is topologically non-trivial, so we need to find the relation between the periodicities of $\theta$ and $\varphi$ induced by this coupling.

Defining the Lagrange multiplier $\theta$ to have a period $T_\theta$, we can expand $d\theta$ as

$$d\theta = \partial_\mu \theta dx^\mu = d\theta_0 + T_\theta \sum_i n_i \omega^i, \tag{2.62}$$

with single-valued $\theta_0$ and integers $n_i$. We then have

$$\int c \wedge d\theta = \int c_\mu \partial_\nu \theta dx^\mu \wedge dx^\nu = \int d^2x \epsilon^{\mu\nu} c_\mu \partial_\nu \theta = T_\theta \sum_{i,j} a_i J^{ij} n_j, \tag{2.63}$$

where we added the intermediate forms for later use. Now when we perform the path integral over $\theta$, the $\theta_0$ part gives $dc = 0$, but the integral also includes a sum over $n_i$, which gives

$$\sum_{n_j} \exp\left(-i\kappa T_\theta a_i J^{ij} n_j\right) \propto \prod_i \sum_{m_i} \delta(\kappa T_\theta a_i - 2\pi m_i), \tag{2.64}$$

constraining $a_i$ to be integer multiples of $\frac{2\pi}{\kappa T_\theta}$, so comparing to (2.60), we see that $c = d\varphi$ where $\varphi$ is periodic with period $T_\varphi = \frac{2\pi}{\kappa T_\theta}$. Therefore, if we want $T_\theta = T_\varphi = 2\pi$, we should take $\kappa = 1/(2\pi)$.

Now that we fixed the constant in (2.61), the dual action can be obtained by instead integrating out $c_\mu$. Since the action is quadratic, the path integral sets $c_\mu$'s equation of motion,

$$bc^\mu = -\epsilon^{\mu\nu}\partial_\nu \theta, \tag{2.65}$$

or in components, substituting $c$ by $d\varphi$, we obtain the map between the two dual fields

$$b\partial_\pm \varphi = \pm\partial_\pm \theta. \tag{2.66}$$

Inserting this into (2.61), we obtain the dual action (2.59).

## 2.6 (2, 2) duality

Consider now a free $(2,2)$ chiral field $P$ with periodicity $P \sim P + 2\pi i$:

$$S = \frac{b}{16\pi} \int d^2x d^4\theta \, |P|^2. \tag{2.67}$$

To dualize the imaginary component of $P$, we will substitute $P + \bar{P} \to 2B$ where $B$ is a generic real superfield. Denoting the imaginary part of $P$ by $\varphi$ and the one-form in $B$ by $c_\mu$, we see this corresponds to the component substitution in the previous section since

$$\begin{aligned} P + \bar{P} &= \cdots - i\theta^+\bar{\theta}^+\partial_+(p - \bar{p}) - i\theta^-\bar{\theta}^-\partial_-(p - \bar{p}) \\ &= \cdots + 2\theta^+\bar{\theta}^+\partial_+\varphi + 2\theta^-\bar{\theta}^-\partial_-\varphi \end{aligned} \tag{2.68}$$

and we expand

$$B = \cdots + \theta^+\bar{\theta}^+ c_+ + \theta^-\bar{\theta}^- c_- \tag{2.69}$$

so $P + \bar{P} \to 2B$ is a $(2,2)$ extension of $d\varphi \to c$.

Carrying out the substitution and including a Lagrange multiplier $F$, the action becomes

$$\begin{aligned} S_0 &= \frac{1}{8\pi} \int d^2x d^4\theta \left[ bB^2 + F\bar{D}_+ D_- B - \bar{F} D_+ \bar{D}_- B \right] \\ &= \frac{1}{8\pi} \int d^2x d^4\theta \left[ bB^2 - D_-\bar{D}_+ F B + \bar{D}_- D_+ \bar{F} B \right] \\ &= \frac{1}{8\pi} \int d^2x d^4\theta \left[ bB^2 - B(Y + \bar{Y}) \right], \end{aligned} \tag{2.70}$$

where we integrated by parts and defined $Y = D_-\bar{D}_+F$ to obtain the second form of the action. Note that $Y$ is twisted chiral by definition. If we integrate out the Lagrange multiplier $F$, its equation of motion is solved by setting $B$ to the sum of a chiral and its conjugate and we recover the original theory. We should check that the factor multiplying the Lagrange multiplier term is the one determined in section 2.5: expand

$$Y + \bar{Y} = \cdots + 2\theta^+\bar{\theta}^+\partial_+\theta - 2\theta^-\bar{\theta}^-\partial_-\theta, \tag{2.71}$$

from which we see

$$\begin{aligned} S_L &= -\frac{1}{8\pi}\int d^2x d^4\theta B(Y+\bar{Y}) \\ &= -\frac{1}{4\pi}\int d^2x \left[c_-\partial_+\theta - c_+\partial_-\theta\right] + \ldots \\ &= -\frac{1}{2\pi}\int d^2x \epsilon^{\mu\nu} c_\mu \partial_\nu \theta + \ldots \end{aligned} \tag{2.72}$$

therefore the periods of $\theta$ and $\varphi$ will be related as $T_\theta T_\varphi = 4\pi^2$, and if one is $2\pi$-periodic, so will the other.

Solving (2.70) for $B$ instead, we find the dual action

$$S_d = -\frac{1}{32\pi b}\int d^2x d^4\theta (Y+\bar{Y})^2 = -\frac{1}{16\pi b}\int d^2x d^4\theta |Y|^2. \tag{2.73}$$

This is a canonical kinetic term for a twisted chiral. Note the duality has mapped

$$b(P+\bar{P}) = Y + \bar{Y}, \tag{2.74}$$

which for their scalar component fields becomes

$$b(p+\bar{p}) = y + \bar{y}, \tag{2.75}$$

$$b\partial_\pm(p-\bar{p}) = \pm\partial_\pm(y-\bar{y}). \tag{2.76}$$

This confirms our assertion that we are performing a T-duality transformation on the imaginary part of $p$. The imaginary part of $p$ parametrizes a circle with radius $\sqrt{b}$, while the circle parametrized by the imaginary part of $y$ has radius $1/\sqrt{b}$.

Such a field $P$ can be axially charged, making its imaginary part a two-dimensional Stueckelberg field. The action is simply

$$S = \frac{b}{32\pi} \int d^2x d^4\theta \left(P + \bar{P} + 2V\right)^2 . \tag{2.77}$$

Using the same substitution as above we have

$$S_0 = \frac{1}{8\pi} \int d^2x d^4\theta \left[b(B+V)^2 - B(Y+\bar{Y})\right] , \tag{2.78}$$

which we can solve for $B$, to find

$$\begin{aligned} S_d &= \frac{1}{8\pi} \int d^2x d^4\theta \left[-\frac{1}{4b}(Y+\bar{Y})^2 + (Y+\bar{Y})V\right] \\ &= -\frac{1}{16\pi b} \int d^2x d^4\theta \bar{Y} Y - \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- Y\Sigma + \text{c.c.}. \end{aligned} \tag{2.79}$$

The last term in the action is the coupling between $Y$ and $\Sigma$ in (2.22). Note that a field $P$ with charge $Q_p$ is dual to $Y$ with coupling $k_y = Q_p$. The duality maps are similar to (2.76) except the $p$ side will now feature covariant derivatives $\mathcal{D}_\pm p = \partial_\pm p + A_\pm$, so both sides of the maps are gauge-invariant.

Alternatively, if we start from a $(2,2)$ chiral $\Phi$ parametrizing a plane, we can redefine $\Phi = e^\Pi$ to dualize the phase of $\Phi$ by similar techniques to the above. Since $(2,2)$ theories are not chiral, no non-trivial Jacobian results from this redefinition. Another way to understand the necessity of making this redefinition is the fact that we the procedure by which we are dualizing can be rephrased as gauging isometries of the theory [9]. However, the phase of $\phi$ does not parametrize an isometry, since it appears in several Yukawa couplings. Setting $\Phi = e^\Pi$ makes these couplings independent of the imaginary part of $\pi$, providing us with an isometry to dualize.

Then, replacing $\Pi + \bar{\Pi} \to 2B$,

$$S_0 = \frac{1}{16\pi} \int d^2x d^4\theta \left[ e^{2B} - 2B(Y + \bar{Y}) \right], \tag{2.80}$$

from which we can once again solve for $B$ to find

$$S_d = -\frac{1}{16\pi} \int d^2x d^4\theta \, (Y + \bar{Y}) \log(Y + \bar{Y}). \tag{2.81}$$

The same procedure can be done if $\Phi$ is charged,

$$S = \frac{1}{16\pi} \int d^2x d^4\theta \, |\Phi|^2 e^{2V} \tag{2.82}$$

$$\Rightarrow S_0 = \frac{1}{16\pi} \int d^2x d^4\theta \left[ e^{2B+2V} - 2B(Y + \bar{Y}) \right], \tag{2.83}$$

$$\begin{aligned} S_d &= -\frac{1}{16\pi} \int d^2x d^4\theta \, \left[ (Y + \bar{Y}) \log(Y + \bar{Y}) - 2(Y + \bar{Y}) V \right] \\ &= -\frac{1}{16\pi} \int d^2x d^4\theta \, (Y + \bar{Y}) \log(Y + \bar{Y}) - \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \, Y\Sigma + \text{c.c.}. \end{aligned} \tag{2.84}$$

In this case, for the theories to be quantum-mechanically equivalent, we must add a term to the twisted superpotential of the dual theory reflecting instanton corrections in the original theory [8], so the full dual action is

$$S_d = -\frac{1}{16\pi} \int d^2x d^4\theta \, (Y + \bar{Y}) \log(Y + \bar{Y}) - \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \, \left[ Y\Sigma + \mu e^{-Y} \right] + \text{c.c.}. \tag{2.85}$$

# CHAPTER 3

# $(2,2)$ GEOMETRY FROM GAUGE THEORY

Using gauge theory, we describe how to construct generalized Kähler geometries with $(2,2)$ two-dimensional supersymmetry, which are analogues of familiar examples like projective spaces and Calabi-Yau manifolds. For special cases, T-dual descriptions can be found which are squashed Kähler spaces. We explore the vacuum structure of these gauge theories by studying the Coulomb branch, which usually encodes the quantum cohomology ring. Some models without Kähler dual descriptions possess unusual Coulomb branches. Specifically, there appear to be an infinite number of supersymmetric vacua.

## 3.1 Introduction

This work concerns sigma models in two dimensions with target space $\mathcal{M}$ and local coordinates $\phi$. Ignoring fermions, the bosonic sigma model action in a target space patch takes the form

$$S = \frac{1}{4\pi\alpha'} \int d^2x \sqrt{h} h^{\alpha\beta} G_{ij} \partial_\alpha \phi^i \partial_\beta \phi^j + i \int \phi^*(B), \tag{3.1}$$

where $h$ is the two-dimensional worldsheet metric, while $G$ and $B$ denote the target space metric and 2-form $B$-field. Requiring extended worldsheet supersymmetry means introducing fermions and also restricting the target space $\mathcal{M}$. Of particular interest are models with chiral $(0,2)$ worldsheet supersymmetry, suitable for the heterotic string, and models with non-chiral $(2,2)$ supersymmetry suitable for both the heterotic and type II strings.

Target spaces which are compatible with $(2,2)$ supersymmetry are called generalized Kähler spaces [12–14]. There are two basic issues one might try to address. The first is classifying the geometric structures required for $\mathcal{M}$ to admit $(2,2)$ supersymmetry, and the corresponding implications for superspace constructions. There has been a great deal of progress along these lines starting with [12]. For a large, but not completely general, class of $(2,2)$ non-linear sigma models, the basic needed superspace ingredients are chiral, twisted chiral and semi-chiral superfields [15]. See [16] for a recent discussion of the defining data for more general $(2,2)$ models.

The second issue is the question of constructing classes of $(2,2)$ target spaces. This question has a somewhat different flavor because acceptable target spaces can include ingredients that require a physical explanation; for example, spaces with orbifold singularities, particularly those with discrete torsion, brane sources, or the use of stringy worldsheet symmetries like T-duality in patching conditions.

The simplest examples of $(2,2)$ sigma models have Kähler target spaces $\mathcal{M}$. Imposing conformal invariance further restricts $\mathcal{M}$ to a Calabi-Yau space. Once one reaches Calabi-Yau 4-folds, there are believed to be an enormous number of such spaces with lower bound estimates of $O(10^{755})$ [17], and a recent Monte-Carlo based estimate of $O(10^{3000})$ [18]! On top of this geometric degeneracy is the usual enormous number of choices of flux, estimated in one case to be $O(10^{272,000})$ [19]. Somewhat surprising is the realization that a large fraction of these Calabi-Yau spaces admit elliptic fibrations and even $K3$-fibrations [20].

Duality between the heterotic string and $K3$-fibered F-theory flux vacua, built from Calabi-Yau 4-folds, suggests that there should exist an enormous number of worldsheet string geometries with non-vanishing $H = dB$ [1]. These are not Calabi-Yau manifolds but rather a kind of torsional background compatible with $(0,2)$ worldsheet supersymmetry. The expected number of such geometries should dwarf the number of currently known Calabi-Yau 3-folds. Yet very few compact examples are known. Unlike the case of Kähler target spaces, there are few if any systematic constructions of flux geometries with $H \neq 0$. We are missing tools like algebraic geometry which might provide us with large classes of such spaces.

This picture motivates us to move away from the familiar Kähler geometries visible under the lamp post, and search for the new ingredients and structures needed to describe more generic string geometries with non-vanishing $H$. Along the way, we will learn more about the physics of NS-branes and anti-branes. By an NS-brane we mean a localized magnetic source for $B$ such that the charge is non-vanishing,

$$\int_{C_3} H \neq 0, \tag{3.2}$$

where $C_3$ encloses the brane. The sign of the charge distinguishes a brane from an anti-brane. When the sigma model (3.1) is conformal and can serve as a classical string background, these NS-branes are the familiar NS5-branes. However, the definition (3.2) applies to both gapped and conformal sigma models.

The approach we will take is to generalize the gauged linear sigma model (GLSM) construction described by Witten [2]. Our generalization is motivated by the $(0,2)$ constructions described in [21–25], and specifically [26]. We will provide analogous constructions for models with $(2,2)$ supersymmetry. The enhanced $(2,2)$ supersymmetry makes a far larger set of tools available for analysis. While the most general model with $(2,2)$ supersymmetry involves semi-chiral superfields, in this work we will restrict our discussion to models constructed from chiral superfields $\Phi$ satisfying

$$\bar{D}_+\Phi = \bar{D}_-\Phi = 0, \tag{3.3}$$

and twisted chiral superfields $Y$ satisfying

$$\bar{D}_+Y = D_-Y = 0. \tag{3.4}$$

Our conventions are described in Chapter 2. We will also only consider abelian gauge theories. In the usual Kähler setting, this corresponds to considering toric spaces $\mathcal{M}$. Generalizing these constructions by considering non-abelian gauge theories, and by including semi-chiral representations is likely to be interesting.

The main new ingredient over the original work of [2] is the inclusion of periodic superfields,

$$Y \sim Y + 2\pi i. \tag{3.5}$$

Such periodic fields appear in mirror descriptions of $(2,2)$ and $(0,2)$ GLSM theories [8, 27, 28], and in earlier GLSM constructions for torsional target spaces [21, 29, 30]. The superfield $Y$ can be used to build field-dependent Fayet-Iliopoulos (FI) couplings,

$$\int d^2x d\theta^+ d\bar{\theta}^- Y\Sigma, \tag{3.6}$$

where $\Sigma$ is the field strength for a vector superfield. This coupling leads to torsion in the target space. As we will see later, including more couplings respecting the periodicity (3.5), like twisted superpotentials involving $e^Y$, gives interesting physical models.

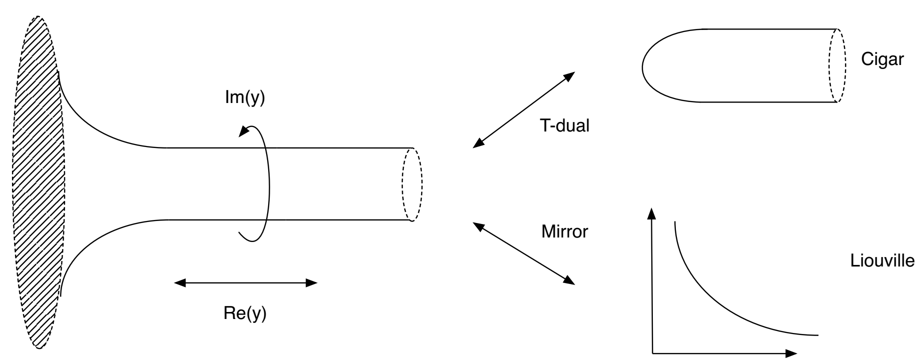


Figure 3.1: The basic trumpet geometry for an NS-brane and its dual realizations.

We can sketch the basic building blocks for our constructions. With one $U(1)$ gauge field and a field-dependent FI coupling (3.6), the target space $\mathcal{M}$ is non-compact. This ingredient is depicted in Figure 3.1. The picture in $Y$ variables is the 'trumpet' geometry, while a T-dual description gives the cigar geometry. Finally a mirror description gives the Liouville theory, which involves a potential energy coupling rather than pure geometry. The blowing up of the Im$(Y)$ circle is the hallmark of the brane in $Y$ variables. This target space is a conformal field theory if one includes an appropriate varying dilaton field. The equivalence between the $(2,2)$ cigar and Liouville descriptions was argued by Giveon and Kutasov [31, 32], building on earlier work and conjectures [33, 34]. A GLSM derivation of the equivalence was provided by Hori and Kapustin [35]. The three pictures of the same physical system make clear the need to include T-duality – the equivalence between large and small circles in string theory – in the patching conditions for the target geometry $\mathcal{M}$. The trumpet is better described in terms of the cigar geometry near the locus where the circle blows up, while the cigar is better described by the Liouville theory when the asymptotic circle becomes small. In addition, the need for the Liouville description makes clear that we must allow potentials as well as metrics and $B$-fields when discussing more general notions of string geometry. This is quite reminiscent of the structure seen in hybrid Landau-Ginzburg phases; see, for example [36, 37].

With multiple abelian gauge fields, the picture becomes richer. Instead of a semi-infinite trumpet, we can build finite-sized cylindrical fixtures. A product of toric varieties is typically fibered over each fixture with varying Kähler parameters. This is schematically depicted in Figure 3.2. The interpretation of this geometry is that one end of the fixture supports a wrapped brane while the other end supports an anti-brane. In a precise sense, these geometries are torsional dual descriptions of compact squashed toric varieties like projective spaces, introduced by Hori and Kapustin [35]. This duality, which generalizes the standard relation between NS5-branes and ALF spaces, is described in section 3.2. As we will see below, it is useful to have both descriptions of the same physical system in order to explore generalizations.

Generic models involving this collection of ingredients cannot, however, be dualized to purely Kähler spaces; they are intrinsically torsional. We discuss two flavors of such models. The first flavor involves allowing the complex structure parameters of a fibered space to vary with $Y$. In a sense, this is the mirror version of the Kähler fibration of Figure 3.2. This leads to generalized Kähler spaces of schematic form depicted in Figure 3.3. We describe

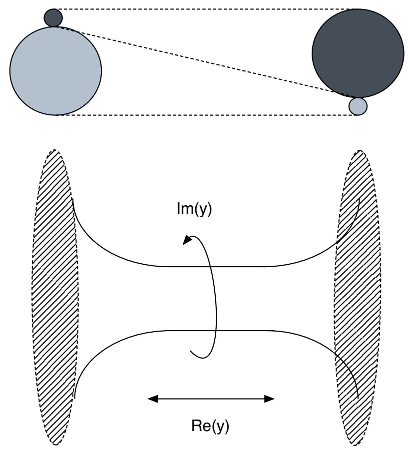


Figure 3.2: A product of toric spaces fibered over the cylindrical fixture.

examples of such models in section 3.3, and discuss the appearance of flat directions when one attempts to construct conformal models – the analogues of Calabi-Yau spaces – as complete intersections in the torsional analogues of toric varieties. We find it plausible that the flat directions we persistently find when a fibered toric space shrinks to zero size reflect the non-perturbative space-time physics supported on the NS-branes. This would match other examples where space-time non-perturbative physics, like the appearance of enhanced gauge symmetry at an ALE singularity, is reflected by a new branch in the associated gauged linear sigma model.

The other generalization leads to wilder structures. Instead of simply allowing couplings like (3.6) which preserve the $U(1)$ isometry that shifts the imaginary part of $Y$, we consider more general field-dependent FI couplings,

$$\int d^2x d\theta^+ d\bar{\theta}^- f(Y, e^Y)\Sigma. \tag{3.7}$$

The inclusion of interactions like this in otherwise topological interactions is familiar from

$\mathcal{N} = 1$ and $\mathcal{N} = 2$ $D = 4$ gauge theory, where superpotential or prepotential interactions can be generated by instantons or strong coupling effects. In $D = 2$, we can include such couplings in the ultraviolet model and such models are described in section 3.3.4. The classical vacuum equations have interesting new properties. We comment on some puzzling but interesting aspects of the quantum vacuum structure on the Coulomb branch of these models in section 3.4. These Coulomb branch vacua are usually related to quantum cohomology rings of the target space $\mathcal{M}$. We find that the inclusion of these more general $U(1)$ breaking couplings leads to an infinite number of discrete Coulomb branch vacua, which is in sharp contrast to the finite number of vacua found in GLSMs describing Kähler spaces. We also find an analogue of the quantum cohomology ring for a class of generalized Kähler examples.

Finally, it is worth noting that the way in which the spaces are constructed has a flavor similar to a recent construction of $G_2$ spaces at the level of geometry and conformal field theory [38–40]. It would be very interesting if this gauged linear construction can be generalized to produce target geometries with $G_2$ holonomy.

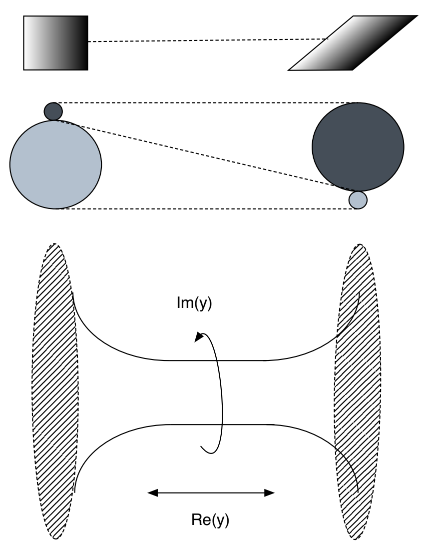


Figure 3.3: The more general case with both complex and Kähler parameters fibered over the fixture.

## 3.2 Dual descriptions

We will consider models built from chiral superfields $\Phi$ with charge $Q$ under an abelian gauge group. All basic conventions are found in Chapter 2. These superfields contain one complex scalar $\phi$. Under a gauge transformation with chiral superfield parameter $\Lambda$, the vector superfield $V$ for the gauge symmetry and $\Phi$ transform as follows:

$$V \to V + \frac{i}{2}(\bar{\Lambda} - \Lambda), \qquad \Phi \to e^{iQ\Lambda}\Phi. \tag{3.8}$$

The field strength superfield defined in (2.10) is denoted $\Sigma$. It contains one complex scalar $\sigma$. Following the notation of [35], we also consider chiral superfields $P$ which are shift charged under the gauge group:

$$P \to P + i\ell\Lambda, \qquad P \sim P + 2\pi i. \tag{3.9}$$

with $\ell \in \mathbb{Z}$. The chiral superfield $P$ also contains one complex scalar $p$. We will write the kinetic terms of $P$ as

$$S^P_{\text{kin}} = \frac{b}{32\pi} \int d^2x d^4\theta \left(P + \bar{P} + 2\ell V\right)^2, \tag{3.10}$$

allowing an overall factor $b$. We could allow the periodicity to vary as well, introducing an additional real parameter $\delta$ and defining $P$ such that $P \sim P + 2\pi i\delta$. However, we can always scale $P$ to have $\delta = 1$, absorbing the change into $b$ and $\ell$. This is what we will do in this work. The last ingredient is twisted chiral neutral superfields $Y$ with periodicity given in (3.5) containing a complex scalar $y$.

Now this is a highly asymmetric treatment of chiral versus twisted chiral superfields. Mirror symmetry exchanges the two kinds of constrained superfield, and it was recognized in very early attempts to prove mirror symmetry that a more symmetric treatment might prove helpful. Following the terminology of Morrison and Plesser [41], we introduce superfields for a twisted GLSM. These fields are denoted by a 'hat' so $\widehat{\Phi}$ is a *twisted chiral* superfield charged under a vector superfield $\widehat{V}$ with *chiral* field strength $\widehat{\Sigma}$. The *chiral* superfields $\widehat{Y}$ are neutral and periodic, just like their twisted chiral cousins in (3.5),

$$\widehat{Y} \sim \widehat{Y} + 2\pi i. \tag{3.11}$$

These are similar to the fields $P$ in (3.9), except we will always take fields $Y$ to not transform under the gauge group while $P$ is shift-charged. As we will see in section 3.3.3, all these field types are very natural for describing general models.

### *3.2.1 Holomorphic data and kinetic terms*

We can group the collection of fields into uncharged periodic fields $(Y, \widehat{Y})$, charged fields $(\Phi, \widehat{\Phi}, P, \widehat{P})$ and gauge fields $(V, \widehat{V})$ with field strengths $(\Sigma, \widehat{\Sigma})$. The data over which we have the most control under RG flow are holomorphic couplings. These include the superpotential,

$$S_W = \frac{1}{8\pi} \int d^2x d\theta^+ d\theta^- \left\{ -i\widehat{t}\widehat{\Sigma} - \widehat{k}\widehat{\Sigma}\widehat{Y} + W(\Phi, P, e^{\widehat{Y}}, \widehat{\Sigma}) + \text{c.c.} \right\}, \tag{3.12}$$

where $\widehat{t}$ determines the FI parameter via (2.15), $\widehat{k}$ is an integer in order to be compatible with the $\widehat{Y}$ periodicity, and $W$ is gauge-invariant and single-valued. There is an analogous twisted chiral superpotential with form,

$$S_{\widetilde{W}} = \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \left\{ -it\Sigma - k\Sigma Y + \widehat{W}(\widehat{\Phi}, \widehat{P}, e^{Y}, \Sigma) + \text{c.c.} \right\}, \tag{3.13}$$

where again $k \in \mathbb{Z}$ and $\widehat{W}$ is single-valued.[1] It is clear from the dependence of the superpotential on $t$ and $Y$ that we could choose to absorb the FI parameters $t$ into a constant shift of the fields $Y$, without affecting the $Y$ kinetic terms. However, the FI parameters $t$ will typically be additively renormalized. While we could choose to absorb this renormalization into a shift of $Y$, for clarity we choose to always keep $t$ explicit in this work. The function

1. Actually the superpotential and twisted superpotential need not be classically gauge-invariant if one includes $\int d^4\theta V\widehat{V}$ couplings, which appear in [41].

$\widehat{W}$ may include terms such as the typical polynomials in the charged fields $\widehat{\Phi}$, polynomials including $\widehat{\Phi}$ as well as $e^{\widehat{P}}$ and $e^{Y}$, and other terms including only $Y$ and $\Sigma$ like $e^{Y}\Sigma$. We will briefly explore these possibilities in what follows.

These are the obvious holomorphic couplings, but there is actually more potential holomorphic data hidden in the kinetic terms because of the periodic fields $(Y, \widehat{Y})$. Let us focus on the $Y$ fields. A general kinetic term for $Y$ takes the form,

$$S^{Y}_{\text{kin}} = -\frac{1}{16\pi}\int d^2x d^4\theta \, \left(\bar{Y} f_1 + f_2 + \text{c.c.}\right), \tag{3.14}$$

where the $f_i$ are gauge-invariant functions of the superfields, and $f_2$ is single-valued. To be sensible under the periodic identification (3.5), $f_1$ must be annihilated by $\int d^4\theta$. This constrains $f_1$ to be a holomorphic function of the twisted chiral superfields; $f_1$ can also depend either holomorphically or anti-holomorphically on the chiral superfields, so that $\bar{D}_+ f_1 = 0$ or $D_- f_1 = 0$. This holomorphic data is likely to prove interesting for non-linear sigma models.

In a linear theory, we are typically interested in kinetic terms that are quadratic in the fields. At the quadratic level, there are no direct couplings of chiral and twisted chiral fields so the $Y$ and $\widehat{Y}$ fields do not kinetically mix. If there are $n$ $Y$-fields then the choice of $Y$ kinetic term corresponds to a choice of metric and $B$-field for $T^n$ encoded in $k_{\mu\bar{\nu}}$

$$S^{Y}_{\text{kin}} = -\frac{1}{16\pi}\int d^2x d^4\theta \, k_{\mu\bar{\nu}} Y_\mu \bar{Y}_{\bar{\nu}}. \tag{3.15}$$

The classical moduli space is the familiar $n^2$-dimensional Narain moduli space,

$$\frac{O(n,n,\mathbb{R})}{O(n)\times O(n)}. \tag{3.16}$$

We will not worry about discrete identifications on the moduli space since those identifications are not generally preserved by interactions. A similar discussion applies to the $\widehat{Y}$

fields.

The charged and uncharged fields also do not mix kinetically at the level of quadratic interactions. For a $\Phi$ field with charge $Q$, we simply assume canonical kinetic terms,

$$S_{\text{kin}}^{\Phi} = \frac{1}{16\pi} \int d^2x d^4\theta \, \bar{\Phi} e^{2QV} \Phi, \tag{3.17}$$

whose component form appears in (2.11), and similarly for a charged $\widehat{\Phi}$ field.

### *3.2.2 Worldsheet duality*

Here we recall the duality dictionary we will subsequently use. More details were presented in Section 2.5. A $(2,2)$ chiral superfield $P$ with periodicity given in (3.9) can be axially charged, making its imaginary part a two-dimensional Stueckelberg field. The action is simply,

$$S = \frac{b}{32\pi} \int d^2x d^4\theta \left(P + \bar{P} + 2V\right)^2, \tag{3.18}$$

where $b$ is a constant that will later be interpreted as a squashing parameter for the models of section 3.3.2. This particular theory has a dual description in terms of a $(2,2)$ twisted chiral superfield $Y$ with the same periodicity, and with action:

$$S_d = -\frac{1}{16\pi b} \int d^2x d^4\theta \, \bar{Y} Y - \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \, Y\Sigma + \text{c.c.}. \tag{3.19}$$

This duality requires the coupling between $Y$ and $\Sigma$ found in (3.6), which will feature heavily in this work.

