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RANDOM WALK AMONG BERNOULLI OBSTACLES

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ABSTRACT

The *random walk among Bernoulli obstacles* model describes a system in which particles move randomly in a space containing random traps. More precisely, obstacles are placed independently at sites in $\mathbb{Z}^d$ ($d \geq 2$) with probability $p \in (0, 1)$, and the random walk is killed if it hits one of these obstacles. Of interest to this study is the random walk's behavior, conditional on the event that it survives for a long time.

The most prominent feature of the model is a strong localization effect: the conditioned random walk will be localized in a tiny region. This is closely related to the so-called *Anderson localization* studied in condensed matter physics.

The past 40 years have seen many important contributions to explicating the localization phenomenon, notably Donsker and Varadhan's large deviation results [27] and Sznitman's method of enlargement of obstacles (see e.g. his monograph [75]). However, the understanding of the random walk's path behavior remains incomplete. This is because, in part, the required analysis usually has to reach a level far beyond the large deviation results.

This thesis investigates the behavior of the random walk path in the obstacle model. The contents are based on joint works [24, 25, 23, 21, 22] with Jian Ding, Ryoki Fukushima, and Rongfeng Sun.

Under the quenched law, we will show that the random walk conditional on survival up to time N will first rush to a small ball that is free of obstacles and of volume asymptotically $d \log_{1/p} N$, and then be localized there until time N.

Under the annealed law, it was known that [66, 12, 63] for any $d \geq 2$, the random walk range is contained in a ball of radius $CN^{\frac{1}{d+2}}$, and for $d = 2$ it also contains a ball of asymptotically the same radius. We will show that the latter is also true for $d \geq 3$, and we give a bound for the boundary size of the random walk range.

Under the annealed law with bias, the model undergoes a phase transition from the sub-ballistic regime to the ballistic regime, depending on the size of the bias. We show the following description of the behavior of the random walk in the sub-ballistic regime: the

random walk is contained in a ball of radius $CN^{\frac{1}{d+2}}$, and the endpoint lies near a fixed point on the boundary of this ball.

CHAPTER 1

INTRODUCTION

1.1 Random walk among Bernoulli obstacles and related models

Independently for each $x \in \mathbb{Z}^d$, an obstacle is placed at x with probability $p \in (0,1)$. This generates the so-called *Bernoulli obstacle* configuration and serves as the role of a random environment. Then, we let $S := (S_n)_{n\geq 0}$ be a discrete time simple symmetric random walk on $\mathbb{Z}^d$. The random walk is killed at the moment it hits an obstacle (called hard obstacles), namely, at the stopping time $\tau_{\mathcal{O}} := \min\{n \geq 0 : S_n \in \mathcal{O}\}$ where $\mathcal{O}$ denotes the set of sites occupied by the obstacles.

The most prominent phenomenon in this model is the random walk's strong localization effect; that is, the random walk will be localized in a tiny region, conditional on survival for a long time.

The precise meaning of "conditional" here depends on the measure being considered. There are two types of measures for this model; the localization effect occurs in both of the settings. We denote by $\mathbb{P}$ and $\mathbf{P}$ the probabilities for the random walk and the obstacles, respectively. The first type is the *quenched law* $\mathbf{P}(S \in \cdot \mid \tau_{\mathcal{O}} > N)$, where we first fix an obstacle configuration, and then consider the conditional law for the random walk. The second type is the *annealed law* $\mu_N := \mathbb{P} \otimes \mathbf{P}((S, \mathcal{O}) \in \cdot \mid \tau_{\mathcal{O}} > N)$, where we consider the conditional law in the product measure; this is equivalent to the average over all environments, in the sense that the marginal probability weight for each environment is proportional to the survival probability in that environment.

Several related models should also be mentioned. There is a general framework containing our setting called the parabolic Anderson model, where the obstacles are replaced by independent and identically distributed random potential $\{\omega(x)\}_{x\in\mathbb{Z}^d}$. One is interested in what happens under the measures

$$\frac{\mathbb{E}\otimes\mathbf{E}\left[\exp\left\{\sum_{k=1}^N \omega(S_k)\right\} : (S,\omega)\in\cdot\right]}{\mathbb{E}\otimes\mathbf{E}\left[\exp\left\{\sum_{k=1}^N \omega(S_k)\right\}\right]} \text{ or } \frac{\mathbf{E}\left[\exp\left\{\sum_{k=1}^N \omega(S_k)\right\} : S\in\cdot\right]}{\mathbf{E}\left[\exp\left\{\sum_{k=1}^N \omega(S_k)\right\}\right]}. \tag{1.1.1}$$

Formally, our model corresponds to the case where ω takes value 0 or $-\infty$. More generally, if ω is non-positive, the above weighted measures can be interpreted as the law of random walk killed with probability $1-e^{\omega(x)}$ when it visits x, conditioned to survive until time N. Thus ω plays the role of *repulsive* impurities. On the other hand, positive ω corresponds to *attractive* impurities. There are various localization results depending on the distribution of ω. See, for example, the recent monograph by König [48] for an up-to-date review. The first measure in (1.1.1) is the annealed law, while the second measure is conditioned on the random potential and is the quenched law.

1.2 Background and related works

The first result dates back to Donsker–Varadhan's work [27] which determined the leading exponential order asymptotic of the annealed survival probability, which can be regarded as the "partition function" of a self-attracting polymer model. The main result of [27] reads:

$$\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}}>N)=\exp\left\{-cN^{\frac{d}{d+2}}(1+o(1))\right\}, \tag{1.2.1}$$

where c is a constant depending only on (d,p).

In fact, Donsker–Varadhan studied this problem in the continuum setting first in [26] as the asymptotics of the moment generating function of the Wiener sausage. This corresponds to a space-time continuum analogue of a random walk among Bernoulli obstacles, known as a *Brownian motion among Poissonian obstacles*, where each obstacle takes a fixed shape (e.g., a ball) and the centers of the obstacles follow a homogeneous Poisson point proces on $\mathbb{R}^d$. This model has been studied extensively and most of the results can be found in

Sznitman's celebrated monograph [75]. The core of the method of Sznitman, called *the method of enlargement of obstacles*, is translated to the discrete setting in [6, 9]. Therefore, most of the results in the continuum setting can be converted to the discrete setting. For this reason, in this section we will not explicate in which setting a result has been proved.

The argument of Donsker–Varadhan indicates that the dominant contribution to the partition function comes from the strategy of finding a ball of optimal radius

$$\varrho_{\mathrm{ann},N} := \Big(\frac{2\lambda_1}{d\log(1/p)}\Big)^{\frac{1}{d+2}} N^{\frac{1}{d+2}}, \tag{1.2.2}$$

which is free of obstacles and the random walk is confined in that ball up to time N. It was later proved that this is what happens under the annealed measure [67, 12, 63].

Theorem A. *For any $d \geq 2$, there exists $\epsilon_1 \in (0,1)$ and $\boldsymbol{\mathcal{X}}_N \in \mathbb{Z}^d$ depending only on the obstacle configuration $\mathcal{O}$, such that $\boldsymbol{\mathcal{X}}_N \in B(0, \varrho_{\mathrm{ann},N})$, the ball of radius $\varrho_{\mathrm{ann},N}$ centered at 0, and*

$$\lim_{N\to\infty} \mu_N\big(S_{[0,N]} \subset B(\boldsymbol{\mathcal{X}}_N, \varrho_{\mathrm{ann},N} + \varrho_{\mathrm{ann},N}^{\epsilon_1})\big) = 1. \tag{1.2.3}$$

Furthermore, for $d = 2$ and for any $\epsilon \in (0,1)$,

$$\lim_{N\to\infty} \mu_N\big(B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_{\mathrm{ann},N}) \subset S_{[0,N]}\big) = 1. \tag{1.2.4}$$

These formulations of confinement are in fact far more precise than what Donsker–Varadhan's argument suggests. Their argument is based on the large deviation principle for the empirical measure, and thus it only indicates that the random walk spends most of the time in the ball. An explanation can be found in [13, Section 2.5].

The same problem in the quenched setting was far more challenging; it was studied first in [68, 74] and the leading exponential order asymptotics for the survival probability was first derived in [68], which says $\mathbb{P}$-a.s. as $n \to \infty$

$$\mathbf{P}(\tau > n) = \exp\{-cn(\log n)^{-2/d}(1 + o(1))\}\,, \tag{1.2.5}$$

where c is a constant depending only on (d, p).

The behavior of the random walk under the quenched law is different from that under the annealed law. This is because in a typical environment the random walk will need to search for an "island" on which to remain. In fact, one strategy for obtaining the lower bound in (1.2.5) is to let the random walk travel the minimal possible steps to the largest ball in $[-n^{1+o(1)}, n^{1+o(1)}]^d$ which is free of obstacles and connected to the origin, and then stay in this ball until time n. By a straightforward computation, the radius of such a ball should be asymptotic to

$$\varrho_{\mathrm{que},n} := (\omega_d^{-1} d \log_{1/p} n)^{1/d}\,.$$

This suggests that under the quenched law, the random walk will be localized in a ball of radius asymptotically $\varrho_{\mathrm{que},n}$. To explicate the random walk behavior under the quenched law rigorously, Sznitman [71] proved the so-called "pinning effect" in 1996: There are $n^{o(1)}$ small balls (called islands) of radii at most $\exp\{(\log n)^\chi\}$ with $\chi \in (0, 1)$ such that the Brownian motion will visit one of the islands randomly, and then remain in that island until time n.

Theorem B. *For any $d \geq 2$, there exists $\mathcal{L}_n \subset \mathbb{R}^d$ with $|\mathcal{L}_n| = n^{o(1)}$ depending only on the obstacle configuration $\mathcal{O}$ such that*

$$\lim_{n\to\infty} \mathbf{P}\Big(\bigcup_{x \in \mathcal{L}_n} \Big\{ H(x) \leq n, \sup_{s \in [H(x), n]} |Z_s - x| \leq e^{(\log n)^\chi} \Big\} \;\Big|\; \tau_{\mathcal{O}} > n \Big) = 1 \quad \mathbb{P}\text{-}a.s., \tag{1.2.6}$$

where $Z_\cdot$ is the Brownian motion and $H(x)$ is the hitting time of $\{y \in \mathbb{R}^d : |x - y| \leq 1\}$.

The balls centered at sites in $\mathcal{L}_n$ are "wonderlands" for the random walk. In fact, they are defined as the good local regions where the probability cost for the random walk to be localized in such regions are near the minimum. For this to happen, a delicate balance between the Dirichlet eigenvalue of the island and the cost of reaching it (before being killed) must occur. Further investigation of these islands, especially the so-called one city theorem that with high probability the random walk is actually only localized in the best island, has been intriguing, but challenging. This is because the probability costs of localizing in these islands are similar – they have the same first exponential order.

1.3 Structure of the Thesis

This thesis is divided into three parts:

Quenched law

The main results we will prove under the quenched measure is that, roughly speaking, the conditional random walk will be localized in a single ball of radius asymptotically $\varrho_{\text{que},n}$, and free of obstacles. This will be provided in Chapters 2-4, which are based on [24, 25, 23]. In Chapter 2, we will prove several path properties of the random walk related to localization and show that the random walk will be localized in a region of volume poly-logarithmic in n. In Chapter 3, we establish the so-called one city theorem, which says that the random walk will be localized in a single ball of radius asymptotically $\varrho_{\text{que},n}$. In Chapter 4, we further show that such a ball is free of obstacles, and we derive the limiting distributions of the random walk at any fixed time conditional on survival.

Annealed law

We will study the annealed law in Chapter 5. Under the annealed measure, the confinement property [66, 12, 63] says that for $d \geq 2$, the random walk range is contained in a ball of radius $\varrho_{\text{ann},N} + \varrho_{\text{ann},N}^a$ for some $a \in (0,1)$ conditional on survival up to time N. On the other hand, the random walk also visits each site of such a ball (except a thin layer near the boundary) for $d = 2$. In fact, Bolthausen used the latter statement in his proof of the localization and he conjectured that this remains true for $d \geq 3$.

In Chapter 5, we show that for any $d \geq 2$, the random walk range asymptotically contains a ball of radius $\varrho_{\mathrm{ann},N} - \varrho_{\mathrm{ann},N}^a$ for some $a \in (0,1)$. Thus, the boundary of the random walk range fluctuates on a scale of at most $\varrho_{\mathrm{ann},N}^a$ for $a \in (0,1)$ around a sphere of radius $\varrho_{\mathrm{ann},N}$. Identifying the precise scale of fluctuation is an extremely interesting, but also challenging question. As a step in this direction, we show that its boundary is of size at most $\varrho_{\mathrm{ann},N}^{d-1}(\log \varrho_{\mathrm{ann},N})^b$ for some $b > 0$. These results are based on [23].

Annealed law with bias

We will study the biased random walk under the annealed measure in Chapter 6. This model has been known to undergo a phase transition: for a large bias, the walk is ballistic whereas, for a small bias, it is sub-ballistic. In 1995, Sznitman [69, 70] proved large deviation results for the endpoint distribution of the unbiased random walk at scales larger than $N^{d/(d+2)}$, indicating that the random walk is contained in a ball of radius $O(N^{d/(d+2)})$ in the sub-ballistic phase. Beyond this scale, the probability cost of the endpoint being far away from the starting point is much smaller than the cost for the random walk to survive. Hence, a more delicate analysis is needed.

In this chapter, we have improved Sznitman's results by providing such large deviation principles at scales between $N^{1/(d+2)}$ and $o(N^{d/(d+2)})$. Furthermore, we give a detailed description of the behavior of the random walk in the sub-ballistic regime: the random walk is contained in a ball of radius $CN^{1/(d+2)}$, and the endpoint lies near the unique point on the boundary of this ball, parallel to the bias. These results are based on [21].

CHAPTER 2

PATH PROPERTIES AND POLY-LOGARITHMIC LOCALIZATION UNDER THE QUENCHED LAW

2.1 Introduction

2.1.1 Model and main results

For $d \geq 2$, we consider a random environment where each vertex of $\mathbb{Z}^d$ is occupied by an obstacle independently with probability $1 - p \in (0,1)$. Given this random environment, we then consider a discrete time simple random walk $(S_t)_{t\in\mathbb{N}}$ started at the origin and killed at the first time τ when it hits an obstacle. In this chapter, we study the quenched behavior of the random walk conditioned on survival for a large time, and we prove the following localization result. For convenience of notation, we use $\mathbb{P}$ (and $\mathbb{E}$) for the probability measure with respect to the random environment, and use $\mathbf{P}$ (and $\mathbf{E}$) for the probability measure with respect to the random walk. We will assume $p > p_c(\mathbb{Z}^d)$, the critical threshold for site percolation, and let $\widehat{\mathbb{P}}$ be the conditional measure for the environment given that the origin is in an infinite open cluster. Our main result in this chapter is the following.

Theorem 2.1.1. *For any fixed $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, there exists a constant $c = c(d,p)$ and a collection of $\mathbb{P}$-measurable subsets $D_n \subset \mathbb{Z}^d$ of cardinality at most $(\log n)^c$ and of distance at least $n(\log n)^{-100d^2}$ from the origin, such that the following holds.*

There exists a random time $T \in [0, cn(\log n)^{-2/d}]$ such that

$$\mathbf{P}\Big(T \leq c|S_T|, \big\{S_t : t \in [T,n]\big\} \subset D_n \mid \tau > n\Big) \to 1 \text{ in } \widehat{\mathbb{P}}\text{-probability}. \tag{2.1.1}$$

2.1.2 A word on proof strategy

We will consider small regions whose volume is poly-logarithmic in n, and consider their principal eigenvalues (formally, the principal eigenvalue for a region is the largest eigenvalue for the transition kernel of the random walk killed upon hitting an obstacle or exiting the region). The starting point of our proof is the crucial intuition that localization more or less amounts to the phenomenon that the order statistics for principal eigenvalues in small regions which are within distance n from the origin have non-small spacings near the edge (i.e., near the extremum). Non-small spacings for principal eigenvalues near the edge plays an important role in controlling the number of (which turns out to be at most poly-logarithmic in n) small regions where the random walk will be localized in: Since the spacings are non-small near the edge, this *roughly speaking* implies that any small region which is not one of the best poly-logarithmic in n regions, is strictly suboptimal compared to the best small region. That is to say, the random walk would prefer to stay in the best small region instead of the union of all the suboptimal regions. In other words, the best poly-logarithmic in n regions are the only possible regions for which the random walk would spend a substantial amount of time. This implies the poly-logarithmic localization as desired. Next, we describe our proof strategy in more detail.

Since principal eigenvalues in small regions are more or less i.i.d., such spacings near the edge are determined by the tail behavior of principal eigenvalues: the heavier the tail is, the larger the spacing is near the edge. To implement this intuition, we consider the survival probability after a poly-logarithmic number of steps starting from each vertex in the box of size n — such survival probabilities are closely related to principal eigenvalues in a region of poly-logarithmic diameter (see Lemma 2.3.2). Here we have to choose the number of steps k_n at least logarithmic in n, otherwise we will have too many starting points with survival probability 1. What is important to us, is the fact that by choosing k_n poly-logarithmic in n, we already get a tail on such survival probabilities which is heavy enough for our purpose.

In light of the above discussions, a key task is to prove that the survival probability, viewed as a random variable measurable with respect to the random environment, has non-light tails. This is incorporated in Section 2.2. Note that there are many balls of radius $10^{-3}(\log_{1/p} n)^{1/d}$ which are free of obstacles and thus have atypically high survival probabilities for random walks started inside. Thus, in light of our interest in spacings only near the edge of the order statistics, it suffices to control the right tail of the survival probability that is far away from its typical value. For vertices started from which the survival probabilities in k_n steps are high, we can then *a priori* prove that the random walk spends at least a positive fraction of steps in a set of cardinality $O(\log n)$ near this vertex (see Proposition 2.2.5). This implies that there exists at least one vertex with large local times conditioned on survival in k_n steps. Therefore, by removing the closest obstacle near this vertex we will be able to add a significant fraction of paths and thus significantly improve the survival probability. Finally, by controlling the cardinality of the preimage of this operation of removing an obstacle, we obtain the desired tail behavior on survival probability, as shown in the proof of Proposition 2.2.3.

With Proposition 2.2.3 at hand, we can then show in Lemma 2.3.3 that there are poly-logarithmic many local regions that are candidates for localization, and any other regions have significantly lower survival probabilities compared to the best candidate regions. Combined with well-known tools from percolation theory, a positive fraction of the candidate regions are connected to the origin by open paths of lengths which are linear in their Euclidean distances from the origin. This is the content of Section 2.3.

Using ingredients from Section 2.3, we prove in Lemma 2.4.2 that conditioned on survival the random walk with probability close to 1 visits one of the candidate regions, for the reason that moving to the best reachable candidate region quickly and staying there afterwards yields a much larger survival probability than never visiting any of the candidate regions. Next, we prove in Proposition 2.4.3 that once the random walk reaches a candidate region it is not efficient to move far away without entering another candidate region. Up to this point, we have derived the poly-logarithmic localization as desired.

Finally, it remains to show that the amount of time for the random walk to reach the region in which it is localized afterwards is at most linear in the Euclidean distance of this region from the origin. To this end, we employ the notion of loop erasure for the random walk, and show that the size of the loop erasure is at most linear and that the total size of (erased) loops is also at most linear. This is the content of Section 2.5.

2.1.3 Notation convention

For notation convenience, we denote by $\mathcal{O}$ the collection of all obstacles (sometimes referred to as closed vertices) and $\mathcal{C}(v)$ the open cluster containing v.

For $A \subset \mathbb{Z}^d$, write $\partial A = \{x \in A^c : y \sim x \text{ for some } y \in A\}$, where $x \sim y$ means that x is a neighbor of y and $\partial_i A = \{x \in A : y \sim x \text{ for some } y \in A^c\}$. We denote by $\xi_A = \inf\{t \geq 0 : S_t \notin A\}$ the first time for the random walk to exit from A, and by $\tau_A = \inf\{t \geq 0 : S_t \in A\}$ the hitting time to A. As having appeared earlier, we write $\tau = \tau_{\mathcal{O}}$ for the survival time of the random walk.

For $m \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$, we denote by $S_{[0,m]} = \{S_0, \ldots, S_m\}$ the range of the first m steps of the random walk. A path is a sequence of vertices $\omega = [\omega_0, \omega_1, ..., \omega_{|\omega|}]$ where $|\omega|$ is its length and ω_i, ω_{i+1} are adjacent for $0 \leq i \leq |\omega| - 1$. We say a path is open if all of its vertices are open. For $u, v \in \mathbb{Z}^d$, we say $u \leftrightarrow v$ if there exists an open path that connects u and v. We define the chemical distance by

$$D(u, v) = \inf\{|\omega| : \omega_0 = u, \omega_{|\omega|} = v, \omega \text{ is open}\}\,. \tag{2.1.2}$$

We denote the ℓ^2-distance by $|u - v| = \big(\sum_{i=1}^d (u_i - v_i)^2\big)^{1/2}$. We denote discrete ℓ^2-ball by $B(v, r) = \{x \in \mathbb{Z}^d : |x - v| \leq r\}$.

We write $A_n \lesssim B_n$ if there exits a constant $C > 0$ depending only on (d, p) such that $A_n \leq C B_n$ for all n, and $A_n \gtrsim B_n$ if $B_n \lesssim A_n$. If $A_n \gtrsim B_n$ and $A_n \lesssim B_n$, we write $A_n \asymp B_n$. A list of frequently used notation is compiled in Appendix 2.A.

2.2 Tail behavior of survival probabilities

The main goal of this section is to prove right tail bounds on the survival probability, as incorporated in Proposition 2.2.3 below (see also the discussions below Proposition 2.2.3 for its proof strategy). To this end, for each vertex $v \in \mathbb{Z}^d$, we let

$$X_v = \mathbf{P}^v(\tau > k_n) \tag{2.2.1}$$

be the probability that the random walk started at v survives for at least k_n steps, where k_n is set as (we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x for $x \in \mathbb{R}$)

$$k_n = \begin{cases} \lfloor (\log n)^3 (\log\log n)^2 \rfloor & \text{if } d = 2; \\ \lfloor (\log n)^{4-2/d} \rfloor & \text{if } d \geq 3. \end{cases} \tag{2.2.2}$$

We remark that there is no fundamental reason for our choice of k_n: it has to be poly-logarithmic in n so that it is "small", and it has to be at least substantially larger than $(\log n)^{2/d}$ so that $\max_{v \in B(0,n)} X_v = o(1)$. We made our particular choice of k_n for convenience of analysis. Note that X_v is a $\mathbb{P}$-measurable random variable. As mentioned in the introduction, it suffices to consider the right tail of X_v far away from its typical value. For reasons that will become clear soon, it is convenient to set the threshold as

$$\beta_\chi = \chi^{k_n/(\log n)^{2/d}},$$

where χ is a positive constant to be selected.

Lemma 2.2.1. *There exists $\chi = \chi(d,p) > 0$ such that*

$$\mathbb{P}(X_v \geq \beta_\chi) \gtrsim n^{-d+1}\,. \tag{2.2.3}$$

Proof. Since $|B(v,r)| \asymp r^d$, there exists $c_{d,p}$ depending only on (d,p) such that

$$\mathbb{P}(B(c_{d,p}(\log n)^{1/d}, v) \subset \mathcal{O}^c) \gtrsim n^{-d+1}. \tag{2.2.4}$$

When all vertices in $B(c_{d,p}(\log n)^{1/d}, v)$ are open, the random walk with initial point v will survive in k_n steps if it stays in $B(c_{d,p}(\log n)^{1/d}, v)$. Next, we estimate the probability for the random walk to stay in a ball. This is a fairly simple and standard argument, which we give only for completeness. It is clear that there exists $c = c(d) > 0$ such that

$$\min_{x \in B(v,r)} \mathbf{P}(S_t \in B(v,2r) \text{ for } 0 \le t < r^2, S_{r^2} \in B(v,r)) \ge c$$

for all $r \ge 1$. Now, set $r = \lfloor 2^{-1} c_{d,p}(\log n)^{1/d} \rfloor$. By having the random walk to stay within $B(v, 2r)$ and to end in $B(v,r)$ for every block of r^2 steps, we obtain

$$\mathbf{P}^v(S_t \in B(v,2r), t = 0, 1, ..., k_n) \ge c^{k_n/(r^2)+1}. \tag{2.2.5}$$

Now, we can choose $\chi = \chi(d,p) > 0$ small enough so that $c^{k_n/(r^2)+1} \ge \beta_\chi$. Combining (2.2.4) and (2.2.5), we complete the proof of the lemma. □

Remark 2.2.2. A sharp version of (2.2.5) with the exact large deviation rate was derived in [27], but we do not need such sharp estimate here.

Lemma 2.2.1 justifies our choice of considering the right tail of X_v only above the threshold β_χ for some small $\chi > 0$, since there is at least one site $v \in B(0,n)$ with $X_v \ge \beta_\chi$ and thus the extremal level set is above β_χ. In what follows, we always choose $\chi > 0$ such that (2.2.3) holds (and it will become clear that eventually we will choose a $\chi > 0$ depending only on (d,p)).

Proposition 2.2.3. *For all $\chi > 0$ and $\beta \ge \beta_\chi$, we have*

$$\mathbb{P}(X_v \ge \beta) \le c_{2,1} k_n^d \mathbb{P}(X_v \ge c_{2,2} \beta \log n) + n^{-(2d+1)}. \tag{2.2.6}$$

where $c_{2,1}, c_{2,2}$ *are positive constants only depends on* (d, p, χ).

The proof of Proposition 2.2.3 consists of two main ingredients:

(a) The random walk spends a positive fraction of steps in a subset of size $O(\log n)$ conditioned on survival (in the case when the survival probability is at least β_χ). Thus, there exists at least one vertex x which is visited for many times on average conditioned on survival.

(b) If we change the environment by removing the closest obstacle around x we will increase the survival probability substantially, and this will lead to the desired tail estimate (2.2.6).

We now describe how we prove (a), i.e., to control the support of the local times for the random walk.

- We first show in Proposition 2.2.5 that conditioned on survival (in the case when the survival probability is at least β_χ), the random walk spends at least $k_n/2$ steps on c-good vertices (c.f. Definition 2.2.4).
- Next we show in Lemma 2.2.8 that each c-good vertex has to be contained in a "connected" component of ϵ-fair boxes (c.f. Definition 2.2.6) of volume at least $\Omega(\log n)$.
- Since ϵ-fair only occurs with small probability by Lemma 2.2.7, we use a percolation type of argument in Lemma 2.2.9 to show that c-good occurs very rarely, and then in Lemma 2.2.10 that the number of c-good vertices is $O(\log n)$.

The "environment changing" argument as in (b) is carried out in the *Proof of Proposition 2.2.3*, which itself is divided into three steps. One can see the discussions at the beginning of *Proof of Proposition 2.2.3* for an outline of its implementation.

2.2.1 Support of local times

This subsection is devoted to proving (a), following the three steps outlined above.

Definition 2.2.4. *A site v in $\mathbb{Z}^d$ is called c-good if*

$$\mathbf{P}^v(\tau > \lfloor (\log n)^{2/d} \rfloor) \geq c \,. \tag{2.2.7}$$

We first show that the random walk tends to spend many steps on c-good vertices.

Proposition 2.2.5. *For any $\chi > 0$, there exists $c = c(\chi) > 0$ such that for all environments:*

$$\mathbf{P}(\tau > k_n, |\{t \leq k_n : S_t \text{ is a } c\text{-good site}\}| \leq k_n/2) \leq \beta_\chi/2 \,.$$

Proof. Let $\zeta_0 = -1$ and for $m \geq 1$ recursively define

$$\zeta_m = \inf\{t \geq \zeta_{m-1} + (\log n)^{2/d}; S_t \text{ is not } c\text{-good site}\} \,.$$

Write $j_n = \lfloor k_n/(2(\log n)^{2/d}) \rfloor$. By strong Markov property, we get that

$$\mathbf{P}(\zeta_{j_n} \leq k_n < \tau) \leq \mathbf{P}(S_{[\zeta_m, \zeta_m + \lfloor (\log n)^{2/d} \rfloor]} \text{ is open } \forall 1 \leq m \leq j_n - 1) \leq c^{j_n - 1} \,.$$

Note that on the event $E = \{\tau > k_n, |\{t \leq k_n : S_t \text{ is a } c\text{-good site}\}| \leq k_n/2\}$, we have $\zeta_{j_n} \leq k_n < \tau$. Thus, we have $\mathbf{P}(E) \leq c^{j_n - 1}$. Choosing an appropriate $c = c(\chi)$ completes the proof of the proposition. □

Next we control the size of c-good vertices. For this purpose, we consider disjoint boxes

$$K_r(x) := \{y \in \mathbb{Z}^2 : \|x - y\|_\infty \leq r\} \tag{2.2.8}$$

for $x \in (v + (2r+1)\mathbb{Z}^d)$ and $r > 0$ to be selected.

Definition 2.2.6. *A box $K_r(x)$ is called ϵ-fair if there exist $u \in K_r(x)$ such that*

$$\mathbf{P}^u(\tau \geq r^2 \ \textit{or}\ \tau > \xi_{K_{2r}(x)}) \geq \epsilon\,. \tag{2.2.9}$$

In what follows, we carry out the last two steps in the outline of proving (a): we show in Lemma 2.2.7 that the ϵ-fair boxes are rare provided $r = r(\epsilon)$ large enough, and in Lemma 2.2.8 we show that a c-good point has to be in a cluster consisting of $\Omega(\log n)$ many ϵ-fair boxes. Combining these two lemmas, we can then bound the probability for a vertex to be c-good as in Lemma 2.2.9, which leads to Lemma 2.2.10 on the $O(\log n)$ bound for the number of c-good vertices in a box of radius k_n.

Lemma 2.2.7. *For any $\epsilon > 0$, there exists $r = r(\epsilon, d, p)$ such that*

$$\mathbb{P}(K_r(x) \ \textit{is}\ \epsilon\textit{-fair}) \leq \epsilon\,. \tag{2.2.10}$$

Proof. Let y be an arbitrary vertex in $K_r(x)$. By the independence of the environment and random walk, we have

$$\mathbb{E}\left[\mathbf{P}^y(|S_{[0,r^2]}| > r^{1/2}, \tau > r^2)\right] = \mathbb{P} \otimes \mathbf{P}^y(|S_{[0,r^2]}| > r^{1/2}, \tau > r^2) \leq p^{r^{1/2}-1}\,. \tag{2.2.11}$$

and

$$\mathbb{E}\left[\mathbf{P}^y(\tau > \xi_{K_{2r}(x)})\right] = \mathbb{P} \otimes \mathbf{P}^y(\tau > \xi_{K_{2r}(x)}) \leq p^r\,. \tag{2.2.12}$$

In addition, note that in every r steps the random walk has a positive probability to visit at least $r^{1/2}$ distinct sites. Thus, there exists a constant $c > 0$ such that

$$\mathbf{P}^y(|S_{[0,r^2]}| \leq r^{1/2}) \leq e^{-cr}\,.$$

Combined with (2.2.11) and (2.2.12) it implies that

$$\sum_{y\in K_r(x)} \mathbb{P}\Big(\mathbf{P}^y(\tau \geq r^2 \text{ or } \tau > \xi_{K_{2r}(x)}) \geq \epsilon\Big) \leq (2r+1)^d \epsilon^{-1}(p^{r^{1/2}-1} + p^r + e^{-cr})\,.$$

Choosing $r = r(\epsilon, d, p)$ large enough completes the proof of the lemma. □

We will always choose

$$\epsilon = \min(c/2, (2d)^{-3^{d+1}}) \text{ and } r = r(d, \epsilon, p) \tag{2.2.13}$$

such that (2.2.10) holds. We fix $v \in \mathbb{Z}^d$ and define the adjacency relation for ϵ-fair boxes $\{K_r(x),\ x \in (v + (2r+1)\mathbb{Z}^d)$ to be the following:

$$K_r(x) \sim K_r(y) \iff \exists x' \in K_r(x), y' \in K_r(y) \ s.t.\ x' \sim y'. \tag{2.2.14}$$

We next show that in order for a vertex v to be c-good, it requires v to be in a cluster consisting of $\Omega(\log n)$ many ϵ-fair boxes — here a cluster is a connected component where each "vertex" corresponds to an ϵ-fair box and the neighboring relation is given by (2.2.14). Thus, c-good is a rare event. To this end, let L_v be the subset of $(v + (2r+1)\mathbb{Z}^d)$ such that $\{K_r(x),\ x \in L_v\}$ is the cluster of ϵ-fair boxes in $B(v, h(\log n)^{1/d})$ which contains v.

Lemma 2.2.8. *For any $c > 0$ and ϵ satisfying* (2.2.13)*, there exist $l = l(d, c, \epsilon)$ and $h = h(d, c, \epsilon)$ such that v is not a c-good vertex if $|L_v| \leq l \log n$.*

Proof. For any $d \geq 2$, there exists a constant $\theta = \theta(d)$ such that

$$\sup_{x\in\mathbb{Z}^d} \mathbf{P}^v(S_t = x) \leq \theta t^{-d/2} \text{ for all } t \geq 1.$$

Let $m = \lfloor(\log n)^{2/d}\rfloor - r^2$. For all $0 < \Delta < 1$, there exists a constant $h = h(\Delta)$ such that

$$\mathbf{P}^v(S_{[0,m]} \subset B(v, 2^{-1}h(\log n)^{1/d})) \geq 1 - \Delta\,. \tag{2.2.15}$$

In addition, if $|L_v| \le l \log n$ we have

$$\begin{aligned}\mathbf{E}^v\left[\sum_{i=1}^m \mathbb{1}_{\{S_i \in \cup_{x\in L_v} K_r(x)\}}\right] &= \sum_{i=1}^m \mathbf{P}^v(S_i \in \cup_{x\in L_v} K_r(x)) \\ &\le \lfloor \Delta m\rfloor + \sum_{i=\lfloor \Delta m\rfloor+1}^m |\cup_{x\in L_v} K_r(x)|\theta i^{-d/2} \\ &\le \Delta m + \theta(m-\lfloor \Delta m\rfloor)(2r+1)^d |L_v| \lfloor \Delta m+1\rfloor^{-d/2} \\ &\le \Delta m + \theta \Delta^{-d/2}(2r+1)^d 2^{d/2} m l\,,\end{aligned}$$

where the last inequality holds when $\log n \ge (2r)^d$. Setting

$$0<\Delta<(c-\epsilon)/(3-3\epsilon) \text{ and } l = 2^{-d/2}\Delta^{1+d/2}\theta^{-1}(2r+1)^{-d}\,,$$

we get that

$$\mathbf{E}^v\left[\sum_{i=1}^m \mathbb{1}_{\{S_i \in \cup_{x\in L_v} K_r(x)\}}\right] \le 2\Delta m\,.$$

This implies that $\mathbf{P}^v(\xi' > m) \le 2\Delta$, where $\xi' = \inf\{t\ge 0 : S_t \notin \cup_{x\in L_v} K_r(x)\}$. Combined with (2.2.15), it yields that with probability at least $1-3\Delta$ the event $\{S_{[0,m]} \subset B(v, 2^{-1}h(\log n)^{1/d}), \xi' \le m\}$ occurs. Further, on this event, we have that $S_{\xi'}$ is not in an ϵ-fair box, and thus $\mathbf{P}^{S_{\xi'}}(\tau > r^2) \le \epsilon$. Therefore,

$$\mathbf{P}^v(\tau > \lfloor (\log n)^{2/d}\rfloor) \le 1-(1-3\Delta)(1-\epsilon) < c\,. \qquad \square$$

Lemma 2.2.9. *There exists $\delta = \delta(c,d,p) > 0$ such that*

$$\mathbb{P}(|L_v| > l\log n) \le n^{-\delta}.$$

Proof. For all $x \in L_v$, the number of points $y \in L_v$ such that $K_{2r}(x)\cap K_{2r}(y) \ne \varnothing$ is at most 3^d. Therefore, there exists a subset I of L_v, such that $|I| \ge |L_v|/3^d$ and that

$K_{2r}(x) \cap K_{2r}(y) = \varnothing$ for different $x, y \in L_v$. Hence events $\{K_r(x) \text{ is } \epsilon\text{-fair}\}$ for $x \in I$ are independent of each other. In addition, the number of connected components of $|L_v|$ boxes is no more than $(2d)^{2|L_v|}$ — this is a fairly standard combinatorial computation and one could see, e.g., [80] for a reference. Therefore,

$$\mathbb{P}(|L_v| > l \log n) \le \sum_{j \ge l \log n} (2d)^{2j} \cdot \epsilon^{j/3^d} \le 2n^{l \log(4d^2 \epsilon^{1/3^d})} .$$

Combined with Lemma 2.2.8 and (2.2.13), it completes the proof of the lemma. □

Lemma 2.2.10. *For all $v \in \mathbb{Z}^d$ and $\kappa > 0$,*

$$\mathbb{P}(K_{k_n}(v) \text{ contains more than } \kappa \log n \text{ } c\text{-good points}) \le n^{-\kappa\delta/(4h)^d} .$$

Proof. Write $q = 2\lceil h(\log n)^{1/d} + 2r\rceil$. Let $K_i = K_{k_n}(v) \cap (i + q\mathbb{Z}^d)$ for $i \in \{1, \ldots, q\}^d$. For any fixed i, the events $\{|L_v| \ge l \log n\}$ for $v \in K_i$ are independent. Thus, for large n

$$\begin{aligned} &\mathbb{P}(K_{k_n}(v) \text{ contains more than } \kappa \log n \text{ } c\text{-good points}) \\ \le &\sum_{i \in \{1,\ldots,q\}^d} \mathbb{P}(|\{v \in K_i : |L_v| \ge l \log n\}| \ge \kappa/(3h)^d) \\ \le &q^d \mathbb{P}(\mathrm{Bin}(k_n^d, n^{-\delta}) \ge \kappa/(3h)^d) , \end{aligned}$$

where the last inequality follows from Lemma 2.2.9 and $\mathrm{Bin}(k_n^d, n^{-\delta})$ is a binomial random variable with probability $n^{-\delta}$ and k_n^d trials. At this point, the desired bound follows from a standard large deviation estimate for Binomial random variables. □

Lemma 2.2.11. *For $v \in \mathbb{Z}^d$, let $G_v = G_v(\alpha, \kappa)$ be the event that*

(1) For every $u \in K_{k_n}(v)$, there exists a closed site within distance $\alpha(\log n)^{1/d}$.

(2) The number of c-good points in $K_{k_n}(v)$ is at most $\kappa \log n$.

Then there exist $\kappa, \alpha > 0$ depending only on (c, d, p) such that

$$\mathbb{P}(G_v) \geq 1 - n^{-(2d+1)} .$$

Proof. Since $|B(x, \alpha(\log n)^{1/d})| \geq (\alpha/d)^d \log n$, we have that

$$\begin{aligned}&\mathbb{P}(\text{There exists } x \in K_{k_n}(v) \text{ such that } B(x, \alpha(\log n)^{1/d}) \cap \mathcal{O} = \varnothing)\\&\quad \leq (2k_n + 1)^d p^{(\alpha/d)^d \log n} .\end{aligned}$$

This addresses the first requirement for the event G_v. The second requirement for G_v is addressed in Lemma 2.2.10. Altogether, we conclude that $\mathbb{P}(G_v) \geq 1 - n^{-(2d+1)}$ with appropriate choices of α and κ, as desired. □

2.2.2 Environment changing argument

We now prove Proposition 2.2.3.

Proof of Proposition 2.2.3. We choose $c = c(\chi)$ as in Proposition 2.2.5. The proof of Proposition 2.2.3 consists of three steps as follows:

1. For each c-good point x, removing its closest obstacle would enlarge the survival probability by a factor at least $\ell_x (\log n)^{2/d-2}(\log\log n)^{-1}$ (where the $\log\log n$ terms only appears when $d = 2$). Here we denote by $\ell_x = \mathbf{E}\left[\sum_{t=0}^{k_n} \mathbb{1}_{\{S_t = x\}} \mid \tau > k_n\right]$ the expected number of visits to x conditioned on survival.

2. Combining Step 1 with Proposition 2.2.5, we show that there exists at least one c-good point x such that removing the closest obstacle near x enlarge the survival probability by a factor of order $\log n$.

3. The operation of removing the closest obstacle has preimage with multiplicity bounded by $O(k_n^d)$, which leads to the term of $c_{2,1} k_n^d$ in (2.2.6).

We now carry out the proof steps outlined as above.

Step 1. For each c-good site $x \in K_{k_n}(v)$, let x' be one of the closed sites nearest to x (with respect to the Euclidean distance) and let x^* be one of the neighbors of x' such that $|x - x^*| < |x - x'|$ (so x^* is open). Let $b = |x - x^*|$, $\mathring{B}_b(x) = B(x, b) \setminus \{x\}$. For $u, v \in \mathbb{Z}^d$, $A \subset \mathbb{Z}^d$ and $r \geq 1$, we define

$$\begin{aligned} \mathcal{K}_{A,r}(u, v) &= \{\omega = [\omega_0, \dots, \omega_r] : \omega_0 = u, \omega_r = v, \omega_i \in A \text{ for } 1 \leq i \leq r-1\}, \\ \mathcal{K}_A(u, v) &= \cup_{r=1}^{\infty} \mathcal{K}_{A,r}(u, v), \quad \mathcal{K}_{A,r}(u) = \cup_{v \in \mathbb{Z}^d} \mathcal{K}_{A,r}(u, v). \end{aligned} \tag{2.2.16}$$

The key in Step 1 is to construct a collection of paths which does not hit any obstacle except x', such that the collection is large in comparison with the number of paths which does not hit any obstacle. To this end, we let W_x be the collection of paths of form $\omega^1 \oplus \pi^1 \oplus [x^*, x', x^*] \oplus \pi^2 \oplus \omega^2$ (here $\oplus$ denotes for the natural concatenation for paths), where $\omega^1, \pi^1, \pi^2, \omega^2$ are ranging over all choices satisfying

- $\omega^1 \in \mathcal{K}_{\mathcal{O}^c}(v, x), \omega^2 \in \cup_{y \in \mathcal{O}^c} \mathcal{K}_{\mathcal{O}^c}(x, y), |\omega^1| + |\omega^2| = k_n$;
- $\pi^1 \in \mathcal{K}_{\mathring{B}_b(x)}(x, x^*), \pi^2 \in \mathcal{K}_{\mathring{B}_b(x)}(x^*, x)$.

In order to complete Step 1, we need to verify the following two ingredients (which we check below).

(a) We prove that if $\gamma \in W_x$, then the above decomposition into four concatenated parts is unique, and there exists no $\tilde{\gamma} \in W_x$ which is a continuation of γ (meaning, that can be written in the form $\gamma \oplus \pi$ for a non trivial π).

(b) We use this observation to obtain a lower bound on the probability of observing a path in W_x in the first steps of the random walk.

<u>**Step 1 (a)**</u>. As x' is visited only once, the separation between π^1 and ω^1 and that between π^2 and ω^2 must correspond respectively to the last visit of x before visiting x and the first

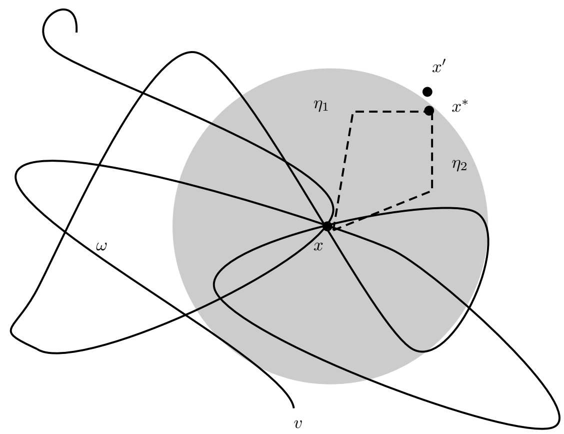

Figure 2.1: A schematic figure for (2.2.17). The gray ball is $B(x, b)$ which is open. The black curve is the random walk path ω, which visits x three times in this picture.

visit of x after visiting x'. This yields uniqueness of the decomposition. The condition $|\omega^1| + |\omega^2| = k_n$ implies that the continuation of a path in W_x cannot belong to W_x.

Step 1 (b). We will abuse the notation by writing

$$\mathbf{P}^v(W) = \mathbf{P}^v([S_0, S_1, ..., S_{|\omega|}] = \omega \text{ for some } \omega \in W)$$

where W is a collection of paths. Then

$$\mathbf{P}^v(W_x) = \sum_{\omega \in \mathcal{K}_{\mathcal{O}^c, k_n}(v)} (2d)^{-k_n - 2} \mathbf{P}^v(\mathcal{K}_{\mathring{B}_b(x)}(x, x^*)) \cdot \mathbf{P}^v(\mathcal{K}_{\mathring{B}_b(x)}(x^*, x)) \cdot \sum_{i \geq 0}^{k_n} \mathbb{1}_x(\omega_i) , \quad (2.2.17)$$

where we have the factor $\sum_{i\geq 0}^{k_n} \mathbb{1}_x(\omega_i)$ because we can insert an $\pi^1 \oplus [x^*, x', x^*] \oplus \pi^2$ at each visit to x along the random walk path (see Figure 2.1). In light of Lemma 2.2.11, we choose α, κ so that $\mathbb{P}(G_v) \geq 1 - n^{-2d-1}$. And we suppose G_v occurs, hence $b \leq \alpha(\log n)^{1/d}$. By [53, Lemma 6.3.7], we get that

$$\mathbf{P}^v(\mathcal{K}_{\mathring{B}_b(x)}(x, x^*)) = \mathbf{P}^v(\mathcal{K}_{\mathring{B}_b(x)}(x^*, x)) \geq \begin{cases} C(\log n)^{-1/2}(\log\log n)^{-1} & \text{if } d = 2; \\ C(\log n)^{1/d-1} & \text{if } d \geq 3 . \end{cases}$$

where $C > 0$ only depends on p and d. In fact, [53, Lemma 6.3.7] gives an estimate on the harmonic measure when the random walk exits a discrete ℓ^2-ball. Combined with the observation that such harmonic measure is unchanged conditioned on the random walk not returning to the starting point, this yields the preceding inequality. Thus, we have

$$\mathbf{P}(\mathcal{K}_{\mathring{B}_b(x)}(x, x^*)) \geq 2^{-1} C k_n^{-1/2} \log n\,.$$

Therefore, (recall that $\ell_x = \mathbf{E}[\sum_{t=0}^{k_n} \mathbb{1}_{\{S_t = x\}} \mid \tau > k_n])$ we obtain that

$$\mathbf{P}(W_x) \geq (2d)^{-3} C^2 k_n^{-1} (\log n)^2 \ell_x X_v\,. \tag{2.2.18}$$

The preceding inequality can be immediately translated into a bound on the survival probability after removing the obstacle at x'.

Step 2. Recall Proposition 2.2.5 and recall that $X_v = \mathbf{P}^v(\tau > k_n)$. We see that on the event $\{X_v \geq \beta_\chi\}$

$$\sum_{x \in K_{k_n}(v): x \text{ is } c\text{-good}} \ell_x \geq k_n/4\,.$$

At the same time, on the event $G_v \cap \{X_v \geq \beta_\chi\}$ there are no more than $\kappa \log n$ c-good points in $K_{k_n}(v)$. Altogether, it follows that there exists a c-good point x in $K_{k_n}(v)$ such that $\ell_x \geq k_n/(4\kappa \log n)$. Combined with (2.2.18), it yields that there exists $x' \in K_{k_n}(v)$ such that for $c_{2,2} = c_{2,2}(p, d) > 0$

$$\mathbf{P}^v(\tau' > k_n) \geq c_{2,2} X_v \log n\,, \text{ where } \tau' = \tau_{\mathcal{O}\setminus\{x'\}} = \inf\{t : S_t \in \mathcal{O} \setminus \{x'\}\}\,. \tag{2.2.19}$$

Step 3. Now, for $\beta \geq \beta_\chi$, let E_v be the event that {there exists a closed site $x' \in K_{k_n}(v)$ such that $\mathbf{P}^v(\tau' > k_n) \geq c_{2,2}\beta \log n$}. We have shown that $G_v \cap \{X_v \geq \beta\} \subset E_v$, as in (2.2.19). Furthermore, note that $x' \in K_{k_n}(v)$ provided $W_x \neq \varnothing$. Thus, all environments where E_v occurs can be obtained by closing one open site in $K_{k_n}(v)$ in one of the environments where

$X_v \geq c_{2,2}\beta \log n$. Write $\mathcal{O}^n = \mathcal{O} \cap B(v, k_n)$ — we restrict the consideration of $\mathcal{O}$ into a finite set $B(v, k_n)$ so that each realization of $\mathcal{O}^n$ has positive probability. Let A, B be collections of subsets of $B(v, k_n)$ such that $E_v = \{\mathcal{O}^n \in A\}$, and $\{X_v \geq c_{2,2}\beta \log n\} = \{\mathcal{O}^n \in B\}$. Let

$$\mathcal{S} = \{(x, O) \in K_{k_n}(v) \times B : x \notin O\}.$$

We define the map $\varphi \colon \mathcal{S} \to 2^{B(v,n)}$ by

$$\varphi(x, O) = O \cup \{x\}.$$

Then for any $(x, O) \in \mathcal{S}$, $\mathbb{P}(\mathcal{O}^n = \varphi(x, O)) = \frac{1-p}{p}\mathbb{P}(\mathcal{O}^n = O)$. Thus,

$$\mathbb{P}(\mathcal{O}^n \in \varphi(\mathcal{S})) \leq \frac{1-p}{p} \sum_{(x,O)\in\mathcal{S}} \mathbb{P}(\mathcal{O}^n = O) \leq \frac{1-p}{p} |K_{k_n}(v)| \mathbb{P}(\mathcal{O}^n \in B).$$

By definition, we have $A \subset \varphi(\mathcal{S})$. Hence,

$$\mathbb{P}(E_v) = \mathbb{P}\{\mathcal{O}^n \in A\} \leq (2k_n + 1)^d \frac{1-p}{p} \mathbb{P}\{X_v \geq c_{2,2}\beta \log n\}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}(X_v \geq \beta) &\leq \mathbb{P}(G_v \cap \{X_v \geq \beta\}) + \mathbb{P}(G_v^c) \\
&\leq \mathbb{P}(E_v) + \mathbb{P}(G_v^c) \\
&\leq (2k_n + 1)^d \frac{1-p}{p} \mathbb{P}\{X_v \geq c_{2,2}\beta \log n\} + n^{-2d-1} \\
&\leq c_{2,1} k_n^d \mathbb{P}\{X_v \geq c_{2,2}\beta \log n\} + n^{-2d-1},
\end{aligned}$$

where $c_{2,1} > 0$ is a constant depending only on (d, p). □

2.3 Candidate regions for localization

Recalling our discussion on proof strategy in Section 2.1.2, in order to show localization it is important to show that all except poly-logarithmic many small regions will be suboptimal compared to some "best" small region. Here, to measure the level of "goodness" for small regions, we will use principal eigenvalues, which in turn is closely related to survival probabilities for random walk as we show in Lemma 2.3.1. Thus, it is natural to introduce the following quantiles which measure goodness from the perspective of survival probabilities (in k_n steps):

$$\begin{aligned} p_0 &:= \sup\{\beta \geq 0, \mathbb{P}(X_v \geq \beta) \geq n^{-d} k_n^{2d} \log n\}, \\ p_\alpha &:= p_0/(c_{2,2} \log n)^\alpha \quad \text{for } \alpha \geq 0, \end{aligned} \tag{2.3.1}$$

where $c_{2,2}$ is chosen such that (2.2.6) holds. Denote

$$\mathcal{U}_\alpha := \{v \in \mathbb{Z}^d : X_v \geq p_\alpha\}\,. \tag{2.3.2}$$

Thus, we have that $\mathcal{U}_0 = \{v \in \mathbb{Z}^d : X_v \geq p_0\}$. Heuristically, the hope is that if α is large enough, all regions outside $\mathcal{U}_\alpha$ will be suboptimal compared to $\mathcal{U}_0$. In order to make this intuition rigorous, it turns out more convenient to consider principal eigenvalues for small regions. To this end, we introduce the following definition.

Definition 2.3.1. *For any site $v \in \mathbb{Z}^d$, we let $\mathcal{C}_R(v)$ be the connected component in $B(v, R) \backslash \mathcal{O}$ that contains v for $R = k_n (\log n)^2$, and let $\lambda(v)$ be the principal eigenvalue of $P|_{\mathcal{C}_R(v)}$ where $P|_{\mathcal{C}_R(v)}$ is the transition matrix of simple random walk on $\mathbb{Z}^d$ restricted to $C_R(v)$.*

In the next lemma, (as announced earlier) we will relate the survival probability X_v to the principal eigenvalue $\lambda(v)$, and thus relating the survival probability in k_n steps (i.e., X_v) to the survival probability to arbitrary number of steps.

Lemma 2.3.2. *For any* $m \geq 1$,

$$\lambda(v)^m \leq \max_x \mathbf{P}^x(\xi_{\mathcal{C}_R(v)} > m) \leq (2R)^{d/2} \lambda(v)^m \,. \tag{2.3.3}$$

In particular,

$$\big(X_v/(2R)^{d/2}\big)^{1/k_n} \leq \lambda(v) \leq \max_{x \in \mathcal{C}_R(v)} (X_x)^{1/k_n}. \tag{2.3.4}$$

We set $\alpha_1 = 3d$ and $\alpha_2 = 4d$. By definition there is a clear separation on the level of goodness (in terms of survival probabilities in k_n steps) for typical regions in $\mathcal{U}_0$, $\mathcal{U}_{\alpha_1}$ and $\mathcal{U}_{\alpha_2}$ where $\mathcal{U}_0$ contains the "most desirable" regions. The level p_{α_1} will be the threshold of candidate regions, while the spacing between p_0 and p_{α_1} is used in Lemma 2.4.2 and the spacing between p_{α_2} and p_{α_1} is used in Lemma 2.4.5. By Lemma 2.3.2, such separation can be translated to that in terms of principal eigenvalues (which then controls survival probabilities for arbitrary number of steps). This motivates the following definition:

$$\mathcal{D}_\lambda := \{v \in \mathbb{Z}^d : \lambda(v) > \lambda\} \text{ and } \mathcal{D}_* := \{v \in \mathbb{Z}^d : \lambda(v) \geq p_{\alpha_1}^{1/k_n}\} \,. \tag{2.3.5}$$

With preceding definitions, $\mathcal{D}_*$ represents candidate regions for localization: indeed, we will show in Section 2.4 that random walk will eventually be localized in neighborhoods that are close to $\mathcal{D}_*$ (see (2.3.11) for a formal definition for the union of islands for localization). The remaining section is devoted to proving a number of structural properties for $\mathcal{D}_*$ (via structure properties of $\mathcal{U}$.), as listed below.

- We prove Lemma 2.3.3 by a crucial application of Proposition 2.2.3, which in turn guarantees that the number of islands in $\mathcal{U}_\alpha$ are at most poly-logarithmic in n — this is important for bounding $|D_n|$.

- We show in Lemma 2.3.4 that vertices in $\mathcal{U}_\alpha$ is either close or far away from each other — this implies that it is costly for the random walk to travel from one good region to another (this is important in the proof of Lemma 2.4.5 later).

- We show in Lemmas 2.3.5 and 2.3.6 (whose proof uses results in percolation theory) that there exists vertices in $\mathcal{U}_0$ which are connected to the origin by open paths with lengths which are linear in their Euclidean distances from the origin — this implies a lower bound on $\mathbf{P}(\tau > n)$ by letting the random walk travel to one vertex in $\mathcal{U}_0$ quickly and stays around it afterwards (see (2.4.10)).

- We use Lemma 2.3.2 to deduce structural properties on $\mathcal{D}_\cdot$ from $\mathcal{U}_\cdot$ — these are incorporated in Corollary 2.3.7 and Lemma 2.3.8.

The proofs of Lemma 2.3.2 and the following four lemmas are postponed to Section 2.3.1.

Lemma 2.3.3. *We have*

$$n^{-d}k_n^{2d}\log n \le \mathbb{P}(X_v \ge p_0) \le n^{-d}k_n^{4d}\,,$$
$$\text{and} \quad \mathbb{P}(X_v \ge p_\alpha) \le n^{-d}k_n^{(\alpha+4)d}\,.$$

Lemma 2.3.4. *For any $\alpha \in \mathbb{N}^*$, with $\mathbb{P}$-probability tending to one there exist no $u, v \in \mathcal{U}_\alpha \cap B(0, 2n)$ such that $2k_n \le |u - v| \le nk_n^{-2(\alpha+5)}$.*

Lemma 2.3.5. *Conditioned on the origin being in an infinite cluster, with $\mathbb{P}$-probability approaching one*

$$\mathcal{U}_0 \cap \mathcal{C}(0) \cap B(0, n/k_n) \neq \varnothing\,. \tag{2.3.6}$$

Lemma 2.3.6. *Let $D(u, v)$ be defined as in (2.1.2). For $p > p_c(\mathbb{Z}^d)$, there exists a constant $\rho > 0$ which only depends on (d, p) such that the following holds with $\mathbb{P}$-probability tending to one. For all $u, v \in B(0, 2n)$*

$$\text{either } \mathcal{C}(u) = \mathcal{C}(0) \text{ or } |\mathcal{C}(u)| \le (\log n)^3\,, \tag{2.3.7}$$

$$D(u, v)\mathbb{1}_{\{u \leftrightarrow v\}} \le \rho \max(|u - v|, (\log n)^3)\,. \tag{2.3.8}$$

Corollary 2.3.7. *We have that*

(1) With $\mathbb{P}$-probability tending to one, for any $v \in B(0,2n) \cap (\mathcal{U}_{\alpha_2} \cup \mathcal{D}_{p_{\alpha_2}^{1/k_n}})$,

$$\big(B(v, nk_n^{-14d}) \setminus B(v,3R)\big) \cap \big(\mathcal{U}_{\alpha_2} \cup \mathcal{D}_{p_{\alpha_2}^{1/k_n}}\big) = \varnothing\,.$$

(2) $k_n^{2d} n^{-d} \le \mathbb{P}(v \in \mathcal{D}_*) \le k_n^{\alpha+6} n^{-d}$.

(3) $p_{\alpha_2}^{1/k_n} \ge 1 - \chi/(\log n)^{2/d}$ for some constant χ depending only on (d,p).

Proof. It follows from (2.3.4) that

$$\begin{aligned} &\{v \in \mathcal{D}_{p_{\alpha_2}^{1/k_n}}\} \subset \cup_{u \in B(v,R)} \{u \in \mathcal{U}_{\alpha_2}\}\,, \\ &\{v \in \mathcal{U}_0\} \subset \{v \in \mathcal{D}_*\} \subset \cup_{u \in B(v,R)} \{u \in \mathcal{U}_{\alpha_1}\}\,. \end{aligned} \tag{2.3.9}$$

Combining with Lemmas 2.3.4 and 2.3.3 yields (1) and (2). Combining Lemma 2.3.3 and Lemma 2.2.1 gives (3). □

The following structural property for $\mathcal{D}_*$ will be useful.

Lemma 2.3.8. *With $\mathbb{P}$-probability tending to one, there exists a subset $\mathsf{V} \subset \mathcal{D}_* \cap \mathcal{C}(0) \cap B(0,2n)$ such that*

$$\begin{aligned} &\lambda(v) = \max\{\lambda(u) : u \in B(v,3R)\} \quad \forall\, v \in \mathsf{V}\,; \\ &\mathcal{D}_* \cap \mathcal{C}(0) \cap B(0,2n) \subset \cup_{v \in \mathsf{V}} B(v,3R)\,; \\ &B(v, nk_n^{-14d}), v \in \mathsf{V} \cup \{0\} \textit{ are disjoint}\,. \end{aligned} \tag{2.3.10}$$

Proof of Lemma 2.3.8. Combining (2.3.9) and Lemmas 2.3.4, 2.3.3 yields the desired result. □

We will prove in Section 2.4 that random walk will eventually be localized in the union of the following islands for some constant $\iota > 0$ to be selected:

$$D_n = \bigcup_{v \in \mathsf{V}} B(v, (\log n)^{\iota} k_n)\,. \tag{2.3.11}$$

Proof of Theorem 2.1.1: volume of the islands. Combining Corollary 2.3.7 (2) and the Markov inequality implies that with $\mathbb{P}$-probability tending to one, $|\mathcal{D}_*\cap B(0,2n)|\leq(\log n)^{100d}$. Then by Lemma 2.3.8,

$$|D_n| \leq (2(\log n)^{\iota} k_n)^d |\mathcal{D}_* \cap B(0,2n)| \leq (\log n)^{\iota+200d},$$

and $D_n \cap B(0, n(\log n)^{-100d^2}) = \varnothing$. □

2.3.1 Proof of Lemmas 2.3.2, 2.3.3, 2.3.4, 2.3.5 and 2.3.6

Proof of Lemma 2.3.2. Recall that $P|_{\mathcal{C}_R(v)}$ is the transition matrix restricted to $\mathcal{C}_R(v)$. Let $\mathbb{1}_x = (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^{C_R(v)}$ be the vector which takes value 1 only in the coordinate corresponding to the site x, and let $\mathbf{1} = (1,1,\dots,1) \in \mathbb{R}^{C_R(v)}$. We have

$$\mathbf{P}^x(\xi_{\mathcal{C}_R(v)} > m) = \mathbb{1}_x^{\mathsf{T}}(P|_{\mathcal{C}_R(v)})^m\mathbf{1} \leq \lambda(v)^m\sqrt{|\mathcal{C}_R(v)|} \leq (2R)^{d/2}\lambda(v)^m.$$

Let μ be the eigenvector of $P|_{\mathcal{C}_R(v)}$ corresponding to $\lambda(v)$, then

$$\sum_{x\in\mathcal{C}_R(v)} \mu(x)\mathbf{P}^x(\xi_{\mathcal{C}_R(v)} > m) = \mu^{\mathsf{T}}(P|_{\mathcal{C}_R(v)})^m\mathbf{1} = \lambda(v)^m \sum_{x\in\mathcal{C}_R(v)} \mu(x)\,.$$

Hence there exists $x \in \mathcal{C}_R(v)$ such that $\mathbf{P}^x(\xi_{\mathcal{C}_R(v)} > m) \geq \lambda(v)^m$. □

Proof of Lemma 2.3.3. By Lemma 2.2.1, since $\beta_\chi \leq n\beta_{\chi/2}$ we can choose $\chi = \chi(d,p)$ such that for large n

$$\mathbb{P}(X_v \geq n\beta_\chi) \geq n^{-d+1}\,. \tag{2.3.12}$$

Thus, by definition of p_0 we see that for any fixed α and sufficiently large n,

$$p_0 \geq n\beta_\chi \text{ and hence } p_\alpha \geq \beta_\chi\,. \tag{2.3.13}$$

This allows us to apply Proposition 2.2.3 with $\beta = p_0$, yielding that

$$\mathbb{P}(X_v \geq p_0) \leq c_{2,1} k_n^d n^{-d} k_n^{2d} \log n + n^{-2d-1} \leq n^{-d} k_n^{4d}.$$

The left continuity of $\mathbb{P}(X_v \geq x)$ gives

$$\mathbb{P}(X_v \geq p_0) \geq n^{-d} k_n^{2d} \log n.$$

Therefore, for $\beta \in (\beta_\chi, p_0)$ and sufficiently large n, (2.2.6) implies

$$\mathbb{P}(X_v \geq \beta) \leq 2c_{2,1} k_n^d \mathbb{P}(X_v \geq c_{2,2} \beta \log n)\,.$$

Since $p_0 = (c_{2,2} \log n)^\alpha p_\alpha$, applying the above inequality α times yields

$$\mathbb{P}(X_v \geq p_\alpha) \leq n^{-d} k_n^{(\alpha+4)d}.$$

□

Proof of Lemma 2.3.4. For any $u, v \in B(0, 2n)$ such that $|u - v| \geq 2k_n$, the events $\{X_v \geq p_\alpha\}$ and $\{X_u \geq p_\alpha\}$ are independent (since in k_n steps, the random walk will not exit the ball of radius k_n). Hence Lemma 2.3.3 yields

$$\mathbb{P}(X_v \geq p_\alpha, X_u \geq p_\alpha) \leq n^{-2d} k_n^{2(\alpha+4)d}.$$

Then we complete the proof by enumerating all possible $(u, v) \in B(0, 2n) \times B(0, 2n)$ such that $2k_n \leq |u - v| \leq n k_n^{-2(\alpha+5)}$. □

Proof of Lemma 2.3.6. By [17, Theorem 3] and [38] (see also [46, Corollary 3]) there exists $C > 0$ which only depends on p such that for all $m \geq 1$

$$\mathbb{P}(|\mathcal{C}(v)| = m) \leq \mathrm{e}^{-Cm^{1/2}}.$$

Then for any $v \in \mathbb{Z}^d$

$$\mathbb{P}((\log n)^3 \leq |\mathcal{C}(v)| < \infty) \leq \sum_{m \geq (\log n)^3} \mathrm{e}^{-Cm^{1/2}} = o(n^{-d}). \tag{2.3.14}$$

This proves (2.3.7).

By [7, Theorem 1], we know that for u, v with $|u - v| \geq (\log n)^3$ (the main arguments in [7] were written for bond percolation, but as the authors suggest one can verify that the proof adapts to site percolation with minimal changes)

$$\mathbb{P}(u \leftrightarrow v, D(u, v) > \rho|u - v|) \leq e^{-C|u-v|} \leq n^{-C(\log n)^2},$$

Hence the event E_n that $D(u, v)\mathbb{1}_{\{u \leftrightarrow v\}} \leq \rho|u - v|$ for all $u, v \in B(0, 3n)$ with $|u - v| \geq (\log n)^3$ has probability tending to one.

On the event E_n, we consider any $u, v \in B(0, 2n)$ such that $u \leftrightarrow v$ with $|u-v| < (\log n)^3$. In the case $\mathcal{C}(u) = \mathcal{C}(0)$, we see from the connectivity that there exists $w \in \mathcal{C}(0)$ such that $\min(|w - v|, |w - u|) \in [(\log n)^3, (\log n)^3 + 2]$. Then by triangle inequality $\max(|w - v|, |w - u|) \leq 4(\log n)^3$. Hence

$$D(u, v) \leq D(u, w) + D(v, w) \leq 5\rho(\log n)^3.$$

In the case that $\mathcal{C}(u) \neq \mathcal{C}(0)$, It follows from (2.3.7) that with $\mathbb{P}$-probability tending to one

$$D(u, v) \leq |\mathcal{C}(u)| \leq (\log n)^3.$$

The proof is completed by adjusting the value of ρ. □

Remark 2.3.9. The work [7] improves earlier results of [35, 38], where the main objective of [35] is to understand certain parabolic problems for the Anderson model with heavy potential.

Proof of Lemma 2.3.5. We say a site $v \in \mathbb{Z}^d$ is reachable if the connected component in $B(v, k_n) \backslash \mathcal{O}$ that contains v is of size at least k_n. Then by (2.3.7), conditioned on origin being in an infinite cluster, with $\mathbb{P}$-probability approaching one all reachable sites are in $\mathcal{C}(0)$. Let $\mathcal{U}_0^* = \{v \in \mathcal{U}_0 : v \text{ is reachable}\}$, it suffices to prove

$$\mathbb{P}(\mathcal{U}_0^* \cap B(0, n/k_n) = \varnothing) \to 0\,.$$

To verify this, we first observe that for each site $v \in \mathbb{Z}^d$ in an infinite cluster, it connects to $\partial B(v, k_n)$ by an open path. Hence the connected component in $B(v, k_n) \backslash \mathcal{O}$ that contains v has at least k_n vertices. As a result,

$$\mathbb{P}(v \text{ is reachable}) \geq \theta(p)\,.$$

Now by FKG inequality and Lemma 2.3.3,

$$\mathbb{P}(v \in \mathcal{U}_0^*) \geq \mathbb{P}(v \in \mathcal{U}_0) \cdot \mathbb{P}(v \text{ is reachable}) \geq \theta(p) n^{-d} k_n^{2d} \log n.$$

Since events $\{v \in \mathcal{U}_0^*\}$ for $v \in (2k_n + 1)\mathbb{Z}^d$ are independent of each other, we have

$$\begin{aligned} \mathbb{P}(\mathcal{U}_0^* \cap B(0, n/k_n) = \varnothing) &\leq \left(1 - \theta(p) n^{-d} k_n^{2d} \log n\right)^{(2d^{-1}\lfloor n/k_n \rfloor (2k_n+1)^{-1} - 1)^d} \\ &\leq n^{-c} \end{aligned}$$

for some constant $c = c(d, p)$. This completes the proof of the lemma. $\square$

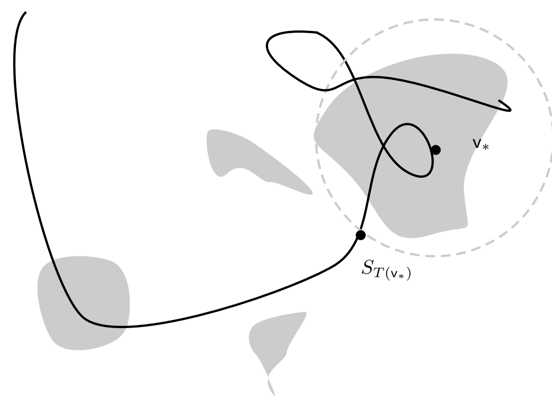

Figure 2.2: The shaded regions are islands in $\mathcal{D}_*$. The site v_* is the representative of the best island that the random walk ever visits.

2.4 Endpoint localization

In this section, we prove that conditioned on survival for a long time the random walk will be localized in an island (which we refer to as target island below) in D_n, where the target island is chosen randomly (from all the poly-logarithmic many islands in D_n) with respect to the random walk. In addition, the target island will be a neighborhood of v_* (see Definition 2.4.1 below and Figure 2.2 for an illustration), which is the best island that the random walk ever visits.

Definition 2.4.1. *On event $\{S_{[0,n]} \cap \mathcal{D}_* \neq \varnothing\}$, we let v_* be the unique site in V (defined in Lemma 2.3.8) such that*

$$S_{t_*} \in B(\mathsf{v}_*, 3R) \quad \text{with} \;\; t_* := \min\{0 \le t \le n : \lambda(S_t) = \max_{0 \le i \le n} \lambda(S_i)\}\,.$$

Otherwise, we set $\mathsf{v}_* := 0$, as in such case the random walk never visits any candidate regions.

For $v \in \mathsf{V}$ and constant $\iota > 0$ to be selected, we define the hitting time of a neighborhood of v by

$$\mathsf{T}(v) = \min\{0 \le t \le n : |S_t - v| \le (\log n)^\iota\}\,. \tag{2.4.1}$$

The endpoint localization is proved by combining the following two ingredients: the random walk will visit $\mathcal{D}_*$ with high probability (as shown in Lemma 2.4.2); the random walk will stay in a neighborhood of v_* after getting close to v_* (as shown in Proposition 2.4.3).

Lemma 2.4.2. *Conditioned on the event that the origin is in an infinite open cluster,*

$$\mathbf{P}(\tau_{\mathcal{D}_*} \le n \mid \tau > n) \to 1 \quad \textit{in } \mathbb{P}\textit{-probability.}.$$

Proposition 2.4.3. *For ι sufficiently large, conditioned on the event that the origin is in an infinite open cluster, we have that*

$$\mathbf{P}(S_{[\mathsf{T}(\mathsf{v}_*),n]} \subset B(\mathsf{v}_*, (\log n)^\iota k_n) \mid \tau > n) \to 1 \quad \textit{in } \mathbb{P}\textit{-probability.} \tag{2.4.2}$$

Proof of Theorem 2.1.1: endpoint localization. Set ι to be a sufficiently large constant as in Proposition 2.4.3. Combining Lemma 2.4.2 and Proposition 2.4.3 gives

$$\mathbf{P}\big(S_n \in D_n \mid \tau > n\big) \to 1\,.$$

□

In order to prove Lemma 2.4.2 and Proposition 2.4.3, we first provide upper bound on the probability for the random walk to survive and also avoid $\mathcal{U}_\alpha$ (or respectively $\mathcal{D}_\lambda$) as in Lemmas 2.4.4 (respectively Lemma 2.4.5). Provided with Lemma 2.4.5, Lemma 2.4.2 follows from a lower bound on $\mathbf{P}(\tau > n)$, which is substantially larger than (the upper bound on) $\mathbf{P}(\tau_{\mathcal{O}\cup\mathcal{D}_*} > n)$. The proof of Proposition 2.4.3 is yet more complicated, which will employ a careful application of Lemmas 2.4.4 and 2.4.5 together with Lemma 2.3.4.

2.4.1 Upper bounds on survival probability

Lemma 2.4.4. *For all $\alpha \ge 0$ and $m \ge 1$, we have that for all $v \in \mathbb{Z}^d$*

$$\mathbf{P}^v(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > m) \le (2R)^{d/2} p_\alpha^{m/k_n}\,.$$

Proof. Write $m = jk_n + i$ where $0 \le i < k_n$ and $j \in \mathbb{N}^*$. By the strong Markov property, we see that for all $v \in \mathbb{Z}^d$

$$\begin{aligned}
\mathbf{P}^v(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > m) &= \sum_{x \in (\mathcal{U}_\alpha \cup \mathcal{O})^c} \mathbf{P}^v(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > m, S_{m-k_n} = x) \\
&\le \sum_{x \in (\mathcal{U}_\alpha \cup \mathcal{O})^c} \mathbf{P}^v(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > m - k_n, S_{m-k_n} = x)\mathbf{P}^x(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > k_n) \\
&\le \mathbf{P}^v(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > m - k_n) \cdot p_\alpha.
\end{aligned}$$

Applying the preceding inequality repeatedly, we get that

$$\mathbf{P}^v(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > m) \le p_\alpha^j \max_{x \in (\mathcal{U}_\alpha \cup \mathcal{O})^c} \mathbf{P}^x(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > i)\,. \tag{2.4.3}$$

Write $\mathcal{C}_{\alpha,R}(x) = \mathcal{C}_R(x) \setminus \mathcal{U}_\alpha$ and let $\lambda_{\alpha,x}$ be the principal eigenvalue of $P|_{\mathcal{C}_{\alpha,R}(x)}$. Then, by the same arguments as for Lemma 2.3.2, we deduce that $\lambda_{\alpha,x} \le (\max_x \mathbf{P}^x(\tau_{C_{\alpha,R}(x)} > k_n))^{1/k_n} \le p_\alpha^{1/k_n}$ and then

$$\mathbf{P}^x(\tau_{\mathcal{U}_\alpha \cup \mathcal{O}} > i) \le (2R)^{d/2} p_\alpha^{i/k_n}\,.$$

Combined with (2.4.3), this completes the proof of the lemma. ☐

Lemma 2.4.5. *With $\mathbb{P}$-probability tending to 1 as $n \to \infty$ the following holds. For any $v \in B(0,n)$ and $\lambda > (p_{\alpha_1}/\log n)^{1/k_n}$ and for all $1 \le m \le n$, we have*

$$\mathbf{P}^v(\tau_{\mathcal{O} \cup \mathcal{D}_\lambda} > m) \le R^{3d} \lambda^m. \tag{2.4.4}$$

Here we recall that $\mathcal{D}_\lambda = \{u \in \mathbb{Z}^d : \lambda(u) > \lambda\}$ and $k_n \ge (\log n)^2$.

Proof. We consider two scenarios for the random walk. In the first scenario, the random walk never enter the region $\mathcal{U}_{\alpha_2}$. In this case, since for any $u \in \mathcal{U}_{\alpha_2}^c$ we have $X_u \le p_{\alpha_2}$, this yields an efficient upper bound. In the second scenario, the random walk enters $\mathcal{U}_{\alpha_2}$ and possibly exits an enlarged neighborhood around $\mathcal{U}_{\alpha_2}$ and re-enters for multiple times. In this case, we are fighting with the following two factors.

- The enumeration on the possible times for exiting and re-entering is large (see (2.4.5)).
- When we estimate the survival probability, we repeatedly use the relation between X_v the principal eigenvalues $\lambda(v)$ as in Lemma 2.3.2, and each time we use such a relation we accumulate a certain error factor. As a result, such error factors will grow in the number of times for the random walk to exit an enlarged neighborhood around $\mathcal{U}_{\alpha_2}$ and then re-enter $\mathcal{U}_{\alpha_2}$.

In order to beat the preceding two factors, we note that every time the random walk exits an enlarged neighborhood of $\mathcal{U}_{\alpha_2}$ and re-enters $\mathcal{U}_{\alpha_2}$, it has to travel for a fair amount of steps outside of $\mathcal{U}_{\alpha_2}$, due to Lemma 2.3.4. This leads to a decrement on the survival probability. Such probability decrement, also growing in the number of "exiting and re-entering", is sufficient to beat the enumeration factor as well as the error factors accumulated when switching between X_v and $\lambda(v)$.

In what follows, we carry out the proof in details following preceding discussions. We define stopping times

$$a_0 = 0 \text{ and } a_j = \inf\{t \geq b_{j-1} : S_t \notin B(S_{b_{j-1}}, R) \text{ or } t = m\} \text{ for } j \geq 1\,,$$

$$b_j = \inf\{t \geq a_j : S_t \in \mathcal{U}_{\alpha_2} \text{ or } t = m\} \text{ for } j \geq 0\,.$$

For all $j \geq 0$ we have $S_t \in \mathcal{C}_R(S_{b_j})$ for $t \in [b_j, a_{j+1} - 1]$ and $S_t \notin \mathcal{U}_{\alpha_2}$ for $t \in [a_j, b_j - 1]$ (see Figure 2.3). By Lemma 2.3.4, we see that with $\mathbb{P}$-probability approaching 1 we have that

$$a_j - b_{j-1} \geq R \text{ and } b_j - a_j \geq R - 2k_n \text{ for } 1 \leq j \leq L - 1,$$

where $L = \inf\{j \geq 0 : b_j \geq m\}$. We have $L \leq m/R + 1$. We denote $\Theta_0 = \{(0, m)\}$ and for

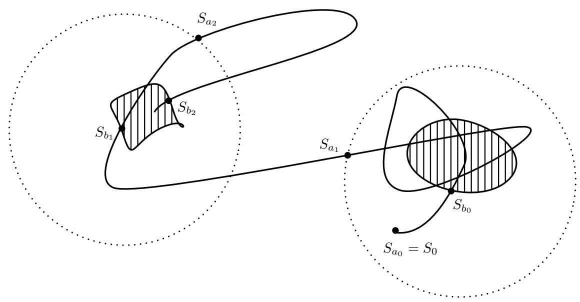

Figure 2.3: The shaded region is $\mathcal{U}_{\alpha_2}$. Two balls are of radius R. The left ball is centered at S_{b_1} and the right one is centered at S_{b_0}. Random walk stays in $B(S_{b_j}, R)$ during time $[b_j, a_{j+1}-1]$ and stays in $\mathcal{U}_{\alpha_2}^c$ during time $[a_j, b_j - 1]$.

$1 \leq l \leq m/R + 1$ define

$$\Theta_l = \{(x,y) \in \mathbb{Z}^{(l+1)} \times \mathbb{Z}^{(l+1)} : x_0 = 0, x_l \leq y_l = m,$$
$$x_j < y_j < x_{j+1}, y_j - x_j \geq R - 2k_n \text{ for } j = 0, 1, ..., l-1\}.$$

Here Θ_l is the collection of all possible entrance times (to $\mathcal{U}_{\alpha_2}$) and exit times (from a ball of radius R centered at the entrance point). Then a straightforward combinatorial computation gives that

$$|\Theta_l| \leq \binom{m}{2l} \leq m^{2l}\,. \tag{2.4.5}$$

For any $m, l \geq 1$ and $(x,y) \in \Theta_l$, we get from (2.3.3) and Lemma 2.4.4 that

$$\begin{aligned}
&\mathbf{P}^v(\tau_{\mathcal{O}\cup\mathcal{D}_\lambda} > m, L = l, a_j = x_j, b_j = y_j \text{ for } 1 \leq j \leq l)\\
\leq &\prod_{j=0}^{l} (2R)^d (p_{\alpha_2})^{(y_j - x_j - 1)/k_n} \lambda^{(x_j - y_{j-1} - 1)}\\
\leq &\lambda^m \prod_{j=0}^{l} (p_{\alpha_2})^{-2/k_n} (2R)^d \Big(\frac{\log n \cdot p_{\alpha_2}}{p_{\alpha_1}}\Big)^{(y_j - x_j)/k_n}.
\end{aligned}$$

Note that $y_j - x_j \geq R - 2k_n \geq k_n \log n$ for $j = 1, 2, ..., l-1$. Hence for large n (recalling

$R = k_n(\log n)^2$ and $p_{\alpha_2} \geq \chi^{k_n/(\log n)^{2/d}}$ as in (2.3.13))

$$\Big(\frac{p_{\alpha_1}}{\log n \cdot p_{\alpha_2}}\Big)^{(y_j - x_j)/k_n} \geq n^{10} \text{ for } 1 \leq j \leq l-1\,.$$

Therefore for $l \geq 2$ and sufficiently large n

$$\mathbf{P}^v(\tau_{\mathcal{O}\cup\mathcal{D}_\lambda} > m, L = l, a_j = x_j, b_j = y_j \text{ for } 0 \leq j \leq l) \leq \lambda^m n^{-7(l-1)}.$$

Summing over $l = 2, 3, \ldots, \lfloor m/R \rfloor + 1$ and applying (2.4.5), we obtain that for $m \leq n$

$$\mathbf{P}^v(\tau_{\mathcal{O}\cup\mathcal{D}_\lambda} > m, L \geq 2) \leq \lambda^m n^{-4}\,. \tag{2.4.6}$$

In addition, for $l = 1$ we have

$$\mathbf{P}^v(\tau_{\mathcal{O}\cup\mathcal{D}_\lambda} > m, L = 1) \leq \lambda^m ((p_{\alpha_2})^{-2/k_n}(2R)^d)^2 \leq 2(2R)^{2d}\lambda^m\,, \tag{2.4.7}$$

and by Lemma 2.4.4 we have

$$\mathbf{P}^v(\tau_{\mathcal{O}\cup\mathcal{D}_\lambda} > m, L = 0) \leq \mathbf{P}^v(\tau_{\mathcal{O}\cup\mathcal{U}_{\alpha_2}} > m) \leq (2R)^{d/2}\lambda^m\,. \tag{2.4.8}$$

Combining (2.4.6), (2.4.7) and (2.4.8) we completes the proof of the lemma. □

2.4.2 Proof of Lemma 2.4.2 and Proposition 2.4.3

Lemma 2.4.6. *The following holds with $\mathbb{P}$-probability tending to 1. For all $u, v, w \in B(0, 2n)$ such that $u \leftrightarrow v, v \leftrightarrow w$ and for any positive number t such that $t - |u-w|_1$ is even, we have*

$$\mathbf{P}^u(S_t = w, \tau > t) \geq (2d)^{-\rho(|u-v|+|v-w|+R)}\lambda(v)^t\,. \tag{2.4.9}$$

Proof. This is an immediate consequence of Lemma 2.3.6 and (2.3.3). □

Proof of Lemma 2.4.2. We first see that reaching $\mathcal{U}_0$ quickly and staying there afterwards gives a lower bound on $\mathbf{P}(\tau > n)$. By Lemmas 2.3.5 and (2.3.8), there exists a site $v_f \in \mathcal{U}_0$ such that $D(0, v_f) \leq \rho n/k_n$. It follows from (2.3.4) that

$$\lambda(v_f) \geq (X_{v_f}/(2R)^{d/2})^{1/k_n} \geq (p_0/(2R)^{d/2})^{1/k_n} .$$

Then Lemma 2.4.6 implies

$$\mathbf{P}(\tau > n) \geq (2d)^{-\rho(n/k_n+1+R)}\lambda(v_f)^{n/k_n} \geq ((2d)^{-2\rho}p_0/(2R)^{d/2})^{n/k_n}. \tag{2.4.10}$$

By Lemma 2.4.5, we get that $\mathbf{P}(\tau_{\mathcal{D}_*\cup\mathcal{O}} > n) \leq R^{3d}p_{\alpha_1}^{n/k_n}$. Altogether, we conclude that

$$\mathbf{P}(\tau_{\mathcal{D}_*} > n \mid \tau > n) = \frac{\mathbf{P}(\tau_{\mathcal{D}_*\cup\mathcal{O}} > n)}{\mathbf{P}(\tau > n)} \leq \frac{R^{3d}p_{\alpha_1}^{n/k_n}}{((2d)^{-2\rho}p_0(2R_n)^{-d/2})^{n/k_n}} = o(1). \qquad \square$$

Proof of Proposition 2.4.3. Note that

$$\begin{aligned}&\mathbf{P}(S_{[\mathsf{T}(\mathsf{v}_*),n]} \not\subset B(\mathsf{v}_*, (\log n)^\iota k_n), \tau > n)\\ =&\sum_{v\in\mathsf{V}}\mathbf{P}(\mathsf{v}_* = v, S_{[T(v),n]} \not\subset B(v, (\log n)^\iota k_n), \tau > n)\,.\end{aligned} \tag{2.4.11}$$

Consider $v \in \mathsf{V} \cap B(0, n)$. Since $T(v)$ is a stopping time for any fixed v, by strong Markov property, we have that

$$\begin{aligned}&\mathbf{P}(\mathsf{v}_* = v, S_{[T(v),n]} \not\subset B(v, (\log n)^\iota k_n), \tau > n)\\ \leq&\mathbf{E}[\mathbb{1}_{\{\tau>n-T(v)\}}\mathbf{P}^{S_{T(v)}}(\xi_{B(v,(\log n)^\iota k_n)} \leq m, \tau_{\mathcal{O}\cup\mathcal{D}_{\lambda(v)}} > m)|_{m=n-T(v)}]\,.\end{aligned} \tag{2.4.12}$$

We now bound the second term on the right hand side of (2.4.12). Suppose that the random walk escapes the ball $B(\mathsf{v}_*, (\log n)^\iota k_n)$ during time $[\mathsf{T}(v), n]$, then there exists a time interval

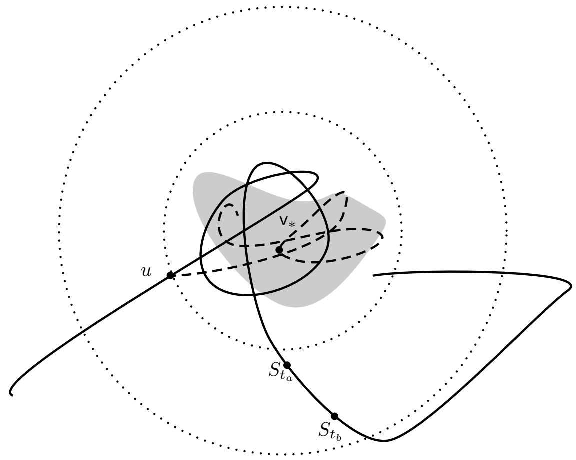

Figure 2.4: The big ball is $B(\mathsf{v}_*, (\log n)^\iota k_n)$ and the small ball is $B(\mathsf{v}_*, (\log n)^\iota)$. If the random walk ever escapes the big ball (solid curve), then it must go through the annulus $B(\mathsf{v}_*, (\log n)^\iota k_n) \setminus B(\mathsf{v}_*, 3R)$ during time $[t_a, t_b]$. But since survival probability in such annulus is very low, the random walk would prefer to stay in the ball (dotted curve).

$[t_a, t_b] \subset [\mathsf{T}(v), n]$, such that

$$t_b - t_a = \lceil (\log n)^\iota k_n/2 \rceil \text{ and } S_t \in B(\mathsf{v}_*, (\log n)^\iota k_n) \setminus B(\mathsf{v}_*, 3R) \text{ for } t \in [t_a, t_b]$$

(see Figure 2.4). Since Corollary 2.3.7 (1) implies that $(B(\mathsf{v}_*, (\log n)^\iota k_n) \setminus B(\mathsf{v}_*, 3R)) \cap \mathcal{U}_{\alpha_2} = \varnothing$, we get

$$S_t \in (\mathcal{O} \cup \mathcal{U}_{\alpha_2})^c \text{ for } t \in [t_a, t_b]\,.$$

Therefore, for all $v \in \mathsf{V}$, $u \in \partial_i B(v, (\log n)^\iota) \cap \mathcal{C}(0)$ and $m \in \mathbb{N}^*$,

$$\begin{aligned}
&\mathbf{P}^u(\xi_{B(v,(\log n)^\iota k_n)} \le m, \tau_{\mathcal{O} \cup \mathcal{D}_{\lambda(v)}} > m) \\
&\quad \le \sum_{t_a=0}^{m - \lceil (\log n)^\iota k_n/2 \rceil} \mathbf{P}^u(\tau_{\mathcal{O} \cup \mathcal{D}_{\lambda(v)}} > m, S_{[t_a,t_b]} \subset (\mathcal{O} \cup \mathcal{U}_{\alpha_2})^c)\,.
\end{aligned}$$

Then we get from Lemmas 2.4.4 and 2.4.5 that

$$\begin{aligned}\mathbf{P}^u(\tau_{\mathcal{O}\cup\mathcal{D}_{\lambda(v)}} > m, S_{[t_a,t_b]} \subset (\mathcal{O}\cup\mathcal{U}_{\alpha_2})^c) &\le (2R)^{10d}\lambda(v)^{m-2-(t_b-t_a)} p_{\alpha_2}^{(t_b-t_a)/k_n} \\ &\le (2R)^{10d}\lambda(v)^{m-2}(\log n)^{-(\log n)^\iota/2}\,.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\mathbf{P}^u(\xi_{B(v,(\log n)^\iota k_n)} \le m, \tau_{\mathcal{O}\cup\mathcal{D}_{\lambda(v)}} > m) \\ &\quad \le (\log n)^{-(\log n)^\iota/4}(2d)^{-4\rho(\log n)^\iota}\lambda(v)^m \le (\log n)^{-(\log n)^\iota/4}\mathbf{P}^u(\tau > m)\,,\end{aligned}$$

where we have used the fact which follows from Lemma 2.4.6 that

$$\mathbf{P}^u(\tau > m) \ge (2d)^{-4\rho(\log n)^\iota}\lambda(v)^m, \quad \forall\, u \in \partial_i B(v,(\log n)^\iota) \cap \mathcal{C}(0)\,.$$

Together with (2.4.12), we get

$$\mathbf{P}(\mathsf{v}_* = v, S_{[T(v),n]} \not\subset B(v,(\log n)^\iota k_n), \tau > n) \le (\log n)^{-(\log n)^\iota/4}\mathbf{P}(\tau > n)\,.$$

Combined with (2.4.11), this completes the proof of the proposition by summing over all $v \in \mathsf{V} \cap B(0,n)$. □

2.5 Path localization

This section is devoted to the proof of path localization. More precisely, we will show that conditioned on survival the amount of time the random walk spends before getting close to the island in which it is eventually localized, is at most linear in the Euclidean distance from that island to the origin.

To this end, we consider the loop erasure for the random walk path, i.e., we consider the following unique decomposition for each path $\omega \in \mathcal{K}_{\mathbb{Z}^d}(0,u)$:

$$\omega = l_0 \oplus [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus ... \oplus [\eta_{|\eta|-1}, \eta_{|\eta|}] \oplus l_{|\eta|} \tag{2.5.1}$$

where $l_i \in \mathcal{K}_{\mathbb{Z}^d \setminus \{\eta_0,...,\eta_{i-1}\}}(\eta_i, \eta_i)$ (recall (2.2.16)) are the loops erased in chronological order and $\eta = [\eta_0, .., \eta_{|\eta|}]$ is the loop-erasure of ω denoted by $\eta = \eta(\omega)$ (see [53, Chapter 9.5] for more details on loop erasure). We first show in Lemma 2.5.4 that in a typical environment the survival probability for the random walk decays exponentially in the length of its loop erasure, which then implies that the loop erasure of the random walk path upon reaching the target island has at most a linear number of steps.

In light of the preceding discussion, it remains to control the lengths of the erased loops which we consider in the following two cases.

- For loops of lengths at most k_n^{50d}: we will first show that for a typical environment for majority of the vertices on any self-avoiding path, the survival probability for the random walk started at those vertices up to time $t \leq k_n^{50d}$ decays quickly in t (Lemma 2.5.3); as a consequence we then show in Lemma 2.5.5 that it is too costly for the small loops to have a total length super-linear in the length of the loop erasure $|\eta|$.
- For loops of lengths at least k_n^{50d}: we will first show in Lemma 2.5.6 that except near the target island the random walk does not encounter any other vertex around which the principal eigenvalue is close to that of the target island; as a result we then show in Lemma 2.5.8 that it is too costly to have any big loop.

In the rest of the section, we carry out the details as outlined above.

Definition 2.5.1. *Let $\mathcal{M}(t)$ be the collection of sites v such that*

$$\mathbf{P}^v(\tau > t) \geq e^{-t/(\log t)^2} . \tag{2.5.2}$$

In addition, we define

$$A_t(\omega) = \{0 \le i \le |\eta| : |l_i| = t, \eta_i \notin \mathcal{M}(t)\}. \tag{2.5.3}$$

Lemma 2.5.2. *There exist positive constants $c_{2,1}, c_{2,2}$ depending only on (d,p) such that for all $t \in \mathbb{N}^*$*

$$\mathbb{P}(v \in \mathcal{M}(t)) \le c_{2,1} e^{-c_{2,2}(\log t)^d}. \tag{2.5.4}$$

Proof. By Lemmas 2.2.8 and 2.2.9, for $n \ge 2$

$$\mathbb{P}(\mathbf{P}^v(\tau > \lfloor (\log n)^{2/d} \rfloor) \ge c) \le n^{-\delta(c)}.$$

By a change of variable, there exist constants $c_{2,1}, c_{2,2}$ depending only on (d,p) such that for all $t \in \mathbb{N}^*$

$$\mathbb{P}(\mathbf{P}^v(\tau > \lfloor (\log t)^2 \rfloor) \ge 1/10) \le c_{2,1} e^{-c_{2,2}(\log t)^d}.$$

Therefore, by a simple union bound we get that

$$\mathbb{P}(\exists u \in K_t(v) \ s.t. \ \mathbf{P}^u(\tau > \lfloor (\log t)^2 \rfloor) \ge 1/10) \le c_{2,3} \exp(-c_{2,4}(\log t)^d),$$

where $c_{2,3}, c_{2,4}$ are constants only depends on (d,p).

On the event $\{\mathbf{P}^u(\tau > \lfloor (\log t)^2 \rfloor) \le 1/10$ for all $u \in K_t(v)\}$ (recall the definition in (2.2.8)), for every $\lfloor (\log t)^2 \rfloor$ steps the random walk has at most $1/10$ probability to survive. Thus,

$$\mathbf{P}^v(\tau > t) \le 10^{-t/(2(\log t)^2)}.$$

This completes the proof of the lemma. □

Lemma 2.5.3. *There exists a costant $t_1^* = t_1^*(d,p)$ such that the following holds with probability tending to one. For all self-avoiding path γ started at origin with length $|\gamma| \ge$*

$n(\log n)^{-100d^2}$ *and* $t \ge t_1^*$,

$$|\gamma \cap \mathcal{M}(t)| \le e^{-(\log t)^{3/2}} |\gamma| \,. \tag{2.5.5}$$

Proof. If $m > n(\log n)^{-100d^2}$ and $\log t \ge (\log m)^{5/9}$, Lemma 2.5.2 yieds

$$\mathbb{P}(\mathcal{M}(t) \cap B(0,m) \ne \varnothing) \le c_{2,1} e^{-c_{2,2}(\log t)^d + d\log(2m)} \le e^{-2^{-1}c_{2,2}(\log t)^d} \,.$$

Hence, it suffices to prove that for large t and m such that $\log t \le (\log m)^{5/9}$, we have

$$\mathbb{P}(\max_{\gamma \in \mathcal{W}_{\mathbb{Z}^d,m}(0)} |\gamma \cap \mathcal{M}(t)| \ge e^{-(\log t)^{3/2}} m) \le \exp\Big(-\exp(-(\log t)^{7/4}) m\Big) \,, \tag{2.5.6}$$

where $\mathcal{W}_{\mathbb{Z}^d,m}(0)$ is the collection of self-avoiding path in $\mathbb{Z}^d$ of length m.

To this end, we denote $\mathcal{V}_i = i + (2t+1)\mathbb{Z}^d$ for $i \in \{1, 2, ..., (2t+1)\}^d$, where $\mathcal{V}_i$ inherits the graph structure from the natural bijection which maps $v \in \mathbb{Z}^d$ to $i + (2t+1)v \in \mathcal{V}_i$. Then events $\{x \in \mathcal{M}(t)\}$ for $x \in \mathcal{V}_i$ are independent. For any self-avoiding path γ, we know that $\{x \in \mathcal{V}_i : \gamma \cap K_t(x) \ne \varnothing\}$ is a lattice animal (i.e., a connected subset) in $\mathcal{V}_i$ of size at most $3^d|\gamma|/t$. Combined with Lemma 2.5.2 and a result on greedy lattice animals proved in [54, Page 281] (see also [57]), this implies

$$\begin{aligned}
&\mathbb{P}(\max_{\gamma \in \mathcal{W}_{\mathbb{Z}^d,m}(0)} |\gamma \cap \mathcal{V}_i \cap \mathcal{M}(t)| \ge \exp(-(\log t)^{5/3}) m) \\
\le &\mathbb{P}(\max_{\gamma \in \mathcal{W}_{\mathbb{Z}^d,m}(0)} |\{x \in \mathcal{V}_i \cap \mathcal{M}(t) : \gamma \cap K_t(x) \ne \varnothing\}| \ge \exp(-(\log t)^{5/3}) m) \\
\le &\exp\Big(-2^{-1} \exp(-(\log t)^{5/3}) m\Big) \,.
\end{aligned}$$

We complete the proof of (2.5.6) by summing over $i \in \{1, 2, ..., (2t+1)\}^d$. □

Lemma 2.5.4. *There exist constants* $c \in (0,1), c', r_0 > 0$ *depending only on* (d,p) *such that for any* $r_1 > r_0$*, the following holds for with* $\mathbb{P}$*-probability at least* $1 - e^{-c' r_1}$*. For all* $r > r_1$

and $m \in \mathbb{N}^*$,

$$\mathbf{P}(|\eta(S_{[0,m]})| \geq r, \tau > m) \leq c^r\,. \tag{2.5.7}$$

Proof. By (2.5.6), we see that there exist constants $C > e^{10}$ and $c', r_0 > 0$ depending only on (d,p) such that for all $r_1 > r_0$, with $\mathbb{P}$-probability at least $1 - \exp\big(-c' r_1\big)$, for all self-avoiding path γ of length at least r_1,

$$|\gamma \cap \mathcal{M}(C)| \leq e^{-(\log C)^{3/2}} |\gamma|\,. \tag{2.5.8}$$

We recursively define stopping times $\zeta_0 = 0$,

$$\zeta_i = \inf\{t > \zeta_{i-1} + C : S_t \notin \mathcal{M}(C)\}\,.$$

On the event $\{|\eta(S_{[0,m]})| \geq r\}$, since we assumed $r > r_1$, we know from (2.5.8) that

$$|S_{[0,m]} \cap \mathcal{M}(C)^c| \geq |\eta(S_{[0,m]}) \cap \mathcal{M}(C)^c| \geq |\eta(S_{[0,m]})|(1 - e^{-(\log C)^{3/2}})\,.$$

Let $j = \lfloor r/(2C) \rfloor$. By definition of ζ_i's and $C > e^{10}$,

$$|S_{[0,\zeta_j]} \cap \mathcal{M}(C)^c| \leq (C+1)j + 1 \leq r\tfrac{C+1}{2C} + 1 < |\eta(S_{[0,m]})|(1 - e^{-(\log C)^{3/2}})\,.$$

Therefore, we get $\zeta_j \leq m\,$. Then by strong Markov property,

$$\begin{aligned}\mathbf{P}(|\eta(S_{[0,m]})| \geq r, \tau > m) &\leq \mathbf{P}(S_{[\zeta_m, \zeta_m + C]} \text{ is open } \forall 1 \leq m \leq j-1)\\ &\leq \Big[\exp(-C/(\log C)^2)\,\Big]^{r/(2C)-2}\,,\end{aligned}$$

completing the proof of the lemma. □

Lemma 2.5.5. *Recall definitions in* (2.2.16) *and* (2.3.5). *There exists a constant* $t_2^* = t_2^*(d,p)$ *such that the following holds with* $\mathbb{P}$*-probability tending to one. For all* $t_2^* \leq t \leq k_n^{50d}$,

$u \in \bigcup_{v\in \mathsf{V}} (\partial_i B(v, (\log n)^\iota) \cap \mathcal{C}(v))$ *and* $m \le n$*,*

$$\mathbf{P}(\{\omega \in \mathcal{K}_{\mathcal{O}^c,m}(0,u) : l_{|\eta|} = \varnothing, |A_t(\omega)| \ge |\eta| t^{-10}\}) \le e^{-n^{1/2}} \mathbf{P}(\mathcal{K}_{\mathcal{O}^c,m}(0,u)) . \tag{2.5.9}$$

Proof. For any ω, we denote

$$\phi(\omega) = \tilde{l}_0 \oplus [\eta_0, \eta_1] \oplus \tilde{l}_1 \oplus [\eta_1, \eta_2] \oplus ... \oplus [\eta_{|\eta|-1}, \eta_{|\eta|}] \oplus \tilde{l}_{|\eta|} \tag{2.5.10}$$

where $\tilde{l}_i = l_i$ if $i \notin A_t(\omega)$ and $\tilde{l}_i = \varnothing$ otherwise. Note that for any $\omega \in \mathcal{K}_{\mathcal{O}^c,m}(0,u)$ such that $|A_t(\omega)| \ge |\eta| t^{-10}$,

$$m - |\phi(\omega)| = t|A_t(\omega)| \ge |\eta| t^{-9} .$$

We consider every $\gamma \in \phi(\mathcal{K}_{\mathcal{O}^c,m}(0,u))$ such that $m-|\gamma| \ge |\eta| t^{-9}$. For large t, since $\eta_i \notin \mathcal{M}(t)$ for $i \in A_t(\omega)$ and $|\{i : \tilde{l}_i = \varnothing\}| \le |\eta|$, we have

$$\begin{aligned} \mathbf{P}(\{\omega \in \mathcal{K}_{\mathcal{O}^c,m}(0,u) : \phi(\omega) = \gamma, l_{|\eta|} = \varnothing\}) &\le \mathbf{P}(\gamma) \binom{|\eta|}{\frac{m-|\gamma|}{t}} e^{-t(\log t)^{-2} \frac{m-|\gamma|}{t}} \\ &\le \mathbf{P}(\gamma) e^{-(m-|\gamma|)(\log t)^{-3}} . \end{aligned}$$

In the last inequality, we used the fact that

$$(\tfrac{m-|\gamma|}{t})! \ge (\tfrac{m-|\gamma|}{et})^{\frac{m-|\gamma|}{t}} \ge (|\eta| t^{-10} e^{-1})^{\frac{m-|\gamma|}{t}} .$$

In addition, it follows from Lemma 2.4.6 and Corollary 2.3.7(3) that

$$\mathbf{P}(\mathcal{K}_{\mathcal{O}^c,m-|\gamma|}(u,u)) \ge (2d)^{-7\rho(\log n)^\iota} e^{-\chi(m-|\gamma|)(\log n)^{-2/d}} . \tag{2.5.11}$$

Note that $(\log t)^3 \le (\log k_n^{50d}) \lesssim (\log\log n)^3 = o((\log n)^{2/d})$ and by Lemma 2.3.8 we have

$m - |\gamma| \geq |\eta| t^{-9} \geq n(\log n)^{-2000d^2}$. Therefore

$$\mathbf{P}(\{\omega \in \mathcal{K}_{\mathcal{O}^c,m}(0,u) : \phi(\omega) = \gamma, l_{|\eta|} = \varnothing\}) \leq \mathbf{P}(\gamma \oplus \mathcal{K}_{\mathcal{O}^c,m-|\gamma|}(u,u)) e^{-2^{-1}|\eta| t^{-10}}.$$

We complete the proof of the lemma by summing over all such γ's (where the pre-factor of $e^{-\sqrt{n}}$ is a crude bound with room to spare). □

Lemma 2.5.6. *Recall the definition of* v_* *and* $\mathsf{T}(\mathsf{v}_*)$ *as in* (2.4.1). *For constant* $q > 0$, *let* $U(t) = \cup_{i=0}^t B(S_i, (\log n)^q) \cap \mathcal{C}(0)$. *Conditioned on the event that the origin is in an infinite open cluster, we have that*

$$\mathbf{P}(U(n) \setminus B(\mathsf{v}_*, 3R) \subset \mathcal{D}^c_{(1-k_n^{-20d})\lambda(\mathsf{v}_*)} \mid \tau > n) \to 1 \quad \text{in } \mathbb{P}\text{-probability.} \tag{2.5.12}$$

Remark 2.5.7. For purpose of the present article, it suffices to take $U(t) = \cup_{i=0}^t S_i$; we strengthened the lemma as it may be useful for future application.