Alternatively, a $(2,2)$ chiral $\Phi$ parametrizing $\mathbb{C}$ can also be dualized. The action for a charged chiral takes the form,

$$S = \frac{1}{16\pi} \int d^2x d^4\theta \, |\Phi|^2 e^{2V}, \tag{3.20}$$

and the dual $Y$ is also a twisted chiral superfield with periodicity (3.5), and with action:

$$\begin{aligned} S_d = & -\frac{1}{16\pi}\int d^2x d^4\theta\,(Y+\bar{Y})\log(Y+\bar{Y}) \\ & -\left[\frac{1}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \,\left(Y\Sigma + \mu e^{-Y}\right) + \text{c.c.}\right]. \end{aligned} \tag{3.21}$$

### *3.2.3 Anomaly and conditions for conformality*

Given a $(2,2)$ gauged linear sigma model defined in the ultraviolet, it is usually a non-trivial issue to decide whether or not it flows to a non-trivial conformal field theory. One way to strengthen the case for a non-trivial infrared fixed point is to construct a candidate $\mathcal{N}=2$ superconformal algebra in the ultraviolet theory. This boils down to finding a non-anomalous right-moving R-symmetry current. In usual $(2,2)$ gauged linear sigma models built using only chiral superfields, this is possible if the sum of the charges vanishes,

$$\sum_i Q_{ia} = 0, \tag{3.22}$$

for all gauge symmetries. This means that the FI parameters of the theory will be invariant under renormalization group flow, and that the curvature two-form for the non-linear model found by symplectic quotient will be trivial in cohomology, so there is a Ricci-flat metric in the same Kähler class. However, this condition is modified once we add $P$ and $Y$ fields. In this section we will find the more general condition.

Let us start by defining our R-symmetries in the gauge theory. Under $U(1)_R$, $\theta^+$ has charge 1, while under $U(1)_L$, $\theta^-$ has charge 1. In a V-A basis, $q_{\theta+}=(1,1)$, and $q_{\theta-}=(1,-1)$. Note $d\theta$ transforms as $\theta^{-1}$, so a chiral superpotential should have $R$ transformation with charges $(2,0)$ in that basis, while a twisted superpotential would have charges $(0,2)$. The existence of an FI parameter then fixes $q_\Sigma=(0,2)$. A chiral superfield with charge $Q$ causes an anomaly under $U(1)_A$, since $q_{\psi_+}-q_{\psi_-}=-2$. This corresponds to a variation of the effective action

$$\delta S = \frac{\beta Q(-2)}{4\pi} \int d^2 x \epsilon^{\mu\nu} F_{\mu\nu}, \tag{3.23}$$

where $\beta$ is the transformation parameter. The same expression is valid for a twisted chiral with charge $\widehat{Q}$ under $U(1)_V$. We can see from (3.21) and (2.22) that reproducing this anomaly fixes the transformation under $U(1)_A$ for a field $Y$ dual to a charged $\Phi$ to be a shift charge of $-2$, so $e^{-Y}$ in (3.21) transforms like a twisted superpotential.

On the other hand, when $Y$ is not dual to a charged $\Phi$, its $U(1)_A$ transformation is not fixed. Defining it generally to have a shift charge $2\gamma$ under $U(1)_A$, or $\pm\gamma$ under $U(1)_{R,L}$, will cause the action (3.13) with $\tilde{W} = 0$ to vary as

$$\delta S = \frac{\beta(2k\gamma)}{4\pi} \int d^2 x \epsilon^{\mu\nu} F_{\mu\nu} \tag{3.24}$$

from the coupling (2.22). The updated condition for a non-anomalous $U(1)_R$ symmetry with $Y^\mu$ fields coupled to gauge fields $\Sigma^a$ with coefficients $k_{\mu a}$ is then obtained by requiring the sum of variations (3.23) and (3.24) to vanish,

$$\sum_i Q_{ia} - \sum_\mu \gamma_\mu k_{\mu a} = 0 \tag{3.25}$$

for some charges $\gamma_\mu$.

What about our fields $P$ with Stueckelberg couplings? At first sight, unlike $Y$ they do not have a coupling that would let us compensate an anomalous variation with a field transformation. However, in many cases we can take a field $Y$ and dualize it into $P$, so the same condition must apply in both pictures. To understand how this works, note that postulating a transformation $y \to y + i\gamma\beta$ means we are modifying the axial R-symmetry current by

$$j_+ = \cdots - \frac{i\gamma}{2}\partial_+(y - \bar{y}), \qquad j_- = \cdots - \frac{i\gamma}{2}\partial_-(y - \bar{y}). \tag{3.26}$$

If we dualize $y$ into $p$, from (2.76), these current modifications become

$$j_+ = \cdots - \frac{i\gamma}{2}D_+(p - \bar{p}), \qquad j_- = \cdots + \frac{i\gamma}{2}D_-(p - \bar{p}). \tag{3.27}$$

These are the current improvement terms constructed in [35]. These terms do not give $p$ a variation under $U(1)_A$, but they modify the current conservation equation. In conclusion, for each field $p_\alpha$ with shift charges $\ell_{\alpha a}$ under the gauge symmetries, we will also have a parameter that modifies the condition for a non-anomalous R-symmetry into

$$\sum_i Q_{ia} - \sum_\alpha \gamma_\alpha \ell_{\alpha a} - \sum_\mu \gamma_\mu k_{\mu a} = 0. \tag{3.28}$$

This is the condition we must work with in models with $Y^\mu \Sigma^a$ couplings. On the other hand, note that a superpotential term of the form $e^{Y^\mu}\Sigma$ precludes any transformation of that field $Y^\mu$ under $U(1)_R$, setting $\gamma_\mu = 0$. This is because $\Sigma$ alone must have the correct transformation for a twisted superpotential as long as the FI coupling is non-trivial.

When we do expect a theory to flow to a non-trivial infrared fixed point, we can construct the protected, right-moving chiral algebra in the ultraviolet whose central charge, in particular, agrees with that of the infrared theory in the absence of accidents [42]. We will construct the right-moving superconformal multiplet for a general class of models here.

A very general model constructed from the ingredients we have considered is the following:

$$
\begin{aligned}
S = \frac{1}{16\pi} \int d^2x d^4\theta & \left( |\Phi_i|^2 e^{2Q_{ia}V_a} + \frac{b_\alpha}{2} \left( P_\alpha + \bar{P}_\alpha + 2\ell_{\alpha a} V_a \right)^2 - \frac{1}{b_\mu} |Y_\mu|^2 - \frac{2\pi}{e_a^2} |\Sigma_a|^2 \right) \\
- \frac{i}{8\pi} \int & d^2x d\theta^+ d\bar{\theta}^- \left( t_a - i k_{\mu a} Y_\mu \right) \Sigma_a + \text{c.c.} \\
+ \frac{1}{8\pi} \int & d^2x d\theta^+ d\theta^- \; W \left( \Phi_i, e^{P_\alpha} \right) + \text{c.c.}.
\end{aligned}
\tag{3.29}
$$

For the moment, let us ignore the superpotential term. This model has a protected, right-moving superconformal multiplet of the form:

$$
\begin{aligned}
\mathcal{J}^0_{--} = & -\frac{1}{8\pi} D_- \left( e^{2Q_i \cdot V} \Phi_i \right) e^{-2Q_i \cdot V} \bar{D}_- \left( e^{2Q_i \cdot V} \bar{\Phi}_i \right) \\
& - \frac{b_\alpha}{8\pi} D_- \left( P_\alpha + \bar{P}_\alpha + 2\ell_\alpha \cdot V \right) \bar{D}_- \left( P_\alpha + \bar{P}_\alpha + 2\ell_\alpha \cdot V \right) \\
& - \frac{1}{8\pi b_\mu} D_- \bar{Y}_\mu \bar{D}_- Y_\mu - \frac{1}{4e_a^2} \Sigma_a \bar{D}_- D_- \bar{\Sigma}_a.
\end{aligned}
\tag{3.30}
$$

This is the superconformal multiplet assigning $R$-charge 0 to the $\Phi_i$ fields. When we include the superpotential, we will modify this multiplet by the addition of a flavor symmetry under which the $\Phi_i$ rotate.

Classically, $\mathcal{J}^0_{--}$ satisfies

$$
\bar{D}_+ \mathcal{J}^0_{--} = 0; \tag{3.31}
$$

however, there is a 1-loop anomaly that modifies this to

$$
\bar{D}_+ \mathcal{J}^0_{--} = \frac{\gamma_a}{4\pi} \bar{D}_- \Sigma_a. \tag{3.32}
$$

Following [35], we determine the anomaly coefficient $\gamma_a$ by point-splitting. The result is

$$
\gamma_a = \sum_i Q_{ia}. \tag{3.33}
$$

As in [35], a modified current can be defined that is 1-loop superconformal. Unlike in that case, we will also take advantage of the freedom to include the $Y_\mu$ fields in that modification. In particular, we use

$$\mathcal{J}_{--} = \mathcal{J}^0_{--} + \frac{\gamma_\alpha}{8\pi}\left[\bar{D}_-, D_-\right]\left(P_\alpha + \bar{P}_\alpha + 2\ell_{\alpha a}V_a\right) + \frac{\gamma_\mu}{8\pi\sqrt{b_\mu}}\left[\bar{D}_-, D_-\right]\left(Y_\mu + \bar{Y}_\mu\right), \quad (3.34)$$

where $\gamma_\alpha$ and $\gamma_\mu$ are chosen such that

$$\gamma_a = \gamma_\alpha \ell_{\alpha a} + \gamma_\mu k_{\mu a}. \quad (3.35)$$

This is the same as (3.28).

Including the superpotential, $\mathcal{J}_{--}$ is no longer $\bar{D}_+$ closed. Instead,

$$\bar{D}_+\mathcal{J}_{--} = \frac{1}{4\pi}\left(\left(\mathcal{D}_-\Phi_i\right)\partial_i + \left(\mathcal{D}_-P_\alpha + \gamma_\alpha D_-\right)\partial_\alpha\right) W\left(\Phi_i, e^{P_\alpha}\right). \quad (3.36)$$

We can partially remedy this by assuming the quasi-homogeneity of $W$ and including in $\mathcal{J}_{--}$ the flavor symmetry under which each chiral field $\Phi_i$ rotates with the corresponding degree:

$$\mathcal{J}_{--} \to \mathcal{J}_{--} + \mathcal{F}_{--}, \qquad \mathcal{F}_{--} = \frac{\alpha_i}{8\pi} D_-\bar{D}_-\left(|\Phi_i|^2 e^{2Q_i\cdot V}\right), \qquad \left(\alpha_i\Phi_i\partial_i + \beta_\alpha\partial_\alpha\right)W = W. \quad (3.37)$$

Then,

$$\bar{D}_+\left(\mathcal{J}_{--} + \mathcal{F}_{--}\right) = \frac{1}{4\pi}\left(\beta_\alpha + \gamma_\alpha\right) D_-\partial_\alpha W. \quad (3.38)$$

Assuming the above is zero, i.e. $\beta_\alpha + \gamma_\alpha = 0$ and we have a non-trivial infrared CFT, we can find its central charge from the supercurrent we have just written.

The leading singularities in the OPE of the basic fields of the model (3.29) are:

$$\begin{aligned}
\phi_i(x)\phi_j(0) &\sim -\delta_{ij}\log\left(x^2\right), & \psi_{i,\pm}(x)\bar{\psi}_{j,\pm}(0) &\sim -\frac{i\delta_{ij}}{x^{\pm\pm}}, \\
p_\alpha(x)p_\beta(0) &\sim -\frac{\delta_{\alpha\beta}}{b_\alpha}\log\left(x^2\right), & \eta_{\alpha,\pm}(x)\bar{\eta}_{\beta,\pm}(0) &\sim -\frac{i\delta_{\alpha\beta}}{b_\alpha x^{\pm\pm}}, \\
y_\mu(x)y_\nu(0) &\sim -b_\mu\delta_{\mu\nu}\log\left(x^2\right), & \chi_{\mu,\pm}(x)\bar{\chi}_{\nu,\pm}(0) &\sim -\frac{ib_\mu\delta_{\mu\nu}}{x^{\pm\pm}}, \\
\sigma_a(x)\sigma_b(0) &\sim -\frac{e_a^2}{2\pi}\delta_{ab}\log\left(x^2\right), & \lambda_{a,\pm}(x)\bar{\lambda}_{b,\pm}(0) &\sim -\frac{e_a^2}{2\pi}\frac{i\delta_{ab}}{x^{\pm\pm}}.
\end{aligned} \tag{3.39}$$

The bottom component of the classical supercurrent $\mathcal{J}^0_{--}$ includes the composite operator $\sum_i \psi_{i,-}\bar{\psi}_{i,-}(x)$, which we define via point-splitting

$$\begin{aligned}
\sum_i \psi_{i,-}\bar{\psi}_{i,-}(x) &:= \lim_{y\to x}\left(\sum_i \psi_{i,-}(y)e^{i\int_y^x Q_{ia}A_a}\bar{\psi}_{i,-}(x) - \frac{i}{(x-y)^{--}}\right) \\
&=: \sum_i \psi_{i,-}\bar{\psi}_{i,-}(x) : -\sum_i Q_{ia}A_{a--}(x) - \sum_i Q_{ia}A_{a++}(x)\lim_{y\to x}\frac{(x-y)^{++}}{(x-y)^{--}}.
\end{aligned} \tag{3.40}$$

The anomaly in the supercurrent is determined by this operator:

$$\bar{D}_-\mathcal{J}^0_{--}\Big| = -\frac{1}{4\pi}\left[\bar{Q}_+, \sum_i \psi_{i,-}\bar{\psi}_{i,-}(x)\right] = -\frac{\sqrt{2}}{4\pi}\sum_i Q_{ia}\lambda_{a,-}(x) = \frac{\sum_i Q_{ia}}{4\pi}\,\bar{D}_-\Sigma_a\Big|. \tag{3.41}$$

Thus, the anomaly

$$\gamma_a = \sum_i Q_{ia}. \tag{3.42}$$

We determine the central charge of the $\mathcal{N}=2$ Virasoro algebra generated by $\mathcal{J}_{--}+\mathcal{F}_{--}$ by considering the leading singularity of the current-current OPE. The $R$-current, the bottom component of the superfield is

$$j_{--} = \frac{i\alpha_i}{4\pi}\phi_i \mathcal{D}_{--}\bar{\phi}_i + \frac{i\gamma_\alpha}{4\pi}\left(\mathcal{D}_{--}p_\alpha - \mathcal{D}_{--}\bar{p}_\alpha\right) - \frac{i\gamma_\mu}{4\pi\sqrt{b_\mu}}\partial_{--}\left(y_\mu - \bar{y}_\mu\right) - \frac{i}{2e_a^2}\sigma_a\partial_{--}\bar{\sigma}_a$$
$$- \frac{1-\alpha_i}{4\pi}\psi_{i,-}\bar{\psi}_{i,-} - \frac{b_\alpha}{4\pi}\eta_{\alpha,-}\bar{\eta}_{\alpha,-} + \frac{1}{4\pi b_\mu}\chi_{\mu,-}\bar{\chi}_{\mu,-}. \tag{3.43}$$

Using the OPEs from above, we see that

$$j_{--}(x)j_{--}(0) \sim -\frac{\left(\sum_i (1-2\alpha_i) - N_{U(1)} + N_P + N_Y + 2\sum_\alpha \frac{\gamma_\alpha^2}{b_\alpha} + 2\sum_\mu \frac{\gamma_\mu^2}{b_\mu}\right)}{(x^{--})^2} + \ldots,$$
$$= -\frac{c/3}{(x^{--})^2} + \ldots. \tag{3.44}$$

The result is

$$c = 3\left(\sum_i (1-2\alpha_i) - N_{U(1)} + N_P + N_Y + 2\sum_\alpha \frac{\gamma_\alpha^2}{b_\alpha} + 2\sum_\mu \frac{\gamma_\mu^2}{b_\mu}\right). \tag{3.45}$$

The first two terms are familiar from standard GLSM model building. The next two terms, $N_P$ and $N_Y$ are the number of $P$ and $Y$ fields, respectively, while the last two terms are modifications due to the non-standard shift transformations required of the $P$ and $Y$ fields in order to cancel all anomalies. In the next section, we will comment on specific examples of models and their supercurrents and central charges, when possible.

## 3.3 A collection of models

We will now describe a series of examples of varying complexity that serve to illustrate some of the possible target geometries described by this construction. Here we are only concerned with the classical geometry that emerges from minimizing the potential energy for a given

GLSM. This corresponds to solving the $D$-term and $F$-term conditions, and quotienting by the gauge group action. For the usual Kähler setting, solving the $D$-term conditions and quotienting by the gauge group action defines a toric variety via symplectic quotient. Further imposing $F$-term conditions gives an algebraic variety.

### *3.3.1 Bounding D-terms*

#### One $U(1)$ action

Turning on a single field-dependent FI-term, we work with the action

$$S = \frac{1}{16\pi}\int d^2x d^4\theta \left(\sum_i |\Phi_i|^2 e^{2Q_i V} - |Y|^2 - \frac{2\pi}{e^2}|\Sigma|^2\right) - \frac{i}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \left(t - ikY\right)\Sigma + \text{c.c.} \tag{3.46}$$

leading to a $D$-term potential imposing

$$\sum_i Q_i|\phi_i|^2 = r - 2k\text{Re}(y). \tag{3.47}$$

For simplicity, let us assume all $Q_i$ are positive, $r \geq 0$ and $k \geq 0$. After quotienting by $U(1)$, the $\phi^i$ configuration space is a weighted projective space with size determined by the right-hand side of (3.47). This space is non-compact since $\text{Re}(y)$ is only bounded from above,

$$\text{Re}(y) \leq \frac{r}{2k}. \tag{3.48}$$

When the inequality in (3.48) is saturated, the weighted projective space collapses to zero size. Although no fields charged under the gauge symmetry have expectation values at this boundary, $\Sigma$ is still massive because of the $Y\Sigma$ coupling which contributes a $|\sigma|^2$ mass term to the physical potential. There is therefore no classical Coulomb branch emitting from $|\phi_i|^2 = 0$. Just like its dual Stueckelberg field $P$, the field $Y$ gives a mass to the gauge field everywhere, so in the limit $e \to \infty$ the gauge field is not dynamical.

The simplest example has $n$ chiral fields $\Phi^i$ of charge 1 and one twisted chiral $Y$ with the coupling (3.46) and a standard kinetic term. After carrying out the symplectic quotient, we obtain a metric describing $\mathbb{C}P^{n-1}$ parametrized by $n-1$ complex coordinates $z^i$ fibered over a cylinder parametrized by $y$,

$$ds^2 = R(y)\left(\frac{dz\cdot d\bar{z}}{1+|z|^2} - \frac{|\bar{z}\cdot dz|^2}{(1+|z|^2)^2}\right) + \left(1+\frac{k^2}{R(y)}\right) dy d\bar{y}, \tag{3.49}$$

$$R(y) = r - 2k\,\mathrm{Re}(y) \tag{3.50}$$

and also a $B$-field

$$B = \frac{k\bar{z}\cdot dz\wedge d\bar{y} + kz\cdot d\bar{z}\wedge dy}{1+|z|^2}. \tag{3.51}$$

We can write $y = a + i\theta$, absorb $r$ into $a$, and define $\rho^2 \equiv R = 2ka$ to rewrite the metric in a different form

$$ds^2 = \rho^2\left(\frac{dz\cdot d\bar{z}}{1+|z|^2} - \frac{|\bar{z}\cdot dz|^2}{(1+|z|^2)^2}\right) + \left(1+\frac{\rho^2}{k^2}\right) d\rho^2 + \left(1+\frac{k^2}{\rho^2}\right) d\theta^2. \tag{3.52}$$

This form makes it clear that $\rho$ has range $(0,\infty)$, with the projective space pinching to zero size at $\rho = 0$, while the circle parametrized by $\theta$ becomes infinitely large at that end. This is the trumpet geometry of Figure 3.1. This geometry is singular with diverging curvature as $\rho$ tends to zero. For example, when $n = 2$, the Ricci scalar is

$$R = -\frac{4}{\rho^2} + \ldots \quad \text{as} \quad \rho \to 0. \tag{3.53}$$

While the space is geometrically singular, the theory has no physical singularity; the resolution of the singularity requires T-duality and we will be discussed in section 3.3.2.

We can also calculate

$$H = dB = 2kd\theta \wedge J_{FS} \tag{3.54}$$

where $J_{FS}$ is the fundamental two-form of Fubini-Study for $\mathbb{C}P^{n-1}$, which integrates to a non-trivial torsion

$$\int_{\mathcal{C}\times S^1} H = 4\pi k \tag{3.55}$$

where $\mathcal{C}$ is the two-cycle dual to $J_{FS}$, where the dual is taken at fixed $\rho$ in the $\mathbb{C}P^{n-1}$ fiber, and the $S^1$ is parametrized by $\theta$.

In this model we can pick an $R$-symmetry transformation for $Y$ such that $n - k\gamma = 0$, so that there is a non-anomalous $U(1)_A$, and we expect a non-trivial infrared CFT. Using (3.45), the central charge of this CFT is calculated to be

$$c = 3n\left(1 + \frac{2n}{k^2}\right). \tag{3.56}$$

### Two $U(1)$ actions

With at least one more $U(1)$ gauge-field, we can bound the range of $\mathrm{Re}(y)$. Introduce a second FI-term which couples the same field $Y$ to the new field strength $\tilde{\Sigma}$,

$$-\frac{\tilde{k}}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \, Y\tilde{\Sigma} + \text{c.c.}, \tag{3.57}$$

with charged fields $\tilde{\phi}$ satisfying

$$\sum_i \tilde{Q}_i |\tilde{\phi}_i|^2 = \tilde{r} - 2\tilde{k}\mathrm{Re}(y). \tag{3.58}$$

As long as $\tilde{k} \leq 0$, $\tilde{r} \geq 0$, and the charges $Q_i, \tilde{Q}_i > 0$, the range of $\mathrm{Re}(y)$ is bounded:

$$\frac{\tilde{r}}{2\tilde{k}} \leq \mathrm{Re}(Y) \leq \frac{r}{2k}. \tag{3.59}$$

This gives the cylindrical fixture of Figure 3.2 with a product of weighted projective spaces fibered over the cylinder. In this basic fixture, the size of each projective space vanishes at one of the ends.

Once again, the simplest models have $n$ fields $\Phi^i$ with charges $(1,0)$ and $\tilde{n}$ fields $\tilde{\Phi}^{\tilde{\imath}}$ with charges $(0,1)$. Taking $k \geq 0, \tilde{k} \leq 0$ as in (3.59), we find the metric and $B$ field

$$\begin{aligned} ds^2 = {} & R(y)\left(\frac{dz \cdot d\bar{z}}{1+|z|^2} - \frac{|\bar{z}\cdot dz|^2}{(1+|z|^2)^2}\right) + \tilde{R}(y)\left(\frac{d\tilde{z}\cdot d\bar{\tilde{z}}}{1+|\tilde{z}|^2} - \frac{|\bar{\tilde{z}}\cdot d\tilde{z}|^2}{(1+|\tilde{z}|^2)^2}\right) \\ & + \left(1 + \frac{k^2}{R(y)} + \frac{\tilde{k}^2}{\tilde{R}(y)}\right) dy d\bar{y}, \end{aligned} \tag{3.60}$$

$$B = \frac{k\bar{z}\cdot dz \wedge d\bar{y} + kz\cdot d\bar{z}\wedge dy}{1+|z|^2} + \frac{\tilde{k}\bar{\tilde{z}}\cdot d\tilde{z}\wedge d\bar{y} + \tilde{k}\tilde{z}\cdot d\bar{\tilde{z}}\wedge dy}{1+|z|^2}. \tag{3.61}$$

We can calculate the $H$ flux from $B$,

$$H = 2d\theta \wedge \left(kJ_{FS} + \tilde{k}\tilde{J}_{FS}\right), \tag{3.62}$$

which is easy to integrate over the two-cycle dual to either $J_{FS}$ or $\tilde{J}_{FS}$, and the $S^1$ formed by $\theta$,

$$\int_{\mathcal{C}\times S^1} H = 4\pi k, \qquad\qquad \int_{\tilde{\mathcal{C}}\times S^1} H = 4\pi\tilde{k}. \tag{3.63}$$

Note that while $k$ and $\tilde{k}$ must have opposite signs in order to bound $Y$, they do not necessarily have the same magnitude, and the same is true of the corresponding $H$-fluxes. This is somewhat surprising because we might have expected the total brane and anti-brane charge to sum to zero for a compact space. However, this does not seem to be a requirement for these geometries. The dual descriptions of models with $k \neq -\tilde{k}$, which will be discussed in section 3.3.2, involve either squashed weighted projective spaces, or spaces with orbifold singularities.

When each fibered projective space is actually a sphere, which happens for $\mathbb{P}^1 \sim S^2$, the resulting space is $S^5 \times S^1$ [26]. Otherwise the space looks singular and, as we will discuss shortly, we will need the T-dual description to see that the collapsing projective space is actually acceptable.

Near each end, the metric has the same asymptotic form as (3.52), with the extra projective space staying at finite size. The spaces discussed in this section so far have been previously studied in a $(0,2)$ context in [26].

## Compact and conformal models?

The model with two $U(1)$ gauge fields gives us a construction of a torsional compact geometry. However, there is no $U(1)_R$ charge assignment for $y$ that allows us to solve (3.25), and so this is a massive model. We can show that compact models will be generically massive if we only allow $\Phi$ and $Y$ fields with couplings of the form $Y\Sigma$ and usual chiral superpotentials.

As we have seen in the previous sections, we obtain one bound on the range of $\mathrm{Re}(y)$ from each $D$-term condition, as long as all fields $\Phi^i$ charged under the corresponding $U(1)$ have positive charges (or all negative charges). The bound is of the form

$$\frac{\sum_i Q_i |\phi^i|^2}{\sum_j Q_j} \geq 0 \Rightarrow \frac{r - 2k\,\mathrm{Re}(y)}{\sum_j Q_j} \geq 0 \Rightarrow \frac{2k\,\mathrm{Re}(y)}{\sum_j Q_j} \leq \frac{r}{\sum_j Q_j}. \tag{3.64}$$

This allows us to see that the direction of this bound depends only on the sign of $\frac{\sum_j Q_j}{k}$. But that sign is precisely what sets the sign of $\gamma$ satisfying (3.25). Therefore if we have two bounds in opposite directions as needed to make the range of $\mathrm{Re}(y)$ compact, there is no $\gamma$ which solves (3.25) for both $U(1)$, and we will be dealing with a massive model.

We may now illustrate this argument with an example of an attempt to evade it, in order to provide some intuition on how compactness is violated. Since the effects of $Y$ are generally not enough to cancel the $U(1)_R$ anomalies from two $U(1)$ gauge fields, we can try to also add a negatively-charged field coupled in a superpotential. This helps with cancelling the anomaly, as it usually does for example in the quintic. However, since the models we have been considering have a point where the size of the ambient projective space vanishes, we would be left with a non-compact direction where $Y$ and the negatively-charged field grow without bound. We can try to be smarter and add an extra $U(1)$.

Building on the double trumpet model, take then three $U(1)$ gauge fields and the following field content: $n$ fields $\Phi^i$ with charges $(1,0,0)$, $\tilde{n}$ fields $\tilde{\Phi}^i$ with charges $(0,1,0)$, a field $S$ with charges $(-Q,-\tilde{Q},0)$ and a field $A$ with charges $(0,0,Q_a)$. Introduce a periodic twisted chiral $Y$, coupled to the three twisted chirals $\Sigma, \tilde{\Sigma}, \Sigma_a$ with coefficients $(k, \tilde{k}, k_a)$. As before we will want $k\tilde{k} < 0$ in order to obtain a bound, so choose $\tilde{k} < 0$. The field $S$ can be used to write a gauge-invariant superpotential $Sf(\Phi)g(\tilde{\Phi})$, with $f$ and $g$ polynomials of degrees $Q$ and $\tilde{Q}$, respectively. The $D$-term constraints read

$$-Q|s|^2 + |\phi|^2 = r - 2k\,\mathrm{Re}(y), \tag{3.65a}$$

$$-\tilde{Q}|s|^2 + |\tilde{\phi}|^2 = \tilde{r} - 2\tilde{k}\,\mathrm{Re}(y), \tag{3.65b}$$

$$Q_a|a|^2 = r_a - 2k_a\,\mathrm{Re}(y). \tag{3.65c}$$

From this it is clear that at the previous extrema of $\mathrm{Re}(y)$, $\phi$ or $\tilde{\phi}$ become zero, liberating $s$ and $y$ in a non-compact direction. One of these can be removed by setting $Q = 0$. Once we have done that, our anomaly cancellation conditions (3.25) read

$$n - \gamma k = 0, \tag{3.66a}$$

$$\tilde{n} - \tilde{Q} - \gamma\tilde{k} = 0, \tag{3.66b}$$

$$Q_a - \gamma k_a = 0. \tag{3.66c}$$

Now we would like to use the bound from the third $U(1)$ to remove the second non-compact direction from the region $\tilde{\phi} = 0$. However, note (3.66a) sets $\gamma k = n > 0 \Rightarrow \gamma > 0$. Using (3.66c), this implies $Q_a/k_a > 0$. Then the constraint imposed on $y$ from (3.65c) will have the same sign as that from (3.65a), and therefore will not help when the second D-term constraint disappears and $S$ becomes non-zero.

### *3.3.2 The Kähler picture*

Both examples in sections 3.3.1 and 3.3.1 have T-duals which are Kähler and described by the squashed toric varieties first discussed in [35]. At the level of an ultraviolet GLSM, squashing is implemented as follows.

Consider a toric GLSM, namely a collection of $n$ chiral superfields charged under a collection of $k$ abelian gauge symmetries. Such a model, in the absence of a superpotential, has $n-k$ remaining flavor symmetries. The squashing construction gauges each of these flavor symmetries while simultaneously adding a Stueckelberg chiral superfield for each. An action for such a model is

$$\begin{aligned} S = \frac{1}{16\pi} \int d^2x d^4\theta & \left( \sum_{i=1}^{n} \sum_{a=1}^{k} \sum_{\alpha=1}^{n-k} |\Phi_i|^2 e^{2Q_i^a V_a + 2R_i^\alpha V_\alpha} - \sum_{a=1}^{k} \frac{2\pi}{e_a^2} |\Sigma_a|^2 \right) \\ - \frac{i}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- & \sum_{a=1}^{k} t^a \Sigma_a + \text{c.c} \\ + \frac{1}{16\pi} \int d^2x d^4\theta & \sum_{\alpha=1}^{n-k} \left( \frac{b_\alpha}{2} \left( P_\alpha + \bar{P}_\alpha + 2V_\alpha \right)^2 - \frac{2\pi}{e_\alpha^2} |\Sigma_\alpha|^2 \right). \end{aligned} \tag{3.67}$$

The charges of the chiral fields under the original $k$ gauge symmetries are $Q_i^a$, while the charges of the flavor symmetries are $R_i^\alpha$. We stipulate that the combined $n \times n$ matrix $(Q_i^a, R_i^\alpha)$ has rank $n$. Note also that there are no FI couplings for the gauged flavor symmetries, as these can be absorbed into a redefinition of the corresponding Stueckelberg fields.

Each squashing has an associated squashing parameter $b_\alpha \in \mathbb{R}$. In the limit $b_\alpha \to \infty$, the

Stueckelberg fields decouple, and the squashing is removed. Since each squashing corresponds to a $U(1)$ isometry, the Stueckelberg fields are periodic: $\mathrm{Im}P_\alpha \sim \mathrm{Im}P_\alpha + 2\pi$. Modifying the charge of $P_\alpha$ to $k_\alpha$ for some $k_\alpha \in \mathbb{Z}$ yields a $\mathbb{Z}_{k_\alpha}$ orbifold [35]. Note that in section 3.3.1 we took $b_\alpha = 1$ for all fields $Y$. Taking the limit of no squashing in the $Y$-picture leads to a $Y$ field that has a vanishing classical kinetic term but gains a metric when we descend to the non-linear sigma model. The torsion of the $Y$ model is unaffected by the value of the squashing parameter.

Applying the T-duality of section 3.2.2 results in the field-dependent FI couplings we have already seen. The T-dual of (3.67) is

$$\begin{aligned} S = &\frac{1}{16\pi}\int d^2x d^4\theta \left(\sum_{i=1}^{N}\sum_{a=1}^{k}\sum_{\alpha=1}^{N-k} |\Phi_i|^2 e^{2Q_i^a V_a + 2R_i^\alpha V_\alpha} - \sum_{a=1}^{k}\frac{2\pi}{e_a^2}|\Sigma_a|^2\right) \\ &- \frac{i}{8\pi}\int d^2x d\theta^+ d\bar\theta^- \sum_{a=1}^{k} t^a \Sigma_a + \mathrm{c.c} \\ &- \frac{1}{16\pi}\int d^2x d^4\theta \sum_{\alpha=1}^{N-k}\left(\frac{\bar Y_\alpha Y_\alpha}{b_\alpha} + \frac{2\pi}{e_\alpha^2}|\Sigma_\alpha|^2\right) \\ &- \frac{1}{8\pi}\int d^2x d\theta^+ d\bar\theta^- \sum_{\alpha=1}^{N-k} Y_\alpha \Sigma_\alpha + \mathrm{c.c.} \end{aligned} \tag{3.68}$$

Dualizing a field $P_\alpha$ with shift-charge $k_\alpha$ would lead to a factor $k_\alpha$ multiplying the last line, just like the couplings $k$ in section 3.3.1. We will also mostly take $b_\alpha = 1$ to connect to the previous discussion.

### One $U(1)$ action

The dual of the trumpet geometry discussed in section 3.3.1 has $n$ chiral fields of charge 1 and a Stueckelberg field of charge $k$ and period $2\pi$ with kinetic action

$$\frac{1}{32\pi}\int d^2x d^4\theta \, \left(P + \bar P + 2kV\right)^2 .$$

This model was discussed in [29]. The total space is topologically $\mathbb{C}^n/\mathbb{Z}_k$, but it is to easier to visualize as $S^{2n-1}$ warped over a half line $R_+$. The orbifold action is through discrete translations along the fiber of the Hopf fibration $U(1) \hookrightarrow S^{2n-1} \to \mathbb{C}P^n$. Note that if $k=1$, this space is birationally equivalent to the total space of the tautological bundle over $\mathbb{C}P^{n-1}$. The classical metric in an affine patch of the $\mathbb{C}P^{n-1}$ base is

$$ds^2 = \rho^2 \left( \frac{dz \cdot d\bar{z}}{1+|z|^2} - \frac{|\bar{z} \cdot dz|^2}{(1+|z|^2)^2} \right) + \left( 1 + \frac{\rho^2}{k^2} \right) d\rho^2 + \frac{\rho^2}{\left(1+\frac{\rho^2}{k^2}\right)} \left( \frac{d\theta}{k} + A_{FS} \right)^2 , \qquad (3.69)$$

where $\theta$ has period $2\pi$ and $A_{FS}$ in this patch is

$$A_{FS} = -\frac{i}{2} \frac{z \cdot d\bar{z} - \bar{z} \cdot dz}{1+|z|^2}. \qquad (3.70)$$

This is, indeed, the T-dual of (3.52) using the $B$-field (3.51).

This type of T-duality also provides us with a recipe for T-dualizing in the UV GLSM along a given circle isometry in a non-linear sigma model. Once we identify the GLSM circle that descends to the circle we would like to dualize, we can gauge the flavor symmetry corresponding to that isometry, and add a corresponding Stueckelberg field $P$. We can then dualize $P$ into a $Y$ as in Section 2.5, and descend to the non-linear sigma model in the resulting theory. Finally, if we want to remove the effects of the new field, we should take the limit of no squashing, or $b \to \infty$. It would be interesting to try this prescription in models with blowing-up circles, such as those of [25].

### Two $U(1)$ actions

This dual has $n$ chirals of charge $(1,0)$ and $\tilde{n}$ chirals of charge $(0,1)$ and a single Stueckelberg with charge $(k,\tilde{k})$ and period $2\pi$ with kinetic term

$$\frac{1}{32\pi} \int d^2x d^4\theta \, \left( P + \bar{P} + 2kV + 2\tilde{k}\tilde{V} \right)^2 .$$

As before, we will assume that $k > 0$ and $\tilde{k} < 0$ in order that the target space be compact. Let $\widehat{k} = \gcd(k, -\tilde{k})$. We can perform an $SL(2,\mathbb{Z})$ transformation on the gauge fields

$$\begin{pmatrix} \widehat{V} \\ \check{V} \end{pmatrix} = \begin{pmatrix} \frac{k}{\widehat{k}} & \frac{\tilde{k}}{\widehat{k}} \\ -\tilde{\ell} & \ell \end{pmatrix} \begin{pmatrix} V \\ \tilde{V} \end{pmatrix}, \qquad k\ell + \tilde{k}\tilde{\ell} = \widehat{k}, \tag{3.71}$$

yielding a theory with $n$ chirals of charge $(\ell, -\frac{\tilde{k}}{\widehat{k}})$ and $\tilde{n}$ chirals of charge $(\tilde{\ell}, \frac{k}{\widehat{k}})$ and a Stueckelberg field of charge $(\widehat{k}, 0)$ and period $2\pi$ with kinetic term

$$\frac{1}{32\pi} \int d^2x d^4\theta \, \left(P + \bar{P} + 2\widehat{k}\widehat{V}\right)^2 .$$

Matching onto the general model, we see this describes a weighted $\mathbb{C}P^{n+\tilde{n}-1}$ with $n$ weights of $-\frac{\tilde{k}}{\widehat{k}}$ and $\tilde{n}$ weights of $\frac{k}{\widehat{k}}$. Further, this space is squashed and orbifolded along the $U(1)$ isometry under which the $n$ chirals have charge $\ell$ and the $\tilde{n}$ chirals have charge $\tilde{\ell}$.

### 3.3.3 More general fibrations

The models described so far have both a Kähler and a torsional description, related by duality. We would like to find models which do not have a Kähler description, and hence live beyond the lamp post.

There is a very natural way to construct such models. Imagine a base GLSM theory with $Y$-fields and some charged $\Phi$ fields. We will fiber a twisted sigma model over this base theory in a way that obstructs dualizing $Y$ back to a chiral superfield. We will fiber the complex structure of the twisted sigma model over $Y$ using superpotential couplings between $Y$ and charged twisted chirals $\widehat{\Phi}$.

As a first example modeled on the quintic, consider a theory with a $U(1)^2$ gauge group with charged chirals and a $\widehat{U}(1)$ gauge group with charged twisted chirals. The charge matrix is given in Table 3.1.

We will not worry about imposing conformal invariance for the moment. Rather, our

| | $U(1)_1$ | $U(1)_2$ | $\widehat{U}(1)$ |
|---|---|---|---|
| $\Phi$ | 1 | 0 | 0 |
| $\tilde{\Phi}$ | 0 | 1 | 0 |
| $\widehat{\Phi}$ | 0 | 0 | 1 |
| $\widehat{S}$ | 0 | 0 | −5 |

Table 3.1: Charge matrix for the quintic fibration over the double trumpet.

interest is in the structure of the resulting generalized Kähler geometries. The chiral content is the same as in section 3.3.1, and so the range of $\mathrm{Re}(y)$ will be bounded. We do not need to assign any $R$-symmetry transformation to $Y$. On the other hand, the twisted chiral content is similar to the usual quintic, except that we allow the twisted superpotential to depend on $e^Y$:

$$\widehat{W} = \widehat{S} f(e^Y, \widehat{\Phi}), \tag{3.72}$$

where $f$ is a polynomial of degree 5 in $\widehat{\Phi}$ whose coefficients are polynomials in $e^Y$.

For concreteness, we can consider deforming the Fermat quintic by a $Y$-dependent monomial, so

$$f(e^Y, \widehat{\Phi}) = \sum_{i=1}^{5} \widehat{\Phi}_i^5 + e^Y \widehat{\Phi}_1 \widehat{\Phi}_2 \widehat{\Phi}_3 \widehat{\Phi}_4 \widehat{\Phi}_5. \tag{3.73}$$

This $Y$-dependence cannot be removed by a field redefinition; the complex structure modulus parametrized by $\widehat{\Phi}_1 \widehat{\Phi}_2 \widehat{\Phi}_3 \widehat{\Phi}_4 \widehat{\Phi}_5$ is now fibered over the $Y$ cylinder. This space is therefore a fibration of the quintic CY 3-fold over the double trumpet model. For a suitable choice of parameters, the complex structure of the fiber can be kept away from degeneration limits. This is the structure illustrated in Figure 3.3.

It is easy to generalize this structure to more general fibrations and bases resulting in compact intrinsically torsional spaces. If, however, one wishes to impose conformal invariance on the resulting space then we encounter the flat direction issue described in section 3.3.1. The other natural possibility is to include $e^Y$ couplings to $\Sigma$ fields. That case is rather interesting and will be discussed in section 3.3.4.

### A comment on squashed Calabi-Yau

Although we run into the flat direction issue when trying to build compact conformal models from $Y$ fixtures, we can certainly repeat the usual hypersurface or complete intersection construction of compact Calabi-Yau spaces for squashed projective spaces in terms of $P$ variables.

Let us describe these models by way of an example: a squashed analogue of the quintic Calabi-Yau three-fold. Specifically, the model will describe a hypersurface inside of a squashed $\mathbb{C}P^4$ that is topologically Calabi-Yau. Recall the squashed $\mathbb{C}P^4$: start with a model with 5 chiral fields $\Phi_i$, each with charge 1 under a single $U(1)_G$ gauge symmetry. We choose to squash the flavor symmetry $U(1)_F$ under which only $\Phi_5$ rotates with charge 1. To do this, we gauge this symmetry and add a chiral Stueckelberg field $P$ with kinetic term

$$\frac{b}{32\pi}\int d^4\theta \; \left(P+\bar{P}+2V_F\right)^2,$$

where $V_F$ is the vector superfield of the gauged flavor symmetry. As usual, we choose $\mathrm{Im}P$ to have period $2\pi$. In order that the squashed $\mathbb{C}P^4$ be smooth, we choose the shift charge of $P$ to equal one; we can effect an orbifold of the ambient space by choosing another integer charge $\ell$.

To carve out a hypersurface, we include another chiral field $S$ with charge $-5$ under the original gauge symmetry and charge 0 under the flavor symmetry, and we add the superpotential

$$W(S,\Phi_i,e^P) = S\left(G(\Phi_1,\ldots,\Phi_4)+e^{-5P}\Phi_5^5\right).$$

The polynomial $G$ is homogeneous of degree 5 and non-singular in the subspace $\Phi_5=0$. A summary of the fields and their charges is provided in Table 3.2.

This model is of the form described in (3.29), and we can build a protected superconformal

|  | $U(1)_G$ | $U(1)_F$ |
|---|---|---|
| $\Phi_{1-4}$ | 1 | 0 |
| $\Phi_5$ | 1 | 1 |
| $S$ | $-5$ | 0 |
| $P$ | 0 | 1 |

Table 3.2: Charge matrix for the squashed quintic model.

multiplet in $\bar{Q}_+$-cohomology with the following parameter choices

$$\gamma = -\beta = 1, \qquad \alpha_5 = -1, \qquad \alpha_i = 0, \;\; i = 1, \dots, 4, \qquad \alpha_S = 1. \tag{3.74}$$

As usual, this choice is ambiguous up to shifts by the charges under the original $U(1)$ gauge action. Now we expect this construction to give the correct central charge for the non-compact total space of $O(-5)$ over the squashed $\mathbb{C}P^4$, prior to turning on the superpotential:

$$c_{\text{non-cpt}} = 9 + \frac{6}{b}. \tag{3.75}$$

This central charge depends on the squashing parameter $b > 0$. However, there is an interesting question of the correct IR description of this theory with the superpotential turned on. By a field redefinition $\phi_5 \to e^{-p}\phi_5$, the effect of the squashing can be removed from the F-term constraints. The only effect of the squashing is a change of D-terms. However, the IR theory is expected to be fully determined by the F-term structure if the metric is compact, so our expectation is that this theory flows to the usual quintic Calabi-Yau conformal field theory with $c = 9$ in the IR.[2]

What we do learn from this construction is that there should be a corresponding relevant operator in $Y$ variables that also produces a compact CFT. The $Y$ description contains two

2. We would like to thank Ilarion Melnikov for clarifying this issue.

$U(1)$ gauge-fields with $D$-term constraints,

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 + |\phi_5|^2 - 5|S|^2 = r_1, \tag{3.76}$$

$$|\phi_5|^2 = r_2 - 2\mathrm{Re}(y). \tag{3.77}$$

A part of that relevant operator is just the superpotential couplings involving fields that are not dualized:

$$W(S, \Phi_i, e^P) = S\left(G(\Phi_1, \ldots, \Phi_4)\right).$$

This interaction still leaves flat directions for the physical potential. To lift those remaining flat directions, we need the dual of the chiral operator $e^{-5P}\Phi_5^5$. However, as is often found in dual descriptions, this nice local chiral operator has no simple local description in $Y$ variables.

### 3.3.4 *Exponential couplings*

We now consider more exotic models that also obstruct a straightforward dualization to a Kähler picture. These models break the $U(1)$ symmetry shifting $\mathrm{Im}(Y)$. That circle is precisely the one we would want to T-dualize to produce a Kähler picture in terms of $P$ variables.

#### One $U(1)$ action

As a first example of such a model, consider a theory with one $Y$, one $U(1)$ gauge-field and a twisted superpotential

$$S = -\frac{1}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \kappa e^Y \Sigma. \tag{3.78}$$

This model is the analogue of Figure 3.1, and the new coupling expressed in component fields is found in (2.23). This coupling explicitly breaks the $U(1)$ isometry which shifts the imaginary part of $Y$. It should also be noted that while the coupling $k$ in previous sections was integer, $\kappa$ can take any real value. This coupling also fixes $Y$ to be invariant under $R$-symmetry transformations, so $Y$ can no longer be used to absorb any possible anomalies.

Writing $y = a + i\theta$, the $D$-term potential condition now reads

$$\sum_i Q_i |\phi_i|^2 = r - 2\kappa e^a \cos\theta = R(y), \tag{3.79}$$

so if all the charges $Q_i$ are positive the space is bounded to the region

$$2\kappa e^a \cos\theta \leq r. \tag{3.80}$$

Topologically, the space of possible $y$ values satisfying this inequality can have one of three shapes, depending only on $r$. These three possibilities are depicted in Figure 3.4.

- If $r = 0$, the condition (3.80) picks out one sign of the cosine for any $a$, and we obtain an infinite strip.

- If $r < 0$, the condition imposes a bound on $a$, and we find a semi-infinite strip.

- If $r > 0$, on the other hand, we essentially obtain the converse of that, an infinite cylinder with a semi-infinite strip taken out.

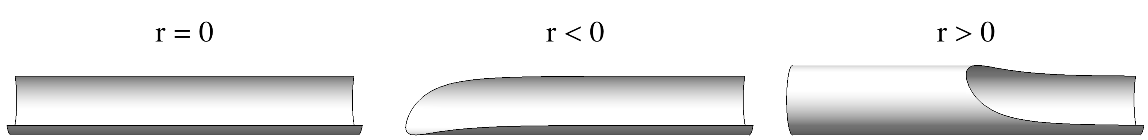


Figure 3.4: The three possibilities.

The metric and $B$-field can be obtained using similar methods from the above examples, and have the form

$$ds^2 = R(y)\left(\frac{dz \cdot d\bar{z}}{1+|z|^2} - \frac{|\bar{z} \cdot dz|^2}{(1+|z|^2)^2}\right) + \left(1 + \frac{\kappa^2 e^{2a}}{R(y)}\right) dy d\bar{y}, \tag{3.81}$$

$$B = \frac{\kappa e^{\bar{y}} \bar{z} \cdot dz \wedge d\bar{y} + \kappa e^{y} z \cdot d\bar{z} \wedge dy}{1+|z|^2}. \tag{3.82}$$

As above, the metric blows up at the boundary $R(y) = 0$. Define

$$\rho^2 = \tilde{r} - 2\kappa e^a \cos\theta, \tag{3.83}$$

$$\alpha = -\kappa e^a \sin\theta, \tag{3.84}$$

so

$$dy d\bar{y} = da^2 + d\theta^2 = \frac{\rho^2 d\rho^2 + d\alpha^2}{\kappa^2 e^{2a}}, \tag{3.85}$$

and the metric becomes, in the limit close to the boundary,

$$ds^2 = R(y) ds_{FS}^2 + \rho^2 d\tilde{s}_{FS}^2 + d\rho^2 + \frac{1}{\rho^2} d\alpha^2, \tag{3.86}$$

which has a similar form to the metric close to the boundaries in trumpet models, as can be seen by comparison with (3.52).

The torsion in this case can be obtained from (3.82) and has the form

$$H = 2\kappa d(e^a \sin\theta) \wedge J_{FS} = -2 d\alpha \wedge J_{FS} \tag{3.87}$$

This three-form integrates to zero in any region where the $\theta$ circle closes, but has a non-zero integral in other regions.

### Several $U(1)$ actions

In order to build a compact space including an exponential coupling to a field strength multiplet, start by taking the field content of section 3.3.1 which led to Figure 3.2. This theory includes two $U(1)$ gauge fields $\Sigma$ and $\tilde{\Sigma}$ coupled to two sets of chiral fields $\Phi^i$ and $\tilde{\Phi}^i$, leading to the D-term conditions:

$$\sum_i Q_i|\phi^i|^2 = r - 2k\mathrm{Re}(y) = R(y), \tag{3.88}$$

$$\sum_i \tilde{Q}_i|\tilde{\phi}^i|^2 = \tilde{r} - 2\tilde{k}\mathrm{Re}(y) = \tilde{R}(y). \tag{3.89}$$

This bounds the range of $\mathrm{Re}(y)$ if all charges $Q_i$ and $\tilde{Q}_i$ are positive, $k > 0$, and $\tilde{k} < 0$. To this configuration, which is dual to a squashed space, we can now add a third $U(1)$ multiplet $\Sigma'$ with its own set of charged fields $\Phi'$, coupled to $Y$ with a superpotential $\kappa e^Y \Sigma'$. The corresponding D-term condition reads

$$\sum_i Q'_i|\phi'_i|^2 = r' - 2\kappa e^a \cos\theta = R'(y). \tag{3.90}$$

The boundaries of the space are set by the $y$ values where any single projective space collapses to zero size. We encounter a problem with new flat directions if any two projective spaces collapse to zero size at the same $y$ value. To see this, note that the $Y$ superpotential can only mass up a single combination of $\Sigma$ fields. If two or more projective spaces collapse at the same point, there will be two distinct $U(1)$ factors for which no charged fields have an expectation value at that point. Therefore, one $\Sigma$ multiplet will be massless resulting in a new flat direction.

It is straightforward to see that to avoid an intersection where two or more projective spaces collapse, we need to be in the case where the exponential allows the full $y$ circle, so $r' > 0$. This is the last case depicted in Figure 3.4. We also need the boundary for the projective space with the exponential coupling to be fully outside the space defined by the other two constraints. The set of $a$ satisfying the condition

$$e^a \geq \frac{r'}{2|\kappa|}, \tag{3.91}$$

contains a boundary point for some $\theta$ where the projective space with the exponential coupling vanishes. We therefore need to impose the condition

$$\log \frac{r'}{2|\kappa|} > \frac{r}{2k}, \tag{3.92}$$

to ensure these boundary points are excluded. We will define the theory at a UV scale $\Lambda$ with bare FI parameters satisfying inequality (3.92). The inequality will then generally be preserved by RG flow because the additive renormalization of the FI parameters makes the right-hand side decrease faster than the left-hand side as we flow down in energy.

The metric and $B$-field are essentially the sum of those given in sections 3.3.1 and 3.3.4. This model then consists of the same ingredients as those of the double trumpet model, with an additional space fibered over the double trumpet whose size depends on the real and imaginary values of $y$. This fibration structure is depicted in Figure 3.5. The integrated torsion will have the same value as found in the double trumpet. However, since the isometry is broken by the exponential coupling, this model cannot be dualized into a Kähler picture like a squashed projective space.

### *3.3.5 Unifying constructions*

We can unify the structures described in sections 3.3.1 and 3.3.4 by writing the $\Sigma^a$ coupling as $f_a(Y)\Sigma^a$, with an $f_a(Y)$ that shifts at most by an integer multiple of $2\pi i$ when $Y$ shifts by $2\pi i$. For simplicity, take each $\phi$ to have charge 1 under one of the gauge symmetries, and

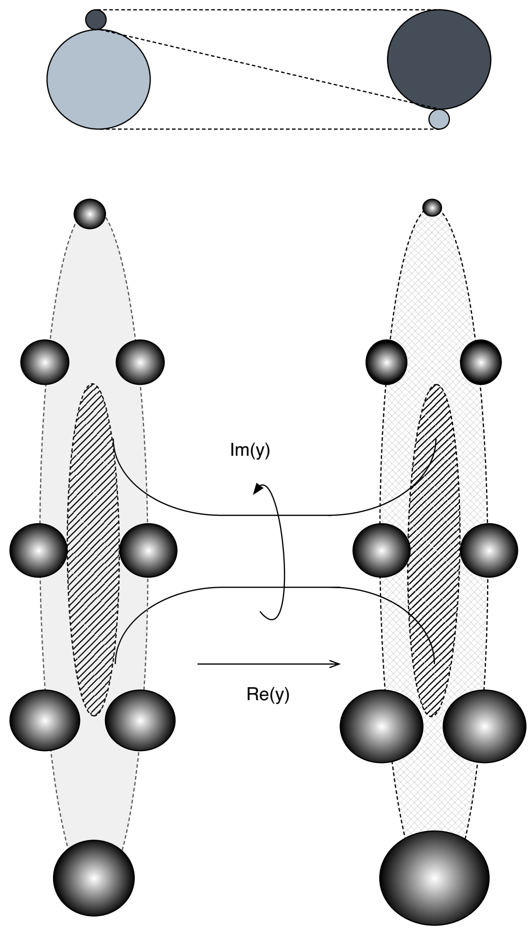


Figure 3.5: A compact example with exponential couplings. The three fibrations are depicted. The first two fibrations are the ones already seen in Figure 3.2. The radius of the third space depends on both the real and imaginary parts of $y$. Note that the radius oscillations become larger as $\mathrm{Re}(y)$ becomes bigger.

0 under the others. Then the $D$-terms constrain

$$\sum_i |\Phi_i^a|^2 e^{2A_a} = r_a - 2\,\mathrm{Re}\, f_a(y) = R_a(y), \tag{3.93}$$

and repeating the analysis yields a metric of the form

$$ds^2 = \sum_a R_a(y) ds^2(\phi^a) + \sum_\mu |dy_\mu|^2 + \sum_a \frac{|df_a|^2}{R_a(y)}, \tag{3.94}$$

Near $R_a = 0$ it makes sense to choose coordinates including $\rho^2 = f_a$, and in that limit the $\rho$ metric will reduce to the familiar form (3.52). The $B$ field can also be written in terms of the functions $f_a$ as

$$B = \sum_a \frac{\phi^a \cdot d\bar{\phi}_a \wedge df_a}{1 + |\phi^a|^2} + \text{c.c.} \tag{3.95}$$

$$\Rightarrow H = \sum_a 2d\,(\text{Im}\, f_a) \wedge J^a_{FS}. \tag{3.96}$$

If the circle from the imaginary part of $y$ closes, we can see from this expression that the integral of $H$ will be non-zero if $f_a$ is not single-valued when we travel around the circle.

## 3.4 The quantum cohomology ring

Our discussion so far has been largely classical. We wanted to describe gauge theories that give rise to classical vacuum equations describing generalized Kähler spaces. In this section, we turn to some quantum aspects of these models. Specifically, we will probe the vacuum structure by calculating the quantum cohomology rings for some of the models we have discussed.

Recall from Section 2.4 that in the Coulomb branch of standard $(2,2)$ GLSM we have the twisted superpotential

$$S_{\widetilde{W}} = \frac{1}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \sum_a \Sigma_a \left( \sum_i Q_i^a \left[ \log\left( \frac{\sum_b Q_i^b \Sigma_b}{\mu} \right) - 1 \right] - it_a(\mu) \right) \tag{3.97}$$

which leads to the vacuum equations

$$\sum_i Q_i^a \log\left( \frac{\sum_b Q_i^b \Sigma_b}{\mu} \right) = it_a(\mu) \Rightarrow \prod_i \left( \frac{\sum_b Q_i^b \Sigma_b}{\mu} \right)^{Q_i^a} = e^{it_a(\mu)}. \tag{3.98}$$

### 3.4.1 Double trumpet

We now analyze the simplest model we presented with a bounded range for $y$, the double trumpet model introduced in section 3.3.1. To study the Coulomb branch, we take $\Sigma, \tilde{\Sigma}$ to have non-zero vacuum expectation values. This causes the fields $\Phi, \tilde{\Phi}$ to become massive, and we can integrate them out. Note that $Y$ is not massed up by these expectation values, since there is no $y\sigma$ potential. The equations determining supersymmetric vacua take the form:

$$k\Sigma + \tilde{k}\tilde{\Sigma} = 0, \tag{3.99a}$$

$$n \log\left(\frac{\Sigma}{\mu}\right) - kY = it, \tag{3.99b}$$

$$\tilde{n} \log\left(\frac{\tilde{\Sigma}}{\mu}\right) - \tilde{k}Y = i\tilde{t}, \tag{3.99c}$$

There are some basic issues to understand in this $Y$ picture. The first issue is one of identifying observables. In conventional Kähler GLSM models, each $\Sigma_a$ field is associated to an FI parameter and therefore to a Kähler class. For non-Kähler models, the map between Coulomb branch operators and observables on the Higgs branch is not a priori clear. For this particular model, each field $\Sigma, \tilde{\Sigma}$ and $Y$ is a twisted chiral superfield with a bottom component that is $(\bar{Q}_+ + Q_-)$-closed, but not exact. The $Y$ field is distinguished from the $\Sigma$, $\tilde{\Sigma}$ fields because it is not a field strength multiplet. Here duality helps us determine the observables because we know this model is equivalent to a Kähler model with a $P$ field. In that picture $\Sigma$, $\tilde{\Sigma}$ are possible observables but not $Y$. Eliminating $Y$ from (3.99b) and (3.99c) gives the relation:

$$-\tilde{k}n \log\left(\frac{\Sigma}{\mu}\right) + k\tilde{n} \log\left(\frac{\tilde{\Sigma}}{\mu}\right) = -i\tilde{k}t + ik\tilde{t} \quad \Rightarrow \quad \Sigma^{-\tilde{k}n}\tilde{\Sigma}^{k\tilde{n}} = \mu^{-\tilde{k}n+k\tilde{n}}e^{-i\tilde{k}t+ik\tilde{t}}. \tag{3.100}$$

We can then use (3.99a) to express this relation in terms of a single $\Sigma$,

$$\left(-\frac{k}{\tilde{k}}\right)^{k\tilde{n}} \Sigma^{-\tilde{k}n+k\tilde{n}} = \mu^{-\tilde{k}n+k\tilde{n}} e^{-i\tilde{k}t+ik\tilde{t}}. \tag{3.101}$$

This is the quantum cohomology ring for the double trumpet model.

We can ask whether this ring is fundamentally different from the ring one would find in a toric case with no $Y$ couplings. While the basis of gauge symmetries we used for this model is the most convenient to see compactness of the $Y$ interval, it is not the best basis to understand the ring. We saw in section 3.3.2 that we can change the basis of gauge fields for this model in order to transform it into the dual of a squashed toric model with $n$ fields of charge $-\tilde{k}/\widehat{k}$ and $\tilde{n}$ fields of charge $k/\widehat{k}$, where $\widehat{k} = \gcd(k, -\tilde{k})$. Note all these charges are positive.

In this basis, $Y$ is only coupled to one field strength multiplet, which is precisely the combination of the original gauge field multiplets appearing in (3.99a). That multiplet will be set to zero by the $Y$ equation of motion. The other field strength multiplet will obey a condition that is a function only of its FI parameter and the charges of the fields integrated out. The ring we constructed above in (3.101) is therefore the quantum cohomology ring of a weighted projective space.