Proof of Lemma 2.5.6. We start with a brief description on the intuition behind (2.5.12). If the random walk hits some local region with the principal eigenvalue close to that near v_* (which is the presumed target island) before time $\mathsf{T}(\mathsf{v}_*)$, then the random walk tends to stay around this local region as opposed to travel all the way to the presumed target island — since by Lemma 2.3.4 the regions with large principal eigenvalues are far away from each other and thus it is costly for the random walk to travel from one to the other.

Let $a = \inf\{t \geq 0 : \max_{u \in U(t) \setminus B(\mathsf{v}_*, 3R)} \lambda(u) > (1 - k_n^{-20d})\lambda(\mathsf{v}_*)\}$. Then there exists $x \in \big(B(S_a, 2(\log n)^q) \cap \mathcal{C}(0)\big) \setminus B(\mathsf{v}_*, 3R)$ such that and $\lambda(x) \geq (1 - k_n^{-20d})\lambda(\mathsf{v}_*)$. We restrict to the event $\{\tau_{\mathcal{D}_*} < n, S_{[\mathsf{T}(\mathsf{v}_*), n]} \subset B(\mathsf{v}_*, (\log n)^\iota k_n), a < n\}$. Hence, we have

$$\lambda(x) \geq (1 - k_n^{-20d})\lambda(\mathsf{v}_*) \geq (1 - k_n^{-20d}) p_{\alpha_1}^{1/k_n} \geq p_{\alpha_2}^{1/k_n}.$$

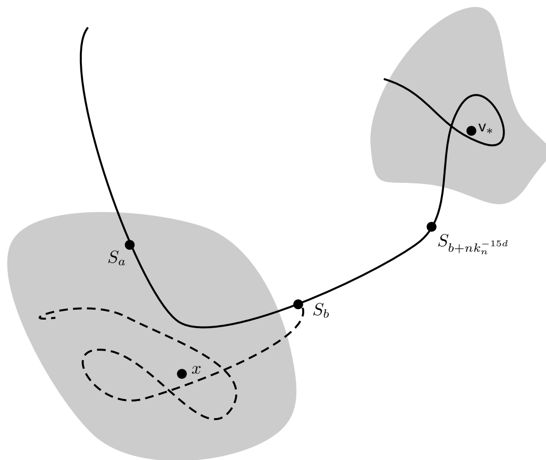

Figure 2.5: The random walk would prefer staying in the neighborhood of x (dotted curve) to going ahead to v_* (solid curve), since survival probability during $[b, b+nk_n^{-15d}]$ is very low.

Since $|x-\mathsf{v}_*| \geq 3R$, Corollary 2.3.7 (1) yields

$$|x-\mathsf{v}_*| \geq nk_n^{-14d} \text{ and } (B(x, nk_n^{-14d}) \setminus B(x, 3R)) \cap \mathcal{U}_{\alpha_2} = \varnothing\,.$$

Let $b = \sup\{t \leq n : S_t \in B(x, 2(\log n)^q)\}$. Then $b + nk_n^{-15d} \leq n$ and

$$S_{[b, b+nk_n^{-15d}]} \notin \mathcal{U}_{\alpha_2}, \text{ and } S_{[b+nk_n^{-15d}, n]} \notin \mathcal{D}_{\lambda(\mathsf{v}_*)}\,.$$

For any $v \in \mathsf{V}$ and x such that $\lambda(x) \geq (1-k_n^{-20d})\lambda(v)$, we deduce from the Markov property and Lemmas 2.4.4, 2.4.5 that

$$\begin{aligned}
&\mathbf{P}(b=m, \mathsf{v}_*=v, S_a \in B(x, 2(\log n)^q), \tau > n, a \leq n, S_{[\mathsf{T}(\mathsf{v}_*), n]} \subset B(\mathsf{v}_*, (\log n)^\iota k_n)) \\
\leq &\mathbf{P}(S_m \in \partial_i B(x, 2(\log n)^q), \tau > m)(2R)^{4d} p_{\alpha_2}^{\lfloor nk_n^{-15d} \rfloor / k_n} \lambda(v)^{n-m-\lfloor nk_n^{-15d} \rfloor} \\
\leq &\mathbf{P}(S_m \in \partial_i B(x, 2(\log n)^q), \tau > m)(2R)^{4d} (\log n)^{-\lfloor nk_n^{-15d} \rfloor} \lambda(v)^{n-m}\,.
\end{aligned} \tag{2.5.13}$$

Next, we give a lower bound on survival probability. By Lemma 2.4.6, $\mathbf{P}^u(\tau > n-m) \geq$

$(2d)^{-10\rho(\log n)^q}\lambda(x)^{n-m}$. Hence

$$\begin{aligned}\mathbf{P}(\tau > n) &\geq \mathbf{P}(S_m \in \partial_i B(x, 2(\log n)^q), \tau > n)\\ &\geq \mathbf{P}(S_m \in \partial_i B(x, 2(\log n)^q), \tau > m)(2d)^{-10\rho(\log n)^q}\lambda(x)^{n-m}\,.\end{aligned}$$

Combined with (2.5.13) and $\lambda(x) \geq (1-k_n^{-20d})\lambda(v)$, since

$$(\lambda(v)/\lambda(x))^{n-m} \leq (\lambda(v)/\lambda(x))^n \leq \exp(-nk_n^{-20d})\,,$$

it yields that

$$\begin{aligned}&\mathbf{P}(b = m, \mathsf{v}_* = v, S_a \in B(x, 2(\log n)^q), a \leq n, S_{[\mathsf{T}(\mathsf{v}_*),n]} \subset B(\mathsf{v}_*, (\log n)^\iota k_n), \tau > n)\\ &\quad\leq e^{-nk_n^{-16d}}\mathbf{P}(\tau > n)\,.\end{aligned}$$

Summing over $0 \leq m \leq n$, $v \in \mathsf{V}$ and $x \in B(0,n)$ such that $\lambda(x) \geq (1-k_n^{-20d})\lambda(v)$, we complete the proof by Lemma 2.4.2 and Proposition 2.4.3. □

Lemma 2.5.8. *For $u \in \partial_i B(v, (\log n)^\iota) \cap \mathcal{C}(v)$ and $\lambda = (1-k_n^{-20d})\lambda(v)$ for some $v \in \mathsf{V}$, we have that for all $m \leq n$*

$$\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u) : l_{|\eta|} = \varnothing, \max_i |l_i| \geq k_n^{50d}\}) \leq e^{-k_n^{20d}}\mathbf{P}(\mathcal{K}_{\mathcal{O}^c,m}(0,u))\,. \quad (2.5.14)$$

Proof. For any ω, we denote

$$\phi(\omega) = \tilde{l}_0 \oplus [\eta_0, \eta_1] \oplus \tilde{l}_1 \oplus [\eta_1, \eta_2] \oplus ... \oplus [\eta_{|\eta|-1}, \eta_{|\eta|}] \oplus \tilde{l}_{|\eta|} \quad (2.5.15)$$

where $\tilde{l}_i = l_i$ if $|l_i| \leq k_n^{50d}$ and $\tilde{l}_i = \varnothing$ otherwise. Then for any $\gamma \in \phi(\mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u))$ with

$|\gamma| \neq m$, we deduce from Lemma 2.4.5 that

$$\begin{aligned}&\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u) : \phi(\omega) = \gamma, l_{|\eta|} = \varnothing, |\{i : |l_i| > k_n^{50d}\}| = j\}) \\ \leq &\mathbf{P}(\gamma) \binom{|\{i : \tilde{l}_i = \varnothing\}|}{j} (m - |\gamma|))^j R^{3jd} \lambda^{m-|\gamma|} .\end{aligned}$$

Summing over $j \leq (m - |\gamma|)/k_n^{50d}$, we get

$$\begin{aligned}&\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u) : \phi(\omega) = \gamma, l_{|\eta|} = \varnothing\}) \\ \leq &\sum_{j=1}^{\lfloor (m-|\gamma|)/k_n^{50d} \rfloor} \mathbf{P}(\gamma) (|\eta|(m - |\gamma|))^j R^{3jd} \lambda^{m-|\gamma|} \\ \leq &\mathbf{P}(\gamma) n^{3(m-|\gamma|)/k_n^{50d}+2} e^{-(m-|\gamma|)/k_n^{20d}} \lambda(v)^{m-|\gamma|} .\end{aligned}$$

Note that $\lambda \leq (1 - k_n^{-20d})\lambda(v)$ and that by Lemma 2.4.6

$$\mathbf{P}(\mathcal{K}_{\mathcal{O}^c, m-|\gamma|}(u,u)) \geq R^{-3\rho(\log n)^\iota} \lambda(v)^{m-|\gamma|} .$$

We then get that

$$\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u) : \phi(\omega) = \gamma, l_{|\eta|} = \varnothing\} \leq \mathbf{P}(\gamma \oplus \mathcal{K}_{\mathcal{O}^c, m-|\gamma|}(u,u)) e^{-k_n^{20d}} .$$

We complete the proof of the lemma by summing over all such γ's. □

Corollary 2.5.9. *There exists a constant $c = c(d,p)$ such that the following holds with $\mathbb{P}$-probability tending to one. If $u \in \partial_i B(v, (\log n)^\iota) \cap \mathcal{C}(v)$ and $\lambda \leq (1 - k_n^{-20d})\lambda(v)$ for some $v \in \mathsf{V}$, then for all $m \in \mathbb{N}^*$,*

$$\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u) : l_{|\eta|} = \varnothing, m > c|\eta|\}) \leq e^{-k_n^{10d}} \mathbf{P}(\mathcal{K}_{\mathcal{O}^c, m}(0,u)) . \tag{2.5.16}$$

Proof. By Lemma 2.3.8, with $\mathbb{P}$-probability tending to one, we have $|u| \geq n(\log n)^{-100d^2}$.

Then by Lemma 2.5.3, there exists $t_1^* = t_1^*(d,p)$ such that for all self-avoiding path γ from 0 to u and $t_1^* \le t \le k_n^{50d}$, we have

$$|\gamma \cap \mathcal{M}(t)| \le t^{-10d} |\gamma| \,.$$

Now, we consider any $\omega \in \mathcal{K}_{(\mathcal{D}_\lambda \cup \mathcal{O})^c, m}(0,u)$ such that $l_{|\eta|} = \varnothing$. If

$$|A_t(\omega)| \le |\eta| t^{-10} \text{ for } t_2^* \le t \le k_n^{50d} \text{ and } \max_{0 \le i \le |\eta|} |l_i| \le k_n^{50d},$$

for some $t_2^* = t_2^*(d,p)$, then for $t^* = \max(t_1^*, t_2^*)$

$$\begin{aligned} m &= |\eta| + \sum_{0 \le i \le |\eta|} |l_i| \\ &\le |\eta| + \sum_{t=1}^{t^*-1} t|\eta| + \sum_{t=t^*}^{k_n^{50d}} t\left(|A_t(\omega)| + |\eta \cap \mathcal{M}(t)|\right) \\ &\le \left(1 + \sum_{t=1}^{t^*-1} t + \sum_{t=t^*}^{\infty} t^{-9} + \sum_{t=t^*}^{\infty} t e^{-(\log t)^{3/2}}\right) |\eta| \,. \end{aligned}$$

Combining Lemmas 2.5.5 and 2.5.8, we complete the proof of the corollary. □

Proof of Theorem 2.1.1: path localization. We will prove that

$$\mathbf{P}\Big(T(\mathsf{v}_*) \le c \min(|S_T(\mathsf{v}_*)|, n(\log n)^{-2/d}), S_{[T(\mathsf{v}_*),n]} \subset D_n \mid \tau > n\Big) \to 1 \,.$$

To this end, applying Lemma 2.5.4 with $r_1 = c_{2,0} n (\log n)^{-2/d}$ and combining with (2.4.10), we get that there exists $c_{2,0} = c_{2,0}(d,p)$ such that

$$\mathbf{P}(|\eta(S_{[0,n]})| > c_{2,0} n (\log n)^{-2/d} \mid \tau > n) \to 0 \,, \tag{2.5.17}$$

which implies $\mathbf{P}(|S_{\mathsf{T}(\mathsf{v}_*)}| \le c_{2,0} n (\log n)^{-2/d} \mid \tau > n) \to 1$. In light of Proposition 2.4.3, it remains to prove that there exists $c = c(d,p)$ such that

$$\mathbf{P}(\mathsf{T}(\mathsf{v}_*) \le c|S_{\mathsf{T}(\mathsf{v}_*)}| \mid \tau > n) \to 1. \tag{2.5.18}$$

By Lemma 2.5.6, it suffices to show (we write $\lambda'_v = (1 - k_n^{-20d})\lambda(v)$ below)

$$\sum_v \sum_u \sum_m \mathbf{P}(\mathsf{T}(v) = m, S_m = u, \mathsf{v}_* = v, \tau_{\mathcal{D}_{\lambda'_v}} > m \mid \tau > n) \to 0\,, \tag{2.5.19}$$

where the summation is over $v \in \mathsf{V}$, $u \in \partial_i B(v, (\log n)^\iota) \cap \mathcal{C}(v)$ and $c|u| \le m \le n$. To this end, we first notice that

$$\begin{aligned}
&\mathbf{P}(\mathsf{T}(v) = m, S_m = u, \mathsf{v}_* = v, \tau_{\mathcal{D}_{\lambda'_v}} > m, \tau > n) \\
\le& \mathbf{P}(S_m = u, u \notin S_{[0,m-1]}, \tau_{\mathcal{O}\cup\mathcal{D}_{\lambda'_v}} > m)\mathbf{P}^u(\tau > n - m)\,.
\end{aligned} \tag{2.5.20}$$

At the same time, by Corollary 2.5.9 and Lemma 2.5.4 (applied with $r_1 = m/c'_{2,1}$), there exist positive constants c'_1, c'_2 depending only on (d,p) such that for any $v \in \mathcal{D}_*$ and $u \in \partial_i B(v, (\log n)^\iota) \cap \mathcal{C}(v)$

$$\begin{aligned}
&\mathbf{P}(S_m = u, u \notin S_{[0,m-1]}, m > c'_1|\eta(S_{[0,m]})|, \tau_{\mathcal{O}\cup\mathcal{D}_{\lambda'_v}} > m) \\
&\quad \le e^{-k_n^{10}}\mathbf{P}(S_m = u, \tau > m),
\end{aligned} \tag{2.5.21}$$

$$\text{and} \qquad \mathbf{P}(m \le c'_1|\eta(S_{[0,m]})|, \tau > m) \le {c'_{2,2}}^m. \tag{2.5.22}$$

Then by Lemma 2.4.6 and Corollary 2.3.7 (3), we have

$$\mathbf{P}(S_m = u, \tau > m) \ge (2d)^{-2\rho(\log n)^\iota - \rho|u|} e^{-\chi m/(\log n)^{2/d}}\,.$$

Thus, there exists $c = c(d,p)$, such that for all $m > c|u|$

$$\mathbf{P}(S_m = u, \tau > m) \geq {c'_{2,2}}^{m/2}\,.$$

Combined with (2.5.21) and (2.5.22), this gives that

$$\begin{aligned}
&\mathbf{P}(S_m = u, u \notin S_{[0,m-1]}, \tau_{\mathcal{O}\cup\mathcal{D}_{\lambda'_v}} > m) \\
\leq&\mathbf{P}(S_m = u, u \notin S_{[0,m-1]}, m > c'_1|\eta(S_{[0,m]})|, \tau_{\mathcal{O}\cup\mathcal{D}_{\lambda'_v}} > m) \\
&+ \mathbf{P}(m \leq c'_1|\eta(S_{[0,m]})|, \tau > m) \\
\leq&e^{-2^{-1}k_n^{10}}\mathbf{P}(S_m = u, \tau > m)\,.
\end{aligned}$$

Combined with (2.5.20), this implies

$$\begin{aligned}
&\mathbf{P}(\mathsf{T}(v) = m, S_m = u, \mathsf{v}_* = v, \tau_{\mathcal{D}_{\lambda'_v}} > m, \tau > n) \\
\leq&e^{-2^{-1}k_n^{10}}\mathbf{P}(S_m = u, \tau > m)\mathbf{P}^u(\tau > n - m) \\
=&e^{-2^{-1}k_n^{10}}\mathbf{P}(S_m = u, \tau > n)\,.
\end{aligned}$$

Summing over u, v and $m \geq c|u|$, we complete the verification of (2.5.19). $\square$

2.A Index of notation

X_v	(2.2.1)
k_n	(2.2.2)
c-good	Definition 2.2.4
ϵ-fair	Definition 2.2.6
$\mathcal{K}(\cdot,\cdot)$	(2.2.16)
p_α's	(2.3.1)
$\mathcal{U}_\alpha$'s	(2.3.2)
$\mathcal{C}_R(v)$	Definition 2.3.1
$\lambda(v), R$	Definition 2.3.1
$\mathcal{D}_*$, $\mathcal{D}$'s	(2.3.5)
V	(2.3.10)
D_n	(2.3.11)
v_*	Definition 2.4.1
$T(\cdot)$	(2.4.1)
η	(2.5.1)
$\mathcal{M}(t)$	Definition 2.5.1
$A_t(\omega)$	Definition 2.5.1

CHAPTER 3

SINGLE BALL LOCALIZATION UNDER THE QUENCHED LAW

3.1 Introduction

3.1.1 Model and main results

In this chapter, we continue studying the quenched behavior of the random walk conditioned on survival for a large time. For $d \geq 2$, we consider a random environment where each vertex of $\mathbb{Z}^d$ is placed with an obstacle independently with probability $1-p$. On this random environment, we then consider a discrete-time simple random walk $(S_t)_{t\in\mathbb{N}}$ started at the origin and killed at time τ when the random walk hits an obstacle for the first time. For convenience of notation, we use $\mathbb{P}$ (and $\mathbb{E}$) for the probability measure with respect to the random environment, and use $\mathbf{P}$ (and $\mathbf{E}$) for the probability measure with respect to the random walk. We will assume $p > p_c(\mathbb{Z}^d)$, the critical threshold for site percolation, and let $\widehat{\mathbb{P}}$ be the conditional measure for the environment given that the origin is in an infinite open cluster. Our main result in this chapter is the following.

Theorem 3.1.1. *For any fixed $d \geq 2$, there exists a constant $C = C(d,p)$ and a $\widehat{\mathbb{P}}$-measurable discrete ball $\mathfrak{B}_n \subseteq \mathbb{Z}^d$ of cardinality at most $(1+\epsilon_n)d\log_{1/p} n$ (where ϵ_n tends to 0 as $n \to \infty$) such that the following holds: for any $t \in [Cn(\log n)^{-2/d}, n]$*

$$\mathbf{P}(S_t \in \mathfrak{B}_n \mid \tau > n) \to 1 \text{ in } \widehat{\mathbb{P}}\text{-probability as } n \to \infty\,. \tag{3.1.1}$$

Here a discrete ball (in $\mathbb{Z}^d$) is the set that contains all the lattice points of some Euclidean ball (in $\mathbb{R}^d$).

3.1.2 From poly-logarithmic localization to sharp localization

The main result in our previous work [24] is that the random walk is confined in at most poly-logarithmic in n many *islands* (where an *island* is a connected subset in $\mathbb{Z}^d$) during time $[\epsilon_n n, n]$ (for some $\epsilon_n \to_{n\to\infty} 0$) and each island has diameter at most poly-logarithmic in n — we will refer to these islands as *pocket islands*. The present article is closely related to [24]: we rely both on the results and techniques in [24] (we note that our proof is otherwise self-contained and in particular does not rely on results in [75]). Provided with [24], our proof of Theorem 3.1.1 is naturally divided into two essentially separate parts as follows.

One-city theorem. The first step toward proving Theorem 3.1.1 is to show that conditioned on survival the random walk is localized in a single pocket island.

Theorem 3.1.2. *Let v_* be the maximizer of the variational problem* (3.3.5) *(in Section 3.3), which is measurable with respect to the environment. Then there exist constants κ, C depending only on (d,p) such that the following holds: Let T be the first time that the random walk visits $B(v_*, (\log n)^{\kappa/2})$ (i.e., a ball centered at v_* of radius $(\log n)^{\kappa/2}$). Then as $n \to \infty$*

$$\mathbf{P}\Big(T \le C|v_*|, \{S_t : t \in [T,n]\} \subseteq B(v_*, (\log n)^{\kappa/2}) \mid \tau > n\Big) \to 1 \text{ in } \widehat{\mathbb{P}}\text{-probability}. \quad (3.1.2)$$

We now explain the variational problem (3.3.5) and our intuition behind the proof of Theorem 3.1.2. The probability of localizing in a pocket island is a product of the searching probability (i.e., the probability for the random walk to travel from the origin to a certain island) and the confinement probability (i.e., the probability for the random walk to staying in this island for $n - o(n)$ steps). In essence, this is the variational problem (3.3.5) which optimizes the sum of two quantities corresponding to the approximation of the logarithms of the confinement probability and the searching probability.

The major challenge here comes from the fact that the confinement probability is much smaller than the searching probability; in fact, the leading exponential terms of the confinement probabilities in all pocket islands are the same and the second order exponential terms of the confinement probabilities would be comparable to the searching probabilities. The optimal island is the maximizer which strikes a balance between the searching probability and the confinement probability. Our goal is to prove that the maximizer of the product of these two probabilities is much larger than the sum of the rest products. One possible approach to attack this is to obtain refined estimates on the confinement probabilities. However, this seems quite challenging. In order to circumvent this difficulty, we note that the searching probability and the confinement probability are roughly independent. Hence it suffices to prove that either of these two probabilities has large fluctuation across different islands, which then implies that the maximizer has a product which is much larger than the sum of the rest products of these two probabilities.

Due to the difficulty of controlling the confinement probability, we instead choose to work with searching probability and show that the logarithm of the searching probability grows (roughly speaking) linearly in the distance from the origin to the island provided that the angle is fixed. Thus, searching probabilities have a large fluctuation since pocket islands occur more or less uniformly in the box under consideration. Due to the fluctuation, the best pocket island will substantially dominate all the others.

The difficulty of implementing such an analysis is that the decomposition (into searching probability and confinement probabilities) must be done in a way that would not introduce an error larger than the fluctuation. This is the reason why we consider the variational problem (3.3.5), which is different from that in [71]. In fact, the optimal island will attain a value close to the minima of the corresponding variational problem in [71], but it may not be the actual optimizer. A major ingredient in proving Theorem 3.1.2 is the proof of a refined version of "logarithm of the searching probability grows linearly in distance" incorporated in Proposition 3.3.3 (and carried out in Sections 3.3 and 3.4).

Intermittent island. While the random walk is confined in a pocket island during time $[o(n), n]$, we will show that at each given time $t \in [o(n), n]$ it is in a much smaller region (called *intermittent island*) inside the pocket island with high probability. In addition, we will show that the intermittent island is asymptotically a ball. These are covered in Sections 3.5 and 3.6. An important observation here is that the total volume for regions with low obstacle density in any pocket island is at most $d \log_{1/p} n(1 + o(1))$. This observation, combined with the celebrated Faber–Krahn inequality (see Section 3.1.3 below), then implies that the asymptotic shape of the intermittent island is a ball.

We would like to add a remark that in this chapter, we use the term *intermittent island* in a manner that is not completely precise. For instance, we have referred to $\mathcal{E}$ in Definition 3.5.1, Ω_ϵ in Definition 3.5.3, B_ϵ in Lemma 3.5.9, $\widehat{B}_n$ in (3.6.1) as intermittent island in informal discussions. The abuse of the terminology is justified by the fact that all of these sets have negligible pair-wise symmetric differences (and of course our mathematical statements are always precisely formulated).

3.1.3 Two important proof ingredients

In Section 3.1.2 we described the high-level structure of our proof in two essentially separate steps; in this subsection, we will discuss two important proof ingredients, which provides a glance at some highlights of our proof. Discussions on more detailed proof ideas can be found at the beginning of Sections 3.3, 3.4, 3.5, 3.6.

Convergence rates for sub-additive functions. Rate of convergence for sub-additive functionals has received much attention in the past. See [3, 45, 4, 5, 8] for progress on bounds for rate of convergence for sub-additive functionals with prominent application in first-passage percolation. In particular, a general theory was given in [4] via the ingenious convex hull approximation property which applies to several processes on lattices including first-passage percolation. For instance, it was shown that in first-passage percolation the expected length of the shortest path connecting 0 and x can be approximated by a function of x that is convex and homogeneous of order 1, with approximation error at most $O(|x|^\nu)$ for $\nu < 1$. Our proof in Section 3.4.4 follows the framework developed in [4] and is dedicated to verifying the convexity hull condition in [4] for our log-weighted Green's functions defined as in (3.4.1) — an incorrect but heuristically useful interpretation of log-weighted Green's function is the logarithm of the probability for the random walk to travel from one point to another point without hitting an obstacle (in fact, this is simply the logarithm of the Green's function without reweighting, which converge to Lyapunov exponents [75]). While this resembles the first-passage percolation problem (as already noted in [75]), our context is more complicated since our function in a vague sense takes average over many (not necessarily self-avoiding) paths rather than takes the length of the single shortest path as in first-passage percolation. Furthermore, the real definition of log-weighted Green's function is even more complicated: for instance, it has to take into account the travel time for the random walk as well as to incorporate the requirement that the random walk has to avoid certain regions. These incur substantial challenges in implementing the proof framework in [4].

Faber–Krahn inequality. A classic result, known as the celebrated Faber–Krahn inequality, states that among sets with given volume balls are the only sets which minimize the first eigenvalue (we remark that Faber–Krahn inequality is the fundamental reason behind the phenomenon that the localization occurs in a ball). Various versions of quantitative Faber–Krahn inequality have been proved in the past [41, 58, 11, 34, 15]. In particular, the following sharp quantitative Faber–Krahn inequality was proved in [15, Main Theorem] (here "sharp" means that the lower bound is achievable (up to constant) for some choice of Ω)

$$|\Omega|^{2/d}\mu_\Omega - |B|^{2/d}\mu_B \geq \sigma_d(\mathcal{A}(\Omega))^2 \text{ for a constant } \sigma_d > 0 \text{ depending only on } d\,, \tag{3.1.3}$$

where B is a Euclidean ball, $|\Omega|$ denotes the volume of Ω, $\mathcal{A}(\Omega)$ is the *Fraenkel asymmetry* defined as (below $\triangle$ denotes the symmetric difference)

$$\mathcal{A}(\Omega) = \inf\left\{\frac{|\Omega \triangle B|}{|B|} : B \text{ is a ball such that } |B| = |\Omega|\right\}\,, \tag{3.1.4}$$

and

$$\mu_\Omega = \frac{1}{2d} \min_{u \in W_0^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, \mathrm{d}x : \|u\|_{L^2(\Omega)=1} \right\}. \tag{3.1.5}$$

Our proof that the localization region is asymptotically a ball uses (3.1.3). In fact, for the purpose of our proof, we do not need the full power of the sharp inequality (3.1.3) — the inequalities in [11, 34] would suffice.

We remark that ours is not the first application of Faber–Krahn type of inequality in the study of localization of random walks. For instance: in [12] a key ingredient was a inequality of this type in two dimensions which was proved in the same paper; in [74] another quantitative version of Faber–Krahn was proved independently; in [63] a quantitative version of isoperimetric inequality (related to Faber–Krahn inequality) from [40] was a key ingredient in the proof.

3.1.4 Organization

The remaining sections of the chapter are organized as follows. In Section 3.2 we review results from [24] and also record a few useful lemmas. In Section 3.3, we give a proof of Theorem 3.1.2 assuming a major ingredient as incorporated in Proposition 3.3.3. Section 3.4 is devoted to the proof of Proposition 3.3.3. In Section 3.5, we prove that there is a ball in the pocket island of cardinality asymptotically $d \log_{1/p} n$ such that the principal eigenvalue of this ball is close to that of the pocket island. Finally, we prove in Section 3.6 that the random walk will be localized in the intermittent island and hence complete the proof of Theorem 3.1.1.

3.1.5 Notation convention

For $v \in \mathbb{Z}^d$, we recall that v is open if there is no obstacle placed at v. We define the ℓ_2-norm $|\cdot|$ by $|v| = (\sum_{i=1}^d v_i^2)^{1/2}$, the ℓ_1-norm $|\cdot|_1$ by $|v|_1 = \sum_{i=1}^d |v_i|$, and the ℓ_∞-norm $|\cdot|_\infty$ by $|v|_\infty = \max_{1 \le i \le d} |v_i|$. For $r > 0, v \in \mathbb{Z}^d$, we define $B(v,r) = \{x \in \mathbb{Z}^d : |x - v| \le r\}$

and $K_r(v) = \{x \in \mathbb{Z}^d : |x - v|_\infty \leq r\}$. For $A \subseteq \mathbb{Z}^d$, $|A|$ denotes the cardinality of A. Write $\partial A = \{x \in A^c : y \sim x \text{ for some } y \in A\}$, where $x \sim y$ means that x is a neighbor of y (i.e. $|x - y|_1 = 1$) and $\partial_i A = \{x \in A : y \sim x \text{ for some } y \in A^c\}$. We let λ_A denote the principal eigenvalue (i.e., the largest eigenvalue) of $P|_A$, which is the transition matrix of simple random walk on $\mathbb{Z}^d$ killed upon exiting A. For Lebesgue measurable set A in $\mathbb{R}^d$, we use $|A|$ to denote the Lebesgue measure of A. For $x, y \in A \subseteq \mathbb{Z}^d$, we define $D_A(x, y)$ to be the length of the shortest path which stays within A and joins x and y.

We denote by $\mathcal{O}$ the collection of all obstacles (some times referred as closed vertices). For $v \in \mathbb{Z}^d$, we denote by $\mathcal{C}(v)$ the open cluster containing v and by $\mathcal{C}(\infty)$ the infinite open cluster. If v is closed, then $\mathcal{C}(v) = \varnothing$. We denote by $\xi_A = \inf\{t \geq 0 : S_t \notin A\}$ the first time for the random walk to exit from A, and by $\tau_A = \inf\{t \geq 0 : S_t \in A\}$ the hitting time to A. In particular, we denote $\tau_x = \tau_{\{x\}}$ for $x \in \mathbb{Z}^d$. As having appeared earlier, we let $\tau = \tau_{\mathcal{O}}$ be the survival time of the random walk. For a subset of non-negative integers I, we denote $S_I = \{S_t : t \in I\}$.

Throughout the rest of the chapter, C, c denote positive constants depending only on (d, p) whose numerical values may vary from line to line (and we do not introduce them anymore). We have in mind that C is a large constant while c is a small constant. For constants with decorations such as c_*, $C_{3,1}$ or κ (which also depend only (d, p)), their values will stay the same in the whole chapter. A list of frequently used notation is compiled in Appendix 3.A.

3.2 Preliminaries

As described in Section 3.1.2, it was proved in [24] that the random walk will be localized in poly-logarithmic in n many balls of radius $(\log n)^\kappa$ (see Theorem 3.2.3), which we refer to as pocket islands (see Lemma 3.2.1 and the discussions that follow for a more formal definition for pocket islands). In this subsection, we will describe the main result of [24] in more detail and record a number of useful lemmas.

Pocket Islands. Let us first recall some notations and definitions from [24]. Let $k_n = (\log n)^{4-2/d}(\log\log n)^{2\mathbb{1}_{d=2}}$. Write $R = k_n(\log n)^2$ and denote by $\mathcal{C}_R(v)$ the connected component in $B(v,R)\backslash\mathcal{O}$ that contains v. Let $\lambda(v) = \lambda_{\mathcal{C}_R(v)}$ be the principal eigenvalue of the transition matrix $P|_{\mathcal{C}_R(v)}$— we note that $(1-\lambda(v))$ is the discrete analogue the first eigenvalue of Dirichlet-Laplacian on $\mathcal{C}_R(v)$ defined in (3.1.5). We call for the attention of the reader that the notation of $\lambda(v)$ and $\lambda_{\{v\}}$ have completely different meanings.

Set

$$\lambda_* = p_{\alpha_1}^{1/k_n} \tag{3.2.1}$$

where p_{α_1} (defined in [24, (3.1)]) is appropriately chosen according to some large quantile of the distribution of survival probability up to k_n steps. Denote $\mathcal{D}_* = \{v \in \mathcal{C}(0) : \lambda(v) \geq \lambda_*\}$. We have that ([24, Corollary 3.7])

$$k_n^{2d} n^{-d} \leq \mathbb{P}(v \in \mathcal{D}_*) \leq k_n^{8d} n^{-d}\,. \tag{3.2.2}$$

Note that the events $\{v \in \mathcal{D}_*\}$ for $v \in \mathbb{Z}^d$ are rare (c.f. (3.2.2)) and are only locally dependent. Thus, the set $\mathcal{D}_*$ can be divided into many isolated islands as incorporated in the next lemma.

Lemma 3.2.1. ([24, Lemma 3.8]) *For every constant $C_{3,0} > 0$, with $\widehat{\mathbb{P}}$-probability tending to one, there exists a skeletal set $\mathsf{V} \subseteq \mathcal{D}_* \cap \mathcal{C}(0) \cap B(0, C_{3,0}n(\log n)^{-2/d})$ such that*

$$\lambda(v) = \max\{\lambda(u) : u \in B(v,3R)\}, \quad \mathcal{D}_* \cap \mathcal{C}(0) \cap B(0, C_{3,0}n(\log n)^{-2/d}) \subseteq \bigcup_{v\in\mathsf{V}} B(v,3R),$$
$$\text{and} \quad B(v, nk_n^{-100d}) \text{ for } v \in \mathsf{V}\cup\{0\} \text{ are disjoint.} \tag{3.2.3}$$

We will fix the values of $\kappa, C_{3,0}$ in Theorem 3.2.3. The balls $B(v,(\log n)^\kappa)$ for $v \in \mathsf{V}$ will be referred to as *pocket islands*.

Path Localization. The following path localization result has been proved in [24]. Conditioned on survival, the random walk will travel to one of the pocket islands (which we refer to as the target island) and then it will be confined in the target island afterwards. In addition, the random walk will avoid getting close to any region that is better than or almost as good as (i.e., has larger or nearly the same principal eigenvalue) the target island, and the random walk will reach the target island at a time at most linear in the distance between the target island and the origin. We next give a more formal statement (see Theorem 3.2.3) on the path localization.

Definition 3.2.2. *For constant $\kappa, C_{3,1} > 0$ to be determined and each $v \in \mathbb{Z}^d$, we define the hitting time of $B(v, (\log n)^{\kappa/2})$*

$$\mathsf{T}_v = \tau_{B(v,(\log n)^{\kappa/2})},$$

and the event

$$\mathsf{E}_v = \left\{\tau_{\mathsf{D}_{\lambda(v)}} > \mathsf{T}_v, S_{[\mathsf{T}_v,n]} \subseteq B(v, (\log n)^{\kappa}), \tau > n\right\},$$

where $\mathsf{D}_\lambda = \{v \in \mathbb{Z}^d : \mathbf{P}^v(\tau > (\log n)^{C_{3,1}}) > [(1 - k_n^{-21d})\lambda]^{(\log n)^{C_{3,1}}}\}$.

Theorem 3.2.3 ([24])**.** *For constants $\kappa, C_{3,0}, C_{3,1}$ sufficiently large ($C_{3,0}$ and $C_{3,1}$ are defined in Lemma 3.2.1 and Definition 3.2.2, respectively) and $c \in (0,1)$ sufficiently small, with $\widehat{\mathbb{P}}$-probability tending to one,*

$$\mathbf{P}\Big(\bigcup_{v\in\mathsf{V}}(\mathsf{E}_v \cap \{\mathsf{T}_v \leq C|S_{\mathsf{T}_v}|\}) \mid \tau > n\Big) \geq 1 - e^{-(\log n)^c}. \tag{3.2.4}$$

Proof. It can be found in the proof of [24, Proposition 4.3 and (5.19)] that for sufficiently large $\kappa, C_{3,0}$, there is a random site $x_n \in \mathsf{V}$ (depending on $\mathcal{O}$ and potentially also depending on $(S_t)_{t=0}^n$) such that the following hold accordingly with $\widehat{\mathbb{P}}$-probability tending to one,

$$\mathbf{P}(S_{[\mathsf{T}_{x_n},n]} \subseteq B(x_n, (\log n)^{\kappa}) \mid \tau > n) \leq e^{-(\log n)^c},$$
$$\mathbf{P}(\mathsf{T}_{x_n} \leq C|S_{\mathsf{T}_{x_n}}| \mid \tau > n) \leq e^{-(\log n)^c}.$$

For the same x_n, [24, Lemma 5.6] states that for any constant q,

$$\mathbf{P}(\lambda(x) \le (1 - k_n^{-20d})\lambda(x_n), \forall x \in \bigcup_{0 \le t \le \mathsf{T}_{x_n}} B(S_t, (\log n)^q) \mid \tau > n) \le e^{-n/(\log n)^C}.$$

At the same time, [24, Lemma 4.5] yields if $\lambda(x) \le (1 - k_n^{-20d})\lambda, \forall x \in B(y, (\log n)^q)$, then

$$\mathbf{P}^y(\tau > (\log n)^{C_{3,1}}) \le (\log n)^C [(1 - k_n^{-20d})\lambda]^{(\log n)^{C_{3,1}}} \le [(1 - k_n^{-21d})\lambda]^{(\log n)^{C_{3,1}}},$$

where the last inequality holds if $C_{3,1}$ is sufficiently large. Combining above three results gives (3.2.4). □

We will choose $\kappa, C_{3,0}, C_{3,1}$ sufficiently large as in this theorem and will assume $\kappa \ge 10C_{3,2} + C_{3,1} + 10$ where $C_{3,2}$ is a constant to be selected in Lemma 3.4.10.

A few useful lemmas. We next record a few lemmas for later use.

Lemma 3.2.4. ([17, Theorem 3] and [38] (see also [46, Corollary 3])) *For standard (Bernoulli) site percolation on $\mathbb{Z}^d$ with parameter $p > p_c(\mathbb{Z}^d)$,*

$$\mathbb{P}(|\mathcal{C}(0)| = m) \le e^{-cm^{(d-1)/d}}. \tag{3.2.5}$$

Lemma 3.2.5. *For n sufficiently large, $\lambda_* \ge 1 - c_*(\log n)^{-2/d} - C(\log n)^{-3/d}$.*

Proof. Recall that $\varrho_n = \lfloor (\omega_d^{-1} d \log_{1/p} n)^{1/d} \rfloor$ as defined in (4.1.2). Let $r = \varrho_n - q$. By [51, (24)] and scaling we see that for any fixed $q > 0$,

$$1 - \lambda_{B(0,r)} \le c_*(\log n)^{-2/d} + O((\log n)^{-3/d}).$$

In addition, $|B(v, r)| \le d \log_p(n^{-1}) - \varrho_n^{d-1}$ for sufficently large q, hence for all $v \in \mathbb{Z}^d$,

$$\mathbb{P}(\lambda(v) \ge \lambda_{B(v,r)}) \ge \mathbb{P}(B(v, r) \text{ is open}) \ge n^{-d} e^{c\varrho_n^{d-1}}.$$

Fix a large q and compare it with (3.2.2). We get $\lambda_* \geq 1 - c_*(\log n)^{-2/d} - C(\log n)^{-3/d}$. $\square$

Lemma 3.2.6. *For n sufficiently large, $\lambda \geq \lambda_*$, and any $v \in \mathbb{Z}^d, t > 0$*

$$\mathbb{P}(v \in \mathsf{D}_\lambda) \leq (\log n)^C n^{-d}\,, \tag{3.2.6}$$

$$\mathbf{P}^v(\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > t) \leq (\log n)^C (1 - (\log n)^{-100d})^t \lambda^t\,. \tag{3.2.7}$$

Proof. If $x \in \mathsf{D}_\lambda$, then there exists $y \in B(x, (\log n)^{C_{3,1}})$ such that $\mathbf{P}^y(\tau > k_n) \geq [(1 - k_n^{-21d})\lambda_*]^{k_n+1} \geq \lambda_*^{k_n}/2$. Hence [24, Lemma 3.3] gives (3.2.6). The bound (3.2.7) is an analogue of [24, Lemma 4.4]. The adaption of the proof is straightforward: we can adapt the proof by just changing all occurrences of $k_n, R, \mathcal{U}_\alpha, p_\alpha$ to $(\log n)^{C_{3,1}}$, $(\log n)^{C_{3,1}}$, $\mathcal{D}_\lambda$, $\lambda^{(\log n)^{C_{3,1}}}$, respectively and noting that $k_n^{-21d} > (\log n)^{-100d}$. $\square$

3.3 One city theorem

3.3.1 Overview

In this section, we will give the proof of Theorem 3.1.2. We first give a heuristic description. There are poly-logarithmic many pocket islands (see (3.2.3),(3.2.4)). For each of them, the probability for localizing in that island is *roughly speaking* the product of the probability of reaching the island (which we refer to as searching probability) and the probability of staying in that island afterwards. Since these two probabilities are roughly independent, the fluctuation of the product across different islands is greater than the fluctuation of either of these two probabilities. We will work on the fluctuation of the searching probability and use it to show that one of the pocket islands will be dominating. Below are the key ingredients for demonstrating that the fluctuation of searching probability is large:

- We expect that the searching probability to a far away vertex v (this is close to the searching probability to a neighborhood around v) is exponentially small in $|v|$, where the rate of decay may depend on the direction $\frac{v}{|v|}$.

- The locations of these pocket islands are roughly independent and uniform in $B(0, n)$.

In fact, we can have a quantitative version for the first ingredient which controls the rate of convergence for the logarithmic of the searching probability. To prove this, we can adapt methods discussed in Section 3.1.3 on the rate of convergence in first-passage percolation (in particular the method in [4]).

However, our situation is more complicated as we have to keep track of the time spent on reaching an island. This is because, when we require the random walk to stay in the island after reaching it, the remaining amount of time is not fixed but depends on how much time the random walk has already spent on reaching the island. This motivates the following definition.

Definition 3.3.1. *Recall that* $\mathsf{T}_v = \tau_{B(v,(\log n)^{\kappa/2})}$. *For* $\lambda > 0$ *we define*

$$\varphi_\star(0, v; \lambda) = -\log \mathbf{E}\big[\lambda^{-\mathsf{T}_v} \mathbb{1}_{n \wedge \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} \geq \mathsf{T}_v}\big] \,. \tag{3.3.1}$$

We wish to make a couple of remarks on our definition of $\varphi_\star$.

- We have a term of $\lambda^{-\mathsf{T}_y}$ in the definition. This is because after reaching the island, every step of survival costs roughly a probability of λ (assuming the principal eigenvalue of the target island is λ), and thus for every step spent on reaching the island we give a reward of $\lambda^{-1} > 1$ to account for the saving on future probability cost. (3.6)

- We do not allow the random walk to enter D_λ. Otherwise, the random walk may stay in a region of eigenvalue greater than λ for excessively large amount of time and lead to an excessively small value of $\varphi_\star(x, y; \lambda)$ (since for every step the random walk gains a prize of λ^{-1}), and thus fails to serve its intended purpose.

Next, we list three ingredients for the proof of Theorem 3.1.2: Lemma 3.3.2 expresses $\mathbf{P}(\mathsf{E}_v)$ as a combination of $\lambda(v)$ and $\varphi_\star(0, v; \lambda(v))$; in Proposition 3.3.3 we approximate $\varphi_\star$

by a linear function g; Lemma 3.3.5 encapsulates our basic intuition that the fluctuation of $g(v;\lambda(v))$ should be large since sites in V are roughly uniformly distributed in $B(0,n)$.

Lemma 3.3.2. *With $\widehat{\mathbb{P}}$-probability tending to one, uniformly for all $v \in \mathsf{V}$*

$$\log \mathbf{P}(\mathsf{E}_v) = n \log \lambda(v) - \varphi_\star(0, v; \lambda(v)) + O((\log n)^C)\,. \tag{3.3.2}$$

Proposition 3.3.3. *There exists a deterministic nonnegative function g such that for some constants $\underline{c}, \bar{C}$ depending only on (d,p) and all $\lambda \in [\lambda_*, 1]$,*

$$g(\,\cdot\,;\lambda) \text{ is convex, homogeneous of order 1 and } \underline{c}|x| \le g(x;\lambda) \le \bar{C}|x|\,. \tag{3.3.3}$$

Also, with $\widehat{\mathbb{P}}$-probability tending to one, uniformly for all $v \in \mathsf{V}$

$$\varphi_\star(0, v; \lambda(v)) = g(v;\lambda(v)) + O(n^{5/6})\,. \tag{3.3.4}$$

Remark 3.3.4. For our purpose, we are interested in the case when $\lambda < 1$ since λ will be the principal eigenvalue of a transition matrix of random walk with killing. Similar results has been proved when $\lambda \ge 1$ in [72], where one considers Brownian motion with Poissonian obstacles and the corresponding $g(\,\cdot\,;\lambda)$ will be proportional to Euclidean distance. We note that our case when $\lambda < 1$ is substantially more challenging to analyze (since for instance, we have to forbid the random walk to enter D_λ which incurs a number of complications in the proof).

Lemma 3.3.5. *With $\mathbb{P}$-probability tending to one, for any distinct $u, v \in \mathsf{V}$ there exists a large constant $C > 0$ depending only on (d,p) such that*

$$|(n \log \lambda(v) - g(v;\lambda(v))) - (n \log \lambda(u) - g(u;\lambda(u)))| \ge n(\log n)^{-C}\,.$$

Let v_* be the maximizer of the following variational problem

$$\max_{v \in \mathsf{V}} n \log \lambda(v) - g(v; \lambda(v)) \,. \tag{3.3.5}$$

Combining preceding three results, we will show in *Proof of Theorem 3.1.2* that conditioned on survival, with $\widehat{\mathbb{P}}$-probability tending to one the random walk travels to the pocket island around v_* and stays there afterwards. We will call the pocket island around v_* the *optimal pocket island*.

Remark 3.3.6. By Theorem 3.1.2, for a typical environment, with $\mathbf{P}$-probability tending to 1 the target island as described in Section 3.2 coincides with the optimal pocket island. Furthermore,

$$\log \mathbf{P}(\tau > n) = n \log \lambda(v_*) - g(v_*; \lambda(v_*)) + O(n^{5/6}) \,. \tag{3.3.6}$$

3.3.2 Proof of Lemmas 3.3.2, 3.3.5 and Theorem 3.1.2

In this subsection, we prove Lemmas 3.3.2, 3.3.5 and then prove Theorem 3.1.2 by combining Lemmas 3.3.2, 3.3.5 and Proposition 3.3.3. The proof of Proposition 3.3.3 is postponed to Section 3.4 and occupies the entire section.

Proof of Lemma 3.3.2. By the strong Markov Property, we get that

$$\mathbf{P}(\mathsf{E}_v) = \mathbf{E}\left[\mathbb{1}_{\tau_{\mathsf{D}_{\lambda(v)}} \cup \mathcal{O} > \mathsf{T}_v} \mathbf{P}^{S_t}(\xi_{B(v,(\log n)^\kappa) \setminus \mathcal{O}} > n - t)|_{t = \mathsf{T}_v}\right] \,. \tag{3.3.7}$$

Note that for $m \in (0, n)$, by (3.2.3) and [24, Lemma 4.5],

$$\mathbf{P}^{S_{\mathsf{T}_v}}(\xi_{B(v,(\log n)^\kappa) \setminus \mathcal{O}} > m) \le \mathbf{P}^{S_{\mathsf{T}_v}}(\lambda(x) \le \lambda(v) \text{ for } x \in S_{[0,m]}, \tau > m) \le (\log n)^C \lambda(v)^m \,.$$

At the same time, since $v, S_{\mathsf{T}_v} \in \mathcal{C}(0)$ and $|S_{\mathsf{T}_v} - v| \le (\log n)^{\kappa/2}$, by [7, Theorem 1.1] (or [24, (3.8)]) we see that any point in $\mathcal{C}_R(v)$ (defined in Section 3.2) and S_{T_v} can be connected by an open path of length at most $C(\log n)^{\kappa/2}$ (Thus the open path is also inside $B(v, (\log n)^\kappa)$). Combined with $\max_x \mathbf{P}^x(\xi_A > t) \geq \lambda_A^t$ (see [24, Lemma 3.2]), it gives that for $m \in (0, n)$

$$\mathbf{P}^{S_{\mathsf{T}_v}}(\xi_{B(v,(\log n)^\kappa)\setminus\mathcal{O}} > m) \geq e^{-(\log n)^C}\lambda(v)^m\,. \tag{3.3.8}$$

Combining preceding three displays completes the proof of the lemma. □

Proof of Lemma 3.3.5. It suffices to prove the following: with $\mathbb{P}$-probability tending to one, for any $u, v \in \mathcal{D}_*$ such that $|u - v| \geq (\log n)^6$,

$$|(n\log\lambda(v) - g(v;\lambda(v))) - (n\log\lambda(u) - g(u;\lambda(u)))| \geq n(\log n)^{-C}\,.$$

Now we verify this statement. Let A be the set of all the possible values for the random variable $\log\lambda(v)$ that are greater than or equal to $\log\lambda_*$ and $C' > 0$ be a large constant to be selected. We have

$$\begin{aligned}&\sum_{\substack{u,v\in B(0,n)\\|u-v|\geq(\log n)^6}} \mathbb{P}\Big(|(n\log\lambda(v) + g(v;\lambda(v))) - (n\log\lambda(u) + g(u;\lambda(u)))| \leq \frac{n}{(\log n)^{C'}}, u, v \in \mathcal{D}_*\Big)\\ &\qquad= \sum_{u,v\in B(0,n),|u-v|\geq(\log n)^6}\ \sum_{a_1,a_2\in A} \mathbb{P}(\log\lambda(v) = a_1, \log\lambda(u) = a_2) I_{a_1,a_2}\,,\end{aligned}$$

where $I_{a_1,a_2} = \mathbb{1}_{|(na_1+g(v;e^{a_1}))-(na_2+g(u;e^{a_2}))|\leq n(\log n)^{-C'}}$. Noting that (3.3.3) holds for all $\lambda \in [\lambda_*, 1]$, we let

$$\mathcal{H} = \{h : \mathbb{R}^d \to \mathbb{R}^+ : \underline{c} \leq h(v)/|v| \leq \bar{C} \text{ for all } v \in \mathbb{Z}^d, h \text{ convex, homogeneous of degree } 1\}. \tag{3.3.9}$$

Then for all $h \in \mathcal{H}, x > 0$, $h^{-1}(\{x\})$ is a convex set in $B(0, x/\underline{c})$ and

$$\sup_{a_1 \in A} \sup_{x \in \mathbb{R}} |\{v \in B(0,n) : |g(v; e^{a_1}) - x| \le n(\log n)^{-C'}\}|$$
$$\le \sup_{h \in \mathcal{H}} \sup_{x \in \mathbb{R}} |\{v \in B(0,n) : |h(v) - x| \le n(\log n)^{-C'}\}|$$
$$= \sup_{h \in \mathcal{H}} \sup_{x \in \mathbb{R}} |\{y \in \mathbb{R}^d : \exists\, v \in B(0,n) \text{ such that } |h(v) - x| \le n(\log n)^{-C'}, |v - y|_\infty \le 1/2\}|,$$

where in the last expression the outmost $|\cdot|$ stands for Lebesgue measure instead of cardinality. Note that $|v - y|_\infty \le 1/2$ implies $|h(v) - h(y)| \le \bar{C} d^{1/2}$ and $h(v) \le \bar{C} n$ for $v \in B(0,n)$. We thus obtain that

$$\sup_{a_1 \in A} \sup_{x \in \mathbb{R}} |\{v \in B(0,n) : |g(v; e^{a_1}) - x| \le n(\log n)^{-C'}\}|$$
$$\le \sup_{h \in \mathcal{H}} \sup_{x \le 2\bar{C} n} \left| h^{-1}([x - n(\log n)^{-C'} - \bar{C} d^{1/2}, x + n(\log n)^{-C'} + \bar{C} d^{1/2}]) \right|$$
$$\le C \underline{c}^{-1} (2\bar{C} n)^{d-1} \cdot (\underline{c}^{-1} n (\log n)^{-C'} + \underline{c}^{-1} 2\bar{C} d^{1/2}) \le C n^d (\log n)^{-C'},$$

where the second inequality follows from the fact that the surface area of any convex set contained in a ball is less than the surface area of that ball (see, e.g., [39, Page 48–50]). Since $\lambda(v)$ and $\lambda(u)$ are independent if $|u - v| \ge (\log n)^6$, we conclude that

$$\sum_{\substack{u,v \in B(0,n) \\ |u-v| \ge (\log n)^6}} \sum_{a_1, a_2 \in A} \mathbb{P}(\log \lambda(v) = a_1, \log \lambda(u) = a_2) \cdot I_{a_1,a_2} \le 2^d n^{2d} (\log n)^{-C'} (\mathbb{P}(v \in \mathcal{D}_*))^2.$$

Now, the results follow from (3.2.2) and choosing C' such that $(\log n)^{C'} \ge k_n^{20d}$. □

Proof of Theorem 3.1.2. Recall that v_* is the maximizer of the function $v \mapsto n \log \lambda(v) - g(v; \lambda(v))$ (c.f. (3.3.5)). Then combining Lemmas 3.3.2, 3.3.5 and Proposition 3.3.3, we get that with $\widehat{\mathbb{P}}$-probability tending to one

$$\log \mathbf{P}(\mathsf{E}_{v_*}) - \log \mathbf{P}(\mathsf{E}_u) \geq 2^{-1} n (\log n)^{-C} \quad \forall u \in \mathsf{V} \setminus \{v_*\}\,.$$

Hence, on the preceding event $\mathbf{P}(\mathsf{E}_{v_*})/\sum_{u \in \mathsf{V}} \mathbf{P}(\mathsf{E}_u) \geq 1 - e^{-n(\log n)^{-C}}$. Combined with (3.2.4), it follows that $\mathbf{P}(\mathsf{E}_{v_*} \mid \tau > n) \geq 1 - e^{-(\log n)^c}$. Define

$$\mathcal{U} := \text{the connected component in } B(v_*, (\log n)^\kappa) \text{ that contains } v_*. \tag{3.3.10}$$

And recall that $T = \mathsf{T}_{v_*}$. On event E_{v_*}, the random walk stays in $B(v_*, (\log n)^\kappa)$ during $[T, n]$. The discussion before (3.3.8) yields $S_T \in \mathcal{U}$. Hence

$$\mathbf{P}\Big(T \leq C|v_*|, \{S_t : t \in [T, n]\} \subseteq \mathcal{U} \mid \tau > n\Big) \leq e^{-(\log n)^c}\,. \tag{3.3.11}$$

This completes the proof of the theorem. □

3.4 Approximation and concentration for $\varphi_\star$-function

This entire section is devoted to the proof of Proposition 3.3.3. To this end, we first introduce a couple of definitions.

Definition 3.4.1. *For $\lambda > 0$, $A \subseteq \mathbb{Z}^d$ and $x, y \in \mathbb{Z}^d$, we define*

$$G_A(x, y; \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} \mathbf{P}^x(S_n = y, S_{[1,n-1]} \subseteq A)\,.$$

Note that in the preceding definition, we have chosen λ^{-n} as opposed to the more conventional λ^n so that it will be consistent with the definition below.

Definition 3.4.2. *We set $r(x, y) = (\log |x - y|)^{2\kappa + 10d}$, $\mathcal{V}_\lambda = (\mathsf{D}_\lambda \cup \mathcal{O})^c$ and define log-*

weighted Green's functions (LWGF)

$$\begin{aligned} \varphi(x,y;\lambda) &= -\log\left(\mathbf{E}^x\left[\lambda^{-\tau_y}\mathbb{1}_{\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}>\tau_y}\right]\right) \\ \varphi_*(x,y;\lambda) &= \min_{x'\in B(x,r(x,y))}\min_{y'\in B(y,r(x,y))}\varphi(x',y';\lambda)\,. \end{aligned} \tag{3.4.1}$$

Note that the name of log-weighted Green's functions came from the factor of $\lambda^{-\tau_y}$ in the preceding definition. Also,

$$\varphi(x,y;\lambda) = -\log G_{\mathcal{V}_\lambda\setminus\{y\}}(x,y;\lambda) = -\log\left(G_{\mathcal{V}_\lambda}(x,y;\lambda)/G_{\mathcal{V}_\lambda}(y,y;\lambda)\right). \tag{3.4.2}$$

The next result justifies the approximation of $\varphi_\star$ by the log-weighted Green's functions, whose proof can be found in Section 3.4.2.

Lemma 3.4.3. *For n sufficiently large and all $v\in\mathbb{Z}^d$ with $|v|\in(n^{2/3},C_{3,0}n(\log n)^{-2/d})$,*

$$\mathbb{P}\Big(\max_{\lambda\in[\lambda_*,1]}|\varphi_*(0,v;\lambda)-\varphi_\star(0,v;\lambda)|\le(\log n)^C\mid G_0\Big)\ge 1-e^{-c(\log n)^2}\,.$$

where

$$G_0 := \{\mathcal{C}(0)\cap B^c_{n^{1/2}}(0)\neq\varnothing,\mathsf{D}_{\lambda_*}\cap B(0,n^{1/2})=\varnothing\}\,. \tag{3.4.3}$$

Here the event G_0 is used as an approximation of $\{0\in\mathcal{C}(\infty)\}$ (see Lemma 3.2.4, (3.2.6)), and in *Proof of Proposition* 3.3.3 we will take advantage of the fact that G_0 only depends on the local environment of 0. We also wish to make a couple of remarks on Definition 3.4.2:

- We first approximate $\varphi_\star$ by φ where we replace T_y by τ_y — this is useful since later we will apply sub-additive arguments and it would be convenient to get rid of reference to n in the definition (recall that $\mathsf{T}_y=\tau_{B(y,(\log n)^{\kappa/2})}$). In addition, we do not restrict $\tau_y<n$, which will be justified in Lemma 3.4.16.
- We further approximate φ by φ_* by allowing to minimize over the starting and ending points in some local ball around x and y, for the purpose of getting around the

complication when x and y are disconnected by obstacles (which occurs with positive $\mathbb{P}$-probability).

In addition, we note that in later subsections we will introduce more approximations of LWGFs to facilitate our analysis.

The rest of this section is organized as follows: In Section 3.4.1 we apply renormalization techniques to control the chemical distances on the cluster $(\mathsf{D}_\lambda \cup \mathcal{O})^c$ as well as some refined geometric properties (c.f. Lemma 3.4.9) for later use. In Section 3.4.2, we justify the approximation of φ_* by the LWGF and prove a few technical lemmas about LWGF. In Section 3.4.3, we prove a concentration inequality for φ_* and sub-additivity of $\mathbb{E}\varphi_*$. Then in Section 3.4.4, we follow the framework developed in [4] for first-passage percolation to prove that $\mathbb{E}\varphi_*$ is approximated by g with approximation error bounded by $O(|v|^\alpha)$ for some $\alpha < 1$. In this step, we have to address a number of challenges that are not seen in the first-passage percolation setup, due to the complication in the definition of our log-weighted Green's function φ_*. Finally, in Section 3.4.5 we prove Proposition 3.3.3, by combining the ingredients in previous subsections.

3.4.1 Percolation process avoiding high survival probability regions

In this subsection, we study connectivity properties for the percolation process on $\mathcal{O}^c \cap \mathsf{D}^c_{\lambda_*}$ where λ_* is defined in (3.2.1) — this will be useful later when analyzing LWGFs. In order to analyze this percolation process with (short-range) correlations, we employ the standard renormalization technique in percolation theory, and reduce it to the analysis of a certain independent percolation process (see Lemmas 3.4.7 and 3.4.8, and discussions following Lemma 3.4.8).

Definition 3.4.4. *Recall that* $K_r(v) = \{x \in \mathbb{Z}^d : |x - v|_\infty \le r\}$. *Let* $L = \lfloor \log n \rfloor^{2d}$ *and consider disjoint boxes*

$$\mathbb{K}_{\mathbf{i}} := K_L((2L+1)\mathbf{i}) \quad \textit{for} \quad \mathbf{i} \in \mathbb{Z}^d\,. \tag{3.4.4}$$

We define the renormalized lattice $\{\mathbb{K}_{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^d\}$ *which inherits the graph structure from the bijection* $\mathbb{K}_{\mathbf{i}} \mapsto \mathbf{i}$.

Definition 3.4.5. *Let* C_D *be a positive constant to be selected in Lemma 3.4.7. We say* $\mathbb{K}_{\mathbf{i}}$ *(or* $\mathbf{i}$*) is white (otherwise black), if the following hold:*

1. *There exists a unique open connected component* $\mathscr{C}_{\mathbf{i}}$ *in* $K_{C_D L}((2L+1)\mathbf{i})$*, such that*

$$|\mathscr{C}_{\mathbf{i}} \cap (\bigcup_{\mathbf{j}:\|\mathbf{j}-\mathbf{i}\|_\infty \le 1} \mathbb{K}_{\mathbf{j}})| \ge L/10\,. \tag{3.4.5}$$

2. *For all* $u, v \in \mathscr{C}_{\mathbf{i}} \cap (\bigcup_{\mathbf{j}:\|\mathbf{j}-\mathbf{i}\|_\infty \le 1} \mathbb{K}_{\mathbf{j}})$ *(recalling definition of* $D.(\cdot,\cdot)$ *in Section 3.1.5),*

$$D_{\mathscr{C}_{\mathbf{i}}}(u, v) \le C_D L\,. \tag{3.4.6}$$

3. $K_{C_D L}((2L+1)\mathbf{i}) \cap \mathsf{D}_{\lambda_*} = \varnothing$*. In addition, for all* $\mathbf{j}$ *satisfying* $\|\mathbf{j}-\mathbf{i}\|_\infty \le 1$*, one has*

$$|\mathscr{C}_{\mathbf{i}} \cap \mathbb{K}_{\mathbf{j}}| \ge L/10\,. \tag{3.4.7}$$

Remark 3.4.6. The requirements in (3.4.5) and (3.4.7) look somewhat odd and repetitive at first glance. We present the conditions in this way since we wish that the component satisfying (3.4.5) is unique, and in addition this component satisfies (3.4.7). This is stronger than the claim that there is a unique connected component satisfying (3.4.7).

The collection of white vertices gives a dependent site percolation process on the renormalized lattice. In the next two lemmas, we will show that the white vertices dominate a supercritical Bernoulli percolation for an appropriate choice of C_D and $n \ge n_0$ where n_0 is a fixed large constant.

Lemma 3.4.7. *There exists a constant* $C_D \ge 4$ *such that for* n *sufficiently large and all* $\mathbf{i} \in \mathbb{Z}^d$

(1) The event $\{\mathbb{K}_{\mathbf{i}}$ *is white*$\}$ *is independent of* $\sigma($ $\{\mathbb{K}_{\mathbf{j}}$ *is white*$\}$ for $\mathbf{j}$ s.t. $|\mathbf{j}-\mathbf{i}| \geq 4C_D)$.

(2) $\mathbb{P}(\mathbb{K}_{\mathbf{i}}$ *is white*$) \geq 1 - n^{-1}$.

Proof. The first item is a direct consequence of the definition. Now, we verify the second item. We first claim that $\mathbb{K}_{\mathbf{i}}$ is white if all of the following hold:

(a) For all $u, v \in \bigcup_{\mathbf{j}:\|\mathbf{j}-\mathbf{i}\|_\infty \leq 1} \mathbb{K}_{\mathbf{j}}$ such that u, v are in the same open connected component,

$$D_{\mathcal{O}^c}(u,v) \leq C_D L\,. \tag{3.4.8}$$

(b) For any $v \in \bigcup_{\mathbf{j}:\|\mathbf{j}-\mathbf{i}\|_\infty \leq 1} \mathbb{K}_{\mathbf{j}}$, either $\mathcal{C}(v) = \mathcal{C}(\infty)$ or $|\mathcal{C}(v)| < L/10$.

(c) For all $\|\mathbf{j}-\mathbf{i}\|_\infty \leq 1$,

$$|\mathcal{C}(\infty) \cap \mathbb{K}_{\mathbf{j}}| \geq L/10\,. \tag{3.4.9}$$

(d) $K_{C_D L}((2L+1)\mathbf{i}) \cap \mathsf{D}_{\lambda_*} = \varnothing$.

To verify this, we observe that (a) implies that

$$\text{all vertices in } \mathcal{C}(\infty) \cap \Big(\bigcup_{j:\|j-i\|_\infty \leq 1} \mathbb{K}_{\mathbf{j}}\Big) \text{ are connected in } K_{C_D L}((2L+1)\mathbf{i}) \cap \mathcal{C}(\infty)\,. \tag{3.4.10}$$

Combining with (b) and (c), we get Property 1 in Definition 3.4.5 where $\mathscr{C}_{\mathbf{i}}$ is the connected component of $\mathcal{C}(\infty) \cap K_{C_D L}((2L+1)\mathbf{i})$ which has non-empty intersection with $\mathbb{K}_{\mathbf{i}}$ — such a connected component is unique by (3.4.10). Combining this with (d) gives Property 3. Property 2 follows from (a).

Now, [7, Theorems 1.1] yields

$$\mathbb{P}((\text{a}) \text{ holds}) \geq 1 - (6L+3)^{2d} e^{-cL^{1/d}}.$$

By Lemma 3.2.4

$$\mathbb{P}((\text{b}) \text{ holds}) \geq 1 - (6L+3)^{2d} e^{-cL^{1/2}}.$$

Combined with [59, Theorem 5], this implies

$$\mathbb{P}((\text{b}) \text{ and } (\text{c}) \text{ holds}) \geq 1 - (6L+4)^{2d} e^{-cL^{1/2}}.$$

Finally, by (3.2.6)

$$\mathbb{P}((\text{d}) \text{ holds}) \geq 1 - (2C_D L + 1)^d (\log n)^C n^{-d}.$$

Altogether, this completes the proof of the lemma. □

Lemma 3.4.8. *For any $\epsilon > 0$ the white vertices stochastically dominates (with respect to subset inclusion) the open vertices in a supercritical independent site percolation with parameter $1-\epsilon$ as long as n is greater than a large constant depending on (d, ϵ).*

Proof. The lemma follows from Lemma 3.4.7 and [55, Theorem 0.0]. □

In what follows, we will call vertices that are open in this Bernoulli$(1-\epsilon)$ percolation tilde-white. Thus, there is a coupling such that a tilde-white vertex is always white, and tilde-white vertices form a Bernoulli percolation with parameter $1-\epsilon$. In what follows, we will work with tilde-white vertices as opposed to white vertices. We will call the tilde-white percolation the *macroscopic* process and call the original site percolation (where open means free of obstacles) *microscopic* process. For instance, we will refer to a microscopic path as a path that consists of vertices in the original lattice and a macroscopic path as a path that consists of vertices in the renormalized lattice. For any $x \in \mathbb{Z}^d$, we denote $\mathbf{i}_x$ be such that $\mathbb{K}_{\mathbf{i}_x}$ is the unique macroscopic box that contains x. For $\mathbf{A} \subseteq \mathbb{Z}^d$, define

$$\mathbb{K}_{\mathbf{A}} := \bigcup_{\mathbf{i} \in \mathbf{A}} \mathbb{K}_{\mathbf{i}} \quad \text{and} \quad \mathscr{C}_{\mathbf{A}} := \bigcup_{\mathbf{i} \in \mathbf{A}} \mathscr{C}_{\mathbf{i}}, \tag{3.4.11}$$

where $\mathscr{C}_{\mathbf{i}}$ is defined as in Definition 3.4.5 if $\mathbf{i}$ is tilde-white and an empty set otherwise. For $\mathbf{i} \in \mathbb{Z}^d$, we denote by $\mathcal{C}_{\mathbb{K}}(\mathbf{i})$ the tilde-white cluster containing $\mathbb{K}_{\mathbf{i}}$. If $\mathbf{i}$ is tilde-black (i.e., not tilde-white), then $\mathcal{C}_{\mathbb{K}}(\mathbf{i}) = \varnothing$.

For $A_1, A_2, A_3 \subseteq \mathbb{Z}^d$, we say A_1 is a vertex cut that separates A_2 and A_3 if any path joining A_2 and A_3 has nonempty intersection with A_1.

Lemma 3.4.9. *For any $x \in \mathbb{Z}^d$ and $r > (\log n)^{3d}$, let $\mathbf{\Lambda}(x,r)$ be an arbitrary (macroscopic) tilde-white connected set such that $\mathbb{K}_{\mathbf{\Lambda}(x,r)}$ is a vertex cut that separates $B(x,r)$ and $(B(x,2r))^c$ (See Figure 3.1). If such cut does not exist, let $\mathbf{\Lambda}(x,r) = \varnothing$. Then for n sufficiently large,*

(1) $\mathbb{P}(\mathbf{\Lambda}(x,r) = \varnothing) \le e^{-rL^{-1}}$.

(2) For any $u \in \mathbb{K}_{\mathbf{\Lambda}(x,r)}$ such that $|\mathcal{C}(u)| \ge L/10$, we have $u \in \mathscr{C}_{\mathbf{\Lambda}(x,r)}$. Any $u, v \in \mathscr{C}_{\mathbf{\Lambda}(x,r)}$ can be join by a path in $\mathscr{C}_{\mathbf{\Lambda}(x,r)}$(thus in $(\mathcal{O} \cup \mathsf{D}_{\lambda_*})^c$) of length at most r^{2d}.

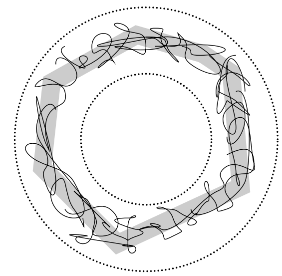

Figure 3.1: Two dotted circles centered at x, with radius r and $2r$ accordingly. Shaded area represents $\mathbf{\Lambda}(x,r)$. Solid curves represent $\mathscr{C}_{\mathbf{\Lambda}(x,r)}$.

Proof. (1) It suffices to prove that in the macroscopic lattice there exists a tilde-white connected set $\mathbf{\Lambda}$ that separates $B(\mathbf{i}_x, R)$ and $(B(\mathbf{i}_x, 3R/2))^c$ for $R = r(2L)^{-1} + 2$, since each microscopic path not intersecting with $\mathbb{K}_{\mathbf{\Lambda}}$ corresponds to a macroscopic path not intersecting with $\mathbf{\Lambda}$. (Here, $B(\mathbf{i}_x, R)$ is the discrete ball in the macroscopic lattice.)

For $q > 0$, we say a subset of $\mathbb{Z}^d$ is $\sharp_q$-connected, if it is connected with respect to the adjacency relation

$$x \overset{\sharp_q}{\sim} y \iff |x-y|_1 \le q, .$$

In addition, we say a subset of $\mathbb{Z}^d$ is $*$-connected, if it is connected with respect to the adjacency relation

$$x \overset{*}{\sim} y \iff |x-y|_\infty \le 1.$$

Now, let A be the union of $B(\mathbf{i}_x, R)$ and all tilde-black $\sharp_{3d}$-connected components that have nonempty intersection with $\bigcup_{\mathbf{j} \in B(\mathbf{i}_x, R)} \{\mathbf{k} : |\mathbf{k} - \mathbf{j}|_1 \le 3d\}$. If $A \not\subseteq B(\mathbf{i}_x, 3R/2 - 10d)$, then there exists a $\sharp_{3d}$-connected tilde-black path connecting $B(\mathbf{i}_x, R+3d)$ and $(B(\mathbf{i}_x, 3R/2-10d))^c$. By definition of tilde-white process, we get from a simple union bound over all $\sharp_{3d}$-connected tilde-black path connecting $B(\mathbf{i}_x, R+3d)$ and $B(\mathbf{i}_x, 3R/2-10d)^c$ that

$$\mathbb{P}(A \not\subseteq B(\mathbf{i}_x, 3R/2-10d)) \leq (2R+6d)^d (6d+1)^{d(3R/2-10d)} \epsilon^{3R/2-10d} \leq e^{-2R},$$

for sufficiently large n (and thus ϵ is sufficiently small). On the event $\{A \subseteq B(\mathbf{i}_x, 3R/2-10d)\}$, we let $A' = \bigcup_{\mathbf{j}\in A}\{\mathbf{k} : |\mathbf{k}-\mathbf{j}|_1 \leq d\}$ and A_0 be the connected component in A' containing $B(\mathbf{i}_x, R)$. Then by [20, Lemma 2.1] (which states that the external outer boundary of a $*$-connected set is $*$-connected), there is a subset of ∂A_0 which is $*$-connected and contains A_0 in its interior. We denote this subset by $\mathbf{\Lambda}_0$ and let $\mathbf{\Lambda} = \bigcup_{\mathbf{j}\in\mathbf{\Lambda}_0}\{\mathbf{k} : |\mathbf{k}-\mathbf{j}|_\infty \leq 1\}$. Then, the set $\mathbf{\Lambda}$ is connected and is contained in $B(\mathbf{i}_x, 3R/2)$. It remains to prove that $\mathbf{\Lambda}$ is tilde-white and contains $A = B(\mathbf{i}_x, R)$ in its interior.

Note that for any $\mathbf{j} \in \mathbf{\Lambda}_0$, the ℓ_1-distance between $\mathbf{j}$ and A is $d+1$. Hence, the set $\mathbf{\Lambda}$ contains $A = B(\mathbf{i}_x, R)$ in its interior. By the $\sharp_{3d}$-connectivity, the ℓ_1-distance between $\mathbf{j}$ and any tilde-black point in A^c is at least $3d-(d+1) \geq d+1$. Hence, the set $\mathbf{\Lambda}$ is also free of tilde-black points.

(2) By Property 1 in Definition 3.4.5, $|\mathcal{C}(u)| \geq L/10$ implies that $u \in \mathscr{C}_{\mathbf{\Lambda}(x,r)}$. Also, since $\mathbf{\Lambda}(x,r)$ is connected, by Properties 1 and 3 in Definition 3.4.5, $\mathscr{C}_{\mathbf{\Lambda}(w,r)}$ is connected. Then the last statement follows from $|\mathscr{C}_{\mathbf{\Lambda}(x,r)}| \leq r^{2d}$. □

Lemma 3.4.10. *Recall* $\mathcal{V}_\lambda = (\mathsf{D}_\lambda \cup \mathcal{O})^c$ *and let* $C_{3,2} = 6d^2$. *Suppose that* $\lambda \geq \lambda_*$ *where* λ_* *is defined in* (3.2.1). *For* n *sufficiently large and any* $x, y \in \mathbb{Z}^d$ *such that* $|x-y| \geq n^{1/10}$, *with* $\mathbb{P}$-*probability at least* $1-e^{-(\log|x-y|)^3}$ *we have*

(1) There is a unique connected component in $\mathcal{V}_\lambda$ *of diameter no less than* $(\log|x-y|)^{C_{3,2}}$ *that intersect with* $B(x, |x-y|^2)$.

(2) For any u, v *in this component, there exists a path in* $\mathcal{V}_\lambda$ *that connects* u, v *and which has length at most* $\max(C|u-v|, (\log|x-y|)^C)$.

Proof. Suppose $B_1, B_2 \subseteq \mathcal{V}_\lambda$ are two connected set and both B_1 and B_2 have diameter at least $(\log|x-y|)^{C_{3,2}}$ and have nonempty intersection with $B(x, |x-y|^2)$. We will prove that with $\mathbb{P}$-probability at least $1 - e^{-(\log|x-y|)^3}$ for any $v_1 \in B_1, v_2 \in B_2$, there exists a path in $\mathcal{V}_\lambda$ that connects v_1, v_2 and is of length less than $\max(C|v_1 - v_2|, (\log|x-y|)^C)$. Provided with this claim, the lemma follows immediately. Next we verify the claim.

Without loss of generality, we suppose $|B_1|, |B_2| \leq (2\log|x-y|)^{C_{3,2}d}$. We suppose $\mathbf{\Lambda}(v, r)$ exists for $r = (\log|x-y|)^{5d}$ and all $v \in B(x, 2|x-y|^2)$, which has probability at least $1 - e^{-(\log|x-y|)^4}$ according to Lemma 3.4.9. Since $C_{3,2} = 6d^2$, it follows that $B_i \cap \mathbb{K}_{\mathbf{\Lambda}(v_i,r)} \neq \varnothing$. By Property 1 in Definition 3.4.5 and that B_i is open, it follows that $B_i \cap \mathscr{C}_{\mathbf{\Lambda}(v_i,r)} \neq \varnothing$.

Since tilde-white percolation is a supercritical independent site percolation, with probability 1 there exists a unique infinite (macroscopic) tilde-white cluster (c.f. [1]), which we denote as $\mathcal{C}_{\mathbb{K}}(\infty)$. Then the following holds with probability at least $1 - e^{-(\log|x-y|)^4}$ (by [7, Theorems 1.1] (or [24, (3.8)]) and Lemma 3.2.4)

(a) For all $\mathbf{i}, \mathbf{j} \in \mathcal{C}_{\mathbb{K}}(\infty) \cap B(\mathbf{i}_x, |x-y|^2)$, $D_{\mathcal{C}_{\mathbb{K}}(\infty)}(\mathbf{i}, \mathbf{j}) \leq \max(C|\mathbf{i} - \mathbf{j}|, (\log|x-y|)^5)$.

(b) For all $\mathbf{i} \in \mathbb{Z}^d \cap B(\mathbf{i}_x, |x-y|^2)$, either $\mathcal{C}_{\mathbb{K}}(\mathbf{i}) = \mathcal{C}_{\mathbb{K}}(\infty)$ or $|\mathcal{C}_{\mathbb{K}}(\mathbf{i})| \leq (\log|x-y|)^9$.

Then it follows from (b) that $\mathbf{\Lambda}(v_i, r) \subseteq \mathcal{C}_{\mathbb{K}}(\infty)$ for $i = 1, 2$. Combined with (a) and the Property 2 in Definition 3.4.5, it yields the claim we described at the beginning of the proof and thus complete the proof of the lemma. □

3.4.2 Approximations of LWGF

In order to prove concentration inequality for φ_* and sub-additivity for $\mathbb{E}\varphi_*$ in Section 3.4.3, we shall introduce $\bar{\varphi}_*$, and φ° as follows. For notation convenience, we will omit λ in φ_* (recalling that $\lambda \geq \lambda_*$).

Definition 3.4.11. *Recall* (3.4.1)*. We define $\bar\varphi$, the truncation of φ, as*

$$\bar\varphi_*(x,y) = \underline{c}|x-y| \vee \left(\varphi_*(x,y) \wedge \bar C|x-y|\right), \tag{3.4.12}$$

and define φ°'s as

$$\begin{aligned}
\varphi^\circ(x,y) &= -\log\left(\mathbf{E}^x\left[\lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y, \xi_{B(x,R_\circ)} > \tau_y\right]\right), \\
\varphi_*^\circ(x,y) &= \min_{x' \in B(x,r(x,y))} \min_{y' \in B(y,r(x,y))} \varphi^\circ(x',y'), \\
\bar\varphi_*^\circ(x,y) &= \underline{c}|x-y| \vee \left(\varphi_*^\circ(x,y) \wedge \bar C|x-y|\right),
\end{aligned} \tag{3.4.13}$$

where $\underline{c}, \bar C > 0$ are two constants to be selected, $R_\circ = |x-y|(\log|x-y|)^{C_\circ}$, $C_\circ = C_{3,1} + 3$. (Recall $r(x,y) = (\log|x-y|)^{2\kappa+10d}$).

We remark that φ° is φ with an additional restriction that the random walk does not exit a big ball centered at x (the superscript $\circ$ stands for restriction to a "ball", and the underline and bar decorations of C in the notation correspond to lower and upper truncations.)

It follows directly from definition that

$$\varphi^\circ(x,y) \geq \varphi(x,y). \tag{3.4.14}$$

Some remarks are in order for Definition 3.4.11.

- As we will show in Lemma 3.4.12 $\varphi_*(x,y)$ is linear in $|x-y|$. Then we can (and we will) choose $\underline{c}, \bar C$ such that φ_* and φ_*° (also $\bar\varphi_*$ and $\bar\varphi_*^\circ$) are close. Such truncations are useful as they allow us to work with functions that are deterministically bounded from below and above.

- We will show in Lemma 3.4.13 that φ and φ° are close to each other and show in Lemma 3.4.14 that concentration of $\bar\varphi_*^\circ(x,y)$ around its expectation is sufficient to guarantee the concentration of $\bar\varphi_*(x,y)$. This is useful since it is more convenient to prove concentration for $\bar{\varphi}^{\circ}_{*}(x,y)$ for the reason that it only depends on a finite number of random variables.

Lemma 3.4.12. *There exist constants $\underline{c}, \bar{C} > 0$ depending only on (d,p) such that for n sufficenlty large and all $x, y \in \mathbb{Z}^d$ with $|x-y| > n^{1/10}$, the following holds with probability at least $1 - e^{-(\log|x-y|)^2}$: for all $\lambda \in [\lambda_*, 1]$*

$$\varphi^{\circ}_{*}(x,y) \le \tfrac{1}{4}\bar{C}|x-y|\,, \tag{3.4.15}$$

$$\varphi_{*}(x,y) \ge 4\underline{c}|x-y|\,. \tag{3.4.16}$$

As a result, for the same (x,y), we have for all $\lambda \in [\lambda_*, 1]$

$$\varphi_*(x,y) = \bar{\varphi}_*(x,y), \varphi^{\circ}_*(x,y) = \bar{\varphi}^{\circ}_*(x,y) \tag{3.4.17}$$

Lemma 3.4.13. *The following holds for all environments. For n sufficenlty large and all $x, y \in \mathbb{Z}^d$ such that $|x-y| > n^{1/10}$. If $\varphi(x,y) \le 2\bar{C}|x-y|$, then*

$$\varphi(x,y) \ge \varphi^{\circ}(x,y) - e^{-|x-y|}\,, \tag{3.4.18}$$

$$\mathbf{E}^x\left[\tau_y \lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y\right] / \mathbf{E}^x\left[\lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y\right] \le |x-y|(\log|x-y|)^{C_\circ}\,. \tag{3.4.19}$$

Lemma 3.4.14. *The following holds for all environments. For n sufficenlty large and all $x, y \in \mathbb{Z}^d$ such that $|x-y| > n^{1/10}$,*

$$\begin{aligned}
&\{\mathbb{E}\left[\bar{\varphi}_*(x,y)\right] - \bar{\varphi}_*(x,y) \le 2^{-1}|x-y|^{2/3}, 4\underline{c}|x-y| \le \bar{\varphi}_*(x,y) \le 4^{-1}\bar{C}|x-y|\} \\
&\subseteq \{\mathbb{E}\left[\bar{\varphi}^{\circ}_*(x,y)\right] - \bar{\varphi}^{\circ}_*(x,y) \le |x-y|^{2/3}, 2\underline{c}|x-y| \le \bar{\varphi}^{\circ}_*(x,y) \le 2^{-1}\bar{C}|x-y|\} \\
&\subseteq \{\mathbb{E}\left[\bar{\varphi}_*(x,y)\right] - \bar{\varphi}_*(x,y) \le 2|x-y|^{2/3}, \underline{c}|x-y| < \bar{\varphi}_*(x,y) < \bar{C}|x-y|\}\,.
\end{aligned}$$

Now, we prove the aforementioned results as well as Lemma 3.4.3. We first record the following corollary, which is an immediate consequence of (3.2.7).

Corollary 3.4.15. *There exists $C_{3,3} = C_{3,3}(d,p)$ such that for n sufficenlty large, all $x, y \in \mathbb{Z}^d$, and $0 < \lambda < 1$*

$$G_{\mathcal{V}_\lambda}(x, y; \lambda) \le (\log n)^{C_{3,3}} . \tag{3.4.20}$$

Proof of (3.4.15) **in Lemma 3.4.12.** We will work on the event that both $\mathbf{\Lambda}(x, r(x,y)/2)$ and $\mathbf{\Lambda}(y, r(x,y)/2)$ are non-empty and the properties described in Lemma 3.4.10 hold for (x, y, λ_*). By Lemmas 3.4.9 and 3.4.10, this event has probability at least $1 - 2e^{(\log|x-y|)^3}$. We note that this event does not depend on λ.

On such a event, since $\mathscr{C}_{\mathbf{\Lambda}(x,r(x,y)/2)}$ and $\mathscr{C}_{\mathbf{\Lambda}(y,r(x,y)/2)}$ are in $(\mathsf{D}_{\lambda_*} \cup \mathcal{O})^c$ and of diameter at least $(\log|x-y|)^{C_{3,2}}$, Lemma 3.4.10 (2) implies there is a path in $\mathcal{V}_{\lambda_*}$ of length at most $C|x-y|$ that connects $B(x, r(x,y))$ and $B(y, r(x,y))$. Therefore, by forcing the random walk to go along such a path we get

$$\max_{\lambda \in [\lambda_*, 1]} \varphi_*^\circ(x, y; \lambda) \le C \log(2d)|x-y| . \qquad \square$$

The next result will be useful in proving (3.4.16).

Lemma 3.4.16. *For n sufficiently large and all $u \in \mathcal{C}(0)$ with $|u| \ge n^{1/2}$, with $\widehat{\mathbb{P}}$-probability at least $1 - 2e^{-(\log|u|)^{2d}}$, for all $\lambda \in [\lambda_*, 1]$*

$$\mathbf{E}\left[\lambda^{-\tau_u} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_u} \mathbb{1}_{\tau_u < C|u|}\right] \ge \mathbf{E}\left[\lambda^{-\tau_u} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_u}\right] (1 - e^{-|u|^{1/101}}) .$$

Proof. By setting $\lambda = 1$ and $u \in B(0,n)$, this lemma reduces to the path localization result $\mathsf{T}_u \le C|S_{\mathsf{T}_u}|$ (see (3.2.4), [24, (5.18)]). The proof of the lemma is a straightforward adaption of arguments in [24], and here we only describe how to implement such an adaption.

For $u, v \in \mathbb{Z}^d$, $A \subseteq \mathbb{Z}^d$ and $r \ge 1$, we define $\mathcal{K}_{A,r}(u,v)$ as in [24, Equation (2.16)]. For a random walk path ω, we define its unique loop erasure decomposition as in [24, Equation (5.1)] where $\eta = \eta(\omega)$ is the loop erasure and we define $\phi(\omega)$ as in [24, Equation (5.10)]. For $t \ge 1$, as in [24], we let $\mathcal{M}(t)$ be the collection of sites v such that $\mathbf{P}^v(\tau > t) \ge e^{-t/(\log t)^2}$.

We also define

$$A_t(\omega) = \{0 \le i \le |\eta| : |l_i| = t, \eta_i \notin \mathcal{M}(t)\}.$$

We will work on the following event (which does not depend on λ)

$$\Big\{|\gamma \cap \mathcal{M}(t)| \le e^{-(\log t)^{3/2}}|\gamma|, \quad \forall \gamma \in \bigcup_{m \ge |u|} \mathcal{W}_m(0), t \ge t_1\Big\} \tag{3.4.21}$$

where $\mathcal{W}_m(0)$ is the set of self-avoiding paths in $\mathbb{Z}^d$ of length m with initial point 0 and t_1 is a constant depending only on (d,p). By [24, Lemma 5.3], the event described in (3.4.21) has $\mathbb{P}$-probability at least $1 - e^{-(\log |u|)^{2d}}$. (Note that we should replace "$n(\log n)^{-100d^2}$" in the proof of [24, Lemma 5.3] by "$|u|$".) By $\lambda \ge \lambda_*$ and (3.2.7) we could see that

$$\max_{u \notin \mathcal{M}(t)} \mathbf{P}^u(\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > t) \le \begin{cases} \lambda^t e^{-2^{-1}t/(\log t)^2}, & t \le k_n^{50d}, \\ \lambda^t e^{-2^{-1}tk_n^{-21d}}, & t \ge k_n^{50d}. \end{cases}$$

As in the proof of [24, Lemma 5.5], there exists a constant $t_2 = t_2(d,p)$ such that for $m \ge 1$, $t \ge t_2$, $\gamma \in \phi(\mathcal{K}_{(\mathsf{D}_\lambda \cup \mathcal{O})^c \setminus \{u\}, m}(0,u))$ with $m - |\gamma| \ge |\eta| t^{-9}$ we have

$$\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathsf{D}_\lambda \cup \mathcal{O})^c \setminus \{u\}, m}(0,u) : \phi(\omega) = \gamma\}) \le \mathbf{P}(\gamma) \binom{|\eta|}{\frac{m-|\gamma|}{t}} \Big(\max_{u \notin \mathcal{M}(t)} \mathbf{P}^u(\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > t)\Big)^{\frac{m-|\gamma|}{t}}.$$

Substituting the bound on $\max_{u \notin \mathcal{M}(t)} \mathbf{P}^u(\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > t)$ and using $\binom{M}{N} \le (eM/N)^N$ gives

$$\mathbf{P}(\{\omega \in \mathcal{K}_{(\mathsf{D}_\lambda \cup \mathcal{O})^c \setminus \{u\}, m}(0,u) : \phi(\omega) = \gamma\}) \le \Big(\mathbf{P}(\gamma)\lambda^{m-|\gamma|}\Big) \cdot e^{-3^{-1}(m-|\gamma|)k_n^{-21d}}.$$

For $t \le |u|^{1/20}$, since $|\eta| \ge |u| \ge n^{1/2}$, we have $(m - |\gamma|)k_n^{-21d} \ge |\eta| t^{-9} k_n^{-21d} \ge |u|^{1/2}$. For $t \ge |u|^{1/20}$, since $m - |\gamma|$ is a multiple of t, $(m - |\gamma|)k_n^{-21d} \ge t \cdot k_n^{-21d} \ge |u|^{1/50} t^{1/2}$. Then summing over all m, γ such that $m - |\gamma| \geq |\eta| t^{-9}$ and $t \geq t_2$ we get

$$\sum_{t \geq t_2} \sum_{m \geq 0} \lambda^{-m} \mathbf{P}(\{\omega \in \mathcal{K}_{(\mathcal{O} \cup \mathsf{D}_\lambda)^c \setminus \{u\}, m}(0, u) : |A_t(\omega)| \geq |\eta| t^{-10}\})$$
$$\leq \mathbf{E}\left[\lambda^{-\tau_u} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_u}\right] e^{-|u|^{1/100}} . \tag{3.4.22}$$

Next, as it was treated in [24, Corollary 5.9], we combine (3.4.22) and (3.4.21) to deduce that on event (3.4.21), we have

$$\mathbf{E}\left[\lambda^{-\tau_u} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_u} \mathbb{1}_{\tau_u \geq C' \cdot \eta(S_{[0,\tau_u]})}\right] \leq \mathbf{E}\left[\lambda^{-\tau_u} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_u}\right] e^{-|u|^{1/100}}$$

where C' is an appropriately chosen constant. Finally, by [24, Lemma 5.4], we see that

$$\mathbf{E}\left[\lambda^{-\tau_u} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_u} \mathbb{1}_{\tau_u < C' \cdot \eta(S_{[0,\tau_u]})} \mathbb{1}_{\eta(S_{[0,\tau_u]}) \geq C''|u|}\right] \leq \sum_{j \geq C''|u|} (c' \lambda_*^{-C'})^j ,$$

for some constant $c' = c(d, p) \in (0, 1)$ and sufficiently large C''. Combining the preceding two inequalities and (3.4.15) completes the proof of the lemma. □

Proof of (3.4.16) **and** (3.4.17) **in Lemma 3.4.12.** Lemma 3.4.16 implies that for any $u \in B(x, r(x, y)), v \in B(y, r(x, y))$ with $\mathbb{P}$-probability at least $1 - 2e^{-(\log|u-v|)^{2d}}$, we have that for all $\lambda \in [\lambda_*, 1]$

$$\varphi(u, v) = -\log \mathbf{E}^u\left[\lambda^{-\tau_v} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_v}\right] \geq -\log \mathbf{E}^u\left[\lambda^{-\tau_v} \mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_v} \mathbb{1}_{\tau_v < C|u-v|}\right] - 1$$
$$\geq -\log \mathbf{P}^u(\tau > \tau_v) - (1 + C|u - v| \log \lambda) .$$

At the same time, since any path connecting u and v has a loop erasure of length at least $|u - v|$, [24, Lemma 5.4] implies that there exists $c' = c'(d, p) \in (0, 1)$ such that

$$\mathbb{P}(\mathbf{P}^u(\tau > \tau_v) \geq (c')^{|v-u|}) \leq e^{-c|v-u|} .$$

We complete the proof of (3.4.16) by combining preceding two inequalities and using a union bound over all $u \in B(x, r(x,y))$, $v \in B(y, r(x,y))$.

Since (3.4.17) is an immediate consequence of (3.4.16) and (3.4.15), we have completed the proof of the lemma. $\square$

Proof of Lemma 3.4.13. We denote $j_* = |x-y|(\log|x-y|)^{C_\circ - 1}$. By (3.2.7), we have for any $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} &\mathbf{E}^x\left[\tau_y \lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y, \tau_y \geq j_*\right] \leq \sum_{j \geq j_*} j \lambda^{-j} \mathbf{P}^x(\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > j) \\ &\leq \sum_{j \geq j_*} j (\log n)^C (1 - (\log n)^{-100d})^j \leq e^{-|x-y| \log|x-y|}. \end{aligned} \tag{3.4.23}$$

Since $\lambda^{-\tau_y} \leq \tau_y \lambda^{-\tau_y}$, we have that $e^{-\varphi(x,y)} - e^{-\varphi^\circ(x,y)}$ equals to

$$\mathbf{E}^x[\lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y \geq \xi_{B(x,j_*)}] \leq \mathbf{E}^x[\lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y \geq j_*] \leq e^{-|x-y| \log|x-y|}.$$

Combined with $\varphi(x,y) \leq 2\bar{C}|x-y|$, it yields

$$\varphi^\circ(x,y) - \varphi(x,y) \leq -\log(1 - e^{-|x-y|\log|x-y| + 2\bar{C}|x-y|}) \leq e^{-|x-y|},$$

verifying (3.4.18). Next, we prove (3.4.19). To this end, we observe

$$\mathbf{E}^x\left[\tau_y \lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y, \tau_y \leq j_*\right] \leq j_* \mathbf{E}^x\left[\lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y\right] = j_* e^{-\varphi(x,y)}.$$

Combined with (3.4.23) and $\varphi(x,y) \leq 2\bar{C}|x-y|$, it yields that

$$\mathbf{E}^x\left[\tau_y \lambda^{-\tau_y}; \tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \tau_y\right] \leq j_* e^{-\varphi(x,y)} + e^{-|x-y|\log|x-y|} \leq |x-y|(\log|x-y|)^{C_\circ} e^{-\varphi(x,y)},$$

which gives (3.4.19) as desired and thus completes the proof of the lemma. $\square$

Proof of Lemma 3.4.14. Combining (3.4.18) and Definition 3.4.2, we get that

$$\varphi_*(x,y) \le 2\bar{C}|x-y| \implies \varphi_*(x,y) \ge \varphi_*^\circ(x,y) - e^{-|x-y|/2}\,.$$

Since $\varphi_*(x,y) \le \varphi_*^\circ(x,y)$, it suffices to prove $\mathbb{E}\left[\bar{\varphi}_*^\circ(x,y)\right] \le \mathbb{E}\left[\bar{\varphi}_*(x,y)\right] + e^{-(\log|x-y|)^2/2}$. Now, denote event $E = \{\bar{\varphi}_*(x,y) = \varphi_*(x,y) \text{ and } \bar{\varphi}_*^\circ(x,y) = \varphi_*^\circ(x,y)\}$. Then by (3.4.17), we have

$$\mathbb{E}\left[\bar{\varphi}_*(x,y)\right] - \mathbb{E}\left[\varphi_*(x,y)\mathbb{1}_E\right] \le \bar{C}|x-y|e^{-(\log|x-y|)^2}\,,$$

$$\mathbb{E}\left[\bar{\varphi}_*^\circ(x,y)\right] - \mathbb{E}\left[\varphi_*^\circ(x,y)\mathbb{1}_E\right] \le \bar{C}|x-y|e^{-(\log|x-y|)^2}\,.$$

Suppose $\varphi_*(x,y) = \varphi(u,v)$, for $u \in B(x, r(x,y))$ and $v \in B(y, r(x,y))$. By (3.4.18), we have

$$\varphi_*(x,y)\mathbb{1}_E = \varphi(u,v)\mathbb{1}_E \ge \varphi^\circ(u,v)\mathbb{1}_E - e^{-|u-v|} \ge \varphi_*^\circ(x,y)\mathbb{1}_E - e^{-|x-y|/2}\,.$$

We complete the proof by combining preceding three inequalities. □

Proof of Lemma 3.4.3. We first note that combining Lemma 3.2.4 and (3.2.6) gives

$$\mathbb{P}(G_0 \,\triangle\, \{0 \in \mathcal{C}(\infty)\}) \to 0\,. \tag{3.4.24}$$

So $\mathbb{P}(G_0)$ is bounded away from 0 for large n. Let $A = B(v, (\log n)^{\kappa/2})$ where κ is chosen in Theorem 3.2.3. Since $\varphi(0,z) \ge \varphi_*(0,v)$ for all $z \in \partial_i A$, we have the upper bound

$$\mathbf{E}\left[\lambda^{-\mathsf{T}_v}\mathbb{1}_{\tau_{\mathsf{D}_\lambda \cup \mathcal{O}} > \mathsf{T}_v}\right] \le \sum_{z \in \partial_i A} e^{-\varphi(0,z)} \le 2^d (\log n)^{d\kappa/2} e^{-\varphi_*(0,v)}\,. \tag{3.4.25}$$

For the lower bound, we will work on the event such that the following holds (which does not depend on λ).

- $\mathbf{\Lambda}(v, (\log n)^{C_{3,2}}) \ne \varnothing$,
- Properties described in Lemma 3.4.10 hold for $(0, n\mathbf{e}_1, \lambda_*)$,

- For all $x \in B(0, r(0,v))$, $y \in B(v, r(0,v))$ and $\lambda \in [\lambda_*, 1]$,

$$\mathbf{E}^x\left[\lambda^{-\tau_y}\mathbb{1}_{\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}>\tau_y}\mathbb{1}_{\tau_y<C|y|}\right] \geq \mathbf{E}^x\left[\lambda^{-\tau_y}\mathbb{1}_{\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}>\tau_y}\right](1-e^{-|y|^{1/101}}). \tag{3.4.26}$$

Combining Lemmas 3.4.9, 3.4.10 and Lemma 3.4.16 (since $|v| \geq n^{2/3}$ ensures $|y| \geq n^{1/2}$), this event has probability at least $1 - e^{-c(\log n)^2}$.

Now, we choose $x \in B(0, r(0,v))$ and $y \in B(v, r(0,v))$ such that $\varphi_*(0,v) = \varphi(x,y)$ and we claim that

$$D_{\mathcal{V}_{\lambda_*}}(0,x) \leq (\log n)^C \text{ and there exists } z_0 \in A \text{ such that } D_{\mathcal{V}_{\lambda_*}}(y,z_0) \leq (\log n)^C. \tag{3.4.27}$$

Provided with this, by forcing random walk to travel from 0 to x at the beginning and travel from y to z_0 at the end (both along the geodesics), we have that for $i \geq 0$,

$$\mathbf{P}(S_{i+D_{\mathcal{V}_{\lambda_*}}(0,x)+D_{\mathcal{V}_{\lambda_*}}(y,z_0)} = z_0, S_{[1,i+D_{\mathcal{V}_{\lambda_*}}(0,x)+D_{\mathcal{V}_{\lambda_*}}(y,z_0)-1]} \subseteq \mathcal{V}_\lambda)$$

is bounded below by $(2d)^{-2(\log n)^C}\mathbf{P}^x(S_i = y, S_{[1,i-1]} \subseteq \mathcal{V}_\lambda)$. Hence,

$$\begin{aligned}\mathbf{E}^x\left[\lambda^{-\tau_y}\mathbb{1}_{\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}>\tau_y}\mathbb{1}_{\tau_y<C|y|}\right] &\leq \sum_{i=0}^{C|y|}\lambda^{-i}\mathbf{P}^x(S_i = y, S_{[1,i-1]} \subseteq \mathcal{V}_\lambda)\\ &\leq (2d)^{2(\log n)^C}\sum_{i=0}^{C|y|}\lambda^{-i}\mathbf{P}(S_i = z_0, S_{[1,i-1]} \subseteq \mathcal{V}_\lambda), \end{aligned} \tag{3.4.28}$$

where in the last step, we have changed the index using $i + D_{\mathcal{V}_{\lambda_*}}(0,x) + D_{\mathcal{V}_{\lambda_*}}(y,z_0) \mapsto i$. Decomposing the random walk path depending on the entrance point in A and using the strong Markov property, we get that

$$\begin{aligned}(3.4.28) &\leq (2d)^{2(\log n)^C} \sum_{z\in\partial_i A}\sum_{i=0}^{C|y|} \lambda^{-i}\mathbf{P}(S_i = z, S_{[1,i-1]} \subseteq \mathcal{V}_\lambda \setminus A)\cdot G_{\mathcal{V}_\lambda}(z, z_0;\lambda)\\ &\leq (2d)^{2(\log n)^C}(\log n)^{C_{3,3}}\mathbf{E}\big[\lambda^{-\mathsf{T}_v}\mathbb{1}_{n\wedge\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}\geq \mathsf{T}_v}\big],\end{aligned}$$

where in the last step, we used Corollary 3.4.15 and $|y| \leq 2C_{3,0}n(\log n)^{-2/d}$ (since $|v| \leq C_{3,0}n(\log n)^{-2/d}$)). Combining it with (3.4.26) gives the lower bound

$$\mathbf{E}\big[\lambda^{-\mathsf{T}_v}\mathbb{1}_{n\wedge\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}\geq \mathsf{T}_v}\big] \geq e^{-(\log n)^C}\mathbf{E}^x\left[\lambda^{-\tau_y}\mathbb{1}_{\tau_{\mathsf{D}_\lambda\cup\mathcal{O}}>\tau_y}\right].$$

Combined with (3.4.25) and (3.4.24), this yields the result of the lemma.

It remains to prove (3.4.27). Since x is connected to y in $\mathcal{V}_\lambda$ and (by (3.4.3)) 0 is connected to $B^c_{n^{1/2}}(0)$ in $\mathcal{V}_\lambda$, Lemma 3.4.10 (2) yields the first part of (3.4.27). The second part also follows directly if $y \in A$. If $y \notin A$, since we assumed $\mathbf{\Lambda}(v, (\log n)^{C_{3,2}}) \neq \varnothing$, applying Lemma 3.4.10 to $\mathscr{C}_{\mathbf{\Lambda}(v,(\log n)^{C_{3,2}})}$ yields that the vertex y is connected to $\mathscr{C}_{\mathbf{\Lambda}(v,(\log n)^{C_{3,2}})}$ in $\mathcal{V}_{\lambda_*}$. Since $A \supseteq B(v, 3(\log n)^{C_{3,2}})$, we get from Lemma 3.4.10 (2) that y is connected to some $z_0 \in \partial_i A$ in $\mathcal{V}_{\lambda_*}$ by a path of length at most $(\log n)^C$. □

3.4.3 *Concentration and Sub-additivity of LWGF*

In this subsection, we will prove two key properties of LWGF: concentration as incorporated in Lemma 3.4.17 and sub-additivity as incorporated in Lemma 3.4.21.

Lemma 3.4.17. *For any $q \geq 2$, $|x - y| > n^{1/10}$ and sufficiently large n*

$$\mathrm{Var}\left[\bar{\varphi}_*(x, y)\right] \leq (\log|x - y|)^C \cdot |x - y|, \tag{3.4.29}$$

$$\mathbb{E}\left[(\mathbb{E}\bar{\varphi}_*(x, y) - \bar{\varphi}_*(x, y))_+^q\right] \leq (\log|x - y|)^{qC}\cdot|x - y|^{q/2}, \tag{3.4.30}$$

where we used the notation $a_+ = a\mathbb{1}_{a\geq 0}$.

We will use the Efron-Stein inequality to bound the second and higher moments (see [14, Theorem 2]). To this end, we will control the increment of $\bar{\varphi}_*(x,y)$ when resampling obstacles in $(B(x,2R_\circ))^c$ and resampling the obstacle at each $w \in B(x,2R_\circ)$. We see that Lemma 3.4.13 implies resampling obstacles in $(B(x,2R_\circ))^c$ only has a very small effect on LWGFs (see (3.4.32) below). It is a more complicated issue to control the influence from resampling $w \in B(x,2R_\circ)$. To this end, we will employ the following two types of bounds.

- We show in Lemma 3.4.19 that on a typical environment, the increment of $\bar{\varphi}_*(x,y)$ due to the resampling at w is at most poly-logarithmic in $|x-y|$. The proof is divided into two cases as follows.
  - For w near x (or y), (recalling that in defining $\bar{\varphi}_*(x,y)$ we have optimized over starting and ending points that are near x and y respectively) we will choose a different starting point (or end point) and consider the set of paths that do not get close to w. See Lemma 3.4.19 Case 1 below.
  - For w away from x and y, we note that the random walk can take a detour using $\mathscr{C}_{\mathbf{\Lambda}(w,(\log|x-y|)^C)}$ for some constant $C>0$ (recall $\mathscr{C}$ as in Definition 3.4.5 and $\mathbf{\Lambda}$ as in Lemma 3.4.9) to avoid getting close to w. See Lemma 3.4.19 Case 2 below.
- We prove in Lemma 3.4.20 a bound on the increment by a direct computation which takes into account how likely the random walk will get close to w.

Remark 3.4.18. Due to the requirement of avoiding D_λ in the definition of LWGFs, by resampling the obstacle at w, it is possible to change the accessibility for more than just the vertex w. However, by (3.4.33) below the change on accessibility is confined in a local ball around w.

In order to implement the preceding outline in detail, we first introduce a few definitions.

For $w \in B(x, 2R_\circ)$, we denote

$$\mathcal{O}_w = (\mathcal{O} \setminus \{w\}) \cup (\mathcal{O}' \cap \{w\}), \quad \mathcal{O}_\circ = (\mathcal{O} \setminus (B(x, 2R_\circ))^c) \cup (\mathcal{O}' \cap (B(x, 2R_\circ))^c)$$

where $\mathcal{O}'$ is an independent copy of $\mathcal{O}$ (that is, $\mathcal{O}_w$ is obtained from $\mathcal{O}$ by re-sampling the environment at w). Write $\varphi(x, y; \mathcal{O})$ to emphasize its dependence on the environment. Let $E_{x,y}$ be the event such that the following hold:

- For any $v \in B(x, |x-y|^2)$, either $\mathcal{C}(v) = \mathcal{C}(\infty)$ or $|\mathcal{C}(v)| \leq (\log|x-y|)^5$.
- For all $w \in B(x, |x-y|^2)$, $\mathbf{\Lambda}(w, r) \neq \varnothing$ for $r := (\log|x-y|)^{C_{3,1}+C_{3,2}+6d}$.
- $\bar{\varphi}_*(x, y; \mathcal{O}) = \varphi_*(x, y; \mathcal{O})$, and for all $w \in B(x, |x-y|^2)$, $\bar{\varphi}_*(x, y; \mathcal{O}_w) = \varphi_*(x, y; \mathcal{O}_w)$.

Then by Lemmas 3.2.4, 3.4.9, (3.4.17) and (3.4.14),

$$\begin{aligned}\mathbb{P}(E_{x,y}^c) \leq & (2|x-y|^2)^d e^{-c(\log|x-y|)^{5/2}} + (2|x-y|^2)^d e^{-r(\log n)^{-2d}} \\ & + 3(2|x-y|^2)^d e^{-(\log|x-y|)^2} \leq e^{-(\log|x-y|)^{3/2}}.\end{aligned}$$

On the event $E_{x,y}$, by (3.4.1) we can choose $u \in B(x, r(x,y))$ and $v \in B(y, r(x,y))$ such that

$$\bar{\varphi}_*(x, y; \mathcal{O}) = \varphi_*(x, y; \mathcal{O}) = \varphi(u, v; \mathcal{O}). \tag{3.4.31}$$

Let

$$\mathcal{V}_{\lambda,w} = (\mathsf{D}_\lambda(\mathcal{O}_w) \cup \mathcal{O}_w)^c.$$

Then

$$\varphi(x, y, \mathcal{O}_w) = -\log\left(\mathbf{E}^x\left[\lambda^{-\tau_y}; \xi_{\mathcal{V}_{\lambda,w}} > \tau_y\right]\right).$$

Now, we claim that on the event $E_{x,y}$,

$$\varphi(u, v; \mathcal{O}) \geq \varphi_*(x, y; \mathcal{O}_\circ) - e^{-|x-y|/2}. \tag{3.4.32}$$

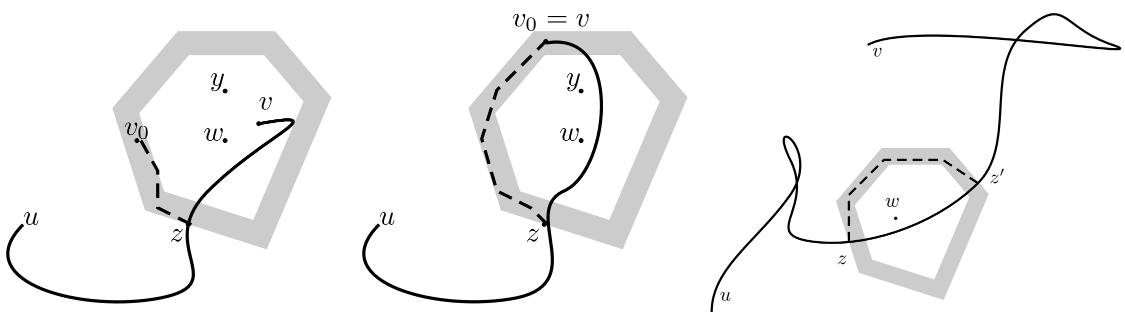

Figure 3.2: From left to right: Case 1(i), Case 1(ii), Case 2 in Lemma 3.4.19. The gray areas represent $\mathbb{K}_{\mathbf{\Lambda}(w,r)}$'s. The solid curve are the original paths. We replace a segment of it by the dotted curve.

We first see that (3.4.13) implies $\varphi(u, v; \mathcal{O}_\circ) \leq \varphi^\circ(u, v; \mathcal{O})$ and Lemma 3.4.13 yields $\varphi^\circ(u, v; \mathcal{O}) \leq \varphi(u, v; \mathcal{O}) + e^{-|u-v|}$. Then (3.4.32) follows from (3.4.1).

Further, for all $w \in B(x, |x-y|^2)$ let A_w denote the union of $\mathbb{K}_{\mathbf{\Lambda}(w,r)}$ and its interior region; i.e. A_w is the complement of the infinite connected component in $(\mathbb{K}_{\mathbf{\Lambda}(w,r)})^c$. Since $\mathcal{V}_\lambda \triangle \mathcal{V}_{\lambda,w} \subseteq B(w, (\log n)^{C_{3,1}})$ and $A_w \supseteq B(w, r) \supseteq B(w, (\log n)^{C_{3,1}})$,

$$\mathcal{V}_\lambda \setminus A_w = \mathcal{V}_{\lambda,w} \setminus A_w \,. \tag{3.4.33}$$

That is, removing or adding obstacle at w can only change the accessibility of the sites in A_w (actually, only in $B(w, r)$) for the random walk.

Lemma 3.4.19. *For n sufficiently large and x, y with $|x-y| > n^{1/10}$, let u, v be chosen as in* (3.4.31). *On the event $E_{x,y}$ we have*

$$\varphi(u, v; \mathcal{O}) \geq \varphi_*(x, y; \mathcal{O}_w) - \log(2d) r^{3d} \quad \forall w \in B(x, 2R_\circ) \,. \tag{3.4.34}$$

Proof. We divide the proof into two cases as follows.

Case 1: $u \in A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$ or $v \in A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$.

We assume that $v \in A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$ (the case $u \in A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$ can be treated similarly). We first claim that there exists $v_0 \in \mathscr{C}_{\mathbf{\Lambda}(w,r)} \cap B(y, r(x, y))$. To see this, we consider the following two scenarios.

(i) $v \in A_w \setminus \mathbb{K}_{\mathbf{\Lambda}(w,r)}$: In this scenario, since $r(x,y) \geq 100r$, we know that $B(y, r(x,y)) \cap \mathbb{K}_{\mathbf{i}}$ is non-empty for some $\mathbf{i} \in \mathbf{\Lambda}(w,r)$. Then we choose an arbitrary v_0 in $\mathbb{K}_{\mathbf{i}} \cap \mathscr{C}_{\mathbf{\Lambda}(w,r)}$. Then Property 1 in Definition 3.4.5 gives $\mathcal{C}(v_0) \geq L/10$ and the claim follows from Lemma 3.4.9 (2).

(ii) $v \in \mathbb{K}_{\mathbf{\Lambda}(w,r)} \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$: In this scenario, choose $v_0 = v$ and we only need to prove $v \in \mathscr{C}_{\mathbf{\Lambda}(w,r)}$ when $v \in \mathbb{K}_{\mathbf{\Lambda}(w,r)}$. Since v is connected to u in $\mathcal{O}^c$, by the first requirement of the event $E_{x,y}$ we see that $\mathcal{C}(v) = \mathcal{C}(\infty)$ and the desired claim follows from Lemma 3.4.9 (2).

Also, for the same reason as in (ii) above, we know that for any $z \in \partial_i(A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)})$ with $G_{\mathcal{V}_\lambda \setminus ((A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}) \cup \{v\})}(u, z; \lambda) > 0$, we have $\mathcal{C}(z) = \mathcal{C}(\infty)$ and thus by Lemma 3.4.9 (2), we get $z \in \mathscr{C}_{\mathbf{\Lambda}(w,r)}$ (since $z \in \partial_i(A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)})$).

Next, we will verify (3.4.34) in this case by combining the preceding discussions with the following decomposition of φ:

$$\varphi(u, v; \mathcal{O}) = -\log \Big(\sum_{z \in \partial_i(A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)})} G_{\mathcal{V}_\lambda \setminus ((A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}) \cup \{v\})}(u, z, \lambda) G_{\mathcal{V}_\lambda \setminus \{v\}}(z, v; \lambda) \Big). \tag{3.4.35}$$

By Corollary 3.4.15, $G_{\mathcal{V}_\lambda \setminus \{v\}}(z, v; \lambda) \leq (\log n)^{C_{3,3}}$. Let v_0 be chosen as above and consider any $z \in \partial_i(A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)})$ with $G_{\mathcal{V}_\lambda \setminus ((A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}) \cup \{v\})}(u, z; \lambda) > 0$. In light of scenario (i),(ii) and the discussion after (ii), we see that $v_0, z \in \mathscr{C}_{\mathbf{\Lambda}(w,r)}$ and thus by Lemma 3.4.9 (2) we get that $G_{\mathcal{V}_{\lambda,w} \setminus \{v\}}(v_0, z; \lambda) \geq (2d)^{-r^{2d}}$. Therefore, combined with (3.4.35) it yields that $\varphi(u, v; \mathcal{O})$ is bounded below by

$$\begin{aligned} -\log \Big((2d)^{r^{2d}} (\log n)^{C_{3,3}} \cdot \sum_{z \in \partial_i(A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)})} G_{\mathcal{V}_\lambda \setminus ((A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}) \cup \{v_0\})}(u, z; \lambda) G_{\mathcal{V}_\lambda \setminus \{v_0\}}(z, v_0; \lambda) \Big) \\ = \varphi(u, v_0; \mathcal{O}_w) - \log(2d) r^{2d} - C_{3,3} \log\log n \geq \varphi_*(x, y; \mathcal{O}_w) - \log(2d) r^{3d}, \end{aligned}$$

where we have used $A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)} \cup \{v_0\} = A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$.

Case 2: Neither u nor v is in $A_w \cup \mathscr{C}_{\mathbf{\Lambda}(w,r)}$.

Let $I_{z,z'} = G_{\mathcal{V}_\lambda \setminus (A_w \cup \{v\})}(u, z; \lambda) G_{\mathcal{V}_\lambda \setminus (A_w \cup \{v\})}(z', v; \lambda)$, we have

$$G_{\mathcal{V}_\lambda \setminus \{v\}}(u, v; \lambda) = \sum_{z,z' \in \partial_i A_w} I_{z,z'} \cdot G_{\mathcal{V}_\lambda \setminus \{v\}}(z, z'; \lambda) + G_{\mathcal{V}_\lambda \setminus (A_w \cup \{v\})}(u, v; \lambda)\,,$$

$$G_{\mathcal{V}_{\lambda,w} \setminus \{v\}}(u, v; \lambda) = \sum_{z,z' \in \partial_i A_w} I_{z,z'} \cdot G_{\mathcal{V}_{\lambda,w} \setminus \{v\}}(z, z'; \lambda) + G_{\mathcal{V}_\lambda \setminus (A_w \cup \{v\})}(u, v; \lambda)\,.$$

Wen claim that that for $z, z' \in \partial_i A_w$ with $I_{z,z'} > 0$,

$$G_{\mathcal{V}_\lambda \setminus \{v\}}(z, z'; \lambda) / G_{\mathcal{V}_{\lambda,w} \setminus \{v\}}(z, z'; \lambda) \le (2d)^{r^{3d}}\,. \tag{3.4.36}$$

Provided with (3.4.36), we see that $G_{\mathcal{V}_\lambda \setminus \{v\}}(u, v; \lambda) / G_{\mathcal{V}_{\lambda,w} \setminus \{v\}}(u, v; \lambda) \le (2d)^{r^{3d}}$, which then yields (3.4.34). It remains to prove (3.4.36). We first note that $G_{\mathcal{V}_\lambda}(z, z'; \lambda) \le (\log n)^{C_{3,3}}$ due to Corollary 3.4.15. At the same time, for the same reason as in Case 1, we know $I_{z,z'} > 0$ implies $\mathcal{C}(z) = \mathcal{C}(z') = \mathcal{C}(\infty)$. Combining this with $z, z' \in \partial_i A_w$ and Lemma 3.4.9 (2) gives $z, z' \in \mathscr{C}_{\mathbf{\Lambda}(w,r)}$. Then Lemma 3.4.9 (2) implies $G_{\mathscr{C}_{\mathbf{\Lambda}(w,r)}}(z, z'; \lambda) \ge (2d)^{-r^{2d}}$. Thus (3.4.36) would follow once we prove $\mathscr{C}_{\mathbf{\Lambda}(w,r)} \subseteq \mathcal{V}_{\lambda,w} \setminus \{v\}$. To this end, note that $\mathscr{C}_{\mathbf{\Lambda}(w,r)} \subseteq \mathcal{V}_\lambda \setminus \{v\}$ (by Definition 3.4.5 and the assumption of this case) and that $\mathscr{C}_{\mathbf{\Lambda}(w,r)} \cap B(w, r - 100L) = \varnothing$ (by the definition of $\mathscr{C}_{\mathbf{\Lambda}(w,r)}$ in Lemma 3.4.9). Since $\mathcal{V}_\lambda \triangle \mathcal{V}_{\lambda,w} \subseteq B(w, (\log n)^4) \subseteq B(w, r - 100L)$, it yields that $\mathscr{C}_{\mathbf{\Lambda}(w,r)} \subseteq \mathcal{V}_{\lambda,w} \setminus \{v\}$. At this point, we complete the verification of (3.4.36).

Combining the two cases above completes the proof of the lemma. □

We remark that (3.4.34) holds for general w provided that the event $E_{x,y}$ occurs, but it is suboptimal for typical w. For a typical w, the influence of resampling the environment at w is much smaller than the bound given in (3.4.34), as incorporated in the next lemma.

Lemma 3.4.20. *For n sufficiently large and x, y with $|x - y| > n^{1/10}$, let u, v be chosen as* *in* (3.4.31)*. For* $w \in B(x, 2R_\circ)$*, we have*

$$\frac{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v, \tau_{A_w} \le \tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v\right]} \ge \frac{1}{2} \min(1, \varphi(u, v; \mathcal{O}_w) - \varphi(u, v; \mathcal{O})) . \tag{3.4.37}$$

Proof. We note that

$$\begin{aligned}
\varphi(u, v; \mathcal{O}_w) &= -\log\left(\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_{\lambda w}} > \tau_v, \tau_{A_w} \le \tau_v\right] + \mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v, \tau_{A_w} > \tau_v\right]\right) \\
&\le -\log\left(\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v, \tau_{A_w} > \tau_v\right]\right) .
\end{aligned}$$

Thus, we have

$$\varphi(u, v; \mathcal{O}_w) - \varphi(u, v; \mathcal{O}) \le -\log\left(\frac{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v, \tau_{A_w} > \tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v\right]}\right) .$$

Therefore, using $-\log t \le (1-t)$ for $1/2 \le t \le 1$ gives (3.4.37). □

Proof of Lemma 3.4.17. It follows from (3.4.31), (3.4.1) and the third requirement of $E_{x,y}$ that

$$\begin{aligned}
\left(\left(\bar{\varphi}_*(x, y; \mathcal{O}_w) - \bar{\varphi}_*(x, y; \mathcal{O})\right)_+\right)^2 &\le \left(\left(\varphi(u, v; \mathcal{O}_w) - \varphi(u, v; \mathcal{O})\right)_+\right)^2 \\
&\le (\log(2d))^2 r^{6d} \times \frac{2\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v, \tau_{A_w} \le \tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v\right]} \\
&\le C r^{6d} \sum_{z \in B(w, 2r)} \frac{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v, \tau_z \le \tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v}; \xi_{\mathcal{V}_\lambda} > \tau_v\right]} ,
\end{aligned} \tag{3.4.38}$$

where in the second inequality, we used Lemmas 3.4.19 and 3.4.20. Now, we define

$$V_+ = \left(\left(\bar{\varphi}_*(x, y; \mathcal{O}_\circ) - \bar{\varphi}_*(x, y; \mathcal{O})\right)_+\right)^2 + \sum_{w \in B(x, 2R_\circ)} \left(\left(\bar{\varphi}_*(x, y; \mathcal{O}_w) - \bar{\varphi}_*(x, y; \mathcal{O})\right)_+\right)^2 . \tag{3.4.39}$$

Combining (3.4.32) and (3.4.38), we have that on event $E_{x,y}$

$$
\begin{aligned}
V_+ &\le \sum_{w\in B(x,2R_\circ)} Cr^{6d} \sum_{z\in B(w,2r)} \frac{\mathbf{E}^u\left[\lambda^{-\tau_v};\xi_{\mathcal{V}_\lambda}>\tau_v,\tau_z\le\tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v};\xi_{\mathcal{V}_\lambda}>\tau_v\right]}+1\\
&\le Cr^{7d}\sum_{z\in\mathbb{Z}^d}\frac{\mathbf{E}^u\left[\lambda^{-\tau_v};\xi_{\mathcal{V}_\lambda}>\tau_v,\tau_z\le\tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v};\xi_{\mathcal{V}_\lambda}>\tau_v\right]}+1\le Cr^{7d}\frac{\mathbf{E}^u\left[\tau_v\lambda^{-\tau_v};\xi_{\mathcal{V}_\lambda}>\tau_v\right]}{\mathbf{E}^u\left[\lambda^{-\tau_v};\xi_{\mathcal{V}_\lambda}>\tau_v\right]}\,.
\end{aligned}
$$

Combining with (3.4.19), we get that on the event $E_{x,y}$

$$V_+\le(\log|x-y|)^C|x-y|\,.$$

Combined with (3.4.31) and $|\bar{\varphi}_*(x,y)|\le\bar{C}|x-y|$, this yields for any $q\ge 2$, and $|x-y|\ge n^{1/10}$

$$\mathbb{E}\left[V_+^{q/2}\right]\le\Big((\log|x-y|)^C|x-y|\Big)^{q/2}\,.$$

In light of the preceding estimate, we complete the proof of the lemma by applying Efron-Stein inequality and [14, Theorem 2]. □

Next, we prove the sub-additivity of our LWGF. Note that the sub-additivity for the logarithm of *unweighted* Green's function (i.e., when $\lambda=1$) was proved in [75, Chapter 5, Lemma 2.1]. The proof for our LWGF shares the same spirit but is a bit more complicated.

Lemma 3.4.21. *For n sufficiently large, any $x,y,z\in\mathbb{Z}^d$ with $|x-y|\ge n^{1/10}$, and all $\lambda\in[\lambda_*,1]$*

$$\mathbb{E}\bar{\varphi}_*(x,y)\le\mathbb{E}\bar{\varphi}_*(x,z)+\mathbb{E}\bar{\varphi}_*(z,y)+e^{(\log|x-y|)^{2/3}}\,. \tag{3.4.40}$$

Proof. For any $u,v,w\in\mathbb{Z}^d$, by the strong Markov property at τ_v, we have

$$G_{\mathcal{V}_\lambda\setminus\{w\}}(u,w;\lambda)\ge G_{\mathcal{V}_\lambda\setminus\{v,w\}}(u,v;\lambda)G_{\mathcal{V}_\lambda\setminus\{w\}}(v,w;\lambda)\,.$$

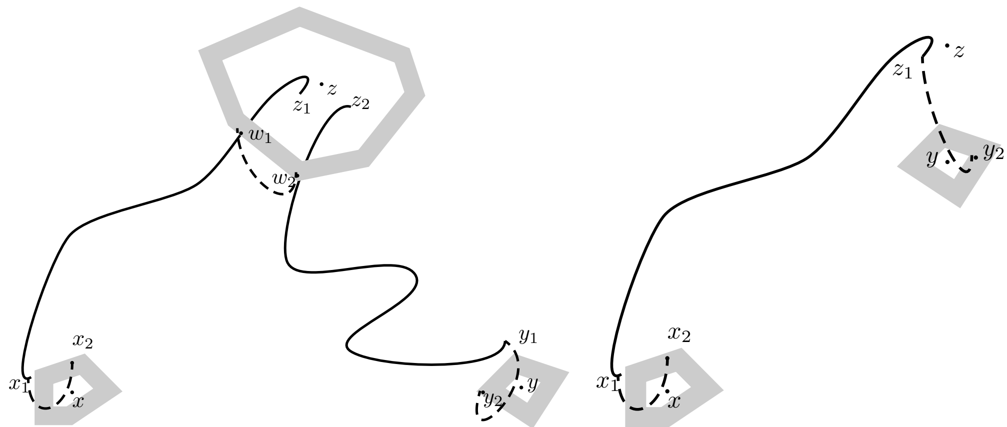

Figure 3.3: Case 1 (left) and Case 2 (right) in Lemma 3.4.21. The gray areas represent $\mathbb{K}_{\mathbf{\Lambda}}$'s. The solid curve are the original paths. We use dotted curves to connect two paths or change the starting/ending points.

Then (3.4.2) and Lemma 3.4.15 implies

$$\varphi(u,w) \le \varphi(u,v) + \varphi(v,w) + C_{3,3} \log\log n\,. \tag{3.4.41}$$

To prove (3.4.40), we have the complication that $\varphi_*(x,y)$ is the minimum over a neighborhood of x to a neighborhood of y (see (3.4.1)) and the size of the neighborhood depends on $|x-y|$. This requires more careful treatment in the proof.

Without lose of generality, we assume $|x-z| \ge |y-z|$. By the definition of $\bar{\varphi}_*$, it suffices to consider the case when $|x-y| \ge \underline{c}\bar{C}^{-1}|x-z|$. The proof divides into the following two cases.

Case 1: $|y-z| \ge e^{(\log|x-y|)^{1/2}}$.

In this case, we let $E_{x,y,z}$ be the event such that the following holds:

- For any $v \in B(x, |x-y|^2)$, either $\mathcal{C}(v) = \mathcal{C}(\infty)$ or $|\mathcal{C}(v)| \le (\log|x-y|)^5$.
- $\bar{\varphi}_*(x,y) = \varphi_*(x,y), \bar{\varphi}_*(x,z) = \varphi_*(x,z)$ and $\bar{\varphi}_*(y,z) = \varphi_*(y,z)$,
- $\mathbf{\Lambda}(x, r(x,y)/2), \mathbf{\Lambda}(y, r(x,y)/2), \mathbf{\Lambda}(z, 2r(x,z))$ are not empty.
- Properties described in Lemma 3.4.10.

By Lemmas 3.2.4, 3.4.9, (3.4.17) and 3.4.10,

$$\mathbb{P}(E^c_{x,y,z}) \le (2|x-y|)^d e^{-c(\log|x-y|^{5/2})} + 3e^{-(\log|x-y|)^2} + e^{-(\log|x-z|)^3} + e^{-(\log|z-y|)^3} + e^{-2^{-1}r(x,y)(\log n)^{-2d}} + e^{-2r(x,z)(\log n)^{-2d}} \le 2e^{-(\log|x-y|)^{3/2}}\,. \tag{3.4.42}$$

On event $E_{x,y,z}$, by (3.4.1) we choose $x_1 \in B(x, r(x,z))$ and $z_1 \in B(z, r(x,z))$, $z_2 \in B(z, r(z,y))$ and $y_1 \in B(y, r(z,y))$ such that

$$\bar\varphi_*(x,z) = \varphi_*(x,z) = \varphi(x_1,z_1) \quad \text{and} \quad \bar\varphi_*(z,y) = \varphi_*(z,y) = \varphi(z_2,y_1).$$

Noting that $\mathbf{\Lambda}(z, 2r(x,z))$ is not empty, we denote by A the union of $\mathbb{K}_{\mathbf{\Lambda}(z,2r(x,z))}$ and its interior region; i.e., A is the complement of the infinite connected component in $\mathbb{K}^c_{\mathbf{\Lambda}(z,2r(x,z))}$. By decomposition of random walk paths, we have

$$G_{\mathcal{V}_\lambda}(x_1,y_1;\lambda) \ge \sum_{w_1,w_2\in\partial_i A} G_{\mathcal{V}_\lambda\setminus A}(x_1,w_1;\lambda) G_{\mathcal{V}_\lambda}(w_1,w_2;\lambda) G_{\mathcal{V}_\lambda\setminus A}(w_2,y_1;\lambda)\,.$$

Also, for w_1, w_2 such that $G_{\mathcal{V}_\lambda\setminus A}(x_1,w_1;\lambda) G_{\mathcal{V}_\lambda\setminus A}(w_2,z_1;\lambda) > 0$, by the first requirement of $E_{x,y,z}$, we have $\mathcal{C}(w_1) = \mathcal{C}(w_2) = \mathcal{C}(\infty)$. Applying Lemma 3.4.9 (2), we get

$$G_{\mathcal{V}_\lambda}(w_1,w_2;\lambda) \ge (2d)^{-(2r(x,z))^{2d}}\,.$$

Therefore, combining the preceding two displayed inequalities gives

$$G_{\mathcal{V}_\lambda}(x_1,y_1;\lambda) \ge (2d)^{-(2r(x,z))^{2d}} \cdot \sum_{w_1,w_2\in\partial_i A} G_{\mathcal{V}_\lambda\setminus A}(x_1,w_1;\lambda) G_{\mathcal{V}_\lambda\setminus A}(w_2,y_1;\lambda)\,.$$

Also, by Corollary 3.4.15,

$$G_{\mathcal{V}_\lambda}(x_1,z_1;\lambda) = \sum_{w\in\partial_i A} G_{\mathcal{V}_\lambda\setminus A}(x_1,w;\lambda) G_{\mathcal{V}_\lambda}(w,z_1;\lambda) \le (\log n)^{C_{3,3}} \sum_{w\in\partial_i A} G_{\mathcal{V}_\lambda\setminus A}(x_1,w;\lambda)\,,$$

and similarly $G_{\mathcal{V}_\lambda}(z_2, y_1; \lambda) \le (\log n)^{C_{3,3}} \sum_{w \in \partial_i A} G_{\mathcal{V}_\lambda \setminus A}(w, z_1; \lambda)$. Combining preceding three inequalities and that $r(x,z) \ge \log(|x-y|/2) \ge c \log n$ gives

$$G_{\mathcal{V}_\lambda}(x_1, y_1; \lambda) \ge e^{-C(r(x,z))^{2d}} G_{\mathcal{V}_\lambda}(x_1, z_1; \lambda) G_{\mathcal{V}_\lambda}(z_2, y_1; \lambda)\,.$$

Then by (3.4.2) and Corollary 3.4.15, we deduce that

$$\varphi(x_1, y_1) \le \bar{\varphi}_*(x,z) + \bar{\varphi}_*(z,y) + C(r(x,z))^{2d}\,.$$

Since $\mathbf{\Lambda}(x, r(x,y)/2), \mathbf{\Lambda}(y, r(x,y)/2) \neq \varnothing$ and $r(x,y)/2 > (\log|x-y|)^{C_{3,2}}$, we choose $x_2 \in \mathscr{C}_{\mathbf{\Lambda}(x, r(x,y)/2)}$ and $y_2 \in \mathscr{C}_{\mathbf{\Lambda}(y, r(x,y)/2)}$ arbitrarily. Then by Lemma 3.4.10, x_1 and x_2 is connected by path in $\mathcal{V}_\lambda$ of length at most $(\log|x-y|)^C$. Then $\varphi(x_1, x_2) \le \log(2d)(\log|x-y|)^C$ (and the same holds for y_1 and y_2). Therefore, on the event $E_{x,y,z}$, (3.4.41) gives

$$\varphi_*(x,y) \le \varphi(x_2,y_2) \le \varphi(x_1,y_1) + 2\log(2d)(\log|x-y|)^C \le \bar{\varphi}_*(x,z) + \bar{\varphi}_*(z,y) + (\log|x-y|)^C\,.$$

Combined with (3.4.42) and $\bar{\varphi}_*(x,y) \le \bar{C}|x-y|$, this implies

$$\mathbb{E}\bar{\varphi}_*(x,y) \le \mathbb{E}\bar{\varphi}_*(x,z) + \mathbb{E}\bar{\varphi}_*(z,y) + (\log|x-y|)^C\,.$$

Case 2: $|y-z| \le e^{(\log|x-y|)^{1/2}}$.

In this case, $|x-z| \ge |x-y|/2$. We let $E_{x,y,z}$ be the event such that the following hold:

- For any $v \in B(x, |x-y|^2)$, either $\mathcal{C}(v) = \mathcal{C}(\infty)$ or $|\mathcal{C}(v)| \le (\log|x-y|)^5$.
- $\bar{\varphi}_*(x,y) = \varphi_*(x,y)$ and $\bar{\varphi}_*(x,z) = \varphi_*(x,z)$,
- $\mathbf{\Lambda}(x, r(x,y)/2), \mathbf{\Lambda}(y, r(x,y)/2)$ are not empty.
- Properties described in Lemma 3.4.10.

By Lemmas 3.2.4, 3.4.9, (3.4.17) and 3.4.10,

$$\mathbb{P}(E^c_{x,y,z}) \le (2|x-y|)^d e^{-c(\log|x-y|^{5/2})} + 3e^{-(\log|x-y|)^2} + 2e^{-(\log|x-z|)^2} + 2e^{-2^{-1}r(x,y)(\log n)^{-2d}} \le e^{-(\log|x-y|)^{3/2}} . \quad (3.4.43)$$

On the event $E_{x,y,z}$, by (3.4.1) we let $x_1 \in B(x, r(x,z))$ and $z_1 \in B(z, r(x,z))$ such that

$$\bar{\varphi}_*(x,z) = \varphi_*(x,z) = \varphi(x_1, z_1) .$$

Since $\mathbf{\Lambda}(x, r(x,y)/2), \mathbf{\Lambda}(y, r(x,y)/2)$ are not empty, by Lemma 3.4.10, there exists $x_2 \in B(x, r(x,y))$ and $y_2 \in B(y, r(x,y))$ such that x_1 is connected to x_2 by path in $\mathcal{V}_\lambda$ of length at most $(\log|x-y|)^C$ and z_1 is connected to y_2 by path in $\mathcal{V}_\lambda$ of length at most $Ce^{-(\log|x-y|)^{1/2}}$. Then $\varphi(x_2, x_1) \le \log(2d)(\log|x-y|)^C$, $\varphi(z_1, y_2) \le C\log(2d)e^{-(\log|x-y|)^{1/2}}$. Therefore, on the event $E_{x,y,z}$, (3.4.41) gives

$$\begin{aligned} \varphi_*(x,y) &\le \varphi(x_2, y_2) \le \varphi(x_1, z_1) + Ce^{(\log|x-y|)^{1/2}} \log(2d) \\ &= \varphi_*(x,z) + Ce^{(\log|x-y|)^{1/2}} \log(2d) . \end{aligned}$$

Combined with (3.4.43) and $\bar{\varphi}_*(x,y) \le \bar{C}|x-y|$, this implies

$$\mathbb{E}\bar{\varphi}_*(x,y) \le \mathbb{E}\bar{\varphi}_*(x,z) + \mathbb{E}\bar{\varphi}_*(z,y) + e^{2(\log|x-y|)^{1/2}} . \qquad \square$$

3.4.4 Rate of convergence for LWGF to linear functions

Let

$$h(x) = \mathbb{E}\left[\bar{\varphi}_*(0,x)\right] \text{ for all } x \in \mathbb{Z}^d . \quad (3.4.44)$$

It has been proved in Lemma 3.4.21 that $\{h(mx)\}_{m\geq 1}$ is sub-additive (with some error term). Hence, one can prove that (see [18, Theorem 23]) for all $x \in \mathbb{Z}^d$ the limit

$$g(x) := \lim_{m\to\infty} h(mx)/m \tag{3.4.45}$$

exists. Furthermore, g could first extends to $\mathbb{Q}^d$ by restricting m to such that $mx \in \mathbb{Z}^d$, and then extends to $\mathbb{R}^d$ by continuity (see [4, Lemma 1.5]). Moreover, it follows directly from definition and Lemma 3.4.21 that g is homogeneous of order 1 and sub-additive, i.e.,

$$g(\alpha x) = \alpha g(x) \text{ for } x \in \mathbb{R}^d, \alpha > 0\,, \text{ and } g(x+y) \leq g(x) + g(y) \text{ for } x, y \in \mathbb{R}^d\,. \tag{3.4.46}$$

Thus, g is convex (these properties will be useful, e.g., in the proof of Lemma 3.3.5). Now, we can state the main conclusion of this subsection.

Lemma 3.4.22. *For n sufficiently large and $|x| \geq n^{1/2}$, we have that*

$$h(x) \leq g(x) + |x|^{4/5}\,. \tag{3.4.47}$$

The proof of Lemma 3.4.22 is a combination of the concentration results proved in Section 3.4.3 and a nontrivial adaption of arguments in [4]. We first record some straightforward properties of h. By Lemma 3.4.21 and [18, Theorem 23], for all $|x| \geq n^{1/10}$

$$g(x) \leq h(x) + |x|^{1/10}\,, \tag{3.4.48}$$

$$\underline{c}|x| \leq g(x) \leq \bar{C}|x|\,, \tag{3.4.49}$$

where $\underline{c}, \bar{C}$ are two constants as in (3.4.12) In addition, g is continuous (see [4, Lemma 1.5]).

One of the main results in [4] is that CHAP (as described in Lemma 3.4.24) implies approximation property of the form (3.4.47). Hence, it suffices to verify the CHAP condition. We will adapt the arguments in [4] in order to verify CHAP (Lemma 3.4.25). However, in our case h is not the length of the shortest path (as in the case for the first-passage percolation), but in a vague sense h is the "average" length of all open paths (which are not necessarily self-avoiding). This incurs some nontrivial challenges, whose treatment requires some new technical ingredients including Lemma 3.4.27 and some concentration results in our case (Lemmas 3.4.29, 3.4.30).

We point out that Lemmas 3.4.24, 3.4.26 and the arguments in *the proof of Lemma 3.4.22* are from [4]. We omit the proof the Lemma 3.4.24 but present the proof the other two, as there is an error term in the sub-additivity of h and hence a few (straightforward) modifications are required there. Also, Lemma 3.4.25 is similar to [4, Proposition 3.4].

Now, we turn to the proof of the main result in this subsection.

Definition 3.4.23. *For $x \in \mathbb{R}^d$, let H_x denote a hyperplane tangent to $\partial\{y : g(y) \leq g(x)\}$ at x. Let H_x^0 denote the hyperplane through 0 parallel to H_x. We define $g_x(y)$ by the unique linear function such that*

$$g_x(y) = 0 \quad \textit{for} \quad y \in H_x^0, \quad \textit{and} \quad g_x(x) = g(x)\,,$$

and define

$$Q_x := \{y \in \mathbb{Z}^d : g_x(y) \leq g(x), h(y) \leq g_x(y) + 10C_Q|x|^{2/3}\}\,, \tag{3.4.50}$$

where $C_Q = 2\bar{C}\underline{c}^{-1} + 10$.

Then one can prove that (see [4, (1.9)])

$$|g_x(y)| \leq g(y) \text{ for all } y \in \mathbb{R}^d\,. \tag{3.4.51}$$

Lemma 3.4.24. ([4, Lemma 1.6]) *Let $M, a > 1$. Suppose that for each $x \in \mathbb{Q}^d$ with $|x| \geq M$, there exists $N \geq 1$, a path γ in $\mathbb{Z}^d$ from 0 to Nx and a sequence of sites $0 = v_0, v_1, ..., v_m = Nx$ in γ such that $m \leq aN$ and $v_i - v_{i-1} \in Q_x$ for all $1 \leq i \leq m$. Then h satisfies the* *convex-hull approximation property (CHAP), meaning for all $x \in \mathbb{Q}^d$ with $|x| \geq M$, we have*

$$x/\alpha \in \mathrm{Co}(Q_x) \quad \textit{for some } \alpha \in [1, a], \tag{3.4.52}$$

where $\mathrm{Co}(\cdot)$ *denotes the convex hull.*

To verify the condition in Lemma 3.4.24, we fix a large integer N such that $Nx \in \mathbb{Z}^d$ and choose v_i's depending on the environment and show that they satisfy the condition in Lemma 3.4.24 with positive $\mathbb{P}$-probability (which suffices as the existence of desired v_i's as stated in Lemma 3.4.24 is a deterministic event).

We choose $u_1 \in B(0, r(0, Nx))$ and $u_2 \in B(Nx, r(0, Nx))$ such that $\varphi_*(0, Nx) = \varphi(u_1, u_2)$. Then we choose v_i for $i = 1, 2, ..., \zeta_1, ..., \zeta_2, ..., \zeta_3$ as well as $\zeta_1, \zeta_2, \zeta_3$ inductively as follows. Set $v_0 = 0$ and for any $i \geq 1$

$$v_i = \arg \min_{u:\ u - v_{i-1} \in \partial_i Q_x} |u - u_1| \tag{3.4.53}$$

until $u_1 \in v_{i-1} + Q_x$, in which case we set $v_i = u_1$ and $\zeta_1 = i$. Next for $i \geq \zeta_1 + 1$ set

$$v_i = \arg \max_{\substack{u:\ u - v_{i-1} \in \partial_i Q_x \\ u \notin \{v_{\zeta_1}, v_{\zeta_1+1}, \dots, v_{i-1}\}}} G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \dots, v_{i-1}\}}(v_{i-1}, u; \lambda) \cdot G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \dots, u\}}(u, u_2; \lambda), \tag{3.4.54}$$

until $u_2 \in v_{i-1} + Q_x$, in which case we set $v_i = u_2$ and $\zeta_2 = i$. Finally for $i \geq \zeta_2 + 1$ set

$$v_i = \arg \min_{u:\ u - v_{i-1} \in \partial_i Q_x} |u - u_3| \tag{3.4.55}$$

until $Nx \in v_{i-1} + Q_x$, in which case we set $v_i = Nx$ and $\zeta_3 = i$. We remark that we do not require either 0 or Nx is in $\mathcal{C}(\infty)$ here.

We verify the condition in Lemma 3.4.24 for $M = n^{1/3}$ and $a = 4$ as a consequence of the following lemma, and then prove Lemma 3.4.22.

Lemma 3.4.25. *For n sufficiently large and any $x \in \mathbb{Q}^d$ with $|x| \ge n^{1/3}$, there exists an integer $N \ge 1$ such that*

$$\mathbb{P}(\zeta_3 \le 4N) > 0\,.$$

Proof of Lemma 3.4.22. We recall that the conditions in Lemma 3.4.24 are on h, which is deterministic. By choosing v_i's depending on environment as in (3.4.53)-(3.4.55), in light of Lemma 3.4.25, we know v_i satisfies conditions in Lemma 3.4.24 for $M = n^{1/3}$ and $a = 4$ with positive $\mathbb{P}$-probability. Hence the conditions in Lemma 3.4.24 hold, because it only requires the existence of such v_i's. Now, choose $q \in [|x|^{\frac{1}{4}}/2, |x|^{\frac{1}{4}}] \cap \mathbb{Q}$. Since $|x/q| \ge |x|^{\frac{3}{4}} \ge n^{1/3}$, we use Lemma 3.4.24 (with x/q in replace of x) and Carathéodory's theorem (which states that every points in a convex hull is a convex combination of at most $d+1$ fixed points in this convex hull. See [79, Theorem 3.10].) to conclude that

$$x/q = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_{d+1} y_{d+1} \tag{3.4.56}$$

with $\alpha_i \ge 0, \sum_{i=1}^{d+1} \alpha_i \in [1,4]$ and $y_i \in Q_{x/q}$. Let $x^* = \sum_{i=1}^{d+1} \lfloor q\alpha_i \rfloor y_i$. Note that $y_i \in Q_{x/q}, g_{x/q} = g_x$ (since g_x only depends on $x/|x|$). By Lemma 3.4.21,

$$\begin{aligned} h(x^*) &\le \sum_{i=1}^{d+1} \lfloor q\alpha_i \rfloor (h(y_i) + e^{(\log 4|x|)^{2/3}}) \\ &\le \sum_{i=1}^{d+1} \lfloor q\alpha_i \rfloor \big(g_x(y_i) + 10C_Q|x/q|^{2/3}\big) + 4(d+1)qe^{(\log 4|x|)^{2/3}} \\ &\le g_x(x^*) + 40C_Q q|x/q|^{2/3} + 4(d+1)qe^{(\log 4|x|)^{2/3}}\,, \end{aligned}$$

where in the last step we used that $g_x(\cdot)$ is a linear function. In addition, since $|x - x^*| \le \sum_{i=1}^{d+1} |y_i| \le (d+1)C_Q|x/q|$, by (3.4.51) and (3.4.49),

$$h(x - x^*) - g_x(x - x^*) \le 2\bar{C}C_Q(d+1)|x/q|\,.$$

Combining the preceding two inequalities, we conclude that

$$\begin{aligned} h(x) &\le g(x) + 40C_Q q|x/q|^{2/3} + 4(d+1)qe^{(\log 4|x|)^{2/3}} + 2\bar{C}C_Q(d+1)|x/q| \\ &\le g(x) + C|x|^{3/4}. \end{aligned}$$

□

The rest of this subsection is devoted to the proof of Lemma 3.4.25. We need a few more definitions: For $y \in \mathbb{Z}^d$, we define

$$\begin{aligned} s_x(y) &:= h(y) - g_x(y), \\ G_x &:= \{y \in \mathbb{Z}^d : g_x(y) > g(x)\}, \\ \Delta_x &:= \{y \in Q_x : y \text{ adjacent to } \mathbb{Z}^d \setminus Q_x, y \text{ not adjacent to } G_x\}, \\ D_x &:= \{y \in Q_x : y \text{ adjacent to } G_x\}. \end{aligned} \tag{3.4.57}$$

Lemma 3.4.26. ([4, Lemma 3.3]) *Recall that* $C_Q = 2\bar{C}\underline{c}^{-1} + 10$. *For all* $x \in \mathbb{Q}^d$ *with* $|x| \ge n^{1/3}$

(1) If $y \in Q_x$, *then* $g(y) \le 2g(x)$, $|y| \le C_Q|x|$ *and* $s_x(y) \ge -|y|^{1/10}$.

(2) If $y \in \Delta_x$, *then* $s_x(y) \ge 8C_Q|x|^{2/3}$.

(3) If $y \in D_x$, *then* $g_x(y) \ge 5g(x)/6$.

Proof. (1) For $y \in Q_x$, we have $s_x(y) \le 10C_Q|x|^{2/3}$ and $g_x(y) \le g(x)$. Then by (3.4.48)

$$g(y) \le h(y) + |y|^{1/10} = g_x(y) + s_x(y) + |y|^{1/10} \le g(x) + 10C_Q|x|^{2/3} + |y|^{1/10}.$$

Hence, by (3.4.49), we have $g(y) \le 2g(x)$ for large x and y. Then by (3.4.49), $|y| \le 2\bar{C}\underline{c}^{-1}|x| < C_Q|x|$. Also, (3.4.48) implies $s_x(y) \ge -|y|^{1/10}$.

(2) If $y \in \Delta_x$, then $y = z + e$ for some $z \in \mathbb{Z}^d \setminus (Q_x \cup G_x)$ and $e \in \{\pm e_1, ..., \pm e_d\}$ where $\{e_i\}_{i=1}^d$ is the standard basis in $\mathbb{R}^d$. Since $s_x(z) \ge 10C_Q|x|^{2/3}$ and $|s_x(e)| \le 2\bar{C}$, using sub-additivity of h (Lemma 3.4.21), linearity of g_x and $|z| \le |y| + 1 \le 2C_Q|x|$ (which follows

from $y \in \Delta_x \subseteq Q_x$ and the first item of the current lemma) we get

$$s_x(y) = h(y) - g_x(y) \geq h(z) - h(e) - e^{(\log|z|)^{2/3}} - (g_x(z) + g_x(e))$$
$$= s_x(z) - s_x(e) - e^{(\log|z|)^{2/3}} \geq 8C_Q|x|^{2/3}.$$

(3) If $y \in D_x$, then $y = z + e$ for some $z \in \mathbb{Z}^d \cap G_x$ and $e \in \{\pm e_1, ..., \pm e_d\}$. Hence

$$g_x(y) = g_x(z) + g_x(e) \geq g(x) - \bar{C} \geq 5g(x)/6. \qquad \square$$

The main goal of Lemmas 3.4.27-3.4.30 is to show that $\varphi_*(0, Nx)$ can be well approximated by $\sum_{i=0}^{\zeta_3-1} \mathbb{E}[\bar{\varphi}_*(v_i, v_{i+1})]$ with high probability, which is needed to prove Lemma 3.4.25. The analog of such an approximation is used in FPP setup to serve a similar purpose in [4]. In FPP, the total length of the shortest path from 0 to Nx is simply equals to the sum of that of non-intersecting path segments that assembles the geodesic. And the concentration can be obtained by using BK's inequality and an exponential concentration bound for each segment. However, our case is a bit more complicated and our proof is different from the arguments in [4]. In particular, we have applied results on lattice greedy animals ([54, 57]).

Lemma 3.4.27. *The following holds for all environments. For n sufficiently large and all $x \in \mathbb{Q}^d$ with $|x| \geq n^{1/3}$,*

(1) $\zeta_2 - \zeta_1 \geq N/(2C_Q)$ *and* $\zeta_1 + \zeta_3 - \zeta_2 \leq 4|x|^{-1/3} r(0, Nx)$.

(2) $\sum_{i=\zeta_1}^{\zeta_2-1} \varphi_*(v_i, v_{i+1}) \leq \varphi_*(0, Nx) + 3d(\zeta_2 - \zeta_1) \log|x|$.

Proof. (1) By Lemma 3.4.26 (1), we have $|v_i - v_{i-1}| \leq C_Q|x|$ for all $\zeta_1 + 1 \leq i \leq \zeta_2$. In addition, we see that $|v_{\zeta_1} - v_{\zeta_2}| \geq N|x| - 2r(0, Nx) \geq N|x|/2$. This implies that $\zeta_2 - \zeta_1 \geq |Nx|/(2C_Q|x|) = N/(2C_Q)$.

For $y \in \mathbb{Z}^d$ with $|y| \leq |x|^{1/3}$. By (3.4.51), $g_x(y) \leq g(y) \leq \bar{C}|x|^{1/3} \leq g(x)$ and $h(y) -$

$g_x(y) \le 2\bar{C}|x|^{1/3} \le 10C_Q|x|^{2/3}$. Then by (3.4.50),

$$B(0, |x|^{1/3}) \subseteq Q_x\,. \tag{3.4.58}$$

Hence, by (3.4.53) we have that for $i \le \zeta_1 - 1$

$$|u_1 - v_i| \le |u_1 - v_{i-1}| - |x|^{1/3}/2\,.$$

Since $|u_1| \le r(0, Nx)$, we have $\zeta_1 \le 2|x|^{-1/3} r(0, Nx)$. Similarly $\zeta_3 - \zeta_2 \le 2|x|^{-1/3} r(0, Nx)$. Thus, we complete the proof of (1).

(2) By considering the last visit to $v_{i-1} + Q_x = \{v_{i-1} + y : y \in Q_x\}$, we get for $\zeta_1 + 1 \le i \le \zeta_2 - 1$, $G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}(v_{i-1}, u_2; \lambda)$ equals to

$$\begin{aligned}
&\sum_{\substack{u:\ u - v_{i-1} \in \partial_i Q_x \\ u \notin \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}} G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}(v_{i-1}, u; \lambda) \cdot G_{\mathcal{V}_\lambda \setminus (\{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\} \cup (v_{i-1} + Q_x))}(u, u_2; \lambda) \\
&\le \sum_{\substack{u:\ u - v_{i-1} \in \partial_i Q_x \\ u \notin \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}} G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}(v_{i-1}, u; \lambda) \cdot G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}, u\}}(u, u_2; \lambda)\,.
\end{aligned}$$

By Lemma 3.4.26 (1), $|\partial_i Q_x| \le (2C_Q|x|)^d < |x|^{2d}$. Then by (3.4.54) we get that (using the simple fact that the sum of non-negative numbers is bounded above by the product of the maximal term and the number of terms in the summation)

$$\begin{aligned}
&G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}(v_{i-1}, u_2; \lambda) \\
&\le G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_{i-1}\}}(v_{i-1}, v_i; \lambda) G_{\mathcal{V}_\lambda \setminus \{v_{\zeta_1}, v_{\zeta_1+1}, \ldots, v_i\}}(v_i, u_2; \lambda)|x|^{2d}.
\end{aligned}$$

Taking logarithm for the above inequality and summing over $\zeta_1 + 1 \le i \le \zeta_2 - 1$, we have

$$\sum_{i=\zeta_1}^{\zeta_2 - 1} \varphi(v_i, v_{i+1}) \le \varphi(u_1, u_2) + 3d(\zeta_2 - \zeta_1 - 1) \log |x|\,,$$

where we used (3.4.2) and Corollary 3.4.15. Since $\varphi_*(v_i, v_{i+1}) \le \varphi(v_i, v_{i+1})$, combining with $\varphi_*(0, Nx) = \varphi(u_1, u_2)$, we completes the proof of (2). $\square$

Next, we prove some concentration results for LWGFs.

Definition 3.4.28. *For fixed $\lambda \in [\lambda_*, 1]$ and $x \in \mathbb{Q}^d$ with $|x| \ge n^{1/3}$, let $\mathcal{G}_x$ be the collection of sites $u \in \mathbb{Z}^d$ such that for all v satisfying $|x|^{1/3} \le |u - v| \le C_Q|x|$,*

$$\mathbb{E}\left[\bar{\varphi}_*^{\circ}(u,v)\right] - \bar{\varphi}_*^{\circ}(u,v) \le |u-v|^{2/3} \quad \text{and} \quad 2\underline{c}|x-y| \le \bar{\varphi}_*^{\circ}(u,v) \le 2^{-1}\bar{C}|u-v|\,. \quad (3.4.59)$$

In the preceding definition we used $\bar{\varphi}_*^{\circ}(u,v)$ to take advantage of the fact that it only depends on local environment, and the fact that the concentration of $\bar{\varphi}_*^{\circ}(u,v)$ around its expectation is sufficient to guarantee the concentration of $\bar{\varphi}_*(u,v)$ (Lemma 3.4.14) (which plays an important role in proving the concentration of $\sum \bar{\varphi}_*(v_i, v_{i+1})$ in Lemma 3.4.30).

Lemma 3.4.29. *For all $u \in \mathbb{Z}^d$, n sufficiently large, and $x \in \mathbb{Q}^d$ with $|x| \ge n^{1/3}$,*

$$\mathbb{P}(u \notin \mathcal{G}_x) \le |x|^{-10d^2}\,.$$

Proof. For any $|u-v| \ge |x|^{1/3}$, since $|x|^{1/3} \ge n^{1/10}$, by Lemma 3.4.17

$$\mathbb{P}(\mathbb{E}\left[\bar{\varphi}_*(u,v)\right] - \varphi_*(u,v) \ge 2^{-1}|u-v|^{2/3}) \le |u-v|^{-60d^2} \le |x|^{-20d^2}\,.$$

By Lemma 3.4.12, $\mathbb{P}(\bar{\varphi}_*(u,v) \ge 4^{-1}\bar{C}|u-v| \text{ } or \text{ } \bar{\varphi}_*(u,v) \le 4\underline{c}|u-v|) \le e^{-3^{-2}(\log|x|)^2}$. Combined with Lemma 3.4.14, it yields that

$$\mathbb{P}(u \notin \mathcal{G}_x) \le (2C_Q|x|)^d(|x|^{-20d^2} + e^{-3^{-2}(\log|x|)^2}) \le |x|^{-10d^2}\,. \qquad \square$$

Lemma 3.4.30. *For n sufficiently large, any $m_0 \ge |x|^{10d}$ and $|x| \ge n^{1/3}$, the following holds with $\mathbb{P}$-probability at least $1 - e^{-m_0|x|^{-5d}}$: For all $m \ge m_0$ and $v_0, v_1, ..., v_m \in \mathbb{Z}^d$ such* *that* $|v_0| \le e^{(\log m)^{1/d}}$ *and* $|x|^{1/3} \le |v_i - v_{i-1}| \le C_Q|x| \quad i = 1, 2, ..., m-1,$ *we have that*

$$\sum_{i=0}^{m-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] \le \sum_{i=0}^{m-1} \varphi_*(v_i, v_{i+1}) + C_Q m|x|^{2/3}.$$

Proof. By Lemma 3.4.26 (1) and (3.4.12), $\bar{\varphi}_*(v_i, v_{i+1}) \le \bar{C}C_Q|x|$. By Corollary 3.4.15, we have that $\varphi_*(v_i, v_{i+1}) \ge -C_{3,3} \log\log n$. Then by (3.4.59) and Lemma 3.4.14, we get the following two inequalities:

$$\sum_{i=0}^{m-1} \bar{\varphi}_*(v_i, v_{i+1}) \le \sum_{i=0}^{m-1} \varphi_*(v_i, v_{i+1}) + |\{i \in \{0, ..., m-1\} : v_i \notin \mathcal{G}_x\}| \cdot (\bar{C}C_Q|x| + C_{3,3} \log\log n),$$

$$\sum_{i=0}^{m-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] - \bar{\varphi}_*(v_i, v_{i+1}) \le 2m(C_Q|x|)^{2/3} + |\{i \in \{0, ..., m-1\} : v_i \notin \mathcal{G}_x\}| \cdot \bar{C}C_Q|x|.$$

Since $C_Q > 10, |x| \ge n^{1/3}$, it suffices to prove that with $\mathbb{P}$-probability at least $1 - e^{-m_0|x|^{-5d}}$

$$|\{i \in \{0, 1, ..., m-1\} : v_i \notin \mathcal{G}_x\}| \le m|x|^{-2}. \tag{3.4.60}$$

To this end, we denote $\mathcal{V}_i = i + |x|^2\mathbb{Z}^d$ for $i \in \{1, 2, ..., |x|^2\}^d$, where $\mathcal{V}_i$ inherits the graph structure from the natural bijection which maps $v \in \mathbb{Z}^d$ to $i + |x|^2 v \in \mathcal{V}_i$. For any integer $M > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\max_{\eta \in \mathcal{W}_M(v_0)} |\eta \cap \mathcal{V}_i \cap \mathcal{G}_x^c| \ge M|x|^{-4d}\Big) \\
\le&\mathbb{P}\Big(\max_{\eta \in \mathcal{W}_M(v_0)} |\{y \in \mathcal{V}_i : \eta \cap K_{|x|^2}(y) \neq \varnothing\} \cap \mathcal{G}_x^c| \ge M|x|^{-4d}\Big).
\end{aligned}$$

where $\mathcal{W}_M(v_0)$ is the set of connected self-avoiding paths in the original lattice of length M with initial point v_0. Since the event $v \in \mathcal{G}_x$ is independent of $\sigma(u \in \mathcal{G}_x, u \in \mathbb{Z}^d : |u - v| \ge |x|^2)$, we have that events $\{v \in \mathcal{G}_x^c\}$ for $v \in \mathcal{V}_i$ are independent. Also, for any $\eta \in \mathcal{W}_M(v_0)$, we know that $\{y \in \mathcal{V}_i : \eta \cap K_{|x|^2}(y) \neq \varnothing\}$ is a lattice animal in $\mathcal{V}_i$ of size at most $3^d M/|x|^2$.

By Lemma 3.4.29,

$$M|x|^{-4d} \geq |x| \cdot ((3^d M|x|^{-2}) \cdot \mathbb{P}(v \in \mathcal{G}_x^c)^{1/d}).$$

Then a result on greedy lattice animals proved in [54, Page 281] (see also [57]) yields

$$\mathbb{P}(\max_{\eta \in \mathcal{W}_M(v_0)} |\eta \cap \mathcal{V}_i \cap \mathcal{G}_x^c| \geq M|x|^{-4d}) \leq e^{-M|x|^{-4d}/2}. \tag{3.4.61}$$

Note that for all $v_0, v_1, ..., v_m$ such that $|v_i - v_{i-1}| \leq C_Q|x|$, there exists a self avoiding path η that goes through $v_0, v_1, ..., v_m$ and has length $dC_Q m|x|$. Thus, (3.4.60) follows by summing (3.4.61) over $i \in \{1, 2, ..., |x|^2\}^d$, $M = dC_Q m|x|$, $m \geq m_0$ and $v_0 \in B(0, e^{(\log m)^{1/d}})$. □

Proof of Lemma 3.4.25. By (3.4.45) and (3.4.49) and the Markov's inequality, we get for sufficiently large N,

$$\mathbb{P}(\bar{\varphi}_*(0, Nx) \leq Ng(x) + N) \geq 1 - \frac{\mathbb{E}\bar{\varphi}_*(0, Nx)}{Ng(x) + N} \geq \frac{1}{2\bar{C}|x| + 2}. \tag{3.4.62}$$

At the same time, by Lemma 3.4.27 (1), we can apply Lemma 3.4.30 to $v_{\zeta_1}, ..., v_{\zeta_2 - 1}$ and $m_0 = \lfloor N/(2C_Q) \rfloor - 1$. Combining with (3.4.17), we get that the following two inequalities hold with positive $\mathbb{P}$-probability for sufficiently large N,

$$\sum_{i=\zeta_1}^{\zeta_2 - 1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] \leq \sum_{i=\zeta_1}^{\zeta_2 - 1} \varphi_*(v_i, v_{i+1}) + C_Q(\zeta_2 - \zeta_1)|x|^{2/3}$$

$$\varphi_*(0, Nx) = \bar{\varphi}_*(0, Nx) \leq Ng(x) + N.$$

On this event, combining preceding two inequality and Lemma 3.4.27 (2), we get that

$$\sum_{i=\zeta_1}^{\zeta_2 - 1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] \leq Ng(x) + N + 2C_Q(\zeta_2 - \zeta_1)|x|^{2/3}. \tag{3.4.63}$$

At the same time, by Lemma 3.4.26 (1), we have $|v_{i+1} - v_i| \leq C_Q|x|$. Combined with

(3.4.12) and Lemma 3.4.27 (1), this implies

$$\sum_{i=0}^{\zeta_1-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] + \sum_{i=\zeta_2}^{\zeta_3-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] \le (\zeta_1 + \zeta_3 - \zeta_2)\bar{C}C_Q|x|$$

$$\le 4\bar{C}C_Q|x|^{2/3} r(0, Nx)\,. \tag{3.4.64}$$

Note that Lemma 3.4.27 (1) implies

$$C_Q(\zeta_2 - \zeta_1)|x|^{2/3} \ge N|x|^{2/3}/2 \text{ and } r(0, Nx) = (\log(Nx))^{2\kappa+10d}\,.$$

It follows that (3.4.64) is upper bounded by $C_Q(\zeta_2 - \zeta_1)|x|^{2/3}$. Then by (3.4.63), we have

$$\sum_{i=0}^{\zeta_3-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] \le Ng(x) + 3C_Q\zeta_3|x|^{2/3}\,. \tag{3.4.65}$$

Now, by Lemma 3.4.26 (1)(2) (recall that $s_x(y) := h(y) - g_x(y)$)

$$\sum_{i=0}^{\zeta_3-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] = \sum_{i=0}^{\zeta_3-1} g_x(v_i - v_{i+1}) + s_x(v_i - v_{i+1})$$

$$\ge g_x(Nx) + |\{1 \le i \le \zeta_3 - 1 : v_i \in \Delta_x\}| \cdot 8C_Q|x|^{2/3} - \zeta_3|x|^{1/10}\,.$$

Combining this with (3.4.65) gives

$$|\{1 \le i \le \zeta_3 - 1 : v_i \in \Delta_x\}| \le \zeta_3/2\,. \tag{3.4.66}$$

At the same time, by Lemma 3.4.26(3)

$$\sum_{i=0}^{\zeta_3-1} \mathbb{E}\left[\bar{\varphi}_*(v_i, v_{i+1})\right] = \sum_{i=0}^{\zeta_3-1} g_x(v_i - v_{i+1}) + s_x(v_i - v_{i+1})$$

$$\ge |\{1 \le i \le \zeta_3 - 1 : v_i \in D_x\}| \cdot 5g(x)/6 - \zeta_3|x|^{1/10}\,.$$

Combining with the lower bound in (3.4.49) and (3.4.65), we get

$$|\{1 \le i \le \zeta_3 - 1 : v_i \in D_x\}| \le 6N/5 + \zeta_3/8 . \tag{3.4.67}$$

Note that $v_i \in \Delta_x \cup D_x$ for $i \neq \zeta_1, \zeta_2, \zeta_3$; that is,

$$|\{1 \le i \le \zeta_3 - 1 : v_i \in \Delta_x\}| + |\{1 \le i \le \zeta_3 - 1 : v_i \in D_x\}| \ge \zeta_3 - 3 . \tag{3.4.68}$$

Combining (3.4.66), (3.4.67) and (3.4.68), we complete the proof of the lemma. □

3.4.5 Proof of Proposition 3.3.3

In this subsection we combine the ingredients in previous subsections and provide the proof for Proposition 3.3.3. We will consider different λ's. Thus, we will use the notation $g(v, \lambda)$ and $h(v, \lambda)$ to denote the functions $g(v)$ and $h(v)$, respectively (as defined at the beginning of Subsection 3.4.4) with an explicit emphasis on the dependence of λ.

Proof of Proposition 3.3.3. Lemma 3.4.22 and (3.4.48) implies for all $\lambda \in [\lambda_*, 1]$ and $v \in \mathsf{V} \subseteq B(0, C_{3,0} n (\log n)^{-2/d}) \setminus B(0, n^{2/3})$ (c.f. (3.2.3))

$$g(v; \lambda) = h(v; \lambda) + O(n^{4/5}) .$$

Let

$$E_v = \Big\{ \max_{\lambda \in [\lambda_*, 1]} |\varphi_\star(0, v; \lambda) - \bar{\varphi}_*(0, v; \lambda)| \le (\log n)^C \Big\} .$$

Then Lemmas 3.4.3 and 3.4.12 imply that

$$\mathbb{P}\Big(\bigcup_{v \in B(0, C_{3,0} n (\log n)^{-2/d}) \setminus B(0, n^{2/3})} E_v \mid G_0 \Big) \ge 1 - e^{-c(\log n)^2} .$$

Now by (3.4.24), it suffices to prove that conditioned on G_0, with $\mathbb{P}$-probability tending to

1, for all $v \in \mathsf{V}$

$$|\bar{\varphi}_*(0, v; \lambda(v)) - h(v; \lambda(v))| \leq n^{2/3}\,. \tag{3.4.69}$$

Note that this does not follow directly from Lemma 3.4.17 because Lemma 3.4.17 can provide concentration neither uniformly on all $v \in B(0, n)$ nor on all $\lambda \in [\lambda_*, 1]$. To prove (3.4.69), we first notice that for any $v \in B(0, C_{3,0}n(\log n)^{-2/d}) \setminus B(0, n^{2/3})$, $\{\lambda(v) \geq \lambda_*\}$ and G_0 are independent. Thus (3.2.2) yields

$$\mathbb{P}(E_v \mid \lambda(v) \geq \lambda_*, G_0) \geq 1 - e^{-c(\log n)^2}\,.$$

Hence,

$$\begin{aligned}
&\mathbb{P}(|\bar{\varphi}_*(0, v; \lambda(v)) - h(v; \lambda(v))| \leq n^{2/3} \mid \lambda(v) \geq \lambda_*, G_0) \\
\geq &\mathbb{P}(|\varphi_\star(0, v; \lambda(v)) - h(v; \lambda(v))| \leq n^{2/3}/2, E_v \mid \lambda(v) \geq \lambda_*, G_0) \\
\geq &\mathbb{P}(|\varphi_\star(0, v; \lambda(v)) - h(v; \lambda(v))| \leq n^{2/3}/2 \mid \lambda(v) \geq \lambda_*, G_0) - e^{-c(\log n)^2}\,.
\end{aligned}$$

Since $\sigma(G_0, \varphi_\star(0, v; \lambda), \lambda \in (0, 1))$ is independent of $\sigma(\lambda(v))$, this is

$$\begin{aligned}
\geq &\min_{\lambda \in [\lambda_*, 1]} \mathbb{P}(|\varphi_\star(0, v; \lambda) - h(v; \lambda)| \leq n^{2/3}/2 \mid G_0) - e^{-c(\log n)^2} \\
\geq &\min_{\lambda \in [\lambda_*, 1]} \mathbb{P}(|\bar{\varphi}_*(0, v; \lambda) - h(v; \lambda)| \leq n^{2/3}/3 \mid G_0) - e^{-c(\log n)^2} - \mathbb{P}(E_v^c \mid G_0) \\
\geq &1 - n^{1/4}\,.
\end{aligned}$$

where in the last step, we used Lemma 3.4.17. Combined with (3.2.2), it gives that

$$\mathbb{P}\big(|\bar{\varphi}_*(0, v; \lambda(v)) - h(v; \lambda(v))| > n^{2/3}, \lambda(v) \geq \lambda_* \mid G_0\big)$$

is bounded from above by $n^{-d-1/5}$. Then using a union bound over v yields that (3.4.69) holds for all $v \in \mathsf{V}$ conditioned on G_0 and thus complete the proof of the lemma. □

3.5 Asymptotic ball

From Theorem 3.1.2, we know that conditioned on survival the random walk is localized in the optimal pocket island of volume poly-logarithmic in n. In this section, we will show that in fact, there is a region (which we refer to as *intermittent island*) contained in the optimal pocket island such that the following holds:

- the intermittent island is asymptotically a discrete Euclidean ball of volume $d \log_{1/p} n$,
- the principal eigenvalues of the intermittent island and the optimal pocket island are close to each other.

The proof consists of the following two steps.

- In Section 3.5.1, we first notice that the region with low density of obstacles (which presumably forms the intermittent island) has low entropy, thus a sharp upper bound on the volume of the intermittent island can be derived. Then we will show that the principal eigenvalues of the intermittent island and the optimal pocket island are close to each other. A key ingredient in this step is to show that the principal eigenfunction of the optimal pocket island is supported on the intermittent island.
- In Section 3.5.2, we observe that the intermittent island achieves nearly largest eigenvalue (Lemma 3.5.7) over all set of the same volume (Lemma 4.2.3). Thus, by Faber–Krahn inequality the intermittent island has to be a discrete ball asymptotically. The proof is carried out in Section 3.5.2, where we use a quantitative version of Faber–Krahn inequality as in (3.1.3). Note that (3.1.3) is in the continuous setup but our problem is discrete. To address this, we use the relation between the continuous eigenvalue and its discrete approximation ([51, (38)] (see also [76, §4], [77, §6] and [60]).

3.5.1 Intermittent island

Recall that $\varrho_n = \lfloor (\omega_d^{-1} d \log_{1/p} n)^{1/d} \rfloor$ as defined in (4.1.2). For $\iota \in (0,1)$, we consider the following disjoint boxes that cover $\mathbb{Z}^d$:

$$K_{\lfloor \iota \varrho_n \rfloor}(x) = \{y \in \mathbb{Z}^d : |x-y|_\infty \le \lfloor \iota \varrho_n \rfloor\} \quad \text{for} \quad x \in (2\lfloor \iota \varrho_n \rfloor + 1)\mathbb{Z}^d \,. \tag{3.5.1}$$

Definition 3.5.1. *For $\rho \in (0,1)$, we say a box $K_{\lfloor \iota \varrho_n \rfloor}(x)$ is $(\iota \varrho_n, \rho)$-empty if*

$$|\mathcal{O} \cap K_{\lfloor \iota \varrho_n \rfloor}(x)| \le \rho |K_{\lfloor \iota \varrho_n \rfloor}(x)| \,.$$

Let $\mathcal{E}(\iota, \rho)$ be the union of $(\iota \varrho_n, \rho)$-empty boxes in (3.5.1) *that intersect with $B(v_*, (\log n)^\kappa)$.*

Lemma 3.5.2. *For any $\iota, \rho \in (\varrho_n^{-1/2}, p^2 d^{-100})$*

$$\widehat{\mathbb{P}}(|\mathcal{E}(\iota,\rho)| \le d \log_{1/p} n + \rho^{1/2} \varrho_n^d) \ge 1 - e^{-\varrho_n} \,.$$

Proof. For all $x \in \mathbb{Z}^d$, a straightforward computation gives that

$$\begin{aligned}
\mathbb{P}(K_{\lfloor \iota \varrho_n \rfloor}(x) \text{ is } (\iota \varrho_n, \rho)\text{-empty}) &\le \sum_{m=0}^{\lfloor \rho |K_{\lfloor \iota \varrho_n \rfloor}(x)| \rfloor} \binom{|K_{\lfloor \iota \varrho_n \rfloor}(x)|}{m} p^{|K_{\lfloor \iota \varrho_n \rfloor}(x)| - m} \\
&\le \rho |K_{\lfloor \iota \varrho_n \rfloor}(x)| \rho^{-\rho |K_{\lfloor \iota \varrho_n \rfloor}(x)|} p^{|K_{\lfloor \iota \varrho_n \rfloor}(x)|(1-\rho)} \,.
\end{aligned}$$

Since $\rho \le p$, we have

$$\mathbb{P}(K_{\lfloor \iota \varrho_n \rfloor}(x) \text{ is } (\iota \varrho_n, \rho)\text{-empty}) \le \exp\{|K_{\lfloor \iota \varrho_n \rfloor}(x)| (\log p - 2\rho \log \rho)\} \,. \tag{3.5.2}$$

Note that $B(v, (\log n)^\kappa)$ contains at most $(\log n)^{2\kappa d}$ boxes of the form (3.5.1) for each $v \in B(0, 2n)$. We let $q = \mathbb{P}(K_{\lfloor \iota \varrho_n \rfloor}(x)$ is $(\iota \varrho_n, \rho)$-empty). By (3.5.2) and $\iota^{-d} = O((\log n)^{1/2})$

(denoting by $\mathrm{Bin}((\log n)^{2\kappa d}, q)$ a Binomial variable with parameter $(\log n)^{2\kappa d}$ and q),

$$\mathbb{P}\left(\mathrm{Bin}\left((\log n)^{2\kappa d}, q\right) \geq \omega_d(2\iota)^{-d}(1+\frac{4\rho\log\rho}{\log(1/p)})\right) \leq n^{-d(1+\rho\log_p \rho)}.$$

Therefore, by a union bound, with $\mathbb{P}$-probability at least $1 - Cn^{-d\rho\log_p\rho} \geq 1 - e^{-2\varrho_n}$, no ball in $\{B(v, (\log n)^\kappa) : v \in B(0, 2n)\}$ contains more than $\omega_d(2\iota)^{-d}(1+\frac{4\rho\log\rho}{\log(1/p)})$ many $(\iota\varrho_n, \rho)$-empty boxes. □

Definition 3.5.3. *Recall that $\mathcal{U}$ is the connected component in $B(v_*, (\log n)^\kappa)$ that contains v_*. Let f be the principal eigenfunction of $P|_\mathcal{U}$ corresponding to $\lambda_\mathcal{U}$ such that $\sum_{v\in\mathcal{U}} f(v) = 1$. We extend f to $\mathbb{Z}^d$ by letting $f(v) = 0$ for $v \in \mathcal{U}^c$. For $\epsilon \in (0, 1)$, denote*

$$\Omega_\epsilon = \{v \in \mathbb{Z}^d : f(v) \geq \epsilon\varrho_n^{-d}\}.$$

We will show in Lemma 3.5.6 that Ω_ϵ, the set of sites where the eigenfunction value is high largely coincide with $\mathcal{E}(\epsilon^2, \epsilon^2)$ defined in Definition 3.5.1 and carries most of the weight of f — this is due to a simple relation between f and $\mathcal{E}(\iota, \rho)$ as in Lemma 3.5.5. Combining with Lemma 3.5.2, we are then able to provide a lower bound of the principal eigenvalue of $P|_{\Omega_\epsilon}$ in Lemma 3.5.7.

Lemma 3.5.4. *With $\widehat{\mathbb{P}}$-probability tending to one*

$$\lambda_\mathcal{U} \geq e^{-c_*(\log n)^{-2/d} - C(\log n)^{-3/d}}. \tag{3.5.3}$$

Then for any $x \in \mathcal{U}$ and $t > 0$,

$$\mathbf{P}^x(\xi_\mathcal{U} > t) \geq e^{-(2\log n)^{\kappa d}} e^{-c_*(\log n)^{-2/d} t(1-C(\log n)^{-1/d})}. \tag{3.5.4}$$

Proof. By definitions of $\mathcal{U}$ and $C_R(v_*)$, $C_R(v_*) \subseteq \mathcal{U}$. Recall (3.2.3) and that $v_* \in \mathsf{V}$. Lemma 3.2.5 gives (3.5.3). Since $\mathcal{U}$ is connected, (3.5.4) follows from $\max_x \mathbf{P}^x(\xi_\mathcal{U} > t) \geq \lambda_\mathcal{U}^t$ (which can be found in [24, Lemma 3.2]) and $|\mathcal{U}| \le (2\log n)^{\kappa d}$. ☐

Lemma 3.5.5. *For any $\iota, \rho \in (0,1)$ we have $\sum_{v\in\mathcal{E}(\iota,\rho)^c} f(v) \le C\rho^{-1}\iota^2$.*

Proof. By (3.5.3) and $e^x \ge 1+x$,

$$\sum_{v\in\mathcal{U}\setminus\mathcal{E}(\iota,\rho)} f(v)\mathbf{P}^v(\xi_{\mathcal{U}} \le \lfloor \iota\varrho_n \rfloor^2) \le \sum_{v\in\mathcal{U}} f(v)\mathbf{P}^v(\xi_{\mathcal{U}} \le \lfloor \iota\varrho_n \rfloor^2)$$
$$= 1 - \lambda_{\mathcal{U}}^{\lfloor \iota\varrho_n \rfloor^2} \le C\iota^2 \,. \tag{3.5.5}$$

At the same time, for $v \in \mathcal{U} \setminus \mathcal{E}(\iota,\rho)$, there exists at least $\rho(\iota\varrho_n)^2$ obstacles in $B(v,\iota\varrho_n)$. Hence,

$$\mathbf{P}^v(\tau \le \lfloor \iota\varrho_n \rfloor^2) \ge \max(\mathbf{P}^v(S_{\lfloor \iota\varrho_n \rfloor^2} \in \mathcal{O}), \mathbf{P}^v(S_{\lfloor \iota\varrho_n \rfloor^2-1} \in \mathcal{O})) \ge c\rho \,. \tag{3.5.6}$$

Substituting this into (3.5.5) yields (noting that $\xi_{\mathcal{U}} \le \tau$)

$$\sum_{v\in\mathcal{U}\setminus\mathcal{E}(\iota,\rho)} f(v) \le (c\rho)^{-1} \sum_{v\in\mathcal{U}\setminus\mathcal{E}(\iota,\rho)} f(v)\mathbf{P}^v(\xi_{\mathcal{U}} \le \lfloor \iota\varrho_n \rfloor^2) \le c^{-1}\rho^{-1}C\iota^2 \,.$$ ☐

Lemma 3.5.6. *Uniformly in all $\epsilon \in (\varrho_n^{-c}, c)$, the following holds with $\widehat{\mathbb{P}}$-probability at least $1 - e^{-\varrho_n}$,*

$$\sum_{v\in\mathcal{U}\setminus\Omega_\epsilon} f(v) \le C\epsilon \,, \tag{3.5.7}$$

$$|\Omega_\epsilon \setminus \mathcal{E}(\epsilon^2,\epsilon^2)| \le C\epsilon\varrho_n^d, \quad |\Omega_\epsilon \cup \mathcal{E}(\epsilon^2,\epsilon^2)| \le d\log_{1/p} n + C\epsilon\varrho_n^d \,. \tag{3.5.8}$$

Proof. Applying Lemmas 3.5.2 and 3.5.5 with $\iota = \rho = \epsilon^2$, we get that with $\widehat{\mathbb{P}}$-probability tending to 1,

$$\sum_{v\in\mathcal{U}\setminus\mathcal{E}(\epsilon^2,\epsilon^2)} f(v) \le C\epsilon^2 \quad \text{and} \quad |\mathcal{E}(\epsilon^2,\epsilon^2)| - d\log_{1/p} n \le \epsilon\varrho_n^d \,. \tag{3.5.9}$$

Hence,

$$\sum_{v\in\mathcal{U}\setminus\Omega_\epsilon} f(v) \le \sum_{v\in\mathcal{U}\setminus\mathcal{E}(\epsilon^2,\epsilon^2)} f(v) + \sum_{v\in\mathcal{E}(\epsilon^2,\epsilon^2)\setminus\Omega_\epsilon} f(v)$$
$$\le \sum_{v\in\mathcal{U}\setminus\mathcal{E}(\epsilon^2,\epsilon^2)} f(v) + |\mathcal{E}(\epsilon^2,\epsilon^2)|\epsilon\varrho_n^{-d} \le C\epsilon\,,$$

yielding (3.5.7). Combining (3.5.9) and the fact that $f(v) \ge \epsilon/\varrho_n^d$ for $v \in \Omega_\epsilon$ gives that

$$|\Omega_\epsilon \setminus \mathcal{E}(\epsilon^2,\epsilon^2)| \le \sum_{v\in\mathcal{U}\setminus\mathcal{E}(\epsilon^2,\epsilon^2)} f(v)\epsilon^{-1}\varrho_n^d \le C\epsilon\varrho_n^d\,.$$

Combined with (3.5.9), it immediately implies that $|\Omega_\epsilon \cup \mathcal{E}(\epsilon^2,\epsilon^2)| \le d\log_{1/p} n + C\epsilon\varrho_n^d$. This completes the proof of (3.5.8) and thus the proof of the lemma. □

Lemma 3.5.7. *Uniformly in any fixed (sequence of) $\epsilon = \epsilon(n) \in (\varrho_n^{-c}, c)$, with $\widehat{\mathbb{P}}$-probability tending to one,*

$$1 - \lambda_{\Omega_\epsilon} \le c_*(\log n)^{-2/d}(1 + C\epsilon)\,.$$

Proof. Let $\bar f = (f - \epsilon/\varrho_n^d)_+$. (Recall that $a_+ = a\mathbb{1}_{a\ge 0}$.) Then $\bar f$ is supported on Ω_ϵ and

$$\sum_{x\sim y}(\bar f(x) - \bar f(y))^2 \le \sum_{x\sim y}(f(x) - f(y))^2\,. \tag{3.5.10}$$

By Cauchy's inequality and (3.5.7), (3.5.8),

$$|f|_2^2 \ge |\Omega_\epsilon|^{-1}\Big(\sum_{x\in\Omega_\epsilon} f(x)\Big)^2 \ge c\varrho_n^{-d}\,.$$

By (3.5.7),

$$|\bar f|_2^2 = \sum_{x\in\Omega_\epsilon}(f(x) - \epsilon\varrho_n^{-d})^2 \ge |f|_2^2 - 2\epsilon\varrho_n^{-d}|f|_1 - \sum_{x\notin\Omega_\epsilon} f^2(x) \ge |f|_2^2 - 2\epsilon\varrho_n^{-d} - \epsilon^2\varrho_n^{-d}\,.$$

Hence, $|\bar{f}|_2^2 \geq |f|_2^2(1 - C\epsilon)$. Combined with (3.5.10) and the fact that

$$1 - \lambda_A = \min\Big\{\frac{1}{4d}\sum_{x\sim y}(g(x) - g(y))^2 : |g|_2^2 = 1, g(x) = 0\ \forall x \notin A\Big\} \quad \forall A \subseteq \mathbb{Z}^d\,,$$

it yields

$$(1 - \lambda_{\Omega_\epsilon}) \leq (1 - \lambda_{\mathcal{U}})(1 + C\epsilon)\,.$$

We complete the proof of the lemma by (3.5.3). □

3.5.2 Asymptotic ball

As discussed earlier, in order to apply the quantitative Faber–Krahn inequality we need to relate the principal eigenvalue in the continuum to that in discrete setup. To this end, we first provide Lemma 3.5.8 which gives an upper bound on the size of the boundary of Ω_ϵ — this will be used in the proof of Lemma 3.5.9. Define

$$\Omega_\epsilon^+ = \{v \in \mathbb{Z}^d : \min_{x\in\Omega_\epsilon} |x - v|_\infty \leq 2\}\,. \tag{3.5.11}$$

Lemma 3.5.8. *Uniformly in any fixed (sequence of) $\epsilon = \epsilon(n) \in (\varrho_n^{-c}, c)$, with $\widehat{\mathbb{P}}$-probability tending to one, we have $|\Omega_\epsilon^+ \setminus \Omega_{\epsilon^2}| \leq C\epsilon\varrho_n^d$.*

Proof. Let $t_1 = 2d$ and $t_2 = 2d + 1$. By Lemma 3.5.4, for $i = 1, 2$ (recalling that P is the transition kernel for the simple random walk with no killing)

$$\sum_{v\in\Omega_{\epsilon^2}} ((P|_{\mathcal{U}})^{t_i} f)(v) = \lambda_{\mathcal{U}}^{t_i} \sum_{v\in\Omega_{\epsilon^2}} f(v) \geq 1 - 3dc_*(\log n)^{-2/d} - \sum_{v\notin\Omega_{\epsilon^2}} f(v)\,. \tag{3.5.12}$$

At the same time, we have

$$\begin{aligned}\sum_{v\in\Omega_{\epsilon^2}}(P^{t_i}f)(v) &= \sum_{u\in\mathbb{Z}^d}(1-\mathbf{P}^u(S_{t_i}\notin\Omega_{\epsilon^2}))f(u)\\ &= 1-\sum_{v\notin\Omega_{\epsilon^2}}\sum_{u\in\mathbb{Z}^d}p_{t_i}(u,v)f(u)\,, \end{aligned}\tag{3.5.13}$$

where $p_t(\cdot,\cdot)$ is the t-step transition probability for simple random walk on $\mathbb{Z}^d$. Combining (3.5.12) and (3.5.13) (noting $\sum_{v\in\Omega_{\epsilon^2}}((P|_{\mathcal{U}})^{t_i}f)(v)\le\sum_{v\in\Omega_{\epsilon^2}}(P^{t_i}f)(v)$), we get that

$$\sum_{v\notin\Omega_{\epsilon^2}}\Big(\sum_{u\in\mathbb{Z}^d}(p_{t_1}(u,v)+p_{t_2}(u,v))f(u)\Big)\le 2(3dc_*(\log n)^{-2/d}+\sum_{v\notin\Omega_{\epsilon^2}}f(v))\le C\epsilon^2\,,\tag{3.5.14}$$

where the second transition follows from (3.5.7). Now, if $v\notin\Omega_{\epsilon^2}$ and $|v-x|_\infty\le 2$ for some $x\in\Omega_\epsilon$, then

$$p_{t_1}(x,v)+p_{t_2}(x,v)\ge(2d)^{-2d}\quad\text{and}\quad f(x)\ge\epsilon\varrho_n^{-d}\,.$$

Substituting these bounds in (3.5.14) yields the desired result. □

Lemma 3.5.9. *Uniformly in any fixed (sequence of) $\epsilon=\epsilon(n)\in(\varrho_n^{-c},c)$, with $\widehat{\mathbb{P}}$-probability tending to one, there exists a discrete ball B_ϵ such that*

$$\begin{aligned}|B_\epsilon\cup\mathcal{E}(\epsilon^2,\epsilon^2)\cup\Omega_\epsilon| &\le d\log_{1/p}n(1+C\epsilon^{1/2})\,,\\ |B_\epsilon\cap\mathcal{E}(\epsilon^2,\epsilon^2)\cap\Omega_\epsilon| &\ge d\log_{1/p}n(1-C\epsilon^{1/2})\,.\end{aligned}\tag{3.5.15}$$

Proof. The proof of the lemma crucially relies on the Faber–Krahn inequality, the application of which requires to approximate a discrete set in $\mathbb{Z}^d$ by a continuous set in $\mathbb{R}^d$. For notation clarity, in the proof we will use boldface to denote a subset in $\mathbb{R}^d$ (which typically has non-zero Lebesgue measure). Following the notation convention, we define

$$\boldsymbol{\Omega}_\epsilon^+=\{y\in\mathbb{R}^d:\min_{x\in\Omega_\epsilon}|y-x|_\infty\le 2\}\,.\tag{3.5.16}$$

Recalling (3.5.11), we see that $\Omega_\epsilon^+ = \mathbf{\Omega}_\epsilon^+ \cap \mathbb{Z}^d$. We will consider $\mu_{\mathbf{\Omega}_\epsilon^+}$ where $\mu_\cdot$ is defined as in (3.1.5). By [51, (38)] (see also [76, §4],[77, §6] and [60]), if $1-\lambda_{\Omega_\epsilon}$ is less than a sufficiently small constant depending on d, then

$$\mu_{\mathbf{\Omega}_\epsilon^+} \le (1-\lambda_{\Omega_\epsilon}) + C(1-\lambda_{\Omega_\epsilon})^2 .$$

Combined with Lemma 3.5.7, it yields that with $\widehat{\mathbb{P}}$-probability tending to 1

$$\mu_{\mathbf{\Omega}_\epsilon^+} \le c_*(\log n)^{-2/d}(1+C\epsilon) . \tag{3.5.17}$$

At the same time, by (3.1.3), we have that

$$\mu_{\mathbf{\Omega}_\epsilon^+}|\mathbf{\Omega}_\epsilon^+|^{2/d} - \mu_{\mathbf{B}}|\mathbf{B}|^{2/d} \ge c_d \mathcal{A}(\mathbf{\Omega}_\epsilon^+)^2 , \tag{3.5.18}$$

where $\mathcal{A}(\mathbf{\Omega}_\epsilon^+)$ is defined as in (3.1.4), and $\mathbf{B} \subseteq \mathbb{R}^d$ is an arbitrary continuous ball. Note that $|\Omega_\epsilon^+| = |\{y \in \mathbb{R}^d : \min_{x\in\Omega_\epsilon^+} |y-x|_\infty \le 1/2\}| = |\{y \in \mathbb{R}^d : \min_{x\in\Omega_\epsilon} |y-x|_\infty \le 2+1/2\}| \ge |\mathbf{\Omega}_\epsilon^+|$. By Lemma 3.5.8 and (3.5.8), we have with $\widehat{\mathbb{P}}$-probability tending to 1

$$|\mathbf{\Omega}_\epsilon^+| \le |\Omega_\epsilon^+| \le |\Omega_\epsilon^+ \setminus \Omega_{\epsilon^2}| + |\Omega_{\epsilon^2}| \le d\log_{1/p} n(1+C\epsilon).$$

Combined with (3.5.17), (3.5.18), it gives that $\mathcal{A}(\mathbf{\Omega}_\epsilon^+)^2 \le C\epsilon$. We denote by $\mathbf{B}_\epsilon$ the ball in $\mathbb{R}^d$ that achieves the minimum in $\mathcal{A}(\mathbf{\Omega}_\epsilon^+)$. Note that

$$|\{y \in \mathbb{R}^d : |y-x|_\infty < 1/2\} \setminus \mathbf{B}_\epsilon| \ge c \quad \text{for all } x \notin \mathbf{B}_\epsilon .$$

Since $\{y \in \mathbb{R}^d : |y-x|_\infty < 1/2\} \setminus \mathbf{B}_\epsilon$ for $x \in \Omega_\epsilon \setminus \mathbf{B}_\epsilon$ are disjoint subsets of $\mathbf{\Omega}_\epsilon^+ \setminus \mathbf{B}_\epsilon$, a volume calculation yields $|\Omega_\epsilon \setminus \mathbf{B}_\epsilon| \le C\epsilon^{1/2}|\mathbf{B}_\epsilon|$. Now let $B_\epsilon = \mathbf{B}_\epsilon \cap \mathbb{Z}^d$ (so B_ϵ is a discrete ball). We have

$$|B_\epsilon| \le d\log_{1/p} n(1+C\epsilon), \quad |\Omega_\epsilon \setminus B_\epsilon| \le C\epsilon^{1/2}|B_\epsilon| . \tag{3.5.19}$$

Also, combining (3.5.17) and (3.5.18) yields $|\boldsymbol{\Omega}_\epsilon^+| \geq (\mu_{\boldsymbol{\Omega}_\epsilon^+}^{-1}\mu_B)^{d/2}|B| \geq d\log_{1/p} n(1-C\epsilon)$. Combined with $|\boldsymbol{\Omega}_\epsilon^+| \leq |\Omega_\epsilon^+|$ and Lemma 3.5.8, it yields that with $\widehat{\mathbb{P}}$-probability tending to 1

$$|\Omega_{\epsilon^2}| \geq |\Omega_\epsilon^+| - |\Omega_\epsilon^+ \setminus \Omega_{\epsilon^2}| \geq d\log_{1/p} n(1 - C\epsilon)\,.$$

We replace ϵ by $\sqrt{\epsilon}$ in the preceding display and get that with $\widehat{\mathbb{P}}$-probability tending to 1

$$|\Omega_\epsilon| \geq d\log_{1/p} n(1 - C\epsilon^{1/2})\,. \tag{3.5.20}$$

Combined with (3.5.19), (3.5.8), it yields that

$$|B_\epsilon \setminus \Omega_\epsilon| \leq |B_\epsilon| + |\Omega_\epsilon \setminus B_\epsilon| - |\Omega_\epsilon| \ \leq C\,\epsilon^{1/2}|B_\epsilon|\,,$$
$$|\mathcal{E}(\epsilon^2,\epsilon^2) \triangle \Omega_\epsilon| \leq |\mathcal{E}(\epsilon^2,\epsilon^2)| + 2|\Omega_\epsilon \setminus \mathcal{E}(\epsilon^2,\epsilon^2)| - |\Omega_\epsilon| \ \leq C\,\epsilon^{1/2}\varrho_n^d\,.$$

Combined with (3.5.19) and (3.5.20), this completes the proof of the lemma. □

3.6 Localization on the intermittent island

This section is devoted to the proof for localization on the intermittent island. Denote

$$\widetilde{\Omega}_\epsilon := B_\epsilon \cap \Omega_\epsilon, \quad \widehat{B}_n := \bigcup_{x\in\widetilde{\Omega}_\epsilon} B(x, \epsilon^{1/10d}\varrho_n) \tag{3.6.1}$$

where B_ϵ is the ball chosen in Lemma 3.5.9. We will prove that the random walk will be in $\widehat{B}_n$ (a neighborhood of $\widetilde{\Omega}_\epsilon$) with high probability at any given time t after hitting $\widetilde{\Omega}_\epsilon$ and thus complete the proof of Theorem 3.1.1. To this end, in Section 3.6.1 we prove a couple of estimates on survival probabilities, building on which we provide the proof of localization in Section 3.6.2.

3.6.1 Survival probability estimates

The main results of this subsection are the following two survival probability estimates: Lemma 3.6.1 says the survival probability of random walk staying outside $\widetilde{\Omega}_\epsilon$ is very low, and Lemma 3.6.3 gives a lower bound on the survival probability of the random walk staying in $\mathcal{U}$ depending on its starting point (via the principal eigenfunction).

Lemma 3.6.1. *Uniformly in any fixed (sequence of) $\epsilon = \epsilon(n) \in (\varrho_n^{-c}, c)$, with $\widehat{\mathbb{P}}$-probability tending to one, for all $x \in \mathbb{Z}^d$ and $t \geq C\epsilon^{-1/d}\varrho_n^2$,*

$$\mathbf{P}^x(\xi_{\mathcal{U}\setminus\widetilde{\Omega}_\epsilon} > t) \leq 2\exp(-c\epsilon^{-1/d}\varrho_n^{-2}t)\,. \tag{3.6.2}$$

Remark 3.6.2. It follows from Lemma 3.6.1 that $B_\epsilon \subseteq B(v_*, (\log n)^{\kappa/2})$. Because (3.2.2) implies $\mathbf{P}^{v_*}(\xi_{\mathcal{C}_R(v_*)} > t) \geq c(n)\lambda_*^t \gg 2\exp(-c\epsilon^{-1/d}\varrho_n^{-2}t) \geq \mathbf{P}^{v_*}(\xi_{\mathcal{U}\setminus\widetilde{\Omega}_\epsilon} > t)$ for sufficiently large t. Hence $\widetilde{\Omega}_\epsilon \cap \mathcal{C}_R(v_*) \neq \varnothing$.

Lemma 3.6.3. *With $\widehat{\mathbb{P}}$-probability tending to one, for any $v \in \mathcal{U}$,*

$$\mathbf{P}^v(\xi_{\mathcal{U}} > t) \geq c\varrho_n^d f(v)\lambda_{\mathcal{U}}^t\,. \tag{3.6.3}$$

Proof of Lemma 3.6.1 and Lemma 3.6.3

Lemma 3.6.1 is a direct consequence of Lemmas 3.6.4 and 3.6.5 (below): Lemma 3.6.4 states that the random walk can not spend two much time in $\mathcal{E}(\epsilon^2,\epsilon^2) \setminus \widetilde{\Omega}_\epsilon$ due to its small size, and Lemma 3.6.5 states that the random walk can not spend two much time in $\mathcal{U} \setminus \mathcal{E}(\epsilon^2,\epsilon^2)$ because in this region the density of the obstacles is too high.

Lemma 3.6.4. *Uniformly in any fixed (sequence of) $\epsilon = \epsilon(n) \in (\varrho_n^{-c}, c)$, with $\widehat{\mathbb{P}}$-probability tending to one, for any $v \in \mathcal{U}$, $m > C\epsilon^{1/d}\varrho_n^2$,*

$$\mathbf{P}^v(|\{0 \leq t \leq m : S_t \in \mathcal{E}(\epsilon^2,\epsilon^2) \setminus \widetilde{\Omega}_\epsilon\}| \geq m/3) \leq \exp(-cm\epsilon^{-1/d}\varrho_n^{-2})\,. \tag{3.6.4}$$

Proof. We denote $q = \lceil |\mathcal{E}(\epsilon^2, \epsilon^2) \setminus \widetilde{\Omega}_\epsilon|^{2/d} \rceil$. Then for sufficiently small constant $\delta > 0$ and any $t \geq \delta^{-1} q$, $x \in \mathbb{Z}^d$

$$\mathbf{P}^x(S_t \in \mathcal{E}(\epsilon^2, \epsilon^2) \setminus \widetilde{\Omega}_\epsilon) \leq C\delta^{d/2} \leq \delta\,.$$

Therefore for all $x \in \mathbb{Z}^d$

$$\mathbf{E}^x \left[|\{0 \leq t \leq \lfloor \delta^{-2} q \rfloor : S_t \in \mathcal{E}(\epsilon^2, \epsilon^2) \setminus \widetilde{\Omega}_\epsilon\}| \right] \leq \delta^{-1} q\,.$$

We assume $m > \delta^{-2} q$ and δ is sufficiently small. For $k = 0, 1, ..., \lfloor m / \lfloor \delta^{-2} q \rfloor \rfloor$, we define

$$\theta_k = \mathbb{1}_{|\{k\lfloor \delta^{-2} q\rfloor \leq t < (k+1)\lfloor \delta^{-2} q \rfloor : S_t \in \mathcal{E}(\epsilon^2, \epsilon^2) \setminus \widetilde{\Omega}_\epsilon\}| \geq \delta^{-3/2} q}\,.$$

Then by the strong Markov property, θ_k's are dominated by i.i.d. Bernoulli random variable with parameter $C\delta^{1/2}$. A standard large deviation computation for Binomial random variables gives

$$\mathbf{P}\Big(\sum_{k=0}^{\lfloor m/\lfloor \delta^{-2} q \rfloor \rfloor} \theta_k \geq \delta^{1/4} \lfloor m / \lfloor \delta^{-2} q \rfloor \rfloor \Big) \leq e^{-c\delta^{5/2} m/q}\,. \tag{3.6.5}$$

Hence, with $\mathbf{P}$-probability at least $1 - e^{-c\delta^{5/2} m/q}$, for $m > \delta^{-2} q$,

$$|\{0 \leq t \leq m : S_t \in \mathcal{E}(\epsilon^2, \epsilon^2) \setminus \widetilde{\Omega}_\epsilon\}| \leq (1 + \sum_{k=0}^{\lfloor m/\lfloor \delta^{-2} q \rfloor \rfloor} \theta_k) \cdot \delta^{-2} q + \lfloor m / \lfloor \delta^{-2} q \rfloor \rfloor \cdot \delta^{-3/2} q \leq 3\delta^{1/4} m\,.$$

Since (3.5.15) yields $q \leq C\epsilon^{1/d} \varrho_n^2$, we complete the proof of the lemma by choosing a sufficiently small constant δ. □

Lemma 3.6.5. *Uniformly in any fixed (sequence of) $\epsilon = \epsilon(n) \in (\varrho_n^{-c}, c)$, with $\widehat{\mathbb{P}}$-probability tending to one, for any $v \in \mathcal{U}$, $m \geq C\epsilon^2 \varrho_n^2$,*

$$\mathbf{P}^v(|\{0 \leq t \leq m : S_t \in \mathcal{U} \setminus \mathcal{E}(\epsilon^2, \epsilon^2)\}| \geq m/3, \xi_{\mathcal{U}} > m) \leq \exp(-cm\epsilon^{-2} \varrho_n^{-2})\,. \tag{3.6.6}$$

Proof. Let $\iota = \rho = \epsilon^2$, $\zeta_0 = 0$ and for $m \geq 1$,

$$\zeta_m := \inf\{t \geq \xi_{m-1} + \lfloor \iota \varrho_n \rfloor^2 : S_t \in \mathcal{U} \setminus \mathcal{E}(\iota, \rho)\}\,.$$

Note that $|\{0 \leq t \leq m : S_t \in \mathcal{U} \setminus \mathcal{E}(\iota, \rho)\}| \geq m/3$ implies $\zeta_{\lfloor 10^{-1}\iota^{-2} m \varrho_n^{-2} \rfloor} < m$. By the strong Markov property at $\zeta_1, \zeta_2, ...$ and (3.5.6)

$$\begin{aligned}
&\mathbf{P}(\xi_{\mathcal{U}} > m, \zeta_{\lfloor 10^{-1}\iota^{-2} m \varrho_n^{-2} \rfloor} < m) \\
&\quad \leq \mathbf{P}(S_{[\zeta_m, \zeta_m + \lfloor \iota \varrho_n \rfloor^2]} \subseteq \mathcal{O}^c, S_{\zeta_m} \in \mathcal{U} \setminus \mathcal{E}(\iota, \rho) \text{ for } m = 1, 2, ..., \lfloor 10^{-1} \iota^{-2} m \varrho_n^{-2} \rfloor - 1) \\
&\quad \leq \exp\{-(10^{-1}\iota^{-2} m \varrho_n^{-2} - 2) c\rho\}\,. \qquad \square
\end{aligned}$$

Now we prove Lemma 3.6.3.