We can apply the same reasoning to any model which is either a squashed Kähler model or a dual description of a squashed Kähler model. This is the case because squashing is a modification which only affects the $D$-terms, but not the $F$-terms which determine the quantum cohomology ring. We can therefore turn squashing off without changing the resulting ring.

### *3.4.2 Exponential models*

The model in section 3.3.4 does not have Coulomb branch vacua because the $e^Y\Sigma$ coupling with only one $U(1)$ factor always sets $\Sigma = 0$.

The Coulomb branch vacuum structure for the model of section 3.3.4 is determined from the critical points of the effective twisted chiral superpotential:

$$S_{\tilde{W}} = \frac{1}{8\pi}\int d^2x d\theta^+ d\bar{\theta}^- \Bigg[\Sigma\left(n\log\left(\frac{\Sigma}{\mu}\right) - n - kY - it(\mu)\right) + \tilde{\Sigma}\left(\tilde{n}\log\left(\frac{\tilde{\Sigma}}{\mu}\right) - \tilde{n} - \tilde{k}Y - i\tilde{t}(\mu)\right) + \Sigma'\left(n'\log\left(\frac{\Sigma'}{\mu}\right) - n' - \kappa e^Y - it'(\mu)\right)\Bigg]. \tag{3.102}$$

These critical points satisfy the equations,

$$k\Sigma + \tilde{k}\tilde{\Sigma} + \kappa e^Y \Sigma' = 0, \tag{3.103a}$$

$$n\log\left(\frac{\Sigma}{\mu}\right) - kY = it(\mu), \tag{3.103b}$$

$$\tilde{n}\log\left(\frac{\tilde{\Sigma}}{\mu}\right) - \tilde{k}Y = i\tilde{t}(\mu), \tag{3.103c}$$

$$n'\log\left(\frac{\Sigma'}{\mu}\right) - \kappa e^Y = it'(\mu). \tag{3.103d}$$

Solving the three latter equations leads to $\Sigma$ solutions for any value of $Y$,

$$\left(\frac{\Sigma}{\mu}\right)^n = e^{it(\mu)+kY}, \qquad \left(\frac{\tilde{\Sigma}}{\mu}\right)^{\tilde{n}} = e^{i\tilde{t}(\mu)+\tilde{k}Y}, \qquad \left(\frac{\Sigma'}{\mu}\right)^{n'} = e^{it'(\mu)+\kappa e^Y}. \tag{3.104}$$

Each of these equations has a finite number of solutions for a fixed $Y$, giving a total of $n\tilde{n}n'$ vacua.

Just like the case considered in (2.57), it is valid to integrate out the charged fields as long as the right-hand sides of the equations appearing in (3.104) are large. In equation (3.103b), the FI parameter $t$ will run according to (2.51),

$$it(\mu) = it(\Lambda) - n\log\frac{\mu}{\Lambda}, \tag{3.105}$$

and therefore $\Sigma/\mu$ will scale with $1/\mu$ as $\mu$ becomes smaller. Equivalently, solutions for $\Sigma$ will be independent of $\mu$. The $n$-dependence dropped out of this argument; it also drops out of analogous arguments which apply to equations (3.103c) and (3.103d). This implies that once we plug in $\Sigma$ into (3.103a), it can also be solved for $Y$ independently of $\mu$. The masses of the fields we integrated out will therefore become large when compared to $\mu$ for all vacuum solutions by the same argument found in section 2.4.

We still have to solve the first equation, (3.103a). To count the number of solutions, it is easier to work with the single-valued field $X = e^Y$. In terms of $X$, the remaining vacuum equation takes the form:

$$ke^{it/n}X^{k/n} + \tilde{k}e^{i\tilde{t}/\tilde{n}}X^{\tilde{k}/\tilde{n}} + \kappa X e^{it'/n' + \kappa X/n'} = 0. \tag{3.106}$$

This equation has no dependence on the scale $\mu$. This can be seen by noting that $\mu$ drops out of equations (3.103) and (3.104). So one can use $t, \tilde{t}$ and $t'$ defined at the scale $\Lambda$ in equation (3.106). As a complex function of $X$, the left-hand side of this equation has an infinite number of zeroes, so we have infinite distinct vacua in the Coulomb branch. Since the masses of the integrated out fields are large in the IR, there is no obvious problem with this analysis.

The structure we have found here is quite surprising and quite different from usual computations of quantum cohomology. It might well be indicative of a more generic vacuum structure found when examining generalized Kähler spaces beyond the lamp post. There are a couple of points to summarize: first, the Higgs branch geometry is compact for this model. In fact, the condition for the space to remain compact found in (3.92) can easily be preserved under RG flow. However, the space is non-Kähler and the structure of instanton corrections, which usually generate quantum cohomology, has yet to be understood in any detail. Similar comments apply to the observables of the theory. Because there is a non-zero $H$, the instanton configurations are likely to be complex field configurations.

| | $U(1)$ | $U(1)_s$ | $U(1)'$ |
|---|---|---|---|
| $\Phi^i$ $(\times n)$ | 1 | 0 | 0 |
| $\Phi_s$ $(\times 1)$ | 1 | 1 | 0 |
| $\Phi'^j$ $(\times n')$ | 0 | 0 | 1 |

Table 3.3: The charge matrix for the example of section 3.4.3.

What we see is an infinite number of Coulomb branch vacua for this model. It is possible that further quantum corrections will lift these vacua, but since these vacua are seen from a holomorphic superpotential, it is not clear from where these quantum corrections might originate. One possibility are strong interactions between $Y$ and $\Sigma$ generating an anomalous dimension for operators like $e^Y$. If we want to interpret the Coulomb branch as an operator ring, capturing quantum corrections to a classical ring of observables associated to the Higgs branch, then we note that $X = e^Y$ must be retained as an operator. The four operators $(\Sigma, \tilde{\Sigma}, \Sigma', X)$ then satisfy the ring relations (3.104) and (3.106).

One other possibility is that the Coulomb branch of this model should not be viewed as purely encoding data interpretable in terms of Higgs branch physics. In conventional GLSM examples, $\Sigma$ fields can be related to Higgs branch fields via equations of motion. In this case, this is still true for the $\Sigma$ fields but the neutral $Y$ field does appear on both branches, which perhaps suggests that the Coulomb branch might be viewed as distinct from the Higgs branch.

### *3.4.3 An explicit example*

Let us remove some of the notational clutter to better understand and interpret what might be going on. Take the case $k = -\tilde{k} = 1$. Additionally, take $\tilde{n} = 1$. We also want a better feel for how the exponential coupling with coefficient $\kappa$ is so dramatically changing the vacuum structure. So we will change basis for $(\Sigma, \tilde{\Sigma})$, as outlined in section 3.4.1, to $(\Sigma, \Sigma_s) = (\Sigma, \tilde{\Sigma} - \Sigma)$. This is a basis in which the $U(1)$ factors transparently describe the dual of a squashed $\mathbb{C}P^n$ model. We have $n$ fields $\Phi^i$ with charge 1 under the first $U(1)$ factor $\Sigma$, which has no $Y$ coupling; there is one field $\Phi_s$ with charge 1 under $\Sigma$ and also charge 1 under $\Sigma_s$. The $\Sigma_s$ gauge symmetry has a $Y$ coupling. The phase of the $\Phi_s$ field is squashed in the $P$ picture. Finally there are fields $\Phi'^j$ with charge 1 under $\Sigma'$. This collection of fields appears in table 3.3.

Supersymmetric vacua are determined by equations (3.103) which become

$$-\Sigma_s + \kappa e^Y \Sigma' = 0, \tag{3.107a}$$

$$n \log\left(\frac{\Sigma}{\mu}\right) + \log\left(\frac{\Sigma+\Sigma_s}{\mu}\right) = it + i\tilde{t}, \tag{3.107b}$$

$$\log\left(\frac{\Sigma+\Sigma_s}{\mu}\right) + Y = i\tilde{t}, \tag{3.107c}$$

$$n' \log\left(\frac{\Sigma'}{\mu}\right) - \kappa e^Y = it'. \tag{3.107d}$$

The first thing we would like to recover is the ring of the dual projective space, which should emerge in the limit $\kappa \to 0$. Setting $\kappa = 0$ forces $\S_s = 0$ from (3.107a). The remaining equations then decouple and in terms of $X = e^Y$ we find:

$$\Sigma^{n+1} = \mu^{n+1} e^{i(t+\tilde{t})}, \qquad \Sigma'^{\,n'} = \mu^{n'} e^{it'}, \qquad \Sigma X = \mu e^{i\tilde{t}}. \tag{3.108}$$

The first two relations are the familiar ones we expect for the $\mathbb{C}P^n \times \mathbb{C}P^{n'-1}$ model. The last relation constrains the operator $X$ in terms of $\Sigma$.

Now we turn on $\kappa$. It still seems natural to use (3.107a) to solve for $\Sigma_s$ with $\Sigma_s = \kappa X \Sigma'$. The rest of the equations give the relations,

$$\Sigma^n(\Sigma + \kappa X \Sigma') = \mu^{n+1} e^{i(t+\tilde{t})}, \qquad \Sigma'^{n'} e^{-\kappa X} = \mu^{n'} e^{it'}, \qquad X\left(\Sigma + \kappa X \Sigma'\right) = \mu e^{i\tilde{t}}, \tag{3.109}$$

which are an intriguing deformation of the ring relations (3.108). Specifically, $\Sigma$ and $\Sigma'$ are now coupled as we might expect from figure 3.5. This appears to be the case even in the classical limit where $r, \tilde{r}, r' \to \infty$, where the ring should correspond to a geometric ring of the generalized Kähler space.

Once we take $\kappa \neq 0$, the number of solutions to the equation determining $X$,

$$e^{it/n} X^{1/n} - e^{i\tilde{t}} X^{-1} + \kappa X e^{it'/n' + \kappa X/n'} = 0, \tag{3.110}$$

moves from finite to infinite. It is worth noting that if we try to perturbatively expand the zero solutions around $\kappa = 0$ to any finite order in $\kappa$, the number of solutions will still be finite. There are an infinite number of solutions that are not analytic around $\kappa = 0$. To make this more natural, we can examine a toy model with only one chiral superfield $X$ and a superpotential of the form

$$W = X - e^{\kappa X}. \tag{3.111}$$

Critical points of this superpotential obey the condition

$$1 - \kappa e^{\kappa X} = 0. \tag{3.112}$$

When $\kappa = 0$, this equation has no solutions. However, if $\kappa > 0$, the solutions are given by

$$X = \frac{1}{\kappa} \log \left( \frac{1}{\kappa} \right). \tag{3.113}$$

There is an infinite set of solutions, one for each branch of the logarithm. They are all non-analytic in $\kappa$, moving to infinite $|X|$ as $\kappa$ is taken to 0.

It would be very interesting to calculate elliptic genera for this class of models, in order to better understand the infinity of vacua we have found. Unfortunately, to compute the elliptic genus in a straightforward way, we need both $U(1)_L$ and $U(1)_R$ R-symmetries to be unbroken, which is not true for these compact models. As we have already discussed, it is challenging to find torsional examples which are both compact and conformal in this $(2,2)$ setting. Such models are possible with $(0,2)$ worldsheet supersymmetry. For non-compact models, the elliptic genus should generically have a non-holomorphic dependence on the torus modular parameter; see, for example [43, 44].

### 3.4.4 Unified structure

This Coulomb branch analysis can also be generalized to the unified case described in section 3.3.5. We allow for several fields $Y^\mu$, and any number of vector superfields $\Sigma_a$, coupled by twisted superpotential couplings of the more general form $f_a(Y)\Sigma_a$. We take the set of charged fields to consist of $n_a$ chiral superfields which are charged with charge 1 only under one gauge symmetry corresponding to $\Sigma_a$. If we integrate out the charged superfields, we obtain an effective twisted superpotential,

$$S_{\tilde{W}} = \frac{1}{8\pi} \int d^2x d\theta^+ d\bar{\theta}^- \sum_a \Sigma_a \left( n_a \left[ \log\left(\frac{\Sigma_a}{\mu}\right) - 1 \right] - f_a(Y) - it_a(\mu) \right). \tag{3.114}$$

Varying the superpotential with respect to $\Sigma_a$ gives equations of the form

$$n_a \log\left(\frac{\Sigma_a}{\mu}\right) - f_a = it_a \quad \Rightarrow \quad \left(\frac{\Sigma_a}{\mu}\right)^{n_a} = e^{it_a + f_a}. \tag{3.115}$$

Varying with respect to $Y^\mu$ gives the conditions

$$\sum_a \partial_\mu f_a \Sigma_a = 0. \tag{3.116}$$

If we are interested in solving for vacua rather than studying rings, we can further substitute solutions to (3.115) giving:

$$\sum_a \partial_\mu f_a e^{it_a/n_a + f_a/n_a} = \partial_\mu \left( \sum_a n_a e^{it_a/n_a + f_a/n_a} \right) = 0. \tag{3.117}$$

As in the previous section, the masses of the fields that were integrated out become arbitrarily large as we flow to lower energies.

We now want to explore the generic number of solutions to (3.117). For simplicity, take one $Y$ field, and consider (3.117) initially as a complex function of the cylinder variable $y$. For compactness $a > 1$. We want to characterize the number of zeros of this function,

$$h(y) = \sum_a \partial_y f_a e^{it_a/n_a + f_a/n_a}. \tag{3.118}$$

The functions $f_a$ take the form

$$f_a = k_a y + \tilde{f}_a, \qquad \tilde{f}_a = \sum_{m=-\infty}^{\infty} c_a^m e^{my},$$

for complex constants $c_a^m$. Some $k_a$ might be negative. We would like to move from the cylinder variable $y \sim y + 2\pi i$ to a single-valued variable. If it were not for the $n_a$ factors appearing in (3.118), we would simply use $x = e^y$. Instead define $n_{\text{lcm}} = \text{lcm}\{n_a\}$. We can then view $h(y)$ as a complex function of $z = e^{y/n_{\text{lcm}}}$ rather than $y$. The penalty for this change of variable is that the equation $h(y) = 0$ is replaced by a finite collection of equations in $z$ obtained by repeatedly shifting $y \to y + 2\pi i$.

To proceed, we need to be able to say something about $f_a$. Usually, we do not want singular couplings in the classical Lagrangian so let us assume that $f_a$ is smooth with no singularities for finite values of $y$. Viewed as a function of $z$, this implies $f_a$ is holomorphic in $z$ away from 0 and $\infty$. It is not particularly strange to also insist that $\tilde{f}_a$ is holomorphic in $z$. At least under this restriction, we can say something more about the number of zeros because $h(z)$ is analytic in the complex plane. Any analytic function with a finite number of zeros can be written in the form $P(z)e^{g(z)}$ where $g(z)$ is also an analytic function. Our $h(z)$ takes the form,

$$h(z) = \sum_a \left(k_a + \partial\tilde{f}_a(z)\right) z^{\frac{k_a n_{\text{lcm}}}{n_a}} e^{\frac{\tilde{f}_a(z)}{n_a}} e^{\frac{it_a}{n_a}}, \qquad \tilde{f}_a = \sum_{m=0}^{\infty} c_a^m z^{m \cdot n_{\text{lcm}}}. \tag{3.119}$$

If all $\tilde{f}_a(z)$ are identical then $h(z)$ can admit a finite number of zero solutions. Otherwise, we generically expect an infinite number of solutions as we saw in the example of section 3.4.3.

# CHAPTER 4

# NEW $(0,2)$ VACUA

Our understanding of $(0,2)$ gauged linear sigma models, and the full set of geometries they can describe, is much less complete than that of $(2,2)$ gauged linear sigma models. In particular, the richness of couplings allowed by chiral gauge theories, and a non-trivial anomaly cancellation, brings with it a myriad of unexplored possibilities. Here we start by reviewing some of the work that has been done in understanding the vacuum structure of $(0,2)$ gauge theories, and also the generalization of quantum cohomology rings to a $(0,2)$ context. We present how these ideas can be applied in $(0,2)$ gauge theories which do not have the field content of a $(2,2)$ theory and are therefore more inherently $(0,2)$, presenting some previously unexplored examples.

## 4.1 Quantum sheaf cohomology rings

Quantum cohomology rings for $(2,2)$ theories are a quantum extension of the de Rham cohomology rings for $(2,2)$ theories, studied in [10]. The existence of a $(0,2)$ generalization, named quantum sheaf cohomology rings, was hypothesized in [27] and shown to exist in [45].

In theories with $(2,2)$ supersymmetry, consider the operators

$$Q_A = \bar{Q}_+ + Q_-, \qquad\qquad Q_B = \bar{Q}_+ + \bar{Q}_-. \tag{4.1}$$

Note that by definition, the lowest component $\phi$ of a chiral superfield obeys $[Q_B, \phi] = 0$, while the lowest component $\widehat{\phi}$ of a twisted chiral superfield obeys $[Q_A, \widehat{\phi}] = 0$. More generally, we name operators $\mathcal{O}$ chiral if they commute with $Q_B$, and twisted chiral if they commute with $Q_A$. The sum and product of chiral operators will still be chiral, and therefore they form the *chiral ring* of the theory. The twisted chiral ring is analogously defined from twisted chiral operators.

The operators $Q_A$ and $Q_B$ may seem like strange operators to consider. After all, they are formed as the sum of operators with different Lorentz transformations. However, they become more natural as part of theories obtained from the original supersymmetric model by a procedure known as *twisting*. Twisting is achieved by shifting the spin connection of the theory by a multiple of the connection corresponding to a $U(1)$ R-symmetry. This changes the spins of operators in the theory. Such an operation is particularly simple to carry out in an Euclidean worldsheet, where the spin connection is simply a $U(1)$ connection. In $(2,2)$ theories, two twists are natural: if we use the vector R-symmetry, we obtain the A-twist, for which the operator $Q_A$ becomes a scalar, while the axial R-symmetry leads to the B-twist where $Q_B$ is a scalar [46]. Since the twists transform the corresponding fermionic charge into a scalar, the charge will be conserved even on a curved Riemann surface which may not have a covariantly constant spinor.

In a twisted theory, we can restrict our definition of physical observables to include only observables in $Q$-cohomology, that is, operators $\mathcal{O}$ satisfying

$$[Q, \mathcal{O}] = 0, \tag{4.2}$$

or $Q$-closed, for which there is no operator $\mathcal{O}'$ such that $[Q, \mathcal{O}'] = \mathcal{O}$, that is, $\mathcal{O}$ is not $Q$-exact. The correlation functions of a product of such operators can be shown to be independent of the metric of spacetime, so the twisted theory with these observables is a topological quantum field theory. In addition, derivatives of these operators are $Q$-exact, so correlation functions involving derivatives of the operators will vanish. This implies that the correlation functions are independent of the positions at which we insert the operators.

In a $(2,2)$ non-linear sigma model, there is a correspondence between the action of the operator $Q_A$ on physical observables and the action of the exterior derivative on differential forms on the manifold described by the non-linear sigma model. The operators in $Q$-cohomology can therefore be mapped to the de Rham cohomology classes of the manifold.

The correlation functions of observables in the twisted theory correspond, classically, to the geometric cohomology ring relations. The correlation functions are corrected by instanton effects, and therefore form a structure called *quantum cohomology ring*.

In a standard $(2,2)$ GLSM, the chiral superfields $\Phi^i$ will be the charged fields, while the twisted chiral superfields $\Sigma^a$ contain the gauge field strengths. The operators in the quantum cohomology ring then correspond to the bottom components of $\Sigma^a$, denoted $\sigma^a$. The equations of motion used when descending from the GLSM to the NLSM allow us to map between $\sigma_a$ and rank $(1,1)$ differential forms in the cohomology ring. It should also be noted that the B-twist is only possible when the axial R-symmetry is non-anomalous, which in these GLSM corresponds to all gauge symmetries having a vanishing sum of charges. On the other hand, the vector R-symmetry is always non-anomalous in these theories, so we can always define the A-twist. Therefore we will focus on the A-twist and its $(0,2)$ analogue in what follows.

In $(0,2)$ theories, since we only have right-moving supersymmetries, the space of operators in $Q = \bar{Q}_+$ cohomology is much larger than the spaces in $Q$-cohomology for any of the charges defined above for $(2,2)$ theories. In fact, instead of a finite-dimensional ring, we would obtain a ring of infinite dimension. In addition, the twisted quantum field theory with those objects as its physical observables is no longer topological. However, in GLSM obeying

$$\sum_{i \in R} Q_i = \sum_{a \in L} Q_a, \tag{4.3}$$

where $R$ and $L$ refer to the set of right-moving and left-moving fermions respectively, we can define a subset of the chiral ring whose correlation functions are more constrained. This subset is obtained by restricting to operators which saturate the BPS bound not only on the right (which is implied by being in $Q$-cohomology) but also on the left. That is, we restrict to operators whose left-moving dimension and R-symmetry charge are related by $\Delta = \frac{1}{2}q$. The theory obtained here is sometimes called the A/2-twist, and while it is not topological, it is a conformal field theory.

In $(0,2)$ non-linear sigma models, the left-moving fermions are no longer related to the scalar fields by supersymmetry, and therefore they no longer live on the tangent bundle of the manifold as they do in $(2,2)$ theories. The fermionic charge $Q$ can be seen to correspond to the geometric operator $\partial$, and the ring of operators defined above corresponds to the sheaf cohomology over the left-moving bundle. The quantum deformation of this ring is then named quantum sheaf cohomology ring.

## 4.2 Coulomb branch of $(0,2)$ theories

In toric $(2,2)$ GLSM, (that is, GLSM without a chiral superpotential term $W$), the Coulomb branch calculation outlined in Section 2.4 can be directly used to arrive at the quantum cohomology ring relations [47]. In $(0,2)$ GLSM which can be obtained as deformations of $(2,2)$ theories, we can still use a slight generalization of the Coulomb branch computation in Section 2.4. This was used for instance in [48] to study the quantum sheaf cohomology rings of certain $(0,2)$ theories.

Since we will use related computations below, we will review the relevant one-loop corrections here in a $(0,2)$ context. The only one-loop correction is the one in (2.50),

$$\frac{1}{e^2}\langle D\rangle = \frac{Q_i}{4\pi}\log\frac{\Lambda^2+M^2}{\mu^2+M^2} \tag{4.4}$$

However, now we will allow more general masses for our chiral fields than the mass $m^2 = Q^2|\sigma|^2$ used in $(2,2)$ Coulomb branch calculations.

For instance, in $(0,2)$ theories which are deformations of $(2,2)$ theories, that is, which have the same field content as a $(2,2)$ theory, we can generalize the $E$-couplings, while keeping them linear in $\Sigma_a$, as

$$\bar{D}_+\Gamma_i = \sqrt{2}M_{ij}^{(\alpha)}\Phi_j, \tag{4.5}$$

where $\alpha$ denotes the charge sector (since $\Gamma^i$ and $\Phi^j$ must have the same charge under all gauge symmetries to be coupled in this way). Recall the $(2,2)$ locus has $M_{ij}^{(\alpha)} = \sum_a Q_{(\alpha)}^a \Sigma_a \delta_{ij}$. With this generalization, the chiral superfields are no longer mass eigenstates. The mass matrix is given by

$$m_{ij}^2 = M_{ki}^{(\alpha)*} M_{kj}^{(\alpha)}. \tag{4.6}$$

If the matrices $M^{(\alpha)}$ are non-singular, all chiral superfields are massed up when the $\Sigma_a$ fields gain expectation values. This leads to an effective superpotential of the form

$$S_{\tilde{W}} = \frac{\sqrt{2}}{8\pi} \int d^2x d\theta^+ \sum_a \Upsilon_{a-} \left( \sum_\alpha Q_{(\alpha)}^a \log \left( \frac{\det M^{(\alpha)}}{\mu^{n_\alpha}} \right) - it_a(\mu) \right), \tag{4.7}$$

where $n_\alpha$ is the number of fields in charge sector $(\alpha)$. This leads to the condition for Coulomb branch vacua

$$\prod_\alpha \left( \frac{\det M^{(\alpha)}}{\mu^{n_\alpha}} \right)^{Q_{(\alpha)}^a} = e^{it_a(\mu)}, \tag{4.8}$$

which generalizes (2.53). As the deformation from $(2,2)$ theories does not change the R-symmetry of the theory, the fields $\sigma_a$ still form the physical observables of the A/2 twist. In this sense, we can obtain the quantum sheaf cohomology ring from a Coulomb branch calculation in a family of $(0,2)$ quantum field theories.

## 4.3 Elliptic genera

The elliptic genus is of a two-dimensional theory is simply the partition function of the theory considered on a two-torus. It is a function of the modular parameter $\tau$, often with interesting modular transformation properties. A recipe for calculating the elliptic genus in $(0,2)$ and $(2,2)$ gauged linear sigma models was given recently in [49, 50]. Typically we construct the flavored elliptic genus, by turning on fugacities corresponding to a Wilson line for each flavor symmetry of the theory. Briefly, the expression for the elliptic genus is given by

$$Z(y_i, \tau) = \oint \frac{du_j}{2\pi i} I(u, y_i, \tau), \tag{4.9}$$

where $y_i$ are the flavor fugacities mentioned above, $\tau$ is the modular parameter, and $u_j$ is the fugacities for the gauge symmetries, which are integrated over. We will now explain how to build $I$ and specify an integration contour for a general $(0,2)$ GLSM. Note that to calculate the elliptic genus, we also need to have a non-anomalous $U(1)_R$ symmetry, which in standard gauge theories just corresponds to a vanishing sum of charges.

$I$ will be formed as the product of a factor for each superfield in the theory. A chiral superfield $\Phi$ with gauge charges $Q_j$ and flavor charges $q_i$ contributes

$$I_\Phi = i\frac{\eta(q)}{\theta_1(q, x_j^{Q_j} y_i^{q_i})} \tag{4.10}$$

where we made use of $x_j = e^{2\pi i u_j}$. A Fermi multiplet on the other hand contributes

$$I_\Gamma = i\frac{\theta_1(q, x_j^{Q_j} y_i^{q_i})}{\eta(q)}. \tag{4.11}$$

It is not hard to see, using $\theta_1(q, x^{-1}) = \theta_1(q, x)$, that the contributions of a chiral and a Fermi with opposite gauge and flavor charges cancel out. This should be expected, as we can mass up the two fields with a superpotential coupling $m\Gamma_-\Phi$. As this procedure does not change the elliptic genus computation for the other fields (the theory gains a new non-anomalous symmetry for each of these pairs, which rotates just the fields in that pair), we can use this trick to generalize elliptic genera computations to massive models. Finally, a vector superfield contributes

$$I_\Upsilon = -2\pi i \eta(q)^2. \tag{4.12}$$

The only poles in the integrand come from the contributions of the chiral superfields, at the locations of the zeros of $\theta_1(q, e^{2\pi i z})$, which occur at $z = m + n\tau$, $m, n \in \mathbb{Z}$. It is therefore useful to keep in mind

$$\frac{d}{dz}\theta_1(q, e^{2\pi i z})\Big|_{z=0} = 2\pi\eta(q)^3, \tag{4.13}$$

so

$$\frac{1}{2\pi i}\oint_{u=m+n\tau} du \frac{1}{\theta_1(q, e^{2\pi i u})} = \frac{(-1)^{m+n} e^{i\pi n^2 \tau}}{2\pi \eta(q)^3}. \tag{4.14}$$

Finally, we must specify the integration contour. For a single $U(1)$ gauge group, we should sum only over poles corresponding to fields with positive charge, or only over fields with negative charge [49]. Note they give the same result up to a sign, since the sum over all poles is zero as the torus is a closed surface. When we use the trick above to compute the elliptic genus for models with non-zero sum of charges, in order not to miss poles coming from the fields we have to add to make the theory conformal, we should calculate the elliptic genus using the sign with larger $|\sum_i Q_i|$.

For a gauge group of higher rank $n$, the poles we should sum over are given by the Jeffrey-Kirwan residue [50]. This procedure is as follows [43]: start by picking an arbitrary vector $\mathbf{v} \in \mathbb{R}^n$ (like with the choice of sign for rank one, the result will not depend on this choice). Now a given pole is to be summed over if $\mathbf{v}$ is contained in the interior of the cone in $\mathbb{R}^n$ defined by the $n$ charge vectors of the fields corresponding to the functions $\theta_1$ that vanish at that pole.

## 4.4 Examples of new vacua

In this section, we will enumerate some examples of $(0,2)$ gauged linear sigma models which have a field content different from the $(2,2)$ content, making them inherently $(0,2)$, without a $(2,2)$ locus. We will generally consider superpotentials that give rise to potentials of the form $|x|^2|y|^2$ for two scalar fields $x$ and $y$, so that only one of these fields can be non-zero. This is akin to the $|\sigma|^2|\phi|^2$ potential that leads to the quantum correction to the twisted superpotential in $(2,2)$ models. Classically, these models have a singularity at the point $xy = 0$, but as we will see the one-loop corrections to the potential will typically create a potential barrier keeping us away from this point. This has similarities to the phenomena explored in [25], but here we search for examples where we have more control over the vacuum structure, putting less emphasis on determining the geometry.

### 4.4.1 Two fields, equal charges

For simplicity, we will study examples with a single $U(1)$ gauge symmetry. The simplest model we will consider has just two chiral superfields, which here we call $\Phi_\pm$, with charges $\pm 1$, coupled by a superpotential to a Fermi $\Gamma_{0-}$ as $m\Gamma_{0-}\Phi_+\Phi_-$. We add Fermi superfields $\Gamma_\pm$ with charges $\pm 1$ and no superpotentials in order to cancel gauge anomalies.

The potential gives mass to each of $\phi_\pm$ when the other field gains an expectation value, of the form

$$M_\pm^2 = m^2|\phi_\mp|^2. \tag{4.15}$$

The one-loop correction in (2.50) then adds logarithmic terms to the $D$-term condition, correcting it to read

$$|\phi_+|^2 - |\phi_-|^2 + \log\frac{\mu^2 + m^2|\phi_+|^2}{\mu^2 + m^2|\phi_-|^2} = r. \tag{4.16}$$

Note that as the sum of charges of the scalar fields in the theory is zero, the FI parameter $r$ does not run. We can now ask in which regime we trust the computation, and where one of the $\phi$ can be integrated out at one loop.