Lemma 3.6.6. *For all $v \in \mathcal{U}$ we have $f(v) \leq C\varrho_n^{-d}$. More generally, for all $l \leq \varrho_n$ we have $f(v) \leq Cl^{-d} \sum_{|u-v| \leq l} f(u)$.*

Proof. For any $v \in \mathcal{U}$, we consider stopping time $T = \lfloor l^2 \rfloor \wedge \zeta_U$, where for $k \geq 1$, $\zeta_k := \inf\{t : |S_t - v|_1 \geq k\}$ and U is independent of both the random walk (S_t) and the environment, and has a uniform distribution on $\{\lfloor l/2 \rfloor, \lfloor l/2 \rfloor + 1, ..., \lfloor l \rfloor - 1\}$. Note that $|S_T - v| \leq |S_T - v|_1 \leq l$ and that both $\mathbf{P}^v(S_{\lfloor l^2 \rfloor} = u) \leq Cl^{-d}$ and $\mathbf{P}^v(S_{\zeta_U} = u) \leq Cl^{-d}$ hold for all $u \in \mathbb{Z}^d$. So

$$\mathbf{P}^v(S_T = u) \leq Cl^{-d} \quad \text{for} \quad |u - v| \leq l\,. \tag{3.6.7}$$

By $P|_{\mathcal{U}} f = \lambda_{\mathcal{U}} f$, we have $\sum_{u: u \sim v} (2d)^{-1} f(u) = \lambda_{\mathcal{U}} f(v)$. Then it follows from the Markov property that $\lambda_{\mathcal{U}}^{-t} f(S_{t \wedge \xi_{\mathcal{U}}})$ is a martingale. Then by (3.6.7) and optional sampling theorem,

$$\begin{aligned}
f(v) = \mathbf{E}[\lambda_{\mathcal{U}}^{-T} f(S_{T \wedge \xi_{\mathcal{U}}})] &\leq \lambda_{\mathcal{U}}^{-\varrho_n^2} \sum_u \mathbb{P}(S_T = u) f(u) + \mathbf{P}(\xi_{\mathcal{U}} \leq T) \cdot 0 \\
&\leq C \lambda_{\mathcal{U}}^{-\varrho_n^2} l^{-d} \sum_{|u-v| \leq l} f(u)\,.
\end{aligned}$$

We complete the proof of the lemma by Lemma 3.5.4. □

Proof of Lemma 3.6.3. Since f is the eigenfunction of eigenvalue $\lambda_{\mathcal{U}}$,

$$\sum_{u\in\mathcal{U}} f(u)\mathbf{P}^u(S_t = v, \xi_{\mathcal{U}} > t) = \lambda_{\mathcal{U}}^t f(v)\,. \tag{3.6.8}$$

Applying Lemma 3.6.6 with $l = \varrho_n$ yields that $f(u) \le C\varrho_n^{-d}$ for all $u \in \mathcal{U}$. Plugging this bound into (3.6.8) and using reversibility yields

$$\lambda_{\mathcal{U}}^t f(v) \le C\varrho_n^{-d} \sum_{u\in\mathcal{U}} \mathbf{P}^u(S_t = v, \xi_{\mathcal{U}} > t) = C\varrho_n^{-d}\mathbf{P}^v(\xi_{\mathcal{U}} > t)\,. \qquad \square$$

3.6.2 Proof of localization

In this subsection, we first prove the localization result for the end point (Lemma 3.6.8), which enables us to give an upper bound of the survival probability in $\mathcal{U}$ with a constant error factor as in Lemma 3.6.9. Next, in Lemma 3.6.10, we prove that the random walk will hit $\widetilde{\Omega}_\epsilon$ in poly-log n steps conditioned on staying in $\mathcal{U}$ and prove a localization result for any fix time point. Finally, we prove Theorem 3.1.1 by combining these ingredients in Lemma 3.6.11.

To start, we consider a random walk starting from $z \in \mathcal{U}$ conditioned on staying in $\mathcal{U}$ up to time t. We assume

$$\text{either } z \in \widetilde{\Omega}_\epsilon, t \ge 0 \quad \text{or} \quad z \in \mathcal{U}, t \ge \varrho_n^\iota\,, \tag{3.6.9}$$

where ι is a large constant to be determined in the following lemma. The following lemma guarantees that under this assumption we either starting from $\widetilde{\Omega}_\epsilon$, or the time t is long enough so that the random walk can reach $\widetilde{\Omega}_\epsilon$.

Lemma 3.6.7. *Consider $\epsilon \in (\varrho_n^{-c}, c)$. For n sufficiently large, there exists $\iota > 0$ such that* *for all* $z \in \mathcal{U}$ *and* $t \geq \varrho_n^\iota$*,*

$$\mathbf{P}^z(S_{[0,t]} \cap \widetilde{\Omega}_\epsilon \neq \varnothing \mid \xi_{\mathcal{U}} > t) \leq C \exp(-c\epsilon^{-1/d} t \varrho_n^{-2}) . \tag{3.6.10}$$

Proof. Lemma 3.6.1 implies $\mathbf{P}^z(S_{[0,t]} \subseteq \mathcal{U} \setminus \widetilde{\Omega}_\epsilon) \leq 2\exp(-c\epsilon^{-1/d} t \varrho_n^{-2})$. Comparing it with (3.5.4) and choosing a sufficiently large ι yields the desired result. □

By our convention of c, we can choose a sufficiently small constant $b > 0$ such that $\epsilon := \varrho_n^{-b}$ satisfies the condition for ϵ in all previous lemmas (from Lemma 3.5.2 to Lemma 3.6.7). We will fix $\epsilon = \varrho_n^{-b}$ henceforth.

Lemma 3.6.8. *Recall* $\widehat{B}_n$ *as in* (3.6.1). *We assume* (3.6.9). *For* $\epsilon > 0$ *sufficiently small and* n *sufficiently large,*

$$\mathbf{P}^z(S_t \in \widehat{B}_n \mid \xi_{\mathcal{U}} > t) \geq 1 - \exp(-c\varrho_n^{b/(10d)}) . \tag{3.6.11}$$

Proof. The proof divides into two steps. In **Step 1** we consider the last visit to $\widetilde{\Omega}_\epsilon$ conditioned on survival: the last excursion should be short because the survival probability (if not coming back to $\widetilde{\Omega}_\epsilon$) decays very fast in light of Lemma 3.6.1 while Lemma 3.6.3 shows that the survival probability starting from $\widetilde{\Omega}_\epsilon$ is comparably large. Then in **Step 2**, we show that the random walk can not go too far in the last few steps.

Step 1. We consider the last visit time j to $\widetilde{\Omega}_\epsilon$ before time t. By Lemma 3.6.1 and the Markov property, we get

$$\mathbf{P}^z(S_j \in \widetilde{\Omega}_\epsilon, S_{[j+1,t]} \subset \mathcal{U} \setminus \widetilde{\Omega}_\epsilon, \xi_{\mathcal{U}} > t) \leq \mathbf{P}^z(S_t \in \widetilde{\Omega}_\epsilon, \xi_{\mathcal{U}} > j) \cdot 2\exp(-c\varrho_n^{-2+b/d}(t-j-1)) .$$

In addition, we deduce from Lemma 3.6.3 and Lemma 3.5.4 that

$$\begin{aligned}\mathbf{P}^z(S_j \in \widetilde{\Omega}_\epsilon, \xi_{\mathcal{U}} > t) &\geq \mathbf{P}^z(S_j \in \widetilde{\Omega}_\epsilon, \xi_{\mathcal{U}} > j) \cdot c\varrho_n^{-b} \lambda_{\mathcal{U}}^{t-j} \\ &\geq \mathbf{P}^z(S_j \in \widetilde{\Omega}_\epsilon, \xi_{\mathcal{U}} > j) \cdot c\varrho_n^{-b} \exp(-C\varrho_n^{-2}(t-j)) \,.\end{aligned}$$

Combining preceding two inequalities, we see that for $j \leq t - \varrho_n^{2-b/(2d)}$,

$$\mathbf{P}^z(S_j \in \widetilde{\Omega}_\epsilon, S_{[j+1,t]} \subset \mathcal{U} \setminus \widetilde{\Omega}_\epsilon \mid \xi_{\mathcal{U}} > t) \leq e^{-c\varrho_n^{-b/(2d)}} \,.$$

Then a union bound over $0 \leq j \leq t - \varrho_n^{2-b/(2d)}$ and Lemma 3.6.7 gives

$$\mathbf{P}^z(S_{[t-\varrho_n^{2-b/(2d)},t]} \cap \widetilde{\Omega}_\epsilon = \varnothing \mid \xi_{\mathcal{U}} > t) \leq e^{-c\varrho_n^{-b/(2d)}} \,. \tag{3.6.12}$$

Step 2. We define stopping time $T_\star = \inf\{j \geq t - \varrho_n^{2-b/(2d)} : S_j \in \widetilde{\Omega}_\epsilon\}$. We claim that

$$\mathbf{P}^z(T_\star < t, \max_{T_\star \leq j \leq \varrho_n^{2-b/(2d)} + T_\star} |S_j - S_{T_\star}| \leq \varrho_n^{1-b/(5d)} \mid \xi_{\mathcal{U}} > t) \geq 1 - \exp(-c\varrho_n^{b/(10d)}) \,. \tag{3.6.13}$$

We note that the desired result is a direct consequence of (3.6.13). It remains to prove (3.6.13). We first see that (3.6.12) implies

$$\mathbf{P}^z(T_\star \geq t \mid \xi_{\mathcal{U}} > t) \leq e^{-c\varrho_n^{-b/(2d)}} \,.$$

In addition, by the strong Markov property at $T_\star$, we get that

$$\begin{aligned}&\mathbf{P}^z\Big(T_\star < t, \max_{T_\star \leq j \leq T_\star + \varrho_n^{2-b/(2d)}} |S_j - S_{T_\star}| > \varrho_n^{1-b/(5d)}, \xi_{\mathcal{U}} > t\Big) \\ &\leq \mathbf{E}^z\Big[\mathbb{1}_{T_\star < t, \xi_{\mathcal{U}} > T_\star} \mathbf{P}^{S_{T_\star}}\Big(\max_{0 \leq j \leq \varrho_n^{2-b/(2d)}} |S_j - S_{T_\star}| > \varrho_n^{1-b/(5d)}\Big)\Big] \\ &\leq C\exp(-c\varrho_n^{b/(10d)})\mathbf{P}^z(T_\star < t, \xi_{\mathcal{U}} > T_\star) \leq C\exp(-c\varrho_n^{b/(10d)})\mathbf{P}^z(\xi_{\mathcal{U}} > T_\star) \,,\end{aligned}$$

where we used [53, Proposition 2.4.5] in the second inequality. At the same time, by the strong Markov property at $T_\star$ and Lemma 3.6.3 and (3.5.3)

$$
\begin{aligned}
\mathbf{P}^z(\xi_{\mathcal{U}} > t) &\ge \mathbf{P}^z(\xi_{\mathcal{U}} > T_\star + \varrho_n^{2-b/(2d)}) \\
&= \mathbf{E}^z\left[\mathbb{1}_{\xi_{\mathcal{U}} > T_\star}\mathbf{P}^{S_{T_\star}}\left(\xi_{\mathcal{U}} > \varrho_n^{2-b/(2d)}\right)\right] \ge c\epsilon(1 - 2c_*\varrho_n^{-b/(2d)})\mathbf{P}^z(\xi_{\mathcal{U}} > T_\star)\,.
\end{aligned}
$$

Combining the last three inequalities, we deduce (3.6.13) as required, and thus complete the proof of the lemma. □

Lemma 3.6.9. *Recall $\widehat{B}_n$ as in* (3.6.1)*. For any $x \in \mathbb{Z}^d, t \ge 0$, $\mathbf{P}^x(\xi_{\mathcal{U}} > t, S_t \in \widehat{B}_n) \le C\lambda_{\mathcal{U}}^t$.*

Proof. By (3.5.3), it suffice to consider $t \ge \varrho_n^2$. By the Markov property at ϱ_n^2,

$$
\begin{aligned}
\mathbf{P}^x(\xi_{\mathcal{U}} > t, S_t \in \widehat{B}_n) &\le \sum_y p_{\varrho_n^2}(x,y)\mathbf{P}^y(\xi_{\mathcal{U}} > t - \varrho_n^2, S_{t-\varrho_n^2} \in \widehat{B}_n) \\
&= p_{\varrho_n^2}(x,\cdot)(P|_{\mathcal{U}})^{t-\varrho_n^2}\mathbb{1}_{\widehat{B}_n} \le C\lambda_{\mathcal{U}}^{t-\varrho_n^2}\,.
\end{aligned}
$$

where in the last inequality we used $|p_{\varrho_n^2}(x,\cdot)|_2 \le 1 \times |p_{\varrho_n^2}(x,\cdot)|_\infty \le C\varrho_n^{-d}$ and $|\widehat{B}_n| \le C\varrho_n^d$. We complete the proof of the lemma by (3.5.3). □

Lemma 3.6.10. *We assume* (3.6.9) *and $m \ge t$. Then for n sufficiently large,*

$$
\mathbf{P}^z(\tau_{\widetilde{\Omega}_\epsilon} \le \varrho_n^\iota \mid \xi_{\mathcal{U}} > m) \ge 1 - \exp(-\varrho_n)\,, \tag{3.6.14}
$$

$$
\mathbf{P}^z(S_t \in \widehat{B}_n \mid \xi_{\mathcal{U}} > m) \ge 1 - \exp(-c\varrho_n^{b/(10d)})\,. \tag{3.6.15}
$$

Proof. We first give a lower bound on $\mathbf{P}^z(\xi_{\mathcal{U}} > m)$. Define stopping time $T'_\star = \inf\{j \ge$

$t - \varrho_n^2 : S_j \in \widetilde{\Omega}_\epsilon\}$. By the strong Markov property at $T'_\star$ and Lemma 3.6.3 and (3.5.3)

$$\begin{aligned}\mathbf{P}^z(\xi_{\mathcal{U}} > m) &\geq \mathbf{P}^z(\xi_{\mathcal{U}} > T'_\star + \varrho_n^2 + m - t) \\ &= \mathbf{E}^z[\mathbb{1}_{\xi_{\mathcal{U}} > T'_\star} \mathbf{P}^{S_{T'_\star}}(\xi_{\mathcal{U}} > \varrho_n^2 + m - t)] \\ &\geq c\epsilon \lambda_{\mathcal{U}}^{\varrho_n^2 + m - t} \mathbf{P}^z(\xi_{\mathcal{U}} > T'_\star) \geq c\epsilon \lambda_{\mathcal{U}}^{m-t} \mathbf{P}^z(\xi_{\mathcal{U}} > t, T'_\star \leq t)\,.\end{aligned}$$

Applying (3.6.12) to t gives $\mathbf{P}^z(T'_\star < t \mid \xi_{\mathcal{U}} > t) \geq 1/2$. Hence

$$\mathbf{P}^z(\xi_{\mathcal{U}} > m) \geq c\epsilon \lambda_{\mathcal{U}}^{m-t} \mathbf{P}^z(\xi_{\mathcal{U}} > t)\,. \tag{3.6.16}$$

We are now ready to prove (3.6.14). First, by Lemma 3.6.8, $\mathbf{P}^z(S_m \notin \widehat{B}_n \mid \xi_{\mathcal{U}} > m) \leq \exp(-c\varrho_n^{b/(10d)})$. Second, using the Markov property at time ϱ_n^ι and (3.6.9),

$$\mathbf{P}^z(\tau_{\widetilde{\Omega}_\epsilon} > \varrho_n^\iota, S_m \in \widehat{B}_n, \xi_{\mathcal{U}} > m) \leq \mathbf{P}^z(\tau_{\widetilde{\Omega}_\epsilon} > \varrho_n^\iota, \xi_{\mathcal{U}} > \varrho_n^\iota) C \lambda_{\mathcal{U}}^{m-\varrho_n^\iota}\,.$$

Then by Lemma 3.6.7, this is bounded above by

$$C \exp(-c\varrho_n^{b/d} \varrho_n^{-2} \varrho_n^\iota) \cdot \mathbf{P}^z(\xi_{\mathcal{U}} > \varrho_n^\iota) \lambda_{\mathcal{U}}^{m-\varrho_n^\iota}\,.$$

Now (3.6.14) follows from setting $t = \varrho_n^\iota$ in (3.6.16) and comparing it with this bound.

We next prove (3.6.15). The case $t = m$ has been treated by Lemma 3.6.8. We assume $t < m$. By the Markov property at time t and Lemma 3.6.9

$$\begin{aligned}\mathbf{P}^z(S_t \notin \widehat{B}_n, S_m \in \widehat{B}_n, \xi_{\mathcal{U}} > m) &\leq C\mathbf{P}^z(S_t \notin \widehat{B}_n, \xi_{\mathcal{U}} > t) \lambda_{\mathcal{U}}^{m-t} \\ &\leq \exp(-c\varrho_n^{b/(10d)}) \mathbf{P}^z(\xi_{\mathcal{U}} > t) \lambda_{\mathcal{U}}^{m-t}\,,\end{aligned}$$

where in the last inequality we used Lemma 3.6.8. Combining with (3.6.16), we get

$$\mathbf{P}^z(S_t \notin \widehat{B}_n, S_m \in \widehat{B}_n \mid \xi_{\mathcal{U}} > m) \leq \exp(-c\varrho_n^{b/(10d)})\,.$$

We complete the proof of (3.6.15) by Lemma 3.6.8. □

Theorem 3.1.1 follows from the following lemma.

Lemma 3.6.11. *With $\widehat{\mathbb{P}}$-probability tending to one, we have that*

$$\mathbf{P}(\tau_{\widetilde{\Omega}_\epsilon} \geq C|v_*| \mid \tau > n) \leq e^{-\varrho_n^c}\,,$$
$$\mathbf{P}(S_{t\vee\tau_{\widetilde{\Omega}_\epsilon}} \notin \widehat{B}_n \mid \tau > n) \leq e^{-\varrho_n^c} \textit{ for all } t \leq n\,.$$

Proof. Recall that $T = \mathsf{T}_{v_*} = \tau_{B(v_*,(\log n)^{\kappa/2})}$ as in Theorem 3.1.2 . By Remark 3.6.2, $\tau_{\widetilde{\Omega}_\epsilon} > T$. We first prove that $\tau_{\widetilde{\Omega}_\epsilon} \leq T + \varrho_n^\iota$. By the strong Markov property at T,

$$\begin{aligned}&\mathbf{P}(\tau > T, T \leq C|S_T|, \tau_{\widetilde{\Omega}_\epsilon} > T + \varrho_n^\iota, S_{[T,n]} \subseteq \mathcal{U})\\ &= \mathbf{E}\big[\mathbb{1}_{\tau>T,T\leq C|S_T|}\mathbf{P}^{S_T}(\tau_{\widetilde{\Omega}_\epsilon} > \varrho_n^\iota, \xi_{\mathcal{U}} > n-T)\big]\,.\end{aligned}$$

By (3.6.14) (since $n - C|S_T| > \varrho_n^\iota$), this is bounded from above by

$$\exp(-\varrho_n)\mathbf{E}\big[\mathbb{1}_{\tau>T,T\leq C|S_T|}\mathbf{P}^{S_T}(\xi_{\mathcal{U}} > n-T)\big] = \exp(-\varrho_n)\mathbf{P}(T \leq C|S_T|, S_{[T,n]} \subseteq \mathcal{U}, \tau > n)\,.$$

Combining with (3.3.11), we have that $\mathbf{P}(\tau_{\widetilde{\Omega}_\epsilon} \leq T + \varrho_n^\iota, T \leq C|S_T| \mid \tau > n) \geq 1 - e^{-\varrho_n^c}$. Hence

$$\mathbf{P}(\tau_{\widetilde{\Omega}_\epsilon} < C|v_*| \mid \tau > n) \geq 1 - e^{-\varrho_n^c}\,.$$

Next, by the strong Markov property at $\tau_{\widetilde{\Omega}_\epsilon}$,

$$\mathbf{P}(\tau > \tau_{\widetilde{\Omega}_\epsilon}, S_{t\vee\tau_{\widetilde{\Omega}_\epsilon}} \notin \widehat{B}_n, S_{[\tau_{\widetilde{\Omega}_\epsilon},n]} \subseteq \mathcal{U}) = \mathbf{E}\big[\mathbb{1}_{\tau>\tau_{\widetilde{\Omega}_\epsilon}}\mathbf{P}^{S_{\tau_{\widetilde{\Omega}_\epsilon}}}(S_{(t-\tau_{\widetilde{\Omega}_\epsilon})_+} \notin \widehat{B}_n, \xi_{\mathcal{U}} > n - \tau_{\widetilde{\Omega}_\epsilon})\big]\,.$$

By (3.6.15) (Since $S_{\tau_{\widetilde{\Omega}_\epsilon}} \in \widetilde{\Omega}_\epsilon$), this is bounded from above by

$$e^{-\varrho_n^c}\mathbf{E}\big[\mathbb{1}_{\tau>\tau_{\widetilde{\Omega}_\epsilon}}\mathbf{P}^{S_{\tau_{\widetilde{\Omega}_\epsilon}}}(\xi_{\mathcal{U}} > n - \tau_{\widetilde{\Omega}_\epsilon})\big] = e^{-\varrho_n^c}\mathbf{P}(S_{[\tau_{\widetilde{\Omega}_\epsilon},n]} \subseteq \mathcal{U}, \tau > n)\,.$$

We complete the proof of the lemma by (3.3.11). □

3.A Index of notation

μ_Ω	(3.1.5)
ϱ_n	(4.1.2)
$k_n, R, \mathcal{C}_R(v)$	Section 3.2
$\mathcal{D}_*, \lambda(v), \lambda_*$	Section 3.2
V	Definition 3.2.1
$\mathsf{T}_v, \mathsf{E}_v, \mathsf{D}_\lambda$	Definition 3.2.2
$\varphi_\star$	Definition 3.3.1
g	(3.3.3)(3.4.45)
v_*	(3.3.5)
$\mathcal{U}$	(3.3.10)
$G_A(x, y; \lambda)$	Definition 3.4.1
$r, \mathcal{V}_\lambda, \varphi, \varphi_*$	Definition 3.4.2
$L, \mathbb{K}_{\mathbf{i}}$	Definition 3.4.4
white/black, $\mathscr{C}_{\mathbf{i}}$	Definition 3.4.5
tilde-white/black $\mathbf{i}_x, \mathcal{C}_{\mathbb{K}}, \mathbb{K}_{\mathbf{A}}, \mathscr{C}_{\mathbf{A}}$	Around (3.4.11)
$\mathbf{\Lambda}(x, r)$	Lemma 3.4.9
$\bar{\varphi}, \varphi^\circ, \varphi^\circ_*, \bar{\varphi}^\circ_*, R_\circ$	Definition 3.4.11
g_x, Q_x	Definition 3.4.23
h	(3.4.44)
s_x, G_x, Δ_x, D_x	(3.4.57)
$\mathcal{G}_x$	Definition 3.4.28
$(\iota r, \rho)$-empty, $\mathcal{E}$	Definition 3.5.1
f, Ω_ϵ	Definition 3.5.3
Ω^+_ϵ	(3.5.11)
B_ϵ	Lemma 3.5.9
$\widetilde{\Omega}_\epsilon, \widehat{B}_n$	(3.6.1)

CHAPTER 4

RANDOM WALK DISTRIBUTION UNDER THE QUENCHED LAW

4.1 Introduction

4.1.1 Model and main results

For $d \geq 2$, let $(S_t)_{t\geq 0}$ be a discrete time simple symmetric random walk on $\mathbb{Z}^d$, with $\mathbf{P}^x$ and $\mathbf{E}^x$ denoting probability and expectation for the random walk with $S_0 = x \in \mathbb{Z}^d$ and the superscript omitted when x is the origin. We place the random walk in a random environment where an obstacle is placed independently at each point $x \in \mathbb{Z}^d$ with probability $1-p \in (0,1)$, with $\mathbb{P}$ and $\mathbb{E}$ denoting probability and expectation for the random environment. We will say x is *closed* if x is occupied by an obstacle, and x is *open* otherwise. Denote by $\mathcal{O}$ the set of sites occupied by the obstacles. The random walk is killed at the moment it hits an obstacle, namely at the stopping time

$$\tau := \tau_{\mathcal{O}} = \inf\{t \geq 0 : S_t \in \mathcal{O}\}\,. \tag{4.1.1}$$

More generally, we denote by τ_A the first hitting time of a set $A \subset \mathbb{Z}^d$. We will assume $p > p_c(\mathbb{Z}^d)$, the critical threshold for site percolation, and let $\widehat{\mathbb{P}}$ be the conditional probability measure for $\mathcal{O}$ given that the origin is in the infinite open cluster. Given an environment under $\widehat{\mathbb{P}}$, we are interested in the behavior of the random walk given that it survives for a long time.

It has been shown in Chapter 3 that conditioned on survival up to time n, the random walk stays in an *island* (determined by the environment) of diameter at most poly-logarithmic in n during time $[o(n), n]$. Furthermore, at any deterministic time $t \in [o(n), n]$, the random walk stays with high probability in a ball of radius asymptotically

$$\varrho_n = \lfloor (\omega_d^{-1} d \log_{1/p} n)^{1/d} \rfloor \,, \tag{4.1.2}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Namely, the following was shown in Chapter 3.

Theorem C. *There exist a constant $C = C(d,p)$, $\boldsymbol{\mathcal{X}}_n \in \mathbb{Z}^d$ within distance $C(\log n)^{-2/d} n$ from the origin, and $\epsilon_n > 0$ tends to 0 as $n \to \infty$ such that*

$$\min_{C|\boldsymbol{\mathcal{X}}_n| \le t \le n} \mathbf{P}(S_t \in B(\boldsymbol{\mathcal{X}}_n, (1+\epsilon_n)\varrho_n) \mid \tau > n) \to 1 \text{ in } \widehat{\mathbb{P}}\text{-probability}\,, \tag{4.1.3}$$

where $B(x,r)$ denotes the Euclidean ball with center x and radius r.

The study of the random walk killed by random obstacles is partially motivated by its relation to the so-called *Anderson localization*. The generator of the killed random walk can be formally written as the random Schrödinger operator $-\frac{1}{2d}\Delta + \infty \cdot \mathbb{1}_{\mathcal{O}}$, where Δ is the discrete Laplacian. For this type of operators, various localization phenomena have been predicted and some of them have been rigorously proved; see e.g., [2, 48]. In particular, the corresponding parabolic problem in our setting is the discrete time initial-boundary value problem

$$\begin{cases} u(n+1,x) - u(n,x) = \frac{1}{2d}\Delta u(n,x), & (n,x) \in \mathbb{Z}_+ \times (\mathbb{Z}^d \setminus \mathcal{O}), \\ u(n,x) = 0, & (n,x) \in \mathbb{Z}_+ \times \mathcal{O}, \\ u(0,x) = \mathbb{1}_{\{0\}}(x), \end{cases} \tag{4.1.4}$$

and the probability $\mathbf{P}(S_t = x, \tau > n)$ represents its unique bounded solution. Since $\mathbf{P}(\tau > n)$ is the total mass of the solution, the conditional probability $\mathbf{P}(S_t = x \mid \tau > n)$ is the normalized mass distribution. Thus Theorem C implies that the dominant proportion of mass tends to localize in a single ball of radius ϱ_n. It is an important problem to further identify the profile of the mass distribution inside the localization region. The first step to tackle this problem is to understand how the environment looks like in the localization region, which is an interesting problem itself. These two problems are listed as the main questions in [48, Section 1.3].

The first main result in this chapter is about the behavior of environment in the localization region. Intuitively, the ball where the random walk will be localized should contain very few obstacles, or even no obstacle. It is proved in [25] that the volume proportion of obstacles inside the localizing ball is at most $o(1)$, but it remains open to show that it actually contains no obstacle at all. Our first main result resolves this question.

Theorem 4.1.1. *There exists a constant $\kappa > 0$ depending only on (d, p) such that with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$,*

$$\mathcal{B}_n := B(\boldsymbol{\mathcal{X}}_n, \varrho_n - \varrho_n^{1-\kappa}) \text{ is open.} \tag{4.1.5}$$

Remark 4.1.2. Under the *annealed* law $\mathbb{P}\otimes\mathbf{P}(\cdot \mid \tau > n)$, a similar ball clearing phenomenon was proved for $d = 2$ in [12] and [67], while the latter work studied a continuum analogue called Brownian motion among Poissonian obstacles. The extension to $d \geq 3$, conjectured in [12], was open for a long time but recently resolved in [23] and [10] independently.

The proof of Theorem 4.1.1 differs substantially from those of the annealed result mentioned in Remark 4.1.2 and requires new ideas. It is based on intricate bounds on the tail distribution of the principal Dirichlet eigenvalue of the random walk in the island of localization, and relies on environment switching arguments that add or remove obstacles. The heart of the proof consists in making judicious choices of which obstacles to add or remove, and estimating how such modifications change the associated principal Dirichlet eigenvalue, which is of independent interest. See Section 4.3 for a more detailed proof outline.

Our second main result gives the limiting law of $\varrho_n^{-1}(S_t - \boldsymbol{\mathcal{X}}_n)$ for $t = n$ or t in the bulk, conditioned on survival up to time n. The result for $t = n$ corresponds to the limiting profile of the solution of (4.1.4). Thanks to Theorem 4.1.1, we are able prove the convergence at the level of local limit theorem. Let ϕ_1 and ϕ_2 be the L^1 and L^2-normalized first eigenfunction of the Dirichlet-Laplacian of the unit ball in $\mathbb{R}^d$, respectively, and let $|\cdot|_1$ and $|\cdot|$ denote ℓ^1 and ℓ^2 norms on $\mathbb{Z}^d$, respectively.

Theorem 4.1.3. *There exists $C > 0$ depending only on (d, p) such that the following hold with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$:*

$$\min_{C|\boldsymbol{\mathcal{X}}_n| \le t \le n} \mathbf{P}(S_t \in \mathcal{B}_n \mid \tau > n) \to 1\,, \tag{4.1.6}$$

$$\sup_{x \in \mathcal{B}_n : |x|_1 + n \text{ is even}} \left| \varrho_n^d \mathbf{P}(S_n = x \mid \tau > n) - 2\phi_1(\tfrac{x - \boldsymbol{\mathcal{X}}_n}{\varrho_n}) \right| \to 0\,, \tag{4.1.7}$$

$$\sup_{x \in \mathcal{B}_n : |x|_1 + t \text{ is even}} \left| \varrho_n^d \mathbf{P}(S_t = x \mid \tau > n) - 2\phi_2^2(\tfrac{x - \boldsymbol{\mathcal{X}}_n}{\varrho_n}) \right| \to 0\,, \tag{4.1.8}$$

where the convergence in (4.1.8) *holds uniformly for all $t \in [C|\boldsymbol{\mathcal{X}}_n|, n - C\varrho_n^2 \log \varrho_n]$.*

Remark 4.1.4. The limiting marginal distribution identified by Theorem 4.1.3 is consistent with the behaviour of a random walk conditioned to stay inside the ball $\mathcal{B}_n$ (see (4.1.5)) up to time n after reaching it, although in our case we expect the walk to make excursions of lengths up to $c \log n$ away from $\mathcal{B}_n$.

The proof of Theorem 4.1.3 is based on eigenfunction decompositions and eigenvalue and eigenfunction estimates.

4.1.2 Organization and notation

The rest of this chapter is organized as follows. We will first collect some results from [24, 25] in Section 4.2, which we will need later in the proof. Then in Section 4.3, we introduce some intermediate results and use them to prove Theorems 4.1.1, 4.1.3. Results introduced in Section 4.3 will then be proved in Section 4.4 and 4.5.

Throughout the rest of this chapter, C, c will denote positive constants depending only on (d, p) whose numerical values may vary from line to line with C typically a large constant and c a small constant. For constants such as $C_{4,1}$, $c_{5,2}$ or κ (which also depend only on (d,p)), their values will stay the same throughout this chapter.

4.2 Preliminaries

We recall here some basic results and tools developed in Chapter 3 that we will need in our proof. The main result of Chapter 3 was that conditioned on quenched survival up to time n, the random walk will be confined in an island of diameter $(\log n)^C$ during the time interval $[Cn(\log n)^{-2/d}, n]$. Furthermore, with high probability the random walk S_t is in a ball whose radius is asymptotically equal to ϱ_n. More precisely, the following was proved in the previous chapter, where

$$S_{[k,l]} = \{S_i \colon k \le i \le l\} \tag{4.2.1}$$

denotes the range of the random walk during the time interval $[k,l]$.

Theorem D ([25, Lemma 6.11 and Remark 6.2.]). *There exist constants* $c_{4,1} = c_{4,1}(d,p)$, $C_{4,1} = C_{4,1}(d,p) > 0$, $v_* = v_*(\mathcal{O}) \in B(0, Cn(\log n)^{-2/d})$ *and*

$$\mathcal{X}_{\mathcal{U}} \in \mathcal{U} := \textit{the connected component in } B(v_*, (\log n)^{C_{4,1}}) \setminus \mathcal{O} \textit{ that contains } v_*$$

such that the following holds: Let

$$\widehat{B}_n := B(\mathcal{X}_{\mathcal{U}}, (1+\varrho_n^{-c_{4,1}})\varrho_n) \setminus \mathcal{O}. \tag{4.2.2}$$

Then with $\widehat{\mathbb{P}}$*-probability tending to one as* $n \to \infty$*, we have* $\widehat{B}_n \subset \mathcal{U}$ *and*

$$\mathbf{P}(\tau_{\widehat{B}_n} < C|\mathcal{X}_{\mathcal{U}}|, S_{[\tau_{\widehat{B}_n}, n]} \subset \mathcal{U} \mid \tau > n) \ge 1 - \exp(-\varrho_n^c)\,; \tag{4.2.3}$$

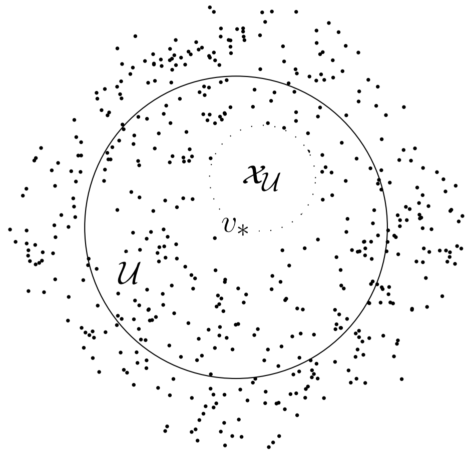

Figure 4.1: The centers v_* and $\mathcal{X}_\mathcal{U}$ in Theorem D. Little dots are obstacles. Lemma 4.2.4 asserts that $\mathcal{X}_\mathcal{U}$ is the center of an almost vacant ball.

furthermore, uniformly in $t \in [C|\mathcal{X}_\mathcal{U}|, n]$,

$$\mathbf{P}(S_t \in \widehat{B}_n \mid \tau > n) \geq 1 - \exp(-\varrho_n^c)\,. \tag{4.2.4}$$

Theorem D asserts that conditioned on survival up to time n, the walk reaches $\mathcal{U}$ with linear speed and then stays confined to $\mathcal{U}$ till time n. Furthermore, at each $t \in [C|\mathcal{X}_\mathcal{U}|, n]$, with high probability, the walk is localized in the ball $\widehat{B}_n$ with center $\mathcal{X}_\mathcal{U}$, which is the same as $\mathcal{X}_n$ in Theorem 4.1.1.

Let us briefly recall how v_*, and hence $\mathcal{U}$, is determined in the theorem above. Basically, there are small number of local regions like $\mathcal{U}$ which have an atypically large eigenvalue for the transition matrix (small survival cost) and are within a rather small distance from the origin (small crossing cost). The set $\mathcal{U}$ is then determined as the region which minimizes the sum of above two costs, see (3.5) in [25].

We will need the following two properties for the set $\mathcal{U}$. The first one asserts that the eigenvalue for $\mathcal{U}$ is atypically large, which is expected from the above definition. The second asserts that there is a ball almost free of obstacles with radius ϱ_n in $\mathcal{U}$, which essentially follows from the first one and an quantitative isoperimetric inequality.

(a) The eigenvalue $\lambda_\mathcal{U}$. For $A \subset \mathbb{Z}^d$, we let λ_A denote the principal (largest) eigenvalue of

$P|_A$, which is the transition matrix of the simple symmetric random walk on $\mathbb{Z}^d$ killed upon exiting A. The following result gives a deterministic lower bound λ_* on $\lambda_{\mathcal{U}}$ and shows that it is very close to 1. In particular, there are at most $(\log n)^C$ many such islands in $B(0,n)$ with eigenvalues larger than λ_*.

Lemma 4.2.1. *There exists*

$$\lambda_* = \lambda_*(n,d,p) \geq 1 - \mu_B \varrho_n^{-2} - C_* \varrho_n^{-3}\,, \tag{4.2.5}$$

where μ_B is the first Dirichlet-eigenvalue of $-\frac{1}{2d}\Delta$ in the unit ball and C_* is a constant depending only on (d,p) such that

$$\lim_{n\to\infty} \widehat{\mathbb{P}}(\lambda_{\mathcal{U}} \geq \lambda_*) = 1\,. \tag{4.2.6}$$

Furthermore, if we denote

$$\mathcal{V} = B(0,(\log n)^{C_{4,1}}) \setminus \mathcal{O}\,, \tag{4.2.7}$$

then for some constants $C, c > 0$ and n sufficiently large,

$$n^{-d}(\log n)^c \leq \mathbb{P}(\lambda_{\mathcal{V}} \geq \lambda_*) \leq n^{-d}(\log n)^C\,. \tag{4.2.8}$$

Proof. This follows from results in [24, 25]. As in [25], we choose the cutoff $\lambda_* := p_{\alpha_1}^{1/k_n}$ where $k_n = (\log n)^{4-2/d}(\log\log n)^{2\mathbb{1}_{d=2}}$ and p_{α_1} (defined in [24, (3.1)]) is appropriately chosen according to some large quantile of the distribution of survival probability up to k_n steps. Then (4.2.6) can be found in [25, Lemma 2.1] and (4.2.5) can be found in [25, Lemma 2.5]. The lower bound in (4.2.8) follows from [25, (2.2)]. The upper bound in (4.2.8) can be proved using [24, Lemma 3.3] and adapting the proof of [24, (3.4)] by changing the value of R there to $(\log n)^{C_{4,1}}$. □

(b) An almost open ball in $\mathcal{U}$. We want to find an open subset of $\mathcal{U}$ which is very close to a ball. This is accomplished by a coarse graining argument that first divides $\mathcal{U}$ into two types of mesoscopic boxes according to the local obstacles density. Intuitively, the random walk tends to stay in the low obstacle density region. The key ingredient in proving (4.2.4) is that the low obstacle density region is very close to a ball. To elaborate on this, it is convenient to shift the center of localization to the origin and work with $\mathcal{V}$ defined in (4.2.7). Roughly speaking, $\mathcal{U}$ is the best among all possible translates of $\mathcal{V}$ in $[-n, n]^d$.

Let $|\cdot|_\infty$ denote the ℓ^∞-norm on $\mathbb{Z}^d$ and denote the ℓ^∞ ball of radius r (or box of side length $2r+1$) by

$$K(v, r) := \{x \in \mathbb{Z}^d : |x - v|_\infty \le r\}\,. \tag{4.2.9}$$

For $\epsilon \in (0, 1)$, which may depend on n, we consider the following disjoint boxes that cover $\mathbb{Z}^d$:

$$K(x, \lfloor \epsilon \varrho_n \rfloor) \; for \; x \in (2\lfloor \epsilon \varrho_n \rfloor + 1)\mathbb{Z}^d\,.$$

Definition 4.2.2. *For $\epsilon \in (0, 1)$, let $\mathcal{E}(\epsilon)$ be the union of boxes $K(x, \lfloor \epsilon \varrho_n \rfloor)$ that intersect $\mathcal{V}$ (defined in (4.2.7)) such that $|\mathcal{O} \cap K(x, \lfloor \epsilon \varrho_n \rfloor)| \le \epsilon |K(x, \lfloor \epsilon \varrho_n \rfloor)|$, where $|K(x, \lfloor \epsilon \varrho_n \rfloor)|$ denotes the cardinality of the set $K(x, \lfloor \epsilon \varrho_n \rfloor)$.*

By the same combinatorial calculation as in the proof of [25, Lemma 5.2], one can show that typically the volume of the low obstacle density region $\mathcal{E}(\epsilon)$ is at most $C\varrho_n^d$. More precisely, we have the following lemma.

Lemma 4.2.3. *There exists a constant $c \in (0, 1)$ such that for any $\epsilon \in (\varrho_n^{-1/2}, c)$,*

$$\mathbb{P}(|\mathcal{E}(\epsilon)| \ge |B(0, \varrho_n)| + \epsilon^{1/2} \varrho_n^d) \le e^{-\varrho_n} n^{-d}\,.$$

Also, there exists $C_{4,2} = C_{4,2}(d, p) > 0$ such that

$$\mathbb{P}(|\mathcal{E}(\epsilon)| \ge C_{4,2} \varrho_n^d) \le n^{-100d}\,.$$

The following lemma is one of the key ingredients in proving Theorem D. It says that if

$\lambda_{\mathcal{V}}$ is grater than or equal to λ_* (which is close to $\lambda_{B(0,\varrho_n)}$, see (4.2.5)), then typically $\mathcal{E}(\epsilon)$ is very close to a ball of radius ϱ_n.

Lemma 4.2.4 ([25, Lemmas 5.2–5.9])**.** *Let λ_* be as in Lemma 4.2.1. There exists $c_{4,2} > 0$ sufficiently small such that if we denote*

$$\varepsilon_n := \varrho_n^{-c_{4,2}}, \tag{4.2.10}$$

and assume that $\lambda_{\mathcal{V}} \geq \lambda_*$ and

$$|\mathcal{E}(\epsilon)| \leq |B(0, \varrho_n)| + \epsilon^{1/2} \varrho_n^d, \text{ for } \epsilon = \varepsilon_n^{1/2}, \varepsilon_n, \varepsilon_n^2, \tag{4.2.11}$$

then there exists $\boldsymbol{x}_{\mathcal{V}} \in \mathcal{V}$ such that

$$|B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n) \triangle \mathcal{E}(\varepsilon_n)| \leq {\varepsilon_n}^{1/4} \varrho_n^d, \tag{4.2.12}$$

where $\triangle$ stands for the symmetric difference of two sets.

It follows immediately that the volume proportion of obstacles in $B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n)$ is very small:

$$|B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n) \cap \mathcal{O}| \leq |B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n) \triangle \mathcal{E}(\varepsilon_n)| + \varepsilon_n |\mathcal{E}(\varepsilon_n)| \leq C {\varepsilon_n}^{1/4} \varrho_n^d, \tag{4.2.13}$$

which is the starting point of our proof. In particular, since $\mathcal{U} \subset B(0, n)$, we can deduce from Lemma 4.2.3 that with high probability, $\mathcal{U}$ satisfies the same volume control as in (4.2.11), and hence the volume proportion of obstacles in $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n)$ is very small.

4.3 Proof Outline

In this section, we list the key intermediate results and prove Theorem 4.1.1 and Theorem 4.1.3 assuming those results.

4.3.1 Ball Clearing

By (4.2.8), we know that there are at most $(\log n)^C$ many balls of radius $(\log n)^{C_{4,1}}$ in $B(0,n)$ with eigenvalues at least λ_*. The fact that $B(\boldsymbol{\mathcal{X}}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa})$ is open will then follow from the following proposition.

Proposition 4.3.1. *Let $\mathcal{V}$ be as in* (4.2.7) *and let λ_* be as in Lemma 4.2.1. Then there exist $\kappa \in (0,1)$ and $C > 0$, such that for all n sufficiently large,*

$$\mathbb{P}(B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \cap \mathcal{O} \neq \varnothing \mid \lambda_{\mathcal{V}} \geq \lambda_*) \leq Ce^{-\varrho_n^{1/3}} . \tag{4.3.1}$$

Proof of Theorem 4.1.1. In view of Theorem D, it suffices to prove (4.1.5) with $\boldsymbol{\mathcal{X}}_n$ replaced by $\boldsymbol{\mathcal{X}}_{\mathcal{U}}$. Let us denote the translate of $\mathcal{V}$ by

$$\mathcal{V}(x) = B(x, (\log n)^{C_{4,1}}) \setminus \mathcal{O} .$$

By Proposition 4.3.1 and (4.2.8),

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \lambda_*, B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \text{ is not open}) \leq Ce^{-\varrho_n^{1/3}} (\log n)^C n^{-d} = o(n^{-d}) .$$

This yields that with $\mathbb{P}$-probability tending to one, for all $x \in B(0,n)$,

$$\text{either } \lambda_{\mathcal{V}(x)} < \lambda_* \text{ or } B(\boldsymbol{\mathcal{X}}_{\mathcal{V}(x)}, \varrho_n - \varrho_n^{1-\kappa}) \text{ is open} .$$

Recall that we proved in Lemma 4.2.1 that $\lambda_{\mathcal{U}} \geq \lambda_*$. Hence, $B(\boldsymbol{\mathcal{X}}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa})$ is open with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$. □

The proof of Proposition 4.3.1 is based on the following heuristics. Suppose $B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa})$ is not completely open, then we consider the operation that removes all obstacles inside $B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa})$. After performing such an operation, the eigenvalue $\lambda_{\mathcal{V}}$ will increase, for example, from $\lambda_{\mathcal{V}}$ to $\lambda_{\mathcal{V}} + \delta$. Such an operation will yield the following inequality:

$$\mathbb{P}(B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \cap \mathcal{O} \neq \varnothing, \lambda_{\mathcal{V}} > \lambda_*) \leq C(n, \delta, d, p)\mathbb{P}(\lambda_{\mathcal{V}} > \lambda_* + \delta) \,. \tag{4.3.2}$$

Now if the tail of the probability distribution of $\lambda_{\mathcal{V}}$ is not very heavy in the sense that

$$\frac{\mathbb{P}(\lambda_{\mathcal{V}} > \lambda_* + \delta)}{\mathbb{P}(\lambda_{\mathcal{V}} > \lambda_*)} \ll 1 \,, \tag{4.3.3}$$

and the factor $C(n, \delta, d, p)$ in (4.3.2) is small compare to (4.3.3), then we have

$$\mathbb{P}(B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \cap \mathcal{O} \neq \varnothing \mid \lambda_{\mathcal{V}} > \lambda_*) \leq C(n, \delta, d, p)\frac{\mathbb{P}(\lambda_{\mathcal{V}} > \lambda_* + \delta)}{\mathbb{P}(\lambda_{\mathcal{V}} > \lambda_*)} \ll 1 \,,$$

which yields Proposition 4.3.1. Therefore it suffices to establish in a more precise manner the two ingredients (4.3.2) and (4.3.3).

Remark 4.3.2. In [23, Proposition 2.2], we have proved an analogue of Proposition 4.3.1 under the annealed polymer measure, where we used operations that modify the obstacle configurations and the random walk paths jointly. The difficulty in the quenched setting is that we need to identify a vacant ball in $\mathcal{U}$, for which we only know $\lambda_{\mathcal{U}} \geq \lambda_*$ from Lemma 4.2.1. This is why Proposition 4.3.1 is formulated in terms of the eigenvalue and as a result, we can perform operations only on the obstacle configurations. Nevertheless, it is worth mentioning that operations that modify obstacle configurations or random walk paths play an important role both in this result and in [23].

The following result makes (4.3.3) precise and shows that the tail of the probability distribution of $\lambda_{\mathcal{V}}$ is not too heavy.

Lemma 4.3.3. *Suppose $\beta \geq 1 - b\varrho_n^{-2}$ for some $b \geq 1$. Then there exists a constant $c_b > 0$ depending only on (b, d, p) such that for all*

$$\epsilon \in \left((\log\log n)^4 \varrho_n^{-d}, c_b\right) , \tag{4.3.4}$$

we have

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta) \leq e^{-\epsilon(\log(1/\epsilon))^{-3}\varrho_n^d}\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta - \epsilon\varrho_n^{-2}) + n^{-10d}\,. \tag{4.3.5}$$

One of the challenges in proving results of the type (4.3.2) is that how much $\lambda_{\mathcal{V}}$ increases after removing the obstacles depends on the local configuration around the obstacles. For example, if the obstacles being removed are near the boundary of $B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n)$ where there are a lot of unremoved obstacles, then the effect of the removal would be very small (more discussions about this issue can be found at the beginning of Section 4.4.2). To quantify the effect of removing certain obstacles, we suppose $\lambda_{\mathcal{V}} \geq \lambda_*$ and (4.2.11) holds. For $\delta \in (0, 1/2)$, which may depend on n, and nonnegative integer k, define

$$B_{\delta,k} = B(\boldsymbol{x}_{\mathcal{V}}, (1 - \delta + 2^{-k}\delta)\varrho_n)\,. \tag{4.3.6}$$

Then for all $k \geq 0$, $B(\boldsymbol{x}_{\mathcal{V}}, (1-\delta)\varrho_n) \subset B_{\delta,k+1} \subset B_{\delta,k} \subset B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n)$. For any $\delta > 0$, if $B(\boldsymbol{x}_{\mathcal{V}}, (1-\delta)\varrho_n)$ is not completely open, then we define (for some constant $c_{4,5} \in (0, 1)$ to be chosen in Lemma 4.4.13)

$$\mathcal{J} = \mathcal{J}_\delta := \min\{k \in \mathbb{N}^* : |\mathcal{O} \cap B_{\delta,k}| \geq c_{4,5}|\mathcal{O} \cap B_{\delta,k-1}|\}\,, \tag{4.3.7}$$

which must be finite due to the assumption that $B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa})$ is not completely open (see Figure 4.2).

The following result makes (4.3.2) more precise and says that removing m obstacles in $B_{\delta,\mathcal{J}-1}$ will increase the eigenvalue $\lambda_{\mathcal{V}}$ by $(m/\varrho_n^d)^{1-1/d}\varrho_n^{-2}$.

Lemma 4.3.4. *Let $C_{4,1}$ be defined as in Theorem D. There exist constants $\kappa \in (0, 1), C > 0$ such that for any $1 \leq m \leq \varrho_n^d, \beta \geq \lambda_*$, $\delta = \varrho_n^{-\kappa}$,*

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta, |\mathcal{O} \cap B_{\delta,\mathcal{J}-1}| = m, (4.2.11)) \leq C\varrho_n^{dC_{4,1}}\Big(\frac{C\varrho_n^d}{m}\Big)^m\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta + (m/\varrho_n^d)^{1-1/d}\varrho_n^{-2})\,. \tag{4.3.8}$$

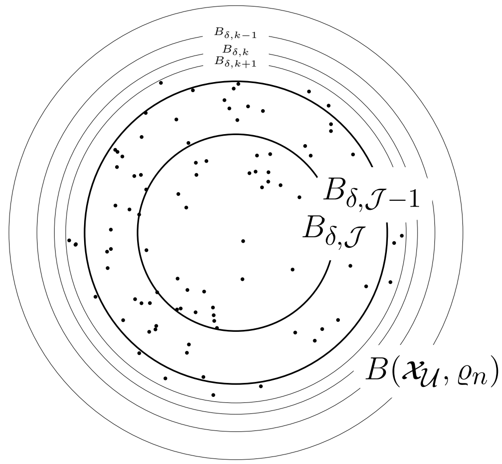

Figure 4.2: The balls $B_{\delta,k}$ and $B_{\delta,\mathcal{J}}$ as defined in (4.3.6) and (4.3.7). Little dots are obstacles.

Both Lemmas 4.3.3 and 4.3.4 are estimates on the tail distribution of $\lambda_{\mathcal{V}}$, but in opposite directions. The common strategy in both proofs is obstacle modification. To prove Lemma 4.3.3, we judiciously add obstacles and show that we get a large gain in probability for the obstacle configuration but little decrease in $\lambda_{\mathcal{V}}$. To prove Lemma 4.3.4, we judiciously remove obstacles and show that we get a large gain in $\lambda_{\mathcal{V}}$ while the probabllity of the obstacle configuration changes little. We will prove Proposition 4.3.1, Lemmas 4.3.3 and 4.3.4 in Section 4.4.

4.3.2 Random Walk Localization

We know from Theorem D that conditioned on survival up to time n, the random walk stays in $\mathcal{U}$ during the time interval $[C|\boldsymbol{x}_{\mathcal{U}}|, n]$ with high probability. To give a more detailed description for the random walk in this time window, it is convenient to consider the random walk conditioned to stay in $\mathcal{U}$ for a long time.

Recall from Theorem D that $\widehat{B}_n := B(\boldsymbol{x}_{\mathcal{U}}, (1 + \varrho_n^{-c_{4,1}})\varrho_n) \setminus \mathcal{O}$ is a ball of radius slightly larger than ϱ_n. Knowing that $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa})$ is open, it is straightforward to deduce the following result from [25, (6.15)]. The details will be given in Section 4.5.2.

Lemma 4.3.5. *There exist constants $C_{4,3}, c > 0$ depending only on (d, p) such that with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$, for any $z \in \mathcal{U}$ and constant $\varepsilon \in (0, 1)$, $m \geq t$, if we* *assume*

$$\textit{either } z \in B(\boldsymbol{x}_{\mathcal{U}}, (1-\varepsilon)\varrho_n) \textit{ and } t \geq 0, \quad \textit{or} \quad z \in \mathcal{U} \textit{ and } t \geq \varrho_n^{C_{4,3}}, \tag{4.3.9}$$

then

$$\mathbf{P}^z(S_t \notin \widehat{B}_n \mid \tau_{\mathcal{U}^c} > m) \leq \exp(-\varrho_n^c) \tag{4.3.10}$$

for all sufficiently large n.

We will strengthen this result in two steps :

(a). We will first prove that conditioned on $\{\tau_{\mathcal{U}^c} > m\}$, the probability that the random walk is at some site x at a fixed time $\varrho_n^2 \leq t \leq m$ is $O(\varrho_n^{-d})$ uniformly in x. Then combining with (4.3.10), we deduce that at any fixed time, with high probability, the random walk will be localized in a ball of radius slightly smaller than ϱ_n.

(b). We will then derive the limiting marginal distribution of the random walk at the end point and at a deterministic time in the bulk, conditioned on $\{\tau_{\mathcal{U}^c} > m\}$.

First we show that conditioned on $\{\tau_{\mathcal{U}^c} > m\}$, the random walk will hit the the deep interior of the ball $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n)$ within $(\log n)^C$ steps. This allows us to focus on the random walk starting from the deep interior of the ball.

Lemma 4.3.6. *There exist $b_2 \in (0,1)$ and $C_{4,4}, c > 0$ depending only on (d,p) ($C_{4,4}$ to be defined in Lemma 4.5.5) such that with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$, for all $m \geq \varrho_n^{C_{4,4}}$ and $u \in \mathcal{U}$,*

$$\mathbf{P}^u(\tau_{B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)} \geq \varrho_n^{C_{4,4}} \mid \tau_{\mathcal{U}^c} > m) \leq \exp(-\varrho_n^c)\,. \tag{4.3.11}$$

The first improvement upon (4.3.10) in Lemma 4.3.5 (see (a) after Lemma 4.3.5) is the following local limit result.

Lemma 4.3.7. *Let b_2 be as in Lemma 4.3.6 and let $u \in B(\mathcal{X}_\mathcal{U}, b_2\varrho_n)$, $m \geq t \geq \varrho_n^2$, and $\epsilon \in (0,1)$. Then with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$, the following holds for all $y \in \mathcal{U}$, $x \in B(\mathcal{X}_\mathcal{U}, (1-\epsilon)\varrho_n)$ such that $|x-u|_1 + t$ is even:*

$$\mathbf{P}^u(S_t = y \mid \tau_{\mathcal{U}^c} > m) \leq C\varrho_n^{-d}, \tag{4.3.12}$$

$$\mathbf{P}^u(S_t = x \mid \tau_{\mathcal{U}^c} > m) \geq c\epsilon^2\varrho_n^{-d}. \tag{4.3.13}$$

Remark 4.3.8. The second assertion (4.3.13) will not be used in the proof of the main results. We include it to complement (4.3.12) and as a precursor to Theorem 4.1.3.

Combined with Lemma 4.3.5, the preceding lemma can then be used to show that the random walk at any fixed time t will be localized in the ball centered at $\mathcal{X}_\mathcal{U}$ with radius $\varrho_n\big(1-o(1)\big)$. More precisely,

Corollary 4.3.9. *Let $\kappa > 0$ be defined as in Proposition 4.3.1. There exists a constant $c = c(d,p) > 0$ such that with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$, for all $u \in B(\mathcal{X}_\mathcal{U}, b_2\varrho_n)$, $m \geq t \geq 0$,*

$$\mathbf{P}^u(S_t \in B(\mathcal{X}_\mathcal{U}, (1-2\varrho_n^{-\kappa})\varrho_n) \mid \tau_{\mathcal{U}^c} > m) \geq 1 - \varrho_n^{-c}. \tag{4.3.14}$$

Lastly, Theorem 4.1.3 will be proved using Corollary 4.3.9 and the following lemma, which says that conditioned on the random walk staying in $\widehat{B}_n$ for sufficiently long time, the distribution of the random walk at the end point (or at a deterministic time in the bulk) will converge in total variation distance to the normalized first eigenfunction (or normalized eigenfunction squared) on $\widehat{B}_n$.

Lemma 4.3.10. *There exist constants $c, C_{4,5} > 0$ depending only on (d,p) such that uniformly in $v, y \in B(\mathcal{X}_\mathcal{U}, (1-2\varrho_n^{-\kappa})\varrho_n)$ and $m, t \geq C_{4,5}\varrho_n^2 \log\log n$ with $|y-v|_1 + m + t$ even,* *we have*

$$\sup_{x\in\mathcal{B}_n:|x-v|_1+m \text{ is even}} \left|\varrho_n^d \mathbf{P}^v(S_m = x \mid \tau_{\widehat{B}_n^c} > m) - 2\phi_1(\tfrac{x-\mathcal{X}_{\mathcal{U}}}{\varrho_n})\right| \le \varrho_n^{-c}\,, \tag{4.3.15}$$

$$\sup_{x\in\mathcal{B}_n:|x-v|_1+m \text{ is even}} \left|\varrho_n^d \mathbf{P}^v(S_m = x \mid S_{m+t}=y, \tau_{\widehat{B}_n^c} > m+t) - 2\phi_2^2(\tfrac{x-\mathcal{X}_{\mathcal{U}}}{\varrho_n})\right| \le \varrho_n^{-c}\,, \tag{4.3.16}$$

where ϕ_1 and ϕ_2 are respectively the L^1 and L^2-normalized first eigenfunction of the Dirichlet-Laplacian of the unit ball in $\mathbb{R}^d$.

Let us prove Theorem 4.1.3 assuming the above lemmas.

Proof of Theorem 4.1.3. To prove (4.1.6), we first show that combining (4.2.3) with Lemma 4.3.6 yields

$$\mathbf{P}(\tau_{B(\mathcal{X}_{\mathcal{U}},b_2\varrho_n)} < C|\mathcal{X}_{\mathcal{U}}|, S_{[\tau_{\widehat{B}_n},n]} \subset \mathcal{U} \mid \tau > n) \ge 1-\exp(-\varrho_n^c)\,. \tag{4.3.17}$$

Indeed, by the strong Markov property at time $\tau_{\widehat{B}_n}$,

$$\begin{aligned}&\mathbf{P}(\tau_{B(\mathcal{X}_{\mathcal{U}},b_2\varrho_n)} > \tau_{\widehat{B}_n} + \lceil \varrho_n^{C_{4,4}}\rceil, \tau_{\widehat{B}_n} < C|\mathcal{X}_{\mathcal{U}}|, S_{[\tau_{\widehat{B}_n},n]} \subset \mathcal{U}, \tau > n)\\ &= \mathbf{E}\big[\mathbb{1}_{\tau>\tau_{\widehat{B}_n},\tau_{\widehat{B}_n}<C|\mathcal{X}_{\mathcal{U}}|}\mathbf{P}^{S_{\tau_{\widehat{B}_n}}}(\tau_{B(\mathcal{X}_{\mathcal{U}},b_2\varrho_n)} > \lceil \varrho_n^{C_{4,4}}\rceil, \tau_{\mathcal{U}^c} > n-\tau_{\widehat{B}_n})\big]\,.\end{aligned}$$

Since $|\mathcal{X}_{\mathcal{U}}| \le Cn(\log n)^{-2/d}$ implies $n - \tau_{\widehat{B}_n} \ge n/2$, it follows from Lemma 4.3.6 that the above quantity can be bounded from above by

$$\begin{aligned}&\exp(-\varrho_n^c)\mathbf{E}\big[\mathbb{1}_{\tau>\tau_{\widehat{B}_n},\tau_{\widehat{B}_n}<C|\mathcal{X}_{\mathcal{U}}|}\mathbf{P}^{S_{\tau_{\widehat{B}_n}}}(\tau_{\mathcal{U}^c} > n-\tau_{\widehat{B}_n})\big]\\ &= \exp(-\varrho_n^c)\mathbf{P}(\tau > \tau_{\widehat{B}_n}, \tau_{\widehat{B}_n} < C|\mathcal{X}_{\mathcal{U}}|, S_{[\tau_{\widehat{B}_n},n]} \subset \mathcal{U})\,.\end{aligned}$$

Combined with (4.2.3), this proves (4.3.17).

Now, let T denote the hitting time of the ball $B(\mathcal{X}_{\mathcal{U}}, b_2\varrho_n)$ to lighten the notation. We consider deterministic time t with $C|\mathcal{X}_\mathcal{U}| \le t \le n$ and $x \in B(v,(1-\epsilon)\varrho_n)$ with $|x|_1 + t$ even. By the strong Markov property,

$$\begin{aligned}&\mathbf{P}(S_{[T,n]} \subset \mathcal{U}, S_t \in B(\mathcal{X}_\mathcal{U}, \varrho_n), t > T, \tau > n)\\&\quad = \mathbf{E}\big[\mathbb{1}_{\tau\wedge t > T}\mathbf{P}^{S_T}(S_{t-T} \in B(\mathcal{X}_\mathcal{U}, \varrho_n), \tau_{\mathcal{U}^c} > n - T)\big]\,.\end{aligned}$$

By Corollary 4.3.9, this equals $\mathbf{P}(S_{[T,n]} \subset \mathcal{U}, \tau > n, t > T)(1 + o(1))$. Combining this with (4.3.17) gives (4.1.6).

Next, we turn to the proof of (4.1.7) and (4.1.8). The basic idea is to restrict the walk to a time interval $[t_1, t_2]$ such that the walk does not exit $\widehat{B}_n$ during this time interval, which then allows us to apply Lemma 4.3.10. To this end, we denote for $0 < t_1 < t_2$,

$$A_{t_1,t_2} := \{S_{t_1}, S_{t_2} \in B(\mathcal{X}_\mathcal{U}, (1 - 2\varrho_n^{-\kappa})\varrho_n), S_{[t_1,t_2]} \subseteq \widehat{B}_n, S_{[C|\mathcal{X}_\mathcal{U}|,n]} \subseteq \mathcal{U}, \tau > n\}\,.$$

We first notice that for any $[t_1, t_2] \subseteq [C|\mathcal{X}_\mathcal{U}|, n]$ with $t_2 - t_1 \le e^{\varrho_n^c}$, combining (4.3.17), (4.3.14), and a union bound for the event in (4.3.10) over all $t \in [t_2, t_1]$ yields that

$$\mathbf{P}(A_{t_1,t_2} \mid \tau > n) \ge 1 - \varrho_n^{-c}\,. \tag{4.3.18}$$

To prove (4.1.8), we choose $t_1 = t - \varrho_n^3, t_2 = t + \varrho_n^3$, and let $I_{v,w}$ be

$$\mathbf{P}(S_{t_1} = v, S_{[C|\mathcal{X}_\mathcal{U}|,t_1]} \subseteq \mathcal{U}, \tau > t_1) \cdot \mathbf{P}^v(S_{t_2-t_1} = w, \tau_{\widehat{B}_n^c} > t_2 - t_1) \cdot \mathbf{P}^w(\tau_{\mathcal{U}^c} > n - t_2)\,.$$

We have

$$\mathbf{P}(S_t = x, A_{t_1,t_2}) = \sum_{v,w\in B(\mathcal{X}_\mathcal{U},(1-2\varrho_n^{-\kappa})\varrho_n)} I_{v,w} \cdot \mathbf{P}^v(S_{t-t_1} = x \mid S_{t_2-t_1} = w, \tau_{\widehat{B}_n^c} > t_2 - t_1)\,.$$

Therefore

$$\sum_{x:|x|_1+t \text{ is even}} \left|\mathbf{P}(S_t = x, A_{t_1,t_2}) - 2\varrho_n^{-d}\phi_2^2(\tfrac{x-\varkappa_{\mathcal{U}}}{\varrho_n})\mathbf{P}(A_{t_1,t_2})\right| \le \sum_{v,w\in B(\varkappa_{\mathcal{U}},(1-2\varrho_n^{-\kappa})\varrho_n)} I_{v,w}$$

$$\times \sum_{x:|x|_1+t \text{ is even}} \left|\mathbf{P}^v(S_{t-t_1} = x \mid S_{t_2-t_1} = w, \tau_{\widehat{B}_n^c} > t_2 - t_1) - 2\varrho_n^{-d}\phi_2^2(\tfrac{x-\varkappa_{\mathcal{U}}}{\varrho_n})\right|$$

$$\le \varrho_n^{-c}\mathbf{P}(A_{t_1,t_2}) .$$

where in the last step, we used (4.3.16). Combining this with (4.3.18) yields (4.1.8).

Finally, choose $t_1 = n-\varrho_n^3, t_2 = n$ and combining (4.3.18) with (4.3.15) yields (4.1.7). $\square$

Lemmas 4.3.5, 4.3.6, and 4.3.7, Corollary 4.3.9 will be proved in Section 4.5.2, and Lemma 4.3.10 will be proved in Section 4.5.3.

4.4 Ball Clearing

In this section, we will first prove Lemmas 4.3.3 and 4.3.4 in Sections 4.4.1 and 4.4.2, respectively, and then conclude the proof of Proposition 4.3.1 in Section 4.4.3.

4.4.1 Proof of Lemma 4.3.3

Proof outline

In this section, we outline the proof of Lemma 4.3.3, which shows that the tail of the distribution of $\lambda_{\mathcal{V}}$ is not too heavy. The basic strategy is obstacle modification.

Let $\ell \in \mathbb{N}^*$ and partition $\mathbb{Z}^d$ into disjoint boxes $K(x,\ell)$ of side length $2\ell+1$ (see (4.2.9)) for $x \in (2\ell+1)\mathbb{Z}^d$. Let $\mathcal{V}$ be as defined in (4.2.7).

Definition 4.4.1. *A box $K(x,\ell)$ is said to be* "truly"-open *if*

$$\max_{u\in K(x,\ell)} \mathbf{P}^u\big(S_{[0,\ell^2]} \subset (K(x,4\ell)\cap \mathcal{V})\big) \ge 1/10 . \tag{4.4.1}$$

Let $C_{4,1}$ be as in Theorem D. We fix ℓ and let $\mathcal{T}$ denote the union of all "truly"-open boxes that intersect with $B(0, (\log n)^{C_{4,1}})$.

We first note that "truly"-open boxes are very rare:

Lemma 4.4.2. *There exists $c = c(d, p) > 0$ such that for all ℓ sufficiently large,*

$$\mathbb{P}\big(K(x, \ell) \text{ is "truly"-open}\big) \le \exp(-c\ell^d)\,. \tag{4.4.2}$$

Proof. To prove (4.4.2), it suffices to show that for all $u \in K(x, \ell)$,

$$\mathbb{P}\big(\mathbf{P}^u(S_{[0,\ell^2]} \cap \mathcal{O} = \varnothing) \ge 1/10\big) \le \exp(-c\ell^d)\,, \tag{4.4.3}$$

which can be found in [24, Definition 2.4, Lemmas 2.8 and 2.9]. □

In light of Lemma 4.4.2, by *closing* a "truly"-open box, namely, changing the obstacle configuration in a "truly"-open box to typical configurations, we could gain much probability for the obstacle configurations.

On the other hand, we have the following result, which says that in a typical environment, we can find a "truly"-open box such that $\lambda_{\mathcal{V}}$ will only decrease slightly after closing this "truly"-open box.

Lemma 4.4.3. *Fix $\ell \ge 1$. Let $C_{4,6} > 1$ be a constant depending only on (d, p) to be chosen in Lemma 4.4.6, and let $b_1 > 0, b_2 \in (0, 1)$ be two arbitrary constants. Let ε_n, $\mathcal{E}(\varepsilon_n)$, and $C_{4,2} > 1$ be as defined in* (4.2.10)*, Definition 4.2.2, and Lemma 4.2.3, respectively. We assume*

$$\min_{x \in B(0,(\log n)^{C_{4,1}})} |B(x, C_{4,6}\varrho_n) \setminus \mathcal{T}| \ge \varrho_n^d, \quad |\mathcal{E}(\varepsilon_n)| \le C_{4,2}\varrho_n^d\,, \tag{4.4.4}$$

and $\lambda_{\mathcal{V}} \ge 1 - b_1\varrho_n^{-2}$. Then for each $z \in B(0, (\log n)^{C_{4,1}})$ such that $|\mathcal{T} \cap B(z, C_{4,6}\varrho_n)| \ge (b_2\varrho_n)^d$, there exists a "truly"-open box $K(x, \ell)$ with $x \in B(z, 20\varrho_n)$ such that

$$\lambda_{\mathcal{V} \setminus K(x,10\ell)} \ge \lambda_{\mathcal{V}} - Cb_1^2 b_2^{-2(d-1)} \ell^{2(d+2)} \varrho_n^{-d-2}\,, \tag{4.4.5}$$

where C is a constant depending only on (d, p).

Lemma 4.4.3 is the key ingredient in the proof of Lemma 4.3.3. We will use Lemma 4.4.3 repeatedly (see Lemma 4.4.9 below) to show that we can find a number of "truly"-open boxes such that $\lambda_{\mathcal{V}}$ will not be decreased much after closing them. The operation of changing these "truly"-open boxes to typical configurations will map the event $\{\lambda_{\mathcal{V}} \geq \beta\}$ to $\{\lambda_{\mathcal{V}} \geq \beta - \delta\}$, where δ depends on ℓ and the number of "truly"-open boxes being closed. Combining with Lemma 4.4.2 will then give an upper bound for $\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta)/\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta - \delta)$ (see Lemma 4.4.10.) The proof of Lemmas 4.4.3, 4.4.9 and 4.4.10 will be provided in Section 4.4.1. In Section 4.4.1, we fix appropriate choices of ℓ and the number of "truly"-open box being closed, and prove Lemma 4.3.3.

Some useful facts

Before embarking on the proof of Lemma 4.4.3, we will show in this section that with high probability, assumption (4.4.4) holds and the choice of z in Lemma 4.4.3 exists.

Definition 4.4.4. *For any $U \subset \mathbb{Z}^d$, let Φ_U be the ℓ^1-normalized principal eigenfunction of $P|_U$, the transition matrix of the random walk restricted to U.*

The following lemma will be used repeatedly, for instance, to bound $\Phi_U(v)$ at sites v close to the boundary of U, or to find sites v from where the walk cannot exit U too quickly.

Lemma 4.4.5. *For $t \in \mathbb{N}^*$,*

$$\sum_{v \in U} \Phi_U(v) \cdot \mathbf{P}^v(\tau_{U^c} \leq t) = 1 - \lambda_U^t\,. \tag{4.4.6}$$

Proof. Let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^U$, then

$$\sum_{v \in U} \Phi_U(v) \cdot \mathbf{P}^v(\tau_{U^c} \leq t) = 1 - \langle \Phi_U, (P|_U)^t \mathbf{1} \rangle = 1 - \lambda_U^t\,. \qquad \square$$

Recall the definition of $\mathcal{E}(\varepsilon_n)$ in Definition 4.2.2 and (4.2.10). The following result says that neither $\mathcal{T}$ nor $\mathcal{E}(\varepsilon_n)$ can be too large in a typical obstacle configuration, namely, (4.4.4) holds.

Lemma 4.4.6. *Let $\ell \in (C, \varrho_n^{1/2})$ for some large constant C. There exists a constant $C_{4,6} > 1$ depending only on (d,p) such that (4.4.4) holds with $\mathbb{P}$-probability at least $1 - n^{-10d}$.*

Proof. Since Lemma 4.2.3 gives the second inequality in (4.4.4), it suffices to show that the first inequality in (4.4.4) holds with high probability. Consider boxes of the form $K(v,\ell), v \in (2\ell+1)\mathbb{Z}^d$. We can partition these boxes into 10^d groups $\{K(v,\ell) : v \in (2\ell+1)(10\mathbb{Z}^d + i)\}$, for $i \in \{0,1,\ldots,9\}^d$ so that the distance between any two boxes in the same group is at least 10ℓ. Recalling the definition of "truly"-open box in (4.4.1), we have that within each group the events that each box is "truly"-open are mutually independent, and have probability less than $e^{-c\ell^d}$ by Lemma 4.4.2. Note that on the event $\{|B(x, C_{4,6}\varrho_n) \setminus \mathcal{T}| < \varrho_n^d\}$, there exists a group where there are at most $\varrho_n^d/[10^d \cdot (2\ell+1)^d]$ many non-"truly"-open box that intersect $B(x, C_{4,6}\varrho_n)$. Also, the number of boxes that intersect $B(x, C_{4,6}\varrho_n)$ in each group is at least $|B(x, C_{4,6}\varrho_n)|/[10^d \cdot (2\ell+1)^d]$. It follows from large deviation estimates for sums of i.i.d. Bernoulli random variables (in each group) and a union bound over those groups that

$$\mathbb{P}\big(|\mathcal{T} \cap B(x, C_{4,6}\varrho_n)| \geq |B(x, C_{4,6}\varrho_n)| - \varrho_n^d\big) \leq n^{-20d},$$

for $C_{4,6}$ and ℓ sufficiently large. Then a union bound over $x \in B(0, (\log n)^{C_{4,1}})$ yields

$$\mathbb{P}\Big(\min_{x \in B(0,(\log n)^{C_{4,1}})} |B(x, C_{4,6}\varrho_n) \setminus \mathcal{T}| \geq \varrho_n^d\Big) \geq 1 - n^{-15d}.$$

This completes the proof of Lemma 4.4.6. □

The following result says that under the assumptions in Lemma 4.4.3, there exits z such that $B(z, C_{4,6}\varrho_n)$ contains enough "truly"-open boxes (for some b_2 depending on b_1 as determined by (4.4.7)).

Lemma 4.4.7. *Let $\ell \in (C, \varrho_n^{1/2})$ for some large constant C. There exists a constant $c_{4,3} = c_{4,3}(d) \in (0,1)$ such that for any $b \geq 1$, if $\lambda_{\mathcal{V}} \geq 1 - b\varrho_n^{-2}$, then there exists $x \in B(0, (\log n)^{C_{4,1}})$ such that*

$$|\mathcal{T} \cap B(x, C_{4,6}\varrho_n)| \geq c_{4,3} b^{-d/2} \varrho_n^d \,. \tag{4.4.7}$$

The following lemma is needed to prove Lemma 4.4.7.

Lemma 4.4.8. *If $K(x,\ell)$ is not a "truly"-open box, then for any starting point $u \in K(x,\ell)$, the survival probability up to ℓ^2 steps is less than $1/2$, namely,*

$$\mathbf{P}^u(\tau_{\mathcal{V}^c} > \ell^2) \leq 1/2 \,. \tag{4.4.8}$$

Proof. We first note that for any $u \in K(x,\ell)$,

$$\mathbf{P}^u(\tau_{\mathcal{V}^c} > \ell^2) \leq \mathbf{P}^u(\max_{t\in[0,\ell^2]} |S_t - u|_\infty \geq 3\ell) + \mathbf{P}^u(S_{[0,\ell^2]} \subset (K(x,4\ell) \setminus \mathcal{O}))\,.$$

If $K(x,\ell)$ is not a "truly"-open box, then the definition of the "truly"-open boxes (Definition 4.4.1) implies

$$\mathbf{P}^u(S_{[0,\ell^2]} \subset (K(x,4\ell) \setminus \mathcal{O})) \leq 1/10\,.$$

In addition, the reflection principle yields

$$\mathbf{P}^u(\max_{t\in[0,\ell^2]} |S_t - u|_\infty \geq 3\ell) \leq d \cdot 2\mathbf{P}(|S_{\ell^2} \cdot \mathbf{e}_1| \geq 3\ell) \leq 2d \cdot \frac{\ell^2/d}{9\ell^2} = 2/9\,.$$

Combining the previous three inequalities gives (4.4.8). □

Proof of Lemma 4.4.7. Since by assumption $\lambda_{\mathcal{V}} \geq 1 - b\varrho_n^{-2}$, (4.4.6) implies that

$$\sum_{v\in\mathcal{V}} \Phi_{\mathcal{V}}(v) \cdot \mathbf{P}^v(\tau_{\mathcal{V}^c} \leq 10^{-3}\varrho_n^2/b) \leq 1 - \lambda_{\mathcal{V}}^{10^{-3}\varrho_n^2/b} \leq 1 - e^{-\lceil 10^{-3}\varrho_n^2/b\rceil b\varrho_n^{-2}} \leq 1/100\,.$$

Hence, there exists $u \in \mathcal{V}$ such that

$$\mathbf{P}^u(\tau_{\mathcal{V}^c} \le \lceil 10^{-3} \varrho_n^2/b \rceil) \le 1/100\,.$$

On the other hand, by (4.4.8) if the random walk hit $\mathcal{T}^c$, then it will get killed with probability at least 1/2 in next ℓ^2 steps. Since $\ell^2 \le \varrho_n \le 10^{-4}\varrho_n^2/b$ for sufficiently large n, we get

$$\mathbf{P}^u(\tau_{\mathcal{V}^c} \le \lceil 10^{-3} \varrho_n^2/b \rceil) \ge \mathbf{P}^u(S_{\lceil 10^{-4}\varrho_n^2/b \rceil} \in \mathcal{T}^c)/2\,.$$

Combining the previous two inequalities gives

$$\mathbf{P}^u(S_{\lceil 10^{-4}\varrho_n^2/b \rceil} \in \mathcal{T}^c) \le 1/50\,.$$

Since we assumed $b \ge 1$ and $C_{4,6} \ge 1$ (chosen in Lemma 4.4.6), we have

$$\mathbf{P}^u(S_{\lceil 10^{-4}\varrho_n^2/b \rceil} \notin B(u, C_{4,6}\varrho_n)) \le 1/50\,.$$

Combining the previous two inequalities with the local limit theorem for the random walk S gives

$$24/25 \le \mathbf{P}^u(S_{\lceil 10^{-4}\varrho_n^2/b \rceil} \in \mathcal{T} \cap B(u, C_{4,6}\varrho_n)) \le |\mathcal{T} \cap B(u, C_{4,6}\varrho_n)| \cdot C(\varrho_n b^{1/2})^{-d}\,.$$

This yields (4.4.7). □

Proof of Lemma 4.4.3 and its corollaries

Proof of Lemma 4.4.3. We need to show that for each for each $z \in B(0, (\log n)^{C_{4,1}})$ such that $|\mathcal{T} \cap B(z, C_{4,6}\varrho_n)| \ge (b_2\varrho_n)^d$, we can find a truly open box $K(x, \ell)$ such that filling $K(x, 10\ell)$ with obstacles will not decrease $\lambda_{\mathcal{V}}$ too much, namely, (4.4.5) holds. We will find such an x near the boundary of $\mathcal{T}$. The change in $\lambda_{\mathcal{V}}$ can then be shown to be small because

$\Phi_{\mathcal{V}}$ is small near x. (Recall that $\Phi_{\mathcal{V}}$ is the ℓ^1-normalized principal eigenfunction of $P|_{\mathcal{V}}$, the transition matrix of the random walk restricted to $\mathcal{V}$.)

Denote by $|\Phi_{\mathcal{V}}|_2$ the ℓ^2-norm of $\Phi_{\mathcal{V}}$, namely $|\Phi_{\mathcal{V}}|_2^2 = \sum_{x\in\mathbb{Z}^d}\Phi_{\mathcal{V}}(x)^2$. By Lemma 4.B.1 in the appendix, for any $x \in \mathcal{V}$ with $\sum_{u\in K(x,11\ell)}\Phi_{\mathcal{V}}^2(u) \le |\Phi_{\mathcal{V}}|_2^2/2$, we have

$$\lambda_{\mathcal{V}} - \lambda_{\mathcal{V}\setminus K(x,10\ell)} \le 4 \sum_{u\in K(x,11\ell)} \Phi_{\mathcal{V}}^2(u)/|\Phi_{\mathcal{V}}|_2^2 \,. \tag{4.4.9}$$

To bound the right hand side of (4.4.9), we first show that the assumption (4.4.4) implies

$$|\Phi_{\mathcal{V}}|_2^2 \ge c\varrho_n^{-d}\,. \tag{4.4.10}$$

To this end, let $\Omega_{\varepsilon_n} := \{v \in \mathbb{Z}^d : \Phi_{\mathcal{V}}(v) \ge \varepsilon_n\varrho_n^{-d}\}$. It was shown in [25, Lemma 5.5] that $\sum_{v\notin\mathcal{E}(\varepsilon_n)}\Phi_{\mathcal{V}}(v) \le C\varepsilon_n$ for some constant C, which implies $|\Omega_{\varepsilon_n}\setminus\mathcal{E}(\varepsilon_n)| \le C\varrho_n^d$. Since $|\mathcal{E}(\varepsilon_n)| \le C_{4,2}\varrho_n^d$, we get $|\Omega_{\varepsilon_n}| \le C\varrho_n^d$ and

$$\sum_{v\in\Omega_{\varepsilon_n}}\Phi_{\mathcal{V}}(v) \ge \sum_{v\in\Omega_{\varepsilon_n}\cup\mathcal{E}(\varepsilon_n)}\Phi_{\mathcal{V}}(v) - |\mathcal{E}(\varepsilon_n)|\cdot\varepsilon_n\varrho_n^{-d} \ge 1 - C\varepsilon_n \ge 1/2\,.$$

Then (4.4.10) follows from $|\Omega_{\varepsilon_n}|\cdot\sum_{v\in\Omega_{\varepsilon_n}}\Phi_{\mathcal{V}}^2(v) \ge \Big(\sum_{v\in\Omega_{\varepsilon_n}}\Phi_{\mathcal{V}}(v)\Big)^2$.

Now, suppose $z \in B(0,(\log n)^{C_{4,1}})$ satisfies $|\mathcal{T}\cap B(z,C_{4,6}\varrho_n)| \ge (b_2\varrho_n)^d$. By (4.4.9) and (4.4.10), to prove (4.4.5), it suffices to find a "truly"-open $K(x,\ell)$ with $x \in B(z,20\varrho_n)$ such that for some constant $C > 0$,

$$\sum_{u\in K(x,11\ell)}\Phi_{\mathcal{V}}^2(u) \le Cb_1^2 b_2^{-2(d-1)}\varrho_n^{-2(d+1)}\ell^{2(d+2)}\,. \tag{4.4.11}$$

Heuristically, $\Phi_{\mathcal{V}}$ is large on $\mathcal{T}$ and small on $\mathcal{T}^c$. So we expect that such a "truly"-open box can be found near the boundary of $\mathcal{T}$.

We define the outer boundary for $D \subseteq \mathbb{Z}^d$ by

$$\partial D := \{x \in D^c : |x-y|_1 = 1 \text{ for some } y \in D\}\,, \tag{4.4.12}$$

and denote by A the points in $\mathbb{Z}^d$ which are close to $\mathcal{T}^c$:

$$A = \{u \in \mathbb{Z}^d : \exists\, v \in \mathcal{T}^c \ s.t.\ |u-v|_\infty \le 100\ell\}\,.$$

For any starting point in $u \in A$, since $\mathcal{T}^c$ is a union of boxes of side length ℓ, the probability that the random walk hits $\mathcal{T}^c$ within ℓ^2 steps is uniformly bounded away from 0. Recalling (4.4.8), which says that starting from any point in $\mathcal{T}^c$, with probability at least 1/2, the random walk will be killed in ℓ^2 steps, we get for some constant $c' = c'(d)$,

$$\mathbf{P}^u(\tau_{\mathcal{V}^c} \le 2\ell^2) \ge c'\,.$$

Then (4.4.6) and the assumption $\lambda_{\mathcal{V}} \ge 1 - b_1\varrho_n^{-2}$ implies

$$\sum_{u\in A} \Phi_{\mathcal{V}}(u)c' \le 1 - \lambda_{\mathcal{V}}^{2\ell^2} \le Cb_1\ell^2\varrho_n^{-2}\,. \tag{4.4.13}$$

We claim that A contains many truly open boxes. Indeed, the cardinality of $A \cap B(z, C_{4,6}\varrho_n)$ can be bounded from below in terms of the cardinality of $\partial\mathcal{T} \cap B(z, C_{4,6}\varrho_n)$. Since $|\mathcal{T} \cap B(z, C_{4,6}\varrho_n)| \ge (b_2\varrho_n)^d$, $|B(z, C_{4,6}\varrho_n) \setminus \mathcal{T}| \ge \varrho_n^d$, and $b_2 \in (0,1)$, Lemma 4.C.1 implies that

$$|\partial\mathcal{T} \cap B(z, C_{4,6}\varrho_n)| \ge c(b_2\varrho_n)^{d-1}\,.$$

Then we can choose $c\ell^{-d}(b_2\varrho_n)^{d-1}$ many "truly"-open boxes such that that the boxes of side length $22\ell+1$ centered at these boxes are disjoint and also in A. Combined with (4.4.13), it implies that there exists a "truly"-open box $K(x,\ell)$ with $x \in B(z, C_{4,6}\varrho_n)$ and

$$\sum_{u \in K(x,11\ell)} \Phi_{\mathcal{V}}(u) \le \frac{Cb_1 \ell^2 \varrho_n^{-2}}{c\ell^{-d}(b_2\varrho_n)^{d-1}} \le Cb_1 b_2^{-(d-1)} \varrho_n^{-(d+1)} \ell^{d+2} .$$

Hence

$$\sum_{u \in K(x,11\ell)} \Phi_{\mathcal{V}}^2(u) \le \Big(\sum_{u \in K(x,11\ell)} \Phi_{\mathcal{V}}(u) \Big)^2 \le Cb_1^2 b_2^{-2(d-1)} \varrho_n^{-2(d+1)} \ell^{2(d+2)} .$$

Thus (4.4.11) follows and this completes the proof of Lemma 4.4.3. □

In the following lemma, we apply Lemma 4.4.3 repeatedly to remove a number of "truly"-open boxes, while $\lambda_{\mathcal{U}}$ only decreases slightly.

Lemma 4.4.9. *Assume* (4.4.4) *holds and* $\lambda_{\mathcal{V}} \ge 1 - b\varrho_n^{-2}$ *for some* $b \ge 1$*. There exists a constant* $C_{4,7} > 2$ *depending only on* (d,p) *such that the following holds. For any sufficiently large* ℓ *with* $\ell \le \varrho_n^{1/2}$ *, there exist* $z \in B(0,(\log n)^{C_{4,1}})$ *and* $\{x_m\}_{m=1}^{M_{\ell,b}} \subset B(z, C_{4,6}\varrho_n)$ *with*

$$M_{\ell,b} := C_{4,7}^{-1} b^{-d-2} \ell^{-2d-2} \varrho_n^d \tag{4.4.14}$$

such that $\{K(x_m,\ell)\}_{m=1}^{M_{\ell,b}}$ *are "truly"-open,* $\{K(x_m,5\ell)\}_{m=1}^{M_{\ell,b}}$ *are disjoint, and for any* $1 \le m \le M_{\ell,b}$*,*

$$\lambda_{\mathcal{V}} - \lambda_{\mathcal{V} \setminus \bigcup_{j=1}^m K(x_j,10\ell)} \le \delta(m,\ell,b) , \tag{4.4.15}$$

where $\delta(m,\ell,b) := C_{4,7} m b^{3-d} \ell^{2(d+1)} \varrho_n^{-d-2}$*.*

Proof. First note that since ℓ is sufficiently large and $\ell \le \varrho_n^{1/2}$, by Lemma 4.4.7, the assumption $\lambda_{\mathcal{V}} \ge 1 - b\varrho_n^{-2}$ implies that we can choose $z_{\mathcal{V}} \in B(0,(\log n)^{C_{4,1}})$ such that

$$|\mathcal{T} \cap B(z_{\mathcal{V}}, C_{4,6}\varrho_n)| \ge c_{4,3} b^{-d/2} \varrho_n^d . \tag{4.4.16}$$

We will choose the x_i's inductively by repeatedly applying Lemma 4.4.3 such that $x_i \in$

$B(z_{\mathcal{V}}, C_{4,6}\varrho_n)$ and $K(x_i,\ell)$ are "truly"-openfor $1 \le i \le M_{\ell,b}$, $K(x_1, 5\ell), \dots, K(x_{M_{\ell,b}}, 5\ell)$ are disjoint and for $1 \le i \le M_{\ell,b}$,

$$\lambda_{\mathcal{V}\setminus\bigcup_{j=1}^{i} K(x_j,10\ell)} \ge \lambda_{\mathcal{V}\setminus\bigcup_{j=1}^{i-1} K(x_j,10\ell)} - C_{4,7} b^{3-d} \ell^{2(d+1)} \varrho_n^{-d-2} . \tag{4.4.17}$$

Set

$$(b_1, b_2, z) = (2b, (c_{4,3}/2)^{1/d} b^{-1/2}, z_{\mathcal{V}}) , \tag{4.4.18}$$

For $i = 1$, we apply Lemma 4.4.3 with parameters in (4.4.18). We now verify the conditions in Lemma 4.4.3. Firstly, (4.4.4) is satisfied by assumption. Secondly, we assumed $\lambda_{\mathcal{V}} \ge 1 - b\varrho_n^{-2}$ and thus $\lambda_{\mathcal{V}} > 1 - b_1\varrho_n^{-2}$. Lastly, (4.4.16) gives $|\mathcal{T} \cap B(z, C_{4,6}\varrho_n)| \ge c_{4,3} b^{-d/2} \varrho_n^d > (b_2\varrho_n)^d$. Hence Lemma 4.4.3 implies that there exists $x_1 \in B(z, C_{4,6}\varrho_n)$ such that $K(x_1,\ell)$ is "truly"-open and (4.4.17) holds for $i = 1$ and sufficiently large $C_{4,7}$.

Suppose that we have chosen $x_1, \dots, x_i$ for $1 \le i \le M_{\ell,b} - 1$ with the aforementioned properties. We apply Lemma 4.4.3 with the same parameters as in (4.4.18) to $\mathcal{V}\setminus\bigcup_{j=1}^{i} K(x_j, 10\ell)$ in place of $\mathcal{V}$. We now verify the conditions in Lemma 4.4.3.

Firstly, since $\mathcal{T}$ and $\mathcal{E}(\varepsilon_n)$ are non-increasing as we close $\bigcup_{j=1}^{i} K(x_j, 10\ell)$ in $\mathcal{V}$, (4.4.4) still holds.

Secondly, combining the hypothesis (4.4.17) and the assumption $\lambda_{\mathcal{V}} \ge 1 - b\varrho_n^{-2}$ yields

$$\lambda_{\mathcal{V}\setminus\bigcup_{j=1}^{i} K(x_j,10\ell)} \ge \lambda_{\mathcal{V}} - i \cdot C_{4,7} b^{3-d} \ell^{2(d+1)} \varrho_n^{-d-2} \ge 1 - 2b\varrho_n^{-2} = 1 - b_1\varrho_n^{-2} ,$$

where in the last inequality we used $i \le M_{\ell,b}$ and $b \ge 1$.

Lastly, closing $\bigcup_{j=1}^{i} K(x_j, 10\ell)$ will at most affect whether sites in $\bigcup_{j=1}^{i} K(x_j, 20\ell)$ are "truly"-open or not. Hence, the reduction in the volume of "truly"-open box volume is at most $i(40\ell + 1)^d$ and hence (with $\mathcal{V}$ replaced by $\mathcal{V} \setminus \bigcup_{j=1}^{i} K(x_j, 10\ell)$) for sufficiently large $C_{4,7}$,

$$|\mathcal{T} \cap B(z, C_{4,6}\varrho_n)| \ge c_{4,3} b^{-d/2} \varrho_n^d - M_{\ell,b}(40\ell + 1)^d \ge c_{4,3} b^{-d/2} \varrho_n^d / 2 = (b_2\varrho_n)^d .$$

Therefore Lemma 4.4.3 implies that there exists $x_{i+1} \in B(z, C_{4,6}\varrho_n)$ such that $K(x_{i+1}, \ell)$ is "truly"-open in $\mathcal{V}\setminus\bigcup_{j=1}^{i} K(x_j, 10\ell)$ and (4.4.17) holds for $i+1$. Also, $K(x_{i+1}, \ell)$ is "truly"-open in $\mathcal{V} \setminus \bigcup_{j=1}^{i} K(x_j, 10\ell)$ implies that $K(x_{i+1}, \ell)$ is "truly"-open in $\mathcal{V}$, and it is disjoint from $\bigcup_{j=1}^{i} K(x_j, 10\ell)$. Hence $K(x_1, 5\ell), \ldots, K(x_{i+1}, 5\ell)$ are disjoint. This completes the proof of Lemma 4.4.9. □

Next we estimate the probability gain achieved by closing the "truly"-open boxes $\{K(x_i, \ell)\}_{i=1}^{M_{\ell,b}}$ identified in Lemma 4.4.9.

Lemma 4.4.10. *Let $M_{\ell,b}$ and $\delta(m, \ell, b)$ be as in Lemma 4.4.9. There exists a constant $C_{4,8} = C_{4,8}(d, p)$ such that the following holds. Suppose $\beta \geq 1 - b\varrho_n^{-2}$ for some constant $b \geq 1$, ℓ is sufficiently large with $\ell \leq \varrho_n^{1/2}$, $m \leq M_{\ell,b}$ and*

$$\ell^d \geq C_{4,8} \log(\varrho_n^d/m)\,. \tag{4.4.19}$$

Then

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta) \leq e^{-cm\ell^d}\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta - \delta(m, \ell, b)) + n^{-10d}\,. \tag{4.4.20}$$

Proof. We will consider an operation that changes some "truly"-open boxes to typical obstacle configurations, which maps the event $\{\lambda_{\mathcal{V}} \geq \beta\}$ to $\{\lambda_{\mathcal{V}} \geq \beta - \delta(m, \ell, b)\}$, allowing us to bound the probability ratio of these two events.

To this end, we define $\mathcal{T}_{U^c}$ and $\mathcal{E}_{U^c}$ for general $U \subset B(0, 2(\log n)^{C_{4,1}})$ by regarding $U^c = \mathcal{O}$, and for any $\beta > 1 - b\varrho_n^{-2}$, consider two classes of subsets of $B(0, 2(\log n)^{C_{4,1}})$:

$$\mathscr{U}(\beta) = \{U \subset B(0, 2(\log n)^{C_{4,1}}) : \lambda_{U \cap B(0,(\log n)^{C_{4,1}})} \geq \beta\}\,,$$
$$\mathscr{G} = \{U \subset B(0, 2(\log n)^{C_{4,1}}) : \min_{x \in B(0,(\log n)^{C_{4,1}})} |B(x, C_{4,6}\varrho_n) \setminus \mathcal{T}_{U^c}| \geq \varrho_n^d, |\mathcal{E}_{U^c}(\varepsilon_n)| \leq C_{4,2}\varrho_n^d\}\,.$$

Then, denoting $\mathcal{V}_+ := B(0, 2(\log n)^{C_{4,1}})\setminus\mathcal{O}$, we can rewrite the event $\{\lambda_{\mathcal{V}} \geq \beta, \text{(4.4.4) holds}\}$ as $\{\mathcal{V}_+ \in \mathscr{U}(\beta) \cap \mathscr{G}\}$.

For any sufficiently large ℓ with $\ell \leq \varrho_n^{1/2}$ and $m \leq M_{\ell,b}$, we define a map Ξ_m for all $U \in \mathscr{U}(\beta) \cap \mathscr{G}$ by

$$\Xi_m(U) := \bigcup_{j=1}^{m} K(x_j, 5\ell)\,, \tag{4.4.21}$$

where $x_1, \ldots, x_m$ are chosen as in Lemma 4.4.9 depending on U (make arbitrary choice when x_i's are not unique.) The idea of the proof is the following. For each $U \in \mathscr{U}(\beta) \cap \mathscr{G}$, we change the obstacle configuration in $\Xi_m(U)$ to typical configurations. The image of $\mathscr{U}(\beta) \cap \mathscr{G}$ has much higher probability under the law of $\mathcal{V}_+$ than that of $\mathscr{U}(\beta) \cap \mathscr{G}$ itself, because $\Xi_m(U)$ contains m "truly"-open boxes at a large probability cost. Combined with (4.4.15), which yields that the image of $\mathscr{U}(\beta) \cap \mathscr{G}$ is a subset of $\mathscr{U}(\beta - \delta(m, \ell, b))$, this gives the desired result.

We now rigorously implement this idea. We first define an equivalence relation on $\mathscr{U}(\beta) \cap \mathscr{G}$ by

$$U \sim U' \iff \Xi_m(U) = \Xi_m(U'), U \setminus \Xi_m(U) = U' \setminus \Xi_m(U')\,,$$

and denote the equivalence class for U in $\mathscr{U}(\beta) \cap \mathscr{G}/\!\sim$ by $[U]$, namely

$$[U] := \{U' \subset B(0, 2(\log n)^{C_{4,1}}) : U' \sim U\}\,.$$

Consider the map

$$\varphi([U]) = \{V : V \setminus \Xi_m(U) = U \setminus \Xi_m(U)\}\,,$$

which contains modifications of U by allowing arbitrary configurations on $\Xi_m(U)$ as long as the set $\Xi_m(U)$ does not change. Applying Claim 4.4.11 below to two families of events $(\{\mathcal{V}_+ \in [U]\})_{[U] \in \mathscr{U}(\beta) \cap \mathscr{G}/\sim}$ and $\{\mathcal{V}_+ \in \varphi([U])\}_{[U] \in \mathscr{U}(\beta) \cap \mathscr{G}/\sim}$, we obtain that

$$\frac{\mathbb{P}\big(\mathcal{V}_+ \in \bigcup_{[U]} \varphi([U])\big)}{\mathbb{P}(\mathcal{V}_+ \in \mathscr{U}(\beta) \cap \mathscr{G})} \geq \inf_{[U]} \frac{\mathbb{P}(\mathcal{V}_+ \in \varphi([U]))}{\mathbb{P}(\mathcal{V}_+ \in [U])} \Big/ \sup_{V \subset B(0, 2(\log n)^{C_{4,1}})} \sum_{[U]} \mathbb{1}_{V \in \varphi([U])}\,. \tag{4.4.22}$$

Claim 4.4.11. *Let $(E_i)_{1\le i\le k}$ and $(F_i)_{1\le i\le k}$ be two families of events. Then we have*

$$\frac{\mathbb{P}(\bigcup_i F_i)}{\mathbb{P}(\bigcup_i E_i)} \ge \frac{\inf_i \mathbb{P}(F_i)/\mathbb{P}(E_i)}{\sup_\omega \sum_i \mathbb{1}_{\omega\in F_i}} .$$

Proof. This follows from

$$\inf_i \mathbb{P}(F_i)/\mathbb{P}(E_i)\cdot\sum_i \mathbb{P}(E_i) \le \sum_i \mathbb{P}(F_i) = \mathbb{E}\Big[\sum_i \mathbb{1}_{\omega\in F_i}\Big] \le \sup_\omega \sum_i \mathbb{1}_{\omega\in F_i}\mathbb{P}\Big(\bigcup_i F_i\Big) .$$

□

Since Lemma 4.4.9 gives

$$\bigcup_{[U]} \varphi([U]) \subset \mathscr{U}(\beta-\delta(m,\ell,b))$$

and Lemma 4.4.6 yields $\mathbb{P}(\mathcal{V}_+\in\mathscr{G})\ge 1-n^{-10d}$, we can bound the left hand side of (4.4.22) by

$$\frac{\mathbb{P}\big(\mathcal{V}_+\in\bigcup_{[U]}\varphi([U])\big)}{\mathbb{P}(\mathcal{V}_+\in\mathscr{U}(\beta)\cap\mathscr{G})} \le \frac{\mathbb{P}(\lambda_{\mathcal{V}}\ge\beta-\delta(m,\ell,b))}{\mathbb{P}(\lambda_{\mathcal{V}}\ge\beta)-n^{-10d}} .$$

Therefore, to prove (4.4.20), it suffices to show that for some constant $c=c(d,p)$,

$$\inf_{[U]}\frac{\mathbb{P}(\mathcal{V}_+\in\varphi([U]))}{\mathbb{P}(\mathcal{V}_+\in[U])}\Big/\sup_{V\subset B(0,2(\log n)^{C_{4,1}})}\sum_{[U]}\mathbb{1}_{V\in\varphi([U])}\ge e^{cm\ell^d} . \tag{4.4.23}$$

We first prove that there exists a constant $c'=c'(d,p)$ such that

$$\inf_{[U]}\frac{\mathbb{P}(\mathcal{V}_+\in\varphi([U]))}{\mathbb{P}(\mathcal{V}_+\in[U])}\ge e^{c'm\ell^d} . \tag{4.4.24}$$

For any fixed $U\in(\mathscr{U}(\beta)\cap\mathscr{G})$, $\Xi_m(U)$ is a union of m boxes (defined in (4.4.21)), which we denote by $K(x_i,5\ell)$ for $1\le i\le m$. Then

$$\mathbb{P}(\mathcal{V}_+\in[U])\le\mathbb{P}(\mathcal{V}_+\setminus\Xi_m(U)=U\setminus\Xi_m(U), K(x_i,\ell)\text{ for all }i\le m\text{ are "truly"-open}) .$$

Note that the events $\{\mathcal{V}_+ \setminus \Xi_m(U) = U \setminus \Xi_m(U)\}$, $\{K(x_i, \ell)$ is "truly"-open$)\}$ for $i \le m$ are mutually independent as they depend on obstacle configurations on disjoint regions. Then by Lemma 4.4.2,

$$\begin{aligned}\mathbb{P}(\mathcal{V}_+ \in [U]) &\le \mathbb{P}(\mathcal{V}_+ \setminus \Xi_m(U) = U \setminus \Xi_m(U)) \cdot \mathbb{P}(K(0,\ell) \text{ is "truly"-open})^m \\ &\le e^{-cm\ell^d}\mathbb{P}(\mathcal{V}_+ \setminus \Xi_m(U) = U \setminus \Xi_m(U)) \\ &= e^{-cm\ell^d}\mathbb{P}\big(\mathcal{V}_+ \in \varphi([U])\big)\,.\end{aligned}$$

This gives (4.4.24).

Now, it only remains to prove that for the constant $c' > 0$ as in (4.4.24),

$$\sup_{V \subset B(0, 2(\log n)^{C_{4,1}})} \sum_{[U]} \mathbb{1}_{V \in \varphi([U])} \le e^{c' m \ell^d / 2}\,. \tag{4.4.25}$$

To this end, note that

$$\sum_{[U]} \mathbb{1}_{V \in \varphi([U])} = |\{[U] : V \setminus \Xi_m(U) = U \setminus \Xi_m(U)\}|\,,$$

where the cardinality of the set of such $[U]$ is bounded by the number of possible choices of $\Xi_m(U) = \bigcup_{i=1}^m K(x_i, 5\ell)$ with $x_1, \dots, x_m$ in $B(z, C_{4,6}\varrho_n)$ for some $z \in B(0, 2(\log n)^{C_{4,1}})$. Therefore, we have

$$\sum_{[U]} \mathbb{1}_{V \in \varphi([U])} \le |B(0, 2(\log n)^{C_{4,1}})| \cdot \binom{|B(0, C_{4,6}\varrho_n)|}{m} \le 2^d (\log n)^{C_{4,1} d} (e(2C_{4,6}\varrho_n)^d/m)^m\,.$$

Since $m \le M_{\ell,b}$ (defined as in (4.4.14)) ensures $\varrho_n^d/m \ge 2$, we can further bound the right hand side above by

$$\exp(C \log\log n + Cm \log(\varrho_n^d/m))$$

for some large constant $C > 0$. Also, by (4.1.2) and $\varrho_n^d/m \ge 2$, we have $\log\log n \le$

$Cm\log(\varrho_n^d/m)$ for some large constant $C > 0$. Therefore, the assumption (4.4.19) implies there exists a constant C' such that

$$C\log\log n + Cm\log(\varrho_n^d/m) \le C_{4,8}^{-1}C'm\ell^d\,.$$

For $C_{4,8}$ sufficiently large, this implies (4.4.25), which completes the proof of Lemma 4.4.10.

□

Proof of Lemma 4.3.3

We only need to apply Lemma 4.4.10 with appropriate choices of ℓ and m. Recall from (4.3.4) that $\epsilon \in ((\log\log n)^4\varrho_n^{-d}, c_b)$ is an arbitrary number for some small constant c_b to be determined. We let

$$\begin{aligned} \ell &= \lfloor \Theta[\log(1/\epsilon)]^{1/d} \rfloor\,, \\ m &= \lfloor \theta \cdot \epsilon(\log(1/\epsilon))^{-3} \cdot \varrho_n^d \rfloor\,, \end{aligned} \tag{4.4.26}$$

where Θ and θ are constants depending only on (d,p,b) to be determined.

First, we verify that all the conditions in Lemma 4.4.10 hold. Since $\epsilon \ge (\log\log n)^4\varrho_n^{-d}$, (4.4.26) implies $\ell < \varrho_n^{1/2}$ and $m \ge 1$. On the other hand, for all $\epsilon < c_b$ with $c_b = c_b(d,p,b,\theta,\Theta)$ sufficiently small, we have

$$\epsilon^2\varrho_n^d \le m \le \epsilon\varrho_n^d \cdot 2\Theta^{3d}\theta \cdot \ell^{-3d}\,.$$

Hence $m \le M_{\ell,b}$ (defined in Lemma 4.4.9), and $\ell^d \ge \Theta^d\log(\varrho_n^d/m)/2$. Then (4.4.19) follows by choosing Θ to be sufficiently large. We thus know that all the conditions in Lemma 4.4.10 hold and hence this lemma yields

$$\mathbb{P}(\lambda_{\mathcal{V}} \ge \beta) \le e^{-cm\ell^d}\mathbb{P}(\lambda_{\mathcal{V}} \ge \beta - \delta(m,\ell,b)) + n^{-10d}\,.$$

Recall $\delta(m, \ell, b)$ from Lemma 4.4.9. Then (4.4.26) yields that for θ and c_b sufficiently small,

$$\delta(m, \ell, b) \le \epsilon \varrho_n^{-2} \quad \text{and} \quad m\ell^d \ge \epsilon(\log(1/\epsilon))^{-3} \cdot \varrho_n^d\,.$$

We thus complete the proof of Lemma 4.3.3. □

4.4.2 Proof of Lemma 4.3.4

As we have discussed in Section 4.3, we want to understand how much the eigenvalue will increase when we remove obstacles inside the ball $B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n)$. The difficulty is that, for $k \ge 0$, removing obstacles in $B_{\delta,k}$ (defined in (4.3.6)) may hardly increase $\lambda_{\mathcal{V}}$, especially when there are many obstacles outside $B_{\delta,k}$ near its boundary and most obstacles in $B_{\delta,k}$ are also near the boundary. However, if we first remove all obstacles in the annulus $B_{\delta,k-1} \setminus B_{\delta,k}$ and then remove all obstacles in $B_{\delta,k}$, then in the second step, the increase in the eigenvalue $\lambda_{\mathcal{V}}$ can be bounded from below in terms of $|B_{\delta,k} \cap \mathcal{O}|$ since all removed obstacles are in $B_{\delta,k-1}$ with distance at least $\delta 2^{-k}$ to the boundary of $B(\boldsymbol{x}_{\mathcal{V}}, \varrho_n)$. The $\mathcal{J}$ defined in (4.3.7) ensures that a significant proportion of obstacles are in the bulk of the ball $B_{\delta,\mathcal{J}-1}$:

Lemma 4.4.12. *Let $\mathcal{J}, c_{4,5}$ be as in* (4.3.7) *and let λ_* be as in* (4.2.1)*. For any $\delta > 0$, we assume $\lambda_{\mathcal{V}} \ge \lambda_*$,* (4.2.11) *holds, and $B(\boldsymbol{x}_{\mathcal{V}}, (1-\delta)\varrho_n) \cap \mathcal{O} \neq \varnothing$. Then there exists a constant $C > 0$ such that*

$$\frac{|\mathcal{O} \cap B_{\delta,\mathcal{J}-1}|}{\varrho_n^d} \le C{\varepsilon_n}^{1/4} c_{4,5}^{\mathcal{J}-1}, \quad \frac{|\mathcal{O} \cap B_{\delta,\mathcal{J}}|}{|\mathcal{O} \cap B_{\delta,\mathcal{J}-1}|} \ge c_{4,5}\,. \tag{4.4.27}$$

Proof. This result follows directly from the definition (4.3.7) and (4.2.13). □

The following lemma gives a lower bound on how much the eigenvalue will increase if we remove all obstacles in $B_{\delta,\mathcal{J}-1}$.

Lemma 4.4.13. *Suppose $\lambda_{\mathcal{V}} \ge \lambda_*$ and* (4.2.11) *holds. (Recall the definition of $\mathcal{J}$ in* (4.3.7) *depending on $c_{4,5}$.) There exist constants $c_{4,5}, \kappa \in (0, 1)$ depending only on (d, p) such that* *for* $\delta = \varrho_n^{-\kappa}$,

$$\lambda_{\mathcal{V}\cup B_{\delta,\mathcal{J}-1}} - \lambda_{\mathcal{V}\cup(B_{\delta,\mathcal{J}-1}\setminus B_{\delta,\mathcal{J}})} \geq \Big(\frac{|\mathcal{O}\cap B_{\delta,\mathcal{J}-1}|}{\varrho_n^d}\Big)^{1-1/d}\varrho_n^{-2}\,. \tag{4.4.28}$$

Lemma 4.4.13 is proved by applying Lemma 4.B.2 in the appendix, which requires the following estimates.

Lemma 4.4.14. *Let* λ_* *and* Φ *be defined as in Lemma 4.2.1 and Definition 4.4.4, respectively. Suppose* $\lambda_{\mathcal{V}} \geq \lambda_*$ *and* (4.2.11) *holds. Then there exists a constant* $b_1 = b_1(d,p) \in (0,1)$ *such that for all* U *with* $\mathcal{V}\subset U\subset \mathcal{V}\cup B(\boldsymbol{x}_{\mathcal{V}},\varrho_n)$*, we have*

$$\sum_{u\in B(\boldsymbol{x}_{\mathcal{V}},b_1\varrho_n)} \Phi_U(u) \geq 1/2\,. \tag{4.4.29}$$

Proof. Recall the definition of $\mathcal{E}(\varepsilon_n)$ from Definition 4.2.2. Note that

$$B(\boldsymbol{x}_{\mathcal{V}},b_1\varrho_n)^c \subset A_1\cup A_2\cup A_3 \tag{4.4.30}$$

where

$$\begin{aligned}
A_1 &:= B(\boldsymbol{x}_{\mathcal{V}},(1+\sqrt{d}\varepsilon_n)\varrho_n)\setminus B(\boldsymbol{x}_{\mathcal{V}},b_1\varrho_n)\,,\\
A_2 &:= \mathcal{E}(\varepsilon_n)\setminus B(\boldsymbol{x}_{\mathcal{V}},(1+\sqrt{d}\varepsilon_n)\varrho_n)\,,\\
A_3 &:= \big(\mathcal{E}(\varepsilon_n)\cup B(\boldsymbol{x}_{\mathcal{V}},(1+\sqrt{d}\varepsilon_n)\varrho_n)\big)^c\,.
\end{aligned}$$

Hence

$$\sum_{u\in B(\boldsymbol{x}_{\mathcal{V}},b_1\varrho_n)} \Phi_U(u) \geq 1 - \sum_{u\in A_3}\Phi_U(u) - |\Phi_U|_\infty(|A_1|+|A_2|)\,. \tag{4.4.31}$$

We first prove that

$$\sum_{u\in A_3}\Phi_U(u) \leq C\varepsilon_n\,. \tag{4.4.32}$$

To this end, we notice that since $U\setminus B(\boldsymbol{x}_{\mathcal{V}},\varrho_n) = \mathcal{V}\setminus B(\boldsymbol{x}_{\mathcal{V}},\varrho_n)$, it follows from the definition of $\mathcal{E}(\varepsilon_n)$ (which depends on $\mathcal{V}$) that that $\mathcal{E}(\varepsilon_n) \setminus B(\boldsymbol{x}_{\mathcal{V}}, (1+\sqrt{d}\varepsilon_n)\varrho_n)$ does not change if we change $\mathcal{V}$ to U. Therefore, for every $u \in A_3$, there exists u' such that $u \in K(u', \lfloor \varepsilon_n \varrho_n \rfloor)$ and $|U^c \cap K(u', \lfloor \varepsilon_n \varrho_n \rfloor)| \geq \varepsilon_n |K(u', \lfloor \varepsilon_n \varrho_n \rfloor)|$. Hence by the local limit theorem

$$\mathbf{P}^u(\tau_{U^c} \leq (\varepsilon_n \varrho_n)^2) \geq c\varepsilon_n \,.$$

Then (4.4.6) gives

$$\sum_{u \in A_3} \Phi_U(u) \leq C{\varepsilon_n}^{-1}(1 - \lambda_U^{(\varepsilon_n \varrho_n)^2}) \,.$$

Now, since $\mathcal{V} \subset U$, we have $\lambda_U \geq \lambda_{\mathcal{V}} \geq \lambda_*$ and thus (4.4.32) follows.

Next, by Lemma 4.A.1, $\lambda_U \geq \lambda_*$ implies $|\Phi_U|_\infty \leq C\varrho_n^{-d}$. Combined with (4.4.31) and (4.4.32), this implies

$$\sum_{u \in B(\boldsymbol{x}_{\mathcal{V}}, b_1 \varrho_n)} \Phi_U(u) \geq \frac{2}{3} - C\varrho_n^{-d}(|A_1| + |A_2|) \,. \tag{4.4.33}$$

By (4.2.12), we have

$$|A_1| + |A_2| \leq C(1 - b_1 + \sqrt{d}\varepsilon_n + {\varepsilon_n}^{1/4})\varrho_n^d \,. \tag{4.4.34}$$

Combining (4.4.33) and (4.4.34), we see that (4.4.29) follows by letting b_1 be a constant sufficiently close to 1. □

Proof of Lemma 4.4.13. We apply Lemma 4.B.2 with

$$\mathcal{B} = \mathcal{V} \cup B_{\delta, \mathcal{J}-1}, \qquad \mathcal{B}_\circ = \mathcal{V} \cup (B_{\delta, \mathcal{J}-1} \setminus B_{\delta, \mathcal{J}}) \,,$$

$B_{R_1} = B_{\delta, \mathcal{J}-1}, B_{R_2} = B_{\delta, \mathcal{J}}$ (defined in (4.3.7)), and $B_{R_3} = B(\boldsymbol{x}_{\mathcal{V}}, b_1 \varrho_n)$ where b_1 is chosen as in Lemma 4.4.14. It follows from $\lambda_{\mathcal{V}} \geq \lambda_*$ and Lemma 4.4.14 that the conditions (4.B.2) and (4.B.3) in Lemma 4.B.2 holds. Hence, by (4.B.4) and the definition of $B_{\delta, \mathcal{J}-1}$ in (4.3.7), we have

$$
\begin{aligned}
\lambda_{\mathcal{V}\cup B_{\delta,\mathcal{J}-1}} - \lambda_{\mathcal{V}\cup(B_{\delta,\mathcal{J}-1}\setminus B_{\delta,\mathcal{J}})} &\geq \frac{c}{R_1^d(\log R_1)^{\mathbb{1}_{d=2}}}\Big(1-\frac{R_2}{R_1}\Big)^{C_d}|\mathcal{B}\setminus\mathcal{B}_\circ|^{(d-2)/d} \\
&\geq \frac{c}{\varrho_n^d(\log\varrho_n)^{\mathbb{1}_{d=2}}}(4^{\mathcal{J}}\delta)^{-C_d}|\mathcal{O}\cap B_{\delta,\mathcal{J}}|^{(d-2)/d}\,.
\end{aligned} \tag{4.4.35}
$$

By Lemma 4.4.12,

$$
\begin{aligned}
\varrho_n^{-d}|\mathcal{O}\cap B_{\delta,\mathcal{J}}|^{(d-2)/d} &\geq c_{4,5}^{(d-2)/d}\varrho_n^{-d}|\mathcal{O}\cap B_{\delta,\mathcal{J}-1}|^{(d-2)/d} \\
&= c_{4,5}^{(d-2)/d}\varrho_n^{-2}(|\mathcal{O}\cap B_{\delta,\mathcal{J}-1}|/\varrho_n^d)^{(d-1)/d-1/d} \\
&\geq cc_{4,5}\varepsilon_n^{-1/4d}\cdot c_{4,5}^{-\mathcal{J}/d}\cdot\varrho_n^{-2}(|\mathcal{O}\cap B_{\delta,\mathcal{J}-1}|/\varrho_n^d)^{(d-1)/d}\,,
\end{aligned} \tag{4.4.36}
$$

where we have used (4.2.13). Recall that we have set $\delta = \varrho_n^{-\kappa}$ and that $\varepsilon_n = \varrho_n^{-c_{4,2}}$ as defined in (4.2.10). Combining (4.4.35) and (4.4.36), we complete the proof of (4.4.28) by choosing $c_{4,5}$ and κ sufficiently small. □

Proof of Lemma 4.3.4. In light of Lemma 4.4.13, we will bound the probability ratio in (4.3.8) by considering the operation of removing all obstacles in $B_{\delta,\mathcal{J}-1}$. First note that the condition (4.2.11), $B_{\delta,k}$ and $\mathcal{J}$ (defined in (4.3.6) and (4.3.7), respectively) only depend on $\mathcal{O}\cap B(0,2(\log n)^{C_{4,1}})$. Therefore, for $\beta\geq\lambda_*$ and $m\geq 1$, we define

$$
\mathscr{U}_{\beta,m} := \Big\{U\subset B(0,2(\log n)^{C_{4,1}}) : \lambda_{U\cap B(0,(\log n)^{C_{4,1}})}\geq\beta, (4.2.11)\text{ holds}, |B_{\delta,\mathcal{J}-1}\setminus U| = m\Big\}
$$

where (4.2.11), $B_{\delta,k}$, $\mathcal{J}$ should be understood as if $\mathcal{O}=U^c$.

Now we consider the map ϕ for $U\in\mathscr{U}_{\beta,m}$ that removes all obstacles in $B_{\delta,\mathcal{J}-1}$, namely,

$$
\phi(U) := U\cup B_{\delta,\mathcal{J}-1}\,.
$$

Then by Lemma 4.4.13,

$$\bigcup_{U\in\mathscr{U}_{\beta,m}} \phi(U)\subset\Big\{U\subset B(0,2(\log n)^{C_{4,1}}):\lambda_{U\cap B(0,(\log n)^{C_{4,1}})}\geq\beta+\Big(\frac{m}{\varrho_n^d}\Big)^{1-1/d}\varrho_n^{-2}\Big\}\,. \tag{4.4.37}$$

Recall that $\mathcal{V}^+ := B(0,2(\log n)^{C_{4,1}})\setminus\mathcal{O}$. For every $U\in\mathscr{U}_{\beta,m}$, there are exactly m closed sites in $B_{\delta,\mathcal{J}-1}$, thus

$$\mathbb{P}(\mathcal{V}^+=U)=\Big(\frac{1-p}{p}\Big)^m\cdot\mathbb{P}(\mathcal{V}^+=\phi(U))\,.$$

Then by Claim 4.4.11, we have that

$$\begin{aligned}\mathbb{P}(\mathcal{V}^+\in\mathscr{U}_{\beta,m})&\leq\Big(\frac{1-p}{p}\Big)^m\cdot\max_{U\in\mathscr{U}_{\beta,m}}|\phi^{-1}(U)|\cdot\mathbb{P}(\mathcal{V}^+\in\bigcup_{U\in\mathscr{U}_{\beta,m}}\phi(U))\\&\leq\Big(\frac{1-p}{p}\Big)^m\cdot\max_{U\in\mathscr{U}_{\beta,m}}|\phi^{-1}(U)|\cdot\mathbb{P}(\lambda_{\mathcal{V}}\geq\beta+(\frac{m}{\varrho_n^d})^{1-1/d}\varrho_n^{-2})\,,\end{aligned}$$

where in the last step we used (4.4.37). The multiplicity $\max_{U\in\mathscr{U}_{\beta,m}}|\phi^{-1}(U)|$ is bounded above uniformly over U by the number of sets of m points contained in a ball of radius ϱ_n centered at some point in $B(0,(\log n)^{C_{4,1}})$, namely,

$$\max_{U\in\mathscr{U}_{\beta,m}}|\phi^{-1}(U)|\leq|B(0,2(\log n)^{C_{4,1}})|\cdot\binom{|B(0,\varrho_n)|}{m}\leq C\varrho_n^{dC_{4,1}}(\frac{e(2\varrho_n)^d}{m})^m\,.$$

We complete the proof of (4.3.8) by combining the preceding two inequalities. □

4.4.3 Proof of Proposition 4.3.1

We first note that Proposition 4.3.1 follows from the following result.

Claim 4.4.15. *Let $\kappa>0$ be defined as in Lemma 4.3.4, and λ_* as in (4.2.5). Then for all*

$\beta \geq \lambda_*$,

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta, B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \cap \mathcal{O} \neq \varnothing) \leq e^{-\varrho_n^{1/3}} \cdot \mathbb{P}(\lambda_{\mathcal{V}} \geq \beta) + 4e^{-\varrho_n} n^{-d} . \tag{4.4.38}$$

Indeed, applying Claim 4.4.15 with $\beta = \lambda_*$ yields

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \lambda_*, B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \cap \mathcal{O} \neq \varnothing) \leq e^{-\varrho_n^{1/3}} \mathbb{P}(\lambda_{\mathcal{V}} \geq \lambda_*) + 4e^{-\varrho_n} n^{-d} . \tag{4.4.39}$$

Combined with (4.2.8), this yields

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \lambda_*, B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n - \varrho_n^{1-\kappa}) \cap \mathcal{O} \neq \varnothing) \leq (e^{-\varrho_n^{1/3}} + 4e^{-\varrho_n} (\log n)^{-c}) \mathbb{P}(\lambda_{\mathcal{V}} \geq \lambda_*) ,$$

which implies (4.3.1).

Next, we prove Claim 4.4.15. We assume $\lambda_{\mathcal{V}} \geq \beta$, and by Lemma 4.2.3, we may also assume that (4.2.11) holds. Then by Lemma 4.2.4, this implies (4.2.13). Let $\kappa > 0$ be defined as in Lemma 4.3.4 and let $\delta = \varrho_n^{-\kappa}$. Recall $B_{\delta, \mathcal{J}-1}$ as in (4.3.6), (4.3.7). Then (4.2.13) gives

$$|\mathcal{O} \cap B_{\delta, \mathcal{J}-1}| \leq |\mathcal{O} \cap B(\boldsymbol{\mathcal{X}}_{\mathcal{V}}, \varrho_n)| \leq C {\varepsilon_n}^{1/4} \varrho_n^d .$$

Now, for each $m = |\mathcal{O} \cap B_{\delta, \mathcal{J}-1}| \in [1, C {\varepsilon_n}^{1/4} \varrho_n^d]$, we denote $q = m/\varrho_n^d$. Then Lemma 4.3.4 yields

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta, |\mathcal{O} \cap B_{\delta, \mathcal{J}-1}| = m, (4.2.11)) \leq C \varrho_n^{dC_{4,1}} (C/q)^{q \varrho_n^d} \mathbb{P}(\lambda_{\mathcal{V}} \geq \beta + q^{1-1/d} \varrho_n^{-2}) , \tag{4.4.40}$$

while applying Lemma 4.3.3 with $\epsilon = q^{1-1/d}$ gives

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta + q^{1-1/d} \varrho_n^{-2}) \leq \exp\big\{ -(1-1/d)^{-3} q^{1-1/d} (\log(1/q))^{-3} \varrho_n^d \big\} \mathbb{P}(\lambda_{\mathcal{V}} \geq \beta) + n^{-10d} .$$

Since $(1-1/d)^{-3} \geq 1$ and for sufficiently large n we have

$$\begin{aligned} & C\varrho_n^{dC_{4,1}}(C/q)^{q\varrho_n^d} \cdot \exp\{-q^{1-1/d}(\log(1/q))^{-3}\varrho_n^d\} \\ & \quad \leq \exp\{C\log\varrho_n + Cm\log(\tfrac{\varrho_n^d}{m}) - m(\tfrac{\varrho_n^d}{m})^{1/d}\log^{-3}(\tfrac{\varrho_n^d}{m})\} \\ & \quad \leq \exp(-m^{1-1/d}\varrho_n^{1/2})\,, \end{aligned}$$

and $C\varrho_n^{dC_{4,1}}(C/q)^{q\varrho_n^d} \leq \exp(C\log\varrho_n + C\varrho_n^d q\log(\frac{1}{q})) \leq n$, we obtain from (4.4.40) that

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta, |\mathcal{O} \cap B_{\delta,\mathcal{J}-1}| = m, (4.2.11)) \leq e^{-\varrho_n^{1/2}}\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta) + n^{1-10d}\,.$$

Summing it over $1 \leq m \leq C{\varepsilon_n}^{1/4}\varrho_n^d$ yields

$$\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta, (4.2.11)) \leq e^{-\varrho_n^{1/2}/2}\mathbb{P}(\lambda_{\mathcal{V}} \geq \beta) + n^{-8d}\,.$$

We complete the proof of (4.4.38) by noticing that Lemma 4.2.3 yields that (4.2.11) holds with probability at least $1 - 3n^{-d}e^{-\varrho_n}$.

4.5 Random Walk Localization

In this section, we first collect a few survival probability estimates from [25] in Section 4.5.1. Then we prove Lemmas 4.3.5, 4.3.6, and 4.3.7, Corollary 4.3.9 in Section 4.5.2, and prove Lemma 4.3.10 in Section 4.5.3.

4.5.1 Survival probability estimates

The following lemma gives upper and lower bounds on the probability that the random walk stays in $\mathcal{U}$ for t steps, which can be found in [25, Lemma 6.3, Lemma 6.9].

Lemma 4.5.1. *Let $\widehat{B}_n$ and $\mathcal{U}$ be as in Theorem D, and let Φ be as in Definition 4.4.4. There exist constants $C, c > 0$ such that the following holds with $\widehat{\mathbb{P}}$-probability tending to one* *as $n \to \infty$: For any $u \in \mathcal{U}$, $t \geq 0$,*

$$\mathbf{P}^u(\tau_{\mathcal{U}^c} > t) \geq c\Phi_{\mathcal{U}}(u)\varrho_n^d \lambda_{\mathcal{U}}^t \,, \tag{4.5.1}$$

$$\mathbf{P}^u(\tau_{\mathcal{U}^c} > t, S_t \in \widehat{B}_n) \leq C\lambda_{\mathcal{U}}^t \,. \tag{4.5.2}$$

The next lemma gives upper bounds on the probability cost for the random walk to stay in a bad region.

Lemma 4.5.2. *Let μ_B be as in* (4.2.5)*. There exist constants $C > 0, b_2 \in (0,1)$ such that with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$, for all $x \in \mathbb{Z}^d$ and $t \geq \varrho_n^2/2$,*

$$\mathbf{P}^x(\tau_{\mathcal{U}^c \cup \mathcal{O} \cup B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)} > t) \leq Ce^{-100\mu_B \varrho_n^{-2} t} \,. \tag{4.5.3}$$

Proof. (4.5.3) can be proved by a straightforward adaptation of the proof of [25, Lemma 6.1], using the fact that $|B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n) \setminus B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)| \leq C(1-b_2)\varrho_n^d$ with $b_2 \in (0,1)$ chosen sufficiently close to 1. □

4.5.2 Upper and lower bounds on transition probabilities

We will prove Lemma 4.3.5, Lemma 4.3.7, and Corollary 4.3.9 in this section. We have proved in (4.1.5) that $B(\boldsymbol{x}_{\mathcal{U}}, (1-\varrho_n^{-\kappa})\varrho_n)$ is open. This immediately leads to the following lower bound on the eigenfunction $\Phi_{\mathcal{U}}$ in the interior of the ball $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n)$.

Lemma 4.5.3. *There exists a constant $c > 0$ such that the following holds with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$: For all $x \in B(\boldsymbol{x}_{\mathcal{U}}, (1-2\varrho_n^{-\kappa})\varrho_n)$,*

$$\Phi_{\mathcal{U}}(x) \geq c\varrho_n^{-d-1} \cdot \mathrm{dist}(x, B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n)^c) \,. \tag{4.5.4}$$

Proof. Note that since $\langle \Phi_{\mathcal{U}}, (P|_{\mathcal{U}})^t \mathbb{1}_x \rangle = \lambda_{\mathcal{U}}^t \Phi_{\mathcal{U}}(x)$ for all x and $\lambda_{\mathcal{U}} \le 1$, we have

$$\Phi_{\mathcal{U}}(x) \ \ge \ \frac{1}{2} \sum_{i=\varrho_n^2, \varrho_n^2+1} \sum_{y \in \mathcal{U}} \Phi_{\mathcal{U}}(y) \mathbf{P}^y(S_i = x, \tau_{\mathcal{U}^c} > i)\,, \tag{4.5.5}$$

where we sum over two values of i because the walk has period 2. Since $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa})$ is open, we know that $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa}) \subset \mathcal{U}$. Hence for all $y \in B(\boldsymbol{x}_{\mathcal{U}}, b_1 \varrho_n)$ ($b_1 = b_1(d,p)$ is chosen in Lemma 4.4.14), [53, Proposition 6.9.4] yields for any $x \in B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n)$,

$$\sum_{i=\varrho_n^2, \varrho_n^2+1} \mathbf{P}^y(S_i = x, \tau_{\mathcal{U}^c} > i) \ge c \cdot \mathrm{dist}(x, \partial B(\boldsymbol{x}_{\mathcal{U}}, (1-2\varrho_n^{-\kappa})\varrho_n))\varrho_n^{-d-1}\,, \tag{4.5.6}$$

where c is a constant depending only on (d,p). Substituting (4.5.6) into (4.5.5) for $y \in B(\boldsymbol{x}_{\mathcal{U}}, b_1 \varrho_n)$ and then using Lemma 4.4.14 gives (4.5.4). □

Proof of Lemma 4.3.5. It is proved in [25, (6.15)] that (4.3.10) holds if $\Phi_{\mathcal{U}}(z) \ge c\epsilon\varrho_n^{-d}$ for $z \in B(\boldsymbol{x}_{\mathcal{U}}, (1-\epsilon)\varrho_n)$ in addition to the assumption (4.3.9) in Lemma 4.3.5. This additional assumption is verified by Lemma 4.5.3. □

That the ball $B(\boldsymbol{x}_{\mathcal{U}}, (1-\varrho_n^{-\kappa})\varrho_n)$ is open implies that if the random walk starts from the interior ball $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$ (b_2 defined in Lemma 4.5.2), then in the next ϱ_n^2 steps, all points in $B(\boldsymbol{x}_{\mathcal{U}}, (1-\epsilon)\varrho_n)$ can be reached with comparable probability. Lemma 4.3.7 will follow from the following lemma, which says that the random walk has a positive probability of visiting the interior of $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$ in any given time interval of length $C\varrho_n^2$.

Lemma 4.5.4. *Let $m \ge t$ and assume*

$$\text{either } u \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) \text{ and } t \ge 0 \quad \text{or} \quad u \in \mathcal{U} \text{ and } t \ge \varrho_n^{C_{4,4}}\,. \tag{4.5.7}$$

Then there exist constants $C_{4,9}, c > 0$ such that with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$,

$$\mathbf{P}^u(\tau_{\mathcal{U}^c} \geq m) \geq c\lambda_{\mathcal{U}}^{m-t}\mathbf{P}^u(\tau_{\mathcal{U}^c} \geq t)\,, \tag{4.5.8}$$

$$\mathbf{P}^u(S_{[t-C_{4,9}\varrho_n^2,t]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) \neq \varnothing \mid \tau_{\mathcal{U}^c} > m) \geq c\,. \tag{4.5.9}$$

The second case in (4.5.7) is harder to deal with since the random walk may start far away from $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$. However, it can be reduced to the first case by using the following lemma, which guarantees that the random walk starting in $\mathcal{U}$ reaches $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$ before time $\varrho_n^{C_{4,4}}$.

Lemma 4.5.5. *Let $C_{4,3} > 0$ be as in Lemma 4.3.5. There exist $c > 0$ and $C_{4,4}$ with $C_{4,4} > C_{4,3} > 0$ such that the following holds with $\widehat{\mathbb{P}}$-probability tending to one as $n \to \infty$: For all $u \in \mathcal{U}$ and $t \geq \varrho_n^{C_{4,4}}$,*

$$\mathbf{P}^u(S_{[0,t]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) = \varnothing \mid \tau_{\mathcal{U}^c} > t) \leq e^{-c\varrho_n^{-2}t}\,. \tag{4.5.10}$$

Proof. Lemma 4.5.2 implies $\mathbf{P}^u(S_{[0,t]} \subseteq \mathcal{U} \setminus B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)) \leq C\exp(-100\mu_B t\varrho_n^{-2})$. Comparing it with [25, (5.4)] (which implies that $\mathbf{P}^u(\tau_{\mathcal{U}^c} > t)$ is bounded from below by $\exp(-2\mu_B\varrho_n^{-2}t - (\log n)^C)$), and choosing a sufficiently large $C_{4,4}$ yields the desired result. □

Proof of Lemma 4.5.4. We first prove the lemma when $t = m$; more precisely, there exists a constant $C_{4,9} = C_{4,9}(d,p)$ such that for t and u satisfying either condition in (4.5.7),

$$\mathbf{P}^u(S_{[t-C_{4,9}\varrho_n^2,t]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) = \varnothing \mid \tau_{\mathcal{U}^c} > t) \leq 1/100\,. \tag{4.5.11}$$

We will prove this by considering the last visit to $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$, and for the rest of the time comparing the survival probability for the random walk outside $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$ to the survival probability in the whole region $\mathcal{U}$ with starting point in $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$.

To reduce the entropy resulting from the many possible last visit times, we chop time into small windows of length ϱ_n^2 and let

$$N_{\text{exit}} := \sup\{k \in \mathbb{N}^* : S_{[t-k\varrho_n^2+1,t]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) = \varnothing\}\,.$$

Since $N_{\text{exit}} = \infty$ is equivalent to no visit to $B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$, we always have $N_{\text{exit}} < \infty$ in the first case in (4.5.7). In the second case, we see from Lemma 4.5.5 that

$$\mathbf{P}^u(N_{\text{exit}} = \infty \mid \tau_{\mathcal{U}^c} > t) \le e^{-c\varrho_n^{-2}t}\,. \tag{4.5.12}$$

Now, we claim that for large $C_{4,9}$,

$$\mathbf{P}^u(N_{\text{exit}} \ge C_{4,9} \mid \tau_{\mathcal{U}^c} > t) \le 1/100\,, \tag{4.5.13}$$

which then implies (4.5.11). It remains to verify (4.5.13). To this end, we define stopping times $T_k = \inf\{j \ge t-(k+1)\varrho_n^2+1 : S_j \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)\}$ for $k \ge 0$. Since on the event $\{N_{\text{exit}} = k\}$, we have $k\varrho_n^2 \le t - T_k \le (k+1)\varrho_n^2$, by the strong Markov property, $\mathbf{P}^u(N_{\text{exit}} = k, \tau_{\mathcal{U}^c} > t)$ equals

$$\mathbf{E}^u[\mathbb{1}_{\tau_{\mathcal{U}^c} > T_k, T_k < t-k\varrho_n^2}\mathbf{P}^{S_{T_k}}(\tau_{\mathcal{U}^c} > t-T_k, S_{[t-T_k-k\varrho_n^2+1,t-T_k]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) = \varnothing)]. \tag{4.5.14}$$

Now we consider all $x \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$ and $t-(k+1)\varrho_n^2 \le m \le t-k\varrho_n^2$ (which include all (x,m) such that $(S_{T_k}, T_k) = (x,m)$ occurs with non-zero probability). On one hand, Lemma 4.5.2 implies

$$\mathbf{P}^x(\tau_{\mathcal{U}^c} > t-m, S_{[t-m-k\varrho_n^2+1,t-m]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) = \varnothing) \le C\exp(-99\mu_B k)\,. \tag{4.5.15}$$

On the other hand, by (4.5.1), Lemmas 4.2.1, 4.5.3, and $\lambda_{\mathcal{U}} > \lambda_*$,

$$\mathbf{P}^x(\tau_{\mathcal{U}^c} > t-m) \geq c\lambda_{\mathcal{U}}^{t-m} \geq c\exp\{-2\mu_B k\}\,. \tag{4.5.16}$$

Combining the preceding two inequalities and (4.5.14) yields that for sufficiently large k,

$$\mathbf{P}^u(N_{\text{exit}} = k, \tau_{\mathcal{U}^c} > t) \leq e^{-50\mu_B k}\mathbf{E}^u[\mathbb{1}_{\tau_{\mathcal{U}^c} > T_k}\mathbf{P}^{S_{T_k}}(\tau_{\mathcal{U}^c} > t - T_k)] = e^{-50\mu_B k}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t)\,.$$

We complete the proof of (4.5.13) by summing over all $k \geq C_{4,9}$ chosen sufficiently large.

Next, we prove (4.5.8). If we define stopping time $T_\star := \inf\{j \geq t - C_{4,9}\varrho_n^2 : S_j \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)\}$, then by the strong Markov property at $T_\star$, (4.5.1) and Lemma 4.5.3,

$$\begin{aligned}\mathbf{P}^u(\tau_{\mathcal{U}^c} \geq m) &\geq \mathbf{P}^u(S_{[t-C_{4,9}\varrho_n^2, t]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) \neq \varnothing, \tau_{\mathcal{U}^c} > m) \\ &\geq \mathbf{E}^u\left[\mathbb{1}_{T_\star < t, \tau_{\mathcal{U}^c} > T_\star} \cdot c\lambda_{\mathcal{U}}^{m-T_\star}\right]\,.\end{aligned} \tag{4.5.17}$$

Since $m - T_\star \leq m - t + C_{4,9}\varrho_n^2$ and $\lambda_{\mathcal{U}} > \lambda_* \geq 1 - \mu_B\varrho_n^{-2} - C_*\varrho_n^{-3}$ (see Lemma 4.2.1), this is further bounded from below by

$$c\lambda_{\mathcal{U}}^{m-t+C_{4,9}\varrho_n^2}\mathbf{P}^u(T_\star < t, \tau_{\mathcal{U}^c} > t) \geq c\lambda_{\mathcal{U}}^{m-t}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t)\,, \tag{4.5.18}$$

where in the last inequality, we used (4.5.11). This gives (4.5.8).

Finally, we prove (4.5.9). First note that by the Markov property at time t and (4.5.2),

$$\mathbf{P}^u(\tau_{\mathcal{U}^c} > m, S_m \in \widehat{B}_n) \leq C\lambda_{\mathcal{U}}^{m-t}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t)\,. \tag{4.5.19}$$

Then by (4.5.17) and (4.5.18), this is less than

$$C\mathbf{P}^u(S_{[t-C_{4,9}\varrho_n^2, t]} \cap B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n) \neq \varnothing, \tau_{\mathcal{U}^c} > m)\,.$$

Combining this and Lemma 4.3.5 yields (4.5.9). ☐

Proof of Lemma 4.3.6. Set $t = \lceil \varrho_n^{C_{4,4}} \rceil$. By the Markov property at time t and (4.5.2),

$$\begin{aligned}\mathbf{P}^u(\tau_{B(\boldsymbol{x}_{\mathcal{U}},b_2\varrho_n)} > t, \tau_{\mathcal{U}^c} > m, S_m \in \widehat{B}_n) &\le C\mathbf{P}^u(\tau_{B(\boldsymbol{x}_{\mathcal{U}},b_2\varrho_n)} > t, \tau_{\mathcal{U}^c} > t)\lambda_{\mathcal{U}}^{m-t} \\ &= Ce^{-ct\varrho_n^{-2}} \cdot \mathbf{P}^u(\tau_{\mathcal{U}^c} > t)\lambda_{\mathcal{U}}^{m-t},\end{aligned}$$

where in the last step, we used Lemma 4.5.5. Combined with the lower bound of $\mathbf{P}^u(\tau_{\mathcal{U}^c} > m)$ given by (4.5.8), it yields

$$\mathbf{P}^u(\tau_{B(\boldsymbol{x}_{\mathcal{U}},b_2\varrho_n)} > t, S_m \in \widehat{B}_n \mid \tau_{\mathcal{U}^c} > m) \le Ce^{-c\varrho_n^{C_{4,4}-2}}.$$

Combining it with Lemma 4.3.5 gives (4.3.11). ☐

Proof of Lemma 4.3.7. By adjusting the constant factor c in (4.3.13), we may assume $\epsilon < 1 - b_2$. Let $u \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$. We first prove that for $t \ge \varrho_n^2$,

$$\min_{\substack{x\in B(\boldsymbol{x}_{\mathcal{U}},(1-\epsilon)\varrho_n) \\ |x-u|_1+t \text{ is even}}} \mathbf{P}^u(\tau_{\mathcal{U}^c} > t, S_t = x) \ge c\epsilon \max_{y\in\mathcal{U}} \mathbf{P}^u(\tau_{\mathcal{U}^c} > t, S_t = y). \tag{4.5.20}$$

To this end, we define stopping time $T_\star = \inf\{j \ge t - (C_{4,9}+1)\varrho_n^2 : S_j \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)\}$(with $T_\star = 0$ for $t \le (C_{4,9}+1)\varrho_n^2$). Then by (4.5.9),

$$\mathbf{P}^u(T_\star \le t - \varrho_n^2 \mid \tau_{\mathcal{U}^c} > t - \varrho_n^2) \ge c. \tag{4.5.21}$$

Then for all $x \in B(\boldsymbol{x}_{\mathcal{U}}, (1-\epsilon)\varrho_n)$ such that $|x-u|_1 + t$ is even,

$$\mathbf{P}^u(\tau_{\mathcal{U}^c} > t, S_t = x) \ge \mathbf{E}^u\big[\mathbb{1}_{\tau_{\mathcal{U}^c} > T_\star, T_\star \le t-\varrho_n^2}\mathbf{P}^{S_{T_\star}}(\tau_{\mathcal{U}^c} > t - T_\star, S_{t-T_\star} = x)\big] \tag{4.5.22}$$

Since $B(\boldsymbol{x}_{\mathcal{U}}, (1-\varrho_n^{-\kappa})\varrho_n) \subset \mathcal{U}$ by Theorem D, it follows from [53, Proposition 6.9.4] that uniformly in $x \in B(\boldsymbol{x}_{\mathcal{U}}, (1-\epsilon)\varrho_n), y \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$ and $\varrho_n^2 \le k \le (C_{4,9}+1)\varrho_n^2$ such that

$|x-u|_1+k$ is even, we have

$$\mathbf{P}^y(\tau_{\mathcal{U}^c} > k, S_k = x) \geq c\epsilon\varrho_n^{-d}\,. \tag{4.5.23}$$

Substituting this bounds into (4.5.22) yields

$$\begin{aligned}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t, S_t = x) &\geq c\epsilon\varrho_n^{-d}\mathbf{P}^u(\tau_{\mathcal{U}^c} > T_\star, T_\star \leq t-\varrho_n^2)\\ &\geq c\epsilon\varrho_n^{-d}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t-\varrho_n^2, T_\star \leq t-\varrho_n^2)\\ &\geq c\epsilon\varrho_n^{-d}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t-\varrho_n^2)\,,\end{aligned}$$

where we used (4.5.21). On the other hand, for all $y \in \mathcal{U}$,

$$\begin{aligned}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t, S_t = y) &= \mathbf{E}^u\big[\mathbb{1}_{\tau_{\mathcal{U}^c} > t-\varrho_n^2}\mathbf{P}^{S_{t-\varrho_n^2}}(\tau_{\mathcal{U}^c} > \varrho_n^2, S_{\varrho_n^2} = y)\big]\\ &\leq C\varrho_n^{-d}\mathbf{P}^u(\tau_{\mathcal{U}^c} > t-\varrho_n^2)\,.\end{aligned} \tag{4.5.24}$$

Combining the two preceding bounds give (4.5.20).

We now prove (4.3.12) and (4.3.13). Combining Lemmas 4.5.1 and 4.5.3 gives that for $m-t \geq 0$,

$$\min_{x \in B(x_{\mathcal{U}},(1-\epsilon)\varrho_n)} \mathbf{P}^x(\tau_{\mathcal{U}^c} > m-t) \geq c\epsilon \max_{y \in \mathcal{U}} \mathbf{P}^y(\tau_{\mathcal{U}^c} > m-t, S_{m-t} \in \widehat{B}_n)\,. \tag{4.5.25}$$

Multiplying each side of (4.5.25) with that of (4.5.20) and using the Markov property at time $m-t$, we obtain

$$\min_{\substack{x \in B(x_{\mathcal{U}},(1-\epsilon)\varrho_n),\\ |x-u|_1+t \text{ is even}}} \mathbf{P}^u(S_t = x, \tau_{\mathcal{U}^c} > m) \geq c\epsilon^2 \max_{y \in \mathcal{U}} \mathbf{P}^u(S_t = y, S_m \in \widehat{B}_n, \tau_{\mathcal{U}^c} > m)\,.$$

Lemma 4.3.5 implies

$$\mathbf{P}^u(S_m \in \widehat{B}_n \mid \tau_{\mathcal{U}^c} > m) \geq 1 - \exp(-\varrho_n^c) \quad \text{and} \quad \max_{y \in \mathcal{U}} \mathbf{P}^u(S_t = y \mid \tau_{\mathcal{U}^c} > m) \geq c\varrho_n^{-d}\,.$$

Therefore,

$$\min_{\substack{x \in B(\boldsymbol{x}_{\mathcal{U}},(1-\epsilon)\varrho_n),\\ |x-u|_1+t \text{ is even}}} \mathbf{P}^u(S_t = x \mid \tau_{\mathcal{U}^c} > m) \geq c\epsilon^2 \max_{y \in \mathcal{U}} \mathbf{P}^u(S_t = y \mid \tau_{\mathcal{U}^c} > m) - \exp(-\varrho_n^c)\,, \tag{4.5.26}$$

then (4.3.12) follows. In addition, (4.5.26) implies

$$\begin{aligned} 1 &\geq \mathbf{P}^u(S_t \in B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n/2) \mid \tau_{\mathcal{U}^c} > m) \\ &\geq c\varrho_n^d \max_{y \in \mathcal{U}} \mathbf{P}^u(S_t = y \mid \tau_{\mathcal{U}^c} > m) - C\varrho_n^d \exp(-\varrho_n^c)\,, \end{aligned} \tag{4.5.27}$$

which yields (4.3.13). □

Proof of Corollary 4.3.9. We first consider the case when $t \geq \varrho_n^2$. Since $|\widehat{B}_n \setminus B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n))| \leq \varrho_n^{d-c}$ for some constant $c \in (0,1)$, by (4.3.12),

$$\mathbf{P}^u(S_t \in \widehat{B}_n \setminus B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n) \mid \tau_{\mathcal{U}^c} > m) \leq C\varrho_n^{-c}\,. \tag{4.5.28}$$

Combined with Lemma 4.3.5, it yields (4.3.14).

Now, we consider the case $t \leq \varrho_n^2$. For $u \in B(\boldsymbol{x}_{\mathcal{U}}, b_2\varrho_n)$, since

$$\text{dist}(u, \widehat{B}_n \setminus B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n)) \geq (1 - b_2)\varrho_n/2\,,$$

by a union bound and the local limit theorem, we have that for any $t \geq 0$,

$$\mathbf{P}^u(S_t \in \widehat{B}_n \setminus B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n)) \leq C|\widehat{B}_n \setminus B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n)|\varrho_n^{-d} \leq \varrho_n^{-c}\,.$$

Then by the Markov property at time t and (4.5.2),

$$\mathbf{P}^u(S_t \in \widehat{B}_n \setminus B(\boldsymbol{x}_{\mathcal{U}}, (1 - 2\varrho_n^{-\kappa})\varrho_n), S_m \in \widehat{B}_n, \tau_{\mathcal{U}^c} > m) \le \varrho_n^{-c} \cdot \lambda_{\mathcal{U}}^{m-t}. \tag{4.5.29}$$

On the other hand, combining Lemma 4.5.1 and Lemma 4.5.3 gives

$$\mathbf{P}^u(\tau_{\mathcal{U}^c} > m) \ge c\lambda_{\mathcal{U}}^m. \tag{4.5.30}$$

Since Lemma 4.2.1 yields $\lambda_{\mathcal{U}}^{-t} \le C$ for $t \le \varrho_n^2$, combining (4.5.29), (4.5.30), and Lemma 4.3.5 gives (4.3.14). □

4.5.3 Distribution of the random walk

This section is devoted to the proof of Lemma 4.3.10. We will prove Lemma 4.3.10 by the eigenfunction expansion of $P|_{\widehat{B}_n}$. Loosely speaking, if we know that the spectral gap is larger than $c\varrho_n^{-2}$, then after time much longer than ϱ_n^2, the principal eigenfunction term should dominate all the other terms. However, there is an issue caused by the periodicity of the random walk, that is, there is a negative eigenvalue with the same modulus as the principal eigenvalue. In order to circumvent this issue, we will deal with even and odd times and sites separately. This corresponds to dealing with $(P|_{\widehat{B}_n})^2$, instead of $P|_{\widehat{B}_n}$, and we will prove necessary estimates for the corresponding eigenvalues and eigenfunctions in Appendix 4.A.

Let $\lambda_i(M)$ denote the i-th largest eigenvalue of the matrix M and let $\Phi_i(M)$ denote the corresponding ℓ^1-normalized eigenvector. For any vector η indexed by $\mathbb{Z}^d$, we let η_{e} and η_{o} be η restricted to even and odd sites in $\mathbb{Z}^d$, respectively. Also, for any matrix M indexed by $\mathbb{Z}^d \times \mathbb{Z}^d$, we let M_{e} and M_{o} be M with both coordinates restricted to even and odd sites, respectively.

Proof of Lemma 4.3.10. Denote $Q = P|_{\widehat{B}_n}$ and $\eta = \Phi_1(Q)$ to simplify the notation.

Since $B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa})$ is open by (4.1.5) , we have

$$B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n - \varrho_n^{1-\kappa}) \subset \widehat{B}_n \subset B(\boldsymbol{x}_{\mathcal{U}}, \varrho_n + \varrho_n^{1-c_{4,1}})\,.$$

Hence the assumption (4.A.9) in Lemma 4.A.4 holds. By the eigendecomposition of Q_{e}^2 and (4.A.6), we have that for any even site $v \in \mathbb{Z}^d$ and $m \geq 0$,

$$\left| \mathbb{1}_v^{\mathsf{T}} (Q_{\mathrm{e}}^2)^m - \lambda_1 (Q_{\mathrm{e}}^2)^m \frac{\eta_{\mathrm{e}}(v)}{|\eta_{\mathrm{e}}|_2^2} \eta_{\mathrm{e}} \right|_1 \leq \sum_{i \geq 2} \lambda_i (Q_{\mathrm{e}}^2)^m \frac{\langle \Phi_i(Q)_{\mathrm{e}}, \mathbb{1}_v \rangle}{|\Phi_i(Q)_{\mathrm{e}}|_2^2} |\Phi_i(Q)_{\mathrm{e}}|_1 \tag{4.5.31}$$

Since the dimension of the matrix Q is at most $|\widehat{B}_n|$, and (4.A.11) implies

$$|\Phi_i(Q)_{\mathrm{e}}|_2^2 = \frac{1}{2} |\Phi_i(Q)|_2^2 \geq \frac{|\Phi_i(Q)|_1^2}{2 \dim(Q)}\,,$$

we can further bound the right hand side of (4.5.31) from above by

$$2 \sum_{i \geq 2} \lambda_i (Q_{\mathrm{e}}^2)^m |\widehat{B}_n| \leq 2 \lambda_1 (Q_{\mathrm{e}}^2)^m |\widehat{B}_n|^2 e^{-c \varrho_n^{-2} m} = \lambda_{\widehat{B}_n}^{2m} |\widehat{B}_n|^2 e^{-c \varrho_n^{-2} m}\,, \tag{4.5.32}$$

where we used (4.A.12) and $\lambda_{\widehat{B}_n} = \lambda_1(\widehat{B}_n)$ as defined before Lemma 4.2.1. Then since $|x^{\mathsf{T}} Q|_1 \leq |x|_1$ for all x, $\mathbb{1}_v^{\mathsf{T}} (Q_{\mathrm{e}}^2)^t = \mathbb{1}_v^{\mathsf{T}} Q^{2t}$, $Q \eta_{\mathrm{e}} = \lambda_{\widehat{B}_n} \eta_{\mathrm{o}}$, we have

$$\left| \mathbb{1}_v^{\mathsf{T}} Q^{2m+1} - \lambda_{\widehat{B}_n}^{2m+1} \frac{\eta_{\mathrm{e}}(v)}{|\eta_{\mathrm{e}}|_2^2} \eta_{\mathrm{o}} \right|_1 \leq \lambda_{\widehat{B}_n}^{2m+1} |\widehat{B}_n|^2 e^{-c \varrho_n^{-2} m}\,. \tag{4.5.33}$$

Fix C' to be some large constant to be determined. Combining (4.5.31) and (4.5.33) with (4.A.11), we get that for any even site $v \in \mathbb{Z}^d$, $m \geq C' \varrho_n^2 \log\log n$ and x such that $|x-v|_1 + m$ is even,

$$\left| \mathbf{P}^v (S_m = x, \tau_{\widehat{B}_n^c} > m) - 2 |\eta|_2^{-2} \lambda_{\widehat{B}_n}^m \eta(v) \eta(x) \right| \leq \lambda_{\widehat{B}_n}^m (\log n)^{-cC'}\,. \tag{4.5.34}$$

Similarly, this also holds for odd site $v \in \mathbb{Z}^d$. Summing (4.5.34) over x and using (4.A.11), we get

$$\mathbf{P}^v(\tau_{\widehat{B}_n^c} > m) = |\eta|_2^{-2}\lambda_{\widehat{B}_n}^m \eta(v)(1 + O(\varrho_n^{-2}))\,. \tag{4.5.35}$$

Hence

$$\Big|\mathbf{P}^v(S_m = x \mid \tau_{\widehat{B}_n^c} > m) - 2\eta(x)\Big| \le C|\eta|_2^2\eta(v)^{-1}(\log n)^{-cC'} + C\eta(x)\varrho_n^{-2}\,. \tag{4.5.36}$$

Since $v \in B(\boldsymbol{x}_{\mathcal{U}}, (1-2\varrho_n^{-\kappa})\varrho_n)$, (4.A.13) yields $\eta(v) \ge c\varrho_n^{-d-\kappa}$. Also (4.A.10) yields $|\eta|_2^2 \le C\varrho_n^{-d}$. Now, choosing a sufficiently large C' and applying (4.A.3) to replace $\eta(x)$ by $\varrho_n^{-d}\phi_1(\frac{x-\boldsymbol{x}_{\mathcal{U}}}{\varrho_n})$, we get (4.3.15).

On the other hand, define for $m \ge 0$, $u, v \in \mathbb{Z}^d$,

$$q_m(u,v) := 2|\eta|_2^{-2}\lambda_{\widehat{B}_n}^m \eta(u)\eta(v)\,.$$

Then combining (4.5.34), (4.5.35) and (4.A.10) yields that for any $v, x, y \in \mathbb{Z}^d$ and $m, t \ge C'\varrho_n^2 \log\log n$ such that both $|x-v|_1 + m$ and $|y-v|_1 + m + t$ are even,

$$\begin{aligned}
&\Big|\mathbf{P}^v(S_m = x, S_{m+t} = y, \tau_{\widehat{B}_n^c} > m+t) - q_m(v,x)q_t(x,y)\Big| \\
=&\Big|\mathbf{P}^v(S_m = x, \tau_{\widehat{B}_n^c} > m)\mathbf{P}^x(S_t = y, \tau_{\widehat{B}_n^c} > t) - q_m(v,x)q_t(x,y)\Big| \\
\le&\lambda_{\widehat{B}_n}^{m+t}(\log n)^{-2cC'} + q_m(v,x)\lambda_{\widehat{B}_n}^t(\log n)^{-cC'} + q_t(x,y)\lambda_{\widehat{B}_n}^m(\log n)^{-cC'} \\
\le&5\lambda_{\widehat{B}_n}^{m+t}(\log n)^{-cC'}\,.
\end{aligned}$$

Combined with (4.5.34), this yields

$$\Big|\mathbf{P}^v(S_m = x \mid S_{m+t} = y, \tau_{\widehat{B}_n^c} > m+t) - 2|\eta|_2^{-2}\eta(x)^2\Big| \le (\log n)^{-cC'+C}\frac{|\eta|_2^2}{\eta(v)\eta(y)}.$$

Since $v, y \in B(\boldsymbol{x}_{\mathcal{U}}, (1-2\varrho_n^{-\kappa})\varrho_n)$, (4.A.13) yields $\eta(v), \eta(y) \ge c\varrho_n^{-d-\kappa}$. Also (4.A.10) yields

$|\eta|_2^2 \le C\varrho_n^{-d}$. Now, choosing a sufficiently large C' and applying (4.A.2) to replace $|\eta|_2^{-2}\eta(x)$ by $\varrho_n^{-d/2}\phi_2(\frac{x-x_{\mathcal{U}}}{\varrho_n})$, we get (4.3.16). □

4.A Estimates for eigenvalues and eigenfunctions

This section collects some basic estimates for eigenfunctions and eigenvalues used in the proof. For $A \subset \mathbb{Z}^d$, we let λ_A denote the principal (largest) eigenvalue of $P|_A$, which is the transition matrix of the simple symmetric random walk on $\mathbb{Z}^d$ killed upon exiting A, and let Φ_A be the ℓ^1-normalized principal eigenfunction of $P|_A$. The following lemma bounds the ℓ_∞-norm of the eigenfunction Φ_D in a domain $D \subset \mathbb{Z}^d$ in terms of the eigenvalue λ_D.

Lemma 4.A.1. *There exists a constant $C > 0$ such that $|\Phi_D|_\infty \le C(1-\lambda_D)^{d/2}$ for all $D \subset \mathbb{Z}^d$.*

Proof. By $P|_D\Phi_D = \lambda_D\Phi_D$, we have $\sum_{u:u\sim v}(2d)^{-1}\Phi_D(u) = \lambda_D\Phi_D(v)$. Then it follows from the Markov property that $\left(\lambda_D^{-t\wedge\tau_{D^c}}\Phi_D(S_{t\wedge\tau_{D^c}})\right)_{t\ge 0}$ is a martingale.

Let $l = (1-\lambda_D)^{-1/2}$. By the optional sampling theorem and the local limit theorem,

$$\begin{aligned}\Phi_D(v) = \mathbf{E}^v[\lambda_D^{-\lfloor l^2\rfloor\wedge\tau_{D^c}}\Phi_D(S_{\lfloor l^2\rfloor\wedge\tau_{D^c}})] &\le \lambda_D^{-l^2}\sum_u \mathbf{P}^v(S_{\lfloor l^2\rfloor}=u)\Phi_D(u) + \mathbf{P}^v(\tau_{D^c}\le\lfloor l^2\rfloor)\cdot 0\\ &\le Cl^{-d} \quad \text{uniformly in } v \in D.\end{aligned}$$

□

Let $\lambda_i(M)$ denote the i-th largest eigenvalue of matrix M and $\Phi_i(M)$ denote the corresponding ℓ^1-normalized eigenvector. The following lemma says that if a large domain in $\mathbb{Z}^d$ is close to a ball, then the first eigenvalue and eigenfunction of this domain are also close to that of the ball.

Lemma 4.A.2. *Suppose $B(0,(1-\epsilon)t) \subset \mathcal{B} \subset B(0,(1+\epsilon)t)$ where ϵ is smaller than some* *constant depending only on d. Then there exist constants $C, c > 0$ such that*

$$\lambda_1(P|_{\mathcal{B}}) - \lambda_2(P|_{\mathcal{B}}) \geq ct^{-2}, \tag{4.A.1}$$

$$\Big|\frac{\Phi_1(\mathcal{B})^2}{|\Phi_1(\mathcal{B})|_2^2} - \phi_{2,t}^2\Big|_1 \leq C(\sqrt{\epsilon} + t^{-1/2}), \tag{4.A.2}$$

$$|\Phi_1(\mathcal{B}) - \phi_{1,t}|_1 \leq C(\sqrt{\epsilon} + t^{-1/2}), \tag{4.A.3}$$

where ϕ_1 and ϕ_2 are respectively the L^1 and L^2-normalized first eigenfunction of the Dirichlet-Laplacian of the unit ball in $\mathbb{R}^d$, and $\phi_{2,t}(\cdot) = t^{-d/2}\phi_2(\cdot/t)$, $\phi_{1,t}(\cdot) = t^{-d}\phi_1(\cdot/t)$.

Proof. First, we see that for $i = 1, 2$, by [77, (3.27) and (6.11)]

$$\lambda_i(P|_{B(0,t)}) = 1 - t^{-2}\mu_i(\mathbf{B}) + O(t^{-3}), \tag{4.A.4}$$

where $\mu_i(\mathbf{B})$ is the i-th eigenvalue of the Dirichlet-Laplacian of the unite ball $\mathbf{B} \subset \mathbb{R}^d$. The min-max theorem implies that $\lambda_i(B(0,(1-\epsilon)t)) \leq \lambda_i(\mathcal{B}) \leq \lambda_i(B(0,(1+\epsilon)t))$. Hence for $i = 1, 2$

$$\lambda_i(P|_{\mathcal{B}}) = 1 - t^{-2}\mu_i(\mathbf{B}) + O(\epsilon t^{-2} + t^{-3}).$$

This implies the first assertion.

Let $\phi_{2,\mathcal{B}}$ be the ℓ^2-normalized first eigenvector of $P|_{\mathcal{B}}$, let ϕ_2 be the L^2-normalized first eigenfunction of the Dirichlet-Laplacian of the unit ball in $\mathbb{R}^d$, and define

$$\tilde{\phi}_{2,t}(x) := t^{d/2} \int_{x/t+[0,1/t]^d} \phi_2(y)\,\mathrm{d}y, \ x \in \mathbb{Z}^d.$$

Then by [77, (6.11)] and [78, (1.5)],

$$\Big|\phi_{2,\mathcal{B}} - \frac{\tilde{\phi}_{2,(1-\epsilon)t}}{|\tilde{\phi}_{2,(1-\epsilon)t}|_2}\Big|_2^2 \leq C\frac{\lambda_1(P|_{\mathcal{B}}) - \lambda_1(P|_{B(0,(1-\epsilon)t)})}{\lambda_1(P|_{\mathcal{B}}) - \lambda_2(P|_{\mathcal{B}})} = O(\epsilon + t^{-1}).$$

Since ϕ_2 is continuously differentiable (see, for example, [36, Corollary 8.11]),

$$|\tilde{\phi}_{2,(1-\epsilon)t} - \phi_{2,t}|_\infty = O(t^{-d/2-1}) \quad \textit{and} \quad |\tilde{\phi}_{2,(1-\epsilon)t}|_2^2 = 1 + O(t^{-1})\,.$$

Altogether, we have $\sum_{x\in\mathbb{Z}^d}(\phi_{2,\mathcal{B}}(x) - t^{-d/2}\phi_2(x/t))^2 = O(\epsilon + t^{-1})$. The second and third assertion follow by combining this with the boundedness of ϕ_2 and Lemma 4.A.1. □

The following two lemmas are needed to deal with the periodicity of the simple random walk. In what follows, for any vector η indexed by sites in $\mathbb{Z}^d$, we let η_{e} and η_{o} be η restricted to even and odd sites in $\mathbb{Z}^d$, respectively.

Lemma 4.A.3. *Let $Q = P|_A$ for some $A \subset \mathbb{Z}^d$. We denote by Q_{e}^2 and Q_{o}^2 the transition matrix Q^2 restricted to even and odd sites in $\mathbb{Z}^d$,respectively. Then*

$$\mathrm{rank}\ Q_{\mathrm{e}}^2 = \mathrm{rank}\ Q_{\mathrm{o}}^2 = \mathrm{rank}\ Q/2\,.$$

For $1 \le i \le \mathrm{rank}\ Q/2$, we have

$$\lambda_i(Q)^2 = \lambda_i(Q_{\mathrm{o}}^2) = \lambda_i(Q_{\mathrm{e}}^2)\,, \tag{4.A.5}$$

and

$$\frac{\Phi_i(Q)_{\mathrm{e}}}{|\Phi_i(Q)_{\mathrm{e}}|_1} = \Phi_i(Q_{\mathrm{e}}^2), \quad \frac{\Phi_i(Q)_{\mathrm{o}}}{|\Phi_i(Q)_{\mathrm{o}}|_1} = \Phi_i(Q_{\mathrm{o}}^2)\,. \tag{4.A.6}$$

Furthermore,

$$\begin{aligned} &|\Phi_i(Q)_{\mathrm{e}}|_2 = |\Phi_i(Q)_{\mathrm{o}}|_2\,, \\ &|\lambda_i(Q)| \le \frac{|\Phi_i(Q)_{\mathrm{e}}|_1}{|\Phi_i(Q)_{\mathrm{o}}|_1} \le |\lambda_i(Q)|^{-1}\,. \end{aligned} \tag{4.A.7}$$

Proof. Note that if λ is an eigenvalue of Q with eigenvector η, then

$$Q\eta_{\mathrm{o}} = \lambda\eta_{\mathrm{e}}, \quad Q\eta_{\mathrm{e}} = \lambda\eta_{\mathrm{o}}\,, \tag{4.A.8}$$

and hence $-\lambda$ is an eigenvalue of Q with eigenvector $\eta_{\mathrm{e}} - \eta_{\mathrm{o}}$. Furthermore, η_{o} is in the null space of the matrix Q_{e}^2, hence η_{e} is an eigenvector of Q_{e}^2 associated with eigenvalue λ. Similarly, η_{o} is an eigenvector of Q_{o}^2 associated with eigenvalue λ. Therefore, we conclude that the nonzero eigenvalues of Q_{e}^2 (or Q_{o}^2) are exactly the square of positive eigenvalues of Q. The corresponding eigenfunctions can be found by restricting eigenfunctions of Q to odd (or even) sites. Hence (4.A.5) and (4.A.6) follow. (4.A.7) follows directly from (4.A.8). $\square$

Lemma 4.A.4. *Let $a \in (0,1)$ and $Q = P|_B$ where B is a subset of $\mathbb{Z}^d$ that satisfies*

$$B(0, n - n^{1-a}) \subset B \subset B(0, n + n^{1-a}) . \tag{4.A.9}$$

Then there exist constants $C, c > 0$ depending only on (a, d) such that for sufficiently large n,

$$\lambda_1(Q) \geq 1 - Cn^{-2} , \qquad |\Phi_1(Q)|_\infty \leq Cn^{-d} \tag{4.A.10}$$

$$|\Phi_1(Q)_{\mathrm{e}}|_2 = |\Phi_1(Q)_{\mathrm{o}}|_2 , \qquad |\Phi_1(Q)_{\mathrm{e}}|_1, |\Phi_1(Q)_{\mathrm{o}}|_1 = 1/2 + O(n^{-2}) \tag{4.A.11}$$

$$\lambda_1(Q_{\mathrm{e}}^2) - \lambda_2(Q_{\mathrm{e}}^2) = \lambda_1(Q_{\mathrm{o}}^2) - \lambda_2(Q_{\mathrm{o}}^2) \geq cn^{-2} . \tag{4.A.12}$$

For $x \in B(0, (1 - 2n^{-c})n)$, we have

$$\Phi_1(Q)(x) \geq cn^{-d-1} \cdot \mathrm{dist}(x, B(0,n)^c) . \tag{4.A.13}$$

Proof. First, (4.A.4) gives $\lambda_1(Q) \geq 1 - Cn^{-2}$. Combining with Lemma 4.A.1, we get (4.A.10). Also, by (4.A.1), we know that $\lambda_1(Q) - \lambda_2(Q) \geq cn^{-2}$. Hence (4.A.11) follows from (4.A.7) and (4.A.12) follows from (4.A.5).

Next, we verify (4.A.13). We first see that $|\Phi_1(Q)|_\infty \leq Cn^{-d}$ yields that for some constant $c' > 0$,

$$\sum_{x \in B(0,(1-c')n)} \Phi_1(Q)(x) \geq 1/2.$$

Then the proof of Lemma 4.5.3 also works here. □

4.B Comparison of eigenvalues on nested domains

In this section, we derive upper and lower bounds on the eigenvalue decrement after we remove a subset from the domain in Lemmas 4.B.1 and 4.B.2, respectively. In particular, we are interested in the case when the subset being removed is very close to the boundary as in Lemma 4.B.2. More precisely, we show the following, where

$$\partial A := \{x \in A^c : |x-y|_1 = 1 \text{ for some } y \in A\}\,.$$

Lemma 4.B.1. *Let $D_2 \subset D_1 \subset \mathbb{Z}^d$ and $q = \sum_{x\in(D_2\cup\partial D_2)} \Phi^2_{D_1}(x)/|\Phi_{D_1}|_2^2$. Then*

$$\lambda_{D_1} - \lambda_{D_1\setminus D_2} \le \frac{2q}{1-q}\,. \tag{4.B.1}$$

Proof. Let $\tilde\Phi_{D_1}(v) = \Phi_{D_1}(v)\mathbb{1}_{v\notin D_2}$, then $\tilde\Phi_{D_1}$ is supported on $D_1 \setminus D_2$ and

$$|\tilde\Phi_{D_1}|_2^2 \ge |\Phi_{D_1}|_2^2(1-q)\,.$$

For any adjacent $x, y \in \mathbb{Z}^d$ such that $(x,y) \notin (\partial D_2^c \times \partial D_2) \cup (\partial D_2 \times \partial D_2^c)$, we have

$$(\tilde\Phi_{D_1}(x) - \tilde\Phi_{D_1}(y))^2 \le (\Phi_{D_1}(x) - \Phi_{D_1}(y))^2\,.$$

Hence,

$$\frac{1}{4d}\sum_{(x,y):|x-y|_1=1}\left[(\tilde\Phi_{D_1}(x) - \tilde\Phi_{D_1}(y))^2 - (\Phi_{D_1}(x) - \Phi_{D_1}(y))^2\right] \le \sum_{x\in\partial D_2}\Phi^2_{D_1}(x) \le q|\Phi_{D_1}|_2^2\,.$$

Recall that

$$1-\lambda_A = \min\Big\{\frac{1}{4d}\sum_{x\sim y}(g(x)-g(y))^2 : |g|_2^2 = 1, g(x)=0\ \forall x\notin A\Big\} \quad \forall A\subseteq \mathbb{Z}^d,$$

where $\Phi_A/|\Phi_A|_2$ is the minimizer. Therefore, we have

$$\begin{aligned}1-\lambda_{D_1\setminus D_2} \le & \frac{\frac{1}{4d}\sum_{x\sim y}(\tilde\Phi_{D_1}(x)-\tilde\Phi_{D_1}(y))^2}{|\tilde\Phi_{D_1}|_2^2} \le \frac{\frac{1}{4d}\sum_{x\sim y}(\Phi_{D_1}(x)-\Phi_{D_1}(y))^2 + q|\Phi_{D_1}|_2^2}{|\Phi_{D_1}|_2^2(1-q)}\\ &= \frac{1-\lambda_{D_1}+q}{1-q} = 1-\lambda_{D_1} + \frac{2q-q\lambda_{D_1}}{1-q}.\end{aligned}$$

Since $\lambda_{D_1}\ge 0$, this yields the desired result. □

Lemma 4.B.2. *Let $B_{R_1}, B_{R_2}, B_{R_3}$ be three concentric balls whose radii $R_1 > R_2 > R_3$ are sufficiently large, and let $\mathcal{B}$, $\mathcal{B}_\circ$ be subsets of $\mathbb{Z}^d$. Suppose $B_{R_1}\subset\mathcal{B}$; and suppose that $\mathcal{B}_\circ$ can be obtained from $\mathcal{B}$ by removing some points in B_{R_2}, that is,*

$$\mathcal{B}_\circ\subset\mathcal{B} \quad \text{and} \quad \mathcal{B}\setminus\mathcal{B}_\circ\subset B_{R_2}.$$

In addition, we assume that $b_1, b_2 > 0$ satisfy

$$\sum_{x\in B_{R_3}}\Phi_{\mathcal{B}}(x), \sum_{x\in B_{R_3}}\Phi_{\mathcal{B}_\circ}(x)\ge b_1, \tag{4.B.2}$$

$$\text{and} \quad \lambda_{\mathcal{B}_\circ}\ge 1-b_2R_1^{-2}. \tag{4.B.3}$$

Then there exist constants $c_b = c_b(b_1,b_2,d) > 0$ and $C_d = C_d(d) > 0$ such that

$$\lambda_{\mathcal{B}}-\lambda_{\mathcal{B}_\circ}\ge\frac{c_b}{R_1^d(\log R_1)^{\mathbb{1}_{d=2}}}\Big(1-\frac{R_2}{R_1}\Big)^{C_d}|\mathcal{B}\setminus\mathcal{B}_\circ|^{(d-2)/d}. \tag{4.B.4}$$

Proof. We first see that by $\mathcal{B}_\circ \subset \mathcal{B}$,

$$(\lambda_{\mathcal{B}} - \lambda_{\mathcal{B}_\circ})\langle \Phi_{\mathcal{B}}, \Phi_{\mathcal{B}_\circ}\rangle = \langle \Phi_{\mathcal{B}}, (P|_{\mathcal{B}} - P|_{\mathcal{B}_\circ})\Phi_{\mathcal{B}_\circ}\rangle$$
$$= \frac{1}{2d} \sum_{x\sim y, x\in \mathcal{B}\setminus\mathcal{B}_\circ} \Phi_{\mathcal{B}}(x)\Phi_{\mathcal{B}_\circ}(y)\,.$$

Since $\mathcal{B} \setminus \mathcal{B}_\circ \subset B_{R_2}$, it follows that

$$\lambda_{\mathcal{B}} - \lambda_{\mathcal{B}_\circ} \geq \frac{\min_{x\in B_{R_2}} \Phi_{\mathcal{B}}(x)}{2d\,|\Phi_{\mathcal{B}}|_2|\Phi_{\mathcal{B}_\circ}|_2} \sum_{y\in\partial(\mathcal{B}\setminus\mathcal{B}_\circ)} \Phi_{\mathcal{B}_\circ}(y)\,. \tag{4.B.5}$$

First we give a lower bound on $\Phi_{\mathcal{B}}$ on B_{R_2}. Note that for all $x \in B_{R_2}$ and $y \in B_{R_3}$, by [53, Proposition 6.9.4] and taking into account the periodicity of the random walk, we have

$$p^{B_{R_1}}_{R_1^2}(y,x) + p^{B_{R_1}}_{R_1^2+1}(y,x) \geq \frac{c}{R_1^d}\Big(1 - \frac{R_2}{R_1}\Big)\Big(1 - \frac{R_3}{R_1}\Big)\,.$$

Combined with (4.B.2) and $\lambda_{\mathcal{B}} \leq 1$, it yields that for all $x \in B_{R_2}$

$$\begin{aligned}
\Phi_{\mathcal{B}}(x) &= \frac{1}{2}\sum_{i=R_1^2, R_1^2+1} \lambda_{\mathcal{B}}^{-i} \sum_{y\in\mathcal{B}} \Phi_{\mathcal{B}}(y)\mathbf{P}^y(S_i = x, \tau_{\mathcal{B}^c} > i)\\
&\geq \frac{1}{2}\sum_{y\in B_{R_3}} \Phi_{\mathcal{B}}(y)\cdot \frac{c}{R_1^d}\Big(1-\frac{R_2}{R_1}\Big)\Big(1-\frac{R_3}{R_1}\Big)\\
&\geq \frac{cb_1}{2R_1^d}\Big(1-\frac{R_2}{R_1}\Big)^2\,.
\end{aligned} \tag{4.B.6}$$

On the other hand, combining (4.B.3) and Lemma 4.A.1 yields

$$|\Phi_{\mathcal{B}}|_2^2 \leq C b_2^{d/2} R_1^{-d}, \quad |\Phi_{\mathcal{B}_\circ}|_2^2 \leq C b_2^{d/2} R_1^{-d}\,. \tag{4.B.7}$$

Combining (4.B.5), (4.B.6), and (4.B.7), we see that to prove (4.B.4), it suffices to prove

$$\sum_{x\in\partial(\mathcal{B}\setminus\mathcal{B}_\circ)} \Phi_{\mathcal{B}_\circ}(x) \geq \frac{c_b}{R_1^d(\log R_1)^{\mathbb{1}_{d=2}}}\Big(1-\frac{R_2}{R_1}\Big)^{C_d}|\mathcal{B}\setminus\mathcal{B}_\circ|^{(d-2)/d}\,, \tag{4.B.8}$$

where $c_b = c_b(b_1, b_2, d)$ and $C_d = C_d(d)$ are positive constants, with C_d to be chosen later in (4.B.15).

To verify (4.B.8), we first consider the case

$$\sum_{x\in\mathcal{B}_\circ} \Phi_{\mathcal{B}_\circ}(x)\mathbf{P}^x(S_{\tau_{\mathcal{B}_\circ^c}} \in \mathcal{B}\setminus\mathcal{B}_\circ) > \frac{b_1}{2}\Big(1-\frac{R_2}{R_1}\Big)^{C_d}. \tag{4.B.9}$$

Note that

$$\begin{aligned}\sum_{x\in\mathcal{B}_\circ} \Phi_{\mathcal{B}_\circ}(x)\mathbf{P}^x(S_{\tau_{\mathcal{B}_\circ^c}} \in \mathcal{B}\setminus\mathcal{B}_\circ) &\leq \sum_{i\geq 0}\sum_{x\in\mathcal{B}_\circ} \Phi_{\mathcal{B}_\circ}(x)\mathbf{P}^x(\tau_{\mathcal{B}_\circ^c} > i, S_i \in \partial(\mathcal{B}\setminus\mathcal{B}_\circ)) \\ &= \sum_{i\geq 0}\lambda_{\mathcal{B}_\circ}^i \sum_{x\in\partial(\mathcal{B}\setminus\mathcal{B}_\circ)} \Phi_{\mathcal{B}_\circ}(x) \\ &= \frac{1}{1-\lambda_{\mathcal{B}_\circ}} \sum_{x\in\partial(\mathcal{B}\setminus\mathcal{B}_\circ)} \Phi_{\mathcal{B}_\circ}(x)\,.\end{aligned} \tag{4.B.10}$$

On the other hand, (4.B.2) implies $|\Phi_{\mathcal{B}_\circ}|_\infty \geq cb_1R_1^{-d}$. Then Lemma 4.A.1 implies

$$\lambda_{\mathcal{B}_\circ} \leq 1 - cb_1^{2/d}R_1^{-2}\,.$$

Substituting this into (4.B.10) and using the assumption that (4.B.9) and the fact that $|\mathcal{B}\setminus\mathcal{B}^\circ| \leq (2R_1)^d$, we obtain (4.B.8).

Now we consider the other case

$$\sum_{x\in\mathcal{B}_\circ} \Phi_{\mathcal{B}_\circ}(x)\mathbf{P}^x(S_{\tau_{\mathcal{B}_\circ^c}} \in \mathcal{B}\setminus\mathcal{B}_\circ) \leq \frac{b_1}{2}\Big(1-\frac{R_2}{R_1}\Big)^{C_d}, \tag{4.B.11}$$

in which case the probability of exiting $\mathcal{B}_\circ$ via $\mathcal{B}\setminus\mathcal{B}_\circ \subset B_{R_2}$ is small. Heuristically, this allows us to approximate the random walk in $\mathcal{B}_\circ$ by the walk in $\mathcal{B}$, and then we are able to get good control on the Green's function which relates to the eigenfunction $\Phi_{\mathcal{B}_\circ}$ as follows. For any $x \in \mathcal{B}_\circ$, by the eigenvalue equation for the resolvent, we have

$$\Phi_{\mathcal{B}_\circ}(x) = (1 - \lambda_{\mathcal{B}_\circ}) \sum_v \Phi_{\mathcal{B}_\circ}(v) G_{\mathcal{B}_\circ}(v, x)\,, \tag{4.B.12}$$

where

$$G_{\mathcal{B}_\circ}(v, x) := \sum_{t=0}^{\infty} \mathbb{P}^v(S_t = x, \tau_{\mathcal{B}_\circ^c} \geq t)\,. \tag{4.B.13}$$

We will get a lower bound on $\Phi_{\mathcal{B}_\circ}(x)$ by restricting the sum over v in (4.B.12) to the annulus

$$\mathcal{A} := B_{R_1-(R_1-R_2)/3} \setminus B_{R_1-2(R_1-R_2)/3} \subset \mathcal{B}_\circ\,.$$

For all $v \in \mathcal{A}$ and $u \in B_{R_3}$,

$$G_{\mathcal{B}_\circ}(u, v) \geq G_{\mathcal{B}}(u, v) - \mathbf{P}^u(S_{\tau_{\mathcal{B}_\circ^c}} \in \mathcal{B} \setminus \mathcal{B}^\circ) \cdot \max_{y \in \mathcal{B} \setminus \mathcal{B}_\circ} G_{\mathcal{B}}(y, v)\,. \tag{4.B.14}$$

Since the function $x \mapsto G_{\mathcal{B}}(x, v)$ is harmonic on $B_{R_1-2(R_1-R_2)/3}$, we can use the Harnack inequality and a standard chaining argument as in [23, (4.62)] to obtain that for a constant C_d depending only on d,

$$\max_{y \in B_{R_2}} \frac{G_{\mathcal{B}}(y, v)}{G_{\mathcal{B}}(u, v)} \leq C_d \Big(1 - \frac{R_2}{R_1}\Big)^{-C_d}\,. \tag{4.B.15}$$

By the strong Markov property at time τ_v and [53, Proposition 6.9.4], we get

$$G_{\mathcal{B}}(u, v) \geq G_{B_{R_1}}(u, v) \geq c\Big(1 - \frac{R_2}{R_1}\Big)\Big(1 - \frac{R_3}{R_1}\Big) R_1^{-d+2}\,. \tag{4.B.16}$$

Combining (4.B.14), (4.B.15), and (4.B.16) yields

$$G_{\mathcal{B}_\circ}(u,v) \geq c\Big(1-\frac{R_2}{R_1}\Big)^2 R_1^{-d+2}\Big[1-\Big(1-\frac{R_2}{R_1}\Big)^{-C_d}\mathbf{P}^u(\tau_{\mathcal{B}_\circ^c}\in\mathcal{B}\setminus\mathcal{B}^\circ)\Big]\,.$$

Combining with (4.B.2) and (4.B.11), we get for all $v\in\mathcal{A}$ and $u\in B_{R_3}$,

$$\begin{aligned}&\sum_{u\in B_{R_3}}\Phi_{\mathcal{B}_\circ}(u)G_{\mathcal{B}_\circ}(u,v)\\ \geq& c\Big(1-\frac{R_2}{R_1}\Big)^2 R_1^{-d+2}\Big(\sum_{u\in B_{R_3}}\Phi_{\mathcal{B}_\circ}(u)-\Big(1-\frac{R_2}{R_1}\Big)^{-C_d}\sum_{u\in B_{R_3}}\Phi_{\mathcal{B}_\circ}(u)\mathbf{P}^u(\tau_{\mathcal{B}_\circ^c}\in\mathcal{B}\setminus\mathcal{B}^\circ)\Big)\\ \geq& cb_1\Big(1-\frac{R_2}{R_1}\Big)^2 R_1^{-d+2}\,.\end{aligned}$$

Therefore, by (4.B.12) and (4.B.3) we get for all $v\in\mathcal{A}$,

$$\Phi_{\mathcal{B}_\circ}(v)\geq(1-\lambda_{\mathcal{B}_\circ})\sum_{u\in B_{R_3}}\Phi_{\mathcal{B}_\circ}(u)G_{\mathcal{B}_\circ}(u,v)\geq cb_1b_2\Big(1-\frac{R_2}{R_1}\Big)^2R_1^{-d}\,.$$

Then by (4.B.12) and (4.B.3) again (summing over $v\in\mathcal{A}$), we get for $x\in\mathcal{B}_\circ$,

$$\begin{aligned}\Phi_{\mathcal{B}_\circ}(x)&\geq c(1-\lambda_{\mathcal{B}_\circ})\cdot b_1b_2\Big(1-\frac{R_2}{R_1}\Big)^2R_1^{-d}\sum_{v\in\mathcal{A}}G_{\mathcal{B}_\circ}(v,x)\\ &\geq cb_1b_2^2\Big(1-\frac{R_2}{R_1}\Big)^2R_1^{-d-2}\mathbf{E}^x\big[|\{1\leq i\leq\tau_{\mathcal{B}_\circ^c}:S_i\in\mathcal{A}\}|;\tau_{\mathcal{B}\setminus\mathcal{B}_\circ}>\tau_{B_{R_1}^c}\big]\,.\end{aligned}$$

Conditioned on $\tau_{\mathcal{B}\setminus\mathcal{B}_\circ}>\tau_{B_{R_1}^c}$, the random walk must cross $\mathcal{A}$. Uniformly in starting and ending points, the first crossing of $\mathcal{A}$ has length at least $c(R_1-R_2)^2$ with positive probability. Hence, we get

$$\Phi_{\mathcal{B}_\circ}(x)\geq cb_1b_2^2\Big(1-\frac{R_2}{R_1}\Big)^4R_1^{-d}\mathbf{P}^x(\tau_{\mathcal{B}\setminus\mathcal{B}_\circ}>\tau_{B_{R_1}^c})\,. \tag{4.B.17}$$

For $d\geq3$, summing over $x\in\partial(\mathcal{B}\setminus\mathcal{B}_\circ)$ in (4.B.17) gives

$$\sum_{x\in\partial(\mathcal{B}\setminus\mathcal{B}_\circ)}\Phi_{\mathcal{B}_\circ}(x)\geq c\Big(1-\frac{R_2}{R_1}\Big)^4R_1^{-d}\mathrm{cap}(\mathcal{B}\setminus\mathcal{B}_\circ)\,, \tag{4.B.18}$$

where for $d \geq 3$,

$$\mathrm{cap}(\mathcal{B} \setminus \mathcal{B}_\circ) := \sum_{x \in \mathcal{B} \setminus \mathcal{B}_\circ} \mathbf{P}^x(S_t \notin \mathcal{B} \setminus \mathcal{B}_\circ, \text{for all } t \geq 1) . \tag{4.B.19}$$

Combining with $\mathrm{cap}(\mathcal{B} \setminus \mathcal{B}_\circ) \geq c|\mathcal{B} \setminus \mathcal{B}_\circ|^{(d-2)/d}$ (see the proof of [52, Proposition 2.5.1]) yields (4.B.8).

For $d = 2$, fix an arbitrary $z \in \mathcal{B} \setminus \mathcal{B}_\circ$. By decomposing a random walk path that starts from z and exists B_{R_1} before returning to z according to its last exit time from $\mathcal{B} \setminus \mathcal{B}_\circ$, we have

$$\begin{aligned} \mathbf{P}^z(\tau_z^+ > \tau_{B_{R_1}^c}) &= \sum_{x \in \partial(\mathcal{B} \setminus \mathcal{B}_\circ)} \mathbf{P}^x(S_1 \in \mathcal{B} \setminus \mathcal{B}_\circ, \tau_z < \tau_{B_{R_1}^c}) \mathbf{P}^x(\tau_{\mathcal{B} \setminus \mathcal{B}_\circ} > \tau_{B_{R_1}^c}) \\ &\leq \sum_{x \in \partial(\mathcal{B} \setminus \mathcal{B}_\circ)} \mathbf{P}^x(\tau_{\mathcal{B} \setminus \mathcal{B}_\circ} > \tau_{B_{R_1}^c}) , \end{aligned}$$

where $\tau_z^+ := \inf\{t \geq 1 : S_t = z\}$. Combining with (4.B.17) and [53, Proposition 6.4.3], which implies that for R_1 sufficiently large,

$$\mathbf{P}^z(\tau_z^+ > \tau_{B_{R_1}^c}) \geq c(\log R_1)^{-1} , \tag{4.B.20}$$

we get (4.B.8). We thus complete the proof of (4.B.4). □

4.C An isoperimetric inequality

The following isoperimetric inequality is needed in the proof of Lemma 4.4.3. It says that if we partition a ball in $\mathbb{Z}^d$ into two parts, then the area of the interface between the two parts can be bounded from below as a function of the volume of the smaller part.

Lemma 4.C.1. *Fix an arbitrary $R \geq 1$. Let $\mathbf{B}(0, R) = \{x \in \mathbb{R}^d : |x|_2 \leq R\}$, and let*

$B := \mathbf{B}(0,R) \cap \mathbb{Z}^d$. Suppose $B = A_1 \cup A_2$ is a partition of B. Then

$$\min(|\partial A_1 \cap A_2|, |\partial A_2 \cap A_1|) \geq c \min(|A_1|, |A_2|)^{1-1/d}, \tag{4.C.1}$$

where $c > 0$ is a constant depending only on d and $\partial A_i := \{x \in A_i^c : |x-y|_1 = 1$ for some $y \in A_i\}$ for $i = 1, 2$.

Proof. For any $x \in \mathbb{Z}^d$, let $x^* := [x - 1/2, x + 1/2]^d \cap \mathbf{B}(0,R)$. Then there exist constants $c, C > 0$ depending only on d such that for all $x \in B$,

$$\mathrm{vol}(x^*) \geq c \quad \text{and} \quad \mathrm{suf}(x^*) \leq C,$$

where $\mathrm{vol}(x^*)$ and $\mathrm{suf}(x^*)$ are volume and surface area of x^*, respectively. For any set $U \subset \mathbb{Z}^d$, we denote $U^* := \bigcup_{x \in U} x^*$. If we denote

$$q := \min\{\mathrm{vol}(A_1), \mathrm{vol}(A_2)\},$$

then

$$q^* := \min\{\mathrm{vol}(A_1^*), \mathrm{vol}(A_2^*)\} \geq cq.$$

By the isoperimetric inequality in $\mathbb{R}^d$,

$$\mathrm{suf}(A_1^*) + \mathrm{suf}(A_2^*) \geq d\mathrm{vol}(\mathbf{B}(0,1))^{1/d} \cdot \left[\mathrm{vol}(A_1^*)^{1-1/d} + \mathrm{vol}(A_2^*)^{1-1/d}\right].$$

Since for any $\alpha \in (0,1)$, the function $x^\alpha - (x+1)^\alpha$ is increasing in $x \in (0, +\infty)$, we have

$$x^{1-1/d} + y^{1-1/d} \geq (x+y)^{1-1/d} + (2 - 2^{1-1/d})y^{1-1/d} \text{ for all } 0 \leq y \leq x.$$

Therefore,

$$\begin{aligned}\mathrm{suf}(A_1^*) + \mathrm{suf}(A_2^*) &\geq d\mathrm{vol}(\mathbf{B}(0,1))^{1/d}\big[\mathrm{vol}(B^*)^{1-1/d} + (2-2^{1-1/d})q^{*1-1/d}\big] \\ &\geq \mathrm{suf}(B^*) + d\mathrm{vol}(\mathbf{B}(0,1))^{1/d}(2-2^{1-1/d})(cq)^{1-1/d}\,.\end{aligned}$$

Note that the interface between A_1^* and A_2^* is contained in both the surface of $(\partial A_1 \cap B)^*$ and $(\partial A_1 \cap B)^*$, and it is counted exactly twice in $\mathrm{suf}(A_1^*) + \mathrm{suf}(A_2^*) - \mathrm{suf}(B^*)$. It follows that

$$\begin{aligned}\mathrm{suf}(A_1^*) + \mathrm{suf}(A_2^*) - \mathrm{suf}(B^*) &\leq 2\min\Big\{\mathrm{suf}\big((\partial A_1 \cap B)^*\big), \mathrm{suf}\big((\partial A_2 \cap B)^*\big)\Big\} \\ &\leq C\min\big(|\partial A_1 \cap B|, |\partial A_2 \cap B|\big)\,.\end{aligned}$$

Combining the previous two inequalities completes the proof of (4.C.1). $\square$

CHAPTER 5

GEOMETRY OF THE RANDOM WALK RANGE UNDER THE ANNEALED LAW

5.1 Introduction

Let $S := (S_n)_{n\geq 0}$ be a discrete time simple symmetric random walk on $\mathbb{Z}^d$. We will use $\mathbf{P}_x$ and $\mathbf{E}_x$ to denote probability and expectation for S with $S_0 = x \in \mathbb{Z}^d$, and we will omit the subscript x when $x = 0$. Independently for each $x \in \mathbb{Z}^d$, an obstacle is placed at x with probability $1 - p$ for some fixed $p \in (0, 1)$, which generates the so-called *Bernoulli obstacle* configuration and plays the role of a random environment. Probability and expectation for the obstacles will be denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively. Let $\mathcal{O}$ denote the set of sites occupied by obstacles. When there is no obstacle at the site $x \in \mathbb{Z}^d$, we will say x is open. The random walk is killed at the moment it hits an obstacle (called hard obstacles), namely, at the stopping time

$$\tau_{\mathcal{O}} := \min\{n \geq 0 : S_n \in \mathcal{O}\}. \tag{5.1.1}$$

More generally, we will use τ_A to denote the first hitting time of a set $A \subset \mathbb{Z}^d$. We will write $\mathbf{E}[f(S)\colon A] = \mathbf{E}[f(S)1_A]$ and $\mathbb{E}[g(\mathcal{O})\colon B] = \mathbb{E}[g(\mathcal{O})1_B]$.

We are interested in $\mathbb{P} \otimes \mathbf{P}((S, \mathcal{O}) \in \cdot \mid \tau_{\mathcal{O}} > N)$, the so-called *annealed law* of $(S, \mathcal{O})$ conditioned on the random walk's survival up to time N. For simplicity, we will denote

$$\mu_N((S, \mathcal{O}) \in \cdot) := \mathbb{P} \otimes \mathbf{P}((S, \mathcal{O}) \in \cdot \mid \tau_{\mathcal{O}} > N). \tag{5.1.2}$$

In particular, we are interested in the law of the random walk range

$$S_{[0,N]} := \{S_i : 0 \leq i \leq N\} \tag{5.1.3}$$

under the conditioned measure μ_N. It is worth noting that the marginal law of μ_N for the random walk admits a representation in terms of the range of the random walk:

$$\mu_N(S \in \cdot) = \frac{\mathbf{E}\left[p^{|S_{[0,N]}|} \colon S \in \cdot\right]}{\mathbf{E}\left[p^{|S_{[0,N]}|}\right]}, \tag{5.1.4}$$

where $|S_{[0,N]}|$ denotes the cardinality of the set $S_{[0,N]}$.

Let us review known results on this model. The first result dates back to Donsker–Varadhan's work [27] which determined the leading order asymptotics of the denominator in (5.1.4), which can be regarded as the "partition function" of a self-attracting polymer model. The main result of [27] reads as

$$\begin{aligned}\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) &= \mathbf{E}\left[p^{|S_{[0,N]}|}\right] \\ &= \exp\left\{-c(d,p) N^{\frac{d}{d+2}}(1+o(1))\right\}, \\ \text{with } c(d,p) :&= \frac{d+2}{2}\left(\log(1/p)\right)^{\frac{2}{d+2}}\left(\frac{2\lambda_1}{d}\right)^{\frac{d}{d+2}},\end{aligned} \tag{5.1.5}$$

where λ_1 is the principal Dirichlet eigenvalue of $-\frac{1}{2d}\Delta$ in the ball of unit volume in $\mathbb{R}^d$ centered at the origin.

The argument of Donsker–Varadhan indicates that the dominant contribution to the partition function comes from the strategy of finding a ball of optimal radius

$$\varrho_N := \left(\frac{2\lambda_1}{d\log(1/p)}\right)^{\frac{1}{d+2}} N^{\frac{1}{d+2}}, \tag{5.1.6}$$

which is free of obstacles and the random walk is confined in that ball up to time N. It has been proved later that this is what happens under the annealed measure in [67] and [12] for $d = 2$ and [63] for $d \geq 3$:

Theorem E (Confinement)**.** *For any $d \geq 2$, there exists $\epsilon_1 \in (0,1)$ and $\mathcal{X}_N \in \mathbb{Z}^d$ depending only on the obstacle configuration $\mathcal{O}$, such that $\mathcal{X}_N \in B(0, \varrho_N)$, the ball of radius ϱ_N centered* *at* 0*, and*

$$\lim_{N\to\infty} \mu_N\big(S_{[0,N]} \subset B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1})\big) = 1. \tag{5.1.7}$$

The law of $\varrho_N^{-1}\boldsymbol{x}_N$ *converges to* $\phi_{B(0,1)}dx$ *as* $N \to \infty$*, where* $\phi_{B(0,1)}$ *is the* L^1*-normalized principal Dirichlet eigenfunction of* $-\frac{1}{2d}\Delta$ *in* $B(0,1)$*. Furthermore, for* $d = 2$ *and for any* $\epsilon \in (0,1)$*,*

$$\lim_{N\to\infty} \mu_N\big(B(\boldsymbol{x}_N, (1-\epsilon)\varrho_N) \subset S_{[0,N]}\big) = 1. \tag{5.1.8}$$

It remains open to show that (5.1.8) also holds for dimensions $d \geq 3$, that the random walk range covers a full ball with radius almost ϱ_N (see [12, Conjecture 1.3]). Our first main result resolves this question.

Theorem 5.1.1 (Ball covering)**.** *Let* $d \geq 2$*, and let* ϱ_N *and* $\boldsymbol{x}_N$ *be as in* (5.1.6) *and Theorem E, respectively. Then there exists* $\epsilon_2 \in (0,1)$*, such that*

$$\lim_{N\to\infty} \mu_N\left(B(\boldsymbol{x}_N, \varrho_N - \varrho_N^{\epsilon_2}) \subset S_{[0,N]}\right) = 1. \tag{5.1.9}$$

Remark 5.1.2. This theorem extends and refines (5.1.8) for general $d \geq 2$. In fact, we will first prove the extension of (5.1.8) to $d \geq 3$ as an intermediate step to the above refined result. The interested reader may jump to Section 5.3 after reading Subsections 5.2.1 and 5.2.2.

We proceed to the second main result of this chapter, which is about the boundary of the range of the random walk under the annealed law. For any set $A \subset \mathbb{Z}^d$, we define its external boundary by

$$\partial A := \{y \in \mathbb{Z}^d \setminus A\colon \|y - x\| = 1 \text{ for some } x \in A\}, \tag{5.1.10}$$

where $\|\cdot\|$ denotes the Euclidean norm. Theorem E and Theorem 5.1.1 together imply that, conditioned on survival up to time N, the rescaled boundary of the random walk range, $\varrho_N^{-1}\partial S_{[0,N]}$, converges in probability to a unit sphere as N tends to infinity, and $\partial S_{[0,N]}$ fluctuates on a scale of at most ϱ_N^{ϵ} with $\epsilon = \max\{\epsilon_1, \epsilon_2\} \in (0,1)$. Identifying the precise scale of fluctuation is an extremely interesting, but also challenging question. The following theorem is a step in this direction, which bounds the size of $\partial S_{[0,N]}$.

Theorem 5.1.3 (Boundary size). *Let $d \geq 2$, and let ϱ_N be defined as in* (5.1.6). *Then there exists $a > 0$, such that*

$$\lim_{N\to\infty} \mu_N\big(|\partial S_{[0,N]}| \leq \varrho_N^{d-1}(\log \varrho_N)^a\big) = 1. \tag{5.1.11}$$

Remark 5.1.4. Our proofs of Theorems 5.1.1 and 5.1.3 assume Theorem E as an input. Strictly speaking, Theorem E has only been proved in the continuum setting for $d \geq 3$ in [63] using the method of enlargement of obstacles. We briefly explain how the argument can be adapted to the discrete setting in Appendix 5.B. In fact, it is possible to prove Theorems 5.1.1 and Theorem E together. In the follow-up paper [21], we present such an argument and further derive an extension of Theorem E for a random walk with small bias conditioned to avoid Bernoulli obstacles.

Remark 5.1.5. After our paper [23] was submitted for publication, Berestycki and Cerf announced an independent work [10], where Theorem 5.1.1 is proved by a different method. In addition, [10, Theorem 1.5] proves a quantitative control on the random walk local time, which together with Theorem 5.1.1 makes it possible to prove Theorem E following the strategy of [12]. For more detail, we refer the reader to the introduction of [10].

In what follows, we will use c, c', C, C' to denote generic constants depending only on d and p, whose values may change from line to line. For $G \subset \mathbb{R}^d$, we write $|G|$ for the number of points in $G \cap \mathbb{Z}^d$ and $\mathrm{vol}(G)$ for the Euclidean volume. A list of frequently used notation is compiled in Appendix 5.C.

5.2 Proof Outline

In this section, we list the main ingredients needed and outline the proof structure. An overview of how the rest of the chapter is organized will be given at the end of the section.

In what follows, many statements are supposed to hold with μ_N-probability tending to one as N tends to infinity, but we often make it implicit for brevity.

5.2.1 Path and Environment Switching

An argument that will be used repeatedly in our proof is path and environment switching. More precisely, if A_1, A_2 are two sets of random walk path configurations, and E_1, E_2 are two sets of obstacle configurations, then we can switch from (A_1, E_1) to (A_2, E_2) and bound

$$\begin{aligned}
&\mu_N((S, \mathcal{O}) \in (A_1, E_1)) \\
&\quad \leq \frac{\mathbb{P} \otimes \mathbf{P}(S \in A_1, \mathcal{O} \in E_1, \tau_{\mathcal{O}} > N)}{\mathbb{P} \otimes \mathbf{P}(S \in A_2, \mathcal{O} \in E_2, \tau_{\mathcal{O}} > N)} \\
&\quad = \frac{\mathbb{P}(\mathcal{O} \in E_1)}{\mathbb{P}(\mathcal{O} \in E_2)} \cdot \frac{\mathbb{E}[\mathbf{P}(S \in A_1, \tau_{\mathcal{O}} > N) \mid \mathcal{O} \in E_1]}{\mathbb{E}[\mathbf{P}(S \in A_2, \tau_{\mathcal{O}} > N) \mid \mathcal{O} \in E_2]}.
\end{aligned} \tag{5.2.1}$$

The first factor determines the probability gain or cost in the environment when we switch from obstacle configurations in E_1 to E_2, while the second factor determines the gain or cost in the random walk when we switch from paths in A_1 to A_2. We will find suitable choices of A_2 and E_2 so that the gain in one factor will beat the cost in the other.

One way to bound the second factor in (5.2.1) is to find a coupling between two obstacle configurations $(\mathcal{O}_1, \mathcal{O}_2)$ with marginal distributions $\mathbb{P}(\cdot \mid \mathcal{O} \in E_1)$ and $\mathbb{P}(\cdot \mid \mathcal{O} \in E_2)$, and then bound $\mathbf{P}(S \in A_1, \tau_{\mathcal{O}_1} > N)/\mathbf{P}(S \in A_2, \tau_{\mathcal{O}_2} > N)$ uniformly with respect to $(\mathcal{O}_1, \mathcal{O}_2)$. This is possible because typically, A_2 and E_2 will be constructed by local modifications of paths in A_1 and obstacle configurations in E_1, respectively.

This type of comparison argument is much more useful in the study of the conditional measure μ_N than a direct analysis, since we only have the crude leading order asymptotics on the partition function in (5.1.5).

5.2.2 Proof Outline for the Weaker Version of Ball Covering Theorem

We first prove (5.1.8) for general $d \geq 2$, which will play an important role in the proof of Theorems 5.1.1 and 5.1.3. The key step is to show that if $x \in \mathcal{O}$, then there is a positive fraction of closed sites in its neighborhood.

Lemma 5.2.1 (Density of obstacles). *For each $x \in \mathbb{Z}^d$, $l > 0$, and $\delta > 0$, let*

$$E_l^\delta(x) := \left\{ x \in \mathcal{O} \text{ and } \frac{|\mathcal{O} \cap B(x,l)|}{|B(x,l)|} < \delta \right\}. \tag{5.2.2}$$

Then there exists $\delta > 0$, such that

$$\mu_N \left(\bigcup_{x \in B(0, 2\varrho_N)} \bigcup_{(\log N)^3 \leq l \leq \varrho_N} E_l^\delta(x) \right) \to 0 \text{ as } N \to \infty \tag{5.2.3}$$

faster than any negative power of N.