If $r > 0$, $\phi_+$ will have a classical expectation value, which will set $\phi_- = 0$. The $D$-term then reads

$$|\phi_+|^2 + \log\frac{\mu^2 + m^2|\phi_+|^2}{\mu^2} = r. \tag{4.17}$$

The left-hand side of this equation is monotonic and surjective in $\mathbb{R}$, so there is always one solution for $|\phi_+|$. As $\mu \to 0$, the solution to this equation will have decreasing $|\phi_+|$, so that for small enough $\mu$ the log term will dominate over $|\phi_+|^2$. Then we see the solution has the behavior

$$M_-^2 = m^2|\phi_+|^2 \sim \mu^2 e^r. \tag{4.18}$$

Hence we can tune the ratio between the mass of the field we integrated out and $\mu$ to be as

|  | $\Phi_+$ | $\Phi_-$ | $\Gamma_0$ | $\Gamma_+$ | $\Gamma_-$ | $\mathcal{A}$ | fugacity |
|---|---|---|---|---|---|---|---|
| $U(1)_{\text{gauge}}$ | 1 | $-1$ | 1 | $-1$ |  | 0 | $x$ |
| $U(1)_0$ | 1 | 1 | $-2$ |  |  | 0 | $y_1$ |
| $U(1)_\gamma$ |  |  |  | 1 | 1 | 0 | $y_2$ |
| $U(1)_a$ |  |  |  | 1 |  | -1 | $y_3$ |

Table 4.1: Symmetry table for the model with only two chiral superfields $\Phi_\pm$.

large as we want, by changing the FI parameter $r$. This ratio does not diverge as $\mu \to 0$, so there might be other important corrections to this description, but this gives us confidence that the theory will likely have a supersymmetric vacuum.

If $r < 0$, the classical vacua have $\langle \phi_- \rangle \neq 0$, setting $\phi_+ = 0$, giving a $D$-term condition of the form

$$|\phi_-|^2 + \log \frac{\mu^2 + m^2|\phi_-|^2}{\mu^2} = -r. \tag{4.19}$$

This gives a similar vacuum from the one we found at $r > 0$, with

$$M_-^2 = m^2|\phi_+|^2 \sim \mu^2 e^{-r}. \tag{4.20}$$

To give further evidence, we would like to calculate the elliptic genus of this GLSM. The symmetries of this model are given in Table 4.1. Note that the anomaly in the last symmetry sets $y_3 = 1$, so we will not include it in the calculation.

Following the algorithm in Section 4.3, the integrand we need to calculate the elliptic genus of this theory is

$$I = (-2\pi i \eta(q))^2 \frac{i}{\eta(q)} \frac{\theta_1(q, y_1^{-2})\theta_1(q, xy_2)\theta_1(q, x^{-1}y_2)}{\theta_1(q, xy_1)\theta_1(q, x^{-1}y_1)}. \tag{4.21}$$

We should take only one of the two poles, as the fields have opposite charges. At $x = y_1^{-1}$, we find

$$Z = -\frac{1}{\eta(q)^2}\theta_1(q, y_1 y_2)\theta_1(q, y_1^{-1}y_2). \tag{4.22}$$

As a test, we can see what happens when we turn off the superpotential. Then we can rotate $\Gamma_0$ in an independent rotation from $\Phi_\pm$, so

$$I = (-2\pi i\eta(q))^2 \frac{i}{\eta(q)} \frac{\theta_1(q, y_4)\theta_1(q, xy_2)\theta_1(q, x^{-1}y_2)}{\theta_1(q, xy_1)\theta_1(q, x^{-1}y_1)}, \tag{4.23}$$

and the partition function becomes

$$Z = \frac{1}{\eta(q)^2} \frac{\theta_1(q, y_4)\theta_1(q, y_1 y_2)\theta_1(q, y_1^{-1}y_2)}{\theta_1(q, y_1^{-2})}. \tag{4.24}$$

This also does not vanish.

### *4.4.2 Two fields, different charges*

An interesting possibility is to have the fields have charges of opposite signs but different magnitudes. We will denote the charges by $Q_1$ and $-Q_2$, with both $Q_i > 0$, and the fields as $\Phi_{1,2}$. We will still turn on a superpotential coupling $m\Gamma_{0-}\Phi_1\Phi_2$. Then the field $\Gamma_{0-}$ must have a non-zero charge, given by $Q_2 - Q_1$. We can always arrange a cancelling gauge anomaly by adding Fermi superfields with charges $\pm 1$, since from $\Phi_{1,2}$ and $\Gamma_{0-}$ we obtain an anomaly coefficient

$$Q_1^2 + Q_2^2 - (Q_2 - Q_1)^2 = 2Q_1 Q_2 > 0. \tag{4.25}$$

Now the $D$-term condition is

$$Q_1|\phi_1|^2 - Q_2|\phi_2|^2 - Q_1 \log \frac{\mu^2 + m^2|\phi_2|^2}{\Lambda^2} + Q_2 \log \frac{\mu^2 + m^2|\phi_1|^2}{\Lambda^2} = r, \tag{4.26}$$

In the branch with $\phi_2 = 0$, we want to check if the solution has $m|\phi_1| \gg \mu$. We have

$$Q_1|\phi_1|^2 + \log \frac{m^{2Q_2}|\phi_1|^{2Q_2}\Lambda^{2Q_1-2Q_2}}{\mu^{2Q_1}} = r. \tag{4.27}$$

When $\mu \to 0$, we have

$$m^2|\phi_1|^2 \sim \mu^{2\frac{Q_1}{Q_2}} \Lambda^{2-2\frac{Q_1}{Q_2}} e^r, \tag{4.28}$$

so $m|\phi_1| \gg \mu$ as $\mu$ gets small if $Q_2 > Q_1$. Note that unlike in the case with $Q_1 = Q_2$, now $M_{\phi_2}/\mu$ grows when we take $\mu$ to zero, and therefore the picture obtained by integrating out $\phi_2$ is trustworthy in the infrared. On the other branch, with $\phi_1 = 0$, we have

$$Q_2|\phi_2|^2 + \log \frac{m^{2Q_1}|\phi_2|^{2Q_1}\Lambda^{2Q_2-2Q_1}}{\mu^{2Q_2}} = -r, \tag{4.29}$$

so when $\mu \to 0$,

$$m^2|\phi_2|^2 \sim \mu^{2\frac{Q_2}{Q_1}} \Lambda^{2-2\frac{Q_2}{Q_1}} e^{-r}, \tag{4.30}$$

so this is trustworthy if $Q_1 > Q_2$. Once again only one vacuum gets picked, though now the charges are the relevant quantities, and we obtain more control when starting with a theory whose sum of charges is non-zero.

### *4.4.3 More fields*

We can have examples with more fields. One class of examples that is simple to build has $k$ chiral fields of charge 1 $\Phi^i$, one field with charge $-n$ $\Sigma$, and $k$ Fermi fields of charge $1-n$ $\Gamma^i$. These can be coupled by $\bar{D}_+\Gamma^i = m\Sigma\Phi^i$. This field content has a possible anomaly

$$\mathcal{A} = k_R - k_L = k + n^2 - k(n-1)^2 = n^2 - kn^2 + 2kn = n(n - kn + 2k). \tag{4.31}$$

When $\mathcal{A} \geq 0$, we can supplement the model with free Fermi fields of charge $\pm 1$ to bring it to zero. This is true for $k = 1$ with any $n$, $k \leq 2$ and $n \leq 4$, $n = k = 3$, or $n \leq 2$ with any $k$. The model with $n = k = 1$ is the same as the model in Section 4.4.1, after dualizing $\Gamma_{0-}$ to transform the superpotential into an $E$ coupling.

The D-term conditions above in these models have the form

$$|\phi|^2 - n|\sigma|^2 - k\log\frac{\mu^2+m^2|\sigma|^2}{\Lambda^2} + n\log\frac{\mu^2+m^2|\phi|^2}{\Lambda^2} = r. \tag{4.32}$$

As in the models of Section 4.4.2, for $n > k$ the branch with $\Phi \neq 0$ is trustworthy, whereas for $k > n$ we can trust the branch with $\Sigma \neq 0$. If $n = k$, the branch that we can trust depends on the sign of the FI parameter $r$.

## $n = k = 3$

For example, with $n = k = 3$, take $\Phi^i$, $i = 1, 2, 3$ with charge 1, one $\Sigma$ of charge $-3$, and three $\Gamma^i$ of charge $-2$, with couplings $\bar{D}_+\Gamma^i = m\Sigma\Phi^i$. Note that this is obtained from the $(2,2)$ $\mathbb{C}P^2$ model by adding a multiple of $U(1)_L$ to the gauge symmetry.

In fact, both possible vacua of these models are quite interesting. If $r > 0$, we find what seems to be a Higgs-like vacuum, corresponding to the $\phi^i$ fields having non-zero expectation values, with the $\phi^i$ describing a $\mathbb{C}P^2$ space. However, there are two key differences. First, while the projective space will decrease in size as we bring down $\mu$, it will never reach zero radius, as $m^2|\phi|^2 \sim \mu^2 e^r$. Secondly, the left-moving fermions have charge $-2$ in the GLSM, instead of 1. This means that the left-moving bundle will be quite different from the bundle we would have in a $(2,2)$ non-linear sigma model.

To learn more about the left-moving bundle, we will analyze the conditions the fermions will obey from the Yukawa couplings. These give rise to a short exact sequence describing the bundle $\mathcal{E}$, which allows us to calculate its Chern classes. We have

$$0 \to \mathcal{O}(-3) \xrightarrow{E} \mathcal{O}(-2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2) \to \mathcal{E} \to 0, \tag{4.33}$$

so

$$c(\mathcal{E}) = \frac{c(\mathcal{O}(-2))^3}{c(\mathcal{O}(-3))} = \frac{(1-2H)^3}{1-3H}, \tag{4.34}$$

| | $\Sigma$ | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | $\mathcal{A}$ | fugacity |
|---|---|---|---|---|---|---|---|---|---|
| $U(1)_{\text{gauge}}$ | $-3$ | 1 | 1 | 1 | $-2$ | $-2$ | $-2$ | 0 | $x$ |
| $U(1)_{1-2}$ | | 1 | $-1$ | | 1 | $-1$ | | 0 | $y_1$ |
| $U(1)_{2-3}$ | | | 1 | $-1$ | | 1 | $-1$ | 0 | $y_2$ |
| $U(1)_L$ | 1 | | | | 1 | 1 | 1 | 3 | $y_3$ |

Table 4.2: Symmetries for the $n = k = 3$ model.

where $H$ is the hyperplane class of $\mathbb{C}P^2$, the only non-trivial class in $H^{(1,1)}(\mathbb{C}P^2)$. It has $H^3 = 0$, allowing us to expand

$$\begin{aligned} c(\mathcal{E}) &= (1 - 6H + 12H^2)(1 + 3H + 9H^2) \\ &= 1 - 3H + 3H^2. \end{aligned} \tag{4.35}$$

These are not the topological numbers of the tangent bundle of $\mathbb{C}P^2$, for which we would expect both signs to be positive. This confirms our suspicions that this GLSM flows to a different geometry from its $(2, 2)$ counterpart. However, they are the right topological numbers for the cotangent bundle of $\mathbb{C}P^2$.

The phase with $r < 0$ is also interesting to consider. Giving $\Sigma$ an expectation value breaks the gauge symmetry to a $\mathbb{Z}_3$ group. This $\mathbb{Z}_3$ will act non-trivially on the fields $\Phi^i$ and $\Gamma^i_-$, giving rise to a Landau-Ginzburg orbifold.

We would like to calculate the elliptic genus of this model. There will be three global symmetries, for which we can use the basis in Table 4.2. The calculation goes just as in the previous sections. The integrand is

$$I = 2\pi\eta(q)^3 \frac{\theta_1(q, x^{-2}y_1y_3)\theta_1(q, x^{-2}y_1^{-1}y_2y_3)\theta_1(q, x^{-2}y_2^{-1}y_3)}{\theta_1(q, x^{-3}y_3)\theta_1(q, xy_1)\theta_1(q, xy_1^{-1}y_2)\theta_1(q, xy_2^{-1})}. \tag{4.36}$$

To calculate the poles, we can choose between the poles from $\Sigma$ or the poles from $\phi^i$. We will pick the latter, since they all have charge 1 and then we do not need to worry about torus-equivalent points. We find

$$
\begin{aligned}
Z = & \frac{\theta_1(q, y_1 y_2 y_3)\theta_1(q, y_1^2 y_2^{-1} y_3)}{\theta_1(q, y_1^{-2} y_2)\theta_1(q, y_1^{-1} y_2^{-1})} + \frac{\theta_1(q, y_1^{-1} y_2^2 y_3)\theta_1(q, y_1^{-2} y_2 y_3)}{\theta_1(q, y_1^2 y_2^{-1})\theta_1(q, y_1 y_2^{-2})} \\
& + \frac{\theta_1(q, y_1 y_2^{-2} y_3)\theta_1(q, y_1^{-1} y_2^{-1} y_3)}{\theta_1(q, y_1 y_2)\theta_1(q, y_1^{-1} y_2^2)}.
\end{aligned}
\tag{4.37}
$$

Now note gauge-invariance implies $y_3^3 = 1$, so let us set $y_3 = 1$ at this point. Having done that, using $\theta_1(q, y) = -\theta_1(q, y^{-1})$ twice in each term allows us to reduce this to $Z = 3$. This agrees precisely with the Euler number of $\mathbb{C}P^2$, lending credence to the geometric picture we have found for this model above.

It would also be interesting to calculate the partition function directly from the Landau-Ginzburg orbifold description above.

### $n = 3,\ k = 2$

However, with $n = k$, we might not be able to trust the effective description, as the mass of the fields we integrate out does not grow in the IR relative to $\mu$. To solve this, we would like $n > k$ in these models, so take for example $k = 2,\ n = 3$. The fields $\Gamma_{i-}$ with $E$-couplings $\bar{D}_+\Gamma^i_- = \Sigma\Phi^i$ still have gauge charge $-2$. We need to add left movers to this model in order to cancel the gauge anomaly. For definiteness, we will add three Fermi superfields with no potential couplings and charge 1. This ensures the gauge anomaly has been canceled, and the sum of charges will also match between right and left-moving sectors. It will be more familiar to write the $D$-term in a way to absorb $\Lambda$ dependence into a running of the FI parameter, so rewrite the D-term condition as

$$
|\phi|^2 - n|\sigma|^2 + \log \frac{(\mu^2 + m^2|\phi|^2)^n}{(\mu^2 + m^2|\sigma|^2)^k \mu^{2(n-k)}} = r + (n-k)\log\frac{\Lambda^2}{\mu^2} = r_{\text{eff}}. \tag{4.38}
$$

If $n > k$, as $\mu \to 0$, $r_{\text{eff}} \to +\infty$. In the branch $\sigma = 0, \phi \neq 0$, we have

$$m^2 |\phi|^2 \sim \mu^2 e^{r_{\text{eff}}/n}, \tag{4.39}$$

so if $n > k$ the mass grows relative to $\mu$ as we let $\mu$ get smaller.

The geometric phase of this model, where the fields $\phi^i$ gain a non-zero expectation value, should now describe a $\mathbb{C}P^1$ model. The left-moving bundle will be formed from a linear combination of the two left-moving fermions with $E$-couplings, as well as the three fermions without potential couplings. As we trust the geometric description, it would be interesting to find the operators in $\bar{Q}_+$-cohomology in this model, to see what can be said in terms of its quantum sheaf cohomology ring.

# CHAPTER 5

# ABELIAN DUALITY IN CHIRAL GAUGE THEORIES

Here we describe the steps needed to generalize abelian duality of cycles in a chiral gauge theory. We will build on the formalism introduced in Section 2.5 and Section 2.6 in order to generalize it to $(0,2)$ theories, an endeavor started in [27]. We will find that dealing with chiral gauge theories makes the process more delicate, and reach an understanding of what the dual in the more generic $(0,2)$ case looks like.

## 5.1 Anomalies and Jacobians

In the derivation of abelian duality for the phases of $(2,2)$ chiral superfields in Section 2.6, one key step involved expressing the chiral superfields in the theory as $\Phi = e^{\Pi}$, and changing the path integration variables to the superfields $\Pi$. This achieved two purposes. Firstly, the isometry we are dualizing acts additively on the field $\Pi$, allowing us to more easily translate between two complex superfields. Secondly and more crucially, the redefinition transforms the action so that it does not depend on the phase we want to dualize, and so it is actually an isometry.

To see this, let us take a closer look at what that change of variables involves in components.

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- + 2\theta^-\theta^+F + \dots, \tag{5.1}$$

$$\Pi = \pi + \sqrt{2}\theta^+\chi_+ + \sqrt{2}\theta^-\chi_- + 2\theta^-\theta^+G + \dots$$

$$\Rightarrow e^{\Pi} = e^{\pi}\left(1 + \sqrt{2}\theta^+\chi_+ + \sqrt{2}\theta^-\chi_- + 2\theta^-\theta^+(G + \chi_+\chi_-) + \dots\right), \tag{5.2}$$

where as usual $\dots$ represents terms involving derivatives of the same fields. Matching powers of $\theta^{\pm}$,

$$\phi = e^{\pi}, \qquad \psi_+ = e^{\pi}\chi_+ = \phi\chi_+, \tag{5.3a}$$

$$F = e^{\pi}(G + \chi_+\chi_-), \qquad \psi_- = e^{\pi}\chi_- = \phi\chi_-. \tag{5.3b}$$

If $\Phi$ is charged under a gauge symmetry, since all components of the superfield $\Phi$ will have the same gauge charge, the theory will now be written in terms of a neutral Dirac fermion $\chi$, instead of a charged Dirac fermion $\psi$. Examining the couplings of a charged $(2,2)$ chiral superfield $\Phi$ in (2.11), there are two Yukawa couplings that depend on the phase of the scalar $\phi$, $\bar{\psi}_+\phi\bar{\lambda}_-$ and $\lambda_+\phi\bar{\psi}_-$. The redefinition above transforms these into $\bar{\psi}_+e^{\pi+\bar{\pi}}\bar{\lambda}_-$ and $\lambda_+e^{\pi+\bar{\pi}}\bar{\psi}_-$, so they are not dependent on the imaginary part of $\pi$ and we can therefore dualize it.

In $(2,2)$ theories we always deal with chirally symmetric Dirac fermions, but the same will not be the case when we make similar redefinitions in $(0,2)$ theories, and we will want to make similar changes of variables on Weyl fermions. For this reason, at this point we must make a digression into the topic of anomalies in quantum field theories. We will start by working on theories containing only chiral fermions and background vector fields. Since these theories are often quadratic in the fermions, we will see that we can integrate them out exactly to obtain an effective action depending only on the background vector fields. We will show how to use this effective action to calculate the Jacobians we need, and then extrapolate the arguments to supersymmetric theories.

Consider then a theory of fermions $\psi$ with background vector fields $A$, where we can integrate out the fermions to obtain an effective action $W[A]$,

$$Z[A] = e^{iW[A]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[A,\psi]}. \tag{5.4}$$

We will take a subset of the symmetries associated to $A$ to be gauged. We assume the theory to be fully gauge-invariant under its gauge symmetry, but have an anomaly in a global chiral symmetry

$$\psi \to \psi' = e^{i\beta}\psi, \tag{5.5}$$

meaning that under that symmetry the measure has a nontrivial Jacobian. Note we are taking $\beta$ to be some element of the Lie algebra of $U(N_L) \times U(N_R)$ acting on the fermions, and not simply the vector $U(1)$ symmetry. Then if we allow the parameter to vary in space, $\beta \to \beta(x)$, we have

$$S[A, e^{i\beta}\psi] - S[A, \psi] = -\int d^d x \beta \partial_\mu J^\mu \tag{5.6}$$

where $J^\mu$ is the current associated with this symmetry. If we take the only derivative coupling in the Lagrangian to be of the form $i\bar{\psi}(d + iA)\psi$, there are no higher-order contributions. This can be counteracted by a variation of $A$,

$$A' = A - d\beta, \tag{5.7}$$

so that

$$S[A', \psi'] = S[A - d\beta, e^{i\beta}\psi] = S[A, \psi]. \tag{5.8}$$

We will use this to construct an algorithm to calculate the Jacobian of the change of variables from $\psi$ to $\psi'$ if we know the form of the effective action $W[A]$.

Writing the measure variation as

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[A,\psi]} = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'e^{iJ[A,\beta]}e^{iS[A,\psi]}, \tag{5.9}$$

so that the action $S$ integrated over $\psi$ is equivalent to the action $S + J$ integrated over $\psi'$, we can finally see that

$$\begin{aligned} e^{iW[A]} &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[A,\psi]} = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'e^{iJ[A,\beta]}e^{iS[A,\psi]} = e^{iJ[A,\beta]}\int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'e^{iS[A',\psi']} \\ &= e^{iJ[A,\beta]+iW[A']}. \end{aligned} \tag{5.10}$$

In the equalities of this derivation, we respectively used (5.4), (5.9), (5.8) and once again equation (5.4). Our result is that

$$\begin{aligned} W[A] &= J[A,\beta] + W[A'] \\ \Rightarrow J[A,\beta] &= W[A] - W[A'] = -\delta W. \end{aligned} \tag{5.11}$$

It is worth stressing that the Jacobian has some properties that may at first seem counter-intuitive. For example, even though we wrote the Jacobian as $e^{iJ}$,

$$J[A,\beta] + J[A,-\beta] \neq 0 \tag{5.12}$$

generally. We can see why this is the case by making a change of variables and following up with its inverse, following the steps in (5.10):

$$\begin{aligned} e^{iW[A]} &= e^{iJ[A,\beta]} \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' e^{iS[A',\psi']} = e^{iJ[A,\beta]} e^{iJ[A',-\beta]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[A,\psi]} \\ &= e^{iW[A] + J[A,\beta] + J[A',-\beta]}, \end{aligned} \tag{5.13}$$

from which we see that the correct relation is

$$J[A,\beta] + J[A - d\beta, -\beta] = 0. \tag{5.14}$$

### *5.1.1 Example: a Dirac fermion*

To understand the derivation above, it is useful to see it in action in the simplest possible example, a Dirac fermion coupled to a gauge field in two dimensions. For additional clarity we will couple different background vector fields to both chiralities, so the action is

$$S = \frac{i}{4\pi} \int d^2x \left[ \bar{\psi}_+ (\partial_- + iA_-)\psi_+ + \bar{\gamma}_- (\partial_+ + iB_+)\gamma_- \right]. \tag{5.15}$$

We can integrate out the fermions,

$$e^{iW} = \int d\mu e^{iS}, \qquad\qquad d\mu := \mathcal{D}\psi_+ \mathcal{D}\bar{\psi}_+ \mathcal{D}\gamma_- \mathcal{D}\bar{\gamma}_-, \tag{5.16}$$

obtaining a non-local effective action in terms of $A$ and $B$ [51],

$$W = \frac{1}{8\pi} \int d^2x \left[ -B_+ \frac{\partial_-}{\partial_+} B_+ - A_- \frac{\partial_+}{\partial_-} A_- + 2MB_+A_- \right] \tag{5.17}$$

where the last term is a local counterterm which as we will see is necessary for gauge-invariance. Note that we could consider counterterms of the forms $A_+A_-$ and $B_+B_-$, which would be more natural if we considered the path integrals over $\psi_+$ and $\gamma_-$ independently. However this would introduce the fields $A_+$ and $B_-$ on which the original action does not depend. Note that the + and − components of vectors do not mix under Lorentz transformations, so they are truly different fields. Furthermore, the local counterterm parametrizes different choices of regularization, of which a small set may preserve the chosen gauge symmetry. This supports the image of the counterterm as coming from the set of path integrals and not a part from each one.

We can now use this effective action and the procedure outlined above to calculate the Jacobian under a fermion rotation. Under a gauge transformation,

$$\psi_+ \to \psi'_+ = e^{i\alpha}\psi_+, \qquad\qquad A_- \to A'_- = A_- - \partial_-\alpha, \tag{5.18}$$

$$\gamma_- \to \gamma'_- = e^{i\beta}\gamma_-, \qquad\qquad B_+ \to B'_+ = B_+ - \partial_+\beta, \tag{5.19}$$

the effective action changes by

$$\delta W = \frac{1}{4\pi} \int d^2x \left[ B_+\partial_-\beta + A_-\partial_+\alpha - MB_+\partial_-\alpha - MA_-\partial_+\beta \right]. \tag{5.20}$$

The path integral should be invariant under gauge transformations, so we see that for gauge invariance we must have $\alpha = \pm\beta$, $M = \pm 1$, corresponding to gauging the vector and axial symmetries respectively. For the remainder of this section we will take the upper sign, corresponding to a gauge field $A = B$. We will keep two separate background gauge fields in intermediate calculations, allowing us to transform $\psi_+$ and $\gamma_-$ separately, and typically set $A = B$ at the end.

We can now find the Jacobian under the chiral symmetry

$$\psi_+ \rightarrow \psi'_+ = e^{i\beta}\psi_+. \tag{5.21}$$

We argued the Jacobian is equal to minus the variation of the effective action under the transformation

$$A_- \rightarrow A_- - \partial_-\beta, \tag{5.22}$$

which gives

$$\begin{aligned} J &= \frac{1}{4\pi}\int d^2x \left[ -A_-\partial_+\beta + B_+\partial_-\beta + \frac{1}{2}\partial_-\beta\partial_+\beta \right] \\ &= -\frac{1}{4\pi}\int d^2x \beta\epsilon^{\mu\nu}F_{\mu\nu} - \frac{1}{8\pi}\int d^2x \partial_\mu\beta\partial^\mu\beta. \end{aligned} \tag{5.23}$$

The first term is the usual form for the abelian covariant anomaly: the extra counterterm $A_-B_+$ added for gauge invariance also modifies the chiral currents. This modification is exactly the same as the improvement term that would take us from the consistent to the covariant anomaly for gauge currents [52, 53].

### *5.1.2 The general case with one gauged $U(1)$*

We can now consider an example with $n_L$ left-moving fermions $\gamma_{a-}$ and $n_R$ right-movers $\psi_{i+}$, and a background $U(1)$ vector field coupled to each fermion:

$$S = \frac{i}{4\pi}\int d^2x \left[ \bar{\psi}_{i+}(\partial_- + iA_{i-})\psi_{i+} + \bar{\gamma}_{a-}(\partial_+ + iB_{a+})\gamma_{a-} \right]. \tag{5.24}$$

We can integrate out the fermions to obtain an effective action

$$W = \frac{1}{8\pi} \int d^2x \left[ -A_{i-} \frac{\partial_+}{\partial_-} A_{i-} - B_{a+} \frac{\partial_-}{\partial_+} B_{a+} + 2M_{ia} B_{a+} A_{i-} \right], \tag{5.25}$$

where $M$ is a matrix determining the counterterms that is defined once we specify which gauge symmetries are turned on. Let us suppose we have only one gauge symmetry, given by

$$\psi_i \to e^{iQ_i\beta} \psi_i, \qquad \gamma_a \to e^{iQ_a\beta} \gamma_a, \tag{5.26}$$

so $A_{i-} = Q_i A_-$ and $B_{a+} = Q_a A_+$.

We can find the Jacobian under a generic symmetry

$$\psi_i \to \psi_i' = e^{iq_i\beta} \psi_i, \qquad \gamma_a \to \gamma_a' = e^{iq_a\beta} \gamma_a, \tag{5.27}$$

by finding the variation of the effective action under a corresponding transformation of the vector fields, $A_i \to A_i - q_i d\beta$ and $B_a - q_a d\beta$,

$$\begin{aligned} \delta W = \frac{1}{4\pi} \int d^2x \Big[ & (q_i - M_{ia} q_a) A_{i-} \partial_+ \beta + (q_a - q_i M_{ia}) B_{a+} \partial_- \beta \\ & + (q_i M_{ia} q_a - \frac{1}{2} q_i q_i - \frac{1}{2} q_a q_a) \partial_- \beta \partial_+ \beta \Big] \\ = \frac{1}{4\pi} \int d^2x \beta \Big[ & (q_i M_{ia} Q_a - Q_a q_a) \partial_- A_+ + (Q_i M_{ia} q_a - Q_i q_i) \partial_+ A_- \\ & - \left( q_i M_{ia} q_a - \frac{1}{2} q_i q_i - \frac{1}{2} q_a q_a \right) \partial_- \partial_+ \beta \Big]. \end{aligned} \tag{5.28}$$

If this symmetry is a gauge transformation, $q = Q$, the theory is non-anomalous only if this vanishes for some choice of local counterterms $M_{ia}$. We see that this is only possible if

$$\sum_i Q_i^2 = \sum_a Q_a^2 = \sum_{i,a} Q_i M_{ia} Q_a = k. \tag{5.29}$$

For the general case, a useful parametrization of $M_{ia}$ is [25]

$$M_{ia} = \frac{1}{2} Q_i \frac{k_L + k_R}{k_L k_R} Q_a + N_{ia}, \tag{5.30}$$

where we defined $k_R = \sum_i Q_i^2$ and $k_L = \sum_a Q_a^2$ and the matrix $N$ is such that $Q_i N_{ia} Q_a = 0$. For the moment we set $N = 0$. We see that (5.29) is automatically satisfied if $k_L = k_R$, while if $k_L \neq k_R$ (5.28) becomes

$$\delta W = \frac{1}{8\pi} \int d^2x \left[ \left( 2Q_i q_i - \frac{k_L + k_R}{k_L} Q_a q_a \right) A_- \partial_+ \beta + \left( 2Q_a q_a - \frac{k_L + k_R}{k_R} Q_i q_i \right) A_+ \partial_- \beta \right] + \mathcal{O}(\beta^2). \tag{5.31}$$

For the gauge symmetry, $q = Q$ and this becomes

$$\delta W = \frac{k_R - k_L}{8\pi} \int d^2x \beta \epsilon^{\mu\nu} F_{\mu\nu}, \tag{5.32}$$

which is the usual consistent anomaly, half the first term of the covariant form (5.23). Note this symmetric form is only possible if $k_L, k_R \neq 0$. If we have for example $k_L = 0$, $B_{a+} = 0$ and therefore

$$\delta W = -\frac{Q_i q_i}{4\pi} \int d^2x \beta \partial_+ A_-. \tag{5.33}$$

If the theory is non-anomalous, $k_L = k_R = k$ and (5.30) becomes

$$M_{ia} = \frac{Q_i Q_a}{k} + N_{ia}, \tag{5.34}$$

where once again $N$ is a generic matrix such that $Q_i N_{ia} Q_a = 0$. The variation (5.28) then becomes

$$J = -\delta W = -\frac{Q_i q_i - Q_a q_a}{4\pi} \int d^2x \beta \epsilon^{\mu\nu} F_{\mu\nu} - \frac{1}{4\pi} \int d^2x \beta \left[ q_i N_{ia} Q_a \partial_- A_+ + Q_i N_{ia} q_a \partial_+ A_- \right]$$
$$- \frac{1}{4\pi} \int d^2x \left( q_i M_{ia} q_a - \frac{1}{2} q_i q_i - \frac{1}{2} q_a q_a \right) \partial_- \beta \partial_+ \beta. \tag{5.35}$$

Note that the form of this Jacobian will be affected by $N_{ia}$. If we impose additionally that the first-order Jacobian is proportional to the anomaly coefficient $Q_i q_i - Q_a q_a$ for all symmetries, we must have

$$\forall_q Q_i N_{ia} q_a = q_i N_{ia} Q_a = 0 \Rightarrow Q_i N_{ia} = N_{ia} Q_a = 0. \tag{5.36}$$

Note that while this strongly constrains $N_{ia}$, it is not necessarily zero, although the effects can only be seen in the second-order Jacobian. In a non-chiral theory, for example, it is more natural to take $M_{ia} = \delta_{ia}$. It is straightforward to check that $N_{ia} = \delta_{ia} - \frac{Q_i Q_a}{k}$ satisfies the conditions in (5.36). Finally, our main result in this section is the Jacobian

$$J = -\frac{Q_i q_i - Q_a q_a}{4\pi} \int d^2x \beta \epsilon^{\mu\nu} F_{\mu\nu} - \frac{1}{4\pi} \int d^2x \left( q_i M_{ia} q_a - \frac{1}{2} q_i q_i - \frac{1}{2} q_a q_a \right) \partial_- \beta \partial_+ \beta. \tag{5.37}$$

For two $U(1)$ gauge fields with charges $Q$ and $\tilde{Q}$ satisfying

$$Q_i Q_i = Q_a Q_a = k, \qquad \tilde{Q}_i \tilde{Q}_i = \tilde{Q}_a \tilde{Q}_a = \tilde{k}, \qquad Q_i \tilde{Q}_i = Q_a \tilde{Q}_a = k_M, \tag{5.38}$$

a possible choice satisfying $Q_i M_{ia} = Q_a, M_{ia} Q_a = Q_i$ and the same for the $\tilde{Q}$ is

$$M_{ia} = \frac{\tilde{k} Q_i Q_a + k \tilde{Q}_i \tilde{Q}_a - k_M Q_i \tilde{Q}_a - k_M \tilde{Q}_i Q_a}{k \tilde{k} - k_M^2}. \tag{5.39}$$

This structure generalizes using determinants of the charge matrices.