The proof of Lemma 2.1 will be based on path and environment switching arguments. Roughly speaking, if for some $x \in \mathcal{O}$, $B(x,l)$ contains few obstacles, then: either the walk visits $B(x,l)$ many times, in which case we remove all the obstacles in $B(x,l)$ and we will show that the gain in the random walk survival probability beats the loss from environment switching; or the walk visits $B(x,l)$ rarely, in which case we switch to typical obstacle configurations in $B(x,l)$ and force the walk to avoid $B(x,l)$, and we will show that the gain in environment switching beats the loss in path switching. A more precise outline and the proof will be given in Section 5.3.

Lemma 5.2.1 implies that if there is an obstacle inside the ball $B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N)$, then the confinement ball $B(\boldsymbol{\mathcal{X}}_N, \varrho_N)$ contains order ϱ_N^d obstacles. This makes it too difficult for the random walk to survive and we can then deduce that the ball $B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N)$ is free of obstacles. More precisely, we have the following result.

Proposition 5.2.2. *Let $d \geq 2$. Then for any $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \mu_N \left(B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N) \cap \mathcal{O} = \emptyset\right) = 1. \tag{5.2.4}$$

Once we have this proposition, the covering property (5.1.8) readily follows. Indeed, if the random walk avoids a site $x \in B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N)$ with positive probability uniformly in N, then we can close that site at little cost, which contradicts Proposition 5.2.2.

5.2.3 Reduction to the Cluster of "Truly"-Open Sites

The key idea in our proof of Theorems 5.1.1 and 5.1.3 is to approximate the range of the random walk, $S_{[0,N]}$, by a set of *"truly"-open sites* $\mathcal{T}$ that depends only on the obstacle configuration $\mathcal{O}$. Unlike sites in $S_{[0,N]}$, we can easily control the environment cost of creating a "truly"-open site, which facilitates the application of the switching argument in (5.2.1).

Definition 5.2.3 ("Truly"-open sites)**.** *Given an obstacle configuration $\mathcal{O}$ and $N \in \mathbb{N}$, a site $x \in \mathbb{Z}^d$ is called "truly"-open if*

$$\mathbf{P}_x\left(\tau_{\mathcal{O}} > (\log N)^5\right) \geq \exp\left\{-(\log N)^2\right\}. \tag{5.2.5}$$

If the origin is "truly"-open, then we let $\mathcal{T}$ denote the connected component of "truly"-open sites inside $B(\boldsymbol{\mathcal{X}}_N, \varrho_N + \varrho_N^{\epsilon_1})$ containing the origin, where ϵ_1 is the constant appearing in Theorem E. Otherwise let $\mathcal{T} = \emptyset$.

Remark 5.2.4. A "truly"-open site is a site whose surrounding environment is atypically favorable for the random walk survival. If the environment is typical, then the probability in (5.2.5) would decay like $\exp\{-c(\log N)^{5+o(1)}\}$ (cf. [75, Theorem 5.1 on p. 196]). Note that whether $x \in \mathbb{Z}^d$ is "truly"-open or not depends only on the obstacle configuration in the l^1-ball of radius $(\log N)^5$ centered at x.

The following two lemmas justifies the approximation of $\partial S_{[0,N]}$ by the boundary of "truly"-open sites $\partial\mathcal{T}$.

Lemma 5.2.5. *Let $d \geq 2$. Then*

$$\lim_{N\to\infty} \mu_N \left(S_{[0,N]} \supset \left\{ x \in \mathcal{T} \colon \operatorname{dist}(x, \partial\mathcal{T}) \geq (\log N)^3 \right\} \right) = 1 \tag{5.2.6}$$

and

$$\lim_{N\to\infty} \mu_N \left(\mathcal{T} \subset \left\{ x \in \mathbb{Z}^d \colon \operatorname{dist}(x, S_{[0,N]}) \leq (\log N)^5 \right\} \right) = 1. \tag{5.2.7}$$

Lemma 5.2.6. *Let $d \geq 2$. Then*

$$\lim_{N\to\infty} \mu_N \big(S_{[0,N]} \subset \mathcal{T} \big) = 1. \tag{5.2.8}$$

Indeed, (5.2.6) and (5.2.8) imply that

$$\mu_N \left(\partial S_{[0,N]} \subset \bigcup_{x \in \partial\mathcal{T}} B(x, (\log N)^5) \right) \to 1 \tag{5.2.9}$$

and therefore, Theorems 5.1.1 and 5.1.3 follow immediately from their analogues for $\mathcal{T}$.

Theorem 5.2.7. *Let $d \geq 2$. Then there exists $\epsilon_2 \in (0,1)$ such that*

$$\lim_{N\to\infty} \mu_N \left(B(\boldsymbol{\mathcal{X}}_N, \varrho_N - \varrho_N^{\epsilon_2}) \subset \mathcal{T} \right) = 1. \tag{5.2.10}$$

Theorem 5.2.8. *Let $d \geq 2$. Then there exists $a > 0$ such that*

$$\lim_{N\to\infty} \mu_N \big(|\partial\mathcal{T}| \leq \varrho_N^{d-1} (\log \varrho_N)^a \big) = 1. \tag{5.2.11}$$

5.2.4 Proof Outline for the Cluster of "Truly"-Open Sites

In this subsection, we provide an outline for the proof of Theorems 5.2.7 and 5.2.8, assuming Lemmas 5.2.5 and 5.2.6.

Note that the random walk is confined in $\mathcal{T}$ by Lemma 5.2.6. The geometry of $\mathcal{T}$ under μ_N is then determined by an entropy-energy balance, namely, the number of possible configurations for $\partial\mathcal{T}$, vs the probability that the random walk stays confined in $\mathcal{T}$ up to time N (equivalently, the principal Dirichlet eigenvalue for the discrete Laplacian on $\mathcal{T}$). By definition, $\mathcal{T}$ is contained in the confinement ball $B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1})$ in Theorem E. On the other hand, Proposition 5.2.2 implies that for any $\epsilon > 0$, $B(\boldsymbol{x}_N, (1-\epsilon)\varrho_N)$ is a ball of "truly"-open sites. Therefore, it follows that

$$\partial\mathcal{T} \subset A(\boldsymbol{x}_N; (1-\epsilon)\varrho_N, \varrho_N + \varrho_N^{\epsilon_1}) := B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1}) \setminus \overline{B(\boldsymbol{x}_N, (1-\epsilon)\varrho_N)}. \qquad (5.2.12)$$

We bound the entropy for $\partial\mathcal{T}$ by proving the following weaker version of Theorem 5.2.8:

Proposition 5.2.9. *Let $d \geq 2$. Then for any $b > 0$,*

$$\lim_{N\to\infty} \mu_N\big(|\partial\mathcal{T}| \leq \varrho_N^{d-1+b}\big) = 1. \qquad (5.2.13)$$

We prove Proposition 5.2.9 by considering the expected number of visits to $\bigcup_{x\in\partial\mathcal{T}} B(x, (\log N)^6)$ by a random walk killed upon hitting $\mathcal{O}$. It suffices to prove that

1. the expectation of the total number of visits to $\bigcup_{x\in\partial\mathcal{T}} B(x, (\log N)^6)$ is bounded from above by $(\log N)^c$ for some $c > 0$;

2. uniformly in $x \in \partial\mathcal{T}$, the expected number of visits to $B(x, (\log N)^6)$ is bounded from below by N^{1-d+b} for any $b > 0$.

Here we consider visits to $(\log N)^6$ neighborhood of $x \in \partial\mathcal{T}$ because if the walk does not visit $B(x, (\log N)^6)$, then we can switch a "truly"-open site next to x to be not "truly"-open by only modifying the obstacle configuration inside $B(x, (\log N)^6)$. The first item above follows from the fact that the random walk will typically be killed soon after visiting $x \in \partial\mathcal{T}$. The second item is proved by the path and environment switching arguments. Roughly speaking, a site $x \in \partial\mathcal{T}$ with atypically low expected number of visits is *costly* for the random walk to visit. Thanks to the confinement of $\partial\mathcal{T}$ to the annulus in (5.2.12), $B(x, (\log N)^6)$ can be visited only by an excursion away from $B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N)$ and we can gain a lot in the random walk probability by switching such excursions to those that stay inside $B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N)$. This implies that the random walk does not visit the $(\log N)^6$ neighborhood of x. We can then gain further in the environment probability by switching a "truly"-open site next to x to be not "truly"-open, which shows that such $x \in \partial\mathcal{T}$ does not exist.

Proposition 5.2.9 provides a good enough bound on the entropy for $\partial\mathcal{T}$ to allow us to strengthen the bound on the fluctuation of $\partial\mathcal{T}$ in (5.2.12) to Theorem 5.2.7. More precisely, if $\mathcal{T}^c$ contains a point in $B(\boldsymbol{\mathcal{X}}_N, \varrho_N - \varrho_N^{\epsilon_2})$, then Lemma 5.2.1 implies that $\mathcal{T}$ differs significantly in volume from the confinement ball $B(\boldsymbol{\mathcal{X}}_N, \varrho_N + \varrho_N^{\epsilon_1})$. Recalling that $\mathcal{T} \subset B(\boldsymbol{\mathcal{X}}_N, \varrho_N + \varrho_N^{\epsilon_1})$ by definition, we can then use the Faber–Krahn inequality to show that the principal Dirichlet eigenvalue on $\mathcal{T}$ deviates so much from that of $B(\boldsymbol{\mathcal{X}}_N, \varrho_N + \varrho_N^{\epsilon_1})$ that the loss in survival probability dominates the entropy for $\partial\mathcal{T}$.

Using Theorem 5.2.7 on the fluctuation of $\partial\mathcal{T}$ as an input in place of the weaker Proposition 5.2.2, we then repeat the proof of Proposition 5.2.9. Now that the excursions of random walk visiting $\partial\mathcal{T}$ are smaller, the switching argument becomes more efficient and we obtain

$2'$. uniformly in $x \in \partial\mathcal{T}$, the expected number of visits to $B(x, (\log N)^6)$ is bounded from below by $N^{1-d}(\log N)^{-c'}$ for some $c' > 0$.

Combining this with the first item above, we obtain Theorem 5.2.8.

Organization of this chapter. The rest of this chapter is organized as follows. Section 5.3 is devoted to the proofs of Proposition 5.2.2 and (5.1.8) for general $d \geq 2$. In Section 5.4, we first prove Lemma 5.2.5 with an additional property for "truly"-open sites, and then prove

Lemma 5.2.6 and derive Proposition 5.2.9 from Proposition 5.2.2. We will in fact formulate a lemma (Lemma 5.4.3) which unifies the derivation of Proposition 5.2.9 from Proposition 5.2.2 and the derivation of Theorem 5.2.8 from Theorem 5.2.7. Lastly, in Section 5.5, we conclude with the proof of Theorem 5.2.7. In Appendix 5.A, we prove some technical estimates on the Dirichlet eigenvalues and eigenfunctions for the generator of the random walk, as well as a lower bound on the survival probability slightly better than in (5.1.5). In Appendix 5.B, we briefly explain how to prove Theorem E by adapting the argument in [63]. Appendix 5.C provides an index of notation.

5.3 Proof of the Weaker Version of Ball Covering Theorem

5.3.1 Proof of Proposition 5.2.2 and the Extension of (5.1.8)

In this subsection, we prove Proposition 5.2.2 and then (5.1.8) for general $d \geq 2$, assuming Lemma 5.2.1, which says that under the conditioned law $\mu_N(\cdot)$, obstacles cannot be too isolated. We need another lemma which states that the size of the random walk range $S_{[0,N]}$ satisfies a weak law of large numbers under μ_N:

Lemma 5.3.1 (Size of random walk range)**.** *For all $\epsilon > 0$, we have*

$$\mu_N\left(\left|\frac{|S_{[0,N]}|}{|B(0,\varrho_N)|} - 1\right| > \epsilon\right) \to 0 \text{ as } N \to \infty \tag{5.3.1}$$

faster than any negative power of N.

Proof of Lemma 5.3.1. For a Brownian motion among Poissonian obstacles, the corresponding result is proved in [33]. It is straightforward to adapt the argument there to the current discrete setting. Indeed, the key point of the argument therein was the following formula for the generating function

$$\int \exp\left\{\xi|S_{[0,N]}|\right\} d\mu_N = \frac{\mathbf{E}\left[\exp\{(\xi - \log(1/p))|S_{[0,N]}|\}\right]}{\mathbf{E}\left[\exp\{-\log(1/p)|S_{[0,N]}|\}\right]}. \tag{5.3.2}$$

One can use (5.1.5) to derive the asymptotics of this for $|\xi| < \log(1/p)$ and then (5.3.1) follows by standard exponential Chebyshev bounds. □

Proof of Proposition 5.2.2. Thanks to Theorem E and Lemma 5.2.1, we may assume that $S_{[0,N]} \subset B(\boldsymbol{\mathcal{X}}_N, \varrho_N + \varrho_N^{\epsilon_1})$, and for any $x \in \mathcal{O} \cap B(0, 2\varrho_N)$,

$$\frac{|\mathcal{O} \cap B(x, \epsilon\varrho_N)|}{|B(x, \epsilon\varrho_N)|} \geq \delta. \tag{5.3.3}$$

Suppose that there is a point $x \in B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N) \cap \mathcal{O}$. Then by (5.3.3), at least δ fraction of sites in $B(x, \epsilon\varrho_N)$ are closed and hence are outside $S_{[0,N]}$. Combined with $S_{[0,N]} \subset B(\boldsymbol{\mathcal{X}}_N, \varrho_N + \varrho_N^{\epsilon_1})$, this implies that the ratio $|S_{[0,\varrho_N]}|/|B(0, \varrho_N)|$ stays strictly less than one, which has μ_N-probability tending to zero as $N \to \infty$ by Lemma 5.3.1. □

Proof of (5.1.8) **for general** $d \geq 2$**.** We derive (5.1.8) as a consequence of the following lemma, which asserts that the random walk visits all x in the confinement ball such that $B(x, (\log N)^3)$ is free of obstacles.

Lemma 5.3.2.

$$\lim_{N\to\infty} \mu_N \left(\bigcup_{x \in B(0, 2\varrho_N)} \left\{ \tau_x > N, \mathcal{O} \cap B(x, (\log N)^3) = \emptyset \right\} \right) = 0. \tag{5.3.4}$$

Since we know from Proposition 5.2.2 that for any $\epsilon > 0$, the ball $B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon/2)\varrho_N)$ is free of obstacles with μ_N-probability tending to one, Lemma 5.3.2 implies that

$$\lim \mu_N \left(B(\boldsymbol{\mathcal{X}}_N, (1-\epsilon)\varrho_N) \subset S_{[0,N]} \right) = 1. \tag{5.3.5}$$

$\square$

Proof of Lemma 5.3.2. By the union bound,

$$\mu_N \left(\bigcup_{x \in B(0,2\varrho_N)} \{\tau_x > N, \mathcal{O} \cap B(x, (\log N)^3) = \emptyset\} \right) \\ \leq \sum_{x \in B(0,2\varrho_N)} \mu_N \left(\tau_x > N, \mathcal{O} \cap B(x, (\log N)^3) = \emptyset \right). \tag{5.3.6}$$

Thus it suffices show that for any $x \in B(0, 2\varrho_N)$,

$$\mu_N \left(\tau_x > N, \mathcal{O} \cap B(x, (\log N)^3) = \emptyset \right) \to 0 \text{ as } N \to \infty \tag{5.3.7}$$

faster than any negative power of N.

To this end, observe that the probability $\mathbf{P}(\tau_x \wedge \tau_{\mathcal{O}} > N)$ is independent of whether $x \in \mathcal{O}$ or not. Therefore, we have

$$\mu_N \left(\tau_x > N, \mathcal{O} \cap B(x, (\log N)^3) = \emptyset \right) \\ = \frac{p}{1-p} \mu_N \left(\mathcal{O} \cap B(x, (\log N)^3) = \{x\} \right), \tag{5.3.8}$$

where we have switched the environment at x conditionally on the random walk event $\{\tau_x > N\}$. Lemma 5.2.1 then shows that the right-hand side decays faster than any negative power of N. $\square$

The rest of this section is devoted to the proof of Lemma 5.2.1.

5.3.2 Proof Outline for Lemma 5.2.1

The proof of Lemma 5.2.1 is based on the environment and path switching argument in (5.2.1). The rough (and not fully accurate) heuristic is as follows. Suppose that the event $E_l^\delta(x)$ occurs, i.e., $x \in \mathcal{O}$ and $|\mathcal{O} \cap B(x, l)|/|B(x, l) < \delta$ for some δ much smaller than the typical obstacle density $1 - p$. Either the random walk spends a lot of time in $B(x, l)$, in which case we remove all obstacles in $B(x,l)$, and we expect the gain in the random walk survival probability to beat the environment cost of removing the obstacles; or the random walk spends little time in $B(x,l)$, in which case we force the random walk not to visit $B(x,l/4)$, change to typical obstacle configurations in $B(x,l/4)$ and remove all obstacles in $B(x,l) \setminus B(x,l/4)$, and we expect the probability gain in the environment to beat the cost in changing the random walk. To make this heuristic rigorous, we need the following ingredients.

Path decomposition. First, we decompose the random walk path $(S_n)_{0\le n\le N}$ into successive crossings between the inner and outer shells of the annulus

$$A(x;l/2,l) := B(x,l) \setminus \overline{B(x,l/2)}. \tag{5.3.9}$$

More precisely, we define $\overline{B(x,l/2)} := B(x,l/2) \cup \partial B(x,l/2)$ and stopping times

$$\sigma_1 := \min\{n \ge 0 \colon S_n \in \overline{B(x,l/2)}\} \wedge N, \tag{5.3.10}$$

and for $k \in \mathbb{N}$,

$$\tau_k := \min\{n > \sigma_k \colon S_n \in B^c(x,l)\} \wedge N; \tag{5.3.11}$$

$$\sigma_{k+1} := \min\{n > \tau_k \colon S_n \in \overline{B(x,l/2)}\} \wedge N. \tag{5.3.12}$$

We will perform path switching on the crossings from $\overline{B(x,l/2)}$ to $B(x,l)^c$, and perform environment switching in the ball $B(x,l)$.

Skeletal approximation of $\mathcal{O} \cap B(x,l)$. When the random walk spends a lot of time in $B(x,l)$, we will remove all the obstacles in $B(x,l)$. We need to estimate the gain in the random walk survival probability as a function of $\mathcal{O} \cap B(x,l)$. This is not a very simple task and one of the difficulties is that the contribution from each obstacle is highly non-uniform, depending on others. Indeed, if an obstacle is well surrounded by others, then we gain little in the survival probability by removing it. In order to make the gain from an obstacle independent from others, we will only count the gain from a skeletal set $\mathcal{X}_l(x) \subset \mathcal{O}$ with properties

$$x \in \mathcal{X}_l(x) \text{ and all sites in } \mathcal{X}_l(x) \text{ are at least distance } l^{1/2d} \text{ away from each other;} \tag{5.3.13}$$

$$\text{each } y \in \mathcal{O} \cap B(x,l) \text{ is within distance } l^{1/2d} \text{ of some site in } \mathcal{X}_l(x). \tag{5.3.14}$$

Such a set can be constructed iteratively. First include $x \in \mathcal{X}_l(x)$ and remove all obstacles in $B(x, l^{1/2d})$. Next pick any one of the remaining obstacles at $y \in B(x,l)$ that is closest to x and add it to $\mathcal{X}_l(x)$, and remove all obstacles in $B(y, l^{1/2d})$. Repeat this procedure until no obstacles remain in $B(x,l)$. Another difficulty is that we gain little in the random walk survival probability by removing obstacles in $B(x,l)$ near $\partial B(x,l)$ since we will only count the gain from the crossings $\{S_{[\sigma_k,\tau_k]}\}_{k\in\mathbb{N}}$, that typically spend little time near $\partial B(x,l)$. Therefore we focus on the obstacles deeply inside $B(x,l)$ by setting

$$\mathcal{X}_l^\circ := \mathcal{X}_l^\circ(x) := \mathcal{X}_l(x) \cap B(x, l/2). \tag{5.3.15}$$

Random walk estimates. For $D \subset \mathbb{Z}^d$, $u \in D$ and $v \in D \cup \partial D$, we denote

$$p_n^D(u,v) := \mathbf{P}_u(S_n = v, S_{[1,n-1]} \subset D). \tag{5.3.16}$$

This is nothing but the discrete space-time Dirichlet heat kernel on D if $v \in D$ and, while it is not for $v \in \partial D$, it always satisfies the discrete heat equation in (n,u).

Remark 5.3.3. Since the symmetric simple random walk has period 2, we have $p_n^D(u,v) = 0$ when $n + |u-v|_1$ is an odd number. In what follows, we adopt a convention that $p_n^D(u,v)$ is understood to be $p_{n+1}^D(u,v)$ if $n + |u-v|_1$ is odd.

Now we are ready to state the gain in the random walk survival probability when we remove obstacles: for $c_{5,0}, c_{5,1} > 0$ to be determined later in Lemma 5.3.6, uniformly in $m \geq (c_{5,0}l)^2$, $u \in B(x, l/2)$ and $v \in \overline{B(x,l)}$,

$$p_m^{B(x,l)\setminus\mathcal{O}}(u,v) \leq e^{-c_1(\lfloor m/(c_{5,0}l)^2 \rfloor - 1)\Gamma(|\mathcal{X}_l^\circ|)} p_m^{B(x,l)}(u,v), \tag{RW1}$$

where for $k \in \mathbb{N}$,

$$\Gamma(k) := \begin{cases} \left(\log \frac{(c_{5,0}l)^{3/2}}{2k}\right)^{-1} \vee 0, & d = 2, \\ \frac{l^{2-d}k}{1+l^{(2-d)/2d}k^{2/d}}, & d \geq 3. \end{cases} \tag{5.3.17}$$

Roughly speaking, this estimate means that if the random walk stays in $B(x,l)$, then in every $(c_{5,0}l)^2$ steps, it has more than $c_1\Gamma(|\mathcal{X}_l^\circ|)$ probability of hitting an obstacle (see Lemma 5.3.7). As mentioned at the beginning of this subsection, we force the crossing to avoid $B(x, l/4)$ when the random walk spends little time in $B(x,l)$. We need another random walk estimate to quantify the effect of this switching and also some others to deal with complementary cases. Those random walk estimates will be tagged as (RW1)–(RW5) and restated and proved in Lemma 5.3.6 in Subsection 5.3.4.

Obstacles deep in the interior of $B(x,l)$**.** Note that in (RW1), the bound is in terms of $|\mathcal{X}_l^\circ(x)|$, the number of skeletal points of $\mathcal{O} \cap B(x,l)$ in $B(x,l/2)$. To ensure that the gain in (RW1) dominates the cost of removing all obstacles in $B(x,l)$, we need that $B(x,l)$ has sufficiently many obstacles deep in its interior, namely, $|\mathcal{X}_l^\circ(x)| \geq \rho|\mathcal{X}_l(x)|$ for some $\rho > 0$. The next lemma guarantees that we can achieve this by slightly changing the radius.

Lemma 5.3.4. *Suppose that* $x \in \mathcal{O}$ *and* $|\mathcal{O} \cap B(x,L)|/|B(x,L)| < \delta$ *for some* $L \in \mathbb{N}$. *Then we can find* $\rho > 0$ *independent of* L *and* δ*, such that there exists* $L^{5/6} \leq l \leq L$ *with* $|\mathcal{O} \cap B(x,l)|/|B(x,l)| < \delta$ *and* $|\mathcal{X}_l^\circ(x)| \geq \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\}$.

Therefore it suffices to prove Lemma 5.2.1 where we replace the event $E_l^{\delta}(x)$ by the event

$$E_l^{\delta,\rho}(x) := E_l^{\delta}(x) \cap \left\{ |\mathcal{X}_l^{\circ}(x)| \geq \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\} \right\} \tag{5.3.18}$$

with $l \in [(\log N)^{5/2}, \varrho_N]$, which will be carried out in the next subsection.

Proof of Lemma 5.3.4. Let

$$j^* := \min\left\{ j \geq 0 \colon l = L/2^j \text{ satisfies } |\mathcal{X}_l^{\circ}(x)| \geq \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\}\right\}. \tag{5.3.19}$$

We will show that $l^* := L/2^{j^*}$ satisfies the desired properties if $\rho > 0$ is small enough.

If $j^* = 0$, then $l^* = L$ works. Otherwise, for all $l = L/2^j$ with $0 \leq j \leq j^* - 1$, we have

$$|\mathcal{X}_l^{\circ}(x)| < \rho \min\{|\mathcal{X}_l(x), \delta l^{d-1/2}\} \leq \rho |\mathcal{X}_l(x)|. \tag{5.3.20}$$

We claim that for all $l \geq 1$,

$$|\mathcal{X}_l(x)| \leq C_d |\mathcal{X}_{2l}^{\circ}(x)| \quad \text{for some } C_d \text{ depending only on } d. \tag{5.3.21}$$

Together with (5.3.20), this implies that

$$|\mathcal{X}_{2l^*}^{\circ}(x)| < \rho C_d |\mathcal{X}_{4l^*}^{\circ}(x)| < \cdots < (\rho C_d)^{j^*-1} |\mathcal{X}_L^{\circ}(x)| \leq (\rho C_d)^{j^*-1} C L^d. \tag{5.3.22}$$

Since $|\mathcal{X}_L^{\circ}(x)|, \ldots, |\mathcal{X}_{2l^*}^{\circ}(x)| \geq 1$ because $x \in \mathcal{O}$, we must have $(\rho C_d)^{-j^*+1} \leq CL^d$. We can then choose $\rho > 0$ small such that $2^{j^*} \leq L^{5/6}$, and hence $l^* = L/2^{j^*} \in [L^{5/6}, L]$.

To bound the volume fraction of obstacles in $B(x, l^*)$ when $j^* \geq 1$, we can apply (5.3.20) at $l = L/2^{j^*-1} = 2l^*$ to obtain

$$\begin{aligned} |\mathcal{O} \cap B(x, l^*)| &\leq |\mathcal{X}^\circ_{2l^*}(x)| \cdot |B(0, (2l^*)^{1/2d})| \\ &< \rho\delta(2l^*)^{d-1/2}|B(0, (2l^*)^{1/2d})| \\ &< \delta|B(x, l^*)|, \end{aligned} \tag{5.3.23}$$

where the last inequality holds if $\rho > 0$ is chosen small.

Lastly, we prove the claim (5.3.21). Note that by our construction of $\mathcal{X}_l(x)$ and $\mathcal{X}^\circ_{2l}(x)$,

$$\begin{aligned} &\bigcup_{y \in \mathcal{X}_l(x)} B(y, l^{1/2d}) \subset \bigcup_{y \in \mathcal{O} \cap B(x,l)} B(y, l^{1/2d}), \\ &\text{where} \quad \mathcal{O} \cap B(x, l) \subset \bigcup_{z \in \mathcal{X}^\circ_{2l}(x)} B(z, (2l)^{1/2d}). \end{aligned} \tag{5.3.24}$$

Therefore by doubling the radii of the balls $\{B(z, (2l)^{1/2d})\}_{z \in \mathcal{X}^\circ_{2l}(x)}$, we obtain

$$\bigcup_{y \in \mathcal{X}_l(x)} B(y, l^{1/2d}) \subset \bigcup_{z \in \mathcal{X}^\circ_{2l}(x)} B(z, 2(2l)^{1/2d}).$$

Since the balls $\{B(y, l^{1/2d})\}_{y \in \mathcal{X}_l(x)}$ are disjoint, a volume calculation then yields (5.3.21). $\square$

5.3.3 Proof of Lemma 5.2.1

As remarked after Lemma 5.3.4, if suffices to prove Lemma 5.2.1 with the event $E^\delta_l(x)$ replaced by $E^{\delta,\rho}_l(x)$, with $l \in [(\log N)^{5/2}, \varrho_N]$. We will prove the following bound on $\mu_N(E^{\delta,\rho}_l(x))$, which immediately implies Lemma 5.2.1 by a union bound over all $x \in B(0, 2\varrho_N)$ and $(\log N)^{5/2} \leq l \leq \varrho_N$.

Lemma 5.3.5. *Let $E^{\delta,\rho}_l(x)$ be defined as in* (5.3.18)*. There exist $c_{5,3} > 0$ depending on d, p and δ such that for all $l \in [\log N, \varrho_N]$, we have*

$$\mu_N(E^{\delta,\rho}_l(x)) \leq \exp\{-c_{5,3} l^{1/2}\}. \tag{5.3.25}$$

Proof of Lemma 5.3.5. Recall the path decomposition introduced in Section 5.3.2, where we identified the successive crossings from $\overline{B(x, l/2)} = B(x, l/2) \cup \partial B(x, l/2)$ to $B(x, l)^c$ during the time intervals $[\sigma_k, \tau_k]$, $k \in \mathbb{N}$. Since these stopping times are truncated by N, the duration $\tau_k - \sigma_k$ can be zero. Henceforth, the word *crossing* refers to $S_{[\sigma_k, \tau_k]}$ with $\tau_k - \sigma_k > 0$. In particular, the last crossing may be incomplete. To carry out the path and environment switching, we distinguish between three cases and in order to describe them, we need some more notation. We denote a sequence of numbers or vectors in bold face as $\boldsymbol{a} = (a_k)_{k \geq 1}$ and introduce the set of interlacing sequences

$$\boldsymbol{I}_N := \{(\boldsymbol{s}, \boldsymbol{t}) \colon 0 \leq s_k \leq t_k \leq s_{k+1} \leq N \text{ for all } k \in \mathbb{N}\}. \tag{5.3.26}$$

For $(\boldsymbol{s}, \boldsymbol{t}) \in \bigcup_{N \geq 1} \boldsymbol{I}_N$, we write

$$K(\boldsymbol{s}, \boldsymbol{t}) := \sup\{k \geq 1 \colon t_k - s_k > 0\} \tag{5.3.27}$$

which represents the number of crossings when $(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{\sigma}, \boldsymbol{\tau})$. Now we are ready to describe the three cases. Recall that $c_{5,0} > 0$ has already been chosen to satisfy Lemma 5.3.6. The constant $\delta > 0$ is to be determined later, depending only on the dimension d and the open probability p.

(1) There are many crossings and more than half of them are short ($\leq (c_{5,0} l)^2$), that is, $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ belongs to

$$\boldsymbol{F}_1 := \left\{ (\boldsymbol{s}, \boldsymbol{t}) \in \boldsymbol{I}_N \colon \left| \{k \geq 1 \colon 0 < t_k - s_k \leq (c_{5,0} l)^2\} \right| > \frac{1}{2} K(\boldsymbol{s}, \boldsymbol{t}) \vee \delta^{1/d} l^d \right\}. \tag{5.3.28}$$

(2) The total time duration of the long crossings ($> (c_{5,0} l)^2$) is long, that is, $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ belongs

to

$$\boldsymbol{F}_2 := \left\{ (\boldsymbol{s}, \boldsymbol{t}) \in \boldsymbol{I}_N \colon \sum_{k \geq 1} (t_k - s_k) 1_{\{t_k - s_k > (c_{5,0} l)^2\}} > \delta^{1/d} c_{5,0}^2 l^{d+2} \right\}. \tag{5.3.29}$$

(3) The number of crossings as well as their total duration are small, that is, $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ belongs to

$$\boldsymbol{F}_3 := \left\{ (\boldsymbol{s}, \boldsymbol{t}) \in \boldsymbol{I}_N \colon K(\boldsymbol{s}, \boldsymbol{t}) \leq 2\delta^{1/d} l^d \text{ and } \sum_{k \geq 1} (t_k - s_k) \leq 2\delta^{1/d} c_{5,0}^2 l^{d+2} \right\}. \tag{5.3.30}$$

These three cases exhaust all possibilities. Indeed, if $(\boldsymbol{s}, \boldsymbol{t}) \notin \boldsymbol{F}_2$, then the number of long crossings is at most $\delta^{1/d} l^d$, and their total duration is at most $\delta^{1/d} c_{5,0}^2 l^{d+2}$. If in addition, $(\boldsymbol{s}, \boldsymbol{t}) \notin \boldsymbol{F}_1$, then the number of short crossings is either less than the number of long crossings, or less than $\delta^{1/d} l^d$; either way, it is bounded by $\delta^{1/d} l^d$, and their total duration is at most $\delta^{1/d} c_{5,0}^2 l^{d+2}$. Combining the short and long crossings, one finds that $(\boldsymbol{s}, \boldsymbol{t}) \in \boldsymbol{F}_3$.

For each of the three cases above, by summing over all possible values of $(\sigma_k, \tau_k) \in \boldsymbol{F}_i$ ($i \in \{1, 2, 3\}$) and the position of the walk at these times, we obtain

$$\begin{aligned} &\mathbf{P}\left(\tau_{\mathcal{O}} > N \text{ and } (\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_i\right) \\ &\quad = \sum_{(\boldsymbol{s}, \boldsymbol{t}) \in \boldsymbol{F}_i} \sum_{\boldsymbol{u}, \boldsymbol{v}} p_{s_1}^{\mathbb{Z}^d \setminus \mathcal{O}}(0, u_1) \\ &\quad\quad \times \prod_{k \geq 1} p_{t_k - s_k}^{B(x,l) \setminus \mathcal{O}}(u_k, v_k) p_{s_{k+1} - t_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x, l/2))}(v_k, u_{k+1}), \end{aligned} \tag{5.3.31}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ range over all the possible starting and ending points of crossings with $(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{s}, \boldsymbol{t})$. In particular, $u_k \in \partial B(x, l/2)$ and $v_k \in \partial B(x, l)$ as long as $s_k < N$ and $t_k < N$ respectively, except possibly $u_1 = 0$ when $0 \in B(x, l/2)$. For simplicity, we assume $\delta^{1/d} l^d \in \mathbb{N}$ in this proof.

Case (1): In this case, we remove all the obstacles inside $B(x, l)$ and lengthen all the short crossings by l^2. We formalize this as the environment and path switching (5.2.1) by setting

$$(A_1, A_2) := (\{(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_1\}, \{\tau_{\mathcal{O}} > N\}); \tag{5.3.32}$$

$$(E_1, E_2) := (E_l^{\delta,\rho}(x), \{\mathcal{O} \cap B(x,l) = \emptyset\}). \tag{5.3.33}$$

Since $|\mathcal{O} \cap B(x,l)| \le \delta |B(x,l)|$ on the event $E_l^{\delta,\rho}(x)$, the cost of environment switching can be estimated as

$$\begin{aligned} \frac{\mathbb{P}(E_l^{\delta,\rho})}{\mathbb{P}(\mathcal{O} \cap B(x,l) = \emptyset)} &\le \frac{\mathbb{P}\left(|\mathcal{O} \cap B(x,l)| \le \delta |B(x,l)|\right)}{\mathbb{P}(\mathcal{O} \cap B(x,l) = \emptyset)} \\ &\le \sum_{i=0}^{\delta|B(x,l)|} \binom{|B(x,l)|}{i} \left(\frac{1-p}{p}\right)^j \\ &\le e^{c\delta(\log \frac{1}{\delta}) l^d} \end{aligned} \tag{5.3.34}$$

by using Stirling's approximation. Alternatively, one can also interpret this as a consequence of Cramer's large deviation principle.

On the other hand, since the short crossings are unlikely to happen, we gain in the random walk probability by lengthening them. More precisely, we will see in Lemma 5.3.6 that for $t_k - s_k \le (c_{5,0} l)^2$, $u_k \in \overline{B(x,l/2)}$ and $v_k \in \partial B(x,l)$,

$$\begin{aligned} p_{t_k - s_k}^{B(x,l) \setminus \mathcal{O}}(u_k, v_k) &\le p_{t_k - s_k}^{B(x,l)}(u_k, v_k) \\ &\le \frac{1}{100} p_{t_k - s_k + l^2}^{B(x,l)}(u_k, v_k), \end{aligned} \tag{RW2}$$

where l^2 is to be understood as $l^2 + 1$ when l is odd as mentioned in Remark 5.3.3. It is easy to see that this change is harmless for the following argument and henceforth we will not mention this parity convention again. Setting

$$I_k := 1_{\{0 < t_k - s_k \le (c_{5,0} l)^2, u_k \in \partial B(x,l/2), v_k \in \partial B(x,l)\}}, \tag{5.3.35}$$

we can bound the product in the right-hand side of (5.3.31) by

$$\begin{aligned}
&\prod_{k\geq 1} p^{B(x,l)\setminus\mathcal{O}}_{t_k-s_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k,u_{k+1})\\
&\quad\leq 100^{-\sum_{k\geq 1} I_k}\prod_{k\geq 1} p^{B(x,l)}_{t_k+l^2I_k-s_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k,u_{k+1})\\
&\quad= 100^{-\sum_{k\geq 1} I_k}\prod_{k\geq 1} p^{B(x,l)}_{\widetilde{t}_k-\widetilde{s}_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{\widetilde{s}_{k+1}-\widetilde{t}_k}(v_k,u_{k+1}),
\end{aligned} \tag{5.3.36}$$

where $\widetilde{s}_k := s_k + l^2\sum_{m<k} I_m$ and $\widetilde{t}_k := t_k + l^2\sum_{m\leq k} I_m$. Let us consider the cases

$$(\boldsymbol{\sigma},\boldsymbol{\tau})\in \boldsymbol{F}_{1,j} := \left\{(\boldsymbol{s},\boldsymbol{t})\in\boldsymbol{F}_1\colon \sum_{k\geq 1} I_k = j\right\}, \tag{5.3.37}$$

that is, exactly j crossings are lengthened, separately for $j\in\{\delta^{1/d}l^d, \delta^{1/d}l^d+1,\dots,N\}$. Summing (5.3.36) multiplied by $p^{B(x,l)}_{s_1}(0,u_1)$ over $(\boldsymbol{s},\boldsymbol{t})\in\boldsymbol{F}_{1,j}$ and $(\boldsymbol{u},\boldsymbol{v})$, we obtain

$$\begin{aligned}
&\mathbf{P}(\tau_{\mathcal{O}}>N \text{ and } (\boldsymbol{\sigma},\boldsymbol{\tau})\in\boldsymbol{F}_{1,j})\\
&= \sum_{(\boldsymbol{s},\boldsymbol{t})\in\boldsymbol{F}_{1,j}}\sum_{\boldsymbol{u},\boldsymbol{v}} p^{B(x,l)}_{s_1}(0,u_1)\prod_{k\geq 1} p^{B(x,l)\setminus\mathcal{O}}_{t_k-s_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k,u_{k+1})\\
&\leq 100^{-j}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\boldsymbol{F}_{1,j}}\sum_{\boldsymbol{u},\boldsymbol{v}} p^{B(x,l)}_{\widetilde{s}_1}(0,u_1)\prod_{k\geq 1} p^{B(x,l)}_{\widetilde{t}_k-\widetilde{s}_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{\widetilde{s}_{k+1}-\widetilde{t}_k}(v_k,u_{k+1}).
\end{aligned} \tag{5.3.38}$$

In order to relate this last line to $\mathbf{P}(\tau_{\mathcal{O}}>N)$, we rewrite (5.3.38) as a summation over

$$(\widetilde{\boldsymbol{s}},\widetilde{\boldsymbol{t}})\in\widetilde{F}_{1,j} := \{(\widetilde{\boldsymbol{s}},\widetilde{\boldsymbol{t}})\colon (\boldsymbol{s},\boldsymbol{t})\in\boldsymbol{F}_{1,j}\}. \tag{5.3.39}$$

Note that each $(\widetilde{\boldsymbol{s}},\widetilde{\boldsymbol{t}})$ may come from different $(\boldsymbol{s},\boldsymbol{t})$'s but with the same number of crossings $K(\boldsymbol{s},\boldsymbol{t})$ and hence its pre-image has cardinality at most $2^{K(\boldsymbol{s},\boldsymbol{t})}\leq 2^{2+2j}$ on $\boldsymbol{F}_{1,j}$. Recalling also that $j \geq \delta^{1/d} l^d$, it follows from (5.3.38) that

$$\begin{aligned} &\mathbf{P}(\tau_{\mathcal{O}} > N \text{ and } (\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_{1,j}) \\ &\quad \leq 4 \cdot 25^{-\delta^{1/d} l^d} \sum_{(\widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}) \in \widetilde{F}_{1,j}} \sum_{\boldsymbol{u}, \boldsymbol{v}} p^{B(x,l)}_{\widetilde{s}_1}(0, u_1) \\ &\qquad \times \prod_{k \geq 1} p^{B(x,l)}_{\widetilde{t}_k - \widetilde{s}_k}(u_k, v_k) p^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x,l/2))}_{\widetilde{s}_{k+1} - \widetilde{t}_k}(v_k, u_{k+1}). \end{aligned} \tag{5.3.40}$$

The sum on the right-hand side is seen to be bounded by $\mathbf{P}(\tau_{\mathcal{O} \setminus B(x,l)} > N + jl^2)$. Indeed, any $(\widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}) \in \widetilde{F}_{1,j}$ has terminal time $\widetilde{s}_{K(\widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}})+1} = N + jl^2$ by construction and hence the above sum represents (a part of) the path decomposition before time $N + jl^2$.

Taking a sum of (5.3.40) over j, we obtain

$$\begin{aligned} &\mathbf{P}(\tau_{\mathcal{O}} > N \text{ and } (\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_1) \\ &\quad \leq 4 \cdot 25^{-\delta^{1/d} l^d} \sum_{j = \delta^{1/d} l^d}^{N} \mathbf{P}\left(\tau_{\mathcal{O} \setminus B(x,l)} > N + jl^2\right) \\ &\quad \leq 4N \cdot 25^{-\delta^{1/d} l^d} \mathbf{P}\left(\tau_{\mathcal{O} \setminus B(x,l)} > N\right). \end{aligned} \tag{5.3.41}$$

Since $\tau_{\mathcal{O} \setminus B(x,l)} = \tau_{\mathcal{O}}$ on $\{\mathcal{O} \cap B(x,l) = \emptyset\}$, recalling (5.3.34) and $l \geq \log N$ and choosing $\delta > 0$ small, we can use (5.2.1) to conclude that

$$\mu_N\left((S, \mathcal{O}) \in \left(\{(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_1\}, E^{\delta,\rho}_l(x)\right)\right) \leq e^{-c\delta^{1/d} l^d}. \tag{5.3.42}$$

Case (2): In this case, we again remove all the obstacles in $B(x,l)$ and leave the crossings unchanged. We apply the same environment and path switching as in the previous case. Since the long crossings have higher probability of hitting the obstacles, removing the obstacles gives us a large gain in the random walk probability. In order to make it precise, note first that $(\boldsymbol{s},\boldsymbol{t}) \in \boldsymbol{F}_2$ implies

$$\sum_{k\geq 1} \left\lfloor \frac{t_k - s_k}{(c_{5,0}l)^2} \right\rfloor \geq \frac{1}{2} \sum_{k\geq 1} \frac{t_k - s_k}{(c_{5,0}l)^2} 1_{\{t_k - s_k \geq (c_{5,0}l)^2\}} > \frac{\delta^{1/d}}{2} l^d. \tag{5.3.43}$$

Given this, we use the aforementioned (see also Lemma 5.3.6)

$$p^{B(x,l)\setminus\mathcal{O}}_{t_k - s_k}(u_k, v_k) \leq e^{-c_1(\lfloor (t_k - s_k)/(c_{5,0}l)^2 \rfloor - 1)\Gamma(|\mathcal{X}_l^\circ|)} p^{B(x,l)}_{t_k - s_k}(u_k, v_k) \tag{RW1}$$

repeatedly to obtain

$$\begin{aligned}
&\mathbf{P}(\tau_{\mathcal{O}} > N \text{ and } (\boldsymbol{\sigma},\boldsymbol{\tau}) \in \boldsymbol{F}_2) \\
&\quad = \sum_{(\boldsymbol{s},\boldsymbol{t})\in \boldsymbol{F}_2} \sum_{\boldsymbol{u},\boldsymbol{v}} p^{B(x,l)\setminus\mathcal{O}}_{s_1}(0,u_1) \prod_{k\geq 1} p^{B(x,l)\setminus\mathcal{O}}_{t_k - s_k}(u_k, v_k) p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k, u_{k+1}) \\
&\quad \leq e^{-c\delta^{1/d} l^d \Gamma(\mathcal{X}_l^\circ)} \sum_{(\boldsymbol{s},\boldsymbol{t})\in \boldsymbol{F}_2} \sum_{\boldsymbol{u},\boldsymbol{v}} p^{B(x,l)\setminus\mathcal{O}}_{s_1}(0,u_1) \\
&\qquad \times \prod_{k\geq 1} p^{B(x,l)}_{t_k - s_k}(u_k, v_k) p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k, u_{k+1}) \\
&\quad \leq e^{-c\delta^{1/d} l^d \Gamma(|\mathcal{X}_l^\circ|)} \mathbf{P}(\tau_{\mathcal{O}\setminus B(x,l)} > N)
\end{aligned} \tag{5.3.44}$$

uniformly in $\mathcal{O}$, and we can replace $\tau_{\mathcal{O}\setminus B(x,l)}$ by $\tau_{\mathcal{O}}$ on $\{\mathcal{O} \cap B(x,l) = \emptyset\}$ as before.

We are going to show that the cost of removing the obstacles in $B(x,l)$ is much smaller than the above gain in the random walk probability. Recall that $|\mathcal{X}_l^\circ| \geq \rho \min(\delta l^{d-1/2}, |\mathcal{X}_l|)$ on the event $E_l^{\delta,\rho}(x)$ and also note that (5.3.13) implies the bound $|\mathcal{X}_l^\circ| \leq C\delta l^{d-1/2}$ for some $C > 0$ depending only on the dimension. In the case $|\mathcal{X}_l^\circ| \in [\rho\delta l^{d-1/2}, C\delta l^{d-1/2}]$, we have that $\Gamma(|\mathcal{X}_l^\circ|)$ is bounded below by a positive constant, recalling the definition of Γ in (5.3.17). Combining (5.3.44) with (5.3.34) and (5.2.1) and choosing δ small, we get

$$\begin{aligned}
&\mu_N\left((S,\mathcal{O}) \in \left(\{(\boldsymbol{\sigma},\boldsymbol{\tau}) \in \boldsymbol{F}_2\}, (E_l^{\delta,\rho}(x) \cap \{|\mathcal{X}_l^\circ| \geq \rho\delta l^{d-1/2}\})\right)\right) \\
&\quad \leq e^{-c\delta^{1/d} l^d}.
\end{aligned} \tag{5.3.45}$$

In the other case $|\mathcal{X}_l^\circ| \in [\rho|\mathcal{X}_l|, \rho\delta l^{d-1/2})$, instead, we have that for sufficiently small δ,

$$\Gamma(|\mathcal{X}_l^\circ|) \geq c\delta^{-2/d} l^{1/2-d} |\mathcal{X}_l^\circ| / |\log\delta| \tag{5.3.46}$$

recalling (5.3.17) again. Indeed, for $d \geq 3$, using $|\mathcal{X}_l^\circ| \leq \rho\delta l^{d-1/2}$ in the denominator in (5.3.17) yields the above bound without $|\log\delta|$; for $d = 2$, the argument of log in (5.3.17) is large and the above bound follows from the fact that $1/\log r = r^{-1}(r/\log r)$ and $r/\log r$ is increasing for r large. Given this lower bound on $\Gamma(|\mathcal{X}_l^\circ|)$, the gain from the random walk becomes

$$\frac{\mathbf{P}(\tau_{\mathcal{O}} > N \text{ and } (\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_2)}{\mathbf{P}(\tau_{\mathcal{O}\setminus B(x,l)} > N)} \leq e^{-c\delta^{-1/d} l^{1/2} |\mathcal{X}_l^\circ| / |\log\delta|}. \tag{5.3.47}$$

Note that this gain is much smaller than the bound (5.3.34) on the cost of environment switching when $|\mathcal{X}_l^\circ|$ is small. Therefore we have to estimate the environment switching cost more carefully and this is done by considering separately the events $\{|\mathcal{X}_l^\circ| = k\}$ for $k < \rho\delta l^{d-1/2}$.

In the case under consideration, $|\mathcal{X}_l^\circ| = k$ implies $|\mathcal{X}_l| \leq \rho^{-1}k$. Recall also that all the obstacles in $B(x,l)$ are contained in $\bigcup_{x\in\mathcal{X}_l} B(x, l^{1/2d})$ by (5.3.14). Therefore on each event $\{|\mathcal{X}_l^\circ| = k\}$, by counting the possible choices of $\mathcal{X}_l$ first and then the configurations inside $\bigcup_{x\in\mathcal{X}_l} B(x, l^{1/2d})$, we can estimate the cost of environment switching as

$$\begin{aligned}
&\frac{\mathbb{P}(E_l^{\delta,\rho} \cap \{\mathcal{X}_l^\circ = k\})}{\mathbb{P}(\mathcal{O} \cap B(x,l) = \emptyset)} \\
&\quad \leq \sum_{i=k}^{\lfloor \rho^{-1}k \rfloor} \binom{|B(x,l)|}{i} \sum_{j=0}^{i|B(x,l^{1/2d})|} \binom{i|B(x,l^{1/2d})|}{j} \left(\frac{1-p}{p}\right)^j \\
&\quad \leq e^{Ckl^{1/2}}.
\end{aligned} \tag{5.3.48}$$

Combining this with (5.3.47) and choosing δ small, we obtain

$$\begin{aligned}\mu_N\Big((S,\mathcal{O})\in\Big(\{(\boldsymbol{\sigma},\boldsymbol{\tau})\in\boldsymbol{F}_2\},(E_l^{\delta,\rho}(x)\cap\{|\mathcal{X}_l^\circ|=k\})\Big)\Big)\\ \le e^{-c\delta^{-1/d}l^{1/2}k/|\log\delta|}\end{aligned}\tag{5.3.49}$$

for each $k<\rho\delta l^{d-1/2}$. Finally we sum (5.3.45) and (5.3.49) for $k\in\{1,2,\dots,\lfloor\rho\delta l^{d-1/2}\rfloor\}$ to obtain

$$\mu_N\Big(E_l^{\delta,\rho}(x)\times\{(\boldsymbol{\sigma},\boldsymbol{\tau})\in\boldsymbol{F}_2\}\Big)\le e^{-c\delta^{1/d}l^{1/2}}.\tag{5.3.50}$$

Case (3): In this case, we remove all the obstacles in $A(x;l/4,l)=B(x,l)\setminus B(x,l/4)$, change the obstacles configuration inside $B(x,l/4)$ to typical configurations and force all the crossings to avoid $B(x,l/4)$ after lengthening them by l^2. Complication arises when the origin is close to $B(x,l/4)$ because then it costs a lot to force the first crossing to avoid $B(x,l/4)$. We first deal with the simpler case $0\notin B(x,l/2)$ by applying the environment and path switching (5.2.1) with

$$(A_1,A_2):=(\{(\boldsymbol{\sigma},\boldsymbol{\tau})\in\boldsymbol{F}_3\},\{\tau_{\mathcal{O}}\wedge\tau_{B(x,l/4)}>N\});\tag{5.3.51}$$

$$(E_1,E_2):=(E^{\delta,\rho}(x),\{\mathcal{O}\cap A(x;l/4,l)=\emptyset\}).\tag{5.3.52}$$

The gain from the environment switching can be estimated as

$$\begin{aligned}\frac{\mathbb{P}(E_l^{\delta,\rho})}{\mathbb{P}(\mathcal{O}\cap A(x;l/4,l)=\emptyset)}&\le\frac{\mathbb{P}\left(|\mathcal{O}\cap B(x,l)|\le\delta|B(x,l)|\right)}{\mathbb{P}(\mathcal{O}\cap A(x;l/4,l)=\emptyset)}\\&\le p^{|B(x,l/4)|}\sum_{i=0}^{\delta|B(x,l)|}\binom{|B(x,l)|}{i}\left(\frac{1-p}{p}\right)^j\\&\le e^{-cl^d}\end{aligned}\tag{5.3.53}$$

by using Stirling's approximation (or Cramer's large deviation principle as before).

On the other hand, if we force the random walk to stay in $A(x;l/4,l)$ instead of $B(x,l)$, the extra cost per step should be measured by the difference of the principal Dirichlet eigenvalues of the discrete Laplacian in $A(x;l/4,l)$ and $B(x,l)$, which is of order l^{-2}. In fact, we will see in Lemma 5.3.6 that uniformly in $u_k \in \partial B(x,l/2)$ and $v_k \in \partial B(x,l)$,

$$\begin{aligned} p_{t_k-s_k}^{B(x,l)\setminus\mathcal{O}}(u_k,v_k) &\le p_{t_k-s_k}^{B(x,l)}(u_k,v_k) \\ &\le e^{c_{5,2}((t_k-s_k)l^{-2}+1)} p_{t_k-s_k+l^2}^{A(x;l/4,l)}(u_k,v_k). \end{aligned} \tag{RW3}$$

If $s_k < N$ and $t_k = N$ for some $k \in \mathbb{N}$, then this (last) crossing may be incomplete and its endpoint v_k may be in $B(x,l/4)$. In that case, the path switching should be done differently and we change the endpoint of the last crossing to $\widetilde{v}_k := v_k + (5l/8)\mathbf{e}_1$. The cost is bounded similarly as

$$p_{t_k-s_k}^{B(x,l)\setminus\mathcal{O}}(u_k,v_k) \le e^{c_{5,2}((t_k-s_k)l^{-2}+1)} p_{t_k-s_k+2l^2}^{A(x;l/4,l)}(u_k,\widetilde{v}_k). \tag{RW4}$$

We define $(\widetilde{s}_k,\widetilde{t}_k)_{k\ge 1}$ as the starting and ending times of switched crossings, similarly to Case (1), and also

$$\begin{aligned} &(\widetilde{v}_k,\widetilde{u}_{k+1}) \\ &\quad := \begin{cases} (v_k+(5l/8)\mathbf{e}_1, v_k+(5l/8)\mathbf{e}_1), & \text{if } s_k<N, t_k=N \text{ and } v_k \in B(x,l/4), \\ (v_k,u_{k+1}), & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3.54}$$

Then using the above estimates and recalling the definition of $\boldsymbol{F}_3$, we can bound the product in the right-hand side of (5.3.31) by

$$\begin{aligned}
&\prod_{k\geq 1} p^{B(x,l)\setminus\mathcal{O}}_{t_k-s_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k,u_{k+1}) \\
&\quad\leq \exp\left\{c_{5,2}\sum_{k\geq 1}\frac{t_k-s_k}{l^2}+K(\boldsymbol{s},\boldsymbol{t})\right\} \\
&\qquad\times\prod_{k\geq 1} p^{A(x;l/4,l)}_{\widetilde{t}_k-\widetilde{s}_k}(\widetilde{u}_k,\widetilde{v}_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{\widetilde{s}_{k+1}-\widetilde{t}_k}(\widetilde{v}_k,\widetilde{u}_{k+1}) \\
&\quad= e^{c\delta^{1/d}l^d}\prod_{k\geq 1} p^{A(x;l/4,l)}_{\widetilde{t}_k-\widetilde{s}_k}(\widetilde{u}_k,\widetilde{v}_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{\widetilde{s}_{k+1}-\widetilde{t}_k}(\widetilde{v}_k,\widetilde{u}_{k+1}).
\end{aligned} \tag{5.3.55}$$

Note that each $(\widetilde{\boldsymbol{s}},\widetilde{\boldsymbol{t}})$ has pre-image of cardinality at most $2^{2\delta^{1/d}l^d}$. Therefore summing (5.3.55) over $(\boldsymbol{s},\boldsymbol{t},\boldsymbol{u},\boldsymbol{v})$ separately according to the number of crossings as in Case (1), we can obtain

$$\begin{aligned}
&\mathbf{P}(\tau_{\mathcal{O}}>N \text{ and } (\boldsymbol{\sigma},\boldsymbol{\tau})\in\boldsymbol{F}_3) \\
&\quad=\sum_{(\boldsymbol{s},\boldsymbol{t})\in\boldsymbol{F}_3}\sum_{\boldsymbol{u},\boldsymbol{v}} p^{B(x,l)\setminus\mathcal{O}}_{s_1}(0,u_1)\prod_{k\geq 1} p^{B(x,l)\setminus\mathcal{O}}_{t_k-s_k}(u_k,v_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{s_{k+1}-t_k}(v_k,u_{k+1}) \\
&\quad\leq e^{c\delta^{1/d}l^d}\sum_{(\widetilde{\boldsymbol{s}},\widetilde{\boldsymbol{t}})}\sum_{\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}}} p^{B(x,l)\setminus\mathcal{O}}_{\widetilde{s}_1}(0,u_1)\prod_{k\geq 1} p^{B(x,l)}_{\widetilde{t}_k-\widetilde{s}_k}(\widetilde{u}_k,\widetilde{v}_k)p^{\mathbb{Z}^d\setminus(\mathcal{O}\cup B(x,l/2))}_{\widetilde{s}_{k+1}-\widetilde{t}_k}(\widetilde{v}_k,\widetilde{u}_{k+1}) \\
&\quad\leq CNe^{c\delta^{1/d}l^d}\mathbf{P}(\tau_{\mathcal{O}\cup B(x,l/4)}>N)
\end{aligned} \tag{5.3.56}$$

uniformly in $\mathcal{O}$ in the case $0\notin B(x,l/2)$. Recalling (5.3.53) and $l\geq\log N$ and using (5.2.1), we conclude that in this case

$$\mu_N\left((S,\mathcal{O})\in\left(\{(\boldsymbol{\sigma},\boldsymbol{\tau})\in\boldsymbol{F}_3\},E^{\delta,\rho}_l(x)\right)\right)\leq e^{-cl^d}. \tag{5.3.57}$$

Finally, we deal with the case $0\in B(x,l/2)$. In this case, the starting point of first crossing may be close to (or even inside) $B(x,l/4)$ and we want to ensure that the random walk gets away from that ball quickly. To this end, we fix a path $\pi(x;l)\subset B(x,l/2)$ of length l from 0 to $n\mathbf{e}_1\in\partial B(x,l/2)$ $(n\in\mathbb{N})$ and modify the environment and path switching as follows (see Figure 5.1):

$$(A_1, A_2) := \left(\{(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \boldsymbol{F}_3\}, \left\{S_{[0,l]} = \pi(x;l), \tau_{\mathcal{O}} \wedge (l + \tau_{B(x,l/4)} \circ \theta_l) > N\right\}\right); \tag{5.3.58}$$

$$(E_1, E_2) := (E^{\delta,\rho}(x), \{\mathcal{O} \cap (A(x;l/4,l) \cup \pi(x;l)) = \emptyset\}), \tag{5.3.59}$$

where $l + \tau_{B(x,l/4)} \circ \theta_l$ is the first hitting time to $B(x, l/4)$ after time l. Let us explain the difference from the previous case $0 \notin B(x, l/2)$. For the environment, we need to keep $\pi(x;l)$ empty, which has a cost of e^{-cl}, but is negligible compared with the original gain e^{cl^d} in (5.3.53).

For the random walk, only the first crossing, that is $S_{[0,\tau_1]}$ in this case, is switched differently. In the present case, note that $v_1 \in \partial B(x, l)$ since by the total duration constraint on $\boldsymbol{F}_3$, we have $t_1 \leq 2\delta^{1/d} c_{5,0}^2 l^{d+2} < N$ for small δ. We switch the paths with $\tau_1 = t_1, S_{\tau_1} = v_1$ to those go from 0 to $n\mathbf{e}_1$ following $\pi(x;l)$ in l steps and then to go from $n\mathbf{e}_1$ to v_1 inside $A(x;l/4,l)$ in $t_1 + 2l^2$ steps afterward. The probability to follow $\pi(x;l)$ in the first l steps is $(2d)^{-l}$ and combining this with the estimate (see Lemma 5.3.6)

$$p_{t_1}^{B(x,l)}(0, v_1) \leq e^{c_{5,2}(\lfloor t_1 l^{-2} \rfloor + 1)} p_{t_1+2l^2}^{A(x;l/4,l)}(n\mathbf{e}_1, v_1), \tag{RW5}$$

we obtain the following bound on the switching cost of the first crossing:

$$\begin{aligned} p_{t_1}^{B(x,l)}(0, v_1) &\leq (2d)^l p_l^{\pi(x;l)}(0, n\mathbf{e}_1) e^{c_{5,2}(\lfloor t_1 l^{-2} \rfloor + 1)} p_{t_1+2l^2}^{A(x;l/4,l)}(n\mathbf{e}_1, v_1) \\ &\leq (2d)^l e^{c_{5,2}(\lfloor t_1 l^{-2} \rfloor + 1)} p_{t_1+2l^2+l}^{A(x;l/4,l) \cup \pi(x;l)}(0, v_1). \end{aligned} \tag{5.3.60}$$

Note that the term $c_{5,2}\lfloor t_1 l^{-2} \rfloor$ already appeared in (5.3.55). The extra cost of $(2d)^l$ is again negligible compared with the e^{cl^d} gain from the environment.

Therefore simply by setting $\widetilde{t}_1 := t_1 + 2l^2 + l$ and changing $(\widetilde{s}_k, \widetilde{t}_k)_{k \geq 2}$ accordingly, we can use (5.3.60) to follow the same argument as before to extend (5.3.57) to the case $0 \in B(x, l/2)$. □

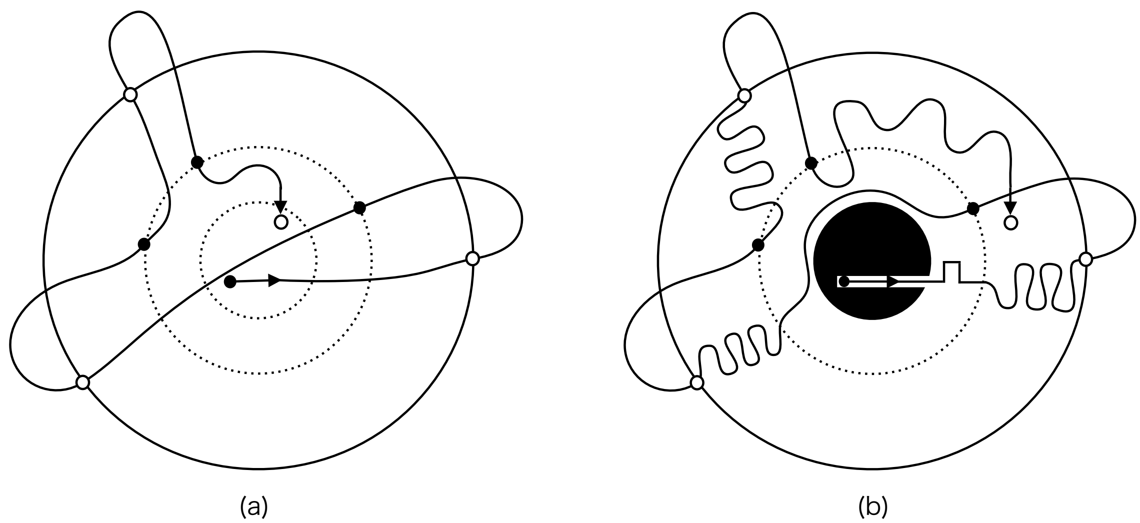

Figure 5.1: A schematic figure of the switching configuration from (a) to (b) in Case (3). The balls are $B(x,l/4)$, $B(x,l/2)$ and $B(x,l)$ from inside. There are 4 crossings from $B(x,l/2)$ to $B(x,l)^c$, including the last incomplete crossing. Both $S_0=0$ and S_N are in $B(x,l/4)$ in (a). The paths from $\bullet$ to $\circ$ are crossings and are lengthened. Observe that we cannot make the second crossing avoid $B(x,l/4)$ without lengthening it as illustrated in (b). The paths from $\circ$ to $\bullet$ are unchanged. The first polygonal segment of the path in (b) represents $\pi(x;l)$.

5.3.4 Random walk estimates

In this subsection, we restate and prove the random walk estimates used in the proof of Lemma 5.2.1. Recall the notation $\overline{B(x,l)} = B(x,l) \cup \partial B(x,l)$ and that $p_n(u,v)$ is understood to be $p_{n+1}(u,v)$ if $n + |u+v|_1$ is odd by the convention introduced in Remark 5.3.3.

Lemma 5.3.6. *Let $\mathcal{X}_l^\circ(x)$ and Γ be defined as in* (5.3.15) *and* (5.3.17)*, respectively. There exist $c_{5,0}, c_1, c_{5,2} > 0$ such that the following hold for all sufficiently large l:*

1. *Uniformly in $m \geq (c_{5,0}l)^2$, $u \in B(x,l/2)$, $v \in \overline{B(x,l)}$,*

$$p_m^{B(x,l)\setminus\mathcal{O}}(u,v) \leq e^{-c_1(\lfloor m/(c_{5,0}l)^2\rfloor - 1)\Gamma(|\mathcal{X}_l^\circ|)} p_m^{B(x,l)}(u,v). \qquad \text{(RW1)}$$

2. *Uniformly in $m \leq (c_{5,0}l)^2$ and $u \in \overline{B(x,l/2)}, v \in \partial B(x,l)$,*

$$p_m^{B(x,l)}(u,v) \leq \frac{1}{100} p_{m+l^2}^{B(x,l)}(u,v). \qquad \text{(RW2)}$$

3. Uniformly in $m > 0$ and $u \in \partial B(x, l/2), v \in \partial B(x, l)$,

$$p_m^{B(x,l)}(u, v) \leq e^{c_{5,2}(ml^{-2}+1)} p_{m+l^2}^{A(x;l/4,l)}(u, v). \tag{RW3}$$

4. Uniformly in $m > 0$ and $u \in \partial B(x, l/2), v \in B(x, l/4)$,

$$p_m^{B(x,l)}(u, v) \leq e^{c_{5,2}(ml^{-2}+1)} p_{m+2l^2}^{A(x;l/4,l)}(u, v + (5l/8)\mathbf{e}_1). \tag{RW4}$$

5. Suppose $0 \in B(x, l/2)$ and let $n \in \mathbb{N}$ be such that $n\mathbf{e}_1 \in \partial B(x, l/2)$. Then uniformly in $m > 0$, and $v \in \partial B(x, l)$,

$$p_m^{B(x,l)}(0, v) \leq e^{c_{5,2}(ml^{-2}+1)} p_{m+2l^2}^{A(x;l/4,l)}(n\mathbf{e}_1, v). \tag{RW5}$$

In the proof of this lemma, we will use the following estimate on the Dirichlet heat kernel: for any $c \in (0, 1)$, there exists $C > 0$ such that uniformly in $r \in \mathbb{N}$, $k \in [cr^2, r^2/c]$ and $u, w \in B(x, r)$,

$$C \leq \frac{p_k^{B(x,r)}(u, w)}{r^{-d-2}\text{dist}(u, \partial B(x, r))\text{dist}(w, \partial B(x, r))} \leq \frac{1}{C}. \tag{5.3.61}$$

This can be found for example in [53, Proposition 6.9.4], where it is stated only for the case $k = r^2$ but the argument therein can easily be adapted to the above uniform estimate.

The first assertion (RW1) will be a direct consequence of the following lemma.

Lemma 5.3.7. *For any $c_{5,0} \in (0, 1)$, there exists $c_1 > 0$ independent of $\mathcal{O}$ such that for any $l \in \mathbb{N}$ and $u, w \in B(x, l)$,*

$$p_{(c_{5,0}l)^2}^{B(x,l)\setminus\mathcal{O}}(u, w) \leq e^{-c_1\Gamma(|\mathcal{X}_l^\circ|)} p_{(c_{5,0}l)^2}^{B(x,l)}(u, w). \tag{5.3.62}$$

Remark 5.3.8. This lemma holds for arbitrary $c_{5,0} \in (0, 1)$. In Lemma 5.3.6, it is only

(RW2) which imposes a restriction on $c_{5,0}$.

Proof of Lemma 5.3.7. We write l_0 for $c_{5,0}l$ in this proof to ease the notation. It suffices to prove

$$\min_{u,w\in B(x,l)} \mathbf{P}_u\big(\tau_{\mathcal{O}} \le l_0^2 \,\big|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\big) \ge c_1\Gamma(|\mathcal{X}_l^\circ|). \tag{5.3.63}$$

for some $c_1 > 0$, since $1 - \lambda \le e^{-\lambda}$. The proof of this relies on the so-called *second moment method*. Let us introduce

$$T := \sum_{m\in[l_0^2/4, 3l_0^2/4]} 1_{\{S_m\in\mathcal{X}_l^\circ\}}. \tag{5.3.64}$$

We will show the following:

$$\mathbf{E}_u\left[T \,\middle|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right] \ge c|\mathcal{X}_l^\circ| l_0^{2-d}, \tag{5.3.65}$$

$$\mathbf{E}_u\left[T^2 \,\middle|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right] \le \frac{C(|\mathcal{X}_l^\circ| l_0^{2-d})^2}{\Gamma(|\mathcal{X}_l^\circ|)}. \tag{5.3.66}$$

Given these bounds, the desired bound follows via the Paley–Zygmunnd inequality as

$$\begin{aligned}
&\mathbf{P}_u\left(\tau_{\mathcal{O}} \le l_0^2 \,\middle|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right) \\
&\quad\ge \mathbf{P}_u\left(T > 0 \,\middle|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right) \\
&\quad\ge \frac{\mathbf{E}_u\left[T \,\middle|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right]^2}{\mathbf{E}_u\left[T^2 \,\middle|\, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right]}.
\end{aligned} \tag{5.3.67}$$

First moment (5.3.65): Note first that uniformly in $m \in [l_0^2/4, 3l_0^2/4]$, $z \in B(x, l/2)$ and u, w as in the statement,

$$\begin{aligned}
\mathbf{P}_u\left(S_m = z, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2\right) &= p_m^{B(x,l)}(u,z) p_{l_0^2-m}^{B(x,l)}(z,w) \\
&\ge \frac{c}{l_0^d} p_{l_0^2}^{B(x,l)}(u,w).
\end{aligned} \tag{5.3.68}$$

Indeed, by (5.3.61), there exists $C > 0$ such that uniformly in m, u, w and z as above,

$$p_m^{B(x,l)}(u,z) \geq \frac{C}{l_0^{d+1}} \mathrm{dist}(u, \partial B(x,l)), \tag{5.3.69}$$

$$p_{l_0^2-m}^{B(x,l)}(z,w) \geq \frac{C}{l_0^{d+1}} \mathrm{dist}(w, \partial B(x,l)), \tag{5.3.70}$$

and

$$p_{l_0^2}^{B(x,l)}(u,w) \leq \frac{1}{Cl_0^{d+2}} \mathrm{dist}(u, \partial B(x,l)) \mathrm{dist}(w, \partial B(x,l)). \tag{5.3.71}$$

The bound (5.3.68) follows from these three bounds. Summing (5.3.68) over $m \in [l_0^2/4, 3l_0^2/4]$ and $z \in \mathcal{X}_l^\circ$ yields the following equivalent of (5.3.65):

$$\begin{aligned} &\mathbf{E}_u \left[T \colon S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2 \right] \\ &\quad \geq c |\mathcal{X}_l^\circ| l_0^{2-d} \mathbf{P}_u \left(S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2 \right). \end{aligned} \tag{5.3.72}$$

Second moment (5.3.66): We begin with

$$\begin{aligned} &\mathbf{E}_u \left[T^2 \colon S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2 \right] \\ &= 2 \sum_{l_0^2/4 \leq i \leq j \leq 3l_0^2/4} \sum_{z_1, z_2 \in \mathcal{X}_l^\circ} \mathbf{P}_u \left(S_i = z_1, S_j = z_2, S_{l_0^2} = w, \tau_{B(x,l)^c} > l_0^2 \right) \\ &= 2 \sum_{\substack{k,m \geq l_0^2/4, \\ k+m \leq 3l_0^2/4}} \sum_{z_1, z_2 \in \mathcal{X}_l^\circ} p_k^{B(x,l)}(u, z_1) p_m^{B(x,l)}(z_1, z_2) p_{l_0^2-k-m}^{B(x,l)}(z_2, w). \end{aligned} \tag{5.3.73}$$

It follows from (5.3.61) as before that for the parameters appearing above,

$$p_k^{B(x,l)}(u, z_1) p_{l_0^2-k-m}^{B(x,l)}(z_2, v) \leq \frac{c}{l_0^d} p_{l_0^2}^{B(x,l)}(u, v). \tag{5.3.74}$$

Substituting this into (5.3.73) and performing the summation over z_2 and k, we find that

$$\begin{aligned}&\mathbf{E}_u\left[T^2\colon S_{l_0^2}=v,\tau_{B(x,l)^c}>l_0^2\right]\\&\quad\le\frac{c}{l_0^{d-2}}p_{l_0^2}^{B(x,l)}(u,v)\sum_{m\le l_0^2/2}\sum_{z\in\mathcal{X}_l^\circ}\mathbf{P}_z(S_m\in\mathcal{X}_l^\circ).\end{aligned}\tag{5.3.75}$$

For the probability appearing in the summation, we claim

$$\sup_{z\in\mathcal{X}_l^\circ}\mathbf{P}_z(S_m\in\mathcal{X}_l^\circ)\le c\begin{cases}m^{-d/2}, & m<l_0^{1/d},\\ l_0^{-1/2}, & m\in[l_0{}^{1/d},|\mathcal{X}_l^\circ|^{2/d}l_0{}^{1/d}],\\ |\mathcal{X}_l^\circ|m^{-d/2}, & m\in(|\mathcal{X}_l^\circ|^{2/d}l_0{}^{1/d},l_0^2/2].\end{cases}\tag{5.3.76}$$

Substituting this bound into (5.3.75), we obtain

$$\mathbf{E}_u\left[T^2\colon S_{l_0^2}=v,\tau_{B(x,l)^c}>l_0^2\right]\le\frac{C(|\mathcal{X}_l^\circ|l_0^{2-d})^2}{\Gamma(|\mathcal{X}_l^\circ|)}p_{l_0^2}^{B(x,l)}(u,v)\tag{5.3.77}$$

which is equivalent to (5.3.66).

It remains to show (5.3.76). First, the *on-diagonal* term $\mathbf{P}_z(S_m=z)\le cm^{-d/2}$ is always smaller than the right-hand side of (5.3.76). Henceforth, we shall focus on the points $w\in\mathcal{X}_l^\circ\setminus\{z\}$ which are at least $l^{1/2d}$ away from z. By a standard Gaussian estimate on the transition probability of the symmetric simple random walk [64, Theorem 6.28],

$$\begin{aligned}&\sum_{w\in\mathcal{X}_l^\circ\setminus\{z\}}\mathbf{P}_z(S_m=w)\\&\quad\le\sum_{n=1}^{\infty}\sum_{w\in\mathcal{X}_l^\circ\cap A(z;nl^{1/2d},(n+1)l^{1/2d})}\frac{C}{m^{d/2}}\exp\left\{-\frac{|w-z|^2}{Cm}\right\}.\end{aligned}\tag{5.3.78}$$

This right-hand side is maximized when the annuli are filled from inside out but since the points in $\mathcal{X}_l^\circ$ are at least $l^{1/2d}$ away from each other, the n-th annulus contains at most

Cn^{d-1} points. This leads us to the bound

$$\sum_{w\in\mathcal{X}_l^\circ\setminus\{z\}} \mathbf{P}_z(S_m = w) \le \frac{C}{m^{d/2}} \sum_{n=1}^{C|\mathcal{X}_l^\circ|^{1/d}} n^{d-1} \exp\left\{-\frac{n^2 l^{1/d}}{Cm}\right\}. \tag{5.3.79}$$

The desired bound (5.3.76) follows by a simple computation considering the cases $m < l^{1/d}$, $m \in [l^{1/d}, |\mathcal{X}_l^\circ|^{2/d} l^{1/d}]$ and $m > |\mathcal{X}_l^\circ|^{2/d} l^{1/d}$ separately. □

Proof of Lemma 5.3.6. We only consider the case when $l, 5l/8$ and n are all even.

The first assertion (RW1) follows immediately from Lemma 5.3.7. Indeed, using (5.3.62) in the Chapman–Kolmogorov identity, we have

$$\begin{aligned} p_m^{B(x,l)\setminus\mathcal{O}}(u,v) &= \sum_{w\in B(x,l)} p_{(c_{5,0}l)^2}^{B(x,l)\setminus\mathcal{O}}(u,w) p_{m-(c_{5,0}l)^2}^{B(x,l)\setminus\mathcal{O}}(w,v) \\ &\le e^{-c\Gamma(|\mathcal{X}_l^\circ|)} \sum_{w\in B(x,l)} p_{(c_{5,0}l)^2}^{B(x,l)}(u,w) p_{m-(c_{5,0}l)^2}^{B(x,l)\setminus\mathcal{O}}(w,v) \end{aligned} \tag{5.3.80}$$

and (RW1) follows by iteration.

Let us proceed to prove the second assertion (RW2). Note first that by a standard Gaussian heat kernel bound [64, Theorem 6.28], for any $u, w \in B(x,l)$ with $|u-w| \ge cl$,

$$\begin{aligned} p_k^{B(x,l)}(u,w) &\le p_k^{\mathbb{Z}^d}(u,w) \\ &\le Ck^{-d/2} \exp\left\{-\frac{|u-w|^2}{Ck}\right\} \\ &\le Cl^{-d} \left(\frac{l^2}{k}\right)^{d/2} \exp\left\{-c\frac{l^2}{k}\right\}. \end{aligned} \tag{5.3.81}$$

On the other hand, for $m \in [l^2, 2l^2]$, $u \in \overline{B(x,l/2)}$ and $w \in \partial B(x,3l/4)$, we have

$$p_m^{B(x,l)}(u,w) \ge Cl^{-d} \tag{5.3.82}$$

by (5.3.61). Combining this with (5.3.81), we obtain the comparison

$$\max_{k \le (c_{5,0} l)^2} \max_{u \in \overline{B(x,l/2)}, w \in \partial B(x,3l/4)} \frac{p_k^{B(x,l)}(u,w)}{p_{k+l^2}^{B(x,l)}(u,w)} \le \frac{1}{100} \tag{5.3.83}$$

for sufficiently small $c_{5,0}$. Suppose $m \le (c_{5,0} l)^2$ for $c_{5,0}$ satisfying the above. By decomposing the random walk path upon the last visit to $\partial B(x, 3l/4)$, we get

$$\begin{aligned} p_m^{B(x,l)}(u,v) &= \sum_{k=1}^{m} \sum_{w \in \partial B(x,3l/4)} p_k^{B(x,l)}(u,w) p_{m-k}^{A(x;3l/4,l)}(w,v) \\ &\le \frac{1}{100} \sum_{k=1}^{m} \sum_{w \in \partial B(x,3l/4)} p_{k+l^2}^{B(x,l)}(u,w) p_{m-k}^{A(x;3l/4,l)}(w,v) \\ &= \frac{1}{100} p_{m+l^2}^{B(x,l)}(u,v). \end{aligned} \tag{5.3.84}$$

This concludes the proof of (RW2).

The proofs of (RW3)–(RW5) rely on the fact that there exists $c_{5,2} > 0$ such that for any $m \in \mathbb{N}$, $u \in \partial B(x, l/2)$ and $w \in A(x; 3l/8, 7l/8)$,

$$p_{m+l^2}^{A(x;l/4,l)}(u,w) \ge \exp\{-c_{5,2}(\lfloor ml^{-2} \rfloor + 1)\} p_m^{B(x,l)}(u,w). \tag{5.3.85}$$

In order to prove this, we bound the left-hand side from below by the probability that $S_{[0,l^2]} \subset A(x; l/4, l)$, $S_{[l^2, m+l^2]} \subset B(w, l/8)$ and $S_{kl^2} \in B(w, l/16)$ for each $k \in \{1, 2, \dots, \lfloor ml^{-2} \rfloor\}$, which can be written as

$$\begin{aligned} \sum_{z_1, \dots, z_{\lfloor ml^{-2} \rfloor} \in B(w,l/16)} p_{l^2}^{A(x;l/4,l)}(u, z_1) \left(\prod_{j=1}^{\lfloor ml^{-2} \rfloor - 1} p_{l^2}^{B(w,l/8)}(z_j, z_{j+1}) \right) \\ \times p_{m - \lfloor ml^{-2} \rfloor l^2 + l^2}^{B(w,l/8)}(z_{\lfloor ml^{-2} \rfloor}, w). \end{aligned} \tag{5.3.86}$$

Since $m - \lfloor ml^{-2} \rfloor l^2 + l^2 \in [l^2, 2l^2]$ and all the points u, z_j's and w are at least $l/16$ away from the corresponding boundaries, by (5.3.61), all the heat kernels appearing in this expression is bounded from below by Cl^{-d} regardless where z_j's are in $B(w, l/16)$. Therefore we find the bound

$$p_{m+l^2}^{A(x;l/4,l)}(u,w) \geq \exp\{-c_{5,2}(\lfloor ml^{-2} \rfloor + 1)\} l^{-d} \tag{5.3.87}$$

for some $c_{5,2} > 0$. Recalling (5.3.81), we have $p_m^{B(x,l)}(u,w) \leq cl^{-d}$ and we conclude the proof of (5.3.85).

Given (5.3.85), the third bound (RW3) can be proved in the same way as in the proof of (RW2) via the last visit decomposition. In order to prove the bound (RW4), we first replace m by $m + l^2$ and choose $u \in \partial B(x, l/2)$ and $w = v + (5l/8)\mathbf{e}_1$ for $v \in B(x, l/4)$ in (5.3.85) to obtain

$$\begin{aligned} &\exp\{c_{5,2}(\lfloor ml^{-2} \rfloor + 1)\} p_{m+l^2}^{A(x;l/4,l)}(u, v + (5l/8)\mathbf{e}_1) \\ &\quad \geq p_{m+l^2}^{B(x,l)}(u, v + (5l/8)\mathbf{e}_1). \end{aligned} \tag{5.3.88}$$

We can further bound the right-hand side from below by $cp_m^{B(x,l)}(u,v)$ using either (5.3.83) ($m \leq (c_{5,0}l)^2$) or the parabolic Harnack inequality from [19] ($m > (c_{5,0}l)^2$). This yields (RW4) by making $c_{5,2}$ larger. The proof of (RW5) is almost the same and left to the reader. □

5.4 Random Walk Range and "Truly"-Open Sites

In this section, we prove various properties of $\mathcal{T}$ and its relation with the random walk range $S_{[0,N]}$, which will pave the way for the proof of Theorems 5.1.1 and 5.2.8. First, we prove Lemma 5.2.5 in Subsection 5.4.1, which shows that the random walk must visit the interior of $\mathcal{T}$, and sites in $\mathcal{T}$ are well-approximated by sites in $S_{[0,N]}$. We then explain in Subsection 5.4.2 how Lemma 5.2.6, Proposition 5.2.9, and the deduction of Theorem 5.2.8 from Theorem 5.2.7 all follow from the same key Lemma 5.4.3 on the probability of visiting certain sites that are costly for survival. The proof of Lemma 5.4.3 is then given in Subsection 5.4.4 using path decomposition and switching, with the basic setup presented earlier in

Section 5.4.3.

5.4.1 Proof of Lemma 5.2.5

In this subsection, we give the proof of Lemma 5.2.5. First we show that "truly"-open sites are rare in the following sense.

Lemma 5.4.1 ("truly"-open sites are rare)**.** *For any $v \in \mathbb{Z}^d$ and all sufficiently large N,*

$$\mathbb{P}(v \textit{ is "truly"-open}) \leq \exp\left\{-(\log N)^2\right\}. \tag{5.4.1}$$

Proof of Lemma 5.4.1. Recall that $x \in \mathbb{Z}^d$ is "truly"-open if

$$\mathbf{P}_x\left(\tau_{\mathcal{O}} > (\log N)^5\right) \geq \exp\left\{-(\log N)^2\right\}. \tag{5.4.2}$$

By using Donsker–Varadhan's asymptotics (5.1.5), we obtain

$$\begin{aligned} &\mathbb{P}\left(\mathbf{P}_x\left(\tau_{\mathcal{O}} > (\log N)^5\right) \geq \exp\left\{-(\log N)^2\right\}\right) \\ &\quad\leq \exp\left\{(\log N)^2\right\}\mathbb{E}\left[\mathbf{P}\left(\tau_{\mathcal{O}} > (\log N)^5\right)\right] \\ &\quad\leq \exp\left\{(\log N)^2\right\}\exp\left\{-c(\log N)^{\frac{5d}{d+2}}\right\}. \end{aligned} \tag{5.4.3}$$

Since the power of $5d/(d+2) > 2$ for $d \geq 2$, we are done. □

Proof of Lemma 5.2.5. The proof of the first assertion (5.2.6) is simple. Indeed, since a "truly"-open site is open by definition, for any site in

$$\left\{x \in \mathcal{T} \colon \operatorname{dist}(x, \partial\mathcal{T}) \geq (\log N)^5\right\}, \tag{5.4.4}$$

its $(\log N)^5$ neighborhood is free of obstacles. Therefore (5.2.6) follows from Lemma 5.3.2.

The second assertion (5.2.7) can be restated as

$$\lim_{N\to\infty} \mu_N \left(\bigcap_{w\in\mathcal{T}} \{\tau_{B(w,(\log N)^5)} \le N\} \right) = 1. \tag{5.4.5}$$

We are going to show that for any $w \in B(0, 3\varrho_N)$,

$$\mu_N \left(w \text{ is "truly"-open}, \tau_{B(w,(\log N)^5)} > N \right) \le \exp\left\{ -(\log N)^2 \right\}, \tag{5.4.6}$$

from which (5.4.5) follows by the union bound. But since whether w is "truly"-open or not depends only on the configuration of obstacles inside $B(w,(\log N)^5)$ and hence is independent of $\mathbf{P}(\tau_{B(w,(\log N)^5)} \wedge \tau_{\mathcal{O}} > N)$, we have

$$\begin{aligned} &\mathbb{P}\otimes\mathbf{P}\left(w \text{ is "truly"-open}, \tau_{B(w,(\log N)^5)} \wedge \tau_{\mathcal{O}} > N \right) \\ &\quad = \mathbb{P}(w \text{ is "truly"-open})\mathbb{E}\left[\mathbf{P}\left(\tau_{B(w,(\log N)^5)} \wedge \tau_{\mathcal{O}} > N \right) \right] \\ &\quad \le \exp\left\{ -(\log N)^2 \right\} \mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}} > N) \end{aligned} \tag{5.4.7}$$

by using Lemma 5.4.1. □

5.4.2 Proof of Lemma 5.2.6, Proposition 5.2.9 and Theorem 5.2.8

The proof of Lemma 5.2.6 and Proposition 5.2.9 turn out to be quite similar, both involving random walk path switching to avoid sites that are costly for survival. As explained in Subsection 5.2.4, to bound $|\partial\mathcal{T}|$ and prove Proposition 5.2.9, it suffices to give an upper bound on the expected total number of visits to $\bigcup_{x\in\partial\mathcal{T}} B(x,(\log N)^6)$, as well as a uniform lower bound on the expected number of visits to $B(x,(\log N)^6)$ over all $x \in \partial\mathcal{T}$. More precisely, define

$$G_{\mathcal{O}}(u,x) := \mathbf{E}_u\left[\sum_{n=0}^{\tau_{\mathcal{O}}} 1_{S_n\in B(x,(\log N)^6)} \right], \quad u \in \mathcal{T}, x \in \partial\mathcal{T}, \tag{5.4.8}$$

which is the expected number of visits to $B(x, (\log N)^6)$ before the walk is killed. We also introduce the set where the above expected number of visits is too small:

$$\mathcal{L}(l) := \bigcup_{x \in \partial \mathcal{T}, G_{\mathcal{O}}(\boldsymbol{x}_N, x) \le \varrho_N^{1-d} \varphi(N,l)} B(x, (\log N)^6), \tag{5.4.9}$$

where

$$\varphi(N,l) := \begin{cases} \varrho_N^{-c_{5,4}\epsilon}, & \text{if } l = \epsilon \varrho_N, \\ (\log N)^{-c_{5,5}}, & \text{if } l = \varrho_N / \log N \end{cases} \tag{5.4.10}$$

with $c_{5,4} \in (0,1)$ and $c_{5,5} > 0$ to be chosen later.

We first claim that the expected number of visits to the neighborhood of $\partial \mathcal{T}$ is not too large.

Lemma 5.4.2. *There exists $c_{5,6} > 0$ such that*

$$\sum_{x \in \partial \mathcal{T}} G_{\mathcal{O}}(\boldsymbol{x}_N, x) \le (\log N)^{c_{5,6}}. \tag{5.4.11}$$

We will then show that on the event of confinement in $B(\boldsymbol{x}_N, \varrho_N + l)$ and $B(\boldsymbol{x}_N, \varrho_N - l/4)$ being free of obstacles, the probability for the random walk to visit $\mathcal{T}^c$ or $\mathcal{L}(l)$ is asymptotically negligible.

Lemma 5.4.3. *There exists $\epsilon_0 > 0$ such that the following holds: let $l := \epsilon \varrho_N$ with $\epsilon \le \epsilon_0$ or $l := \varrho_N / \log N$, and assume that*

$$\lim_{N \to \infty} \mu_N \left(\tau_{B(\boldsymbol{x}_N, \varrho_N + l)} > N, \mathcal{O} \cap B(\boldsymbol{x}_N, \varrho_N - l/4) = \emptyset \right) = 1. \tag{5.4.12}$$

Then

$$\lim_{N \to \infty} \mu_N \left(\tau_{\mathcal{T}^c \cup \mathcal{L}(l)} \le N \right) = 0. \tag{5.4.13}$$

Let us present three consequences of these two lemmas before giving proofs.

Proof of Lemma 5.2.6. Since Theorem E and Proposition 5.2.2 imply

$$\lim_{N\to\infty} \mu_N \left(\tau_{B(\boldsymbol{x}_N,(1+\epsilon)\varrho_N)} > N, \mathcal{O} \cap B(\boldsymbol{x}_N, (1-\epsilon/4)\varrho_N) = \emptyset \right) = 1 \tag{5.4.14}$$

for any $\epsilon > 0$, Lemma 5.2.6 immediately follows from Lemma 5.4.3 with $l = \epsilon\varrho_N$. □

Proof of Proposition 5.2.9. By Lemma 5.2.5 (cf. (5.4.5)), we know that the random walk does visit $B(v, (\log N)^6)$ for each $v \in \partial\mathcal{T}$, and together with Lemma 4.7, this implies that we must have $\mathcal{L}(\epsilon\varrho_N) = \emptyset$. This means that we have a uniform lower bound $\min_{x\in\partial\mathcal{T}} G_{\mathcal{O}}(\boldsymbol{x}_N, x) \geq \varrho_N^{1-d-c_{5,4}\epsilon}$ and hence

$$\sum_{x\in\partial\mathcal{T}} G_{\mathcal{O}}(\boldsymbol{x}_N, x) \geq |\partial\mathcal{T}|\varrho_N^{1-d-c_{5,4}\epsilon}. \tag{5.4.15}$$

Combining with Lemma 5.4.2, we conclude that $|\partial\mathcal{T}| \leq \varrho_N^{d-1+c_{5,4}\epsilon}(\log N)^{c_{5,6}}$, and since $\epsilon > 0$ can be taken arbitrarily small, Proposition 5.2.9 follows. □

Proof of Theorem 5.2.8 assuming Theorem 5.2.7. Observe that once we have proved Theorem 5.2.7, we may take $l = \varrho_N/\log N$ in Lemma 5.4.3. Then the same argument as above yields Theorem 5.2.8. □

We close this subsection with the proof of Lemma 5.4.2 which is fairly simple. The proof of Lemma 5.4.3 is much more involved and will take up the next two subsections.

Proof of Lemma 5.4.2. Let us define the stopping times

$$\xi_1 := \inf\left\{n \geq 0 \colon \operatorname{dist}(S_n, \partial\mathcal{T}) < (\log N)^6\right\} \tag{5.4.16}$$

and for $k \geq 1$,

$$\xi_{k+1} := \inf\left\{n \geq \xi_k + 2(\log N)^{10} \colon \operatorname{dist}(S_n, \partial\mathcal{T}) < (\log N)^6\right\}. \tag{5.4.17}$$

We can then bound the left-hand side of (5.4.11) by

$$\sum_{x\in\partial\mathcal{T}} G_{\mathcal{O}}(\boldsymbol{x}_N, x) \le (\log N)^C \mathbf{E}\left[\max\{k\colon \xi_k < \tau_{\mathcal{O}}\}\right] \tag{5.4.18}$$

for some $C > 0$. Observe that whenever the random walk visits $(\log N)^6$ neighborhood of $\partial\mathcal{T}$, there is more than $c(\log N)^{-6d}$ probability of exiting $\mathcal{T}$ within the next $(\log N)^{12}$ steps by the local central limit theorem. And once the random walk exits $\mathcal{T}$, it will hit $\mathcal{O}$ in the next $(\log N)^{12}$ steps with high probability by the definition of $\mathcal{T}$. Therefore $\max\{k\colon \xi_k < \tau_{\mathcal{O}}\}$ is stochastically dominated by a geometric random variable with parameter $c(\log N)^{-6d}$ and the desired bound follows. □

5.4.3 Path decomposition

In order to prove Lemma 5.4.3, what will be relevant is the behavior of the random walk near $\partial\mathcal{T}$. Since $v \in \mathbb{Z}^d$ is "truly"-open if its $(\log N)^5$ neighborhood is open and we assume $\mathcal{O} \cap B(\boldsymbol{x}_N, \varrho_N - l/4) = \emptyset$, we know that $\partial\mathcal{T}$ lies near $\partial B(\boldsymbol{x}_N, \varrho_N)$. This motivates us to decompose the random walk paths according to the crossings of a thin annulus near the boundary of the confinement ball $B(\boldsymbol{x}_N, \varrho_N)$.

Similarly to (5.3.9)– (5.3.12), we decompose a random walk path $(S_n)_{0\geq n\leq N}$ by using successive crossings between the inner and outer shells of the annulus

$$A(\boldsymbol{x}_N; \varrho_N - 2l, \varrho_N - l) = B(\boldsymbol{x}_N, \varrho_N - l) \setminus \overline{B(\boldsymbol{x}_N, \varrho_N - 2l)}, \tag{5.4.19}$$

where we will choose $l > 0$ to be either $\epsilon\varrho_N$ or $\varrho_N/\log N$. To this end, we introduce the stopping times

$$\sigma_1 := \min\{n \geq 0\colon S_n \in \overline{B(\boldsymbol{x}_n, \varrho_N - 2l)}\}\wedge N, \tag{5.4.20}$$

and for $k \in \mathbb{N}$,

$$\tau_k := \min\{n > \sigma_k \colon S_n \in B(\boldsymbol{x}_n, \varrho_N - l)^c\} \wedge N, \tag{5.4.21}$$

$$\sigma_{k+1} := \min\{n > \tau_k \colon S_n \in \overline{B(\boldsymbol{x}_n, \varrho_N - 2l)}\} \wedge N. \tag{5.4.22}$$

In what follows, we will decompose the random walk paths into the pieces $(S_{[\tau_k, \tau_{k+1}]})_{k \geq 1}$ and the role of $(\sigma_k)_{k \geq 1}$ is auxiliary. More precisely, the paths that visit a costly site $v \in \mathcal{T}^c \cup \mathcal{L}(l)$ (cf. (5.4.9)) during $[\tau_k, \tau_{k+1}]$ are going to be switched to the paths that stay inside $B(\boldsymbol{x}_N, \varrho_N - l/2)$ during $[\tau_k, \tau_{k+1}]$.

We use bold face letters to denote sequences of numbers as in Subsection 5.3.3. For a non-decreasing sequence $\boldsymbol{t} = (t_k)_{k \geq 1}$ of integers, by slightly abusing our notation in Subsection 5.3.3, we write

$$K(\boldsymbol{t}) := \sup\{k \geq 1 \colon t_{k+1} - t_k > 0\} \tag{5.4.23}$$

which represents the number of crossings from $\partial B(\boldsymbol{x}_N, \varrho_N - l)$ to $\partial B(\boldsymbol{x}_N, \varrho_N - 2l)$ and back to $\partial B(\boldsymbol{x}_N, \varrho_N - l)$ when $\boldsymbol{t} = \boldsymbol{\tau}$. We have the following control on $K(\boldsymbol{\tau})$.

Lemma 5.4.4. *There exists $c_{5,3} > 0$ depending only on the dimension d such that if*

$$\lim_{N \to \infty} \mu_N \left(\tau_{B(\boldsymbol{x}_N, \varrho_N + l)^c} > N, \mathcal{O} \cap B(\boldsymbol{x}_N, \varrho_N - l/4) = \emptyset \right) = 1 \tag{5.4.24}$$

for some $l \in [\varrho_N / \log N, c_{5,3} \varrho_N]$, then

$$\lim_{N \to \infty} \mu_N \left(K(\boldsymbol{\tau}) \leq N l^{-2} \right) = 1. \tag{5.4.25}$$

Proof of Lemma 5.4.4. Let us fix $l \in [\varrho_N / \log N, c_{5,3} \varrho_N]$ and suppose that $K(\boldsymbol{\tau}) > N l^{-2}$, or equivalently, $\sigma_{\lfloor Nl^{-2} \rfloor} < N$. Then we find that

$$\begin{aligned}\mathbf{P}&\left(\tau_{B(\boldsymbol{x}_N,\varrho_N+l)^c} > N, K(\boldsymbol{\tau}) > Nl^{-2}\right) \\ &\leq \sup_{u\in\partial B(\boldsymbol{x}_N,\varrho_N-l)} \mathbf{P}_u\left(\tau_{B(\boldsymbol{x}_N,\varrho_N-2l)} < \tau_{B(\boldsymbol{x}_N,\varrho_N+l)^c}\right)^{Nl^{-2}-1}\end{aligned} \tag{5.4.26}$$

by using the strong Markov property at each τ_k with $k \leq Nl^{-2}-1$. Since

$$\sup_{u\in\partial B(\boldsymbol{x}_N,\varrho_N-l)} \mathbf{P}_u\left(\tau_{B(\boldsymbol{x}_N,\varrho_N-2l)} < \tau_{B(\boldsymbol{x}_N,\varrho_N+l)^c}\right) \tag{5.4.27}$$

is bounded away from one for all large N, by choosing $c_{5,3}$ sufficiently small and recalling (5.1.5), we find that the right-hand side of (5.4.26) decays faster than $\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}} > N)$. □

It is also useful to know that the random walk does not end up near the boundary of the confinement ball at time N.

Lemma 5.4.5.

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} \mu_N\left(S_N \in B(\boldsymbol{x}_N, (1-\epsilon)\varrho_N)\right) = 1. \tag{5.4.28}$$

Proof of Lemma 5.4.5. By Theorem E and Proposition 5.2.2, it suffices to show that

$$\begin{aligned}\mathbb{P}\otimes\mathbf{P}\Big(S_N \notin B(\boldsymbol{x}_N,(1-\epsilon)\varrho_N) \mid \tau_{\mathcal{O}\cup B(\boldsymbol{x}_N,\varrho_N+\varrho_N^{\epsilon_1})^c} > N,& \\ \mathcal{O}\cap B(\boldsymbol{x}_N,(1-\epsilon)\varrho_N) = \emptyset\Big)&\end{aligned} \tag{5.4.29}$$

tends to zero as $N\to\infty$ and $\epsilon\to 0$. Let us write $\mathcal{B} = B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1}) \setminus \mathcal{O}$ in this proof to ease the notation. We use the eigenfunction expansion to get

$$\mathbf{E}\left[f(S_N)\colon \tau_{\mathcal{B}^c} > N\right] = \sum_{k\geq 1}\left(1-\lambda_{\mathcal{B}}^{\mathrm{RW},(k)}\right)^N \left\langle \phi_{\mathcal{B}}^{\mathrm{RW},(k)}, f\right\rangle \phi_{\mathcal{B}}^{\mathrm{RW},(k)}(0) \tag{5.4.30}$$

for any bounded function f, where $\lambda_{\mathcal{B}}^{\mathrm{RW},(k)}$ and $\phi_{\mathcal{B}}^{\mathrm{RW},(k)}$ are the k-th smallest eigenvalue and corresponding eigenfunction with $\|\phi_{\mathcal{B}}^{\mathrm{RW},(k)}\|_2 = 1$ for the generator of the random walk killed upon exiting $\mathcal{B}$. On the event $\mathcal{O} \cap B(\boldsymbol{x}_N, (1-\epsilon)\varrho_N) = \emptyset$, each of the eigenvalues $\lambda_{\mathcal{B}}^{\mathrm{RW},(k)}$ and eigenfunctions $\phi_{\mathcal{B}}^{\mathrm{RW},(k)}$ ($k \in \mathbb{N}$), after proper rescaling, should be close to the eigenvalues of the Dirichlet Laplacian on the unit ball. Based on this observation, one can in fact prove (see Lemma 5.A.2 in Appendix 5.A) that

$$\lambda_{\mathcal{B}}^{\mathrm{RW},(2)} - \lambda_{\mathcal{B}}^{\mathrm{RW},(1)} \geq c_{5,7}\varrho_N^{-2}, \tag{EV}$$

$$\left\|\phi_{\mathcal{B}}^{\mathrm{RW},(1)}\right\|_\infty \leq c_{5,8}\varrho_N^{-d/2}, \tag{EF}$$

which are well-known for the eigenvalues and eigenfunctions for the continuum Laplacian. It follows from (EV) that the terms with $k \geq 2$ in (5.4.30) are negligible for bounded f by a standard argument, (see, for example, the proof of Lemma 5.5.2 in Appendix 5.A.) By setting $f = 1_{B(\boldsymbol{x}_N,(1-\epsilon)\varrho_N)^c}$ and $f = 1$ in (5.4.30), we find that

$$\mathbf{P}\left(S_N \notin B(\boldsymbol{x}_N, (1-\epsilon)\varrho_N) \mid \tau_{\mathcal{B}^c} > N\right) = \frac{\left\langle \phi_{\mathcal{B}}^{\mathrm{RW},(1)}, 1_{B(\boldsymbol{x}_N,(1-\epsilon)\varrho_N)^c}\right\rangle}{\left\langle \phi_{\mathcal{B}}^{\mathrm{RW},(1)}, 1\right\rangle}(1+o(1)) \tag{5.4.31}$$

as $N \to \infty$. Since (EF) and the fact $\phi_{\mathcal{B}}^{\mathrm{RW},(1)} \geq 0$ imply that

$$\left\langle \phi_{\mathcal{B}}^{\mathrm{RW},(1)}, 1_{B(\boldsymbol{x}_N,(1-\epsilon)\varrho_N)^c}\right\rangle \leq c\epsilon\varrho_N^{d/2}, \tag{5.4.32}$$

$$\left\langle \phi_{\mathcal{B}}^{\mathrm{RW},(1)}, 1\right\rangle \geq c_{5,8}^{-1}\varrho_N^{d/2}\|\phi_{\mathcal{B}}^{\mathrm{RW},(1)}\|_2^2 = c_{5,8}^{-1}\varrho_N^{d/2}, \tag{5.4.33}$$

the right-hand side of (5.4.31) vanishes as $\epsilon \to 0$. □

Remark 5.4.6. With a little more effort, it is possible to show that the eigenfunction $\phi_{\mathcal{B}}^{\mathrm{RW},(1)}$ converges, after proper rescaling, to the eigenfunction of the Dirichlet Laplacian on the unit ball in L^2. See, for example, [30] for a further discussion on related problems.

5.4.4 *Proof of Lemma 5.4.3*

Proof of Lemma 5.4.3. Referring to (5.4.12) and Lemmas 5.4.4 and 5.4.5, let us fix $\epsilon \in (0, c_{5,3})$, $l \in \{\varrho_N/\log N, \epsilon\varrho_N\}$ and introduce *good events*

$$E_{5.4.12} := \{\mathcal{O} \cap B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l/4) = \emptyset\}, \tag{5.4.34}$$

$$E_{5.4.4} := \left\{K(\boldsymbol{\sigma}, \boldsymbol{\tau}) \le Nl^{-2}\right\}, \tag{5.4.35}$$

$$E_{5.4.5} := \{\boldsymbol{\mathcal{X}}_N \in B(0, (1-2\epsilon)\varrho_N), S_N \in B(\boldsymbol{\mathcal{X}}_N, (1-2\epsilon)\varrho_N)\} \tag{5.4.36}$$

and define $E_{\text{good}} := E_{5.4.12} \cap E_{5.4.4} \cap E_{5.4.5}$, for which we have $\lim_{N\to\infty} \mu_N(E_{\text{good}}) = 1$. We are going to prove

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{T}^c} \le N < \tau_{\mathcal{O}}, E_{\text{good}}\right) \le c(\log N)^{-1}\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) \tag{5.4.37}$$

and for any $v \in B(0, 3\varrho_N)$,

$$\begin{aligned}&\mathbb{P} \otimes \mathbf{P}\left(v \in \partial\mathcal{T}, G_{\mathcal{O}}(\boldsymbol{\mathcal{X}}_N, v) \le \varrho_N^{1-d}\varphi(N,l), \tau_{B(v,(\log N)^6)} \le N < \tau_{\mathcal{O}}, E_{\text{good}}\right)\\ &\quad \le c(\log N)^{-1}\mathbb{P} \otimes \mathbf{P}\left(v \in \partial\mathcal{T}, \tau_{\mathcal{O}} \wedge \tau_{B(v,(\log N)^6)} > N\right).\end{aligned} \tag{5.4.38}$$

First, the bound (5.4.37) implies that $\lim_{N\to\infty} \mu_N(\tau_{\mathcal{T}^c} \le N) = 0$. Second, since $\mathbf{P}(\tau_{\mathcal{O}} \wedge \tau_{B(v,(\log N)^6)} > N)$ is independent of the obstacle configuration in $B(v, (\log N)^6)$, in particular whether each site $w \in B(v,1)$ is "truly"-open or not, Lemma 5.4.1 implies

$$\begin{aligned}&\mathbb{P} \otimes \mathbf{P}\left(v \in \partial\mathcal{T}, \tau_{\mathcal{O}} \wedge \tau_{B(v,(\log N)^6)} > N\right)\\ &\quad \le \sum_{w \in B(v,1)} \mathbb{P} \otimes \mathbf{P}\left(w \text{ is "truly"-open}, \tau_{\mathcal{O}} \wedge \tau_{B(v,(\log N)^6)} > N\right)\\ &\quad \le \exp\left\{-(\log N)^2\right\}\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right).\end{aligned} \tag{5.4.39}$$

Therefore, by substituting this bound into (5.4.38) and summing over $v \in B(0, 3\varrho_N)$, it follows that $\lim_{N\to\infty} \mu_N(\tau_{\mathcal{L}(l)} \le N) = 0$.

The strategy of the proofs of (5.4.37) and (5.4.38) is to show by a path switching argument that it is better for the random walk not to visit $\mathcal{T}^c$, or $B(v, (\log N)^6)$ with $v \in \partial \mathcal{T}$ and $G_{\mathcal{O}}(\boldsymbol{x}_N, v) \le \varrho_N^{1-d} \varphi(N, l)$. Note that on the event $E_{5.4.12}$, we have

$$\mathcal{T}^c \cup B(v, (\log N)^6) \subset B(\boldsymbol{x}_N, \varrho_N - l/2)^c \tag{5.4.40}$$

and hence it is natural to use the path decomposition in terms of the crossings from $\partial B(\boldsymbol{x}_N, \varrho_N - l)$ to $B(\boldsymbol{x}_N, \varrho_N - 2l)$ introduced in Subsection 5.4.3. The random walk can visit $\mathcal{T}^c \cup B(v, (\log N)^6)$ only on a crossing $[\tau_k, \sigma_{k+1}]$ ($k \in \mathbb{N}$) and if it happens, we want to switch such a crossing to one that avoids $\mathcal{T}^c \cup B(v, (\log N)^6)$. However, it turns out to be easier to switch the path on the entire interval $[\tau_k, \tau_{k+1}]$. More precisely, for $u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ and $u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l) \cup B(\boldsymbol{x}_N, (1 - 2\epsilon)\varrho_N)$, we are going to compare

$$\begin{aligned} &p_t^{\text{visit}}(u, u') \\ &:= \begin{cases} \mathbf{P}_u\left(\tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} < \tau_1 = t < \tau_{\mathcal{O}}, S_t = u'\right), & \text{if } u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l), \\ \mathbf{P}_u\left(\tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} < t < \tau_{\mathcal{O}} \wedge \tau_1, S_t = u'\right), & \text{if } u' \in B(\boldsymbol{x}_N, (1 - 2\epsilon)\varrho_N) \end{cases} \end{aligned} \tag{5.4.41}$$

with

$$\begin{aligned} &p_t^{\text{avoid}}(u, u') \\ &:= \begin{cases} \mathbf{P}_u\left(\tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} > \tau_1 = t, S_t = u'\right), & \text{if } u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l), \\ \mathbf{P}_u\left(\tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} \wedge \tau_1 > t, S_t = u'\right), & \text{if } u' \in B(\boldsymbol{x}_N, (1 - 2\epsilon)\varrho_N). \end{cases} \end{aligned} \tag{5.4.42}$$

These are the probabilities that conditionally on $S_{\tau_k} = u$, the random walk path during $[\tau_k, \tau_{k+1}]$ either visits or avoids $\mathcal{T}^c$ and $B(v; (\log N)^6)$ and ends at u' at time $\tau_{k+1} = \tau_k + t$. The two cases $u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ and $u' \in B(\boldsymbol{x}_N, (1 - 2\epsilon)\varrho_N)$ correspond to $k < K(\boldsymbol{\tau})$ and $k = K(\boldsymbol{\tau})$ respectively, where for the latter case recall that we are working on the event $E_{5.4.5}$. The key comparison estimate we will prove is the following: if $v \in \partial\mathcal{T}$, $G_{\mathcal{O}}(\boldsymbol{x}_N, v) \le \varrho_N^{1-d}\varphi(N,l)$ and $E_{5.4.12}$ holds, then

$$p_t^{\text{visit}}(u,u') \le \varrho_N^{-d}(\log N)^{-3} p_{t+2\varrho_N^2}^{\text{avoid}}(u,u'). \tag{5.4.43}$$

Let us first see how we can deduce the desired bounds (5.4.37) and (5.4.38) from (5.4.43). We assume (5.4.40) and $G_{\mathcal{O}}(\boldsymbol{x}_N, v) \le \varrho_N^{1-d}\varphi(N,l)$. For $j \ge 1$ and $\boldsymbol{k} = (k_i)_{i=1}^j \subset \mathbb{N}^j$, consider the event that the crossings with indices $m \in \boldsymbol{k}$ visit $\mathcal{T}^c \cup B(v, (\log N)^6)$ and others do not. Its probability can be bounded as

$$\begin{aligned}
&\sum_K \sum_{t_1 < t_2 < \ldots < t_{K+1} = N} \sum_{u_1, \ldots, u_{K+1}} p_{t_1}^{B(\boldsymbol{x}_N, \varrho_N - l)}(0, u_1) \\
&\quad \times \prod_{m=1}^K \left(p_{t_{m+1}-t_m}^{\text{avoid}}(u_m, u_{m+1}) 1_{m \notin \boldsymbol{k}} + p_{t_{m+1}-t_m}^{\text{visit}}(u_m, u_{m+1}) 1_{m \in \boldsymbol{k}} \right) \\
&\quad \le \varrho_N^{-dj}(\log N)^{-3j} \mathbf{P}\left(\tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} > N + 2j\varrho_N^2 \right),
\end{aligned} \tag{5.4.44}$$

where in the first line, $u_1, \ldots, u_K \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$, $u_{K+1} \in B(\boldsymbol{x}_N, (1-2\epsilon)\varrho_N)$ and for each $m \in \boldsymbol{k}$, we have used (5.4.43) to get the extra multiplicative factor $\varrho_N^{-d}(\log N)^{-3}$ by lengthening the crossing duration by $2\varrho_N^2$. Recalling that we are assuming $E_{5.4.4}$, we may restrict our consideration to $K \le Nl^{-2} \le \varrho_N^d(\log N)^2$. Therefore there are at most $K^j \le \varrho_N^{dj}(\log N)^{2j}$ choices of $\boldsymbol{k} = (k_i)_{i=1}^j$ and thus it follows that

$$\begin{aligned}
&\mathbf{P}\left(\text{exactly } j \text{ crossings visit } \mathcal{T}^c \cup B(v, (\log N)^6), E_{5.4.4}, \tau_{\mathcal{O}} > N \right) \\
&\quad \le (\log N)^{-j} \mathbf{P}\left(\tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} > N \right).
\end{aligned} \tag{5.4.45}$$

Since $\mathcal{O} \subset \mathcal{T}^c$, summing over $j \ge 1$, we obtain (5.4.37) and (5.4.38).

It remains to prove (5.4.43). Recall first that (5.4.40) holds on the event $E_{5.4.12}$. In particular, during $[\sigma_1, \tau_1]$, the random walk stays inside $B(\boldsymbol{x}_N, \varrho_N - l)$ and can visit neither $\mathcal{O}$ nor $\mathcal{T}^c \cup B(v, (\log N)^6)$. Based on this observation, both cases in (5.4.41) can be described as follows: the random walk starting from $u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ visits $\mathcal{T}^c \cup B(v, (\log N)^6)$ and $\overline{B(\boldsymbol{x}_N, \varrho_N - 2l)}$ in this order without hitting $\mathcal{O}$, and then stays inside $B(\boldsymbol{x}_N, \varrho_N - l)$ before it ends at u' at time t. Therefore, using the strong Markov property at σ_1, the first hitting time of $\overline{B(\boldsymbol{x}_N, \varrho_N - 2l)}$, we obtain

$$\begin{aligned} &p_t^{\text{visit}}(u, u') \\ &\quad = \mathbf{E}_u \left[p_{t-\sigma_1}^{B(\boldsymbol{x}_N, \varrho_N - l)}(S_{\sigma_1}, u') \colon \tau_{\mathcal{T}^c \cup B(v, (\log N)^6)} < \sigma_1 < t \wedge \tau_{\mathcal{O}} \right]. \end{aligned} \tag{5.4.46}$$

Similarly, by (5.4.40), for the random walk to avoid $\mathcal{T}^c \cup B(v, (\log N)^6)$, it suffices to stay inside $B(\boldsymbol{x}_N, \varrho_N - l/2)$ and hence using the Markov property at time ϱ_N^2, we obtain

$$\begin{aligned} &p_{t+2\varrho_N^2}^{\text{avoid}}(u, u') \\ &\geq \mathbf{E}_u \left[p_{t+\varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(S_{\varrho_N^2}, u') \colon \tau_{B(\boldsymbol{x}_N, \varrho_N - l/2)} \wedge \tau_1 > \varrho_N^2, S_{\varrho_N^2} \in B(\boldsymbol{x}_N, \varrho_N/2) \right]. \end{aligned} \tag{5.4.47}$$

When we replace p^{visit} by p^{avoid}, we basically switch the path $S_{[0,\sigma_1]}$ to paths of length ρ_N^2 that stays inside $B(\boldsymbol{x}_N, \varrho_N - l/2)$, does not exit $B(\boldsymbol{x}_N, \varrho_N - l)$ after hitting $\overline{B(\boldsymbol{x}_N, \varrho_N - 2l)}$ and ends in $B(\boldsymbol{x}_N, \varrho_N/2)$ at time ϱ_N^2. After time ϱ_N^2, we let the random walk continue to evolve until it first exits (after time ϱ_N^2) from $B(\boldsymbol{x}_N, \varrho_N - l)$ at time $t + 2\varrho_N^2$.

We will prove in Lemma 5.4.7 below the following four estimates (RW6)–(RW9) in order to control the gain from this switching. The first three estimates show that we gain a lot by switching the first piece $S_{[0,\sigma_1]}$: On the event $\{\boldsymbol{x}_N \in B(0, (1-2\epsilon)\varrho_N)\} \cap \{\mathcal{O} \cap B(\boldsymbol{x}_N, \varrho_N - l/4) = \emptyset\}$, for any $v \in \partial \mathcal{T}$, we have

$$\sup_{u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)} \mathbf{P}_u\left(\tau_{\mathcal{T}^c} < \sigma_1 < \tau_{\mathcal{O}}\right) \leq \exp\left\{-(\log N)^2\right\}, \tag{RW6}$$

$$\sup_{u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)} \mathbf{P}_u\left(\tau_{B(v, (\log N)^6)} < \sigma_1 < \tau_{\mathcal{O}}\right) \leq G_{\mathcal{O}}(0, v)^2 \frac{\varrho_N^{d-1}}{l} (\log N)^{8d}, \tag{RW7}$$

and there exists $c_{5,9} > 0$ such that

$$\begin{aligned} &\inf_{u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)} \mathbf{P}_u \left(\tau_{B(\boldsymbol{x}_N, \varrho_N - l/2)^c} \wedge \tau_1 \geq \varrho_N^2, S_{\varrho_N^2} \in B(\boldsymbol{x}_N, \varrho_N/2) \right) \\ &\quad \geq c_{5,9} \frac{l}{\varrho_N}. \end{aligned} \tag{RW8}$$

Substituting the last estimate (RW8) into (5.4.47), we find that uniformly in $u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$,

$$p^{\mathrm{avoid}}_{t+2\varrho_N^2}(u, u') \geq c_{5,9} \frac{l}{\varrho_N} \inf_{y \in B(\boldsymbol{x}_N, \varrho_N/2)} p^{B(\boldsymbol{x}_N, \varrho_N - l)}_{t+\varrho_N^2}(y, u'). \tag{5.4.48}$$

The fourth estimate controls the possible cost caused by switching the second piece $S_{[\sigma_1, \tau_1]}$: There exists $c_{5,10} > 0$ such that for all $t > \sigma_1$, $u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ and $w \in \partial B(\boldsymbol{x}_N, \varrho_N - 2l)$,

$$\begin{aligned} &p^{B(\boldsymbol{x}_N, \varrho_N - l)}_{t-\sigma_1}(w, u') \\ &\quad \leq c_{5,10} \left(\frac{\varrho_N}{l} \right)^{c_{5,10}} \exp\left\{ c_{5,10} \frac{\sigma_1}{\varrho_N^2} \right\} \inf_{y \in B(\boldsymbol{x}_N, \varrho_N/2)} p^{B(\boldsymbol{x}_N, \varrho_N - l)}_{t+\varrho_N^2}(y, u'). \end{aligned} \tag{RW9}$$

We defer the proofs of (RW6)–(RW9) to the next subsection and now complete the proof of (5.4.43). Note that the cost in (RW9) becomes large if σ_1 is large. We first exclude the case where σ_1 is atypically large by using a tail bound for σ_1. For typical values of σ_1, the cost in (RW9) is not too large and we can use (RW6)–(RW8) to prove (5.4.43). Let us first consider the case $\sigma_1 \geq c_{5,11} l^2 \log N$ in (5.4.46), where $c_{5,11} > 1$ is to be determined later. By using (RW9), we find that

$$\begin{aligned} &\mathbf{E}_u \left[p^{B(\boldsymbol{x}_N, \varrho_N - l)}_{t-\sigma_1}(S_{\sigma_1}, u') \colon c_{5,11} l^2 \log N \leq \sigma_1 < t \wedge \tau_{\mathcal{O}} \right] \\ &\quad \leq c_{5,10} \left(\frac{\varrho_N}{l} \right)^{c_{5,10}} \inf_{y \in B(\boldsymbol{x}_N, \varrho_N/2)} p^{B(\boldsymbol{x}_N, \varrho_N - l)}_{t+\varrho_N^2}(y, u') \\ &\qquad \times \mathbf{E}_u \left[\exp\left\{ c_{5,10} \frac{\sigma_1}{\varrho_N^2} \right\} : c_{5,11} l^2 \log N \leq \sigma_1 < t \wedge \tau_{\mathcal{O}} \right]. \end{aligned} \tag{5.4.49}$$

Observe that up to time σ_1, the walk is confined in an annulus of width $3l$ and hence $\mathbf{P}_u(\sigma_1 \geq n) \leq \exp\{-cnl^{-2}\}$. This bound yields that if $l \leq \epsilon \varrho_N$ for sufficiently small $\epsilon > 0$, then

$$\begin{aligned}
&\mathbf{E}_u\left[\exp\left\{c_{5,10}\frac{\sigma_1}{\varrho_N^2}\right\}: c_{5,11}l^2\log N \le \sigma_1 < t < \tau_{\mathcal{T}^c}\right] \\
&\quad\le \sum_{n\ge c_{5,11}l^2\log N}\exp\left\{c_{5,10}\frac{n}{\varrho_N^2} - c\frac{n}{l^2}\right\} \\
&\quad\le \sum_{n\ge c_{5,11}l^2\log N}\exp\left\{-(c-c_{5,10}\epsilon^2)\frac{n}{l^2}\right\} \\
&\quad\le N^{-cc_{5,11}/2}.
\end{aligned} \tag{5.4.50}$$

We choose $c_{5,11}$ so large that the above right-hand side is less than ϱ_N^{-3d}. Then by substituting the above into (5.4.49) and comparing with (5.4.48), we obtain

$$\begin{aligned}
&\mathbf{E}_u\left[p_{t-\sigma_1}^{B(\boldsymbol{x}_N,\varrho_N-l)}(S_{\sigma_1},u')\colon c_{5,11}l^2\log N \le \sigma_1 < t < \tau_{\mathcal{O}}\right] \\
&\quad\le \varrho_N^{-2d}\inf_{y\in B(\boldsymbol{x}_N,\varrho_N/2)} p_{t+\varrho_N^2}^{B(\boldsymbol{x}_N,\varrho_N-l)}(y,u') \\
&\quad\le \frac{1}{2}\varrho_N^{-d}(\log N)^{-3}p_{t+2\varrho_N^2}^{\text{avoid}}(u,u').
\end{aligned} \tag{5.4.51}$$

Next, we consider the case $\sigma_1 < c_{5,11}l^2\log N$ in (5.4.46). In this case, we have a deterministic upper bound on the exponential factor in (RW9) and hence

$$\begin{aligned}
&\mathbf{E}_u\left[p_{t-\sigma_1}^{B(\boldsymbol{x}_N,\varrho_N-l)}(S_{\sigma_1},u')\colon \tau_{\mathcal{T}^c\cup B(v,(\log N)^6)} < \sigma_1 < c_{5,11}l^2\log N\wedge t\wedge\tau_{\mathcal{O}}\right] \\
&\quad\le c_{5,10}\left(\frac{\varrho_N}{l}\right)^{c_{5,10}}\exp\left\{c_{5,10}c_{5,11}\left(\frac{l}{\varrho_N}\right)^2\log N\right\}\inf_{y\in B(\boldsymbol{x}_N,\varrho_N/2)} p_{t+\varrho_N^2}^{B(\boldsymbol{x}_N,\varrho_N-l)}(y,u') \\
&\qquad\times \mathbf{P}_u\left(\tau_{\mathcal{T}^c\cup B(v,(\log N)^6)} < \sigma_1 < c_{5,11}l^2\log N\wedge t\wedge\tau_{\mathcal{O}}\right).
\end{aligned} \tag{5.4.52}$$

Now we use (RW6) and (RW7) to see that on the event $\{G_{\mathcal{O}}(\boldsymbol{x}_N, v) < \varrho_N^{1-d}\varphi(N,l)\}$,

$$\begin{aligned}
&\mathbf{P}_u\left(\tau_{\mathcal{T}^c\cup B(v,(\log N)^6)} < \sigma_1 < c_{5,11}l^2\log N \wedge t\wedge \tau_{\mathcal{O}}\right)\\
&\quad\le \mathbf{P}_u\left(\tau_{\mathcal{T}^c} < \sigma_1 < t\wedge\tau_{\mathcal{O}}\right) + \mathbf{P}_u\left(\tau_{B(v,(\log N)^6)} < \sigma_1 < t\wedge \tau_{\mathcal{O}}\right)\\
&\quad\le 2\varrho_N^{2-d}\frac{\varphi(N,l)^2}{l^2}(\log N)^{8d}.
\end{aligned} \tag{5.4.53}$$

Substituting this into (5.4.52) and comparing with (5.4.48) as in the previous case, we arrive at

$$\begin{aligned}
&\mathbf{E}_u\left[p_{t-\sigma_1}^{B(\boldsymbol{x}_N,\varrho_N-l)}(S_{\sigma_1},u')\colon \tau_{\mathcal{T}^c\cup B(v,(\log N)^6)} < \sigma_1 < c_{5,11}l^2\log N\wedge t\wedge\tau_{\mathcal{O}}\right]\\
&\quad\le c_{5,12}\frac{(\log N)^{c_{5,12}}\varphi(N,l)^2}{\varrho_N^d}\exp\left\{c_{5,12}\left(\frac{l}{\varrho_N}\right)^2\log N\right\}p^{\text{avoid}}_{t+2\varrho_N^2}(u,u')\\
&\quad\le \frac{1}{2}\varrho_N^{-d}(\log N)^{-3}p^{\text{avoid}}_{t+2\varrho_N^2}(u,u')
\end{aligned} \tag{5.4.54}$$

by setting

$$\varphi(N,l) = \begin{cases} N^{-c_{5,12}\epsilon}, & \text{if } l = \epsilon\varrho_N,\\[2ex] (\log N)^{-c_{5,12}-4}, & \text{if } l = \varrho_N/\log N. \end{cases} \tag{5.4.55}$$

Gathering (5.4.54) and (5.4.51), we get (5.4.43) and we are done. □

5.4.5 Random walk estimates II

In this subsection, we prove the random walk estimates (RW6)–(RW9) used in Subsection 5.4.2. Recall the definition of the stopping times σ_k and τ_k ($k\in\mathbb{N}$) in (5.4.20)–(5.4.22).

Lemma 5.4.7. *Suppose that* $l \in [\varrho_N/\log N, c_{5,3}\varrho_N]$, $\boldsymbol{x}_N \in B(0,(1-2\epsilon)\varrho_N)$ *and* $\mathcal{O}\cap B(\boldsymbol{x}_N,\varrho_N - l/4) = \emptyset$. *Then the following hold:*

1. *For $u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ and $v \in \partial \mathcal{T}$,*

$$\mathbf{P}_u\left(\tau_{\mathcal{T}^c} < \sigma_1 < \tau_{\mathcal{O}}\right) \leq \exp\left\{-(\log N)^2\right\}, \tag{RW6}$$

$$\mathbf{P}_u\left(\tau_{B(v,(\log N)^6)} < \sigma_1 < \tau_{\mathcal{O}}\right) \leq G_{\mathcal{O}}(0,v)^2 \frac{\varrho_N^{d-1}}{l}(\log N)^{8d}. \tag{RW7}$$

2. *There exists $c_{5,9} > 0$ such that for $u \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$,*

$$\mathbf{P}_u\left(\tau_{B(\boldsymbol{x}_N,\varrho_N - l/2)^c} \wedge \tau_1 \geq \varrho_N^2, S_{\varrho_N^2} \in B(\boldsymbol{x}_N, \varrho_N/2)\right) \geq c_{5,9}\frac{l}{\varrho_N}. \tag{RW8}$$

3. *There exists $c_{5,10} > 0$ such that uniformly in $0 \leq m < n$, $w \in \partial B(\boldsymbol{x}_N, \varrho_N - 2l)$ and $u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$,*

$$\begin{aligned} &p_{n-m}^{B(\boldsymbol{x}_N,\varrho_N - l)}(w,u') \\ &\leq c_{5,10}\left(\frac{\varrho_N}{l}\right)^{c_{5,10}} \exp\left\{c_{5,10} m \varrho_N^{-2}\right\} \inf_{y \in B(\boldsymbol{x}_N,\varrho_N/2)} p_{n+\varrho_N^2}^{B(\boldsymbol{x}_N,\varrho_N - l)}(y,u'). \end{aligned} \tag{RW9}$$

Let us explain the intuitions behind these bounds before delving into the proof. The first assertion (RW6) follows readily from the definition of the "truly"-open set. The second assertion (RW7) is based on the following observation. The probability for the random walk to visit $B(v,(\log N)^6)$ without hitting $\mathcal{O}$ is bounded by $G_{\mathcal{O}}(u,v)$. Then it has to come back to w but by the time reversal, the probability is again bounded by $G_{\mathcal{O}}(w,v)$. Finally, the factor l^{-1} comes from the fact that the random walk hits $B(x, \varrho_N - 2l)$ for the first time at w. We need the extra poly-logarithmic factor to change the starting points in $G_{\mathcal{O}}(u,v)$ and $G_{\mathcal{O}}(w,v)$ to $\boldsymbol{x}_N$ by using the elliptic Harnack inequality. The third assertion (RW8) is a slight modification of (5.3.61). The fourth assertion (RW9) basically says that it is easier for the random walk to go from $y \in B(\boldsymbol{x}_N, \varrho_N/2)$ to $u' \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ than from $w \in \partial B(\boldsymbol{x}_N, \varrho_N - 2l)$, without exiting $B(\boldsymbol{x}_N, \varrho_N - l)$. There are two reasons why we have a large factor on the right-hand side. First, if w and u' are close to each other and $n-m$ is of order l^2, then it is in fact better to start from w; second, if both m and n are large, then we have to include the cost for the random walk to stay in $B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l)$ for extra time $m + \varrho_N^2$.

Proof of Lemma 5.4.7. The left-hand side of (RW6) is bounded by

$$\mathbf{P}_u\left(\tau_{\mathcal{T}^c} < \sigma_1 < \tau_{\mathcal{O}}\right) \le \sup_{x \in \mathcal{T}^c} \mathbf{P}_x(\sigma_1 < \tau_{\mathcal{O}}) \tag{5.4.56}$$

by the strong Markov property applied at $\tau_{\mathcal{T}^c}$. Since we assume $\mathcal{O} \cap B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l/4) = \emptyset$, we have $\mathcal{T} \subset B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l/2)$ and hence $\sigma_1 > (\log N)^5$ whenever the random walk starts from $\mathcal{T}^c$. The bound (RW6) follows from the definition of $\mathcal{T}$.

Next, the left-hand side of (RW7) is bounded by

$$\sum_{w \in \partial B(\boldsymbol{\mathcal{X}}_N, \varrho_N - 2l)} \sum_{y \in B(v, (\log N)^6)} \mathbf{P}_u\left(\tau_y < \sigma_1 < \tau_{\mathcal{O}}, S_{\sigma_1} = w\right). \tag{5.4.57}$$

By reversing the time on $[\tau_y, \sigma_1]$, we have that for each $y \in B(v, (\log N)^6)$,

$$\begin{aligned} &\mathbf{P}_u\left(\tau_y < \sigma_1 < \tau_{\mathcal{O}}, S_{\sigma_1} = w\right) \\ &\quad \le \mathbf{P}_u\left(\tau_{B(v,(\log N)^6)} < \sigma_1 \wedge \tau_{\mathcal{O}}\right) \mathbf{P}_w\left(\tau_{B(v,(\log N)^6)} < \sigma_1 \wedge \tau_{\mathcal{O}}\right). \end{aligned} \tag{5.4.58}$$

We further bound the second factor on the right-hand side by using the strong Markov property at τ_1 as

$$\begin{aligned} &\sum_{z \in \partial B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l)} \mathbf{P}_w\left(S_{\tau_1} = z, \tau_1 < \sigma_1\right) \mathbf{P}_z\left(\tau_{B(v,(\log N)^6)} < \sigma_1 \wedge \tau_{\mathcal{O}}\right) \\ &\quad \le \mathbf{P}_w\left(\tau_1 < \sigma_1\right) \max_{z \in \partial B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l)} \mathbf{P}_z\left(\tau_{B(v,(\log N)^6)} < \sigma_1 \wedge \tau_{\mathcal{O}}\right) \\ &\quad \le \frac{c}{l} \max_{z \in \partial B(\boldsymbol{\mathcal{X}}_N, \varrho_N - l)} \mathbf{P}_z\left(\tau_{B(v,(\log N)^6)} < \sigma_1 \wedge \tau_{\mathcal{O}}\right), \end{aligned} \tag{5.4.59}$$

where in the last inequality we have used a gambler's ruin type estimate (see [53, (6.14)] for a similar estimate). Substituting this into (5.4.58) and summing over $y \in B(v, (\log N)^6)$, we find that

$$\begin{aligned}
&\mathbf{P}_u \left(\tau_{B(v,(\log N)^6)} < \sigma_1 < \tau_{\mathcal{O}}, S_{\sigma_1} = w \right) \\
&\quad \le \frac{c}{l} (\log N)^{6d} \max_{z \in \partial B(\boldsymbol{x}_N, \varrho_N - l)} \mathbf{P}_z \left(\tau_{B(v,(\log N)^6)} < \sigma_1 \wedge \tau_{\mathcal{O}} \right)^2 \\
&\quad \le \frac{c}{l} (\log N)^{6d} \max_{z \in \partial B(\boldsymbol{x}_N, \varrho_N - l)} G_{\mathcal{O}}(z, v)^2.
\end{aligned} \tag{5.4.60}$$

We are going to shift the variable $z \in \partial B(\boldsymbol{x}_N, \varrho_N - l)$ to $\boldsymbol{x}_N$ by applying the following elliptic Harnack inequality to the function $G_{\mathcal{O}}(\cdot, v)$, which is harmonic in $B(\boldsymbol{x}_N, \varrho_N - l/2)$: There exists $c_{5,13} > 0$ such that for any $x \in \mathbb{Z}^d$, $r \in \mathbb{N}$ sufficiently large and any non-negative harmonic function f on $B(x, r)$,

$$\sup_{B(x,0.9r)} f(y) \le c_{5,13} \inf_{B(x,0.9r)} f(y). \tag{5.4.61}$$

See [53, Theorem 6.3.9], for example.

To compare $G_{\mathcal{O}}(z, v)$ with $G_{\mathcal{O}}(\boldsymbol{x}_N, v)$, we will apply (5.4.61) iteratively as follows. First, let $l_1 = l$ and z_1 be the point on $\partial B(z, l_j)$ closest to $\boldsymbol{x}_N$. Applying (5.4.61) to $G_{\mathcal{O}}(\cdot, v)$ on the ball $B(z_1, l_1)$ gives $G_{\mathcal{O}}(z, v) \le c G_{\mathcal{O}}(z_1, v)$. We can now iterate this procedure. For $j \ge 2$, let $l_j := 2^{j-1} l$ and z_{j+1} be the point on $\partial B(z_j, l_j)$ closest to $\boldsymbol{x}_N$, and apply (5.4.61) to $G_{\mathcal{O}}(\cdot, v)$ on the ball $B(z_j, l_j)$. The iteration is stopped at the first J such that $z_J \in B(\boldsymbol{x}_N, 2\varrho_N/3)\}$. See Figure 5.2. Noting that $J \le c \log(\rho_N / l)$, we have

$$\begin{aligned}
G_{\mathcal{O}}(z, v) &\le c_{5,13} G_{\mathcal{O}}(z_1, v) \\
&\le \cdots \\
&\le c_{5,13}^J G_{\mathcal{O}}(z_J, v) \\
&\le \left(\frac{\varrho_N}{l} \right)^c G_{\mathcal{O}}(\boldsymbol{x}_N, v),
\end{aligned} \tag{5.4.62}$$

where in the last inequality, we have applied (5.4.61) in $B(\boldsymbol{x}_N, \varrho_N - l/2)$ to bound $G_{\mathcal{O}}(z_J, v)$

by $c_{5,13} G_{\mathcal{O}}(\boldsymbol{x}_N, v)$. Since $l \geq \varrho_N / \log N$, by substituting (5.4.62) into (5.4.60) and summing over $w \in \partial B(\boldsymbol{x}_N, \varrho_N - 2l)$, we get the desired bound (RW7).

In order to prove the lower bound (RW8), we let the random walk obey the following strategy: pick $u' \in \partial B(\boldsymbol{x}_N, \varrho_N - 3l/2) \cap B(u, l)$ and

1. $S_{l^2} \in B(u', l/4)$ without exiting $A(\boldsymbol{x}_N; \varrho_N - 2l, \varrho_N - l/2)$;

2. $S_{\varrho_N^2} \in B(\boldsymbol{x}_N, \varrho_N/2)$ without exiting $B(\boldsymbol{x}_N, \varrho_N - l)$.

In this way, the condition $\tau_{B(\boldsymbol{x}_N, \varrho_N - l/2)^c} \wedge \tau_1 \geq \varrho_N^2$ holds and hence the left-hand side of (RW8) is bounded from below by the probability of the above strategy. With the help of (5.3.61), one can find $c > 0$ such that

$$\mathbf{P}_u \left(S_{l^2} \in B(u', l/4), S_{[0,l^2]} \subset A(\boldsymbol{x}_N; \varrho_N - 2l, \varrho_N - l/2) \right) \geq c, \tag{5.4.63}$$

$$\inf_{y \in B(u', l/4)} \mathbf{P}_y \left(S_{\varrho_N^2 - l^2} \in B(\boldsymbol{x}_N, \varrho_N/2), S_{[0, \varrho_N^2 - l^2]} \subset B(\boldsymbol{x}_N, \varrho_N - l) \right) \geq c \frac{l}{\varrho_N}. \tag{5.4.64}$$

Collecting these bounds, we get (RW8).

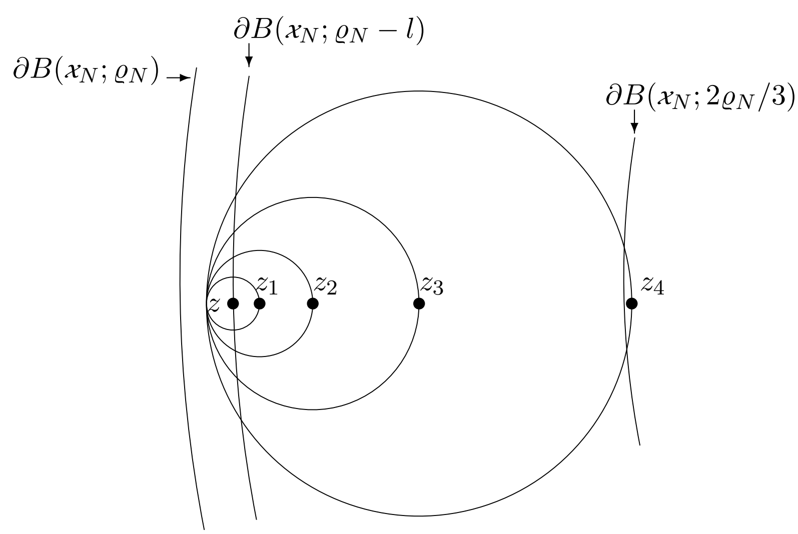

Figure 5.2: The sequence $(z_j)_{j=1}^J$ constructed in the proof of (RW7). The balls around z_j have geometrically growing radii and the construction terminates at $J = 4$ in this picture.

Finally we prove (RW9). In the case $n+\varrho_N^2 \le 2\varrho_N^2$, we have

$$
\begin{aligned}
&p_{n+\varrho_N^2}^{B(\boldsymbol{x}_N,\varrho_N-l)}(y,u') \\
&\quad\ge \mathbf{E}_u\Big[p_{n+\varrho_N^2-\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}}^{B(\boldsymbol{x}_N,\varrho_N-l)}\left(S_{\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}},y\right) : \\
&\qquad\qquad \tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)} < (n-m)\wedge \tau_{B(\boldsymbol{x}_N,\varrho_N-l)^c}\Big] \\
&\quad\ge cl\varrho_N^{-d-1}\mathbf{P}_u\left(\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)} < (n-m)\wedge\tau_{B(\boldsymbol{x}_N,\varrho_N-l)^c}\right),
\end{aligned}
\tag{5.4.65}
$$

where we have used the strong Markov property at $\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}$ and applied (5.3.61) to the transition probability appearing inside the expectation, noting that $\varrho_N^2 \le n+\varrho_N^2 - \tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)} \le 2\varrho_N^2$ under the conditions $n+\varrho_N^2 < 2\varrho_N^2$ and $\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)} < (n-m)\wedge\tau_{B(\boldsymbol{x}_N,\varrho_N-l)^c}$. Similarly, we have

$$
\begin{aligned}
&p_{n-m}^{B(\boldsymbol{x}_N,\varrho_N-l)}(w,u') \\
&\quad= \mathbf{E}_u\Big[p_{n-m-\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}}^{B(\boldsymbol{x}_N,\varrho_N-l)}\left(S_{\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}},w\right) : \\
&\qquad\qquad \tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)} < (n-m)\wedge \tau_{B(\boldsymbol{x}_N,\varrho_N-l)^c}\Big] \\
&\quad\le cl^{-d}\mathbf{P}_u\left(\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)} < (n-m)\wedge\tau_{B(\boldsymbol{x}_N,\varrho_N-l)^c}\right),
\end{aligned}
\tag{5.4.66}
$$

where we have used the strong Markov property at $\tau_{B(\boldsymbol{x}_N,\varrho_N-3l/2)}$ and the estimate

$$
\begin{aligned}
&p_{n-m-\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}}^{B(\boldsymbol{x}_N,\varrho_N-l)}\left(S_{\tau_{\partial B(\boldsymbol{x}_N,\varrho_N-3l/2)}},w\right) \\
&\quad\le \sup_{k\in\mathbb{N},|x-y|\ge l/2} p_k^{B(\boldsymbol{x}_N,\varrho_N-l)}(x,y) \\
&\quad\le Cl^{-d},
\end{aligned}
\tag{5.4.67}
$$

which follows in the same way as in (5.3.81). Combining the above two bounds, we get (RW9) in this case.

In the other case $n+\varrho_N > 2\varrho_N^2$, we use the following parabolic Harnack inequality from [19, $H(C_H)$ in Theorem 1.7]: For all $x_0\in\mathbb{Z}^d$, $s\in\mathbb{R}$, $r>200$ and every non-negative

$u(t,x)$ that satisfies the discrete heat equation on $\mathbb{Z} \cap [s, s+100r^2] \times B(x_0, r)$,

$$\begin{aligned} &\sup_{(t_1,x_1)\in\mathbb{Z}\cap[s+0.01r^2,s+0.1r^2]\times B(x_0,0.99r)} u(t_1,x_1) \\ &\leq 100 \inf_{(t_2,x_2)\in\mathbb{Z}\cap[s+0.11r^2,s+100r^2]\times B(x_0,0.99r)} u(t_2,x_2). \end{aligned} \tag{5.4.68}$$

We use this to first shift the spatial variable w to $y \in B(\boldsymbol{x}_N, \varrho_N/2)$ and then the time variable in the transition probability kernel in (RW9). To this end, we construct a sequence $(w_j)_{j=1}^J$ in the same way as in the proof of (RW7) (see Figure 5.3): let $w_0 := w$, and for $j \geq 0$ let $l_j := 2^{j-1} l$ and w_{j+1} be the point on $\partial B(w_j, l_j)$ closest to $\boldsymbol{x}_N$ and

$$J := \min\{j \geq 0\colon w_j \in B(\boldsymbol{x}_N, 2\varrho_N/3)\}. \tag{5.4.69}$$

Note that $\varrho_N/3 \leq l_J \leq 2\varrho_N/3$ and therefore $J \leq c\log(\rho_N/l)$. As a first step, we switch from $w = w_0$ to w_1 and use the bound

$$p_{n-m}^{B(\boldsymbol{x}_N,\varrho_N-l)}(w,u') \leq c_{5,14}\frac{\varrho_N}{l} p_{n-m+l_1^2}^{B(\boldsymbol{x}_N,\varrho_N-l)}(w_1,u') \tag{5.4.70}$$

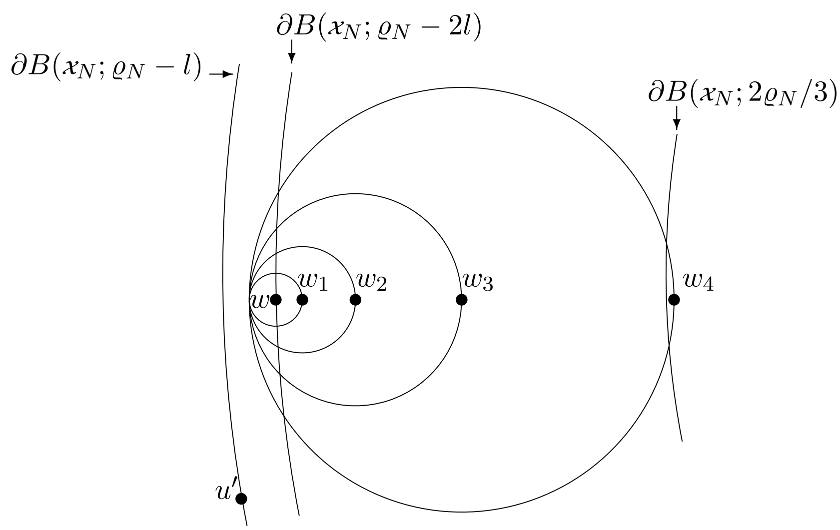

Figure 5.3: The sequence $(w_j)_{j=1}^J$ constructed in the proof of (RW9). The balls around w_j have geometrically growing radii and the construction terminates at $J = 4$ in this picture.

with some $c_{5,14} > 0$, which can be verified by applying either (5.4.67) to the left-hand side and (5.3.61) to the right-hand side when $n - m < \varrho_N^2$, or (5.4.68) with $x_0 = w_1$, $s = n - m - 0.05l_1^2$ and $r = l_1$ to the function $u(t,x) := p_t^{B(\boldsymbol{x}_N, \varrho_N - l)}(x, u')$ restricted to $B(w_1, l_1)$, when $n - m > \varrho_N^2$. Next, noting that w_j keeps distance at least $l_{j+1}/2$ from $\partial B(w_{j+1}, l_{j+1})$ for any $j < J$, we can apply (5.4.68) with $x_0 = w_j$, $s = n - m + 0.95l_j^2$ and $r = l_j$ to the function $u(t,x) := p_t^{B(\boldsymbol{x}_N, \varrho_N - l)}(x, u')$ restricted to $B(w_j, l_j)$ to obtain

$$p_{n-m+l_j^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(w_j, u') \leq 100 p_{n-m+l_{j+1}^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(w_{j+1}, u') \quad \text{for } j \geq 1, \tag{5.4.71}$$

$$p_{n-m+l_J^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(w_J, u') \leq 100 p_{n-m+\varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(y, u'), \tag{5.4.72}$$

where in the second bound, we have used $l_J^2 \leq 4\varrho_N^2/9$. By using (5.4.70)–(5.4.72) and recalling the upper bound on J, we get

$$\begin{aligned} p_{n-m}^{B(\boldsymbol{x}_N, \varrho_N - l)}(w, u') &\leq c^J \frac{\varrho_N}{l} p_{n-m+l_J^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(w_J, u') \\ &\leq \left(\frac{\varrho_N}{l}\right)^c p_{n-m+\varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(y, u'). \end{aligned} \tag{5.4.73}$$

Now another iteration of (5.4.68) leads us to

$$\begin{aligned} p_{n-m+\varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(y, u') &\leq 100 p_{n-m+2\varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(y, u') \\ &\leq \cdots \\ &\leq 100^{\lfloor m\varrho_N^{-2} \rfloor - 1} p_{n-m+\lceil m\varrho_N^{-2} \rceil \varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(y, u') \\ &\leq 100^{\lfloor m\varrho_N^{-2} \rfloor} p_{n+\varrho_N^2}^{B(\boldsymbol{x}_N, \varrho_N - l)}(y, u'), \end{aligned} \tag{5.4.74}$$

where in the k-th inequality, we choose $x_0 = \boldsymbol{x}_N$, $s = n - m + (k - 0.05)\varrho_N^2$, $r = \varrho_N - l$ and $u(t,x) := p_t^{B(\boldsymbol{x}_N, \varrho_N - l)}(x, u')$ in (5.4.68). The desired estimate follows from (5.4.73) and (5.4.74). $\square$

5.5 Proof of the Refined Version of Ball Covering Theorem

In this section, we prove Theorems 5.2.7. As we discussed in Subsection 5.4.2, below Lemma 5.4.3, Theorem 5.2.8 follows from Theorem 5.2.7. To strengthen Proposition 5.2.2 to Theorem 5.2.7, we first show that the volume of $\mathcal{T}$ is very close to $|B(0, \varrho_N)|$ with the help of the Faber–Krahn inequality and a control on the number of the possible shapes of $\mathcal{T}$ provided by Proposition 5.2.9. Now if there is a non-"truly"-open site $x \in B(\boldsymbol{\mathcal{X}}_N, \varrho_N - \varrho_N^{\epsilon_2})$, then there exists a closed site near x. Lemma 5.2.1 then implies that there are in fact many closed sites around x, which leads us to a contradiction.

In order to carry out the proof, we will compare $\lambda_{\mathcal{T}}^{\mathrm{RW},(k)}$, the k-th smallest eigenvalue for the generator of the random walk killed upon exiting $\mathcal{T}$, with its counterpart for the continuum Laplacian, and then apply the Faber–Krahn inequality. We define the *continuous hull* of $T \subset \mathbb{Z}^d$ by

$$\widetilde{T} := \{x \in \mathbb{R}^d : \mathrm{dist}_\infty(x, T) < 2\} \tag{5.5.1}$$

and denote the k-th smallest Dirichlet eigenvalue of $-\frac{1}{2d}\Delta$ in $\widetilde{T}$ and the corresponding eigenfunction by $\lambda_{\widetilde{T}}^{(k)}$ and $\phi_{\widetilde{T}}^{(k)}$, respectively. We will prove the following comparison in Appendix 5.A.

Lemma 5.5.1. *There exists $c > 0$ such that for any $x \in \mathbb{Z}^d$ and $T \supset B(x, \varrho_N/2)$,*

$$\lambda_T^{\mathrm{RW},(1)} \geq \lambda_{\widetilde{T}}^{(1)} - c\varrho_N^{-4}. \tag{5.5.2}$$

Moreover, for each $k \in \mathbb{N}$, there exists $\gamma_k > 0$ such that

$$\left|\lambda_{B(0,\varrho_N)}^{\mathrm{RW},(k)} - \lambda_{B(0,\varrho_N)}^{(k)}\right| \leq \gamma_k \varrho_N^{-3}. \tag{5.5.3}$$

Let us recall a lower bound on the survival probability which is well-known in the continuum setting [63, (33)] and for the two dimensional continuous time random walk [12, Proposition 2.1]. Lemma 5.5.2 below is the analogue in our discrete time setting, which we prove in Appendix 5.A for completeness. Recall that we denote the Euclidean volume of $G \subset \mathbb{R}^d$ by $\mathrm{vol}(G)$.

Lemma 5.5.2. *There exists $c > 0$ such that for all sufficiently large $N > 0$,*

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) \geq \exp\left\{\mathrm{vol}(B(0,\varrho_N)) \log p - N\lambda^{(1)}_{B(0,\varrho_N)} - c\varrho_N^{d-1}\right\}. \tag{5.5.4}$$

Using this lemma, we can show that the volume of $\widetilde{\mathcal{T}}$ is very close to $\mathrm{vol}(B(0,\varrho_N))$.

Lemma 5.5.3. *For any $b \in (0,1)$,*

$$\lim_{N\to\infty} \mu_N\left(\left|\mathrm{vol}(\widetilde{\mathcal{T}}) - \mathrm{vol}(B(0,\varrho_N))\right| < \varrho_N^{(d-1)/2+2b}\right) = 1. \tag{5.5.5}$$

Proof of Lemma 5.5.3. Referring to Theorem E and Propositions 5.2.2 and 5.2.9, we introduce the set of possible shapes of $\mathcal{T}$ for $b \in (0,1)$:

$$\begin{aligned}\mathbb{T}_b := \{T \subset \mathbb{Z}^d \colon B(x, 0.9\varrho_N) \subset T \subset B(x, \varrho_N + \varrho_N^{\epsilon_1}) \text{ for} \\ \text{some } x \in B(0,\varrho_N),\ \text{and } |\partial T| \leq \varrho_N^{d-1+b}\}\end{aligned} \tag{5.5.6}$$

so that $\lim_{N\to\infty} \mu_N(\mathcal{T} \in \mathbb{T}_b) = 1$. The cardinality of this set is bounded by

$$|\mathbb{T}_b| \leq \exp\{c\varrho_N^{d-1+2b}\} \tag{5.5.7}$$

simply by considering the choice of ϱ_N^{d-1+b} points from $B(0, 3\varrho_N)$. Having controlled the entropy, we estimate next the probability $\mu_N(\mathcal{T} = T)$ for each $T \in \mathbb{T}_b$. With the help of the eigenfunction expansion, one finds the upper bound

$$\begin{aligned}\mathbb{P} \otimes \mathbf{P}\left(\tau_{T^c} > N, \mathcal{T} = T\right) &\leq \mathbb{P}(\mathcal{O} \cap T = \emptyset)\mathbf{P}(\tau_{T^c} > N) \\ &\leq |T|^{1/2} \exp\left\{|T| \log p - N\lambda_T^{\mathrm{RW},(1)}\right\}\end{aligned} \tag{5.5.8}$$

for general $T \subset \mathbb{Z}^d$. See, for example, [48, (2.21)]. Observe that since $\widetilde{T} \subset \bigcup_{x\in T}(x + [-2,2]^d)$, we have $|T| \geq \mathrm{vol}(\widetilde{T}) - c|\partial T|$ and hence for $T \in \mathbb{T}_b$,

$$|T| \geq \mathrm{vol}(\widetilde{T}) - c\varrho_N^{d-1+b}. \tag{5.5.9}$$

On the other hand, by (5.5.2) and the Faber–Krahn inequality (see, for example, [16, pp.87–92]), for any $T \in \mathbb{T}_b$, we have

$$\begin{aligned} \lambda_T^{\mathrm{RW},(1)} &\geq \lambda_{\widetilde{T}}^{(1)} - c\varrho_N^{-4} \\ &\geq \lambda_{|\widetilde{T}|} - c\varrho_N^{-4}, \end{aligned} \tag{5.5.10}$$

where for $r > 0$, we denote by λ_r the principal Dirichlet eigenvalue of $-\frac{1}{2d}\Delta$ in a ball with volume r. Substituting (5.5.9) and (5.5.10) into (5.5.8), we obtain

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P}\left(\tau_{T^c} > N, \mathcal{T} = T\right) \\ &\quad \leq |T|^{1/2} \exp\left\{\mathrm{vol}(\widetilde{T}) \log p - N\lambda_{|\widetilde{T}|} + c\varrho_N^{d-1+b}\right\}. \end{aligned} \tag{5.5.11}$$

Suppose that $T \in \mathbb{T}_b$ satisfies $|\mathrm{vol}(\widetilde{T}) - \mathrm{vol}(B(0, \varrho_N))| \geq \varrho_N^{(d-1)/2+2b}$. Then, since the function

$$r \mapsto r \log p - N\lambda_r = r \log p - \frac{N\lambda_1}{r^{2/d}} \tag{5.5.12}$$

is twice-differentiable and maximized at $\mathrm{vol}(B(0, \varrho_N))$ (cf. (5.1.6)), one finds by the Taylor expansion that

$$\begin{aligned} &\mathrm{vol}(\widetilde{T}) \log p - N\lambda_{|\widetilde{T}|} \\ &\quad \leq \mathrm{vol}(B(0, \varrho_N)) \log p - N\lambda_{B(0,\varrho_N)}^{(1)} - c\left|\mathrm{vol}(\widetilde{T})| - \mathrm{vol}(B(0, \varrho_N))\right|^2 \\ &\quad \leq \mathrm{vol}(B(0, \varrho_N)) \log p - N\lambda_{B(0,\varrho_N)}^{(1)} - c\varrho_N^{d-1+4b}. \end{aligned} \tag{5.5.13}$$

Substituting this into (5.5.11) and comparing with Lemma 5.5.2, we obtain $\mu_N(\mathcal{T} = T) \leq \exp\{-c\varrho_N^{d-1+4b}\}$. Thanks to (5.5.7), we can use the union bound to conclude the proof of

Lemma 5.5.3. □

Proof of Theorems 5.2.7. Thanks to Lemma 5.2.1, we can restrict our consideration to the event

$$\bigcap_{x\in B(0,2\varrho_N)} \bigcap_{(\log N)^3\le l\le \varrho_N} \left\{ x\in\mathcal{O} \text{ and } \frac{|\mathcal{O}\cap B(x,l)|}{|B(x,l)|}<\delta \right\}, \tag{5.5.14}$$

that is, any $x\in B(0,2\varrho_N)$ is either open or has δ-fraction of closed sites in its l-neighborhood for all $l\in[(\log N)^3,\varrho_N]$. In addition, by Lemma 5.5.3 with $b=\epsilon_1/2$ where ϵ_1 is as in Theorem E, we can further assume that

$$\left|\mathrm{vol}(\widetilde{\mathcal{T}})-\mathrm{vol}(B(0,\varrho_N))\right| < \varrho_N^{(d-1)/2+\epsilon_1}. \tag{5.5.15}$$

Let $\epsilon_2>(d-1+\epsilon_1)/d$ and suppose that there exists $x\in B(\boldsymbol{\mathcal{X}}_N,\varrho_N-\varrho_N^{\epsilon_2})\setminus\mathcal{T}$. Then there exists at least one closed site $y\in B(x,(\log N)^5)$, and by (5.5.14), more than δ fraction of the points in the ball $B(y,\varrho_N^{\epsilon_2}/2)\subset B(\boldsymbol{\mathcal{X}}_N,\varrho_N)$ must be closed. Recalling (5.5.9) and that $\widetilde{\mathcal{T}}\subset B(\boldsymbol{\mathcal{X}}_N,\varrho_N+2\varrho_N^{\epsilon_1})$ by the definition of $\mathcal{T}$, we find that

$$\begin{aligned}
\mathrm{vol}(\widetilde{\mathcal{T}}) &\le |\mathcal{T}|+\varrho_N^{d-1+b}\\
&\le |B(\boldsymbol{\mathcal{X}}_N,\varrho_N+2\varrho_N^{\epsilon_1})|-\delta|B(y,\varrho_N^{\epsilon_2}/2)|+\varrho_N^{d-1+\epsilon_1}\\
&\le \mathrm{vol}(B(0,\varrho_N))+c\varrho_N^{d-1+\epsilon_1}-c'\varrho_N^{d\epsilon_2}\\
&\le \mathrm{vol}(B(0,\varrho_N))-c\varrho_N^{d-1+\epsilon_1},
\end{aligned} \tag{5.5.16}$$

which contradicts (5.5.15). □

Remark 5.5.4 (Finer asymptotics of survival probability)**.** There is a conjecture on the precise second order asymptotics of the survival probability in the literature: there exists $a_1>0$ such that

$$\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}}>N)=\exp\left\{-c(d,p)N^{\frac{d}{d+2}}-a_1N^{\frac{d-1}{d+2}}+o(N^{\frac{d-1}{d+2}})\right\}. \tag{5.5.17}$$

See [56] and the bottom of p. 76 in [13] for more information. Lemma 5.5.2 gives a lower bound of this form while the currently best known upper bound is

$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) \leq \exp\left\{-c(d,p)N^{\frac{d}{d+2}} + N^{\frac{d-\epsilon}{d+2}}\right\} \tag{5.5.18}$$

for some $\epsilon \in (0,1)$. See [13, (2.40)] or [75, Theorem 5.6 on page 208].

Based on what we have proved, we can get a refined upper bound on the survival probability. Theorem 5.2.8 implies that $\lim_{N\to\infty} \mu_N(\mathcal{T} \in \mathbb{T}_{0+}) = 1$ with

$$\begin{aligned}\mathbb{T}_{0+} := \Big\{T \subset \mathbb{Z}^d \colon B(x, 0.9\varrho_N) \subset T \subset B(x, \varrho_N + \varrho_N^{\epsilon_1}) \text{ for}\\ \text{some } x \in B(0, \varrho_N) \text{ and } |\partial T| \leq \varrho_N^{d-1}(\log N)^a\Big\}.\end{aligned} \tag{5.5.19}$$

Just as in (5.5.7), we have

$$|\mathbb{T}_{0+}| \leq \exp\{c\varrho_N^{d-1}(\log N)^{a+1}\} \tag{5.5.20}$$

and then by Lemma 5.2.6 and a variant of (5.5.11), we obtain

$$\begin{aligned}
&\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)\\
&\quad \sim \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{T}^c} > N, \mathcal{T} \in \mathbb{T}_{0+})\\
&\quad \leq |\mathbb{T}_{0+}| \sup_{T \in \mathbb{T}_{0+}} |T|^{1/2} \exp\left\{\operatorname{vol}(\widetilde{T}) \log p - N\lambda_{B_{\widetilde{T}}} + c\varrho_N^{d-1}(\log N)^{a+1}\right\}\\
&\quad \leq \exp\left\{\operatorname{vol}(B(0,\varrho_N)) \log p - N\lambda_{B(0,\varrho_N)} + c\varrho_N^{d-1}(\log N)^{a+1}\right\}\\
&\quad = \exp\left\{-c(d, \log(1/p))N^{\frac{d}{d+2}} + cN^{\frac{d-1}{d+2}}(\log N)^{a+1}\right\}.
\end{aligned} \tag{5.5.21}$$

5.A Estimates for eigenvalues and eigenfunctions

In this section, we collect some estimates on eigenvalues and eigenfunctions, including Lemma 5.5.1, and then prove (EV), (EF) (used in the proof of Lemma 5.4.5) and Lemma 5.5.2.

Recall that $\lambda_T^{(k)}$ and $\phi_T^{(k)}$ are the k-th smallest Dirichlet eigenvalue and corresponding eigenfunction with $\|\phi_T^{(k)}\|_2 = 1$ for $-\frac{1}{2d}\Delta$ in $T \subset \mathbb{R}^d$ and $\lambda_T^{\mathrm{RW},(k)}$ and $\phi_T^{\mathrm{RW},(k)}$ are their discrete space counterparts.

We begin with the following comparison lemma which includes Lemma 5.5.1.

Lemma 5.A.1. *For any $d \geq 1$ and $k \geq 1$, there exists $\gamma_k > 0$ such that for all sufficiently large $R > 0$, the following bounds hold:*

$$\left|\lambda_{B(0,R)}^{\mathrm{RW},(k)} - \lambda_{B(0,R)}^{(k)}\right| \leq \gamma_k R^{-3}, \tag{5.A.1}$$

$$\min_{|y|\leq 1} \phi_{B(0,R)}^{\mathrm{RW},(1)}(y) \geq \gamma_1^{-1} R^{-d/2}, \tag{5.A.2}$$

and for any $x \in \mathbb{Z}^d$ and $T \supset B(x, \varrho_N/2)$,

$$\lambda_T^{\mathrm{RW},(1)} \geq \lambda_{\widetilde{T}}^{(1)} - \gamma_1 \varrho_N^{-4}. \tag{5.A.3}$$

Proof of Lemma 5.A.1. The first assertion can be found in [77, (3.27) and (6.11)]. The second assertion follows from [12, Lemma 2.1(b)], which states that

$$\sup_{y\in B(0,R)} \left|\phi_{B(0,R)}^{(1)}(y) - \phi_{B(0,R)}^{\mathrm{RW},(1)}(y)\right| \leq cR^{-d/2-1}, \tag{5.A.4}$$

and the fact that $\min_{y\in\mathbb{Z}^d:|y|\leq 1} \phi_{B(0,R)}^{(1)}(y) \geq c^{-1}R^{-d/2}$. The third assertion follows from [77, (6.9)], which states that

$$\lambda_{\widetilde{T}}^{(1)} \leq \lambda_T^{\mathrm{RW},(1)}(1 + c\varrho_N^{-2}), \tag{5.A.5}$$

and the bound $\lambda_T^{\mathrm{RW},(1)} \leq c\varrho_N^{-2}$ that follows from the assumption $T \supset B(x, \varrho_N/2)$. □

Next we restate and prove (EV) and (EF).

Lemma 5.A.2. *There exist $c_{5,7}, c_{5,8} > 0$ such that if $B(x, (1-\epsilon)\varrho_N) \subset \mathcal{B} \subset B(x, \varrho_N + \varrho_N^{\epsilon_1})$ for some $x \in \mathbb{Z}^d$ and $\epsilon > 0$ sufficiently small depending only on the dimension d, then the* *following bounds hold:*

$$\lambda_{\mathcal{B}}^{\mathrm{RW},(2)} - \lambda_{\mathcal{B}}^{\mathrm{RW},(1)} \geq c_{5,7}\varrho_N^{-2}, \tag{EV}$$

$$\left\|\phi_{\mathcal{B}}^{\mathrm{RW},(1)}\right\|_\infty \leq c_{5,8}\varrho_N^{-d/2}. \tag{EF}$$

Proof of Lemma 5.A.2. In order to show the first assertion (EV), recall first that the continuum eigenvalue satisfies the scaling relation $\lambda_{B(0,R)}^{(k)} = R^{-2}\lambda_{B(0,1)}^{(k)}$. Combining this with (5.A.1), we get the following bounds:

$$\begin{aligned}
\lambda_{\mathcal{B}}^{\mathrm{RW},(1)} &\leq \lambda_{B(0,\varrho_N+\varrho_N^{\epsilon_1})}^{\mathrm{RW},(1)} \\
&\leq \lambda_{B(0,\varrho_N+\varrho_N^{\epsilon_1})}^{(1)} + \gamma_1\varrho_N^{-3} \\
&\leq \varrho_N^{-2}\lambda_{B(0,1)}^{(1)} + c\varrho_N^{-3+\epsilon_1},
\end{aligned} \tag{5.A.6}$$

and

$$\begin{aligned}
\lambda_{\mathcal{B}}^{\mathrm{RW},(2)} &\geq \lambda_{B(0,(1-\epsilon)\varrho_N)}^{\mathrm{RW},(2)} \\
&\geq \lambda_{B(0,(1-\epsilon)\varrho_N)}^{(2)} - \gamma_2\varrho_N^{-3} \\
&\geq \varrho_N^{-2}\lambda_{B(0,1)}^{(2)} - c\epsilon\varrho_N^{-2}.
\end{aligned} \tag{5.A.7}$$

Since $\lambda_{B(0,1)}^{(1)} < \lambda_{B(0,1)}^{(2)}$, the desired bound (EV) follows for sufficiently small $\epsilon > 0$.

Next, we show the second assertion (EF). By the eigenvalue equation for the semigroup, it follows for any $x \in \mathcal{B}$ that

$$\begin{aligned}
\phi_{\mathcal{B}}^{\mathrm{RW},(1)}(x) &= \left(1-\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\right)^{-\lfloor 1/\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\rfloor} \sum_{y\in\mathcal{B}} p_{\lfloor 1/\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\rfloor}^{\mathcal{B}}(x,y)\phi_{\mathcal{B}}^{\mathrm{RW},(1)}(y) \\
&\leq \left(1-\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\right)^{-\lfloor 1/\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\rfloor} \left\|p_{\lfloor 1/\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\rfloor}^{\mathcal{B}}(x,\cdot)\right\|_2 \left\|\phi_{\mathcal{B}}^{\mathrm{RW},(1)}\right\|_2,
\end{aligned} \tag{5.A.8}$$

where we have used the Schwarz inequality in the second line. Then by the symmetry of the transition kernel $p^{\mathcal{B}}$, the Chapman–Kolmogorov identity and the normalization $\|\phi_{\mathcal{B}}^{(1)}\|_2 = 1$, we can further rewrite (5.A.8) as

$$\begin{aligned} |\phi_{\mathcal{B}}^{\mathrm{RW},(1)}(x)| &\le \left(1-\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\right)^{-\lfloor 1/\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\rfloor} p^{\mathcal{B}}_{2\lfloor 1/\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\rfloor}(x,x)^{1/2} \\ &\le c\left(\lambda_{\mathcal{B}}^{\mathrm{RW},(1)}\right)^{d/4}, \end{aligned} \tag{5.A.9}$$

where we used the local limit theorem as an upper bound. Since we have $\lambda_{\mathcal{B}}^{\mathrm{RW},(1)} \ge c\varrho_N^{-2}$ similarly to (5.A.6), the last line is bounded by $c\varrho_N^{-d/2}$. □

In order to prove Lemma 5.5.2, we use the eigenfunction expansion for the semigroup generated by the random walk killed upon exiting $B(0,\varrho_N)$, whose generator we denote by Q_{ϱ_N}. Due to the periodicity of the random walk, it is convenient to consider the semigroup generated by $Q_{\varrho_N}^2$. Let $\mathbb{Z}_{\mathrm{e}}^d$ ($\mathbb{Z}_{\mathrm{o}}^d$) and 1_{e} (1_{o}) denote the set of even (odd) sites and its indicator function, respectively.

Lemma 5.A.3. *For any positive integer $k \le |B(0,\varrho_N)|/2$, the k-th largest eigenvalues of $Q_{\varrho_N}^2$ is $(1-\lambda_{B(0,\varrho_N)}^{\mathrm{RW},(k)})^2$. Moreover the eigenspace corresponding to $(1-\lambda_{B(0,\varrho_N)}^{\mathrm{RW},(1)})^2$ is spanned by $\phi_{B(0,\varrho_N)}^{\mathrm{RW},(1)}1_{\mathrm{e}}$ and $\phi_{B(0,\varrho_N)}^{\mathrm{RW},(1)}1_{\mathrm{o}}$.*

Proof. For any eigenvalue ζ and corresponding eigenfunction ϕ_ζ of Q_{ϱ_N}, we have

$$Q_{\varrho_N}1_{\mathrm{e}}\phi_\zeta = \zeta 1_{\mathrm{o}}\phi_\zeta \tag{5.A.10}$$

and the same holds with 1_{e} and 1_{o} interchanged. It follows that

$$Q_{\varrho_N}(1_{\mathrm{e}}\phi_\zeta - 1_{\mathrm{o}}\phi_\zeta) = -\zeta(1_{\mathrm{e}}\phi_\zeta - 1_{\mathrm{o}}\phi_\zeta), \tag{5.A.11}$$

that is, $-\zeta$ is also an eigenvalue. Since Q_{ϱ_N} has $|B(0,\varrho_N)|$ eigenvalues counting multiplicity, the first assertion about the eigenvalues follows.

The second assertion about the eigenfunction is a consequence of the following two facts:

$\lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)}$ is a simple eigenvalue and $Q^2_{\varrho_N}$ leaves $\ell^2(\mathbb{Z}^d_{\mathrm{e}})$ and $\ell^2(\mathbb{Z}^d_{\mathrm{o}})$ invariant. □

Proof of Lemma 5.5.2. Let us start with the case that N is an even integer. In this case, $S_N \in \mathbb{Z}^d_{\mathrm{e}}$ and hence

$$\begin{aligned}
&\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) \\
&\quad \geq \mathbb{P}(\mathcal{O} \cap B(0,\varrho_N) = \emptyset)\mathbf{P}\left(\tau_{B(0,\varrho_N)^c} > N, S_N \in \mathbb{Z}^d_{\mathrm{e}}\right) \\
&\quad = p^{|B(0,\varrho_N)|} Q^{N/2}_{\varrho_N} 1_{\mathrm{e}}(0).
\end{aligned} \tag{5.A.12}$$

Let us denote by P the orthogonal projection onto the first eigenspace of $Q^2_{\varrho_N}$. Then

$$Q^{N/2}_{\varrho_N} 1_{\mathrm{e}}(0) = \left(1 - \lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)}\right)^N P 1_{\mathrm{e}}(0) + Q^{N/2}_{\varrho_N}(\mathrm{id} - P) 1_{\mathrm{e}}(0). \tag{5.A.13}$$

Using $1 - \lambda \geq \exp\{-\lambda - \lambda^2\}$ for small $\lambda > 0$, $\phi^{\mathrm{RW},(1)}_{B(0,\varrho_N)} \geq 0$, (5.A.2) and Lemma 5.A.3, we can bound the first term below by

$$\begin{aligned}
&\left(1 - \lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)}\right)^N \left\langle \phi^{\mathrm{RW},(1)}_{B(0,\varrho_N)}, 1_{\mathrm{e}} \right\rangle \phi^{\mathrm{RW},(1)}_{B(0,\varrho_N)}(0) \\
&\quad \geq \left(1 - \lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)}\right)^N \phi^{\mathrm{RW},(1)}_{B(0,\varrho_N)}(0)^2 \\
&\quad \geq \exp\left\{-N\lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)} - cN\varrho_N^{-3}\right\}.
\end{aligned} \tag{5.A.14}$$

On the other hand, it also follows from Lemma 5.A.3 that the operator norm of $Q^N_{\varrho_N}(\mathrm{id} - P)$ is bounded by $(1 - \lambda^{\mathrm{RW},(2)}_{B(0,\varrho_N)})^N$. Combining this with (EV) and $1 - \lambda \leq \exp\{-\lambda\}$, we can bound the second term in (5.A.13) by

$$\begin{aligned}
\left| Q^N_{\varrho_N}(\mathrm{id} - P) 1_{\mathrm{e}}(0) \right| &\leq \left(1 - \lambda^{\mathrm{RW},(2)}_{B(0,\varrho_N)}\right)^N \|1_{\mathrm{e}}\|_{\ell^2(B(0,\varrho_N))} \\
&\leq \exp\left\{-N\lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)} - cN\varrho_N^{-2}\right\}.
\end{aligned} \tag{5.A.15}$$

Since (5.A.15) is negligible compared with (5.A.14) for large N, we obtain

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) \geq \exp\left\{|B(0, \varrho_N)| \log(1/p) - N\lambda^{\mathrm{RW},(1)}_{B(0,\varrho_N)} - cN\varrho_N^{-3}\right\}. \tag{5.A.16}$$

Substituting the bound $||B(0, \varrho_N)| - \mathrm{vol}(B(0, \varrho_N))| \leq c\varrho_N^{d-1}$ and (5.A.1) into the above, we arrive at the desired bound.

Finally when N is an odd integer, we start with

$$\begin{aligned}
&\mathbb{P}(\mathcal{O} \cap B(0, \varrho_N) = \emptyset)\mathbf{P}\left(\tau_{B(0,\varrho_N)^c} > N\right) \\
&\quad = \sum_{|y|=1} \mathbb{P}(\mathcal{O} \cap B(0, \varrho_N) = \emptyset)\mathbf{P}_y\left(\tau_{B(0,\varrho_N)^c} > N - 1\right).
\end{aligned} \tag{5.A.17}$$

Then the rest of the argument is the same as before. We have the sum of $\frac{1}{2d}\phi^{\mathrm{RW},(1)}_{B(0,\varrho_N)}(y)$ over $\{|y| = 1\}$ instead of $\phi^{\mathrm{RW},(1)}_{B(0,\varrho_N)}(0)$ in (5.A.14), but (5.A.2) gives us the same lower bound. □

5.B On the proof of Theorem E

In this section, we briefly explain how to prove Theorem E by adapting the argument in [63]. Roughly speaking, it consists of the following five steps:

1. use the method of enlargement of obstacles in [75] to define a *clearing set* $\mathscr{U}_{\mathrm{cl}}$,
2. prove volume and eigenvalue constraints for $\mathscr{U}_{\mathrm{cl}}$,
3. apply a quantitative Faber–Krahn inequality to show that $\mathscr{U}_{\mathrm{cl}}$ is close to a ball with radius ϱ_N,
4. prove a sharp lower bound on the partition function in terms of a random eigenvalue,
5. use the fact that the region outside $\mathscr{U}_{\mathrm{cl}}$ has much larger eigenvalue to deduce that it is too costly for the random walk to get away from the ball.

In what follows, we explain these steps in more detail using the same parameters as in [63] as much as possible. It turns out that it is only Step 3 that requires an extra argument in the discrete setting.

Steps 1 and 2 can be carried out exactly as in [63] since the method of enlargement of obstacles used in that paper has been translated to the discrete setting in [9]. This method allows us to define the clearing set $\mathscr{U}_{\mathrm{cl}}$ as a union of large boxes (lattice animal) that are almost free of obstacles (cf. [63, (51)–(52)]). Since $\mathscr{U}_{\mathrm{cl}}^c$ has rather high density of obstacles, we can effectively discard it when we consider the eigenvalue (cf. [63, (23), (55)]):

$$\left|\lambda_{\mathscr{U}_{\mathrm{cl}}}^{\mathrm{RW},(1)} - \lambda_{B(0,2N)\setminus\mathcal{O}}^{\mathrm{RW},(1)}\right| \le \varrho_N^{-\rho} \tag{5.B.1}$$

for some $\rho > 0$. Combining the above two properties, in the same way as [63, Proposition 1], we can prove that for some $\alpha_1 > 0$, the μ_N-probability of

$$|\mathscr{U}_{\mathrm{cl}}| \log \tfrac{1}{p} + N\lambda_{\mathscr{U}_{\mathrm{cl}}}^{\mathrm{RW},(1)} \le N^{\frac{d}{d+2}} \left(c(d,p) + N^{-\frac{\alpha_1}{d+2}}\right) \tag{5.B.2}$$

tends to one as $N \to \infty$.

As for Step 3, if $\mathscr{U}_{\mathrm{cl}}$ were a subset of $\mathbb{R}^d$, then the quantitative Faber–Krahn inequality [63, Theorem A] would imply that $\mathscr{U}_{\mathrm{cl}}$ satisfying (5.B.2) must be close to a ball with radius ϱ_N in the symmetric difference. But we are in the discrete setting and hence we need to find a set in $\mathbb{R}^d$ whose volume and (continuum) eigenvalue are almost the same as $|\mathscr{U}_{\mathrm{cl}}|$ and $\lambda_{\mathscr{U}_{\mathrm{cl}}}^{\mathrm{RW},(1)}$, respectively. By the results in [77], the set $\mathscr{U}_{\mathrm{cl}}^+ = \left\{x \in \mathbb{R}^d \colon \mathrm{dist}_{\ell^\infty}(x, \mathscr{U}_{\mathrm{cl}}) \le 2\right\}$ has the eigenvalue $\lambda_{\mathscr{U}_{\mathrm{cl}}^+}^{(1)}$ almost the same as $\lambda_{\mathscr{U}_{\mathrm{cl}}}^{\mathrm{RW},(1)}$. Also, since $\mathscr{U}_{\mathrm{cl}}$ is a union of large boxes, the volume of $\mathscr{U}_{\mathrm{cl}}^+$ is close to $|\mathscr{U}_{\mathrm{cl}}|$. In this way, we can conclude that there exists a ball $B(\boldsymbol{x}_N, \varrho_N)$ that almost coincides with $\mathscr{U}_{\mathrm{cl}}$. The outside of this ball has rather high density of obstacles and hence just as in [63, Lemma 1], we have

$$\lambda_{B(0,2N)\setminus B(\boldsymbol{x}_N,\varrho_N)}^{\mathrm{RW},(1)} \ge \varrho_N^{-2+\alpha_3} \tag{5.B.3}$$

for some $\alpha_3 > 0$.

Step 4 relies on a simple functional analytic argument and there is no difficulty in adapting it to the discrete setting. It corresponds to [63, Lemma 2] and provides the following lower bound on the partition function:

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) \geq N^{-c}\mathbb{E}\left[\exp\left\{-N\lambda^{\mathrm{RW},(1)}_{B(0,2N)\setminus\mathcal{O}}\right\}\right]. \tag{5.B.4}$$

See also [33, Lemma 2] for a slightly simplified argument.

Finally, Step 5 roughly goes as follows. For $0 \leq k < l \leq N$, consider the event that S_k is away from the confinement ball $B(\boldsymbol{x}_N, \varrho_N)$ and returns to it at time l for the first time after k. In this situation, the survival probability between k and l decays like $\exp\{-(l-k)\lambda^{\mathrm{RW},(1)}_{B(0,2N)\setminus B(\boldsymbol{x}_N,\varrho_N)}\}$ and hence using Steps 3 and 4, we obtain

$$\begin{aligned}\mu_N &\left(S_k \notin B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1}), k + \tau_{B(\boldsymbol{x}_N,\varrho_N)} \circ \theta_k = l\right)\\ &\lesssim N^c \frac{\mathbb{E}\left[\exp\left\{-(N-l+k)\lambda^{\mathrm{RW},(1)}_{B(0,2N)\setminus\mathcal{O}} + (l-k)\varrho_N^{-2+\alpha_3}\right\}\right]}{\mathbb{E}\left[\exp\left\{-N\lambda^{\mathrm{RW},(1)}_{B(0,2N)\setminus\mathcal{O}}\right\}\right]}.\end{aligned} \tag{5.B.5}$$

Note that we may impose $\lambda^{\mathrm{RW},(1)}_{B(0,2N)\setminus\mathcal{O}} \leq 2c(d,p)\varrho_N^{-2}$ both in the numerator and denominator, in view of (5.1.5). Now if $l-k \geq \varrho_N^{-2+2\alpha_3}$, then the term $(l-k)\varrho_N^{-2+\alpha_3}$ in the numerator causes a large additional cost and hence the right-hand side decays stretched exponentially in N. If $l-k < \varrho_N^{-2+2\alpha_3}$, then we choose $\epsilon_1 > 1-\alpha_3$ (so that $|S_l - S_k| \gg (l-k)^{1/2}$) and use the Gaussian heat kernel bound for the random walk, instead of the eigenvalue bound, to derive a stretched exponential decay. Summing over k and l, we conclude that the random walk does not make a crossing from $B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1})^c$ to $B(\boldsymbol{x}_N, \varrho_N)$. The case that the random walk does not return to $B(\boldsymbol{x}_N, \varrho_N)$ after visiting $B(\boldsymbol{x}_N, \varrho_N + \varrho_N^{\epsilon_1})^c$ but this can be dealt with in a similar way, by changing l to the last visit to $B(\boldsymbol{x}_N, \varrho_N)$ before k.

5.C Index of notation

CHAPTER 6

BIASED RANDOM WALK IN THE SUBCRITICAL PHASE

6.1 Introduction

6.1.1 Model and main results

Let $(S := (S_n)_{n\geq 0}, \mathbf{P})$ be a simple symmetric random walk on $\mathbb{Z}^d$ starting at the origin and denote the corresponding expectation by $\mathbf{E}$. When we start the random walk from $x \in \mathbb{Z}^d \setminus \{0\}$, we indicate the starting point by subscript as $\mathbf{P}_x$ or $\mathbf{E}_x$. We place an obstacle at each site $x \in \mathbb{Z}^d$ independently with probability $1-p$ for some $p \in (0,1)$ and write $\mathcal{O}$ for the set of sites occupied by the obstacles. Probability and expectation for the random obstacles configuration will be denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively. For a random variable X and an event A, we write $\mathbb{E}[X : A]$ for $\mathbb{E}[X \cdot 1_A]$, and this convention applies to other probability measures. We are interested in the behavior of the random walk with bias $h \in \mathbb{R}^d$ conditioned to avoid $\mathcal{O}$ for a long time, that is, the hitting time $\tau_{\mathcal{O}}$ of $\mathcal{O}$ is large.

Definition 6.1.1. *The annealed law with bias $h \in \mathbb{R}^d$ is defined by*

$$\mu_N^h((S,\mathcal{O}) \in \cdot) = \frac{\mathbb{E} \otimes \mathbf{E}\left[e^{\langle h, S_N\rangle} : \tau_{\mathcal{O}} > N, (S,\mathcal{O}) \in \cdot\right]}{\mathbb{E} \otimes \mathbf{E}\left[e^{\langle h, S_N\rangle} : \tau_{\mathcal{O}} > N\right]}. \tag{6.1.1}$$

When $h = 0$, we omit the superscript and write $\mu_N^0 = \mu_N$ for simplicity.

Remark 6.1.2. In the definition of μ_N^h, we can perform the $\mathbb{E}$-expectation conditionally on the random walk to get the expression

$$\mu_N^h(S \in \cdot) = \frac{\mathbf{E}\left[\exp\left\{\langle h, S_N\rangle - |S_{[0,N]}| \log \frac{1}{p}\right\} : S \in \cdot\right]}{\mathbf{E}\left[\exp\left\{\langle h, S_N\rangle - |S_{[0,N]}| \log \frac{1}{p}\right\}\right]}, \tag{6.1.2}$$

where $S_{[0,N]} = \{S_0, S_1, \ldots, S_N\}$ is the range of the random walk. This can be viewed as a model of self-attractive polymer with an external force h.

In the case $h = 0$, the leading order asymptotics of the partition function was determined by Donsker–Varadhan [27] as follows:

$$\begin{aligned} \mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) &= \exp\left\{-c(d,p)N^{\frac{d}{d+2}} + o(N^{\frac{d}{d+2}})\right\} \text{ with} \\ c(d,p) &= \inf_{\boldsymbol{U}\subset\mathbb{R}^d}\left\{\boldsymbol{\lambda_U} + \mathrm{vol}(\boldsymbol{U})\log\frac{1}{p}\right\} \end{aligned} \tag{6.1.3}$$

as $N \to \infty$, where $\boldsymbol{\lambda_U}$ denotes the smallest eigenvalue of the continuum Laplacian $-\frac{1}{2d}\Delta$ with the Dirichlet boundary condition outside $\boldsymbol{U} \subset \mathbb{R}^d$. Here and in what follows, we use boldface to denote subsets of $\mathbb{R}^d$ and the eigenvalues of continuum Laplacian. For instance, we write $\boldsymbol{B}(x;r) \subset \mathbb{R}^d$ for the Euclidean ball with center x and radius r and $B(x;r) := \boldsymbol{B}(x;r) \cap \mathbb{Z}^d$. By the classical Faber–Krahn inequality, the above infimum is achieved by $U = \boldsymbol{B}(0;\varrho_1)$ for some $\varrho_1 = \varrho_1(d,p)$ (but in fact the center is arbitrary). This indicates that the best strategy to achieve $\{\tau_{\mathcal{O}} > N\}$ is for the random walk to spend most of the time in a vacant (i.e., free of obstacles) ball of radius

$$\varrho_N = \varrho_1 N^{1/(d+2)}. \tag{6.1.4}$$

Subsequently, more refined picture under μ_N has been proved in [66, 12, 63, 23, 10]: there exists a random center

$$\boldsymbol{\mathcal{X}}_N(\mathcal{O}) \in B(0,\varrho_N) \tag{6.1.5}$$

such that for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mu_N\left(B(\boldsymbol{\mathcal{X}}_N,(1-\epsilon)\varrho_N) \subset S_{[0,N]} \subset B(\boldsymbol{\mathcal{X}}_N,(1+\epsilon)\varrho_N)\right) = 1. \tag{6.1.6}$$

Note that the left inclusion in particular implies that the ball $B(\boldsymbol{\mathcal{X}}_N;(1-\epsilon)\varrho_N)$ is vacant.

The model with non-zero bias first appeared in the physics literature [37] where a phase transition of the asymptotic velocity was discussed. A rigorous proof of this ballisticity transition was given in [69, 70], as a consequence of a large deviation principle for $\mu_N(S_N/N \in \cdot)$. We shall provide a more detailed overview on related works in Section 6.1.2.

In this chapter, we study the sub-ballistic phase of μ_N^h in detail. In order to state the results, we need to introduce the so-called Lyapunov exponent (or norm) which measures the cost for the random walk to make a long crossing among the obstacles. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we write $[x] := (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor) \in \mathbb{Z}^d$.

Definition 6.1.3. *The annealed Lyapunov exponent $\beta \colon \mathbb{R}^d \to [0, \infty)$ is defined by*

$$\beta(x) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > \tau_{[nx]}), \tag{6.1.7}$$

and its dual norm β^* is defined by

$$\beta^*(h) = \sup \{\langle h, x\rangle \colon \beta(x) = 1\}. \tag{6.1.8}$$

The set of critical points $h_c \in \mathbb{R}^d$ for the aforementioned ballisticity transition are characterized by this dual norm as $\beta^*(h_c) = 1$. The existence of the limit in (6.1.7) follows from the subadditive ergodic theorem. It can be further shown that

$$\lim_{|x|\to\infty} \frac{1}{|x|} |\beta(x) + \log \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > \tau_x)| = 0. \tag{6.1.9}$$

See [75, Theorem 3.4 on p.244] for the corresponding result in the continuum setting.

Now we are ready to state the first main result of this chapter, that is the large deviation principle under the annealed law without bias in the scale between ϱ_N and $o(\varrho_N^d)$. For scales between ϱ_N^d to N, the large deviation principle is proved in [69, 70]. We write $B(y; r) \subset \mathbb{Z}^d$ for the Euclidean ball centered at $y \in \mathbb{R}^d$ and radius $r > 0$, and dist_β for the distance with respect to the Lyapunov norm $\beta(\cdot)$.

Theorem 6.1.4. *Let $d \geq 2$.*

1. *Let $\varphi(N)$ be such that $\varrho_N \ll \varphi(N) \ll \varrho_N^d$. Then for any $x \in \mathbb{R}^d$,*

$$\mu_N(S_N = [\varphi(N)x]) = \exp\{-\beta(x)\varphi(N)(1 + o(1))\}, \tag{6.1.10}$$

as $N \to \infty$.

2. *For any* $x \in \mathbb{R}^d$,

$$\mu_N(S_N = [\varrho_N x]) = \exp\left\{-\mathrm{dist}_\beta(x, B(0,2))\varrho_N(1+o(1))\right\}, \tag{6.1.11}$$

as $N \to \infty$.

The form of the rate functions reflects the following facts. First, we can let the random walk reach any point $y \in B(0, 2\varrho_N)$ with a negligible cost by shifting the center $\boldsymbol{\mathcal{X}}_N$ of the vacant ball in (6.1.6) so that $B(\boldsymbol{\mathcal{X}}_N, \varrho_N)$ contains 0 and y. This is why the rate function is zero inside $B(0,2)$ in (6.1.11). Next, when $[\varphi(N)x] \notin B(0, 2\varrho_N)$, it turns out that the best strategy is still to have a vacant ball of radius almost ϱ_N. Thus the cost for the random walk to reach $[\varphi(N)x]$ comes solely from the crossing from $B(0, 2\varrho_N)$ to $[\varphi(N)x]$, and it is measured by the Lyapunov norm β. This explains the form of rate function in (6.1.11). In (6.1.10), the size of $B(0, 2\varrho_N)$ is negligible compared with $\varphi(N)$ and hence it does not affect the asymptotics.

The second main result in this chapter is a detailed description of the behavior of the random walk under μ_N^h with a sub-critical drift. As μ_N^h is obtained by tilting μ_N by $e^{\langle h, S_N\rangle}$, the competition between the gain $\langle h, S_N\rangle$ and the cost for the displacement in Theorem 6.1.4 determines the behavior of S_N. The following theorem describes not only the endpoint but also the whole path behavior.

Theorem 6.1.5. *Let* $d \geq 2$. *Suppose* $\beta^*(h) < 1$. *Then for any* $\epsilon > 0$,

$$\lim_{N\to\infty} \mu_N^h\left(B(\varrho_N \mathbf{e}_h, (1-\epsilon)\varrho_N) \subset S_{[0,N]} \subset B(\varrho_N \mathbf{e}_h, (1+\epsilon)\varrho_N)\right) = 1, \tag{6.1.12}$$

where $\mathbf{e}_h := h/|h|$. *Furthermore,*

$$\lim_{N\to\infty} \frac{1}{\varrho_N} S_N = 2\mathbf{e}_h \text{ in } \mu_N^h\text{-probability}, \tag{6.1.13}$$

Remark 6.1.6. With a little more effort, we can replace ϵ in the above theorem by ϱ_N^{-c} for a small $c > 0$. However, since our argument does not seem to give a good control on c, we decided not to present the proof of this refinement.

Remark 6.1.7. Our argument for Theorem 6.1.5 provides a proof of (6.1.12) in the case $h = 0$ as well. See Remarks 6.7.3 and 6.8.2. In this case, it can be regarded as a combination of the ideas from [67, 63] and from [12].

The first assertion (6.1.12) says that the result (6.1.6) remains true under μ_N^h if $\beta^*(h) < 1$ but the center $\boldsymbol{\mathcal{X}}_N$ of the ball becomes $\varrho_N \mathbf{e}_h$. The second assertion (6.1.13) is natural since this strategy maximizes the weight $e^{\langle h, S_N \rangle}$ in (6.1.1) under the constraint in (6.1.12).

6.1.2 Related works

We give a brief overview on the earlier works related to our results. The problem of diffusing particle among the traps has been discussed in the continuum and discrete settings in parallel and most of the results hold in both cases without change. For this reason, we often refrain from indicating in which setting the results are proved.

This type of model with non-zero bias first appeared in the physics literature [37] where a ballisticity transition was discussed. On the mathematical side, the first result seems to be [28] where a phase transition for the free energy of μ_N^h is proved. In particular, it is proved that when $d \geq 2$ and the bias is small, the partition function of μ_N^h has the same asymptotics as (6.1.3). Then in 1990s, Sznitman studied this model and its quenched counterpart in a series of works. We summarize some of the related results. In [69, 70], the annealed Lyapunov exponent with an additional parameter $\lambda \geq 0$ was defined as follows:

$$\beta_\lambda(x) = -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \otimes \mathbf{E}\left[e^{-\lambda \tau_{[nx]}} : \tau_{\mathcal{O}} > \tau_{[nx]} \right]. \tag{6.1.14}$$

Then, improving upon earlier results in [66], it is proved for $\varrho_N^d \leq \varphi(N) \leq N$ that the law of $S_N/\varphi(N)$ under μ_N satisfies a large deviation principle at rate $\varphi(N)$ with rate function

$$\begin{cases} J(x) = \sup\{\beta_\lambda(x) - \lambda \colon \lambda \geq 0\} & \text{if } \varphi(N) = N \text{ ([69, (0.2), (0.3)])}, \\ \beta(x) & \text{if } \varrho_N^d \ll \varphi(N) \ll N \text{ ([69, (0.4)])}, \\ \beta(x) & \text{if } \varphi(N) = \varrho_N^d \text{ and } d \geq 2 \text{ ([70, (0.4)])}. \end{cases}$$

We discuss the case $d = 1$ and $\varphi(N) = \varrho_N^d$ later.

By a standard tilting argument (see [29, Theorem II.7.2], for example), the above large deviation principle can be transferred to those for $\mu_N^h(S_N/\varphi(N) \in \cdot)$ at the same rate with rate function

$$\begin{cases} J^h(x) = J(x) - \langle h, x\rangle - \inf_{y\in\mathbb{R}^d}\{J(y) - \langle h, y\rangle\} & \text{if } \varphi(N) = N \text{ ([70, Theorem 2.1])}, \\ \beta(x) - \langle h, x\rangle & \text{if } \beta^*(h) < 1 \text{ and } \varrho_N^d \ll \varphi(N) \ll N, \\ \beta(x) - \langle h, x\rangle & \text{if } \beta^*(h) < 1, d \geq 2 \text{ and } \varphi(N) = \varrho_N^d \\ & \text{ ([70, Theorem 2.2])}. \end{cases}$$

The first one in particular implies the phase transition of the velocity at $\beta^*(h) = 1$ (see [75, Corollary 4.10 on p.262] for the precise statement). The third one implies that in the subcritical phase, the endpoint of the walk is of distance $o(\varrho_N^d)$ from the origin. This also extends the result of [28] to the whole subcritical phase. See also [31, 32] for the results in the discrete setting. Later Ioffe and Velenik studied the ballistic phase in more detail. An interested reader is referred to [42]. Among other things, it is proved in [44] that the walk is ballistic at criticality. Thus what has been left open is the precise scaling limit under the subcritical drift. Theorem 6.1.5 fills this missing piece and completes the picture.

Let us also mention that more is known in dimension one. The ballisticity transition follows from the results in [69]. The results corresponding to Theorems 6.1.4 and 6.1.5 are proved in [61] and [62], respectively, but with some notable differences. First, unlike in our

Theorem 6.1.4, the rate function in [62] in the scale ϱ_N has a vanishing gradient at $|x| = 2$. The reason for this singularity in $d = 1$ is that the costs for $\tau_{\mathcal{O}} > N$ and $S_N = [\varrho_N x]$ cannot be separated. In order for the random walk to reach $[\varrho_N x]$, the interval $[0, [\varrho_N x]]$ must be free of obstacles and that certainly helps to have $\tau_{\mathcal{O}} > N$. When $d \geq 2$, the size of the vacant ball is essentially determined by the leading term in (6.1.3) which is much larger than ϱ_N, and hence we can separate the cost for $S_N = [\varrho_N x]$ as we explained after Theorem 6.1.4. Second, in [61], the path behavior is studied not only on the macroscopic scale ϱ_N but also on the microscopic scale $O(1)$. On the latter scale, the result roughly says that the walk behaves as if it is conditioned to stay away from a wall with a random position, which lies at the first obstacle to the left of the origin. Though the macroscopic scaling result was later extended to the so-called "soft obstacles" in [65], the microscopic scaling problem remains open in that case. Finally around the critical bias, the asymptotic speed is proved to be continuous in the hard obstacles case in [43, Theorem 5.1], whereas [49, Corollary 1.1] implies that it is discontinuous in the case of soft obstacles. Later in [47], it is proved that the walk with the critical bias among hard obstacles scales like $N^{1/2}$.

Remark 6.1.8. The behavior on the microscopic scale in higher dimensions is a very interesting open problem. The difficulty in the soft obstacles and higher dimensional cases is that, unlike in the case of one-dimensional hard obstacles, a single obstacle cannot play the role of a wall. One needs to understand the geometry of the obstacles configuration around the starting point of the random walk under the effect of the conditioning on the long time survival.

6.2 Outline of proofs

In this section, we explain the outline of the proofs and the organization of the rest of this chapter. The main conceptual difficulty is that we are studying events whose probability decay much slower than the partition functions. This is particularly easy to see in Theorem 6.1.4: from (6.1.3), we know the asymptotics of the partition function

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right) = \exp\left\{-c(d,p)N^{\frac{d}{d+2}} + o\left(N^{\frac{d}{d+2}}\right)\right\}, \tag{6.2.1}$$

but the error term is much larger than the leading terms in Theorem 6.1.4. Since we have little further information about the error term, it is difficult to prove Theorem 6.1.4 by computing the asymptotics of $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N, S_N = x)$ explicitly. Instead, we will use comparison arguments to say something about the path measure, comparing different strategies to achieve $\{\tau_{\mathcal{O}} > N, S_N = x\}$. Roughly speaking, it turns out that when $|x| = o(\varrho_N^d)$, one of the best strategies for the random walk, which gives the dominant contribution, is to stay in a vacant ball of radius almost ϱ_N for a long time and then go to x during the small time interval near the end. As a result, the costs for surviving for a long time and reaching x without hitting the obstacles in $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N, S_N = x)$ can be separated. The former cost counterbalance the partition function and the latter cost gives the rate function in Theorem 6.1.4. For a technical reason, to be explained in Remark 6.5.12, we will work under a slightly different conditioning: Let τ_x^N be the first hitting time of x after time N and define

$$\mu_{N,x}(\cdot) = \mathbb{P} \otimes \mathbf{P}(\cdot \mid \tau_{\mathcal{O}} > \tau_x^N). \tag{6.2.2}$$

Now let us describe in more detail how the rest of the chapter is organized.

In Section 6.4, we show the lower bound in Theorem 6.1.4. In particular, it implies a lower bound on the partition function of $\mu_{N,x}$ since $\{S_N = x, \tau_{\mathcal{O}} > N\} \subset \{\tau_{\mathcal{O}} > \tau_x^N\}$. The proof is based on the construction of a specific strategy to achieve $S_N = x$ and $\tau_{\mathcal{O}} > N$. We first use (6.1.6) to find a vacant (i.e., free of obstacles) ball "shifted toward x" and let the random walk stay there most of the time. Then in the final $O(\mathrm{dist}_\beta(x, B(0, 2\varrho_N)))$ time, we let the random walk go to x. The first part has a probability comparable to $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)$, while the probability of the second part decays exponentially in $\mathrm{dist}_\beta(x, 2\varrho_N)$. We use the FKG inequality to separate these two parts.

In Section 6.5, we show that under $\mu_{N,x}$ with $|x| = o(\varrho_N^d)$, there exists a vacant ball of radius almost ϱ_N, just as in (6.1.6). We also show that it is hard for the random walk to survive outside the vacant ball. For the proofs, we will first use a coarse graining scheme from [25] (or alternatively the method of enlargement of obstacles in [73]) to show that there exists an almost vacant ball. Then we use a density dichotomy lemma from [23] to conclude that the ball is completely vacant.

In Section 6.6, we show that the random walk under $\mu_{N,x}$ with $|x| = o(\varrho_N^d)$ will spend only little time outside the vacant ball. More precisely, we first prove that the time spent before reaching and after leaving the vacant ball cannot be too long. Second, we prove that the random walk path between the first and last visit to the vacant ball is confined in a slightly larger and concentric ball. The proofs rely on the results in Section 6.5 and a path switching argument in the same spirit as in [23].

In Section 6.7, we essentially show that the cost for $\tau_{\mathcal{O}} > \tau_x^N$ can be separated into three parts: (i) crossings from the origin to the vacant ball, (ii) staying near the vacant ball, and (iii) crossing from the vacant ball to x. Due to the confinement proved in Section 6.6, part (ii) is independent from other parts and has probability comparable to $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)$. If parts (i) and (iii) are nearly independent, then the costs are measured by the distances from the origin and x to the vacant ball with respect to the Lyapunov norm, respectively. As it is not easy to control the dependence between (i) and (iii), we will modify them in the proof. See Proposition 6.7.1 for the precise formulation. Adapting the same argument, we also prove that when $|x|$ is close to $2\varrho_N$, then the whole random walk path is confined in a small neighborhood of the vacant ball under $\mu_{N,x}$.

In Section 6.8, we prove the upper bound in Theorem 6.1.4. This follows almost directly from the first result in Section 6.7 since $\{S_N = x, \tau_{\mathcal{O}} > N\} \subset \{\tau_{\mathcal{O}} > \tau_x^N\}$.

In Section 6.9, we prove Theorem 6.1.5. The law of large numbers (6.1.13) can be deduced from Theorem 6.1.4 and large deviation results in [69, 70] via a standard tilting argument, but we will present a more direct argument. In order to prove the confinement (6.1.12), we use (6.1.13) to relate μ_N^h to the random walk law conditioned to avoid obstacles up to time N and end around $2\varrho_N \mathbf{e}_h$. By using the results in Section 6.6, this latter law can further be related to $\mu_{N,x}$ with x close to $2\varrho_N \mathbf{e}_h$, for which the confinement is proved in Section 6.7.

6.3 Notation and preliminaries

We will prove various intermediate propositions with error terms depending on $|x|$. To simplify the notation, we define

$$\delta_{N,x} = \varrho_N^{-1/5} \vee (|x|/\varrho_N^d), \tag{6.3.1}$$

which goes to zero polynomially fast in N when $|x| \le \varrho_N^{d-\xi}$ for some $\xi > 0$. The exponent $-1/5$ is rather arbitrary and has no significance.

We use c and c' to denote a positive constant whose value may change from line to line. When we need to keep the value of a constant within a proof, we use the upper case letters C, $C_{6,1}$ and C_2. We write $c_{X.Y}$ for a constant defined in Theorem/Proposition/Lemma $X.Y$, if it is referred to in other places.

Next, we collect some notation and estimates for the simple symmetric random walk on $\mathbb{Z}^d$. For $U \subset \mathbb{Z}^d$, we denote by λ_U the smallest Dirichlet eigenvalue of the discrete Laplacian $-\frac{1}{2d}\Delta$. Then, we have the following tail estimate for the exit time τ_{U^c} from U:

$$\begin{aligned} \mathbf{P}_x(\tau_{U^c} > n) &\le |U|^{1/2}(1-\lambda_U)^n \\ &\le |U|^{1/2} \exp\{-n\lambda_U\}. \end{aligned} \tag{6.3.2}$$

See [48, (2.21)] for the continuous time analogue. A similar bound with $|U|^{1/2}$ replaced by $c(1+n\lambda_U)^{d/2}$ also holds, see [75, (1.9) in Section 3]. Combining this with a Faber–Krahn type inequality $\lambda_U \ge c|U|^{-2/d}$ for the eigenvalue of Laplacian on $\mathbb{Z}^d$ [50, Remark 3.2.6 and

Theorem 3.2.7], we can deduce that

$$\mathbf{P}_x(\tau_{U^c} > n) \le c \exp\left\{-cn|U|^{-2/d}\right\}. \tag{6.3.3}$$

For $U \subset \mathbb{Z}^d$, $n \ge 0$ and $x, y \in U$, we write $p_n^U(x,y)$ for the transition probability of the random walk killed upon exiting from U. Whenever we use this notation, we tacitly assume that $|y|_1$ has the same parity as $n + |x|_1$.

Lemma 6.3.1. *There exists $c > 0$ such that for any $R \ge 2$, $n \ge R^2/2$ and $x, y \in B(0,R)$,*

$$p_n^{B(0,R)}(x,y) \ge \frac{c}{R^d} d_R(x) d_R(y) \exp\left\{-\frac{n}{cR^2}\right\}, \tag{6.3.4}$$

where $d_R(z) = R^{-1}\mathrm{dist}_{\ell^1}(z, \partial B(0,R))$. If $x = y \in B(0,R)$, the same bound holds for all $n \ge 0$.

Proof. When $n \in [R^2/2, R^2]$, this is a consequence of [53, Proposition 6.9.4]. For $n \ge R^2$, we use the Chapman–Kolmogorov identity to obtain

$$p_n^{B(0,R)}(x,y) \ge \sum_{z \in B(0,R/2)} p_{R^2/2}^{B(0,R)}(x,z) p_{n-R^2/2}^{B(0,R)}(z,y). \tag{6.3.5}$$

The second factor is bounded from below by $cd_R(y)R^{-d}\exp\{-c^{-1}nR^{-2}\}$, uniformly in $z \in B(0,R/2)$, by the second part of [53, Proposition 6.9.4] and [53, Corollaries 6.9.5 and 6.9.6]. Then, we use the result for $n = R^2/2$ to obtain

$$\sum_{z \in B(0,R/2)} p^{B(0,R)}(x,z) \ge cd_R(x). \tag{6.3.6}$$

Combining the above two estimates, we obtain (6.3.4).