As a final check, we would like to show that the form of the Jacobian is not dependent on simple deformations of the initial action. For instance, we could have two fermions $\psi_{j+}$ and $\gamma_{b-}$ coupled by a Yukawa coupling of the form

$$S_{Yuk} = m \int d^2x \left[ \psi_{j+} e^{i\varphi} \gamma_{b-} + \text{c.c.} \right], \tag{5.40}$$

where $\varphi$ is a compact scalar with charge $Q_\varphi = -Q_j - Q_b$. Throughout this section we will take $b,\ j$ to refer to these particular pair of fermions, while $a,\ i$ refer to all fermions in the theory as usual. This coupling gives a mass to this pair of fermions, so the effective action obtained by integrating them out will be local (in an expansion in powers of $1/m$). The terms left at an energy scale $\mu \ll m$ are [25, 54]

$$W = \frac{1}{4\pi} \int d^2x \left[ \frac{1}{2} \partial_+ \varphi \partial_- \varphi - B_{b+} \partial_- \varphi - A_{j-} \partial_+ \varphi + P_{ia} A_{i-} B_{a+} \right], \tag{5.41}$$

where the counterterm with constants $P_{ia}$, summed over all fermions with indices $i,\ a$, couples the vector fields $A_{j-}$ and $B_{b+}$ to each other and also to the other background fields in the theory, just like $M_{ia}$ in the massless case. Under a change of variables

$$\psi_j \to \psi_j' = e^{iq_j\beta} \psi_j, \qquad \gamma_b \to \gamma_b' = e^{iq_b\beta} \gamma_b, \tag{5.42}$$

the Jacobian can now be obtained from the variation $\delta A_{j-} = -q_j \partial_- \beta$, $\delta B_{b+} = -q_b \partial_+ \theta$, $\delta\varphi = -(q_j + q_b)\beta$. We find

$$\begin{aligned} J = -\delta W = \frac{1}{4\pi} \int d^2x \Bigg( & \left[ \frac{1}{2} (q_j + q_b)^2 - q_j q_b P_{jb} \right] \partial_+ \beta \partial_- \beta \\ & + \left[ Q_j \left( q_j + q_b \right) - Q_i q_b P_{ib} \right] \beta \partial_+ A_- \\ & + \left[ Q_b \left( q_j + q_b \right) - q_j Q_a P_{ja} \right] \beta \partial_- A_+ \Bigg), \end{aligned} \tag{5.43}$$

which agrees with the massless case (5.37) if

$$P_{ia} = M_{ia} + \delta_{ij}\delta_{ab}. \tag{5.44}$$

### 5.1.3 $(0,2)$ *Jacobians*

Finally, the result we are interested in is the generalization of the Jacobians above to the types of redefinitions we are interested in doing to fields in $(0,2)$ GLSM. Coupling chiral multiplets to vector superfields $A_i$, $V_{i-}$ and Fermi multiplets to $A_a$ (chiral multiplets have couplings to both components of vector fields, as they include a scalar field), our starting point is the basic GLSM action with background vector fields

$$S = \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[-\frac{i}{2}\bar{\Phi}_i e^{2A_i}\left(\partial_- + iV_{i-}\right)\Phi^i + \text{c.c.} - \bar{\Gamma}_{a-} e^{2A_a}\Gamma^{a}_{-}\right]. \tag{5.45}$$

As in the analysis using the component fermions, we know the Jacobian as long as we can compute the effective action in terms of the background vector fields.

Some considerations make matters simpler. Namely, the redefinitions we will be interested in making can be formulated in terms of finite chiral gauge transformations on the fields. We know that the scalar field path integral measure is gauge-invariant, so we do not need to concern ourselves with the effective action coming from the scalar field. This is fortunate, as scalar fields in two dimensions are badly IR-divergent, and it would be challenging to obtain a scale-independent effective action including the scalar field effects (one could write down an effective action including effects up to a certain scale $\mu$, and verify that it is indeed gauge-invariant at any finite scale, as e.g. in [25]). For similar reasons, we are free to add any gauge-invariant combination of the background vector fields. We will use this to remove all quadratic terms in $V_{i-}$.

Then, taking the fields to not be massed up by potential couplings, the effective action has the form [25]

$$W = \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[\bar{D}_+ A_i \frac{\partial_-}{\partial_+} D_+ A_i - \bar{D}_+ A_a \frac{\partial_-}{\partial_+} D_+ A_a + 2M_{ia} A_a V_{i-} - 2A_i V_{i-}\right]. \quad (5.46)$$

Armed with this expression, we can calculate the Jacobian for any change of variables that can be expressed as a transformation of fields by multiplications of chiral fields. In particular, we will be interested in changes of variables of the form

$$\Phi^i \to \Pi_i = \log \Phi^i, \qquad \Gamma^a_- \to \Lambda^a_- = e^{-x_{ia}\Pi^i}\Gamma^a_-. \quad (5.47)$$

Note the change of variables on the right-moving fermion is of the form

$$\psi \to \chi = \phi^{-1}\psi = e^{-\pi}\psi. \quad (5.48)$$

The effects of these changes on the fermion action can be counteracted by changes in the vector fields corresponding to gauge transformations with parameter $i\Pi_i$ for $A_i$, $V_{i-}$ and $ix_{ia}\Pi_i$ for $A_a$,

$$A_i \to A'_i = A_i + \frac{\Pi_i + \bar{\Pi}_i}{2} \qquad V_{i-} \to V'_{i-} = V_{i-} + \frac{\partial_-(\Pi_i - \bar{\Pi}_i)}{2i}, \quad (5.49a)$$

$$A_a \to A'_a = A_a + \frac{x_{ia}}{2}(\Pi_i + \bar{\Pi}_i) \quad (5.49b)$$

from which we can find a Jacobian

$$J = \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[ -\frac{i}{4}\left(\delta_{ij} + x_{ia}x_{ja} - M_{ia}x_{ja} - M_{ja}x_{ia}\right)\bar{\Pi}_i\partial_-\Pi_j + \text{c.c.} \quad (5.50)$$
$$- i\left(M_{ia} - x_{ia}\right)\left(\Pi_i - \bar{\Pi}_i\right)\partial_- A_a + \left(\delta_{ij} - x_{ia}M_{ja}\right)\left(\Pi_i + \bar{\Pi}_i\right)V_{j-}\Big].$$

As a simple check, it is easy to verify that in the non-chiral $(2,2)$ case, for which $M_{ia} = \delta_{ia}$ and the redefinition $\Phi = e^{\Pi}$ of Section 2.6 corresponds to $x_{ia} = \delta_{ia}$, this expression vanishes.

We can simplify the expression by substituting the background vector fields for the gauge fields in the theory, and using $Q_i M_{ia} = Q_a,\ M_{ia} Q_a = Q_i,\ \Upsilon_- = \frac{i}{\sqrt{2}} \bar{D}_+ (\partial_- A + i V_-)$:

$$
\begin{aligned}
J = \frac{1}{8\pi} \int d^2x d^2\theta^+ \Big[ &- \frac{i}{4} \left( \delta_{ij} + x_{ia} x_{ja} - M_{ia} x_{ja} - M_{ja} x_{ia} \right) \bar{\Pi}_i \partial_- \Pi_j + \text{c.c.} \\
&- i \left( Q_i - x_{ia} Q_a \right) \left( \Pi_i - \bar{\Pi}_i \right) \partial_- A + \left( Q_i - x_{ia} Q_a \right) \left( \Pi_i + \bar{\Pi}_i \right) V_- \Big]. \\
= -\frac{i}{32\pi} \int & d^2x d^2\theta^+ \left( \delta_{ij} + x_{ia} x_{ja} - M_{ia} x_{ja} - M_{ja} x_{ia} \right) (\Pi_i + \bar{\Pi}_i) \partial_- (\Pi_j - \bar{\Pi}_j) \\
&- \frac{\sqrt{2}(Q_i - x_{ia} Q_a)}{8\pi} \int d^2x d\theta^+ \Pi_i \Upsilon_- + \text{c.c.} \\
= -\frac{i}{32\pi} \int & d^2x d^2\theta^+ K_{ij} (\Pi_i + \bar{\Pi}_i) \partial_- (\Pi_j - \bar{\Pi}_j) - \frac{\sqrt{2}\tilde{Q}_i}{8\pi} \int d^2x d\theta^+ \Pi_i \Upsilon_- + \text{c.c.}, \qquad (5.51)
\end{aligned}
$$

where we defined the new matrix and vector

$$
K = \mathbb{1} + X X^T - M X^T - X M^T, \qquad \tilde{Q} = Q_R - X Q_L, \qquad (5.52)
$$

in terms of the matrices $M$ and $X$ with entries $M_{ia},\ x_{ia}$, and the vectors of charges $Q_R$ and $Q_L$. This is our final result of this section. We see that $\Pi^i$ becomes a field-dependent FI parameter, with its imaginary part having the form of a field-dependent theta coupling. If the redefinition leaves the theory with an anomalous fermion content, this coupling should cancel that anomaly [25, 55].

### *5.1.4 Non-abelian redefinitions*

When the $E$ or $J$ couplings are not given by simple monomials, the redefinitions necessary to absorb the phases present in the Yukawa couplings into the fermions are not simply $\mathbb{C}^*$, but $GL(n, \mathbb{C})$. Like we did above, the best way to find the correct Jacobians is by coupling background $U(N_R) \times U(N_L)$ vector fields into the action, regularizing the action such that the gauge symmetry is preserved in the effective action obtained by integrating out the matter fields. The Jacobian can then be found as a variation of the effective action, where in the last step we set the background fields to zero. Just like the $\mathbb{C}^*$ Jacobians could be obtained from the $U(1)$ versions and $(0,2)$ supersymmetry, as we could calculate the exact effective action, also here we can then obtain the $GL(n,\mathbb{C})$ actions from the $U(n)$ Jacobians by imposing $(0,2)$ supersymmetry.

Note that if we are redefining $\Gamma_a = U_{ab}F_b$, the kinetic terms of the Fermi superfields become

$$\bar{\Gamma}_a e^{2Q_a A}\Gamma^a = \bar{F}_c U^\dagger_{ca} e^{2Q_a A} U_{ab} F^b \tag{5.53}$$

and therefore they will remain diagonal if the redefinition matrices obey

$$U^\dagger_{ca} e^{2Q_a A} U_{ab} = \text{diagonal}. \tag{5.54}$$

What is the effective action for the $U(N_R)\times U(N_L)$ background vector fields? The solution for this problem has long ago been found by Polyakov and Wiegmann [56]. Writing $B_- = -ig^{-1}\partial_- g$ and $B_+ = -ih^{-1}\partial_+ h$, the effective action is given by

$$\begin{aligned} W &= F[g] + F[h^{-1}] + W_{ct}, \\ F[g] &= \frac{1}{8\pi}\int d^2x \,\mathrm{tr}\left(\partial_\mu g \partial^\mu g^{-1}\right) + \frac{1}{12\pi}\int d^3x \epsilon^{\mu\nu\lambda}\,\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g\right), \end{aligned} \tag{5.55}$$

where $W_{ct}$ is a local term of the form $M_{ab}B_+^a B_-^b$ fixed by gauge-invariance once we have specified which of the background vector fields correspond to a dynamic gauge field. Crucially for our purposes, these actions satisfy the Polyakov-Wiegmann identity,

$$F[gh] = F[g] + F[h] + \frac{1}{4\pi}\int d^2x \,\mathrm{tr}\left(g^{-1}\partial_+ g \partial_- h h^{-1}\right). \tag{5.56}$$

A rotation of the right-movers, $\psi_+ \to g\psi_+$, gives the same change in the classical action as a background vector field change to $B_- \to B_-^g = g^{-1}(B_- - i\partial_-)g$, which if we have written $B_- = -ih^{-1}\partial_- h$ is simply $h \to hg$. The change in the effective action is thus given by

$$\delta W = F[g] + \frac{i}{4\pi}\int d^2x\, \mathrm{tr}(\partial_+ g g^{-1} B_-) + \delta W_{ct}, \tag{5.57}$$

where $\delta W_{ct}$ should complete the second term in the way necessary to obtain a gauge-covariant expression. For the left-movers, we find the symmetric expression.

We can start by seeing that what this gives in the abelian case. Note that if $g$ is abelian, the WZ part of the Jacobian vanishes, and the result is a fully local expression. If we redefine the right-handed fermions by $\psi_i \to e^{i\theta_i}\psi_i$, so

$$g_{ii} = e^{i\theta_i} \Rightarrow i\,\mathrm{tr}(\partial_+ g g^{-1} B_-) = -Q_i \partial_+ \theta_i A_-, \qquad \mathrm{tr}(\partial_\mu g \partial^\mu g^{-1}) = \partial_\mu \theta_i \partial^\mu \theta_i. \tag{5.58}$$

Adding the contribution from $W_{ct}$, we find the total Jacobian

$$J = -\delta W = -\frac{1}{8\pi}\int d^2x \partial_\mu \theta_i \partial^\mu \theta_i + \frac{Q_i}{4\pi}\int d^2x \theta_i F_{-+}. \tag{5.59}$$

Similarly for an independent rotation of the left-handed fermions $\gamma_a \to e^{ix_{ia}\theta_i}\gamma_a$, giving

$$g_{aa} = e^{ix_{ia}\theta_i} \Rightarrow i\,\mathrm{tr}(\partial_- g g^{-1} B_+) = -Q_a x_{ia} \partial_- \theta_i A_+, \quad \mathrm{tr}(\partial_\mu g \partial^\mu g^{-1}) = x_{ia} x_{ja} \partial_\mu \theta_i \partial^\mu \theta_j, \tag{5.60}$$

so the Jacobian for this rotation alone gives

$$J = -\delta W = -\frac{x_{ia} x_{ja}}{8\pi}\int d^2x \partial_\mu \theta_i \partial^\mu \theta_j - \frac{x_{ia} Q_a}{4\pi}\int d^2x \theta_i F_{-+}. \tag{5.61}$$

If we rotate both the right and left-movers, we find the sum of the two Jacobians above, as well as a quadratic term from the $B_+ B_-$ counterterm which depends in general on the matrix $N_{ia}$ defined when discussing abelian redefinitions. See the comments below (5.36) for more details on specific cases.

## 5.2 (0, 2) duality

### 5.2.1 Duality map

As in Section 2.5, to perform the duality we will replace the derivatives of the phase we wish to dualize by unconstrained one-forms, and introduce Lagrange multipliers that when integrated out bring us back to the original action.

The simplest context in which to understand the duality map is for a chiral field $P$ with periodic imaginary part, $P \sim P + 2\pi i$. Its canonical kinetic term is given by

$$\begin{aligned} S &= -\frac{ib}{16\pi}\int d^2x d^2\theta^+ \bar{P}\partial_- P + \text{c.c.} \\ &= -\frac{ib}{16\pi}\int d^2x d^2\theta^+ (P+\bar{P})\partial_-(P-\bar{P}). \end{aligned} \tag{5.62}$$

For a $(0,2)$ superfield, the appropriate substitutions are $P+\bar{P} \to 2B,\ -i\partial_-(P-\bar{P}) \to 2C_-$, where $B$ is a real superfield and $C_-$ is a vector superfield. We can see that these are the correct redefinitions as in components they include

$$P + \bar{P} = \cdots - i\theta^+\bar{\theta}^+\partial_+(p-\bar{p}) \to 2B = \cdots + 2\theta^+\bar{\theta}^+ c_+, \tag{5.63a}$$

$$i\partial_-(\bar{P}-P) = i\partial_-(\bar{p}-p) + \ldots \to 2C_- = 2c_- + \ldots, \tag{5.63b}$$

similarly to the $(2,2)$ substitutions in (2.68) and (2.69) and their component analogues. When we make the substitution, we should include a Lagrange multiplier $F_-$ to keep $-i\partial_- P = C_- - i\partial_- B$ chiral and $i\partial_-\bar{P} = C_- + i\partial_- B$ antichiral:

$$\begin{aligned} S_L &= \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[-F_-\bar{D}_+(C_- - i\partial_- B) + \bar{F}_- D_+(C_- + i\partial_- B)\right] \\ &= \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[-C_-(Y+\bar{Y}) - iB\partial_-(Y-\bar{Y})\right], \end{aligned} \tag{5.64}$$

where we defined in the second line $Y = \bar{D}_+ F_-$, which by definition obeys $\bar{D}_+ Y = 0$. We can see that this term has the correct factor derived in Section 2.5:

$$\begin{aligned} S_L &= -\frac{1}{8\pi}\int d^2x d^2\theta^+ \left[C_-(Y+\bar{Y}) + iB\partial_-(Y-\bar{Y})\right] \\ &= -\frac{1}{4\pi}\int d^2x \left[c_-\partial_+\theta - c_+\partial_-\theta\right] \\ &= -\frac{1}{2\pi}\int d^2x \epsilon^{\mu\nu} c_\mu \partial_\nu \theta \end{aligned} \tag{5.65}$$

where we defined $\theta = \operatorname{Im} y$. Therefore the dual $Y$ theory will also have $Y \sim Y + 2\pi i$. Finally, carrying out the substitutions and including the Lagrange multiplier term, the action is

$$S_0 = \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[2bBC_- - C_-(Y+\bar{Y}) - iB\partial_-(Y-\bar{Y})\right]. \tag{5.66}$$

To integrate out $Y$, we should go back to the description in terms of unconstrained fields $F_-$, and then integrate out those. We recover the original action (5.62). If on the other hand we integrate $B$ and $C$ out, we obtain

$$S_d = -\frac{i}{16\pi b}\int d^2x d^2\theta^+ (Y+\bar{Y})\partial_-(Y-\bar{Y}). \tag{5.67}$$

Once again we recognize the usual T-duality mapping between a circle of radius $\sqrt{b}$ and one of radius $1/\sqrt{b}$.

To continue familiarizing ourselves with the notation of $(0,2)$ duality, it is now helpful to dualize a charged chiral field in a theory with $(2,2)$ field content, namely one chiral $\Phi$ and a Fermi $\Gamma_-$ hat both have charge 1. The action will be

$$S = \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[-\frac{i}{2}\bar{\Phi} e^{2A}(\partial_- + iV_-)\Phi + \text{c.c.} - \bar{\Gamma}_- e^{2A}\Gamma_-\right]. \tag{5.68}$$

To avoid a Jacobian, we change variables to $\Pi$ and $\Lambda_-$ where $\Phi = e^{\Pi}$ and $\Lambda_- = e^{-\Pi}\Gamma_-$. We can then make a similar substitution to the one above, $\Pi + \bar{\Pi} \to 2B$, $-i\partial_-(\Pi - \bar{\Pi}) \to 2C_-$.

Including the appropriate Lagrange multiplier term, the action is

$$S_0 = \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[e^{2B+2A}(C_- + V_-) - \bar{\Lambda}_- e^{2B+2A}\Lambda_- - C_-(Y+\bar{Y}) - iB\partial_-(Y-\bar{Y})\right]. \tag{5.69}$$

Integrating out $Y$, we return to the original action. If instead we integrate out $B$ and $C_-$, we find

$$\begin{aligned} S_d &= \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[(Y+\bar{Y})V_- - \bar{\Lambda}_-(Y+\bar{Y})\Lambda_- + iA\partial_-(Y-\bar{Y}) \\ &\qquad - \frac{i}{2}\log(Y+\bar{Y})\partial_-(Y-\bar{Y})\Big] \\ &= \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[-\frac{i}{2}\log(Y+\bar{Y})\partial_-(Y-\bar{Y}) - \bar{\Lambda}_-(Y+\bar{Y})\Lambda_-\Big] \\ &\quad -\frac{\sqrt{2}}{8\pi}\int d^2x d\theta^+ Y\Upsilon_- + \text{c.c.}, \end{aligned} \tag{5.70}$$

where we used $\Upsilon_- = \frac{i}{\sqrt{2}}\bar{D}_+(\partial_- A + iV_-)$ to write the couplings between $Y$ and the gauge field as a superpotential. Note this is not quite the $(0,2)$ version of the dual action we found in Section 2.6 for a $(2,2)$ charged chiral, for two reasons: one is that we did not dualize the Fermi superfield $\Lambda_-$, which we will see how to do in Section 5.2.2. The other is that we have not taken into account the analogous non-perturbative corrections. We will analyze those in Section 5.2.3.

Before that analysis, we can consider our first case with a non-trivial Jacobian. To see that, let us dualize the phase of an uncharged field $\Phi$, with action

$$S = -\frac{i}{16\pi}\int d^2x d^2\theta^+ \bar{\Phi}\partial_-\Phi + \text{c.c.} \tag{5.71}$$

To dualize the phase of $\Phi$, we redefine $\Phi = e^{\Pi}$. Here we see the first subtlety from working with chiral fermions, as this redefinition imposes a Jacobian. Since there are no left-moving fermions in this theory, the matrices $M$ and $X$ of Section 5.1 are 0, and the Jacobian from

(5.51) is simply

$$J = -\frac{i}{32\pi}\int d^2xd^2\theta^+(\Pi+\bar{\Pi})\partial_-(\Pi-\bar{\Pi}) \tag{5.72}$$

Adding this to the action, we can substitute $\Pi+\bar{\Pi}\to 2B$, $-i\partial_-(\Pi-\bar{\Pi})\to 2C_-$. The resulting action is

$$S_0 = \frac{1}{8\pi}\int d^2xd^2\theta^+\left[e^{2B}C_- + BC_- - C_-(Y+\bar{Y}) - iB\partial_-(Y-\bar{Y})\right]. \tag{5.73}$$

Integrating out $Y$, we find the original action. If we instead integrate out $B$ and $C_-$, we find that the Jacobian forces us to solve a transcendental equation for $B$,

$$e^{2B}+B = Y+\bar{Y}. \tag{5.74}$$

Taking reality constraints into account, this equation has a single solution given by

$$B = Y+\bar{Y}-\frac{1}{2}W\left(2e^{2(Y+\bar{Y})}\right), \tag{5.75}$$

where $W$ is the Lambert W-function. We can plug $B$ back into the action, which becomes

$$S_d = \frac{1}{8\pi}\int d^2xd^2\theta^+\left[-i(Y+\bar{Y})\partial_-(Y-\bar{Y}) + \frac{i}{2}W\left(2e^{2(Y+\bar{Y})}\right)\partial_-(Y-\bar{Y})\right]. \tag{5.76}$$

Finally, it is useful at this point to consider the dual of a general $(0,2)$ GLSM at this level. Our starting action is

$$S = \frac{1}{8\pi}\int d^2xd^2\theta^+\left[-\frac{i}{2}\bar{\Phi}_ie^{2Q_iA}(\partial_- + iQ_iV_-)\Phi^i + \text{c.c.} - \bar{\Gamma}_{a-}e^{2Q_aA}\Gamma^a_-\right]. \tag{5.77}$$

Carrying out the general redefinition in (5.47), there is a Jacobian. The action becomes

$$S = \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[ -\frac{i}{2} e^{\bar{\Pi}_i + \Pi_i + 2Q_i A} \partial_-(\Pi_i - \bar{\Pi}_i) + Q_i e^{\bar{\Pi}_i + \Pi_i + 2Q_i A} V_- - \bar{\Lambda}_{a-} e^{x_{ia}(\Pi_i + \bar{\Pi}_i) + 2Q_a A} \Lambda_-^a - \frac{i}{4} K_{ij} \left(\Pi_i + \bar{\Pi}_i\right) \partial_- \left(\Pi_j - \bar{\Pi}_j\right) + iA\tilde{Q}_i \partial_- \left(\Pi_i - \bar{\Pi}_i\right) + \tilde{Q}_i \left(\Pi_i + \bar{\Pi}_i\right) V_- \Big]. \quad (5.78)$$

Using the substitutions on $\Pi_i$ that we are by now accustomed to, this becomes

$$S_0 = \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[ e^{2B_i + 2Q_i A} \left(C_{i-} + Q_i V_-\right) - \bar{\Lambda}_{a-} e^{2x_{ia} B_i + 2Q_a A} \Lambda_-^a + K_{ij} B_i C_{j-} - 2A\tilde{Q}_i C_{i-} + 2\tilde{Q}_i B_i V_- - C_{i-}(Y_i + \bar{Y}_i) - iB_i \partial_-(Y_i - \bar{Y}_i) \Big]. \quad (5.79)$$

The duality map obtained when integrating out $B, C_-$ now becomes

$$e^{2B_i + 2Q_i A} + K_{ij} B_j = Y_i + \bar{Y}_i + 2\tilde{Q}_i A, \quad (5.80a)$$

$$2e^{2B_i + 2Q_i A} \left(C_{i-} + Q_i V_-\right) - 2x_{ia} \bar{\Lambda}_{a-} e^{2x_{ia} B_i + 2Q_a A} \Lambda_-^a + K_{ij} C_{j-} = i\partial_-(Y_i - \bar{Y}_i) - 2\tilde{Q}_i V_-. \quad (5.80b)$$

The form of the right-hand sides of these equations strongly suggests that the dual field $Y$ is now charged. We will establish this on more solid ground in the following sections. For now, to find the classical dual, we should solve for $B_i$ and plug the solution back into the action. Using $B_i$ as a stand-in for the functions of $(Y, A)$ that solve this system of equations, the dual action is given by

$$S_d = \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[ Q_i e^{2B_i + 2Q_i A} V_- - \bar{\Lambda}_{a-} e^{2x_{ia} B_i + 2Q_a A} \Lambda_-^a + 2\tilde{Q}_i B_i V_- - iB_i \partial_-(Y_i - \bar{Y}_i) \Big]. \quad (5.81)$$

### 5.2.2 Fermi superfields

Since Fermi multiplets do not have a propagating scalar degree of freedom, the duality map will be different for them. As in Section 2.2, the starting action is

$$S = -\frac{1}{8\pi}\int d^2x d^2\theta^+ \bar{\Gamma}_- f(\Phi, A)\Gamma_- + \frac{\sqrt{2}}{8\pi}\int d^2x d\theta^+ \Gamma_- J + \frac{\sqrt{2}}{8\pi}\int d^2x d\bar{\theta}^+ \bar{\Gamma}_- \bar{J}, \tag{5.82}$$

where $f$ is a function of the chiral and vector superfields, and $\Gamma_-$ obeys $\bar{D}_+\Gamma_- = 2E$, with $E$ a function of chiral fields. For usual gauged Fermi superfields, $f(\Phi, A) = e^{2QA}$, but it will be useful for us to allow more general functions. Note that if both $E$ and $J$ are non-zero, this is not by itself $(0,2)$-supersymmetric: we need $E \cdot J = 0$, where the dot product is over all Fermi superfields. To dualize with $J$-couplings, we need to present them as $D$-terms and so we write

$$J = \frac{1}{\sqrt{2}}\bar{D}_+\mathcal{J}_-, \qquad \bar{J}_- = -\frac{1}{\sqrt{2}}D_+\bar{\mathcal{J}}_- \tag{5.83}$$

where $\mathcal{J}_- = \mathcal{J}_-(\Phi^i, F_-)$ and $F_-$ is an unconstrained potential field for some $\Phi$ such that $\Phi = \bar{D}_+F_-$. In terms of $\mathcal{J}_-$ the action is

$$S = \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[-\bar{\Gamma}_- f(\Phi, A)\Gamma_- - \Gamma_-\mathcal{J}_- + \bar{\Gamma}_-\bar{\mathcal{J}}_-\right]. \tag{5.84}$$

A priori we only know that $E \cdot \mathcal{J}$ is chiral, but it may be non-zero. However, we can use the "gauge symmetry" $\mathcal{J}_a \sim \mathcal{J}_a + \bar{D}_+\chi_a$ to choose $\mathcal{J}_a$ orthogonal to $E^a$.[1]

Now we substitute $\Gamma_- \to N_-$, with $N_-$ an unconstrained anticommuting superfield, adding a Lagrange multiplier $S_-$ to enforce the constraint $\bar{D}_+N_- = \sqrt{2}E$. The action

1. We can easily prove this statement. Since $E\cdot\mathcal{J}$ is chiral, we can always find a $\beta$ such that $E\cdot\mathcal{J} = \bar{D}_+\beta$. On the other hand, $E\cdot\mathcal{J} \sim E\cdot\mathcal{J} + E\cdot\bar{D}_+\chi = \bar{D}_+(\beta + E\cdot\chi)$. Thus, we simply need to choose $\chi_a$ such that $E\cdot\chi = -\beta$, up to $\bar{D}_+$-exact terms.

becomes

$$S_0 = \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[ -\bar{N}_- f(\Phi,A) N_- - N_-\mathcal{J}_- + \bar{N}_-\bar{\mathcal{J}}_- - S_-(\bar{D}_+N_- - \sqrt{2}E) + \bar{S}_-(D_+\bar{N}_- + \sqrt{2}\bar{E})\Big] \tag{5.85}$$

If we integrate out $S$, we find the chirality constraint for $N_-$, so $N_- = \Gamma_-$ and we recover the original action.