Next, let $x = y \in B(0,R)$, which forces $n \in 2\mathbb{N}^*$. For $n \in \{0,2\}$, the left-hand side of (6.3.4) is larger than $(2d)^{-2}$. For $n \in [4, R^2/2]$, we can find a ball $B(z, \sqrt{n})$ such that

$\text{dist}_{\ell^1}(x, \partial B(z, \sqrt{n})) = \text{dist}_{\ell^1}(x, \partial B(0, R))$. Applying the first part of [53, Proposition 6.9.4] to this ball, we obtain (6.3.4) in this case. □

6.4 Proof of the lower bound in Theorem 6.1.4

We first show a lower bound on the survival probability with a fixed endpoint, which in particular implies the lower bounds in Theorem 6.1.4.

Proposition 6.4.1. *There exists $c_{6.4.1} > 0$ such that when $\epsilon > 0$ is small depending on d and p and $|x| \leq \epsilon \varrho_N^d$,*

$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N, S_N = x) \geq \exp\left\{-\text{dist}_\beta\left(x, B(0, 2\varrho_N)\right) - c_{6.4.1}\epsilon(|x| \vee \varrho_N)\right\} \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) \tag{6.4.1}$$

for all sufficiently large N. Furthermore, under the same condition,

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N\right) \geq \exp\left\{-c(d,p)N^{\frac{d}{d+2}} - \text{dist}_\beta\left(x, B(0, 2\varrho_N)\right) - c_{6.4.1}\delta_{N,x}N^{\frac{d}{d+2}}\right\} \tag{6.4.2}$$

for all sufficiently large N, where $c(d,p)$ and $\delta_{N,x}$ are defined in (6.2.1) *and* (6.3.1)*, respectively.*

Proof. We start by introducing several objects used in this proof. Let us first assume $|x| \geq 2\varrho_N$ and let $y \in B(0, (2-4\epsilon)\varrho_N)$ be such that $\beta(x-y) = \text{dist}_\beta\left(x, B(0, (2-4\epsilon)\varrho_N)\right)$. Then for $M > 0$ to be chosen later in (6.4.11) depending only on d and p, define

$$n = N - M|x-y| - \varrho_N^2 \geq N - 2M\varrho_N^d. \tag{6.4.3}$$

Roughly speaking, we consider the following strategy: There is a ball of radius ϱ_n centered around $\frac{1}{2}y$ which is free of obstacles, and we let the random walk (i) stay inside that ball up to time n, (ii) get close to y in the next ϱ_N^2 steps, (iii) go to x in the remaining $M|x-y|$ steps (See Figure 6.1). The cost up to (ii) is comparable to $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)$ while the cost for

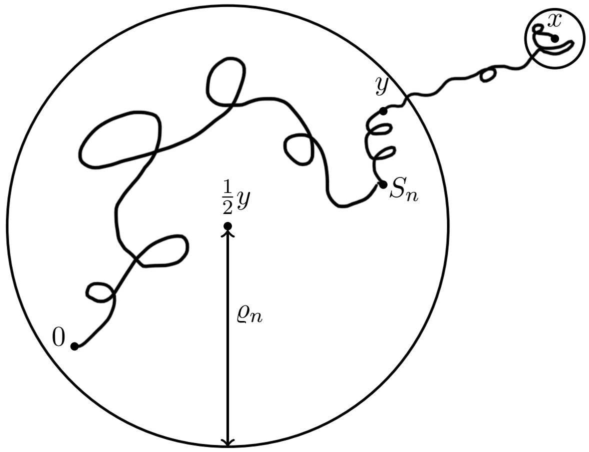

Figure 6.1: The strategy to achieve the large deviation lower bound in Proposition 6.4.1. Since $\lim_{N\to\infty} N/n = 1$ for n defined in (6.4.3), the large ball has radius almost ϱ_N.

(iii) is measured by $\exp\{-\beta(x-y)\}$. The following argument makes this outline rigorous.

It is proved in [23] that for $\S_n$ in (6.1.6) and any $\epsilon > 0$,

$$\mathbb{P} \otimes \mathbf{P} \left(\mathcal{O} \cap B(\S_n, (1-\epsilon)\varrho_n) = \emptyset \mid \tau_{\mathcal{O}} > n \right) \to 1 \tag{6.4.4}$$

as $n \to \infty$. Moreover, we know from [67, 63] that the distribution of $\varrho_n^{-1}\S_n$ converges to $\phi_1(x)\,\mathrm{d}x$, where ϕ_1 is the L^1-normalized principal eigenfunction of the Dirichlet Laplacian in $B(0,1) \subset \mathbb{R}^d$. Since ϕ_1 is positive and continuous inside $B(0,1)$, there exists $c(\epsilon) > 0$ such that

$$\mathbb{P} \otimes \mathbf{P} \left(\S_n \in B(\tfrac{1}{2}y, \epsilon\varrho_n) \,\middle|\, \tau_{\mathcal{O}} > n \right) \geq c(\epsilon) \tag{6.4.5}$$

for all $n \geq 1$. Recall also that [23, Lemma 4.5] shows

$$\mathbb{P} \otimes \mathbf{P} \left(S_n \in B(\S_n, (1-4\epsilon)\varrho_n) \mid \tau_{\mathcal{O}} > n \right) \to 1 \tag{6.4.6}$$

as $n \to \infty$ and $\epsilon \to 0$. Summarizing the above considerations, when $\epsilon > 0$ is sufficiently small, we have

$$\begin{aligned}&\mathbb{P}\otimes\mathbf{P}\left(S_n\in B(\tfrac{1}{2}y,(1-3\epsilon)\varrho_n),\mathcal{O}\cap B(\tfrac{1}{2}y,(1-2\epsilon)\varrho_n)=\emptyset,\tau_{\mathcal{O}}>n\right)\\&\quad\geq\frac{c(\epsilon)}{2}\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>n\right)\end{aligned}\tag{6.4.7}$$

for all sufficiently large n. On the event on the left-hand side, we further let the random walk go to y inside $B(\frac{1}{2}y,(1-2\epsilon)\varrho_n)$ during the time interval $[n,n+\varrho_N^2]$. Then using the Markov property at time n and the random walk estimate (6.3.4), we obtain

$$\mathbb{P}\otimes\mathbf{P}\left(S_{n+\varrho_N^2}=y,\tau_{\mathcal{O}}>n+\varrho_N^2\right)\geq\frac{c(\epsilon)}{\varrho_n^d}\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>n\right)\tag{6.4.8}$$

for all sufficiently large n.

Next we let the random walk go from y to x during the time interval $[n+\varrho_N^2,N]$ without hitting the obstacles. By imposing an extra condition $\mathcal{O}\cap B(x,R)=\emptyset$ for $R=|x-y|^{1/2d}$, the probability of this last piece is bounded from below by

$$\begin{aligned}&\mathbb{P}\otimes\mathbf{P}_y\left(S_{M|x-y|}=x,\tau_{\mathcal{O}}>M|x-y|\right)\\&\quad\geq\mathbb{E}\otimes\mathbf{E}_y\left[p^{B(x,R)}_{M|x-y|-\tau_x}(x,x)\colon\mathcal{O}\cap B(x,R)=\emptyset,\tau_x<\tau_{\mathcal{O}}\wedge M|x-y|\right]\\&\quad\geq p^{|B(x,R)|}\min_{k\in2\mathbb{Z}:0\leq k\leq M|x-y|}p_k^{B(x,R)}(x,x)\mathbb{P}\otimes\mathbf{P}_y\left(\tau_x<\tau_{\mathcal{O}}\wedge M|x-y|\right),\end{aligned}\tag{6.4.9}$$

where we have applied the FKG inequality to $1_{\{\mathcal{O}\cap B(x,R)=\emptyset\}}$ and $\mathbf{P}_y(\tau_x<\tau_{\mathcal{O}}\wedge M|x-y|)$, which are decreasing functions in the obstacle field. Due to the random walk estimate (6.3.4), the above $p_k^{B(x,R)}(x,x)$ is bounded from below by $cR^{-d}\exp\{-c^{-1}M|x-y|/R^2\}$. Since we have chosen $R=|x-y|^{1/2d}$, it follows that

$$p^{|B(x,R)|}\min_{k\in2\mathbb{Z}:k\leq M|x-y|}p_k^{B(x,R)}(x,x)\geq\exp\{-cM|x-y|^{1-1/d}\}.\tag{6.4.10}$$

To bound the third factor in the third line of (6.4.9), we use a result in [49, Theorem 1.1] which says that $\mathbb{E} \otimes \mathbf{E}[\tau_z \mid \tau_{\mathcal{O}} > \tau_z] \leq C|z|$. From this and the Markov inequality, it follows that

$$\begin{aligned} \mathbb{P} \otimes \mathbf{P}_y \left(\tau_x < \tau_{\mathcal{O}} \wedge M|x-y| \right) &\geq \frac{1}{2} \mathbb{P} \otimes \mathbf{P}_y (\tau_{\mathcal{O}} > \tau_x) \\ &\geq \exp\{-(1+\epsilon)\beta(x-y)\} \end{aligned} \tag{6.4.11}$$

when ϵ is small and M, N are large, where in the second inequality we have used (6.1.9) and that $|x - y| \geq c\epsilon\varrho_N$ for $|x| \geq 2\varrho_N$.

Finally, since $\mathbf{P}(S_{n+\varrho_N^2} = y, \tau_{\mathcal{O}} > n)$ and $\mathbf{P}_y(S_N = x, \tau_{\mathcal{O}} > M|x-y|)$ are both decreasing in $\mathcal{O}$, we can use the FKG inequality to deduce from (6.4.8)–(6.4.11) that

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P} \left(S_N = x, \tau_{\mathcal{O}} > N \right) \\ &\quad \geq \mathbb{E} \left[\mathbf{P}(S_{n+\varrho_N^2} = y, \tau_{\mathcal{O}} > n + \varrho_N^2) \mathbf{P}_y(S_{M|x-y|} = x, \tau_{\mathcal{O}} > M|x-y|) \right] \\ &\quad \geq \exp\left\{-(1+\epsilon)\beta(x-y) - c\epsilon(|x| \vee \varrho_N)\right\} \mathbb{P} \otimes \mathbf{P} \left(\tau_{\mathcal{O}} > n \right) \end{aligned} \tag{6.4.12}$$

for all sufficiently large N. Since $\beta(x - y) \leq c|x|$, this concludes the proof of (6.4.1) in the case $|x| \geq 2\varrho_N$.

Let us turn to the case $|x| < 2\varrho_N$. If we assume a slightly stronger condition $|x| \leq (2 - 4\epsilon)\varrho_N$, then we have $y = x$ and $n = N - \varrho_N^2$ and hence (6.4.8) gives us the desired bound. If $(2 - 4\epsilon)\varrho_N \leq |x| \leq 2\varrho_N$, then we set y as before and let $n = N - |y - x|_1$. We follow the same argument up to (6.4.8). Then instead of (6.4.11), we fix a path $\pi(y, x)$ connecting y and x with $|y - x|_1$ steps and use

$$\begin{aligned} \mathbb{P} \otimes \mathbf{P}_y \left(S_{|y-x|_1} = x, \mathcal{O} \cap \pi(y,x) = \emptyset \right) &= \left(\frac{p}{2d} \right)^{|y-x|_1} \\ &\geq \exp\left\{-c\epsilon\varrho_N\right\}. \end{aligned} \tag{6.4.13}$$

Then following the same argument as above, we obtain (6.4.1) in this case.

The second assertion (6.4.2) follows from (6.4.1) and the bound

$$\mathbb{P}\otimes\mathbf{P}(\tau_{\mathcal{O}}>N)\geq\exp\left\{-c(d,p)N^{\frac{d}{d+2}}-cN^{\frac{d-1}{d+2}}\right\} \tag{6.4.14}$$

proved in [12, Proposition 2.1] by making $c_{6.4.1}$ larger. □

As a consequence of Proposition 6.4.1, we have a crude upper bound on τ_x^N, the first hitting time of x after N:

Corollary 6.4.2. *There exists $c_{6.4.2}>0$ such that when $\epsilon>0$ is small depending on d and p and $|x|\leq\epsilon\varrho_N^d$,*

$$\mu_{N,x}\left(\tau_x^N>2N\right)\leq\exp\left\{-c_{6.4.2}N^{\frac{d}{d+2}}\right\} \tag{6.4.15}$$

for all sufficiently large N.

Proof. By (6.1.3),

$$\begin{aligned}\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>\tau_x^N>2N\right)&\leq\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>2N\right)\\&\leq\exp\left\{-(c(d,p)+o(1))(2N)^{\frac{d}{d+2}}\right\},\end{aligned} \tag{6.4.16}$$

as $N\to\infty$. Comparing this with (6.4.2), we get (6.4.15). □

Remark 6.4.3. Due to Corollary 6.4.2, we may effectively discard the event $\{\tau_x^N>2N\}$ from our consideration. Thus in what follows, we will tacitly assume $\tau_x^N\leq 2N$. Since we are considering the discrete time random walk, this in particular implies that all the points of $\mathbb{Z}^d$ appearing hereafter can be assumed to be in $B(0,2N)$. In particular, we will replace the set of obstacles $\mathcal{O}$ by $\mathcal{O}\cup B(0,2N)^c$.

6.5 Existence of a vacant ball

The main result in this section is the existence of a ball of radius almost ϱ_N which is free of obstacles under the measure $\mu_{N,x}$ with $|x|=o(\varrho_N^d)$.

Proposition 6.5.1. *There exist* $\boldsymbol{\mathcal{X}}_N(\mathcal{O}) \in B(0, \varrho_N)$ *and* $c_{6.5.1} > 0$ *such that when* $\epsilon > 0$ *is small depending on* d *and* p *and* $|x| \le \epsilon \varrho_N^d$*, the* $\mu_{N,x}$*-probability of the events*

$$\left\{\mathcal{O} \cap B(\boldsymbol{\mathcal{X}}_N, (1 - \delta_{N,x}^{c_{6.5.1}})\varrho_N) = \emptyset\right\} \tag{6.5.1}$$

and

$$\left\{B(\boldsymbol{\mathcal{X}}_N, (1 - \delta_{N,x}^{c_{6.5.1}})\varrho_N) \subset S_{[0,\tau_x^N]}\right\} \tag{6.5.2}$$

are greater than $1 - \exp\{-(\log N)^2\}$ *for all sufficiently large* N*, where* $\delta_{N,x}$ *is defined in* (6.3.1).

We deduce Proposition 6.5.1 from the following two lemmas. The first one asserts that there is a ball of radius ϱ_N which is almost free of obstacles; the second one asserts that every obstacle is well surrounded by others.

Lemma 6.5.2. *There exists* $c_{6.5.2} > 0$ *and* $\boldsymbol{\mathcal{X}}_N(\mathcal{O}) \in B(0, \varrho_N)$ *such that when* $\epsilon > 0$ *is small depending on* d *and* p *and* $|x| \le \epsilon \varrho_N^d$*,*

$$\mu_{N,x}\left(|\mathcal{O} \cap B(\boldsymbol{\mathcal{X}}_N, \varrho_N)| \ge \delta_{N,x}^{c_{6.5.2}} N^{\frac{d}{d+2}}\right) \le \exp\left\{-c_{6.5.2}\delta_{N,x} N^{\frac{d}{d+2}}\right\} \tag{6.5.3}$$

for all sufficiently large N*.*

Lemma 6.5.3.

$$E_l^\delta(v) = \left\{v \in \mathcal{O} \text{ and } \frac{|\mathcal{O} \cap B(v, l)|}{|B(v, l)|} < \delta\right\}. \tag{6.5.4}$$

Then there exists $c_{5.2.1} > 0$ *such that for sufficiently large* N*,*

$$\mu_{N,x}\left(\bigcup_{v \in B(0,2N)} \bigcup_{(\log N)^3 \le l \le \varrho_N} E_l^{c_{5.2.1}}(v)\right) \le \exp\left\{-c_{5.2.1}(\log N)^3\right\}. \tag{6.5.5}$$

Proof of Proposition 6.5.1. If there is an obstacle deep inside the ball $B(\boldsymbol{\mathcal{X}}_N; \varrho_N)$ found in Lemma 6.5.2, then there are in fact many obstacles by Lemma 5.2.1, which contradicts (6.5.3). The other part (6.5.2) can also be deduced from these lemmas in the same way as [23, Lemma 3.2]. □

In the proof of (6.1.12), we need to know that it is hard for the random walk to stay outside the vacant ball. The next lemma gives us such an estimate. For technical reasons, we will consider a slightly smaller ball

$$\begin{aligned} B^-(z) &= B(z, (1 - 2\delta_{N,x}^{c_{6.5.1}})\varrho_N), \\ B^- &= B^-(\boldsymbol{x}_N). \end{aligned} \tag{6.5.6}$$

Recall from Remark 6.4.3 that we enlarged the obstacles to $\mathcal{O} \cup B(0, 2N)^c$.

Lemma 6.5.4. *There exists $c_{6.5.4} > 0$ such that when $\epsilon > 0$ is small depending on d and p and $|x| \leq \epsilon \varrho_N^d$, for $t \geq \delta_{N,x}^{c_{6.5.4}} (\log N)^2 \varrho_N^2$, $\mu_{N,x}$-probability of the event (which depends only on $\mathcal{O}$)*

$$\left\{ \sup_{y \in B(0,2N)} \mathbf{P}_y \left(S_{[0,t]} \cap \left(\mathcal{O} \cup B^- \right) = \emptyset \right) \leq \exp \left\{ -\delta_{N,x}^{-c_{6.5.4}} \varrho_N^{-2} t \right\} \right\} \tag{6.5.7}$$

is greater than $1 - \exp\{-(\log N)^2\}$ for all sufficiently large N.

The proofs of Lemmas 6.5.2 and 6.5.4 are given in Section 6.5.1.

Lemma 5.2.1 is an analogue of [23, Lemma 2.1]. We will provide an outline of the argument in Section 6.5.2.

6.5.1 Proofs of Lemmas 6.5.2 and 6.5.4

We prove Lemmas 6.5.2 and 6.5.4 using some concepts and results from the recent paper [25], which proves a quenched localization result for the random walk conditioned to avoid $\mathcal{O}$. Let us explain the outline of the proof before delving into the details. Recall that we denote by λ_U the smallest Dirichlet eigenvalue of the discrete Laplacian $-\frac{1}{2d}\Delta$ in U, which is different from the notation in [25].

For $\iota, \rho > 0$, we introduce a set $\mathcal{E}(\iota,\rho) \subset B(0,2N)$ in Definition 6.5.5 which is a collection of large (but $o(\varrho_N)$) boxes where the density of obstacles is low. We will show that $\mathcal{E}(\iota,\rho)$ is close to a ball with radius ϱ_N in symmetric difference. Then it follows that the ball is almost free of obstacles. The purpose of this coarse graining is two-fold: it identifies the regions that actually contribute to the vacant ball (note that the number of open sites in $B(0;2N)$ is of order $N^d \gg \varrho_N^d$); and it reduces the entropy of the set of obstacle configurations and allows estimates of the form $\mathbb{P}(|\mathcal{E}(\iota,\rho)| = V) = (1-p)^{V(1+o(1))}$, as if there is only one configuration of $\mathcal{E}(\iota,\rho)$ with given volume V.

If we could show that $\mathcal{E}(\iota,\rho)$ correctly identifies where the random walk is localized in the sense that, the Dirichlet eigenvalue does not change much if we restrict the walk to $\mathcal{E}(\iota,\rho)$, namely,

$$\lambda_{B(0,2N)\setminus\mathcal{O}} \sim \lambda_{\mathcal{E}(\iota,\rho)}, \tag{6.5.8}$$

then we could (formally) write

$$\begin{aligned} \mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N\right) &\lesssim \mathbb{E}\left[\exp\left\{-N\lambda_{B(0,2N)\setminus\mathcal{O}}\right\}\right] \\ &\lesssim \sum_{V=1}^{(4N+1)^d} \mathbb{E}\left[\exp\left\{-N\lambda_{\mathcal{E}(\iota,\rho)}\right\} : |\mathcal{E}(\iota,\rho)| = V\right] \\ &\approx \sup_{U\subset\mathbb{Z}^d} \exp\left\{-N\lambda_U - |U|\log\tfrac{1}{1-p}\right\}. \end{aligned} \tag{6.5.9}$$

The last approximation is justified by the fact that V can only take $O(N^d)$ many values, which is of lower order than the exponential asymptotics. We can then apply a quantitative Faber-Krahn inequality to show that the dominant contribution comes from configurations of $\mathcal{E}$ that are close to a ball with radius ϱ_N.

Unfortunately, it is not easy to prove (6.5.8) directly. Instead, we make a detour by comparing $\mathcal{E}(\iota,\rho)$ with a low level set Ω_η of the principal eigenfunction in $B(0,2N)\setminus\mathcal{O}$ (see Definition 6.5.7). It is relatively easy to prove that λ_{Ω_η} well approximates $\lambda_{B(0,2N)\setminus\mathcal{O}}$ when η is small, and it is also relatively easy to prove that the eigenfunction is small on

$\mathcal{E}(\iota,\rho)$, and as a consequence $\mathcal{E}(\iota,\rho)$ almost contains Ω_η. See Lemma 6.5.9 for the precise formulation. That lemma essentially allows us to carry out the argument around (6.5.9) with $\mathcal{E}(\iota,\rho)$ replaced by Ω_η.

Now let us turn to the formal proof.

Definition 6.5.5 (Definition 5.1 in [25]). *For $\iota,\rho>0$, a box of the form*

$$K_{\lfloor \iota\varrho_N\rfloor}(x)=x+[-\lfloor \iota\varrho_N\rfloor,\lfloor \iota\varrho_N\rfloor)^d \text{ for } x\in(2\lfloor \iota\varrho_N\rfloor+1)\mathbb{Z}^d \tag{6.5.10}$$

is said to be $(\iota\varrho_N,\rho)$-empty if

$$|\mathcal{O}\cap K_{\lfloor \iota\varrho_N\rfloor}(x)|\le \rho|K_{\lfloor \iota\varrho_N\rfloor}(x)|. \tag{6.5.11}$$

Let $\mathcal{E}(\iota,\rho)$ denote the intersection between $B(0,2N)$ and the union of $(\iota\varrho_N,\rho)$-empty boxes.

We will choose $\rho>0$ small so that $(\iota\varrho_N,\rho)$-empty boxes are rare. As a consequence, we have a rather good control on the volume of $\mathcal{E}(\iota,\rho)$.

Lemma 6.5.6. *For any $\iota,\rho\in(0,1)$ and $V>0$,*

$$\mathbb{P}(|\mathcal{E}(\iota,\rho)|=V)\le \exp\left\{-V\left(\log\tfrac{1}{p}+2\rho\log\rho-\frac{\log(3N)}{\lfloor 2\iota\varrho_N\rfloor^d}\right)\right\}. \tag{6.5.12}$$

Proof. This can be proved in the same way as [25, Lemma 5.2]. □

Definition 6.5.7 (Definition 5.3 in [25]). *Let f be the eigenfunction corresponding to the eigenvalue $\lambda_{B(0;2N)\setminus\mathcal{O}}$ such that $\|f\|_1=1$. We extend f to $\mathbb{Z}^d$ by letting $f(v)=0$ for $v\in B(0,2N)^c\cup\mathcal{O}$ and define*

$$\Omega_\eta=\{v\in\mathbb{Z}^d\colon f(v)\ge \eta|\mathcal{E}(\iota,\rho)|^{-1}\},$$

where we will fix the parameters

$$\eta = 2\delta_{N,x}, \quad \rho = \eta^2, \quad \iota = \eta^{5/2}. \tag{6.5.13}$$

Our η plays the role of ϵ in [25]. Noting that $\delta_{N,x} \in [\varrho_N^{-1/5}, \epsilon]$ by (6.3.1) and the assumptions of Lemmas 6.5.2 and 6.5.4, we have

$$\left| 2\rho \log \rho - \frac{\log(3N)}{\lfloor 2\iota\varrho_N \rfloor^d} \right| \leq c\eta \tag{6.5.14}$$

in (6.5.12) for all sufficiently large N when ϵ is small.

As mentioned before, we are going to prove that Ω_η largely coincides with $\mathcal{E}(\iota, \rho)$. We start with an *a priori* bound on the eigenvalue $\lambda_{B(0,2N)\setminus\mathcal{O}}$ under $\mu_{N,x}$ with $|x| \leq \varrho_N^d$, which is a consequence of Proposition 6.4.1.

Corollary 6.5.8. *There exists constant* $c_{6.5.8} > 0$ *such that for all* $|x| \leq \varrho_N^d$,

$$\mu_{N,x}(\lambda_{B(0,2N)\setminus\mathcal{O}} \geq c_{6.5.8}\varrho_N^{-2}) \leq \exp\left\{ -c_{6.5.8}^{-1} N^{\frac{d}{d+2}} \right\}. \tag{6.5.15}$$

Proof. It follows from (6.3.2) that

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P}\left(\lambda_{B(0,2N)\setminus\mathcal{O}} \geq C\varrho_N^{-2}, \tau_{\mathcal{O}} > \tau_x^N \right) \\ &\quad \leq \mathbb{E}\left[cN^{d/2}(1 - \lambda_{B(0,2N)\setminus\mathcal{O}})^N : \lambda_{B(0,2N)\setminus\mathcal{O}} \geq C\varrho_N^{-2} \right] \\ &\quad \leq cN^{d/2} \exp\left\{ -C\varrho_1^{-2} N^{\frac{d}{d+2}} \right\}. \end{aligned} \tag{6.5.16}$$

Comparing this with (6.4.2) and choosing $C > 0$ sufficiently large, we obtain the desired result. □

In what follows, we often assume the condition

$$\lambda_{B(0;2N)\setminus\mathcal{O}} \leq c_{6.5.8}\varrho_N^{-2} \tag{6.5.17}$$

appearing in Corollary 6.5.8.

Lemma 6.5.9 (Lemmas 5.6 and 5.7 in [25])**.** *Assume* (6.5.17)*. Then there exists* $c_{6.5.9} > 0$ *such that*

$$|\Omega_{\eta^2} \setminus \mathcal{E}(\iota, \rho)| \le c_{6.5.9}\eta |\mathcal{E}(\iota, \rho)|, \tag{6.5.18}$$

$$\lambda_{\Omega_\eta} \le \lambda_{B(0;2N)\setminus\mathcal{O}}(1 + c_{6.5.9}\eta). \tag{6.5.19}$$

Proof. This can be proved in the same way as [25, Lemmas 5.6 and 5.7]. □

Lemmas 6.5.6 and 6.5.9 provide controls on the volume and eigenvalue of Ω_η since $\Omega_\eta \subset \Omega_{\eta^2}$. The reason for considering Ω_{η^2} will be explained shortly. We will show that Ω_η is approximately a ball of radius ϱ_N by applying the quantitative Faber–Krahn inequality for the continuum Laplacian eigenvalue in [15]. In order to apply it in our discrete setting, we need to approximate $\Omega_\eta \subset \mathbb{Z}^d$ by a continuous set in $\mathbb{R}^d$ with the volume and eigenvalue controlled. Recall our convention of using boldface letters to denote a subset of $\mathbb{R}^d$ as well as the smallest Dirichlet eigenvalue of the continuum Laplacian $-\frac{1}{2d}\Delta$. For the eigenvalue approximation, we use a classical result about the comparison between discrete and continuum eigenvalues in [51]. To this end, we need to introduce slightly enlarged sets

$$\Omega_\eta^+ = \left\{ v \in \mathbb{Z}^d : \min_{x \in \Omega_\eta} |x - v|_\infty < 2 \right\}, \tag{6.5.20}$$

$$\boldsymbol{\Omega}_\eta^+ = \bigcup_{v \in \Omega_\eta^+} \left(v + [-\tfrac{1}{2}, \tfrac{1}{2}]^d \right). \tag{6.5.21}$$

Then [51, (38)] asserts

$$\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} \le \lambda_{\Omega_\eta} + c\lambda_{\Omega_\eta}^2. \tag{6.5.22}$$

The passage from Ω_η to Ω_η^+ is potentially problematic since it can increase the volume substantially when Ω_η has many tiny holes. The following lemma is to solve this problem by showing that Ω_η^+ is not much larger than a slightly lower level set Ω_{η^2}, for which the volume bound (6.5.18) holds.

Lemma 6.5.10 (Lemma 5.8 in [25])**.** *Assume* (6.5.17)*. Then there exists* $c_{6.5.10} > 0$ *such that*

$$|\Omega_\eta^+ \setminus \Omega_{\eta^2}| \le c_{6.5.10}\eta|\mathcal{E}(\iota,\rho)|. \tag{6.5.23}$$

Proof. This can be proved in the same way as [25, Lemma 5.8]. □

Now we are ready to prove Lemmas 6.5.2 and 6.5.4.

Proof of Lemma 6.5.2. In view of Corollary 6.5.8, we may assume (6.5.17). The proof is divided into three steps. The last two steps are similar to the proof of [25, Lemma 5.9] and hence we omit some technical details.

Step 1: We first prove that $|\boldsymbol{\Omega}_\eta^+|$ is not much larger than $|\boldsymbol{B}(0;\varrho_N)|$, the volume of the Euclidean ball $\boldsymbol{B}(0;\varrho_N)$, under $\mu_{N,x}$ with high probability. To this end, we use (6.3.2) to obtain

$$\begin{aligned}&\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, |\mathcal{E}(\iota,\rho)| = V\right)\\ &\le \mathbb{E}\left[CN^{d/2}\exp\left\{-N\lambda_{B(0;2N)\setminus\mathcal{O}}\right\} : |\mathcal{E}(\iota,\rho)| = V\right].\end{aligned} \tag{6.5.24}$$

By (6.5.17), (6.5.19), (6.5.22) and the classical Faber–Krahn inequality, when N is sufficiently large, we have

$$\begin{aligned}\lambda_{B(0;2N)\setminus\mathcal{O}} &\ge \boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+}(1-c\eta)\\ &\ge |\boldsymbol{\Omega}_\eta^+|^{-2/d}\boldsymbol{\lambda}_{\boldsymbol{B}}(1-c\eta),\end{aligned} \tag{6.5.25}$$

where $\boldsymbol{B}$ is the ball with unit volume in $\mathbb{R}^d$ centered at the origin. Moreover, by Lemmas 6.5.9 and 6.5.10 and the fact that $|\Omega_\eta^+| = |\boldsymbol{\Omega}_\eta^+|$, it follows that on $\{|\mathcal{E}(\iota,\rho)| = V\}$,

$$\begin{aligned}|\boldsymbol{\Omega}_\eta^+| &\le |\mathcal{E}(\iota,\rho)| + |\Omega_{\eta^2}\setminus\mathcal{E}(\iota,\rho)| + |\Omega_\eta^+\setminus\Omega_{\eta^2}|\\ &\le |V|(1+c\eta).\end{aligned} \tag{6.5.26}$$

Substituting (6.5.25) and (6.5.26) into (6.5.24) and using Lemma 6.5.6 and (6.5.14), we find that

$$\begin{aligned}
&\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, |\mathcal{E}(\iota,\rho)| = V\right) \\
&\quad \leq CN^{d/2} \exp\left\{-NV^{-2/d}\boldsymbol{\lambda}_{\boldsymbol{B}}(1-c\eta)\right\} \mathbb{P}\left(|\mathcal{E}(\iota,\rho)| = V\right) \\
&\quad \leq CN^{d/2} \exp\left\{-NV^{-2/d}\boldsymbol{\lambda}_{\boldsymbol{B}}(1-c\eta) + V(\log p + \eta)\right\} \\
&\quad \leq \exp\left\{-\left(c(d,p) - C\eta + c\left(\frac{V}{|\boldsymbol{B}(0;\varrho_N)|} - 1\right)^2\right) N^{\frac{d}{d+2}}\right\},
\end{aligned} \tag{6.5.27}$$

where we used a second order Taylor expansion for the function $V \mapsto NV^{-2/d}\boldsymbol{\lambda}_{\boldsymbol{B}(0;r)} + V \log \frac{1}{p}$ at $V = |\boldsymbol{B}(0;\varrho_N)|$, where it takes its minimal value $c(d,p)N^{\frac{d}{d+2}}$ (see the discussion following (6.1.3)).

Now if we suppose $|\boldsymbol{\Omega}_\eta^+| \geq |\boldsymbol{B}(0;\varrho_N)| + \eta^{1/3}\varrho_N^d$ and $|\mathcal{E}(\iota,\rho)| = V$, then by (6.5.26) we have $V \geq |\boldsymbol{B}(0;\varrho_N)| + \frac{1}{2}\eta^{1/3}\varrho_N^d$ and hence

$$\left(\frac{V}{|\boldsymbol{B}(0;\varrho_N)|} - 1\right)^2 \geq c\eta^{2/3}. \tag{6.5.28}$$

Since the number of possible values of V is bounded by $(4N+1)^d$ and $\eta = 2\delta_{N,x}$, comparing (6.5.27) with (6.4.2) shows that

$$\begin{aligned}
&\mu_{N,x}\left(|\boldsymbol{\Omega}_\eta^+| \geq |\boldsymbol{B}(0;\varrho_N)| + \eta^{1/3}\varrho_N^d\right) \\
&\quad \leq \sum_{V:(6.5.28)} \frac{\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > \tau_x^N, |\mathcal{E}(\iota,\rho)| = V)}{\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > \tau_x^N)} \\
&\quad \leq (4N+1)^d \exp\left\{-c(2\delta_{N,x})^{2/3}N^{\frac{d}{d+2}} + \mathrm{dist}_\beta\left(x, B(0,2\varrho_N)\right) + \epsilon\delta_{N,x}N^{\frac{d}{d+2}}\right\} \\
&\quad \leq \exp\left\{-\delta_{N,x}N^{\frac{d}{d+2}}\right\}
\end{aligned} \tag{6.5.29}$$

for all sufficiently large N when $\epsilon > 0$ is small. This gives the bound we need on $|\boldsymbol{\Omega}_\eta^+|$.

Step 2: Next, we prove that with high probability under $\mu_{N,x}$, $\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+}$ is not much larger than

$\boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)}$. As a consequence, we will also see that $|\boldsymbol{\Omega}_\eta^+|$ is not much smaller than $|\boldsymbol{B}(0;\varrho_N)|$. If we replace the condition $|\mathcal{E}(\iota,\rho)| = V$ in (6.5.24) by $\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} \geq \boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)}(1+\eta^{1/2})$, then by using the first line of (6.5.25), we get

$$\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, \boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} \geq \boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)}(1+\eta^{1/2})\right) \leq \exp\left\{-\left(c(d,p)+\tfrac{1}{2}\eta^{1/2}\right)N^{\frac{d}{d+2}}\right\}. \tag{6.5.30}$$

Recalling $\eta = 2\delta_{N,x}$ and comparing the above with (6.4.2) again, we conclude that

$$\mu_{N,x}\left(\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} \geq \boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)}(1+\eta^{1/2})\right) \leq \exp\left\{-\delta_{N,x}^{1/2}N^{\frac{d}{d+2}}\right\} \tag{6.5.31}$$

for all sufficiently large N when $\epsilon > 0$ is small. On the complementary event $\{\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} < \boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)}(1+\eta^{1/2})\}$, the classical Faber–Krahn inequality implies

$$|\boldsymbol{\Omega}_\eta^+| \geq |\boldsymbol{B}(0;\varrho_N)|(1-c\eta^{1/2}), \tag{6.5.32}$$

which complements the upper bound in (6.5.29).

Step 3: We prove that $\boldsymbol{\Omega}_\eta^+$ is well approximated by a ball of radius almost ϱ_N by using the quantitative Faber–Krahn inequality in [15]. Let us recall that [15, MAIN THEOREM] asserts

$$\inf\left\{\frac{|\boldsymbol{B'}\triangle\boldsymbol{\Omega}|}{|\boldsymbol{B'}|} : \boldsymbol{B'} \text{ is any ball with } |\boldsymbol{B'}| = |\boldsymbol{\Omega}|\right\}^2 \leq c\left(|\boldsymbol{\Omega}|^{2/d}\boldsymbol{\lambda}_{\boldsymbol{\Omega}} - \boldsymbol{\lambda}_{\boldsymbol{B}}\right), \tag{6.5.33}$$

where $\boldsymbol{B}$ is the ball with unit volume in $\mathbb{R}^d$ centered at the origin (see (6.5.25)). By (6.5.29), (6.5.32) and (6.5.31) in the previous steps, we may assume

$$\big||\boldsymbol{\Omega}_\eta^+| - |\boldsymbol{B}(0;\varrho_N)|\big| \leq \eta^{1/3}\varrho_N^d, \tag{6.5.34}$$

$$\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} \leq \boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)}(1+\eta^{1/2}). \tag{6.5.35}$$

In particular, it follows that

$$\begin{aligned} |\boldsymbol{\Omega}_\eta^+|^{2/d}\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} - \boldsymbol{\lambda}_{\boldsymbol{B}} &= |\boldsymbol{\Omega}_\eta^+|^{2/d}\boldsymbol{\lambda}_{\boldsymbol{\Omega}_\eta^+} - |\boldsymbol{B}(0;\varrho_N)|^{2/d}\boldsymbol{\lambda}_{\boldsymbol{B}(0;\varrho_N)} \\ &\le c\eta^{1/3}. \end{aligned} \tag{6.5.36}$$

Substituting this into (6.5.33), we can find a ball $\boldsymbol{B}_\eta$ such that $|\boldsymbol{B}_\eta| = |\boldsymbol{\Omega}_\eta^+|$ and $|\boldsymbol{\Omega}_\eta^+ \triangle \boldsymbol{B}_\eta| \le c\eta^{1/6}\varrho_N^d$. Setting $\boldsymbol{x}_N$ as the center of $\boldsymbol{B}_\eta$ and using (6.5.34) again, we find

$$|\boldsymbol{B}(\boldsymbol{x}_N;\varrho_N) \triangle \boldsymbol{B}_\eta| \le c\eta^{1/6}\varrho_N^d. \tag{6.5.37}$$

Step 4: Finally, we prove that $B_\eta = \boldsymbol{B}_\eta \cap \mathbb{Z}^d$ is almost free of obstacles. To this end, we first show that

$$|B_\eta \setminus \mathcal{E}(\iota,\rho)| \le |B_\eta \setminus \Omega_\eta^+| + |\Omega_\eta^+ \setminus \mathcal{E}(\iota,\rho)| \tag{6.5.38}$$

is small. Recall that (6.5.29) shows $|\mathcal{E}(\iota,\rho)| \le c\varrho_N^d$ with $\mu_{N,x}$ probability greater than $1 - \exp\{-\delta_{N,x}N^{\frac{d}{d+2}}\}$. Under this condition, the second term on the right-hand side is smaller than $c\eta\varrho_N^d$ due to Lemmas 6.5.9 and 6.5.10. For the first term, note first that

$$|B_\eta \setminus \Omega_\eta^+| = |B_\eta| - |\Omega_\eta^+| + |\Omega_\eta^+ \setminus B_\eta|. \tag{6.5.39}$$

Since we know $|\boldsymbol{B}_\eta| = |\boldsymbol{\Omega}_\eta^+|$, $|\Omega_\eta^+| = |\boldsymbol{\Omega}_\eta^+|$ and $|B_\eta| \le |\boldsymbol{B}_\eta| + c\varrho_N^{d-1}$, we only need to prove that $|\Omega_\eta^+ \setminus B_\eta|$ is small. Since $\boldsymbol{\Omega}_\eta^+$ is a union of cubes $\{y + [-\frac{1}{2},\frac{1}{2}]^d \colon y \in \Omega_\eta^+\}$ and

$|(y+[-\frac{1}{2},\frac{1}{2}]^d)\setminus \boldsymbol{B}_\eta| \geq \frac{1}{2}$ for any $y \in \Omega_\eta^+ \setminus B_\eta$, it follows that

$$
\begin{aligned}
|\Omega_\eta^+ \setminus B_\eta| &= \sum_{y\in\Omega_\eta^+\setminus B_\eta} |y+[-\tfrac{1}{2},\tfrac{1}{2}]^d| \\
&\leq 2\sum_{y\in\Omega_\eta^+} |(y+[-\tfrac{1}{2},\tfrac{1}{2}]^d)\setminus \boldsymbol{B}_\eta| \\
&\leq 2|\boldsymbol{\Omega}_\eta^+ \setminus \boldsymbol{B}_\eta|.
\end{aligned}
\tag{6.5.40}
$$

This last line is shown to be bounded by $c\eta^{1/6}\varrho_N^d$ in Step 3. Therefore we conclude that (6.5.38) is bounded by $c\eta^{1/6}\varrho_N^d$. Recalling (6.5.37) and Definition 6.5.5, one can easily check that this implies (6.5.3). □

In fact, (6.5.29) in Step 1 shows that $|\mathcal{E}(\iota,\rho)|$ is close to $|B(0,\varrho_N)|$ and hence by arguing as in (6.5.39), we find the following:

Corollary 6.5.11. *Under the same assumption as in Lemma 6.5.2, there exists* $c_{6.5.11} > 0$ *such that for* ι,ρ *as in* (6.5.13),

$$
\mu_{N,x}\left(|\mathcal{E}(\iota,\rho)\triangle B(\boldsymbol{x}_N,\varrho_N)| \geq \delta_{N,x}^{c_{6.5.11}} N^{\frac{d}{d+2}}\right) \leq \exp\left\{-c_{6.5.11}\delta_{N,x}N^{\frac{d}{d+2}}\right\}
\tag{6.5.41}
$$

for all sufficiently large N.

Proof of Lemma 6.5.4. We only sketch the argument since the proof is almost identical to [25, Lemma 6.1]. Thanks to Corollary 6.5.11, it suffices to prove that (6.5.7) holds on the event

$$
\left\{|\mathcal{E}(\iota,\rho)\triangle B(\boldsymbol{x}_N,\varrho_N)| < \delta_{N,x}^{c_{6.5.11}} N^{\frac{d}{d+2}}\right\}.
\tag{6.5.42}
$$

On this event, we have

$$
\left|\mathcal{E}(\iota,\rho)\setminus B^-\right| \leq \delta_{N,x}^{C}\varrho_N^d
\tag{6.5.43}
$$

for some $C \in (0,1)$, where B^- is defined in (6.5.6).

Let us first assume that $S_{[0,t]}$ visits $(\mathcal{E}(\iota,\rho)\setminus B^-)^c$ more than $t/4$ times, which is natural since $|\mathcal{E}(\iota,\rho)\setminus B^-|$ has a small volume by (6.5.43). Then we can extract at least $\frac{1}{4}\delta_{N,x}^{-5}\varrho_N^{-2}t$ different times k_i's in $[0,t]$ which satisfy

- $S_{k_i}\notin\mathcal{E}(\iota,\rho)\setminus B^-$ and
- $k_{i+1}-k_i\geq\delta_{N,x}^5\varrho_N^2$

for all $i\leq\frac{1}{4}\delta_{N,x}^{-5}\varrho_N^{-2}t-1$. Recalling the definition of $\mathcal{E}(\iota,\rho)$, one can prove that for each k_i, the probability for the random walk to avoid $\mathcal{O}\cup B^-$ until next k_{i+1} is smaller than $1-c\rho$ (see [25, (5.6)]). Therefore we obtain

$$\begin{aligned}&\sup_{y\in B(0,2N)}\mathbf{P}_y\left(S_{[0,t]}\text{ visits }(\mathcal{E}(\iota,\rho)\setminus B^-)^c\text{ more than }t/4\text{ times and }\tau_{\mathcal{O}\cup B^-}>t\right)\\&\leq(1-c\rho)^{\frac{1}{4}\delta_{N,x}^{-5}\varrho_N^{-2}t-1}\\&\leq\exp\left\{-c\delta_{N,x}^{-3}\varrho_N^{-2}t\right\}\end{aligned}\tag{6.5.44}$$

by recalling $\rho=(2\delta_{N,x})^2$.

It remains to show that under the condition (6.5.43),

$$\sup_{y\in B(0,2N)}\mathbf{P}_y\left(S_{[0,t]}\text{ visits }(\mathcal{E}(\iota,\rho)\setminus B^-)^c\text{ less than }t/4\text{ times}\right)\leq\exp\left\{-\delta_{N,x}^{-c}\varrho_N^{-2}t\right\}\tag{6.5.45}$$

for some $c>0$. To this end, we divide $[0,t]$ into sub-intervals of size $M^2\delta_{N,x}^{2C/d}\varrho_N^2$ for some large $M>0$ to be determined later. We call a sub-interval *successful* if the random walk spends more than half of the time in $(\mathcal{E}(\iota,\rho)\setminus B^-)^c$. If half of the intervals are successful, then the random walk visits $(\mathcal{E}(\iota,\rho)\setminus B^-)^c$ at least $t/4$ times.

For any $u\geq M\delta_{N,x}^{2C/d}\varrho_N^2$, a well-known bound on the transition probability $\mathbf{P}_z(S_u=y)\leq$

$cM^{-d/2}\delta_{N,x}^{-C}\varrho_N^{-d}$ and (6.5.43) imply that

$$\sup_{z\in\mathbb{Z}^d} \mathbf{P}_z(S_u \in \mathcal{E}(\iota,\rho) \setminus B^-) \le cM^{-d/2}. \tag{6.5.46}$$

From this and the so-called *first moment method* (applied to the number of visits to $\mathcal{E}(\iota,\rho)\cup B^-$), one can easily deduce that the probability for an interval $\delta_{N,x}^{2C/d}\varrho_N^2[kM^2,(k+1)M^2]$ to be successful is more than 1/2 for large M (see [25, Lemma 6.4]). Since there are $M^{-2}\delta_{N,x}^{-2C/d}\varrho_N^{-2}t$ intervals that intersect $[0,t]$, a simple large deviation estimate yields

$$\sup_{z\in\mathbb{Z}^d} \mathbf{P}_z(\text{half of those intervals are not successful}) \le \exp\left\{-cM^{-2}\delta_{N,x}^{-2C/d}\varrho_N^{-2}t\right\}. \tag{6.5.47}$$

Combining (6.5.45) and (6.5.44), we get Lemma 6.5.4. □

6.5.2 Sketch proof of Lemma 5.2.1

This is an analogue of [23, Lemma 2.1] and can be proved by almost the same argument. We recall the outline and indicate where we need an additional argument.

The proof of [23, Lemma 2.1] is based on an environment and path switching argument. Suppose that $v \in \mathcal{O}$ and $|\mathcal{O}\cap B(v,l)| < \delta|B(v,l)|$. First, if the random walk frequently visits $B(v,l/2)$, then we simply remove all the obstacles in $B(v,l)$. This causes a cost in $\mathbb{P}$-probability but not too much since δ is small. On the other hand, we gain a lot in $\mathbf{P}$-probability since $B(v,l/2)$ is visited frequently and it turns out that the gain beats the cost. It follows that

$$E_l^\delta(v) \cap \{B(v,l/2) \text{ is visited frequently}\} \tag{6.5.48}$$

is much less likely than $\{\mathcal{O}\cap B(v,l)=\emptyset\}$ under $\mathbb{P}\otimes\mathbf{P}$ and hence μ_N. Second, if the random walk rarely visits $B(v,l)$, then we deform the random walk paths to avoid $B(v,l/2)$. This causes a cost in $\mathbf{P}$-probability but not too much since the random walk visits $B(v,l/2)$ only rarely. On the other hand, after this operation, we can change the configuration of

$\mathcal{O} \cap B(v, l/2)$ to a typical one. As we started from an atypical low density configuration, we gain a lot in $\mathbb{P}$-probability and it turns out that the gain beats the cost. It follows that

$$E_l^{\delta}(v) \cap \{B(v, l/2) \text{ is visited rarely}\} \tag{6.5.49}$$

is much less likely than $\{|\mathcal{O} \cap B(v, l)| \geq \delta |B(v, l)\}$ under μ_N.

When $0 \in B(v, l/2)$, the argument in the second case, where the random walk rarely visits $B(v, l)$, requires a modification since we cannot change the starting point of the random walk. In this situation, we create a one-dimensional path from 0 to $\partial B(0, l/2)$ free of obstacles and force the random walk to follow that path. The only difference in the setting of the present article is that we have the same problem when $x \in B(v, l/2)$, since we cannot change the endpoint. But this can be treated in the same way as the case $0 \in B(v, l/2)$.

Finally, in the following remark, we explain a technical point which forces us to work under $\mu_{N,x}(\cdot) = \mathbb{P} \otimes \mathbf{P}(\cdot \mid \tau_{\mathcal{O}} > \tau_x^N)$ instead of $\mathbb{P} \otimes \mathbf{P}(\cdot \mid \tau_{\mathcal{O}} > N, S_N = x)$.

Remark 6.5.12. In the case that $B(v, l/2)$ is rarely visited, we deform the random walk path to avoid $B(v, l/2)$, which may lengthen the path. Therefore the condition $S_N = x$ is not preserved by the above argument but $\tau_{\mathcal{O}} > \tau_x^N$ is. This is why we work with $\mu_{N,x}$.

6.6 Time spent outside the vacant ball

In this section, we prove several results concerning the behavior of the random walk outside the vacant ball B^- defined in (6.5.6). The first one, Proposition 6.6.1 to be proved in Section 6.6.1, shows that the random walk does not spend too much time before the first visit and after the last visit to the ball B^-. The second one, Proposition 6.6.2 to be proved in Section 6.6.2, shows that between the first and last visit to B^-, the random walk is confined in a slightly larger ball. By the same argument, we show in Corollary 6.6.4 that the random walk returns to B^- frequently between the first and the last visit to B^-.

Let us write $\tau_{B^-}^{\leftarrow}$ for the last visit to B^- before τ_x^N, which is the first hitting time of B^-

by the time-reversed random walk.

6.6.1 First and last visits to the vacant ball

Proposition 6.6.1. *Let* $\delta_{N,x}$ *and* $\boldsymbol{\mathcal{X}}_N$ *be as in* (6.3.1) *and Proposition 6.5.1, respectively. There exist* $c_{6.6.1} > 0$ *such that when* $\epsilon > 0$ *is small depending on* d *and* p *and* $|x| \le \epsilon \varrho_N^d$,

$$\mu_{N,x}\left(\tau_{B^-} \ge \delta_{N,x}^{c_{6.6.1}} |\boldsymbol{\mathcal{X}}_N|_1 \varrho_N^2\right) \le \exp\left\{-\tfrac{1}{2}(\log N)^2\right\} \tag{6.6.1}$$

and

$$\mu_{N,x}\left(\tau_x^N - \tau_{B^-}^{\leftarrow} \ge \delta_{N,x}^{c_{6.6.1}} |x - \boldsymbol{\mathcal{X}}_N|_1 \varrho_N^2\right) \le \exp\left\{-\tfrac{1}{2}(\log N)^2\right\} \tag{6.6.2}$$

for all sufficiently large N.

Proof. We give a proof of (6.6.1). One can prove (6.6.2) similarly by considering the time-reversed random walk. Thanks to Proposition 6.5.1 and Lemma 6.5.4, we may assume that there exists $z \in B(0, 2N)$ such that

$$\mathcal{O} \cap B(z, (1 - \delta_{N,x}^{c_{6.5.1}})\varrho_N) = \emptyset, \tag{6.6.3}$$

$$\mathbf{P}_y\left(S_{[0,t]} \cap \left(\mathcal{O} \cup B^-(z)\right) = \emptyset\right) \le \exp\left\{-\delta_{N,x}^{-c_{6.5.4}} \varrho_N^{-2} t\right\} \tag{6.6.4}$$

for all $y \in B(0, 2N)$ and $t \ge \delta_{N,x}^{c_{6.5.4}} (\log N)^2 \varrho_N^2$. In particular, it follows that

$$\mathbf{P}\left(\tau_{\mathcal{O}} \wedge \tau_{B^-(z)} > N/2\right) \le \exp\left\{-c\delta_{N,x}^{-c_{6.5.4}} N^{\frac{d}{d+2}}\right\}. \tag{6.6.5}$$

Comparing with Proposition 6.4.1 and using that $\lim_{\epsilon\to 0} \lim_{N\to\infty} \delta_{N,x} = 0$, we find that the random walk hits $B^-(z)$ with high probability:

$$\mu_{N,x}\left((6.6.3),\ (6.6.4),\ \tau_{B^-(z)} > N/2\right) \le \exp\left\{-c\delta_{N,x}^{-c_{6.5.4}} N^{\frac{d}{d+2}}\right\} \tag{6.6.6}$$

for all sufficiently large N when ϵ is small. Now let us fix $c_{6.6.1} < c_{6.5.4}$, $z \in B(0,2N)$, $y \in B^-(z)$ and $n \in [\delta_{N,x}^{c_{6.6.1}}|z|_1\varrho_N^2, N/2]$. To prove (6.6.1), it suffices to show that

$$\mu_{N,x}\left((6.6.3), (6.6.4), \tau_{B^-(z)} = n, S_n = y\right) \le \exp\left\{-(\log N)^3\right\} \tag{6.6.7}$$

since the number of possible choices of (y,z,n) is polynomial in N. We will only consider $|z|_1 \ge \varrho_N/2$ since otherwise $\tau_{B^-(z)} = 0$ almost surely under $\mu_{N,x}$. By using the Markov property at time n and (6.6.4), we find that

$$\begin{aligned}\mathbf{P}\left(\tau_{B^-(z)} = n, S_n = y, \tau_{\mathcal{O}} > \tau_x^N\right) &\le \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_{B^-(z)} = n, S_n = y\right)\mathbf{P}_y\left(\tau_{\mathcal{O}} > \tau_x^{N-n}\right)\\ &\le \exp\left\{-\delta_{N,x}^{-c_{6.5.4}}n\varrho_N^{-2}\right\}\mathbf{P}_y\left(\tau_{\mathcal{O}} > \tau_x^{N-n}\right).\end{aligned} \tag{6.6.8}$$

In order to compare this with the partition function, let us fix a nearest neighbor path $\pi(0,z)$ of length $|z|_1$ connecting 0 and z and consider the events

$$E_1 = \{S_{[0,|z|_1]} = \pi(0,z)\}, \tag{6.6.9}$$

$$E_2 = \left\{S_{[|z|_1, n-\varrho_N^2]} \subset B(z, \varrho_N/2)\right\}, \tag{6.6.10}$$

$$E_3 = \left\{S_{[n-\varrho_N^2, n]} \subset B(z; (1-\delta_{N,x}^{c_{6.5.1}})\varrho_N), S_n = y\right\}. \tag{6.6.11}$$

Note that on the event $\{(6.6.3), \mathcal{O} \cap \pi(0,z) = \emptyset\}$, we have $E_1 \cap E_2 \cap E_3 \subset \{S_n = y, \tau_{\mathcal{O}} > n\}$. Therefore by the Markov property and $|z|_1 \le \delta_{N,x}^{-c_{6.6.1}}n\varrho_N^{-2}$, we get

$$\begin{aligned}&\mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N\right)\\ &\quad\ge \mathbf{P}(E_1)\mathbf{P}(E_2 \mid S_{|z|_1} = z)\inf_{w \in B(z,\varrho_N/2)}\mathbf{P}(E_3 \mid S_{n-\varrho_N^2} = w)\mathbf{P}_y\left(\tau_{\mathcal{O}} > \tau_x^{N-n}\right)\\ &\quad\ge \left(\frac{1}{2d}\right)^{|z|_1}\exp\left\{-c(n-\varrho_N^2-|z|_1)\varrho_N^{-2}\right\}\frac{c}{\varrho_N^{d+1}}\mathbf{P}_y\left(\tau_{\mathcal{O}} > \tau_x^{N-n}\right)\\ &\quad\ge \exp\left\{-c\delta_{N,x}^{-c_{6.6.1}}n\varrho_N^{-2}\right\}\mathbf{P}_y\left(\tau_{\mathcal{O}} > \tau_x^{N-n}\right),\end{aligned} \tag{6.6.12}$$

where we have used the random walk estimate (6.3.4) for the second and third factors in the second line.

Now we use a slight variant of the switching argument in [23]. We first use the Markov property at time n and (6.6.8) to obtain

$$\begin{aligned}&\mathbb{P}\otimes\mathbf{P}\left((6.6.3),\,(6.6.4),\tau_{B^-(z)}=n,S_n=y,\tau_{\mathcal{O}}>\tau_x^N\right)\\&\quad\le\exp\left\{-c\delta_{N,x}^{-c_{6.5.4}}n\varrho_N^{-2}\right\}\mathbb{E}\left[\mathbf{P}_y\left(\tau_{\mathcal{O}}>\tau_x^{N-n}\right):(6.6.3)\right].\end{aligned}\tag{6.6.13}$$

Then we "switch" a given $\mathcal{O}$ satisfying (6.6.3) by removing the obstacles on $\pi(0,z)$. Since $\mathbf{P}_y\left(\tau_{\mathcal{O}}>\tau_x^{N-n}\right)$, (6.6.3) and $\{\pi(0,z)\cap\mathcal{O}=\emptyset\}$ are all decreasing in $\mathcal{O}$, we can use the FKG inequality to obtain

$$\begin{aligned}&\mathbb{E}\left[\mathbf{P}_y\left(\tau_{\mathcal{O}}>\tau_x^{N-n}\right):(6.6.3)\right]\\&\quad\le p^{-|z|_1}\mathbb{E}\left[\mathbf{P}_y\left(\tau_{\mathcal{O}}>\tau_x^{N-n}\right):(6.6.3),\,\mathcal{O}\cap\pi(0,z)=\emptyset\right].\end{aligned}\tag{6.6.14}$$

Substituting this and (6.6.12) into (6.6.13) and recalling $c_{6.6.1}<c_{6.5.4}$ and $|z|_1\le\delta_{N,x}^{-c_{6.6.1}}n\varrho_N^{-2}$ again, we find

$$\begin{aligned}&\mathbb{P}\otimes\mathbf{P}\left((6.6.3),\,(6.6.4),\tau_{B^-(z)}=n,S_n=y,\tau_{\mathcal{O}}>\tau_x^N\right)\\&\quad\le\exp\left\{-c\delta_{N,x}^{-c_{6.5.4}}n\varrho_N^{-2}\right\}\mathbb{E}\left[\mathbf{P}_y\left(\tau_{\mathcal{O}}>\tau_x^{N-n}\right):(6.6.3),\,\mathcal{O}\cap\pi(0,z)=\emptyset\right].\\&\quad\le\exp\left\{-c'\delta_{N,x}^{-c_{6.5.4}}n\varrho_N^{-2}\right\}\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>\tau_x^N\right)\end{aligned}\tag{6.6.15}$$

for all sufficiently large N. Recalling $n\ge\delta_{N,x}^{c_{6.6.1}}|z|_1\varrho_N^2$ and $|z|_1\ge\varrho_N/2$, this implies (6.6.7) and we are done. □

6.6.2 Confinement between the first and last visits to the vacant ball

Let us introduce a ball concentric to B^- with a larger radius by

$$\begin{aligned} B^+(z) &= B(z, (1+\delta_{N,x}^{c_{6.5.4}/2}(\log N)^3)\varrho_N), \\ B^+ &= B^+(\boldsymbol{x}_N). \end{aligned} \tag{6.6.16}$$

Note that by our definition of $\delta_{N,x}$ in (6.3.1), this is much larger than B^-, see (6.5.6), when $|x|$ is close to ϱ_N^d. We will explain the reason in Remark 6.6.3. In the following proposition, we show that $S_{[\tau_{B^-},\tau_{B^-}^{\leftarrow}]}$ is confined in B^+ with high probability under $\mu_{N,x}$.

Proposition 6.6.2. *When $\epsilon > 0$ is small depending on d and p and $|x| \le \epsilon\varrho_N^d$,*

$$\mu_{N,x}\left(S_{[\tau_{B^-},\tau_{B^-}^{\leftarrow}]} \not\subset B^+\right) \le \exp\left\{-\tfrac{1}{3}(\log N)^2\right\} \tag{6.6.17}$$

for all sufficiently large N.

Proof. Throughout this proof, we assume that (6.5.1) and (6.5.7) hold, that is, there exists a vacant ball of radius almost ϱ_N and the outside is dangerous for the random walk.

Suppose that $S_{[\tau_{B^-},\tau_{B^-}^{\leftarrow}]} \not\subset B^+$. Then since we know $0 \le \tau_{B^-} < \tau_B^{\leftarrow} < \tau_{\mathcal{O}}^N$ from Proposition 6.6.1, there exist $[t_1, t_2] \subset [\tau_{B^-}, \tau_{B^-}^{\leftarrow}]$ such that $S_{t_1}, S_{t_2} \in \partial B^-$,

$$S_{[t_1,t_2]} \cap \left(\mathcal{O} \cup B^-\right) = \emptyset \text{ and } S_{[t_1,t_2]} \cap (B^+)^c \neq \emptyset. \tag{6.6.18}$$

Therefore, by using the union bound and the Markov property, we have

$$\begin{aligned} &\mathbf{P}\left(S_{[\tau_{B^-},\tau_{B^-}^{\leftarrow}]} \not\subset B^+, \tau_{\mathcal{O}} > \tau_x^N\right) \\ &\quad \le \sum_{t_1,t_2,x_1,x_2} \mathbf{P}\left(S_{t_1} = x_1, \tau_{\mathcal{O}} > t_1\right) \mathbf{P}_{x_1}((6.6.18), S_{t_2-t_1} = x_2) \mathbf{P}_{x_2}\left(\tau_{\mathcal{O}} > \tau_x^{N-t_2}\right), \end{aligned} \tag{6.6.19}$$

where the above sum runs over $0 \le t_1 < t_2 \le 2N$ and $x_1, x_2 \in \partial B^-$. We are going to show that the middle term on the right-hand side of (6.6.19) is much smaller than the probability of

$$S_{[t_1,t_2+\varrho_N^2]} \subset B(\boldsymbol{x}_N, (1-\delta_{N,x}^{c_{6.5.1}})\varrho_N) \text{ and } S_{t_2+\varrho_N^2} = x_2. \tag{6.6.20}$$

Since we assumed (6.5.1), this in particular implies $S_{[t_1,t_2+\varrho_N^2]} \cap \mathcal{O} = \emptyset$. Let us first get a lower bound on the probability of (6.6.20). Using (6.3.4), we obtain

$$\min_{x_1,x_2\in\partial B^-} \mathbf{P}_{x_1}((6.6.20)) \geq \frac{c}{\varrho_N^{d+2}} \exp\left\{-c^{-1}(t_2-t_1)\varrho_N^{-2}\right\}. \tag{6.6.21}$$

Next we get an upper bound on the probability of (6.6.18) which splits into two cases.

Case 1: $t_2 - t_1 \geq \delta_{N,x}^{c_{6.5.4}}(\log N)^2\varrho_N^2$. In this case, we only consider the first condition in (6.6.18). Then since we are assuming (6.5.7), we have

$$\begin{aligned} \mathbf{P}_{x_1}((6.6.18)) &\leq \exp\left\{-\delta_{N,x}^{-c_{6.5.4}}\varrho_N^{-2}(t_2-t_1)\right\} \\ &\leq \exp\left\{-(\log N)^2\right\}\mathbf{P}_{x_1}((6.6.20)) \end{aligned} \tag{6.6.22}$$

for all sufficiently large N when ϵ is small.

Case 2: $t_2 - t_1 \leq \delta_{N,x}^{c_{6.5.4}}(\log N)^2\varrho_N^2$. In this case, we only consider the second condition in (6.6.18), which implies that the maximal displacement of the random walk on $[t_1, t_2]$ is larger than $\delta_{N,x}^{c_{6.5.4}/2}(\log N)^3\varrho_N$. Then, the Gaussian heat kernel estimate and the reflection principle yield

$$\begin{aligned} \mathbf{P}_{x_1}((6.6.18)) &\leq \exp\left\{-c\frac{(\delta_{N,x}^{c_{6.5.4}/2}(\log N)^3\varrho_N)^2}{t_2-t_1}\right\} \\ &\leq \exp\left\{-(\log N)^2\right\}\mathbf{P}_{x_1}((6.6.20)) \end{aligned} \tag{6.6.23}$$

for all sufficiently large N when ϵ is small.

Substituting (6.6.22) and (6.6.23) into (6.6.19), we find that

$$\begin{aligned}
&\mathbf{P}\left(S_{[\tau_{B^-},\tau^{\leftarrow}_{B^-}]} \not\subset B^+, \tau_{\mathcal{O}} > \tau_x^N\right) \\
&\quad \le \exp\left\{-(\log N)^2\right\} \sum_{t_1,t_2,x_1,x_2} \mathbf{P}\left(S_{t_1} = x_1, S_{t_2-t_1+\varrho_N^2} = x_2, \tau_{\mathcal{O}} > \tau_x^{N+\varrho_N^2}\right) \qquad (6.6.24)\\
&\quad \le cN^{2+2d} \exp\left\{-(\log N)^2\right\} \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N\right),
\end{aligned}$$

where in the last line, we have used that $t_1, t_2 \in [0, 2N]$ and $x_1, x_2 \in B(0, 2N)$ (recall Remark 6.4.3). Integrating both sides with respect to $\mathbb{P}$, we complete the proof of (6.6.17).

□

Remark 6.6.3. The super-polynomial rate of decay in (6.6.17) will be used later in the proof of Theorem 6.1.5. To achieve this, as well as to counterbalance the factor N^{2+2d} in the last step of the proof, we had to include $(\log N)^2$ factor in the condition for $t_2 - t_1$ in Case 1 since $\delta_{N,x}$ can be as large as ϵ. Then in Case 2, we needed an extra $(\log N)^3$ factor in the displacement. This is why we included $(\log N)^3$ in (6.6.16).

By the same argument, we can show that the random walk returns to B^- frequently. This result will be used later to replace our condition $\{\tau_{\mathcal{O}} > \tau_x^N\}$ by $\{\tau_{\mathcal{O}} > N, S_N = x\}$ when x is close to $2\varrho_N \mathbf{e}_h$.

Corollary 6.6.4. *For any* $|x| \le 3\varrho_N$*,*

$$\mu_{N,x}\left(\exists k \in [\tau_{B^-}, \tau^{\leftarrow}_{B^-} - \varrho_N^2], S_{[k,k+\varrho_N^2]} \cap B^- = \emptyset\right) \le \exp\left\{-\tfrac{1}{3}(\log N)^2\right\} \qquad (6.6.25)$$

for all sufficiently large N*.*

Proof. This can be proved in the same way as Proposition 6.6.2. We again assume that (6.5.1) and (6.5.7) hold. If $S_{[k,k+\varrho_N^2]} \cap B^- = \emptyset$, then we take t_1 (and t_2) to be the last (resp. first) visit to B^- before k (resp. after $k+\varrho_N^2$). This probability can be bounded by $\exp\{-\delta_{N,x}^{-c_{6.5.4}} \varrho_N^{-2}(t_2-$

$t_1)\}$ by (6.5.7). Comparing this with (6.6.21) and recalling that $\delta_{N,x} = \varrho_N^{-1/5}$ when $|x| \le 3\varrho_N$, we obtain (6.6.25) as before. □

6.7 Cost for the first and last pieces

In this section, we estimate the cost for the random walk to move from 0 to B^- and B^- to x. Although it is natural to expect that they are measured by the Lyapunov distances $\mathrm{dist}_\beta(0, B^-)$ and $\mathrm{dist}_\beta(x, B^-)$, we will formulate the bound under the additional restriction that $\boldsymbol{\mathcal{X}}_N$ is fixed to be a generic point and it requires some preparation. The motivation for this formulation will be clear in Corollary 6.7.2.

For each $|x| \le \epsilon\varrho_N^d$ and $z \in B(0, 2N)$, we introduce

$$t_{\mathrm{out}}(x, z) = \delta_{N,x}^{c_{6.6.1}}(|z|_1 + |x - z|_1)\varrho_N^2. \tag{6.7.1}$$

and define a *good* event by

$$G(z) = \Big\{\mathcal{O} \cap B(z, (1 - \delta_{N,x}^{c_{6.5.1}})\varrho_N) = \emptyset, \tag{6.7.2}$$

$$B^-(z) \subset S_{[\tau_{B^-(z)}, \tau^{\leftarrow}_{B^-(z)}]} \subset B^+(z) \setminus \mathcal{O}, \tag{6.7.3}$$

$$\tau^{\leftarrow}_{B^-(z)} - \tau_{B^-(z)} \ge N - t_{\mathrm{out}}(x, z)\Big\}. \tag{6.7.4}$$

This event $G(z)$ morally corresponds to $\{\boldsymbol{\mathcal{X}}_N = z\}$ but is more explicit in the strategy of the random walk and the obstacle configuration. Thanks to Propositions 6.5.1, 6.6.1 and 6.6.2, we know that

$$\mu_{N,x}\left(\bigcup_{z \in B(0;2N)} G(z)\right) \ge 1 - \exp\left\{-\frac{1}{5}(\log N)^2\right\} \tag{6.7.5}$$

for all sufficiently large N.

In this section, we are going to find an upper bound on

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, G(z) \,\middle|\, \tau_{\mathcal{O}} > N\right) \tag{6.7.6}$$

by considering the following event that contains $\{\tau_{\mathcal{O}} > \tau_x^N\} \cap G(z)$:

$$\left\{S_{[0,\tau_{B^+(z)}]} \cap \mathcal{O} = \emptyset\right\} \cap \left\{S_{[\tau_{B^-(z)}, \tau^{\leftarrow}_{B^-(z)}]} \cap \mathcal{O} = \emptyset, (6.7.3)\right\} \cap \left\{S_{[\tau^{\leftarrow}_{B^+(z)}, \tau_x^N]} \cap \mathcal{O} = \emptyset\right\}, \tag{6.7.7}$$

where note that we stop the first piece of random walk and the last reversed walk upon hitting $B^+(z)$, which is before hitting $B^-(z)$. If we further specify the times $\tau_{B^-(z)}$ and $\tau^{\leftarrow}_{B^-(z)}$ and locations of the random walk at these times, then the second event is independent of the other two events and has $\mathbb{P} \otimes \mathbf{P}$-probability not much larger than $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N)$ by (6.7.4). If the first and the third events in (6.7.7) were independent, then their $\mathbb{P} \otimes \mathbf{P}$-probabilities would decay exponentially in $\mathrm{dist}_\beta(0, B^-)$ and $\mathrm{dist}_\beta(x, B^-)$, respectively.

However, the first and the third events in (6.7.7) are not independent under $\mathbb{P}$ since the corresponding pieces of random walk path may overlap. For this reason, we will consider shorter pieces of the random walk path so that their survival depend on disjoint parts of environment. Let us denote the ball with respect to the Lyapunov norm in Definition 6.1.3 by

$$\mathcal{B}\left(u; r\right) = \left\{v \in \mathbb{Z}^d \colon \beta(u - v) \le r\right\} \tag{6.7.8}$$

and introduce

$$r(z) = \mathrm{dist}_\beta(0, B^+(z)), \tag{6.7.9}$$

$$r(x, z) = \mathrm{dist}_\beta(x, B^+(z) \cup \mathcal{B}\left(0; r(z)\right)). \tag{6.7.10}$$

Then by stopping the random walk and the time-reversed walk upon exiting $\mathcal{B}\left(0; r(z)\right)$ and $\mathcal{B}\left(x; r(x, z)\right)$, respectively, we find shortened paths which stay in disjoint sets (see Figure 6.2).

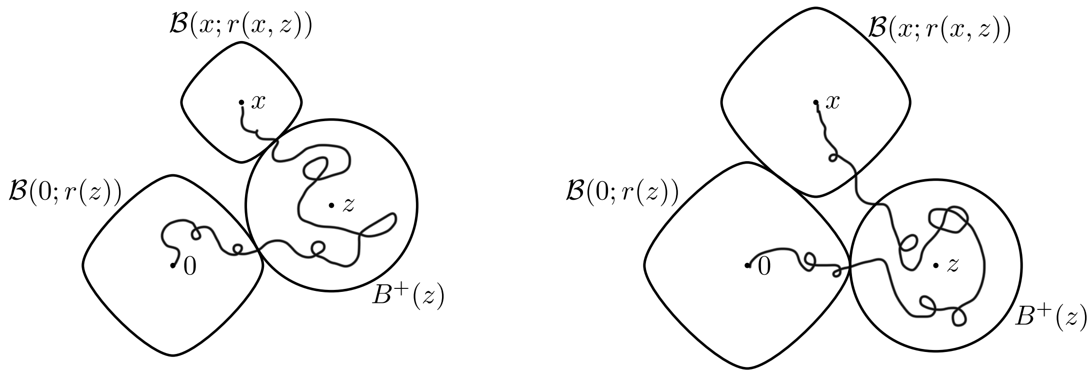

Figure 6.2: The convex shapes around 0 and x are the balls with respect to the Lyapunov norm with radius $r(0, z)$ and $r(x, z)$, respectively. The Euclidean ball centered at z is $B^+(z)$ that has radius $(1 + \delta_{N,x}^{c_{6.5.4}/2}(\log N)^3)\varrho_N$. In the left picture, $\mathcal{B}\,(x; r(x, z))$ touches $B^+(z)$ while in the right picture it touches $\mathcal{B}\,(0; r(z))$.

We are now ready to state the main result of this section.

Proposition 6.7.1. *When $\epsilon > 0$ is small depending on d and p, $|x| \le \epsilon \varrho_N^d$ and $z \in B(0, 2N)$,*

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, G(z) \,\middle|\, \tau_{\mathcal{O}} > N\right) \\ &\quad \le \exp\left\{-(1-\epsilon)(r(z) + r(x,z)) + \delta_{N,x}^{c_{6.6.1}/2}(|z|_1 + |x-z|_1) + (\log N)^2\right\} \end{aligned} \tag{6.7.11}$$

for all sufficiently large N.

Proof. Let us fix $k, l \in [0, 2N]$ satisfying

$$\begin{aligned} l - k &\ge N - t_{\text{out}}(x, z) \\ &= N - \delta_{N,x}^{c_{6.6.1}}(|z|_1 + |x-z|_1)\varrho_N^2, \end{aligned} \tag{6.7.12}$$

and define

$$G(z; k, l) = \left\{\mathcal{O} \cap B(z, (1 - \delta_{N,x}^{c_{6.5.1}})\varrho_N) = \emptyset, B^-(z) \subset S_{[k,l]} \subset B^+(z) \setminus \mathcal{O}\right\}. \tag{6.7.13}$$

We further introduce $x_1, x_2 \in B^-(z)$ and start by rewriting

$$\begin{aligned}
&\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, \tau_{B^-} = k, S_k = x_1, G(z), \tau_{B^-}^{\leftarrow} = l, , S_l = x_2\right) \\
&\quad = \mathbb{P} \otimes \mathbf{P}\Big(\tau_{B^-(z)} = k, S_k = x_1, S_{[0,k]} \cap \mathcal{O} = \emptyset, G(z; k, l), \\
&\qquad\qquad \tau_{B^-(z)}^{\leftarrow} = l, S_l = x_2, S_{[l, \tau_x^N]} \cap \mathcal{O} = \emptyset\Big).
\end{aligned} \tag{6.7.14}$$

We will take a sum over k, l, x_1, x_2 in the end. We are going to estimate the costs for the three pieces $S_{[0,\tau_{B^+(z)}]}$, $S_{[k,l]}$ and $S_{[\tau_{B^+(z)}^{\leftarrow}, \tau_x^N]}$ to avoid $\mathcal{O}$ separately. Noting that $\tau_{B^+(z)} > \tau_{\mathcal{B}(0;r(z))^c}$, we have

$$\mathbf{P}\left(\tau_{B^-(z)} = k, S_k = x_1, S_{[0,k]} \cap \mathcal{O} = \emptyset\right) \leq \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_{\mathcal{B}(0;r(z))^c}\right) \tag{6.7.15}$$

and similarly, by considering the time-reversed random walk,

$$\mathbf{P}\left(\tau_{B^-(z)}^{\leftarrow} = l, S_l = x_2, S_{[l,\tau_x^N]} \cap \mathcal{O} = \emptyset\right) \leq \mathbf{P}_x\left(\tau_{\mathcal{O}} > \tau_{\mathcal{B}(x;r(x,z))^c}\right). \tag{6.7.16}$$

Since $\mathcal{B}(x;r)$, $\mathcal{B}(0;r(z)))$ and $B^+(z)$ are disjoint, the right-hand sides of the above two inequalities and $\mathbf{P}(G(z;k,l) \mid S_k = x_1, S_l = x_2)$ are independent under $\mathbb{P}$. Therefore it follows from (6.7.14) that

$$\begin{aligned}
&\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, \tau_{B^-} = k, S_k = x_1, G(z), \tau_{B^-}^{\leftarrow} = l, , S_l = x_2\right) \\
&\quad \leq \mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_{\mathcal{B}(0;r(z))^c}\right) \cdot \mathbb{P} \otimes \mathbf{P}\big(G(z;k,l) \mid S_k = x_1, S_l = x_2\big) \\
&\qquad \cdot \mathbb{P} \otimes \mathbf{P}_x\left(\tau_{\mathcal{O}} > \tau_{\mathcal{B}(x;r(x,z))^c}\right).
\end{aligned} \tag{6.7.17}$$

We begin with the first and third factors. From [69, (0.5)] and the union bound, it follows that for any $\epsilon \in (0,1)$,

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_{\mathcal{B}(0;r(z))^c}\right) \le cN^d \exp\left\{-(1-\epsilon)r(z)\right\}, \tag{6.7.18}$$

$$\mathbb{P} \otimes \mathbf{P}_x\left(\tau_{\mathcal{O}} > \tau_{\mathcal{B}(x;r(x,z))^c}\right) \le cN^d \exp\left\{-(1-\epsilon)r(x,z)\right\}, \tag{6.7.19}$$

where cN^d is a crude upper bound on $|\partial\mathcal{B}(0;r(z))|$ and $|\partial\mathcal{B}(x;r(x,z))|$. For the second factor in (6.7.17), we use (6.7.12) and the local central limit theorem to obtain

$$\begin{aligned}\mathbf{P}\left(G(z;k,l) \mid S_k = x_1, S_l = x_2\right) &\le \mathbf{P}_{x_1}\left(\tau_{\mathcal{O}} > N - t_{\text{out}}(x,z) \mid S_{l-k} = x_2\right) \\ &\le N^c \mathbf{P}_{x_1}\left(\tau_{\mathcal{O}} > N - t_{\text{out}}(x,z)\right).\end{aligned} \tag{6.7.20}$$

We further add a piece of random walk loop satisfying

$$S_0 = S_{t_{\text{out}}(x,z)} = x_1, S_{[0,t_{\text{out}}(x,z)]} \subset B(z,(1-\delta_{N,x}^{c_{6.5.1}})\varrho_N) \tag{6.7.21}$$

in order to recover $\tau_{\mathcal{O}} > N$. Due to (6.7.2), the additional cost can be controlled by the random walk estimate (6.3.4) and the right-hand side of (6.7.20) is bounded by

$$N^c \exp\left\{c\delta_{N,x}^{c_{6.6.1}}(|z|_1 + |x-z|_1)\right\} \mathbf{P}_{x_1}\left(\tau_{\mathcal{O}} > N\right). \tag{6.7.22}$$

By the translation invariance of $\mathbb{P}$ and the union bound over $x_1 \in B(0,2N)$, it follows that

$$\begin{aligned}&\mathbb{P} \otimes \mathbf{P}\left(G(z;k,l) \mid S_k = x_1, S_l = x_2\right) \\ &\le N^c \exp\left\{C_{6,1}\delta_{N,x}^{c_{6.6.1}}(|z|_1 + |x-z|_1)\right\} \mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > N\right).\end{aligned} \tag{6.7.23}$$

Substituting (6.7.18), (6.7.19) and (6.7.23) into (6.7.17), we arrive at

$$\begin{aligned}&\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, \tau_{B^-} = k, S_k = x_1, G(z), \tau_{B^-}^{\leftarrow} = l,, S_l = x_2 \;\middle|\; \tau_{\mathcal{O}} > N\right) \\ &\le N^c \exp\left\{-(1-\epsilon)(r(z) + r(x,z)) + C_{6,1}\delta_{N,x}^{c_{6.6.1}}(|z|_1 + |x-z|_1)\right\}.\end{aligned} \tag{6.7.24}$$

Summing over $k, l \leq 2N$ and $x_1, x_2 \in B^-(z)$, and choosing ϵ so small that $C_{6,1}\delta_{N,x}^{c_{6.6.1}} \leq \delta_{N,x}^{c_{6.6.1}/2}$, we obtain (6.7.11). □

Proposition 6.7.1 measures not only the cost for the random walk to visit $B^-(z)$, but also the cost as z varies. In the following corollary, we use it to show that if $|x|$ is close to $2\varrho_N$, then z must be near $\frac{1}{2}x$. In addition, we show that the whole random walk path is confined in a ball slightly larger than B^+. This will be used in the proof of (6.1.12).

Corollary 6.7.2. *Let $h \in \mathbb{R}^d$ and $\mathbf{e}_h = h/|h|$. When $\epsilon > 0$ is small depending on d and p and $x \in B(2\varrho_N \mathbf{e}_h, \epsilon\varrho_N)$,*

$$\mu_{N,x}\left(\bigcup_{z \in B(\varrho_N \mathbf{e}_h, \epsilon^{1/4}\varrho_N)} G(z) \cap \left\{S_{[0,\tau_x^N]} \subset B(\varrho_N \mathbf{e}_h, (1+\epsilon^{1/5})\varrho_N)\right\}\right) \geq 1 - \exp\left\{-\tfrac{1}{4}(\log N)^2\right\} \tag{6.7.25}$$

for all sufficiently large N.

Proof. Note first that when $x \in B(2\varrho_N \mathbf{e}_h; \epsilon\varrho_N)$, Proposition 6.4.1 yields

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N \;\middle|\; \tau_{\mathcal{O}} > N\right) \geq \exp\left\{-c\epsilon\varrho_N\right\}. \tag{6.7.26}$$

Let us prove that we can discard $G(z)$ with z not close to $\varrho_N \mathbf{e}_h$. For any $x \in B(2\varrho_N \mathbf{e}_h, \epsilon\varrho_N)$ and $z \notin B(\varrho_N \mathbf{e}_h, \epsilon^{1/4}\varrho_N)$, we have

$$r(z) + r(x, z) \geq c\epsilon^{1/2}\varrho_N. \tag{6.7.27}$$

Substituting this into (6.7.11) and comparing with (6.7.26), we find that for any $z \notin B(\varrho_N \mathbf{e}_h; \epsilon^{1/4}\varrho_N)$,

$$\mu_{N,x}\left(G(z)\right) \leq \exp\left\{-c\epsilon^{1/2}\varrho_N\right\}. \tag{6.7.28}$$

Next we prove the confinement part. By Proposition 6.6.2 and what we have just proved, we may assume that

$$G(z) \text{ holds for some } z \in B(\varrho_N \mathbf{e}_h, \epsilon^{1/4}\varrho_N) \text{ and } S_{[\tau_{B^-(z)}, \tau^{\leftarrow}_{B^-(z)}]} \subset B^+(z). \tag{6.7.29}$$

Therefore, the random walk can exit $B(\varrho_N \mathbf{e}_h; (1+\epsilon^{1/5})\varrho_N)$ only during the time interval $[0, \tau_{B^-(z)}]$ or $[\tau^{\leftarrow}_{B^-(z)}, \tau_x^N]$. We first consider the former case. In this case, we use the strong Markov property at the first exit time from $B(\varrho_N \mathbf{e}_h; (1+\epsilon^{1/5})\varrho_N)$. Starting from the exit time, we repeat the proof of Proposition 6.7.1. Then the cost for the first piece of the random walk becomes the Lyapunov distance between $B(\varrho_N \mathbf{e}_h; (1+\epsilon^{1/5})\varrho_N)^c$ and $B^+(z)$, which is larger than $c\epsilon^{1/5}\varrho_N$. Thus it follows for any $z \in B(\varrho_N \mathbf{e}_h; \epsilon^{1/4}\varrho_N)$ that

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, G(z), S_{[0,\tau_{B^-}]} \not\subset B(\varrho_N \mathbf{e}_h, (1+\epsilon^{1/5})\varrho_N) \;\middle|\; \tau_{\mathcal{O}} > N\right) \\ &\quad \le \exp\left\{-c\epsilon^{1/5}\varrho_N\right\} \end{aligned} \tag{6.7.30}$$

and comparing with (6.7.26), we conclude that

$$\mu_{N,x}\left(G(z), S_{[0,\tau_{B^-}]} \not\subset B(\varrho_N \mathbf{e}_h, (1+\epsilon^{1/5})\varrho_N)\right) \le \exp\left\{-c\epsilon^{1/5}\varrho_N\right\}. \tag{6.7.31}$$

By the same argument, we get the same bound for the probability that the random walk exits from $B(\varrho_N \mathbf{e}_h; (1+\epsilon^{1/5})\varrho_N)$ during the time interval $[\tau^{\leftarrow}_{B^-(z)}, \tau_x^N]$ and we are done. □

Remark 6.7.3. As long as $|x| \le (2+\epsilon)\varrho_N$, we can follow the same argument as above to prove that the random walk path $S_{[0,N]}$ is confined in some $B(z; (1+\epsilon^{1/5})\varrho_N)$ which contains both 0 and x, and also $S_{[0,N]}$ covers a slightly smaller ball with the same center.

6.8 Proof of the upper bound in Theorem 6.1.4

In this section, we prove the following proposition, which in particular implies the upper bound in Theorem 6.1.4. Combined with the lower bound proved in Section 6.4, it completes the proof of Theorem 6.1.4.

Proposition 6.8.1. *There exists* $c_{6.8.1} > 0$ *such that when* $\epsilon > 0$ *is small depending on* d *and* p *and* $(2+\epsilon)\varrho_N \le |x| \le \epsilon \varrho_N^d$,

$$\mathbb{P} \otimes \mathbf{P} \left(S_N = x \mid \tau_{\mathcal{O}} > N \right) \le \exp\left\{ -(1-\epsilon^{c_{6.8.1}}) \mathrm{dist}_\beta(x, B(0, 2\varrho_N)) \right\} \tag{6.8.1}$$

for all sufficiently large N.

Proof. By the fact $\{S_N = x, \tau_{\mathcal{O}} > N\} \subset \{\tau_{\mathcal{O}} > \tau_x^N\}$ and (6.7.5), we have

$$\mathbb{P} \otimes \mathbf{P} \left(S_N = x, \tau_{\mathcal{O}} > N \right) \le (1+o(1)) \mathbb{P} \otimes \mathbf{P} \left(\tau_{\mathcal{O}} > \tau_x^N, \bigcup_{z \in B(0;2N)} G(z) \right) \tag{6.8.2}$$

as $N \to \infty$. Therefore, it suffices to bound $\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > \tau_x^N, \bigcup_{z \in B(0;2N)} G(z) \mid \tau_{\mathcal{O}} > N)$ by the right-hand side of (6.8.1). We can bound this probability by using Proposition 6.7.1 and the union bound as follows:

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P} \left(\tau_{\mathcal{O}} > \tau_x^N, \bigcup_{z \in B(0;2N)} G(z) \,\middle|\, \tau_{\mathcal{O}} > N \right) \\ &\quad \le \sum_{z \in B(0;2N)} \mathbb{P} \otimes \mathbf{P} \left(\tau_{\mathcal{O}} > \tau_x^N, G(z) \,\middle|\, \tau_{\mathcal{O}} > N \right) \\ &\quad \le c N^d \exp\left\{ - \inf_{z \in B(0,2N)} \left[(1-\epsilon)(r(z) + r(x,z)) - \delta_{N,x}^{c_{6.6.1}/2} (|z|_1 + |x-z|_1) + (\log N)^2 \right] \right\}. \end{aligned} \tag{6.8.3}$$

Let us first consider the case $|x| \le \varrho_N^{d-1/2}$ and prove that for any $z \in B(0, 2N)$,

$$r(z) + r(x,z) \ge \mathrm{dist}_\beta(x, B(0, 2\varrho_N)) + o(\varrho_N) \tag{6.8.4}$$

as $N \to \infty$. We may assume that $r(x,z) = \mathrm{dist}_\beta(x, B^+(z))$, that is, we are in the situation of the left picture in Figure 6.2. Otherwise, we can decrease the left-hand side of (6.8.4) by moving z to the point where $\mathcal{B}(0; r(z))$ touches $\mathcal{B}(x; r(x,z))$. Then for any $r > 0$, we have

$$\begin{aligned} &\min\left\{r(z) + \mathrm{dist}_\beta(x, B(z, \varrho_N)) \colon z \in B(0, 2N), r(z) = r\right\} \\ &\quad \geq \min\left\{\beta(u) + \beta(x - (u+v)) \colon u \in \partial\mathcal{B}(0; r), v \in B(0, 2(1 + \delta_{N,x}^{c_{6.5.4}/2}(\log N)^3))\varrho_N)\right\} \end{aligned} \tag{6.8.5}$$

by choosing u and $u+v$ so that $\beta(u) = \mathrm{dist}_\beta(0, B^+(z))$ and $\beta(x-(u+v)) = \mathrm{dist}_\beta(x, B^+(z))$, respectively. Since $\beta(\cdot)$ is a norm, the above is further bounded from below by

$$\min\left\{\beta(x - v) \colon v \in B(0, 2(1 + \delta_{N,x}^{c_{6.5.4}/2}(\log N)^3)\varrho_N)\right\} \geq \mathrm{dist}_\beta(x, B(0, 2\varrho_N)) + o(\varrho_N) \tag{6.8.6}$$

as $N \to \infty$, in the case $|x| \leq \varrho_N^{d-1/2}$. Since $r > 0$ was arbitrary, this proves (6.8.4).

Next, by the assumption $|x| \geq (2+\epsilon)\varrho_N$ and (6.8.4), we have $r(x) + r(x,z) \geq c\epsilon|x|$ and hence for $|z| \leq 2|x|$,

$$|z|_1 + |x - z|_1 \leq c\epsilon^{-1}(r(z) + r(x,z)). \tag{6.8.7}$$

This bound remains valid for $|z| > 2|x|$ since $r(z) \geq c|z|$ and $|z|_1 + |x-z|_1 \leq 3|z|$ in this case. Substituting (6.8.7) and (6.8.4) into (6.8.3), we obtain the desired bound since in the case $|x| \leq \varrho_N^{d-1/2}$, we have $\lim_{N\to\infty} \delta_{N,x} = 0$.

In the other case $|x| > \varrho_N^{d-1/2}$, it is easily seen that the size of $B^+(z)$ is negligible compared with $r(x) + r(x,z)$. Then it follows that

$$\begin{aligned} r(z) + r(x,z) &= (\beta(z) + \beta(x-z))(1 + o(1)) \\ &= \mathrm{dist}_\beta(x, B(0, 2\varrho_N))(1 + o(1)) \end{aligned} \tag{6.8.8}$$

and that

$$|z| + |x - z| \leq c\,(r(z) + r(x,z)) \tag{6.8.9}$$

as $N \to \infty$. Using this bound instead of (6.8.4), we can complete the proof as before. □

Remark 6.8.2. In the case $h = 0$, Theorem 6.1.4 shows that $\mu_N(|S_N| \le (2+\epsilon)\varrho_N) \to 1$ as $N \to \infty$. Combining this with Remark 6.7.3, we get a proof of (6.1.6).

6.9 Proof of Theorem 6.1.5

In this section we prove Theorem 6.1.5.

Proof of Theorem 6.1.5. Let us start by proving (6.1.13). We can deduce it from Proposition 6.8.1 and the large deviation results in [69, 70] by a standard exponential tilting argument [29, Theorem II.7.2]. But we provide a more direct argument which elucidates the role of the assumption $\beta^*(h) < 1$.

Let us recall that by [70, Theorem 2.2],

$$\lim_{N\to\infty} \mu_N^h \left(|S_N| \le \epsilon \varrho_N^d \right) = 1. \tag{6.9.1}$$

Choosing $x = 2\varrho_N \mathbf{e}_h$ in (6.4.1), we find the following lower bound on the partition function of μ_N^h:

$$\begin{aligned} \mathbb{E} \otimes \mathbf{E} \left[e^{\langle h, S_N \rangle} \colon \tau_{\mathcal{O}} > N \right] &\ge \mathbb{E} \otimes \mathbf{E} \left[e^{\langle h, S_N \rangle} \colon \tau_{\mathcal{O}} > N, S_N = 2\varrho_N \mathbf{e}_h \right] \\ &\ge e^{(2|h| - c_{6.4.1}\epsilon)\varrho_N} \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N). \end{aligned} \tag{6.9.2}$$

On the other hand, we have

$$\mathbb{E} \otimes \mathbf{E} \left[e^{\langle h, S_N \rangle} \colon \tau_{\mathcal{O}} > N, S_N = x \right] \le e^{(2|h| - c\epsilon^{1/2})\varrho_N} \mathbb{P} \otimes \mathbf{P} \left(\tau_{\mathcal{O}} > N \right) \tag{6.9.3}$$

for all sufficiently small $\epsilon > 0$ and $x \in B(0, 2N)$ satisfying either of the following conditions:

1. $\langle h, x \rangle < (2|h| - \epsilon^{1/2})\varrho_N$, in which case we simply drop the constraints $S_N = x$ and
2. $|x| > (2 + \epsilon^{1/2})\varrho_N$, in which case by Proposition 6.8.1, the subcriticality assumption

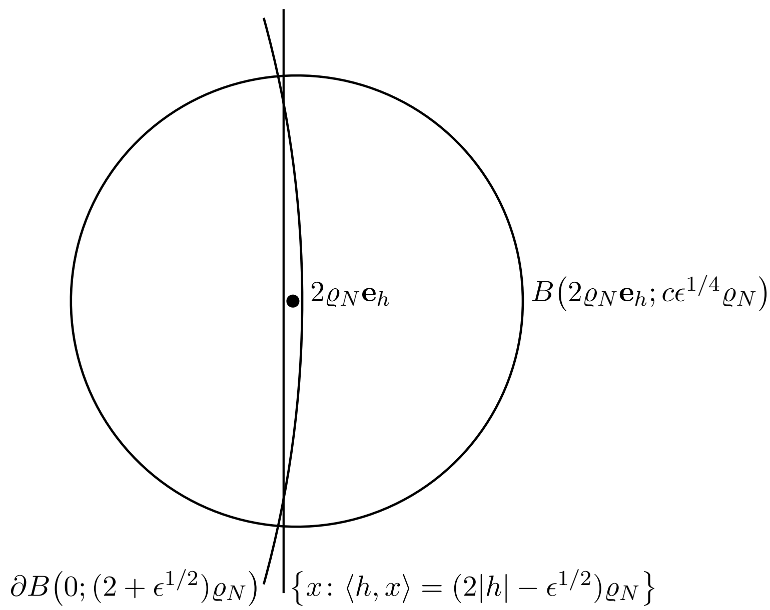

Figure 6.3: The balls and hyperplane appearing in the proof of (6.1.13).

$\beta^*(h) < 1$, and taking $y \in B(0, 2\varrho_N)$ such that $\beta(x-y) = \mathrm{dist}_\beta(x, B(0, 2\varrho_N))$,

$$\begin{aligned} \langle h, x\rangle - \mathrm{dist}_\beta(x, B(0, 2\varrho_N)) &\le 2\varrho_N|h| + \langle h, x-y\rangle - \beta(x-y) \\ &\le 2\varrho_N|h| + \beta(x-y)(\beta^*(h) - 1). \end{aligned} \tag{6.9.4}$$

Comparing (6.9.3) with (6.9.2) and summing over $x \in B(0, 2N)$, we obtain

$$\lim_{N\to\infty} \mu_N^h\left(\langle h, S_N\rangle \ge (2|h| - \epsilon^{1/2})\varrho_N \text{ and } |S_N| \le (2+\epsilon^{1/2})\varrho_N\right) = 1 \tag{6.9.5}$$

for sufficiently small $\epsilon > 0$. Since

$$\left\{x\colon \langle h, x\rangle \ge (2|h| - \epsilon^{1/2})\varrho_N\right\} \cap B(0, (2+\epsilon^{1/2})\varrho_N) \subset B(2\varrho_N\mathbf{e}_h; c\epsilon^{1/4}\varrho_N) \tag{6.9.6}$$

for small ϵ (see Figure 6.3), the proof of (6.1.13) is completed.

Next we turn to the proof of (6.1.12). Since we have already proved (6.1.13), we have

$$\mu_N^h(A) = \sum_{x \in B(2\varrho_N \mathbf{e}_h, \epsilon\varrho_N)} \mu_N^h(S_N = x, A) + o(1) \tag{6.9.7}$$

for any event A as $N \to \infty$. Let us now introduce the *pinned* measure

$$\widetilde{\mu}_{N,x}(\cdot) = \mathbb{P} \otimes \mathbf{P}(\cdot \mid S_N = x, \tau_{\mathcal{O}} > N). \tag{6.9.8}$$

We assume the following lemma for the moment.

Lemma 6.9.1. *There exists $c_{6.9.1} > 0$ such that for any $|x| \le 3\varrho_N$ and any event A,*

$$\widetilde{\mu}_{N,x}(A) \le N^{c_{6.9.1}} \mu_{N,x}(A) \tag{6.9.9}$$

for all sufficiently large N.

This lemma implies that the summand in (6.9.7) can be estimated as

$$\begin{aligned}
\mu_N^h(S_N = x, A) &= e^{\langle h,x\rangle} \widetilde{\mu}_{N,x}(A) \frac{\mathbb{P} \otimes \mathbf{P}(S_N = x, \tau_{\mathcal{O}} > N)}{\mathbb{E} \otimes \mathbf{E}\left[e^{\langle h,S_N\rangle} \colon \tau_{\mathcal{O}} > N\right]} \\
&\le e^{\langle h,x\rangle} \widetilde{\mu}_{N,x}(A) \frac{\mathbb{P} \otimes \mathbf{P}(S_N = x, \tau_{\mathcal{O}} > N)}{\mathbb{E} \otimes \mathbf{E}\left[e^{\langle h,S_N\rangle} \colon S_N = x, \tau_{\mathcal{O}} > N\right]} \\
&= \widetilde{\mu}_{N,x}(A) \\
&\le N^{c_{6.9.1}} \mu_{N,x}(A).
\end{aligned} \tag{6.9.10}$$

Now we choose A to be the event

$$\left(\bigcup_{z \in B(\varrho_N \mathbf{e}_h, \epsilon^{1/4}\varrho_N)} G(z) \cap \left\{ S_{[0,\tau_x^N]} \subset B(\varrho_N \mathbf{e}_h, (1+\epsilon^{1/5})\varrho_N) \right\} \right)^c \tag{6.9.11}$$

Since the $\mu_{N,x}$ probability of this event decays super-polynomially by Corollary 6.7.2, so does the left-hand side of (6.9.10). Coming back to (6.9.7) and recalling (6.7.3) in the definition of $G(z)$, we conclude that

$$\mu_N^h\left(B(\varrho_N\mathbf{e}_h,\left(1-\epsilon^{1/3}\right)\varrho_N)\subset S_{[0,N]}\subset B(\varrho_N\mathbf{e}_h,\left(1+\epsilon^{1/3}\right)\varrho_N)\right)\to 1 \tag{6.9.12}$$

as $N\to\infty$. Since this holds for all sufficiently small $\epsilon>0$, we complete the proof of (6.1.12).

□

Proof of Lemma 6.9.1. We are going to prove

$$\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>\tau_x^N\right)\le N^c\mathbb{P}\otimes\mathbf{P}\left(S_N=x,\tau_{\mathcal{O}}>N\right). \tag{6.9.13}$$

From this and $\{S_N=x,\tau_{\mathcal{O}}>N\}\subset\{\tau_{\mathcal{O}}>\tau_x^N\}$, we can deduce (6.9.9). The proof of (6.9.13) relies on a path switching argument: we show that a path with $\tau_{\mathcal{O}}>\tau_x^N$ can be shortened to satisfy $S_N=x$ and $\tau_{\mathcal{O}}>N$ without paying too much cost. To this end, let us define a good event by

$$G'=\Big\{\mathcal{O}\cap B(\boldsymbol{x}_N,(1-\delta_{N,x}^{c_{6.5.1}})\varrho_N)=\emptyset, \tag{6.9.14}$$

$$\tau_{B^-}\vee\left(\tau_x^N-\tau_{B^-}^{\leftarrow}\right)\le\epsilon N, \tag{6.9.15}$$

$$S_{[\tau_{B^-},\tau_{B^-}^{\leftarrow}]}\subset B^+, \tag{6.9.16}$$

$$\forall k\in[\tau_{B^-},\tau_{B^-}^{\leftarrow}-\varrho_N^2],S_{[k,k+\varrho_N^2]}\cap B^-\neq\emptyset.\Big\} \tag{6.9.17}$$

Under the assumption $|x|\le\epsilon\varrho_N^d$, by Propositions 6.5.1 , 6.6.1 and 6.6.2 and Corollary 6.6.4, we have

$$\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>\tau_x^N\right)=(1+o(1))\mathbb{P}\otimes\mathbf{P}\left(\tau_{\mathcal{O}}>\tau_x^N,G'\right) \tag{6.9.18}$$

as $N\to\infty$, and hence it suffices to bound the right-hand side.

Now suppose that $\tau_x^N=N+l<\tau_{\mathcal{O}}$ and G' holds, where we may assume $l\le N$ by Corollary 6.4.2. Then from (6.9.15) and (6.9.17), it follows that there exists $m\in[l+\varrho_N^2,l+2\varrho_N^2]$ such that $S_{\tau_{B^-}+m}\in B^-(z)$. We make a case distinction according to $\tau_{B^-}=n$

$(0 \leq n \leq \epsilon N)$ and use the Markov property at time n and $n+m$ to obtain

$$\begin{aligned} &\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N = N + l, G'\right) \\ &\quad \leq \sum_{n \leq \epsilon N} \sum_{m \in [l+\varrho_N^2, l+2\varrho_N^2]} \sum_{y,z \in B^-} \mathbb{E}\left[p_n^{\mathbb{Z}^d \setminus \mathcal{O}}(0,y) p_m^{B^+}(y,z) p_{N+l-m-n}^{\mathbb{Z}^d \setminus \mathcal{O}}(z,x) \colon (6.9.14)\right], \end{aligned} \tag{6.9.19}$$

where $p_n^U(x,y)$ stands for the transition probability of the random walk killed upon existing from U (see below (6.3.3)). We are going to shorten the time in $p_m^{B^+}(y,z)$. Since $p_m^{B^+}(y,z) \leq 1$, $p_{m-l}^{B^-}(y,z) \geq c\varrho_N^{-d-1}$ by [53, Proposition 6.9.4], and $B^- \subset \mathbb{Z}^d \setminus \mathcal{O}$ by (6.9.14), we have

$$\begin{aligned} p_m^{B^+}(y,z) &\leq N^c p_{m-l}^{B^-}(y,z) \\ &\leq N^c p_{m-l}^{\mathbb{Z}^d \setminus \mathcal{O}}(y,z). \end{aligned} \tag{6.9.20}$$

Substituting this into (6.9.19) and summing over $l \leq N$, we obtain

$$\mathbb{P} \otimes \mathbf{P}\left(\tau_{\mathcal{O}} > \tau_x^N, G'\right) \leq N^c \mathbb{P} \otimes \mathbf{P}\left(S_N = x, \tau_{\mathcal{O}} > N\right) \tag{6.9.21}$$

and we are done. □

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