We can now solve the equations of motion for $N_-$, giving

$$N_- = f(\Phi,A)^{-1}\left(D_+\bar{S}_- + \bar{\mathcal{J}}_-\right). \tag{5.86}$$

The action then becomes

$$\begin{aligned} S_d &= \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[ f(\Phi,A)^{-1}\left(\bar{D}_+S_- + \mathcal{J}_-\right)\left(D_+\bar{S}_- + \bar{\mathcal{J}}_-\right) + \sqrt{2}S_-E + \sqrt{2}\bar{S}_-\bar{E}\right] \\ &= -\frac{1}{8\pi}\int d^2x d^2\theta \bar{\Lambda}_- f(\Phi,A)^{-1}\Lambda_- + \frac{\sqrt{2}}{8\pi}\int d^2x d\theta^+\Lambda_- E + \frac{\sqrt{2}}{8\pi}\int d^2x d\bar{\theta}^+\bar{\Lambda}_-\bar{E}, \end{aligned} \tag{5.87}$$

where we defined

$$\Lambda_- = \bar{D}_+S_- + \mathcal{J}_- = f(\Phi,A)\bar{N}_- = f(\Phi,A)\bar{\Gamma}_-, \tag{5.88}$$

which automatically satisfies $\bar{D}_+\Lambda_- = \sqrt{2}J$. Depending on the form we chose for $\mathcal{J}$, there might be additional terms of the form $E\mathcal{J}_-$, but if we dualize all Fermi multiplets, these cancel since $E\cdot\mathcal{J}_- = 0$. Note we exchanged a field of charge $Q$ by one of charge $-Q$, and $E$ was exchanged with $J$. This implements the duality we found in (2.42).

Note we could have absorbed some of the fields in $E$ into $\Gamma_-$ to make it neutral before dualizing. This is what happens in $(2,2)$ theories, where the fermion $\psi_-$ is replaced by $\phi\chi_-$ before dualizing. This redefinition has a nonzero Jacobian, but in a $(2,2)$ theory that would cancel with the Jacobians that occur when dualizing the chiral fields.

### 5.2.3 Duality symmetry map

When we carry out the duality maps in the previous sections, it may be necessary to take into account non-perturbative corrections in the original theory to make the theories truly dual. This was already the case for $(2,2)$ theories [8]. The need for non-perturbative corrections can be thought of as coming from the fact that the redefinition $\phi = e^{\pi}$ is removing the points at which $\phi = 0$ from the theory. As the vortex instanton solution in two dimensions has $\phi = 0$ for the charged fields at the core of the instanton, those solutions have been removed. If we want the theories to be truly equivalent, we must then add back the instanton corrections to the theory. The form of the non-perturbative corrections is strongly restricted by symmetry, so in this section we will analyze the map of symmetries between the original theory and the dual.

To use symmetry arguments, we need to find the transformations of the dual fields under symmetries of the original theory. For this, we can use the duality maps in (5.80), translated to be written in terms of the fields of the original theory. This becomes

$$|\Phi^i|^2 e^{2Q_iA} + K_{ij}\log|\Phi^j| = Y_i + \bar{Y}_i + 2\tilde{Q}_iA, \tag{5.89a}$$

$$e^{2Q_iA}\left(-\frac{i}{2}(\bar{\Phi}_i\partial_-\Phi_i - \Phi_i\partial_-\bar{\Phi}_i) + Q_i|\Phi_i|^2V_-\right)$$
$$-\sum_a x_{ia}e^{2Q_aA}\bar{\Gamma}_{a-}\Gamma^a_- - \frac{i}{4}K_{ij}\partial_-\left(\log\Phi_j - \log\bar{\Phi}_j\right) = \frac{i}{2}\partial_-(Y_i - \bar{Y}_i) - \tilde{Q}_iV_-. \tag{5.89b}$$

Expanding in components,

$$\Phi^i = \phi^i + \sqrt{2}\theta^+\psi^i_+ - i\theta^+\bar{\theta}^+\partial_+\phi^i, \tag{5.90a}$$

$$Y_i = y_i + \sqrt{2}\theta^+\xi_{i+} - i\theta^+\bar{\theta}^+\partial_+y_i, \tag{5.90b}$$

$$\phi_i = \rho_i e^{i\varphi_i}, \tag{5.90c}$$

$$y_i = a_i + i\theta_i, \tag{5.90d}$$

we can relate the original component fields and the dual fields,

$$\rho_i^2 + \sum_j K_{ij} \log \rho_j = 2a_i, \tag{5.91a}$$

$$\bar{\phi}_i \psi_+^i + \sum_j \frac{K_{ij}}{2} \frac{\psi_+^j}{\phi^j} = \xi_{i+}, \tag{5.91b}$$

$$\rho_i^2(\partial_+\varphi_i + Q_i A_+) - \bar{\psi}_{i+}\psi_+^i + \sum_j \frac{K_{ij}}{2}\partial_+\varphi_j = \partial_+\theta_i + \tilde{Q}_i A_+, \tag{5.91c}$$

$$\rho_i^2(\partial_-\varphi_i + Q_i A_-) - \sum_a x_{ia}\bar{\gamma}_{a-}\gamma_-^a + \sum_j \frac{K_{ij}}{2}\partial_-\varphi_j = -\partial_-\theta_i - \tilde{Q}_i A_-. \tag{5.91d}$$

It is worth once again highlighting the differences from the $(2,2)$ duality: there $K = 0$ and $x_{ia} = \delta_{ia}$.

We can now use the duality maps above to argue what the transformations of $y$ should be under any symmetry of the original theory. The current for a symmetry of the original theory under which the chiral fields vary by $q_i$ and the Fermi superfields vary by $q_a$ is given by

$$J_+ = \sum_i (q_i\bar{\psi}_{i+}\psi_{i+} + iq_i\phi_i\partial_+\bar{\phi}_i), \qquad J_- = -\sum_i iq_i\bar{\phi}_i\partial_-\phi_i + \sum_a q_a\bar{\gamma}_{a-}\gamma_{a-}. \tag{5.92}$$

The fermion normalization for the right-movers in our original gauge theories comes from

$$\begin{aligned} S &= \frac{1}{8\pi}\int d^2x d^2\theta^+ \left[-\frac{i}{2}\bar{\Phi}\partial_-\Phi + \frac{i}{2}\Phi\partial_-\bar{\Phi}\right] \\ &= \frac{i}{4\pi}\int d^2x\bar{\psi}_+\partial_-\psi_+ - \frac{1}{4\pi}\int d^2x\partial_\mu\phi\partial^\mu\bar{\phi}. \end{aligned} \tag{5.93}$$

To find the $\bar{\psi}_+\psi_+$ OPE, calculate the variation of the path integral with one $\psi_+$ insertion,

$$\begin{aligned} 0 &= \int \mathcal{D}\psi_+\mathcal{D}\bar{\psi}_+ \frac{\delta}{\delta\bar{\psi}_+(x)}\left[e^{iS}\bar{\psi}_+(y)\right] \\ &= \int \mathcal{D}\psi_+\mathcal{D}\bar{\psi}_+ e^{iS}\left[-\frac{1}{4\pi}\partial_-\psi_+(x)\bar{\psi}_+(y) + \delta(x-y)\right] \end{aligned} \tag{5.94}$$

which implies

$$\partial_- \langle \psi_+(x)\bar{\psi}_+(y)\rangle = 4\pi\delta(x-y) \Rightarrow \langle \psi_+(x)\bar{\psi}_+(y)\rangle = -\frac{1}{x^+ - y^+}. \tag{5.95}$$

Given this OPE, we have

$$J_+(x)\psi_+(y) \sim -\frac{q_i\psi_+(x)}{x^+ - y^+}, \qquad J_-^R(x)\psi_+(y) \sim 0. \tag{5.96}$$

From $J$ we can construct the conserved charge,

$$Q = \frac{1}{2\pi}\int dx^1 J_0 = \frac{1}{4\pi}\int dx^1 (J_+ + J_-) \tag{5.97}$$

so that

$$[Q, \psi_+] = q_i\psi_+ \Rightarrow \delta\psi_+ = iq_i\psi_+ \tag{5.98}$$

as required.

Doing the same for $\theta$, using the duality map, we find

$$J_+(x)\partial_+\theta(y) \sim -q_i : \bar{\psi}_+\psi_+ : (x) : \bar{\psi}_+\psi_+ : (y) \sim -\frac{q_i}{(x^+ - y^+)^2}, \tag{5.99}$$

$$J_-(x)\partial_-\theta(y) \sim x_{ia}q_a : \bar{\gamma}_-\gamma_- : (x) : \bar{\gamma}_-\gamma_- : (y) \sim \frac{x_{ia}q_a}{(x^+ - y^+)^2} \tag{5.100}$$

and all other OPEs regular, so that

$$J_+(x)\theta(y) \sim -\frac{q_i}{x^+ - y^+}, \qquad J_-(x)\theta(y) \sim \frac{x_{ia}q_a}{x^- - y^-} \tag{5.101}$$

which from the same argument gives

$$\delta\theta = q_i - x_{ia}q_a, \tag{5.102}$$

that is, $Y$ has shift charge $q_i - x_{ia}q_a$ under this generic symmetry transformation. Note the scalar contribution cancels, so we can use the fermion charges to find the transformation under any symmetry. This includes the gauge symmetry as well as R-symmetries.

## 5.3 Duality in chiral gauge theories and gauge invariance

Before tackling the case of $(0,2)$ gauge theories, we will test the Jacobians we found in a simpler setting. We will work with theories with one fermion of each chirality and one periodic boson. This is just enough to see the Jacobians in action. One advantage of this setup is that we can bosonize the fermions into bosons, leading to a purely bosonic system. One thing to note is that the duality maps we have found in (5.80) do not seem gauge-invariant whenever $K_{ij} \neq 0$ for any gauge transformation. In the purely bosonic system, on the other hand, gauge invariance should be manifest.

### *5.3.1 Bosonization maps, with currents and mass terms*

Take a free Dirac fermion, coupled to some vector fields,

$$
\begin{aligned}
S &= -\frac{i}{4\pi}\int d^2x \left[\bar{\Psi}\gamma^\mu \left(\partial_\mu + ia_\mu + i\gamma_5 b_\mu\right)\Psi\right] \\
&= \frac{i}{4\pi}\int d^2x \left[\bar{\psi}_+ \left(\partial_- + ia_- + ib_-\right)\psi_+ + \bar{\gamma}_- \left(\partial_+ + ia_+ - ib_+\right)\gamma_-\right] \qquad (5.103)
\end{aligned}
$$

Note there is a simple duality here, $\gamma \leftrightarrow \bar{\gamma}$, $a \leftrightarrow b$, which interchanges the vector and axial current. We can bosonize this action using the methods of [57], by gauging either the vector or axial isometries. This is also equivalent to interchanging $a \leftrightarrow b$. Gauging the vector symmetry, we find a bosonic action

$$
S_b = \frac{1}{8\pi}\int d^2x \left[(\partial_+\theta - 2b_+)(\partial_-\theta - 2b_-) - 4b_+b_- - 2\theta(\partial_-a_+ - \partial_+a_-)\right]. \qquad (5.104)
$$

Since we obtained the bosonization by gauging the vector symmetry, this action is not invariant under the axial symmetry whose field strength is $b$. If we want to gauge the axial symmetry, we should also bosonize using that symmetry. We would then obtain the action above with $a \leftrightarrow b$.

As a test of this result we can take a compact scalar with action

$$S = \frac{1}{4\pi} \int d^2x \left[ R^2 \partial_+ \theta \partial_- \theta + 2L_+ \partial_- \theta + 2R_- \partial_+ \theta \right] \tag{5.105}$$

and integrate out $\theta$, obtaining an effective action

$$W = -\frac{1}{4\pi R^2} \int d^2x \left[ L_+ \frac{\partial_-}{\partial_+} L_+ + R_- \frac{\partial_+}{\partial_-} R_- + 2L_+ R_- \right]. \tag{5.106}$$

We can apply this to the special case

$$S = \frac{1}{4\pi} \int d^2x \left[ R^2 (\partial_+ \theta + qA_+)(\partial_- \theta + qA_-) - Q\theta \epsilon^{\mu\nu} F_{\mu\nu} \right] \tag{5.107}$$

and integrate out the scalar $\theta$ to obtain a non-local action in terms of the gauge field. We find

$$\begin{aligned} W = \frac{1}{8\pi} \int d^2x \Big[ & - \left( \frac{Q^2}{2R^2} + \frac{q^2 R^2}{2} \right) \left( A_- \frac{\partial_+}{\partial_-} A_- + A_+ \frac{\partial_-}{\partial_+} A_+ - 2A_+ A_- \right) \\ & + qQ \left( A_- \frac{\partial_+}{\partial_-} A_- - A_+ \frac{\partial_-}{\partial_+} A_+ \right) \Big]. \end{aligned} \tag{5.108}$$

The last term gives the right gauge variation if both $q$ and $Q$ are turned on. Note this action is invariant under the T-duality $(R, q, Q) \to (1/R, Q, q)$. For a non-anomalous gauging of a Dirac fermion, we saw above that we have $R^2 = 1/2$, and either $q = -2$ or $Q = 1$. Indeed for either of these choices the factor in the first line is $\frac{Q^2}{2R^2} + \frac{q^2 R^2}{2} = 1$.

It will also be useful to recall the T-duality map for more general vector fields,

$$S = \frac{1}{4\pi}\int d^2x \left[R^2(\partial_-\theta + a_-)(\partial_+\theta + a_+) + 2\epsilon^{\mu\nu}\partial_\mu\theta b_\nu\right] \tag{5.109}$$

$$\Rightarrow S_d = \frac{1}{4\pi}\int d^2x \left[\frac{1}{R^2}(\partial_-\varphi + b_-)(\partial_+\varphi + b_+) + 2\epsilon^{\mu\nu}(\partial_\mu\varphi + b_\mu)a_\nu\right], \tag{5.110}$$

with duality maps

$$R^2(\partial_-\theta + a_-) = \partial_-\varphi + b_-, \qquad R^2(\partial_+\theta + a_+) = -\partial_+\varphi - b_+. \tag{5.111}$$

Another line that will be helpful is to start with

$$S = \frac{1}{4\pi}\int d^2x \left[\rho^2\partial_-\theta\partial_+\theta + \partial_+\theta L_- + \partial_-\theta R_+\right] \tag{5.112}$$

$$\Rightarrow S_d = \frac{1}{4\pi}\int d^2x \left[\frac{1}{\rho^2}(\partial_-\varphi - L_-)(\partial_+\varphi + R_+)\right], \tag{5.113}$$

with maps

$$\rho^2\partial_-\theta = \partial_-\varphi - L_-, \qquad \rho^2\partial_+\theta = -\partial_+\varphi - R_+. \tag{5.114}$$

Another test of this process can be done by adding a quartic coupling to the fermionic action (here we turn off $b_\mu$),

$$S = \frac{1}{4\pi}\int d^2x \left[-i\bar{\Psi}\gamma^\mu\left(\partial_\mu + ia_\mu\right)\Psi - \frac{1}{2}g^2\bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi\right]. \tag{5.115}$$

Note that

$$\bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi = -4\bar{\gamma}_-\gamma_-\bar{\psi}_+\psi_+ \tag{5.116}$$

It is customary to rewrite this coupling using an auxiliary field,

$$S = \frac{1}{4\pi}\int d^2x \left[ -i\bar{\Psi}\gamma^\mu \left(\partial_\mu + ia_\mu + igc_\mu\right)\Psi + \frac{1}{2}c_\mu c^\mu \right]. \tag{5.117}$$

The bosonized theory can then be found as above,

$$S_b = \frac{1}{8\pi}\int d^2x \left[\partial_+\theta\partial_-\theta - 2\theta(\partial_- a_+ - \partial_+ a_-) - 2g\theta(\partial_- c_+ - \partial_+ c_-) - c_+ c_-\right], \tag{5.118}$$

and finally we can integrate out $c_\mu$ to find

$$S_b = \frac{1}{8\pi}\int d^2x \left[(1-4g^2)\partial_+\theta\partial_-\theta - 2\theta(\partial_- a_+ - \partial_+ a_-)\right] \tag{5.119}$$

so $R^2 = \frac{1}{2} - 2g^2$. Note if we turn on $b_\mu$, it just comes through as before, so

$$S_b = \frac{1}{8\pi}\int d^2x \left[(1-4g^2)\partial_+\theta\partial_-\theta - 2(b_+\partial_-\theta + b_-\partial_+\theta) - 2\theta(\partial_- a_+ - \partial_+ a_-)\right], \tag{5.120}$$

Note also that $\gamma \leftrightarrow \bar{\gamma}$ now takes $a \leftrightarrow b$, $g^2 \leftrightarrow -g^2$.

Finally we can add a mass term,

$$\begin{aligned} S = \frac{1}{4\pi}\int d^2x \big[ & i\bar{\psi}_+ \left(\partial_- + ia_- + ib_-\right)\psi_+ + i\bar{\gamma}_- \left(\partial_+ + ia_+ - ib_+\right)\gamma_- \\ & - M\psi_+\bar{\gamma}_- - M^*\gamma_-\bar{\psi}_+ \big]. \end{aligned} \tag{5.121}$$

Here the main argument is based on symmetry transformations. Under a chiral transformation $\psi \to e^{i\alpha}\psi$, $\gamma \to e^{-i\alpha}\gamma$, $\bar{\psi}\gamma \to e^{-2i\alpha}\bar{\psi}\gamma$, so the bosonic action must reflect this. We saw the dual scalar has charge $-2$ under this symmetry, so we must have

$$S_b = \frac{1}{8\pi}\int d^2x \left[(\partial_+\theta - 2b_+)(\partial_-\theta - 2b_-) - 2\theta(\partial_- a_+ - \partial_+ a_-) + AMe^{-i\theta} + AM^* e^{i\theta}\right], \tag{5.122}$$

where $A$ is an undetermined constant. For the couplings we had above, where $M = me^{i\varphi}$, we have

$$S_b = \frac{1}{8\pi}\int d^2x\,[(\partial_+\theta - 2b_+)(\partial_-\theta - 2b_-) - 2\theta(\partial_- a_+ - \partial_+ a_-) + Am\cos(\varphi - \theta)]\,. \quad (5.123)$$

If we instead want a mass term of the form $M\psi\gamma+$ c.c., we need a scalar that is charged under the vector transformation, so we cannot write it locally in terms of $\theta$, but only in terms of its dual scalar.

### 5.3.2 *Component duality*

We can try to dualize a phase $\theta$ present in a Yukawa coupling,

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2 D_\mu\theta D^\mu\theta + i\bar{\psi}_+ D_-\psi_+ + i\bar{\gamma}_- D_+\gamma_- + m\psi_+ e^{i\theta}\gamma_- + m\bar{\gamma}_- e^{-i\theta}\bar{\psi}_+\right]. \quad (5.124)$$

Before tackling the general problem, we can consider a neutral $\theta$, with $Q = Q_\psi = -Q_\gamma$. If we integrate out the fermions, the effective action is given by

$$W = \frac{1}{8\pi}\int d^2x\left[-2\rho^2 D_\mu\theta D^\mu\theta + \partial_+\theta\partial_-\theta - 2Q\theta F_{-+}\right] \quad (5.125)$$

If we redefine the fermions by

$$\gamma_- \to \lambda_- = e^{i\alpha\theta}\gamma -, \qquad \psi_+ \to \chi_+ = e^{i\beta\theta}\psi_+. \quad (5.126)$$

with $\alpha + \beta = 1$, the Yukawa coupling loses its $\theta$ dependence and integrating out the massive fermions would not give any contributions to the effective action. The Jacobian can be found either from this reasoning or the general formula (5.37). We have

$$S = \frac{1}{4\pi}\int d^2x\Big[-\left(\rho^2+\frac{1}{2}\right)\partial_\mu\theta\partial^\mu\theta - 2Q\epsilon^{\mu\nu}A_\mu\partial_\nu\theta + i\bar{\chi}_+(\partial_- + iQA_- - i\beta\partial_-\theta)\chi_+$$
$$+ i\bar{\lambda}_-(\partial_+ - iQA_+ - i\alpha\partial_+\theta)\lambda_- + m\chi_+\lambda_- + m\bar{\lambda}_-\bar{\chi}_+\Big]. \tag{5.127}$$

To dualise we write $\partial_\mu \to c_\mu$ with an appropriate Lagrange multiplier,

$$S_d = \frac{1}{4\pi}\int d^2x\Big[\left(\rho^2+\frac{1}{2}\right)c_+c_- - 2Q\epsilon^{\mu\nu}A_\mu c_\nu + 2\varphi\epsilon^{\mu\nu}\partial_\mu c_\nu + i\bar{\chi}_+(\partial_- + iQA_- - i\beta c_-)\chi_+$$
$$+ i\bar{\lambda}_-(\partial_+ - iQA_+ - i\alpha c_+)\lambda_- + m\chi_+\lambda_- + m\bar{\lambda}_-\bar{\chi}_+\Big]. \tag{5.128}$$

Integrating out $c_+$ and $c_-$ gives

$$\left(\rho^2+\frac{1}{2}\right)c_+ + \beta\bar{\chi}_+\chi_+ = -\partial_+\varphi - QA_+, \tag{5.129}$$
$$\left(\rho^2+\frac{1}{2}\right)c_- + \alpha\bar{\lambda}_-\lambda_- = \partial_-\varphi + QA_-. \tag{5.130}$$

These are gauge-invariant if $\varphi$ has charge $Q$, which is confirmed by its two-point function with the gauge current as in Section 5.2.3: we get $Q_\varphi = Q(\alpha+\beta) = Q$. Finally we can write the dual action,

$$S_d = \frac{1}{4\pi}\int d^2x\Big[-\frac{1}{\rho^2+1/2}D_\mu\varphi D^\mu\varphi + i\bar{\chi}_+\left(\partial_- + iQA_- - i\frac{\beta}{\rho^2+1/2}D_-\varphi\right)\chi_+$$
$$-\frac{\alpha\beta}{\rho^2+1/2}\bar{\chi}\chi\bar{\lambda}\lambda + i\bar{\lambda}_-\left(\partial_+ - iQA_+ + i\frac{\alpha}{\rho^2+1/2}D_+\varphi\right)\lambda_- + m\chi_+\lambda_-$$
$$+ m\bar{\lambda}_-\bar{\chi}_+\Big]. \tag{5.131}$$

Note that integrating out the fermions before or after dualizing should lead to the same results. This is easy to check in the case $\alpha\beta = 0$, as integrating out the fermions does not bring any contributions to the effective action. The one difference is the fermions would not appear in the duality maps if they were integrated out first, so the argument based on the correlation function for the charge of $\varphi$ would be less straightforward.

Let's examine more closely the subtleties in the case when the phase is charged. We start with a path integral $\mathcal{D}A\mathcal{D}\theta\mathcal{D}\psi\mathcal{D}\gamma$. We will take now $Q_\psi = Q_\gamma = 1$, $Q_\theta = -2$. The Jacobian can be found as above. The action is, writing $k = \alpha - \beta$

$$S_0 = \frac{1}{4\pi}\int d^2x\Big[-\left(\rho^2 + \frac{k^2}{2}\right)c_\mu c^\mu - 2\rho^2 Q_\theta A_\mu c^\mu - \rho^2 Q_\theta^2 A_\mu A^\mu + 2k\epsilon^{\mu\nu}A_\mu c_\nu + 2\varphi\epsilon^{\mu\nu}\partial_\mu c_\nu$$
$$+ i\bar{\chi}_+(\partial_- + iA_- - i\beta c_-)\chi_+ + i\bar{\lambda}_-(\partial_+ + iA_+ - i\alpha c_+)\lambda_-$$
$$+ m\chi_+\lambda_- + m\bar{\lambda}_-\bar{\chi}_+\Big]. \tag{5.132}$$

The equations of motion for $c_\mu$ are

$$\left(\rho^2 + \frac{k^2}{2}\right)c_- + \rho^2 Q_\theta A_- + \alpha\bar{\lambda}_-\lambda_- = \partial_-\varphi - kA_-, \tag{5.133}$$
$$\left(\rho^2 + \frac{k^2}{2}\right)c_+ + \rho^2 Q_\theta A_+ + \beta\bar{\chi}_+\chi_+ = -\partial_+\varphi + kA_+. \tag{5.134}$$

Solving them, we find the dual action

$$S_d = \frac{1}{4\pi}\int d^2x\Big[\frac{1}{\rho^2 + k^2/2}\left(\partial_+\varphi - kA_+ + \rho^2 Q_\theta A_+ + \beta\bar{\chi}\chi\right)$$
$$\times\left(\partial_-\varphi - kA_- - \rho^2 Q_\theta A_- - \alpha\bar{\lambda}\lambda\right) - \rho^2 Q_\theta^2 A_\mu A^\mu$$
$$+ i\bar{\chi}_+(\partial_- + iA_-)\chi_+ + i\bar{\lambda}_-(\partial_+ + iA_+)\lambda_- + m\chi_+\lambda_- + m\bar{\lambda}_-\bar{\chi}_+\Big]$$
$$= \frac{1}{4\pi}\int d^2x\Big[\frac{1}{\rho^2 + k^2/2}D_+\varphi D_-\varphi + \frac{k^2\rho^2 Q_\theta^2/2}{\rho^2 + k^2/2}A_+A_- - \frac{\rho^2 Q_\theta}{\rho^2 + k^2/2}\varphi F_{-+}$$
$$+ i\bar{\chi}_+\left(\partial_- + iA_- + \frac{\beta}{\rho^2 + k^2/2}D_-\varphi - \frac{\beta\rho^2 Q_\theta}{\rho^2 + k^2/2}A_-\right)\chi_+$$
$$+ i\bar{\lambda}_-\left(\partial_+ + iA_+ - \frac{\alpha}{\rho^2 + k^2/2}D_+\varphi - \frac{\alpha\rho^2 Q_\theta}{\rho^2 + k^2/2}A_+\right)\lambda_-$$
$$+ m\chi_+\lambda_- + m\bar{\lambda}_-\bar{\chi}_+ - \frac{\alpha\beta}{\rho^2 + k^2/2}\bar{\chi}\chi\bar{\lambda}\lambda\Big]. \tag{5.135}$$

This action is not explicitly gauge-invariant. However, it might still be correct, as the path integral over the fermions is non-trivial. To analyze this, it is helpful to bosonize the fermions.

### 5.3.3 Bosonized duality

#### Vector redefinition

First consider the case when $\theta$ is uncharged, $Q_\theta = 0$, so $Q = Q_\psi = Q_\gamma$. Then the bosonization gives

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2\partial_\mu\theta\partial^\mu\theta - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + Am\cos(\theta-\varphi) - Q\varphi F_{-+}\right]. \tag{5.136}$$

We want to dualize the phase $\theta$, but to do so we first need to redefine $\omega = \varphi - \theta$, in terms of which the action becomes

$$\begin{aligned} S &= \frac{1}{4\pi}\int d^2x\left[-\rho^2\partial_\mu\theta\partial^\mu\theta - \frac{1}{2}\partial_\mu(\omega+\theta)\partial^\mu(\omega+\theta) + Am\cos(\omega) - Q(\omega+\theta)F_{-+}\right] \\ &= \frac{1}{4\pi}\int d^2x\Bigg[-\left(\rho^2+\frac{1}{2}\right)\left(\partial_\mu\theta + \frac{1/2}{\rho^2+1/2}\partial_\mu\omega\right)\left(\partial^\mu\theta + \frac{1/2}{\rho^2+1/2}\partial^\mu\omega\right) - Q\theta F_{-+} \\ &\qquad - \left(\frac{1}{2} - \frac{1/4}{\rho^2+1/2}\right)\partial_\mu\omega\partial^\mu\omega + Am\cos(\omega) - Q\omega F_{-+}\Bigg]. \end{aligned} \tag{5.137}$$

Note we can fermionise $\omega$ at this point, and we obtain (5.127). Now we can directly dualise $\theta \to \zeta$,

$$\begin{aligned} S_d = \frac{1}{4\pi}\int d^2x\Bigg[&-\frac{1}{\rho^2+1/2}\left(\partial_\mu\zeta + QA_\mu\right)\left(\partial^\mu\zeta + QA^\mu\right) - \left(\frac{1}{2} - \frac{1/4}{\rho^2+1/2}\right)\partial_\mu\omega\partial^\mu\omega \\ &+ Am\cos(\omega) - Q\omega F_{-+} + \frac{1}{\rho^2+1/2}\epsilon^{\mu\nu}D_\mu\zeta\partial_\nu\omega\Bigg]. \end{aligned} \tag{5.138}$$

To compare to our earlier results, we should now fermionise $\omega$. We have

$$S_f = \frac{1}{4\pi}\int d^2x\Big[-\frac{1}{\rho^2+1/2}\left(\partial_\mu\zeta - QA_\mu\right)\left(\partial^\mu\zeta - QA^\mu\right) + i\bar{\chi}_+\left(D_- - \frac{i}{2}\frac{1}{\rho^2+1/2}D_-\zeta\right)\chi_+ \\ + i\bar{\lambda}_-\left(D_+ - \frac{i}{2}\frac{1}{\rho^2+1/2}D_+\zeta\right)\lambda_- + \frac{1/4}{\rho^2+1/2}\bar{\lambda}_-\lambda_-\bar{\chi}_+\chi_+ \\ + m\bar{\chi}_+\lambda_- + m\bar{\lambda}_-\chi_+\Big]. \tag{5.139}$$

This agrees with (5.131) (with $\alpha = \beta = 1/2$ and a switch $\lambda \leftrightarrow \bar{\lambda}$)!

## Chiral redefinition

The next step to understand are general chiral rotations. This is less trivial to implement directly in the bosonic picture, since it involves using $\theta$ to rotate in the direction dual to the bosonized fermion. We can do it with $m = 0$, where we can explicitly dualize it. When $m \neq 0$, the variation of the action will be given by $j_\mu\partial^\mu\theta$, so we can carry over the results from $m = 0$. The bosonized action is

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2\partial_\mu\theta\partial^\mu\theta - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \varphi F_{-+}\right]. \tag{5.140}$$

Dualizing $\varphi$ we find

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2\partial_\mu\theta\partial^\mu\theta - 2(\partial_\mu\eta + A_\mu)(\partial^\mu\eta + A^\mu)\right]. \tag{5.141}$$

Here we can take $\eta = \zeta + \frac{1}{2}(\alpha - \beta)\theta$, then dualize back $\zeta$. We find

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2\partial_\mu\theta\partial^\mu\theta - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \varphi F_{-+} - (\alpha-\beta)\epsilon^{\mu\nu}\partial_\mu\theta\partial_\nu\varphi\right]. \tag{5.142}$$

Now we can finally take $\varphi = \omega + k\theta$, $k = \alpha + \beta$, obtaining the right action.

$$
\begin{aligned}
S = \frac{1}{4\pi}\int d^2x\Big[ &- \rho^2 \partial_\mu\theta\partial^\mu\theta - \frac{1}{2}\partial_\mu(\omega + k\theta)\partial^\mu(\omega + k\theta) - (\omega + k\theta)F_{-+} \\
&- (\alpha - \beta)\epsilon^{\mu\nu}\partial_\mu\theta\partial_\nu\omega\Big] \\
= \frac{1}{4\pi}\int d^2x\Big[ &- \left(\rho^2 + \frac{k^2}{2}\right)\left(\partial_\mu\theta + \frac{k/2}{\rho^2 + k^2/2}\partial_\mu\omega\right)\left(\partial^\mu\theta + \frac{k/2}{\rho^2 + k^2/2}\partial^\mu\omega\right) - k\theta F_{-+} \\
&- (\alpha - \beta)\epsilon^{\mu\nu}\partial_\mu\theta\partial_\nu\omega - \left(\frac{1}{2} - \frac{k^2/4}{\rho^2 + k^2/2}\right)\partial_\mu\omega\partial^\mu\omega - \omega F_{-+}\Big]. \qquad (5.143)
\end{aligned}
$$

Note the order of the redefinitions does not influence the final result. Dualising $\theta$ from this action, we find

$$
\begin{aligned}
S_d = \frac{1}{4\pi}\int d^2x\Big[ &- \frac{1}{\rho^2 + k^2/2}\left(\partial_\mu\zeta + kA_\mu - \frac{l}{2}\partial_\mu\omega\right)\left(\partial^\mu\zeta + kA^\mu - \frac{l}{2}\partial^\mu\omega\right) \\
&+ \frac{k}{\rho^2 + k^2/2}\epsilon^{\mu\nu}\left(\partial_\mu\zeta + kA_\mu\right)\partial_\nu\omega - \left(\frac{1}{2} - \frac{k^2/4}{\rho^2 + k^2/2}\right)\partial_\mu\omega\partial^\mu\omega - \omega F_{-+}\Big] \\
= \frac{1}{4\pi}\int d^2x\Big[ &- \frac{1}{\rho^2 + k^2/2}D_\mu\zeta D^\mu\zeta + \frac{l}{\rho^2 + k^2/2}D_\mu\zeta\partial^\mu\omega - \left(\frac{1}{2} - \frac{\alpha\beta}{\rho^2 + k^2/2}\right)\partial_\mu\omega\partial^\mu\omega \\
&- \omega F_{-+} + \frac{k}{\rho^2 + k^2/2}\epsilon^{\mu\nu}D_\mu\zeta\partial_\nu\omega\Big] \qquad (5.144) \\
= \frac{1}{4\pi}\int d^2x\Big[ &- \left(\frac{1}{2} - \frac{\alpha\beta}{\rho^2 + k^2/2}\right)\left(\partial_\mu\omega - \frac{l}{\rho^2 + l^2/2}D_\mu\zeta\right)\left(\partial^\mu\omega - \frac{l}{\rho^2 + l^2/2}D^\mu\zeta\right) \\
&- \frac{\rho^2}{(\rho^2 + k^2/2)(\rho^2 + l^2/2)}D_\mu\zeta D^\mu\zeta - \omega F_{-+} + \frac{k}{\rho^2 + k^2/2}\epsilon^{\mu\nu}D_\mu\zeta\partial_\nu\omega\Big].
\end{aligned}
$$

where $l = \alpha - \beta$. To complete the circle we should now fermionise $\omega$, finding finally

$$
\begin{aligned}
S_f = \frac{1}{4\pi}\int d^2x\Big[ &- \frac{1}{\rho^2 + k^2/2}D_\mu\zeta D^\mu\zeta + i\bar{\chi}_+\left(\partial_- + iA_- - i\frac{\beta D_-\zeta}{\rho^2 + k^2/2}\right)\chi_+ \\
&+ i\bar{\lambda}_-\left(\partial_+ + iA_+ - i\frac{\alpha D_+\zeta}{\rho^2 + k^2/2}\right)\lambda_- + \frac{\alpha\beta}{\rho^2 + k^2/2}\bar{\lambda}_-\lambda_-\bar{\chi}_+\chi_+\Big]. \qquad (5.145)
\end{aligned}
$$

This is once again the right action.

### Chiral charged redefinition

Here we will not turn on any Yukawa, and let $\theta$ have a general charge $Q_\theta = Q$. We have the bosonized action

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2 D_\mu\theta D^\mu\theta - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \varphi F_{-+}\right]. \tag{5.146}$$

We can now follow the same steps we did above with this different starting action. The same redefinitions lead to

$$S = \frac{1}{4\pi}\int d^2x\left[-\rho^2 D_\mu\theta D^\mu\theta - \frac{1}{2}\partial_\mu(\omega+k\theta)\partial^\mu(\omega+k\theta) - (\omega+k\theta)F_{-+} - l\epsilon^{\mu\nu}\partial_\mu\theta\partial_\nu\omega\right]. \tag{5.147}$$

It should now be noted that since $\varphi = \omega + k\theta$ is gauge-invariant, $Q_\omega = -kQ$. Therefore when splitting the mixed terms in $\omega$ and $\theta$, it is useful to do so introducing $A_\mu$ terms to make the derivatives covariant,

$$\begin{aligned}S = \frac{1}{4\pi}\int d^2x\Big[&-\left(\rho^2+\frac{k^2}{2}\right)D_\mu\theta D^\mu\theta - \frac{1}{2}D_\mu\omega D^\mu\omega - kD_\mu\theta D^\mu\omega - (\omega+k\theta)F_{-+}\\ &- l\epsilon^{\mu\nu}\partial_\mu\theta\partial_\nu\omega\Big].\\ = \frac{1}{4\pi}\int d^2x\Big[&-\left(\rho^2+\frac{k^2}{2}\right)\left(D_\mu\theta + \frac{k/2}{\rho^2+k^2/2}D_\mu\omega\right)\left(D^\mu\theta + \frac{k/2}{\rho^2+k^2/2}D^\mu\omega\right) - k\theta F_{-+}\\ &- l\epsilon^{\mu\nu}\partial_\mu\theta\partial_\nu\omega - \left(\frac{1}{2} - \frac{k^2/4}{\rho^2+k^2/2}\right)D_\mu\omega D^\mu\omega - \omega F_{-+}\Big].\end{aligned} \tag{5.148}$$

It is instructive as above to fermionize $\omega$ at this point. We find, with $a = A + ld\theta/2$ and $b = -kd\theta/2$,

$$\begin{aligned}S = \frac{1}{4\pi}\int d^2x\Big[&-\rho^2 D_\mu\theta D^\mu\theta - \frac{k^2}{2}\partial_\mu\theta\partial^\mu\theta - k\theta F_{-+} + i\bar{\chi}_+(\partial_- + iA_- - i\beta\partial_-\theta)\chi_+\\ &+ i\bar{\lambda}_-(\partial_+ + iA_+ + i\alpha\partial_-\theta)\lambda_-\Big].\end{aligned} \tag{5.149}$$

The second and third terms here are the Jacobian of our redefinition, agreeing with the Jacobians we had calculated directly from the fermion path integral. The one remaining question is how to dualize this theory and how it is compatible with gauge-invariance.

Dualizing $\theta$ in the bosonic picture, we obtain

$$\begin{aligned}
S = \frac{1}{4\pi}\int d^2x\Big[ &-\frac{1}{\rho^2+k^2/2}D_\mu\xi D^\mu\xi + \frac{l}{\rho^2+k^2/2}D_\mu\xi D^\mu\omega - \left(\frac{1}{2}-\frac{\alpha\beta}{\rho^2+k^2/2}\right)D_\mu\omega D^\mu\omega \\
&- \left(Q\xi+\omega-\frac{1}{2}lQ\omega\right)F_{-+} + \frac{k}{\rho^2+k^2/2}\epsilon^{\mu\nu}D_\mu\xi D_\nu\omega\Big] \\
= \frac{1}{4\pi}\int d^2x\Big[ &-\frac{1}{\rho^2+k^2/2}D_\mu\xi D^\mu\xi + \frac{l}{\rho^2+k^2/2}D_\mu\xi D^\mu\omega - \left(\frac{1}{2}-\frac{\alpha\beta}{\rho^2+k^2/2}\right)D_\mu\omega D^\mu\omega \\
&+ \frac{\rho^2}{k(\rho^2+k^2/2)}\left(Q_\omega\xi - Q_\xi\omega\right)F_{-+} + \frac{k}{\rho^2+k^2/2}\epsilon^{\mu\nu}\partial_\mu\xi\partial_\nu\omega\Big]. \qquad (5.150)
\end{aligned}$$

with $Q_\xi = k - \frac{1}{2}klQ$. Note that $Q\xi + \omega - \frac{1}{2}lQ\omega = -\frac{1}{k}(Q_\omega\xi - Q_\xi\omega)$ is a gauge-invariant combination, so this action is explicitly gauge-invariant. We can now try to close the duality circle. If we fermionize $\omega$ from this action, we find

$$\begin{aligned}
S = \frac{1}{4\pi}\int d^2x\Big[ &-\frac{1}{\rho^2+k^2/2}D_\mu\xi D^\mu\xi + \frac{lQ_\omega}{\rho^2+k^2/2}D_\mu\xi A^\mu - \left(\frac{1}{2}-\frac{\alpha\beta}{\rho^2+k^2/2}\right)Q_\omega^2 A_\mu A^\mu \\
&- \frac{Q\rho^2}{\rho^2+k^2/2}\xi F_{-+} + i\bar\chi_+\left(D_- - i\frac{\alpha\beta k Q}{\rho^2+k^2/2}A_- - i\frac{\beta}{\rho^2+k^2/2}D_-\xi\right)\chi_+ \\
&+ \frac{\alpha\beta}{\rho^2+k^2/2}\bar\lambda\lambda\bar\chi\chi + i\bar\lambda_-\left(D_+ + i\frac{\alpha\beta k Q}{\rho^2+k^2/2}A_+ - i\frac{\alpha}{\rho^2+k^2/2}D_+\xi\right)\lambda_-\Big] \\
= \frac{1}{4\pi}\int d^2x\Big[ &-\frac{1}{\rho^2+k^2/2}D_\mu\xi D^\mu\xi + \frac{lQ_\omega}{\rho^2+k^2/2}D_\mu\xi A^\mu - \left(\frac{1}{2}-\frac{\alpha\beta}{\rho^2+k^2/2}\right)Q_\omega^2 A_\mu A^\mu \\
&+ i\bar\chi_+\left(\partial_- + iA_- + i\frac{\beta(Q\rho^2-k)}{\rho^2+k^2/2}A_- - i\frac{\beta}{\rho^2+k^2/2}\partial_-\xi\right)\chi_+ \\
&+ i\bar\lambda_-\left(\partial_+ + iA_+ - i\frac{\alpha(Q\rho^2+k)}{\rho^2+k^2/2}A_+ - i\frac{\alpha}{\rho^2+k^2/2}\partial_+\xi\right)\lambda_- \\
&- \frac{Q\rho^2}{\rho^2+k^2/2}\xi F_{-+} + \frac{\alpha\beta}{\rho^2+k^2/2}\bar\lambda\lambda\bar\chi\chi\Big]. \qquad (5.151)
\end{aligned}$$

Note we used $Q_\chi = 1+\beta Q$, $Q_\lambda = 1-\alpha Q$ when writing the covariant derivatives. We can check that this is the same action we find when dualizing $\theta$ from (5.149):

$$S_d = \frac{1}{4\pi}\int d^2x \Big[\frac{1}{\rho^2 + k^2/2}(\partial_+\xi + Q\rho^2 A_+ + kA_+ + \beta\bar{\chi}_+\chi_+) \\ \times (\partial_-\xi - Q\rho^2 A_- + kA_- + \alpha\bar{\lambda}_-\lambda_-) \\ + Q^2\rho^2 A_+ A_- + i\bar{\chi}_+(\partial_- + iA_-)\chi_+ + i\bar{\lambda}_-(\partial_+ + iA_+)\lambda_-\Big]. \tag{5.152}$$

We can check term by term that these dualized actions indeed agree. It should be noted that neither is explicitly gauge-invariant.

From the duality maps in the latter picture, however,

$$\rho^2(\partial_-\theta + QA_-) + \frac{k^2}{2}\partial_-\theta - \alpha\bar{\lambda}_-\lambda_- = \partial_-\xi + kA_-, \tag{5.153}$$

$$\rho^2(\partial_+\theta + QA_+) + \frac{k^2}{2}\partial_+\theta + \beta\bar{\chi}_+\chi_+ = -\partial_+\xi - kA_+, \tag{5.154}$$

we would say from the arguments in Section 5.2.3

$$Q_\xi = \alpha Q_\lambda + \beta Q_\chi = \alpha(1-\alpha Q) + \beta(1+\beta Q) = k - \alpha^2 Q + \beta^2 Q = k - klQ, \tag{5.155}$$

or, noting that $\bar{\lambda}\lambda = \bar{\gamma}\gamma$, $\bar{\chi}\chi = \bar{\psi}\psi$, we might also have said simply that $Q_\xi = k$. In the bosonized theory on the other hand we found $k - \frac{1}{2}klQ$. Which one is right?

To straighten this out, we note that if we integrate out the fermions, the action in terms of $\xi$ and the gauge fields should be explicitly gauge-invariant. (5.149) only depends on the original action and the Jacobian, and dualizing it leads to (5.152), so if we integrate out the fermions from (5.152) we will obtain an answer that is independent of the bosonization process.

To do this we can start by integrating out a fermion with a generic quadratic coupling,

$$S = \frac{1}{4\pi}\int d^2x \left[i\bar{\psi}_+(\partial_- + iR_-)\psi_+ + i\bar{\gamma}_-(\partial_+ + iL_+)\gamma_- - g^2\bar{\gamma}\gamma\bar{\psi}\psi\right]. \tag{5.156}$$

As above, we reduce this to a known problem by means of an auxiliary field $c$,

$$S = \frac{1}{4\pi}\int d^2x\left[i\bar{\psi}_+(\partial_- + iR_- + ic_-)\psi_+ + i\bar{\gamma}_-(\partial_+ + iL_+ + ic_+)\gamma_- + g^{-2}c_+c_-\right]. \quad (5.157)$$

Integrating out the fermions, we find (with the same regularization we had above)

$$S = \frac{1}{4\pi}\int d^2x\Big[-\frac{1}{2}(R_- + c_-)\frac{\partial_+}{\partial_-}(R_- + c_-) - \frac{1}{2}(L_+ + c_+)\frac{\partial_-}{\partial_+}(L_+ + c_+) + (R_- + c_-)(L_+ + c_+) + g^{-2}c_+c_-\Big]. \quad (5.158)$$

We can now integrate out $c$, obtaining

$$c_- = \frac{g^2}{2g^2+1}\left(\frac{\partial_-}{\partial_+}L_+ - R_-\right), \qquad c_+ = \frac{g^2}{2g^2+1}\left(\frac{\partial_+}{\partial_-}R_- - L_+\right) \quad (5.159)$$

and

$$S = \frac{1}{4\pi(2g^2+1)}\int d^2x\Big[-\frac{1}{2}R_-\frac{\partial_+}{\partial_-}R_- - \frac{1}{2}L_+\frac{\partial_-}{\partial_+}L_+ + R_-L_+\Big]. \quad (5.160)$$

We can now apply this to (5.152). The couplings are given by

$$g^2 = -\frac{\alpha\beta}{\rho^2 + k^2/2} \Rightarrow 2g^2 + 1 = \frac{\rho^2 + l^2/2}{\rho^2 + k^2/2}, \quad (5.161)$$

$$L_+ = A_+ - \frac{\alpha}{\rho^2 + k^2/2}(\partial_+\xi + Q\rho^2 A_+ + kA_+) = \frac{-\alpha\partial_+\xi + (\rho^2 - kl/2 - \alpha Q\rho^2)A_+}{\rho^2 + k^2/2}, \quad (5.162)$$

$$R_- = A_- - \frac{\beta}{\rho^2 + k^2/2}(\partial_-\xi - Q\rho^2 A_- + kA_-) = \frac{-\beta\partial_-\xi + (\rho^2 + kl/2 + \beta Q\rho^2)A_-}{\rho^2 + k^2/2}. \quad (5.163)$$

Note to this we must sum the remaining terms,

$$S_{nf} = \frac{1}{4\pi}\int d^2x\left[\frac{1}{\rho^2+k^2/2}(\partial_+\xi + Q\rho^2 A_+ + kA_+)(\partial_-\xi - Q\rho^2 A_- + kA_-) + Q^2\rho^2 A_+ A_-\right]. \tag{5.164}$$

We want to show that the effective action is gauge-invariant. There are only three terms we can write that are gauge-invariant,

$$\begin{aligned} S &= \frac{1}{4\pi}\int d^2x\left[C_0 D_+\xi D_-\xi + C_+ D_+\xi\frac{\partial_-}{\partial_+}D_+\xi + C_- D_-\xi\frac{\partial_+}{\partial_-}D_-\xi\right] \\ &= \frac{1}{4\pi}\int d^2x\Big[(C_0+C_++C_-)\partial_+\xi\partial_-\xi + q(C_0+2C_+)A_+\partial_-\xi + q(C_0+2C_-)A_-\partial_+\xi \\ &\qquad + q^2 C_0 A_+A_- + q^2 C_+ A_+\frac{\partial_-}{\partial_+}A_+ + q^2 C_- A_-\frac{\partial_+}{\partial_-}A_-\Big]. \end{aligned} \tag{5.165}$$

Matching these to the effective action we have

$$C_0 + C_+ + C_- = \frac{\rho^2}{(\rho^2+k^2/2)(\rho^2+l^2/2)}, \tag{5.166}$$

$$q(C_0+2C_+) = \frac{2\alpha\rho^2 - Q\rho^2 kl/2 + Q\rho^4}{(\rho^2+k^2/2)(\rho^2+l^2/2)}, \tag{5.167}$$

$$q(C_0+2C_-) = \frac{2\beta\rho^2 - Q\rho^2 kl/2 - Q\rho^4}{(\rho^2+k^2/2)(\rho^2+l^2/2)}, \tag{5.168}$$

$$q^2 C_+ = -\frac{1}{2}\frac{(\rho^2 - kl/2 - \alpha Q\rho^2)^2}{(\rho^2+k^2/2)(\rho^2+l^2/2)}, \tag{5.169}$$

$$q^2 C_- = -\frac{1}{2}\frac{(\rho^2 + kl/2 + \beta Q\rho^2)^2}{(\rho^2+k^2/2)(\rho^2+l^2/2)}, \tag{5.170}$$

$$q^2 C_0 = \frac{Q^2\rho^2 k^2/2 + k^2}{\rho^2+k^2/2} + \frac{(\rho^2 + kl/2 + \beta Q\rho^2)(\rho^2 - kl/2 - \alpha Q\rho^2)}{(\rho^2+k^2/2)(\rho^2+l^2/2)}. \tag{5.171}$$

We can then check that these equations are satisfied for $q = k - \frac{1}{2}Qkl$, as the bosonized action indicated.

Since the factor of $\frac{1}{2}Qkl$ can be a half-integer, it will cause some awkward situations.

However we can track down where it stems from in detail by studying the bosonized action. This is related to the charges the bosonized fields have under the different fermion rotations. We saw that under a chiral rotation of the fermions, the boson has charge $-2$. Meanwhile, under a vector rotation, the T-dual boson has charge 1. This means that under a purely left or purely right rotation, the T-dual boson has charge $1/2$!

It is not clear how to find the correct charge from the methods of Section 5.2.3, which makes it impossible to accurately find the non-perturbative corrections to the dual actions. This will be a target of future work.

## 5.4 Examples

### *5.4.1 Two chirals*

As a first example, take a model with two chiral fields of charges $\pm 1$, which we name $\Phi_\pm$, two Fermi superfields of charge $\pm 1$, $\Gamma_\pm$ with $\bar{D}_+\Gamma_\pm = 0$, and an uncharged Fermi superfield $\Gamma_0$ with $\bar{D}_+\Gamma_0 = m\Phi_+\Phi_-$. This system was studied in Section 4.4.1.

We would like to find the action obtained from dualizing the phases of $\phi_\pm$. Since they appear in the $E$-coupling of $\Gamma_0$, we must redefine $\Gamma_0$ using the chiral superfields before dualizing, to $\Lambda_0 = e^{-\Pi_+ - \Pi_-}\Gamma_0$, so $x_{+0} = x_{-0} = 1$. Other redefinitions, as well as the second-order coupling in the effective action, have enough degrees of freedom to achieve a vanishing Jacobian, if we let

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}. \tag{5.172}$$

Then

$$Q_R - XQ_L = 0, \qquad \mathbb{1} + XX^T - MX^T - XM^T = 0, \tag{5.173}$$

and indeed the Jacobian vanishes.

With no Jacobian, the chiral fields $\Phi_\pm$ are dualized into neutral twisted chirals $Y_\pm$, and the classical dual can be found using the maps above,

$$S_d = \frac{1}{8\pi}\int d^2x d^2\theta^+ \Big[ -\frac{i}{2}\log(Y_\pm+\bar{Y}_\pm)\partial_-(Y_\pm-\bar{Y}_\pm) - (Y_\pm+\bar{Y}_\pm)\bar{\Lambda}_\pm\Lambda_\pm \\ - (Y_+ + \bar{Y}_+)(Y_- + \bar{Y}_-)\bar{\Lambda}_0\Lambda_0 \pm (Y_\pm+\bar{Y}_\pm)V_- \pm iA\partial_-(Y_\pm - \bar{Y}_\pm)\Big]. \tag{5.174}$$

### 5.4.2 $\mathbb{C}P^1 \times \mathbb{C}P^1$

A particularly interesting case to tackle involving non-abelian redefinitions of fields is the deformations of $\mathbb{C}P^1 \times \mathbb{C}P^1$ for which a ring was predicted in [27]. We take a $(2,2)$ field content, but take the Fermi multiplets to have $E$ couplings of the form

$$\bar{D}_+\Gamma_1 = \sqrt{2}\left(\Sigma + \epsilon_1\tilde{\Sigma}\right)\Phi_1, \qquad \bar{D}_+\tilde{\Gamma}_1 = \sqrt{2}\tilde{\Sigma}\tilde{\Phi}_1, \tag{5.175}$$

$$\bar{D}_+\Gamma_2 = \sqrt{2}\left(\Sigma + \epsilon_2\tilde{\Sigma}\right)\Phi_2, \qquad \bar{D}_+\tilde{\Gamma}_2 = \sqrt{2}\tilde{\Sigma}\tilde{\Phi}_2. \tag{5.176}$$

Since the $E$ couplings are not simple monomials, we might try a non-abelian field redefinition. However, this option leads to mixing in the kinetic terms, so it is easier to proceed with an abelian field redefinition, $\Gamma_i = \Phi_i F_i$ and $\tilde{\Gamma}_i = \tilde{\Phi}_i\tilde{F}_i$. This gives

$$\bar{D}_+F_1 = \sqrt{2}\left(\Sigma + \epsilon_1\tilde{\Sigma}\right), \qquad \bar{D}_+\tilde{F}_1 = \sqrt{2}\tilde{\Sigma}, \tag{5.177}$$

$$\bar{D}_+F_2 = \sqrt{2}\left(\Sigma + \epsilon_2\tilde{\Sigma}\right), \qquad \bar{D}_+\tilde{F}_2 = \sqrt{2}\tilde{\Sigma}. \tag{5.178}$$

Since the combined redefinition has no Jacobian, the dual fields do not transform under the gauge symmetry.

The only thing that remains is to find the non-perturbative corrections to the superpotential. It is useful to assign a transformation to the parameters $\epsilon_i$ as well. We have for a general symmetry:

| $\theta$ | $\Phi_i$ | $\tilde{\Phi}_i$ | $\Sigma$ | $\tilde{\Sigma}$ | $\Gamma_i$ | $\tilde{\Gamma}_i$ | $\epsilon_i$ |
|---|---|---|---|---|---|---|---|
| $Q_\theta$ | $Q_i$ | $\tilde{Q}_i$ | $Q_\sigma$ | $\tilde{Q}_\sigma$ | $Q_i + Q_\sigma - Q_\theta$ | $\tilde{Q}_i + Q_\sigma - Q_\theta$ | $Q_\sigma - \tilde{Q}_\sigma$ |

This gives simple transformations for the dual fields,

| $Y_i$ | $\tilde{Y}_i$ | $\Lambda_i$ | $\tilde{\Lambda}_i$ |
|---|---|---|---|
| $-Q_\sigma$ | $-\tilde{Q}_\sigma$ | $Q_\theta - Q_\sigma$ | $Q_\theta - \tilde{Q}_\sigma$ |

The possible combinations are therefore $\Lambda_i e^{-Y_j}$, $\tilde{\Lambda}_i e^{-\tilde{Y}_j}$ and $\epsilon_k \Lambda_j e^{-\tilde{Y}_i}$. Other terms with $1/\epsilon_i$ dependences are also allowed by symmetry, but we know that when $\epsilon_i \to 0$ the non-perturbative corrections to the superpotential are given by

$$W_{np} = \Lambda_i e^{-Y_i} + \tilde{\Lambda}_i e^{-\tilde{Y}_i}, \tag{5.179}$$

so our result must reduce to this in that limit, so $1/\epsilon$ should not be allowed. In general, then, we have the non-perturbative superpotential

$$W_{np} = \Lambda_i e^{-Y_i} + \tilde{\Lambda}_i e^{-\tilde{Y}_i} + \left[\epsilon_1(a\Lambda_1 + b\Lambda_2) + \epsilon_2(a\Lambda_2 + b\Lambda_1)\right](e^{-\tilde{Y}_1} + e^{-\tilde{Y}_2}) \tag{5.180}$$

using the $\tilde{Y}_1 \to \tilde{Y}_2$, $1 \leftrightarrow 2$ symmetries to constrain the possible coefficients. To find $a$ and $b$, we can compare the dual theory with the original theory in the Coulomb branch where $\Sigma$ and $\tilde{\Sigma}$ have large expectation values.

Recall that a coupling $\bar{D}_+ \tilde{\Gamma}_1 = \sqrt{2}\tilde{\Sigma}\tilde{\Phi}_1$ generates, upon integrating out $\tilde{\Gamma}_1$ and $\tilde{\Phi}_1$, a coupling

$$-\frac{\tilde{Q}_1}{8\pi i}\int d^2x \log \Sigma \Upsilon_- + \text{c.c.}. \tag{5.181}$$

Generalizing to our case, in the Coulomb branch of the original theory we find an effective potential

$$W = \frac{1}{8\pi}\int d^2x d\theta^+ \left[i\sum_i \log(\Sigma + \epsilon_i\tilde{\Sigma}) + t\right]\Upsilon_- + \frac{1}{8\pi}\int d^2x d\theta^+ \left[2i\log\tilde{\Sigma} + \tilde{t}\right]\tilde{\Upsilon}_- + \text{c.c.}. \tag{5.182}$$

In the dual theory, on the other hand, we have the perturbative superpotential

$$W_p = \Lambda_i(\Sigma + \epsilon_i\tilde{\Sigma}) + \tilde{\Lambda}_i\tilde{\Sigma}_i. \tag{5.183}$$

We can use $W_p + W_{np}$ to integrate out $\Lambda_i$ and $\tilde{\Lambda}_i$, obtaining the relations

$$\tilde{Y}_i = -\log\tilde{\Sigma}, \tag{5.184}$$

$$\begin{aligned} Y_1 &= -\log\left[\Sigma + \epsilon_1\tilde{\Sigma} + (a\epsilon_1 + b\epsilon_2)\left(e^{-\tilde{Y}_1} + e^{-\tilde{Y}_2}\right)\right] \\ &= -\log\left[\Sigma + ((2a+1)\epsilon_1 + 2b\epsilon_2)\tilde{\Sigma}\right], \end{aligned} \tag{5.185}$$

$$Y_2 = -\log\left[\Sigma + (2a\epsilon_1 + (2b+1)\epsilon_2)\tilde{\Sigma}\right]. \tag{5.186}$$

Plugging this into

$$W = \frac{1}{8\pi}\int d^2x d\theta^+ \left(-iY_1 - iY_2 + t\right)\Upsilon_- + \frac{1}{8\pi}\int d^2x d\theta^+ \left(-i\tilde{Y}_1 - i\tilde{Y}_2 + \tilde{t}\right)\tilde{\Upsilon}_- + \text{c.c.} \tag{5.187}$$

and comparing with (5.182), we see they agree for several different choices of $a$ and $b$, including the choice $a = b = 0$, so there are no additional superpotential corrections from turning on the deformations $\epsilon_i$.

Setting $\tilde{X} = e^{-\tilde{Y}_1}$ and $X = e^{-Y_1}$, we can express these in terms of the usual ring relations,

$$\tilde{X}^2 = e^{i\tilde{t}}, \qquad X\left(X + (\epsilon_2 - \epsilon_1)\tilde{X}\right) = e^{it}. \tag{5.188}$$

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